量子场论 I

第十九课 QED计算技巧(二)

课件下载 https://github.com/zqhuang/SYSU_QFTI

γ矩阵的迹

这节课我们尝试来完成Compton散射的散射振幅的计算,为此我们先回顾下之前学过的有用的结论。

- 奇数个不包含γ^5的γ矩阵的乘积的迹为零
- $$\mathrm{Tr}\left(\not{u}_1\not{u}_2\right)=4u_1u_2$$
- $$\mathrm{Tr}\left(\not{u}_1\not{u}_2\not{u}_3\not{u}_4\right)=4(u_1u_2)(u_3u_4)-4(u_1u_3)(u_2u_4)+4(u_1u_4)(u_2u_3)$$

带形式下标的γ矩阵的性质

默认重复指标求和:

$$\gamma^\mu \gamma_\mu = 4$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu = 4g^{\alpha\beta}$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\rho \gamma_\mu = -2\gamma^\rho \gamma^\beta \gamma^\alpha$$

更常用的是这些式子的推论:

$$\gamma^\mu \not{A} \gamma_\mu = -2\not{A}$$

$$\gamma^\mu \not{A} \not{B} \gamma^\beta \gamma_\mu = 4AB$$

$$\gamma^\mu \not{A} \not{B} \not{C} \gamma_\mu = -2\not{C} \not{B} \not{A}$$

此外对只含γ矩阵乘积的等式，只要保持乘积的顺序不变，张量的指标升降规则仍然适用。例如

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

$$\gamma_\mu \gamma_\nu \gamma^\mu = -2\gamma_\nu$$

$$\gamma_\mu \gamma^\alpha \gamma_\beta \gamma^\mu = 4\delta^\alpha_\beta$$

Dirac费米子的自旋求和

$$\sum_{\text{spins}} u_p \bar{u}_p = \frac{\not{p} + m}{2\omega}$$

$$\sum_{\text{spins}} v_{-p} \bar{v}_{-p} = \frac{\not{p} - m}{2\omega}$$

Dirac费米子的四维动量的性质

设Dirac费米子的质量为m，四维动量为p，则有

$$p^2 = m^2$$

由此可推出

$$\not{p}(\not{p}+m) = m(\not{p}+m)$$

$$\not{p}(\not{p}-m) = -m(\not{p}-m)$$

$$(\not{p}+m)\gamma^\mu(\not{p}+m) = 2p^\mu(\not{p}+m)$$

$$(\not{p}-m)\gamma^\mu(\not{p}-m) = 2p^\mu(\not{p}-m)$$

光子的四维动量的性质

设光子四维动量为k，则有

$$k^2 = 0$$

由此可推出

$$\not{k}^2 = 0$$

$$\not{k}\gamma^\mu\not{k} = 2k^\mu\not{k}$$

Compton散射

我们把入射电子和光子的四维动量记为p, k，出射电子和光子的四维动量记为p', k'

$$
\begin{aligned}
|\mathcal{M}|^2 &= \frac{q^4}{16\omega_\gamma\omega'_\gamma}\sum_{\text{spins}}(\mathrm{e}_{k'})_\mu(\mathrm{e}^*_k)_\nu(\mathrm{e}^*_{k'})_\alpha(\mathrm{e}_k)_\beta \\
&\times\left(\bar{u}_{p'}\left(\gamma^\mu\frac{i}{\not{p}+\not{k}-m}\gamma^\nu+\gamma^\nu\frac{i}{\not{p}-\not{k}'-m}\gamma^\mu\right)u_p\right)^\dagger \\
&\times\left(\bar{u}_{p'}\left(\gamma^\alpha\frac{i}{\not{p}+\not{k}-m}\gamma^\beta+\gamma^\beta\frac{i}{\not{p}-\not{k}'-m}\gamma^\alpha\right)u_p\right) \\
&= \frac{q^4}{16\omega_\gamma\omega'_\gamma}\sum_{\text{spins}}(-g_{\alpha\mu})(-g_{\nu\beta})\left(\bar{u}_{p'}\left(\gamma^\mu\frac{i}{\not{p}+\not{k}-m}\gamma^\nu+\gamma^\nu\frac{i}{\not{p}-\not{k}'-m}\gamma^\mu\right)u_p\right)^\dagger \\
&\times\left(\bar{u}_{p'}\left(\gamma^\alpha\frac{i}{\not{p}+\not{k}-m}\gamma^\beta+\gamma^\beta\frac{i}{\not{p}-\not{k}'-m}\gamma^\alpha\right)u_p\right)
\end{aligned}
$$

Compton散射

利用$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$，以及$1\times 1$矩阵的迹等于自身，上式可写成

$$
\begin{aligned}
|\mathcal{M}|^2 &= \frac{q^4}{16\omega_\gamma\omega'_\gamma}\sum_{\text{spins}} \bar{u}_p\left(\gamma^\nu\frac{1}{\not p+\not k-m}\gamma^\mu+\gamma^\mu\frac{1}{\not p-\not k'-m}\gamma^\nu\right)u_{p'} \\
&\quad \times\bar{u}_{p'}\left(\gamma_\mu\frac{1}{\not p+\not k-m}\gamma_\nu+\gamma_\nu\frac{1}{\not p-\not k'-m}\gamma_\mu\right)u_p \\
&= \frac{q^4}{16\omega_\gamma\omega'_\gamma}\sum_{\text{spins}} \mathrm{Tr}\left[\bar{u}_p\left(\gamma^\nu\frac{1}{\not p+\not k-m}\gamma^\mu+\gamma^\mu\frac{1}{\not p-\not k'-m}\gamma^\nu\right)u_{p'}\right. \\
&\quad \left.\times\ \bar{u}_{p'}\left(\gamma_\mu\frac{1}{\not p+\not k-m}\gamma_\nu+\gamma_\nu\frac{1}{\not p-\not k'-m}\gamma_\mu\right)u_p\right] \\
&= \frac{q^4}{16\omega_\gamma\omega'_\gamma}\sum_{\text{spins}} \mathrm{Tr}\left[u_p\bar{u}_p\left(\gamma^\nu\frac{1}{\not p+\not k-m}\gamma^\mu+\gamma^\mu\frac{1}{\not p-\not k'-m}\gamma^\nu\right)u_{p'}\bar{u}_{p'}\right. \\
&\quad \left.\times\left(\gamma_\mu\frac{1}{\not p+\not k-m}\gamma_\nu+\gamma_\nu\frac{1}{\not p-\not k'-m}\gamma_\mu\right)\right]
\end{aligned}
$$

Compton散射

利用$u\bar{u}$的自旋求和性质，得到

$$|\mathcal{M}|^2 = \frac{q^4}{64\omega_\gamma \omega'_\gamma \omega_e \omega'_e} K$$

其中

$$\begin{aligned} K \quad = \quad & \mathrm{Tr}\left[(\not{p} + m) \left(\gamma^\nu \frac{1}{\not{p} + \not{k} - m} \gamma^\mu + \gamma^\mu \frac{1}{\not{p} - \not{k}' - m} \gamma^\nu \right) (\not{p}' + m) \right. \\ & \left. \times \left(\gamma_\mu \frac{1}{\not{p} + \not{k} - m} \gamma_\nu + \gamma_\nu \frac{1}{\not{p} - \not{k}' - m} \gamma_\mu \right) \right] \end{aligned}$$

Compton散射

利用$\not{A}^2 = A^2$, 以及$p^2 = p'^2 = m^2$, $k^2 = k'^2 = 0$, Feynman符号在分母的项可以化简为

$$\frac{1}{\not{p} + \not{k} - m} = \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2} = \frac{\not{p} + \not{k} + m}{2pk}$$

$$\frac{1}{\not{p} - \not{k}' - m} = \frac{\not{p} - \not{k}' + m}{(p-k')^2 - m^2} = -\frac{\not{p} - \not{k}' + m}{2pk'}$$

得到

$$\begin{aligned} K \quad &= \quad \mathrm{Tr}\left[(\not{p} + m)\left(\gamma^\nu \frac{\not{p} + \not{k} + m}{2pk}\gamma^\mu - \gamma^\mu \frac{\not{p} - \not{k}' + m}{2pk'}\gamma^\nu\right)(\not{p}' + m)\right. \\ &\quad \left. \times \left(\gamma_\mu \frac{\not{p} + \not{k} + m}{2pk}\gamma_\nu - \gamma_\nu \frac{\not{p} - \not{k}' + m}{2pk'}\gamma_\mu\right)\right] \end{aligned}$$

Compton散射

然后利用Dirac费米子的四维动量的性质，得到

$$
\begin{aligned}
K &= \mathrm{Tr}\left[(\not{p}+m)\left(\frac{\gamma^\nu \not{k}\gamma^\mu + 2\gamma^\mu p^\nu}{2pk} + \frac{\gamma^\mu \not{k}'\gamma^\nu - 2\gamma^\nu p^\mu}{2pk'}\right)(\not{p}'+m)\right. \\
&\quad \times \left.\left(\gamma_\mu \frac{\not{p}+\not{k}+m}{2pk}\gamma_\nu - \gamma_\nu \frac{\not{p}-\not{k}'+m}{2pk'}\gamma_\mu\right)\right]
\end{aligned}
$$

把$(\not{p}+m)$轮换到最后，再次利用Dirac费米子的四维动量的性质

$$
\begin{aligned}
K &= \mathrm{Tr}\left[\left(\frac{\gamma^\nu \not{k}\gamma^\mu + 2\gamma^\mu p^\nu}{2pk} + \frac{\gamma^\mu \not{k}'\gamma^\nu - 2\gamma^\nu p^\mu}{2pk'}\right)(\not{p}'+m)\right. \\
&\quad \times \left.\left(\frac{\gamma_\mu \not{k}\gamma_\nu + 2\gamma_\mu p_\nu}{2pk} + \frac{\gamma_\nu \not{k}'\gamma_\mu - 2\gamma_\nu p_\mu}{2pk'}\right)(\not{p}+m)\right]
\end{aligned}
$$

Compton散射

把K展成四项： $K = \frac{K_1}{4(pk)^2} + \frac{K_2}{4(pk)(pk')} + \frac{K_3}{4(pk)(pk')} + \frac{K_4}{4(pk')^2}$ 其中

$$
\begin{aligned}
K_1 &= \mathrm{Tr}\left[(\gamma^\nu \not{k}\gamma^\mu + 2\gamma^\mu p^\nu)(\not{p}' + m)(\gamma_\mu \not{k}\gamma_\nu + 2\gamma_\mu p_\nu)(\not{p} + m)\right] \\
K_2 &= \mathrm{Tr}\left[(\gamma^\nu \not{k}\gamma^\mu + 2\gamma^\mu p^\nu)(\not{p}' + m)(\gamma_\nu \not{k}'\gamma_\mu - 2\gamma_\nu p_\mu)(\not{p} + m)\right] \\
K_3 &= \mathrm{Tr}\left[(\gamma^\mu \not{k}'\gamma^\nu - 2\gamma^\nu p^\mu)(\not{p}' + m)(\gamma_\mu \not{k}\gamma_\nu + 2\gamma_\mu p_\nu)(\not{p} + m)\right] \\
K_4 &= \mathrm{Tr}\left[(\gamma^\mu \not{k}'\gamma^\nu - 2\gamma^\nu p^\mu)(\not{p}' + m)(\gamma_\nu \not{k}'\gamma_\mu - 2\gamma_\nu p_\mu)(\not{p} + m)\right]
\end{aligned}
$$

计算K_1

把K_1展开，因为是求迹运算，只须保留偶次项。

$$
\begin{aligned}
K_1 &= \mathrm{Tr}\left[(\gamma^\nu \not{k}\gamma^\mu + 2\gamma^\mu p^\nu)(\not{p}' + m)(\gamma_\mu \not{k}\gamma_\nu + 2\gamma_\mu p_\nu)(\not{p} + m)\right] \\
&= \mathrm{Tr}\left(\gamma^\nu \not{k}\gamma^\mu \not{p}'\gamma_\mu \not{k}\gamma_\nu \not{p}\right) + m^2\mathrm{Tr}\left(\gamma^\nu \not{k}\gamma^\mu \gamma_\mu \not{k}\gamma_\nu\right) + 4m^2\mathrm{Tr}\left(\gamma^\mu \not{p}'\gamma_\mu \not{p}\right) + 4m^4\mathrm{Tr}\left(\gamma^\mu \gamma_\mu\right) \\
&\quad +2\mathrm{Tr}\left(\gamma^\nu \not{k}\gamma^\mu \not{p}'\gamma_\mu p_\nu \not{p}\right) + 2m^2\mathrm{Tr}\left(\gamma^\nu \not{k}\gamma^\mu \gamma_\mu p_\nu\right) + 2\mathrm{Tr}\left(\gamma^\mu p^\nu \not{p}'\gamma_\mu \not{k}\gamma_\nu \not{p}\right) + 2m^2\mathrm{Tr}\left(\gamma^\mu p^\nu \gamma_\mu \not{k}\gamma_\nu\right) \\
&= -2\mathrm{Tr}\left(\gamma^\nu \not{k}\not{p}'\not{k}\gamma_\nu \not{p}\right) - 8m^2\mathrm{Tr}\left(\not{p}'\not{p}\right) + 64m^4 \\
&\quad -4\mathrm{Tr}\left(\not{p}\not{k}\not{p}'\not{p}\right) + 8m^2\mathrm{Tr}\left(\not{p}\not{k}\right) - 4\mathrm{Tr}\left(\not{p}'\not{k}\not{p}\not{p}\right) + 8m^2\mathrm{Tr}\left(\not{k}\not{p}\right) \\
&= 4\mathrm{Tr}\left(\not{k}\not{p}'\not{k}\not{p}\right) - 32m^2(pp') + 64m^4 - 4m^2\mathrm{Tr}\left(\not{k}\not{p}'\right) + 16m^2\mathrm{Tr}\left(\not{p}\not{k}\right) - 4m^2\mathrm{Tr}\left(\not{p}'\not{k}\right) \\
&= 32(pk)(p'k) - 32m^2(pp') + 64m^4 + 64m^2(pk) - 32m^2(p'k) \\
&= 32(pk)(p'k) + 32m^4 + 32m^2(pk)
\end{aligned}
$$

计算K_4

只要把k和$-k'$互换即得到K_4

$$K_4 = 32(pk')(p'k') + 32m^4 - 32m^2(pk')$$

计算K_2

$$
\begin{aligned}
K_2 &= \mathrm{Tr}\left[(\gamma^\nu \not{k} \gamma^\mu + 2\gamma^\mu p^\nu)(\not{p}' + m)(\gamma_\nu \not{k}' \gamma_\mu - 2\gamma_\nu p_\mu)(\not{p} + m)\right] \\
&= \mathrm{Tr}\left(\gamma^\nu \not{k} \gamma^\mu \not{p}' \gamma_\nu \not{k}' \gamma_\mu \not{p}\right) + m^2 \mathrm{Tr}\left(\gamma^\nu \not{k} \gamma^\mu \gamma_\nu \not{k}' \gamma_\mu\right) - 4\mathrm{Tr}\left(\gamma^\mu p^\nu \not{p}' \gamma_\nu p_\mu \not{p}\right) - 4m^2 \mathrm{Tr}\left(\gamma^\mu p^\nu \gamma_\nu p_\mu\right) \\
&\quad -2\mathrm{Tr}\left(\gamma^\nu \not{k} \gamma^\mu \not{p}' \gamma_\nu p_\mu \not{p}\right) - 2m^2 \mathrm{Tr}\left(\gamma^\nu \not{k} \gamma^\mu \gamma_\nu p_\mu\right) + 2\mathrm{Tr}\left(\gamma^\mu p^\nu \not{p}' \gamma_\nu \not{k}' \gamma_\mu \not{p}\right) + 2m^2 \mathrm{Tr}\left(\gamma^\mu p^\nu \gamma_\nu \not{k}' \gamma_\mu\right) \\
&= -2\mathrm{Tr}\left(\not{p}' \gamma^\mu \not{k} \not{k}' \gamma_\mu \not{p}\right) + 4m^2 \mathrm{Tr}\left(k^\mu \not{k}' \gamma_\mu\right) - 4\mathrm{Tr}\left(\not{p} \not{p}' \not{p} \not{p}\right) - 4m^2 \mathrm{Tr}\left(\not{p}^2\right) \\
&\quad +4\mathrm{Tr}\left(\not{p}' \gamma^\mu \not{k} p_\mu \not{p}\right) - 8m^2 \mathrm{Tr}\left(k^\mu p_\mu\right) + 2\mathrm{Tr}\left(\gamma^\mu \not{p}' \not{p} \not{k}' \gamma_\mu \not{p}\right) + 8m^2 \mathrm{Tr}\left(\not{p} \not{k}'\right) \\
&= -8(kk')\mathrm{Tr}\left(\not{p}' \not{p}\right) + 4m^2 \mathrm{Tr}\left(\not{k}' \not{k}\right) - 4m^2 \mathrm{Tr}\left(\not{p} \not{p}'\right) - 16m^4 \\
&\quad +4\mathrm{Tr}\left(\not{p}' \not{p} \not{k} \not{p}\right) - 32m^2 (pk) - 4\mathrm{Tr}\left(\not{k}' \not{p} \not{p}' \not{p}\right) + 32m^2 (pk') \\
&= -32(kk')(pp') + 16m^2(kk') - 16m^2(pp') - 16m^4 \\
&\quad +32(pp')(pk) - 16m^2(p'k) - 32m^2(pk) - 32(pp')(pk') + 16m^2(p'k') + 32m^2(pk') \\
&= 32(pp')(pk) - 32(kk')(pp') - 32(pp')(pk') - 32m^4 - 16m^2(pk) + 16m^2(pk') \\
&= -32m^4 - 16m^2(pk) + 16m^2(pk')
\end{aligned}
$$

计算K_3

同样，在K_2的结果中把k和$-k'$互换即得到K_3

$$K_3 = -32m^4 + 16m^2(pk') - 16m^2(pk) = K_2$$

总和

最后总和为

$$K = 8\left[\frac{pk'}{pk} + \frac{pk}{pk'} + 2m^2\left(\frac{1}{pk} - \frac{1}{pk'}\right) + m^4\left(\frac{1}{pk} - \frac{1}{pk'}\right)^2\right]$$

即

$$|\mathcal{M}|^2 = \frac{q^4}{8\omega_p\omega'_p\omega_k\omega'_k}\left[\frac{pk'}{pk} + \frac{pk}{pk'} + 2m^2\left(\frac{1}{pk} - \frac{1}{pk'}\right) + m^4\left(\frac{1}{pk} - \frac{1}{pk'}\right)^2\right]$$