

量子场论 I

第十八课 QED计算技巧(二)

课件下载 https://github.com/zqhuang/SYSU_QFTI

γ 矩阵的迹

这节课我们尝试来完成Compton散射的散射振幅的计算,为此我们先回顾下之前学过的有用的结论。

- ▶ 奇数个不包含 γ^5 的 γ 矩阵的乘积的迹为零



$$\text{Tr}(\psi_1 \psi_2) = 4u_1 u_2$$



$$\text{Tr}(\psi_1 \psi_2 \psi_3 \psi_4) = 4(u_1 u_2)(u_3 u_4) - 4(u_1 u_3)(u_2 u_4) + 4(u_1 u_4)(u_2 u_3)$$

带形式下标的 γ 矩阵的性质

默认重复指标求和:

$$\gamma^\mu \gamma_\mu = 4$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu = 4g^{\alpha\beta}$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\rho \gamma_\mu = -2\gamma^\rho \gamma^\beta \gamma^\alpha$$

更常用的是这些式子的推论:

$$\gamma^\mu \not{A} \gamma_\mu = -2\not{A}$$

$$\gamma^\mu \not{A} \not{B} \gamma^\beta \gamma_\mu = 4AB$$

$$\gamma^\mu \not{A} \not{B} \not{C} \gamma_\mu = -2\not{C} \not{B} \not{A}$$

此外对只含 γ 矩阵乘积的等式, 只要保持乘积的顺序不变, 张量的指标升降规则仍然适用。例如

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

$$\gamma_\mu \gamma_\nu \gamma^\mu = -2\gamma_\nu$$

$$\gamma_\mu \gamma^\alpha \gamma_\beta \gamma^\mu = 4\delta_\beta^\alpha$$

Dirac费米子的自旋求和

$$\sum_{\text{spins}} u_p \bar{u}_p = \frac{\not{p} + m}{2\omega}$$

$$\sum_{\text{spins}} v_{-p} \bar{v}_{-p} = \frac{\not{p} - m}{2\omega}$$

Dirac费米子的四维动量的性质

设Dirac费米子的质量为 m ，四维动量为 p ，则有

$$p^2 = m^2$$

由此可推出

$$\not{p}(\not{p} + m) = m(\not{p} + m)$$

$$\not{p}(\not{p} - m) = -m(\not{p} - m)$$

$$(\not{p} + m)\gamma^\mu(\not{p} + m) = 2p^\mu(\not{p} + m)$$

$$(\not{p} - m)\gamma^\mu(\not{p} - m) = 2p^\mu(\not{p} - m)$$

光子的四维动量的性质

设光子四维动量为 k ，则有

$$k^2 = 0$$

由此可推出

$$\not{k}^2 = 0$$

$$\not{k} \gamma^\mu \not{k} = 2k^\mu \not{k}$$

Compton 散射

我们把入射电子和光子的四维动量记为 p, k , 出射电子和光子的四维动量记为 p', k'

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{q^4}{16\omega_\gamma \omega'_\gamma} \sum_{\text{spins}} (e_{k'})_\mu (e_k^*)_\nu (e_{k'}^*)_\alpha (e_k)_\beta \\ &\quad \times \left(\bar{u}_{p'} \left(\gamma^\mu \frac{i}{\not{p} + \not{k} - m} \gamma^\nu + \gamma^\nu \frac{i}{\not{p} - \not{k}' - m} \gamma^\mu \right) u_p \right)^\dagger \\ &\quad \times \left(\bar{u}_{p'} \left(\gamma^\alpha \frac{i}{\not{p} + \not{k} - m} \gamma^\beta + \gamma^\beta \frac{i}{\not{p} - \not{k}' - m} \gamma^\alpha \right) u_p \right) \\ &= \frac{q^4}{16\omega_\gamma \omega'_\gamma} \sum_{\text{spins}} (-g_{\alpha\mu})(-g_{\nu\beta}) \left(\bar{u}_{p'} \left(\gamma^\mu \frac{i}{\not{p} + \not{k} - m} \gamma^\nu + \gamma^\nu \frac{i}{\not{p} - \not{k}' - m} \gamma^\mu \right) u_p \right)^\dagger \\ &\quad \times \left(\bar{u}_{p'} \left(\gamma^\alpha \frac{i}{\not{p} + \not{k} - m} \gamma^\beta + \gamma^\beta \frac{i}{\not{p} - \not{k}' - m} \gamma^\alpha \right) u_p \right) \end{aligned}$$

Compton 散射

利用 $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$, 以及 1×1 矩阵的迹等于自身, 上式可写成

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{q^4}{16\omega_\gamma \omega'_\gamma} \sum_{\text{spins}} \bar{u}_p \left(\gamma^\nu \frac{1}{\not{p} + \not{k} - m} \gamma^\mu + \gamma^\mu \frac{1}{\not{p} - \not{k}' - m} \gamma^\nu \right) u_{p'} \\ &\quad \times \bar{u}_{p'} \left(\gamma_\mu \frac{1}{\not{p} + \not{k} - m} \gamma_\nu + \gamma_\nu \frac{1}{\not{p} - \not{k}' - m} \gamma_\mu \right) u_p \\ &= \frac{q^4}{16\omega_\gamma \omega'_\gamma} \sum_{\text{spins}} \text{Tr} \left[\bar{u}_p \left(\gamma^\nu \frac{1}{\not{p} + \not{k} - m} \gamma^\mu + \gamma^\mu \frac{1}{\not{p} - \not{k}' - m} \gamma^\nu \right) u_{p'} \right. \\ &\quad \times \left. \bar{u}_{p'} \left(\gamma_\mu \frac{1}{\not{p} + \not{k} - m} \gamma_\nu + \gamma_\nu \frac{1}{\not{p} - \not{k}' - m} \gamma_\mu \right) u_p \right] \\ &= \frac{q^4}{16\omega_\gamma \omega'_\gamma} \sum_{\text{spins}} \text{Tr} \left[u_p \bar{u}_p \left(\gamma^\nu \frac{1}{\not{p} + \not{k} - m} \gamma^\mu + \gamma^\mu \frac{1}{\not{p} - \not{k}' - m} \gamma^\nu \right) u_{p'} \bar{u}_{p'} \right. \\ &\quad \times \left. \left(\gamma_\mu \frac{1}{\not{p} + \not{k} - m} \gamma_\nu + \gamma_\nu \frac{1}{\not{p} - \not{k}' - m} \gamma_\mu \right) \right] \end{aligned}$$

Compton 散射

利用 $u\bar{u}$ 的自旋求和性质，得到

$$|\mathcal{M}|^2 = \frac{q^4}{64\omega_\gamma\omega'_\gamma\omega_e\omega'_e} K$$

其中

$$\begin{aligned} K = & \text{Tr} \left[(\not{p} + m) \left(\gamma^\nu \frac{1}{\not{p} + \not{k} - m} \gamma^\mu + \gamma^\mu \frac{1}{\not{p} - \not{k}' - m} \gamma^\nu \right) (\not{p}' + m) \right. \\ & \left. \times \left(\gamma_\mu \frac{1}{\not{p} + \not{k} - m} \gamma_\nu + \gamma_\nu \frac{1}{\not{p} - \not{k}' - m} \gamma_\mu \right) \right] \end{aligned}$$

Compton 散射

利用 $\not{A}^2 = A^2$, 以及 $p^2 = p'^2 = m^2$, $k^2 = k'^2 = 0$, Feynman 符号在分母的项可以化简为

$$\frac{1}{\not{p} + \not{k} - m} = \frac{\not{p} + \not{k} + m}{(p + k)^2 - m^2} = \frac{\not{p} + \not{k} + m}{2pk}$$

$$\frac{1}{\not{p} - \not{k}' - m} = \frac{\not{p} - \not{k}' + m}{(p - k')^2 - m^2} = -\frac{\not{p} - \not{k}' + m}{2pk'}$$

得到

$$\begin{aligned} K &= \text{Tr} \left[(\not{p} + m) \left(\gamma^\nu \frac{\not{p} + \not{k} + m}{2pk} \gamma^\mu - \gamma^\mu \frac{\not{p} - \not{k}' + m}{2pk'} \gamma^\nu \right) (\not{p}' + m) \right. \\ &\quad \left. \times \left(\gamma_\mu \frac{\not{p} + \not{k} + m}{2pk} \gamma_\nu - \gamma_\nu \frac{\not{p} - \not{k}' + m}{2pk'} \gamma_\mu \right) \right] \end{aligned}$$

Compton散射

然后利用Dirac费米子的四维动量的性质，得到

$$\begin{aligned} K = & \text{Tr} \left[(\not{p} + m) \left(\frac{\gamma^\nu \not{k} \gamma^\mu + 2\gamma^\mu p^\nu}{2pk} + \frac{\gamma^\mu \not{k}' \gamma^\nu - 2\gamma^\nu p^\mu}{2pk'} \right) (\not{p}' + m) \right. \\ & \left. \times \left(\gamma_\mu \frac{\not{p} + \not{k} + m}{2pk} \gamma_\nu - \gamma_\nu \frac{\not{p} - \not{k}' + m}{2pk'} \gamma_\mu \right) \right] \end{aligned}$$

把 $(\not{p} + m)$ 轮换到最后，再次利用Dirac费米子的四维动量的性质

$$\begin{aligned} K = & \text{Tr} \left[\left(\frac{\gamma^\nu \not{k} \gamma^\mu + 2\gamma^\mu p^\nu}{2pk} + \frac{\gamma^\mu \not{k}' \gamma^\nu - 2\gamma^\nu p^\mu}{2pk'} \right) (\not{p}' + m) \right. \\ & \left. \times \left(\frac{\gamma_\mu \not{k} \gamma_\nu + 2\gamma_\mu p_\nu}{2pk} + \frac{\gamma_\nu \not{k}' \gamma_\mu - 2\gamma_\nu p_\mu}{2pk'} \right) (\not{p} + m) \right] \end{aligned}$$

Compton 散射

把 K 展成四项: $K = \frac{K_1}{4(pk)^2} + \frac{K_2}{4(pk)(pk')} + \frac{K_3}{4(pk)(pk')} + \frac{K_4}{4(pk')^2}$ 其中

$$K_1 = \text{Tr} [(\gamma^\nu \not{k} \gamma^\mu + 2\gamma^\mu p^\nu)(\not{p}' + m)(\gamma_\mu \not{k} \gamma_\nu + 2\gamma_\mu p_\nu)(\not{p} + m)]$$

$$K_2 = \text{Tr} [(\gamma^\nu \not{k} \gamma^\mu + 2\gamma^\mu p^\nu)(\not{p}' + m)(\gamma_\nu \not{k}' \gamma_\mu - 2\gamma_\nu p_\mu)(\not{p} + m)]$$

$$K_3 = \text{Tr} [(\gamma^\mu \not{k}' \gamma^\nu - 2\gamma^\nu p^\mu)(\not{p}' + m)(\gamma_\mu \not{k} \gamma_\nu + 2\gamma_\mu p_\nu)(\not{p} + m)]$$

$$K_4 = \text{Tr} [(\gamma^\mu \not{k}' \gamma^\nu - 2\gamma^\nu p^\mu)(\not{p}' + m)(\gamma_\nu \not{k}' \gamma_\mu - 2\gamma_\nu p_\mu)(\not{p} + m)]$$

计算 K_1

把 K_1 展开，因为是求迹运算，只须保留偶次项。

$$\begin{aligned} K_1 &= \text{Tr} \left[(\gamma^\nu \not{k} \gamma^\mu + 2\gamma^\mu p^\nu)(\not{p}' + m)(\gamma_\mu \not{k} \gamma_\nu + 2\gamma_\mu p_\nu)(\not{p} + m) \right] \\ &= \text{Tr} \left(\gamma^\nu \not{k} \gamma^\mu \not{p}' \gamma_\mu \not{k} \gamma_\nu \not{p} \right) + m^2 \text{Tr} \left(\gamma^\nu \not{k} \gamma^\mu \gamma_\mu \not{k} \gamma_\nu \right) + 4m^2 \text{Tr} \left(\gamma^\mu \not{p}' \gamma_\mu \not{p} \right) + 4m^4 \text{Tr} \left(\gamma^\mu \gamma_\mu \right) \\ &\quad + 2\text{Tr} \left(\gamma^\nu \not{k} \gamma^\mu \not{p}' \gamma_\mu p_\nu \not{p} \right) + 2m^2 \text{Tr} \left(\gamma^\nu \not{k} \gamma^\mu \gamma_\mu p_\nu \right) + 2\text{Tr} \left(\gamma^\mu p^\nu \not{p}' \gamma_\mu \not{k} \gamma_\nu \not{p} \right) + 2m^2 \text{Tr} \left(\gamma^\mu p^\nu \gamma_\mu \not{k} \gamma_\nu \right) \\ &= -2\text{Tr} \left(\gamma^\nu \not{k} \not{p}' \not{k} \gamma_\nu \not{p} \right) - 8m^2 \text{Tr} \left(\not{p}' \not{p} \right) + 64m^4 \\ &\quad - 4\text{Tr} \left(\not{p} \not{k} \not{p}' \not{p} \right) + 8m^2 \text{Tr} \left(\not{p} \not{k} \right) - 4\text{Tr} \left(\not{p}' \not{k} \not{p} \not{p} \right) + 8m^2 \text{Tr} \left(\not{k} \not{p} \right) \\ &= 4\text{Tr} \left(\not{k} \not{p}' \not{k} \not{p} \right) - 32m^2 (pp') + 64m^4 - 4m^2 \text{Tr} \left(\not{k} \not{p}' \right) + 16m^2 \text{Tr} \left(\not{p} \not{k} \right) - 4m^2 \text{Tr} \left(\not{p}' \not{k} \right) \\ &= 32(pk)(p'k) - 32m^2(pp') + 64m^4 + 64m^2(pk) - 32m^2(p'k) \\ &= 32(pk)(p'k) + 32m^4 + 32m^2(pk) \end{aligned}$$

计算 K_4

只要把 k 和 $-k'$ 互换即得到 K_4

$$K_4 = 32(pk')(p'k') + 32m^4 - 32m^2(pk')$$

计算 K_2

$$\begin{aligned}
 K_2 &= \text{Tr} \left[(\gamma^\nu \not{k} \gamma^\mu + 2\gamma^\mu p^\nu)(\not{p}' + m)(\gamma_\nu \not{k}' \gamma_\mu - 2\gamma_\nu p_\mu)(\not{p} + m) \right] \\
 &= \text{Tr} \left(\gamma^\nu \not{k} \gamma^\mu \not{p}' \gamma_\nu \not{k}' \gamma_\mu \not{p} \right) + m^2 \text{Tr} \left(\gamma^\nu \not{k} \gamma^\mu \gamma_\nu \not{k}' \gamma_\mu \right) - 4\text{Tr} \left(\gamma^\mu p^\nu \not{p}' \gamma_\nu p_\mu \not{p} \right) - 4m^2 \text{Tr} \left(\gamma^\mu p^\nu \gamma_\nu p_\mu \right) \\
 &\quad - 2\text{Tr} \left(\gamma^\nu \not{k} \gamma^\mu \not{p}' \gamma_\nu p_\mu \not{p} \right) - 2m^2 \text{Tr} \left(\gamma^\nu \not{k} \gamma^\mu \gamma_\nu p_\mu \right) + 2\text{Tr} \left(\gamma^\mu p^\nu \not{p}' \gamma_\nu \not{k}' \gamma_\mu \not{p} \right) + 2m^2 \text{Tr} \left(\gamma^\mu p^\nu \gamma_\nu \not{k}' \gamma_\mu \right) \\
 &= -2\text{Tr} \left(\not{p}' \gamma^\mu \not{k} \not{k}' \gamma_\mu \not{p} \right) + 4m^2 \text{Tr} \left(k^\mu \not{k}' \gamma_\mu \right) - 4\text{Tr} \left(\not{p} \not{p}' \not{p} \not{p} \right) - 4m^2 \text{Tr} \left(\not{p}^2 \right) \\
 &\quad + 4\text{Tr} \left(\not{p}' \gamma^\mu \not{k} p_\mu \not{p} \right) - 8m^2 \text{Tr} \left(k^\mu p_\mu \right) + 2\text{Tr} \left(\gamma^\mu \not{p}' \not{p} \not{k}' \gamma_\mu \not{p} \right) + 8m^2 \text{Tr} \left(\not{p} \not{k}' \right) \\
 &= -8(kk') \text{Tr} \left(\not{p}' \not{p} \right) + 4m^2 \text{Tr} \left(\not{k}' \not{k} \right) - 4m^2 \text{Tr} \left(\not{p} \not{p}' \right) - 16m^4 \\
 &\quad + 4\text{Tr} \left(\not{p}' \not{p} \not{k} \not{p} \right) - 32m^2(pk) - 4\text{Tr} \left(\not{k}' \not{p} \not{p}' \not{p} \right) + 32m^2(pk') \\
 &= -32(kk')(pp') + 16m^2(kk') - 16m^2(pp') - 16m^4 \\
 &\quad + 32(pp')(pk) - 16m^2(p'k) - 32m^2(pk) - 32(pp')(pk') + 16m^2(p'k') + 32m^2(pk') \\
 &= 32(pp')(pk) - 32(kk')(pp') - 32(pp')(pk') - 32m^4 - 16m^2(pk) + 16m^2(pk') \\
 &= -32m^4 - 16m^2(pk) + 16m^2(pk')
 \end{aligned}$$

计算 K_3

同样，在 K_2 的结果中把 k 和 $-k'$ 互换即得到 K_3

$$K_3 = -32m^4 + 16m^2(pk') - 16m^2(pk) = K_2$$

总和

最后总和为

$$K = 8 \left[\frac{pk'}{pk} + \frac{pk}{pk'} + 2m^2 \left(\frac{1}{pk} - \frac{1}{pk'} \right) + m^4 \left(\frac{1}{pk} - \frac{1}{pk'} \right)^2 \right]$$

即

$$|\mathcal{M}|^2 = \frac{q^4}{8\omega_p\omega'_p\omega_k\omega'_k} \left[\frac{pk'}{pk} + \frac{pk}{pk'} + 2m^2 \left(\frac{1}{pk} - \frac{1}{pk'} \right) + m^4 \left(\frac{1}{pk} - \frac{1}{pk'} \right)^2 \right]$$