Gravitational Wave Lensing Magnification Simulation

QK Z

March 2025

1 Introduction

Gravitational wave lensing occurs when gravitational waves propagate near massive astrophysical objects, causing amplification and distortion due to gravitational lensing effects. The lensing phenomenon in wave optics regimes arises when gravitational wave wavelengths become comparable to or larger than the characteristic size of the lensing mass distribution, requiring a wave-optical treatment rather than geometric optics.

For a point mass lens scenario, the lensing effect is quantified by the amplification factor $F(\omega, y)$, derived from solving the wave equation in curved spacetime. This factor characterizes how gravitational waves are magnified or distorted by lenses such as black holes or dense stellar objects. Precisely calculating this amplification is crucial for detecting and interpreting gravitational wave events observed by current (LIGO and Virgo) and future third-generation detectors like the Einstein Telescope and Cosmic Explorer.

In this work, we focus on the amplification factor $F(\omega, y)$ for a point mass lens, specifically leveraging the confluent hypergeometric function as defined by Takahashi and Nakamura. Our objective is to construct a Physics-Informed Neural Network (PINN) that accurately models this hypergeometric function and the related amplification, enabling efficient predictions across various parameters without repeatedly solving complex integral equations.

2 Background

2.1 Gravitational Waves

Gravitational waves are ripples in the fabric of spacetime generated by accelerated massive astrophysical objects. Examples include the coalescence of black holes, neutron star collisions, and other highly energetic cosmic events. Since their first direct detection by the Laser Interferometer Gravitational-Wave Observatory (LIGO), gravitational waves have become a powerful tool for studying compact objects and for testing General Relativity in strong-field regimes.

2.2 Gravitational Lensing Effect

Gravitational lensing refers to the phenomenon in which the gravitational field of a massive object bends the surrounding spacetime, thereby deflecting the path of light or gravitational waves passing near it. Analogous to an optical lens, a gravitational lens can magnify and distort signals, and, under suitable conditions, it can create multiple observable copies of an event.

2.2.1 Stellar Mass Lenses

Stellar mass lenses are objects with masses on the order of one solar mass () up to tens of solar masses (e.g., black holes or neutron stars). When a gravitational wave passes near such a lens, its wavefront experiences:

- **Signal Amplification**: The lens can transiently increase the observed amplitude of the gravitational wave.
- **Time Delay**: The different paths of the wave through curved spacetime arrive at the detector at slightly different times.
- **Waveform Splitting**: If the lens is sufficiently massive and aligned, multiple, resolvable waveforms (copies) of the signal may form.

2.2.2 Multiple Lensing Bodies

When a gravitational wave traverses a densely populated region (e.g., a globular cluster or galactic center), multiple massive objects may act in succession as lenses. The net effect can be:

- **Cumulative Amplification**: Several lenses can add their individual lensing effects, further magnifying (or distorting) the signal.
- **Feature Confusion**: The signal distortion from each lens can combine to mask or complicate the original gravitational wave signature, motivating advanced numerical or machine-learning-based techniques to recover the underlying physical information.

3 Magnification Definition

The magnification (or amplification) factor describes how the lens modifies the amplitude and phase of an unlensed gravitational wave. Let denote the frequency-domain representation of the original signal in the absence of a lens. When a lens is present, the observed (lensed) signal can be written as:

$$\tilde{h}\mathrm{lensed}(\omega) = F(\omega), \tilde{h}\mathrm{unlensed}(\omega), \tag{1}$$

where is the *complex* magnification factor. Its modulus represents amplitude magnification (or attenuation), and its argument encodes the additional phase shift introduced by lensing.

4 Derivation of the Point-Mass Formula (Equation 10)

The starting point for gravitational-wave lensing in the wave-optics regime is the perturbed Friedmann-Lemaître-Robertson-Walker metric (FLRW). When ignoring cosmological expansion details and focusing on a local, thin-lens approximation, one solves the wave equation:

$$\left(\nabla^2 + \omega^2\right) \tilde{h} = 16\pi\rho$$

where $\tilde{h}$ is the frequency-domain metric perturbation and ρ is the energy density.

$$\nabla^2 \tilde{\phi} + \omega^2 \tilde{\phi} = 4\omega^2 \tilde{U} \tilde{\phi} \tag{2}$$

where is the Newtonian gravitational potential of the lens, and is the scalar part of the gravitational waveform (satisfying linearized General Relativity conditions).

4.1 Fresnel-Kirchhoff Integral

Under the thin-lens and wave-optics assumptions, one arrives at the Fresnel-Kirchhoff diffraction integral:

$$F(\omega, y) = \frac{\omega}{2\pi i} \int d^2x; \exp\left[i, \omega, T(x, y)\right], \tag{3}$$

where is the time-delay function that depends on the lensing potential. The variables are typically nondimensionalized by choosing characteristic scales such as the Einstein radius, ensuring dimensionless frequency and dimensionless impact parameter .

4.2 Point-Mass Potential

For a point-mass lens of mass , placed at the origin of the lens plane, the lensing potential can be expressed as . By substituting this potential into the integral and evaluating it (often via stationary phase or related integral techniques), one obtains an analytic expression for the magnification factor in terms of the confluent hypergeometric function : The amplification factor $F(\omega, y)$ is given by:

$$F(\omega, y) = e^{i\phi_m} \Gamma(1 - i\omega/2)\ {}_1F_1(i\omega/2, 1, i\omega y^2/2), \tag{4}$$

where:

- ω is the dimensionless frequency,
- y is the impact parameter,
- ${}_1F_1(a, b, z)$ is the confluent hypergeometric function,

- ϕ_m is a phase term given by:

$$\phi_m = \frac{(x_m - y)^2}{2} - \ln x_m, \tag{5}$$

where $x_m = \frac{y+\sqrt{y^2+4}}{2}$.

This is often quoted as in many references on wave-optics gravitational lensing. It succinctly captures the lensing amplification for a single point-mass lens, showing how the frequency and dimensionless impact parameter dictate the magnitude and phase of the wave.

5 Physics-Informed Neural Network (PINN)

Our PINN is trained without direct data loss, using only the physical constraint derived from the governing equation:

$$z\frac{d^2F}{dz^2} + (1-z)\frac{dF}{dz} - \frac{i\omega}{2}F = 0, \tag{6}$$

where $z = i\omega y^2/2$.

Using the chain rule, we express derivatives in terms of ω:

$$\frac{dF}{dz} = \frac{\frac{dF}{d\omega}}{\frac{dz}{d\omega}}, \tag{7}$$

$$\frac{d^2F}{dz^2} = \frac{\frac{d^2F}{d\omega^2}}{\left(\frac{dz}{d\omega}\right)^2}. \tag{8}$$

Since $\frac{dz}{d\omega} = iy^2/2$, we obtain:

$$\frac{dF}{dz} = -\frac{2i}{y^2}\frac{dF}{d\omega}, \tag{9}$$

$$\frac{d^2F}{dz^2} = -\frac{4}{y^4}\frac{d^2F}{d\omega^2}. \tag{10}$$

Substituting these into the governing equation:

$$-\frac{4z}{y^4}\frac{d^2F}{d\omega^2} - \frac{2i(1-z)}{y^2}\frac{dF}{d\omega} - \frac{i\omega}{2}F = 0. \tag{11}$$

6 Loss Function for Training

In this study, we compare three different training approaches for the Physics-Informed Neural Network (PINN):

6.1 Data-Driven Training (MLP with Data Loss)

The first approach trains the neural network purely based on data loss. Given a set of training data points $\{(\omega_j, F_j^{\text{true}})\}$, the mean squared error (MSE) loss function is defined as:

$$\mathcal{L}_{\text{data}} = \sum_j \left| F(\omega_j) - F_j^{\text{true}} \right|^2. \tag{12}$$

This method relies entirely on labeled data and does not incorporate physical constraints.

6.2 Hybrid Training (MLP with Physics and Data Loss)

The second approach combines the data loss with the physics-based loss function. The total loss function is:

$$\mathcal{L} = \lambda_{\text{data}} \mathcal{L}_{\text{data}} + \lambda_{\text{phys}} \mathcal{L}_{\text{phys}}, \tag{13}$$

where λ_{data} and λ_{phys} are weighting factors. The physics-based loss $\mathcal{L}_{\text{phys}}$ is derived from the governing differential equation:

6.3 Physics Loss Calculation in Detail

We define the complex-valued function as:

$$F(\omega) = F_r(\omega) + iF_i(\omega). \tag{14}$$

Using automatic differentiation, we compute first and second derivatives:

$$F_r' = \frac{dF_r}{d\omega}, \quad F_i' = \frac{dF_i}{d\omega}, \tag{15}$$

$$F_r'' = \frac{d^2F_r}{d\omega^2}, \quad F_i'' = \frac{d^2F_i}{d\omega^2}. \tag{16}$$

We introduce z as:

$$z = i\frac{\omega y^2}{2}, \tag{17}$$

which gives:

$$\frac{dF}{dz} = \frac{dF/d\omega}{dz/d\omega} = \frac{dF/d\omega}{iy^2/2} = -\frac{2i}{y^2}\frac{dF}{d\omega}, \tag{18}$$

$$\frac{d^2F}{dz^2} = \frac{d^2F/d\omega^2}{(dz/d\omega)^2} = \frac{d^2F/d\omega^2}{(iy^2/2)^2} = -\frac{4}{y^4}\frac{d^2F}{d\omega^2}. \tag{19}$$

Thus, the governing equation:

$$z\frac{d^2F}{dz^2} + (1-z)\frac{dF}{dz} - \frac{i\omega}{2}F = 0 \tag{20}$$

expands as:

$$\text{term}_1 = -\frac{4}{y^4} z \frac{d^2F}{d\omega^2}, \tag{21}$$

$$\text{term}_2 = (1-z)\left(-\frac{2i}{y^2}\frac{dF}{d\omega}\right), \tag{22}$$

$$\text{term}_3 = -\frac{i\omega}{2}F. \tag{23}$$

Splitting into real and imaginary parts:

$$\text{term}_{1r} = \frac{2\omega}{y^2}F_i'', \quad \text{term}_{1i} = -\frac{2\omega}{y^2}F_r'', \tag{24}$$

$$\text{term}_{2r} = \frac{2}{y^2}F_i' - \omega F_r', \quad \text{term}_{2i} = -\frac{2}{y^2}F_r' - \omega F_i', \tag{25}$$

$$\text{term}_{3r} = \frac{\omega}{2}F_i, \quad \text{term}_{3i} = -\frac{\omega}{2}F_r. \tag{26}$$

The residual function is given by:

$$R_r = \text{term}_{1r} + \text{term}_{2r} + \text{term}_{3r}, \tag{27}$$

$$R_i = \text{term}_{1i} + \text{term}_{2i} + \text{term}_{3i}. \tag{28}$$

Finally, the physics loss is computed as:

$$\mathcal{L}_{\text{phys}} = \mathbb{E}\left[R_r^2 + R_i^2\right]. \tag{29}$$

6.4 Physics-Informed Training (MLP with Physics Loss Only)

The third approach eliminates the data loss entirely and relies purely on the physics-based loss function:

$$\mathcal{L} = \mathcal{L}_{\text{phys}}. \tag{30}$$

This method enforces physical consistency but may require a well-posed differential equation and careful regularization to achieve stable convergence.

6.5 Boundary Condition Incorporation

To ensure physical plausibility, especially in the low-frequency regime, we introduce an additional boundary condition (BC) loss term. Theoretically, as the dimensionless frequency $\omega \to 0$, the amplification factor should approach unity, i.e., $F(\omega, y) \to 1 + 0i$, reflecting the unlensed behavior.

We impose this constraint in the training process by selectively penalizing deviations from this asymptotic behavior in the region where $z = \frac{i\omega y^2}{2}$ is sufficiently small. Specifically, we define the BC loss as:

$$\mathcal{L}_{bc} = \mathbb{E}_{\omega,y}\left[\left|F_r(\omega, y) - 1\right|^2 + \left|F_i(\omega, y)\right|^2\right], \quad \text{for } z < \epsilon, \tag{31}$$

where ϵ is a small positive threshold (e.g., $\epsilon = 0.02$ in our implementation), and F_r, F_i denote the real and imaginary parts of the predicted amplification factor.

This constraint effectively anchors the network's predictions at low-frequency limits, ensuring that the learned model preserves asymptotic correctness even in the absence of sufficient training data in that regime. The final loss becomes:

$$\mathcal{L} = \lambda_{\text{phys}}\mathcal{L}_{\text{phys}} + \lambda_{\text{data}}\mathcal{L}_{\text{data}} + \lambda_{bc}\mathcal{L}_{bc}, \tag{32}$$

where λ_{bc} is a tunable weight to balance the contribution of the boundary condition during optimization (set to 2.0 in our experiment).

7 Experimental Setup And Results

We generated **40,000** training data points by sampling the dimensionless frequency in the range and discretizing the impact parameter at the values . Hence, each batch of data consists of input pairs along with their corresponding target outputs evaluated by the point-mass lensing formula (Equation (10)).

We trained three different neural network models under the following strategies:

1. **ModelA: Data-Only Training (MLP + Data Loss)**
2. **ModelB: Hybrid Training (MLP + Physics and Data Loss)**
3. **Model C: Physics-Only Training (MLP + Physics Loss)**

The data loss is a Mean Squared Error (MSE) comparing predicted to the target . The physics loss is derived from the Kummer-type differential equation that governs the magnification factor . Combining these losses in different proportions provides insight into how the physical constraints and direct data supervision affect the final accuracy.

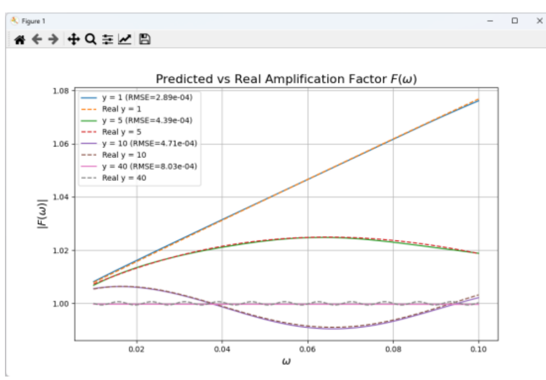

Figure 1: Training result using data loss only.

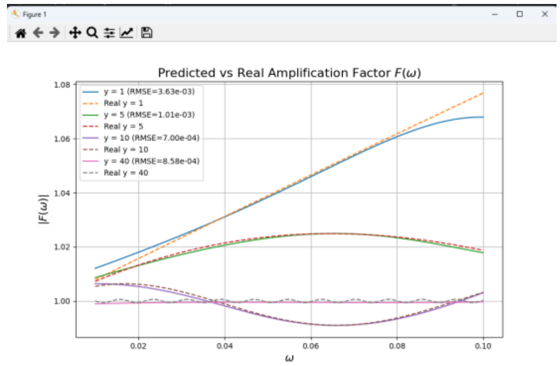

Figure 2: Training result using both data and physics loss.

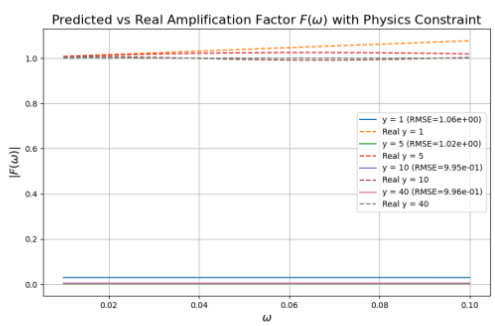

Figure 3: Training result using physics loss only.

From the given three graphs, it can be seen that Figure 1 only used data loss with minimal error, while Figure 2 used both data loss and physical loss with increasing error. When only physical losses are used in Figure 3, the obtained magnification tends towards 0, which is obviously illogical.