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Vector Valued Rational Interpolants I.

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Summary. It is shown explicitly that the Samelson inverse may be used to define vector-valued Thiele type rational interpolants, and the equivalence with Claessens' ε-algorithm is established. For the special case of vector valued Padé approximants, the general form of McCleod's theorem follows as a corollary.

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1. Introduction

Wynn (1963) raised the question of rational interpolation of vectors. He noted that the ε-algorithm, applied to vector valued quantities and implemented with Samelson inverses, can give exact results in the same circumstances as the scalar ε-algorithm does. A precise form of one of Wynn's results was proved by McCleod (1971), assuming initially that the elements of the ε-array belong to an associative division algebra over the complex numbers. The specific form often considered (e.g. Smith and Ford, 1981) uses complex valued vectors $\{\mathbf{s}^{(i)}\}$ which satisfy an $n+1$ term recurrence relation with constant real coefficients $\{\beta_i\}$:

$$\sum_{i=0}^{n} \beta_i \mathbf{s}^{(m+i)} = \mathbf{a} \sum_{i=0}^{n} \beta_i, \qquad m=0, 1, 2, \ldots$$

with the result that

$$\varepsilon_{2n}^{(m)} = \mathbf{a} \quad \text{if } \sum_{i=0}^{n} \beta_i \neq 0.$$

In this paper, we provide a practical Thiele-fraction method for rational interpolation of vectors, based on the Samelson inverse. We show that it is equivalent to Claessens' ε algorithm implemented with Samelson inverses. It follows that its confluent form, vector-valued Padé approximation, is equivalent to the standard ε-algorithm, and we show that McCleod's theorem is also valid for complex-valued $\{\beta_i\}$.

This is the first of two papers on vector-valued rational interpolation. We are concerned only with methods in which the vector-valued rational interpolant satisfies some fundamental principles: –

(i) If, for some fixed k, the k^{th} components of the vectors $\mathbf{v}^{(i)}$ are the only non-zero components,

$$\text{i.e. } v_j^{(i)}=0, \quad j=1,2,\ldots,k-1,k+1,\ldots,d.$$

Then the vector-valued interpolant reduces to the corresponding rational fraction interpolant.

(ii) If all the components of each vector $\mathbf{v}^{(i)}$ are proportional, i.e.

$$v_k^{(i)}=\lambda_k \mu_i, \quad k=1,2,\ldots,d, \; i=0,1,\ldots,n,$$

and the scalar values μ_i at points x_i are interpolated by the corresponding scalar rational interpolant $r(x)$, then the vector-valued interpolant has components $\{\lambda_k r(x), \; k=1,2,\ldots,d\}$. Note that property (i) is actually a special case of property (ii).

(iii) The value of the vector rational interpolant does not depend on the order in which the interpolation points are used to construct the interpolant.

(iv) There is some sense in which a specified rational interpolant is unique.

(v) The poles of the d components of the interpolant normally occur at common positions in the x-plane.

In this paper, we establish these principles for generalised inverse rational interpolant (GIRIs). We ignore the degeneracies which are known to plague all rational interpolation schemes, in the hope that they can be circumvented as in the scalar case (Meinguet, 1970; Graves-Morris, 1981). We use the order notation throughout in the sense of accuracy-through-order, i.e. $(1-x)^{-2}=1+2x+\mathcal{O}(x^2)$, and no bounding properties are implied.

2. GIRFs and GIRIs

A set of points $\{x_i, \; i=0,1,2,\ldots,n\colon \; x_i\in\mathbb{R}\}$ and a set $\{\mathbf{v}^{(i)}, \; i=0,1,2,\ldots,n\colon \mathbf{v}^{(i)}\in\mathbb{C}^d\}$ are given: each complex valued vector $\mathbf{v}^{(i)}$ is associated with a distinct real point x_i. We show that these vectors can normally be interpolated by the rational function

$$\mathbf{R}(x)=\mathbf{N}(x)/D(x) \tag{2.1}$$

in the sense that $\mathbf{N}(x)$ is a d-dimensional vector of polynomials, $D(x)$ is a polynomial and

$$R_j(x_i)=N_j(x_i)/D(x_i)=v_j^{(i)}, \quad i=0,1,2,\ldots,n, \; j=1,2,3,\ldots,d. \tag{2.2}$$

The construction process is based on the use of the Samelson inverse for vectors:

$$\mathbf{v}^{-1}=\mathbf{v}^*/|\mathbf{v}|^2. \tag{2.3}$$

Construction 1 of generalised inverse rational interpolants (GIRIs).

Initialisation. Define

$$\mathbf{b}^{(0)}=\mathbf{v}^{(0)}, \tag{2.4}$$

and, for $i=1,2,\dots,n$,

$$\mathbf{R}^{(1)}(x_i)=\frac{x_i-x_0}{\mathbf{v}^{(i)}-\mathbf{b}^{(0)}}. \tag{2.5}$$

Iteration. For $k=1,2,\dots,n-1$, define

$$\mathbf{b}^{(k)}=\mathbf{R}^{(k)}(x_k), \tag{2.6}$$

and for $i=k+1,k+2.\dots,n$,

$$\mathbf{R}^{(k+1)}(x_i)=(x_i-x_k)/(\mathbf{R}^{(k)}(x_i)-\mathbf{b}^{(k)}). \tag{2.7}$$

Termination. Define

$$\mathbf{b}^{(n)}=\mathbf{R}^{(n)}(x_n). \tag{2.8}$$

The resulting construct is

$$\mathbf{R}(x)=\mathbf{b}^{(0)}+\frac{x-x_0}{\mathbf{b}^{(1)}}\,\frac{}{+}\,\frac{x-x_1}{\mathbf{b}^{(2)}}\,\frac{}{+}\dots\frac{}{+}\,\frac{x-x_{n-1}}{\mathbf{b}^{(n)}}. \tag{2.9}$$

By tail-to-head rationalisation, detailed later, we see that $\mathbf{R}(x)$ takes the form (2.1).

Example 1. The first three rows of Table 1 are the data: four vectors $\mathbf{v}^{(i)}$ are specified, each associated with a point x_i. Construction 1 is used to derive $\mathbf{b}^{(0)}, \mathbf{b}^{(1)}, \dots, \mathbf{b}^{(3)}$ from this data.

Table 1. Construction of a GIRI

i	0	1	2	3
x_i	-1	0	1	2
$\mathbf{v}^{(i)}$	$(0,0,0)$	$(\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$	$(\frac{7}{5},-\frac{1}{5},0)$	$(\frac{27}{17},0,\frac{6}{17})$
$\dfrac{x_i-x_0}{\mathbf{v}^{(i)}-\mathbf{b}_0}$	–	$(1,-1,-1)$	$(\frac{7}{5},-\frac{1}{5},0)$	$(\frac{9}{5},0,\frac{2}{5})$
$\dfrac{x_i-x_1}{\mathbf{R}^{(1)}(x_i)-\mathbf{b}_1}$	–	–	$(\frac{2}{9},\frac{4}{9},\frac{5}{9})$	$(\frac{4}{9},\frac{5}{9},\frac{7}{9})$
$\dfrac{x_i-x_2}{\mathbf{R}^{(2)}(x_i)-\mathbf{b}_2}$	–	–	–	$(2,1,2)$

The result of the calculation displayed in Table 1 is

$$\mathbf{R}(x)=\mathbf{0}+\frac{x+1}{(1,-1,-1)}\,\frac{}{+}\,\frac{x}{(\frac{2}{9},\frac{4}{9},\frac{5}{9})}\,\frac{}{+}\,\frac{x-1}{(2,1,2)}. \tag{2.10}$$

From (2.5), we find

$$\mathbf{v}^{(i)}=\mathbf{b}^{(0)}+\frac{x_i-x_0}{\mathbf{R}^{(1)}(x_i)}, \tag{2.11}$$

and from (2.7) we find, for $k=1,2,\ldots,n-1$,

$$\mathbf{R}^{(k)}(x_i)=\mathbf{b}^{(k)}+\frac{x_i-x_k}{\mathbf{R}^{(k+1)}(x_i)}. \tag{2.12}$$

We see that the conditions

$$\mathbf{R}^{(k)}(x_{k-1})\neq\mathbf{0}, \qquad k=1,2,\ldots,n-1, \tag{2.13}$$

as well as the existence of all inverses explicitly used, are sufficient to show that $\mathbf{R}(x)$ interpolates correctly according to (2.2). However, in this paper we will assume that the exceptional cases, like (2.13) not holding, do not occur.

The Samelson inverse of $\mathbf{v}$, defined by (2.3), is the vector $\mathbf{v}^{-1}=\mathbf{w}$ with the property that $\|\mathbf{w}\|_2$ is a minimum, subject to $\mathbf{w}\cdot\mathbf{v}=1$. It has the desirable properties that

$$(\mathbf{v}^{-1})^{-1}=\mathbf{v} \tag{2.14}$$

and

$$\mathbf{v}\cdot\mathbf{v}^{-1}=1. \tag{2.15}$$

Although $\mathbf{v}^{-1}$ is uniquely specified by (2.3), in no sense is it a unique inverse of $\mathbf{v}$. Consequently, the GIRI $\mathbf{R}(x)$ of (2.9) is uniquely specified but is in no sense a unique rational interpolant. We seek to characterise the GIRIs so as to establish some uniqueness properties for them. We start by defining the numerator and denominator polynomials obtained by tail-to-head rationalisation of (2.9).

Definition. A vector valued rational fraction $\mathbf{R}(x)$,

$$\mathbf{R}(x)=\mathbf{N}(x)/D(x), \tag{2.16}$$

as defined in (2.1), is said to be of type $[\mathbf{l}/m]$ if

$$\partial N_j \leqq l \qquad j=1,2,\ldots,d, \tag{2.17a}$$

$$\partial N_j = l \qquad \text{for some } j,\ 1\leqq j\leqq d, \tag{2.17b}$$

$$\partial D = m. \tag{2.17c}$$

Definition. A vector valued rational fraction $\mathbf{R}(x)$, (2.16), as defined in (2.1), is *irreducible* if $D(x)$ is real and if there is no non-trivial real polynomial which is a common factor of the $d+1$ polynomials $\mathbf{N}(x)$ and $D(x)$.

Note that we have allowed for the possibility of "complex" common factors for $\mathbf{N}(x)$ and $D(x)$. Recall that the right-hand side of

$$\frac{1}{x-i}=\frac{x+i}{x^2+i}$$

is the "standard form" of the left-hand side. It is obvious that every representation of $\mathbf{R}(x)$ may be reduced to its irreducible form, which is essentially unique.

Construction 2

Specification. Numerator polynomials $\mathbf{P}(x)$ and a denominator polynomial $Q(x)$ are defined (recursively) by tail-to-head rationalisation of

$$\mathbf{S}(x)=\mathbf{b}^{(0)}+\frac{x-x_0}{\mathbf{b}^{(1)}}+\frac{x-x_1}{\mathbf{b}^{(2)}}+\ldots+\frac{x-x_{n-1}}{\mathbf{b}^{(n)}}. \tag{2.18}$$

Initialisation. $\mathbf{p}(x)=\mathbf{b}^{(n)}$; $q(x)=1$.

Recursion. For $j=n, n-1, \ldots, 1$, let

$$\mathbf{S}^{(j)}(x)=\mathbf{b}^{(j)}+\frac{x-x_j}{\mathbf{b}^{(j+1)}}+\frac{x-x_{j+1}}{\mathbf{b}^{(j+2)}}+\ldots+\frac{x-x_{n-1}}{\mathbf{b}^{(n)}} \tag{2.19}$$

have the representation

$$\mathbf{S}^{(j)}(x)=\mathbf{p}(x)/q(x) \tag{2.20}$$

where $q(x)$ is a real polynomial with the property that

$$q(x)\,|\,|\mathbf{p}(x)|^2. \tag{2.21}$$

Define $Q(x)$ by

$$|\mathbf{p}(x)|^2=Q(x)\,q(x) \tag{2.22}$$

and $\mathbf{P}(x)$ by

$$\mathbf{P}(x)=\mathbf{b}^{(j-1)}Q(x)+(x-x_{j-1})\,\mathbf{p}^*(x). \tag{2.23}$$

Then $\mathbf{S}^{(j-1)}(x)$ has a representation of type (2.20),

$$S^{(j-1)}(x)=\mathbf{P}(x)/Q(x). \tag{2.24}$$

Termination. $\mathbf{S}(x)=\mathbf{S}^{(0)}(x)$.

Justification. The recursion procedure is only valid if (2.21) holds so that (2.22) defines a real polynomial $Q(x)$. From the definition (2.23),

$$\begin{aligned}|\mathbf{P}(x)|^2=&|\mathbf{b}^{j-1}|^2\,[Q(x)]^2+(x-x_{j-1})^2\,|\mathbf{p}(x)|^2\\&+(x-x_{j-1})\,Q(x)\,[\mathbf{b}^{(j-1)}\cdot\mathbf{p}(x)+\mathbf{b}^{(j-1)*}\cdot\mathbf{p}^*(x)]\end{aligned} \tag{2.25}$$

and (2.25) implies that

$$Q(x)\,|\,|\mathbf{P}(x)|^2. \tag{2.26}$$

Definitions. We define a rational fraction $\mathbf{R}(x)=\mathbf{N}(x)/D(x)$ to be a generalised inverse rational fraction (GIRF) if $D(x)$ is real and $D(x)\,|\,|\mathbf{N}(x)|^2$. A GIRI is defined to be a GIRF with the interpolating property (2.2). A GIRI or GIRF, $\mathbf{R}(x)=\mathbf{N}(x)/D(x)$, is said to be *reduced* if $D(x)$ is real and all possible real polynomial common factors of $\mathbf{N}(x)$ and $D(x)$ have been removed, consistently with $D(x)\,|\,|\mathbf{N}(x)|^2$.

The Justification of construction 2 shows that Thiele type interpolants (2.9), (2.18) are GIRIs.

Characterisation Theorem. *Consider a GIRI of the form*

$$\mathbf{R}(x)=\frac{\mathbf{N}(x)}{D(x)}=\mathbf{b}^{(0)}+\frac{x-x_0}{\mathbf{b}^{(1)}}\,\frac{x-x_1}{+\;\mathbf{b}^{(2)}}\,\frac{}{+\ldots+}\,\frac{x-x_{n-1}}{\mathbf{b}^{(n)}}, \tag{2.27}$$

where $\mathbf{N}(x)$ *and* $D(x)$ *have been found by construction 2. If* n *is even,* $\mathbf{R}(x)$ *is normally of type* $[\mathbf{n}/n]$. *If* n *is odd,* $\mathbf{R}(x)$ *is normally of type* $[\mathbf{n}/n-1]$.

Proof. The proof is recursive.

For the case of n even, we may assume that

$$\frac{\mathbf{p}(x)}{q(x)}=\mathbf{b}^{(1)}+\frac{x-x_1}{\mathbf{b}^{(2)}}\,\frac{x-x_2}{+\;\mathbf{b}^{(3)}}\,\frac{}{+\ldots+}\,\frac{x-x_{n-1}}{\mathbf{b}^{(n)}} \tag{2.28}$$

holds with

(i) $\partial\mathbf{p}=n-1$,

(ii) $\partial q=n-2$,

(iii) $q(x)\,|\,|\mathbf{p}(x)|^2$.

Using the formalism (2.19)–(2.26) with $j=1$, we find from (2.22) that $\partial Q=n$, from (2.23) that $\partial\mathbf{P}=n$ and from (2.26) that factorisation holds. For the case of n being odd, the method is the same, but the corresponding formulas are

$$\partial\mathbf{p}=\partial q=\partial Q=n-1,$$
$$\partial\mathbf{P}=n.$$

Example 2. The rational forms encountered in a tail-to-head evaluation of $\mathbf{R}(x)$ in (2.9) are defined by

$$\mathbf{R}^{(n)}(x)=\mathbf{b}^{(n)} \tag{2.29}$$

and for $j=n-1, n-2, \ldots, 0$ by

$$\mathbf{R}^{(k)}(x)=\mathbf{b}^{(k)}+\frac{x-x_k}{\mathbf{R}^{(k+1)}(x)}. \tag{2.30}$$

For the case of example 1, their GIRF representations are

$$\mathbf{R}^{(3)}(x)=(2,1,2),$$

$$\mathbf{R}^{(2)}(x)=\left(\frac{2x}{9},\frac{x+3}{9},\frac{2x+3}{9}\right),$$

$$\mathbf{R}^{(1)}(x)=\frac{(3x^2+2x+2,\,x-2,\,x^2+x-2)}{x^2+2x+2},$$

$$\mathbf{R}^{(0)}(x)=\frac{\frac{1}{2}(x+1)}{5x^2-3x+3}(3x^2+2x+2,\,x-2,\,x^2+x-2).$$

The computation of $\mathbf{R}^{(0)}(x)$ from $\mathbf{R}^{(1)}(x)$ shows the elimination of the factor x^2+2x+2. It is also clear that $\mathbf{R}^{(0)}(x)$ actually interpolates the data of Table 2.

Example 3. This is an example of degeneracy which can occur but is not discussed further in this paper. Its analogue is familiar in the scalar case.

$$\begin{aligned}\mathbf{R}(x)&=\frac{x-1}{(2,4,2)+\frac{x-2}{(\frac{1}{6},\frac{1}{3},\frac{1}{6})}}\\&=\frac{x-1}{(x,2x,x)}.\end{aligned}$$

Using the construction (2.22), (2.23), we find

$$\mathbf{N}(x)=(x-1)(x,2x,x),$$

$$D(x)=6x^2,$$

and

$$R(x)=\mathbf{N}(x)/D(x).$$

We see that the zero of $\mathbf{R}^{(1)}(x)$ at $x=0$ shows up as a pole of each component of $\mathbf{R}(x)$. Under such circumstances, $\mathbf{N}(x)$ and $D(x)$ have a common factor, and cancellation beyond that required by the characterisation theorem can occur. Example 3 exhibits the case of exceptional cancellation.

3. Uniqueness and Superdiagonal Interpolants

Uniqueness Theorem. *Any two GIRIs,* $\mathbf{R}(x)$ *and* $\mathbf{r}(x)$ *which interpolate the same set of (finite valued) vectors at* $n+1$ *distinct points, i.e.*

$$\mathbf{R}(x_i)=\mathbf{r}(x_i)=\mathbf{v}^{(i)}, \qquad i=0,1,2,\dots,n \tag{3.1}$$

and of the same type (i.e. $\mathbf{n}/n$ *if* n *is even or* $\mathbf{n}/n-1$ *if* n *is odd) are equal.*

Proof. Express $\mathbf{R}(x)$ and $\mathbf{r}(x)$ in their reduced forms

$$\mathbf{R}(x)=\mathbf{P}(x)/Q(x), \tag{3.2}$$

$$\mathbf{r}(x)=\mathbf{p}(x)/q(x) \tag{3.3}$$

with real polynomials $Q(x)$ and $q(x)$ satisfying

$$Q(x)\,|\,|\mathbf{P}(x)|^2, \qquad q(x)\,|\,|\mathbf{p}(x)|^2,$$

$$\partial Q=n-f \quad (n \text{ even}) \text{ or } \partial Q=n-f-1 \ (n \text{ odd}),\ f\geqq 0, \tag{3.4a}$$

$$\partial \mathbf{P}\leqq n-f, \tag{3.4b}$$

$$\partial q=n-g \quad (n \text{ even}) \text{ or } \partial q=n-g-1 \ (n \text{ odd}),\ g\geqq 0, \tag{3.4c}$$

$$\partial \mathbf{p}\leqq n-g. \tag{3.4d}$$

In fact, (3.4) allows slightly greater generality than (2.17) requires. Let $t(x)$ be the greatest common factor of $q(x)$ and $Q(x)$. $t(x)$ is necessarily real.

Define polynomials $q_r(x)$ and $Q_r(x)$ by

$$Q(x)=t(x)\,Q_r(x), \tag{3.5a}$$

$$q(x)=t(x)\,q_r(x) \tag{3.5b}$$

so that $Q_r(x)$ and $q_r(x)$ have no non-trivial common factor. Define $\mathbf{T}(x)$ by

$$\mathbf{T}(x)=\mathbf{P}(x)\,q_r(x)-\mathbf{p}(\mathbf{x})\,Q_r(x) \tag{3.6}$$

and note that

$$\mathbf{T}(x_i)=\mathbf{0}, \qquad i=0,1,\dots,n. \tag{3.7}$$

Define $Z(x)$ by

$$Z(x)=(x-x_0)(x-x_1)\dots(x-x_n) \tag{3.8}$$

and a polynomial valued vector $\mathbf{V}(x)$ by

$$\mathbf{T}(x)=Z(x)\,\mathbf{V}(x). \tag{3.9}$$

Then, from (3.6)

$$|\mathbf{T}(x)|^2=|\mathbf{P}(x)|^2\,q_r(x)^2+|\mathbf{p}(x)|^2\,Q_r(x)^2 \\ -2q_r(x)\,Q_r(x)\,\mathrm{Re}[\mathbf{p}(x)\cdot\mathbf{P}^*(x)]. \tag{3.10}$$

From (3.10),

$$Q_r(x)\,|\,|\mathbf{T}(x)|^2 \quad \text{and} \quad q_r(x)\,|\,|\mathbf{T}(x)|^2. \tag{3.11}$$

Because the vectors $\{\mathbf{v}^{(i)}\}$ are finite, we deduce from (3.9) and (3.11) that

$$Q_r(x)\,|\,|\mathbf{V}(x)|^2 \quad \text{and} \quad q_r(x)\,|\,|\mathbf{V}(x)|^2. \tag{3.12}$$

Unless $\mathbf{V}(x)\equiv 0$, it follows from (3.12) that

$$2\partial\mathbf{V}\geqq\partial Q_r+\partial q_r. \tag{3.13}$$

Case of n even. The following inequalities follow from (3.4), (3.5), (3.6), (3.9) and (3.13)

$$\partial Q_r=n-f-\partial t,$$

$$\partial q_r=n-g-\partial t,$$

$$\partial\mathbf{T}\leqq 2n-f-g-\partial t,$$

$$\partial\mathbf{V}^2\leqq 2(n-f-g-\partial t-1). \tag{3.14}$$

$$\partial\mathbf{V}^2\geqq 2n-f-g-2\partial t. \tag{3.15}$$

Case of n odd. Similarly,

$$\partial Q_r=n-1-f-\partial t,$$

$$\partial q_r=n-1-g-\partial t,$$

$$\partial\mathbf{T}\leqq 2n-f-g-\partial t-1,$$

$$\partial\mathbf{V}^2\leqq 2(n-f-g-\partial t-2), \tag{3.16}$$

$$\partial\mathbf{V}^2\geqq 2n-2-f-g-2\partial t. \tag{3.17}$$

In each case (3.14)–(3.17) show a contradiction, and so $\mathbf{V}(x)\equiv 0$.

The uniqueness theorem is important because it shows that the interpolant $\mathbf{R}(x)$ produced by construction 1 and rationalised by construction 2 is unique up to irrelevant common factors, regardless of the actual ordering of the interpolation points used in the construction process.

In (3.2), $(n+1)d$ parameters are needed to specify the d polynomials constituting $\mathbf{P}(x)$, and $\partial Q+1$ parameters specify $Q(x)$. Since the polynomial interpolant of each component of the set of vectors $\{\mathbf{v}^{(i)}, i=0,1,\dots,n\}$ needs $n+1$ parameters, leading to a total of $d(n+1)$ parameters for vector polynomial interpolation of all the data, it may seem that (3.1), which needs an extra $\partial Q+1$ parameters, is wasteful. However, we may view the factorisation conditions following (3.3) as expressing the fact that the remainder of $|\mathbf{P}(x)|^2$, after division by $Q(x)$, is zero, which is equivalent to ∂Q constraint equations. The relative renormalisation of $\mathbf{P}(x)$ and $Q(x)$ is arbitrary, and so we see that there are no spare parameters in the reduced formulation (3.2)–(3.4).

Following custom, we call $[\mathbf{n}/n]$ type GIRIs *diagonal* interpolants, and note that they interpolate on $x_0, x_1, \dots, x_n$ for n even. We have also used $[\mathbf{n}/n-1]$ type GIRIs for the case of interpolation on an even number of points, when n is odd. We can easily extend these ideas to other off-diagonal interpolants of type

$$[\mathbf{2\kappa+j}/2\kappa], \qquad \kappa=0,1,2,\dots \text{ and } j=2,3,4,\dots. \tag{3.18}$$

Let $\boldsymbol{\pi}^{(j-1)}(x)$ be the ordinary vector of $j-1^{\text{th}}$ degree polynomials which interpolate according to

$$\boldsymbol{\pi}^{(j-1)}(x_i)=\mathbf{v}^{(i)}, \qquad i=0,1,\dots,j-1. \tag{3.19}$$

Let $\mathbf{S}(x)$ be the $[\mathbf{2\kappa}/2\kappa]$ type GIRI which interpolates the values

$$\mathbf{S}(x_i)=\frac{\mathbf{v}^{(i)}-\boldsymbol{\pi}^{(j-1)}(x_i)}{\prod_{l=0}^{j-1}(x_i-x_l)}, \qquad i=j,\ j+1,\dots,2\kappa+j. \tag{3.20}$$

Then $\mathbf{R}(x)$, defined by

$$\mathbf{R}(x)=\boldsymbol{\pi}^{(j-1)}(x)+\mathbf{S}(x)\prod_{l=0}^{j-1}(x-x_l). \tag{3.21}$$

is the $[\mathbf{2\kappa+j}/2\kappa]$ type GIRI for the cases $j=2,3,4,\dots$. We note that the GIRI properties of order, interpolation and factorisation all hold. It also seems possible to extend these ideas to subdiagonal GIRIs of type $[\mathbf{2\kappa-1}/2\kappa]$ for

$$\mathbf{R}^{(-1)}(x)=\frac{1}{\mathbf{b}^{(0)}}+\frac{x-x_0}{\mathbf{b}^{(1)}}+\frac{x-x_1}{\mathbf{b}^{(2)}}+\dots+\frac{x-x_{n-1}}{\mathbf{b}^{(n)}}, \tag{3.22}$$

but we have not pursued such developments yet.

General Uniqueness Theorem. *Any two GIRIs, of the same type* $[\mathbf{2\kappa+j}/2\kappa]$ *which interpolate the same set of finite vectors at* $2\kappa+j+1$ *distinct points,*

$$\mathbf{R}(x_i)=\mathbf{r}(x_i)=\mathbf{v}^{(i)}, \qquad i=0,1,\dots,2\kappa+j,$$

are equal.

Proof. The proof is essentially the same as for the previous uniqueness theorem. Using the formalism (3.1)–(3.13), but omitting (3.4), we find

$$
\begin{aligned}
&\partial Q = 2\kappa - f, \quad f \geqq 0,\\
&\partial \mathbf{P} = 2\kappa + j - f,\\
&\partial q = \partial \kappa - g, \quad g \geqq 0,\\
&\partial \mathbf{p} \leqq 2\kappa + j - g,\\
&\partial Q_r = 2\kappa - f - \partial t,\\
&\partial q_r = 2\kappa - g - \partial t,\\
&\partial \mathbf{T} \leqq 4\kappa + j - f - g - \partial t,\\
&\partial Z = 2\kappa + j + 1,\\
&\partial \mathbf{V}^2 \leqq 2(2\kappa - f - g - \partial t - 1),\\
&\partial \mathbf{V}^2 \geqq 4\kappa - f - g - 2\partial t.
\end{aligned}
$$

The final inequalities are contradictory, and it follows that $\mathbf{V}(x) \equiv 0$.

We may lay out the various interpolants in a GIRI table, Table 2.

Table 2. The GIRI table

2κ \ j					
0	0/0	1/0	2/0	3/0	...
2		2/2	3/2	4/2	...
4			4/4	5/4	
				. . .	. . .

The antidiagonals of the table, along which $2\kappa + j = N =$ constant, contain GIRIs, all of which interpolate on $x_0, x_1, \ldots, x_N$. The obvious correspondence is to be made with the ordinary rational interpolation table. Properties (iii) and (iv) of Sect. 1 have now been proved, property (v) is obvious, and properties (i) and (ii) follow from (2.3).

4. The ε-Algorithm and Claessens' Identity

Claessens (1978) derived the generalisation of Wynn's star identity and the ε-algorithm to scalar rational interpolation. We may summarise his results using a table.

Table 3. Elements of the scalar rational interpolation table. l, m are the orders of the numerators and denominator polynomials respectively

m \ l			
		$l/m-1$	
	$l-1/m$	l/m	$l+1/m$
		$l/m+1$	

$\equiv$

	N	
W	C	E
	S	

In Table 3, we identify the various interpolants in the star by compass points. Denoting the interpolation points by $x_0, x_1, \ldots, x_{l+m+1}$, Claessens' identity is

$$(x-x_{l+m})[(E-C)^{-1}-(S-C)^{-1}]=(x-x_{l+m+1})[(N-C)^{-1}-(W-C)^{-1}]. \quad (4.1)$$

It is convenient that the formula is convention invariant in the sense that it is formally unchanged if the l and m axes are reversed, as in Claessens' paper. Claessens' ε-algorithm is

$$(x-x_{k+j+1})[\varepsilon_{k+1}^{(j)}-\varepsilon_{k-1}^{(j+1)}][\varepsilon_k^{(j+1)}-\varepsilon_k^{(j)}]=1 \quad (4.2)$$

with boundary conditions

$$\varepsilon_{-1}^{(j)}=0, \qquad j=0,1,2,\ldots \quad (4.3)$$

and

$$\varepsilon_0^{(j)}=[j/0], \quad j=0,1,2,\ldots. \quad (4.4)$$

The upper half of the table can also be found using

$$\varepsilon_{2\kappa}^{(-\kappa-1)}=0, \quad \kappa=0,1,2,\ldots.$$

The identification with rational interpolants is made in even columns, and is

$$\varepsilon_{2\kappa}^{(j)}=[\mathbf{2\kappa+j}/2\kappa]. \quad (4.5)$$

If we set $x_j=0$, $j=0,1,2,\ldots$ for the confluent case and put $x=1$, (4.2) becomes the ε-algorithm and (4.1) becomes Wynn's identity. In this section, we identify the elements of Table 3 with those of the GIRI table, and so (4.1) and (4.2) generalise directly to the vector case.

For the vector case, the new forms of (4.2)-(4.4) are

$$\varepsilon_{2\kappa+1}^{(j)}=\varepsilon_{2\kappa-1}^{(j+1)}+\frac{(x-x_{j+2\kappa+1})^{-1}}{\varepsilon_{2\kappa}^{(j+1)}-\varepsilon_{2\kappa}^{(j)}}, \quad (4.6)$$

$$\varepsilon_{2\kappa+2}^{(j)}=\varepsilon_{2\kappa}^{(j+1)}+\frac{(x-x_{j+2\kappa+2})^{-1}}{\varepsilon_{2\kappa+1}^{(j+1)}-\varepsilon_{2\kappa+1}^{(j)}} \quad (4.7)$$

with

$$\varepsilon_{-1}^{(j)}=\mathbf{0}, \quad j=0,1,2,\ldots, \quad (4.8)$$

$$\varepsilon_0^{(j)}=[\mathbf{j}/0], \quad j=0,1,2,\ldots. \quad (4.9)$$

It then follows from (4.6) that Eastern elements in the even columns of the ε-table may be constructed using

$$(x-x_{j+2\kappa})[(E-C)^{-1}-(S-C)^{-1}]=(x-x_{j+2\kappa+1})[(N-C)^{-1}-(W-C)^{-1}] \quad (4.10)$$

where $C=\varepsilon_{2\kappa}^{(j)}$.

Next we prove the identification (4.11) of the GIRIs with the elements of the even columns of the ε-table.

Identification Theorem. *Claessens ε-algorithm, as expressed by* (4.6)–(4.8) *with the Samelson inverse, constructs GIRIs, identified by*

$$\varepsilon_{2\kappa}^{(j)} = [\mathbf{j}+\mathbf{2\kappa}/2\kappa], \qquad j,\ \kappa \geqq 0. \tag{4.11}$$

Proof. The zeroth column, $k=\kappa=0$, of the ε-table contains the entries (4.8), and there is nothing to prove. We may express the Newton interpolating polynomials as

$$[\mathbf{j}/0] = \sum_{i=0}^{j} \mathbf{c}^{(i)} \omega_i(x) \tag{4.12}$$

where

$$\omega_0(x) = 1 \tag{4.13}$$

and

$$\omega_i(x) = \prod_{l=0}^{i-1} (x - x_l), \qquad i = 1, 2, \ldots. \tag{4.14}$$

Hence we find that

$$[\mathbf{j}+\mathbf{1}/0] - [\mathbf{j}/0] = \mathbf{c}^{(j+1)} \omega_{j+1}(x) \tag{4.15}$$

and that the first column of the ε-table contains entries

$$\varepsilon_1^{(j)} = [\mathbf{c}^{(j+1)} \omega_{j+2}(x)]^{-1}, \qquad j = 0, 1, 2, \ldots.$$

These entries in odd suffix columns are not identified with interpolants. For the second column, the entries are defined by (4.6) to be

$$\varepsilon_2^{(j)} = \sum_{i=0}^{j+1} \mathbf{c}^{(i)} \omega_i(x) + \omega_{j+2}(x) \left[\frac{1}{\mathbf{c}^{(j+2)}} - \frac{x - x_{j+2}}{\mathbf{c}^{(j+1)}} \right]^{-1}. \tag{4.16}$$

As $x \to \infty$, the first expression of the RHS of (4.16) has asymptotic behaviour $\mathbf{c}^{(j+1)} x^{j+1}$, which cancels exactly with the leading term of the second expression.

It follows that

$$\varepsilon_2^{(j)} = \mathcal{O}(x^j) \qquad \text{as } x \to \infty,$$

that the degree of the numerator and denominator of $\varepsilon_2^{(j)}$ are $j+2$ and 2 respectively, and the factorisation property is obvious. The explicit form (4.7) shows that

$$\varepsilon_2^{(j)}(x_i) = \varepsilon_0^{(j+1)}(x_i), \qquad i = 0, 1, \ldots, j+2.$$

We deduce that the elements of the second column are GIRIs:

$$\varepsilon_2^{(j)} = [\mathbf{j}+\mathbf{2}/2], \qquad j = 0, 1, 2, \ldots.$$

To extend the proof to the k^{th} column, $k > 2$, we make the following inductive hypotheses, each true for $j = 0, 1, 2, \ldots$:

(i) $\varepsilon_{2\kappa-1}^{(j)}=[x\varepsilon_{2\kappa-2}^{(j+1)}]^{-1}+\mathcal{O}(x^{-j-3})=\mathcal{O}(x^{-j-2})$ as $x\to\infty$, (4.17a)

(ii) $\varepsilon_{2\kappa}^{(j)}=\mathcal{O}(x^j)$ as $x\to\infty$, (4.17b)

(iii) $\varepsilon_{2\kappa-1}^{(j)}=\boldsymbol{\pi}^{(j;2\kappa-2)}(x)/\omega_{j+2\kappa}(x)$, (4.17c)

where $\boldsymbol{\pi}^{(j;2\kappa-2)}(x)$ is a vector polynomial of degree $(2\kappa-2)$

(iv) $\varepsilon_{2\kappa}^{(j)}(x_i)=[\mathbf{2\kappa+j/0}](x_i)$, $i=0,1,\ldots,2\kappa+j$, (4.17d)

(v) $\varepsilon_{2\kappa}^{(j)}$ is a GIRI of type $[\mathbf{j+2\kappa}/2\kappa]$. (4.17e)

To prove the theorem, we must show that each of these results holds with $\kappa\to\kappa+1$.

From (4.6), as it stands, (i) and (ii), we have

$$\varepsilon_{2\kappa+1}^{(j)}=\mathcal{O}([x\varepsilon_{2\kappa}^{(j+1)}]^{-1})=\mathcal{O}(x^{-j-2}),\quad j=0,1,2,\ldots \tag{4.18}$$

as $x\to\infty$, so proving (i) for $\kappa\to\kappa+1$.

From (4.7),

$$\varepsilon_{2\kappa+2}^{(j)}=\varepsilon_{2\kappa}^{(j+1)}+\frac{(x-x_{2\kappa+j+2})^{-1}}{\varepsilon_{2\kappa+1}^{(j+1)}-\varepsilon_{2\kappa+1}^{(j)}} \tag{4.19}$$

and by substituting (4.18) into (4.19), we see that the leading terms in each of the two expressions in the RHS of (4.19) cancel. Therefore (ii) is proved for $\kappa\to\kappa+1$. Next, we consider (4.6) again as

$$\varepsilon_{2\kappa+1}^{(j)}=\frac{\boldsymbol{\pi}^{(j+1;2\kappa-2)}(x)}{\omega_{j+2\kappa+1}(x)}+\frac{(x-x_{j+2\kappa+1})^{-1}}{\varepsilon_{2\kappa}^{(j+1)}-\varepsilon_{2\kappa}^{(j)}}. \tag{4.20}$$

Using (v), we express $\varepsilon_{2\kappa}^{(j+1)}$ and $\varepsilon_{2\kappa}^{(j)}$ as GIRIs:

$$\varepsilon_{2\kappa}^{(j+1)}=\mathbf{P}(x)/Q(x),$$
$$\varepsilon_{2\kappa}^{(j)}=\mathbf{p}(x)/q(x).$$

Following the same argument as for the uniqueness proofs in Sect. 3, we find that

$$[\varepsilon_{2\kappa}^{(j+1)}-\varepsilon_{2\kappa}^{(j)}]^{-1}=c_{j,\kappa}\boldsymbol{\Pi}^{(j;2\kappa)}(x)/\omega_{j+2\kappa+1}(x) \tag{4.21}$$

where $c_{j,\kappa}$ is a constant, independent of x, and $\boldsymbol{\Pi}^{(j;2\kappa)}(x)$ is a vector polynomial of degree 2κ. We sustitute (4.21) into (4.20) to find that

$$\varepsilon_{2\kappa+1}^{(j)}=\boldsymbol{\pi}^{(j;2\kappa)}(x)/\omega_{j+2\kappa+2}(x), \tag{4.22}$$

where $\boldsymbol{\pi}^{(j;2\kappa)}(x)$ is a vector polynomial of order 2κ, so proving (iii) for $\kappa\to\kappa+1$.

From (4.19) and (4.22), we have

$$\varepsilon_{2\kappa+2}^{(j)}=\varepsilon_{2\kappa}^{(j+1)}+\frac{\omega_{2\kappa+j+2}(x)}{(x-x_{2\kappa+j+2})\boldsymbol{\pi}^{(j+1;2\kappa)}-\boldsymbol{\pi}^{(j;2\kappa)}(x)}. \tag{4.23}$$

Therefore

$$\varepsilon_{2\kappa+2}^{(j)}(x_i)=\varepsilon_{2\kappa}^{(j+1)}(x_i)=\mathbf{v}^{(i)}, \qquad i=0,1,\ldots,2\kappa+j+1. \tag{4.24}$$

But, to prove (iv) with $\kappa\to\kappa+1$, we must also prove that (4.24) holds for $i=2\kappa+j+2$. We use (4.10) with

$$S=\varepsilon_{2\kappa+2}^{(j)}; \qquad E=\varepsilon_{2\kappa}^{(j+2)}; \qquad C=\varepsilon_{2\kappa}^{(j+1)}.$$

Since the value $\mathbf{v}^{(j+2\kappa+2)}$ is not used for the construction of W, C or N, it normally follows from (4.10) that

$$E(x_i)=S(x_i) \quad \text{for } i=2\kappa+j+2. \tag{4.25}$$

Results (4.24) and (4.25) establish (iv) for $\kappa\to\kappa+1$.

Finally, we must establish the character of $\varepsilon_{2\kappa+2}^{(j)}(x)$.

Define polynomials $\pi(x)$, $\mathbf{p}(x)$ and $q(x)$ by

$$\pi(x)=(x-x_{j+2\kappa+2})\,\pi^{(j+1;2\kappa)}(x)-\pi^{(j;2\kappa)}(x), \tag{4.26}$$

$$\varepsilon_{2\kappa}^{(j+1)}=\mathbf{p}(x)/q(x) \tag{4.27}$$

with

$$\partial\pi=2\kappa+1, \tag{4.28a}$$

$$\partial\mathbf{p}=j+2\kappa+1, \tag{4.28b}$$

$$\partial q=2\kappa \tag{4.28c}$$

to simplify (4.23). Let $x=\xi$ be any of the 2κ zeros of $q(x)$. Either from (4.7) or from experience with the scalar ε-algorithm, we know that a pole of $\varepsilon_{2\kappa}^{(j+1)}$ at $x=\xi$ is associated with the coincidence

$$\varepsilon_{2\kappa-1}^{(j+2)}(\xi)=\varepsilon_{2\kappa-1}^{(j+1)}(\xi). \tag{4.29}$$

Using (4.6), we see that

$$\varepsilon_{2\kappa+1}^{(j)}(\xi)=\varepsilon_{2\kappa-1}^{(j+1)}(\xi), \tag{4.30}$$

$$\varepsilon_{2\kappa+1}^{(j+1)}(\xi)=\varepsilon_{2\kappa-1}^{(j+2)}(\xi). \tag{4.31}$$

From (4.29)–(4.31), we see that both terms of the RHS of (4.19) have poles at $x=\xi$. In the scalar algorithm, these poles normally have equal and opposite residues and so cancel; otherwise they would be common to $\varepsilon_{2\kappa+2}^{(j)}$. This cancellation occurs in exactly the same way in the vector case; we can also argue this point by noting that only $\varepsilon_{2\kappa+2}^{(j)}$ and $\varepsilon_{2\kappa+1}^{(j+1)}$ in (4.19) depend on $\mathbf{v}^{(j+2\kappa+2)}$.

We write (4.23) as

$$\varepsilon_{2\kappa+2}^{(j)}=\frac{\mathbf{p}(x)}{q(x)}+\frac{\omega_{j+2\kappa+2}(x)\,\pi^*(x)}{|\pi(x)|^2}. \tag{4.32}$$

The argument about the zeros of $q(x)$ shows that

$$q(x)\,|\,|\pi(x)|^2,$$

and hence we may define a polynomial $Q(x)$ by

$$|\boldsymbol{\pi}(x)|^2 = q(x)\,Q(x).$$

We express (4.32) as a GIRF,

$$\varepsilon_{2\kappa+2}^{(j)} = \tilde{\boldsymbol{\pi}}(x)/Q(x)$$

with

$$\begin{aligned}\partial Q &= 2(\partial\pi) - \partial q = 2\kappa+2\\ \partial\tilde{\boldsymbol{\pi}} &\leqq \max\{\partial\mathbf{p} + \partial Q,\, 2\kappa+j+2+\partial\boldsymbol{\pi}\} - \partial q\\ &= j+2\kappa+3.\end{aligned} \tag{4.33}$$

In (4.32), we expect the same cancellation of leading terms as in (4.17), and it is proved using (4.18). Consequently (4.33) is altered to

$$\partial\tilde{\boldsymbol{\pi}} = j+2\kappa+2$$

and (v) is established for $\kappa \to \kappa+1$.

The proof of the equivalence between Claessens' ε-algorithm and the Thiele fraction approach to GIRIs is now complete.

5. McCleod's Theorem

Suppose that a sequence of vectors

$$\mathscr{S} \equiv \{\mathbf{s}^{(i)},\, i=0,1,2,\ldots : \mathbf{s}^{(i)} \in \mathbb{C}^d\} \tag{5.1}$$

is given, and that the vectors of $\mathscr{S}$ *satisfy*

$$\sum_{i=0}^{n} \beta_i \mathbf{s}^{(i+j)} = \left(\sum_{i=0}^{n} \beta_i\right)\mathbf{a}, \qquad j=0,1,2,\ldots, \tag{5.2}$$

which is an $n+1$ *term recurrence relation if* $\beta_0 \neq 0$, $\beta_n \neq 0$. *McCleod's theorem states that the ε-algorithm, initialised by*

$$\varepsilon_0^{(j)} = \mathbf{s}^{(j)}, \qquad j=0,1,2,\ldots \tag{5.3}$$

and implemented with Samelson inverses leads to

$$\varepsilon_{2n}^{(m)} = \mathbf{a}, \qquad \text{if } \sum_{i=0}^{n} \beta_i \neq 0, \tag{5.4}$$

$$\varepsilon_{2n}^{(m)} = \mathbf{0}, \qquad \text{if } \sum_{i=0}^{n} \beta_i = 0, \tag{5.5}$$

provided that β_i *are real constants and unless zero divisors are encountered in the construction.*

We consider the associated sequence $\mathscr{T}$ of vectors

$$\mathbf{t}^{(i)} = \mathbf{s}^{(i)} - \mathbf{a}. \tag{5.6}$$

Note that the ε-table of $\mathscr{T}$ has even columns displaced by **a** from those of the ε-table of $\mathscr{S}$, but the odd columns are the same. Consequently, the result that the ε-table for $\mathscr{T}$, initialised by

$$\varepsilon_0^{(j)} = \mathbf{t}^{(j)}, \qquad j = 0, 1, 2, \ldots \tag{5.7}$$

leads to

$$\varepsilon_{2n}^{(m)} = \mathbf{0}, \qquad m = 0, 1, 2, \ldots \tag{5.8}$$

provided that the homogeneous difference equation

$$\sum_{i=0}^{n} \beta_i \mathbf{t}^{(i+j)} = \mathbf{0} \tag{5.9}$$

is satisfied and

$$\sum_{i=0}^{n} \beta_i \neq 0 \tag{5.10}$$

is an equivalent form of McCleod's result (5.4). In fact, (5.5) is a degenerate case.

We make the connection with rational interpolation by considering the formal series

$$\mathbf{F}(x) = \mathbf{t}^{(0)} + \sum_{i=1}^{\infty} x^i [\mathbf{t}^{(i)} - \mathbf{t}^{(i-1)}] \tag{5.11}$$

and a generating function

$$\mathbf{G}(x) = \sum_{i=0}^{\infty} x^i \mathbf{t}^{(i)}, \tag{5.12}$$

which are properly defined in (5.15), (5.16).

From (5.2), we find that

$$\sum_{i=0}^{n} \sum_{j=0}^{m} \beta_i \mathbf{t}^{(i+j)} x^j = \mathbf{0}, \tag{5.13}$$

and we rearrange terms to obtain

$$\sum_{i=0}^{m} x^i \mathbf{t}^{(i)} = \frac{\sum_{k=1}^{n} [\mathbf{t}^{(k-1)} x^{k-1} - \mathbf{t}^{(m+k)} x^{m+k}] \sum_{i=k}^{n} \beta_i x^{-i}}{\sum_{i=0}^{n} \beta_i x^{-i}}. \tag{5.14}$$

We define $\mathbf{G}(x)$ by

$$\mathbf{G}(x) = \sum_{k=1}^{n} \mathbf{t}^{(k-1)} x^{k-1} \sum_{i=k}^{n} \beta_i x^{-i} \Big/ \sum_{i=0}^{n} \beta_i x^{-i} \tag{5.15}$$

and $\mathbf{F}(x)$ by

$$\begin{aligned} \mathbf{F}(x) &= \mathbf{t}^{(0)} + \mathbf{G}(x) - \mathbf{G}(0) - x\mathbf{G}(x) \\ &= (1-x)\,\mathbf{G}(x). \end{aligned} \tag{5.16}$$

We define $q(x)$ by

$$q(x) = \sum_{i=0}^{n} \beta_i x^{n-i}. \tag{5.17}$$

If $q(x)$ has n simple zeros in $|x|>1$, the usual interpretation of (5.15) is that the n poles of $\mathbf{G}(x)$ contribute n decaying geometric components to the sequence $\mathscr{S}$ [Baker and Graves-Morris, 1981, Chap. 3, Sect. 2]. We define $\boldsymbol{\pi}(x)$ by

$$\mathbf{F}(x)=\boldsymbol{\pi}(x)/q(x) \tag{5.18}$$

and put $\mathbf{F}(x)$ in standard form as

$$\mathbf{F}(x)=\frac{\boldsymbol{\pi}(x)\,q^*(x)}{q(x)\,q^*(x)}. \tag{5.19}$$

The R.H.S. of (5.19) is an osculatory GIRI (vector Padé approximant) of type $[\mathbf{2n}/2n]$, and so the identification theorem of Sect. 4 implies that

$$\mathbf{F}(x)=[\mathbf{2n}/2n](x)=\varepsilon_{2n}^{(0)}(x). \tag{5.20}$$

Provided $q(1)\neq 0$, it therefore follows from (5.16) that

$$\mathbf{F}(1)=\varepsilon_{2n}^{(0)}(1)=\mathbf{0}$$

for the $2n^{\text{th}}$ column of the ε-table for $\mathscr{T}$. Using (5.6), we have the result (5.4) for the sequence $\mathscr{S}$. The interpretation in terms of decaying geometric components is that

$$\mathbf{t}^{(\infty)}=\mathbf{F}(1)=\mathbf{0}; \qquad \mathbf{s}^{(\infty)}=\mathbf{a}.$$

This is our constructive proof of McCleod's result (5.4), which is valid for the case of complex β_i.

The exceptional case occurs if

$$q(1)=\sum_{i=0}^{n}\beta_i=0.$$

Assuming that $q(x)$ has a simple zero at $x=1$, we find from (5.18) and (5.19) that

$$[\mathbf{2n-2}/2n-2]_{\mathbf{F}}(1)=\mathbf{F}(1),$$

and therefore zero divisors necessarily occur in the construction of the $k=2n-1$ column of the ε-table. It follows from (5.2) and (5.6) that the vectors

$$\mathbf{v}^{(i)}=\mathbf{t}^{(i)}-\mathbf{t}^{(i-1)}=\mathbf{s}^{(i)}-\mathbf{s}^{(i-1)}$$

satisfy an n-term recurrence relation, with the interpretation that the degeneracy occurs because there are at most $n-1$ active geometric components in $\mathscr{S}$.

6. Conclusion

Although the question about the nature of the recurrence coefficients in McCleod's theorem has been answered, we leave many problems open. Does the GIRI table extend below the first subdiagonal? Apart from the ε-algorithm (and presumably the η-algorithm) do any of the other identities, like the three term recurrence relation, generalise in a practical way to the vector case? Does Cordellier's identity (1979) treat degenerate cases, and are there blocks? Do the interpolation points have to be real?

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