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On the choice of preconditioner for minimum residual methods for non-Hermitian matrices

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ABSTRACT

We consider the solution of left preconditioned linear systems $P^{-1}Cx = P^{-1}c$, where $P, C \in \mathbb{C}^{n\times n}$ are non-Hermitian, $c \in \mathbb{C}^n$, and C, P, and $P^{-1}C$ are diagonalisable with spectra symmetric about the real line. We prove that, when P and C are self-adjoint with respect to the same Hermitian sesquilinear form, the convergence of a minimum residual method in a particular nonstandard inner product applied to the preconditioned linear system is bounded by a term that depends only on the spectrum of $P^{-1}C$. The inner product is related to the spectral decomposition of P. When P is self-adjoint with respect to a nearby Hermitian sesquilinear form to C, the convergence of a minimum residual method in this nonstandard inner product applied to the preconditioned linear system is bounded by a term involving the eigenvalues of $P^{-1}C$ and a constant factor. The size of this factor is related to the nearness of the Hermitian sesquilinear forms. Numerical experiments indicate that for certain matrices eigenvalue-dependent convergence is observed both for the nonstandard method and for standard GMRES.

1. Introduction

For systems of linear equations

$$Bx = b, \tag{1}$$

$B \in \mathbb{C}^{n\times n}$, $x, b \in \mathbb{C}^n$, with a Hermitian coefficient matrix, there are standard iterative methods of choice: for positive definite problems the conjugate gradient method [1] and for indefinite problems the MINRES method [2]. (SYMMLQ [2] is a lesser used alternative in the indefinite case.) For these methods there exist convergence bounds given in terms of polynomial approximation problems on the eigenvalue spectrum that are often reasonably tight [3, p. 51]. A key consequence is that it is clear what one is trying to achieve by preconditioning in these cases, namely nicely distributed eigenvalues or reduced spectral condition number.

By contrast, for non-Hermitian matrices B there are many possible and competing methods. GMRES [4] is probably the most popular minimum residual method, but unless the required number of iterations is kept small the increased work of orthogonalisation in the Arnoldi process and the storage requirements can grow to infeasible levels. To avoid this increased work and storage, several methods based on the non-Hermitian Lanczos method are also widely employed [5–10], though none minimises the residual and we do not consider them here; the convergence of these methods is even less well understood than that of minimum residual methods such as GMRES. For all of these methods, preconditioning is almost always employed in the hope of obtaining acceptably rapid convergence.

It remains a fundamental and largely open mathematical question as to what one is trying to achieve with preconditioning for non-Hermitian matrices, and the known results are largely negative. For example, Greenbaum, Pták, and Strakoš [11]

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proved that, given any complete set of eigenvalues and any monotonically reducing convergence curve, it is possible to construct a matrix with the given eigenvalues and an initial residual (starting iterate) such that GMRES will achieve precisely the given convergence curve. This is in stark contrast to the situation for Hermitian matrices. A subsequent characterisation of the set of matrices with prescribed eigenvalues and right-hand side vectors for which a given convergence curve is obtained was provided in [12]. Other convergence bounds and discussions can be found in, for example, [3,13–15], but it is difficult to discern from these desirable properties of a preconditioner.

A minimum residual method, such as GMRES, for (1) computes iterate vectors $\{x_k\}$, $k = 1, 2, \ldots$, from a starting guess x_0. Each iterate is such that $x_k - x_0$ lies in the Krylov subspace

$$\mathcal{K}_k(B, r_0) = \text{span}\{r_0, Br_0, B^2r_0, \ldots, B^{k-1}r_0\},$$

and the corresponding residual $r_k = b - Bx_k$ satisfies $r_k = p(B)r_0$, where $p \in \Pi_k$, the set of real polynomials of degree k or lower, and p satisfies the consistency condition $p(0) = 1$ (see, for example, [3, Section 2.4] for details). If B has diagonalisation $B = Z\Lambda Z^{-1}$, the minimum residual property ensures that

$$\frac{\|r_k\|_2}{\|r_0\|_2} \leq \|Z\|_2\|Z^{-1}\|_2 \min_{\substack{p\in\Pi_k,\\ p(0)=1}} \max_{\lambda\in\sigma(B)} |p(\lambda)|, \tag{2}$$

where $\sigma(B)$ is the spectrum of B. When the condition number $\kappa_2(Z) = \|Z\|_2\|Z^{-1}\|_2$ is small, convergence typically depends on the eigenvalues of B. However, when $\kappa_2(Z) \gg 1$, i.e., when B is significantly nonnormal, the GMRES residuals often decrease in norm much faster than (2) suggests, because of the size of $\kappa_2(Z)$ (see Section 5 for some examples). We note that since (2) is a worst-case upper bound it is certainly true that for a given right-hand side vector convergence may be faster than that predicted by (2), even for normal matrices for which $\kappa_2(Z) = 1$ (see, for example, [16, p. 492] [17–20]). This second issue is beyond the scope of this paper, although it would be interesting to consider whether the ideas here can be applied to bounds that incorporate the initial residual r_0.

We present convergence bounds for a minimum residual method, in a specific nonstandard inner product that we describe below, for the left preconditioned system

$$P^{-1}Cx = P^{-1}c, \tag{3}$$

where we assume that $P, C \in \mathbb{C}^{n\times n}$, $b = P^{-1}c \in \mathbb{C}^n$, P is nonsingular, and that C, P, and $B = P^{-1}C$ have spectra that are symmetric about the real line and are diagonalisable. Thus, the eigenvalues of the matrix B may be complex, but complex eigenvalues must appear in complex conjugate pairs. The condition on the spectra might seem restrictive but, for example, complex eigenvalues of a real matrix always appear in complex conjugate pairs, and so our theory covers, for example, real diagonalisable matrices.

Our first bound applies when P is self-adjoint with respect to the same Hermitian sesquilinear form $\langle\cdot,\cdot\rangle_X$ as C. In this case, convergence of a minimum residual method in a nonstandard inner product that is related to the spectral decomposition of P is bounded by a term that involves only the eigenvalues of $P^{-1}C$. Our main result is that P need not be self-adjoint with respect to $\langle\cdot,\cdot\rangle_X$; when P is self-adjoint with respect to a nearby Hermitian sesquilinear form, say $\langle\cdot,\cdot\rangle_{\mathcal{X}}$, the convergence of a minimum residual method in a particular inner product, which again depends on the spectral decomposition of P, is bounded by a term involving the eigenvalues and a constant factor. This factor is related to the nearness of $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_{\mathcal{X}}$, and is small when they are close. If the eigenvalues of P are good approximations of the eigenvalues of C, then convergence of this minimum residual method should be fast. Although we focus on eigenvalue bounds, other bounds for GMRES convergence are available, notably based on the field of values [21,22] and pseudospectra [23]. A similar result to Theorem 2 in Section 4 below holds for pseudospectral bounds [24, Theorem 6.10], and it is possible that the analysis of this paper might also be applicable to the field-of-values approach.

Our convergence results can be interpreted as choosing a more descriptive norm with which to measure convergence. Although one should always be wary when considering convergence of a minimum residual method with respect to a different norm, since convergence can be distorted [25], even left preconditioning alters the norm with respect to which residuals are measured since, for any $r \in \mathbb{C}^n$, $\|P^{-1}r\|_2^2 = r^*P^{-*}P^{-1}r = \|r\|^2_{P^{-*}P^{-1}}$. Regardless, we observe from the numerical examples we present that a minimum residual method in the Euclidean inner product exhibits similar convergence behaviour to the nonstandard method. This indicates that it might be desirable that a preconditioner is self-adjoint with respect to a Hermitian sesquilinear form nearby that with respect to which C is self-adjoint and has eigenvalues near to those of C even when a standard minimum residual method is used.

In Section 2, we introduce background material and discuss a minimum residual method in a particular inner product for which convergence is bounded by a term involving only the eigenvalues of the coefficient matrix. The convergence of a minimum residual method in a nonstandard inner product is linked to that of a minimum residual method in the Euclidean inner product in Section 3, where we additionally introduce a bound for standard GMRES that shows why the $\kappa_2(Z)$ term can make (2) unduly pessimistic. Our main results regarding preconditioning are given in Section 4, and the results of numerical tests are described in Section 5. We summarise our results in Section 6.

Throughout this paper, we denote the diagonalisations of B, C, and P by

$$B = Z\Lambda Z^{-1}, \tag{4}$$

$C = S\Lambda_C S^{-1}$, and $P = \mathcal{S}\Lambda_P \mathcal{S}^{-1}$, respectively. Without loss of generality, real eigenvalues are labelled first in each diagonalisation so, for example,

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots \lambda_\alpha, \lambda_{\alpha+1}, \overline{\lambda_{\alpha+1}}, \ldots, \lambda_\beta, \overline{\lambda_\beta}), \tag{5}$$

where $\lambda_1, \ldots, \lambda_\alpha$ are the real eigenvalues of B and $\lambda_{\alpha+1}, \ldots, \lambda_\beta$ are the nonreal eigenvalues in the upper half-plane. Our results can be generalised to nondiagonalisable matrices, although one must deal with matrix polynomials evaluated at Jordan blocks rather than scalar polynomials evaluated at eigenvalues. We use D^T to denote the transpose of a matrix $D \in \mathbb{C}^{n\times n}$ and D^* to denote the conjugate transpose, and similarly for vectors, while the complex conjugate of a scalar α is $\overline{\alpha}$. The spectrum and 2-norm condition number are represented by $\sigma(D)$ and $\kappa_2(D)$, respectively. By r_k we denote the kth residual vector of a minimum residual method in the Euclidean inner product, ρ_k represents the kth residual vector of a minimum residual method in a general nonstandard inner product, and $\widetilde{\rho}_k$ and $\widehat{\rho}_k$ denote the kth residual vectors of minimum residual methods in the particular inner products $\langle\cdot,\cdot\rangle_M$ and $\langle\cdot,\cdot\rangle_{\mathcal{M}}$, respectively, which are defined below. For every minimum residual method we use r_0 to represent the initial residual, since this is independent of the inner product.

2. Self-adjointness in nonstandard inner products

Any nondegenerate Hermitian sesquilinear form [26, Section 2.1] is related to an invertible Hermitian matrix $H \in \mathbb{C}^{n\times n}$ by $\langle x, y\rangle_H = y^*Hx$, where $x, y \in \mathbb{C}^n$. The H-adjoint of any $B \in \mathbb{C}^{n\times n}$ is defined to be the unique matrix $B^\star \in \mathbb{C}^{n\times n}$ that satisfies $\langle Bx, y\rangle_H = \langle x, B^\star y\rangle_H$ for all $x, y \in \mathbb{C}^n$. It follows that

$$B^\star = H^{-1}B^*H. \tag{6}$$

An H-normal matrix satisfies $B^\star B = BB^\star$, while an H-self-adjoint matrix is one for which $B = B^\star$ or

$$HB = B^*H, \tag{7}$$

so HB is Hermitian. Clearly, H-self-adjoint matrices are H-normal. Since (6) implies that a self-adjoint matrix is similar to its conjugate transpose, the spectrum of a self-adjoint matrix is symmetric with respect to the real line.

The Hermitian sesquilinear forms $\langle\cdot,\cdot\rangle_H$ with respect to which a given diagonalisable $B = Z\Lambda Z^{-1}$ is self-adjoint are all defined by matrices H of the form [26, Theorem 5.1.1]

$$H = Z^{-*}YZ^{-1}, \quad Y = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_\alpha, Y_{\alpha+1}, \ldots, Y_\beta), \tag{8}$$

where $\epsilon_i = \pm 1$, $i = 1, \ldots, \alpha$, and $Y_j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $j = \alpha+1, \ldots, \beta$. The well-known result that a real matrix B is self-adjoint with respect to an inner product if and only if it is diagonalisable with real eigenvalues is clear from this factorisation of H. The order of the eigenvectors in Z and their scaling can vary, as can the signs $\epsilon_1, \ldots, \epsilon_\alpha$, and so H is not unique.

An inner product $\langle\cdot,\cdot\rangle_W$ is a Hermitian sesquilinear form for which $W \in \mathbb{C}^{n\times n}$ is Hermitian positive definite (HPD). With any inner product $\langle\cdot,\cdot\rangle_W$ we can associate a vector norm for $x \in \mathbb{C}^n$ and the corresponding induced matrix norm for any $D \in \mathbb{C}^{n\times n}$:

$$\|x\|_W = \sqrt{\langle x, x\rangle_W} \quad \text{and} \quad \|D\|_W = \max_{x\neq 0} \frac{\|Dx\|_W}{\|x\|_W}.$$

Any W-matrix norm can be related to the I-norm since

$$\|D\|_W = \|FDF^{-1}\|_I, \tag{9}$$

where F is any nonsingular matrix satisfying $W = F^*F$.

When B is self-adjoint with respect to a Hermitian sesquilinear form $\langle\cdot,\cdot\rangle_H$ defined by (8), it is also normal with respect to the inner product, $\langle\cdot,\cdot\rangle_M$, where

$$M = Z^{-*}Z^{-1}. \tag{10}$$

Similarly to H, M is non-unique since Z is not unique. However, when $B = Z\Lambda Z^{-1}$ is real, both H and M can be chosen to be real [24, Section 2.2.1].

We now turn our attention to minimum residual (MR) methods in nonstandard inner products. We call W-MR a method for which iterates are selected from Krylov subspaces of increasing dimension and for which the corresponding residual, ρ_k, is minimised with respect to $\|\cdot\|_W$, so

$$\|\rho_k\|_W = \min_{x-x_0\in\mathcal{K}_k(B,r_0)} \|b - Bx\|_W.$$

Thus, minimum residual methods in the Euclidean inner product are I-MR methods. Further details of minimum residual methods in nonstandard inner products can be found in [22].

If $B \in \mathbb{C}^{n\times n}$ has diagonalisation (4) and $W = F^*F$ for some invertible $F \in \mathbb{C}^{n\times n}$, then ρ_k, the kth residual vector of W-MR, satisfies

$$\frac{\|\rho_k\|_W}{\|r_0\|_W} \leq \kappa_2(FZ) \min_{\substack{p\in\Pi_k,\\ p(0)=1}} \max_{\lambda\in\sigma(B)} |p(\lambda)|, \tag{11}$$

with different factorisations of W leading to different bounds. When $W = I$, i.e., for I-MR, F is unitary, and we recover the standard bound (2) that includes the (potentially large) condition number of the eigenvectors, while for the particular method M-MR, with M given by (10), $\kappa_2(FZ) = \kappa_2(Z^{-1}Z) = 1$. Such M-MR methods have been explored in [27–29] and the references therein, although generally in the case that B is M-self-adjoint (rather than just M-normal). The self-adjointness of B with respect to $\langle\cdot,\cdot\rangle_M$ means that a MINRES-type algorithm with three-term (or coupled two-term) recurrences can be used [24, Section 3.2] [30].

When $W = M$, the bound (11) becomes

$$\frac{\|\widetilde{\rho}_k\|_M}{\|r_0\|_M} \leq \min_{\substack{p\in\Pi_k,\\ p(0)=1}} \max_{\lambda\in\sigma(B)} |p(\lambda)|, \tag{12}$$

as was noted in [31, Eq. (3.8)] for the case that B is self-adjoint with respect to an inner product. In this way, (12) generalises the Euclidean-norm result that convergence of a minimum residual method applied to a normal matrix is bounded by a term that depends only on the eigenvalues of the matrix. We stress, however, that the effect of the eigenvectors has not disappeared; it is present in the norm $\|\cdot\|_M$. Convergence of M-MR may differ significantly, therefore, from that of I-MR (standard GMRES), although for the preconditioned linear systems in Section 5 we observe that both methods converge similarly. We also caution that, as mentioned above, bounds of the type (11) and (12) are worst-case upper bounds. For a particular right-hand side, convergence may be faster than they predict and may not be governed by all the eigenvalues of B. This presents no difficulty when (12) ensures fast convergence but (12), like (2), may be less informative when it predicts slow convergence. Consequently, we look to preconditioning in the next section to ensure that convergence is rapid. Since M is not uniquely defined (see the discussion in Section 2) the deviation of the computed relative residuals from the right-hand side of (12) may depend on the inner product used, although investigation of this aspect is beyond the scope of this paper.

3. Relationship between the convergence of minimum residual methods in different inner products

When using minimum residual methods in nonstandard inner products, a natural consideration is whether this knowledge provides insight into the convergence of minimum residual methods in the Euclidean, or indeed any other, inner product. Several results linking the convergence of minimum residual methods in different inner products exist (see, for example, [28,22,32–34]). We briefly present in this section bounds that relate residuals of a minimum residual method in $\langle\cdot,\cdot\rangle_M$, with M defined by (10), to those of a minimum residual method in the Euclidean inner product $\langle\cdot,\cdot\rangle_I$. These can be extended in a straightforward manner to bounds for minimum residual methods in any inner products.

Consider the application of I-MR and M-MR to the general system (1) (with diagonalisable coefficient matrix B). Let the columns of $\widetilde{V}_k \in \mathbb{C}^{n\times k}$ be the M-unitary basis vectors (M-Arnoldi vectors) of the kth Krylov subspace generated by M-MR, and let $V_k \in \mathbb{C}^{n\times k}$have as its columns the I-unitary basis vectors (I-Arnoldi vectors) of I-MR. Similarly to the kth I-MR residual (see, for example, [3, p. 38]) the kth M-MR residual satisfies

$$\|\widetilde{\rho}_k\|_M = \min_{z\in\mathbb{R}^k} \| \, \|r_0\|_M \, e_1 - \widetilde{H}_k z\|_2, \tag{13}$$

where $\widetilde{H}_k \in \mathbb{C}^{(k+1)\times k}$ is the upper Hessenberg matrix generated by the Arnoldi method in the M-inner product. Since both M-MR and I-MR form bases of the same nested Krylov subspaces $\mathcal{K}_k(B, r_0)$, assuming that these methods do not terminate at or before the kth step, we must have

$$V_k = \widetilde{V}_k R_k, \tag{14}$$

where $R_k \in \mathbb{R}^{k\times k}$ is upper triangular and nonsingular; this is a QR decomposition of V_k in the $\langle\cdot,\cdot\rangle_M$ inner product. The connection (14) allows the I-MR residual to be related to the corresponding M-MR residual for the same linear system (and starting vector).

Lemma 1. *Let I-MR and M-MR be applied to $Bx = b$, with B having diagonalisation (4). If $M = Z^{-*}Z^{-1}$ as in (10), the kth M-MR residual $\widetilde{\rho}_k$ and the kth I-MR residual r_k satisfy*

$$\frac{1}{s_{\max}(Z^{-1}V_{k+1})}\|\widetilde{\rho}_k\|_M \leq \|r_k\|_I \leq \frac{1}{s_{\min}(Z^{-1}V_{k+1})}\|\widetilde{\rho}_k\|_M, \tag{15}$$

where V_{k+1} has as its columns the I-Arnoldi vectors and $s_{\min}(Z^{-1}V_{k+1})$ and $s_{\max}(Z^{-1}V_{k+1})$ are the minimum and maximum singular values of $Z^{-1}V_{k+1}$, respectively.

Proof. If $z_k = \arg\min_z \| \|r_0\|_M e_1 - \widetilde{H}_k z\|_2$, then, from [32, Section 4],

$$\|r_k\|_I = \min_{y\in\mathbb{R}^k} \| \|r_0\|_M e_1 - \widetilde{H}_k y\|_{R_{k+1}^{-*}R_{k+1}^{-1}}.$$

Using (13), we then find that

$$\|r_k\|_I \le \| \|r_0\|_M e_1 - \widetilde{H}_k z_k\|_{R_{k+1}^{-*}R_{k+1}^{-1}} \le s_{\max}(R_{k+1}^{-1})\|\widetilde{\rho}_k\|_M. \tag{16}$$

From (14) and the M-orthogonality of $\widetilde{V}_{k+1}$, we find that $(V_{k+1})^*MV_{k+1} = R_{k+1}^*R_{k+1}$. Combining this with the factorisation $M = Z^{-*}Z^{-1}$ gives that $s_k(R_{k+1}^{-1})^2 = 1/s_{n-k}^2(Z^{-1}V_{k+1})$, where $s_k(R_{k+1}^{-1})$ is the kth largest singular value of R_{k+1}^{-1}, and so (16) is equivalent to

$$\|r_k\|_I \le \frac{1}{s_{\min}(Z^{-1}V_{k+1})}\|\widetilde{\rho}_k\|_M. \tag{17}$$

The lower bound is obtained by applying a corresponding argument. □

When $k = 0$, (15) shows that $\|r_0\|_M \le s_{\max}(Z^{-1}V_1)\|r_0\|_I$. Combining this with (12) and (15) gives that the kth I-MR residual for (1) is bounded by

$$\begin{aligned}\frac{\|r_k\|_I}{\|r_0\|_I} &\le \frac{s_{\max}(Z^{-1}V_1)}{s_{\min}(Z^{-1}V_{k+1})}\min_{\substack{p\in\Pi_k,\\ p(0)=1}}\max_{\lambda\in\sigma(B)}|p(\lambda)|\\ &\le \kappa_2(Z^{-1}V_{k+1})\min_{\substack{p\in\Pi_k,\\ p(0)=1}}\max_{\lambda\in\sigma(B)}|p(\lambda)|,\end{aligned} \tag{18}$$

where we have used [35, Theorem 7.3.9] to relate $s_{\max}(Z^{-1}V_1)$ to $s_{\max}(Z^{-1}V_{k+1})$. Although this bound is equal to (2) at step n, if not before, if $\kappa_2(Z^{-1}V_{k+1}^l) \ll \kappa_2(Z^{-1})$, (2) will be a poor predictor of convergence.

The bound (15) depends on the iteration number, k. Bounds that are independent of the iteration number can also be derived [24, Section 3.2.4], but these may be pessimistic, particularly for small k.

4. Preconditioning

We have seen that, when B is M-normal, with M defined by (10), (1) can be solved by M-MR with convergence, measured in the M-norm, bounded by a term that depends only on the eigenvalues of B (see (12)). As for I-MR, for any $B \in \mathbb{C}^{n\times n}$, there may be right-hand side vectors for which (12) is not descriptive. However, it is clear that a sufficient condition for rapid convergence of M-MR is that B has nicely distributed eigenvalues. Here, we investigate effective preconditioners P for I-MR in light of normality with respect to inner products and self-adjointness with respect to Hermitian sesquilinear forms. We concentrate on left preconditioning, but show how the results may be adapted to right preconditioning.

In this section, we consider the left preconditioned system (3), where $B = P^{-1}C$ and $b = P^{-1}c$. If $P^{-1}C(=B)$ has diagonalisation (4), it is itself self-adjoint with respect to some Hermitian sesquilinear form $\langle\cdot,\cdot\rangle_H$. It is additionally M-normal, where $M = Z^{-*}Z^{-1}$, and (12) implies that a sufficient condition for fast convergence of M-MR is that the eigenvalues of $B = P^{-1}C$ are nicely distributed. Hermitian sesquilinear forms with respect to which a preconditioned matrix is self-adjoint have been characterised for certain matrices (see [36, Section 13] and the references therein). However, self-adjointness can *always* be preserved if C and P are both self-adjoint with respect to the same Hermitian sesquilinear form, say $\langle\cdot,\cdot\rangle_X$, since then

$$(XP)(P^{-1}C) = XC = C^*X = C^*P^{-*}P^*X = (P^{-1}C)^*(XP). \tag{19}$$

The symmetry and invertibility of X and the X-self-adjointness and invertibility of P guarantee that XP is Hermitian and nonsingular. Thus, $P^{-1}C$ is self-adjoint with respect to the Hermitian sesquilinear form $\langle\cdot,\cdot\rangle_{XP}$. Additionally, $B = P^{-1}C$ has a diagonalisation $B = Z\Lambda Z^{-1}$ such that, if $M = Z^{-*}Z^{-1}$, then $B = P^{-1}C$ is M-normal, and by (12) the preconditioned M-MR residuals, $P^{-1}\widetilde{\rho}_k$, for (3) satisfy

$$\frac{\|P^{-1}\widetilde{\rho}_k\|_M}{\|P^{-1}r_0\|_M} \le \min_{\substack{p\in\Pi_k,\\ p(0)=1}}\max_{\lambda\in\sigma(P^{-1}C)}|p(\lambda)|.$$

Thus, convergence of M-MR is bounded by a term that depends only on eigenvalues. A similar result holds for right preconditioning: if P and C are both X-self-adjoint, CP^{-1} is XC^{-1}-self-adjoint, and the following convergence results for left preconditioning can be adapted to right preconditioning by instead using $\langle\cdot,\cdot\rangle_{XC^{-1}}$. Clearly, one would not form XC^{-1}, but the convergence bounds one obtained could be related to bounds for I-MR convergence through the results of Section 3.

The requirement that P be self-adjoint with respect to the same Hermitian sesquilinear form as C is easily satisfied for Hermitian coefficient matrices by choosing a Hermitian preconditioner. It can also be satisfied when P and C are both persymmetric and therefore self-adjoint with respect to $\langle\cdot,\cdot\rangle_{\widehat{Y}}$, where $\widehat{Y}$ is the reverse identity matrix

$$\widehat{Y} = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}. \tag{20}$$

Examples of persymmetric preconditioners for persymmetric matrices include circulant preconditioners for Toeplitz matrices [37–40] and the preconditioner in Example 1 in Section 5 below. We note that, when P and C are certain Kronecker sums of Toeplitz matrices and circulant matrices, as in [41], P and C are also self-adjoint with respect to the same Hermitian sesquilinear form.

Unfortunately, for most other applications a preconditioner that is self-adjoint with respect to the same Hermitian sesquilinear form $\langle\cdot,\cdot\rangle_X$ as C is not known. However, if a preconditioner is self-adjoint with respect to a Hermitian sesquilinear form that is close to $\langle\cdot,\cdot\rangle_X$, say $\langle\cdot,\cdot\rangle_{\mathcal{X}}$, we might expect that we can find a matrix $\mathcal{Z}$ such that $\mathcal{H} = \mathcal{X}P = \mathcal{Z}^{-*}Y\mathcal{Z}^{-1}$ and $P^{-1}C = (\mathcal{Z}-E)\Lambda(\mathcal{Z}-E)^{-1}$ is a diagonalisation of $P^{-1}B$, with E a small perturbation. If this is the case, then the convergence of $\mathcal{M}$-MR, with $\mathcal{M} = \mathcal{Z}^{-*}\mathcal{Z}^{-1}$, is bounded by a term that depends primarily on eigenvalues, and a sufficient condition for fast convergence of this method is that $P^{-1}C$ has nicely distributed eigenvalues.

Theorem 2. *Let $Cx = c$, where $C \in \mathbb{C}^{n\times n}$ is nonsingular and $c \in \mathbb{C}^n$, be left preconditioned by the nonsingular matrix $P \in \mathbb{C}^{n\times n}$, and let $P^{-1}C$ have diagonalisation*

$$P^{-1}C = (\mathcal{Z}-E)\Lambda(\mathcal{Z}-E)^{-1}, \tag{21}$$

where $\mathcal{Z} \in \mathbb{C}^{n\times n}$ is nonsingular, $E \in \mathbb{C}^{n\times n}$, and $\Lambda \in \mathbb{C}^{n\times n}$ is of the form (5). If $\mathcal{M} = \mathcal{Z}^{-*}\mathcal{Z}^{-1}$, then preconditioned $\mathcal{M}$-MR residuals $P^{-1}\widehat{\rho}_k$ satisfy

$$\frac{\|P^{-1}\widehat{\rho}_k\|_{\mathcal{M}}}{\|P^{-1}r_0\|_{\mathcal{M}}} \le \kappa_2(I-\mathcal{Z}^{-1}E) \min_{\substack{p\in\Pi_k,\\ p(0)=1}} \max_{\lambda\in\sigma(P^{-1}C)} |p(\lambda)|. \tag{22}$$

Moreover, if $\|\mathcal{Z}^{-1}E\|_2 < 1$, then

$$\frac{\|P^{-1}\widehat{\rho}_k\|_{\mathcal{M}}}{\|P^{-1}r_0\|_{\mathcal{M}}} \le \frac{1+\|\mathcal{Z}^{-1}E\|_2}{1-\|\mathcal{Z}^{-1}E\|_2} \min_{\substack{p\in\Pi_k,\\ p(0)=1}} \max_{\lambda\in\sigma(P^{-1}C)} |p(\lambda)|. \tag{23}$$

Proof. We have that

$$\begin{aligned}\|P^{-1}\widehat{\rho}_k\|_{\mathcal{M}} &= \|p_k(P^{-1}C)P^{-1}r_0\|_{\mathcal{M}} \\ &\le \|\mathcal{Z}^{-1}(\mathcal{Z}-E)p_k(\Lambda)(\mathcal{Z}-E)^{-1}\mathcal{Z}\|_2\|P^{-1}r_0\|_{\mathcal{M}},\end{aligned}$$

where we have used (9) and (21). Thus,

$$\|P^{-1}\widehat{\rho}_k\|_{\mathcal{M}} \le \|I-\mathcal{Z}^{-1}E\|_2\|(I-\mathcal{Z}^{-1}E)^{-1}\|_2\|p_k(\Lambda)\|_2\|P^{-1}r_0\|_{\mathcal{M}},$$

from which (22) follows.

If $\|\mathcal{Z}^{-1}E\|_2 < 1$, then [35, p. 301], $\|(I-\mathcal{Z}^{-1}E)^{-1}\|_2 \le 1/\left(1-\|\mathcal{Z}^{-1}E\|_2\right)$, which gives (23). □

Remark. Provided E is small, both (12) and Theorem 2 should give similar bounds, and the inner products $\langle\cdot,\cdot\rangle_M$ in (12) and $\langle\cdot,\cdot\rangle_{\mathcal{M}}$ in Theorem 2 should be close. However, only the latter highlights a potentially desirable property of a preconditioner, namely that it is self-adjoint with respect to a Hermitian sesquilinear form nearby that with respect to which the coefficient matrix is self-adjoint.

Let us examine Theorem 2 in greater detail. We first address how to choose the matrix $\mathcal{Z}$: we will employ the matrix $\mathcal{S}$ to denote the eigenvector matrix of P. Since $P = \mathcal{S}\Lambda_P\mathcal{S}^{-1}$ is $\mathcal{X}$-self-adjoint, from (8) we find that

$$\mathcal{X}P = (\mathcal{S}^{-*}Y\mathcal{S}^{-1})(\mathcal{S}\Lambda_P\mathcal{S}^{-1}) = \mathcal{S}^{-*}Y\Lambda_P\mathcal{S}^{-1}.$$

Recall that the real eigenvalues are labelled first in Λ_P. This and the structure of Y in (8) mean that $Y\Lambda_P = \left(\Lambda_P^{\frac{1}{2}}\right)^* Y\Lambda_P^{\frac{1}{2}}$,

and so $\mathcal{X}P = \mathcal{Z}^{-*}Y\mathcal{Z}^{-1}$, where $\mathcal{Z} = \mathcal{S}\Lambda_P^{-\frac{1}{2}}$. Since diagonal matrices commute, $P = \mathcal{S}\Lambda_P\mathcal{S}^{-1} = \mathcal{Z}\Lambda_P\mathcal{Z}^{-1}$, which shows that $\mathcal{Z}$ can be taken to be an eigenvector matrix of P.

We now consider the situation when E in Theorem 2 is small. With the particular choice that $\mathcal{Z}$ is an eigenvector matrix of P, we find that

$$\mathcal{Z}^{-1}P^{-1}C\mathcal{Z} = \Lambda_P(\mathcal{Z}^{-1}C\mathcal{Z}).$$

The last term will be almost diagonal when $\mathcal{Z}$ (or $\mathcal{S}$) is close to an eigenvector matrix of C, in which case E will be small. Convergence of M-MR then depends on the spectrum of $P^{-1}C$, which will be clustered when P and C have close spectra.

Furthermore, if the number of real eigenvalues of C and P is almost the same and the eigenvectors of P and C are close, then P and C will be self-adjoint with respect to nearby Hermitian sesquilinear forms. In particular, let $X = S^{-*}US^{-1}$ define the Hermitian sesquilinear form with respect to which C is self-adjoint and $\mathcal{X} = \mathcal{S}^{-*}\mathcal{U}\mathcal{S}^{-1}$ define the Hermitian sesquilinear form with respect to which P is self-adjoint, where U and $\mathcal{U}$ are of the same form as Y in (8). Then

$$X - \mathcal{X} = S^{-*}US^{-1} - \mathcal{S}^{-*}\mathcal{U}\mathcal{S}^{-1} = \mathcal{S}^{-*}\left(\mathcal{S}^{*}S^{-*}US^{-1}\mathcal{S} - \mathcal{U}\right)\mathcal{S}^{-1},$$

so that, if $S^{-1}\mathcal{S} = I + G$, $G \in \mathbb{C}^{n\times n}$, then

$$\mathcal{S}^{*}(X - \mathcal{X})\mathcal{S} = (I + G)^{*U}(I + G) - \mathcal{U}. \tag{24}$$

If G is small, we would expect $(I + G)^{*U}(I + G) \approx U$. The difference between U and $\mathcal{U}$ is related to how well P captures the number of real and complex conjugate eigenvalues of C. When this is well approximated by P, $X - \mathcal{X}$ is small. In other words, when P approximates the eigenvalues and eigenvectors of C well, P and C are self-adjoint with respect to nearby Hermitian sesquilinear forms.

One might find that P is self-adjoint with respect to a Hermitian sesquilinear form that is near to that with respect to which C is self-adjoint if P is, for example, a different discretisation of the same differential operator in the context of partial differential equations. This may also help explain why certain general preconditioners, such as incomplete factorisations, often perform well when the coefficient matrix is non-Hermitian. Examples of both of these types of preconditioners are explored in the next section. Another example may be found in [42], in which the authors discuss how preconditioners for the Euler equations are improved when the eigenvectors of the preconditioner are considered.

Applying $\mathcal{M}$-MR involves inner products with $\mathcal{M}$, which may be costly to compute even if this matrix is known. Moreover, since $\mathcal{M}$ in Theorem 2 and the eigenvector matrix $\mathcal{S}$ of P are related, when $\mathcal{S}$ is ill-conditioned $\mathcal{M}$ tends to be also. We would not advise using $\mathcal{M}$-MR in either circumstance. However, we observe from the examples below that I-MR convergence can be very similar to $\mathcal{M}$-MR convergence even when $\mathcal{M}$ is poorly conditioned. The bounds in Section 3 link the convergence of minimum residual methods in nonstandard inner products to those in standard inner products.

5. Numerical experiments

In this section, we examine some well-known preconditioners in light of the results of Section 4. To unpreconditioned systems we apply M-GMRES, where $M = S^{-*}S^{-1}$, with S the matrix of eigenvectors of the coefficient matrix C computed by Matlab. For both the unpreconditioned and preconditioned systems we use the in-built Matlab GMRES function, which is based on the GMRES method of Saad and Schultz [4], to compute I-MR residuals. Relative preconditioned $\mathcal{M}$-MR residuals, with $\mathcal{M}$ as in Theorem 2, are computed by $\mathcal{M}$-GMRES. We compute $\mathcal{S}$, a matrix of eigenvectors of P, using the Matlab function `eig`, and set $\mathcal{M}$ in Theorem 2 to be $\mathcal{M} = \mathcal{S}^{-*}\mathcal{S}^{-1}$ (see the previous section for a discussion). For M-MR, $\mathcal{M}$-MR, and I-MR, the stopping criterion is a reduction in the norm of the relative (preconditioned) residuals of 10^6, measured in the M-, $\mathcal{M}$-, or I-norm, respectively. We reiterate that it is in general infeasible to compute with nonstandard inner products, but is done here for illustrative purposes.

Tables 1–3 give the condition numbers of C, P, and $P^{-1}C$ for our examples, computed by the Matlab function `cond`. That these condition numbers are smaller than 10^{16} shows that these matrices are nonsingular. Also included in Tables 1–3 are the condition numbers of the eigenvector matrices S, $\mathcal{S}$, and Z of C, P, and $P^{-1}C$, respectively. The nonsingularity of these eigenvector matrices implies that C, P, and $P^{-1}C$ are diagonalisable. Moreover, since we choose $\mathcal{Z} = \mathcal{S}$, it follows that $\mathcal{Z}$ in Theorem 2 is nonsingular. Both P and C (and thus $P^{-1}C$) are real, and so their spectra are symmetric about the real axis. Thus, the conditions of Theorem 2 are satisfied for the following examples.

To estimate the standard eigenvalue-based GMRES bound (2), we require an approximation μ_k to the polynomial term

$$\mu_k \approx \min_{\substack{p\in\Pi_k,\\ p(0)=1}} \max_{\lambda\in\sigma(P^{-1}C)} |p(\lambda)|. \tag{25}$$

We choose to enclose the eigenvalues in an ellipse (which for all examples lies in the right half-plane) and use Chebyshev polynomials to approximate (25) as in [14, Section 6.11].

Example 1. We examine a persymmetric streamline upwind Petrov–Galerkin (SUPG) finite element discretisation of the convection–diffusion operator in [43, Example 3.3]. The differential equation satisfies

$$-\frac{1}{200}\nabla^2 u + w\cdot\nabla u = f \text{ in } \Omega = (-1,1)\times(-1,1), \quad u = g \text{ on } \partial\Omega, \tag{26}$$

Table 1
Condition numbers of C, P, and $P^{-1}C$, and their eigenvector matrices, as well as $\lambda_{\min} = \min_{\lambda\in\sigma(P^{-1}C)} |\lambda|$ and $\lambda_{\max} = \max_{\lambda\in\sigma(P^{-1}C)} |\lambda|$, for the SUPG matrix and multigrid preconditioner in Example 1.

$\kappa_2(C)$	$\kappa_2(P)$	$\kappa_2(P^{-1}C)$	$\kappa_2(S)$	$\kappa_2(\mathcal{S})$	$\kappa_2(Z)$	$\lambda_{\min}$	$\lambda_{\max}$
89.1	89.1	1.2	1.7×10^8	1.6×10^8	3.8×10^9	0.96	1.0

Table 2
Condition numbers of C, P, and $P^{-1}C$, and their eigenvector matrices, as well as $\lambda_{\min} = \min_{\lambda\in\sigma(P^{-1}C)} |\lambda|$ and $\lambda_{\max} = \max_{\lambda\in\sigma(P^{-1}C)} |\lambda|$, for the advection–diffusion problem in Example 2.

$\kappa_2(C)$	$\kappa_2(P)$	$\kappa_2(P^{-1}C)$	$\kappa_2(S)$	$\kappa_2(\mathcal{S})$	$\kappa_2(Z)$	$\lambda_{\min}$	$\lambda_{\max}$
2.6×10^3	1.5×10^3	3.8	74	88	2.8×10^{10}	0.59	1.0

Table 3
Condition numbers of C, P, and $P^{-1}C$, and their eigenvector matrices, as well as $\lambda_{\min} = \min_{\lambda\in\sigma(P^{-1}C)} |\lambda|$ and $\lambda_{\max} = \max_{\lambda\in\sigma(P^{-1}C)} |\lambda|$, for the ILU(0) test problems in Example 3.

Matrix	$\kappa_2(C)$	$\kappa_2(P)$	$\kappa_2(P^{-1}C)$	$\kappa_2(S)$	$\kappa_2(\mathcal{S})$	$\kappa_2(Z)$	$\lambda_{\min}$	$\lambda_{\max}$
Poisson2D	133	9.0	31	1.2	1.3	5.0×10^3	0.057	1.4
steam3	5.0×10^10	5.0×10^10	1.4	5.6	5.6	3.2×10^7	0.87	1.1

with a wind $w = (-\sin\frac{\pi}{6}, \cos\frac{\pi}{6})^T$ at a 30° angle to the left of vertical and boundary conditions

$$\begin{aligned} &u(x,-1) = 1, \quad 0 < x \le 1 \\ &u(1,y) = 1, \quad -1 \le y < 1 \\ &u(x,y) = 0, \quad \text{elsewhere on } \partial\Omega. \end{aligned} \tag{27}$$

The convection–diffusion equation is discretised using Ifiss 3.1 [44] with the default settings of Q1 finite elements on a 16×16 grid and SUPG with the default stabilisation parameter (defined in [43, Eq. (3.44)]). This gives rise to a matrix of dimension 289.

The geometric multigrid (GMG) preconditioner P of Ramage [45] has been shown to be effective for SUPG convection–diffusion problems for which the viscosity is not too small. For (26) and (27), P is also persymmetric, so P and C are both self-adjoint with respect to the same Hermitian sesquilinear form, defined by the reverse identity matrix (20), i.e., $E = 0$ in Theorem 2. The multigrid preconditioner is also computed by Ifiss with the default settings of line Gauss–Seidel smoothing in two directions with one pre-smoothing step and one post-smoothing step.

Fig. 1(a) shows the convergence of I-MR for the unpreconditioned system and the preconditioned system, as well as the standard eigenvalue bound (2) for GMRES applied to the preconditioned system. We see that GMRES convergence for the preconditioned system is very rapid, and that P is effective, but that the standard bound is not so descriptive of this fast convergence. Fig. 1(b) shows M-MR convergence for the unpreconditioned system and $\mathcal{M}$-MR convergence, with $\mathcal{M} = \mathcal{S}^{-*}\mathcal{S}^{-1}$, for the preconditioned system. We additionally plot μ_k, the Chebyshev polynomial approximation (25). The first observation we make is that the M-MR and I-MR convergence curves for the unpreconditioned system differ. Since M-MR convergence is bounded by a term that depends only on the eigenvalues of C (see (12)), this indicates that I-MR convergence for the unpreconditioned problem does not depend solely on eigenvalues (as discussed for a different convection–diffusion problem in [19]). In contrast, both the $\mathcal{M}$-MR and I-MR convergence curves for the preconditioned system decrease in a similar fashion, indicating that convergence of I-MR applied to the preconditioned system does depend on eigenvalues. Fig. 1(c) and (d) show that the eigenvalues of P approximate those of C well. Thus, this effective preconditioner is self-adjoint with respect to the same Hermitian sesquilinear form as C, and has eigenvalues that are nearby.

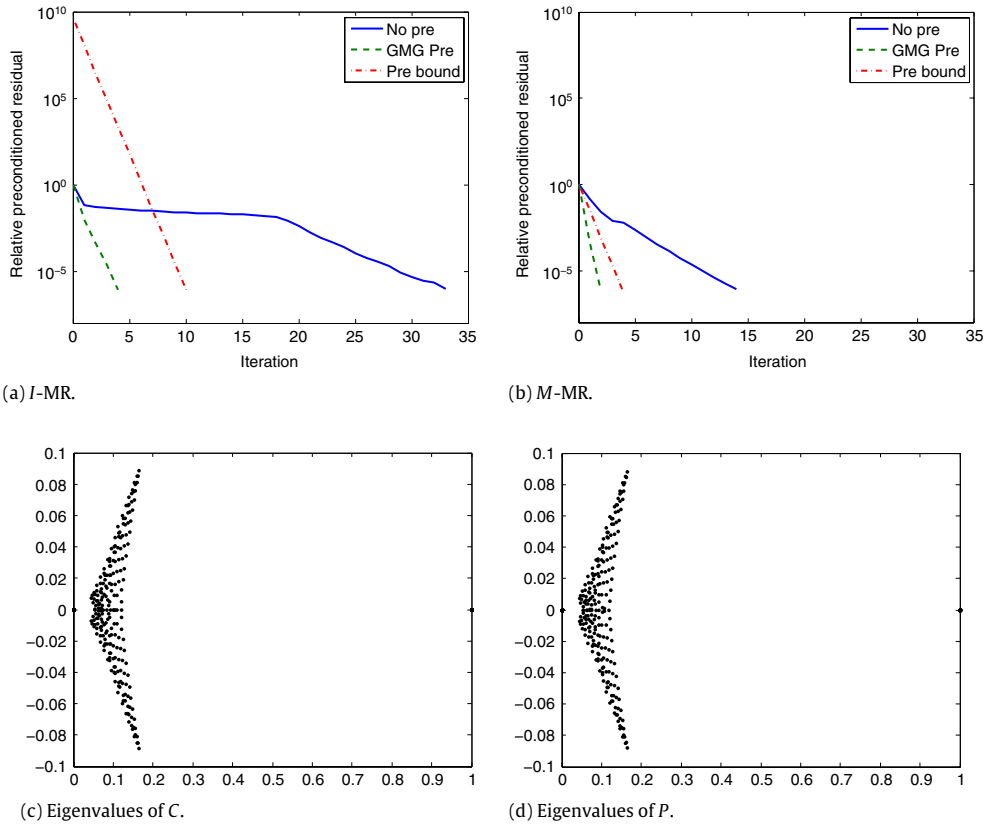
Example 2. The next example – for which P and C are self-adjoint with respect to different Hermitian sesquilinear forms – examines convergence for a finite element discretisation of a different convection–diffusion equation, the fourth example in [43, Chapter 3]. The partial differential equation is again of the form (26), but is now characterised by a recirculating wind,

$$w = (2y(1-x^2), -2x(1-y^2))^T,$$

and boundary conditions

$$\begin{aligned} &u(1,y) = 1 \quad -1 \le y \le 1 \\ &u(x,y) = 0 \quad \text{elsewhere on } \partial\Omega. \end{aligned}$$

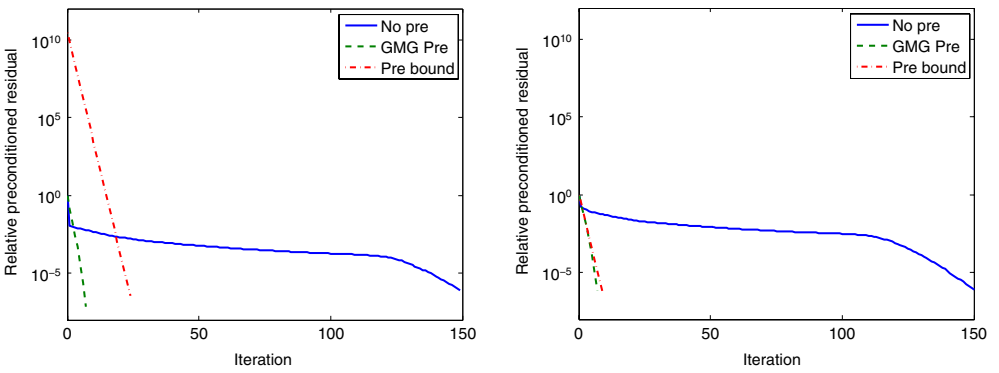
We again apply the Ramage geometric multigrid preconditioner using Ifiss, with the same settings as in Example 1. Thus, C has dimension 289.



(a) I-MR. (b) M-MR.

(c) Eigenvalues of C. (d) Eigenvalues of P.

Fig. 1. (a) $\|r_k\|_I/\|r_0\|_I$ for $Cx = c$ (solid line), $\|P^{-1}r_k\|_I/\|P^{-1}r_0\|_I$ for $P^{-1}Cx = P^{-1}b$ (dashed line), and (2) for $P^{-1}Cx = P^{-1}c$ (dash–dotted line) for I-MR, (b) $\|\widetilde{\rho}_k\|_M/\|r_0\|_M$ for $Cx = c$ (solid line) for M-MR with $M = S^{-*}S^{-1}$, $\|P^{-1}\widehat{\rho}_k\|_{\mathcal{M}}/\|P^{-1}r_0\|_{\mathcal{M}}$ for $P^{-1}Cx = P^{-1}c$ (dashed line) for $\mathcal{M}$-MR with $\mathcal{M} = \mathcal{S}^{-*}\mathcal{S}^{-1}$, and (25) for $P^{-1}Cx = P^{-1}c$ (dash–dotted line), (c) eigenvalues of C, and (d) eigenvalues of P for the SUPG matrix and multigrid preconditioner in Example 1.



(a) I-MR. (b) $\mathcal{M}$-MR.

Fig. 2. (a) $\|r_k\|_I/\|r_0\|_I$ for $Cx = c$ (solid line), $\|P^{-1}r_k\|_I/\|P^{-1}r_0\|_I$ for $P^{-1}Cx = P^{-1}c$ (dashed line), and (2) for $P^{-1}Cx = P^{-1}c$ (dash–dotted line) for I-MR and (b) $\|\widetilde{\rho}_k\|_M/\|r_0\|_M$ for $Cx = c$ (solid line) for M-MR with $M = S^{-*}S^{-1}$, $\|P^{-1}\widehat{\rho}_k\|_{\mathcal{M}}/\|P^{-1}r_0\|_{\mathcal{M}}$ for $P^{-1}Cx = P^{-1}c$ (dashed line) for $\mathcal{M}$-MR with $\mathcal{M} = \mathcal{S}^{-*}\mathcal{S}^{-1}$, and (25) for $P^{-1}Cx = P^{-1}c$ (dash–dotted line) for the SUPG matrix and multigrid preconditioner in Example 2.

Fig. 2(a) shows I-MR convergence for both the unpreconditioned and preconditioned systems, as well as the bound (2) for the preconditioned system. Again, we see that (2) does not accurately describe convergence for the preconditioned system; this is unsurprising, since the condition number of Z, the matrix of eigenvectors of $P^{-1}C$ computed by Matlab, is of the order of 10^{10} (see Table 2). Fig. 2(b) shows M-MR convergence for the unpreconditioned system, $\mathcal{M}$-MR convergence for the preconditioned system, and (25) for the preconditioned system. The latter closely matches the $\mathcal{M}$-MR convergence curve for the preconditioned system, from which we deduce that eigenvalues essentially determine the convergence of $\mathcal{M}$-MR, which could indicate that E is small. Since these eigenvalues are reasonably nicely distributed (as indicated by Table 2), convergence is fast.

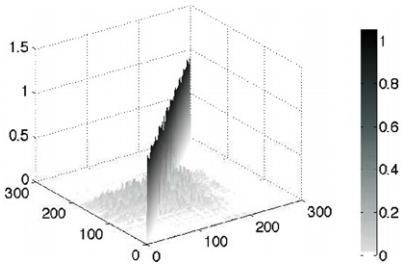


Fig. 3. $\left|\mathcal{S}^{-1}P^{-1}C\mathcal{S}\right|$ for the SUPG matrix and multigrid preconditioner in Example 2, where $\mathcal{M} = \mathcal{S}^{-*}\mathcal{S}^{-1}$. This provides an indirect measure of the size of E in Theorem 2.

Fig. 3 shows that $|\mathcal{S}^{-1}P^{-1}C\mathcal{S}|$ is close to diagonal, where $|B|$ represents the operation that takes the absolute values of all the elements of B. This indicates that $\mathcal{S}$ is a good approximation of a matrix of eigenvectors of $P^{-1}C$ and that E in Theorem 2 is small; indeed, if $E = 0$, the matrix $\mathcal{S}$ diagonalises $P^{-1}C$. We comment that the direct computation of E appears to be fraught with numerical difficulties because of the ill-conditioning of the eigenvectors of $P^{-1}C$. This does not indicate a deficiency in the theory of Section 4, but does highlight the well-known difficulties of obtaining eigenvalues and eigenvectors of highly nonnormal matrices.

For this example, both the M-MR convergence curve for the unpreconditioned system and the $\mathcal{M}$-MR convergence curve for the preconditioned system are close to the corresponding I-MR convergence curves. The bound (12) shows that M-MR convergence is bounded by a term that depends only on eigenvalues, and Theorem 2 shows that the same is essentially true of $\mathcal{M}$-MR convergence when E is small. For both the unpreconditioned and preconditioned systems, therefore, I-MR convergence seems to be determined largely by eigenvalues.

Example 3. Incomplete LU (ILU) factorisations constitute one of the best known classes of general-purpose preconditioners. It is known that ILU preconditioners tend to cluster eigenvalues, but for nonnormal matrices clustered eigenvalues do not necessarily lead to fast GMRES convergence. We examine ILU(0) preconditioning [46], as implemented in Matlab, applied to Poisson2D and steam3 matrices from the University of Florida Sparse Matrix Collection [47]. The matrix Poisson2D has 367 rows, while steam3 is smaller, with 80 rows. We show that it is indeed the eigenvalues of the preconditioned systems that lead to rapid convergence; that is, the eigenvectors are not so important for these matrices.

The left-hand plots of Fig. 4 again show I-MR convergence for the unpreconditioned and preconditioned systems, as well as the standard eigenvalue bound (2). As in previous examples, the bound (2) does not accurately reflect the rapid convergence of the preconditioned systems. This appears to be principally due to the large eigenvector condition number of $P^{-1}C$ (see Table 3). M-MR convergence for the unpreconditioned system, $\mathcal{M}$-MR convergence for the preconditioned system, and (25) are shown on the right-hand plots of Fig. 4. Similarly to I-MR, M-MR for the unpreconditioned system stagnates, while $\mathcal{M}$-MR and I-MR converge similarly (and rapidly) for the preconditioned systems.

We also note that the $\mathcal{M}$-MR convergence curve for the preconditioned systems and the Chebyshev polynomial approximations decrease in a similar fashion; this indicates that the eigenvalues of the preconditioned system primarily determine convergence, i.e., that E in Theorem 2 is small. Further evidence that E is small is given by Fig. 5, which shows that the matrix $|\mathcal{S}^{-1}P^{-1}B\mathcal{S}|$ is almost diagonal for both test problems. Since I-MR residuals for the preconditioned system behave similarly to $\mathcal{M}$-MR residuals, it appears that even for the standard method in the Euclidean inner product it is eigenvalues that determine GMRES convergence, for this example, despite the large eigenvector condition number.

6. Conclusions

In the context of preconditioning non-Hermitian linear systems for solution by minimum residual methods, we have shown that, when the preconditioner is self-adjoint with respect to a Hermitian sesquilinear form that is nearby that with respect to which the coefficient matrix is self-adjoint, convergence of a minimum residual method in a specific nonstandard inner product essentially depends on the eigenvalues of the preconditioned coefficient matrix. This has been verified numerically for a range of linear systems and preconditioners. Eigenvalue-dependent convergence is also observed for the same examples for a standard minimum residual method in the Euclidean inner product. It would be desirable to have a method for obtaining preconditioners for which eigenvalue-dependent convergence of I-MR was obtained; this could form the basis of future research in this area. We have also presented a GMRES convergence bound (18) that highlights situations in which (2) may be pessimistic. Heuristics based on this bound could be useful for predicting GMRES convergence.

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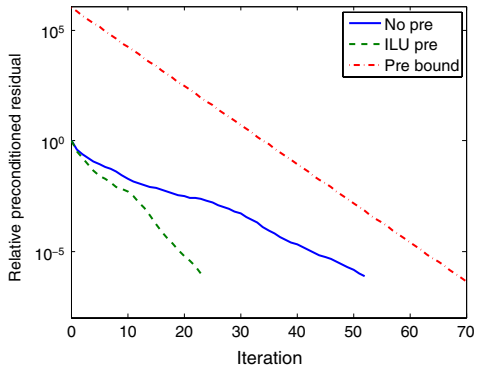
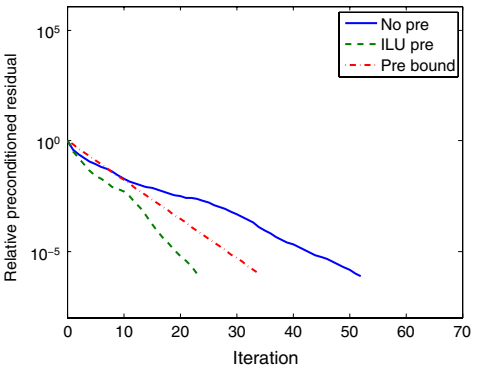
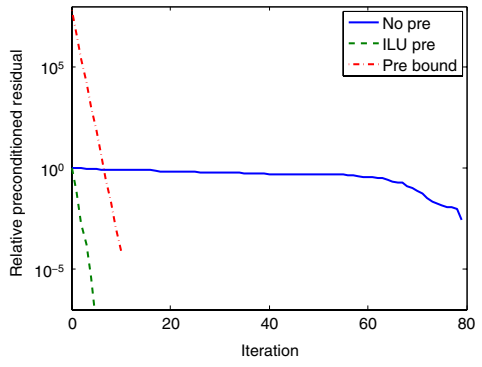


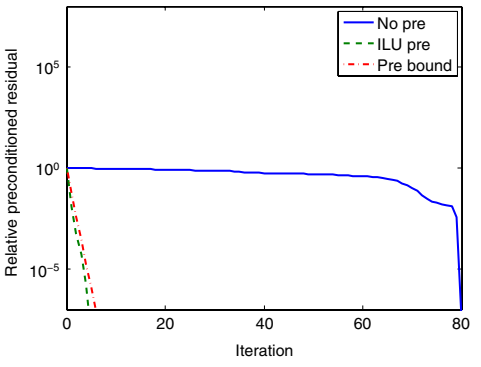
(a) Poisson2D: I-MR.



(b) Poisson2D: $\mathcal{M}$-MR.

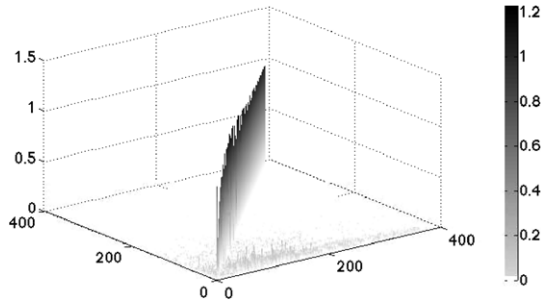


(c) steam3: I-MR.

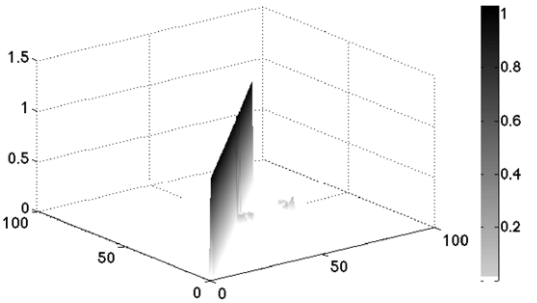


(d) steam3: $\mathcal{M}$-MR.

Fig. 4. The left-hand plots show $\|r_k\|_I/\|r_0\|_I$ for $Cx = c$ (solid line), $\|P^{-1}r_k\|_I/\|P^{-1}r_0\|_I$ for $P^{-1}Cx = P^{-1}c$ (dashed line), and (2) for $P^{-1}Cx = P^{-1}c$ (dash–dotted line) for I-MR. The right-hand plots show $\|\widetilde{\rho}_k\|_M/\|r_0\|_M$ for $Cx = c$ (solid line) for M-MR with $M = S^{-*}S^{-1}$, $\|P^{-1}\widehat{\rho}_k\|_{\mathcal{M}}/\|P^{-1}r_0\|_{\mathcal{M}}$ for $P^{-1}Cx = P^{-1}c$ (dashed line) for $\mathcal{M}$-MR with $\mathcal{M} = \mathcal{S}^{-*}\mathcal{S}^{-1}$, and (25) for $P^{-1}Cx = P^{-1}c$ (dash–dotted line) for the ILU(0) test problems in Example 3.



(a) Poisson2D.



(b) steam3.

Fig. 5. $\left|\mathcal{S}^{-1}P^{-1}C\mathcal{S}\right|$ for the ILU(0) test problems in Example 3, where $\mathcal{M} = \mathcal{S}^{-*}\mathcal{S}^{-1}$. This provides an indirect measure of the size of E in Theorem 2.

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