# SOLVING ELLIPTIC PROBLEMS BY DOMAIN DECOMPOSITION METHODS WTIH APPLICATIONS

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## I. INTRODUCTION

Solving elliptic problems using *domain decomposition* is an idea which seems to originate with the classic Schwarz alternating method. Actually, many well-known methods for solving linear systems originating from the approximation of elliptic problems (e.g., block over-relaxation, block Gauss or Cholesky direct methods, etc.) take advantage of a subdomain decomposition.

The main goal of this paper is to discuss several variants of the Schwarz method. Used as preconditioners, they provide efficient tools for solving difficult nonlinear problems on complicated 2-D and 3-D geometries (cf: [1] (to which we also refer for more details on some of the methods discussed in this paper)).

## II. FORMULATION OF THE MODEL PROBLEM. GENERALITIES

Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ ($N = 2,3$ in practice) with a smooth boundary $\Gamma = \partial\Omega$. Consider the following problem

(2.1) $\qquad -\Delta y = f \quad \text{in } \Omega, \qquad y = g \quad \text{on } \Gamma$

(where f and g are *given* functions). In the remainder of this paper we consider the following two types of decomposition of $\Omega$ by subdomains $\Omega_{ii}$:

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(a) With

$$(2.2) \qquad \Omega = \overline{\bigcup_i \overset{\circ}{\Omega}_{ii}}$$

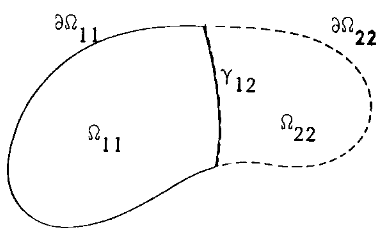
where $\overline{X}$ and $\overset{\circ}{X}$ denote, respectively, the *closure* and the *interior* of a set X, and also with

$$(2.3) \qquad \Omega_{ii} \cap \Omega_{jj} = \emptyset \quad \forall i,j,\ i \neq j,$$

we define $\gamma_{ij}$ (possibly empty) by

$$\gamma_{ij} = \gamma_{ji} = \partial\Omega_{ii} \cap \partial\Omega_{jj} \quad \text{if } i \neq j .$$

Figure 1 illustrates this decomposition for the case $i = 1,2$ in (2.2).



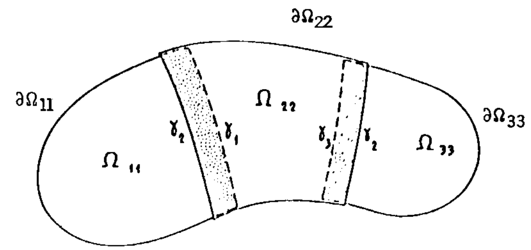

A decomposition of type (a)

Figure 1

(b) With (2.2) still true, but (2.3) replaced, for $i \neq j$, by

$$(2.4) \qquad \begin{cases} \textit{either } \Omega_{ii} \cap \Omega_{jj} = \emptyset, \\ \\ \textit{or } \Omega_{ii} \cap \Omega_{jj} = \Omega_{ij} (= \Omega_{ji}) \textit{ with } \int_{\Omega_{ij}} dx > 0 , \end{cases}$$

(where $dx = dx_1 dx_2$). We also define $\gamma_i = \partial\Omega_{ii} - (\partial\Omega_{ii} \cap \Gamma)$. Figure 2 illustrates this situation (with i = 1,2,3); a more general decomposition of this type is shown on Figure 3.




A decomposition of type (b)

Figure 2

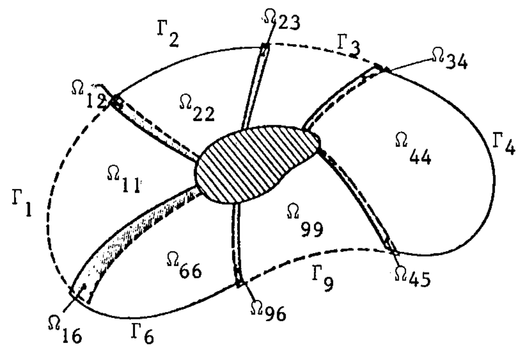



Figure 3

To conclude this section, we introduce some *Sobolev* functional spaces, which we use in later sections, namely

$$(2.5) \qquad H^1(\Omega) = \{v \mid v \in L^2(\Omega),\ \frac{\partial v}{\partial x_i} \in L^2(\Omega) \quad \forall i = 1,\dots,N\} ,$$

$$(2.6) \qquad H_0^1(\Omega) = \{v \mid v \in H^1(\Omega),\ v = 0 \quad on\ \Gamma\} .$$

Note the space $H^1(\Omega)$ is a *Hilbert space* for the inner product

$$(2.7) \qquad (u,v)_{H^1(\Omega)} = \int_\Omega uv \, dx + \int_\Omega \underset{\sim}{\nabla} u \cdot \underset{\sim}{\nabla} v \, dx \; ;$$

moreover $H^1(\Omega)$ is a *closed* subspace of $H^1(\Omega)$, and if $\Omega$ is bounded (at least in one direction of $\mathbb{R}^N$), then $H_o^1(\Omega)$ is a Hilbert space if associated with the inner product

$$(2.8) \qquad (u,v)_{H_o^1(\Omega)} = \int_\Omega \underset{\sim}{\nabla} u \cdot \underset{\sim}{\nabla} v \, dx \; ,$$

and the corresponding norm

$$||v||_{H_o^1(\Omega)} = \left(\int_\Omega |\underset{\sim}{\nabla} v|^2 dx\right)^{1/2}$$

is equivalent to the norm induced by $H^1(\Omega)$. Suppose that $L^2(\Omega)$ has been identified to its dual space; then if $H^{-1}(\Omega)$ is the dual space of $H_o^1(\Omega)$ we have

$$(2.9) \qquad H_o^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega) .$$

Furthermore, operator $-\Delta$ is an *isomorphism* from $H_o^1(\Omega)$ onto $H^{-1}(\Omega)$; we shall denote by $<.,.>$ the duality pairing between $H^{-1}(\Omega)$, such that

$$(2.10) \qquad <f,v> = \int_\Omega fv \, dx \quad \forall f \in L^2(\Omega), \quad \forall v \in H_o^1(\Omega) \; .$$

If $f \in H^{-1}(\Omega)$, we have

$$(2.11) \qquad ||f||_{-1} = \sup_{v \in H_o^1(\Omega)-\{0\}} \frac{|<f,v>|}{||v||_{H_o^1(\Omega)}}$$

For more properties and details on Sobolev spaces we refer to e.g., [2]-[4]. In the next sections, similar definitions will also be used for the subdomains $\Omega_{ii}$, $\Omega_{ij}$.

## III. THE SCHWARZ ALTERNATING METHOD

The Schwarz alternating method was introduced around 1865 to solve Dirichlet problems by H.A. Schwarz, who proved its convergence using the *Maximum Principle*. It can be summarized as follows.

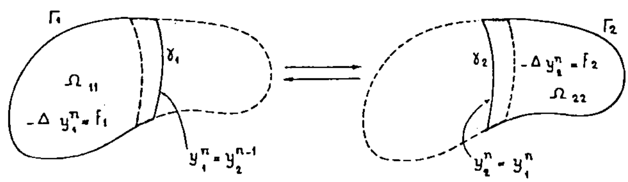
With $\Omega$ split as in Figure 4 into subdomains $\Omega_{11}$ and $\Omega_{22}$, let $f_i$ and $g_i$ denote the restriction of f and g from (2.1) to $\Omega_{ii}$ and $\partial\Omega_{ii} \cap \Gamma$, respectively. Following the notation of Sec. II, the Schwarz algorithm is then defined by

*Let* $\eta$ *be a function defined on* $\gamma_1$*, then for* $n \geq 0$ *construct a sequence* $\{y_1^n, y_2^n\}$ *as follows*

$$(3.1) \qquad \begin{cases} -\Delta y_1^n = f_1 \text{ in } \Omega_{11}, & y_1^n = g_1 \text{ on } \partial\Omega_{11} \cap \Gamma , \\ \\ y_1^n|_{\gamma_1} = y_2^{n-1}|_{\gamma_1} \text{ if } n \geq 1, & y_1^o|_{\gamma_1} = \eta , \end{cases}$$

$$(3.2) \qquad \begin{cases} -\Delta y_2^n = f_2 \text{ in } \Omega_{22}, & y_2^n = g_2 \text{ on } \partial\Omega_{22} \cap \Gamma , \\ y_2^n|_{\gamma_2} = y_1^n|_{\gamma_2} . \end{cases}$$

The mechanism of the above method is shown on Figure 4.




Mechanism of the Schwarz alternating method

Figure 4

For more properties of the Schwarz alternating method see [5].

## IV. DOMAIN DECOMPOSITION METHODS WITH LAGRANGE MULTIPLIERS

### IV.1. A first method

Problem (2.1) is equivalent to the following *minimization* problem

$$(4.1) \qquad \textit{Find } y \in V_g \textit{ such that } J(y) \leq J(z), \ \forall z \in V_g ,$$

where

$$(4.2)\quad V_g = \{z \mid z \in H^1(\Omega),\ z = g \text{ on } \Gamma\},\ J(z) = \frac{1}{2}\int_\Omega |\nabla z|^2 dx - \int_\Omega fz dx.$$

We consider now a decomposition of $\Omega$ as done in case (a), Sec. II (we only consider a two-subdomain decomposition as in Figure 1); we associate to this decomposition the functionals $J_i$: $H^1(\Omega_{ii}) \to \mathbb{R}$ defined (with $f_i = f|_{\Omega_{ii}}$, $g_i = g|_{\partial\Omega_{ii}\cap\Gamma}$) by

$$(4.3)\qquad J_i(z_i) = \frac{1}{2}\int_{\Omega_{ii}} |\nabla z_i|^2 dx - \int_{\Omega_{ii}} f_i z_i dx,\quad i = 1,2\ .$$

We clearly have equivalence (in that $y_i = y|_{\Omega_{ii}}$, $\forall i = 1,2$) between (2.1), (4.1) and the following minimization problem

$$(4.4)\qquad \begin{cases} \textit{Find } \{y_1,y_2\} \in W \textit{ such that} \\ \\ J_1(y_1) + J_2(y_2) \leq J_1(z_1) + J_2(z_2) \quad \forall\{z_1,z_2\} \in W \end{cases}$$

where $W = \{\{z_1,z_2\} \in H^1(\Omega_{11})\times H^1(\Omega_{22}),\ \ z_1 = z_2 \text{ on } \gamma_{12}$, $z_i|_{\partial\Omega_{ii}\cap\Gamma} = g_i,\ \forall i = 1,2\}$.

To overcome the main difficulty in (4.4), namely the linear relation

$$(4.5)\qquad z_1|_{\gamma_{12}} = z_2|_{\gamma_{12}}\ ,$$

we introduce a *Lagrange multiplier*, via the *lagrangian functional* L defined by

$$(4.6)\qquad L(z_1,z_2;\mu) = J_1(z_1) + J_2(z_2) + \int_{\gamma_{12}} \mu(z_2 - z_1)d\gamma\ .$$

We can prove the following

<u>Proposition 4.1</u>. *Define the spaces* V *and* $\Lambda$ *by*

$$(4.7)\begin{cases} V = \{\{z_1,z_2\},\ \ \forall i = 1,2, z_i \in H^1(\Omega_{ii}),\ \ z_i|_{\partial\Omega_{ii}\cap\Gamma} = g_i\}, \\ \Lambda = L^2(\gamma_{12}) \end{cases}$$

*and suppose that* $\{y_1, y_2; \lambda\}$ *is a saddle-point of* L *over* $V\times\Lambda$ *i.e.* $\{y_1, y_2; \lambda\} \in V\times\Lambda$ *and*

$$(4.8)\quad L(y_1,y_2;\mu) \leq L(y_1,y_2;\lambda) \leq L(z_1,z_2;\lambda), \quad \forall\{z_1,z_2;\mu\}\in V\times\Lambda .$$

*We have then,* $\forall i = 1,2,$

$$y_i = y|_{\Omega_{ii}} \ \ and\ \ \lambda = \frac{\partial y_1}{\partial n_1}\Big|_{\gamma_{12}} = -\frac{\partial y_2}{\partial n_2}\Big|_{\gamma_{12}}$$

*where* y *is the solution of* (2.1) *and* $\underset{\sim}{n}_i$ *the outward unit normal vector at* $\partial\Omega_{ii}$. *The reciprocal property holds if* y *is sufficiently smooth.*

From Proposition 4.1, we can replace the solution of (2.1), (4.1) by the solution of the saddle-point problem (4.8). It follows then from [6, Chapter 2 and Appendix 2] and [7] that (4.8) can be solved by a *saddle-point solver* like Uzawa's algorithm (and its conjugate gradient variants). The basic Uzawa algorithm corresponding to (4.8) is defined by

(4.9) *Let* $\lambda^o \in \Lambda$ *be arbitrarily given,*

*then for* $n \geq 0$, *assuming* $\lambda^n$ *known, we compute* $y_1^n, y_2^n$ *and* $\lambda^{n+1}$ *by*

$$(4.10)\quad \begin{cases} -\Delta y_1^n = f_1 \ in\ \Omega_{11}, \\ y_1^n = g_1 \ on\ \partial\Omega_{11}\cap\Gamma, \quad \dfrac{\partial y_1^n}{\partial n_1} = \lambda^n \ on\ \gamma_{12}, \end{cases}$$

$$(4.11)\quad \begin{cases} -\Delta y_2^n = f_2 \ in\ \Omega_{22}, \\ y_2^n = g_2 \ on\ \partial\Omega_{22}\cap\Gamma, \quad \dfrac{\partial y_2^n}{\partial n_2} = -\lambda^n \ on\ \gamma_{12}, \end{cases}$$

$$(4.12)\quad \lambda^{n+1} = \lambda^n + \rho(y_2^n - y_1^n)|_{\gamma_{12}}, \quad \rho > 0 .$$

The following can be shown about the convergence of algorithm (4.9)-(4.12).

Proposition 4.2.: *Suppose that the saddle-point problem* (4.8) *has a solution; then* $\forall \overset{\circ}{\lambda} \in \Lambda$, *we have, for* i = 1,2,

$$\lim_{n\to+\infty} y_i^n = y_i = y|_{\Omega_{ii}} \quad \textit{strongly in } H^1(\Omega_{ii}) \tag{4.13}$$

(*where* y *is the solution of* (2.1), (4.1)), *if*

$$0 < \rho < 2/c^2 \tag{4.14}$$

*where, in* (4.14), $c = \sup\limits_{\{z_1,z_2\}\in Z-\{0\}} \dfrac{||z_2-z_1||_{L^2(\gamma_{12})}}{||\{z_1,z_2\}||_Z}$, *and*

*where*

$$Z = \{\{z_1,z_2\} \in H^1(\Omega_{11}) \times H^1(\Omega_{22}),\quad z_i|_{\partial\Omega_{ii}\cap\Gamma} = 0 \quad \forall i = 1,2\},$$

$$||\{z_1,z_2\}||_Z = (\sum_{i=1}^{2} \int_{\Omega_{ii}} |\underset{\sim}{\nabla} z_i|^2\, dx)^{1/2} .$$

Algorithm (4.9)-(4.12) is in fact a *gradient algorithm* with the *constant step* $\rho$, applied to the solution of the *dual problem* associated to (4.8), i.e.

$$\textit{Find } \lambda \in \Lambda \textit{ such that } J^*(\lambda) \leq J^*(\mu) \quad \forall \mu \in \Lambda \tag{4.15}$$

where $J^*(\mu) = -\inf\limits_{\{z_1,z_2\}\in V} L(z_1,z_2;\mu)$. It is easy to see that $J(\mu) = a(\mu,\mu)/2 - \ell(\mu)$, where $a(.,.)$ is *bilinear, continuous symmetric* over $\Lambda\times\Lambda$ (but *not* strongly-elliptic) and where $\ell(\cdot)$ is *linear continuous* over $\Lambda$. The above properties clearly suggest a *conjugate gradient* algorithm for solving (4.15), and therefore (2.1); to derive such an algorithm we define a linear *boundary* operator A as follows:

*Let* $\lambda \in \Lambda$; *to* $\lambda$ *we associate, for* i = 1,2, $y_i(\lambda) = y_i$ *as the solution in* $H^1(\Omega_{ii})$ *of*

$$\begin{cases} \Delta y_i = 0 \textit{ in } \Omega_{ii},\quad y_i = 0 \textit{ on } \partial\Omega_{ii}\cap\Gamma,\quad \dfrac{\partial y_i}{\partial n_i} = (-1)^{i-1}\lambda \\ \textit{on } \gamma_{12}, \end{cases} \tag{4.16}$$

*and then define* A *by*

$$(4.17)\qquad A\lambda = (y_1(\lambda) - y_2(\lambda))|_{\gamma_{12}} .$$

Using *Green's formula,* we should prove that $\forall\lambda,\mu \in \Lambda$ we have

$$(4.18)\qquad \int_{\gamma_{12}} A\lambda\mu \, d\gamma = \sum_{i=1}^{2} \int_{\gamma_{12}} y_i(\lambda) \frac{\partial y_i}{\partial n_i}(\mu)\, d\gamma$$

$$= \sum_{i=1}^{2} \int_{\Omega_{ii}} \nabla y_i(\lambda) \cdot \nabla y_i(\mu)\, dx .$$

It follows from (4.18) that A is *symmetric* and *positive definite* over $\Lambda$; it is also *continuous* but *not* $\Lambda$*-elliptic.* We should observe that the bilinear form $a(.,.)$ considered above satisfies

$$(4.19)\qquad a(\lambda,\mu) = \int_{\gamma_{12}} A\lambda\mu \, d\gamma, \quad \forall\lambda,\mu \in \Lambda .$$

The main result concerning the dual problem (4.15) is the following

Proposition 4.3.: *Let* $\lambda$ *be the solution of the dual problem* (4.15); *then* $\lambda$ *is also the solution of the following linear problem in* $\Lambda$:

$$(4.20)\qquad A\lambda = (y_{o2} - y_{o1})|_{\gamma_{12}}$$

*where* $\{y_{o1}, y_{o2}\} \in H^1(\Omega_{11}) \times H^1(\Omega_{22})$ *is defined (with* $i = 1,2$*) by*

$$(4.21)\quad \begin{cases} -\nabla y_{oi} = f_i \text{ in } \Omega_{ii}, \quad y_{oi} = g_i \text{ on } \partial\Omega_{ii} \cap \Gamma, \\ \dfrac{\partial y_{oi}}{\partial n_i} = 0 \text{ on } \gamma_{12}. \end{cases}$$

From the above results we derive the following *conjugate gradient* variant of algorithm (4.9)-(4.12):

Step 0: *Initialization*

(4.22) $\lambda^o \in \Lambda$, *arbitrarily given,*

(4.23) $$g^o = A\lambda^o - (y_{o2} - y_{o1})|_{\gamma_{12}} ,$$

(4.24) $$w^o = g^o ;$$

*then for* $n \geq 0$, *assuming that* $\{\lambda^n, w^n, g^n\} \in \Lambda^3$ *are known, compute* $\lambda^{n+1}$, $w^{n+1}$, $g^{n+1}$ *by*

Step 1: *Descent*

(4.25) $$\lambda^{n+1} = \lambda^n - \rho_n w^n \text{ with } \rho_n = \frac{\int_{\gamma_{12}} g^n w^n d\gamma}{\int_{\gamma_{12}} (Aw^n) w^n d\gamma} = \frac{\int_{\gamma_{12}} |g^n|^2 d\gamma}{\int_{\gamma_{12}} (A\, w^n) w^n d\gamma}$$

Step 2: *New descent direction*

(4.26) $$g^{n+1} = g^n - \rho_n Aw^n ,$$

(4.27) $$w^{n+1} = g^{n+1} + \gamma_n w^n, \text{ with } \gamma_n = \frac{||g^{n+1}||^2_\Lambda}{||g^n||^2_\Lambda}$$

*Do* $n = n+1$ *and go to* (4.25).

Despite its abstract formalism algorithm (4.22)-(4.27) is no more complicated to implement than algorithm (4.9)-(4.12) and like this later requires only the solution at each iteration of a Poisson problem on each $\Omega_{ii}$; indeed $g^o$ is obtained from $\lambda^o$ (with $i = 1,2$) by

(4.28) $$\begin{cases} -\Delta y_i^o = f_i \text{ in } \Omega_{ii}, \quad y_i^o = g_i \text{ on } \partial\Omega_{ii} \cap \Gamma, \\ \dfrac{\partial y_i^o}{\partial n_i} = (-1)^{i-1}\lambda^o \text{ on } \gamma_{12}, \end{cases}$$

(4.29) $$g^o = (y_1^o - y_2^o)|_{\gamma_{12}} .$$

Concerning Steps 1 and 2, the key point is the calculation of $Aw^n$, which is done as follows:

*We define* $\{\chi_1^n,\chi_2^n\} \in H^1(\Omega_{11}) \times H^1(\Omega_{22})$ *by*

$$(4.30)\qquad \begin{cases} \Delta\chi_i^n = 0 \text{ in } \Omega_{ii}, \quad \chi_i^n = 0 \text{ on } \partial\Omega_{ii} \cap \Gamma, \\ \dfrac{\partial\chi_i^n}{\partial n_i} = (-1)^{i-1} w^n \text{ on } \gamma_{12}, \end{cases}$$

*for* i = 1,2, *and then*

$$(4.31)\qquad Aw^n = (\chi_1^n - \chi_2^n)|_{\gamma_{12}} ;$$

thus one iteration of (4.22)-(4.27) is no more costly than one iteration of (4.9)-(4.12). Another conjugate gradient variant of (4.9)-(4.12) is given in [8]; it is however *twice as costly per iteration* as (4.22)-(4.27).

## IV.2. A second method

The methods discussed in Sec. IV.1 behave quite well as long as the Poisson problem to be solved has Dirichlet boundary conditions (or is at least Dirichlet boundary conditions dominated); actually methods with better convergence properties are obtained if we take as unknown the trace of the solution of (2.1) on $\gamma_{12}$ and we adjust its values in order to satisfy $(\partial y_1/\partial n_1 + \partial y_2/\partial n_2 = 0)$ which appears then as a *linear constraint*, the corresponding *Lagrange multiplier* being $\lambda = y|_{\gamma_{12}}$. The following algorithm is founded on the above principle (and can be viewed too as an *Uzawa's algorithm)*:

$$(4.32)\qquad \text{Given } \lambda^0 \in \Lambda_g ,$$

*then for* $n \geq 0$, *assuming* $\lambda^n$ *known, we compute* $y_1^n$, $y_2^n$, *by*:

$$(4.33)\qquad \begin{cases} -\Delta y_i^n = f_i \text{ in } \Omega_{ii}, \quad y_i^n = g_i \text{ on } \partial\Omega_{ii} \cap \Gamma, \quad y_i^n = \lambda^n \\ \text{on } \gamma_{12};\ i = 1,2 \end{cases}$$

*to compute* $\lambda^{n+1}$ *from* $y_1^n$, $y_2^n$ *we first introduce*

$\delta^{n+1/2} = \lambda^{n+1} - \lambda^n$ *and we solve on* $\Lambda_o$ *the variational problem*

$$(4.34)\quad \begin{cases} \textit{Find } \delta^{n+1/2} \in \Lambda_o, \textit{ such that } \forall\mu \in \Lambda_o \\ \sum_{i=1}^{2} \int_{\Omega_{ii}} \nabla \tilde{\delta}_i^{n+1/2} \cdot \nabla \tilde{\mu}_i dx \\ \qquad = -\rho \sum_{i=1}^{2} \left( \int_{\Omega_{ii}} \nabla y_i^n \cdot \nabla \tilde{\mu}_i dx - \int_{\Omega_{ii}} f_i \tilde{\mu}_i dx \right) ; \end{cases}$$

in (4.32)-(4.34) we have

$$(4.35)\quad \Lambda_o = \{\mu | \mu \in L^2(\gamma_{12}), \quad \mu = \tilde{\mu}|_{\gamma_{12}} \textit{ with } \tilde{\mu} \in H_0^1(\Omega)\} ,$$

$$(4.36)\quad \begin{cases} \Lambda_g = \{\mu | \mu \in L^2(\gamma_{12}), \quad \mu = \tilde{\mu}|_{\gamma_{12}} \textit{ with } \tilde{\mu} \in H^1(\Omega), \\ \tilde{\mu} = g \textit{ on } \Gamma\} \end{cases}$$

and, in (4.34), $\tilde{q}_i$ is defined, for i = 1,2, from $q \in \Lambda_o$ as follows:

$$(4.37)\quad \tilde{q}_i \in H^1(\Omega_{ii}), \quad \tilde{q}_i = 0 \textit{ on } \partial\Omega_{ii} - \gamma_{12}, \quad \tilde{q}_i = q \textit{ on } \gamma_{12} .$$

In practice to obtain *unique* extensions $q_1$, $q_2$ of q we introduce, for i = 1,2, $H_i$ ($\subset H^1(\Omega_{ii})$) by

$$(4.38)\quad H_i = \{v_i | v_i \in H^1(\Omega_{ii}), \quad v_i = 0 \textit{ on } \partial\Omega_{ii} - \gamma_{12}\} ,$$

and then $P_i$ such that

$$(4.39)\quad H_i = H_o^1(\Omega_{ii}) \oplus P_i ;$$

if $q \in \Lambda_o$ is given we shall take for $\tilde{q}_i$ the *unique element* of $P_i$ such that $\tilde{q}_i|_{\gamma_{12}} = q$. We observe using Green's formula that we have for the right-hand side in (4.34)

$$(4.40)\qquad \sum_{i=1}^{2}\left(\int_{\Omega_{ii}} \nabla y_i^n\cdot\nabla\tilde{\mu}_i\,dx - \int_{\Omega_{ii}} f_i\tilde{\mu}_i dx\right) = \int_{\gamma_{12}} \left(\frac{\partial y_1^n}{\partial n_1}+\frac{\partial y_2^n}{\partial n_2}\right)\mu d\gamma \quad \forall\mu\in\Lambda_o\ ;$$

in (4.40) the left-hand side is easier to implement (in combination with the usual *finite element approximations*) than the right-hand side, since it does not require the explicit knowledge of the normal derivatives $\frac{\partial y_1^n}{\partial n_1}$, $\frac{\partial y_2^n}{\partial n_2}$ along $\gamma_{12}$.

To conclude Sec. IV.2 we shall describe a *conjugate gradient* variant of (4.32)-(4.34) defined as follows:

Step 0: *Initialization*

$$(4.41)\qquad \lambda^o\in\Lambda_g,\ \textit{arbitrarily given,}$$

*solve then for* i = 1,2

$$(4.42)\qquad \begin{cases} -\Delta y_i^o = f_i \ \textit{in}\ \Omega_{ii},\quad y_i^o = g_i\ \textit{on}\ \partial\Omega_{ii}-\gamma_{12}\ , \\ y_i^o = \lambda^o\ \textit{on}\ \gamma_{12}, \end{cases}$$

*and then*

$$(4.43)\qquad \begin{cases} \textit{Find}\ g^o\in\Lambda_o\ \textit{such that}\ \forall\mu\in\Lambda_o \\ \displaystyle\sum_{i=1}^{2}\int_{\Omega_{ii}} \nabla\tilde{g}_i^o\cdot\nabla\tilde{\mu}_i\,dx = \sum_{i=1}^{2}\left(\int_{\Omega_{ii}} \nabla y_i^o\cdot\nabla\tilde{\mu}_i\,dx - \int_{\Omega_{ii}} f_i\tilde{\mu}_i dx\right) \end{cases}$$

*and set*

$$(4.44)\qquad w^o = g^n\ . \qquad \square$$

For $n \geq 0$, suppose that $\lambda^n$, $g^n$, $w^n$ are known and compute $\lambda^{n+1}$, $g^{n+1}$, $w^{n+1}$ as follows:

Step 1: *Descent*

*Define* $\{\chi_i^n\}_{i=1}^2 \in H^1(\Omega_{11}) \times H^1(\Omega_{22})$ *by*

$$(4.45)\quad \Delta\chi_i^n = 0 \ \text{in}\ \Omega_{ii},\quad \chi_i^n = 0 \ \text{on}\ \partial\Omega_{ii} - \gamma_{12},\quad \chi_i^n = w^n \ \text{on}\ \gamma_{12}\ ,$$

*and then*

$$(4.46)\quad \lambda^{n+1} = \lambda^n - \rho_n w^n,\ \text{with}\ \rho_n = \frac{\sum_{i=1}^{2} \int_\Omega |\nabla \tilde{g}_i^n|^2\, dx}{\sum_{i=1}^{2} \int_\Omega \nabla \chi_i^n \cdot \nabla \tilde{w}_i^n\, dx}$$

Step 2: *Descent direction*

*Solve*

$$(4.47)\quad \begin{cases} \text{Find } g^{n+1} \in \Lambda_o \text{ such that } \forall \mu \in \Lambda_o \\ \sum_{i=1}^{2} \int_{\Omega} \nabla \tilde{g}_i^{n+1} \cdot \nabla \tilde{\mu}_i dx = \sum_{i=1}^{2} \left( \int_{\Omega} \nabla \tilde{g}_i^n \cdot \nabla \tilde{\mu}_i dx - \rho_n \int_{\Omega} \nabla \chi_i^n \cdot \nabla \tilde{\mu}_i dx \right), \end{cases}$$

*and set*

$$(4.48)\quad w^{n+1} = g^{n+1} + \gamma_n w^n,\ \text{with}\ \gamma_n = \frac{\sum_{i=1}^{2} \int_{\Omega_{ii}} |\nabla \tilde{g}^{n+1}|^2\, dx}{\sum_{i=1}^{2} \int_{\Omega_{ii}} |\nabla \tilde{g}^n|^2\, dx}\ .$$

*Do* n = n+1 *and go to* (4.45).

The notation used in (4.41)-(4.48) are those used previously in (4.32)-(4.34). Algorithm (4.41)-(4.48) (in fact *finite element* variants of it) has proved being very efficient for solving elliptic problems using domain decomposition. A generalization of the method of Sec. IV.2 will be discussed in the sequel; it concerns the solution of the full potential equation for transonic flows.

## V. A NEW DOMAIN DECOMPOSITION METHOD

### V.1. Motivation. An equivalence results

The two main goals of the method which follows are:

(i) Find a *conjugate gradient* variant of the *Schwarz's algorithm* (3.1), (3.2); and

(ii) Develop some experience of matching methods using *least squares*.

We suppose that $\Omega$ has been split in several subdomains according to situation (b) of Sec. II (whose notation is kept) and we suppose for simplicity that i,j = 1,2 in (2.2), (2.4). We introduce, for i = 1,2, the following spaces

(5.1) $$V_i = \{v_i \in L^2(\gamma_i),\quad v_i = \tilde{v}_i|_{\gamma_i} \text{ with } \tilde{v}_i \in H^1(\Omega_{ii}),\quad \tilde{v}_i = g_i \text{ on } \partial\Omega_{ii} \cap \Gamma\}\ ;$$

we have then the following (obvious):

Proposition 5.1: *The minimization problem*

*Find* $\{u_1,u_2\} \in V_1 \times V_2$ *such that*

(5.2) $$J(u_1,u_2) \leq J(v_1,v_2),\quad \forall \{v_1,v_2\} \in V_1 \times V_2\ ,$$

*where*

(5.3) $$J(v_1,v_2) = \frac{1}{2}\int_{\Omega_{12}} \{|\nabla(y_2 - y_1)|^2 + (y_2 - y_1)^2\}\, dx$$

*with* $y_i$ *a solution in* $H^1(\Omega_{ii})$ *(for* i = 1,2*) of*

(5.4) $$-\Delta y_i = f_i \text{ in } \Omega_{ii},\quad y_i = g_i \text{ on } \partial\Omega_{ii} \cap \Gamma,\quad y_i = v_i \text{ on } \gamma_i,$$

*has a unique solution such that*

(5.5) $$u_i = y_i,\quad \forall i = 1,2, \text{ where } y \text{ is the solution of } (2.1).$$

Problem (5.2) clearly has the structure of an *optimal control* *problem* (see [9]) with: $v_1$, $v_2$ the *control variables*; $y_1$, $y_2$ the state variables; (5.4) the *state equations*; $J(.,.)$ the *cost function*. Actually other cost functions than (5.3) may be used, such as

$$\int_{\Omega_{12}} |y_2 - y_1|^2 \, dx \ ;$$

it seems, however, that $J(.,.)$ defined by (5.3) is optimally suited for solving (2.1), via (5.2).

## V.2. Conjugate gradient solution of (5.2). Further comments.

Any conjugate gradient algorithm for solving (5.2) is actually very close to the algorithms discussed in Sec. IV; it is clear that the natural metrics to use on the $V_i$'s are those induced by the $H^1(\Omega_{ii})$'s; using these metrics is rather technical, therefore we shall consider only the case where the $V_i$'s are equipped with the $L^2(\gamma_i)$-norms (*it is nonsense for the continuous problems, but not for the discrete ones*); we refer to [1] for a discussion of the case where the norm on $V_i$ is the one induced by $H^1(\Omega_{ii})$.

Description of a conjugate gradient algorithm: For $i = 1,2$, define $V_{oi}$ by

$$(5.6)\quad \begin{cases} V_{oi} = \{v_i \in L^2(\gamma_i), \quad v_i = \tilde{v}_i|_{\gamma_i} \ \textit{with} \ \tilde{v}_i \in H^1(\Omega_{ii}), \\ \tilde{v}_i = 0 \ \textit{on} \ \partial\Omega_{ii} \cap \Gamma\} \ ; \end{cases}$$

the conjugate gradient algorithm is then defined as follows

Step 0: *Initialization*

$$(5.7)\qquad u^o = \{u_1^o, u_2^o\} \in V_1 \times V_2 \ \textit{is arbitrarily given}$$

*define then* $g^o = \{g_i^o\}_{i=1}^2 \in V_{01} \times V_{02}$ *such that*

$$(5.8)\qquad \sum_{i=1}^{2} \int_{\gamma_i} g_i^o z_i \, d\gamma = (J'(u^o), z), \quad \forall z = \{z_i\}_{i=1}^2 \in V_{01} \times V_{02} \ ,$$

*where* J' *denotes the differential of* J; *set then*

$$(5.9) \qquad w^o = g^o \ .$$

*Assuming that* $u^n = \{u_i^n\}_{i=1}^2$, $g^n = \{g_i^n\}_{i=1}^2$, $w^n = \{w_i^n\}_{i=1}^2$ *are known, compute* $u^{n+1}$, $g^{n+1}$, $w^{n+1}$ *by*

Step 1: *Steepest descent*

$$(5.10) \qquad u^{n+1} = u^n - \rho_n w^n \ \textit{with} \ \rho_n = \operatorname*{Arg\,Min}_{\rho\in\mathbb{R}} J(u^n - \rho w^n) \ .$$

Step 2: *New descent direction*

*Compute* $g^{n+1} = \{g_i^{n+1}\}_{i=1}^2 \in V_{01} \times V_{02}$ *such that*

$$(5.11) \qquad \sum_{i=1}^{2} \int_{\gamma_i} g_i^{n+1} z_i \, d\gamma = (J'(u^{n+1}), z) \ \ \forall z = \{z_i\}_{i=1}^2 \in V_{01} \times V_{02},$$

*and*

$$(5.12) \qquad w^{n+1} = g^{n+1} + \lambda_n w^n \ \textit{with} \ \lambda_n = \frac{\sum_{i=1}^{2} \int_{\gamma_i} |g_i^{n+1}|^2 \, d\gamma}{\sum_{i=1}^{2} \int_{\gamma_i} |g_i^n|^2 \, d\gamma}$$

*Do* n = n+1, *go to* (5.10).

*Some remarks concerning algorithm* (5.7)-(5.12): It appears clearly from the above description that the two non-trivial steps in algorithm (5.7)-(5.12) are finding $\rho_n$ and the calculation of $J'(u^{n+1})$ in (5.11). Fortunately, since the cost function is a quadratic functional of $\{v_1, v_2\}$ we can compute $\rho_n$ exactly; similarly using a recurrent relation satisfied by the $g^n$, we can reduce to 2 the number of Poisson's problems like (5.4) which must be solved on each $\Omega_{ii}$ at each iteration (instead of *one* for the Schwarz's method). The calculation of $\rho_n$ and $J'(u^n)$ is discussed with many details in [1]. We shall find also in [1] a rather complete discussion of the solution of a *finite element approximation* of (2.1) by discrete variants of algorithm (5.7)-(5.12) and by other conjugate gradient algorithms. One may find also in [1] a discussion of the *direct* solution of (2.1) (in fact its *discrete* variants) via the least squares formulation (5.2); this solution technique involves matrices which approximate some *boundary operators* and is quite close to methods based on the concept of *capacitance matrices* (see [10] for more details and further references on capacitance matrix methods). The implementation of those direct methods on an array processor system is also discussed in [1] (with an evaluation of the computing performances).

## VI. NUMERICAL EXPERIMENTS

We consider in this paper one family of numerical experiments, only; more results are discussed in [1], and particularly the solution of *nonlinear problems* in Fluid Mechanics by iterative methods using the above domain decomposition methods as *preconditioners*. The test problem is the following mixed linear elliptic boundary value problem

$$(6.1) \qquad \Delta\phi = 0 \text{ in } \Omega, \quad \phi = g_o \text{ on } \Gamma_o, \quad \frac{\partial\phi}{\partial n} = g_1 \text{ on } \Gamma_1$$

where $\Omega$ is the nozzle of Fig. 5, on which are also shown $\Gamma_0$, $\Gamma_1$ and a finite element triangulation. The calculations have been done using a 3 subdomain decomposition with overlapping as shown on Fig. 6. We have compared the performances of the

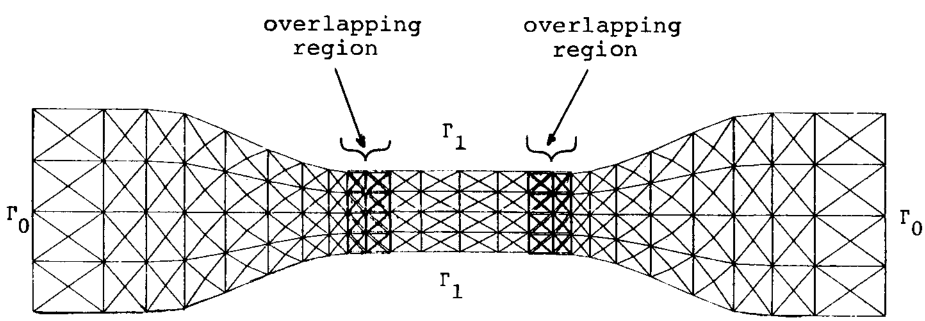



Figure 5

A convergent-divergent nozzle and its finite element triangulation

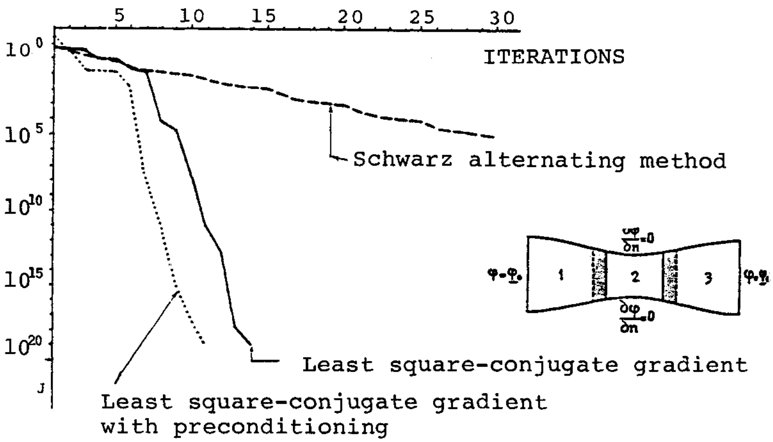



Figure 6

Schwarz method, of a discrete variant of algorithm (5.7)-(5.12) and of a preconditioned discrete variant of (5.7)-(5.12) (described in details in [1]). The results shown on Figure 6 clearly show the better performances of the method based on a least squares coupling on the overlapping region, combined with a conjugate gradient algorithm.

## VII. APPLICATION TO THE NUMERICAL SIMULATION OF TRANSONIC FLOWS ON LARGE COMPUTATIONAL DOMAINS

### VII.1. Synopsis

The numerical treatment of *nonlinear boundary value problems* involving *unbounded* regions of $\mathbb{R}^N$ is a nontrivial task. In the particular case of *two-dimensional transonic potential flows* around an airfoil we shall discuss a method combining a standard formulation close to the airfoil and an *exponential stretching transformation* to take the far field flow into account. The coupling between the corresponding local solutions is achieved through the method discussed in Sec. IV.2.

### VII.2. Mathematical Formulation

The *potential transonic flows of compressible inviscid fluids* are governed by

(7.1) $\quad \underset{\sim}{\nabla}\cdot\rho\underset{\sim}{u} = 0 \quad \textit{in } \Omega \; ,$

where $\Omega$ is the *flow region*, and where

(7.2) $$\rho = \rho_o\left(1 - \frac{\gamma-1}{\gamma+1}\,\frac{|\underset{\sim}{u}|^2}{c_*^2}\right)^{1/(\gamma-1)}, \quad \underset{\sim}{u} = \underset{\sim}{\nabla}\phi$$

where $\phi$ is the *velocity potential*; $\rho$ is the *density* of the fluid; $\gamma$ is the *ratio specific heats* ($\gamma = 1.4$ in air); $c_*$ is the *critical velocity*. For an airfoil B (see Figure 7) the flow is assumed *uniform* at infinity (visualized by $\Gamma_\infty$) and tangential at $\Gamma_B$.

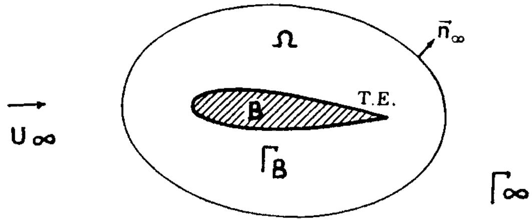



Figure 7

The corresponding boundary conditions are

(7.3) $$\frac{\partial\phi}{\partial n} = \underset{\sim}{u}_\infty\cdot\underset{\sim}{n}_\infty \text{ on } \Gamma_\infty, \quad \frac{\partial\phi}{\partial n} = 0 \text{ on } \Gamma_B \; .$$

Since Neumann boundary conditions are involved, the potential is determined only to within an arbitrary constant. To remedy this we can prescribe the value of $\phi$ at some point within $\Omega \cup \Gamma$. For example we may use

(7.4) $\quad \phi = 0$ *at the trailing edge* T.E. *of* B.

In addition to the above boundary conditions, the *circulation* of $\underset{\sim}{u} = \underset{\sim}{\nabla}\phi$ has to be adjusted in order to have $\underset{\sim}{u}$ satisfying the well-known *Kutta-Joukowsky condition*. In fact, this condition is not specific of transonic flows since it occurs also for other types of potential flows; we refer to e.g. [11], [12] for the numerical treatment of the Kutta-Joukowsky condition for 2-D and 3-D flows. Transonic flows of compressible inviscid fluids contain shocks; these shocks have to satisfy various conditions such that *Raukine-Hugoniot* and *entropy conditions* for which we refer to [13], [14] (and to [15], [16] for their numerical implementation).

## VII.3. Least squares formulation and conjugate gradient solution of the continuous problem.

We do not consider the practical implementation of the Kutta-Joukowsky and entropy conditions, concentrating only on a variational formulation of (7.1)-(7.3) and of an associate nonlinear least squares formulation.

Consider the situation shown in Fig. 7. Under the assumptions that the airfoil and flow are symmetric and that the flow is subsonic at infinity; the Kutta-Joukowsky condition is then *automatically* satisfied. For practical reasons the airfoil is imbedded in a "large" bounded domain, we obtain then as flow formulation

$$(7.5) \quad \underset{\sim}{\nabla}\cdot\rho(\phi)\underset{\sim}{\nabla}\phi = 0 \ \textit{in}\ \Omega, \quad \textit{with}\ \rho(\phi) = \rho_o\left(1 - \frac{\gamma-1}{\gamma+1}\,\frac{|\underset{\sim}{\nabla}\phi|^2}{c_*^2}\right)^{1/(\gamma-1)}$$

and

$$(7.6) \quad \rho\,\frac{\partial\phi}{\partial n} = 0 \ \textit{on}\ \Gamma_B, \quad \rho\,\frac{\partial\phi}{\partial n} = \rho_\infty \underset{\sim}{u}_\infty\cdot\underset{\sim}{n}_\infty \ \textit{on}\ \Gamma_\infty\ ;$$

on $\Gamma$ $(= \Gamma_B \cup \Gamma_\infty)$ we define g by $g = 0$ *on* $\Gamma_B$, $g = \rho_\infty \underset{\sim}{u}_\infty\cdot n$ *on* $\Gamma_\infty$; we clearly have $\rho\,\frac{\partial\phi}{\partial n} = g$ on $\Gamma$ with $\int_\Gamma g\,d\gamma = 0$. An equivalent formulation of (7.5), (7.6) is

$$(7.7) \quad \int_\Omega \rho(\phi)\underset{\sim}{\nabla}\phi\cdot\underset{\sim}{\nabla}v\,dx = \int_\Gamma gv\,d\gamma \quad \forall v\in H^1(\Omega),\ \phi\in W^{1,\infty}(\Omega)/\mathbb{R}\ ,$$

where, for $p \geq 1$, $W^{1,p}(\Omega) = \{v|v\in L^p(\Omega),\ \frac{\partial v}{\partial x_i}\in L^p(\Omega)\ \forall i\}$ (with $W^{1,2}(\Omega) = H^1(\Omega)$); $\phi$ is determined only to within an arbitrary constant. The space $W^{1,\infty}(\Omega)$ is a natural choice for $\phi$ since physical flows require (among other properties) a *positive* *density* $\rho$; therefore from (7.5), $\phi$ has to satisfy

$$|\nabla\phi| \leq \delta < \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} c_* \text{ a.e. on } \Omega.$$

Least squares formulation of (7.7): Since, for a genuine transonic flow, (7.7) is not equivalent to a standard problem of the Calculus of Variations (as it is the case for subsonic flows) we remedy this situation by introducing the following nonlinear least squares formulation:

$$(7.8) \quad \min_{\xi \in X} J(\xi) \quad (X: \textit{ a set of feasible transonic flow solutions})$$

and

$$(7.9) \quad J(\xi) = \frac{1}{2}\int_\Omega |\nabla y(\xi)|^2 \, dx ,$$

where $y(\xi)$ (= y) is the solution of the state equation

$$(7.10) \begin{cases} \int_\Omega \nabla y \cdot \nabla v \, dx = \int_\Omega \rho(\xi)\nabla\xi\cdot\nabla v \, dx - \int_\Gamma gv \, d\gamma \quad \forall v \in H^1(\Omega), \\ y \in H^1(\Omega)/\mathbb{R} \end{cases}$$

Conjugate gradient solution of the least squares problem (7.8), (7.9): We follow [1], [15] [16]; a preconditioned conjugate gradient algorithm for solving (7.8), (7.9) -with $X = H^1(\Omega)$ - is

Step 0: *Initialization*

$$(7.11) \quad \phi^0 \in H^1(\Omega), \textit{ given}$$

*then, compute* $g^0$ *by*

$$(7.12) \quad \int_\Omega \nabla g^0 \cdot \nabla v \, dx = \langle J'(\phi^0), v\rangle \quad \forall v \in H^1(\Omega), \quad g^0 \in H^1(\Omega),$$

*and set*

$$(7.13) \quad z^0 = g^0 . \qquad \square$$

*Then for* $n \geq 0$, *assuming* $\phi^n$, $g^n$, $z^n$ *known, compute* $\phi^{n+1}$, $g^{n+1}$, $z^{n+1}$ *by*

Step 1: *Descent*

(7.14) $\phi^{n+1} = \phi^n - \lambda_n z^n$ *with* $\lambda_n = \underset{\lambda\in\mathbb{R}}{\text{Arg Min}}\ J(\phi^n - \lambda z^n)$ .

Step 2: *New descent direction. Define* $g^{n+1}$ *by*

(7.15) $\begin{cases} g^{n+1} \in H^1(\Omega), & \int_\Omega \nabla g^{n+1}\cdot\nabla v\, dx = \langle J'(\phi^{n+1}), v\rangle \\ \forall v \in H^1(\Omega) , & \end{cases}$

*compute*

(7.16) $\gamma_n = \int_\Omega \nabla g^{n+1}\cdot\nabla(g^{n+1} - g^n)\,dx / \int_\Omega |\nabla g^n|^2\, dx$ ,

(7.17) $z^{n+1} = g^{n+1} + \gamma_n z^n$ . □

(7.18) *Do* $n = n+1$, *go to* (7.14).

We should prove that $J'(\xi)$ can be identified with the linear functional

(7.19) $\eta \to \int_\Omega \rho(\xi)\nabla y\cdot\nabla\eta\, dx - 2K\alpha \int_\Omega (\rho(\xi))^{2-\gamma}\nabla\xi\cdot\nabla y\nabla\xi\cdot\nabla\eta\, dx$

where $y$ is defined from $\xi$ by (7.10) (we have used the notation

$$K = \frac{1}{c_*^2}\frac{\gamma-1}{\gamma+1},\quad \alpha = 1/(\gamma-1) \text{ and supposed that } \rho_0 = 1).$$

## VII.4. Transonic flow simulations on large two-dimensional bounded computational domains by analytical transformation and decomposition methods.

Motivation: A genuine transonic flow is partly supersonic. If the supersonic region is large compared to the size of the airfoil, the computational domain itself has to be large compared to the size of the supersonic region, implying that standard finite element methods become inadequate. In the case of 2-D flows, we discuss below a method in which the far field is treated by an *exponential stretching* method which allows the use of very large computational domains.

Transformation of problem (7.5), (7.6): We consider again the situation depicted on Fig. 7. Let 0 be the origin of coordinates; it is then reasonable to take for $\Gamma_\infty$ the circle $\{x | x \in \mathbb{R}^2,\ \Omega = R_\infty\}$, with $r = (x_1^2 + x_2^2)^{1/2}$. Suppose now that the disk of center 0 and radius $R_o$ is sufficiently large to contain the airfoil B in its interior; we introduce then

$$\Omega_1 = \{x | x \in \mathbb{R}^2,\quad 0 \le r < R_o,\quad x \notin B\},$$

$$\Omega_2 = \{x | x \in \mathbb{R}^2,\quad R_o < r < R_\infty\}$$

With $\{r,\theta\}$ a standard *polar coordinates* system associated to 0, we define the following new variables

$$(7.20)\qquad \begin{cases} \xi_1 = x_1,\quad \xi_2 = x_2 \quad \forall x = \{x_1,x_2\} \in \Omega_1\ , \\ \xi_1 = R_o(1 + \operatorname{Log} \frac{r}{R_o}),\quad \xi_2 = \theta \quad \forall \{r,\theta\} \in \Omega_2\ ; \end{cases}$$

we use the notation $\xi = \{\xi_1,\xi_2\}$. We introduce now $\tilde{\Omega}_1 = \Omega_1$ and

$$\tilde{\Omega}_2 = \{\xi | \xi \in \mathbb{R}^2,\quad R_o < \xi_1 < R_o(1 + \operatorname{Log} \frac{R_\infty}{R_o}),\quad 0 < \xi_2 < 2\pi\}\ ;$$

we define then

$$\tilde{H}^1 = \{\tilde{v} = \{\tilde{v}_1,\tilde{v}_2\} \in H^1(\tilde{\Omega}_1) \times H^1(\tilde{\Omega}_2),\quad \tilde{v}_1(R_o \cos \xi_2,\ R_o \sin \xi_2)$$
$$= \tilde{v}_2(R_o,\xi_2)\ \ \forall\ \xi_2 \in ]0,2\pi[\ ,\ \tilde{v}_2(\xi_1,0) = \tilde{v}_2(\xi_1,2\pi)$$
$$\forall\ \xi_1 \in ]R_o,R_o(1 + \operatorname{Log} \frac{R_\infty}{R_o})[\}\ .$$

The space $\tilde{H}^1$ is clearly *isomorphic* to $H^1(\Omega)$ via the transformation $\tilde{v} \to v$: $\tilde{H}^1 \to H^1(\Omega)$ defined by

$$(7.21)\quad v(x) = \begin{cases} \tilde{v}_1(x_1,x_2)\ \ \textit{if}\ x = \{x_1,x_2\} \in \Omega_1\ , \\ \tilde{v}_2(R_o(1 + \operatorname{Log} \frac{r}{R_o})\theta\ )\ \ \textit{if}\ x = \{r,\theta\} \in \Omega_2\ . \end{cases}$$

Using the above properties, the continuity equation (7.10) is now formulated by

$$
(7.22)\quad \begin{cases} \textit{Find } \tilde{\phi} \in \tilde{H}^1 \textit{ such that} \\ \int_{\tilde{\Omega}} \rho(\tilde{\phi}_1)\underset{\sim}{\nabla}\tilde{\phi}_1\cdot\underset{\sim}{\nabla}\tilde{v}_1 d\xi + \int_{\tilde{\Omega}}\tilde{\rho}(\tilde{\phi}_2)(R_o \frac{\partial\tilde{\phi}_2}{\partial\xi_1}\frac{\partial\tilde{v}_2}{\partial\xi_1} \\ \quad + \frac{1}{R_o}\frac{\partial\tilde{\phi}_2}{\partial\xi_2}\frac{\partial\tilde{v}_2}{\partial\xi_2})d\xi \\ = R_\infty\rho_\infty \int_0^{2\pi}(u_{1\infty}\cos\xi_2 + u_{2\infty}\sin\xi_2)\tilde{v}_2(R_o(1+\mathrm{Log}\,\frac{R_\infty}{R_o}), \\ \quad \xi_2)d\xi_2 \quad \forall\tilde{v} \in \tilde{H}^1 , \end{cases}
$$

with, in (7.22), $\{u_{1\infty}, u_{2\infty}\} = \underset{\sim}{u}_\infty$, and

$$
(7.23)\quad \tilde{\rho}(\tilde{\phi}_2) = \rho_o[1 - \frac{\gamma-1}{\gamma+1}\frac{1}{c_*^2}(|\frac{\partial\tilde{\phi}}{\partial\xi_1}|^2 + R_o^{-2}|\frac{\partial\tilde{\phi}}{\partial\xi_2}|^2) \times e^{(2[R_o-\xi_1))/R_o}]^{1/(\gamma-1)} .
$$

The discretization of (7.22), (7.23) can be done as follows:

In $\tilde{\Omega}_1$ (which is part of the *physical* space) use a standard finite element approximation taking into account the possible complexities of the geometry; on the other hand since $\tilde{\Omega}_2$ is a *rectangle* one may use on it a *finite difference* discretization which can be obtained in fact via a finite element approximation on a *uniform* triangular or rectangular grid.

From the above decomposition of the computational domain (in $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$) it is natural to solve the approximate problems by iterative methods taking this decomposition into account, and also the fact that finite differences can be used in $\tilde{\Omega}_2$ (since finite differences allow special solvers on $\tilde{\Omega}_2$). Such a method is discussed below.

On the solution of (7.22), (7.23) by a decomposition method

Solving (7.22), (7.23) by a nonlinear least squares conjugate gradient method similar to the one discussed in Sec. VII.3 is quite easy in principle. The least squares formulation would be

$$(7.24)\qquad \underset{\tilde{\eta}\in\tilde{H}^1}{\text{Min}}\ J(\tilde{\eta})\ ,$$

with

$$(7.25)\qquad J(\tilde{\eta}) = \frac{1}{2}\int_{\tilde{\Omega}_1} |\nabla\tilde{y}_1|^2\, d\xi + \frac{1}{2}\int_{\tilde{\Omega}_2} \Big(R_o\Big|\frac{\partial\tilde{y}_2}{\partial\xi_1}\Big|^2 + R_o^{-1}\Big|\frac{\partial\tilde{y}_2}{\partial\xi_2}\Big|^2\Big)\, d\xi$$

where, in (7.25), y is obtained from $\tilde{\eta}$ via the solution of the state equation

$$(7.26)\qquad \begin{cases} \textit{Find } \tilde{y}\in\tilde{H}^1 \textit{ such that} \\ \displaystyle\int_{\tilde{\Omega}_1} \nabla\tilde{y}_1\cdot\nabla\tilde{v}_1 d\xi + \int_{\tilde{\Omega}_2} \Big(R_o\,\frac{\partial\tilde{y}_2}{\partial\xi_1}\frac{\partial\tilde{v}_2}{\partial\xi_1} + R_o^{-1}\,\frac{\partial\tilde{y}_2}{\partial\xi_2}\frac{\partial\tilde{v}_2}{\partial\xi_2}\Big)d\xi \\ \displaystyle = \int_{\tilde{\Omega}_1} \rho(\tilde{\eta}_1)\nabla\tilde{\eta}_1\cdot\nabla\tilde{v}_1 d\xi + \int_{\tilde{\Omega}_2} \tilde{\rho}(\tilde{\eta}_2)\Big(R_o\,\frac{\partial\tilde{y}_2}{\partial\xi_1}\frac{\partial\tilde{v}_2}{\partial\xi_1} \\ \displaystyle + R_o^{-1}\,\frac{\partial\tilde{\eta}_2}{\partial\xi_2}\frac{\partial\tilde{v}_2}{\partial\xi_2}\Big)d\xi - R_\infty\rho_\infty\int_0^{2\pi}(u_{1\infty}\cos\xi_2 \\ \displaystyle + u_{2\infty}\sin\xi_2)\tilde{v}_2\Big(R_o\Big(1+\mathrm{Log}\,\frac{R_\infty}{R_o}\Big),\xi_2\Big)d\xi_2 \quad \forall\tilde{v}\in\tilde{H}^1\ , \end{cases}$$

where $\tilde{\rho}$ is still defined by (7.23). Solving (7.24)-(7.26) by a conjugate gradient algorithm using as preconditioning operator the elliptic operator associated to the bilinear form occurring in the left hand side of (7.26) is quite easy, and the crucial point is then to have efficient solvers for problems like

$$(7.27)\qquad \begin{cases} \textit{Find } \tilde{u}\in\tilde{H}^1 \textit{ such that} \\ \displaystyle\int_{\tilde{\Omega}_1} \nabla\tilde{u}_1\cdot\nabla\tilde{v}_1\, d\xi + \int_{\tilde{\Omega}_2} \Big(R_o\,\frac{\partial\tilde{u}_2}{\partial\xi_1}\frac{\partial\tilde{v}_2}{\partial\xi_1} + R_o^{-1}\,\frac{\partial\tilde{u}_2}{\partial\xi_2}\frac{\partial\tilde{v}_2}{\partial\xi_2}\Big)d\xi \\ \displaystyle = L(\tilde{v})\quad \forall\tilde{v}\in\tilde{H}^1 \end{cases}$$

where L is linear and continuous over $\tilde{H}^1$; we can always suppose that $L(\tilde{v}) = L_1(\tilde{v}_1) + L_2(\tilde{v}_2)$ where $L_i$ is for $i = 1,2$ linear and continuous over $H^1(\tilde{\Omega}_i)$. Solving linear problems like (7.27) (in fact discrete variants of (7.27)) by methods taking into account their special structure can be done by either direct or iterative techniques. Among these methods we have chosen to describe one which is a generalization of the conjugate gradient algorithm (4.41)-(4.48).

We denote by $\gamma_{12}$ the circle of radius $R_o$ and centre 0, and by $\Lambda$ the space defined by

$$(7.28) \qquad \Lambda = \{\mu \mid \mu \in L^2(\gamma_{12}),\quad \mu = \tilde{\mu}_1|_{\gamma_{12}} \ \textit{where}\ \tilde{\mu}_1 \in H^1(\tilde{\Omega}_1)\}\ ;$$

we observe that if $\mu \in \Lambda$, then $\mu$ is a *periodic* function of the *polar angle* $\theta$. We define also the following spaces

$$(7.29) \qquad \tilde{V}_{01} = \{\tilde{v}_1 \mid \tilde{v}_1 \in H^1(\tilde{\Omega}_1),\quad \tilde{v}_1 = 0 \ \textit{on}\ \gamma_{12}\}\ ,$$

$$(7.30) \qquad \begin{cases} \tilde{V}_2 = \{\tilde{v}_2 \mid \tilde{v}_2 \in H^1(\tilde{\Omega}_2),\quad \tilde{v}_2(\xi_1,0) = \tilde{v}_2(\xi_1,2\pi) \\ \forall \xi_1 \in \left]R_o,\ R_o\left(1 + \operatorname{Log}\frac{R_\infty}{R_o}\right)\right[\}\ , \end{cases}$$

$$(7.31) \qquad \tilde{V}_{02} = \{\tilde{v}_2 \mid \tilde{v}_2 \in \tilde{V}_2,\quad \tilde{v}_2(R_o,\xi_2) = 0 \quad \forall \xi_2 \in ]0,2\pi[\}\ .$$

It is also convenient to define the bilinear forms

$$a_i :\ H^1(\tilde{\Omega}_i) \times H^1(\tilde{\Omega}_i) \to \mathbb{R},\ i = 1,2 \text{ by}$$

$$(7.32) \qquad a_1(v_1,w_1) = \int_{\tilde{\Omega}_1} \nabla v_1 \cdot \nabla w_1\, d\xi \quad \forall v_1,w_1 \in H^1(\tilde{\Omega}_1)\ ,$$

$$(7.33) \qquad a_2(v_2,w_2) = \int_{\tilde{\Omega}_2} \left(R_o \frac{\partial v_2}{\partial \xi_1}\frac{\partial w_2}{\partial \xi_1} + R_o^{-1}\frac{\partial v_2}{\partial \xi_2}\frac{\partial w_2}{\partial \xi_2}\right) d\xi$$

$$\forall v_2,w_2 \in H^1(\tilde{\Omega}_2)$$

respectively. The solution of the linear variational problem (7.27) by a domain decomposition method can be obtained through the following conjugate gradient algorithm which is also a generalization of algorithm (4.41)-(4.48).

Step 0: *Initialization*

(7.34) $\lambda^o \in \Lambda$, *arbitrarily given,*

*solve then the linear variational problems below*

$$(7.35)\quad \begin{cases} \text{Find } y_1^o \in H^1(\tilde{\Omega}_1),\ y_1^o = \lambda^o \text{ on } \gamma_{12}, \text{ such that} \\ a_1(y_1^o, z_1) = L_1(z_1) \quad \forall z_1 \in \tilde{V}_{01}\ , \end{cases}$$

$$(7.36)\quad \begin{cases} \text{Find } y_2^o \in \tilde{V}_2,\ \ y_2^o(R_o,\xi_2) = \lambda^o(R_o \cos \xi_2, R_o \sin \xi_2) \\ \qquad \forall \xi_2 \in \ ]0,2\pi[\ , \\ a_2(y_2^o, z_2) = L_2(z_2) \quad \forall z_2 \in \tilde{V}_{02}\ , \end{cases}$$

*and then*

$$(7.37)\quad \begin{cases} \text{Find } g^o \in \Lambda \quad \text{such that} \\ \sum_{i=1}^{2} a_i(\tilde{g}_i^o, \tilde{\mu}_i) = \sum_{i=1}^{2} [a_i(y_i^o, \tilde{\mu}_i) - L_i(\tilde{\mu}_i)] \quad \forall \mu \in \Lambda, \end{cases}$$

and set

(7.38) $w^o = g^o$. □

*For* $n \geq 0$, *suppose that* $\lambda^n$, $g^n$, $w^n$ *are known and compute* $\lambda^{n+1}$, $g^{n+1}$, $w^{n+1}$ *as follows*

Step 1: *Descent*

*Define* $\{\chi_i^n\}_{i=1}^2$ *by*

$$(7.39)\quad \begin{cases} \chi_1^n \in H^1(\tilde{\Omega}_1), \quad \chi_1^n = w^n \ \textit{on}\ \gamma_{12}, \\ a_1(\chi_1^n, z_1) = 0 \quad \forall z_1 \in \tilde{V}_{01}, \end{cases}$$

$$(7.40)\quad \begin{cases} \chi_2^n \in \tilde{V}_2, \quad \chi_2^n(R_o, \xi_2) = w^n(R_o \cos \xi_2, R_o \sin \xi_2) \\ \qquad\qquad \forall \xi_2 \in ]0, 2\pi[ \quad , \\ a_2(\chi_2^n, z_2) = 0 \quad \forall z_2 \in \tilde{V}_{02}, \end{cases}$$

*then* $\rho_n$ *and* $\lambda^{n+1}$ *by*

$$(7.41)\qquad \lambda^{n+1} = \lambda^n - \rho_n w^n \ \textit{with}\ \rho_n = \sum_{i=1}^{2} a_i(\tilde{g}_i^n, \tilde{g}_i^n) / \sum_{i=1}^{2} a_i(\chi_i^n, \tilde{w}_i^n)$$

*respectively.*

<u>Step 2</u>: *Calculation of the new descent direction*

*Solve*

$$(7.42)\quad \begin{cases} \textit{Find}\ g^{n+1} \in \Lambda \ \textit{such that}\ \forall \mu \in \Lambda \\ \sum_{i=1}^{2} a_i(\tilde{g}_i^{n+1}, \tilde{\mu}_i) = \sum_{i=1}^{2} a_i(\tilde{g}_i^n, \tilde{\mu}_i) - \rho_n \sum_{i=1}^{2} a_i(\chi_i^n, \tilde{\mu}_i) \ , \end{cases}$$

*compute*

$$(7.43)\qquad \gamma_n = \frac{\sum_{i=1}^{2} a_i(\tilde{g}_i^{n+1}, \tilde{g}_i^{n+1})}{\sum_{i=1}^{2} a_i(\tilde{g}_i^n, \tilde{g}_i^n)} \ ,$$

*set finally*

$$(7.44)\qquad w^{n+1} = g^{n+1} + \gamma_n w^n \ . \qquad \square$$

*Do* n = n+1 *and go to* (7.39).

The above algorithm involves functions extending on $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ the trace functions $\mu$, $g^n$, $w^n$; more generally to define $\tilde{q}_1$,

$\tilde{q}_2$ from $q \in \Lambda$, we proceed as follows:

We introduce first $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$ such as

$$(7.45) \qquad \tilde{\Lambda}_1 \subset H^1(\tilde{\Omega}_1), \quad H^1(\tilde{\Omega}_1) = \tilde{V}_{01} \oplus \tilde{\Lambda}_1 ,$$

$$(7.46) \qquad \tilde{\Lambda}_2 \subset \tilde{V}_2, \quad \tilde{V}_2 = \tilde{V}_{02} \oplus \tilde{\Lambda}_2 ;$$

there is an infinity of such $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$ satisfying (7.45), (7.46); in practice the discrete variants of the space $\tilde{\Lambda}_1$ (resp. $\tilde{\Lambda}_2$) will consist of functions vanishing outside a narrow neighborhood of $\gamma_{12}$ in $\tilde{\Omega}_1$ (resp. of $\{\xi = \{\xi_1, \xi_2\} | \xi_1 = R_o, \quad \xi_2 \in ]0, 2\pi[$ in $\tilde{\Omega}_2$); closely related spaces are described in [1].

Once $\tilde{\Lambda}_1, \tilde{\Lambda}_2$ have been defined, we define in turn $\tilde{q}_1, \tilde{q}_2$ as the unique elements of $\tilde{\Lambda}_1, \tilde{\Lambda}_2$ such as

$$(7.47) \qquad \tilde{q}_1|_{\gamma_{12}} = q ,$$

$$(7.48) \qquad \tilde{q}_2(R_o, \xi_2) = q(R_o \cos \xi_2, R_o \sin \xi_2) \quad \forall \xi_2 \in ]0, 2\pi[ .$$

Numerical results obtained using the above decomposition method will be presented elsewhere.

## VIII. FURTHER COMMENTS. CONCLUSION

The methods described in this paper are, obviously, closely related to the *Schwarz alternating method*; they are also closely related to those methods using the concepts of *low rank correction, capacitance matrix*, etc., and also to boundary integral methods (see [10], [17]; actually Reference [10] is also concerned with a family of domain decomposition methods).

The methods discussed in this paper have been restricted to the solution of *Poisson problems*; in fact they have been successfully used by the present authors for solving the *Stokes problem* for incompressible viscous fluids, using either finite element or spectral methods of approximation.

As a final comment we would like to point out that the comparison done in [10] between the methods discussed there and the methods in this paper is not very relevant since, as mentioned in our introduction, the least squares-overlapping approach was designed primarily as a technique to match, possibly different approximations, different mathematical modellings, etc., and not only to be an elliptic solver.

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