ON THE CONVERGENCE OF A RELAXATION METHOD WITH NATURAL CONSTRAINTS ON THE ELLIPTIC OPERATOR*

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ONE OF the most perfect methods for the solution of a network approximation of Poisson's equation in a rectangle is by a change of variables [1]. This method requires arithmetic operations of the order h^{-2} ln h^{-1} ln ε^{-1} for finding the solution of the network equation to within ε; here h is the network step.

The combination of this method with others [2, 3] enables us with a number of operations of the same order to find the solution of network approximations of elliptic equations with operators of fixed sign.

In [4, 5] a relaxation method, based on completely new ideas, is put forward for the solution of a network approximation of Poisson's equation in a square, which requires arithmetic operations of the order of h^{-2} ln ε^{-1} to decrease the norm of the discrepancy by a factor of ε.

In the present paper the application of this method is described with the same order of evaluation of the number of operations in the case of an arbitrary elliptic operator with continuous coefficients on the natural assumption that the point 0 is not a point of the spectrum.

In Section 5 we show that with this definite approach to the problem we can eliminate the factor ln ε^{-1} in the evaluation of the number of operations after which the method becomes optimal as regards the order of the number of operations.

Because of the generality and coarseness of the investigations carried out, the evaluations of the number of operations with practically

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admissible values of h and ε, which we have obtained, may be inferior to the evaluations obtained in [1 - 3]. This, however, is atoned for at least by the fact that the method is used for an essentially wider circle of equations. For instance it is used in the case of the equation $\Delta u + \lambda u = f$ with large positive $\lambda(x_1, x_2)$. Previously no methods of solving this equation with good asymptotics for the number of operations were known.

The basic idea of the method and the proof of the convergence are based on general considerations which suggest the possibility of using the method in the case of arbitrary, and in particular non-linear, equations and other classes of boundary conditions.

1. Definitions and notation

Suppose that in the square $\overline{\Omega}: [0 \leqslant x_1, x_2 \leqslant 1]$ we are solving the equation

$$Lu = L_2u + L_1u + L_0u = f \tag{1.1}$$

with boundary condition $u|_{\Gamma} = 0$; here

$$L_2u = \sum_{j_1, j_2=1}^{2} a_{j_1j_2} u_{x_{j_1}x_{j_2}},$$

$$L_1u = 2a_{10}u_{x_1} + 2a_{20}u_{x_2}, \qquad L_0u = a_{00}u, \qquad a_{j_1j_2} = a_{j_2j_1}.$$

The condition of ellipticity means that for $(x_1, x_2) \in \overline{\Omega}$ and all ξ_1, ξ_2 real, the relation

$$0 < m_0 \sum_{j=1}^{2} \xi_j^2 \leqslant \sum_{j_1, j_2=1}^{2} a_{j_1j_2} \xi_{j_1} \xi_{j_2}. \tag{1.2}$$

is satisfied.

Let $h^{-1} = n$ be some integer. We shall denote by $\overline{\Omega^h}$ the set of points $(i_1h, i_2h) \in \overline{\Omega}$, by Γ^h the set of points $(i_1, h, i_2h) \in \Gamma$, and by Ω^h the set $\overline{\Omega^h} \setminus \Gamma^h$. We shall also call the points (i_1h, i_2h) the nodes (i_1, i_2) of the network $\overline{\Omega}^h$, Ω^h.

If u^h is a function, which is defined on the network, then we shall denote by $u^h_{i_1 i_2}$ its value at the node (i_1, i_2).

We shall denote by $\bar{\Phi}^h$ the space of functions u^h, defined on $\bar{\Omega}^h$, and Φ^h the subspace of functions on $\bar{\Phi}^h$ which satisfy the condition $u^h|_{\Gamma^h} = 0$.

Let us define a set of operators on $\bar{\Phi}^h$. Let

$$(\Delta_1{}^h u^h)_{i_1 i_2} = (\nabla_1{}^h u^h)_{i_1+1,\, i_2} = (u^h_{i_1+1,\, i_2} - u^h_{i_1 i_2})/h,$$

$$(\Delta_2{}^h u^h)_{i_1 i_2} = (\nabla_2{}^h u^h)_{i_1,\, i_2+1} = (u^h_{i_1,\, i_2+1} - u^h_{i_1 i_2})/h,$$

$$D_{j0}{}^h = D_{0j}{}^h = (\nabla_j{}^h + \Delta_j{}^h)\,/\,2,$$

$$d_{jj}{}^h = D_{jj}{}^h = \Delta_j{}^h \nabla_j{}^h, \qquad D_{12}{}^h = D_{21}{}^h = D_{10}{}^h D_{20}{}^h,$$

$$d_{12}{}^h = d_{21}{}^h = \Delta_1{}^h \Delta_2{}^h, \qquad \Delta_h = d_{11}{}^h + d_{22}{}^h.$$

We shall denote by $D_{00}{}^h = d_{00}{}^h$ the identical network operator.

Let l^h be some linear operator and ψ^h a function which is defined on the set of nodes $M^h \subset \bar{\Omega}^h$. We shall denote by $M^h(l^h)$ the set of points at which the values of $l^h \psi^h$ are defined.

We shall assume that

$$\|u^h\|_{\breve{M}^h} = \max_{(i_1,\, i_2) \in M^h} |u_{i_1 i_2}|.$$

From now on in summation with respect to (i_1, i_2) we shall only indicate the region of summation. Let

$$(u^h, v^h)_{M^h} = h^2 \sum_{M^h} u^h_{i_1 i_2} v^h_{i_1 i_2},$$

$$\|u^h\|^2_{M^h} = (u^h, u^h)_{M^h}, \qquad \|l^h u^h\|^2_{M^h} = h^2 \sum_{M^h(l^h)} (l^h u^h)^2_{i_1 i_2},$$

$$\| u^h \|^2_{M^h,\,1} = \sum_{j=1}^{2} \| \Delta_j{}^h u^h \|^2_{M^h}, \qquad \|u^h\|^2_{M^h,\,2} = \sum_{j_1 j_2 = 1}^{2} \| d^h_{j_1 j_2} u^h \|^2_{M^h},$$

$$\widetilde{\| u^h \|}^2_{M^h,\,2} = \sum_{j=1} \| d_{jj}{}^h u^h \|_{M^h},$$

$$\overline{\| u^h \|}^2_{M^h,\,1} = \| u^h \|^2_{M^h} + \| u^h \|^2_{M^h,\,1}, \qquad \overline{\| u^h \|}^2_{M^h,\,2} = \overline{\| u^h \|}^2_{M^h,\,1} + \| u^h \|^2_{M^h,\,2}.$$

In the case where $M^h = \bar{\Omega}^h$, in denoting the norms, we shall simply put h instead of $\bar{\Omega}^h$.

If ψ is a function which is defined on the set enclosing $\bar{\Omega}^h$, then by $\{\psi\}_h$ we shall denote the function which is defined on $\bar{\Omega}^h$ and coincides there with ψ.

We shall put

$$A_{j_1j_2} = \sup_{\bar{\Omega}} |a_{j_1j_2}|, \qquad c_1 = \max A_{j0},$$

$$a^h_{j_1j_2} = \{a_{j_1j_2}\}_h, \qquad f^h = \{f\}_h.$$

Let $\omega(\rho) > 0$ be a function which is defined for $\rho > 0$ and does not decrease with increase in ρ, is continuous for $\rho > 0$ and satisfies the conditions $|a_{j_1j_2}(P) - a_{j_1j_2}(Q)| \leqslant \omega(\rho(P, Q))$ for $j_1, j_2 = 0, 1, 2$, $P, Q \in \bar{\Omega}$, $\omega(y_1 + y_2) \leqslant \omega(y_1) + \omega(y_2)$ for $0 < y_1, y_2$.

In future we shall denote by C_i, c_i absolute constants, by $C_i(d_1, \ldots, d_k)$ quantities which depend only on the parameters $d_1, \ldots, d_k$, and by B_i, b_i constants which can be expressed effectively in terms of the quantities m_0, $A_{j_1j_2}$ and the values of the function $\omega(y)$ on $(0, \sqrt{2}]$.

Constants which can be calculated effectively from the same quantities as B_i, b_i and also from the quantities $\bar{B}_0$, $\bar{b}_0$, defined in Section 2 will be denoted by $\bar{B}_i$, $\bar{b}_i$. Quantities, which can be calculated effectively from quantities given earlier and certain parameters $\lambda_1, \ldots, \lambda_k$, will be denoted respectively by $B_i(\lambda_1, \ldots \lambda_k)$, $b_i(\lambda_1, \ldots, \lambda_k)$, $\bar{B}_i(\lambda_1, \ldots, \lambda_k)$, $\bar{b}_i(\lambda_i, \ldots, \lambda_k)$.

In future we shall assume that the condition

$$\lim_{\rho\to 0} \omega(\rho) = 0$$

is satisfied and, consequently, the coefficients $a_{j_1j_2}$ will be continuous.

In fact we may observe that all arguments and evaluations of Sections 2 - 4 also hold in the case where some relation of the form

$$\lim_{\rho\to 0} \omega(\rho) \leqslant \bar{b} > 0,$$

is satisfied, i.e. they are also true for some classes of equations with discontinuous coefficients.

We shall put

$$L^{h,2} = \sum_{j_1, j_2=1}^{2} a_{j_1 j_2}^{h} D_{j_1 j_2}^{h}, \qquad L^{h,1} = 2 \sum_{j=1}^{2} a_{j0}{}^{h} D_{j0}{}^{h},$$

$$L^{h,0} = a_{00}{}^{h} D_{00}{}^{h}, \qquad L^{h} = L^{h,2} + L^{h,1} + L^{h,0}.$$

If l^h is an operator, defined on the space of functions $\bar{\Phi}^h$, then we put

$$\|l^h\|_h = \sup_{\psi^h \in \Phi^h} \| l^h \psi^h \|_h / \|\psi^h\|.$$

2. Supplementary relations

In this paragraph we quote or prove a number of relations of a general character which are used in the basic part of the paper.

Lemma 2.1

If $u^h \in \Phi^h$, the relation

$$\| u^h \|_h{}^C \leqslant C_1 \| \Delta_h u^h \|_h \leqslant 2C_1 \widetilde{\| u^h \|}_{h,2}.$$

is satisfied.

The left-hand side of the relation is proved in [6], the right-hand side is obvious.

Lemma 2.2

If $u^h \in \Phi^h$, then for any γ the inequality

$$\|u^h\|_{h,1} \leqslant (1/2\gamma) \|u^h\|_h + \gamma \|u^h\|_{h,2}.$$

is satisfied.

This well-known lemma follows from the identity $\|u^h\|_{h,1}^2{}^2 =$ $(-\Delta_h u^h, u^h)_h$ and inequality

$$|(u^h, v^h)_h| \leqslant (u^h, u^h)_h / 2\gamma + \gamma (v^h, v^h)_h / 2, \tag{2.1}$$

which is valid for any $\gamma > 0$.

Lemma 2.3

(proved in [7]). If $u^h \in \Phi^h$, then

$$\|u^h\|_{h,2} \leqslant B_1 \|L^{h,2} u^h\|_h.$$

Lemma 2.4

If $u^h \in \Phi^h$, then

$$\|u^h\|_{h,2} \leqslant B_2(\|L^h u^h\|_h + \|u^h\|_h).$$

Proof. From Lemma 2.3 and the equality $L^{h,2} = L^h - L^{h,1} - L^{h,0}$ it follows that

$$\|u^h\|_{h,2} \leqslant B_1(\|L^h u^h\|_h + \|L^{h,1} u^h\|_h + \|L^{h,0} u^h\|_h).$$

We shall replace the quantity $\|L^{h,1} u^h\|_h$ by the larger quantity $c_1\|u^h\|_{h,1}$ and add to both sides of the inequality $\overline{\|u^h\|}_{h,1}$. We evaluate $\|u^h\|_{h,1}$ on the right-hand side by means of Lemma 2.2, taking $\gamma < ((c_1 + 1)B_1)^{-1}$.

On evaluating the terms on the right-hand side in terms of $\overline{\|u^h\|}_{h,2}$ and transferring the terms containing the last quantity to the left-hand side we obtain the necessary inequality.

Theorem 2.1

Suppose that 0 is not a point of the spectrum of the operator with the boundary condition $u|_\Gamma = 0$, i.e. there exists no function $u \in W_2^{(2)}$, $\|u\|_{L_2} \neq 0, u|_\Gamma = 0$ such that

$$\int_\Omega fLu\, dx_1\, dx_2 = 0$$

for any smooth f. Then numbers $\bar{B}_0$, $\bar{b}_0 > 0$ exist, defined by the operator L, such that if $h \leqslant \bar{b}_0$, $u^h \in \Phi^h$

$$\bar{B}_0\|u^h\|_h \leqslant \|L^h u^h\|_h.$$

We shall indicate the general method of the proof. Let us assume that the statement of the theorem is not true. Then a sequence of functions u^{h_k} can be found such that $\|u^{h_k}\|_{h_k} = 1$, $\|L^{h_k} u^{h_k}\|_{h_k} \to 0$, $h_k \to 0$ as

$k \to \infty$. On the basis of Lemma 2.4 we find that for this sequence the quantities $\|\overline{u^{h_k}}\|_{h_k, 2}$ are uniformly bounded.

We shall determine the functions u^{h_k} for all (x_1, x_2), using constructions from Section 6, Chap. 1, of [8]. From theorems proved there, it follows that from the sequence of these functions we can choose a sequence of functions, weakly converging to some function $u \in W_2^{(2)}$, $\|u\|_{L_2} \neq 0$, $u|_\Gamma = 0$ such that $\int\limits_\Omega fLu\, dx_1 dx_2 = 0$ for any smooth f. We thus arrive at a contradiction with the condition of the theorem.

On the basis of Lemma 2.4 and Theorem 2.1 we conclude that if $h \leqslant \bar{b}_0$, $u^h \in \Phi^h$

$$\|\overline{u^h}\|_{h,2} \leqslant \bar{B}_3 \|L^h u^h\|_h. \tag{2.2}$$

Lemma 2.5

If $u_0 = 0$ then

$$\sum_{k=1}^{N} h(u_k/((k+1)h))^2 \leqslant 10 \sum_{k=1}^{N} h((u_k - u_{k-1})/h)^2.$$

The proof is similar to the proof of the well-known inequality

$$\int\limits_0^x u^2/t^2\, dt \leqslant \text{const} \int\limits_0^x (u_t')^2\, dt.$$

We have the chain of relations

$$s = \sum_{k=1}^{N} (u_k/(k+1))^2 \leqslant \sum_{k=1}^{N} u_k^2 (1/k - 1/(k+1)) \leqslant \sum_{k=1}^{N} (u_k^2 - u_{k-1}^2)/k.$$

On the basis of (2.1) we have $|(u_k^2 - u_{k-1}^2)/k| \leqslant 5(u_k - u_{k-1})^2 + 0.05((u_k + u_{k-1})/k)^2$; but $(u_k + u_{k-1})^2/k^2 \leqslant 2u_{k-1}^2/k^2 + 2u_k^2/k^2 \leqslant 2(u_{k-1}/k)^2 + 8(u_k/(k+1))^2$. After summation with respect to k we obtain

$$s \leqslant s/2 + 5 \sum_{k=1}^{N} (u_k - u_{k-1})^2.$$

Transposing $s/2$ to the left-hand side and multiplying both sides by $2/h$ we obtain the required inequality.

Lemma 2.6

If $u^h_{i_1 i_2} = 0$ for $i_1 = 0$ then we have the following inequalities

$$\sum_{i_1=1}^{N_1} \sum_{i_2=0}^{N_2} h^2 (u^h_{i_1 i_2}/((i_1+1)h))^2 \leqslant 10 \sum_{i_1=1}^{N_1} \sum_{i_2=0}^{N_2} h^2 |\nabla_1{}^h u^h_{i_1 i_2}|^2,$$

$$\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} h^2 (\nabla_2{}^h u^h_{i_1 i_2}/((i_1-1)h)^2)^2 \leqslant 10 \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} h^2 (\nabla_1{}^h \nabla_2{}^h u^h_{i_1 i_2})^2.$$

If $u^h_{i_1 i_2} = 0$ with $i_1 = 0$ and $i_2 = 0$ then

$$\sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} h^2 (u^h_{i_1 i_2}/((i_1+1)(i_2+1)h^2))^2 \leqslant 100 \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} h^2 (\nabla_1{}^h \nabla_2{}^h u^h_{i_1 i_2})^2.$$

The first two inequalities are obtained by direct summation over the variable i_2 of the inequality of Lemma 2.6. The third inequality is obtained by repeated application of Lemma 2.6 with respect to the variables i_1 and i_2.

Let us assume that $R^h_{i_1 i_2} = \sqrt{(i_1+1)^2 h^2 + (i_2+1)^2 h^2}$. By $\|u^h\|_h{}^{(b)}$ we shall denote the norm with a weight function $(R^h)^{-2b}$, i.e. $\|u^h\|_h{}^{(b)} = \|u^h (R^h)^{-b}\|_h$. We have $R^h_{i_1 i_2} \geqslant \max(|i_1+1|h, \ |i_2+1|h)$. For $u^h_{i_1 i_2}$ such that $u^h_{i_1 i_2} = 0$ if $i_1 = 0$ and $i_2 = 0$, from the second inequality of Lemma 2.6 and the similar inequality after the permutation of the variables i_1, i_2 we obtain

Corollary 1

$$\| u^h \|_{h,1}^{(1)} \leqslant 2\sqrt{10} \, \| d_{12}{}^h u^h \|_h \leqslant 2\sqrt{10} \, \| u^h \|_{h,2}.$$

From the third inequality we obtain

Corollary 2

$$\| u^h \|_h{}^{(2)} \leqslant 10 \, \| d_{12}{}^h u^h \|_h \leqslant 10 \, \| u^h \|_{h,2}.$$

Lemma 2.7

Suppose that inequality (2.2) is satisfied. Then for some $b > 0$

$$\overline{\| u^h \|_{h,2}^{(b)}} \leqslant \overline{B}_3{}' \, \| L^h u^h \|_h^{(b)}$$

This lemma is a difference analogue of the relations obtained in [9].

Proof. The expression $L^h(\varphi^h\psi^h)$ can be put in the form

$$L^h(\varphi^h\psi^h) = \varphi^h L(\psi^h) + s_1(\varphi^h, \psi^h) + s_2(\varphi^h, \psi^h).$$

Here $s_1(\varphi^h, \psi^h)$ is the sum of the products of the values of the functions $\Delta_j{}^h\varphi^h$ and the values of the functions $\Delta_k{}^h\psi^h$, and $s_2(\varphi^h, \psi^h)$ is the sum of the products of the values of the function ψ^h and the values of the functions $\Delta_j{}^h\varphi^h$, $d_{ij}{}^h\varphi^h$. Here all the differences of the functions φ^h and ψ^h, in these expressions, are determined by the values of φ^h and ψ^h at nodes (k_1, k_2) such that $|k_1 - i_1|$, $|k_2 - i_2| \leqslant 1$. In this expression we shall put $u^h = \varphi^h\psi^h$, where $\varphi^h = (R^h)^b$. It is not difficult to prove that the following relation is satisfied uniformly with respect to i_1, $i_2 > 0$ and $0 \leqslant b \leqslant 1$: all the difference ratios of the function $(R^h)^b$ of the first and second order, which enter into the expressions for the values of the functions $s_1(\varphi^h, \psi^h)\,|_{(i_1, i_2)}$, $s_2(\varphi^h, \psi^h)\,|_{(i_1, i_2)}$, do not exceed $C_2 b(R^h_{i_1 i_2})^{b-1}$ and $C_3 b(R^h_{i_1 i_2})^{b-2}$, respectively in modulus. For any h $|R^h| \leqslant 2\sqrt{2}$ for all points of $\bar{\Omega}^h$. From these relations and the equality $L^h\psi^h = (R^h)^{-b}(L^h u^h - s_1((R^h)^b, \psi^h) - s_2((R^h)^b, \psi^h))$ there follows the evaluation

$$\| L^h\psi^h \|_h \leqslant \| L^h u^h \|_h^{(b)} + c_2 b (\| \psi^h \|_{h,1}^{(1)} + \| \psi^h \|_h^{(2)}).$$

Using the inequality (2.2) and corollaries of Lemma 2.6 we obtain

$$\overline{\| \psi^h \|}_{h,2} \leqslant \bar{B}_3 \| L^h\psi^h \|_h \leqslant \bar{B}_3 (\| L^h u^h \|_h^{(b)} + c_2 b (2\sqrt{10} + 10) \| \psi^h \|_{h,2}).$$

From this with sufficiently small $b > 0$ we have $\overline{\|\psi^h\|}_{h,2} \leqslant \bar{B}_3^{(2)} \|L^h u^h\|_h^{(b)}$. The equality

$$d_{11}{}^h u^h = (R^h)^b d_{11}{}^h\psi^h + (\Delta_1{}^h((R^h)^b))\Delta_1{}^h\psi^h +$$
$$+ (\nabla_1{}^h((R^h)^b))\nabla_1{}^h\psi^h + (d_{11}((R^h)^b))\psi^h.$$

is also valid.

Hence there follows the evaluation

$$\| d_{11}{}^h u^h \|_h^{(b)} \leqslant c_3 (\| d_{11}{}^h\psi^h \|_h + \| \Delta_1{}^h\psi^h \|_h^{(1)} + \| \psi^h \|_h^{(2)}).$$

From this inequality, the corollaries of Lemma 2.6 and the preceding evaluation we have

$$\| d_{11}{}^h u^h \|_h^{(b)} \leqslant c_4 \| \psi^h \|_{h,2} \leqslant \bar{B}_3^{(3)} \| L^h u^h \|_h^{(b)}.$$

After carrying out similar evaluations for the remaining difference ratios we obtain the statement of the lemma. From now on we shall assume that $0 < b \leqslant 1$.

Directly expressing the quantities $D^{2h}_{j_1 j_2}(\{u^h\}_{2h})$ and $d^{2h}_{j_1 j_2}(\{u^h\}_{2h})$ in terms of the quantities $D^h_{j_1 j_2} u^h$ and $d^h_{j_1 j_2} u^h$, respectively we establish the truth of the inequalities

$$\begin{gathered}
\| \{u^h\}_{2h} \|_{2h} \leqslant 2 \| u^h \|_h, \\
\| \Delta_i^{2h} (\{u^h\}_{2h}) \|_{2h} \leqslant \sqrt{2} \| \Delta_i{}^h u^h \|_h, \\
\| D_{j0}{}^{2h} (\{u^h\}_{2h}) \|_{2h} \leqslant \sqrt{2} \| D_{j0}{}^h u^h \|_h, \\
\| d_{jj}{}^{2h} (\{u^h\}_{2h}) \|_{2h} \leqslant \sqrt{3} \| d_{jj} u^h \|_h, \\
\| d_{12}{}^{2h} (\{u^h\}_{2h}) \|_{2h} \leqslant 2 \| d_{12}{}^h u^h \|_h, \\
\| D_{12}{}^{2h} (\{u^h\}_{2h}) \|_{2h} \leqslant 2 \| D_{12}{}^h u^h \|_h.
\end{gathered} \tag{2.3}$$

From now on we need interpolation operators Π^h, which set the function $u^h \in \bar{\Phi}^h$ in correspondence with the function $u^{2h} \in \bar{\Phi}^{2h}$ in accordance with the formula

$$u^h_{i_1 i_2} = (\Pi^h u^{2h})_{i_1 i_2} = \sum_{|i_1 - 2t_1|,\ |i_2 - 2t_2| \leqslant C_4} \alpha^{t_1 t_2}_{i_1 i_2} u^{2h}_{t_1 t_2};$$

here we shall assume that the inequality

$$\max_{i_1,\, i_2} \sum_{t_1,\, t_2} | \alpha^{t_1 t_2}_{i_1 i_2} | \leqslant C_5.$$

is satisfied for all h.

Without making any special reservations we shall also always assume that all the interpolation operators satisfy the condition: if $u^{2h} \in \Phi^{2h}$, then $\Pi^h u^{2h} \in \Phi^h$.

The inequality

$$\| \Pi^h u^{2h} \|_h \leqslant C_6(C_4,\ C_5) \| u^{2h} \|_{2h}. \tag{2.4}$$

is obviously valid.

Lemma 2.9

Let

$$Pu^h = \sum_{i_1=k_1}^{l_1} \sum_{i_2=k_2}^{l_2} \alpha_{i_1 i_2} u^h_{i_1 i_2}, \tag{2.5}$$

and let $Pu^h = 0$ if $u^h_{i_1 i_2}$ is any polynomial of degree $m - 1$ in the set of variables i_1, i_2. Then the expression Pu^h can be put in the form

$$Pu^h = P_m u^h = \sum_{\substack{k_1 \leqslant i_1 \leqslant l_1 - s_1,\ 0 \leqslant s_1,\ s_2 \\ k_2 \leqslant i_2 \leqslant l_2 - s_2,\ s_1 + s_2 \leqslant m}} \beta^{s_1 s_2}_{i_1 i_2} (((h\Delta_1^h)^{s_1} (h\Delta_2^h)^{s_2}) u^h)_{i_1 i_2}. \tag{2.6}$$

Proof. Any quantity $u^h_{i_1 i_2}$ in the region G: $\{k_1 \leqslant i_1 \leqslant l_1,\ k_2 \leqslant i_2 \leqslant l_2,\ i_1 + i_2 \leqslant l_1 + l_2 - m\}$ can be expressed linearly in terms of the quantity $(((h\Delta_1^h)^{s_1}(h\Delta_2^h)^{s_2}) u^h)_{i_1 i_2}$ and the values of $u^h_{t_1 t_2}$, which satisfy the relations $i_1 \leqslant t_1 \leqslant i_1 + s_1 \leqslant l_1$, $i_2 \leqslant t_2 \leqslant i_2 + s_2 \leqslant l_2$, $s_1 + s_2 = m$, $i_1 + i_2 < t_1 + t_2$. Substituting successively in (2.5) the values of $u^h_{i_1 i_2}$ at the points (k_1, k_2), $(k_1 + 1, k_2)\dots$, $(k_1, k_2 + 1)$, $(k_1 + 1, k_2 + 1)$, $\dots \in G$, we obtain the relation

$$Pu^h = P_m u^h + \sum_{k_1 \leqslant i_1 \leqslant l_1,\ k_2 \leqslant i_2 \leqslant l_2,\ l_1 + l_2 - m < i_1 + i_2} \gamma_{i_1 i_2} u^h_{i_1 i_2}.$$

Let some coefficient $\gamma_{i_1^0 i_2^0}$ be different from zero. There exists [10] a polynomial $Q^0_{i_1 i_2}$ of degree $m - 1$ in the set of variables i_1, i_2, which becomes 1 at the point (i_1^0, i_2^0) and 0 at all the remaining points (i_1, i_2) such that $i_1 \leqslant l_1$, $i_2 \leqslant l_2$, $l_1 + l_2 - m < i_1 + i_2$. Since $(\Delta_1^h)^{s_1}(\Delta_2^h)^{s_2} Q^0 = 0$ for $s_1 + s_2 = m$, then $\gamma_{i_1^0 i_2^0} = PQ^0 = 0$, and we arrive at a contradiction.

Thus we have $Pu^h = P_m u^h$.

Let P^h be an operator on $\bar{\Phi}^{kh}$, which translates $\bar{\Phi}^{kh}$ to $\bar{\Phi}^h$ and is defined by the relation

$$(P^h u^h)_{i_1 i_2} = \sum_{|kt_1 - i_1|,\ |kt_2 - i_2| \leqslant C_7} \gamma^{t_1 t_2}_{i_1 i_2} u^{kh}_{t_1 t_2}, \qquad k = 1, 2.$$

Let this operator vanish in the case where $u^{kh}_{i_1 i_2}$ is any polynomial of degree $m-1$ in the set of variables i_1, i_2 and

$$\max_{i_1, i_2} \sum_{t_1, t_2} |\gamma^{t_1 t_2}_{i_1 i_2}| \leqslant C_8$$

for all h.

Lemma 2.9

The inequality

$$\| P^h u^h \|_h \leqslant C_9(C_7, C_8, m) \sum_{s_1+s_2=m} \| ((kh\Delta_1{}^{kh})^{s_1}(kh\Delta_2{}^{kh})^{s_2}) u^{kh} \|_{kh}.$$

is valid.

Proof. We represent each value of $(P^h u^h)_{l_1 l_2}$ in terms of the differences of the values on the network $\bar{\Omega}^{kh}$ in the form (2.6). It is obvious that all the quantities $\beta^{s_1 s_2}_{i_1 i_2}$, corresponding to different points (l_1, l_2), are bounded uniformly with respect to l_1, l_2, i_1, i_2, s_1, s_2 by some number $C_{10}(C_7, C_8, m)$. Hence the truth of the lemma follows.

Lemma 2.10

Let $z^{2h} \in \Phi^{2h}$ and $\Pi_2{}^h$ be an interpolation operator which is exact for second-degree polynomials. Then the inequality

$$\| L^h(\Pi_2{}^h z^{2h}) \|_h \leqslant \bar{B}_4 \| L^{2h} z^{2h} \|_{2h}.$$

is satisfied.

Proof. The operator $h^{j_1+j_2} D^h_{j_1 j_2} \Pi^h$ vanishes on polynomials of degree $j_1 + j_2 - 1$ and satisfies the conditions imposed on the operator P^h. On the basis of Lemma 2.9 and inequality (2.3) we have $\| D^h_{j_1 j_2} (\Pi^h z^{2h}) \|_h \leqslant C_{11}((j_1, j_2) \| z^{2h} \|_{2h, j_1+j_2}$. Hence from the obvious inequality

$$\| L^h v^h \|_h \leqslant \sum_{j_1, j_2=0}^{2} A_{j_1 j_2} \| D^h_{j_1 j_2} v^h \|_h$$

and (2.2) we obtain the statement of the lemma.

3. Evaluation of the decrease in the error on one step of the iterative process

The aim of this paper is to obtain an effective method of solving the system of equations

$$L^h u^h = f^h, \qquad u^h|_{\Gamma^h} = 0. \tag{3.1}$$

The iterative process which we have considered contains as its basic feature the reduction of the approximate solution of system (3.1) to a multiple solution of similar systems corresponding to coarser networks.

The plan of the process is as follows. Suppose that we are able to decrease the norm of the discrepancy $\|L_{2h}u^{2h} - f^{2h}\|_{2h}$ ε_0 times after $\Phi(\varepsilon_0, 2h)$ operations with any function f^{2h} and any initial approximation $\bar{u}^{2h}$. Because of the identity

$$L^h u^h - f^h = L^h(u^h - \bar{u}^h) - (f^h - L^h\bar{u}^h)$$

the problem of decreasing the discrepancy $L^h u^h - f^h$ with initial approximation $\bar{u}^h$ is equivalent to the problem of decreasing the discrepancy $L^h v^h - \varphi^h$ with $\varphi^h = f^h - L^h\bar{u}^h$ and initial approximation $\bar{v}^h = 0$.

The step of the iterative process, which permits us to decrease the norm of the discrepancy $\|L^h v^h - \varphi^h\|_h$, equal to $\|\varphi^h\|_h$ in the case of an initial approximation $\bar{v}^h = 0$, $\sqrt[t]{\varepsilon_0}$ times $(t > 1)$, consists of the following three stages.

1. We carry out m iterations by the formula

$$v^{h,k+1} = v^{h,k} + \tau(L^h v^{h,k} - \varphi^h), \quad v^{h,k}|_{\Gamma^h} = 0, \tag{3.2}$$

with initial approximation $v^{h,0} = 0$. We assume that $z^{h,k} = L^h v^{h,k} - \varphi^h$.

2. We find the approximation $w^{2h} \in \Phi^{2h}$ to the solution v^{2h} of the equation $L^{2h}v^{2h} = -\{z^{h,m}\}_{2h} = G^{2h}$ with boundary condition $v^{2h}|_{\Gamma^{2h}} = 0$ so that the relation

$$\|L^{2h}w^{2h} - G^{2h}\|_{2h} \leqslant \varepsilon_0\|G^{2h}\|_{2h}.$$

is satisfied. Here it is assumed that $(2h)^{-1}$ is an integer.

3. We interpolate the function w^{2h} from the network $\bar{\Omega}^{2h}$ to the

network $\bar{\Omega}^h$ by means of some interpolation operator $\Pi_2{}^h$, which is exact exact for polynomials of the second degree. We assume that

$$V^h = v^{h,m} + \Pi_2{}^h w^{2h}.$$

Theorem 3.1

For any $t > 1$ we can find numbers m and ε_0, depending on t, values which determine the values of the functions $\bar{B}_i$, and numbers C_4, C_5, corresponding to $\Pi_2{}^h$, such that the inequality

$$\|L^h V^h - \varphi^h\|_h \leqslant \sqrt[t]{\varepsilon_0}\|\varphi^h\|_h.$$

is satisfied for $h \leqslant \bar{b}_1(t)$.

Proof. The idea of the construction of the iterative process [4, 5] and the proof of the theorem are as follows.

We present the function φ^h in the form $\varphi^{h,0} + \varphi^{h,1}$, where $\varphi^{h,0}$ is smooth in some prearranged sense and $\varphi^{h,1}$ varies strongly.

On the first step of the iteration with a defined choice of parameter τ there is a substantial decrease in the strongly oscillating part of the discrepancy. On the second and third steps, on account of the nearness of the solutions of the equations $L^{2h}u^{2h} = f^{2h}$ and $L^h u^h = f^h$, with a smooth right-hand side the "smooth" part of the discrepancy is substantially decreased.

For convenience of investigation it turns out to be useful to introduce into the discussion an operator $Q^{h,0}$ which is the smooth self-conjugate part of the operator L^h.

We put the operator L^h in the form $Q^{h,0} + Q^{h,1}$, where

$$Q^{h,0} = \sum_{j=1}^{2} D_{j0}^{h/2}(a_{jj}^h D_{j0}^{h/2}) + \sum_{j_1 \neq j_2} D_{j_1 0}^h (a_{j_1 j_2}^h D_{j_2 0}^h) + A^h d_{00}{}^h.$$

In concrete cases the evaluations of the number of operations are essentially improved on account of the reasonable choice of the function A^h, but for simplicity we further assume that $A^h \equiv 0$.

Lemma 3.1

The inequality

$$\|Q^{h,1}\|_h \leqslant B_5\omega(h/2)/h^2.$$

is valid.

Proof. We have the valid inequality

$$\begin{aligned}((Q^{h,0}-L^{h,2})u^h)_{i_1i_2} &= \\ &= \frac{1}{2}\Big(\sum_{j=1}^{2}(((\Delta_j^{h/2}a_{jj}^{h/2})_{(i_1h,\,i_2h)}\Delta_j^h+(\nabla_j^{h/2}a_{jj}^{h/2})_{(i_1h,\,i_2h)}\nabla_j^h)u^h)_{i_1i_2}+ \\ &+ (\Delta_1^h a_{12}^h)_{i_1i_2}(D_{20}^h u^h)_{i_1+1,\,i_2}+(\nabla_1^h a_{12}^h)_{i_1i_2}(D_{20}^h u^h)_{i_1-1,\,i_2}+ \\ &+(\Delta_2^h a_{12}^h)_{i_1i_2}(D_{10}^h u^h)_{i_1,\,i_2+1}+(\nabla_2^h a_{12}^h)_{i_1i_2}(D_{10}u^h)_{i_1,\,i_2-1}\Big)\end{aligned} \tag{3.3}$$

Hence on the basis of the obvious inequalities $\|D_{j0}^h\|_h \leqslant 1/h$, $\|\Delta_j^h\|_h$, $\|\nabla_j^h\|_h \leqslant 2/h$, $|\Delta_j^{h/2}a_{jj}^{h/2}|$, $|\nabla_j^{h/2}a_{jj}^{h/2}|$, $|\Delta_j^h a_{12}^h|$, $|\nabla_j^h a_{12}^h| \leqslant 2\omega(h/2)/h$ we obtain the evaluation $\|Q^{h,0}-L^{h,2}\|_h \leqslant 12\omega(h/2)/h^2$. It is obvious that $\|L^{h,1}\|_h \leqslant 2(A_{10}+A_{20})/h$, $\|L^{h,0}\|_h \leqslant A_{00}$.

From these evaluations on using the equality $Q^{h,1}=(L^{h,2}-Q^{h,0})+L^{h,1}+L^{h,0}$ we obtain the relation

$$\|Q^{h,1}\|_h \leqslant 12\omega(h/2)/h^2+2(A_{10}+A_{20})/h+A_{00}.$$

Hence we have the following lemma.

Lemma 3.2

The operator $-Q^{h,0}$ is symmetrical on Φ^h and if $\omega(h/2) < m_0$ its eigenvalues λ satisfy the inequality

$$0<\lambda<B_6/h^2.$$

Proof. The symmetry of the operator $-Q^{h,0}$ is verified directly. We evaluate its eigenvalues. With $u^h \in \Phi^h$ we carry out Abel's transformation and are convinced of the truth of the equality

$$\begin{aligned}(-Q^{h,0}u^h,u^h)_h = \; & h^2\Big(\sum_{\overline{\Omega}(\Delta_1^h)} a_{11}\Big(\Big(i_1+\frac{1}{2}\Big)h,\, i_2h\Big)(\Delta_1^h u^h)_{i_1i_2}^2+ \\ &+\sum_{\overline{\Omega}(\Delta_2^h)} a_{22}\Big(i_1h,\,\Big(i_2+\frac{1}{2}\Big)h\Big)(\Delta_2^h u^h)_{i_1i_2}^2+\end{aligned}$$

$$+ 2 \sum_{\overline{\Omega}^h(D_{12}^h)} a_{12}(i_1h, i_2h)\,(D_{10}^h u^h)_{i_1i_2}(D_{20}^h u^h)_{i_1i_2}.$$

Let us assume that $a_{12}' = a_{12}$; $a_{jj}' = a_{jj} - m_0$ if $j = 1, 2$. Using obvious inequalities

$$\frac{1}{2}\left(a_{11}\left(\left(i_1+\frac{1}{2}\right)h, i_2h\right)(\Delta_1^h u^h)^2_{i_1i_2} + a_{11}\left(\left(i_1-\frac{1}{2}\right)h, i_2h\right)(\nabla_1^h u^h)^2_{i_1i_2}\right) \geqslant$$

$$\geqslant \frac{1}{2}\left(a_{11}(i_1h, i_2h) - \omega(h/2)\right)\left((\Delta_1^h u^h)^2_{i_1i_2} + (\nabla_1^h u^h)^2_{i_1i_2}\right) \geqslant$$

$$\geqslant \left(a_{11}(i_1h, i_2h) - \omega(h/2)\right)(D_{10}^h u^h)^2_{i_1i_2}.$$

and similar inequalities for differences in the variable x_2 we have

$$(-Q^{h,0}u^h, u^h)_h \geqslant S_1 + S_2, \tag{3.4}$$

where

$$S_1 = (m_0 - \omega(h/2))\,\| u^h \|^2_{h,1},$$

$$S_2 = h^2 \sum_{\Omega^h} \sum_{j_1, j_2=1}^{2} \left(a'_{j_1j_2}(D^h_{j_10}u^h)(D^h_{j_20}u^h)\right)_{i_1i_2}.$$

On account of the inequality (1.2) the quadratic form

$$\sum_{j_1, j_2=1}^{2} a'_{j_1j_2}\xi_{j_1}\xi_{j_2}$$

is non-negative and so $S_2 \geqslant 0$. Of $\omega(h/2) < m_0$ then $S_1 > 0$ and so $(-Q^{h,0}u^h, u^h)_h > 0$. Then all the eigenvalues of the operator $-Q^{h,0}$ will be positive. On the basis of the obvious inequalities $\|d_{jj}^h\|_h \leqslant 4/h^2$, $\|D_{12}^h\|_h \leqslant 1/h^2$ we have $\|L^{h,2}\|_h \leqslant (4(A_{11}+A_{22}) + 2A_{12})/h^2$. Hence from the inequality obtained in the proof of Lemma 3.1 it follows that

$$\|-Q^{h,0}\|_h \leqslant (4(A_{11}+A_{22}) + 2A_{12} + 12\omega(h/2))/h^2.$$

Since the modulus of the eigenvalue of the operator does not exceed the norm of the operator the statement of the lemma is true.

Since the operator $-Q^{h,0}$ is positive and symmetrical we can find a system of orthonormal eigenfunctions $\psi^{h,1}, \ldots, \psi^{h,N}$ of this operator which is complete on Φ^h. It is obvious that $N = (1/h - 1)^2$. Expanding

φ^h with respect to the system of functions $\psi^{h,j}$, we have

$$\varphi^h = \sum_{j=1}^{N} (\varphi^h, \psi^{h,j})_h \psi^{h,j}.$$

Let λ_j be an eigenvalue of the operator $-Q^{h,0}$, corresponding to the eigenfunction $\psi^{h,j}$. We take some θ within the limits $0 < \theta < 1$ and denote by $\Lambda^{h,0}_{\theta}$ the subspace of linear combinations of the functions $\psi^{h,j}$ for which $\lambda_j < \theta B_6/h^2$, and by $\Lambda^{h,1}_{\theta}$ the subspace of linear combinations of the remaining functions $\psi^{h,j}$. We put $\varphi^h = \varphi^{h,0} + \varphi^{h,1}$, where $\varphi^{h,0} \in \Lambda^{h,0}_{\theta}$, $\varphi^{h,1} \in \Lambda^{h,1}_{\theta}$. It is obvious that $\|\varphi^h\|_h^2 = \|\varphi^{h,0}\|_h^2 + \|\varphi^{h,1}\|_h^2$. The functions $\varphi^{h,0} \in \Lambda^{h,0}_{\theta}$ play the role of "smooth functions" with small θ.

We also put $\tau = h^2/B_6$. From (3.2) it is easy to obtain the fact that the discrepancies $z^{h,k}$ of the approximations $v^{h,k}$ satisfy the relation $z^{h,k+1} = (E + \tau L^h) z^{h,k}$, where $z^{h,0} = -\varphi^h$. Thus we have

$$z^{h,m} = -(E + \tau L^h)^m \varphi^h. \tag{3.5}$$

Let us put $T^{h,0} = (E + \tau Q^{h,0})$, $T^{h,1} = \tau Q^{h,1}$. The operator $T^{h,0}$ is symmetrical with eigenvalues $\mu_j = 1 - \tau\lambda_j$, which lie on $[0, 1]$, and therefore

$$\|T^{h,0}\|_h \leqslant 1. \tag{3.6}$$

On the basis of Lemma 3.1 we have

$$\|T^{h,1}\|_h \leqslant (B_5/B_6)\,\omega(h/2) = B_7\omega(h/2). \tag{3.7}$$

From (3.5) there follows the equality $-z^{h,m} = g^{h,0} + g^{h,1}$, where $g^{h,0} = (T^{h,0})^m \varphi^{h,0}$, $g^{h,1} = (T^{h,0})^m \varphi^{h,1} + ((T^{h,0} + T^{h,1})^m - (T^{h,0})^m)\varphi^h$. Removing the brackets in the expression $(T^{h,0} + T^{h,1})^m - (T^{h,0})^m$ and using the evaluations (3.6), (3.7) after cancelling the term $(T^{h,0})^m$, we obtain the inequality

$$\begin{aligned} &\|(T^{h,0} + T^{h,1})^m - (T^{h,0})^m\|_h \leqslant \\ &\quad \leqslant (1 + B_7\omega(h/2))^m - 1 \leqslant \exp(B_7 m\omega(h/2)) - 1. \end{aligned} \tag{3.8}$$

The subspaces $\Lambda^{h,k}_{\theta}$ are proper for the operator $T^{h,0}$. The eigenvalues $\mu_j = 1 - \tau\lambda_j$ of the operator $T^{h,0}$ on the subspace $\Lambda^{h,1}_{\theta}$ satisfy the inequality $0 \leqslant \mu_j \leqslant 1 - \theta$. Therefore $\|T^{h,0}\psi^h\|_h \leqslant (1 - \theta)\|\psi^h\|_h$ if

$\psi^h \in \Lambda_\theta^{h,1}$ and consequently

$$\|(T^{h,0})^m \varphi^{h,1}\|_h \leqslant (1-\theta)^m \|\varphi^{h,1}\|_h \leqslant (1-\theta)^m \|\varphi^h\|_h.$$

From this inequality and (3.8) the inequality

$$\|g^{h,1}\|_h \leqslant q(\theta, m, h)\, \|\varphi^h\|_h, \tag{3.9}$$

follows, where

$$q(\theta, m, h) = (1-\theta)^m + \exp\,(B_7 m \omega(h/2)) - 1. \tag{3.10}$$

Since $\|T^{h,0} + T^{h,1}\|_h < \exp\,(B_7 \omega(h/2))$ because of (3.6) and (3.7)

$$\|-z^{h,m}\|_h < \exp\,(B_7 m\omega(h/2))\, \|\varphi^h\|_h. \tag{3.11}$$

On account of (2.3), (3.11) the function $G^{2h} = \{-z^{h,m}\}_{2h}$ satisfies the inequality $\|G^{2h}\|_{2h} < 2\exp\,(B_7 m\omega(h/2))\, \|\varphi^h\|_h$. Thus the function $\bar{v}^{2h} \equiv 0$ is an approximation to the solution of the equation $L^{2h} v^{2h} = G^{2h}$ with discrepancy in norm less than $2\exp\,(B_7 m\omega(h/2))\, \|\varphi^h\|_h$. Therefore, on the basis of the original assumption, on the second stage of the iteration step, without performing more than $\Phi(\varepsilon_0, 2h)$ operations, we can find the function $w^{2h} \in \Phi^{2h}$ which satisfies the inequality

$$\|L^{2h} w^{2h} - G^{2h}\|_{2h} \leqslant 2\varepsilon_0 \exp\,(B_7 m\omega(h/2))\, \|\varphi^h\|_h. \tag{3.12}$$

Let $w^{2h,0}$ be the solution of the system $L^{2h} w^{2h,0} = \{g^{h,0}\}_{2h}$, $w^{2h,0}|_{\Gamma^{2h}} = 0$.

We put the function $V^h = v^{h,m} + \Pi_2^h w^{2h}$ - the final approximation of the iteration step - in the form $V^{h,0} + V^{h,1}$, where $V^{h,0} = v^{h,m} + \Pi_2^h w^{2h,0}$, $V^{h,1} = \Pi_2^h (w^{2h} - w^{2h,0})$. Since $L^{2h} w^{2h} - G^{2h} = L^{2h}(w^{2h} - w^{2h,0}) - \{g^{h,1}\}_{2h}$, the relation (3.12) can be put in the form

$$\|L^{2h}(w^{2h} - w^{2h,0}) - \{g^{h,1}\}_{2h}\|_{2h} \leqslant 2\varepsilon_0 \exp\,(B_7 m\omega(h/2))\, \|\varphi^h\|_h.$$

Hence from (2.3) and (3.9) it follows that

$$\|L^{2h}(w^{2h} - w^{2h,0})\|_{2h} \leqslant 2(\varepsilon_0 \exp\,(B_7 m\omega(h/2)) + q(\theta, m, h))\, \|\varphi^h\|_h.$$

Using Lemma 2.10 we obtain the inequality

$$\|L^h V^{h,1}\|_h \leqslant \bar{B}_8(\varepsilon_0 \exp\,(B_7 m\omega(h/2)) + q(\theta, m, h))\, \|\varphi^h\|_h. \tag{3.13}$$

From the definition of $V^{h,0}$ and $g^{h,0}$ we have

$$L^hV^{h,0}-\varphi^h=(L^h(\Pi_2^h w^{2h,0})-g^{h,0})-g^{h,1}. \tag{3.14}$$

Lemma 3.3

Let

$$L^hw^h=g^h\in\Lambda_\theta^{h,0},\qquad w^h\in\Phi^h.$$

Then for $h\leqslant \bar{b}_2(\theta)>0,\ s_1+s_2=3,\ \theta\leqslant\theta_0$ the inequality

$$\|((\Delta_1^h)^{s_1}(\Delta_2^h)^{s_2})w^h\|_h\leqslant(\bar{B}_9\sqrt[\varkappa]{\theta}/h)\|g^h\|_h,$$

is satisfied, where $\varkappa\geqslant 2$.

Proof. The eigenvalues λ_j of the operator $-Q^{h,0}$ on the subspace $\Lambda_\theta^{h,0}$, which is its proper subspace, satisfy the inequality $0<\lambda_j<\theta B_6/h^2$, and therefore $\|-Q^{h,0}g^h\|_h<(\theta B_6/h^2)\|g^h\|_h$. Hence there follows the inequality $(-Q^{h,0}g^h,g^h)_h<(\theta B_6/h^2)\|g^h\|_h^2$. If $\omega(h/2)\leqslant m_0/2$ from (3.4) the relation $(-Q^{h,0}g^h,g^h)_h\geqslant S_1\geqslant(m_0/2)\|g^h\|_{h,1}^2$. follows. Hence we have

$$\|\Delta_j^hg^h\|_h\leqslant\|g^h\|_{h,1}\leqslant(B_{10}\sqrt{\theta}/h)\|g^h\|_h \tag{3.15}$$

for $j=1,2,B_{10}=\sqrt{2B_6/m_0}.$

Let $\chi(x)$ be smooth, $0\leqslant\chi\leqslant 1$, $\chi(x)=0$ if $x\leqslant 1/3$, $\chi(x)=1$ if $x\geqslant 2/3$. We shall put $\chi_j^h=\chi(i_j/l)\chi((1-i_jh)/lh)$, where $6\leqslant l\leqslant 1/\sqrt{\theta}$, l is an integer.

As in the proof of Lemma 2.7 we have

$$L^h(\chi_1^h\Delta_1^hw^h)=\chi_1^hL^h(\Delta_1^hw^h)+s_1(\chi_1^h,\Delta_1^hw^h)+s_2(\chi_1^h,\Delta_1^hw_1^h).$$

The values of the operator $L^h\Delta_1^h-\Delta_1^hL^h$ are defined at all points of Ω^h, where $\chi_1^h\neq 0$. As in (3.3) they can be put in the form of linear combinations of the values of the operators Δ_j^h, $d_{j_1j_2}^h$ with their coefficients evaluated in terms of the quantity $\omega(h/2)/h$. Hence the inequality

$$\|\chi_1^h(L^h\Delta_1^h-\Delta_1^hL^h)w^h\|_h\leqslant B_{11}(\omega(h/2)/h)\|\overline{w^h}\|_{h,2}.$$

follows.

The moduli of the first difference ratios of the function χ_1^h does not exceed c_5/lh, and so the inequality $\|s_1(\chi_1^h, \Delta_1^n w^h)\|_h \leqslant (c_6/lh)\|w^h\|_{h,2}$. is valid.

The first and second difference ratios of the function χ_1^h in s_2, can differ from zero only in the regions $\Omega^{1,h}$: $\{0 \leqslant i_1 h < lh\}$ and $\Omega^{2,h}$: $\{1 - lh < i_1 h \leqslant 1\}$.

We put $\Omega^{3,h} = \Omega^h \setminus (\Omega^{1,h} \cup \Omega^{2,h})$. From the definition and the given properties of the function χ_1^h there follows the equality

$$\|s_2(\chi_1^h, w^h)\|_h^2 = \|s_2(\chi_1^h, w^h)\|^2_{\Omega^{1,h}} + \|s_2(\chi_1^h, w^h)\|^2_{\Omega^{2,h}}.$$

The moduli of the second difference ratios of the function χ_1^h do not exceed the value $c_7/(lh)^2$. Therevore the evaluation

$$\|s_2(\chi_1^h, w^h)\|_{\Omega^{1,h}} \leqslant (c_8/(lh)^2)\|w^h\|_{\Omega^{1,h}}.$$

is valid. From points of the region $\Omega^{1,h}$ we have $(i_1 + 1)h \leqslant lh$. Therefore the inequality

$$\|w^h\|_{\Omega^{1,h}} \leqslant lh\|w^h / ((i_1+1)h)\|_{\Omega^{1,h}}.$$

is valid. From these inequalities and the first inequality of Lemma 2.6 it follows that

$$\|s_2(\chi_1^h, w^h)\|_{\Omega^{1,h}} \leqslant (\sqrt{10}c_8/lh)\|\nabla_1^h w^h\|_{\Omega^{1,h}} \leqslant$$
$$\leqslant (\sqrt{10}\, c_8/lh)\|w^h\|_{h,1}.$$

The quantity $\|s_2(\chi_1^h, w^h)\|_{\Omega^{2,h}}$. is similarly evaluated. Combining the evaluations obtained above, for sufficiently small values of h we obtain the inequality $\|L^h(\chi_1^h\Delta_1^h w^h)\|_h \leqslant \|\Delta_1^h L^h w^h\|_h + (B_{11}'/lh)\|\overline{w^h}\|_{h,2}$. But $L^h w^h = g^h$. Therefore from this and the inequalities (2.2) and (3.15) the evaluation $\|L^h(\chi_1^h\Delta_1^h w^h)\|_h \leqslant ((B_{10} + B_{11}'\bar{B}_3)/lh)\|g^h\|_h$. follows. Hence on the basis of inequality (2.2) we have $\|\chi_1^h\Delta_1^h w^h\|_{h,2} \leqslant (\bar{B}_3(B_{10} + B_{11}'\bar{B}_3)/lh)\|g^h\|_h$. Consequently $\|(\Delta_1^h)^{s_1}(\Delta_2^h)^{s_2}w^h\|_{\Omega^{3,h}} \leqslant (B_{11}^{(2)}/lh)\|g^h\|_h$ if $s_1 + s_2 = 3$, $s_1 \geqslant 1$. From the relation $\Delta_2^h L^h w^h = \Delta_2^h g^h$ we can express the difference ratio $(\Delta_2^h)^3 w^h$ in terms of $\Delta_2^h g^h$ and the previously evaluated difference ratios of the function w^h. Hence we obtain the evaluation $\|(\Delta_2^h)^3 w^h\|_{\Omega^{3,h}} \leqslant (B_{11}^{(3)}/lh) \times \|g^h\|_h$. We can similarly obtain the evaluation of the norms of the difference ratios over the region $\Omega^{4,h}$: $lh \leqslant i_2 h \leqslant 1 - lh$. For a proof of the statement of the lemma it is sufficient to evaluate the norms of three differences, corresponding to squares with side $(l + 2)h$, adjacent to the vertices of the original square $\bar{\Omega}_h$.

We shall now occupy ourselves with the evaluation of this sum for the square $\Omega^{5,h}$: $0 \leqslant i_1, i_2 \leqslant l+2$.

Let $\Omega^{6,h}$ be the square $0 \leqslant i_1, i_2 \leqslant 3l_0$, where $l_0 = [3 + 1/\sqrt{\theta}]$. We put $\chi^h = (1 - \chi(i_1/3l_0))(1 - \chi(i_2/3l_0))$, $\chi^h w^h = W^h$ As before, we have

$$L^h(W^h) = \chi^h L^h(w^h) + s_1(\chi^h, w^h) + s_2(\chi^h, w^h). \tag{3.16}$$

The moduli of the first and second difference ratios of the function χ^h, in the expressions s_1 and s_2, differ from zero only if $i_1, i_2 \geqslant l_0 - 1$ and do not exceed $c_9 / l_0 h$ and $c_{10} / (l_0 h)^2$. respectively. Hence it follows that they do not exceed $c_{11}(R^h)^{-1}$ and $c_{12}(R^h)^{-2}$ respectively. We then have

$$\|s_k(\chi^h, w^h)\|_h \leqslant c_{13} \|w^h\|^{(k)}_{h,2-k} \qquad \text{if} \qquad k = 1, 2.$$

From these relations, the corollaries of Lemma 2.6 and inequality (2.2) we obtain the evaluation

$$\| L^h(W^h)\|_h \leqslant \bar{B}_{12} \| L^h w^h \|_h = \bar{B}_{12} \| g^h \|_h.$$

We apply to both sides of the relation (3.16) in turn the operators Δ_1^h and Δ_2^h. After similar evaluations of the terms on the right-hand side we obtain the inequality $\| L^h(W^h)\|_{h,1} \leqslant (\bar{B}_{12}' / l_0 h) \| g^h \|_h$. We now consider on the network with step $H = 1/3\ l_0$ the operators given by the equalities

$$L_0^{0,H} = \sum_{j_1, j_2=1}^{2} a_{j_1 j_2}(0,0) D^H_{j_1 j_2},$$

$$L_0^H u^H|_{(i_1, i_2)} = \sum_{j_1, j_2=1}^{2} a_{j_1 j_2}(i_1 h, i_2 h)\,(D^H_{j_1 j_2} u^H)|_{(i_1, i_2)} +$$

$$+ (h/H) 2 \sum_{j=1}^{2} a_{j0}(i_1 h, i_2 h)\,(D_{j0}^H u^H)|_{(i_1, i_2)} + (h/H)^2 a_{00}(i_1 h, i_2 h) u^H_{i_1 i_2}.$$

By an obvious method we obtain the evaluation

$$\|(L_0^H - L_0^{0,H})u^H\|_{H^{(b)}} \leqslant$$
$$\leqslant B_{12}^{(2)}(\omega(3l_0 h\sqrt{2})\|u^H\|_{H,2}^{(b)} + (h/H)\|u^H\|_{H,1}^{(b)} + (h/H)^2\|u^H\|_{H^{(b)}}).$$

On account of Lemma 2.7 with $u^H \in \Phi^H$ we have the inequality $\|\overline{u^H}\|_{H,2}^{(b)} \leqslant \bar{B}_3'\|L_0^{0,H}u^H\|_H^{(b)}$; here $\bar{B}_3' = B_3'(L_0^{0,H})$. From this and the previous inequality for small h we have

$$\|L_0^{0,H}u^H\|_H^{(b)} \leqslant 2\|L_0^H u^H\|_H^{(b)}.$$

The weaker inequality

$$\|u^H\|_{H,2} \leqslant 2\bar{B}3'\|L_0^H u^H\|_H^{(b)}.$$

is also valid.

We now multiply both sides of this inequality by the number $(H/h)^{1+b}$ and change the scale so that the square Ω^H becomes the square $\Omega^{6,h}$. Then the last inequality becomes

$$\|u^h\|_{\Omega^{6,h},2}^{(b)} \leqslant 2\bar{B}_3'\|L^h u^h\|_{\Omega^{6,h}}^{(b)}$$

for all functions which vanish on the boundary of $\Omega^{6,h}$. In particular we have

$$\|W^h\|_{\Omega^{6,h},2}^{(b)} \leqslant 2\bar{B}_3'\|L^h W^h\|_{\Omega^{6,h}}^{(b)}.$$

Since $R^h = O(l_0 h)$ for all points of the region $\Omega^{6,h}$, the inequality

$$\|L^h W^h\|_{\Omega^{6,h}}^{(b)} \leqslant c_{14}(l_0 h)^{1-b}\|L^h W^h\|_{\Omega^{6,h}}^{(1)}.$$

is valid.

From the last inequalities, Corollary 1 of Lemma 2.6 and the evaluation for the norm $\| L^h W^h \|_{\Omega^{6,h},1}$ we have $\|W^h\|_{\Omega^{5,h},2}^{(b)} \leqslant B_{12}^{(3)}\|g^h\|_h(l_0 h)^{-b}$. Since $R = O(lh)$ in the region $\Omega^{5,h}$ from this we have

$$\|W^h\|_{\Omega^{5,h},2} \leqslant B_{12}^{(4)}\|g^h\|_h(l/l_0)^b$$

As a consequence of the inequality $\| \Delta_j^h u^h \|_{M^h} \leqslant (2/h)\| u^h \|_{M^h}$ from this with $s_1 + s_2 = 3$ we have

$$\|(\Delta_1^h)^{s_1}(\Delta_2^h)^{s_2}w^h\|_{\Omega^{5,h}} = \|(\Delta_1^h)^{s_1}(\Delta_2^h)^{s_2}W^h\|_{\Omega^{5,h}} \leqslant 2B_{12}^{(4)}\|g^h\|_h(l/l_0)^b/h.$$

Similarly we carry out evaluations for the remaining squares adjacent to the vertices of $\bar{\Omega}^h$.

We now choose l from the condition $(l\sqrt{\theta})^b = 1/l$, and put that $1/l = \overset{\times}{\sqrt{\theta}}$. Then from the last evaluation and the evaluations of the norms of three differences over the regions $\Omega^{3,h}$, $\Omega^{4,h}$ the lemma follows.

Lemma 3.4

Let

$$L^{2h}w^{2h} = \{g^h\}_{2h}, \qquad g^h \in \Lambda_\theta^{h,0}, \qquad w^{2h} \in \Phi^{2h}.$$

Then for $h \leqslant \bar{b}_3(\vartheta)$ the inequality

$$\| L^h(\Pi_2^h w^{2h}) - g^h \|_h \leqslant \bar{B}_{13}\overset{\times}{\sqrt{\theta}} \| g^h \|_h.$$

is satisfied.

Proof. Let Π_0^h be some interpolation operator which is exact on polynomials of zero degree.

The following equality is valid:

$$\begin{aligned} L^h(\Pi_2^h w^{2h}) - g^h = (L^h(\Pi_2^h w^{2h}) - \Pi_0^h(L^{2h}w^{2h})) + \\ + \Pi_0^h(L^{2h}w^{2h} - \{g^h\}_{2h}) + (\Pi_0^h(\{g^h\}_{2h}) - g^h). \end{aligned} \tag{3.17}$$

The expression $(\Pi_0^h(\{g^h\}_{2h}) - g^h)_{i_1 i_2}$ vanishes in those cases where g^h is a polynomial of zero degree. Therefore from Lemma 2.9 there follows the inequality

$$\| \Pi_0^h(\{g^h\}_{2h}) - g^h \|_h \leqslant C_{12} h \| g^h \|_{h,1}.$$

After using (3.15) we obtain the evaluation

$$\|\Pi_0^h(\{g^h\}_{2h}) - g^h\|_h \leqslant B_{14}\overset{\times}{\sqrt{\theta}}\|g^h\|_h. \tag{3.18}$$

The equality

$$L^h(\Pi_2^h w^{2h}) - \Pi_0^h(L^{2h}w^{2h}) = \sum_{j_1, j_2=0}^{2} \overset{h}{\sigma}_{j_1 j_2},$$

is valid, where $\overset{h}{\sigma}_{j_1 j_2} = \overset{h,1}{\sigma}_{j_1 j_2} + \overset{h,2}{\sigma}_{j_1 j_2}$; in their turn

$$\sigma_{j_1j_2}^{h,1} = a_{j_1j_2}^{h}(D_{j_1j_2}^{h}(\Pi_2^h w^{2h}) - \Pi_0^h(D_{j_1j_2}^{2h} w^{2h})),$$

$$\sigma_{j_1j_2}^{h,2} = a_{j_1j_2}^{h}\Pi_0^h(D_{j_1j_2}^{2h} w^{2h}) - \Pi_0^h(a_{j_1j_2}^{h} D_{j_1j_2}^{2h} w^{2h}).$$

The interpolation operator $\Pi_2{}^h$ is exact for second-degree polynomials, the operator $\Pi_0{}^h$ for polynomials of zero degree, the values of the operators $D_{j_1j_2}^{h}$, $D_{j_1j_2}^{2h}$ from polynomials of degree $j_1 + j_2$ at nodes of the networks Ω^h, Ω^{2h} coincide with the values of the corresponding derivatives of these polynomials. Therefore the expression $h^{j_1+j_2}(\sigma_{j_1j_2}^{h,1})_{i_1i_2}$ vanishes in the case where w^{2h} is a polynomials of degree $j_1 + j_2$. Hence after using Lemma 2.9 we obtain the evaluation

$$\|h^{j_1+j_2}\sigma_{j_1j_2}^{h,1}\|_h \leqslant B_{15} \sum_{s_1+s_2=j_1+j_2+1} \|((2h\Delta_1{}^{2h})^{s_1}(2h\Delta_2{}^{2h})^{s_2})\, w^{2h}\|_{2h}. \tag{3.19}$$

As a consequence of the linearity of the operator $\Pi_0{}^h$ the equality

$$(\sigma_{j_1j_2}^{h,2})_{i_1{}^0 i_2{}^0} = (\Pi_0{}^h(((a_{j_1j_2}^{h})_{i_1{}^0 i_2{}^0} - a_{j_1j_2}^{h}) D_{j_1j_2}^{2h} w^{2h})_{i_1{}^0 i_2{}^0}.$$

is satisfied.

Since the moduli of the quantities $(a_{j_1j_2}^{h})_{i_1{}^0 i_2{}^0} - a_{j_1j_2}^{h}$, in the expression do not exceed $\omega(h\sqrt{2})$, then from this and from (2.4) we have

$$\|\sigma_{j_1j_2}^{h,2}\|_h \leqslant C_{13}\omega(h\sqrt{2})\,\|D_{j_1j_2}^{2h} w^{2h}\|_{2h}. \tag{3.20}$$

From the equation $L^{2h}w^{2h} = \{g^h\}_{2h}$, the inequality $\|\{g^h\}_{2h}\|_{2h} \leqslant 2\|g^h\|_h$ and inequality (2.2) it follows that $\|\overline{w^{2h}}\|_{2h,2} \leqslant 2\overline{B}_3\|g^h\|_h$. Hence from (3.19) and (3.20) it follows that if $j_1 + j_2 < 2$ the inequalities

$$\|\sigma_{j_1j_2}^{h}\|_h \leqslant \overline{B}_{16}(j_1, j_2)\,\omega(h\sqrt{2})\,\|g^h\|_h.$$

are satisfied.

From the inequalities (2.3) and (3.15) we obtain

$$\|\Delta_i{}^{2h}(\{q^h\}_{2h})\|_{2h} \leqslant (B_{10}\sqrt{2\overset{\times}{\theta}} / h)\ \|g^h\|_h.$$

Just as the truth of the statement of Lemma 3.3 follows from the equation $L^h w^h = g^h$ so from the equation $L^{2h} w^{2h} = \{g^h\}_{2h}$ and the last inequality it follows that

$$\| ((\Delta_1^{2h})^{s_1} (\Delta_2^{2h})^{s_2}) w^{2h} \|_{2h} \leqslant (\bar{B}_{17} \sqrt[\varkappa]{\theta} / h) \| g^h \|_h \quad \text{for } s_1 + s_2 = 3.$$

Hence from (3.19) and (3.20) we obtain the fact that $\| \sigma^h_{j_1 j_2} \|_h \leqslant (\bar{B}_{18} \sqrt[\varkappa]{\theta} / h) \| g^h \|_h$ if $j_1 + j_2 = 2$.

From the inequalities given earlier it follows that if $h \leqslant \bar{b}_3(\theta)$ the inequality

$$\| L^h (\Pi_2^h w^{2h}) - \Pi_0^h (L^{2h} w^{2h}) \|_h \leqslant \bar{B}_{19} \sqrt[\varkappa]{\theta} \| g^h \|_h.$$

is satisfied.

Since the second term in the right-hand side of equality (3.17) is zero by the condition of the lemma, from this and (3.18) Lemma 3.4 follows.

Turning to the evaluation of the right-hand side of (3.14) on the basis of Lemma 3.4 we have the inequality

$$\| L^h (\Pi_2^h w^{2h, 0}) - g^{h, 0} \|_h \leqslant \bar{B}_{13} \sqrt[\varkappa]{\theta} \| g^{h, 0} \|_h.$$

On the basis of (3.6) we have $\| g^{h, 0} \|_h \leqslant \| \varphi^h \|_h$. Hence from (3.9) and (3.14) it follows that

$$\| L^h V^{h, 0} - \varphi^h \|_h \leqslant (\bar{B}_{13} \sqrt[\varkappa]{\theta} + q(\theta, m, h)) \| \varphi^h \|_h.$$

Since $L^h V^h - \varphi^h = (L^h V^{h, 0} - \varphi^h) + L^h V^{h, 1}$, then from the last inequality and from (3.10), (3.13) we obtain

$$\| L^h V^h - \varphi^h \|_h \leqslant (\bar{B}_{13} \sqrt[\varkappa]{\theta} + (1 + \bar{B}_8)((1 - \theta)^m + \\ + \exp (B_7 m \omega (h/2)) - 1) + \bar{B}_8 \varepsilon_0 \exp (B_7 m \omega (h/2))) \| \varphi^h \|_h. \tag{3.21}$$

We take some $C_{14} > 1$. We then choose $\varepsilon_0 > 0$, for instance, from the condition $\bar{B}_8 C_{14} \varepsilon_0 \leqslant \sqrt[t]{\varepsilon_0} / 4$, then θ from the condition $\bar{B}_{13} \sqrt[\varkappa]{\theta} \leqslant \sqrt[t]{\varepsilon_0} / 4$, then m from the condition $(1 + \bar{B}_8)(1 - \theta)^m \leqslant \sqrt[t]{\varepsilon_0} / 4$ and finally, $\bar{b}_1(t)$

from the conditions $\exp(B_7 m\omega(\bar{b}_1/2)) \leqslant C_{14}$, $(1+\bar{B}_8)(\exp(B_7 m\omega(\bar{b}_1/2))-1) \leqslant \sqrt[t]{\varepsilon_0}/4$, $\bar{b}_1(t) = 2^{-k_0}$, k_0 is an integer, $\bar{b}_1(t) \leqslant \bar{b}_2(\theta), \bar{b}_3(\theta)$. With these conditions, from inequality (3.21) the truth of the statement of the theorem for $h \leqslant \bar{b}_1(t)$ follows.

The number m, for which the given evaluation is carried out, can be expressed in terms of $\bar{\beta}_0$ and known quantities and does not depend on h if h is sufficiently small.

The theorem is proved.

4. The evaluation of the total number of arithmetic operations

We now consider as a whole the problem of decreasing by a factor of ε_0 the discrepancy $L^h u^h - f^h$ with a given initial approximation and $h = 2^{-l}$. Let $t > 1$ be an integer. Repeating the iteration step, considered in Theorem 3.1 t times, we can decrease the norm of the discrepancy ε_0 times. To carry out each step we require no more than $\Phi(\varepsilon_0, 2h) + \bar{B}_{20}(2h)^{-2}$ arithmetic operations. The quantity $\bar{B}_{20}(2h)^{-2}$ evaluates from above the number of operations expended on carrying out m iterations by the formula (3.2), and interpolation from a network with step $2h$ on the network with step h. Hence follows the inequality $\Phi(\varepsilon_0, h) \leqslant t(\Phi(\varepsilon_0, 2h) + \bar{B}_{20}(2h)^{-2})$.

Let $\Phi(k_0)$ be the number of arithmetic operations which is sufficient to decrease ε_0 times the discrepancy on a network with step $h = 2^{-k_0}$. In the case where the network equation is solved by the method of elimination this number does not depend on ε_0.

The function Z_k - the solution of the equation $Z_{k+1} = t(Z_k + \bar{B}_{20}4^k)$ with initial condition $Z_{k_0} = \Phi(k_0)$ - majorizes the function $\Phi(\varepsilon_0, 2^{-k})$. The strongest evaluations in order as $h \to 0$ of the number of arithmetic operations are obtained with $t = 2$, $t = 3$. In this case

$$Z_k = \frac{t\bar{B}_{20}4^k}{4-t} + \left(\Phi(k_0) - \frac{t\bar{B}_{20}4^{k_0}}{4-t}\right)t^{k-k_0} \leqslant \bar{B}_{21}(t)4^k.$$

Hence we have $\Phi(\varepsilon_0, h) \leqslant \bar{B}_{21}(t) / h^2$

Thus in the case where the solution of the network equation, on a network with step $h = 2^{-k}$ and $k_0 < k \leqslant l$, is reduced each time to a t-fold ($t = 2, 3$) solution of a network equation on a network with step $2h$, in accordance with the above procedure, the total number of arithmetic operations, which is sufficient to decrease the norm of the discrepancy ε_0 times, turns out to be a quantity of the order of h^{-2}.

If $t > 3$ the number of operations is of large order, for instance, if $t = 4$ we have $\Phi(\varepsilon_0, h) = O(h^{-2} \ln h^{-1})$, and if $t > 4$ we have $\Phi(\varepsilon_0, h) = O(h^{-\log_2 t})$.

The method we have considered possesses free parameters τ, m and t, and in practice the optimum choice of these quantities, in the case of each concrete class of elliptic equations, deserves special consideration.

For instance [4, 5], in the case of Poisson's equation it is more reasonable to take τ somewhat larger or generally to change the iterative process of the first stage. The theoretical evaluations of the number of operations are, in a number of cases, essentially improved if the operator Π_2^h is exact for polynomials of the third degree.

It is possible that the parameters m and t with their optimal choice must, as with the parameter τ, depend on the step of the auxiliary network, to which they correspond, and also on the concrete value of the discrepancy of the approximation.

In several cases, for instance in the case of the equation $\Delta u = f$, the increase in the number m in comparison with the one calculated does not lead to an increase in the norm of the discrepancy of the approximation. In the case of the equation $\Delta u + \lambda u = f$ with large positive λ we do not exclude the possibility that the evaluation (3.21) may be attained in order. Then the increase in the number m in comparison with that calculated can lead to a deterioration in the discrepancy of the approximation. Finally, in view of their generality and roughness, our considerations are of a theoretical character and must be presented rather as a theorem on the existence of a good method of integration.

If we need to decrease the discrepancy ε times, where $\varepsilon < \varepsilon_0$ then we must aim at this by repeating $\overline{|\log_{\varepsilon_0} \varepsilon|}$ times the process described above of decreasing the discrepancy ε_0 times. Here $\bar{y} = y$ if y is an integer and $\bar{y} = [y] + 1$ if y is not an integer. Thus to decrease the norm of the discrepancy $\|L^h u^h - f^h\|_h$ ε times it is sufficient to carry out arithmetic operations of the order $h^{-2} \ln \varepsilon^{-1}$.

5. On the optimal method of integration as regards the order of the number of operations

Let U be the exact solution of problem (1.1) and U^h the exact solution of problem (3.1).

On the basis of Lemma 2.1 and inequality (2.2) the following relations are valid:

$$\begin{gathered}\|u^h - U^h\|_h{}^C \leqslant 2C_1\|\overline{u^h - U^h}\|_{h,2} \leqslant \bar{B}_{22}\|L^h u^h - f^h\|_h,\\ \|\{U\}_h - U^h\|_h{}^C \leqslant 2C_1\|\overline{\{U\}_h - U^h}\|_{h,2} \leqslant \bar{B}_{22}\|L^h(\{U\}_h) - f^h\|_h.\end{gathered} \tag{5.1}$$

If the norm $\|L^h(\{U\}_h) - f^h\|_h$ of the error of approximation has an order $O(h^p)$, then, according to (5.1), the error of the exact solution of system (3.1) also has an order $O(h^p)$.

In the case where the quantities $L^h(\{U\}_h) - f^h\|_h$ and $\|\{U\}_h - U^h\|_h$ are of the same order, obviously we can always get rid of the factor $\ln \varepsilon^{-1}$ in evaluating the number of arithmetic operations obtained in Section 4.

With the removal of this factor we make use of the following considerations.

1. In this case there is no special meaning in finding the solution of equation (3.1) so that the norm of the discrepancy is not less in order than the error of the solution.

2. The solutions of equation (3.1) for various h are near in the case of smooth $a_{j_1 j_2}$ and f and so for the initial approximation of the iterative process on a network with step h we can take the approximate solution on a network with step $2h$.

Let us consider as an example the case of a model problem for the Poisson equation $\Delta u = f$, where the results obtained on the basis of these considerations are of the most complete character.

Let $\bar{W}_2{}^2(4, 0)$ be the class of functions f which satisfy the conditions

$$\sum_{j=1}^{2}\iint_{\Omega} |f_{x_jx_j}|^2\, dx_1\, dx_2 \leqslant A^2, \qquad f|_\Gamma = 0.$$

Lemma 5.1

If $f \in \overline{W}_2{}^2(A,\ 0)$ then

$$\overline{\|\{U\}_h - U^h\|}_{h,2} \leqslant C_{15}Ah^2.$$

Proof. Let

$$f \sim \sum_{n_1,\, n_2=1}^{\infty} c_{n_1n_2} \sin \pi n_1 x_1 \sin \pi n_2 x_2.$$

We put $c_{n_1n_2} = d_{n_1n_2}((\pi n_1)^2 + (\pi n_2)^2)^{-1}$ From the condition $f \in \overline{W}_2{}^2(A;\ 0)$ there follows the inequality

$$\sum_{n_1,\, n_2=1} |d_{n_1n_2}|^2 \leqslant 4A^2.$$

Applying to the function $\{U\}_h$ the Laplace difference opeator Δ_h, we obtain

$$\Delta_h(\{U\}_h) = \sum_{n_1,\, n_2=1}^{\infty} d_{n_1n_2}\, 4\left(\sin^2\frac{\pi n_1 h}{2} + \sin^2\frac{\pi n_2 h}{2}\right) h^{-2} \sin \pi n_1 i_1 h \sin \pi n_2 i_2 h.$$

Using the relations $\sin \pi knih = 0$, $\sin \pi(n_j + kn)ih = (-1)^k \sin \pi n_j ih$ for the reduction of similar terms we obtain the equality

$$\Delta_h(\{U\}_h - U^h) = \Delta_h(\{U\}_h) - f^h = \sum_{n_1,\, n_2=1}^{n-1} Y_{n_1n_2} \sin \pi n_1 i_1 h \sin \pi n_2 i_2 h, \qquad (5.2)$$

where

$$Y_{n_1n_2} = \sum_{k_1,\, k_2=0}^{\infty} (-1)^{k_1+k_2} d_{n_1+k_1n,\, n_2+k_2n}\, q_{n_1+k_1n,\, n_2+k_2n},$$

$$q_{m_1m_2} = \left(4\left(\sin^2\frac{\pi m_1 h}{2} + \sin^2\frac{\pi m_2 h}{2}\right)\Big/ ((\pi m_1)^2 + (\pi m_2)^2) - 1\right) \times$$
$$\times ((\pi m_1)^2 + (\pi m_2)^2)^{-1}.$$

On the basis of the inequality

$$0 \geqslant 4\left(\sin^2\frac{x}{2}+\sin^2\frac{y}{2}\right)(x^2+y^2)^{-1}-1 \geqslant -\frac{x^2+y^2}{6}$$

we have the evaluation $0 \geqslant q_{m_1m_2} \geqslant -h^2/6$ Also the obvious evaluation

$$0 \geqslant q_{n_1+k_1n,\, n_2+k_2n} \geqslant -h^2/\left(\pi^2(k_1^2+k_2^2)\right)$$

is valid.

From (5.2) and Bunyakovskii's inequality by using these evaluations we have

$$|Y_{n_1n_2}|^2 \leqslant C_{16}^2h^4 \sum_{k_1,\,k_2=0}^{\infty} |d_{n_1+k_1n,\, n_2+k_2n}|^2,$$

where

$$C_{16}^2 = {}^1/_{36} + \sum_{0\leqslant k_1,\, k_2,\, 0<k_1+k_2}^{\infty} \pi^{-4}(k_1^2+k_2^2)^{-2}$$

By virtue of the fact that the functions $2\sin\pi n_1i_1h\sin\pi n_2i_2h$ constitute an orthonormal system the equality

$$\|\Delta_h(\{U\}_h-U^h)\|_h^2 = \frac{1}{4}\sum_{n_1,\,n_2=1}^{n} |Y_{n_1n_2}|^2.$$

holds.

From the above evaluations there follows the inequality

$$\|\Delta_h(\{U\}_h-U^h)\|_h \leqslant C_{16}Ah^2.$$

Hence from (2.2) the truth of the next lemma follows.

Lemma 5.2

The following inequality is valid:

$$\|((\Delta_1^h)^{s_1}(\Delta_2^h)^{s_2}U^h\|_h \leqslant C_{17}A \text{ npu } s_1+s_2=4.$$

The proof is carried out by expanding the functions to be evaluated in a Fourier sum.

Let $h_k = 2^{-k}$. We take some $\varepsilon_1 > 0$. The search for the approximate solution of equation (3.1) with $h = h_l$ is reduced to the successive calculations of functions $v_k^{h_k}$ $(k = k_0, \ldots, l)$, which satisfy the relations

$$\| \Delta_{h_k} v_k^{h_k} - f^{h_k} \|_{h_k} \leqslant \varepsilon_1 A h_k^2. \tag{5.3}$$

We shall describe the passage from the function $v_k^{h_k}$ to the function $v_{k+1}^{h_{k+1}}$. Let Π_3^h be the interpolation operator from the network $\bar{\Omega}_{2h}$ onto the network $\bar{\Omega}_h$, which is exact for polynomials of the third degree, Π_1^h - exact for polynomials of the first degree, and $V_k^{h_k}$ - the solution of the system $\Delta_{h_k} V_k^{h_k} = f^{h_k}$ with boundary condition $V_k^{h_k} |_{\Gamma^{h_k}} = 0$. We now have the inequality

$$\Delta_{h_{k+1}}(\Pi_3^{h_{k+1}} v_k^{h_k}) - f^{h_{k+1}} = \Delta_{h_{k+1}}(\Pi_3^{h_{k+1}}(v_k^{h_k} - V_k^{h_k})) +$$
$$+ (\Delta_{h_{k+1}}(\Pi_3^{h_{k+1}} V_k^{h_k}) - \Pi_1^{h_{k+1}}(\Delta_{h_k} V_k^{h_k})) + (\Pi_1^{h_{k+1}}(\Delta_{h_k} V_k^{h_k}) - f^{h_{k+1}}).$$

The expression $(\Delta_{h_{k+1}} \Pi_3^{h_{k+1}} u^{h_{k+1}})_{i_1 i_2}$ vanishes on polynomials of the first degree.

Making use of Lemma 2.10 we obtain

$$\|\Delta_{h_{k+1}}(\Pi_3^{h_{k+1}}(v_k^{h_k} - V_k^{h_k}))\|_{h_{k+1}} \leqslant C_{18} \|v_k^{h_k} - V_k^{h_k}\|_{h_k, 2}.$$

For the right-hand side of this inequality on the basis of (2.2) and (5.3) we have the evaluation

$$\|v_k^{h_k} - V_k^{h_k}\|_{h_k, 2} \leqslant C_{19} \|\Delta_{h_k} v^{h_k} - f^{h_k}\|_{h_k} \leqslant C_{19} \varepsilon_1 A h_k^2.$$

The expression

$$\left(\Delta_{h_{k+1}}\left(\Pi_3^{h_{k+1}} u^{h_k}\right) - \Pi_1^{h_{k+1}}(\Delta_{h_k} u^{h_k})\right)_{i_1 i_2}$$

vanishes on polynomials of the third degree.

On the basis of Lemmas 2.9 and 5.2 we have

$$\|\Delta_{h_{k+1}}(\Pi_3^{h_{k+1}} V_k^{h_k}) - \Pi_1^{h_{k+1}}(\Delta_{h_k} V_k^{h_k})\|_{h_{k+1}} \leqslant$$

$$\leqslant C_{20} h_k^2 \sum_{s_1+s_2=4} \|((\Delta_1^{h_k})^{s_1}(\Delta_2^{h_k})^{s_2}) V_k^{h_k}\|_{h_k} \leqslant C_{21} A h_k^2.$$

The expression $(\Pi_1{}^{h_{k+1}}(\Delta_{h_k} V_k{}^{h_k}) - f^{h_{k+1}})_{i_1 i_2} = \Pi_1{}^{h_{k+1}} f^{h_k} - f^{h_{k+1}}$ vanishes when $f_{h_{n+1}}$ is a polynomial of the first degree.

On the basis of inequality (2.2) and Lemmas 2.9 and 5.1 we have

$$\|\Pi_1{}^{h_{k+1}}(\Delta_{h_k} V_k{}^{h_k}) - f^{h_{k+1}}\|_{h_{k+1}} \leqslant C_{22} h_{k+1} \|f^{h_{k+1}}\|_{h_{k+1},2} \leqslant$$
$$\leqslant C_{23} h_{k+1}^2 \|\Delta_{h_{k+1}} f^{h_{k+1}}\|_{h_{k+1}} \leqslant C_{24} A h_{k+1}^2.$$

From the inequalities which we have obtained it follows that

$$\|\Delta_{h_{k+1}}(\Pi_3{}^{h_{k+1}} v_k{}^{h_k}) - f^{h_{k+1}}\|_{h_{k+1}} \leqslant (C_{25}\varepsilon_1 + C_{26}) A h_{k+1}^2.$$

We carry out the iteration on the network $\bar{\Omega}^{h_{k+1}}$ with initial approximation $\Pi_3{}^{h_{k+1}} v_k{}^{h_k}$ in order to decrease the norm of the discrepancy to a quantity which is not more than $\varepsilon_1 A h_{k+1}^2$. The approximation obtained as a result of carrying out these iterations is taken for $v_{k+1}^{h_{k+1}}$.

On the basis of the results of Section 4 for carrying out these iterations $\overline{|\log_{\varepsilon_0}[\varepsilon_1 / (C_{25}\varepsilon_1 + C_{26})]|}\,\Phi(\varepsilon_0, h_{k+1})$ operations are sufficient.

If we begin the process of constructing functions $v_{h_k}^k$ from the function $v_{k_0}^{h_k}$ such that $\|\Delta_{h_{k_0}} v_{k_0}^{h_{k_0}} - f^{h_{k_0}}\|_{h_{k_0}} \leqslant \varepsilon_1 A h_{k_0}^2$, then to find a function $v_l{}^h$, which satisfies the inequality $\|\Delta_h v_l{}^h - f^h\|_h \leqslant \varepsilon_1 A h^2$,

$$\bar{\Phi}(k_0) + \sum_{k=k_0+1}^{l} \overline{\left|\log_{\varepsilon_0} \frac{\varepsilon_1}{C_{25}\varepsilon_1 + C_{26}}\right|}\, \Phi(\varepsilon_0, h_k) \leqslant C_{27} \ln \frac{C_{28}}{\varepsilon_1} h^{-2}$$

operations are sufficient. Here we consider that the number of operations $\bar{\Phi}(k_0)$ which is sufficient to find $v_{k_0}{}^{h_{k_0}}$, does not depend on ε_1 and h.

The relation $C_{2,2}(A / \sqrt{2}, 0) \subset \bar{W}_2{}^2(A, 0)$, holds, where $C_{2,2}(D, 0)$ is the class of functions f which satisfy the conditions $f|_\Gamma = 0$, $|f_{x_1^{j_1} x_2^{j_2}}| \leqslant D$ with $j_1 + j_2 = 2$.

The solution of the Poisson equation $\Delta u = f$ is put in the form of Green's formula

$$u(x_1, x_2) = \int_0^1 \int_0^1 G(x_1, x_2, y_1, y_2) f(y_1, y_2)\, dy_1\, dy_2.$$

From this presentation, after arguments similar to those carried out in Section 3 of [11], the truth of the following statement is clear. If for the solution of the equation $\Delta u = f$ we use information about the values of the function f only at N points, then for any point $(x_1^0, x_2^0) \in \Omega$ a function $f(x_1^0, x_2^0; x_1, x_2) \in C_{2,2}(A / \sqrt{2}, 0) \subset \overline{W}_2^2(A, 0)$ can be found such that the error of the value of $u(x_1^0, x_2^0)$ corresponding to it will not be less in modulus than the value of $C_{29}(x_1^0, x_2^0) A / N > 0$.

To find the function v_l^h by the method we have described we require to calculate the values of the right-hand side of f^h at $N \sim h^{-2}$ points and perform $O(N)$ arithmetic operations.

At the same time the inequalities

$$\begin{aligned} \| v_l^h - \{U\}_h \|_h^C \leqslant C_1 \sqrt{2}\, \overline{\| v_l^h - \{U\}_h \|}_{h,2} \leqslant C_{30} (\| \Delta_h (v_l^h - U^h) \|_h + \\ + \| \Delta_h (U^h - \{U\}_h) \|_h \leqslant C_{30} (\varepsilon_1 + C_{16}) A h^2 = O(A / N) \end{aligned} \tag{5.4}$$

are satisfied.

Thus the proposed method is optimal, on the class of functions considered, as regards the order of the number of computed values of the right-hand part of f and the number of arithmetic operations.

With an error of the order $O(A / N)$ we have obtained not only the solution itself of the original problem on the whole of Ω^h, but also its difference ratios up to the second order inclusive.

From (5.4) there follows the inequality

$$\| K_h(v_l^h) - K_h(\{U\}_h) \|_C \leqslant C_{31} A h^2;$$

here K_h is an operator for multilinear interpolation from $\overline{\Omega}^h$ onto $\overline{\Omega}$ (see, for instance, [8]).

On the basis of the inequalities obtained earlier in Section 5 we have

$$\| K_{h_m}(\{U\}_{h_m}) - K_{h_{m-1}}(\{U\}_{h_{m-1}}) \|_C \leqslant C_{32} A h_m^2.$$

Hence we obtain $\| K_h(V_l^h) - K_{h_m}(\{U\}_{h_m}) \|_C \leqslant C_{33}Ah^2$. On passing to the limit as $m \to \infty$ we obtain $\| K_h(V_l^h) - U \|_C \leqslant C_{34}Ah^2$.

Thus on solving the problem by our method we have also constructed a table with which, for a finite number of operations, the approximate solution is interpolated at any point $(x_1, x_2) \in \overline{\Omega}$ with error $O(1/N)$.

In conclusion we make some remarks about the possibility of generalizing the results obtained. The method proposed in Section 4 of the consecutive solution of the problem on networks with steps $h_k = 2^{-k}$ is of a general nature and can also be used for the solution of other problems, in particular integral equations.

All the statements proved in Section 3 remain true in the solution of the Dirichlet problem in a ring in polar coordinates. Here the proof of Lemma 3.3 is essentially simplified. It is clear that they remain true also in the case of the solution of the Dirichlet network problem in a region, such that this region and the network are transformed, by a smooth transformation of coordinates, into a square with a rectangular network or a ring with a polar network.

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