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Local convergence of Newton-like methods for degenerate eigenvalues of nonlinear eigenproblems: II. Accelerated algorithms

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Abstract The computation of a defective eigenpair of nonlinear algebraic eigenproblems of the form $T(\lambda)v = 0$ is challenging due to its ill-posedness and the linear convergence of classical single-vector Newton-like methods. In this paper, we propose and study new accelerated Newton-like methods for defective eigenvalues which exhibit quadratic local convergence at the cost of solving two linear systems per iteration. To the best of our knowledge, the accelerated algorithms are the most efficient methods for solving defective eigenpairs. The analyses are illustrated by numerical experiments.

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1 Introduction

This paper is the second part of our study of the local convergence of single-vector Newton-like methods for degenerate eigenvalues of nonlinear algebraic eigenproblem of the form

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$$T(\lambda)v = 0, \tag{1}$$

where $T(\cdot) : U \to \mathbb{C}^{n\times n}$ is holomorphic on a domain $U \subset \mathbb{C}$, $\lambda \in U$ is an eigenvalue such that $T(\lambda)$ is singular, and $v \in \text{null}\, T(\lambda)\backslash\{0\}$ is an associated eigenvector. The algebraic multiplicity of λ, denoted as $alg_T(\lambda)$, is the smallest integer j for which $\frac{d^j}{d\mu^j}\det T(\mu)|_{\mu=\lambda} \neq 0$. The geometric multiplicity, $geo_T(\lambda)$, is defined as dim (null $T(\lambda)$). An eigenvalue λ is defective if $geo_T(\lambda) < alg_T(\lambda)$, and non-defective (simple or semi-simple) if $geo_T(\lambda) = alg_T(\lambda)$.

Defective eigenvalues typically originate from applications where certain critical stability conditions are met. For example, a defective λ arises in the study of some special critical delays of a time-delay system, for which λ is a double pure imaginary root of the characteristic equation of the system [3]. Another example involving a defective $\lambda = 0$ comes from the modeling of a chlorine-dioxide–iodine–malonic acid (CDIMA) reaction–diffusion system for the symmetry-breaking instability (Turing bifurcation) in the absence of immobile components [8].

Numerical computation of a defective eigenpair is an ill-posed problem by nature. Defective eigenvalues and corresponding eigenvectors and generalized eigenvectors (principal vectors) are extremely sensitive to perturbations. A very small perturbation usually scatters a defective eigenvalue into a group of tightly clustered, poorly conditioned simple eigenvalues. The highest relative accuracy one can achieve for the computation of a defective eigenpair (λ, v) is $\mathcal{O}\left(\epsilon^{1/m}\right)$, where ϵ is the machine precision, and m is the length of the longest Jordan chain associated with λ [6,7]. Therefore, the computation of defective eigenpairs is relevant only for those with short Jordan chains, say, 2–4 in length, and only if modest accuracy is sufficient for the problem.

In part I of this study [9], we showed that the local convergence of single-vector Newton-like methods towards defective eigenvalues is generally linear. Since linear convergence is much less satisfactory than the typical quadratic convergence these methods exhibit for non-defective eigenvalues, it is important to explore new accelerated algorithms to improve the convergence rate. The purpose of this paper is to propose and study accelerated methods that converge quadratically in the defective case. Our new algorithms require the solution of two linear systems per iteration; they are equally or more efficient than the block variant of Newton's method [5] in this setting, since the latter requires the solution of $alg_T(\lambda) \geq 2$ linear systems per iteration.

The rest of the paper is organized as follows. In Sect. 2, we review some preliminary definitions and theory for the study of accelerated methods. In Sect. 3, we propose an accelerated inverse iteration and show its quadratic local convergence towards defective eigenvalues; an accelerated single-vector Jacobi–Davidson method is also presented and discussed. Numerical experiments in Sect. 4 illustrate the convergence rates of the new algorithms for defective eigenproblems. Section 5 is the conclusion of this paper.

2 Preliminaries

In part I of this study [9], we reviewed some preliminary results about defective eigenvalues and the local convergence of Newton's method towards singular roots

[2]. To make part II self-contained, we briefly summarize some of them, and make appropriate comments relevant to the accelerated algorithms.

2.1 Defective eigenvalues

An eigenvalue λ of $T(\cdot)$ is defective if and only if $alg_T(\lambda) > geo_T(\lambda)$, i.e., it has a Jordan chain of length ≥ 2. This is a generalization of the existence of a nontrivial Jordan block for a defective matrix. The vectors involved in a Jordan chain are described in the following definition.

Definition 1 Let λ be a defective eigenvalue of the holomorphic operator $T(\cdot)$ and $\varphi_{\cdot,0}$ a corresponding right eigenvector. Then the nonzero vectors $\varphi_{\cdot,1}, \ldots, \varphi_{\cdot,m-1}$ are called generalized eigenvectors if

$$\sum_{j=0}^{\ell} \frac{1}{j!} T^{(j)}(\lambda)\varphi_{\cdot,\ell-j} = 0, \quad \ell = 1, \ldots, m-1, \tag{2}$$

where $T^{(j)}(\lambda) = \frac{d^j}{d\mu^j} T(\mu)|_{\mu=\lambda}$. The ordered collection $\{\varphi_{\cdot,0}, \varphi_{\cdot,1}, \ldots, \varphi_{\cdot,m-1}\}$ is called a right Jordan chain corresponding to λ. If (2) is satisfied for some $m = m_*$ and no more vectors can be introduced such that (2) is satisfied for $m = m_* + 1$, then m_* is called the length of the Jordan chain and a partial multiplicity of λ.

Similarly, let $\psi_{\cdot,0}$ be a left eigenvector of λ. One can define a left Jordan chain $\psi_{\cdot,0}, \psi_{\cdot,1}, \ldots, \psi_{\cdot,m-1}$ by replacing $T^{(j)}\varphi_{\cdot,\ell-j}$ in (2) with $\psi^*_{\cdot,\ell-j} T^{(j)}$.

Let $\{\varphi_{k,0}, \ldots, \varphi_{k,m_k-1}\}$ be a right Jordan chain with $m_k \geq 2$, for which $T(\lambda)\varphi_{k,1} + T'(\lambda)\varphi_{k,0} = 0$ from Definition 1. Premultiplying by $\psi^*_{j,0}$, we have

$$\psi^*_{j,0} T(\lambda)\varphi_{k,1} + \psi^*_{j,0} T'(\lambda)\varphi_{k,0} = \psi^*_{j,0} T'(\lambda)\varphi_{k,0} = 0 \tag{3}$$

for $j = 1, \ldots, J$. Similarly, $\psi^*_{j,0} T'(\lambda)\varphi_{k,0} = 0$ also holds for all $\psi_{j,0}$ with $m_j \geq 2$ and $k = 1, \ldots, J$. As a result, we have the following proposition.

Proposition 2 *Let λ be a defective eigenvalue of the holomorphic operator $T(\cdot) : U \to \mathbb{C}^{n\times n}$, and φ and ψ be a corresponding unit right and a unit left eigenvector, respectively. Let $x = \varphi\cos\alpha + \varphi_\perp \sin\alpha$ and $y = \psi\cos\beta + \psi_\perp \sin\beta$, where $\alpha, \beta < \frac{\pi}{2}$, $\varphi_\perp \perp \text{null}\,(T(\lambda))$ and $\psi_\perp \perp \text{null}\,((T(\lambda))^*)$ are unit vectors, and thus $\|x\| = \|y\| = 1$. Assume that $y^*T'(\lambda)x \neq 0$. Let $\rho = \rho_F(x; T, y)$ be the Rayleigh functional value closest to λ such that $y^*T(\rho)x = 0$. Then for sufficiently small α,*

$$|\rho - \lambda| \leq \frac{2\|T(\lambda)\|\,|\sin\alpha \sin\beta|}{|y^*T'(\lambda)x|}$$

$$= \frac{2\|T(\lambda)\|\,|\sin\alpha \sin\beta|}{\left|\overline{\cos\beta}\sin\alpha\psi^*T'(\lambda)\varphi_\perp + \overline{\sin\beta}\cos\alpha\psi^*_\perp T'(\lambda)\varphi + \overline{\sin\beta}\sin\alpha\psi^*_\perp T'(\lambda)\varphi_\perp\right|}. \tag{4}$$

Assume in addition that $\psi^* T'(\lambda)\varphi_\perp$, $\psi_\perp^* T'(\lambda)\varphi$ and $\psi_\perp^* T'(\lambda)\varphi_\perp$ are bounded away from zero. Then, for sufficiently small α, $|\rho - \lambda| \le \mathcal{O}(\sin\alpha)$.

The above proposition shows that the Rayleigh functional value approximating a defective λ is generally of first order accuracy (in terms of eigenvector approximation error), whether it is one-sided or two-sided. As we showed in part I [9], the lack of second order accuracy in eigenvalue approximation is the major reason for Rayleigh functional iteration (RFI), single-vector Jacobi–Davidson method and their two-sided variants to exhibit only linear convergence in the defective case.

We end this section by reviewing the resolvent $T(\mu)^{-1}$ near a defective λ. The structure of $T(\mu)^{-1}$ is described as follows (cf. Theorem A.10.2 of [4]).

Theorem 3 *Let $T : U \to \mathbb{C}^{n\times n}$ be holomorphic in a neighborhood of a defective eigenvalue λ of T, and J and $m_1, \ldots, m_J$ be the geometric and partial multiplicities of λ. Suppose that $\{\varphi_{k,s}\}$, $k = 1, \ldots, J$, $s = 0, \ldots, m_k - 1$ is a canonical system of right Jordan chains of T corresponding to λ. Then*

(i) there is a unique canonical system of left Jordan chains $\{\psi_{k,s}\}$, $k = 1, \ldots, J$, $s = 0, \ldots, m_k - 1$ such that

$$T(\mu)^{-1} = \sum_{k=1}^{J} \sum_{h=0}^{m_k-1} \frac{\sum_{s=0}^{h} \langle \cdot, \psi_{k,s}\rangle \varphi_{k,h-s}}{(\mu-\lambda)^{m_k-h}} + Q(\mu), \tag{5}$$

in a neighborhood of λ, where Q is holomorphic in this neighborhood;

(ii) the left and the right Jordan chains satisfy the following normalization conditions

$$\sum_{s=0}^{\ell} \sum_{\sigma=s+1}^{m_k+s} \frac{1}{\sigma!} \langle T^{(\sigma)}(\lambda)\varphi_{k,m_k+s-\sigma}, \psi_{j,\ell-s}\rangle = \delta_k^j \delta_\ell^0, \tag{6}$$
$$\sum_{s=0}^{\ell} \sum_{\sigma=s+1}^{m_k+s} \frac{1}{\sigma!} \langle T^{(\sigma)}(\lambda)\varphi_{k,\ell-s}, \psi_{j,m_k+s-\sigma}\rangle = \delta_k^j \delta_\ell^0,$$

where $\psi_{j,p} = 0$, $\varphi_{k,q} = 0$ for $p \ge m_j$, $q \ge m_k$ by convention.

2.2 Convergence of Newton's method near singular roots

Consider a solution $z^* \in \mathbb{C}^n$ of the nonlinear system $F(z) = 0$, for which the Jacobian $F'(z^*)$ is singular. Let $\mathcal{N}_1 = \text{null } F'(z^*)$, $\mathcal{M}_2 = \text{range } F'(z^*)$, so that $\text{codim}(\mathcal{M}_2) = \dim(\mathcal{N}_1)$. Choose complementary subspaces $\mathcal{M}_1, \mathcal{N}_2$ such that $\mathbb{C}^n = \mathcal{N}_1 \oplus \mathcal{M}_1 = \mathcal{N}_2 \oplus \mathcal{M}_2$. Let $P_{\mathcal{M}_i}$ be the projections onto $\mathcal{M}_i$ along $\mathcal{N}_i$, and $P_{\mathcal{N}_i} = I - P_{\mathcal{M}_i}$. The Jacobian $F'(z)$ can be decomposed as

$$\begin{aligned} F'(z) &= A_F(z) + B_F(z) + C_F(z) + D_F(z), \quad \text{where} \\ A_F(z) &= P_{\mathcal{M}_2} F'(z) P_{\mathcal{M}_1}, \quad B_F(z) = P_{\mathcal{M}_2} F'(z) P_{\mathcal{N}_1} \\ C_F(z) &= P_{\mathcal{N}_2} F'(z) P_{\mathcal{M}_1}, \quad D_F(z) = P_{\mathcal{N}_2} F'(z) P_{\mathcal{N}_1}. \end{aligned} \tag{7}$$

Then $A_{F^*} = P_{\mathcal{M}_2} A_F(z^*) P_{\mathcal{M}_1}$ is a bijection when considered as an operator from $\mathcal{M}_1$ into $\mathcal{M}_2$. The Taylor expansions of the expressions in (7) at z^* are

$$A_F(z) = A_{F^*} + \sum_{j=a}^{\ell} A_{(j)}(z) + \mathcal{O}_{\ell+1}(e_z), \quad B_F(z) = \sum_{j=b}^{\ell} B_{(j)}(z) + \mathcal{O}_{\ell+1}(e_z),$$
$$C_F(z) = \sum_{j=c}^{\ell} C_{(j)}(z) + \mathcal{O}_{\ell+1}(e_z), \quad D_F(z) = \sum_{j=d}^{\ell} D_{(j)}(z) + \mathcal{O}_{\ell+1}(e_z), \tag{8}$$

where $e_z = z - z^*$ is the error of z, and

$$A_{(j)}(z) = \frac{1}{j!} P_{\mathcal{M}_2} F^{(j+1)}(z^*)(e_z^j, P_{\mathcal{M}_1}\cdot), \; B_{(j)}(z) = \frac{1}{j!} P_{\mathcal{M}_2} F^{(j+1)}(z^*)(e_z^j, P_{\mathcal{N}_1}\cdot),$$
$$C_{(j)}(z) = \frac{1}{j!} P_{\mathcal{N}_2} F^{(j+1)}(z^*)(e_z^j, P_{\mathcal{M}_1}\cdot), \; D_{(j)}(z) = \frac{1}{j!} P_{\mathcal{N}_2} F^{(j+1)}(z^*)(e_z^j, P_{\mathcal{N}_1}\cdot).$$

Here $F^{(j+1)}(z^*)(\cdot, \cdots, \cdot)$, a tensor with $j+1$ arguments, is the $(j+1)$st derivative of F at z^*, and e_z^j means all the first j arguments of $F^{(j+1)}$ are e_z. Also,

$$j = a, b, c, d > 0 \;\text{ and }\; j = \bar{a}, \bar{b}, \bar{c}, \bar{d} > 0 \tag{9}$$

are the smallest integers, independent of z, for which the *vectors* $A_{(j)}(z)e_z$, $B_{(j)}(z)e_z$, $C_{(j)}(z)e_z$, $D_{(j)}(z)e_z$, and the *matrices* $A_{(j)}(z)$, $B_{(j)}(z)$, $C_{(j)}(z)$, $D_{(j)}(z)$ are not identically zero, respectively. Obviously, $\bar{a} \le a$, $\bar{b} \le b$, $\bar{c} \le c$ and $\bar{d} \le d$. With the above definitions, the major result of the local convergence of Newton's method near singular solutions is summarized as follows.

Theorem 4 (Theorem 5.9 in [2]) *Let z^* be a singular root of $F(z) = 0$, and $e_z = z - z^*$. Define the operator $\widetilde{D}_{(j)} = \frac{1}{j!} P_{\mathcal{N}_2} F^{(j+1)}(z^*)((P_{\mathcal{N}_1} e_z)^j, P_{\mathcal{N}_1}\cdot)$, and let $j = \tilde{d}$ be the smallest integer, independent of z, for which $\widetilde{D}_{(j)} e_z \not\equiv 0$. Assume that $\tilde{d} = d \le c$, $\bar{d} \le \bar{c}$ and $\widetilde{D}_{(\bar{d})}$ is nonsingular for all z with $P_{\mathcal{N}_1} e_z \ne 0$. Let z_0 be the initial Newton iterate, and define $\eta = \min(a, b, c, d)$. Then, for sufficiently small $\delta > 0$ and $\theta > 0$, $F'(z_0)$ is nonsingular for all*

$$z_0 \in W(\delta, \theta) \equiv \left\{ z : \, 0 < \|e_z\| \le \delta, \; \|P_{\mathcal{M}_1} e_z\| \le \theta \|P_{\mathcal{N}_1} e_z\| \right\}$$

all subsequent Newton iterates remain in $W(\delta, \theta)$; in addition, $z_k \to z^*$ with

$$\|P_{\mathcal{M}_1}(z_k - z^*)\| \le C \|P_{\mathcal{M}_1}(z_{k-1} - z^*)\|^{\eta+1}$$

for some constant $C > 0$ independent of k, and

$$\lim_{k\to\infty} \frac{\|P_{\mathcal{N}_1}(z_k - z^*)\|}{\|P_{\mathcal{N}_1}(z_{k-1} - z^*)\|} = \frac{d}{d+1}.$$

Moreover, $F'(z)$ is nonsingular if and only if the Schur complement

$$S_F(z) : \mathcal{N}_1 \to \mathcal{N}_2 \equiv D_F(z) - C_F(z)A_F^{-1}(z)B_F(z) \tag{10}$$

is nonsingular. In this case,

$$\begin{aligned} F'(z)^{-1} &= \big(A_F(z) + B_F(z) + C_F(z) + D_F(z)\big)^{-1} \\ &= P_{\mathcal{M}_1}\big(A_F^{-1}(z) + A_F^{-1}(z)B_F(z)S_F^{-1}(z)C_F(z)A_F^{-1}(z)\big)P_{\mathcal{M}_2} \\ &\quad - P_{\mathcal{M}_1}A_F^{-1}(z)B_F(z)S_F^{-1}(z)P_{\mathcal{N}_2} \\ &\quad - P_{\mathcal{N}_1}S_F^{-1}(z)C_F(z)A_F^{-1}(z)P_{\mathcal{M}_2} + P_{\mathcal{N}_1}S_F^{-1}(z)P_{\mathcal{N}_2}. \end{aligned} \tag{11}$$

Theorem 4 states that as Newton's method converges towards a singular solution z^*, the error component of $e_k = z_k - z^*$ lying in $\mathcal{M}_1$ converges at least quadratically, whereas the error component lying in $\mathcal{N}_1$ converges only linearly. This observation, together with the expression of the inverse Jacobian (11), are fundamental for our analysis of accelerated algorithms.

3 Accelerated algorithms

In this section, we first study an accelerated inverse iteration for defective eigenvalues, which is inspired by the accelerated Newton's method for nonlinear system of equations near singular roots [2]. This algorithm requires the solution of two linear systems in each iteration, and it exhibits quadratic local convergence. We then propose an accelerated single-vector Jacobi–Davidson (JD) method based on a minor modification of the accelerated inverse iteration, and experiments show that it also converges quadratically.

3.1 Accelerated inverse iteration

Let λ be a defective eigenvalue of the holomorphic $T(\cdot)$ with a single right and corresponding left Jordan chains $\{\varphi_{1,0}, \ldots, \varphi_{1,m-1}\}$ and $\{\psi_{1,0}, \ldots, \psi_{1,m-1}\}$ as described in Theorem 3. To simplify the notation, we use MATLAB expressions to denote eigenpair approximations written in the form of column vectors; e.g., $[x; \mu]$ is used to represent $\begin{bmatrix} x \\ \mu \end{bmatrix} \in \mathbb{C}^{n+1}$. Let $v = \varphi_{1,0}$ be the desired right eigenvector, and $F([v; \lambda]) = \begin{bmatrix} T(\lambda)v \\ u^*v - 1 \end{bmatrix} = 0$ be the augmented system of (1), and $[x_k; \mu_k]$ be the eigenpair approximation in the kth iteration. Consider the following accelerated Newton's method

$$\left.\begin{aligned} [w_k; v_k] &= [x_k; \mu_k] - F'([x_k; \mu_k])^{-1}F([x_k; \mu_k]); && \text{(half-step iterate)} \\ [x_{k+1}; \mu_{k+1}] &= [w_k; v_k] - mF'([w_k; v_k])^{-1}F([w_k; v_k]) && \text{(full-step iterate)} \\ &= m\big([w_k; v_k] - F'([w_k; v_k])^{-1}F([w_k; v_k])\big) - (m-1)[w_k; v_k], \end{aligned}\right\} \tag{12}$$

where $F([x_k;\mu_k]) = \begin{bmatrix} T(\mu_k)x_k \\ u^*x_k - 1 \end{bmatrix} = \begin{bmatrix} T(\mu_k)x_k \\ 0 \end{bmatrix}$, $\|F([x_k;\mu_k])\| = \|T(\mu_k)x_k\|$ is the residual norm of $[x_k;\mu_k]$, and $F'([x_k;\mu_k]) = \begin{bmatrix} T(\mu_k) & T'(\mu_k)x_k \\ u^* & 0 \end{bmatrix}$ is the Jacobian of F at $[x_k;\mu_k]$. In other words, the full-step eigenpair approximation $[x_{k+1};\mu_{k+1}]$ is a special linear combination of the half-step iterate $[w_k;\nu_k]$ and the next standard iterate $[z_k;\xi_k] = [w_k;\nu_k] - F'([w_k;\nu_k])^{-1}F([w_k;\nu_k])$. We will see later in this section that the length of the Jordan chain m is the only value for the linear combination coefficients in (12) for which we can guarantee the quadratic convergence of the algorithm.

It can be shown by the structure of the block inverse of the Jacobian that (12) is equivalent to the following accelerated inverse iteration:

$$\left.\begin{array}{ll}
1.\ p_k = T^{-1}(\mu_k)T'(\mu_k)x_k; & \text{(half-step intermediate vector)} \\
2.\ w_k = \dfrac{p_k}{u^*p_k}; & \text{(half-step eigenvector approximation)} \\
3.\ \nu_k = \mu_k - \dfrac{1}{u^*p_k}; & \text{(half-step eigenvalue approximation)} \\
4.\ q_k = T^{-1}(\nu_k)T'(\nu_k)w_k; & \text{(full-step intermediate vector)} \\
5.\ x_{k+1} = -(m-1)w_k + m\dfrac{q_k}{u^*q_k}; & \text{(full-step eigenvector approximation)} \\
6.\ \mu_{k+1} = \nu_k - \dfrac{m}{u^*q_k}. & \text{(full-step eigenvalue approximation)}
\end{array}\right\} \quad (13)$$

In this section, we establish the locally quadratic convergence of the accelerated algorithm (12) or (13) towards the desired defective eigenpair. To this end, we first make an assumption about the eigenpair approximation (μ_k, x_k).

Assumption 5 Let λ be a defective eigenvalue of the holomorphic $T(\cdot)$ with a single right and left Jordan chain as in Theorem 3. Assume that the eigenpair approximation (μ_k, x_k) is such that its eigenvalue approximation error $|\mu_k - \lambda|$ is proportional to its eigenvector approximation error $\angle(x_k, v)$.

This assumption is reasonable due to the following observation. In practice, it is usually not known a priori if the desired eigenpair is defective, and thus we use inverse iteration (Newton's method), RFI, or JD to compute the eigenpair. For inverse iteration, it is shown in part I of this study [9] that its local convergence towards a defective eigenpair is linear; in fact, we showed that the error $e_k = [x_k;\mu_k] - [v;\lambda]$ has a component in the one-dimensional space $\mathcal{L} = \operatorname{span}\left\{\begin{bmatrix} \alpha v + \beta v^{\perp} \\ 1 \end{bmatrix}\right\}$ for some scalars α and β, where $v^{\perp}$ lies in the orthogonal complement of null $T(\lambda)$. Since $\mathcal{L} \subset \mathcal{N}_1 = \operatorname{null} F'([v;\lambda])$, the error component lying in $\mathcal{L}$ converges linearly as Newton's method proceeds. Note that this basis vector of $\mathcal{L}$ has nontrivial components representing both eigenvalue and eigenvector approximation errors. Therefore, the two error components become proportional to each other after sufficiently many Newton steps. This assumption also holds for RFI where $\mu_k = \rho$ is the Rayleigh functional value, since Proposition 2 shows that $|\rho - \lambda| \leq \mathcal{O}(\sin\angle(x_k, v))$.

Our main goal is to show that the accelerated method (12) converges quadratically towards the defective eigenpair $[v; \lambda]$. We take three steps to complete this analysis. In step 1, we make a few assumptions about the half-step iterate $[w_k; \nu_k]$, and we study the values of b, c, d in the Taylor expansion of $F([w_k; \nu_k])$ at $[v; \lambda]$; see Sect. 2.2 for their definitions. This step is critical to establish the quadratic convergence. In step 2, we show that the Jacobian at any eigenpair approximation sufficiently close to $[v; \lambda]$ is nonsingular, so that all half-step and full-step iterates are well-defined. In step 3, we write the full-step iterate error as a linear combination of the error of $[w_k; \nu_k]$ and that of the next standard Newton iterate $[z_k; \xi_k] = [w_k; \nu_k] - F'([w_k; \nu_k])^{-1} F([w_k; \nu_k])$; we analyze the projections of the full-step iterate error onto the complementary subspaces $\mathcal{M}_1$ and $\mathcal{N}_1$, and show that all the projected errors are $\mathcal{O}(s_k^2)$, where $s_k = \sin \angle(x_k, v)$.

From now on, for simplicity, we omit the subscript k of s_k, $[x_k; \mu_k]$, $[w_k; \nu_k]$ and $[z_k; \xi_k]$, when there is no risk of confusion; we keep the subscript $k+1$ of $[x_{k+1}; \mu_{k+1}]$ though, to clearly identify the full-step iterate.

In step 1, we first show that the half-step iterate $[w; \nu]$ has a special property as follows. Since Newton's method converges linearly, the error of $[w; \nu]$ is proportional to that of $[x; \mu]$, yet the residual norm of $[w; \nu]$ is significantly smaller than that of $[x; \mu]$; namely, $\|T(\mu)x\| = \mathcal{O}(s)$ and $\|T(\nu)w\| = \mathcal{O}(s^m)$, where $s = \sin \angle(x, v)$ is the eigenvector approximation error, and m is the length of the single Jordan chain associated with λ.

Lemma 1 *Let λ be a defective eigenvalue of the holomorphic operator $T(\cdot)$ associated with only one left and one right Jordan chains $\{\varphi_{1,0}, \ldots, \varphi_{1,m-1}\}$ and $\{\psi_{1,0}, \ldots, \psi_{1,m-1}\}$, for which $\langle T''(\lambda)\varphi_{1,0}, \psi_{1,0}\rangle \neq 0$ or $\langle T'(\lambda)\varphi_{1,0}, \psi_{1,1}\rangle \neq 0$. Let $[x; \mu]$ be an eigenpair approximation satisfying $P_{\mathcal{N}_1}([x; \mu] - [\varphi_{1,0}; \lambda]) \neq 0$ and* Assumption 5*, $s = \sin \angle(x, \varphi_{1,0})$, and $[w; \nu]$ be the half-step iterate in the accelerated inverse iteration* (13)*. Then, under the assumptions of* Theorem 4*, $\|T(\nu)w\| = \mathcal{O}(s^m)$ for sufficiently small s.*

Proof We first establish two critical relations, namely, $\|u^* T^{-1}(\mu) T'(\mu) x\| = \mathcal{O}(s^{-(m-1)})$ and $\|T'(\mu)x - T'(\mu)w\| = \mathcal{O}(s)$. Assume without loss of generality that $x = c\varphi_{1,0} + sg$, where $\|\varphi_{1,0}\| = \|g\| = 1$ and $g \perp \varphi_{1,0}$.

We first show that $\|u^* T^{-1}(\mu) T'(\mu) x\| = \mathcal{O}(s^{-(m-1)})$. From (5), we have

$$
\begin{aligned}
p = T^{-1}(\mu)T'(\mu)x &= \sum_{j=0}^{m-1} \sum_{i=0}^{m-j-1} \frac{\langle T'(\mu)x, \psi_{1,j}\rangle}{(\mu-\lambda)^{m-j-i}} \varphi_{1,i} + G(\mu)T'(\mu)x \\
&= \sum_{i=0}^{m-1} \frac{\langle T'(\mu)x, \psi_{1,0}\rangle}{(\mu-\lambda)^{m-i}} \varphi_{1,i} + \sum_{j=1}^{m-1} \sum_{i=0}^{m-j-1} \frac{\langle T'(\mu)x, \psi_{1,j}\rangle}{(\mu-\lambda)^{m-j-i}} \varphi_{1,i} + G(\mu)T'(\mu)x. \quad (14)
\end{aligned}
$$

Recall from (3) that $\langle T'(\lambda)\varphi_{1,0}, \psi_{1,0}\rangle = 0$, and $|\mu - \lambda| = \mathcal{O}(s)$ by Assumption 5. First, assume that $\langle T''(\lambda)\varphi_{1,0}, \psi_{1,0}\rangle = \mathcal{O}(1)$. Then

$$\begin{aligned}\langle T'(\mu)x, \psi_{1,0}\rangle &= c\langle T'(\mu)\varphi_{1,0}, \psi_{1,0}\rangle + s\langle T'(\mu)g, \psi_{1,0}\rangle \\ &= c\langle T'(\lambda)\varphi_{1,0}, \psi_{1,0}\rangle + c(\mu-\lambda)\langle T''(\lambda)\varphi_{1,0}, \psi_{1,0}\rangle + s\langle T'(\lambda)g, \psi_{1,0}\rangle \\ &\quad + \mathcal{O}\left((\mu-\lambda)^2\right) + \mathcal{O}\left(s(\mu-\lambda)\right) \\ &= 0 + \mathcal{O}(\mu-\lambda) + \mathcal{O}(s) = \mathcal{O}(s). \end{aligned} \tag{15}$$

The dominant term in $\sum_{i=0}^{m-1} \langle T'(\mu)x, \psi_{1,0}\rangle(\mu-\lambda)^{-(m-i)}\varphi_{1,i}$, which appears in the last equality of (14), is thus

$$\frac{\langle T'(\mu)x, \psi_{1,0}\rangle}{(\mu-\lambda)^m}\varphi_{1,0} = \frac{\mathcal{O}(s)}{\mathcal{O}(s^m)}\varphi_{1,0} = \mathcal{O}(s^{-(m-1)})\varphi_{1,0}.$$

Now, assume alternatively that $\langle T'(\lambda)\varphi_{1,0}, \psi_{1,1}\rangle \neq 0$. It follows that

$$\langle T'(\mu)x, \varphi_{1,1}\rangle = \langle T'(\lambda)\varphi_{1,0}, \psi_{1,1}\rangle + \mathcal{O}(s) = \mathcal{O}(1)$$

for small s. Consider the sum of the terms corresponding to $j = 1$ in (14),

$$\sum_{i=0}^{m-2} \frac{\langle T'(\mu)x, \psi_{1,1}\rangle}{(\mu-\lambda)^{m-1-i}}\varphi_{1,i}. \tag{16}$$

The dominant term in (16) is

$$\frac{\langle T'(\mu)x, \psi_{1,1}\rangle}{(\mu-\lambda)^{m-1}}\varphi_{1,0} = \frac{\mathcal{O}(1)}{\mathcal{O}(s^{m-1})}\varphi_{1,0} = \mathcal{O}(s^{-(m-1)})\varphi_{1,0}.$$

For every $j \geq 2$, the sum of corresponding terms in (14) are $\mathcal{O}(s^{-(m-2)})$, and they are therefore not dominant. To summarize, if either $\langle T''(\lambda)\varphi_{1,0}, \psi_{1,0}\rangle \neq 0$ or $\langle T'(\lambda)\varphi_{1,0}, \psi_{1,1}\rangle \neq 0$, the dominant term in (14) can be written as $\mathcal{O}(s^{-(m-1)})\varphi_{1,0}$, and $\|u^*T^{-1}(\mu)T'(\mu)x\| = \mathcal{O}(s^{-(m-1)})$ follows.

To complete the proof, it is sufficient to show $\|T'(\mu)x - T'(\mu)w\| = \mathcal{O}(s)$. In fact, under the assumptions of Theorem 4 and that $P_{\mathcal{N}_1}([x;\mu]-[v;\lambda]) \neq 0$, we showed in [9, Section 4.1] that both the eigenvalue and eigenvector approximation errors decrease linearly from $[x;\mu]$ to $[w;v]$. Since $s = \sin\angle(x, \varphi_{1,0})$, we have $\|w - \varphi_{1,0}\| = \mathcal{O}(s)$. Therefore

$$\begin{aligned}\|T'(\mu)x - T'(\mu)w\| &\leq \|T'(\mu)\|\|x-w\| \\ &\leq \|T'(\mu)\|(\|x-\varphi_{1,0}\| + \|w-\varphi_{1,0}\|) = \mathcal{O}(s).\end{aligned}$$

Finally, note from (13) that

$$\begin{aligned}T(v)w &= T\left(\mu - [u^*T^{-1}(\mu)T'(\mu)x)]^{-1}\right)w \\ &= T(\mu)w - \frac{T'(\mu)w}{u^*T^{-1}(\mu)T'(\mu)x} + \mathcal{O}\left((u^*T^{-1}(\mu)T'(\mu)x)^{-2}\right)\end{aligned}$$

$$= T(\mu)\frac{T^{-1}(\mu)T'(\mu)x}{u^*T^{-1}(\mu)T'(\mu)x} - \frac{T'(\mu)w}{u^*T^{-1}(\mu)T'(\mu)x} + \mathcal{O}(s^{2(m-1)})$$
$$= \frac{T'(\mu)x - T'(\mu)w}{u^*T^{-1}(\mu)T'(\mu)x} + \mathcal{O}(s^{2(m-1)}).$$

It then immediately follows that

$$\|T(v)w\| = \frac{\|T'(\mu)x - T'(\mu)w\|}{|u^*T^{-1}(\mu)T'(\mu)x|} + \mathcal{O}(s^{2(m-1)})$$
$$= \frac{\mathcal{O}(s)}{\mathcal{O}(s^{-(m-1)})} + \mathcal{O}(s^{2(m-1)}) = \mathcal{O}(s^m) \tag{17}$$

for $m \geq 2$. This completes the proof. □

Lemma 1 gives a critical preliminary result to find the values of b, c, and d in the Taylor expansion of $F([w; v])$ at $[v; \lambda]$. To complete this step, we need to review the structure of $\mathcal{N}_1 = \text{null } F'([v; \lambda])$, $\mathcal{M}_2 = \text{range } F'([v; \lambda])$, and their complements discussed in part I of this study [9, Section 4.1].

Proposition 6 *Let λ be a defective eigenvalue of $T(\cdot)$ with J right and corresponding J left Jordan chains $\{\varphi_{1,0}, \ldots, \varphi_{1,m_1-1}\}, \ldots, \{\varphi_{J,0}, \ldots, \varphi_{J,m_J-1}\}$, and $\{\psi_{1,0}, \ldots, \psi_{1,m_1-1}\}, \ldots, \{\psi_{J,0}, \ldots, \psi_{J,m_J-1}\}$. Assume that $m_i \geq 2$ for all $1 \leq i \leq J$. Let $Y^*T(\lambda)X = \begin{bmatrix} 0_J & 0 \\ 0 & \Sigma_{n-J} \end{bmatrix}$ be a singular value decomposition of $T(\lambda)$, where $X = [X_J \;\; X_{n-J}]$ and $Y = [Y_J \;\; Y_{n-J}]$ are unitary matrices, X_J and Y_J contain orthonormal basis vectors of $\text{null } T(\lambda) = \text{span}\{\varphi_{1,0}, \ldots, \varphi_{J,0}\}$ and $\text{null } T(\lambda)^* = \text{span}\{\psi_{1,0}, \ldots, \psi_{J,0}\}$, respectively. Let $v \in \text{null } T(\lambda)$ be an eigenvector and u a nonzero vector such that $u^*v = 1$. Then*

$$\mathcal{M}_2 \equiv \text{range } F'([v; \lambda]) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \oplus \text{range}\left(\begin{bmatrix} T(\lambda)X_{n-J} \\ u^*X_{n-J} \end{bmatrix}\right), \tag{18}$$

and we can let the complementary space be

$$\mathcal{N}_2 = \begin{bmatrix} (T(\lambda)X_{n-J})^\perp \\ 0 \end{bmatrix}, \tag{19}$$

where $(T(\lambda)X_{n-J})^\perp$ contains columns orthogonal to $\text{range}\,(T(\lambda)X_{n-J})$. Let $\mathbf{e_i}$ be the standard basis vector with one in the i-th entry and zeros elsewhere. Let $u^*X_J = [\gamma_1, \ldots, \gamma_J]$, and assume without loss of generality that $\gamma_J \neq 0$. Then

$$\mathcal{N}_1 = \text{null } F'([v; \lambda]) = \text{span}\left\{\begin{bmatrix} X\mathbf{e_1} - \frac{\gamma_1}{\gamma_J}X\mathbf{e_J} \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} X\mathbf{e_{J-1}} - \frac{\gamma_{J-1}}{\gamma_J}X\mathbf{e_J} \\ 0 \end{bmatrix}\right\}$$
$$\oplus \text{span}\left\{\begin{bmatrix} \chi X\mathbf{e_J} + X_{n-J}f \\ 1 \end{bmatrix}\right\}, \tag{20}$$

where $\chi = \frac{1}{\gamma_J} u^* X_{n-J} \Sigma_{n-J}^{-1} Y_{n-J}^* T'(\lambda) v \in \mathbb{C}$ *and* $f = -\Sigma_{n-J}^{-1} Y_{n-J}^* T'(\lambda) v \in \mathbb{C}^{n-J}$. *The complementary space* $\mathcal{M}_1$ *of* $\mathcal{N}_1$ *can be defined as*

$$\mathcal{M}_1 = \text{span}\left\{\begin{bmatrix} X\mathbf{e_J} \\ 0 \end{bmatrix}\right\} \oplus \text{range}\left(\begin{bmatrix} X_{n-J} \\ 0 \end{bmatrix}\right). \tag{21}$$

To complete step 1 and determine the values of b, c and d, we make the following assumption.

Assumption 7 For the half-step iterate $[w_k; \nu_k]$ generated by the accelerated algorithm (12) or (13), assume there is no special relation between the vector $F([w_k; \nu_k]) = \begin{bmatrix} T(\nu_k)w_k \\ 0 \end{bmatrix}$ and the space $\mathcal{M}_2$ so that $\left\{\angle(F([w_k; \nu_k]), \mathcal{M}_2)\right\}$ $(k = 0, 1, \dots)$ does not have a subsequence converging to 0 or $\frac{\pi}{2}$ as the algorithm proceeds and converges, i.e., as $s_k = \sin\angle(x_k, v) \to 0$.

Assumption 7 is reasonable. In fact, we see from (17) that $T(\nu)w$ has a significant component in the direction of $T'(\mu)(x - w)$. Due to the normalization conditions $u^*x = u^*w = 1$, we have $x - w \perp u$, and therefore

$$\begin{aligned} T'(\mu)(x-w) &= T'(\lambda)(x-w) + (\mu-\lambda)T''(\lambda)(x-w) + \mathcal{O}\left((\mu-\lambda)^2\|x-w\|\right) \\ &= T'(\lambda)(x-w) + \mathcal{O}(s^2) \end{aligned}$$

is dominated by $T'(\lambda)(x-w) \in T'(\lambda)\,(\text{span}\{u\})^{\perp}$, which bears no obvious connection with range $(T(\lambda)X_{n-J})$. Since $T(\nu)w$ depends on $[x; \mu]$ and the choice of u, there is no particular reason that $\left\{\angle\left(T(\nu_k)w_k, \text{range}\,(T(\lambda)X_{n-J})\right)\right\}$ has a subsequence converging to 0 or $\frac{\pi}{2}$ as the accelerated algorithm (13) proceeds and converges, i.e., as $s_k = \sin\angle(x_k, \text{null}\,T(\lambda)) \to 0$. In fact, we generated $[x_0; \mu_0]$ and u randomly and repeated the experiments hundreds of times for different test problems, and we have not seen a counterexample.

Moreover, we have $\angle(F([w; \nu]), \mathcal{M}_2) = \angle\left(T(\nu)w, \text{range}(T(\lambda)X_{n-J})\right)$. To see this, let $g = \begin{bmatrix} T(\lambda)X_{n-J}g_1 \\ g_2 \end{bmatrix} \in \mathcal{M}_2$, where $g_1 \in \mathbb{C}^{n-J}$ and $g_2 \in \mathbb{C}$. Then

$$\begin{aligned} \angle(F([w;\nu]), g) &= \cos^{-1}\frac{\langle F([w;\nu]), g\rangle}{\|F([w;\nu])\|\|g\|} \\ &= \cos^{-1}\frac{\langle T(\nu)w, T(\lambda)X_{n-J}g_1\rangle}{\|T(\nu)w\|\sqrt{\|T(\lambda)X_{n-J}g_1\|^2 + |g_2|^2}} \\ &\geq \cos^{-1}\frac{\langle T(\nu)w, T(\lambda)X_{n-J}g_1\rangle}{\|T(\nu)w\|\|T(\lambda)X_{n-J}g_1\|} = \angle(T(\nu)w, T(\lambda)X_{n-J}g_1), \end{aligned}$$

where, for any given g_1, the equality holds if and only if $g_2 = 0$. It follows that $\angle(F([w; \nu]), \mathcal{M}_2) = \angle\left(T(\nu)w, \text{range}(T(\lambda)X_{n-J})\right)$. Recall the definitions of $P_{\mathcal{M}_2}$ and $P_{\mathcal{N}_2}$, where $\mathcal{M}_2$ and $\mathcal{N}_2$ are orthogonal complements; see Proposition 6. From Lemma 1 and Assumption 7, we have

$$\left\| P_{\mathcal{M}_2} F([w;\nu]) \right\| = \mathcal{O}(s^m) \text{ and } \left\| P_{\mathcal{N}_2} F([w;\nu]) \right\| = \mathcal{O}(s^m). \tag{22}$$

To complete Step 1, let

$$e_{wv} = [w;\nu] - [v;\lambda],$$

and recall the Taylor expansion of $F([w;\nu])$ at $[v;\lambda]$:

$$F([w;\nu]) \equiv \begin{bmatrix} T(\nu)w \\ u^* y - 1 \end{bmatrix} = \sum_{j=0}^{\ell} \frac{1}{(j+1)!} F^{(j+1)}([v;\lambda])(e_{wv}^j, e_{wv}) + \mathcal{O}(\|e_{wv}\|^{\ell+2}),$$

where $F^{(j+1)}([v;\lambda])(\cdot,\cdots,\cdot) : \mathbb{C}^{n+1}\times\ldots\times\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is the $(j+1)$st derivative of F at $[v;\lambda]$ (a tensor with $j+1$ arguments), and e_{wv}^j means that the first j arguments of $F^{(j+1)}$ are all e_{wv}. In particular, $F^{(j+1)}([v;\lambda])(e_{wv}^j, \cdot)$ is a $(n+1)\times(n+1)$ matrix. We use this expansion to discuss several cases in which the possible values of b, c, and d in the Taylor expansion of $F([w;\nu])$ at $[v;\lambda]$ can be determined.

Lemma 2 *Let $[v;\lambda]$ be a defective eigenpair of $T(\cdot)$ with one right and one left Jordan chain of length m, $[x;\mu]$ a corresponding eigenpair approximation satisfying Assumption 5 and $P_{\mathcal{N}_1}([x;\mu]-[v;\lambda]) \neq 0$, and $[w;\nu]$ the half-step eigenpair approximation in (12) computed by one step of Newton's method. The Taylor expansion of $F([w;\nu])$ at $[v;\lambda]$ is*

$$\begin{aligned} F([w;\nu]) \equiv \begin{bmatrix} T(\nu)w \\ 0 \end{bmatrix} = \begin{bmatrix} T(\lambda)v \\ 0 \end{bmatrix} + A_{F^*} e_{wv} + \sum_{j=a}^{\ell} \frac{1}{j+1} A_{(j)}([w;\nu]) e_{wv} \\ + \sum_{j=b}^{\ell} \frac{1}{j+1} B_{(j)}([w;\nu]) e_{wv} + \sum_{j=c}^{\ell} \frac{1}{j+1} C_{(j)}([w;\nu]) e_{wv} \\ + \sum_{j=d}^{\ell} \frac{1}{j+1} D_{(j)}([w;\nu]) e_{wv} + \mathcal{O}(\|e_{wv}\|^{\ell+2}). \\ (\max(a,b,c,d) \le \ell). \end{aligned}$$

Then, under the assumptions of Theorem 4, *$b = 1$ if $m > 2$, and $c \ge d = m-1$.*

Proof Recall from (22) that $\left\| P_{\mathcal{M}_2} F([w;\nu]) \right\| = \mathcal{O}(s^m)$ and $\left\| P_{\mathcal{N}_2} F([w;\nu]) \right\| = \mathcal{O}(s^m)$ under our assumptions. In addition, by Assumption 5, $[x;\mu]$ has eigenvalue and eigenvector approximation errors both on the order of $\mathcal{O}(s)$, and the two errors are represented by certain components lying in $\mathcal{N}_1$ and $\mathcal{M}_1$, resepectively; see (20) and (21). It follows that

$$P_{\mathcal{N}_1} e_{x\mu} = \mathcal{O}(s) \quad \text{and} \quad P_{\mathcal{M}_1} e_{x\mu} = \mathcal{O}(s), \quad \text{where} \quad e_{x\mu} = [x;\mu] - [v;\lambda].$$

By Theorem 4, we have $P_{\mathcal{N}_1} e_{wv} = \mathcal{O}(s)$ and $P_{\mathcal{M}_1} e_{wv} = \mathcal{O}(s^2)$. To find the value of b for $m > 2$, note that

$$\begin{aligned} P_{\mathcal{M}_2} F([w;v]) &= A_{F^*} e_{wv} + \sum_{j=a}^{\ell} \frac{A_{(j)}([w;v]) e_{wv}}{j+1} + \sum_{j=b}^{\ell} \frac{B_{(j)}([w;v]) e_{wv}}{j+1} \\ &= A_{F^*}(P_{\mathcal{M}_1} e_{wv}) + \sum_{j=a}^{\ell} \frac{A_{(j)}([w;v])(P_{\mathcal{M}_1} e_{wv})}{j+1} \\ &\quad + \sum_{j=b}^{\ell} \frac{B_{(j)}([w;v])(P_{\mathcal{N}_1} e_{wv})}{j+1} \\ &= \mathcal{O}(s^2) + \mathcal{O}(s^{a+2}) + \mathcal{O}(s^{b+1}) = \mathcal{O}(s^m). \end{aligned}$$

For any $m > 2$, since $\mathcal{O}(s^{a+2}) = \mathcal{O}(s^3)$, we must have $b = 1$ to cancel out the $\mathcal{O}(s^2)$ terms in the last equality above.

Similarly, to see the relation between c and d, we have

$$\begin{aligned} P_{\mathcal{N}_2} F([w;v]) &= \sum_{j=c}^{\ell} \frac{C_{(j)}([w;v]) e_{wv}}{j+1} + \sum_{j=d}^{\ell} \frac{D_{(j)}([w;v]) e_{wv}}{j+1} + \mathcal{O}(\|e_{wv}\|^{\ell+2}) \\ &= \mathcal{O}(s^{c+2}) + \mathcal{O}(s^{d+1}) + \mathcal{O}(s^{\ell+2}) = \mathcal{O}(s^m). \end{aligned} \tag{23}$$

Clearly, (23) holds only if exactly one of the following cases is true:

1. If $c \geq m-1$, then we must have $d = m-1$ ($m \geq 2$) to maintain a $\mathcal{O}(s^m)$ term on the left-hand side of (23).
2. If $c = m-2$, then we must have $d \geq m-1$ ($m \geq 3$, $b = 1$) so that there is no term of order lower than m on the left-hand side of (23).
3. If $c \leq m-3$, then we must have $d = c+1$ ($m \geq 4$, $b = 1$) so that the two terms of order lower than m on the left-hand side of (23) are cancelled out.

However, recall from Theorem 4 that $c \geq d$ is needed to establish the linear convergence of Newton's method. Therefore, only the first case is valid in our setting, and the lemma is thus proved. □

Lemma 2 completes step 1, where we derived the possible values of b, c, and d for the Taylor expansion of $F([w;v])$ at $[v;\lambda]$. In step 2, we show that all the iterates of the accelerated inverse iteration are well-defined. In fact, the nonsingularity of the Jacobian at the half-step and full-step iterates can be guaranteed by the following Lemma.

Lemma 3 *Let $[x;\mu]$ be an eigenpair approximation sufficiently close, but not equal to $[v;\lambda]$. Then the Jacobian $F'([x;\mu]) = \begin{bmatrix} T(\mu) & T'(\mu)x \\ u^* & 0 \end{bmatrix}$ is nonsingular.*

Proof By assumption, the eigenvalues of $T(\cdot)$ are isolated, and therefore $T(\mu)$ is nonsingular if μ is sufficiently close, but not equal to λ. It follows that $\begin{bmatrix} T(\mu) \\ u^* \end{bmatrix}$ has

full column rank. Assume that the Jacobian $\begin{bmatrix} T(\mu) & T'(\mu)x \\ u^* & 0 \end{bmatrix}$ is singular, then there exists $y \in \mathbb{C}^n$ such that $\begin{bmatrix} T(\mu) \\ u^* \end{bmatrix} y = \begin{bmatrix} T'(\mu)x \\ 0 \end{bmatrix}$. It follows that $y = T(\mu)^{-1}T'(\mu)x$ and $u^*y = 0$. However, if x is sufficiently close to v in direction, and $|\mu - \lambda|$ sufficiently small, then the unnormalized new eigenvector approximation y is closer to v in direction, and $\|y\| \gg \|v\|$. Given that $u^*v = 1$, it is impossible to have $u^*y = 0$. Therefore the Jacobian must be nonsingular. The same argument applies verbatim to half-step iterates $[w; \nu]$. □

In step 3, we first show that the error of the full-step iterate can be written as a linear combination of the half-step iterate error and the error of the next standard Newton iterate $[z; \xi] = [w; \nu] - F'([w; \nu])^{-1}F([w; \nu])$; this derivation follows that in [2, Section 4]. Then, we show that the projections of the full-step iterate error onto $\mathcal{N}_1$ and $\mathcal{M}_1$ are both $\mathcal{O}(s^2)$.

To decompose the error of the new full-step iterate $[x_{k+1}; \mu_{k+1}]$, define $[z; \xi] = [w; \nu] - F'([w; \nu])^{-1}F([w; \nu])$, i.e., the iterate obtained by applying one step of Newton's method to the half-step iterate. Then we have from (12) that $[x_{k+1}; \mu_{k+1}] = m[z; \xi] - (m-1)[w; \nu]$. The error of $[x_{k+1}; \mu_{k+1}]$ can be analyzed by studying $e_{z\xi} = [z; \xi] - [v; \lambda]$ and $e_{w\nu} = [w; \nu] - [v; \lambda]$. Following the derivation in [2, Section 4], we have

$$\begin{aligned} e_{z\xi} &= [z;\ \xi] - [v; \lambda] = e_{w\nu} - F'([w; \nu])^{-1}F([w; \nu]) \\ &= F'([w; \nu])^{-1} \left\{ \sum_{j=a}^{\ell} \frac{j}{(j+1)} A_{(j)}([w; \nu])e_{w\nu} + \sum_{j=b}^{\ell} \frac{j}{(j+1)} B_{(j)}([w; \nu])e_{w\nu} \right. \\ &\quad \left. + \sum_{j=c}^{\ell} \frac{j}{(j+1)} C_{(j)}([w; \nu])e_{w\nu} + \sum_{j=d}^{\ell} \frac{j}{(j+1)} D_{(j)}([w; \nu])e_{w\nu} + \mathcal{O}(s^{\ell+2}) \right\}. \end{aligned} \tag{24}$$

To study the errors of $[x_{k+1}; \mu_{k+1}]$ projected onto $\mathcal{N}_1$ and $\mathcal{M}_1$, recall that $c \geq d = m - 1$, $P_{\mathcal{M}_1}e_{w\nu} = \mathcal{O}(s^2)$ and $P_{\mathcal{N}_1}e_{w\nu} = \mathcal{O}(s)$. Therefore

$$\begin{aligned} [x_{k+1}; \mu_{k+1}] - [v; \lambda] &= m[z;\ \xi] - (m-1)[w; \nu] - [v; \lambda] = me_{z\xi} - (m-1)e_{w\nu} \\ &= mF'([w; \nu])^{-1} \Big\{ \frac{a}{a+1} A_{(a)}([w; \nu])e_{w\nu} + \frac{b}{b+1} B_{(b)}([w; \nu])e_{w\nu} \\ &\quad + \frac{c}{c+1} C_{(c)}([w; \nu])e_{w\nu} + \frac{d}{d+1} D_{(d)}([w; \nu])e_{w\nu} \\ &\quad + P_{\mathcal{M}_2}AB(s) + P_{\mathcal{N}_2}CD(s) \Big\} \\ &\quad - (m-1)F'([w; \nu])^{-1}F'([w; \nu])e_{w\nu} \\ &= F'([w; \nu])^{-1} \Big\{ -(m-1)A_{F^*}e_{w\nu} + \Big(\frac{ma}{a+1} - (m-1) \Big) A_{(a)}([w; \nu])e_{w\nu} \\ &\quad + \Big(\frac{mb}{b+1} - (m-1) \Big) B_{(b)}([w; \nu])e_{w\nu} + P_{\mathcal{M}_2}AB(s) + P_{\mathcal{N}_2}CD(s) \Big\}, \end{aligned} \tag{25}$$

where $AB(s) = \sum_{j=a+1}^{\ell} \frac{jA_{(j)}([w;v])e_{wv}}{j+1} + \sum_{j=b+1}^{\ell} \frac{jB_{(j)}([w;v])e_{wv}}{j+1} = \mathcal{O}(s^{\min(a+3,b+2)})$, and $CD(s) = \sum_{j=c+1}^{\ell} \frac{jC_{(j)}([w;v])e_{wv}}{j+1} + \sum_{j=d+1}^{\ell} \frac{jD_{(j)}([w;v])e_{wv}}{j+1} = \mathcal{O}(s^{\min(c+3,d+2)})$ $= \mathcal{O}(s^{d+2})$. In the last equality of (25), $C_{(c)}([w;v])e_{wv} = C_{(c)}([w;v])(P_{\mathcal{M}_1}e_{wv}) = \mathcal{O}(s^{c+2})$ does not appear explicitly, because it is assimilated into $P_{\mathcal{N}_2}CD(s)$, where $d \le c$. More importantly, since $d = m-1$, $D_{(d)}([w;v])e_{wv}$ vanishes because its coefficient is $m\frac{d}{d+1} - (m-1) = 0$. The cancellation of $D_{(d)}([w;v])e_{wv}$ is critical in the proof of the quadratic convergence of the accelerated method.

To finish step 3, we show that the errors of $[x_{k+1};\mu_{k+1}]$ projected onto $\mathcal{N}_1$ and $\mathcal{M}_1$ are both $\mathcal{O}(s^2)$. To this end, recall from (11) the expression of $F'([w;v])^{-1}$, and apply it to (25). With some effort, we have

$$
\begin{aligned}
[x_{k+1};\mu_{k+1}] - [v;\lambda] &\equiv e^{k+1}_{\mathcal{M}\mathcal{M}} + e^{k+1}_{\mathcal{M}\mathcal{N}} + e^{k+1}_{\mathcal{N}\mathcal{M}} + e^{k+1}_{\mathcal{N}\mathcal{N}}, \quad \text{where} \\
e^{k+1}_{\mathcal{M}\mathcal{M}} &= P_{\mathcal{M}_1}\big\{A_F^{-1}([w;v]) \\
&\quad + A_F^{-1}([w;v])B_F([w;v])S_F^{-1}([w;v])C_F([w;v])A_F^{-1}([w;v])\big\} \\
&\quad \times\Big\{-(m-1)A_{F^*}e_{wv} + \frac{a+1-m}{a+1}A_{(a)}([w;v])e_{wv} \\
&\quad + \frac{b+1-m}{b+1}B_{(b)}([w;v])e_{wv} + P_{\mathcal{M}_2}AB(s)\Big\}, \qquad (26) \\
e^{k+1}_{\mathcal{M}\mathcal{N}} &= -P_{\mathcal{M}_1}A_F^{-1}([w;v])B_F([w;v])S_F^{-1}([w;v])P_{\mathcal{N}_2}CD(s), \qquad (27) \\
e^{k+1}_{\mathcal{N}\mathcal{M}} &= -P_{\mathcal{N}_1}S_F^{-1}([w;v])C_F([w;v])A_F^{-1}([w;v])P_{\mathcal{M}_2} \\
&\quad \times\Big\{-(m-1)A_{F^*}e_{wv} + \frac{a+1-m}{a+1}A_{(a)}([w;v])e_{wv} \\
&\quad + \frac{b+1-m}{b+1}B_{(b)}([w;v])e_{wv} + P_{\mathcal{M}_2}AB(s)\Big\}, \quad \text{and} \qquad (28) \\
e^{k+1}_{\mathcal{N}\mathcal{N}} &= P_{\mathcal{N}_1}S_F^{-1}([w;v])P_{\mathcal{N}_2}CD(s). \qquad (29)
\end{aligned}
$$

We show that all the projected errors shown above are $\mathcal{O}(s^2)$. First, note from (8) that the dominant term of $A_F([w;v])$ is

$$A_{F^*} = P_{\mathcal{M}_2}F'([v;\lambda])P_{\mathcal{M}_1},$$

which means $\|A_F([w;v])\| = \mathcal{O}(1)$ and $\|A_F^{-1}([w;v])\| = \mathcal{O}(1)$. In addition, by (10), and by the assumption in Theorem 4 that $\bar{c} \ge \bar{d}$, we have

$$\|S_F([w;v])\| = \mathcal{O}(s^{\min(\bar{d},\bar{b}+\bar{c})}) = \mathcal{O}(s^{\bar{d}}) \text{ and } \|S_F([w;v])^{-1}\| = \mathcal{O}(s^{-\bar{d}}).$$

From (11), the operator involved in $e^{k+1}_{\mathcal{M}\mathcal{M}}$ in (26) satisfies

$$
\begin{aligned}
&\|A_F^{-1}([w;v]) + A_F^{-1}([w;v])B_F([w;v])S_F^{-1}([w;v])C_F([w;v])A_F^{-1}([w;v])\| \\
&\quad \le \|A_F^{-1}([w;v])\| + \|A_F^{-1}([w;v])\|^2\|B_F([w;v])\|\|S_F^{-1}([w;v])\|\|C_F([w;v])\| \\
&\quad = \mathcal{O}(1) + \mathcal{O}(s^{\bar{b}+\bar{c}-\bar{d}}) = \mathcal{O}(1),
\end{aligned}
$$

and therefore

$$\|e^{k+1}_{\mathcal{M}\mathcal{M}}\| = \mathcal{O}(1)\left(\mathcal{O}(s^2) + \mathcal{O}(s^{a+2}) + \mathcal{O}(s^{b+1})\right) = \mathcal{O}(s^2).$$

For $e^{k+1}_{\mathcal{M}\mathcal{N}}$ in (27), since $\|A_F^{-1}([w;v])B_F([w;v])S_F^{-1}([w;v])\| \le \|A_F^{-1}([w;v])\| \times \|B_F([w;v])\|\|S_F^{-1}([w;v])\| = \mathcal{O}(s^{\bar{b}-\bar{d}})$, we have from (11) that

$$\|e^{k+1}_{\mathcal{M}\mathcal{N}}\| \le \mathcal{O}(s^{\bar{b}-\bar{d}})\mathcal{O}(s^{d+2}) = \mathcal{O}(s^{\bar{b}+2}) = \mathcal{O}(s^3).$$

Similarly, note that $\|S_F^{-1}([w;v])C_F([w;v])A_F^{-1}([w;v])\| \le \|S_F^{-1}([w;v])\| \times \|C_F([w;v])\|\|A_F^{-1}([w;v])\| \le \mathcal{O}(s^{\bar{c}-\bar{d}})$. Therefore, from (11) and (28), we have

$$\|e^{k+1}_{\mathcal{N}\mathcal{M}}\| \le \mathcal{O}(s^{\bar{c}-\bar{d}})\left(\mathcal{O}(s^2) + \mathcal{O}(s^{a+2}) + \mathcal{O}(s^{b+1})\right) = \mathcal{O}(s^2).$$

The last error term $e^{k+1}_{\mathcal{N}\mathcal{N}}$ (29) can be directly bounded by $\|S_F^{-1}([w;v])\|$

$$\|e^{k+1}_{\mathcal{N}\mathcal{N}}\| \le \mathcal{O}(s^{-\bar{d}})\mathcal{O}(s^{d+2}) = \mathcal{O}(s^2).$$

The above analysis completes step 3, and the result is summarized below.

Theorem 8 *Let λ be a defective eigenvalue of the holomorphic $T(\cdot)$ with one right and one left Jordan chain of length m, and $v = \varphi_{1,0}$ the corresponding right eigenvector. Let $\delta, \theta > 0$ be some appropriately small constants, $[x_0;\mu_0] \in W(\delta,\theta)$ be an eigenpair approximation of $[v;\lambda]$ satisfying Assumption 5 and $P_{\mathcal{N}_1}([x_0;\mu_0] - [v;\lambda]) \neq 0$. Under the assumptions of Theorem 4 and Assumption 7, for sufficiently small $\sin\angle(x_0, v)$, all iterates of the accelerated method (12) or (13) are well-defined, and $[x_k;\mu_k]$ converges towards $[v;\lambda]$ quadratically.*

We see that $c \ge d = m-1$ and $\bar{c} \ge \bar{d}$ are critical assumptions leading to the quadratic convergence of the accelerated methods. In addition, $[x_{k+1};\mu_{k+1}] = m\big([w;v] - F'([w;v])^{-1}F([w;v])\big) - (m-1)[w;v]$ is the unique linear combination of the two consecutive iterates that cancels out the error components lying in $\mathcal{N}_1$ that converge linearly if standard Newton's method is used. A violation of any of these conditions would lead to a deceleration of the convergence rate back to linear.

Theorem 8 is an extension of the major results in [2, Section 7] to the special setting of eigenvalue computation under less stringent assumptions. Specifically, [2] studied two accelerated Newton's methods for solving a general nonlinear system of equations for a singular root. It was assumed there that $a = \bar{a}, b = \bar{b}, c = \bar{c}$ and $d = \bar{d}$ for any intermediate and full step iterates. Under certain additional hypothesis, the first accelerated method requires the computation of three Newton's directions per iteration, assuming that $c \ge d$ and $b \ge \min(2,d)$; the second method solves for two Newton's directions per iteration, assuming that $b \ge 2$ and $c \ge d \ge 2$. In fact, for the computation of degenerate eigenvalues discussed here, these hypotheses of [2] do not hold. For instance, we have $b = 1 < \min(2,d) = 2$ for $m > 2$, and $c \ge d = 1$ for

$m = 2$. Our assumption in Theorem 8 on the values of c, d, $\bar{c}$, and $\bar{d}$ is made for the half-step iterate $[w; v]$ alone, and it is less stringent than those in [2].

3.2 Accelerated single-vector JD

$$
\left.\begin{array}{l}
\text{1. Choose a vector } y_k, \text{ e.g., } y_k = T'(\mu_k)x_k, \text{ let } \Pi_k^{(1)} = I - \dfrac{T'(\mu_k)x_k y_k^*}{y_k^* T'(\mu_k)x_k}, \\
\Pi_k^{(2)} = I - \dfrac{x_k u^*}{u^* x_k}, \text{ and solve the correction equation} \\
\Pi_k^{(1)} T(\mu_k) \Pi_k^{(2)} \Delta x_k = -\left(T(\mu_k) - \dfrac{y_k^* T(\mu_k)x_k}{y_k^* T'(\mu_k)x_k} T'(\mu_k)\right) x_k \text{ for } \Delta x_k \perp u; \\
\text{2. Set } w_k = x_k + \Delta x_k; \\
\text{3. Choose a vector } z_{k1}, \text{ e.g., } z_{k1} = T'(\mu_k)w_k \text{ and compute the RF value} \\
\quad v_k = \rho_F(w_k; T, z_{k1}); \\
\text{4. Define } \widetilde{\Pi}_k^{(1)} = I - \dfrac{T'(v_k)w_k z_{k1}^*}{z_{k1}^* T'(v_k)w_k}, \widetilde{\Pi}_k^{(2)} = I - \dfrac{w_k u^*}{u^* w_k}, \text{ and solve the} \\
\quad \text{correction equation } \widetilde{\Pi}_k^{(1)} T(v_k) \widetilde{\Pi}_k^{(2)} \Delta w_k = -T(v_k)w_k \text{ for } \Delta w_k \perp u; \\
\text{5. Set } x_{k+1} = -(m-1)w_k + m(w_k + \Delta w_k) = w_k + m\Delta w_k; \\
\text{6. Choose a vector } z_{k2}, \text{ e.g., } z_{k2} = T'(v_k)x_{k+1} \text{ and compute the RF} \\
\quad \text{value } \mu_{k+1} = \rho_F(x_{k+1}; T, z_{k2}).
\end{array}\right\} \tag{30}
$$

One could develop other accelerated Newton-like methods from the accelerated inverse iteration (13). In this section, we propose an accelerated single-vector Jacobi–Davidson method based on a minor modification of (13). In contrast to inverse iteration and RFI, JD computes normalized new eigenvector approximations directly without forming the unnormalized version; as a result, the way the accelerated inverse iteration updates eigenvalue approximations cannot be realized by JD; see Steps 3 and 6 in (13). To work around this difficulty, we use the Rayleigh functional value as new eigenvalue approximations, and we have the new algorithm (30).

Following the derivation of the exact solution of JD correction equations, we can show that $\Delta x_k = \dfrac{T^{-1}(\mu_k)T'(\mu_k)x_k}{u^* T^{-1}(\mu_k)T'(\mu_k)x_k} - x_k$ at Step 1 of (30), and thus $w_k = \dfrac{T^{-1}(\mu_k)T'(\mu_k)x_k}{u^* T^{-1}(\mu_k)T'(\mu_k)x_k}$ at Step 2; also, $\Delta w_k = \dfrac{T^{-1}(v_k)T'(v_k)w_k}{u^* T^{-1}(v_k)T'(v_k)w_k} - w_k$ at Step 4, and $x_{k+1} = \dfrac{mT^{-1}(v_k)T'(v_k)w_k}{u^* T^{-1}(v_k)T'(v_k)w_k} - (m-1)w_k$ at Step 5. Assuming that x_0 is normalized such that $u^* x_0 = 1$, then all the subsequent eigenvector approximations satisfy $u^* w_k = u^* x_{k+1} = 1$. To enhance the numerical stability, however, these normalization conditions should be explicitly enforced.

We see that the only mathematical difference between the accelerated inverse iteration (13) and the accelerated single-vector JD (30) is the way new eigenvalue approximations are computed. Consequently, algorithms (30) and (12) are not mathematically equivalent. Nevertheless, given the close similarity between the two methods, it is nat-

ural to expect that the accelerated JD exhibits quadratic convergence for defective eigenpairs. This convergence rate is illustrated by numerical experiments in Sect. 4.

3.3 Defective eigenvalues with multiple Jordan chains

To make our discussion more complete, we consider in this section defective eigenvalues with multiple ($J \geq 2$) Jordan chains. Our main conclusion is that the accelerated algorithms generally converge quadratically with m set as the length of the longest Jordan chains. The analysis of the accelerated inverse iteration can be extended almost verbatim to this scenario.

First, consider a defective λ with J Jordan chains of identical length $m = m_1 = \ldots = m_J \geq 2$. Suppose that each Jordan chain satisfy the assumptions made in Lemma 1 for a single Jordan chain. Then the convergence analysis of the accelerated inverse iteration easily follows the lines of the derivation for the case involving a single Jordan chain. This result is quite natural, and thus we omit the details.

The analysis of the scenario involving multiple Jordan chains of different lengths is almost the same as well. In fact, assume without loss of generality that $J = 2$ and $m_1 > m_2$. Given an initial iterate $[x_0; \mu_0]$, let v be a desired eigenvector such that the accelerated inverse iteration (13) starting with $[x_0; \mu_0]$ converges to $[v; \lambda]$. Assume that $P_{\mathcal{N}_1}([x_0; \mu_0] - [v; \lambda]) \neq 0$. Suppose that Assumption 5 and the assumptions in Theorem 4 hold. Let $[x; \mu] = [x_k; \mu_k]$, $s = \sin \angle(x, v)$, and $[w; \nu]$ be the half-step iterate following $[x; \mu]$. Then the new eigenvector approximation is

$$\begin{aligned} p = T^{-1}(\mu)T'(\mu)x &= \sum_{k=1}^{2} \sum_{j=0}^{m_k-1} \sum_{i=0}^{m_k-j-1} \frac{\langle T'(\mu)x, \psi_{k,j}\rangle}{(\mu-\lambda)^{m_k-j-i}} \varphi_{k,i} + G(\mu)T'(\mu)x \\ &= \sum_{i=0}^{m_1-1} \frac{\langle T'(\mu)x, \psi_{1,0}\rangle}{(\mu-\lambda)^{m_1-i}} \varphi_{1,i} + \sum_{j=1}^{m_1-1} \sum_{i=0}^{m_1-j-1} \frac{\langle T'(\mu)x, \psi_{1,j}\rangle}{(\mu-\lambda)^{m_1-j-i}} \varphi_{1,i} \\ &\quad + \sum_{i=0}^{m_2-1} \frac{\langle T'(\mu)x, \psi_{2,0}\rangle}{(\mu-\lambda)^{m_2-i}} \varphi_{2,i} + \sum_{j=1}^{m_2-1} \sum_{i=0}^{m_2-j-1} \frac{\langle T'(\mu)x, \psi_{2,j}\rangle}{(\mu-\lambda)^{m_2-j-i}} \varphi_{2,i} \\ &\quad + G(\mu)T'(\mu)x. \end{aligned} \tag{31}$$

Now assume that either both $\langle T''(\lambda)\varphi_{1,0}, \psi_{1,0}\rangle \neq 0$ and $\langle T''(\lambda)\varphi_{2,0}, \psi_{1,0}\rangle \neq 0$, or both $\langle T'(\lambda)\varphi_{1,0}, \psi_{1,1}\rangle \neq 0$ and $\langle T'(\lambda)\varphi_{2,0}, \psi_{1,1}\rangle \neq 0$. Following the proof of Lemma 1, we can show that the dominant term in (31) can be written as

$$\mathcal{O}\left((\mu-\lambda)^{-(m_1-1)}\right)\varphi_{1,0} + \mathcal{O}\left((\mu-\lambda)^{-\max(m_2-1,1)}\right)\varphi_{2,0},$$

where the second term is proportional to the first in magnitude if and only if $m_1 = 2$ and $m_2 = 1$; otherwise the first term dominates. In either case, we can show that $\|T(\nu)w\| = \mathcal{O}(s^{m_1})$ for the half-step iterate $[w; \nu]$. Similarly, one can establish $c \geq d = m_1 - 1$ as shown in Lemma 2. Also, all half-step and full-step iter-

ates are well-defined by Lemma 3. The quadratic convergence of the accelerated inverse iteration with $m = m_1$ can be established by showing that the projections of $e_{k+1} = [x_{k+1}; \mu_{k+1}] - [v; \lambda]$ onto $\mathcal{N}_1$ and $\mathcal{M}_1$ are both $O(s^2)$. Finally, the analysis can be extended easily to the general case involving $J \geq 3$ Jordan chains, in which the parameter m in the accelerated algorithms should be set as the length of the longest Jordan chains. We summarize the result as follows.

Theorem 9 *Let λ be a defective eigenvalue of the holomorphic $T(\cdot)$ with J right and corresponding left Jordan chains. Assume without loss of generality that $\{\varphi_{1,0}, \ldots, \varphi_{1,m_1-1}\}$ is one of the longest Jordan chains. Let $\delta, \theta > 0$ be some appropriately small constants, $[x_0; \mu_0] \in W(\delta, \theta)$ be the initial iterate, and v be a right eigenvector such that the accelerated method* (12) *or* (13) *starting with $[x_0; \mu_0]$ generates well-defined iterates $[x_k; \mu_k]$, and it converges to $[v; \lambda]$. Suppose that $[x_0; \mu_0]$ satisfies* Assumption 5 *and $P_{\mathcal{N}_1}([x_0; \mu_0] - [v; \lambda]) \neq 0$. Further suppose that $\langle T''(\lambda)\varphi_{i,0}, \psi_{1,0}\rangle \neq 0$ or $\langle T'(\lambda)\varphi_{i,0}, \psi_{1,1}\rangle \neq 0$ for $i = 1, \ldots, J$, and that* Assumption 7 *holds. Then, under the assumptions of* Theorem 4, *for sufficiently small $\sin \angle(x_0, v)$, $[x_k; \mu_k]$ converges towards $[v; \lambda]$ quadratically.*

4 Numerical experiments

We present numerical results to illustrate the quadratic convergence of the accelerated algorithms for defective eigenvalues. Two problems from the NLEVP collection [1] and four artificial problems are chosen for this purpose. A description of these problems is given in Table 1. For example, the problem *time_delay*, which defines a function $T(\mu) \in C^{3\times 3}$, is a truly nonlinear problem taken from NLEVP. It has a defective eigenvalue $\lambda = 3\pi i$, and the algebraic and the geometric multiplicities of λ are $alg_T(\lambda) = 2$ and $geo_T(\lambda) = 1$, respectively; since there is only one Jordan chain, the length of the shortest and the longest Jordan chains are identical, i.e., $\min\{m_i\} = \max\{m_i\} = 2$; see [3]. For more details about the four artificial problems, see [9, Section 4.2].

We run accelerated inverse iteration (13) and accelerated JD (30) with appropriate initial eigenpair approximations, and we find that both methods converge at least superlinearly. To obtain a reliable estimate of the order of convergence, we generate a sequence of iterates $\left\{[x_0^{(j)}; \mu_0^{(j)}]\right\}$ $(1 \leq j \leq q)$, for which $|\mu_0^{(j+1)} - \lambda| = \frac{1}{2}|\mu_0^{(j)} - \lambda|$,

Table 1 Description of the test problems for defective eigenvalues

Problem	Source	Type	Size	Eigenvalue	$alg_T(\lambda)$	$geo_T(\lambda)$	$\min\{m_i\}$	$\max\{m_i\}$
time_delay	NLEVP	nep	3	$3\pi i$	2	1	2	2
jordan3	Artificial	lep	256	2	3	1	3	3
df_art_m1m2	Artificial	nep	256	0	5	4	1	2
df_art_m1m3	Artificial	nep	256	0	5	3	1	3
df_art_m2m3	Artificial	nep	256	0	5	2	2	3
mirror	NLEVP	pep	9	0	9	7	1	2

Table 2 Estimated order of convergence η of accelerated algorithms for defective λ with a single Jordan chain ($J = geo_T(\lambda) = 1$)

Problem	$\angle(x_0^{(1)}, v)$	Accelerated inverse iter.		Accelerated JD	
		# Init. approx.	Est. η	# Init. approx.	Est. η
time_delay	10^{-3}	18	2.011	11	2.032
*jordan*3	2.5×10^{-3}	9	2.028	9	2.058

Table 3 Estimated order of convergence η of accelerated algorithms for defective λ with multiple Jordan chains ($J = geo_T(\lambda) \geq 2$)

Problem	$\angle(x_0^{(1)}, v)$	Accelerated inverse iter.		Accelerated JD	
		# Init. approx.	Est. η	# Init. approx.	Est. η
*df_art_m*1*m*3	2.5×10^{-3}	8	1.998	8	2.010
*df_art_m*2*m*3	2.5×10^{-3}	8	2.017	8	2.012
*df_art_m*1*m*2	1.25×10^{-3}	7	1.988	7	1.989
mirror	10^{-2}	10	2.006	10	2.001

and $s_0^{(j)} = \sin\angle(x_0^{(j)}, \mathcal{V})$ satisfies $s_0^{(j+1)} = \frac{1}{2}s_0^{(j)}$, where $\mathcal{V}$ is the space spanned by all eigenvectors associated with the *longest* Jordan chains. Then we apply one iteration of the accelerated algorithms to $\left\{(\mu_0^{(j)}, x_0^{(j)})\right\}$ and obtain a sequence of new iterates $\left\{(\mu_1^{(j)}, x_1^{(j)})\right\}$. Let $s_1^{(j)} = \sin\angle(x_1^{(j)}, \mathcal{V})$. Then $s_1^{(j)} = \left(\frac{1}{2}\right)^{\eta} s_0^{(j)}$ if the order of local convergence is η. An estimate of η can be obtained by calculating the slope of the linear least squares interpolation of $\left\{(\log s_0^{(j)}, \log s_1^{(j)})\right\}$ $(1 \leq j \leq q)$. This approach is more descriptive to estimate η than the standard criterion $s_{k+1}/s_k^{\eta} \leq C$, because the former is based on q independent pairs $(s_0^{(j)}, s_1^{(j)})$ (typical value of q can be as large as 10 to 20), whereas a small number of iterations, typically only 3 to 5, are available to estimate η by the standard criterion. Note that $s_k = \sin\angle(x_k, \mathcal{V})$ is an error estimate different from $e_k = [x_k; \mu_k] - [v; \lambda]$ we used in the analysis. Nevertheless, we found empirically that the use of $\sin\angle(x_k, \mathcal{V})$ with our approach leads to most consistent estimate of η.

The results presented in Table 2 show that both methods converge quadratically for the two problems where the defective λ has a single Jordan chain. We see that both the accelerated inverse iteration and accelerated JD exhibit quadratic local convergence.

To illustrate the convergence of accelerated methods for defective eigenvalues with multiple Jordan chains, we test four problems, namely, *df_art_m*1*m*2, *df_art_m*1*m*3, *df_art_m*2*m*3, and *mirror*. We first run the algorithms with an initial eigenpair $[x_0; \mu_0]$, and we find that $\|T(\mu_k)x_k\|$ decreases superlinearly. Then we follow our descriptive approach to estimate the order of local convergence η. For both methods, the parameter m is set to be the length of the longest Jordan chain.

Table 3 summarizes the numerical results for the accelerated algorithms. Again, both methods exhibit quadratic convergence.[1]

5 Conclusion

The local convergence of single-vector Newton-like methods for defective eigenvalues is generally linear. To speed up the convergence rates, we proposed and studied an accelerated inverse iteration, showing that it exhibits quadratic local convergence towards defective eigenvalues at the cost of solving two linear systems per iteration. A detailed analysis is performed for a defective λ with a single Jordan chain, but it can be extended without difficulty to the case involving multiple Jordan chains. We also proposed an accelerated Jacobi–Davidson method that bears a close connection with the accelerated inverse iteration. Numerical experiments illustrate the quadratic convergence of the accelerated methods. To the best of our knowledge, they are the most efficient methods for solving a defective eigenvalue of algebraic eigenproblems of the form $T(\lambda)v = 0$, including the standard linear problem $Av = \lambda v$, polynomial, rational, and truly nonlinear eigenproblems.

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[1] Note that *mirror* is a special problem for which classical Rayleigh functional iteration (RFI) converges quadratically [9, Section 4.4], and thus there is actually no need to use accelerated algorithms.