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An efficient two-step iterative method for solving a class of complex symmetric linear systems[☆]

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ABSTRACT

In this paper, a new two-step iterative method called the two-step parameterized (TSP) iteration method for a class of complex symmetric linear systems is developed. We investigate its convergence conditions and derive the quasi-optimal parameters which minimize the upper bound of the spectral radius of the iteration matrix of the TSP iteration method. Meanwhile, some more practical ways to choose iteration parameters for the TSP iteration method are proposed. Furthermore, comparisons of the TSP iteration method with some existing ones are given, which show that the upper bound of the spectral radius of the TSP iteration method is smaller than those of the modified Hermitian and skew-Hermitian splitting (MHSS), the preconditioned MHSS (PMHSS), the combination method of real part and imaginary part (CRI) and the parameterized variant of the fixed-point iteration adding the asymmetric error (PFPAE) iteration methods proposed recently. Inexact version of the TSP iteration (ITSP) method and its convergence properties are also presented. Numerical experiments demonstrate that both TSP and ITSP are effective and robust when they are used either as linear solvers or as matrix splitting preconditioners for the Krylov subspace iteration methods and they have comparable advantages over some known ones for the complex symmetric linear systems.

1. Introduction

We consider the system of linear equations of the form:

$$Ax \equiv (W + iT)x = b, \tag{1}$$

where $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them, for example, W, being positive definite, $b \in \mathbb{R}^n$ and $i = \sqrt{-1}$. Such systems widely arise from many applications in scientific and engineering computing. For background information on these kinds of the problems in scientific and engineering applications, see, e.g., [1–6].

A large number of iteration methods have been proposed for solving the linear system (1) in the recent literatures. Based on the Hermitian and skew-Hermitian splitting (HSS) of the coefficient matrix of (1): $A = H + S$, where $H = \frac{1}{2}(A + A^*) = W$ and $S = \frac{1}{2}(A - A^*) = iT$, Bai et al. [7] initially established the HSS iteration method for solving the non-Hermitian positive definite linear systems. Here, A^* denotes the conjugate transpose of the matrix A. However, at each iteration step of the HSS

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iteration method, we need to solve a shift skew-Hermitian linear system. To overcome this difficult, Bai et al. [1] skillfully designed a modified HSS (MHSS) method and its preconditioned version which is called the PMHSS iteration method [2].

The PMHSS iteration method: Let $\alpha > 0$ be a positive constant and $V \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Given an initial guess $x^{(0)}$. For $k = 0, 1, 2, \ldots,$ until $x^{(k)}$ converges, compute

$$\begin{cases} (\alpha V + W)x^{(k+\frac{1}{2})} = (\alpha V - iT)x^{(k)} + b, \\ (\alpha V + T)x^{(k+1)} = (\alpha V - iW)x^{(k+\frac{1}{2})} - ib. \end{cases} \tag{2}$$

In order to improve the convergence rate of the PMHSS iteration method, many researchers have developed some efficient iteration methods recently. In [8], Li et al. proposed the lopsided PMHSS (LPMHSS) iteration method and showed that the IPMHSS iteration method performs better than the PMHSS one when the real part of A is dominant. In the sequal, Zeng and Ma [9] derived a one-step iteration method for solving the system (1) which is named as the parameterized single-step HSS (PSHSS) iteration method and based on the SHSS one put forward by Li and Wu [10]. Then the PSHSS iteration method was generalized to the parameterized single-step preconditioned variant of HSS (PSPHSS) iteration method by Xiao et al. [11] which contains the PSHSS one as a special case.

Note that the HSS-like iteration methods for complex linear systems have been extended in many literatures, see [12,13], and some of them have been used for solving other systems of equations, like Sylvester equations, refer to [14–19] for more details.

Recently, Hezari et al. [4] designed a scale-splitting (SCSP) iteration method by multiplying a complex number $(\alpha - i)$ through both sides of the complex system (1) and proved that it is convergent to the unique solution of the linear system (1) for a loose restriction on the iteration parameter α. Moreover, Zheng et al. [20] combined the idea of symmetry of the PMHSS method and the technique of scaling to reconstruct complex linear system (1) to propose a double-step scale splitting (DSS) iteration method, which is proved to be unconditionally convergent, and converges faster than the PMHSS iteration method. Subsequently, by combining real and imaginary parts of A in (1), Wang et al. [21] derived the combination method of real part and imaginary part which is simply called the CRI iteration method as follows:

The CRI method: Let $\alpha > 0$ be a positive constant. Given an initial guess $x^{(0)}$. For $k = 0, 1, 2, \ldots,$ until $x^{(k)}$ converges, compute

$$\begin{cases} (\alpha T + W)x^{(k+\frac{1}{2})} = (\alpha - i)Tx^{(k)} + b, \\ (\alpha W + T)x^{(k+1)} = (\alpha + i)Wx^{(k+\frac{1}{2})} - ib. \end{cases} \tag{3}$$

They proved that the upper bound of the spectral radius of the CRI iteration method is smaller than that of the PMHSS one. Very recently, Xiao and Wang [22] developed a new single-step iteration method called the parameterized variant of the fixed-point iteration adding the asymmetric error (PFPAE) iteration method for solving (1). They analyzed the convergence properties of the PFPAE iteration method and derived its quasi-optimal parameters. Numerical results showed that this new method outperforms than some existing ones.

The PFPAE iteration method: Let $\alpha > 0$ and $\omega > 0$ be two positive constants. Given an initial guess $x^{(0)}$. For $k = 0, 1, 2, \ldots,$ until $x^{(k)}$ converges, compute

$$(\omega W + T)x^{(k+\frac{1}{2})} = [(1-\alpha)(\omega W + T) - i\alpha(\omega T - W)]x^{(k)} + \alpha(\omega - i)b. \tag{4}$$

On the other hand, to avoid complex arithmetic, let $x = u + iv$ and $b = p + iq$ with $u, v, p, q \in \mathbb{R}^n$, then system (1) is equivalent to a 2-by-2 block real linear system

$$\begin{pmatrix} W & -T \\ T & W \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, \tag{5}$$

with u and v being unknown vectors, which can be formally regarded as a special case of the generalized saddle point problem and was introduced in [23] more recently. For the classical saddle point problems, Bai et al. [24] established the generalized successive overrelaxation (GSOR) iteration method, and proved its convergence under suitable restrictions on the iteration parameters and determined its optimal iteration parameters. After that, Bai and Wang [25] proposed the inexact variants of the GSOR iteration method, which contain the parameterized inexact Uzawa (PIU) iteration method, and they studied the convergence properties of the PIU method. In particular, they discussed its quasi-optimal iteration parameters and the corresponding quasi-optimal convergence factor for the saddle point problems. Recently, Salkuyeh et al. [26] applied the GSOR iteration method to the system (5), and derived its convergence condition and optimal parameter. In order to further improve the efficiency of the GSOR iteration method, Hezari et al. [5] designed the preconditioned GSOR (PGSOR) iteration method for (5), they proved that the minimum value of the spectral radius of the PGSOR iteration method is smaller than that of the GSOR one. After that, Liang and Zhang [27] developed the symmetric SOR (SSOR) iteration method and its accelerated variant for (5), and their optimal iteration parameters were also obtained. For more iteration methods for linear systems (1) or (5), we can refer to References [3,23,28–35].

To further accelerate the convergence rates of the PFPAE and the DSS iteration methods for solving the system (1), we use the idea of PFPAE method and the technique of scaling to reconstruct complex linear system (1), but twice, to design the two-step parameterized (TSP) iteration method. Theoretical analysis shows that the TSP iteration method is convergent under suitable conditions and converges faster than some proposed ones.

The outline of this paper is organized as follows. Section 2 introduces the TSP iteration method and its algorithmic description. The convergence properties of the TSP iteration method which include its convergence conditions and the quasi-optimal parameters, and some practical ways of choosing iteration parameters are derived in Section 3. The comparisons of the TSP iteration method with the MHSS, PMHSS, PFPAE and CRI ones are given in Section 4. In Section 5 we establish the inexact TSP iteration method and study its convergence property. Numerical experiments are presented in Section 6 to illustrate the effectiveness of TSP and ITSP both as solvers and as preconditioners for the generalized minimal residual (GMRES) method. Comparisons between the results obtained with the proposed methods and those obtained with the PMHSS, PSPHSS, CRI, DSS and PFPAE ones are given to show that TSP and ITSP are superior to the aforementioned ones. Finally, in Section 7, we end this paper with concluding remarks.

Throughout this paper, for a square matrix H, $\sigma(H)$ and $\rho(H)$ denote the spectrum and the spectral radius of H, respectively. $\|.\|_2$ stands for the Euclidean norm of a vector or a matrix, and $diag(a_1, a_2, \dots, a_n)$ denotes a diagonal matrix with diagonal elements $a_1, a_2, \dots, a_n$. Moreover, we indicate by $\kappa(A) = \|A^{-1}\|_2 \cdot \|A\|_2$ the spectral condition number of the matrix A.

2. The two-step parameterized (TSP) iteration method

In this section, to solve complex symmetric linear system (1) more efficient than some existing ones, based on the PFPAE and the double-step scale splitting (DSS) iteration methods [20,22], we combine the two-parameter acceleration technique used in [24,36] with the general two-step strategy and theory applied in [37,38], and establish a new two-step iteration method which is referred to as the two-step parameterized (TSP) iteration method.

By multiplying the complex number $\alpha(\omega - i)$ through both sides of the complex system (1) we obtain the following equivalent system:

$$\alpha(\omega - i)A = \alpha[(\omega W + T) + i(\omega T - W)]x = \alpha(\omega - i)b. \tag{6}$$

Similarly, premultiplying the complex system (1) with $\alpha(1 - \delta i)$ results in

$$\alpha(1 - \delta i)A = \alpha[(\delta T + W) + i(T - \delta W)]x = \alpha(1 - \delta i)b. \tag{7}$$

It is noteworthy that the motivation of developing Eqs. (6) and (7) stems from [1,2] in which Bai et al. scaled the complex symmetric linear system (1) by a complex number. We rewrite Eqs. (6) and (7) as the following two fix-point equations:

$$(\omega W + T)x = [(1 - \alpha)(\omega W + T) - i\alpha(\omega T - W)]x + \alpha(\omega - i)b, \tag{8}$$

$$(\delta T + W)x = [(1 - \alpha)(\delta T + W) - i\alpha(T - \delta W)]x + \alpha(1 - \delta i)b. \tag{9}$$

Now, by alternately iterating between the two systems of fixed-point equations (8) and (9), we can establish the following two-step parameterized (TSP) iteration method for solving the complex symmetric linear system (1).

The two-step parameterized (TSP) iteration method: Let $\alpha > 0$, $\omega > 0$ and $\delta > 0$ be three positive constants. Given an initial guess $x^{(0)}$. For $k = 0, 1, 2, \dots$, until $x^{(k)}$ converges, compute

$$\begin{cases} (\omega W + T)x^{(k+\frac{1}{2})} = [(1 - \alpha)(\omega W + T) - i\alpha(\omega T - W)]x^{(k)} + \alpha(\omega - i)b, \\ (\delta T + W)x^{(k+1)} = [(1 - \alpha)(\delta T + W) - i\alpha(T - \delta W)]x^{(k+\frac{1}{2})} + \alpha(1 - \delta i)b. \end{cases} \tag{10}$$

Noting that the TSP iteration method can be rewritten in a fixed-point form:

$$x^{(k+1)} = \mathcal{T}(\alpha, \omega, \delta)x^{(k)} + N(\alpha, \omega, \delta)^{-1}b, \tag{11}$$

where

$$\mathcal{T}(\alpha, \omega, \delta) = [(1 - \alpha)I - i\alpha(\delta T + W)^{-1}(T - \delta W)][(1 - \alpha)I - i\alpha(\omega W + T)^{-1}(\omega T - W)] \tag{12}$$

is the iteration matrix of the TSP iteration method, and

$$\begin{aligned} N(\alpha, \omega, \delta) = (\omega W + T)\{&[(\alpha - \alpha^2)(1 + \omega\delta - \delta i) - i\alpha(\delta + \alpha\omega)]T \\ &+ [(\alpha - \alpha^2)(\omega - i - \delta\omega i) + \alpha(\omega + \alpha\delta)]W\}^{-1}(\delta T + W) \end{aligned}$$

can be taken as a preconditioner for the complex symmetric matrix A, which is referred as the TSP preconditioner.

Remark 2.1. *If we take $\alpha = 1$ and $\omega = \delta$, the TSP iteration method reduces to the DSS iteration method [20], and the PFPAE iteration method [22] is exactly the first step of the TSP one.*

It follows from the iteration scheme (10) of the TSP iteration method that at each step of the TSP iteration, we need to solve two linear systems with $\omega W + T$ and $\delta T + W$ as the coefficient matrices. Since W and T are positive semi-definite matrices and at least one of them is positive definite, both $\omega W + T$ and $\delta T + W$ are positive definite. Thus we can solve them exactly by the Cholesky factorization or inexactly by the conjugate gradient (CG) method.

3. Convergence analysis of the TSP iteration method for complex symmetric linear systems

The TSP iteration method converges to the solution $x = A^{-1}b$ for arbitrary initial guess vector $x^{(0)}$ and right-hand side vector b if and only if $\rho(\mathcal{T}(\alpha, \omega, \delta)) < 1$. In this section, we investigate the convergence of the TSP iteration method for solving the complex symmetric linear systems.

Theorem 3.1. *Let W and T be symmetric positive semi-definite with at least one of them being positive definite. If $0 < \alpha < \frac{4}{2+\rho^2((\delta T+W)^{-1}(T-\delta W))+\rho^2((\omega W+T)^{-1}(\omega T-W))}$, then*

$$\rho(\mathcal{T}(\alpha, \omega, \delta)) \le \gamma(\alpha, \omega, \delta) = (1-\alpha)^2 + \frac{\alpha^2(\rho^2((\delta T+W)^{-1}(T-\delta W)) + \rho^2((\omega W+T)^{-1}(\omega T-W)))}{2} < 1,$$

i.e., the TSP iteration method converges to the exact solution of the linear system (1). Moreover, the minimum point $\alpha^*(\omega, \delta)$ and the minimum value of $\gamma(\alpha^*(\omega, \delta), \omega, \delta)$ of the upper bound $\gamma(\alpha, \omega, \delta)$ are, respectively, as

$$\alpha^*(\omega, \delta) = \frac{2}{2+\rho^2((\delta T+W)^{-1}(T-\delta W)) + \rho^2((\omega W+T)^{-1}(\omega T-W))}$$

and

$$\gamma(\alpha^*(\omega, \delta), \omega, \delta) = \frac{\rho^2((\delta T+W)^{-1}(T-\delta W)) + \rho^2((\omega W+T)^{-1}(\omega T-W))}{2+\rho^2((\delta T+W)^{-1}(T-\delta W)) + \rho^2((\omega W+T)^{-1}(\omega T-W))}.$$

Proof. From (12), it holds that

$$\begin{aligned}
\rho(\mathcal{T}(\alpha, \omega, \delta)) &= \rho([(1-\alpha)I - i\alpha(\delta T+W)^{-1}(T-\delta W)][(1-\alpha)I - i\alpha(\omega W+T)^{-1}(\omega T-W)]) \\
&\le \|(1-\alpha)I - i\alpha(\delta T+W)^{-1}(T-\delta W)\|_2 \|(1-\alpha)I - i\alpha(\omega W+T)^{-1}(\omega T-W)\|_2 \\
&= \sqrt{(1-\alpha)^2 + \alpha^2\rho^2((\delta T+W)^{-1}(T-\delta W))}\sqrt{(1-\alpha)^2 + \alpha^2\rho^2((\omega W+T)^{-1}(\omega T-W))} \\
&\le \frac{1}{2}[2(1-\alpha)^2 + \alpha^2(\rho^2((\delta T+W)^{-1}(T-\delta W)) + \rho^2((\omega W+T)^{-1}(\omega T-W)))] \\
&= (1-\alpha)^2 + \frac{\alpha^2(\rho^2((\delta T+W)^{-1}(T-\delta W)) + \rho^2((\omega W+T)^{-1}(\omega T-W)))}{2} =: \gamma(\alpha, \omega, \delta).
\end{aligned} \tag{13}$$

To get $\rho(\mathcal{T}(\alpha, \omega, \delta)) < 1$, it is enough to have

$$\gamma(\alpha, \omega, \delta) = (1-\alpha)^2 + \frac{\alpha^2(\rho^2((\delta T+W)^{-1}(T-\delta W)) + \rho^2((\omega W+T)^{-1}(\omega T-W)))}{2} < 1,$$

which yields that

$$0 < \alpha < \frac{4}{2+\rho^2((\delta T+W)^{-1}(T-\delta W)) + \rho^2((\omega W+T)^{-1}(\omega T-W))}.$$

From $\frac{\partial \gamma(\alpha,\omega,\delta)}{\partial \alpha} = 0$, we obtain

$$\alpha^*(\omega, \delta) = \frac{2}{2+\rho^2((\delta T+W)^{-1}(T-\delta W)) + \rho^2((\omega W+T)^{-1}(\omega T-W))}.$$

By simple calculations, we have

$$\begin{cases}
\dfrac{\partial \gamma(\alpha, \omega, \delta)}{\partial \alpha} \le 0, & \text{if } 0 < \alpha \le \alpha^*(\omega, \delta), \\
\dfrac{\partial \gamma(\alpha, \omega, \delta)}{\partial \alpha} > 0, & \text{if } \alpha > \alpha^*(\omega, \delta),
\end{cases}$$

which implies that $\alpha^*(\omega, \delta)$ is the minimal point. Taking $\alpha^*(\omega, \delta)$ into $\gamma(\alpha, \omega, \delta)$, we obtain the minimum value of $\gamma(\alpha, \omega, \delta)$ which is given by (13). ■

From Theorem 3.1, it can be seen that the minimum upper bound of the spectral radius $\rho(\mathcal{T}(\alpha, \omega, \delta))$ of the TSP iteration method is $\gamma(\alpha^*(\omega, \delta), \omega, \delta)$, which is strictly monotonic increasing about $\rho((\delta T+W)^{-1}(T-\delta W))$ and $\rho((\omega W+T)^{-1}(\omega T-W))$, so we need to chose the parameters ω and δ to minimize $\rho((\omega W+T)^{-1}(\omega T-W))$ and $\rho((\delta T+W)^{-1}(T-\delta W))$, respectively. It follows from Lemma 2.4 of [5] that the optimal value of the parameter ω which minimizes $\rho((\omega W+T)^{-1}(\omega T-W))$ is

$$\omega^* = \frac{1-\mu_{\min}\mu_{\max} + \sqrt{(1+\mu_{\min}^2)(1+\mu_{\max}^2)}}{\mu_{\min}+\mu_{\max}}.$$

Now, we investigate the spectral radius $\rho((\delta T+W)^{-1}(T-\delta W))$ and get the corresponding optimal value of the parameter δ.

Theorem 3.2. *Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semi-definite, respectively, and δ be a positive constant. If λ is an eigenvalue of the matrix $(\delta T+W)^{-1}(T-\delta W)$, then we have*

(1) There exists an eigenvalue μ of $S = W^{-1}T$ such that

$$\lambda = \frac{\mu-\delta}{\delta\mu+1}. \tag{14}$$

(2) The spectral radius $\rho((\delta T+W)^{-1}(T-\delta W))$ of the matrix $(\delta T+W)^{-1}(T-\delta W)$ satisfies

$$\rho((\delta T+W)^{-1}(T-\delta W)) = \max\left\{\frac{\mu_{\max}-\delta}{\delta\mu_{\max}+1}, \frac{\delta-\mu_{\min}}{\delta\mu_{\min}+1}\right\}.$$

(3) The optimal value of the parameter δ which minimizes $\rho((\delta T+W)^{-1}(T-\delta W))$ is

$$\delta^* = \frac{\mu_{\min}\mu_{\max}-1+\sqrt{(1+\mu_{\min}^2)(1+\mu_{\max}^2)}}{\mu_{\min}+\mu_{\max}},$$

where $\mu_{\min}$ and $\mu_{\max}$ denote the minimum and maximum eigenvalues of the matrix $S = W^{-1}T$, respectively.

Proof. Let (λ, x) be an eigenpair of the matrix $(\delta T+W)^{-1}(T-\delta W)$, then

$$(T-\delta W)x = \lambda(\delta T+W)x. \tag{15}$$

Premultiplying (15) with W^{-1} results in

$$(S-\delta I)x = \lambda(\delta S+I)x.$$

It is obvious that $\lambda\delta \neq 1$. Otherwise, $(\delta+\frac{1}{\delta})x = 0$, a contradiction. Thus

$$Sx = \frac{\delta+\lambda}{1-\lambda\delta}x.$$

Then, there exists an eigenvalue μ of $S = W^{-1}T$ such that

$$\mu = \frac{\delta+\lambda}{1-\lambda\delta},$$

then (14) follows. In addition, it is not difficult to verify that the function

$$g(\mu) = \frac{\mu-\delta}{\delta\mu+1}$$

is an increasing function with respect to μ, thus

$$\rho((\delta T+W)^{-1}(T-\delta W)) = \max_{\mu\in\sigma(W^{-1}T)}\left|\frac{\mu-\delta}{\delta\mu+1}\right| = \max\left\{\frac{\mu_{\max}-\delta}{\delta\mu_{\max}+1}, \frac{\delta-\mu_{\min}}{\delta\mu_{\min}+1}\right\}.$$

In the sequel, we prove (3). If $\mu_{\max} \le \delta$, then for any $\mu \in \sigma(W^{-1}T)$, it holds that $\mu-\delta \le 0$ and

$$\rho((\delta T+W)^{-1}(T-\delta W)) = \max_{\mu\in\sigma(W^{-1}T)}\left|\frac{\mu-\delta}{\delta\mu+1}\right| = \max_{\mu\in\sigma(W^{-1}T)}\frac{\delta-\mu}{\delta\mu+1} = \frac{\delta-\mu_{\min}}{\delta\mu_{\min}+1}$$

as $\frac{\delta-\mu}{\delta\mu+1}$ is monotonic decreasing about δ. If $\mu_{\max} \ge \delta$, then $\mu_{\max}-\delta \ge 0$. First, we assume that $\mu_{\min} \neq 0$. Below we distinguish two cases to prove.

(i) $\mu_{\min} \le \delta$, it holds that $\mu_{\min}-\delta \le 0$ and therefore

$$\rho((\delta T+W)^{-1}(T-\delta W)) = \max_{\mu\in\sigma(W^{-1}T)}\left|\frac{\mu-\delta}{\delta\mu+1}\right| = \max\left\{\frac{\mu_{\max}-\delta}{\delta\mu_{\max}+1}, \frac{\delta-\mu_{\min}}{\delta\mu_{\min}+1}\right\}.$$

(ii) $\delta \le \mu_{\min}$, then $\mu_{\min}-\delta \ge 0$ and

$$\rho((\delta T+W)^{-1}(T-\delta W)) = \max_{\mu\in\sigma(W^{-1}T)}\left|\frac{\mu-\delta}{\delta\mu+1}\right| = \max_{\mu\in\sigma(W^{-1}T)}\frac{\mu-\delta}{\delta\mu+1} = \frac{\mu_{\max}-\delta}{\delta\mu_{\max}+1}$$

as $\frac{\mu-\delta}{\delta\mu+1}$ is monotonic increasing about δ. We obtain the following results by summarizing the above discussions:

$$\rho((\delta T+W)^{-1}(T-\delta W)) = \begin{cases} \dfrac{\delta-\mu_{\min}}{\delta\mu_{\min}+1}, & \text{for } \delta \ge \mu_{\max}, \\ \max\left\{\dfrac{\mu_{\max}-\delta}{\delta\mu_{\max}+1}, \dfrac{\delta-\mu_{\min}}{\delta\mu_{\min}+1}\right\}, & \text{for } \mu_{\min} \le \delta \le \mu_{\max}, \\ \dfrac{\mu_{\max}-\delta}{\delta\mu_{\max}+1}, & \text{for } \delta \le \mu_{\min}. \end{cases}$$

Then, by straightforward computation, we can obtain

$$\frac{\partial \rho((\delta T+W)^{-1}(T-\delta W))}{\partial \delta}=\begin{cases}\dfrac{1+\mu_{\min}^2}{(\delta\mu_{\min}+1)^2}, & \text{for } \delta \geq \mu_{\max},\\ \dfrac{1+\mu_{\min}^2}{(\delta\mu_{\min}+1)^2}, & \text{for } \delta^* \leq \delta \leq \mu_{\max},\\ -\dfrac{1+\mu_{\max}^2}{(\delta\mu_{\max}+1)^2}, & \text{for } \mu_{\min} \leq \delta \leq \delta^*,\\ -\dfrac{1+\mu_{\max}^2}{(\delta\mu_{\max}+1)^2}, & \text{for } \delta \leq \mu_{\min}.\end{cases}$$

It then follows from the monotonicity of $\rho((\delta T+W)^{-1}(T-\delta W))$ with respect to δ that $\rho((\delta T+W)^{-1}(T-\delta W))$ attains its minimum where δ is located in the interval $[\mu_{\min}, \mu_{\max}]$ and satisfies

$$\frac{\mu_{\max}-\delta}{\delta\mu_{\max}+1}=\frac{\delta-\mu_{\min}}{\delta\mu_{\min}+1},$$

which can be transformed into

$$\delta^2(\mu_{\max}+\mu_{\min})-2\delta(\mu_{\max}\mu_{\min}-1)-(\mu_{\max}+\mu_{\min})=0.$$

Under the condition that $\delta>0$, we solve the above equation and derive

$$\delta^*=\frac{\mu_{\min}\mu_{\max}-1+\sqrt{(1+\mu_{\min}^2)(1+\mu_{\max}^2)}}{\mu_{\min}+\mu_{\max}}. \tag{16}$$

For the case that $\mu_{\min}=0$, by applying the similar method used in the above discussions, we have

$$\rho((\delta T+W)^{-1}(T-\delta W))=\begin{cases}\delta, & \text{for } \delta \geq \mu_{\max},\\ \max\left\{\dfrac{\mu_{\max}-\delta}{\delta\mu_{\max}+1},\delta\right\}, & \text{for } 0 \leq \mu \leq \mu_{\max}.\end{cases}$$

Then we deduce that $\rho((\delta T+W)^{-1}(T-\delta W))$ attains its minimum when δ satisfies

$$\frac{\mu_{\max}-\delta}{\delta\mu_{\max}+1}=\delta,$$

which leads to

$$\delta^*=\frac{\sqrt{1+\mu_{\max}^2}-1}{\mu_{\max}},$$

which is exactly the value in (16) when $\mu_{\min}=0$. ■

Theorem 3.3. *Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semi-definite, respectively. Then, the parameters $\alpha^*(\omega^*,\delta^*)$, ω^* and δ^* which minimize the upper bound $\gamma(\alpha(\omega,\delta),\omega,\delta)$ are*

$$\omega^*=\frac{1-\mu_{\max}\mu_{\min}+\sqrt{(1+\mu_{\min}^2)(1+\mu_{\max}^2)}}{\mu_{\max}+\mu_{\min}},\quad \delta^*=\frac{\mu_{\max}\mu_{\min}-1+\sqrt{(1+\mu_{\min}^2)(1+\mu_{\max}^2)}}{\mu_{\max}+\mu_{\min}},$$
$$\alpha^*(\omega^*,\delta^*)=\frac{2}{2+\rho^2((\delta^*T+W)^{-1}(T-\delta^*W))+\rho^2((\omega^*W+T)^{-1}(\omega^*T-W))} \tag{17}$$

and the corresponding optimal upper bound $\gamma(\alpha(\omega,\delta),\omega,\delta)$ of the spectral radius $\rho(\mathcal{T}(\alpha,\omega,\delta))$ of the TSP iteration method is

$$\begin{aligned}\gamma(\alpha^*(\omega^*,\delta^*),\omega^*,\delta^*)&=\frac{\rho^2((\delta^*T+W)^{-1}(T-\delta^*W))+\rho^2((\omega^*W+T)^{-1}(\omega^*T-W))}{2+\rho^2((\delta^*T+W)^{-1}(T-\delta^*W))+\rho^2((\omega^*W+T)^{-1}(\omega^*T-W))}\\&=\frac{2+\mu_{\max}^2+\mu_{\min}^2-2\sqrt{(1+\mu_{\max}^2)(1+\mu_{\min}^2)}}{2[1+\mu_{\max}^2\mu_{\min}^2+\mu_{\max}^2+\mu_{\min}^2+(\mu_{\max}\mu_{\min}-1)\sqrt{(1+\mu_{\max}^2)(1+\mu_{\min}^2)}]},\end{aligned} \tag{18}$$

which satisfies

$$\gamma(\alpha^*(\omega^*,\delta^*),\omega^*,\delta^*)\leq\frac{\mu_{\max}-\mu_{\min}}{2\mu_{\max}(1+\mu_{\min}^2)}\equiv\Phi(\mu_{\max},\mu_{\min}), \tag{19}$$

where

$$\rho((\delta^*T+W)^{-1}(T-\delta^*W))=\frac{\delta^*-\mu_{\min}}{\delta^*\mu_{\min}+1},\ \rho((\omega^*W+T)^{-1}(\omega^*T-W))=\frac{1-\omega^*\mu_{\min}}{\omega^*+\mu_{\min}}$$

and $\mu_{\min}$ and $\mu_{\max}$ are the minimum and maximum eigenvalues of the matrix $S=W^{-1}T$, respectively.

Proof. Taking ω^* and δ^* into $\rho((\omega^*W+T)^{-1}(\omega^*T-W))$ and $\rho((\delta^*T+W)^{-1}(T-\delta^*W))$, respectively, results in

$$\rho((\omega^*W+T)^{-1}(\omega^*T-W))=\frac{1-\omega^*\mu_{\min}}{\omega^*+\mu_{\min}}=\frac{\mu_{\max}\sqrt{1+\mu_{\min}^2}-\mu_{\min}\sqrt{1+\mu_{\max}^2}}{\sqrt{1+\mu_{\min}^2}+\sqrt{1+\mu_{\max}^2}}:=a_1,$$

$$\rho((\delta^*T+W)^{-1}(T-\delta^*W))=\frac{\delta^*-\mu_{\min}}{\delta^*\mu_{\min}+1}=\frac{\sqrt{1+\mu_{\max}^2}-\sqrt{1+\mu_{\min}^2}}{\mu_{\max}\sqrt{1+\mu_{\min}^2}+\mu_{\min}\sqrt{1+\mu_{\max}^2}}:=a_2.$$

It is not difficult to check that $a_1\cdot\frac{1}{a_2}=1$, that is $a_1=a_2$. Then it follows that

$$\begin{aligned}\gamma(\alpha^*(\omega^*,\delta^*),\omega^*,\delta^*)&=\frac{\rho^2((\delta^*T+W)^{-1}(T-\delta^*W))+\rho^2((\omega^*W+T)^{-1}(\omega^*T-W))}{2+\rho^2((\delta^*T+W)^{-1}(T-\delta^*W))+\rho^2((\omega^*W+T)^{-1}(\omega^*T-W))}\\&=\frac{\rho^2((\delta^*T+W)^{-1}(T-\delta^*W))}{1+\rho^2((\delta^*T+W)^{-1}(T-\delta^*W))}\\&=\frac{2+\mu_{\max}^2+\mu_{\min}^2-2\sqrt{(1+\mu_{\max}^2)(1+\mu_{\min}^2)}}{2[1+\mu_{\max}^2\mu_{\min}^2+\mu_{\max}^2+\mu_{\min}^2+(\mu_{\max}\mu_{\min}-1)\sqrt{(1+\mu_{\max}^2)(1+\mu_{\min}^2)}]}<1.\end{aligned}$$

Since $\sqrt{(1+\mu_{\max}^2)(1+\mu_{\min}^2)}\geq 1+\mu_{\min}^2$, we have

$$\begin{aligned}\gamma(\alpha^*(\omega^*,\delta^*),\omega^*,\delta^*)&=\frac{2+\mu_{\max}^2+\mu_{\min}^2-2\sqrt{(1+\mu_{\max}^2)(1+\mu_{\min}^2)}}{2[1+\mu_{\max}^2\mu_{\min}^2+\mu_{\max}^2+\mu_{\min}^2+(\mu_{\max}\mu_{\min}-1)\sqrt{(1+\mu_{\max}^2)(1+\mu_{\min}^2)}]}\\&\leq\frac{2+\mu_{\max}^2+\mu_{\min}^2-2(1+\mu_{\min}^2)}{2[1+\mu_{\max}^2\mu_{\min}^2+\mu_{\max}^2+\mu_{\min}^2+\mu_{\max}\mu_{\min}\sqrt{(1+\mu_{\max}^2)(1+\mu_{\min}^2)}-(1+\mu_{\min}^2)]}\\&\leq\frac{2+\mu_{\max}^2+\mu_{\min}^2-2(1+\mu_{\min}^2)}{2[1+\mu_{\max}^2\mu_{\min}^2+\mu_{\max}^2+\mu_{\min}^2+(\mu_{\max}\mu_{\min}-1)(1+\mu_{\min}^2)]}\\&=\frac{\mu_{\max}-\mu_{\min}}{2\mu_{\max}(1+\mu_{\min}^2)}\equiv\Phi(\mu_{\max},\mu_{\min}),\end{aligned}$$

which derives (19). ■

Remark 3.1. *Although ω^*, δ^* and $\alpha^*(\omega^*,\delta^*)$ minimize only the upper bound of the spectral radius of the TSP iteration matrix, one can utilize them for the TSP iteration method in practice, as $\gamma(\alpha^*(\omega^*,\delta^*),\omega^*,\delta^*)$ is a good approximation to its minimize value of spectral radius, which will be stressed by numerical results in* Section 6. *Besides, we obtain the form of the optimal upper bound $\gamma(\alpha^*(\omega^*,\delta^*),\omega^*,\delta^*)$ in (18). While it is complicated, so we deduce another upper bound $\Phi(\mu_{\max},\mu_{\min})$ for $\rho(\mathcal{T}(\alpha^*(\omega^*,\delta^*),\omega^*,\delta^*))$ in (19), which is more simple and intuitive.*

Now, according to Theorem 3.2, we can obtain the bounds for the parameter $\alpha^*(\omega^*,\delta^*)$ and the upper bound $\gamma(\alpha^*(\omega^*,\delta^*),\omega^*,\delta^*)$ for the TSP iteration method.

Theorem 3.4. *Let the conditions of Theorem 3.3 be satisfied. Then*

$$\rho(\mathcal{T}(\alpha^*(\omega^*,\delta^*),\omega^*,\delta^*))\leq\gamma(\alpha^*(\omega^*,\delta^*),\omega^*,\delta^*)<\frac{1}{2}\text{ and }\alpha^*(\omega^*,\delta^*)\in\left(\frac{1}{2},1\right).$$

Proof. From Corollary 2.2 of [22], we have $\rho((\omega^*W+T)^{-1}(\omega^*T-W))<1$, which together with (17) and (18), it can be seen that the proof will be completed if the following inequality

$$\rho((\delta^*T+W)^{-1}(T-\delta^*W))=\frac{\delta^*-\mu_{\min}}{\delta^*\mu_{\min}+1}<1$$

holds true. By taking δ^* in Theorem 3.3, it has

$$\frac{\sqrt{(1+\mu_{\min}^2)(1+\mu_{\max}^2)}-(1+\mu_{\min}^2)}{\mu_{\min}+\mu_{\max}} < \frac{\mu_{\max}\mu_{\min}^2+\mu_{\max}+\mu_{\min}\sqrt{(1+\mu_{\min}^2)(1+\mu_{\max}^2)}}{\mu_{\min}+\mu_{\max}},$$

which is equivalent to

$$(1-\mu_{\min})\sqrt{(1+\mu_{\min}^2)(1+\mu_{\max}^2)} < (1+\mu_{\max})(1+\mu_{\min}^2). \tag{20}$$

(i) If $\mu_{\min} \geq 1$, then inequality (20) holds true.
(ii) If $0 \leq \mu_{\min} < 1$, then it follows from (20) that

$$(1-\mu_{\min})^2(1+\mu_{\min}^2)(1+\mu_{\max}^2) < (1+\mu_{\max})^2(1+\mu_{\min}^2)^2,$$

i.e.,

$$(1-\mu_{\min})^2(1+\mu_{\max}^2) < (1+\mu_{\max})^2(1+\mu_{\min}^2),$$

which is always true, then $\rho((\delta^*T+W)^{-1}(T-\delta^*W)) < 1$. It is not difficult to prove that the functions $f(x,y)=\frac{x^2+y^2}{2+x^2+y^2}$ and $g(x,y)=\frac{2}{2+x^2+y^2}$ are strictly monotonic increasing and decreasing about x and y, respectively, then

$$\gamma(\alpha^*(\omega^*,\delta^*),\omega^*,\delta^*)=\frac{\rho^2((\delta^*T+W)^{-1}(T-\delta^*W))+\rho^2((\omega^*W+T)^{-1}(\omega^*-W))}{2+\rho^2((\delta^*T+W)^{-1}(T-\delta^*W))+\rho^2((\omega^*W+T)^{-1}(\omega^*T-W))} < \frac{1+1}{2+1+1}=\frac{1}{2}$$

and

$$\frac{1}{2} < \alpha^*(\omega^*,\delta^*)=\frac{2}{2+\rho^2((\delta^*T+W)^{-1}(T-\delta^*W))+\rho^2((\omega^*W+T)^{-1}(\omega^*T-W))} < 1. \quad \blacksquare$$

Note that the parameters ω^*, δ^* and $\alpha^*(\omega^*,\delta^*)$ rely on the extreme eigenvalues of $S=W^{-1}T$, then it is difficult for us to obtain them when the problem size is large enough. Now, we propose a more practical way for the choice of iteration parameters for the TSP iteration method.

- If $\mu_{\min}=0$, then from $\rho((\delta^*T+W)^{-1}(T-\delta^*W)) < 1$ and $\rho((\omega^*W+T)^{-1}(\omega^*T-W)) < 1$, we have

$$\omega^*=\frac{1+\sqrt{1+\mu_{\max}^2}}{\mu_{\max}} > 1,\ \ \delta^*=\frac{1}{\sqrt{1+\mu_{\max}^2}+1} < \frac{1}{2},$$
$$\alpha^*(\omega^*,\delta^*)=\frac{2}{2+\rho^2((\delta^*T+W)^{-1}(T-\delta^*W))+\rho^2((\omega^*W+T)^{-1}(\omega^*T-W))} > \frac{1}{2}.$$

  Thus, we may simply choose $\alpha=\delta=0.5$ and $\omega=1$ for the TSP iteration method.

- If $\mu_{\min}\neq 0$ and $\mu_{\min}$ is close to 0, then

$$\omega^*\approx\frac{1+\sqrt{1+\mu_{\max}^2}}{\mu_{\max}} > 1,\ \ \delta^*\approx\frac{1}{\sqrt{1+\mu_{\max}^2}+1} < \frac{1}{2}.$$

  Similar to the discussions in the case of $\mu_{\min}=0$, we can take $\alpha=\delta=0.5$ and $\omega=1$ for the TSP iteration method. Additionally, if $\mu_{\max}$ is rather large, with assumptions $\omega=\delta=1$, then it follows from Lemma 2.4 of [5] and Theorem 3.2 that $\rho((\delta^*T+W)^{-1}(T-\delta^*W))\approx 1$ and $\rho((\omega^*W+T)^{-1}(\omega^*T-W))\approx 1$. Thus, we derive $\alpha^*\approx\frac{1}{2}$ by virtue of Theorem 3.4.

Besides, it is noteworthy that although choosing optimal parameters is very complicated and almost impossible for most iterative methods, many researchers have devoted to estimating the nearly optimal parameters and have obtained many valuable results; see [39–41]. By adopting a reasonable and simple optimization principle, Chen in [40] derived a cubic polynomial equation to estimate the parameter α for the HSS method. Motivated by the results in [40], next we give another practical way for the choice of iteration parameters of the TSP iteration method.

We need to solve two sub-systems of linear equations in the TSP iteration method, which have coefficient matrices $\omega W+T$ and $\delta T+W$, respectively. If either of them are solved inefficiently, then the convergence speed of the TSP iteration method will be deteriorated. TSP iteration method in (10) is equivalent to

$$\begin{cases}(\omega I+W^{-\frac{1}{2}}TW^{-\frac{1}{2}})\tilde{x}^{(k+\frac{1}{2})}=[(1-\alpha)(\omega I+W^{-\frac{1}{2}}TW^{-\frac{1}{2}})-i\alpha(\omega W^{-\frac{1}{2}}TW^{-\frac{1}{2}}-I)]\tilde{x}^{(k)}+\alpha(\omega-i)\tilde{b},\\(\delta W^{-\frac{1}{2}}TW^{-\frac{1}{2}}+I)\tilde{x}^{(k+1)}=[(1-\alpha)(\delta W^{-\frac{1}{2}}TW^{-\frac{1}{2}}+I)-i\alpha(W^{-\frac{1}{2}}TW^{-\frac{1}{2}}-\delta I)]\tilde{x}^{(k+\frac{1}{2})}+\alpha(1-\delta i)\tilde{b},\end{cases}$$

where $\tilde{x}^{(k)} = W^{\frac{1}{2}}x^{(k)}$ and $\tilde{b} = W^{-\frac{1}{2}}x^{(k)}$. Hence, the TSP iteration method may have fast convergence rate if ω and δ minimize the function $\tau(\omega,\delta) := |(\kappa(\omega I + W^{-\frac{1}{2}}TW^{-\frac{1}{2}}) - \kappa(\delta W^{-\frac{1}{2}}TW^{-\frac{1}{2}} + I))|$. Since $W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$ is symmetric positive semi-definite, there exists an unitary matrix such that $U^*W^{-\frac{1}{2}}TW^{-\frac{1}{2}}U = diag(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n)$ with $\tilde{\lambda}_i \geq 0$ $(i = 1, 2, \ldots, n)$. By direct calculations, we have

$$\begin{aligned}\tau(\omega,\delta) &= |\|(\omega I + W^{-\frac{1}{2}}TW^{-\frac{1}{2}})^{-1}\|_2 \cdot \|\omega I + W^{-\frac{1}{2}}TW^{-\frac{1}{2}}\|_2 - \|(\delta W^{-\frac{1}{2}}TW^{-\frac{1}{2}} + I)^{-1}\|_2 \cdot \|\delta W^{-\frac{1}{2}}TW^{-\frac{1}{2}} + I\|_2|,\\ &= \left|\frac{\omega + \tilde{\lambda}_{\max}}{\omega + \tilde{\lambda}_{\min}} - \frac{\delta\tilde{\lambda}_{\max} + 1}{\delta\tilde{\lambda}_{\min} + 1}\right|,\end{aligned}$$

where $\tilde{\lambda}_{\max}$ and $\tilde{\lambda}_{\min}$ are the minimal and the maximal eigenvalues of the matrix $W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$. If

$$\|(\omega I + W^{-\frac{1}{2}}TW^{-\frac{1}{2}})^{-1}\|_2 \cdot \|\omega I + W^{-\frac{1}{2}}TW^{-\frac{1}{2}}\|_2 = \|(\delta W^{-\frac{1}{2}}TW^{-\frac{1}{2}} + I)^{-1}\|_2 \cdot \|\delta W^{-\frac{1}{2}}TW^{-\frac{1}{2}} + I\|_2,$$

then $\tau(\omega,\delta) = 0$. It follows that

$$\frac{\omega + \tilde{\lambda}_{\max}}{\omega + \tilde{\lambda}_{\min}} = \frac{\delta\tilde{\lambda}_{\max} + 1}{\delta\tilde{\lambda}_{\min} + 1},$$

which leads to $(\omega\delta - 1)(\tilde{\lambda}_{\max} - \tilde{\lambda}_{\min}) = 0$. We assume that $\tilde{\lambda}_{\max} \neq \tilde{\lambda}_{\min}$ as it appears in many cases. Thus, $\omega\delta = 1$, and for a broad class of problems we can choose proper parameters $\omega \geq 1$ and $\delta \leq 1$ satisfying $\omega\delta = 1$ for the TSP iteration method by combining the previous discussions. However, it is very difficult for us to choose the optimal parameter α now, so it may be determined by performing numerical experiments. In summary, we can adopt

- $\alpha = \delta = 0.5, \omega = 1$;
- $\omega = \delta = 1, \alpha = 0.5$;
- proper α and $\omega \geq 1$, $\delta \leq 1$ satisfying $\omega\delta = 1$;

for the TSP iteration method in the practical implements.

4. Comparison the TSP iteration method with some known ones

This section is devoted to comparing the TSP iteration method with the PFPAE, PSHSS, CRI, MHSS and PMHSS ones.

Theorem 4.1. *Assume that the conditions of Theorem 3.3 are satisfied. Let ω be a given positive constant. Then $\rho((\delta T + W)^{-1}(T - \delta W)) < \rho((\omega W + T)^{-1}(\omega T - W)) = \eta$, if one of the following conditions holds:*

- $\mu_{\min} \geq \frac{1}{\eta}$ *and* $\eta \geq \mu_{\max}$;
- $\delta > \frac{\mu_{\max} - \eta}{\mu_{\max}\eta + 1}$ *for* $\mu_{\min} \geq \frac{1}{\eta}$ *and* $\eta < \mu_{\max}$;
- $0 < \delta < \frac{\eta + \mu_{\min}}{1 - \eta\mu_{\min}}$ *for* $\mu_{\min} < \frac{1}{\eta}$ *and* $\eta \geq \mu_{\max}$;
- $\frac{\mu_{\max} - \eta}{\mu_{\max}\eta + 1} < \delta < \frac{\eta + \mu_{\min}}{1 - \eta\mu_{\min}}$ *for* $\mu_{\min} < \frac{1}{\eta}$, $\eta < \mu_{\max}$ *and* $\frac{\mu_{\max}\mu_{\min} + 1}{\mu_{\max} - \mu_{\min}} > \frac{1 - \eta^2}{2\eta}$.

Proof. According to Theorem 3.2, it can be seen that

$$\rho((\delta T + W)^{-1}(T - \delta W)) = \max\left\{\frac{\mu_{\max} - \delta}{\delta\mu_{\max} + 1}, \frac{\delta - \mu_{\min}}{\delta\mu_{\min} + 1}\right\},$$

then $\rho((\delta T + W)^{-1}(T - \delta W)) < \rho((\omega W + T)^{-1}(\omega T - W)) = \eta$ is equivalent to

$$\begin{cases}\delta - \mu_{\min} < \delta\mu_{\min}\eta + \eta,\\ \mu_{\max} - \delta < \eta\delta\mu_{\max} + \eta,\end{cases}$$

that is,

$$\begin{cases}(1 - \eta\mu_{\min})\delta < \eta + \mu_{\min},\\ \mu_{\max} - \eta < \delta(\mu_{\max}\eta + 1).\end{cases} \tag{21}$$

If $\mu_{\min} \geq \frac{1}{\eta}$ and $\eta \geq \mu_{\max}$, then (21) holds true. If $\mu_{\min} \geq \frac{1}{\eta}$ and $\eta < \mu_{\max}$, then the first inequality of (21) holds true. Solving the second inequality of (21) results in

$$\delta > \frac{\mu_{\max} - \eta}{\mu_{\max}\eta + 1}.$$

Besides, if $\mu_{\min} < \frac{1}{\eta}$ and $\eta \geq \mu_{\max}$, then it is easy to obtain the second inequality of (21). It follows from the first inequality of (21) that

$$0 < \delta < \frac{\eta + \mu_{\min}}{1 - \eta\mu_{\min}}.$$

Finally, if $\mu_{\min} < \frac{1}{\eta}$ and $\eta < \mu_{\max}$, then from (21), we derive

$$\frac{\mu_{\max} - \eta}{\mu_{\max}\eta + 1} < \delta < \frac{\eta + \mu_{\min}}{1 - \eta\mu_{\min}}. \tag{22}$$

(22) holds if $\frac{\mu_{\max}-\eta}{\mu_{\max}\eta+1} < \frac{\eta+\mu_{\min}}{1-\eta\mu_{\min}}$, which can be written as

$$\frac{\mu_{\max}\mu_{\min} + 1}{\mu_{\max} - \mu_{\min}} > \frac{1 - \eta^2}{2\eta}.$$

Hence the conclusions of this theorem are obtained by summarizing the above discussions. ■

Note that the PFPAE iteration method is a single-step method. For a fair comparison, we should make comparisons between TSP iteration method with the two-step variant of the PFPAE method:

$$\begin{cases} (\omega W + T)x^{(k+\frac{1}{2})} = [(1-\alpha)(\omega W + T) - i\alpha(\omega T - W)]x^{(k)} + \alpha(\omega - i)b, \\ (\omega W + T)x^{(k+1)} = [(1-\alpha)(\omega W + T) - i\alpha(\omega T - W)]x^{(k+\frac{1}{2})} + \alpha(\omega - i)b. \end{cases}$$

It is not difficult to verify that $(\delta(\alpha^*(\omega), \omega))^2$ is the upper bound of the spectral radius of the two-step variant of the PFPAE method by using the optimal parameter $\alpha^*(\omega)$. Similarly, $(\delta^I(\alpha^*(\omega), \omega))^2$ is the upper bound of the spectral radius of the two-step variant of the PSHSS method when the corresponding optimal parameter $\alpha^*(\omega)$ is used. Here, $\delta(\alpha^*(\omega), \omega)$ and $\delta^I(\alpha^*(\omega), \omega)$ are the minimum value of the spectral radius of the PFPAE iteration method defined as in Theorem 3 of [22] and the minimum upper bound of the spectral radius of the PSHSS iteration method defined as in Lemma 2.2 of [9], respectively.

The following theorem gives the comparison the TSP iteration method with the PFPAE and the PSHSS ones.

Theorem 4.2. *Assume that the conditions of Theorem 3.3 are satisfied. Let ω be a given positive constant. If one of the conditions in Theorem 4.1 holds, then the upper bound $\gamma(\alpha^*(\omega, \delta), \omega, \delta)$ of the spectral radius of the TSP iteration method defined as in Theorem 3.1 satisfies*

$$\gamma(\alpha^*(\omega, \delta), \omega, \delta) < (\delta(\alpha^*(\omega), \omega))^2 = \frac{\rho^2((\omega W + T)^{-1}(\omega T - W))}{1 + \rho^2((\omega W + T)^{-1}(\omega T - W))} \leq (\delta^I(\alpha^*(\omega), \omega))^2.$$

Proof. From Theorem 4.1, we see that if one of the conditions in Theorem 4.1 holds, then $\rho((\delta T + W)^{-1}(T - \delta W)) < \rho((\omega W + T)^{-1}(\omega T - W))$, and therefore

$$\begin{aligned} \gamma(\alpha^*(\omega, \delta), \omega, \delta) &= \frac{\rho^2((\delta T + W)^{-1}(T - \delta W)) + \rho^2((\omega W + T)^{-1}(\omega T - W))}{2 + \rho^2((\delta T + W)^{-1}(T - \delta W)) + \rho^2((\omega W + T)^{-1}(\omega T - W))} \\ &< \frac{2\rho^2((\omega W + T)^{-1}(\omega T - W))}{2 + 2\rho^2((\omega W + T)^{-1}(\omega T - W))} = \frac{\rho^2((\omega W + T)^{-1}(\omega T - W))}{1 + \rho^2((\omega W + T)^{-1}(\omega T - W))} = (\delta(\alpha^*(\omega), \omega))^2. \end{aligned}$$

Moreover, according to Remark 3 of [22], we have

$$\gamma(\alpha^*(\omega, \delta), \omega, \delta) < \frac{\rho^2((\omega W + T)^{-1}(\omega T - W))}{1 + \rho^2((\omega W + T)^{-1}(\omega T - W))} \leq (\delta^I(\alpha^*(\omega), \omega))^2,$$

which implies that the TSP iteration method may be more efficient than the PSHSS one. ■

Now we turn to compare the TSP iteration method with the CRI one in the following theorem.

Theorem 4.3. *Let the conditions of Theorem 3.3 be satisfied. Then*

$$\gamma(\alpha^*(\omega^*, \delta^*), \omega^*, \delta^*) < \zeta(\alpha^*) = \frac{1}{2}.$$

That is, the minimum upper bound of the spectral radius of the TSP iteration method is smaller than that of the CRI one. Here, $\zeta(\alpha^*)$ denotes the minimum upper bound of the spectral radius of the CRI iteration method defined as in Theorem 2.1 of [21].

Proof. It can be seen that the upper bound of the spectral radius of the CRI iteration method is $\zeta(\alpha) = \frac{\alpha^2+1}{(\alpha+1)^2}$ by virtue of Theorem 2.1 of [21]. By direct computations, we get

$$\frac{\partial\zeta(\alpha)}{\partial\alpha} = \frac{2(\alpha - 1)}{(\alpha + 1)^3}.$$

It is not difficult to see that

$$\frac{\partial \zeta(\alpha)}{\partial \alpha} \le 0, \text{ for } 0 < \alpha \le 1, \text{ and } \frac{\partial \zeta(\alpha)}{\partial \alpha} \ge 0, \text{ for } \alpha \ge 1.$$

Hence, $\alpha = 1$ is the optimal iteration parameter which minimizes $\zeta(\alpha)$, which leads to

$$\zeta(\alpha_*) = \zeta(1) = \frac{1^2+1}{(1+1)^2} = \frac{1}{2},$$

which together with Theorem 3.4 results in the conclusion of this theorem. ■

Finally, we make comparisons between the TSP iteration method with the MHSS and the PMHSS ones in [1,2] in the following two theorems, respectively.

Theorem 4.4. *Let the conditions of Theorem 3.3 be satisfied. Then*

$$\gamma(\alpha^*(\omega^*, \delta^*), \omega^*, \delta^*) < \frac{\sqrt{2}}{2} \le \gamma(\alpha_\star),$$

which shows that the minimum upper bound of the spectral radius of the TSP iteration method is smaller than that of the MHSS one. Here, $\gamma(\alpha_\star)$ denotes the minimum upper bound of the spectral radius of the MHSS iteration method defined as in Corollary 2.1 of [1].

Proof. It follows from Corollary 2.1 of [1] that the minimum upper bound of the spectral radius of the MHSS iteration method is $\gamma(\alpha_\star) = \frac{\sqrt{\lambda_{\min}+\lambda_{\max}}}{\sqrt{\lambda_{\min}}+\sqrt{\lambda_{\max}}} = \frac{\sqrt{\kappa(W)+1}}{\sqrt{\kappa(W)}+1}$, where $\kappa(W) = \frac{\lambda_{\max}}{\lambda_{\min}} \ge 1$ with $\lambda_{\max}$ and $\lambda_{\min}$ being the maximum and minimum eigenvalues of the matrix W, respectively. Define the function $f(x) = \frac{\sqrt{x+1}}{\sqrt{x}+1}$ $(x \ge 1)$. After simple calculations, it has

$$\frac{df(x)}{dx} = \frac{\sqrt{x}-1}{2\sqrt{x(x+1)}(\sqrt{x}+1)^2} \ge 0 \text{ for } x \ge 1,$$

which implies that the function $f(x)$ is monotonic increasing about $x \ge 1$. Thus, we have

$$\gamma(\alpha_\star) \ge f(1) = \frac{\sqrt{1+1}}{\sqrt{1}+1} = \frac{\sqrt{2}}{2}.$$

Then the conclusion of this theorem follows by combining Theorem 3.4 with the above discussions. ■

Theorem 4.5. *Let the conditions of Theorem 3.3 be satisfied. Then*

$$\rho(\mathcal{T}(\alpha^*, \omega^*, \delta^*)) \le \gamma(\alpha^*(\omega^*, \delta^*), \omega^*, \delta^*) < \frac{1}{2} \le \rho(L(W; \bar{\alpha})) = \phi(\bar{\alpha}),$$

where $\rho(L(W; \bar{\alpha})) = \phi(\bar{\alpha})$ denotes the minimum spectral radius of the PMHSS iteration method established in [2] for $V = W$. (The preconditioned matrix V for PMHSS method is always taken as W in actual implementations.)

Proof. From [2], we know that the iteration matrix of the PMHSS iteration method is

$$L(V; \alpha) = (\alpha V + T)^{-1}(\alpha V + iW)(\alpha V + W)^{-1}(\alpha V - iT). \tag{23}$$

Taking $V = W$ in (23) leads to $L(W; \alpha) = \frac{\alpha+i}{\alpha+1}(\alpha W + T)^{-1}(\alpha W - iT)$, then it holds that

$$\begin{aligned} \rho(L(W; \alpha)) &= \frac{\sqrt{\alpha^2+1}}{\alpha+1}\rho((\alpha I + W^{-\frac{1}{2}}TW^{-\frac{1}{2}})^{-1}(\alpha I - iW^{-\frac{1}{2}}TW^{-\frac{1}{2}})) \\ &= \frac{\sqrt{\alpha^2+1}}{\alpha+1}\max_{1\le i\le n}\left\{\frac{\sqrt{\alpha^2+\tilde{\lambda}_i^2}}{\alpha+\tilde{\lambda}_i}\right\} \equiv \phi(\alpha), \end{aligned}$$

where $\tilde{\lambda}_i \ge 0$ $(i = 1, 2, \ldots, n)$ is the eigenvalue of the matrix $W^{-1}T$. Define the function $h(x, \alpha) = \frac{\sqrt{\alpha^2+x^2}}{\alpha+x}$ $(x \ge 0)$. Since

$$\frac{\partial h(x, \alpha)}{\partial x} = \frac{\alpha(x-\alpha)}{(\alpha+x)^2\sqrt{\alpha^2+x^2}} \text{ for } x \ge 0,$$

it follows that

$$\frac{\partial h(x, \alpha)}{\partial x} \le 0, \text{ for } 0 \le x \le \alpha, \text{ and } \frac{\partial h(x, \alpha)}{\partial x} \ge 0, \text{ for } x \ge \alpha,$$

which means that $x = \alpha$ is the minimum value point of the function $h(x, \alpha)$. Hence, we have

$$\phi(\alpha) \equiv \frac{\sqrt{\alpha^2+1}}{\alpha+1} \max_{1\le i\le n} \left\{ \frac{\sqrt{\alpha^2+\tilde{\lambda}_i^2}}{\alpha+\tilde{\lambda}_i} \right\} \ge \frac{\sqrt{\alpha^2+1}}{\alpha+1} \cdot \frac{\sqrt{\alpha^2+\alpha^2}}{\alpha+\alpha} = \frac{\sqrt{2}}{2} \frac{\sqrt{\alpha^2+1}}{\alpha+1}.$$

Since the function $g(x) = \sqrt{x}\,(x \ge 0)$ is a monotonic increasing function, it holds that $\frac{\sqrt{\alpha^2+1}}{\alpha+1}$ is monotonic decreasing about $\alpha \in (0, 1]$ and increasing about $\alpha \in [1, +\infty)$ in terms of the proof of Theorem 4.3. Thus

$$\rho(L(W; \bar{\alpha})) = \phi(\bar{\alpha}) \ge \frac{\sqrt{2}}{2} \frac{\sqrt{\bar{\alpha}^2+1}}{\bar{\alpha}+1} \ge \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{1^2+1}}{1+1} = \frac{1}{2} > \gamma(\alpha^*(\omega^*, \delta^*), \omega^*, \delta^*) \ge \rho(\mathcal{T}(\alpha^*, \omega^*, \delta^*)),$$

which completes the proof of this theorem. ■

5. The inexact TSP iteration method

The two half-steps at each step of the HSS method require finding solutions with the matrices $\alpha I + H$ and $\alpha I + S$, which is very costly and impractical in actual implementations. To overcome this disadvantage and further improve the efficiency of the HSS method, Bai et al. [7] further proposed to solve the linear systems with coefficient matrices $\alpha I + H$ and $\alpha I + S$ inexactly by iterative methods, e.g., solving the linear systems with coefficient matrix $\alpha I + H$ by the CG and those with coefficient matrix $\alpha I + S$ by the Lanczos or the CG for normal equations (CGNE) method, to some prescribed accuracies, and obtained two special but quite practical inexact HSS (IHSS) iterations, briefly called as IHSS(CG, Lanczos) and IHSS(CG, CGNE). In [42], Bai et al. studied the convergence properties of both IHSS(CG, Lanczos) and IHSS(CG, CGNE) in-depth and investigated the optimal numbers of inner iteration steps in detail by considering both global convergence speed and overall computation workload. In particular, they showed that the asymptotic convergence rates of these two methods are essentially the same. In addition, the convergence rates of them tend to the convergence rate of HSS when the tolerances of the inner iterations tend to zero as the outer iterations increase. They also investigated their computational efficiencies.

In the process of the TSP iteration method, we must solve two linear equations with coefficient matrices $\omega W + T$ and $\delta T + W$. Since they are symmetric positive definite, we can solve these linear equations by employing the CG method. This results in the following inexact TSP iteration for solving the system of linear equations (1).

The inexact TSP (ITSP) iteration method

Given an initial guess $\bar{x}^{(0)}$, for $k = 0, 1, 2, \ldots,$ until $\bar{x}^{(k)}$ converges, solve $\bar{x}^{(k+\frac{1}{2})}$ approximately from

$$(\omega W + T)\bar{x}^{(k+\frac{1}{2})} \approx [(1-\alpha)(\omega W + T) - i\alpha(\omega T - W)]\bar{x}^{(k)} + \alpha(\omega - i)b,$$

by employing the CG method with $\bar{x}^{(k)}$ as the initial guess; then solve $\bar{x}^{(k+1)}$ approximately from

$$(\delta T + W)\bar{x}^{(k+1)} \approx [(1-\alpha)(\delta T + W) - i\alpha(T - \delta W)]\bar{x}^{(k+\frac{1}{2})} + \alpha(1 - \delta i)b,$$

by employing the CG method with $\bar{x}^{(k+\frac{1}{2})}$ as the initial guess, where ω and δ are given positive constants.

To simplify numerical implementation and convergence analysis, we may rewrite the above ITSP iteration method as the following equivalent scheme.

Given an initial guess $\bar{x}^{(0)}$, for $k = 0, 1, 2, \ldots,$ until $\bar{x}^{(k)}$ converges,

1. Compute $r^{(k)} = b - A\bar{x}^{(k)}$ and set $\bar{r}^{(k)} = \alpha(\omega - i)r^{(k)}$;
2. Solve $(\omega W + T)\bar{z}^{(k)} = \bar{r}^{(k)}$ by the CG method to compute the approximate solution $\bar{z}^{(k)}$ until it is such that the residual $\bar{p}^{(k)} = \bar{r}^{(k)} - (\omega W + T)\bar{z}^{(k)}$ satisfies $\|\bar{p}^{(k)}\| \le \varepsilon_k \|\bar{r}^{(k)}\|$;
3. Compute $\bar{x}^{(k+\frac{1}{2})} = \bar{x}^{(k)} + \bar{z}^{(k)}$;
4. Compute $r^{(k+\frac{1}{2})} = b - A\bar{x}^{(k+\frac{1}{2})}$ and set $\bar{r}^{(k+\frac{1}{2})} = \alpha(1 - \delta i)r^{(k+\frac{1}{2})}$;
5. Solve $(\delta T + W)\bar{z}^{(k+\frac{1}{2})} = \bar{r}^{(k+\frac{1}{2})}$ by the CG method to compute the approximate solution $\bar{z}^{(k+\frac{1}{2})}$ until it is such that the residual $\bar{q}^{(k+\frac{1}{2})} = \bar{r}^{(k+\frac{1}{2})} - (\delta T + W)\bar{z}^{(k+\frac{1}{2})}$ satisfies $\|\bar{q}^{(k+\frac{1}{2})}\| \le \eta_k \|\bar{r}^{(k+\frac{1}{2})}\|$;
6. Compute $\bar{x}^{(k+1)} = \bar{x}^{(k+\frac{1}{2})} + \bar{z}^{(k+\frac{1}{2})}$.

Here, $\|.\|$ is a norm of a vector, and $\{\varepsilon_k\}$ and $\{\eta_k\}$ are two prescribed tolerances.

Next, we analyze the convergence of the ITSP method. To this end, we consider a vector norm $|||x|||_{M_2} = \|M_2 x\|_2$ $(\forall x \in \mathbb{C}^n)$ defined in [7], which induces the matrix norm $|||X|||_{M_2} = \|M_2 X M_2^{-1}\|_2$ $(\forall X \in \mathbb{C}^{n\times n})$ and let

$$M_1 = \frac{1}{\alpha(\omega - i)}(\omega W + T),\ N_1 = \frac{1}{\alpha(\omega - i)}[(1-\alpha)(\omega W + T) - i\alpha(\omega T - W)],$$
$$M_2 = \frac{1}{\alpha(1 - \delta i)}(\delta T + W),\ N_2 = \frac{1}{\alpha(1 - \delta i)}[(1-\alpha)(\delta T + W) - i\alpha(T - \delta W)],$$
$$\bar{\sigma} = \|N_2 M_1^{-1} N_1 M_2^{-1}\|_2,\ \bar{\rho} = \|M_2 M_1^{-1} N_1 M_2^{-1}\|_2,\ \bar{\mu} = \|N_2 M_1^{-1}\|_2,\ \bar{\theta} = \|A M_2^{-1}\|_2,\ \bar{\nu} = \|M_2 M_1^{-1}\|_2.$$

According to Theorem 3.1 of [7], we have the following convergence theorem.

Theorem 5.1. *Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semi-definite, respectively, and let ω, δ and α be positive constants. If $\{\bar{x}^{(k)}\}$ is an iterative sequence generated by the ITSP iteration method and if $x^* \in \mathbb{C}^n$ is the exact solution of the system (1), then we have*

$$|||\bar{x}^{(k+1)} - x^*||| \le \psi(k)|||\bar{x}^{(k)} - x^*|||,$$

where $|||x||| = \frac{1}{\alpha}\sqrt{\frac{1}{1+\delta^2}}\|x\|_{(\delta T+W)}$ and

$$\psi(k) = \sqrt{\kappa(W)}\rho(\mathcal{T}(\alpha,\omega,\delta)) + \alpha\sqrt{\omega^2+1}\bar{\mu}_1\bar{\theta}_1\varepsilon_k + \alpha\sqrt{\delta^2+1}\bar{\theta}_1(\kappa(\delta T+W)\bar{\rho}_1 + \alpha\sqrt{\omega^2+1}\bar{\theta}_1\bar{\nu}_1\varepsilon_k)\eta_k$$

with

$$\bar{\mu}_1 = \|[(1-\alpha)(\delta T+W) - i\alpha(T-\delta W)](\omega W+T)^{-1}\|_2,\ \bar{\theta}_1 = \|A(\delta T+W)^{-1}\|_2,$$
$$\bar{\rho}_1 = \sqrt{(1-\alpha)^2 + \alpha^2\rho^2((\omega W+T)^{-1}(\omega T-W))},\ \bar{\nu}_1 = \|(\delta T+W)(\omega W+T)^{-1}\|_2.$$

In particular, when

$$\sqrt{\kappa(W)}\rho(\mathcal{T}(\alpha,\omega,\delta)) + \alpha\sqrt{\omega^2+1}\bar{\mu}_1\bar{\theta}_1\varepsilon_{\max} + \alpha\sqrt{\delta^2+1}\bar{\theta}_1(\kappa(\delta T+W)\bar{\rho}_1 + \alpha\sqrt{\omega^2+1}\bar{\theta}_1\bar{\nu}_1\varepsilon_{\max})\eta_{\max} < 1,$$

then the sequence $\{\bar{x}^{(k)}\}$ converges to x^*, where $\varepsilon_{\max} = \max_k\{\varepsilon_k\}$ and $\eta_{\max} = \max_k\{\eta_k\}$.

Proof. From Theorem 3.1 of [7], it can be seen that the ITSP iteration method is convergent if $\bar{\sigma} + \bar{\mu}\bar{\theta}\varepsilon_{\max} + \bar{\theta}(\bar{\rho} + \bar{\theta}\bar{\nu}\varepsilon_{\max})\eta_{\max} < 1$. Define $\bar{S} = W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$. Since

$$\begin{aligned}
\bar{\sigma} &= \|[(1-\alpha)(\delta T+W) - i\alpha(T-\delta W)](\omega W+T)^{-1}[(1-\alpha)(\omega W+T) - i\alpha(\omega T-W)](\delta T+W)^{-1}\|_2\\
&= \|W^{\frac{1}{2}}[(1-\alpha)(\delta\bar{S}+I) - i\alpha(\bar{S}-\delta I)](\omega I+\bar{S})^{-1}[(1-\alpha)(\omega I+\bar{S}) - i\alpha(\omega\bar{S}-I)](\delta\bar{S}+I)^{-1}W^{-\frac{1}{2}}\|_2\\
&\le \|W^{\frac{1}{2}}\|_2\|W^{-\frac{1}{2}}\|_2\rho(\mathcal{T}(\alpha,\omega,\delta)) = \sqrt{\kappa(W)}\rho(\mathcal{T}(\alpha,\omega,\delta)),\\
\bar{\rho} &= \|(\delta T+W)(\omega W+T)^{-1}[(1-\alpha)(\omega W+T) - i\alpha(\omega T-W)](\delta T+W)^{-1}\|_2\\
&\le \|(\delta T+W)\|_2\|(\delta T+W)^{-1}\|_2\|[(1-\alpha)I - i\alpha(\omega W+T)^{-1}(\omega T-W)]\|_2\\
&= \kappa(\delta T+W)\sqrt{(1-\alpha)^2+\alpha^2\rho^2((\omega W+T)^{-1}(\omega T-W))} = \kappa(\delta T+W)\bar{\rho}_1,\\
\bar{\mu} &= \|N_2M_1^{-1}\|_2 = \left\|\frac{\omega-i}{1-\delta i}[(1-\alpha)(\delta T+W) - i\alpha(T-\delta W)](\omega W+T)^{-1}\right\|_2\\
&= \sqrt{\frac{\omega^2+1}{\delta^2+1}}\left\|[(1-\alpha)(\delta T+W) - i\alpha(T-\delta W)](\omega W+T)^{-1}\right\|_2 = \sqrt{\frac{\omega^2+1}{\delta^2+1}}\bar{\mu}_1,\\
\bar{\theta} &= \|AM_2^{-1}\|_2 = \|\alpha(1-\delta i)A(\delta T+W)^{-1}\|_2 = \alpha\sqrt{1+\delta^2}\|A(\delta T+W)^{-1}\|_2 = \alpha\sqrt{1+\delta^2}\bar{\theta}_1,
\end{aligned}$$

$$\bar{\nu} = \|M_2M_1^{-1}\|_2 = \left\|\frac{\omega-i}{1-\delta i}(\delta T+W)(\omega W+T)^{-1}\right\|_2 = \sqrt{\frac{\omega^2+1}{\delta^2+1}}\|(\delta T+W)(\omega W+T)^{-1}\|_2 = \sqrt{\frac{\omega^2+1}{\delta^2+1}}\bar{\nu}_1,$$

it holds that $\bar{\sigma} + \bar{\mu}\bar{\theta}\varepsilon_k + \bar{\theta}(\bar{\rho} + \bar{\theta}\bar{\nu}\varepsilon_k)\eta_k \le \psi(k)$, and if $\sqrt{\kappa(W)}\rho(\mathcal{T}(\alpha,\omega,\delta)) + \alpha\sqrt{\omega^2+1}\bar{\mu}_1\bar{\theta}_1\varepsilon_{\max} + \alpha\sqrt{\delta^2+1}\bar{\theta}_1(\kappa(\delta T+W)\bar{\rho}_1 + \alpha\sqrt{\omega^2+1}\bar{\theta}_1\bar{\nu}_1\varepsilon_{\max})\eta_{\max} < 1$, then $\bar{\sigma} + \bar{\mu}\bar{\theta}\varepsilon_{\max} + \bar{\theta}(\bar{\rho} + \bar{\theta}\bar{\nu}\varepsilon_{\max})\eta_{\max} < 1$. Therefore, the ITSP iteration method is convergent. The proof is complete. ■

6. Numerical experiments

In this section, we use three examples which are of the complex linear system of the form (1) with W and T that are symmetric positive definite. By using these examples, we illustrate the feasibility and effectiveness of TSP and ITSP as solvers and as preconditioners for the GMRES method, and we also compare the performance of the TSP with those of PMHSS, PSPHSS, CRI, DSS and PFPAE when they are used as solvers and as preconditioners for the GMRES method, from the point of view of both the number of iteration steps (denoted as "IT") and the elapsed CPU time in seconds (denoted as "CPU"). Numerical comparisons of their inexact versions are also performed. Here, the preconditioned matrix V in the PMHSS and the PSPHSS iteration methods is taken as W. The experimentally found optimal parameters of the tested iteration methods used in actual computations are obtained experimentally by minimizing the corresponding iteration steps.

All the numerical experiments are computed in MATLAB (version R2016a) on a personal computer with Intel (R) Pentium (R) CPU G3240T 2.70 GHz, 4.0 GB memory and XP operating system. In actual computations, we use the zero vector as the initial vector, and all runs are terminated if RES $< 10^{-6}$ or the number of the prescribed iteration steps $k_{\max} = 500$ is exceeded which is indicated by "–", where

$$\text{RES} = \frac{\|b - Ax^{(k)}\|_2}{\|b\|_2} < 10^{-6}.$$

Table 1
Numerical results of Example 6.1 for the six iteration methods when $(\varpi, \mu) = (\pi, 0.02)$.

Method	m	16	32	48	64
PMHSS	$\alpha_{\exp}$	0.8	0.9	0.9	0.9
	IT	69	74	75	76
	CPU	0.0421	1.0196	4.7032	14.5300
	RES	9.27e−07	9.11e−07	9.63e−07	8.87e−07
PSPHSS	$\alpha_{\exp}$	1.2	0.9	0.8	0.8
	$\omega_{\exp}$	1.3	1.3	1.3	1.3
	IT	54	54	54	54
	CPU	0.0258	0.8363	3.9054	11.6278
	RES	8.69e−07	8.20e−07	8.58e−07	9.05e−07
CRI	$\alpha_{\exp}$	1	1	1	1
	IT	30	29	28	28
	CPU	0.0251	0.4127	1.8654	5.8212
	RES	8.79e−07	7.74e−07	8.96e−07	7.21e−07
DSS	$\alpha_{\exp}$	0.12	0.09	0.09	0.08
	IT	40	47	51	51
	CPU	0.0251	0.6716	3.2774	10.6399
	RES	9.18e−07	9.31e−07	9.28e−07	8.62e−07
PFPAE	$\alpha_{\exp}$	0.65	0.65	0.65	0.65
	$\omega_{\exp}$	1.3	1.4	1.4	1.4
	IT	50	51	50	50
	CPU	0.0182	0.5474	2.4550	7.3635
	RES	8.44e−07	9.61e−07	9.75e−07	8.87e−07
TSP	$\omega_{\exp}$	0.42	0.41	0.41	0.41
	$\delta_{\exp}$	0.15	0.12	0.11	0.1
	$\alpha_{\exp}$	0.95	0.94	0.93	0.93
	IT	19	20	20	20
	CPU	0.0120	0.2769	1.2834	3.7676
	RES	8.06e−08	5.35e−07	6.25e−07	5.24e−07

Table 2
When $(\sigma_1, \sigma_2) = (10, 100)$, numerical results of Example 6.2 for different iteration methods.

Method	m	16	32	48	64
PMHSS	$\alpha_{\exp}$	0.74	0.9	0.9	1
	IT	63	74	77	79
	CPU	0.0415	0.9304	4.8985	16.3990
	RES	9.12e−07	8.77e−07	9.52e−07	8.62e−07
PSPHSS	$\alpha_{\exp}$	0.9	0.78	0.75	0.8
	$\omega_{\exp}$	1.2	1.25	1.25	1.25
	IT	52	56	57	58
	CPU	0.0272	0.8221	4.0334	13.0598
	RES	7.68e−07	7.91e−07	9.89e−07	9.88e−07
CRI	$\alpha_{\exp}$	1	1	1	1
	IT	40	38	38	37
	CPU	0.0262	0.5293	2.6112	8.1566
	RES	7.35e−07	9.80e−07	7.39e−07	8.52e−07
DSS	$\alpha_{\exp}$	0.17	0.08	0.055	0.042
	IT	42	81	116	151
	CPU	0.0275	1.1052	7.8307	33.9649
	RES	7.92e−07	9.71e−07	9.25e−07	9.79e−07
PFPAE	$\alpha_{\exp}$	0.68	0.66	0.66	0.66
	$\omega_{\exp}$	1.22	1.35	1.35	1.35
	IT	49	53	53	53
	CPU	0.0186	0.5452	2.8217	9.4808
	RES	9.18e−07	8.37e−07	8.00e−07	9.35e−07
TSP	$\omega_{\exp}$	0.45	0.45	0.42	0.43
	$\delta_{\exp}$	0.2	0.15	0.12	0.1
	$\alpha_{\exp}$	0.95	0.95	0.95	0.95
	IT	21	22	22	22
	CPU	0.0143	0.3261	1.5740	4.7386
	RES	6.58e−07	7.22e−07	9.34e−07	9.52e−07

Table 3
Numerical results of Example 6.3 for different iteration methods.

Method	m	16	32	48	64
PMHSS	α_{exp}	0.5	0.5	0.5	0.5
	IT	61	60	60	60
	CPU	0.0373	0.8555	3.8803	15.0931
	RES	8.09e−07	9.92e−07	9.78e−07	9.78e−07
PSPHSS	α_{exp}	0.26	0.65	0.60	0.75
	ω_{exp}	3	2	1.6	1.4
	IT	21	30	37	42
	CPU	0.0132	0.4505	2.5947	9.7400
	RES	7.44e−07	8.68e−07	8.23e−07	8.90e−07
CRI	α_{exp}	1	1	1	1
	IT	37	38	35	36
	CPU	0.0221	0.5107	2.2349	7.7561
	RES	8.58e−07	8.52e−07	7.68e−07	9.44e−07
DSS	α_{exp}	0.23	0.23	0.22	0.23
	IT	28	28	26	27
	CPU	0.0179	0.3826	1.6330	5.6305
	RES	6.63e−07	9.46e−07	8.21e−07	8.84e−07
PFPAE	α_{exp}	0.95	0.85	0.78	0.8
	ω_{exp}	3	1.9	1.6	1.4
	IT	21	29	36	41
	CPU	0.0081	0.3024	1.7502	6.7512
	RES	7.72e−07	8.85e−07	8.86e−07	9.05e−07
TSP	ω_{exp}	1.78	1	0.7	0.6
	δ_{exp}	0.17	0.2	0.2	0.22
	α_{exp}	0.98	0.95	0.95	0.95
	IT	9	12	15	17
	CPU	0.0056	0.1602	0.9186	3.5408
	RES	6.47e−07	6.67e−07	3.90e−07	4.28e−07

Table 4
Numerical results of Example 6.1 for the six preconditioned GMRES methods when $(\varpi, \mu) = (\pi, 0.02)$.

Method	m	16	32	48	64
PMHSS-GMRES	α_{exp}	12	12	12	12
	IT	7	8	8	8
	CPU	0.0035	0.0450	0.2623	0.7534
	RES	9.02e−07	3.55e−08	4.20e−08	4.50e−08
PSPHSS-GMRES	α_{exp}	0.5	1.5	1.5	2
	ω_{exp}	11.5	10.5	10.5	10
	IT	7	8	8	8
	CPU	0.0033	0.0459	0.2618	0.7877
	RES	9.02e−07	3.55e−08	4.20e−08	4.50e−08
CRI-GMRES	α_{exp}	1	1	1	1
	IT	8	8	8	8
	CPU	0.0060	0.0865	0.4113	1.2255
	RES	1.58e−07	2.86e−07	3.40e−07	3.64e−07
DSS-GMRES	α_{exp}	1	0.96	1	1
	IT	8	8	8	8
	CPU	0.0067	0.0976	0.5208	1.6115
	RES	1.58e−07	2.87e−07	3.40e−07	3.64e−07
PFPAE-GMRES	α_{exp}	0.95	1	0.9	0.95
	ω_{exp}	12	10	10	15
	IT	7	8	8	8
	CPU	0.0034	0.0424	0.2548	0.8171
	RES	9.02e−07	3.55e−08	4.21e−08	4.50e−08
TSP-GMRES	ω_{exp}	10	13	15	17
	δ_{exp}	0.15	0.15	0.15	0.15
	α_{exp}	1	1	1	1
	IT	5	5	5	5
	CPU	0.0027	0.0410	0.2654	0.8615
	RES	1.97e−07	1.98e−07	1.91e−07	2.63e−07

Table 5
When $(\sigma_1, \sigma_2) = (10, 100)$, numerical results of Example 6.2 for different preconditioned GMRES methods.

Method	m	16	32	48	64
PMHSS-GMRES	α_{exp}	6	4	10	10
	IT	9	9	10	10
	CPU	0.0038	0.0473	0.2946	0.8625
	RES	4.97e−07	9.54e−07	8.18e−08	9.39e−08
PSPHSS-GMRES	α_{exp}	1	2	4	4
	ω_{exp}	5	2	6	6
	IT	9	10	10	10
	CPU	0.0039	0.0478	0.3051	0.8814
	RES	4.97e−07	9.54e−07	8.18e−08	9.39e−08
CRI-GMRES	α_{exp}	1	1	1	1
	IT	9	9	9	9
	CPU	0.0056	0.0878	0.4604	1.4938
	RES	2.87e−07	7.98e−07	8.50e−07	9.11e−07
DSS-GMRES	α_{exp}	1	1	1	1
	IT	9	9	9	9
	CPU	0.0060	0.1181	0.5581	1.8745
	RES	2.87e−07	7.98e−07	8.50e−07	9.11e−07
PFPAE-GMRES	α_{exp}	0.95	1	0.9	0.9
	ω_{exp}	12	4	11	11
	IT	9	9	10	10
	CPU	0.0062	0.0489	0.2917	0.8671
	RES	5.17e−07	9.54e−07	8.18e−08	9.39e−08
TSP-GMRES	ω_{exp}	3	4.5	4.2	4.2
	δ_{exp}	0.17	0.33	0.26	0.26
	α_{exp}	1	1	1	1
	IT	6	6	6	6
	CPU	0.0034	0.0465	0.3024	0.9963
	RES	4.38e−07	9.23e−07	8.10e−07	5.54e−07

Table 6
Numerical results of Example 6.3 for different preconditioned GMRES methods.

Method	m	16	32	48	64
PMHSS-GMRES	α_{exp}	1	1	5	5
	IT	6	7	8	8
	CPU	0.0030	0.0403	0.2381	0.7444
	RES	7.98e−07	8.27e−07	9.25e−08	6.09e−07
PSPHSS-GMRES	α_{exp}	0.5	0.5	2	2
	ω_{exp}	0.5	0.5	3	3
	IT	6	7	8	8
	CPU	0.0027	0.0393	0.2417	0.7336
	RES	7.98e−07	8.27e−07	9.25e−08	6.09e−07
CRI-GMRES	α_{exp}	1	1	1	1
	IT	7	7	7	8
	CPU	0.0047	0.0800	0.3773	1.2880
	RES	3.87e−08	8.96e−07	9.62e−07	2.24e−07
DSS-GMRES	α_{exp}	1	1	1	1
	IT	7	7	7	8
	CPU	0.0048	0.0941	0.4588	1.6625
	RES	3.87e−08	8.96e−07	9.62e−07	2.24e−07
PFPAE-GMRES	α_{exp}	0.95	1	0.9	0.9
	ω_{exp}	12	4	11	11
	IT	6	7	8	8
	CPU	0.0028	0.0400	0.2403	0.7447
	RES	2.54e−07	2.55e−07	9.67e−08	6.52e−07
TSP-GMRES	ω_{exp}	7	5	5	5
	δ_{exp}	0.2	0.28	0.28	0.35
	α_{exp}	1	1	1	1
	IT	4	5	5	6
	CPU	0.0024	0.0396	0.2663	0.9813
	RES	2.43e−07	8.66e−08	5.53e−07	2.45e−08

Table 7
Numerical results of the TSP iteration method for Examples 6.1–6.3.

Example	Method	m	16	32	48	64
No. 6.1		IT	31	30	30	30
	TSP-1	CPU	0.0202	0.4138	1.8778	5.5638
		RES	6.56e−07	7.74e−07	8.13e−07	8.51e−07
		IT	36	38	39	39
	TSP-2	CPU	0.0211	0.5294	2.4799	7.4387
		RES	7.44e−07	8.07e−07	6.98e−07	7.69e−07
		IT	27	26	26	26
	TSP-3	CPU	0.0197	0.3891	1.6439	5.4283
	(0.65, 1.4, 0.7143)	RES	9.81e−07	9.61e−07	7.52e−07	6.78e−07
No. 6.2		IT	32	31	31	31
	TSP-1	CPU	0.0199	0.4313	1.9495	5.8990
		RES	8.69e−07	8.96e−07	7.45e−07	7.37e−07
		IT	33	38	39	40
	TSP-2	CPU	0.0223	0.5121	2.4577	7.6281
		RES	7.96e−07	7.69e−07	9.59e−07	8.62e−07
		IT	29	28	28	27
	TSP-3	CPU	0.0189	0.4150	1.7635	5.6037
	(0.65, 1.4, 0.7143)	RES	8.74e−07	9.31e−07	7.03e−07	9.44e−07
No. 6.3		IT	27	27	27	27
	TSP-1	CPU	0.0191	0.3676	1.7182	5.0508
		RES	8.74e−07	8.65e−07	8.61e−07	8.76e−07
		IT	33	33	33	33
	TSP-2	CPU	0.0215	0.4611	2.0929	6.2505
		RES	6.72e−07	6.60e−07	6.54e−07	6.51e−07
		IT	22	22	22	22
	TSP-3	CPU	0.0146	0.3415	1.3506	4.5417
	(0.65, 1.4, 0.7143)	RES	6.04e−07	5.94e−07	5.88e−07	6.40e−07

Example 6.1. We consider the following complex symmetric linear system [1,2]:

$$[(-\varpi^2 M + K) + i(\varpi C_V + C_H)]x = b,$$

where M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, respectively, and ϖ is the driving circular frequency. We take $C_H = \mu K$ with μ a damping coefficient, $M = I$, $C_V = 10I$, and K the five-point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0, 1] \times [0, 1]$ with the mesh-size $h = 1/(m+1)$. The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$, with $V_m = h^{-2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}$. Hence, K is an $n \times n$ block-tridiagonal matrix, with $n = m^2$. In addition, we set the right-hand side vector b to be $b = (1+i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. As before, we normalize the system by multiplying both sides through by h^2.

Example 6.2. Consider the complex Helmholtz equation [5,9,22]:

$$-\triangle u + \sigma_1 u + i\sigma_2 u = f,$$

with σ_1 and σ_2 being real coefficient functions. Here, u satisfies Dirichlet boundary conditions in the square $D = [0, 1]\times[0, 1]$. By discretizing this equation with finite differences on an $m \times m$ grid with mesh size $h = 1/(m+1)$, we obtain a complex linear system

$$[(K + \sigma_1 I) + i\sigma_2 I]x = b,$$

where the matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form

$$K = I \otimes B_m + B_m \otimes I \text{ with } B_m = \frac{1}{h^2} \cdot \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}.$$

Actually, K is the five-point centered difference matrix approximating the negative Laplacian operator $L = -\Delta$. In our tests, let the right-hand side vector $b = (1+i)A\mathbf{1}$ with $\mathbf{1}$ being the vector of all entries be equal to 1. In addition, we normalize the complex linear system by multiplying both sides by h^2.

Example 6.3 (*[1,2,5,21]*)**.** Consider the linear system of equations $(W + iT)x = b$, with

$$T = I \otimes V + V \otimes I \text{ and } W = 10(I \otimes V_c + V_c \otimes I) + 9(e_1 e_m^T + e_m e_1^T) \otimes I,$$

where $V = tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}$, $V_c = V - e_1 e_m^T - e_m e_1^T \in \mathbb{R}^{m\times m}$ and e_1 and e_m are the first and last unit vectors in $\mathbb{R}^m$, respectively. We take the right-hand side vector b to be $b = (1+i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to $\mathbf{1}$.

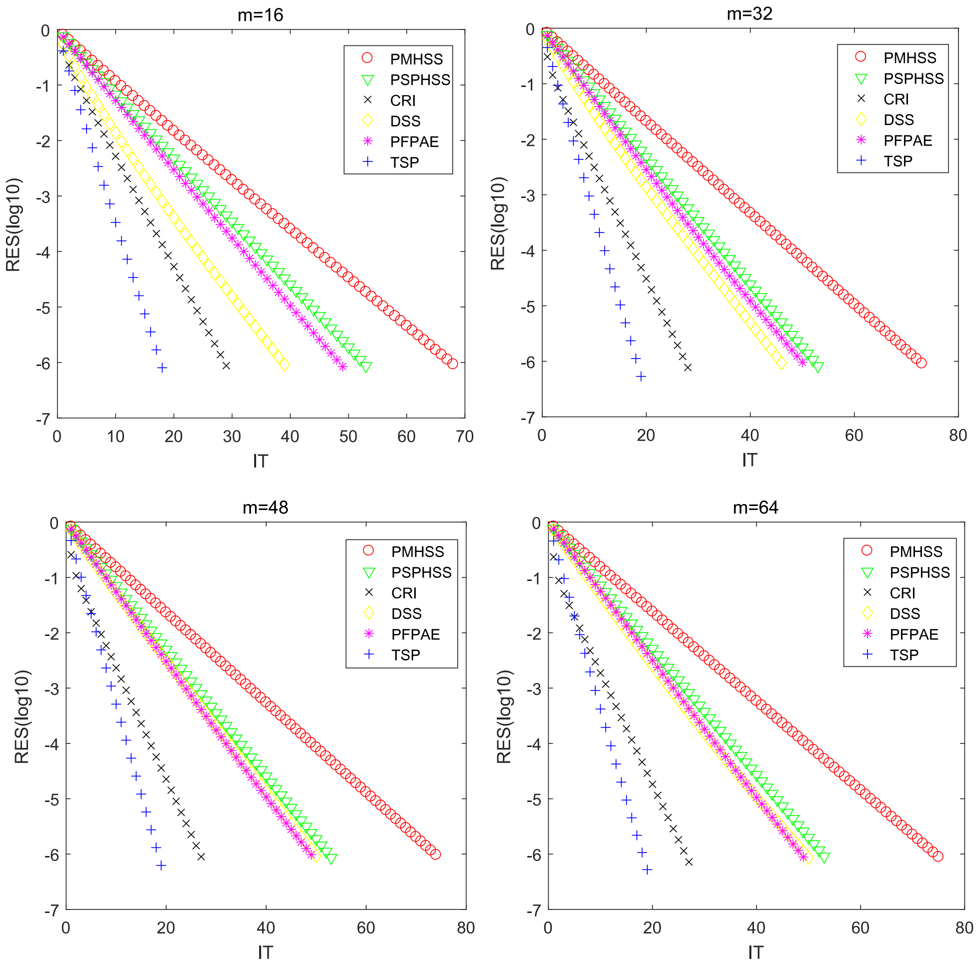

Fig. 1. Comparisons of the residual errors of the tested iteration methods for Example 6.1.

Here T and W correspond to the five-point centered difference matrices approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions and periodic boundary conditions, respectively, on a uniform mesh in the unit square $[0, 1] \times [0, 1]$ with the mesh-size $h = \frac{1}{m+1}$.

6.1. The experimental results of exact implementations

For all the tested exact iteration methods and the corresponding exact preconditioned GMRES methods, we apply the Cholesky factorization of the coefficient matrices for solving the sub-systems.

The experimentally found optimal parameters, IT and CPU times of the tested exact iteration methods for Examples 6.1–6.3 with respect to different problem sizes are listed in Tables 1–3 and those of the tested exact preconditioned GMRES methods for Examples 6.1–6.3 are reported in Tables 4–6, where we use (.)-GMRES to denote the (.) preconditioned GMRES method. From Tables 1–3, we can conclude some observations as follows. Firstly, the TSP iteration method outperforms the other ones in terms of both the IT and the CPU times. Secondly, the IT of the tested iteration methods are not too sensitive m in the sense the iterations barely change. The exception is in Table 2 where the IT of the DSS iteration method is greatly influenced by m. Thirdly, the DSS iteration method requires less IT than that of the PFPAE one in Tables 1 and 3, while it needs more CPU times as in the PFPAE iteration method one sub-system with the same coefficient matrix is required and there are two sub-systems need to be solved at each step of the DSS one. By comparing the numerical results in Tables 4–6, we observe that the TSP-GMRES method has advantage over the CRI- and the DSS-GMRES ones from the point of view of IT and CPU times, and it needs the least IT compared with the other preconditioned GMRES methods. However, the TSP-GMRES method costs a little more CPU times than those of the PMHSS-, PSPHSS- and PFPAE-GMRES ones as $m \geq 48$. This is because two sub-systems require to be solved at each step of the TSP-GMRES method, while in the PMHSS-, PSPHSS- and PFPAE-GMRES methods only one sub-system is required. In addition, we can see that the IT of all the tested exact preconditioned GMRES methods are mildly affected by changes of m.

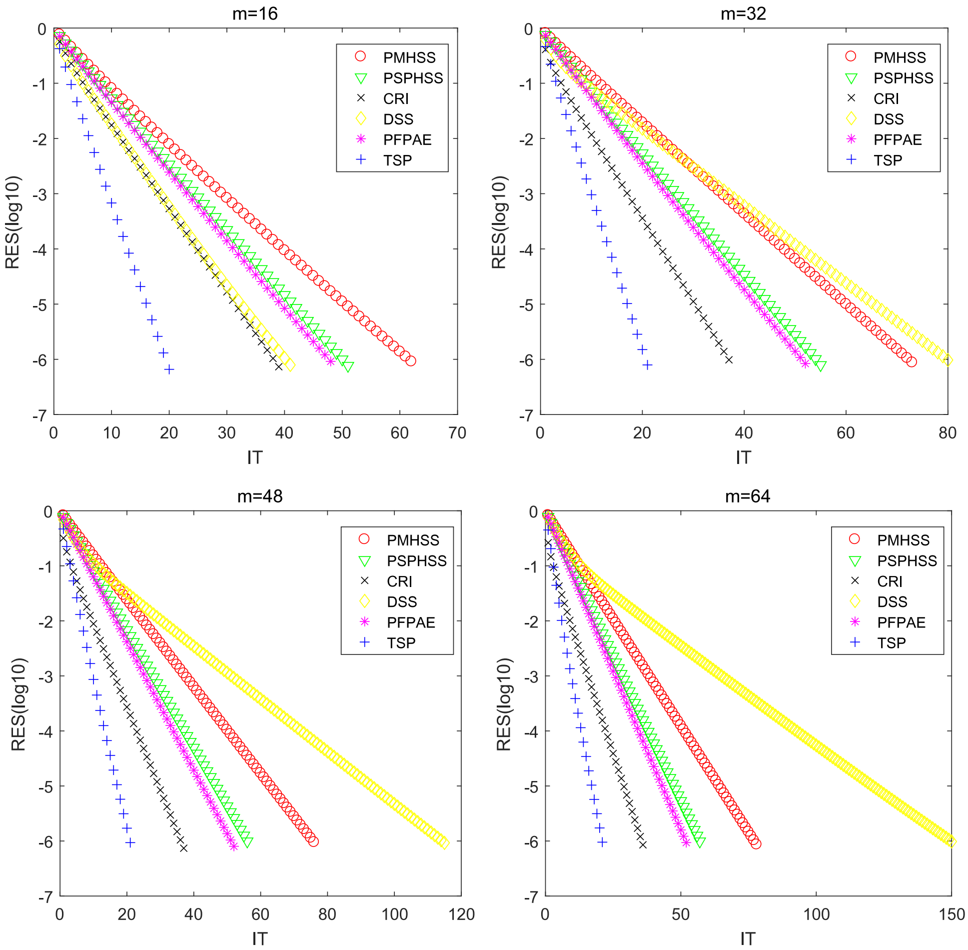

Fig. 2. Comparisons of the residual errors of the tested iteration methods for Example 6.2.

In practical applications, it is a tough thing to calculate the optimal parameters for the TSP iteration method when the problem size is too large. In Table 7 we list the IT, CPU and RES of the TSP iteration method for Examples 6.1–6.3 with the varying of problems size. The iteration parameters for the TSP iteration method are $(\alpha, \omega, \delta) = (0.5, 1, 0.5)$ (denoted by 'TSP-1'), $(\alpha, \omega, \delta) = (0.5, 1, 1)$ (denoted by 'TSP-2') and $(\alpha, \omega, \delta) = (0.65, 1.4, 0.7143)$ (denoted by 'TSP-3') based on the discussions at the end of Section 3. The numerical results of Table 7 show that $(\alpha, \omega, \delta) = (0.5, 1, 0.5)$ and $(\alpha, \omega, \delta) = (0.65, 1.4, 0.7143)$ can be considered as reasonable approximations of the optimal parameters. By comparing the results in Tables 1–3 with those in Table 7, we observe that the TSP iteration method with the parameters $(\alpha, \omega, \delta) = (0.5, 1, 1)$ performs better than the PMHSS and the PSPHSS ones, while the TSP iteration method with the parameters $(\alpha, \omega, \delta) = (0.5, 1, 0.5)$ or $(\alpha, \omega, \delta) = (0.65, 1.4, 0.7143)$ is superior to the other ones in terms of computing efficiency and its IT remains unchanged even reduces. Thus, we may simply choose $(\alpha, \omega, \delta) = (0.5, 1, 0.5)$ and $(\alpha, \omega, \delta) = (0.65, 1.4, 0.7143)$ in practical computation as substitutions for the TSP iteration method.

The graphs of RES(log10) against number of iterations in Tables 1–3 for four different sizes are displayed in Figs. 1–3, respectively. Figs. 1–3 clearly show that among these iteration methods, the TSP iteration one is the most effective method as its residual reduces the fastest.

The convergence speed of the iteration method depends largely on the spectral radius of the iteration matrix. In order to better investigate the performance of the tested iteration methods, the comparisons of spectral radii of the six different tested iteration matrices with the experimentally found optimal parameters, derived by the PMHSS, PSPHSS, CRI, DSS, PFPAE and TSP iteration methods, are performed in Fig. 4. From Fig. 4, it is easy to find that the spectral radius of the TSP iteration method is much smaller than those of the other ones.

In Fig. 5, we compare the spectral radius of the TSP iteration method in conjunction with the experimentally found optimal parameters (denoted by 'TSP-opt'), with the quasi-optimal ones obtained in Theorem 3.3 (denoted by 'TSP-exp'), with $(\alpha, \omega, \delta) = (0.5, 1, 0.5)$ (denoted by 'TSP-1'), with $(\alpha, \omega, \delta) = (0.5, 1, 1)$ (denoted by 'TSP-2') and with $(\alpha, \omega, \delta) = (0.65, 1.4, 0.7143)$ (denoted by 'TSP-3') for all examples. As seen, the quasi-optimal ones derived in Theorem 3.3 and

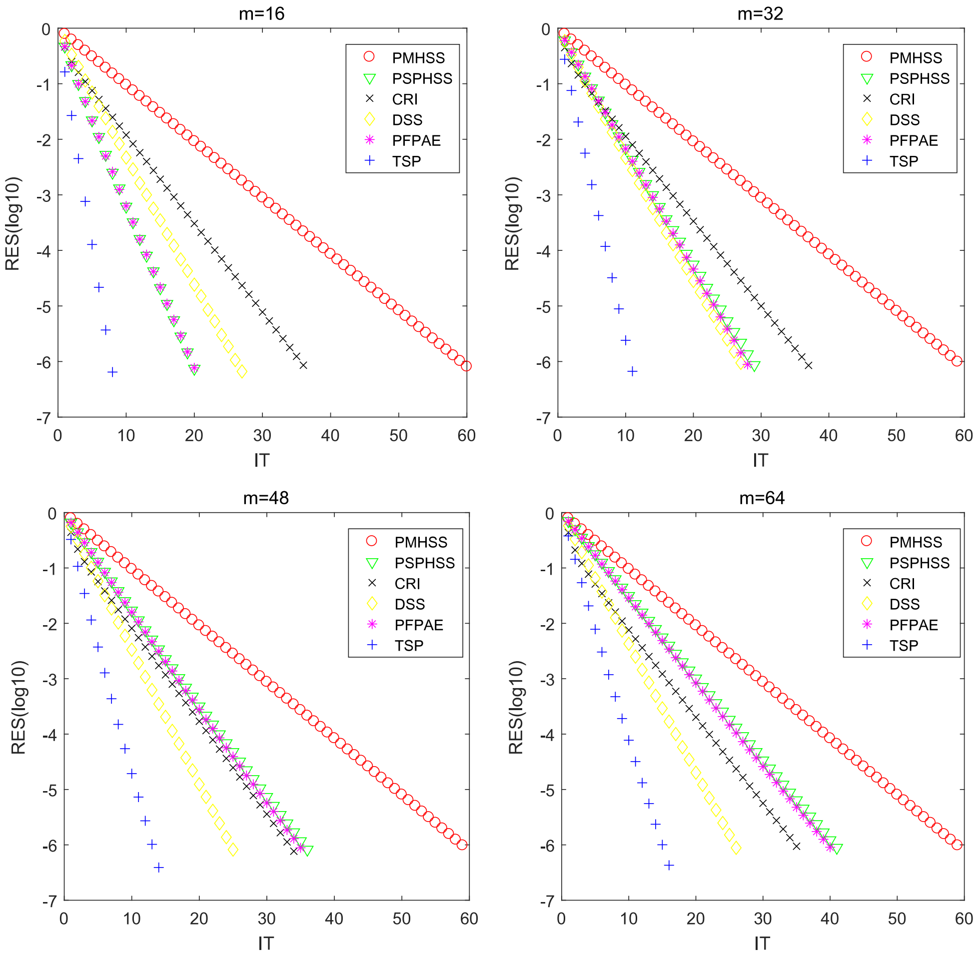

Fig. 3. Comparisons of the residual errors of the tested iteration methods for Example 6.3.

$(\alpha, \omega, \delta) = (0.65, 1.4, 0.7143)$, where $\omega = 1.4$ and $\delta = 0.7143$ are obtained by using the technique applied in [40], can be considered as reasonable approximations of the optimal values of the parameters of the TSP iteration method.

6.2. The experimental results of inexact implementations

We adopt the CG method as the inner solver for the tested inexact iteration methods as well as the corresponding inexact preconditioned GMRES methods. For the inexact iteration methods and inexact preconditioned GMRES methods, the stopping criteria for the CG iteration method are 10^{-2} and 10^{-7}, respectively.

We report the experimentally found optimal parameters, IT and the CPU times of the inexact iteration methods for Examples 6.1–6.3 in Tables 8–10, respectively. For the inexact preconditioned GMRES methods, the same items as those in Tables 4–6 are exhibited in Tables 11–13.

Through numerical experiments in Tables 8–10, it can be clearly seen that all tested inexact iteration methods can successfully compute approximate solutions satisfying the prescribed stopping criterion, and the ITSP iteration method always outperforms the other five ones in terms of IT and CPU times, and the advantage of the ITSP iteration method becomes more pronounced as the system size increases.

By comparing the results in Tables 11–13, it can be seen that the ITSP-GMRES method achieves faster convergence speed than the other five inexact preconditioned GMRES methods except for the case of $m = 64$ in Table 13. Besides, we also find that the convergence behavior of the ITSP-GMRES method is insensitive to the change of m, which indicates that the ITSP-GMRES method is not unduly problem size dependent.

7. Concluding remarks

A numerical solution for complex symmetric linear systems is an important and practically challenging problem in scientific computing and engineering applications. As one feasible way for solving this kind of problem, we establish a new

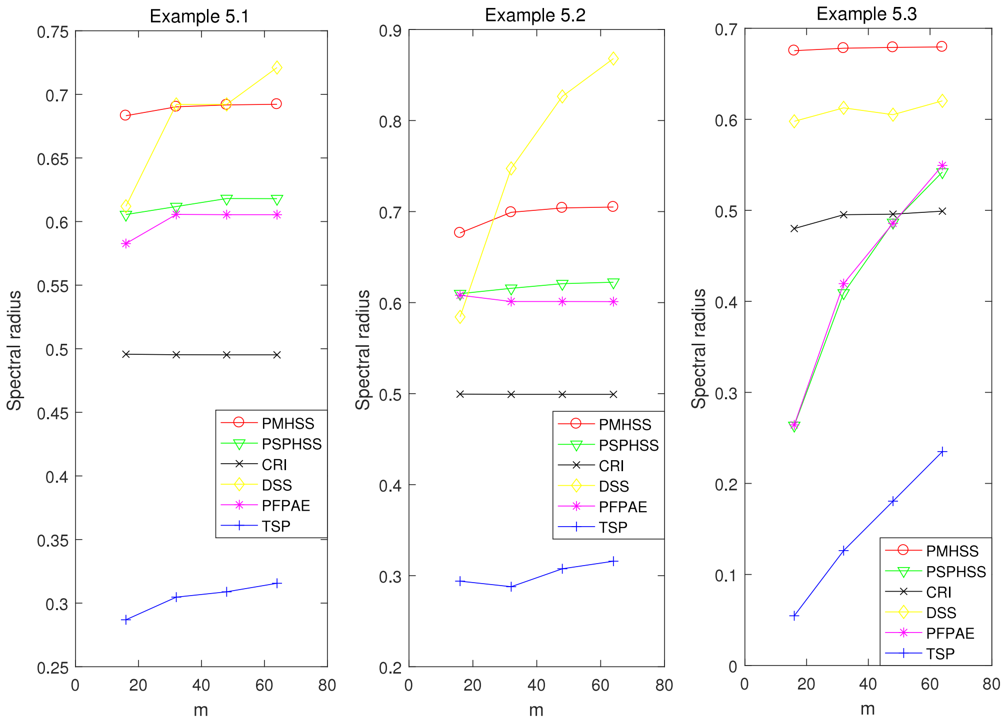

Fig. 4. The spectral radii of the iteration matrices of the tested iteration methods.

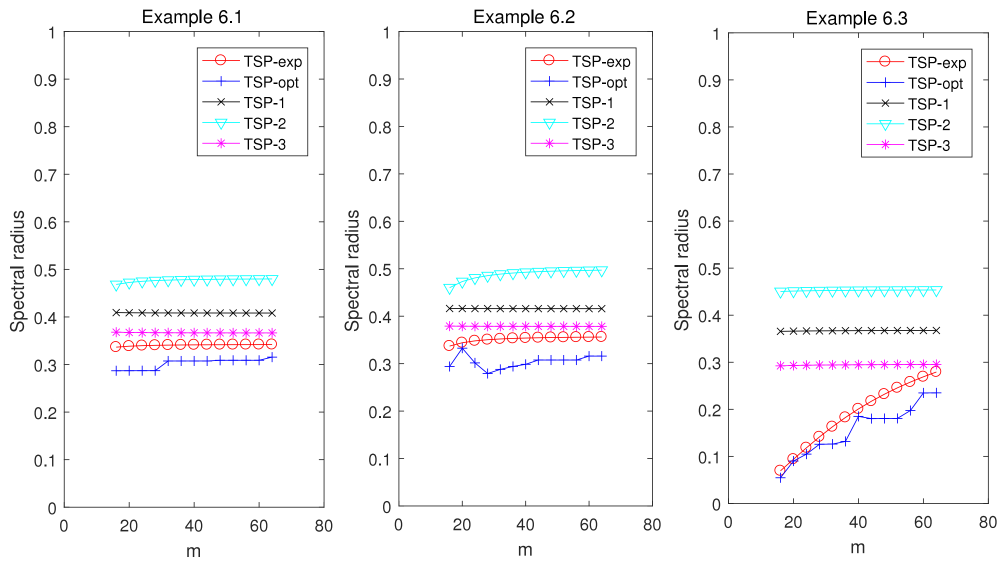

Fig. 5. The spectral radii of the TSP iteration matrix with different parameters.

iteration method called the two-step parameterized (TSP) iteration method by combining the two-parameter acceleration technique used in [24,36] with the general two-step strategy and theory applied in [37,38], and derive its inexact variant ITSP iteration method in this paper. The TSP iteration method is a further generalization of the DSS iteration method proposed in [20] and its first step is exactly the PFPAE iteration method newly developed in [22]. Theoretically, we discuss the convergence condition of the TSP and the ITSP iteration methods, compute the quasi-optimal iteration parameters that minimize the upper bound of the spectral radius of the TSP iteration method and the corresponding quasi-optimal

Table 8
Numerical results of Example 6.1 for the tested inexact iteration methods when $(\varpi, \mu) = (\pi, 0.02)$.

Method	m	16	32	48	64
IPMHSS	$\alpha_{\exp}$	0.8	0.9	0.9	0.9
	IT	69	74	75	76
	CPU	0.1785	2.0047	12.0925	40.5837
	RES	9.33e−07	9.52e−07	9.88e−07	9.27e−07
IPSPHSS	$\alpha_{\exp}$	1.1	0.85	0.8	0.8
	$\omega_{\exp}$	1.3	1.3	1.3	1.3
	IT	53	54	54	54
	CPU	0.0811	1.0479	7.2744	28.3053
	RES	9.50e−07	8.00e−07	8.59e−07	9.06e−07
ICRI	$\alpha_{\exp}$	1	1	1	1
	IT	30	29	28	28
	CPU	0.0813	1.0511	7.1880	26.4427
	RES	8.79e−07	7.72e−07	8.92e−07	7.12e−07
IDSS	$\alpha_{\exp}$	0.11	0.09	0.08	0.08
	IT	41	48	51	51
	CPU	0.1019	1.3790	8.7841	35.0523
	RES	7.28e−07	9.60e−07	8.79e−07	8.50e−07
IPFPAE	$\alpha_{\exp}$	0.65	0.65	0.65	0.65
	$\omega_{\exp}$	1.3	1.4	1.4	1.4
	IT	50	51	50	50
	CPU	0.0828	0.9541	6.2611	24.6392
	RES	8.47e−07	9.61e−07	9.75e−07	8.88e−07
ITSP	$\omega_{\exp}$	0.42	0.48	0.5	0.5
	$\delta_{\exp}$	0.15	0.12	0.11	0.1
	$\alpha_{\exp}$	0.95	0.94	0.93	0.93
	IT	19	18	18	18
	CPU	0.0555	0.5471	3.8499	14.9646
	RES	7.99e−07	7.70e−07	6.14e−07	5.69e−07

Table 9
When $(\sigma_1, \sigma_2) = (10, 100)$, numerical results of Example 6.2 for different inexact iteration methods.

Method	m	16	32	48	64
IPMHSS	$\alpha_{\exp}$	0.74	0.9	0.9	1
	IT	63	74	77	79
	CPU	0.1725	1.8744	11.0587	41.8175
	RES	9.91e−07	8.83e−07	9.56e−07	8.65e−07
IPSPHSS	$\alpha_{\exp}$	0.9	0.75	0.75	0.8
	$\omega_{\exp}$	1.2	1.25	1.25	1.25
	IT	52	56	57	58
	CPU	0.0773	0.8579	5.2239	18.9519
	RES	7.76e−07	8.19e−07	9.91e−07	9.90e−07
ICRI	$\alpha_{\exp}$	1	1	1	1
	IT	40	38	38	37
	CPU	0.1014	0.9979	5.9074	22.2308
	RES	7.34e−07	9.80e−07	7.40e−07	8.51e−07
IDSS	$\alpha_{\exp}$	0.17	0.08	0.041	0.027
	IT	43	84	106	129
	CPU	0.1050	2.0926	12.0974	64.5551
	RES	9.07e−07	9.95e−07	9.20e−07	9.12e−07
IPFPAE	$\alpha_{\exp}$	0.68	0.68	0.68	0.66
	$\omega_{\exp}$	1.22	1.32	1.37	1.35
	IT	49	53	53	53
	CPU	0.0667	0.7491	4.3588	15.7308
	RES	9.44e−07	9.17e−07	9.10e−07	9.37e−07
ITSP	$\omega_{\exp}$	0.45	0.45	0.43	0.43
	$\delta_{\exp}$	0.2	0.15	0.12	0.12
	$\alpha_{\exp}$	0.95	0.95	0.95	0.95
	IT	21	22	22	22
	CPU	0.0558	0.7032	3.9154	14.0933
	RES	6.98e−07	7.26e−07	8.27e−07	6.85e−07

Table 10
Numerical results of Example 6.3 for different inexact iteration methods.

Method	m	16	32	48	64
IPMHSS	$\alpha_{\exp}$	0.5	0.5	0.5	0.5
	IT	61	61	61	61
	CPU	0.2769	3.9190	31.6788	120.1593
	RES	8.25e−07	8.16e−07	7.96e−07	8.66e−07
IPSPHSS	$\alpha_{\exp}$	0.26	0.63	0.60	0.75
	$\omega_{\exp}$	3	2	1.6	1.4
	IT	21	30	37	42
	CPU	0.0489	1.0132	9.8120	43.3447
	RES	7.48e−07	8.74e−07	8.35e−07	9.06e−07
ICRI	$\alpha_{\exp}$	1	1	1	1
	IT	37	38	35	36
	CPU	0.1388	2.0345	17.7485	58.3309
	RES	8.60e−07	8.55e−07	7.66e−07	9.49e−07
IDSS	$\alpha_{\exp}$	0.22	0.23	0.21	0.22
	IT	28	29	27	28
	CPU	0.1190	2.0178	12.7516	51.6989
	RES	9.97e−07	8.22e−07	7.76e−07	9.22e−07
IPFPAE	$\alpha_{\exp}$	0.95	0.85	0.78	0.8
	$\omega_{\exp}$	3	1.9	1.65	1.4
	IT	21	29	37	41
	CPU	0.0451	1.0410	8.6678	38.2567
	RES	7.75e−07	9.03e−07	9.40e−07	9.30e−07
ITSP	$\omega_{\exp}$	1.78	1	0.7	0.6
	$\delta_{\exp}$	0.17	0.2	0.2	0.22
	$\alpha_{\exp}$	0.98	0.95	0.95	0.95
	IT	9	12	15	17
	CPU	0.0349	0.7826	6.3124	29.3475
	RES	6.64e−07	7.32e−07	4.13e−07	4.45e−07

Table 11
Numerical results of Example 6.1 for the six inexact preconditioned GMRES methods when $(\varpi, \mu) = (\pi, 0.02)$.

Method	m	16	32	48	64
IPMHSS-GMRES	$\alpha_{\exp}$	12	12	12	12
	IT	7	8	8	8
	CPU	0.0358	0.3910	2.7124	10.7721
	RES	9.02e−07	6.57e−08	8.87e−08	1.05e−07
IPSPHSS-GMRES	$\alpha_{\exp}$	0.5	1.5	1.5	2
	$\omega_{\exp}$	11.5	10.5	10.5	10
	IT	7	8	8	8
	CPU	0.0375	0.4278	2.7024	10.3080
	RES	9.03e−07	6.53e−08	8.86e−07	1.09e−07
ICRI-GMRES	$\alpha_{\exp}$	1	1	1	1
	IT	8	8	8	8
	CPU	0.0473	0.7241	5.0608	19.8389
	RES	1.59e−07	2.96e−07	3.45e−07	3.67e−07
IDSS-GMRES	$\alpha_{\exp}$	1	0.96	1	1
	IT	8	8	8	8
	CPU	0.0622	0.7038	5.1340	19.5154
	RES	1.59e−07	2.89e−07	3.45e−07	3.67e−07
IPFPAE-GMRES	$\alpha_{\exp}$	0.95	1	0.9	0.95
	$\omega_{\exp}$	12	10	10	15
	IT	7	8	8	8
	CPU	0.0318	0.4065	2.6815	10.6683
	RES	9.02e−07	6.88e−08	7.93e−08	6.66e−08
ITSP-GMRES	$\omega_{\exp}$	10	13	15	17
	$\delta_{\exp}$	0.15	0.15	0.15	0.15
	$\alpha_{\exp}$	1	1	1	1
	IT	5	5	5	5
	CPU	0.0205	0.3585	2.6289	10.1991
	RES	2.09e−07	2.24e−07	2.07e−07	2.89e−07

Table 12
When $(\sigma_1, \sigma_2) = (10, 100)$, numerical results of Example 6.2 for different inexact preconditioned GMRES methods.

Method	m	16	32	48	64
IPMHSS-GMRES	α_{exp}	6	4	10	10
	IT	9	9	10	10
	CPU	0.0237	0.4405	3.0903	11.8873
	RES	4.98e−07	9.55e−07	8.31e−08	9.66e−08
IPSPHSS-GMRES	α_{exp}	1	2	4	4
	ω_{exp}	5	2	6	6
	IT	9	9	10	10
	CPU	0.0259	0.4430	3.1305	12.0587
	RES	4.98e−07	9.55e−07	8.38e−08	9.67e−08
ICRI-GMRES	α_{exp}	1	1	1	1
	IT	9	9	9	9
	CPU	0.0435	0.6845	4.9278	19.4529
	RES	2.88e−07	8.03e−07	8.55e−07	9.29e−07
IDSS-GMRES	α_{exp}	1	1	1	1
	IT	9	9	9	9
	CPU	0.0476	0.7526	4.9820	19.3097
	RES	2.88e−07	8.03e−07	8.55e−07	9.29e−07
IPFPAE-GMRES	α_{exp}	0.95	1	0.9	0.9
	ω_{exp}	12	4	11	11
	IT	9	9	10	10
	CPU	0.0280	0.4399	3.1609	12.0731
	RES	5.13e−07	9.55e−07	9.29e−08	1.61e−07
ITSP-GMRES	ω_{exp}	3	4.5	4.2	4.2
	δ_{exp}	0.17	0.33	0.26	0.26
	α_{exp}	1	1	1	1
	IT	6	6	6	6
	CPU	0.0235	0.4170	3.0714	11.9967
	RES	4.41e−07	9.28e−07	8.13e−07	5.56e−07

Table 13
Numerical results of Example 6.3 for different inexact preconditioned GMRES methods.

Method	m	16	32	48	64
IPMHSS-GMRES	α_{exp}	1	1	5	5
	IT	6	7	8	8
	CPU	0.0199	0.4151	3.4058	13.1030
	RES	7.99e−07	8.29e−07	1.53e−07	6.24e−07
IPSPHSS-GMRES	α_{exp}	0.5	0.5	2	2
	ω_{exp}	0.5	0.5	3	3
	IT	6	7	8	8
	CPU	0.0201	0.4072	3.5128	13.4490
	RES	7.99e−07	8.30e−07	1.53e−08	6.25e−07
ICRI-GMRES	α_{exp}	1	1	1	1
	IT	7	7	7	8
	CPU	0.0611	0.7807	5.6635	24.6543
	RES	1.65e−07	9.18e−07	9.81e−07	2.94e−07
IDSS-GMRES	α_{exp}	1	1	1	1
	IT	7	7	7	8
	CPU	0.0612	0.7480	5.7501	24.7855
	RES	1.65e−07	9.18e−07	9.81e−07	2.94e−07
IPFPAE-GMRES	α_{exp}	0.95	1	0.9	0.9
	ω_{exp}	12	4	11	11
	IT	6	7	8	8
	CPU	0.0316	0.4224	3.4773	13.3317
	RES	2.69e−07	2.60e−07	1.63e−07	6.65e−07
ITSP-GMRES	ω_{exp}	7	5	5	5
	δ_{exp}	0.2	0.2	0.28	0.35
	α_{exp}	1	1	1	1
	IT	4	5	5	6
	CPU	0.0171	0.4118	3.2251	15.7375
	RES	2.56e−07	9.83e−08	5.67e−07	9.63e−08

convergence factor. Based on Theorem 3.4 and applying the strategy proposed by Chen in [40], we give some practical ways for the choice of iteration parameters of the TSP iteration method:

- $\alpha = \delta = 0.5, \omega = 1$;
- $\omega = \delta = 1, \alpha = 0.5$;
- proper α and $\omega \geq 1, \delta \leq 1$ satisfying $\omega\delta = 1$;

which can be considered as reasonable approximations of the optimal parameters of the TSP iteration method. Also, we show that the TSP iteration method has smaller quasi-optimal convergence factor than the MHSS [1], PMHSS [2] and CRI [21] ones, in particular, when the quasi-optimal iteration parameters are employed. In addition, Theorem 4.2 indicates that the TSP iteration method may be more efficient than the PFPAE [22] and the PSHSS [9] ones. Further, numerical experiments illustrate that TSP and ITSP are effective linear solvers and matrix splitting preconditioners for solving the complex symmetric linear systems, and outperform some existing ones in terms of both number of iteration steps and computing times. Moreover, in actual computations, using TSP and ITSP as matrix splitting preconditioner to the GMRES method often yields much less iteration steps and computing times than using them as linear iterative solvers.

Lastly, we should point out that we only derive the quasi-optimal parameters for the TSP iteration method which minimize the upper bound of the spectral radius of it. The choice of the optimal parameters of the TSP iteration method has not been obtained in this paper and is a challenging problem that deserves further study. Hence, searching the optimal parameters for the TSP iteration method is a further work for us.

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