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MULTI-GRID DYNAMIC ITERATION FOR PARABOLIC EQUATIONS

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Abstract.

We study the method which is obtained when a multi-grid method (in space) is first applied directly to a parabolic intitial-boundary value problem, and discretization in time is done only afterwards. This approach is expected to be well-suited to parallel computation. Further, time marching can be done using different time step-sizes in different parts of the spatial domain.

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1. Introduction.

Multi-grid methods are known to be very efficient solvers for elliptic equations. Various approaches have also been given to extend multi-grid techniques to parabolic problems [1], [6], [10]. A common feature of these approaches is that multi-grid methods are applied only *after* the equation has been discretized in time. In the present note we shall rather apply multi-grid (in space) directly to the evolution equation. In the resulting set of (stiff) ordinary differential equations, each of which corresponds to a node on a spatial grid, each differential equation can then be discretized in time *independently*, with step-sizes chosen according to the smoothness of the solution at the corresponding node. For the case when the *same* step-size is used at all the nodes of a level, we regain the method proposed by Hackbusch [6] to which our results will apply in particular.

The present approach is closely related to wave-form relaxation methods [11], [14] which have recently been used successfully in circuit simulation. There a large system of ordinary differential equations is decomposed into subsystems, and each subsystem is integrated independently by reading inputs from other subsystems from their values on the previous iteration. A convergence analysis is given by Miekkala & Nevanlinna [12], who call such methods *dynamic iteration methods*. We shall take up their approach in our analysis. As

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in [12], we shall restrict the analysis to linear (model) equations with time-independent coefficients. There is, however, no such limitation in the formulation of the method, and one may expect the algorithm to be applicable to wider classes of equations.

The basic method can be described briefly thus: In the parabolic equation, replace the time differentiation operator $\partial/\partial t$ by a complex constant, s say. Apply a multi-grid method to the elliptic problem obtained in this way. In the resulting algorithm, reinterpret s as time derivative $\partial/\partial t$.

More formally, a two-grid cycle for the linear initial value problem

$$\frac{du}{dt} + L_h u = f, \quad u(0) = u_0, \tag{1.1}$$

with L_h a discretization of a linear elliptic differential operator L, reads as follows:

- *Gauss-Seidel pre-smoothing.*

Let $L_h = D - A - B$, where D is the diagonal of L_h, A and B are strictly lower and upper triangular, respectively.

Given $u^{(0)} = u^{(0)}(t)$, solve for $k = 0, 1, \ldots, \nu_1 - 1$ the initial value problem

$$\frac{du^{(k+1)}}{dt} + (D - A)u^{(k+1)} - Bu^{(k)} = f, \quad u^{(k+1)}(0) = u_0, \quad t \in [0, \Delta T]. \tag{1.2}$$

- *Coarse-grid correction.*

Compute the defect

$$d = \frac{du^{(\nu_1)}}{dt} + L_h u^{(\nu_1)} - f = B(u^{(\nu_1 - 1)} - u^{(\nu_1)}). \tag{1.3}$$

On the coarse grid (with grid-size H), solve the initial value problem

$$\frac{dv}{dt} + L_H v = rd, \qquad v(0) = 0, \tag{1.4}$$

where L_H is a coarse-grid discretization of L, and r is the restriction from the fine to the coarse grid. Correct

$$\bar{u} = u^{(\nu_1)} - pv, \tag{1.5}$$

where p is the coarse-to-fine prolongation.

- *Gauss-Seidel post-smoothing.*

Apply ν_2 iterations (1.2), with initial iterate $\bar{u}$. ■

Denoting the components of $u^{(k+1)}(t)$ by $u_j^{(k+1)}(t)$, the equations (1.2) take the form

$$\frac{d}{dt} u_j^{(k+1)} + d_j u_j^{(k+1)} = rhs,$$

where the right-hand side depends only on $f, u^{(k)}$ and already known components of $u^{(k+1)}$. These equations can be integrated separately, possibly by choosing different time step-sizes at different spatial nodes j.

To describe the virtues and drawbacks of the above approach, we can adapt the words of [12]: It suits excellently for parallel computation (e.g. in the case of red-black ordering of the nodes for five-point stencils L_h). Known time histories of nearby problems can be efficiently utilized. Savings in the computation are to be expected, especially if the solution is most of the time inactive in large parts of the spatial domain. On the other hand, one has to save the time histories of the node variables and, in case of desynchronization of time marching, one has to introduce an interpolation procedure to provide the values at correct time instances.

We shall show that the smoothing properties of red-black Gauss-Seidel dynamic iteration are not quite as good as for the stationary problem, but still sufficiently good to achieve the typical multi-grid convergence speed. Essentially, the quantity

$$\eta_0(\nu) = \frac{\nu^\nu}{(\nu+1)^{\nu+1}} \tag{1.6}$$

(ν: number of smoothing iterations), which typically arises in smoothing rates and two-grid convergence rates for elliptic problems (see [7], Ch. 6, [16], Ch. 8), is replaced by $\sqrt{[\eta_0(2\nu-1)]}$. A table of these values is given below. Unlike Gauss-Seidel, the smoothing rates of damped Jacobi dynamic iteration remain the same as for the stationary problem, but are still worse than those of Gauss-Seidel for reasonable values of ν. We remark that, in contrast, the *convergence* rates of both Gauss-Seidel and damped Jacobi dynamic iteration are the same as for the stationary iteration for all the problems considered here ([12], Cor. 4.1).

In Section 2 we set up the general frame for our analysis. In Section 3 we study the smoothing rates of red-black Gauss-Seidel and damped Jacobi for the standard five-point spatial discretization of the heat equation on a square. In Section 4 we give a two-grid convergence analysis for the one-dimensional heat equation. In Section 5 we extend some of the more general convergence results for elliptic equations to the parabolic case. The effect of time-discretization, when the same step-size is used throughout, is studied in Section 6. Finally, we give in Section 7 a numerical example which illustrates the use of adaptive time-grids.

Table 1.

ν	1	2	3	4
$\eta_0(\nu)$	0.25	0.148	0.105	0.082
$\sqrt{[\eta_0(2\nu-1)]}$	0.5	0.325	0.259	0.222

2. Preparations.

Let $u^{(k+1)}$ denote the result after one Gauss-Seidel dynamic iteration (1.2) applied to $u^{(k)}$. Then the error to the solution u of (1.1), $e^{(k)} = u^{(k)} - u$, solves the initial value problem

$$\frac{d}{dt} e^{(k+1)} + (D-A)e^{(k+1)} - Be^{(k)} = 0, \quad e^{(k+1)}(0) = 0. \tag{2.1}$$

Hence

$$e^{(k+1)} = \mathcal{S} e^{(k)}, \tag{2.2}$$

where $\mathcal{S}$ is the integral operator

$$\mathcal{S}e(t) = \int_0^t K_S(t-\tau)e(\tau)d\tau, \qquad (t > 0) \tag{2.3}$$

whose kernel $K_S(t)$ $(= e^{-(D-A)t}B)$ has the Laplace transform

$$S(s) = (sI + D - A)^{-1}B. \tag{2.4}$$

In the case of damped Jacobi dynamic iteration one has instead of (2.4)

$$S(s) = (sI + \omega^{-1}D)^{-1}(\omega^{-1}D - L_h). \tag{2.5}$$

Similarly, the errors $e^i = u^i - u$ of the complete two-grid iteration described in Section 1 satisfy

$$e^{i+1} = \mathcal{M} e^i, \tag{2.6}$$

where $\mathcal{M}$ is the integral operator

$$\mathcal{M}e(t) = \int_0^t K_M(t-\tau)e(\tau)d\tau, \qquad (t > 0) \tag{2.7}$$

whose kernel $K_M(t)$ is given via its Laplace transform

$$M(s) = S(s)^{\nu_2}(I - p(s+L_H)^{-1}r(s+L_h))S(s)^{\nu_1}. \tag{2.8}$$

We refer to $M(s)$ as the transfer function of the convolution operator $\mathcal{M}$. Note that, for each s, $S(s)$ and $M(s)$ are the smoothing and two-grid iteration matrix, respectively, corresponding to the *elliptic* problem $su + Lu = f$.

Our main tools in analyzing the smoothing operator $\mathcal{S}$ and the two-grid operator $\mathcal{M}$ are the following two results, which we formulate for $\mathcal{M}$. More generally than (2.8), we suppose here that

$$\begin{cases} M(s) = (m_{ij}(s))_{i,j=1,\ldots,n}, \text{ where the entries } m_{ij}(s) \text{ are rational functions} \\ \text{of } s \text{ vanishing at infinity, all of whose poles have negative real part.} \end{cases} \tag{2.9}$$

Under this assumption, the convolution operator $\mathcal{M}$ with transfer function $M(s)$ is a bounded operator on $L^p(\mathbb{R}_+, \mathbb{C}^n)$ for $1 \leqslant p \leqslant \infty$.

Its spectral radius

$$\varrho(\mathcal{M}) = \lim_{k \to \infty} \|\mathcal{M}^k\|^{1/k},$$

satisfies the following.

PROPOSITION 1. *Assume* (2.9), *and consider* $\mathcal{M}$ *as an operator on* $L^p(\mathbb{R}_+, \mathbb{C}^n)$ $(1 \leqslant p \leqslant \infty)$. *Then*

$$\varrho(\mathcal{M}) = \max_{\operatorname{Re} s \geqslant 0} \varrho(M(s)).$$

This result has been used extensively in [12] for the convergence analysis of dynamic iteration methods. It is a consequence of a theorem of Paley and Wiener (see [15], p. 60, and [4], p. 68) which states that $\lambda - \mathcal{M}$ is invertible as an operator on $L^p(\mathbb{R}_+, \mathbb{C}^n)$ if and only if the matrices $\lambda - M(s)$ are invertible for $\operatorname{Re} s \geqslant 0$ (and $s = \infty$). We remark that in our situation ($m_{ij}(s)$ rational) the proof of this result becomes elementary.

In an L^2 setting an analogous result holds also for the norm.

PROPOSITION 2. *Assume* (2.9), *and consider* $\mathcal{M}$ *as an operator on* $L^2(\mathbb{R}_+, \mathbb{C}^n)$. *Then*

$$\|\mathcal{M}\| = \max_{\operatorname{Re} s \geqslant 0} \|M(s)\|.$$

This result is a consequence of Parseval's formula, cf. e.g. [2], p. 8. We conclude this section with some remarks.

REMARK. Since $\mathcal{M}e(T)$ does not depend on $e(t)$ for $t > T$, we also have

$$\|\mathcal{M}\|_{L^2((0,T),\mathbb{C}^n)} \leqslant \max_{\mathrm{Re}\, s \geqslant 0} \|M(s)\| \quad \text{for every } T > 0.$$

REMARK. With exponential scaling in the norm,

$$\|v\| = \|e^{-\alpha t} v(t)\|_{L^2(\mathbb{R}_+,\mathbb{C}^n)},$$

the maximum in Propositions 1 and 2 would have to be taken over $\mathrm{Re}\, s \geqslant \alpha$.

REMARK. If equation (1.1) is replaced by

$$\frac{du}{dt} + L_h u + cu = f,$$

then the corresponding two-grid operator $\mathcal{M}_c$ satisfies

$$\varrho(\mathcal{M}_c) = \max_{\mathrm{Re}\, s \geqslant 0} \varrho(M(s+c)) = \max_{\mathrm{Re}\, s \geqslant c} \varrho(M(s)).$$

The spectral radii (and L^2 norms) obviously decrease with growing c, as it is also known for the Helmholtz equation in the stationary case.

3. Smoothing rates for the heat equation on a square.

We consider the heat equation on a square with Dirichlet boundary conditions,

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \Omega \times (0,\infty), \qquad \Omega = (0,1)\times(0,1) \tag{3.1}$$

$$u = 0 \text{ on } \partial\Omega, \qquad u = u_0 \text{ at } t = 0.$$

For semi-discretization in space we choose the standard 5-point formula

$$L_h \triangleq \frac{1}{h^2}\begin{pmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{pmatrix}, \qquad h = 1/N. \tag{3.2}$$

Recall that the eigenvectors e_{mn} $(1 \leqslant m,n \leqslant N-1)$ of L_h are given by

$$e_{mn}(x,y) = 2\sin(m\pi x)\sin(n\pi y) \quad \text{for} \quad (x,y) = (kh, lh), \quad 1 \leqslant k,l \leqslant N-1. \tag{3.3}$$

To study the smoothing effect of dynamic iteration methods we consider the orthogonal projection Q onto the space of high frequencies,

$\mathrm{span}\{e_{m'n'}, e_{m'n}, e_{mn'} | m,n < N/2, m' = N-m, n' = N-n\}$. (see [16], p. 92).

We now choose red-black ordering of the nodes and consider the dynamic Gauss-Seidel iteration (1.2). Let the corresponding iteration operator $\mathcal{S}$ be defined by (2.3), (2.4). Its smoothing rate is given next.

PROPOSITION 3. *Red-black Gauss-Seidel dynamic iteration satisfies on* $L^p(\mathbb{R}_+, \mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$

$$\varrho(Q\mathcal{S}^\nu) \leqslant \tfrac{1}{2}\sqrt{[\eta_0(2\nu-1)]} \quad \text{for} \quad \nu \geqslant 1$$

with η_0 *defined in* (1.6). *This is the best possible bound which is independent of* h.

REMARK. The corresponding bounds for the Poisson equation are smaller for $\nu \geqslant 2$ [16]:

$$\varrho(QS(0)^\nu) \leqslant \begin{cases} 2^{-2\nu} & \text{for } \nu = 1, 2 \\ \tfrac{1}{2}\eta_0(2\nu-1) & \text{for } \nu \geqslant 3. \end{cases}$$

PROOF. We use Proposition 1. A straightforward extension of the analysis for the Poisson equation ([16], p. 89) shows that $S(s)$ leaves the subspace spanned by $(e_{mn}, e_{m'n'}, e_{m'n}, e_{mn'})$, $(m, n < N/2)$ invariant, and its restriction has with respect to this basis the matrix representation

$$S_{mn}(s) = \begin{pmatrix} S_a & 0 \\ 0 & S_b \end{pmatrix}, \tag{3.4}$$

where

$$S_a = \frac{a}{2}\begin{pmatrix} 1+a & -(1+a) \\ 1-a & -(1-a) \end{pmatrix}, \qquad S_b = \frac{b}{2}\begin{pmatrix} 1+b & -(1+b) \\ 1-b & -(1-b) \end{pmatrix} \tag{3.5}$$

$$a = \frac{1}{1+sh^2/4} \cdot \tfrac{1}{2}(\cos m\pi h + \cos n\pi h), \quad b = \frac{1}{1+sh^2/4} \cdot \tfrac{1}{2}(-\cos m\pi h + \cos n\pi h). \tag{3.6}$$

We thus have to study the spectral radius of $Q_{mn}S_{mn}(s)^\nu$, where $Q_{mn} = \operatorname{diag}(0, 1, 1, 1)$ is the restriction of the projection Q.

Using $S_a^\nu = a^{2\nu-2}S_a$, $S_b^\nu = b^{2\nu-2}S_b$ we obtain

$$\varrho(Q_{mn}S_{mn}(s)^\nu) = \max\{\tfrac{1}{2}|a^{2\nu-1}(1-a)|, |b|^{2\nu}\}. \tag{3.7}$$

For $\operatorname{Re} s \geqslant 0$ and $h \in (0, h_0]$, a of (3.6) can vary in the disc $D = \{a \in \mathbb{C}; |a - \tfrac{1}{2}| < \tfrac{1}{2}\}$. For $\nu \geqslant 1$ one has

$$\tfrac{1}{2}\max_{a \in \bar{D}} |a^{2\nu-1}(1-a)| = \tfrac{1}{2}\max_{\operatorname{Re} z \geqslant 0} \left|\frac{z}{(1+z)^{2\nu}}\right| = \tfrac{1}{2}\sqrt{[\eta_0(2\nu-1)]}, \tag{3.8}$$

which is greater or equal $2^{-2\nu} \geqslant |b|^{2\nu}$. The maximum is attained at $z = \pm i/\sqrt{(2\nu-1)}$ which for $h \to 0$ corresponds to $sh^2/4 = \pm i/\sqrt{(2\nu-1)}$ in (3.6.). ■

In contrast to red-black Gauss-Seidel, the smoothing rate for damped Jacobi dynamic iteration (2.3), (2.5) is the same as for the stationary problem.

PROPOSITION 4. *Damped Jacobi dynamic iteration with relaxation parameter ω satisfies on $L^p(\mathbb{R}_+, \mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$*

$$\varrho(Q\mathscr{S}^\nu) = \varrho(QS(0)^\nu) \leqslant \mu^{-\nu}, \quad \text{with } \mu = \max\{|1-\omega/2|, |1-2\omega|\}.$$

PROOF. The equality follows from Proposition 1 and the fact that here

$$S(s) = (1+\omega sh^2/4)^{-1} \cdot S(0).$$

The inequality is that for Poisson's equation given in [16], p. 32. ■

REMARK. For $\nu \leqslant 4$ the smoothing rates of Gauss-Seidel are still better than those for damped Jacobi.

4. Two-grid analysis of the one-dimensional heat equation.

In this section we give a two-grid model problem analysis for the *one-dimensional* heat equation which, in contrast to the two-dimensional case, is still manageable with moderate effort. We have, however, reason to believe that the results are also typical for the two-dimensional situation (see below).

We consider the heat equation (3.1) with Dirichlet boundary conditions on $\Omega = (0, 1)$, with the following data:

$$L_h \mathrel{\hat{=}} h^{-2}[-1 \quad 2 \quad -1], \quad h = 1/N, \quad H = 2h,$$
$$r \mathrel{\hat{=}} \tfrac{1}{4}[\,1 \quad 2 \quad 1],$$

p: piecewise linear interpolation.

We then have

PROPOSITION 5. *The two-grid iteration operator $\mathscr{M} = \mathscr{M}(\nu_1, \nu_2)$ for the one-dimensional heat equation, with red-black Gauss-Seidel smoothing, considered as an operator on $L^p((0, \infty), \mathbb{C}^{N-1})$, $(1 \leqslant p \leqslant \infty)$, satisfies*

$$\varrho(\mathscr{M}) \leqslant \tfrac{1}{2}\sqrt{[\eta_0(2\nu-1)]} \quad \text{for} \quad \nu = \nu_1 + \nu_2 \geqslant 1.$$

This is the best possible bound which is independent of the grid-size h.

PROOF. We write

$$M(s) = S(s)^{\nu_2} K(s) S(s)^{\nu_1}, \quad \text{with } K(s) = I - p(s+L_H)^{-1} r(s+L_h).$$

We have to study $\varrho(M(s)) = \varrho(K(s)S(s)^{\nu})$, $\nu = \nu_1 + \nu_2$.
Let e_m $(m = 1, \ldots, N-1)$ denote the eigenvectors of L_h,

$$e_m(x) = \sin(m\pi x), x = kh, 1 \leqslant k \leqslant N-1 \quad (h = 1/N).$$

Both $K(s)$ and $S(s)^{\nu}$ leave the subspace spanned by (e_m, e_{N-m}) invariant, and their resctrictions have with respect to this basis the matrix representation (see e.g. [7], Ch. 2 for similar calculations)

$$K_m(s) = \begin{pmatrix} s_m^2 & c_m^2 \\ s_m^2 & c_m^2 \end{pmatrix} + \frac{sh^2/2}{sh^2/2 + 2c_m^2 s_m^2} \begin{pmatrix} c_m^2 s_m^2 & -c_m^4 \\ -s_m^4 & c_m^2 s_m^2 \end{pmatrix},$$

$$S_m(s)^{\nu} = \frac{(c_m^2 - s_m^2)^{2\nu-1}}{(1+sh^2/2)^{2\nu}} \left[\begin{pmatrix} c_m^2 & c_m^2 \\ -s_m^2 & -s_m^2 \end{pmatrix} + \frac{sh^2/2}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right],$$

where $c_m = \cos(mh\pi/2)$, $s_m = \sin(mh\pi/2)$, and $m = 1, \ldots, N/2-1$. (We omit the degenerate case $m = N/2$, which does not lead to difficulties.) One then has for $\nu \geqslant 1$ and $z = sh^2/2$

$$M_m(s) = K_m(s) S_m(s)^{\nu} = \tfrac{1}{2}(c_m^2 - s_m^2)^{2\nu-1} \frac{z}{(1+z)^{2\nu}} \left[-(c_m^2 - s_m^2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \right.$$

$$\left. + \frac{z + 4c_m^2 s_m^2}{z + 2c_m^2 s_m^2} \begin{pmatrix} c_m^2 & c_m^2 \\ -s_m^2 & -s_m^2 \end{pmatrix} \right]$$

whose spectral radius equals

$$\varrho(M_m(s)) = \left| \tfrac{1}{2}(c_m^2 - s_m^2)^{2\nu} \frac{z}{(1+z)^{2\nu}} \cdot \frac{z}{z + 2c_m^2 s_m^2} \right|.$$

We obtain

$$\varrho(\mathcal{M}) = \max_{\mathrm{Re}\, s \geqslant 0} \varrho(M(s)) = \max_{\mathrm{Re}\, s \geqslant 0} \max_m \varrho(M_m(s)) \leqslant \max_{\mathrm{Re}\, z \geqslant 0} \frac{1}{2} \left| \frac{z}{(1+z)^{2\nu}} \right|$$

$$= \tfrac{1}{2} \sqrt{[\eta_0(2\nu-1)]},$$

where the last maximum is attained for $z = \pm i/\sqrt{(2\nu-1)}$ (cf. (3.8)). ∎

REMARK. On $L^2(\mathbb{R}_+, \mathbb{R}^{N-1})$ with Euclidean norm on $\mathbb{R}^{N-1}$ one has the (non-

optimal) estimate

$$\|\mathcal{M}\| \leqslant \tfrac{1}{2}\sqrt{2}\sqrt{[\eta_0(2\nu-2)]} \quad \text{for} \quad \nu = \nu_1+\nu_2, \quad \text{if} \quad \nu_1, \nu_2 \geqslant 1.$$

(The optimal bound is greater than $\frac{1}{2}\sqrt{2}\sqrt{[\eta_0(2\nu-1)]}$).

Both in Propositions 3 and 5 the maximum is attained for $h \to 0$ and for the extreme frequencies, i.e. $m = n = 1$ (in contrast to the elliptic problem). If one *supposes* this to be true also for the two-grid operator $\mathcal{M}$ of the *two-dimensional* heat equation on a square, with nine-point restriction and prolongation, then a straightforward analysis yields

$$\varrho(\mathcal{M}) = \varrho(Q\mathcal{S}^\nu)+O(h^2).$$

In view of Proposition 3, it thus appears reasonable to conjecture that the spectral radius estimate of Proposition 5 remains valid also in the two-dimensional case. Similar behaviour may be expected for the L^2 norm of the two-grid operator.

5. More general convergence results.

We now extend some of the convergence proofs of Hackbusch [7], Ch. 6 to parabolic problems. We let $\nu_2 = 0$ and rewrite (2.8) as

$$M(s) = (I-p(s+L_H)^{-1}r(s+L_h))L_h^{-1}\cdot L_h S(s)^\nu.$$

Suppose we can show the two properties

$$\|(I-p(s+L_H)^{-1}r(s+L_h))L_h^{-1}\| \leqslant C\cdot h^2 \tag{5.1}$$

and

$$\|L_h S(s)^\nu\| \leqslant \eta(\nu)\cdot h^{-2}, \eta(\nu)\to 0 \quad \text{as} \quad \nu\to\infty, \tag{5.2}$$

with C and $\eta(\nu)$ independent of h and s with $\operatorname{Re} s \geqslant 0$.

(For simplicity we assume that the norms in (5.1) and (5.2) are Euclidean norms.) For sufficiently large ν we then have $\|M(s)\| \leqslant \varrho < 1$ for $\operatorname{Re} s \geqslant 0$, and Propositions 1 and 2 yield convergence of two-grid dynamic iteration in $L^p(\mathbb{R}_+, \mathbb{R}^n)$.

Concerning the approximation property (5.1) we have the following.

PROPOSITION 6. *Assume* $rp = I$, $r = p^*$, L_H *symmetric positive definite. Then*

$$\|(I-p(s+L_H)^{-1}r(s+L_h))L_h^{-1}\| \leqslant \|L_h^{-1}-pL_H^{-1}r\| \quad \textit{for } \operatorname{Re} s \geqslant 0.$$

REMARK. Observe that the term on the right-hand side is the one which is estimated for the approximation property of the elliptic problem ($s = 0$), see [7], Ch. 6.3.

PROOF. Since $rp = I$, we have

$$p(s+L_H)^{-1}r = (I+s\cdot pL_H^{-1}r)^{-1}\cdot pL_H^{-1}r.$$

It follows that

$$\begin{aligned} &I-p(s+L_H)^{-1}r(s+L_h))L_h^{-1} = \\ &= [(I+sL_h^{-1})^{-1}L_h^{-1}-(I+s\cdot pL_H^{-1}r)^{-1}pL_H^{-1}r](I+sL_h^{-1}) = \\ &= (I+s\cdot pL_H^{-1}r)^{-1}\cdot(L_h^{-1}-pL_H^{-1}r). \end{aligned}$$

Under the assumptions made, $pL_H^{-1}r$ is symmetric positive semi-definite, and hence

$$\|(I+s\cdot pL_H^{-1}r)^{-1}\| \leqslant 1 \quad \text{for} \quad \operatorname{Re} s \geqslant 0. \qquad \blacksquare$$

We now turn to the smoothing property (5.2).

We assume that L_h can be written in the form (cf. [7], p. 126)

$$L_h = d\begin{bmatrix} I & -b \\ -a & I \end{bmatrix}, \quad L_h \text{ symmetric positive definite.} \tag{5.3}$$

PROPOSITION 7. *Gauss-Seidel dynamic iteration with* (5.3) *satisfies for* $v \geqslant 1$

$$\|L_hS(s)^v\| \leqslant d\cdot\sqrt{[\eta_0(2v-1)]} \quad \text{for} \quad \operatorname{Re} s \geqslant 0.$$

REMARK. The static iteration satisfies ([7], Prop. 6.2.22)

$$\|L_hS(0)^v\| \leqslant d\cdot\eta_0(v-\tfrac{1}{2}).$$

PROOF. With $z = s/d$ we have

$$S(s)^v = \frac{1}{(1+z)^{2v}}\begin{bmatrix} 0 & (1+z)bc^{v-1} \\ 0 & c^v \end{bmatrix}, \text{ where } c = ab = b^Tb, \text{ and}$$

$$L_hS(s)^v = d\cdot\frac{1}{(1+z)^{2v}}\begin{bmatrix} 0 & (1+z)bc^{v-1}(I-c) \\ 0 & -zc^v \end{bmatrix}.$$

It follows that

$$\|L_h S(s)^\nu\|^2 = d^2 \frac{1}{|1+z|^{4\nu}} \|\,|1+z|^2 c^{2\nu-1}(I-c)^2 + |z|^2 c^{2\nu}\|.$$

Since $0 \leqslant c \leqslant I$ (see [7], p. 127), we have

$$\|L_h S(s)^\nu\| \leqslant d \cdot \max_{\mathrm{Re}\, z \geqslant 0} \max_{0 \leqslant x \leqslant 1} |1+z|^{-2\nu} \left(|1+z|^2 x^{2\nu-1}(1-x)^2 + |z|^2 x^{2\nu}\right)^{1/2}.$$

One finds that the maximum is attained at $x = 1$, $z = \pm i/\sqrt{(2\nu-1)}$, which gives the estimate. ■

In contrast to Gauss-Seidel, the smoothing estimate for damped Jacobi dynamic iteration is again the same as for the stationary problem.

PROPOSITION 8. *Assume L_h symmetric positive definite with constant diagonal dI, and $\omega \leqslant d/\|L_h\|$. Then damped Jacobi dynamic iteration* (2.5) *with relaxation parameter ω satisfies for $\nu \geqslant 1$*

$$\|L_h S(s)^\nu\| \leqslant \|L_h S(0)^\nu\| \leqslant \frac{d}{\omega} \cdot \eta_0(\nu) \quad \text{for} \quad \mathrm{Re}\, s \geqslant 0.$$

PROOF. The result follows from

$$S(s) = (1 + s\omega/d)^{-1} S(0)$$

and Proposition 6.2.11 in [7]. ■

6. Time discretization.

In this section we consider the special case of time-discretization of (1.2) and (1.4) in such a way that all the differential equations are discretized by the *same* method, with the *same* constant step-size Δt. Hackbusch's [6] method is obtained when the resulting algorithm is used over a time interval of length $k\Delta t, k$ fixed. The results below yield optimal bounds for the convergence rate which are independent of $k, \Delta t$ and h. Further, the case of constant step-size gives, as conditio sine qua non, an indication to the choice of suitable ODE methods also for more general cases.

For simplicity of presentation, let us begin with the implicit Euler scheme, applied to (1.2). The discretization of the smoothing step then reads

$$(6.1) \qquad \frac{1}{\Delta t}(u_{n+1}^{(k+1)} - u_n^{(k+1)}) + (D-A)u_{n+1}^{(k+1)} - Bu_{n+1}^{(k)} = f((n+1)\Delta t), \quad n \geqslant 0.$$

The error of the solution $u = (u_n)_0^\infty$ in the Euler discretization of (1.1),

$e^{(k)} = u^{(k)} - u$, then satisfies (6.1) with $f \equiv 0$. Hence

$$e^{(k+1)} = \mathcal{S}_{\Delta t} e^{(k)}, \tag{6.2}$$

where $\mathcal{S}_{\Delta t}$ is the discrete convolution operator

$$\mathcal{S}_{\Delta t} e_n = \sum_{j=0}^{n} S_{n-j} e_j \quad (n \geq 0) \tag{6.3}$$

whose kernel $(S_n)_0^\infty$ is given via its generating power series (discrete Laplace transform)

$$\sum_{n=0}^{\infty} S_n \zeta^n = \left(\frac{1-\zeta}{\Delta t} + D - A \right)^{-1} B = S\left(\frac{1-\zeta}{\Delta t} \right). \tag{6.4}$$

with $S(s)$ of (2.4).

When also the coarse-grid equation (1.4) is discretized by implicit Euler with the same step-size Δt, one obtains for the error $e^i = u^i - u$ of the fully discrete two-grid iteration

$$e^{i+1} = \mathcal{M}_{\Delta t} e^i, \tag{6.6}$$

where $\mathcal{M}_{\Delta t}$ is the discrete convolution operator

$$\mathcal{M}_{\Delta t} e_n = \sum_{j=0}^{n} M_{n-j} e_j \qquad (n \geq 0) \tag{6.7}$$

whose kernel $(M_n)_0^\infty$ has the discrete Laplace transform

$$\sum_{0}^{\infty} M_n \zeta^n = M\left(\frac{1-\zeta}{\Delta t} \right) \quad \text{(implicit Euler)}, \tag{6.8a}$$

with $M(s)$ of (2.8).

If instead one uses the trapezoidal rule (Crank-Nicolson method) for time discretization of (1.2), (1.4), one still has (6.7), with (6.8a) replaced by

$$\sum_{0}^{\infty} M_n \zeta^n = M\left(\frac{2}{\Delta t} \frac{1-\zeta}{1+\zeta} \right) \quad \text{(trapezoidal rule)}. \tag{6.8b}$$

The k-step backward differentiation formula yields

$$\sum_{0}^{\infty} M_n \zeta^n = M\left(\frac{1}{\Delta t} \sum_{j=0}^{k} \frac{1}{j} (1-\zeta)^j \right) \quad \text{(BDF)}. \tag{6.8c}$$

For a general linear multistep method $\sum_0^k \alpha_j y_{n+j} = \Delta t \cdot \sum_0^k \beta_j f_{n+j}$ one has

$$\sum_{0}^{\infty} M_n \zeta^n = M\left(\frac{1}{\Delta t} \delta(\zeta) \right), \quad \text{with } \delta(\zeta) = \frac{\alpha_k + \alpha_{k-1}\zeta + \cdots + \alpha_0 \zeta^k}{\beta_k + \beta_{k-1}\zeta + \cdots + \beta_0 \zeta^k}. \tag{6.8d}$$

(For formulas (6.7), (6.8) to hold it is assumed that one does not iterate on the k given starting values, which are denoted by negative subscripts; hence $e^i_{-k} = \cdots = e^i_{-1} = 0$.)

We have the following result on two-grid convergence of the fully discrete method.

PROPOSITION 9. *Consider* $\mathcal{M}$ *of* (2.7), (2.8) *as an operator on* $l^p(\mathbb{R}_+, \mathbb{C}^N)$, $(1 \leqslant p \leqslant \infty)$, *and* $\mathcal{M}_{\Delta t}$ *of* (6.7), (6.8) *as an operator on* $l^p(\mathbb{N}, \mathbb{C}^n)$.

a) If the linear multistep method is A-stable, then the spectral radius of the fully discrete two-grid operator is bounded by that of the semidiscrete two-grid operator,

$$\varrho(\mathcal{M}_{\Delta t}) \leqslant \varrho(\mathcal{M}) = \max_{\operatorname{Re} s \geqslant 0} \varrho(M(s)).$$

b) If the linear multistep method is A(α)-stable, then

$$\varrho(\mathcal{M}_{\Delta t}) \leqslant \max_{s \in \Sigma_{\pi-\alpha}} \varrho(M(s))$$

with the sector $\Sigma_{\pi-\alpha} = \{s; |\arg s| \leqslant \pi - \alpha\} \cup \{0\}$.

These are the optimal bounds which hold without restriction on the ratio $\Delta t/h^2$. *On* l^2 *the corresponding statements hold also for the norm.*

PROOF. Part a) is the special case $\alpha = \pi/2$ of b), combined with Proposition 1. To prove b), we use the discrete version of the Paley-Wiener theorem (cf. [17], p. 246) and find in complete analogy to Proposition 1,

$$\varrho(\mathcal{M}_{\Delta t}) = \max_{|\zeta| \leqslant 1} \varrho\left(M\left(\frac{1}{\Delta t}\delta(\zeta)\right)\right).$$

The result then follows by noting that the linear multistep method is $A(\alpha)$-stable if and only if

$$\delta(\zeta) \in \textstyle\sum_{\pi-\alpha} \cup \{\infty\} \quad \text{for} \quad |\zeta| \leqslant 1. \qquad \blacksquare$$

In previous sections we have seen that the number $\max\{|z/(1+z)^{2\nu}|; \operatorname{Re} z \geqslant 0\} = \sqrt{[\eta_0(2\nu - 1)]}$ essentially determines the smoothing behaviour of Gauss-Seidel dynamic iteration. For a time-discretization with an $A(\alpha)$-stable multistep method the relevant quantity becomes $\max\{|z/(1+z)^{2\nu}|; z \in \sum_{\pi-\alpha}\}$. For backward differentiation formulas these numbers are given below.

Finally we show in Proposition 10 that also for algebraically stable Runge-Kutta methods (e.g. collocation methods with Radau points) the spectral radius and (suitable) norm of the corresponding two-grid operator are bounded by those of the semi-discrete two-grid operator $\mathcal{M}$. We shall use the techniques of [9].

Table 2. $\max\{|z/(1+z)^{2\nu}|;\, z \in \sum_{\pi-\alpha}\}$ *for BDF methods.*

order	α	ν 1	2	3	4
1,2	90°	0.5	0.325	0.259	0.222
3	88°	0.518	0.345	0.280	0.243
4	73°	0.707	0.600	0.585	0.596
5	51°	1.349	2.000	3.173	5.133
6	18°	10.2	105	1101	11512

Consider an m-stage implicit RK-method, applied to (2.1):

$$(6.9)\quad \begin{cases} e_{ni}^{(k+1)} = e_n^{(k+1)} + \Delta t \cdot \sum_{j=1}^{m} a_{ij}[-(D-A)e_{nj}^{(k+1)} + Be_{nj}^{(k)}], & i = 1,\ldots,m;\ n \geqslant 0, \\ e_{n+1}^{(k+1)} = e_n^{(k+1)} + \Delta t \cdot \sum_{j=1}^{m} b_j[-(D-A)e_{nj}^{(k+1)} + Be_{nj}^{(k)}], & e_0^{(k+1)} = 0. \end{cases}$$

(Note that only the internal stages $e_{nj}^{(k)}$ of the previous iteration are used.) The method is algebraically stable [3], [5] if its coefficients satisfy

$$(6.10)\quad \begin{cases} b_i > 0 \qquad i = 1,\ldots,m \\ (b_i a_{ij} + b_j a_{ji} - b_i b_j)_{i,j=1}^{m} \quad \text{is a positive semi-definite matrix.} \end{cases}$$

We introduce the generating power series

$$E(\zeta) = \sum_{n=0}^{\infty} E_n \zeta^n, \quad \text{with } E_n = (e_{n1},\ldots,e_{nm})^T,$$

$$e(\zeta) = \sum_{n=0}^{\infty} e_n \zeta^n,$$

and denote the RK coefficient matrices

$$\mathscr{A} = (a_{ij})_{i,j=1}^{m}, \quad \mathscr{B} = (b_j)_{i,j=1}^{m} = \mathbb{1} b^T,$$

$$\text{with } b = (b_1,\ldots,b_m)^T, \qquad \mathbb{1} = (1,\ldots,1)^T.$$

Multiplying (6.9) by ζ^n and summing up, we then have

$$E^{(k+1)}(\zeta) = \mathbb{1} \otimes e^{(k+1)}(\zeta) + \Delta t[-\mathscr{A} \otimes (D-A) \cdot E^{(k+1)}(\zeta) + \mathscr{A} \otimes B \cdot E^{(k)}(\zeta)]$$

$$\zeta^{-1} e^{(k+1)}(\zeta) = e^{(k+1)}(\zeta) + \Delta t[-b^T \otimes (D-A) \cdot E^{(k+1)}(\zeta) + b^T \otimes B \cdot E^{(k)}(\zeta)].$$

Eliminating $e^{(k+1)}(\zeta)$, we obtain

$$(6.11)\quad [I + \Delta t \cdot R(\zeta) \otimes (D-A)]E^{(k+1)}(\zeta) = \Delta t \cdot R(\zeta) \otimes B \cdot E^{(k)}(\zeta),$$

where

$$R(\zeta) = \mathscr{A} + \frac{\zeta}{1-\zeta}\mathscr{B} \tag{6.12}$$

depends only on the coefficients of the Runge-Kutta method. With the transformation $z = 1/s$,

$$\tilde{S}(z) = S(1/z) = [I + z(D-A)]^{-1} \cdot zB,$$

with $S(s)$ of (2.4), we can rewrite (6.11) as

$$E^{(k+1)}(\zeta) = \tilde{S}(\Delta t \cdot R(\zeta)) \cdot E^{(k)}(\zeta),$$

in analogy to (6.3), (6.4). Extending this analysis, one finds for the generating power series of the internal stages after a two-grid cycle (cf. (6.7), (6.8))

$$E^{i+1}(\zeta) = \tilde{M}(\Delta t \cdot R(\zeta)) \cdot E^{i}(\zeta), \quad \text{with} \tag{6.13}$$

$$\tilde{M}(z) = M(1/z), \qquad M(s) \quad \text{of (2.8).} \tag{6.14}$$

Introducing the two-grid operator $\mathscr{M}_{\Delta t}$ as before, we have for $E^i = (E_n^i)_{n=0}^{\infty}$

$$E^{i+1} = \mathscr{M}_{\Delta t} E^i, \quad \text{with} \tag{6.15}$$

$$\mathscr{M}_{\Delta t} E_n = \sum_{j=0}^{n} M_{n-j} E_j, \qquad \sum_{n=0}^{\infty} M_n \zeta^n = \tilde{M}(\Delta t \cdot R(\zeta)). \tag{6.16}$$

We consider $\mathscr{M}_{\Delta t}$ as an operator on $l^p(\mathbb{N}, (\mathbb{C}^N)^m)$ for $1 \leqslant p \leqslant \infty$.

Assuming (6.10), we endow $(\mathbb{C}^N)^m$ in the case $p = 2$ with the norm

$$\|w\|^2 = \sum_{i=1}^{m} b_i \|w_i\|^2 \quad \text{for} \quad w = (w_i)_{i=1}^{m}, \qquad w_i \in \mathbb{C}^N, \tag{6.17}$$

where the norm on the right-hand side is some inner product norm on $\mathbb{C}^N$. We then have the following.

PROPOSITION 10. *Suppose that the Runge-Kutta method is algebraically stable.*

a) Consider $\mathscr{M}$ of (2.7), (2.8) *as an operator on $L^p(\mathbb{R}_+, \mathbb{C}^N)$ $(1 \leqslant p \leqslant \infty)$, and $\mathscr{M}_{\Delta t}$ of* (6.14), (6.16) *as an operator on $l^p(\mathbb{N}, (\mathbb{C}^N)^m)$. Then the spectral radii satisfy*

$$\varrho(\mathscr{M}_{\Delta t}) \leqslant \varrho(\mathscr{M}) \quad \textit{for} \quad \Delta t > 0.$$

b) Consider $\mathcal{M}$ and $\mathcal{M}_{\Delta t}$ respectively as operators on $L^2(\mathbb{R}_+, \mathbb{C}^N)$ and $l^2(\mathbb{N}, (\mathbb{C}^N)^m)$, where $\mathbb{C}^N$ and $(\mathbb{C}^N)^m$ are endowed with the norms in (6.17). *Then*

$$\|\mathcal{M}_{\Delta t}\| \leqslant \|\mathcal{M}\| \qquad \text{for} \quad \Delta t > 0.$$

PROOF. We begin by studying $R(\zeta)$ of (6.12). As in [9], we rewrite (6.12) as

$$R(\zeta) = \mathcal{C} + \frac{1}{2}\frac{1+\zeta}{1-\zeta}\mathcal{B}, \quad \text{with} \quad \mathcal{C} = \mathcal{A} - \tfrac{1}{2}\mathcal{B}.$$

With respect to the inner product on $\mathbb{C}^m$,

$$\langle u, v\rangle = \sum_{i=1}^{m} b_i u_i \bar{v}_i, \qquad u, v \in \mathbb{C}^m$$

condition (6.10) implies that $\mathcal{B} = \mathcal{B}^*$ and $\mathcal{C} + \mathcal{C}^*$ are positive semi-definite, because

$$\langle \mathcal{B}v, v\rangle = \left|\sum_i b_i v_i\right|^2$$

$$\langle (\mathcal{C} + \mathcal{C}^*)v, v\rangle = \sum_{i,j} (b_i a_{ij} + b_j a_{ji} - b_i b_j) v_i \bar{v}_j.$$

Since $\operatorname{Re}((1+\zeta)/(1-\zeta)) \geqslant 0$ for $|\zeta| \leqslant 1$, $\zeta \neq 1$, we obtain

(6.19) $\qquad \operatorname{Re}\langle R(\zeta)v, v\rangle \geqslant 0 \quad \text{for} \quad v \in \mathbb{C}^m \quad \text{and} \quad |\zeta| \leqslant 1, \qquad \zeta \neq 1.$

a) As in the proof of Proposition 9,

$$\varrho(\mathcal{M}_{\Delta t}) = \max_{|\zeta| \leqslant 1} \varrho(\tilde{M}(\Delta t \cdot R(\zeta)).$$

By (6.19), the eigenvalues of $R(\zeta)$ have nonnegative real part, and hence

$$\varrho(\tilde{M}(\Delta t R(\zeta))) \leqslant \max_{\operatorname{Re} z \geqslant 0} \varrho(\tilde{M}(z)) = \max_{\operatorname{Re} s \geqslant 0} \varrho(M(s)) = \varrho(\mathcal{M}),$$

using Proposition 1 for the last equality.

b) From Parseval's formula,

$$\|\mathcal{M}_{\Delta t}\| \leqslant \max_{|\zeta| \leqslant 1} \|\tilde{M}(\Delta t R(\zeta))\|,$$

where the norm on the right-hand side is that given by (6.17). By a matrix-valued version of a theorem of von Neumann (see [13], Thm. 1, [8], Thm. 4) (6.19) implies

$$\|\tilde{M}(\Delta t R(\zeta))\| \leqslant \max_{\mathrm{Re}\, z \geqslant 0} \|\tilde{M}(z)\| = \max_{\mathrm{Re}\, s \geqslant 0} \|M(s)\| = \|\mathcal{M}\|,$$

where the last equality holds by Proposition 2. ■

7. Numerical experiment.

In order to illustrate the possibility of choosing different time step-sizes at different spatial nodes, we consider the one-dimensional equation

$$\frac{\partial u}{\partial t} = 0.1 \frac{\partial^2 u}{\partial x^2} + e^{-(x-t)^2}, \qquad 0 \leqslant x \leqslant 10, \qquad 0 \leqslant t \leqslant 12,$$

with homogeneous Neumann boundary conditons, and initial value $u(x, 0) = 0$. We divided the time interval into subintervals of length $\Delta T = 2$, and applied on each subinterval a full multi-grid V-cycle version of the algorithm of Section 1, with $\nu_1 = 2$, $\nu_2 = 1$ and the data of Section 4. For time discretization we used the implicit Euler method. To keep the number of required (linear)

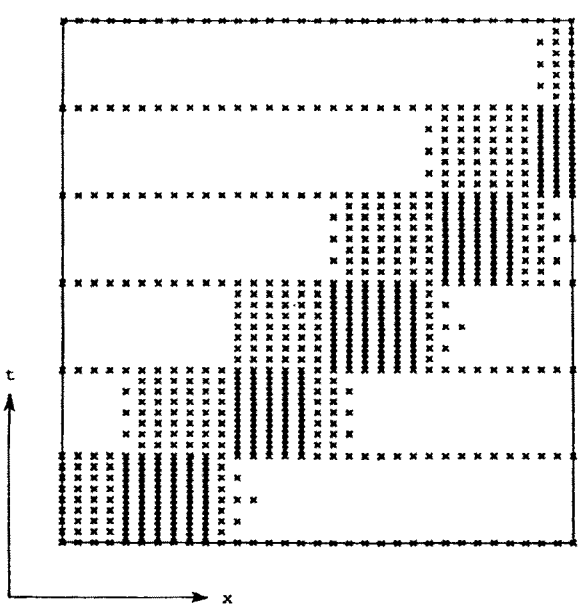

Fig. 1. Grid-points in the (x, t)-plane.

interpolations low we admitted, independently at each spatial node, only step-sizes of interval length divided by powers of 2. The step-size was controlled by comparing the numerical solutions corresponding to step-sizes Δt and $\Delta t/2$ as usual, with a test also for the error of linear interpolation. At each level this was done only for the first smoothing iteration, and the step-sizes chosen there were also taken for the subsequent smoothing steps. The use of the correction scheme (instead of the full approximation scheme) was advantageous for allowing larger time steps at coarser levels. In Figure 1 we have indicated the positions of grid-points in the (x, t)-plane which were chosen for the finest level ($N = 32$), with a tolerance of 10^{-2} for time integration in the Gauss-Seidel steps. The numerical solution agreed well with that obtained from other methods (which are, of course, more efficient for one-dimensional problems).

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