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Accelerated Hermitian and skew-Hermitian splitting iteration methods for saddle-point problems

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We establish a class of accelerated Hermitian and skew-Hermitian splitting (AHSS) iteration methods for large sparse saddle-point problems by making use of the Hermitian and skew-Hermitian splitting (HSS) iteration technique. These methods involve two iteration parameters whose special choices can recover the known preconditioned HSS iteration methods, as well as yield new ones. Theoretical analyses show that the new methods converge unconditionally to the unique solution of the saddle-point problem. Moreover, the optimal choices of the iteration parameters involved and the corresponding asymptotic convergence rates of the new methods are computed exactly. In addition, theoretical properties of the preconditioned Krylov subspace methods such as GMRES are investigated in detail when the AHSS iterations are employed as their preconditioners. Numerical experiments confirm the correctness of the theory and the effectiveness of the methods.

Keywords: saddle-point problem; Hermitian and skew-Hermitian splitting; splitting iteration method; preconditioning.

1. Introduction

We consider the iterative solution of large sparse saddle-point problems of the form

$$Ax \equiv \begin{bmatrix} B & E \\ -E^* & O \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv b, \tag{1.1}$$

where $B \in \mathbb{C}^{p\times p}$ is Hermitian positive definite, $O \in \mathbb{C}^{q\times q}$ is zero, $E \in \mathbb{C}^{p\times q}$ has full column rank, $p \geqslant q$, $f \in \mathbb{C}^p$ and $g \in \mathbb{C}^q$. These assumptions guarantee the existence and uniqueness of the solution of the system of linear equations (1.1).

Saddle-point problems of the form (1.1) correspond to the Kuhn–Tucker conditions for linearly constrained quadratic programming problems. Such systems typically result from mixed or hybrid

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finite-element approximations of second-order elliptic problems, elasticity problems or the Stokes equations (see, e.g. Brezzi & Fortin, 1991) and from Lagrange multiplier methods (see, e.g. Fortin & Glowinski, 1983). A number of structured preconditioners (Elman & Schultz, 1986; Elman *et al.*, 2002; Murphy *et al.*, 2000; Perugia & Simoncini, 2000; Bai, 2006) and iterative methods (Eisenstat *et al.*, 1983; Bramble *et al.*, 1997; Golub & Wathen, 1998; Hu & Zou, 2002, 2006; Bai *et al.*, 2005b) have been studied in the literature for these problems. We refer to Benzi *et al.* (2005) for a comprehensive survey (see also Wathen & Silvester, 1993; Golub & Vanderstraeten, 2000; Golub *et al.*, 2002; Golub & Greif, 2003; Bai & Li, 2003, and the references therein).

Recently, Benzi & Golub (2004) discussed the convergence and the preconditioning properties of the 'Hermitian and skew-Hermitian splitting' (HSS) iteration method (Bai *et al.*, 2003), when it is used to solve the saddle-point problem (1.1). To better reveal the theoretical property of this HSS iteration method, by scaling the problem (1.1) with a block-diagonal matrix (or a preconditioning matrix) and applying the HSS iteration technique (Bai *et al.*, 2003; Wang & Bai, 2001) to the correspondingly resulted linear system, Bai *et al.* (2004) studied a class of 'preconditioned Hermitian and skew-Hermitian splitting' (PHSS) iteration methods for solving the saddle-point problem (1.1). Both theory and experiments have shown that these methods are very efficient (see also Benzi *et al.*, 2003, and Bertaccini *et al.*, 2005). The PHSS iteration methods are only single-parameter ones. However, their two-parameter variants, called the 'accelerated Hermitian and skew-Hermitian splitting' (AHSS) iteration methods, should be meaningful and practical from the viewpoint of both theory and applications, because, without introducing extra computational workload, the AHSS iteration methods can not only algorithmically generalize the PHSS iteration methods to obtain rapidly convergent iterative schemes but also considerably decrease their numerical sensitivity with respect to the iteration parameters, in particular, when the two parameters involved are close to their optimal values.

In this paper, we present a class of AHSS iteration methods for solving the large sparse saddle-point problem (1.1). These methods are two-parameter generalizations of the PHSS iteration methods studied in Bai *et al.* (2004), and they can recover the PHSS methods as well as yielding new ones by suitable choices of the two arbitrary parameters. Theoretical analyses show that for all positive parameters, like PHSS, the AHSS iteration methods converge unconditionally to the unique solution of the saddle-point problem (1.1), and their convergence speeds are only dependent on the positive singular values of the scaled matrix of the submatrix E and are independent of the eigenvectors of the matrices involved. In addition, we compute the optimal iteration parameters and the corresponding optimal convergence factor of the AHSS iteration method, which shows that the asymptotic convergence rate of the optimal AHSS iteration method is much faster than that of the optimal PHSS iteration method and, therefore, the former will be more effective than the latter in actual applications. This fact is further confirmed by numerical experiments which use PHSS and AHSS as both solvers and preconditioners to GMRES (Saad & Schultz, 1986; Saad, 1996). We also discuss in detail the use of AHSS as a preconditioner to Krylov subspace methods such as GMRES, and demonstrate the asymptotic convergence rate of the preconditioned GMRES as well as the real positivity of the preconditioned matrix.

The organization of this paper is as follows: in Section 2, we present the algorithmic description of the AHSS iteration method; in Section 3, we prove the unconditional convergence property, exactly compute the optimal iteration parameters and the corresponding optimal convergence factor of the AHSS iteration method; the utilization of AHSS as a preconditioner to Krylov subspace methods such as GMRES is investigated in Section 4; some numerical results are given in Section 5; in Section 6, we further extend the AHSS iteration method to the generalized saddle-point problems and, finally, in Section 7, to end this paper we discuss our conclusions.

2. The AHSS iterations

For simplicity and without loss of generality, we first transform the saddle-point problem (1.1) into an equivalent form (see Bai *et al.*, 2004). To this end, we let $W \in \mathbb{C}^{p\times p}$ be a non-singular matrix such that $W^*BW = I_{p\times p}$, the p-by-p identity matrix. For example, we may take $W = B^{-\frac{1}{2}}$ or $W = L^{-1}$ if $B = L^*L$ is the Cholesky factorization of B, where L is a lower triangular matrix (see Golub & Van Loan, 1996, Chapter 4). Let $Z \in \mathbb{C}^{q\times q}$ be another non-singular matrix and denote by

$$\overline{E} = W^*EZ, \quad C = (Z^*)^{-1}Z^{-1}, \tag{2.1}$$

$$T = \begin{bmatrix} W & O \\ O & Z \end{bmatrix}, \quad \overline{A} \equiv T^*AT = \begin{bmatrix} I & \overline{E} \\ -\overline{E}^* & O \end{bmatrix} \tag{2.2}$$

and

$$\overline{x} \equiv \begin{bmatrix} \overline{y} \\ \overline{z} \end{bmatrix} = T^{-1}x = \begin{bmatrix} W^{-1}y \\ Z^{-1}z \end{bmatrix}, \quad \overline{b} \equiv \begin{bmatrix} \overline{f} \\ \overline{g} \end{bmatrix} = T^*b = \begin{bmatrix} W^*f \\ Z^*g \end{bmatrix},$$

where I is the identity matrix. Then the saddle-point problem (1.1) is equivalent to the linear system

$$\overline{A}\overline{x} = \overline{b}. \tag{2.3}$$

The coefficient matrix $\overline{A} \in \mathbb{C}^{(p+q)\times(p+q)}$ can be, respectively, split into its Hermitian and skew-Hermitian parts as $\overline{A} = \overline{H} + \overline{S}$, where

$$\overline{H} = \frac{1}{2}(\overline{A} + \overline{A}^*) = \begin{bmatrix} I & O \\ O & O \end{bmatrix}, \quad \overline{S} = \frac{1}{2}(\overline{A} - \overline{A}^*) = \begin{bmatrix} O & \overline{E} \\ -\overline{E}^* & O \end{bmatrix}.$$

Note that $\overline{H}$ is Hermitian positive ‘semidefinite’.

Now, by applying the HSS iteration technique to (2.3), we then obtain the iteration scheme

$$\begin{cases} (\Lambda + \overline{H})\overline{x}^{(k+\frac{1}{2})} = (\Lambda - \overline{S})\overline{x}^{(k)} + \overline{b}, \\ (\Lambda + \overline{S})\overline{x}^{(k+1)} = (\Lambda - \overline{H})\overline{x}^{(k+\frac{1}{2})} + \overline{b}, \end{cases} \tag{2.4}$$

where

$$\Lambda = \begin{bmatrix} \alpha I_{p\times p} & O \\ O & \beta I_{q\times q} \end{bmatrix}, \quad \text{with } \alpha \text{ and } \beta \text{ positive constants.}$$

The iteration scheme (2.4) is evidently different from the existing HSS iteration methodology as it includes two arbitrary parameters. It is obvious that when $\alpha \neq \beta$,

$$\mathcal{Q}(\alpha, \beta) = (\Lambda + \overline{S})^{-1}(\Lambda - \overline{S}) \tag{2.5}$$

is ‘no longer’ a Cayley transform and thus not unitary, even if $\alpha, \beta > 0$ and $\overline{S}$ is skew-Hermitian; see Remark (i) in Section 3. By making use of the definitions of $\overline{H}$ and $\overline{S}$, after straightforward computations, we can rewrite (2.4) in the original variable as

$$\begin{bmatrix} \alpha B & E \\ -E^* & \beta C \end{bmatrix} \begin{bmatrix} y^{(k+1)} \\ z^{(k+1)} \end{bmatrix} = \begin{bmatrix} \frac{\alpha(\alpha-1)}{\alpha+1}B & -\frac{\alpha-1}{\alpha+1}E \\ E^* & \beta C \end{bmatrix} \begin{bmatrix} y^{(k)} \\ z^{(k)} \end{bmatrix} + \begin{bmatrix} \frac{2\alpha}{\alpha+1}f \\ 2g \end{bmatrix}, \tag{2.6}$$

which results in the following AHSS iteration method for solving the saddle-point problem (1.1).

The AHSS iteration method: Given an initial guess $x^{(0)} = (y^{(0)^*}, z^{(0)^*})^* \in \mathbb{C}^n$. For $k = 0, 1, 2, \ldots,$ until $\{x^{(k)}\} = \{(y^{(k)^*}, z^{(k)^*})^*\} \subset \mathbb{C}^n$ converges, compute the next iterate $x^{(k+1)} = (y^{(k+1)^*}, z^{(k+1)^*})^*$ by solving the linear system (2.6), where α and β are given positive constants.

We remark that when $\alpha = \beta > 0$, the above AHSS iteration method naturally reduces to the PHSS iteration method in Bai *et al.* (2004) (see also Benzi & Golub, 2004). When $\alpha \neq \beta$, different choices of α and β can yield many new methods for the saddle-point problem (1.1). Here, we refer to Hu & Zou (2002, 2006) and Bai *et al.* (2005b) for a similar approach of using the two-parameter acceleration technique in the algorithmic designs of the Uzawa-type iteration methods.

From (2.6), we easily know that the AHSS iteration method can be equivalently rewritten as

$$\begin{bmatrix} y^{(k+1)} \\ z^{(k+1)} \end{bmatrix} = \mathcal{L}(\alpha, \beta) \begin{bmatrix} y^{(k)} \\ z^{(k)} \end{bmatrix} + \mathcal{N}(\alpha, \beta) \begin{bmatrix} f \\ g \end{bmatrix}, \tag{2.7}$$

where

$$\mathcal{L}(\alpha, \beta) = \begin{bmatrix} \alpha B & E \\ -E^* & \beta C \end{bmatrix}^{-1} \begin{bmatrix} \frac{\alpha(\alpha-1)}{\alpha+1} B & -\frac{\alpha-1}{\alpha+1} E \\ E^* & \beta C \end{bmatrix} \tag{2.8}$$

and

$$\mathcal{N}(\alpha, \beta) = \begin{bmatrix} \alpha B & E \\ -E^* & \beta C \end{bmatrix}^{-1} \begin{bmatrix} \frac{2\alpha}{\alpha+1} I & O \\ O & 2I \end{bmatrix}.$$

Here, $\mathcal{L}(\alpha, \beta)$ is the iteration matrix of the AHSS iteration. In fact, (2.7) may also result from the splitting

$$A = M(\alpha, \beta) - N(\alpha, \beta) \tag{2.9}$$

of the coefficient matrix A, with

$$M(\alpha, \beta) = \begin{bmatrix} \frac{\alpha+1}{2} B & \frac{\alpha+1}{2\alpha} E \\ -\frac{1}{2} E^* & \frac{\beta}{2} C \end{bmatrix}, \quad N(\alpha, \beta) = \begin{bmatrix} \frac{\alpha-1}{2} B & -\frac{\alpha-1}{2\alpha} E \\ \frac{1}{2} E^* & \frac{\beta}{2} C \end{bmatrix}.$$

In actual computations, at each iterate of the AHSS iterations, we need to solve a linear system with the coefficient matrix

$$M'(\alpha, \beta) = \begin{bmatrix} \alpha B & E \\ -E^* & \beta C \end{bmatrix} \quad \text{or, equivalently, } M(\alpha, \beta). \tag{2.10}$$

As these matrices are real positive, or in other words, positive definite, we may solve this linear system inexactly by another iteration procedure, e.g. HSS (Bai *et al.*, 2003), RPCG (Bai & Li, 2003) or BTSS (Bai *et al.*, 2005a).[1] This results in an 'inexact accelerated Hermitian/skew-Hermitian splitting' (IAHSS) iteration method for the saddle-point problem (1.1), and has been previously discussed in Bai *et al.* (2003). As a matter of fact, via the AHSS iteration technique, the problem of solving a

[1]RPCG and BTSS are abbreviations of the terms "restrictively preconditioned conjugate gradient" and "block triangular and skew-Hermitian splitting", respectively.

positive-semidefinite linear system is transformed to the one of solving a sequence of positive-definite linear systems; the latter may often possess better numerical properties than the former.

By making use of block-triangular factorization of the matrix $M(\alpha,\beta)$, we can straightforwardly obtain the following algorithmic version of the AHSS iteration method.

Given an initial guess $x^{(0)} = (y^{(0)^*}, z^{(0)^*})^*\in \mathbb{C}^n$ and two positive constants α and β. For $k = 0, 1, 2, \ldots$, until $\{x^{(k)}\} = \{(y^{(k)^*}, z^{(k)^*})^*\} \subset \mathbb{C}^n$ converges, compute the next iterate $x^{(k+1)} = (y^{(k+1)^*}, z^{(k+1)^*})^*$ according to the following procedure:

Step 1: Compute the current residual vector

$$r^{(k)} = f - (By^{(k)} + Ez^{(k)}), \quad s^{(k)} = g + E^* y^{(k)};$$

Step 2: Compute the auxiliary vector

$$u^{(k)} = \frac{2}{\alpha+1} r^{(k)}, \quad v^{(k)} = E^* B^{-1} u^{(k)} + 2s^{(k)};$$

Step 3: Compute the update vector

$$\left(\beta C + \frac{1}{\alpha} E^* B^{-1} E\right) w^{(k)} = v^{(k)}, \quad Bt^{(k)} = u^{(k)} - Ew^{(k)};$$

Step 4: Form the next iterate

$$y^{(k+1)} = y^{(k)} + t^{(k)}, \quad z^{(k+1)} = z^{(k)} + w^{(k)}.$$

So, at each of the AHSS iteration steps, we have to solve two subsystems of linear equations with the coefficient matrix B and one subsystem of linear equations with the coefficient matrix $(\beta C + \frac{1}{\alpha} E^* B^{-1} E)$ that is the Schur complement of the matrix $M'(\alpha,\beta)$ defined in (2.10). This amount of computing cost is the same as for the PCG (Bai & Li, 2003; Bai *et al.*, 2005b) and the Uzawa-like (Hu & Zou, 2002, 2006; Benzi *et al.*, 2005; Bai *et al.*, 2005b) methods for solving the saddle-point problem (1.1). It seems that solving the linear systems of the form $(\beta C + \frac{1}{\alpha} E^* B^{-1} E) w = v$ at each of the AHSS iteration steps is costly and impractical in actual applications. However, by recalling that C is an arbitrary Hermitian positive-definite matrix, we can choose it so that the matrix $\beta C + \frac{1}{\alpha} E^* B^{-1} E$ possesses simple and easily invertible form (e.g. a (block) diagonal matrix, or a product of a (block) lower triangular matrix with a (block) upper triangular matrix) and, hence, the abovementioned linear systems are cheaply solvable.

3. Convergence analysis

By straightforward computations, we can obtain explicit expressions about the eigenvalues of the iteration matrix $\mathcal{L}(\alpha,\beta)$ in (2.8). This result is precisely described in the following lemma.

LEMMA 3.1 Consider the saddle-point problem (1.1). Let $B \in \mathbb{C}^{p\times p}$ be Hermitian positive definite, $E \in \mathbb{C}^{p\times q}$ be of full column rank and $\alpha, \beta > 0$ be given constants. Assume that $C \in \mathbb{C}^{q\times q}$ is a Hermitian positive-definite matrix. If $\overline{\sigma}_k (k = 1, 2, \ldots, q)$ are the positive singular values of the matrix $\overline{E} \in \mathbb{C}^{p\times q}$ in (2.1), then the eigenvalues of the iteration matrix $\mathcal{L}(\alpha,\beta)$, defined in (2.8), of the AHSS iteration method are $\frac{\alpha-1}{\alpha+1}$ with multiplicity $p-q$, and

$$\frac{1}{(\alpha+1)(\alpha\beta+\overline{\sigma}_k^2)}(\alpha(\alpha\beta-\overline{\sigma}_k^2)\pm\sqrt{(\alpha\beta+\overline{\sigma}_k^2)^2-4\alpha^3\beta\overline{\sigma}_k^2}), \quad k=1,2,\ldots,q.$$

Proof. The proof is essentially analogous to the proofs of Lemmas 3.1 and 3.2 in Bai *et al.* (2004), with only technical modifications. For details, we refer the readers to Bai & Golub (2004). □

We remark that the singular values of the matrix $\overline{E}\in\mathbb{C}^{p\times q}$ are exactly the square roots of the eigenvalues of either the matrix $C^{-1}E^*B^{-1}E$ or, equivalently, the matrix $E^*B^{-1}EC^{-1}$.

LEMMA 3.2 Let the conditions in Lemma 3.1 be satisfied. If $\overline{\sigma}_k$ $(k=1,2,\ldots,q)$ are the positive singular values of the matrix $\overline{E}\in\mathbb{C}^{p\times q}$ in (2.1), then the iteration matrix $\mathcal{L}(\alpha,\beta)$ of the AHSS iteration has

- $p-q$ eigenvalues λ with absolute value $|\lambda|=\frac{|\alpha-1|}{\alpha+1}$;
- $2q$ eigenvalues λ such that for $k=1,2,\ldots,q$
  - if $\alpha\beta+\overline{\sigma}_k^2>2\alpha\sqrt{\alpha\beta}\bar{\sigma}_k$, then there is a corresponding eigenvalue λ such that

$$|\lambda|=\frac{\alpha}{\alpha+1}\left(\frac{|\alpha\beta-\overline{\sigma}_k^2|}{\alpha\beta+\overline{\sigma}_k^2}+\sqrt{\frac{1}{\alpha^2}-\frac{4\alpha\beta\overline{\sigma}_k^2}{(\alpha\beta+\overline{\sigma}_k^2)^2}}\right),$$

    and a further corresponding eigenvalue such that

$$|\lambda|=\frac{\alpha}{\alpha+1}\left|\frac{|\alpha\beta-\overline{\sigma}_k^2|}{\alpha\beta+\overline{\sigma}_k^2}-\sqrt{\frac{1}{\alpha^2}-\frac{4\alpha\beta\overline{\sigma}_k^2}{(\alpha\beta+\overline{\sigma}_k^2)^2}}\right|;$$

  - if $\alpha\beta+\overline{\sigma}_k^2\leqslant 2\alpha\sqrt{\alpha\beta}\bar{\sigma}_k$, then there are two corresponding eigenvalues λ such that $|\lambda|=\sqrt{\frac{\alpha-1}{\alpha+1}}$.

Proof. According to Lemma 3.1, we know that when $\alpha\beta+\overline{\sigma}_k^2>2\alpha\sqrt{\alpha\beta}\bar{\sigma}_k$,

$$|\lambda|=\frac{1}{(\alpha+1)(\alpha\beta+\overline{\sigma}_k^2)}|\alpha(\alpha\beta-\overline{\sigma}_k^2)\pm\sqrt{(\alpha\beta+\overline{\sigma}_k^2)^2-4\alpha^3\beta\overline{\sigma}_k^2}|$$

$$=\begin{cases}\frac{1}{(\alpha+1)(\alpha\beta+\overline{\sigma}_k^2)}(\alpha|\alpha\beta-\overline{\sigma}_k^2|+\sqrt{(\alpha\beta+\overline{\sigma}_k^2)^2-4\alpha^3\beta\overline{\sigma}_k^2}) \\[2ex] \frac{1}{(\alpha+1)(\alpha\beta+\overline{\sigma}_k^2)}|\alpha|\alpha\beta-\overline{\sigma}_k^2|-\sqrt{(\alpha\beta+\overline{\sigma}_k^2)^2-4\alpha^3\beta\overline{\sigma}_k^2}|\end{cases}$$

$$=\begin{cases}\frac{\alpha}{\alpha+1}\left(\frac{|\alpha\beta-\overline{\sigma}_k^2|}{\alpha\beta+\overline{\sigma}_k^2}+\sqrt{\frac{1}{\alpha^2}-\frac{4\alpha\beta\overline{\sigma}_k^2}{(\alpha\beta+\overline{\sigma}_k^2)^2}}\right) \\[2ex] \frac{\alpha}{\alpha+1}\left|\frac{|\alpha\beta-\overline{\sigma}_k^2|}{\alpha\beta+\overline{\sigma}_k^2}-\sqrt{\frac{1}{\alpha^2}-\frac{4\alpha\beta\overline{\sigma}_k^2}{(\alpha\beta+\overline{\sigma}_k^2)^2}}\right|,\end{cases}$$

and when $\alpha\beta + \overline{\sigma}_k^2 \leqslant 2\alpha\sqrt{\alpha\beta}\bar{\sigma}_k$,

$$\begin{aligned}
|\lambda| &= \frac{1}{(\alpha+1)(\alpha\beta+\overline{\sigma}_k^2)}\sqrt{(\alpha(\alpha\beta-\overline{\sigma}_k^2))^2 + (\sqrt{4\alpha^3\beta\overline{\sigma}_k^2 - (\alpha\beta+\overline{\sigma}_k^2)^2})^2} \\
&= \frac{1}{(\alpha+1)(\alpha\beta+\overline{\sigma}_k^2)}\sqrt{(\alpha^2-1)(\alpha\beta+\overline{\sigma}_k^2)^2} \\
&= \frac{\sqrt{\alpha^2-1}}{\alpha+1} = \sqrt{\frac{\alpha-1}{\alpha+1}}.
\end{aligned}$$

The conclusion then follows directly from the structure of the eigenvalues of the matrix $\mathcal{L}(\alpha,\beta)$ described in Lemma 3.1. □

Based on Lemma 3.2, we are now ready to prove the convergence of the AHSS iteration method for solving the saddle-point problem (1.1).

THEOREM 3.1 Consider the saddle-point problem (1.1). Let $B \in \mathbb{C}^{p\times p}$ be Hermitian positive definite, $E \in \mathbb{C}^{p\times q}$ be of full column rank and $\alpha, \beta > 0$ be given constants. Assume that $C \in \mathbb{C}^{q\times q}$ is a Hermitian positive-definite matrix. Then

$$\rho(\mathcal{L}(\alpha,\beta)) = \begin{cases} \max\left\{\frac{1-\alpha}{1+\alpha}, \frac{\alpha}{\alpha+1}\left(\frac{|\alpha\beta-\overline{\sigma}_k^2|}{\alpha\beta+\overline{\sigma}_k^2} + \sqrt{\frac{1}{\alpha^2} - \frac{4\alpha\beta\overline{\sigma}_k^2}{(\alpha\beta+\overline{\sigma}_k^2)^2}}\right)\right\} & \text{for} \quad \alpha \leqslant 1, \\ \max\left\{\sqrt{\frac{\alpha-1}{\alpha+1}}, \frac{\alpha}{\alpha+1}\left(\frac{|\alpha\beta-\overline{\sigma}_k^2|}{\alpha\beta+\overline{\sigma}_k^2} + \sqrt{\frac{1}{\alpha^2} - \frac{4\alpha\beta\overline{\sigma}_k^2}{(\alpha\beta+\overline{\sigma}_k^2)^2}}\right)\right\} & \text{for} \quad \alpha > 1, \end{cases}$$
$$< 1 \quad \forall\, \alpha, \beta > 0,$$

i.e. the AHSS iteration converges to the exact solution of the saddle-point problem (1.1).

Proof. Obviously, we have

$$\frac{|\alpha-1|}{\alpha+1} < 1 \quad \forall\, \alpha > 0 \quad \text{and} \quad \sqrt{\frac{\alpha-1}{\alpha+1}} < 1 \quad \forall\, \alpha > 1.$$

Because for $k = 1, 2, \ldots, q$, when $\alpha\beta + \overline{\sigma}_k^2 > 2\alpha\sqrt{\alpha\beta}\bar{\sigma}_k$, it holds that

$$\begin{aligned}
\frac{\alpha}{\alpha+1}\left|\frac{|\alpha\beta-\overline{\sigma}_k^2|}{\alpha\beta+\overline{\sigma}_k^2} - \sqrt{\frac{1}{\alpha^2} - \frac{4\alpha\beta\overline{\sigma}_k^2}{(\alpha\beta+\overline{\sigma}_k^2)^2}}\right| &\leqslant \frac{\alpha}{\alpha+1}\left(\frac{|\alpha\beta-\overline{\sigma}_k^2|}{\alpha\beta+\overline{\sigma}_k^2} + \sqrt{\frac{1}{\alpha^2} - \frac{4\alpha\beta\overline{\sigma}_k^2}{(\alpha\beta+\overline{\sigma}_k^2)^2}}\right) \\
&< \frac{\alpha}{\alpha+1}\left(\frac{|\alpha\beta-\overline{\sigma}_k^2|}{\alpha\beta+\overline{\sigma}_k^2} + \frac{1}{\alpha}\right) \\
&< \frac{\alpha}{\alpha+1}\left(1 + \frac{1}{\alpha}\right) = 1,
\end{aligned}$$

by making use of Lemma 3.2 we easily see that $\rho(\mathcal{L}(\alpha,\beta)) < 1$ holds for for all $\alpha, \beta > 0$. □

The optimal iteration parameters and the corresponding optimal asymptotic convergence factor of the AHSS iteration method are described in the following theorem.

THEOREM 3.2 Consider the saddle-point problem (1.1). Let $B \in \mathbb{C}^{p\times p}$ be Hermitian positive definite, $E \in \mathbb{C}^{p\times q}$ be of full column rank and $\alpha, \beta > 0$ be given constants. Assume that $C \in \mathbb{C}^{q\times q}$ is a Hermitian positive-definite matrix. If $\sigma_k (k = 1, 2, \ldots, q)$ are the positive singular values of the matrix $W^* E Z \in \mathbb{C}^{p\times q}$, and $\sigma_{\min} = \min_{1\leqslant k\leqslant q}\{\sigma_k\}$ and $\sigma_{\max} = \max_{1\leqslant k\leqslant q}\{\sigma_k\}$, then, for the AHSS iteration method of the saddle-point problem (1.1), the optimal values of the iteration parameters α and β are given by

$$\{\alpha^*, \beta^*\} = \arg\min_{\alpha,\beta>0} \rho(\mathcal{L}(\alpha, \beta)) = \left\{\tau, \frac{\sigma_{\min}\sigma_{\max}}{\tau}\right\}$$

and correspondingly,

$$\rho(\mathcal{L}(\alpha^*, \beta^*)) = \frac{\sqrt{\sigma_{\max}} - \sqrt{\sigma_{\min}}}{\sqrt{\sigma_{\max}} + \sqrt{\sigma_{\min}}} \equiv \frac{\sqrt[4]{\kappa} - 1}{\sqrt[4]{\kappa} + 1},$$

where $\kappa = \sigma^2_{\max}/\sigma^2_{\min}$ is the condition number of the matrix $C^{-1}E^*B^{-1}E$ and

$$\tau = \frac{\sigma_{\min} + \sigma_{\max}}{2\sqrt{\sigma_{\min}\sigma_{\max}}} \equiv \frac{1}{2}\left(\sqrt[4]{\kappa} + \frac{1}{\sqrt[4]{\kappa}}\right).$$

Proof. We first observe that the following two facts hold true:

(F_1) when $\alpha \leqslant 1$, $\alpha\beta + \sigma_k^2 > 2\alpha\sqrt{\alpha\beta}\sigma_k$, $k = 1, 2, \ldots, q$;

(F_2) when $\alpha > 1$,

(a) $\alpha\beta + \sigma_k^2 > 2\alpha\sqrt{\alpha\beta}\sigma_k$ iff[2] $\sigma_k \in (0, \sigma_-) \cup (\sigma_+, +\infty)$, $k \in \{1, 2, \ldots, q\}$;

(b) $\alpha\beta + \sigma_k^2 \leqslant 2\alpha\sqrt{\alpha\beta}\sigma_k$ iff $\sigma_k \in [\sigma_-, \sigma_+]$, $k \in \{1, 2, \ldots, q\}$;

(c) $\frac{\alpha-1}{\alpha+1} < \sqrt{\frac{\alpha-1}{\alpha+1}}$,

where $\sigma_- = \sqrt{\alpha\beta}(\alpha - \sqrt{\alpha^2 - 1})$ and $\sigma_+ = \sqrt{\alpha\beta}(\alpha + \sqrt{\alpha^2 - 1})$.

Let

$$\theta(\alpha, \beta, \sigma) = \frac{\alpha}{\alpha + 1}\left(\frac{|\alpha\beta - \sigma^2|}{\alpha\beta + \sigma^2} + \sqrt{\frac{1}{\alpha^2} - \frac{4\alpha\beta\sigma^2}{(\alpha\beta + \sigma^2)^2}}\right).$$

Then based on the Facts (F_1) and (F_2), we easily see that

$$\rho(\mathcal{L}(\alpha, \beta)) = \begin{cases} \max\left\{\frac{1-\alpha}{1+\alpha}, \max\limits_{1\leqslant k\leqslant q} \theta(\alpha, \beta, \sigma_k)\right\} & \text{for} \quad \alpha \leqslant 1, \\ \max\left\{\sqrt{\frac{\alpha-1}{\alpha+1}}, \max\limits_{\substack{\sigma_k<\sigma_- \text{ or } \sigma_k>\sigma_+ \\ k\in\{1,2,\ldots,q\}}} \theta(\alpha, \beta, \sigma_k)\right\} & \text{for} \quad \alpha > 1. \end{cases} \tag{3.1}$$

[2] "iff" is used to represent "if and only if."

For fixed α and β with $\alpha, \beta > 0$, we define two functions $\theta_1, \theta_2 : (0, +\infty) \to (0, +\infty)$ by

$$\theta_1(t) = \frac{\alpha\beta - t}{\alpha\beta + t}, \quad \theta_2(t) = \frac{1}{\alpha^2} - \frac{4\alpha\beta t}{(\alpha\beta + t)^2}.$$

After straightforward calculations, we obtain

$$\frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} = -\frac{2\alpha\beta}{(\alpha\beta + t)^2}, \quad \frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} = \frac{4\alpha\beta(t - \alpha\beta)}{(\alpha\beta + t)^3}.$$

It then follows that:

(i) $\max\limits_{1 \leqslant k \leqslant q} \theta(\alpha, \beta, \sigma_k) = \max\{\theta(\alpha, \beta, \sigma_{\min}), \theta(\alpha, \beta, \sigma_{\max})\}$,
for $\alpha \leqslant 1$, and $\sigma_{\min}^2 \leqslant \alpha\beta \leqslant \sigma_{\max}^2$;

(ii) $\max\limits_{\substack{\sigma_k < \sigma_- \text{ or } \sigma_k > \sigma_+ \\ k \in \{1,2,\ldots,q\}}} \theta(\alpha, \beta, \sigma_k) = \max\{\theta(\alpha, \beta, \sigma_{\min}), \theta(\alpha, \beta, \sigma_{\max})\}$,
for $\alpha > 1$, and $\sigma_{\min} < \sigma_-$ or $\sigma_{\max} > \sigma_+$.

Therefore, when $\alpha \leqslant 1$, the optimal parameters α^* and β^* must satisfy $\sigma_{\min}^2 \leqslant \alpha^*\beta^* \leqslant \sigma_{\max}^2$ and either of the following three conditions:

(A_1) $\frac{1-\alpha^*}{1+\alpha^*} = \theta(\alpha^*, \beta^*, \sigma_{\min}) \geqslant \theta(\alpha^*, \beta^*, \sigma_{\max})$;

(A_2) $\frac{1-\alpha^*}{1+\alpha^*} = \theta(\alpha^*, \beta^*, \sigma_{\max}) \geqslant \theta(\alpha^*, \beta^*, \sigma_{\min})$;

(A_3) $\theta(\alpha^*, \beta^*, \sigma_{\min}) = \theta(\alpha^*, \beta^*, \sigma_{\max}) \geqslant \frac{1-\alpha^*}{1+\alpha^*}$,

and when $\alpha > 1$, the optimal parameters α^* and β^* must satisfy $\sigma_{\min} < \sigma_-^*$ or $\sigma_{\max} > \sigma_+^*$ and either of the following three conditions:

(B_1) $\sqrt{\frac{\alpha^*-1}{\alpha^*+1}} = \theta(\alpha^*, \beta^*, \sigma_{\min}) \geqslant \theta(\alpha^*, \beta^*, \sigma_{\max})$;

(B_2) $\sqrt{\frac{\alpha^*-1}{\alpha^*+1}} = \theta(\alpha^*, \beta^*, \sigma_{\max}) \geqslant \theta(\alpha^*, \beta^*, \sigma_{\min})$;

(B_3) $\theta(\alpha^*, \beta^*, \sigma_{\min}) = \theta(\alpha^*, \beta^*, \sigma_{\max}) \geqslant \sqrt{\frac{\alpha^*-1}{\alpha^*+1}}$,

where $\sigma_-^* = \sqrt{\alpha^*\beta^*}(\alpha^* - \sqrt{(\alpha^*)^2 - 1})$ and $\sigma_+^* = \sqrt{\alpha^*\beta^*}(\alpha^* + \sqrt{(\alpha^*)^2 - 1})$. By straightforwardly solving the inequalities (A_1)–(A_3) and (B_1)–(B_3), we deduce that α^* and β^* satisfy

$$\alpha^*\beta^* = \sigma_{\min}\sigma_{\max} \quad \text{and} \quad \alpha^*\sqrt{\sigma_{\min}\sigma_{\max}} \leqslant \frac{1}{2}(\sigma_{\min} + \sigma_{\max}).$$

Thereby,

$$\rho(\mathcal{L}(\alpha^*, \beta^*)) = \rho\left(\mathcal{L}\left(\alpha^*, \frac{\sigma_{\min}\sigma_{\max}}{\alpha^*}\right)\right)$$
$$= \frac{\alpha^*}{\alpha^* + 1}\left(\frac{\sigma_{\max} - \sigma_{\min}}{\sigma_{\max} + \sigma_{\min}} + \sqrt{\frac{1}{(\alpha^*)^2} - \frac{4\sigma_{\min}\sigma_{\max}}{(\sigma_{\min} + \sigma_{\max})^2}}\right).$$

Now, we further minimize $\rho(\mathcal{L}(\alpha^*, \frac{\sigma_{\min}\sigma_{\max}}{\alpha^*}))$ with respect to α^* in the interval $(0, \frac{\sigma_{\min}+\sigma_{\max}}{2\sqrt{\sigma_{\min}\sigma_{\max}}})$ and obtain expressions for α^* and β^*.

To this end, we let

$$\varrho := \frac{\sigma_{\max} - \sigma_{\min}}{\sigma_{\max} + \sigma_{\min}} \quad \text{and} \quad \vartheta := \frac{4\sigma_{\min}\sigma_{\max}}{(\sigma_{\min} + \sigma_{\max})^2}.$$

Then we have

$$\begin{aligned}\omega(t) &:= \rho\left(\mathcal{L}\left(t, \frac{\sigma_{\min}\sigma_{\max}}{t}\right)\right) \\ &= \frac{t}{t+1}\left(\frac{\sigma_{\max} - \sigma_{\min}}{\sigma_{\max} + \sigma_{\min}} + \sqrt{\frac{1}{t^2} - \frac{4\sigma_{\min}\sigma_{\max}}{(\sigma_{\min} + \sigma_{\max})^2}}\right) \\ &= \frac{\varrho t}{t+1} + \frac{1}{t+1}\sqrt{1 - \vartheta t^2}\end{aligned}$$

and

$$\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = \frac{\varrho\sqrt{1 - \vartheta t^2} - \vartheta t - 1}{(t+1)^2\sqrt{1 - \vartheta t^2}}.$$

Because of $\vartheta + \varrho^2 = 1$, the only stationary point of $\omega(t)$ is $t_s = -1$. Noticing that $\omega(0) = 1$, $\omega(1) = \varrho$ and

$$\omega(\tau) \equiv \omega\left(\frac{\sigma_{\min} + \sigma_{\max}}{2\sqrt{\sigma_{\min}\sigma_{\max}}}\right) = \frac{\sqrt{\sigma_{\max}} - \sqrt{\sigma_{\min}}}{\sqrt{\sigma_{\max}} + \sqrt{\sigma_{\min}}},$$

we immediately obtain

$$\alpha^* = \tau \quad \text{and} \quad \beta^* = \frac{\sigma_{\min}\sigma_{\max}}{\tau}.$$

Then by substituting α^* and β^* into (3.1), we obtain $\rho(\mathcal{L}(\alpha^*, \beta^*)) = \frac{\sqrt{\sigma_{\max}} - \sqrt{\sigma_{\min}}}{\sqrt{\sigma_{\max}} + \sqrt{\sigma_{\min}}}$. □

By recalling that the optimal convergence factor of the PHSS iteration method (Bai *et al.*, 2004) is only about $\varrho = \frac{\sigma_{\max} - \sigma_{\min}}{\sigma_{\max} + \sigma_{\min}}$, we see that the AHSS iteration method is much faster than the PHSS iteration method when the optimal iteration parameters are employed; see also Remark 4.3 in Bai *et al.* (2006b). Moreover, observing that both methods have the same computing cost at each of their iteration steps, we can conclude that the AHSS method is much more effective than the PHSS method in actual applications. [3] In addition, from Theorem 3.2 we also observe that the Hermitian positive-definite matrix $C \in \mathbb{C}^{q \times q}$ should be chosen so that the linear system with coefficient matrix C is easily solvable and the singular values of the matrix $W^*EZ \in \mathbb{C}^{p \times q}$ are tightly clustered or, in other words, C should be a good preconditioner to the matrix $E^*B^{-1}E \in \mathbb{C}^{q \times q}$.

Besides, we can observe the following facts and results from Sections 2 and 3. For their detailed illustrations and proofs, we refer to Bai & Golub (2004).

(i) The matrix $\mathcal{Q}(\alpha, \beta)$ defined in (2.5) is a Cayley transform iff $\alpha = \beta$. Because of $\|\mathcal{Q}(\alpha, \beta)\|_2 = \sqrt{1 + 2\delta(\delta + \sqrt{1 + \delta^2})}$, where $\delta = \frac{|\alpha - \beta|}{\alpha\beta + \sigma_{\min}^2}$, with $\sigma_{\min}$ the smallest positive singular value of the

[3] The optimal convergence factor of the 'generalized successive overrelaxation' (GSOR) method, the fastest one among all the Uzawa-like methods, is ϱ, too; see Bai *et al.* (2005b). Also, at each iteration step, the AHSS method costs the same as the GSOR method.

matrix $\overline{E}$, the Cayley transform $\mathcal{Q}(\alpha) := \mathcal{Q}(\alpha, \alpha)$ is the solution of the minimization problem $\min_{\alpha,\beta>0} \|\mathcal{Q}(\alpha, \beta)\|_2$;

(ii) When we take $(\alpha, \beta) = (1, \sigma_{\min}\sigma_{\max})$, $(\sigma_{\min}\sigma_{\max}, 1)$ and $(\sqrt{\sigma_{\min}\sigma_{\max}}, \sqrt{\sigma_{\min}\sigma_{\max}})$, it holds that $\rho(\mathcal{L}(\alpha, \beta)) \equiv \varrho := \frac{\sigma_{\max}-\sigma_{\min}}{\sigma_{\max}+\sigma_{\min}}$, which is approximately equal to the optimal convergence factor of the PHSS iteration method (Bai *et al.*, 2004, 2006b);

(iii) From Theorem 3.2 in Section 3 and that in Bai *et al.* (2004), as well as Remark 4.3 in Bai *et al.* (2006b), we know that the optimal convergence rate of AHSS iteration method is $\frac{2}{\sqrt[4]{\kappa}}$ and that of PHSS iteration method is approximately $\frac{2}{\sqrt{\kappa}}$. Therefore, AHSS converges a lot faster than PHSS when the optimal iteration parameters are employed and $\kappa \gg 1$.

4. Krylov subspace acceleration

Besides its use as a solver, the AHSS iteration can also be used as a preconditioner to accelerate Krylov subspace methods such as GMRES or its restarted variant GMRES(ℓ) (Saad & Schultz, 1986; Saad, 1996).

From (2.9) we see that the saddle-point problem (1.1) is equivalent to

$$\mathbf{A}x = \mathbf{b}, \tag{4.1}$$

with $\mathbf{A} := M(\alpha, \beta)^{-1}A$ and $\mathbf{b} := M(\alpha, \beta)^{-1}b$. This equivalent linear system can be solved by GMRES. Hence, the positive-definite matrix $M(\alpha, \beta)$ can be seen as a preconditioner to GMRES. Equivalently, we can say that GMRES is employed to accelerate the convergence of AHSS applied to the saddle-point problem (1.1).

Assume that the coefficient matrix $\mathbf{A}$ of the system of linear equations (4.1) is diagonalizable, i.e. there exist a non-singular matrix $\mathbf{X} \in \mathbb{C}^{(p+q)\times(p+q)}$ and a diagonal matrix $\mathbf{D} \in \mathbb{C}^{(p+q)\times(p+q)}$ such that $\mathbf{A} = \mathbf{X}\mathbf{D}\mathbf{X}^{-1}$. Then it is well-known from Saad & Schultz (1986, Theorem 4) that the residual norm $\|\mathbf{r}^{(k)}\|_2$ at the kth step of the preconditioned GMRES is bounded by $\|\mathbf{r}^{(k)}\|_2 \leqslant \kappa(\mathbf{X})\epsilon^{(k)}\|\mathbf{r}^{(0)}\|_2$, where $\kappa(\mathbf{X})$ is the Euclidean condition number of $\mathbf{X}$ and $\epsilon^{(k)} := \min_{P\in\mathcal{P}_k} \max_{\lambda_i\in\sigma(\mathbf{A})} |P(\lambda_i)|$. Here, $\mathcal{P}_k$ denotes the set of all polynomials $P(\lambda)$ of degree not greater than k such that $P(0) = 1$, and $\sigma(\mathbf{A})$ denotes the spectrum of the matrix $\mathbf{A}$. Since all eigenvalues of the matrix $\mathbf{A}$ are contained in a circle with centre $(1, 0)$ and radius $\rho(\mathcal{L}(\alpha, \beta))$, a special case of Theorem 5 in Saad & Schultz (1986) implies that $\epsilon^{(k)} \leqslant (\rho(\mathcal{L}(\alpha, \beta)))^k$. In particular, when the optimal parameters α^* and β^* given in Theorem 3.2 are employed, all eigenvalues of the preconditioned matrix $M(\alpha^*, \beta^*)^{-1}A$ are located in a circle with centre $(1, 0)$ and radius $\rho(\mathcal{L}(\alpha^*, \beta^*)) = \frac{\sqrt{\sigma_{\max}}-\sqrt{\sigma_{\min}}}{\sqrt{\sigma_{\max}}+\sqrt{\sigma_{\min}}}$. Therefore, if the optimal AHSS iteration method is applied to preconditioned GMRES, it will improve the numerical behaviour of GMRES considerably. Here, we remark that we are comparing the numerical behaviour of unpreconditioned GMRES with GMRES preconditioned by AHSS with optimal parameters. Clearly, there are other known preconditioners that will drastically improve the numerical behaviour, e.g. the constrained preconditioners (Keller *et al.*, 2000; Benzi & Ng, 2006) and the block-diagonal preconditioners (Perugia & Simoncini, 2000; Bai *et al.*, 2004) applied to the equivalent symmetric form of (1.1).

If the coefficient matrix $\mathbf{A}$ of the system of linear equations (4.1) is positive definite, then it is known from Eisenstat *et al.* (1983) and Saad & Schultz (1986, p. 866) that the error bound $\|\mathbf{r}^{(k)}\|_2 \leqslant (1 - \frac{(\lambda_{\min}(\mathbf{H}))^2}{\lambda_{\max}(\mathbf{A}^*\mathbf{A})})^{\frac{k}{2}}\|\mathbf{r}^{(0)}\|_2$ holds for the preconditioned GMRES, where $\mathbf{H}$ denotes the Hermitian part of the matrix $\mathbf{A}$, and $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote, respectively, the smallest and the largest eigenvalues of the corresponding matrix. This gives a guarantee for the convergence of the restarted preconditioned GMRES iteration, say GMRES(ℓ), for all ℓ when $\mathbf{A}$ is positive definite.

THEOREM 4.1 Let $B \in \mathbb{C}^{p\times p}$ be Hermitian positive definite, $E \in \mathbb{C}^{p\times q}$ be of full column rank, $p \geqslant q$, and α and β be two given positive constants. Assume that $C \in \mathbb{C}^{q\times q}$ is a Hermitian positive-definite matrix. Then there exist positive reals $\widetilde{\alpha}$ and $\widetilde{\beta}$ such that the matrix $\mathbf{A} = M(\alpha,\beta)^{-1}A$ is positive definite for all $\alpha \in (0,\widetilde{\alpha})$ and $\beta \in (0,\widetilde{\beta})$.

Proof. Because of $\mathbf{H} = \frac{1}{2}(\mathbf{A}^* + \mathbf{A}) = \frac{1}{2}(A^*M(\alpha,\beta)^{-*} + M(\alpha,\beta)^{-1}A)$, we can obtain

$$
\begin{aligned}
\widetilde{\mathbf{H}} &:= M(\alpha,\beta)\mathbf{H}(M(\alpha,\beta))^* \\
&= \frac{1}{2}(M(\alpha,\beta)A^* + A(M(\alpha,\beta))^*) \\
&= \frac{1}{2}\left(\begin{bmatrix} \frac{\alpha+1}{2}B & \frac{\alpha+1}{2\alpha}E \\ -\frac{1}{2}E^* & \frac{\beta}{2}C \end{bmatrix}\begin{bmatrix} B & -E \\ E^* & O \end{bmatrix} + \begin{bmatrix} B & E \\ -E^* & O \end{bmatrix}\begin{bmatrix} \frac{\alpha+1}{2}B & -\frac{1}{2}E \\ \frac{\alpha+1}{2\alpha}E^* & \frac{\beta}{2}C \end{bmatrix}\right) \\
&= \frac{1}{2}\begin{bmatrix} \frac{\alpha+1}{2}B^2 + \frac{\alpha+1}{2\alpha}EE^* & -\frac{\alpha+1}{2}BE \\ -\frac{1}{2}E^*B + \frac{\beta}{2}CE^* & \frac{1}{2}E^*E \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{\alpha+1}{2}B^2 + \frac{\alpha+1}{2\alpha}EE^* & -\frac{1}{2}BE + \frac{\beta}{2}EC \\ -\frac{\alpha+1}{2}E^*B & \frac{1}{2}E^*E \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} (\alpha+1)B^2 + \frac{\alpha+1}{\alpha}EE^* & -\frac{\alpha+2}{2}BE + \frac{\beta}{2}EC \\ -\frac{\alpha+2}{2}E^*B + \frac{\beta}{2}CE^* & E^*E \end{bmatrix} \\
&:= \widetilde{\mathbf{H}}_1 + \frac{1}{\alpha}\widetilde{\mathbf{H}}_2 + \alpha\widetilde{\mathbf{H}}_3 + \beta\widetilde{\mathbf{H}}_4,
\end{aligned}
$$

where

$$
\widetilde{\mathbf{H}}_1 = \frac{1}{2}\begin{bmatrix} B^2 + EE^* & -BE \\ -E^*B & E^*E \end{bmatrix},
$$

$$
\widetilde{\mathbf{H}}_2 = \frac{1}{2}\begin{bmatrix} EE^* & O \\ O & O \end{bmatrix},
$$

$$
\widetilde{\mathbf{H}}_3 = \frac{1}{4}\begin{bmatrix} 2B^2 & -BE \\ -E^*B & O \end{bmatrix},
$$

$$
\widetilde{\mathbf{H}}_4 = \frac{1}{4}\begin{bmatrix} O & EC \\ CE^* & O \end{bmatrix}.
$$

Let $\widetilde{\mathbf{H}}_1 = \widetilde{\mathbf{H}}_a + \widetilde{\mathbf{H}}_b$, with

$$
\widetilde{\mathbf{H}}_a = \frac{1}{2}\begin{bmatrix} B^2 & -BE \\ -E^*B & E^*E \end{bmatrix}, \quad \widetilde{\mathbf{H}}_b = \frac{1}{2}\begin{bmatrix} EE^* & O \\ O & O \end{bmatrix}.
$$

Then we easily see that $\widetilde{\mathbf{H}}_1$ is Hermitian positive semidefinite, as both $\widetilde{\mathbf{H}}_a$ and $\widetilde{\mathbf{H}}_b$ are Hermitian positive semidefinite. Moreover, we can demonstrate that $\widetilde{\mathbf{H}}_1$ is non-singular and, hence, Hermitian positive definite; see Bai & Golub (2004).

Now, let $\tilde{h}_1 := \lambda_{\min}(\widetilde{\mathbf{H}}_1)$, $\tilde{h}_2 := \lambda_{\min}(\widetilde{\mathbf{H}}_2)$, $\tilde{h}_3 := \|\widetilde{\mathbf{H}}_3\|_2$ and $\tilde{h}_4 := \|\widetilde{\mathbf{H}}_4\|_2$. Then we know that $\tilde{h}_1$ is a positive constant independent of both α and β and $\tilde{h}_2 = 0$. By straightforward computations, we have

$$\begin{aligned}
\lambda_{\min}(\widetilde{\mathbf{H}}) &= \min_{x\neq 0} \frac{x^*\widetilde{\mathbf{H}}x}{x^*x} \\
&\geqslant \min_{x\neq 0} \left\{ \frac{x^*\widetilde{\mathbf{H}}_1 x}{x^*x} + \frac{1}{\alpha}\frac{x^*\widetilde{\mathbf{H}}_2 x}{x^*x} - \alpha \left|\frac{x^*\widetilde{\mathbf{H}}_3 x}{x^*x}\right| - \beta \left|\frac{x^*\widetilde{\mathbf{H}}_4 x}{x^*x}\right| \right\} \\
&\geqslant \min_{x\neq 0} \frac{x^*\widetilde{\mathbf{H}}_1 x}{x^*x} + \frac{1}{\alpha}\cdot \min_{x\neq 0} \frac{x^*\widetilde{\mathbf{H}}_2 x}{x^*x} - \alpha \cdot \max_{x\neq 0} \left|\frac{x^*\widetilde{\mathbf{H}}_3 x}{x^*x}\right| - \beta \cdot \max_{x\neq 0} \left|\frac{x^*\widetilde{\mathbf{H}}_4 x}{x^*x}\right| \\
&\geqslant \lambda_{\min}(\widetilde{\mathbf{H}}_1) + \frac{1}{\alpha}\lambda_{\min}(\widetilde{\mathbf{H}}_2) - \alpha\|\widetilde{\mathbf{H}}_3\|_2 - \beta\|\widetilde{\mathbf{H}}_4\|_2 \\
&= \tilde{h}_1 - \alpha\tilde{h}_3 - \beta\tilde{h}_4.
\end{aligned}$$

Therefore, there exist two positive reals $\widetilde{\alpha}$ and $\widetilde{\beta}$ such that $\tilde{h}_1 - \alpha\tilde{h}_3 - \beta\tilde{h}_4 > 0$ holds for all $\alpha \in (0, \widetilde{\alpha})$ and $\beta \in (0, \widetilde{\beta})$. This immediately shows that $\widetilde{\mathbf{H}}$, hence $\mathbf{H}$, is Hermitian positive definite for all $\alpha \in (0, \widetilde{\alpha})$ and $\beta \in (0, \widetilde{\beta})$. □

We remark that all eigenvalues of the matrix $\mathbf{A} = M(\alpha, \beta)^{-1}A$ are real and positive for all $\alpha \in (0, 1]$ and $\beta \in (0, +\infty)$. As a matter of fact, because

$$\mathbf{A} = M(\alpha, \beta)^{-1}A = M(\alpha, \beta)^{-1}(M(\alpha, \beta) - N(\alpha, \beta)) = I - \mathcal{L}(\alpha, \beta),$$

by Theorem 3.1 we know that $\rho(\mathcal{L}(\alpha, \beta)) < 1$ holds for $\forall\, \alpha, \beta > 0$. Therefore, all eigenvalues of the matrix $\mathbf{A}$ are located in a circle centred at the point (1, 0) and having radius $\rho(\mathcal{L}(\alpha, \beta)) < 1$, and hence, have positive real parts.

It follows that we only need to demonstrate that all eigenvalues of the matrix $\mathbf{A}$ are real. To this end, from Lemma 3.1 we only need to verify the validity of the inequalities

$$(\alpha\beta + \overline{\sigma}_k^2)^2 - 4\alpha^3\beta\overline{\sigma}_k^2 \geqslant 0, \quad k = 1, 2, \ldots, q,$$

for all $\alpha \in (0, 1]$ and $\beta > 0$ or, equivalently,

$$\overline{\sigma}_k^2 + \alpha\beta \geqslant 2\alpha\sqrt{\alpha\beta}\,\bar{\sigma}_k, \quad k = 1, 2, \ldots, q,$$

where $\overline{\sigma}_k$, $k = 1, 2, \ldots, q$, are the positive singular values of the matrix W^*EZ. In fact, these inequalities hold true because

$$\overline{\sigma}_k^2 - 2\alpha\sqrt{\alpha\beta}\,\bar{\sigma}_k + \alpha\beta = (\overline{\sigma}_k - \alpha\sqrt{\alpha\beta})^2 + \alpha\beta(1 - \alpha^2), \quad k = 1, 2, \ldots, q.$$

This fact establishes the basis for further accelerating the AHSS iteration method by the Chebyshev semi-iterative technique (see Varga, 1962, Chapter 5).

According to the application of the preconditioning matrix $M(\alpha, \beta)$, it seems complicated and costly to solve the associated 'generalized residual equation' $M(\alpha, \beta)z = r$ at each iteration steps of the corresponding Krylov subspace method. However, by choosing the arbitrary Hermitian positive-definite matrix C, we can make its Schur complement $\beta C + \frac{1}{\alpha}E^*B^{-1}E$ to have simple and easily invertible form (e.g. a (block) diagonal matrix, or a product of a (block) lower triangular matrix with a (block) upper triangular matrix) such that the linear system with the coefficient matrix $\beta C + \frac{1}{\alpha}E^*B^{-1}E$ is cheaply solvable. It then follows that the abovementioned generalized residual equation can be solved economically and easily through the AHSS iteration procedure described in Section 2.

5. Numerical results

In this section, we use one example to exhibit the superiority of both PHSS and AHSS methods to GMRES(ℓ) and GMRES, and further examine the effectiveness and show the advantages of the AHSS method over the PHSS method when they are used as solvers as well as preconditioners to GMRES(ℓ) and GMRES for the saddle-point problem (1.1), from the point of view of both number of total iteration steps (denoted by 'IT') and elapsed CPU time in seconds (denoted by 'CPU'). Here, the integer ℓ in GMRES(ℓ) denotes that the algorithm is restarted after every ℓ iterations.

To this end, we need to choose the matrix C in both PHSS and AHSS. A natural choice of the matrix C is $C = E^T \widehat{B}^{-1} E$, where $\widehat{B}$ is a good approximation to the matrix block B. We recall that the PHSS iteration method (Bai *et al.*, 2004) is a special case of the AHSS iteration method when $\alpha = \beta$. In order to emphasize the dependence of the AHSS (or PHSS) iteration upon the acceleration parameters (α, β) (or α), we sometimes also use the notation AHSS(α, β) (or PHSS(α)) instead of AHSS (or PHSS).

In actual computations, we choose the right-hand side vector b so that the exact solution of the saddle-point problem (1.1) is $(1, 1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$. Besides, all runs are started from an initial vector $x^{(0)}$ which is generated randomly with normal distribution by the MATLAB code randn, terminated if the current iterations satisfy RES $\equiv \frac{\|b-Ax^{(k)}\|_2}{\|b-Ax^{(0)}\|_2} \leqslant 10^{-8}$, or if the maximum prescribed number of iterations $k_{\max}$ is exceeded, and performed in MATLAB with machine precision 10^{-16}.

EXAMPLE 5.1 (Bai *et al.*, 2004) Consider the saddle-point problem (1.1), in which

$$B = \begin{bmatrix} I \otimes \Upsilon + \Upsilon \otimes I & O \\ O & I \otimes \Upsilon + \Upsilon \otimes I \end{bmatrix} \in \mathbb{R}^{2m^2 \times 2m^2}, \quad E = \begin{bmatrix} I \otimes \Psi \\ \Psi \otimes I \end{bmatrix} \in \mathbb{R}^{2m^2 \times m^2}$$

and

$$\Upsilon = \frac{\mu}{h^2} \cdot \mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{m \times m}, \quad \Psi = \frac{1}{h} \cdot \mathrm{tridiag}(-1, 1, 0) \in \mathbb{R}^{m \times m},$$

with $\otimes$ the Kronecker product symbol, $\mathrm{Re} = \frac{h}{2\mu}$, $h = \frac{1}{m+1}$ and $\mu > 0$ the viscosity constant.

For this example, we take $k_{\max} = 5m$, and $\widehat{B} = \frac{2\mu}{h^2} I + I \otimes \Upsilon$ the block-diagonal matrix of B. We remark that the number of variables of the corresponding saddle-point problem (1.1) is $n := p + q \equiv 3m^2$. Moreover, because of the specific structure of this problem and of the matrix C, we can easily prove that the optimal iteration parameters α^* and (α^*, β^*), and the corresponding optimal convergence factors $\rho(\mathcal{L}(\alpha^*))$ and $\rho(\mathcal{L}(\alpha^*, \beta^*))$ of the PHSS and the AHSS iteration methods are, respectively, independent of the viscosity constant μ.

In Table 1, we list the Euclidean condition number κ of the matrix $C^{-1}E^T B^{-1} E$, and α^* and (α^*, β^*) as well as the corresponding $\rho(\mathcal{L}(\alpha^*))$ and $\rho(\mathcal{L}(\alpha^*, \beta^*))$ for the PHSS and the AHSS iterations, respectively, for various problem sizes m for Example 5.1. It is clear that both PHSS(α^*) and AHSS(α^*, β^*) methods have reasonably small convergence factors, and the asymptotic convergence rate of the AHSS method is much faster than that of the PHSS method when the optimal parameter(s) is (are) employed. In particular, we see that when m increases, the optimal parameters α^* and β^* are increasing, and the corresponding optimal convergence factors $\rho(\mathcal{L}(\alpha^*))$ and $\rho(\mathcal{L}(\alpha^*, \beta^*))$ of the PHSS and the AHSS methods are increasing gradually as well.

In Figs 1–3, we plot the eigenvalue distributions of the original matrix A, the PHSS(α^*)-preconditioned matrix $M(\alpha^*)^{-1}A$ and the AHSS(α^*, β^*)-preconditioned matrix $M(\alpha^*, \beta^*)^{-1}A$, respectively, for Example 5.1, when $m = 16$ and $\mu = 1$ as well as $m = 32$ and $\mu = \frac{1}{80}$.

TABLE 1 *Parameter(s) versus spectral radius for Example* 5.1

	m	8	16	24	32	48
	κ	14.1738	47.3972	99.8972	171.7262	373.1762
PHSS	α^*	1.4151	1.8718	2.2447	2.5657	3.1113
	$\rho(\mathcal{L}(\alpha^*))$	0.4146	0.5510	0.6194	0.6626	0.7166
AHSS	α^*	1.2278	1.5026	1.7390	1.9482	2.3115
	β^*	1.6309	2.3317	2.8974	3.3789	4.1879
	$\rho(\mathcal{L}(\alpha^*, \beta^*))$	0.3198	0.4481	0.5194	0.5671	0.6293

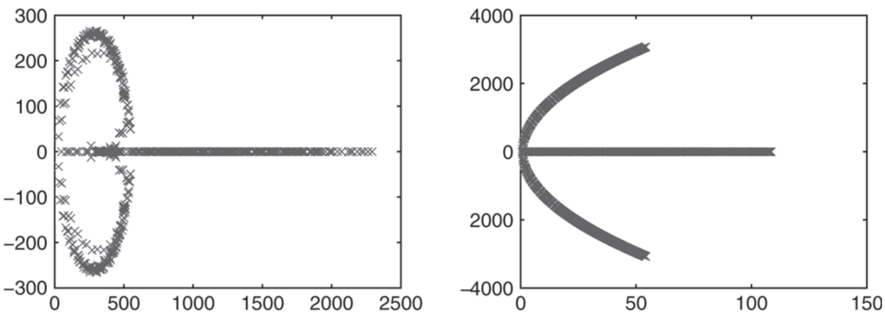

FIG. 1. Spectrum of the original matrix A for Example 5.1: $m = 16$ and $\mu = 1$ (left); and $m = 32$ and $\mu = 1/80$ (right).

Clearly, the matrices A are very ill-conditioned because their spectrums have very large ranges along both the real and the imaginary axis and some of the eigenvalues lye close to the origin. Compared to the matrix A for $m = 16$ and $\mu = 1$, the eigenvalues of the matrix A for $m = 32$ and $\mu = \frac{1}{80}$ have much smaller real parts but significantly larger imaginary parts, and are more clustered near the origin. Therefore, when m is increasing and μ is decreasing, the matrix A becomes more ill-conditioned.

However, both $M(\alpha^*)^{-1}A$ and $M(\alpha^*, \beta^*)^{-1}A$ are well-conditioned because their spectra have considerably smaller ranges along both the real and the imaginary axis and are also clustered on circles with centre $(1, 0)$ and radii about 0.5, which are obviously located in the left complex half-plane and far away from the origin. When m increases and μ decreases, the eigenvalue pictures of the matrices $M(\alpha^*)^{-1}A$ and $M(\alpha^*, \beta^*)^{-1}A$ change slightly and show, roughly speaking, almost the same shapes, respectively. Moreover, the spectrum of $M(\alpha^*, \beta^*)^{-1}A$ is more clustered than that of $M(\alpha^*)^{-1}A$. These observations imply that when PHSS(α^*) and AHSS(α^*, β^*) are employed to preconditioned GMRES(ℓ) and GMRES, the numerical behaviours of the resulting methods can be improved considerably. In addition, the AHSS(α^*, β^*)-preconditioned GMRES(ℓ) (or GMRES) will perform much better than the PHSS(α^*)-preconditioned GMRES(ℓ) (or GMRES). These facts are further confirmed by the numerical results listed in Tables 2–4.

In Tables 2–4, we list numerical results with respect to IT, CPU and RES (if the convergence criterion is not achieved within $k_{\max}$ iteration steps) for the testing methods for Example 5.1. In these tables, we use the highlighted numbers to indicate the minimum and the second smallest CPU times in each column, which may show how well AHSS and AHSS-GMRES do. Tables 2 and 3 list numerical results for a fixed μ ($\mu = 1$ for Table 2 and $\mu = \frac{1}{80}$ for Table 3, respectively) and varying m, while Table 4 lists those for a fixed $m = 32$ and varying μ.

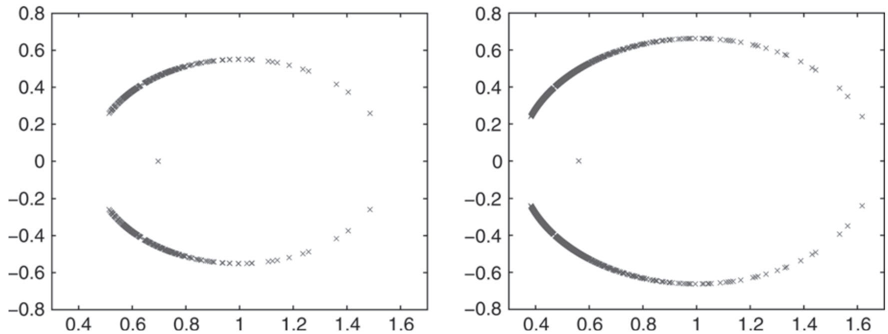

FIG. 2. Spectrum of the preconditioned matrix $M(\alpha^*)^{-1}A$ by PHSS for Example 5.1: $m = 16$ and $\mu = 1$ (left); and $m = 32$ and $\mu = 1/80$ (right).

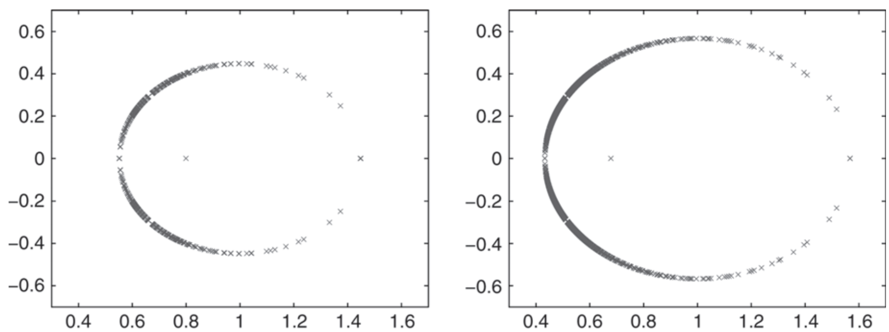

FIG. 3. Spectrum of the preconditioned matrix $M(\alpha^*, \beta^*)^{-1}A$ by AHSS for Example 5.1: $m = 16$ and $\mu = 1$ (left); and $m = 32$ and $\mu = 1/80$ (right).

From these tables, we see that GMRES(ℓ) ($\ell = 5$, 10 and 20) and GMRES cannot achieve the stopping criterion within the largest admissible number $k_{\max}$ of iteration steps. However, both PHSS(α^*) and AHSS(α^*, β^*) succeed in quickly producing approximate solutions of high quality in all cases. Moreover, AHSS(α^*, β^*) always outperforms PHSS(α^*), both in terms of the number of iteration steps and CPU times; see Figs 4–5 concerning the reduction rates of the relative solution errors of these two iteration methods, where ERR $\equiv \frac{\|x^{(k)}-x^*\|_2}{\|x^{(0)}-x^*\|_2}$. As preconditioners, both PHSS(α^*) and AHSS(α^*, β^*) can largely reduce the number of iteration steps and the CPU times of the preconditioned GMRES(ℓ) ($\ell = 5$, 10 and 20) and GMRES methods and, hence, they can considerably improve the numerical properties of GMRES(ℓ) ($\ell = 5$, 10 and 20) and GMRES. In addition, the AHSS(α^*, β^*) preconditioner performs much better than the PHSS(α^*) preconditioner. These observations are still true even when m becomes larger (see Tables 2 and 3) and μ becomes smaller (see Table 4).

We remark that numerical comparisons between the HSS/PHSS (Bai *et al.*, 2003, 2004; Benzi & Golub, 2004) and the block-diagonal/constraint (Perugia & Simoncini, 2000; Murphy *et al.*, 2000; Keller *et al.*, 2000) preconditioners were given in Bai *et al.* (2004) and Benzi & Ng (2006), which show the robust and effective advantages of the former over the latter, when they are used to precondition the GMRES-like methods for solving large, sparse, ill-conditioned and strongly non-Hermitian saddle-point problems.

TABLE 2 *IT, CPU and RES for Example* 5.1 ($\mu = 1$)

m		8	16	24	32	48
PHSS(α^*)	IT	21	32	40	47	58
	CPU	0.079	0.984	4.454	13.50	69.656
AHSS(α^*, β^*)	IT	18	25	31	35	43
	CPU	0.076	0.797	**3.406**	**9.984**	**52.875**
GMRES(5)	IT	40	80	120	160	240
	CPU	—	—	—	—	—
	RES	8.20×10^{-03}	6.75×10^{-03}	7.32×10^{-03}	7.45×10^{-03}	7.39×10^{-03}
PHSS-GMRES(5)	IT	20	30	38	46	57
	CPU	0.049	0.828	5.391	20.484	83.141
AHSS-GMRES(5)	IT	18	25	30	36	44
	CPU	0.046	0.704	4.141	15.188	63.563
GMRES(10)	IT	40	80	120	160	240
	CPU	—	—	—	—	—
	RES	5.56×10^{-03}	6.66×10^{-03}	6.80×10^{-03}	7.29×10^{-03}	6.52×10^{-03}
PHSS-GMRES(10)	IT	19	29	36	42	54
	CPU	0.046	0.719	4.609	16.985	71.891
AHSS-GMRES(10)	IT	17	24	30	35	43
	CPU	0.031	0.641	3.797	14.094	57.297
GMRES(20)	IT	40	80	120	160	240
	CPU	—	—	—	—	—
	RES	4.88×10^{-03}	4.66×10^{-03}	6.19×10^{-03}	6.92×10^{-03}	6.52×10^{-03}
PHSS-GMRES(20)	IT	19	28	35	42	52
	CPU	0.031	0.703	4.390	16.234	66.344
AHSS-GMRES(20)	IT	16	24	30	35	43
	CPU	**0.025**	**0.609**	3.688	13.391	54.718
GMRES	IT	40	80	120	160	240
	CPU	—	—	—	—	—
	RES	3.72×10^{-04}	3.91×10^{-04}	4.43×10^{-04}	5.26×10^{-03}	2.79×10^{-03}
PHSS-GMRES	IT	19	27	34	40	50
	CPU	0.047	0.656	4.062	14.890	62.672
AHSS-GMRES	IT	16	24	30	34	43
	CPU	**0.026**	**0.594**	**3.594**	**12.670**	**51.704**

6. Further extensions

For generalized saddle-point problems of the form

$$Ax \equiv \begin{bmatrix} B & E \\ -E^* & C_0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv b, \tag{6.1}$$

where $B \in \mathbb{C}^{p\times p}$ is positive definite, $C_0 \in \mathbb{C}^{q\times q}$ is Hermitian positive semidefinite, $E \in \mathbb{C}^{p\times q}$ has full column rank, $p \geqslant q$, $f \in \mathbb{C}^p$ and $g \in \mathbb{C}^q$, we can utilize the AHSS iteration technique to obtain the following iteration method.

TABLE 3 *IT, CPU and RES for Example* 5.1 *($\mu = \frac{1}{80}$)*

m		8	16	24	32	48
PHSS(α^*)	IT	25	38	47	55	69
	CPU	0.125	1.187	5.062	**14.750**	79.469
AHSS (α^*, β^*)	IT	21	29	36	42	51
	CPU	0.079	0.907	**3.860**	**11.156**	**60.735**
GMRES(5)	IT	40	80	120	160	240
	CPU	—	—	—	—	—
	RES	1.55×10^{-01}	1.47×10^{-01}	1.43×10^{-01}	1.33×10^{-01}	1.21×10^{-02}
PHSS-GMRES(5)	IT	39	49	63	69	81
	CPU	0.064	1.375	7.266	30.031	112.891
AHSS-GMRES(5)	IT	38	45	57	54	67
	CPU	0.062	1.250	6.782	23.594	94.438
GMRES(10)	IT	40	80	120	160	240
	CPU	—	—	—	—	—
	RES	5.60×10^{-02}	8.70×10^{-02}	9.64×10^{-02}	9.48×10^{-02}	9.01×10^{-03}
PHSS-GMRES(10)	IT	24	36	50	72	67
	CPU	0.047	0.937	6.359	28.937	84.735
AHSS-GMRES(10)	IT	20	35	49	67	59
	CPU	**0.031**	0.906	6.234	26.453	75.406
GMRES(20)	IT	40	80	120	160	240
	CPU	—	—	—	—	—
	RES	3.66×10^{-02}	5.27×10^{-02}	6.77×10^{-02}	7.17×10^{-02}	7.29×10^{-02}
PHSS-GMRES(20)	IT	25	38	47	49	63
	CPU	0.045	0.954	5.781	18.859	76.079
AHSS-GMRES(20)	IT	20	29	38	46	52
	CPU	**0.031**	**0.719**	4.609	17.750	64.297
GMRES	IT	40	80	120	160	240
	CPU	—	—	—	—	—
	RES	1.98×10^{-02}	1.60×10^{-02}	2.26×10^{-02}	5.16×10^{-02}	3.27×10^{-02}
PHSS-GMRES	IT	23	35	43	50	62
	CPU	0.047	0.844	5.093	18.641	72.109
AHSS-GMRES	IT	20	29	35	41	51
	CPU	0.032	**0.719**	**4.173**	15.281	**60.578**

The AHSS iteration method for the generalized saddle-point problem is defined as follows: Given an initial guess $x^{(0)} = (y^{(0)^*}, z^{(0)^*})^* \in \mathbb{C}^n$ and two positive constants α and β. For $k = 0, 1, 2, \ldots,$ until $\{x^{(k)}\} = \{(y^{(k)^*}, z^{(k)^*})^*\} \subset \mathbb{C}^n$ converges, compute the next iterate $x^{(k+1)} = (y^{(k+1)^*}, z^{(k+1)^*})^*$ according to the following procedure:

Step 1: Compute the partial vectors $y^{(k+\frac{1}{2})}$ and $z^{(k+\frac{1}{2})}$ by solving the linear subsystems

$$\begin{cases} (\alpha I + B) y^{(k+\frac{1}{2})} = \alpha y^{(k)} - E z^{(k)} + f, \\ (\beta I + C_0) z^{(k+\frac{1}{2})} = E^* y^{(k)} + \beta z^{(k)} + g; \end{cases}$$

TABLE 4 *IT, CPU and RES for Example* 5.1 *($m = 32$)*

μ Re		1 0.01515	1/20 0.30303	1/40 0.60606	1/80 1.21212	1/160 2.42424	1/1600 24.2424
PHSS(α^*)	IT	47	52	54	55	57	62
	CPU	13.50	18.641	19.344	**14.750**	20.609	22.187
AHSS(α^*, β^*)	IT	35	39	41	42	43	47
	CPU	**9.984**	**13.969**	**14.703**	**11.156**	**15.531**	**16.968**
GMRES(5)	IT	160	160	160	160	160	160
	CPU	—	—	—	—	—	—
	RES	7.45×10^{-03}	1.27×10^{-01}	1.30×10^{-01}	1.34×10^{-01}	1.33×10^{-01}	1.29×10^{-01}
PHSS-GMRES(5)	IT	46	52	62	69	109	160
	CPU	20.484	22.50	26.891	30.031	47.156	–
AHSS-GMRES(5)	IT	36	43	54	54	69	118
	CPU	15.188	18.562	23.375	23.594	29.891	51.109
GMRES(10)	IT	160	160	160	160	160	160
	CPU	—	—	—	—	—	—
	RES	7.29×10^{-03}	8.74×10^{-02}	8.34×10^{-02}	9.48×10^{-02}	6.42×10^{-02}	5.81×10^{-02}
PHSS-GMRES(10)	IT	42	47	49	72	51	75
	CPU	16.985	17.906	18.656	28.937	19.188	28.516
AHSS-GMRES(10)	IT	35	39	46	67	48	71
	CPU	14.094	14.703	17.609	26.453	18.157	27.125
GMRES(20)	IT	160	160	160	160	160	160
	CPU	—	—	—	—	—	—
	RES	6.92×10^{-03}	8.74×10^{-02}	8.34×10^{-02}	7.17×10^{-02}	6.42×10^{-02}	5.81×10^{-02}
PHSS-GMRES(20)	IT	42	47	49	49	51	75
	CPU	16.234	17.969	18.735	18.859	19.188	28.516
AHSS-GMRES(20)	IT	35	39	46	46	48	71
	CPU	13.391	14.750	17.719	17.750	18.157	27.032
GMRES	IT	160	160	160	160	160	160
	CPU	—	—	—	—	—	—
	RES	5.26×10^{-03}	3.37×10^{-02}	2.53×10^{-02}	5.16×10^{-02}	1.02×10^{-02}	2.32×10^{-03}
PHSS-GMRES	IT	40	45	48	50	52	60
	CPU	14.890	16.657	17.734	18.641	19.281	22.203
AHSS-GMRES	IT	34	38	40	41	43	48
	CPU	**12.670**	**14.125**	**14.781**	15.281	**15.907**	**17.797**

Step 2: Compute the partial vectors $y^{(k+1)}$ and $z^{(k+1)}$ by solving the linear subsystems

$$\begin{bmatrix} \alpha I & E \\ -E^* & \beta I \end{bmatrix} \begin{bmatrix} y^{(k+1)} \\ z^{(k+1)} \end{bmatrix} = \begin{bmatrix} (\alpha I - B) y^{(k+\frac{1}{2})} + f \\ (\beta I - C_0) z^{(k+\frac{1}{2})} + g \end{bmatrix}.$$

Analogously to Benzi & Golub (2004) and Bai *et al.* (2006a), we can prove that the above AHSS iteration method also converges unconditionally to the exact solution of the generalized saddle-point problem (6.1). Moreover, similarly to Theorem 4.1, we can demonstrate that there exist positive reals $\widetilde{\alpha}$

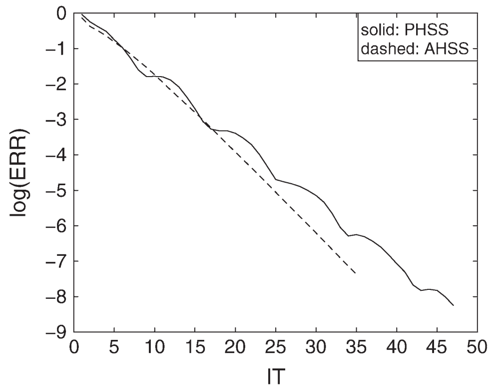

FIG. 4. The curves of relative solution error versus iteration number when $m = 32$ and $\mu = 1$.

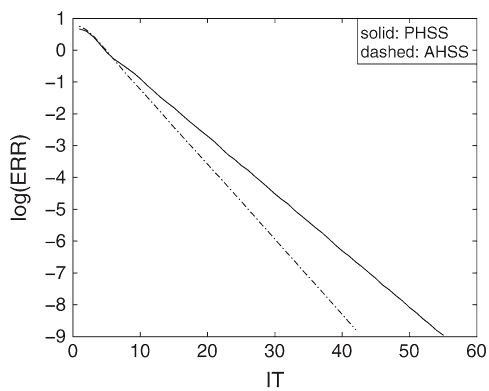

FIG. 5. The curves of relative solution error versus iteration number when $m = 32$ and $\mu = 1/80$.

and $\widetilde{\beta}$ such that the matrix $M(\alpha, \beta)^{-1}A$ is positive definite for all $\alpha \in (0, \widetilde{\alpha})$ and $\beta \in (0, \widetilde{\beta})$, where

$$M(\alpha, \beta) := \frac{1}{2}\begin{bmatrix} \alpha I + B & \frac{1}{\alpha}(\alpha I + B)E \\ -\frac{1}{\beta}(\beta I + C_0)E^* & \beta I + C_0 \end{bmatrix}$$

is the corresponding AHSS preconditioner.

It is easily seen that the above AHSS iteration method is a two-parameter generalization of the HSS and the PSS iteration methods[4] studied in Benzi & Golub (2004) and Pan *et al.* (2006); see also Bai *et al.* (2004). In Section 3, we have precisely computed its optimal iteration parameters and the corresponding optimal convergence factor for the special case $B = I$ and $C_0 = O$. In general, however, the precise determination of these optimal quantities is a very difficult problem, even if we assume that B is Hermitian positive definite and $C_0 = O$ (see Bai *et al.*, 2006b; Benzi & Ng, 2006).

If the $(1, 1)$-block B in the coefficient matrix A in the generalized saddle-point problem (6.1) is Hermitian and singular, but is positive definite on the null space of E^*, we can still apply the AHSS

[4]PSS is an abbreviation of the term “Positive-definite and skew-Hermitian splitting”.

iteration technique, combined with an extrapolate strategy, to compute an approximation to its solution (see Benzi & Golub, 2004). In particular, for $C_0 = O$, as an alternative the corresponding saddle-point problem can also be solved iteratively by directly applying the AHSS iteration technique to its equivalent variant

$$\begin{bmatrix} \widehat{B} & E \\ -E^* & O \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \widehat{f} \\ g \end{bmatrix},$$

where $\widehat{B} = B + E\Omega^{-1}E^*$, $\widehat{f} = f - E\Omega^{-1}g$ and $\Omega \in \mathbb{C}^{q\times q}$ is a prescribed Hermitian positive-definite matrix.

7. Conclusion

For large sparse saddle-point problems, by technically utilizing preconditioning and the HSS techniques, we have presented a class of very effective splitting iteration schemes, called the AHSS iteration methods. Theoretically, we have proved the unconditional convergence, computed the optimal iteration parameters and the corresponding optimal convergence factors and described the convergence properties of the preconditioned Krylov subspace methods, for this AHSS method. Also, we have shown that the AHSS method has considerably smaller asymptotic convergence factor than the PHSS method studied in Bai *et al.* (2004), in particular, when the optimal iteration parameter(s) is (are) employed. Further, we have confirmed numerically that both PHSS and AHSS methods are superior to GMRES and its restarted variants, and AHSS always performs much better than PHSS both as solver and as a preconditioner. Note that the computer storage for the AHSS method is much smaller than that for the GMRES method. Therefore, the AHSS is a very powerful and attractive iterative method for solving large sparse saddle-point problems.

Lastly, we should point out that further work, such as the establishment of practical formulas for computing the optimal iteration parameters and comparisons with known and the constructions of new inner iterations for the AHSS method, deserve in-depth studies, so that we may obtain more robust, more practical and more effective inexact variants of the AHSS iteration method for solving large sparse saddle-point problems.

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