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Several variants of the Hermitian and skew-Hermitian splitting method for a class of complex symmetric linear systems

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SUMMARY

This paper is concerned with several variants of the Hermitian and skew-Hermitian splitting iteration method to solve a class of complex symmetric linear systems. Theoretical analysis shows that several Hermitian and skew-Hermitian splitting based iteration methods are unconditionally convergent. Numerical experiments from an n-degree-of-freedom linear system are reported to illustrate the efficiency of the proposed methods. Copyright © 2014 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Many problems in the area of scientific computing require the solution of the large linear system, that is to say,

$$Ax = b, \tag{1}$$

where A is a complex symmetric matrix of the form

$$A = W + iT. \tag{2}$$

Here, $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ are symmetric, and $i = \sqrt{-1}$ denotes the imaginary unit. One can readily verify that A is non-Hermitian. System such as (1) is important and arises in a variety of scientific and engineering applications, including structural dynamics [1–3], diffuse optical tomography [4], FFT-based solution of certain time-dependent PDEs [5], lattice quantum chromodynamics [6], molecular dynamics and fluid dynamics [7], quantum chemistry, and eddy current problem [8, 9]. One can see References [10, 11] for more examples and additional references.

In recent years, various authors have contributed to the development of the iteration methods for complex symmetric linear system. There are two general approaches to solve the complex linear system (1). One is to deal with one of the several $2n \times 2n$ equivalent real formulations to avoid solving the complex linear system [7, 10, 12–16], and the other is to tackle the $n \times n$ linear system (1) directly, such as Conjugate Orthogonal Conjugate Gradient [17], Complex Symmetric (CSYM) [18], and Quasi-Minimal Residual [19].

Recently, based on the Hermitian and skew-Hermitian parts of the coefficient matrix A: $A = H + S$ with $H = \frac{1}{2}(A + A^*) = W$ and $S = \frac{1}{2}(A - A^*) = iT$, Bai *et al.* [2] skillfully designed a modified Hermitian and skew-Hermitian splitting (MHSS) method to solve the complex symmetric linear system (1) and described in the following.

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The MHSS method. Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty}$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - iT)x^{(k)} + b, \\ (\alpha I + T)x^{(k+1)} = (\alpha I + iW)x^{(k+\frac{1}{2})} - ib, \end{cases} \tag{3}$$

where α is a given positive constant and I is the identity matrix.

Theoretical analysis in [2] shows that the MHSS method converges unconditionally to the unique solution of the complex symmetric linear system (1) when $W \in \mathbb{R}^{n \times n}$ is symmetric positive definite and $T \in \mathbb{R}^{n \times n}$ is symmetric positive semidefinite. The corresponding optimum parameter $\alpha = \sqrt{\lambda_{\min}(W)\lambda_{\max}(W)}$ is obtained to minimize an upper bound on the spectral radius of the iteration matrix associated with (3).

The potential advantage of the MHSS method over the HSS method [20] for solving the complex symmetric linear system (1) is that only two linear subsystems with coefficient matrices $\alpha I + W$ and $\alpha I + T$, both being real and symmetric positive definite, need to be solved at each step. Therefore, in this case, these two linear subsystems can be solved either exactly by a sparse Cholesky factorization or inexactly by conjugated gradient scheme. That is to say, the MHSS method successfully avoids solving a shifted skew-Hermitian linear subsystem with coefficient matrix $\alpha I + iT$.

To generalize the concept of this method and accelerate its convergence rate, using the symmetric positive definite matrix V instead of the identity matrix I in (3) yields the preconditioned MHSS (PMHSS) method [3, 13, 21]. It is not difficult to find that the convergence properties of the PMHSS method are similar to the MHSS method. In [22], based on Hermitian positive semidefinite matrix $-(-iT)^2 = (-iT)^*(-iT)$, Bai exquisitely designed the skew-normal splitting (SNS) and skew-scaling splitting (SSS) methods. Because the coefficient matrices of both SNS and SSS are exactly the same, the SNS method is simply reviewed and precisely described as follows.

The SNS method. Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty}$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I - iT)x^{(k+\frac{1}{2})} = (\alpha W - T^2)x^{(k)} - iTb, \\ (\alpha W + T^2)x^{(k+1)} = (\alpha I + iT)x^{(k+\frac{1}{2})} - iTb, \end{cases} \tag{4}$$

where α is a given positive constant.

Theoretical analysis in [22] shows that the SNS (SSS) method converges unconditionally to the unique solution of the complex symmetric linear system (1). The corresponding optimal parameter is derived to minimize an upper bound on the spectral radius of the iteration matrix associated with (4).

Whereas if $W \in \mathbb{R}^{n \times n}$ is a symmetric indefinite matrix and $T \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix, then the matrix $\alpha I + W$ $(\alpha V + W, \alpha W + T^2)$ may be indefinite or singular, and the convergence speeds of the MHSS (PMHSS) or SNS (SSS) method may be unacceptably slow because the spectral radius of the iteration matrix of the corresponding iteration method may be very close to 1, or even larger than 1. That is to say, the MHSS (PMHSS) or SNS (SSS) method may be invalid. In this situation, it is necessary to derive new iteration methods to solve the complex symmetric linear system (1) with real symmetric indefinite matrix W and real symmetric positive definite matrix T. According to the promising behaviors and elegant mathematical properties of the HSS method, we make use of the HSS method again to solve the complex symmetric linear system (1), that is, this paper is to establish several variants of the HSS method for real symmetric indefinite matrix W and real symmetric positive definite matrix T. In [22], Bai skillfully designed the SNS (SSS) method by using Hermitian positive semidefinite matrix $-(iT)^2 = (iT)^*(iT)$. In this paper, to avoid the real symmetric indefinite matrix W, our approach is to design several variants of the HSS method for solving the complex symmetric linear system (1) by making use of symmetric positive definite matrix W^2. The results show that the methods studied in this paper are not only suitable for W-the real symmetric indefinite matrix but also suitable for any nonsingular symmetric matrix.

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The remainder of the paper is organized as follows. In Section 2, based on the HSS of the coefficient matrix, we establish a Hermitian normal splitting (HNS) method for solving the complex symmetric linear system (1) and discuss its convergence properties. In Section 3, a simplified HNS (SHNS) method is derived and the accompanying eigenvalue distribution of the preconditioned matrix is discussed. Some implementation aspects are briefly discussed in Section 4. The results of the numerical experiments from an n-degree-of-freedom (n-DOF) linear system are reported in Section 5; numerical comparisons show that the proposed SHNS method is much faster for the symmetric positive definite matrix W than the SSS method [22]. Finally, in Section 6, we give some conclusions to end the paper.

2. THE HERMITIAN-NORMAL EQUATIONS

In [23], Bayliss *et al.* equivalently transformed (1) into the normal equations $A^*Ax = A^*b$ and employed a preconditioned conjugate gradient (CG) method to solve the normal equations, where A^* denotes the Hermitian conjugate of A. Although the coefficient matrix A^*A of the normal equations is Hermitian positive definite, the resulting iteration method tends to converge slowly [23, 24]. Even if (1) can be rewritten as a real linear system of twice the size, but the real system is much harder to solve by CG type methods than the original complex linear system [24]. In this case, it is necessary to depend on the preconditioner for the real linear system, as it is shown in [13]. In [24], Incomplete LU is used and that is not good enough.

Because $W \in \mathbb{R}^{n\times n}$ is a symmetric indefinite matrix, then W^2 is symmetric positive definite. Based on this, we multiply (2) on the left by W to obtain a splitting of what we call the Hermitian-normal equations

$$WAx = (W^2 + iWT)x = Wb. \tag{5}$$

Further, we have the following two equivalent forms, that is to say,

$$(\alpha T + iWT)x = (\alpha T - W^2)x + Wb, \tag{6a}$$

$$(\alpha T + W^2)x = (\alpha T - iWT)x + Wb. \tag{6b}$$

It is interesting to note that the new term Tx does not bring about extra work when we solve (6a) before (6b). In this way, we obtain the corresponding HNS method.

The HNS method. Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty}$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I + iW)x^{(k+\frac{1}{2})} = (\alpha T - W^2)x^{(k)} + Wb, \\ (\alpha T + W^2)x^{(k+1)} = (\alpha I - iW)x^{(k+\frac{1}{2})} + Wb, \end{cases} \tag{7}$$

where α is a given positive constant.

By eliminating the intermediate vector $x^{(k+\frac{1}{2})}$, we obtain the following iteration in fixed point form as

$$x^{(k+1)} = M_\alpha x^{(k)} + N_\alpha Wb, k = 0, 1, 2, \ldots$$

where

$$M_\alpha = (\alpha T + W^2)^{-1}(\alpha I - iW)(\alpha I + iW)^{-1}(\alpha T - W^2)$$

and

$$N_\alpha = 2\alpha(\alpha T + W^2)^{-1}(\alpha I + iW)^{-1}.$$

Obviously, M_α is the iteration matrix of the HNS iteration method.

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In addition, if we introduce the matrix

$$B_\alpha = \frac{1}{2\alpha}(\alpha I + iW)(\alpha T + W^2) \text{ and } C_\alpha = \frac{1}{2\alpha}(\alpha I - iW)(\alpha T - W^2),$$

then

$$WA = B_\alpha - C_\alpha \text{ and } M_\alpha = B_\alpha^{-1} C_\alpha. \tag{8}$$

Therefore, one can readily verify that the HNS iteration method can be induced by the matrix splitting $WA = B_\alpha - C_\alpha$. Another approach to obtain the HNS iteration method is that we can transform $Ax = b$ into $-iAx = -ib$, that is, $(T - iW)x = -ib$. Using the SNS method (4) for the linear system $(T - iW)x = -ib$ yields the HNS iteration method (7).

By surveying the iteration matrix M_α, it is easy to observe that for any $\alpha > 0$, all the eigenvalues of M_α by absolute value are less than 1 and are included in the interval $(-1, 1)$. Hence, we have the following proposition.

Proposition 2.1
Let W be a real symmetric indefinite matrix and T be a real symmetric positive definite matrix. Then, all the eigenvalues of M_α are included in the interval $(-1, 1)$ for all $\alpha > 0$.

Proof
By the similarity invariance of the spectrum of the matrix, we get that the iteration matrix M_α is similar to

$$\hat{M}_\alpha = (\alpha I - iW)(\alpha I + iW)^{-1}(\alpha T - W^2)(\alpha T + W^2)^{-1}.$$

Let $Q_\alpha = (\alpha I - iW)(\alpha I + iW)^{-1}$. Then,

$$\begin{aligned} Q_\alpha^* Q_\alpha &= [(\alpha I - iW)(\alpha I + iW)^{-1}]^*(\alpha I - iW)(\alpha I + iW)^{-1} \\ &= [(\alpha I + iW)^{-1}]^*(\alpha I - iW)^*(\alpha I - iW)(\alpha I + iW)^{-1} \\ &= [(\alpha I + iW)^*]^{-1}(\alpha I + iW)(\alpha I - iW)(\alpha I + iW)^{-1} \\ &= (\alpha I - iW)^{-1}(\alpha I + iW)(\alpha I - iW)(\alpha I + iW)^{-1} \\ &= (\alpha I - iW)^{-1}(\alpha I - iW)(\alpha I + iW)(\alpha I + iW)^{-1} \\ &= I. \end{aligned}$$

That is to say, Q_α is a unitary matrix. Because T and W^2 are symmetric positive definite, there exists an invertible matrix S such that $T = SS^*$ and $W^2 = S\Lambda S^*$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ with λ_i $(i = 1, 2, \cdots, n) \in \lambda(W^2 T^{-1})$ positive. Thus,

$$(\alpha T - W^2)(\alpha T + W^2)^{-1} = S(\alpha I - \Lambda)(\alpha I + \Lambda)^{-1} S^{-1}.$$

It follows that

$$\|(\alpha I - \Lambda)(\alpha I + \Lambda)^{-1}\|_2 = \max_i \left| \frac{\alpha - \lambda_i}{\alpha + \lambda_i} \right| < 1,$$

which completes the proof. □

Concerning the convergence property of the HNS iteration method (7), we have the following theorem.

Theorem 2.1
Let W be a real symmetric indefinite matrix and T be a real symmetric positive definite matrix, and let α be a positive constant. Then, the spectral radius $\rho(M_\alpha)$ of the HNS iteration matrix is bounded by

$$\sigma(\alpha) \equiv \max_{\mu_i \in \mu(W^{-1} T W^{-1})} \left| \frac{\alpha \mu_i - 1}{\alpha \mu_i + 1} \right|,$$

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where $\mu(W^{-1}TW^{-1})$ is the spectrum of the matrix $W^{-1}TW^{-1}$. Therefore, it follows that

$$\rho(M_\alpha) \leqslant \sigma(\alpha) < 1 \text{ for } \forall \alpha > 0.$$

that is, the HNS iteration method (7) converges unconditionally to the unique solution of the complex symmetric linear system (1).

Proof
Clearly,

$$\begin{aligned}
\rho(M_\alpha) &= \rho\left(\hat{M}_\alpha\right) \\
&= \rho\left((\alpha I - iW)(\alpha I + iW)^{-1}W(\alpha W^{-1}TW^{-1} - I)(\alpha W^{-1}TW^{-1} + I)^{-1}W^{-1}\right) \\
&= \rho\left(W^{-1}(\alpha I - iW)(\alpha I + iW)^{-1}W(\alpha W^{-1}TW^{-1} - I)(\alpha W^{-1}TW^{-1} + I)^{-1}\right) \\
&= \rho\left((\alpha W^{-1} - iI)(\alpha W^{-1} + iI)^{-1}(\alpha W^{-1}TW^{-1} - I)(\alpha W^{-1}TW^{-1} + I)^{-1}\right).
\end{aligned}$$

Let $U_\alpha = (\alpha W^{-1} - iI)(\alpha W^{-1} + iI)^{-1}$. Then,

$$\begin{aligned}
U_\alpha^* U_\alpha &= \left[(\alpha W^{-1} - iI)(\alpha W^{-1} + iI)^{-1}\right]^* (\alpha W^{-1} - iI)(\alpha W^{-1} + iI)^{-1} \\
&= \left[(\alpha W^{-1} + iI)^{-1}\right]^* (\alpha W^{-1} - iI)^*(\alpha W^{-1} - iI)(\alpha W^{-1} + iI)^{-1} \\
&= \left[(\alpha W^{-1} + iI)^*\right]^{-1} (\alpha W^{-1} + iI)(\alpha W^{-1} - iI)(\alpha W^{-1} + iI)^{-1} \\
&= (\alpha W^{-1} - iI)^{-1}(\alpha W^{-1} + iI)(\alpha W^{-1} - iI)(\alpha W^{-1} + iI)^{-1} \\
&= (\alpha W^{-1} - iI)^{-1}(\alpha W^{-1} - iI)(\alpha W^{-1} + iI)(\alpha W^{-1} + iI)^{-1} \\
&= I.
\end{aligned}$$

That is to say, U_α is a unitary matrix. Therefore, $\|U_\alpha\|_2 = 1$. It is easy to see that $W^{-1}TW^{-1}$ is a symmetric positive definite matrix. Consequently,

$$\begin{aligned}
\rho(M_\alpha) &\leqslant \left\|(\alpha W^{-1} - iI)(\alpha W^{-1} + iI)^{-1}(\alpha W^{-1}TW^{-1} - I)(\alpha W^{-1}TW^{-1} + I)^{-1}\right\|_2 \\
&\leqslant \left\|(\alpha W^{-1} - iI)(\alpha W^{-1} + iI)^{-1}\right\|_2 \|(\alpha W^{-1}TW^{-1} - I)(\alpha W^{-1}TW^{-1} + I)^{-1}\|_2 \\
&= \left\|(\alpha W^{-1}TW^{-1} - I)(\alpha W^{-1}TW^{-1} + I)^{-1}\right\|_2 \\
&= \max_{\mu_i \in \mu(W^{-1}TW^{-1})} \left|\frac{\alpha\mu_i - 1}{\alpha\mu_i + 1}\right|.
\end{aligned}$$

Because all the eigenvalues of the matrix $W^{-1}TW^{-1}$ are positive and α is a positive constant, it is easy to see that $\rho(M_\alpha) \leqslant \sigma(\alpha) < 1$. □

If the extreme eigenvalues of the symmetric positive definite matrix $W^{-1}TW^{-1}$ are known, then the value of α, which minimizes the upper bound $\sigma(\alpha)$, can be obtained. This fact is precisely stated as the following theorem.

Theorem 2.2
Let the conditions of Theorem 2.1 be satisfied. Let $\mu_{\min}$ and $\mu_{\max}$ be the extreme eigenvalues of the symmetric positive definite matrix $W^{-1}TW^{-1}$, respectively. Then,

$$\alpha^\star = \arg\min_\alpha \left\{\max_{\mu_{\min} \leqslant \mu \leqslant \mu_{\max}} \left|\frac{1 - \alpha\mu}{1 + \alpha\mu}\right|\right\} = \frac{1}{\sqrt{\mu_{\min}\mu_{\max}}},$$

and

$$\sigma(\alpha^\star) = \frac{\sqrt{\mu_{\max}} - \sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}} + \sqrt{\mu_{\min}}} = \frac{\sqrt{\kappa(W^{-1}TW^{-1})} - 1}{\sqrt{\kappa(W^{-1}TW^{-1})} + 1},$$

where $\kappa(W^{-1}TW^{-1}) = \frac{\mu_{\max}}{\mu_{\min}}$ is the spectral condition number of the matrix $W^{-1}TW^{-1}$.

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Proof
We define the function $f(\mu)$ with respect to μ. That is to say,

$$f(\mu) = \frac{1-\alpha\mu}{1+\alpha\mu}.$$

It follows that

$$f'(\mu) = -\frac{2\alpha}{(1+\alpha\mu)^2} < 0.$$

Further, we have

$$\begin{aligned}\sigma(\alpha) &= \max\left\{\left|\frac{\alpha\mu_{\min}-1}{\alpha\mu_{\min}+1}\right|, \left|\frac{\alpha\mu_{\max}-1}{\alpha\mu_{\max}+1}\right|\right\}\\ &= \max\left\{\left|\frac{\frac{1}{\alpha}-\mu_{\min}}{\frac{1}{\alpha}+\mu_{\min}}\right|, \left|\frac{\frac{1}{\alpha}-\mu_{\max}}{\frac{1}{\alpha}+\mu_{\max}}\right|\right\}.\end{aligned}$$

To compute an approximate optimal $\alpha > 0$ such that the convergence factor $\rho(M_\alpha)$ of the HNS iteration is minimized, we can minimize the upper bound $\sigma(\alpha)$ of $\rho(M_\alpha)$ instead. If $\sigma(\alpha^\star)$ is such a minimum point, then it must satisfy $\frac{1}{\alpha^\star} - \mu_{\min} > 0$, and $\frac{1}{\alpha^\star} - \mu_{\max} < 0$, and

$$\frac{\frac{1}{\alpha^\star}-\mu_{\min}}{\frac{1}{\alpha^\star}+\mu_{\min}} = \frac{\mu_{\max}-\frac{1}{\alpha^\star}}{\mu_{\max}+\frac{1}{\alpha^\star}}.$$

Therefore,

$$\alpha^\star = \frac{1}{\sqrt{\mu_{\min}\mu_{\max}}},$$

and the result of $\sigma(\alpha^\star)$ holds. □

3. THE SIMPLIFIED HERMITIAN-NORMAL EQUATIONS

To obtain the simplified Hermitian-normal equations, the linear system (1) can be rewritten in the following two equivalent forms:

$$\begin{aligned}\left(W + \frac{i}{\alpha}W^2\right)x &= \left(-iT + \frac{i}{\alpha}W^2\right)x + b,\\ \left(iT + \frac{i}{\alpha}W^2\right)x &= \left(-W + \frac{i}{\alpha}W^2\right)x + b.\end{aligned}$$

Simple computation shows that

$$(\alpha I + iW)iWx = (\alpha T - W^2)x + i\alpha b, \tag{9a}$$
$$(\alpha T + W^2)x = (\alpha I - iW)iWx - i\alpha b. \tag{9b}$$

From (9a) and (9b), we can obtain the following iteration form:

$$\begin{cases}(\alpha I + iW)iWx^{(k+\frac{1}{2})} = (\alpha T - W^2)x^{(k)} + i\alpha b,\\ (\alpha T + W^2)x^{(k+1)} = (\alpha I - iW)iWx^{(k+\frac{1}{2})} - i\alpha b.\end{cases} \tag{10}$$

It is easy to see that in actual implementation, the new term $iWx^{(k+\frac{1}{2})}$ in (10) causes no extra work. In this case, making use of $x^{(k+\frac{1}{2})}$ instead of $iWx^{(k+\frac{1}{2})}$ in (10) yields the SHNS method, which is described as follows.

The SHNS method. Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty}$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I + iW)x^{(k+\frac{1}{2})} = (\alpha T - W^2)x^{(k)} + i\alpha b, \\ (\alpha T + W^2)x^{(k+1)} = (\alpha I - iW)x^{(k+\frac{1}{2})} - i\alpha b. \end{cases} \tag{11}$$

where α is a given positive constant.

Obviously, we can rewrite (1) as

$$(T - iW)x = -ib,$$

or

$$Tx = iWx - ib. \tag{12}$$

From (12), the following two equivalent forms are obtained:

$$\left(iW - \frac{1}{\alpha}W^2\right)x = \left(T - \frac{1}{\alpha}W^2\right)x + ib, \tag{13a}$$

$$\left(T + \frac{1}{\alpha}W^2\right)x = \left(iW + \frac{1}{\alpha}W^2\right)x - ib. \tag{13b}$$

From (13a) and (13b), we still have (10) by the simple transformation. Therefore, the SHNS method is also derived.

Comparing the SHNS method with the HNS method, it is easy to see that the coefficient matrices of the SHNS method are the same as those of the HNS method. Consequently, the iteration matrix of the SHNS method is also the matrix M_α. The only difference between the SHNS method and the HNS method is on the constant vector terms. Specifically, the two constant vector terms of the HNS method are exactly the same: Wb, and the two constant vector terms of the SHNS method is complex conjugate: $\pm i\alpha b$. Thus, we can draw a conclusion that the SHNS method has the same convergence rate but is much cheaper than the HNS method.

It is noted that the HNS or SHNS method needs to solve two linear subsystems with the n-by-n matrices $\alpha T + W^2$ and $\alpha I + iW$. Because the coefficient matrix, $\alpha T + W^2$, is Hermitian positive definite, its solution does not cause significant difficulties, for example, we can employ the CG method to solve it. However, the coefficient matrix, $\alpha I + iW$ is skew-Hermitian, in this case, its solution can be much more problematic, but we can employ some Krylov subspace methods (such as generalized minimal residual [GMRES]) to solve it at each step of the HNS or SHNS iteration. There also exists a situation where skew-Hermitian matrix is structured to make the shifted skew-Hermitian system easy to solve; see [25] for an example arising in image processing and [26] for arising in fluid mechanics, as well as Section 5.

Based on the aforementioned discussion, the convergence of the SHNS method can be described by the following theorem.

Theorem 3.1

Let W be a real symmetric indefinite matrix and T be a real symmetric positive definite matrix, and let α be a positive constant. Then, the spectral radius $\rho(M_\alpha)$ of the SHNS iteration matrix is bounded by

$$\sigma(\alpha) \equiv \max_{\mu_i \in \mu(W^{-1}TW^{-1})} \left| \frac{\alpha\mu_i - 1}{\alpha\mu_i + 1} \right|,$$

where $\mu(W^{-1}TW^{-1})$ is the spectrum of the matrix $W^{-1}TW^{-1}$. Therefore, it follows that

$$\rho(M_\alpha) \leqslant \sigma(\alpha) < 1 \text{ for } \forall \alpha > 0.$$

that is, the SHNS iteration method (11) converges unconditionally to the unique solution of the complex symmetric linear system (1).

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Analogously to Theorem 2.2, we also have the following theorem.

Theorem 3.2
Let the conditions of Theorem 3.1 be satisfied. Let $\mu_{\min}$ and $\mu_{\max}$ be the extreme eigenvalues of the symmetric positive definite matrix $W^{-1}TW^{-1}$, respectively. Then,

$$\alpha^{\star} = \arg\min_{\alpha}\left\{\max_{\mu_{\min}\leqslant\mu\leqslant\mu_{\max}}\left|\frac{1-\alpha\mu}{1+\alpha\mu}\right|\right\} = \frac{1}{\sqrt{\mu_{\min}\mu_{\max}}},$$

and

$$\sigma(\alpha^{\star}) = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}} = \frac{\sqrt{\kappa(W^{-1}TW^{-1})}-1}{\sqrt{\kappa(W^{-1}TW^{-1})}+1},$$

where $\kappa(W^{-1}TW^{-1}) = \frac{\mu_{\max}}{\mu_{\min}}$ is the spectral condition number of the matrix $W^{-1}TW^{-1}$.

Some remarks on Theorems 2.1 and 3.1 are given in the succeeding text.

- It is easy to see that if W is a real nonsingular symmetric matrix and T is a real symmetric positive definite matrix in Theorems 2.1 and 3.1, then the corresponding results still hold.
- From Theorems 2.1 and 3.1, one can see that the convergence rates of both the SHNS and HNS methods are bounded by $\sigma(\alpha)$, which only depends on the spectrum of the symmetric positive definite matrix $W^{-1}TW^{-1}$ but does not depend on the spectrum of the skew-Hermitian part iW, the spectrum of the coefficient matrix A, nor on the eigenvectors of the matrices W, T, A, and $W^{-1}TW^{-1}$.
- It should be noted that the iteration parameter $\alpha^{\star}$ in Theorems 2.2 and 3.2 only minimizes the upper bound $\sigma(\alpha)$ on the spectral radius $\rho(M_\alpha)$ of both SHNS and HNS iteration matrix M_α, but not $\rho(M_\alpha)$ itself. The form of the iteration parameter $\alpha^{\star}$ is similar to that of the HSS iteration method and its variants (see [27–31]).
- In [22], based on $H = \frac{1}{2}(A + A^*) = W$ and $S = \frac{1}{2}(A^* - A) = -iT$, the SNS and SSS methods are successfully designed, and their iteration matrix is the same. That is to say,

  $$\bar{M}_\alpha = (\alpha W + T^2)^{-1}(\alpha I + iT)(\alpha I - iT)^{-1}(\alpha W - T^2).$$

  Let $P = -S^{-1}HS^{-1}$. To minimize the upper bound on the spectral radius $\rho(\bar{M}_\alpha)$, the optimal parameter is derived as follows:

  $$\alpha^{\star} = \sqrt{\lambda_{\min}(P)\lambda_{\max}(P)},$$

  where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively, are the minimum and the maximum eigenvalues of the matrix P. From Theorem 2.2, it is not difficult to find that the optimal parameter of the corresponding iteration matrix $\bar{M}_\alpha$ is $\frac{1}{\alpha^{\star}}$, not $\alpha^{\star}$. To elaborate this situation, we consider the following simple example. Let

  $$W = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \text{ and } T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

  By simple computations, we obtain that $\lambda_{\min}(P) = 1$ and $\lambda_{\max}(P) = 4$. If the optimal parameter $\alpha^{\star} = \sqrt{\lambda_{\min}(P)\lambda_{\max}(P)} = 2$, then $\rho\left(\bar{M}_{\alpha^{\star}}\right) = 0.7778$ and $\sigma(\alpha^{\star}) = 0.3333$, which contradicts with $\rho\left(\bar{M}_{\alpha^{\star}}\right) \leqslant \sigma(\alpha^{\star}) < 1$.
- From Theorem 2.2 or 3.2, in theory, for the HNS or SHNS method, there exists a formula to compute the optimal parameter α, that is to say, $\alpha^{\star} = \frac{1}{\sqrt{\mu_{\min}\mu_{\max}}}$, where $\mu_{\min}$ and $\mu_{\max}$, respectively, are the extreme eigenvalues of the symmetric positive definite matrix $W^{-1}TW^{-1}$. In practice, to further improve the efficiency of the HNS or SHNS method, it is desirable to determine or find a good estimate of the optimal parameter that minimizes the convergence factor. Whereas finding the actual optimal estimate α for the HNS or SHNS method is a hard task because its solution strongly depends on the particular structures and properties of the coefficient matrix A, as well as the splittings matrices W and T[32], and needs further in-depth

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study, both from theory and computation point of view. Fortunately, some good references are available on how to determine a good estimate for the optimal parameter for the corresponding iteration method. For example, by analyzing two-by-two real matrices, Bai *et al.* [28] obtained the optimal parameter to minimize the spectral radius of the iteration matrix of the corresponding HSS method, used these results to determine the optimal parameters for linear systems associated with certain two-by-two block matrices and to estimate the optimal parameter of the HSS method for general nonsymmetric positive definite linear systems. Subsequently, the optimal parameters of its variants such as the Preconditioned Hermitian and skew-Hermitian splitting (PHSS) [27] and Accelerated Hermitian and skew-Hermitian splitting (AHSS) [29, 31] iteration methods for the saddle point linear systems are presented.

From (8), it is easy to see that the splitting matrix B_α can be used as a preconditioner matrix for the complex matrix WA. Note that the multiplicative factor $\frac{1}{2\alpha}$ has no effect on the preconditioned system, and therefore, it can be dropped. The corresponding preconditioning matrix can be of the form as follows:

$$B_\alpha = (\alpha I + iW)(\alpha T + W^2).$$

Application of the alternating preconditioner within GMRES requires solving a linear system of the form $B_\alpha z = r$ at each iteration. This is performed by first solving $(\alpha I + iW)v = r$ for v, followed by $(\alpha T + W^2)z = v$. A direct implication of Theorem 2.1 or Theorem 3.1 is that all the eigenvalues of the HNS-preconditioned matrix lie in the interior of the disk of radius 1 centered at the point $(1, 0)$, which implies that the preconditioned matrix is positive definite. If the spectral radius is small, all the eigenvalues of the preconditioned matrix are clustered around $(1, 0)$. As is well known, these are desirable properties when a Krylov subspace method such as GMRES is used to accelerate the basic iteration by the preconditioner B_α. With respect to this point, one can see Reference [31] for details.

The spectral distribution of $B_\alpha^{-1}WA$ is described in the following theorem.

Theorem 3.3
Let W be a real nonsingular symmetric matrix and T be a real symmetric positive definite matrix. Then, all the eigenvalues of $B_\alpha^{-1}WA$ satisfy $|1 - |\lambda|| < 1$.

Proof
As previously mentioned,

$$WA = B_\alpha - C_\alpha. \tag{14}$$

Not that $M_\alpha = B_\alpha^{-1}C_\alpha$. From (14), we have

$$B_\alpha^{-1}WA = I - B_\alpha^{-1}C_\alpha = I - M_\alpha.$$

Based on Theorem 2.1 or Theorem 3.1, it is easy to know that the absolute value of all the eigenvalues of M_α is less than 1. It follows that all the eigenvalues of $B_\alpha^{-1}WA$ satisfy $|1 - |\lambda|| < 1$ from [11]. □

4. IMPLEMENTATION ASPECTS

In the process of the HNS (SHNS) iteration method, it is necessary to solve two systems of linear equations whose coefficient matrices are $\alpha I + iW$ and $\alpha T + W^2$. This is a tough task, which may entail a high computational cost and be even impractical in actual implementations, particularly when the original problem arises from the discrete three-dimensional partial differential equation. To overcome this disadvantage and improve the computational efficiency of the HNS (SHNS) iteration method, similar to the inexact HSS (IHSS) iteration method [20, 33], we can employ the inexact HNS (IHNS) iteration method or the inexact SHNS (ISHNS) iteration method to solve the corresponding two subproblems. Because $\alpha T + W^2$ is symmetric positive definite, we can solve this system of linear equations by employing the CG method, and some Krylov subspace methods

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[34–36] to solve the system of linear equations with coefficient matrix $\alpha I + iW$. The convergence property of the IHNS (ISHNS) method can be established in an analogous fashion to that of the IHSS iteration method, by making use of Theorems 3.1 and 3.2 in [20]. Because the ISHNS method is similar to the IHNS method, the IHNS iteration scheme is only described in the following algorithm.

Algorithm 1
k = 0;
while (not convergent)
$r^{(k)} = Wb - WAx^{(k)}$;
approximately solve $(\alpha I + iW)z^{(k)} = r^{(k)}$ by employing GMRES method, such that the residual $p^{(k)} = r^{(k)} - (\alpha I + iW)z^{(k)}$ of the iteration satisfies $\|p^{(k)}\| \leqslant \eta_k \|r^{(k)}\|$;
$x^{(k+1/2)} = x^{(k)} + z^{(k)}$;
$r^{(k+1/2)} = Wb - WAx^{(k+1/2)}$;
approximately solve $(\alpha T + W^2)z^{(k+1/2)} = r^{(k+1/2)}$ by employing CG method, such that the residual $q^{(k+1/2)} = r^{(k+1/2)} - (\alpha T + W^2)z^{(k+1/2)}$ of the iteration satisfies $\|q^{(k+1/2)}\| \leqslant \tau_k \|r^{(k+1/2)}\|$;
$x^{(k+1)} = x^{(k+1/2)} + z^{(k+1/2)}$;
$k = k + 1$;
end

When solving (1) by Algorithm 1, some stopping tolerances (or number of inner iterations) are involved that have to be properly tuned when IHNS is used instead of HNS. In general, the tolerances for the inner iteration methods may be different, which vary in the light of the accuracy attained in the outer iteration scheme. Therefore, the IHNS iteration is actually a nonstationary iteration method for solving the system of linear equations (1). That is to say, IHNS cannot be used as a preconditioner for GMRES, and flexible GMRES [37] should be used instead.

It is not difficult to show that when the tolerances of the inner iterations tend to be zero with the outer iterate index increasing, the asymptotic convergence rate of the IHNS iteration approaches that of the HNS iteration. In particular, if the inner systems are solved exactly, the tolerances are all zeros, then the IHNS iteration essentially becomes the HNS iteration. To obtain a convergent IHNS, the tolerances are not required to approach zero with the outer iterate index increasing. Algorithm 1 implies that one can make choices of the tolerances $\{\eta_k\}$ and $\{\tau_k\}$ for convergence. Apparently, we find that there is a trade-off between inner and outer iteration with the choices of $\{\eta_k\}$ and $\{\tau_k\}$, which we can attest in the numerical examples. In the numerical examples, we find that the variation of $\{\eta_k\}$ and $\{\tau_k\}$ are small, and the change of iterations is big. This means that the IHNS method may be sensitive to the inner stopping tolerance. Therefore, finding the proper inner stopping tolerance may be important for the IHNS method. However, the optimal tolerances $\{\eta_k\}$ and $\{\tau_k\}$ are not easily determined. For more details, we refer to [20].

Next, we discuss the computational complexity of the HNS and IHNS methods. Similar to the IHSS iteration method [20, 33], we need to estimate their computer times (via operation counts) and computer memories. Assume that ϕ is the number of operations required to compute WAy for a given vector $y \in \mathbb{C}^n$, φ is the number of operations required to compute Wb, and $\chi_k(WT)$ and $\chi_k(iW)$ are the numbers of operations required to solve inner systems $q^{(k+1/2)} = r^{(k+1/2)} - (\alpha T + W^2)z^{(k+1/2)}$ and $p^{(k)} = r^{(k)} - (\alpha I + iW)z^{(k)}$ inexactly with the tolerances $\{\eta_k\}$ and $\{\tau_k\}$, respectively. Straightforward calculations show that the total work to compute each step of the IHNS iteration is $\mathcal{O}(4n + 2\phi + \chi_k(WT) + 2\varphi + \chi_k(iW))$.

In addition, a simple calculation shows that the memory is required to store $x^{(k)}$, $r^{(k)}$, and $z^{(k)}$. In Algorithm 1, some auxiliary vectors are only required. Moreover, it is not necessary to store W and T explicitly as matrices, as all we need are two subroutines that perform the matrix-vector multiplications with respect to these two matrices. Therefore, the total amount of computer memory is $\mathcal{O}(n)$.

5. NUMERICAL EXPERIMENTS

In this section, based on the aforementioned discussion, we give some numerical experiments to demonstrate the performance of the SHNS method, the HNS-preconditioner and the IHNS method. In addition, all experiments were performed on a personal computer with 2.7 GHz central processing unit [Intel Celeron G1620], 2.0 G memory, and Windows XP operating system.

5.1. The simplified Hermitian normal splitting method

In this subsection, we firstly investigate the performance of the SHNS method for the symmetric indefinite matrix W in (1). Secondly, we compare the SHNS and SSS methods for the symmetric positive definite matrix W in (1).

Now, the following complex symmetric linear system [2, 3, 10] is considered.

$$\left[(-\omega^2 M + K) + i(\omega C_V + C_H)\right] x = b, \tag{15}$$

where M and K are the inertia and stiffness matrices, respectively; C_V and C_H are the viscous and hysteretic damping matrices, respectively; and ω is the driving circular frequency. The complex symmetric system of linear equations (15) arises in direct frequency domain analysis of an n-DOF linear system. For more detail, we refer to [1–3, 10].

In our numerical computations, we take $C_H = \mu K$ with $\mu = 0.02$ being a damping coefficient, $\omega = 2\pi$, $C_V = \frac{1}{2}M$, and K the five-point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0,1] \times [0,1]$ with the mesh size $h = \frac{1}{m+1}$. In this case, the matrix $K \in \mathbb{R}^{n \times n}$ possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$ with $V_m = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m \times m}$ [2]. Hence, the total number of variables is $n = m^2$. In addition, the right-hand side vector b is to be adjusted such that $b = Ae$ ($e = (1, 1, \ldots, 1)^T$).

Subsequently, we study the SHNS iteration method (11) to solve the system of linear equations (15). To make the matrix $-\omega^2 M + K$ indefinite, some values of the inertia matrix M or the driving circular frequency ω are necessary to be selected. For simplicity, some values of the inertia matrix M are listed to make the matrix $-\omega^2 M + K$ indefinite in Tables I–V. The initial guess for all tests is zero. The tests are performed in MATLAB 7.0. The SHNS iteration method terminates if the relative residual error satisfies $\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} < 10^{-6}$. '$\alpha$' denotes the corresponding optimal iteration parameter. 'IT' denotes the number of iteration. 'CPU(s)' denotes the time (in seconds) required to solve a problem. 'T(s)$_\alpha$' denotes the time (in seconds) required to obtain the optimal iteration parameter α.

In Tables I–V, we present some iteration results to illustrate the convergence behavior of the SHNS method. In Tables I–V, we find that for the same grid, with the inertia matrix M increasing,

Table I. α, IT, and CPU for the simplified Hermitian normal splitting method with $n = 64$.

	M	I	$2I$	$5I$	$10I$	$15I$
$n = 64$	α	601.5546	142.9860	45.8121	4.7327	88.0815
	T(s)$_\alpha$	0.016	0.015	0.094	0.096	0.015
	IT	122	116	121	120	112
	CPU(s)	0.2750	0.1410	0.1560	0.1560	0.1570

Table II. α, IT, and CPU for the simplified Hermitian normal splitting method with $n = 256$.

	M	$2I$	$5I$	$10I$	$15I$	$20I$
$n = 256$	α	97.9316	453.9358	417.1668	436.4075	120.1608
	T(s)$_\alpha$	0.093	0.078	0.078	0.078	0.063
	IT	93	94	96	97	99
	CPU(s)	3.2810	3.3280	3.3900	3.6880	3.6710

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Table III. α, IT, and CPU for the simplified Hermitian normal splitting method with $n = 1024$.

	M	$2I$	$5I$	$10I$	$15I$	$20I$
$n = 1024$	α	54.5445	281.1549	308.8877	603.0693	87.1292
	$T(s)_\alpha$	8.156	8.515	8	7.907	7.531
	IT	80	79	80	80	80
	CPU(s)	100.23	103.391	99.3440	101.343	102.8600

Table IV. α, IT, and CPU for the simplified Hermitian normal splitting method.

	M	$5I$	$10I$	$15I$	$20I$	$25I$
$n = 1369$	α	248.3911	541.6339	72.6549	921.3546	942.8609
	$T(s)_\alpha$	12.391	12.187	11.672	11.922	11.468
	IT	77	77	77	77	78
	CPU(s)	181.391	184.953	187.359	186.36	188.75
$n = 1764$	α	222.0773	727.0064	148.0792	722.0376	267.5696
	$T(s)_\alpha$	27.484	26.891	26.828	26.421	25.922
	IT	75	75	75	75	75
	CPU(s)	395.204	393.718	384.266	381.406	387.312
$n = 2209$	α	200.59	661.8478	356.5255	298.2251	379.781
	$T(s)_\alpha$	53.938	52.469	52.547	51.719	53.266
	IT	73	73	73	74	74
	CPU(s)	682.062	686.516	687.313	685.141	682.937

Table V. IT for the simplified Hermitian normal splitting and conjugate gradient methods with $n = 4096$.

	M	$3I$	$5I$	$10I$	$15I$	$20I$
SHNS	α	359.71	150.4159	503.7095	993.7841	986.4061
	IT	69	69	69	69	69
CG	IT	—	—	—	—	—

SHNS = simplified Hermitian normal splitting; CG = conjugate gradient.

the change of the optimal iteration parameter α is not easily described. Yet, the change of the number of iterations is relatively stable. In Tables III–V, it is easy to see that the number of iterations has almost no change. What is more, for increasing grid size, the number of iterations reduces, which implies that the SHNS method may be suitable for the large sparse complex symmetric linear system (1).

In Table V, some iteration results are also listed to compare the SHNS method for (1) and CG for the corresponding normal equations $A^*Ax = A^*b$. In Table V, '—' denotes that the iteration numbers of CG is more than 500. In our numerical experiments, we find that the CG method to solve the corresponding normal equations does not converge indeed. In this case, the CPU time is not reported. Iteration results in Table V show that the coefficient matrix A^*A of the normal equations is Hermitian positive definite, but the resulting iteration method converges slowly. These further confirmed that the strategy in [23] may be undesirable.

In the sequel, we consider the SHNS and SSS methods to solve the complex symmetric linear system (1). To simultaneously make use of both the SHNS method and the SSS method to solve the complex symmetric linear system (1), some values of the inertia matrix M are listed to make the matrix $-\omega^2 M + K$ positive definite in Table VI. Here, we take $C_H = \mu K$ with $\mu = 0.02$, $\omega = \pi$, and $C_V = \frac{1}{2}M$. In Table VI, we present some iteration results to illustrate the convergence behaviors of the SHNS and SSS methods.

Table VI. α, IT, and CPU of simplified Hermitian normal splitting and skew-scaling splitting for $n = 1024$.

	M	$\frac{1}{5}I$	$\frac{2}{5}I$	$\frac{3}{5}I$	$\frac{4}{5}I$	I
SHNS	α	13885	10261	7843	6041.4	4608.3
	IT	85	83	82	81	80
	CPU(s)	107.3430	105.9840	103.4690	102.0160	101.3750
SSS	α	0.3142	0.4486	0.5492	0.6321	0.7080
	IT	142	143	144	145	146
	CPU(s)	220.766	222.422	222.7500	224.5310	227.4840

SHNS = simplified Hermitian normal splitting; SSS = skew-scaling splitting.

In Table VI, with the matrix W being symmetric positive definite, the optimal parameter, the number of iterations, and CPU's time of the SHNS method reduce when the inertia matrix M increases. In this case, with the inertia matrix M increasing, the efficiency of the SHNS method is better. However, the optimal parameter, the number of iterations, and CPU's time of the SSS method increase when the inertia matrix M increases. In our numerical experiments, when the SHNS and SSS methods are used to solve the system of linear equations (1), it is not difficult to get that the SHNS method outperforms the SSS method under certain conditions. Compared with the SSS method, the SHNS method applied to solve the complex symmetric linear system (1) may be the top priority under certain conditions.

In our numerical computations, from Tables I–VI, fixing the mesh size with the inertia matrix M increasing or fixing the inertia matrix M with the mesh size increasing, we find that the iteration parameter α is sensitive and irregular when the SHNS method is applied to solve the complex symmetric linear system (1).

It is noted that in our numerical experiments, the execution times in the Tables I–IV and VI seem to be large for the sizes of the considered systems, compared with the resulting times provided in [13]. The large execution times may be generated from the following three aspects: (1) our codes (our codes may be necessary to optimize in the numerical experiments); (2) the computer (the computer hardware and software may be improved); (3) the problem. In this paper, our goal is to design the HNS (SHNS) method for solving the complex symmetric linear system with the Hermitian part of the original coefficient matrix being real symmetric indefinite. That is to say, the matrix W involved in this article is a real symmetric indefinite matrix. In this case, the matrix $\alpha I + W$ ($\alpha V + W$, $\alpha W + T^2$) may be indefinite or singular, and the convergence speeds of the MHSS (PMHSS) or SNS (SSS) method may be unacceptably slow because the spectral radius of the iteration matrix of the corresponding iteration method may be very close to 1, or even larger than 1. Therefore, the MHSS (PMHSS) or SNS (SSS) methods may be invalid. To avoid this disadvantage, the HNS (SHNS) method for solving the complex symmetric linear system with the Hermitian part W of the original coefficient matrix being real symmetric indefinite is established. Theoretical and numerical results show that the proposed methods in this paper are not only suitable for the real symmetric indefinite matrix W, but suitable for the general nonsingular symmetric matrix W.

5.2. *The Hermitian normal splitting preconditioner*

It is well known that the spectral properties of the preconditioned matrix give important insight in the convergence behavior of the preconditioned Krylov subspace methods. In particular, for the symmetric linear systems, it is desirable that the number of distinct eigenvalues, or at least the number of clusters, is small, because in this case, convergence will be rapid. If there are only a few distinct eigenvalues, then optimal methods such as CG, minimal residual method (MINRES), or GMRES will terminate (in exact arithmetic) after a small and precisely defined number of steps. Thus, based on the aforementioned idea, to illustrate the results in Section 3, it is necessary to test the eigenvalue distribution of the preconditioned matrix $B_\alpha^{-1}WA$. To this end, we take $C_H = \mu K$, $\mu = 0.02$, $\omega = \pi$, and $C_V = \frac{1}{2}M$. All the matrices tested are 64×64 unless otherwise mentioned,

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that is to say, the mesh is 8×8 gird. Figures 1–3 plot the eigenvalue distribution of the matrix $B_\alpha^{-1}WA$.

Figure 1 plots the eigenvalue distribution for the matrix $B_\alpha^{-1}WA$ with $M = 1.5I$ and $\alpha = 453.8016$. We find that the matrix W is symmetric positive definite when $M = 1.5I$. From Figure 1, we see that the numerical results are consistent with the theoretical results in Theorem 3.3.

Figure 2 plots the eigenvalue distribution for the matrix $B_\alpha^{-1}WA$ with $M = 5I$ and $\alpha = 72.4304$. We find that the matrix W is symmetric indefinite when $M = 5I$, but the matrix $B_\alpha^{-1}WA$ is positive stable. From Figure 2, it is easy to find that the numerical results are in correspondence with the results of Theorem 3.3.

Figure 3 plots the eigenvalue distribution for the matrix $B_\alpha^{-1}WA$ with $M = 65I$ and $\alpha = 74.9513$. We find that the matrix W is a symmetric negative definite matrix when $M = 65I$, but the matrix $B_\alpha^{-1}WA$ is positive stable. From Figure 3, it is easy to find that the numerical results hold the results of Theorem 3.3.

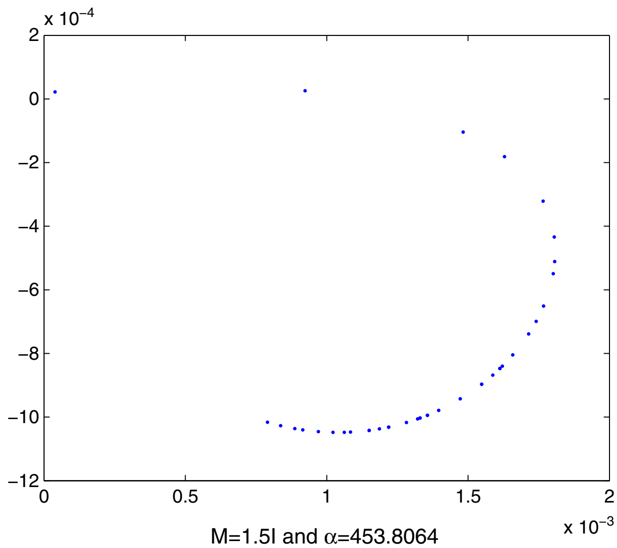

Figure 1. The eigenvalue distribution for $M = 1.5I$ and $\alpha = 453.8016$.

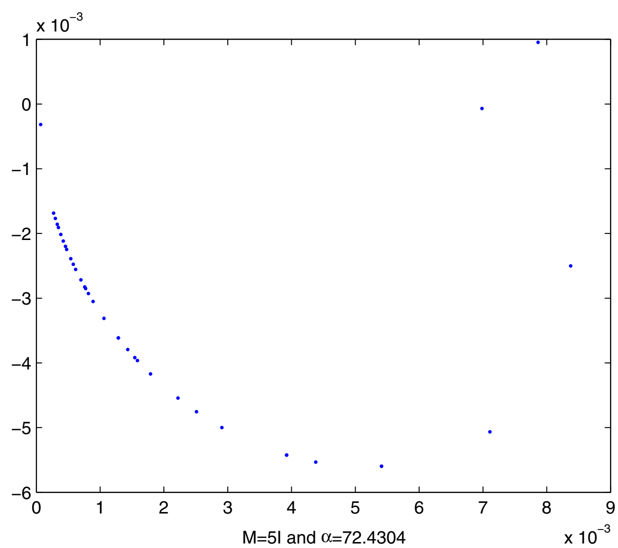

Figure 2. The eigenvalue distribution for $M = 5I$ and $\alpha = 72.4304$.

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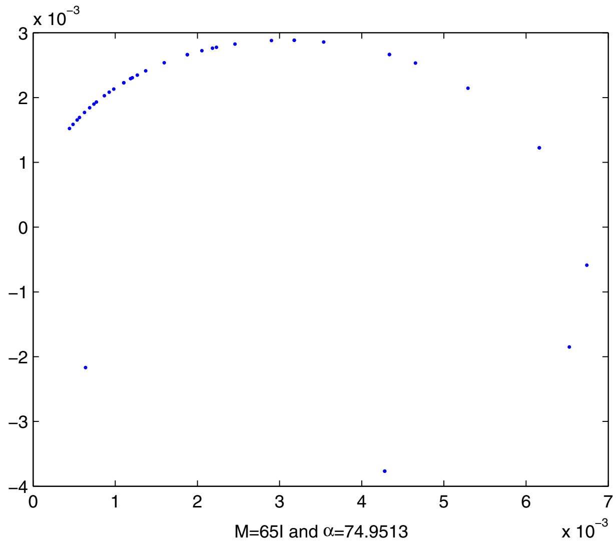

Figure 3. The eigenvalue distribution for $M = 65I$ and $\alpha = 74.9513$.

To efficiently solve (1) as much as possible, we can adopt some Krylov subspace methods (such as MINRES, Symmetric LQ Method (SYYMLQ), and GMRES) for it. It is reason that when using MINRES, SYYMLQ, or GMRES to solve (1), the coefficient matrix is not necessary to keep positive definite. To investigate the performance of the HNS-preconditioner and the SSS-preconditioner for the symmetric positive definite W, here, iteration methods based on Krylov subspace methods such as GMRES(m) and biconjugate gradient stabilized (BiCGSTAB) are cheap to be implemented. In general, the choice of the restart parameter $m(m \ll n)$ is no general rule, which mostly depends on a matter of experience in practice. In our numerical computations, for the sake of simplicity, the value of the restart parameter m is 10. The purpose of these experiments is just to investigate the influence of the eigenvalue distribution on the convergence behaviors of GMRES(10) and BiCGSTAB.

In our implementations, the right-hand side vector b is adjusted such that $b = (1+i)Ae$. The GMRES(10) and BiCGSTAB methods terminate if the relative residual error satisfies $\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} < 10^{-6}$. In Tables VII–XI, '$\alpha$' denotes the corresponding optimal iteration parameter. 'IT' denotes the number of iteration. 'HNS-GMRES(m)' denotes the HNS-preconditioned GMRES(m) method. 'SSS-GMRES(m)' denotes the SSS-preconditioned GMRES(m) method. 'HNS-BiCGSTAB' denotes the HNS-preconditioned BiCGSTAB method and 'SSS-BiCGSTAB' denotes the SSS-preconditioned BiCGSTAB method.

In Tables VII and VIII, we present some iteration results for the symmetric positive definite W to compare the HNS-preconditioner with the SSS-preconditioner. In Tables VII and VIII, we find that the number of iterations of the HNS-preconditioned GMRES(m)/BiCGSTAB increases with the symmetric positive definite inertia matrix M increasing, whereas the number of iterations of the SSS-preconditioned GMRES(m)/BiCGSTAB descends with the symmetric positive definite inertia matrix M increasing. In Tables VII and VIII, the SSS-preconditioner is slightly more efficient than

Table VII. α, IT, and CPU for the symmetric positive definite W and $n = 1024$.

	M	I	$\frac{6}{5}I$	$\frac{7}{5}I$	$\frac{8}{5}I$	$\frac{9}{5}I$
HNS-GMRES(10)	α	4608.3	3418.0	2399.0	1506.9	712.2639
	IT	4(4)	5(4)	6(10)	7(4)	9(4)
	CPU(s)	39.5	51.344	72.406	79.109	113.156
SSS-GMRES(10)	α	0.7080	0.7778	0.8417	0.9011	1.1616
	IT	2(6)	2(5)	2(5)	2(5)	2(4)
	CPU(s)	18.844	17.813	18.109	18.016	16.797

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Table VIII. α, IT, and CPU for the symmetric positive definite W and $n = 1024$.

	M	I	$\frac{6}{5}I$	$\frac{7}{5}I$	$\frac{8}{5}I$	$\frac{9}{5}I$
HNS-BiCGSTAB	α	4608.3	3418.0	2399.0	1506.9	712.2639
	IT	14.5	15.5	19	21.5	27.5
	CPU(s)	30.813	32.547	39.75	45.922	57.891
SSS-BiCGSTAB	α	0.7080	0.7778	0.8417	0.9011	1.1616
	IT	8.5	8.5	8	7.5	7.5
	CPU(s)	17.875	17.734	16.75	16.016	15.734

Table IX. α, IT, and CPU for the symmetric positive definite W and $n = 1024$.

	M	I	$\frac{6}{5}I$	$\frac{7}{5}I$	$\frac{8}{5}I$	$\frac{9}{5}I$
HNS-GMRES(10)	α	1	1	1	1	1
	IT	10(4)	10(7)	10(4)	10(8)	10(7)
	CPU(s)	109.64	112.688	109.64	114.204	112.891

Table X. α, IT, and CPU for the symmetric positive definite W and *HNS-GMRES(10)*.

	M	I	$\frac{6}{5}I$	$\frac{7}{5}I$	$\frac{8}{5}I$	$\frac{9}{5}I$
$n = 1369$	α	5317.2	3945.3	2770.3	1741.1	823.9578
	IT	3(5)	5(1)	6(4)	4(9)	13(7)
	CPU(s)	51.922	83.579	110.5	78.782	259.75
$n = 1764$	α	6025.0	4471.6	3140.8	1974.7	935.2897
	IT	4(3)	6(3)	6(3)	6(7)	9(7)
	CPU(s)	150.469	241.578	237.5	259.813	385.422

Table XI. α, IT, and CPU for the symmetric indefinite W and $n = 1024$.

	M	$5I$	$10I$	$20I$	$30I$	$50I$
HNS-GMRES(10)	α	27.2982	90.5170	201.3379	1136.4	454.2701
	IT	10(4)	10(5)	8(10)	11(4)	14(6)
	CPU(s)	124.61	125.469	94.813	156.688	162.797
HNS-BiCGSTAB	IT	51	40.5	29.5	26	32
	CPU(s)	114.484	89.281	66.672	58.297	71.594

the HNS-preconditioner from iteration number and CPU time, but this does not mean that the SSS-preconditioner is superior to the HNS-preconditioner because both in different ways try to improve the convergence speed of the corresponding Krylov subspace methods. It is not difficult to find that the HNS-preconditioner is applied to solve $WAx = Wb$, and the SSS-preconditioner is applied to solve $TAx = Tb$.

In Table IX, some iteration results are listed for $\alpha = 1$ when the HNS-preconditioned GMRES(m) is applied to solve the complex symmetric linear system (1). Compared iteration results of Tables VII and IX, $\alpha = 1$ may not be a good choice when the HNS-preconditioner is applied to solve the complex symmetric linear system (1). Based on this survey, Table X does not list the iteration results for $\alpha = 1$ but lists some iteration results for the corresponding optimal parameter with $n = 1369$ and $n = 1764$.

In Table XI, some iteration results are listed for the symmetric indefinite matrix W and $n = 1024$. To some extent, the HNS-preconditioner can be applied to solve the complex symmetric linear system (1) with the symmetric indefinite matrix W. Comparing the performance of BiCGSTAB to the performance of GMRES(10) is not within our stated goals, but having results using more than one Krylov solver allows us to confirm the consistency of convergence behavior for most problems.

5.3. *The inexact Hermitian normal splitting method*

In this subsection, we employ the IHNS method to solve the system of linear equations (1) in the actual implementation. Specifically, it is to solve the system with coefficient matrix $\alpha T + W^2$ iteratively by the CG method and solve the system with coefficient matrix $\alpha I + iW$ iteratively by the GMRES method (or other Krylov subspace methods, such as BiCGSTAB and conjugate gradient for normal equations (CGNE)) in each outer iteration. Some good references are available on theoretical analysis for using some Krylov subspace methods as the corresponding inner solvers (such as CG, GMRES, Lanczos, and CGNE) in Algorithm 1. One can see Reference [33] for details.

In our computations, the inner CG and GMRES iterations are terminated if the current residuals of the inner iterations satisfy

$$\frac{\|p^{(j)}\|_2}{\|r^{(k)}\|_2} \leqslant 0.1\tau^{(k)} \text{ and } \frac{\|q^{(j)}\|_2}{\|r^{(k)}\|_2} \leqslant 0.1\tau^{(k)},$$

(cf. Algorithm 1) where $p^{(j)}$ and $q^{(j)}$ are, respectively, the residuals of the jth inner CG and GMRES, $r^{(k)}$ is the kth outer IHNS iteration, and τ is a tolerance. In Tables XII and XIII, we list numerical results for $C_H = \mu K$, $\mu = 0.02$, $\omega = \pi$, and $C_V = \frac{1}{2}M$.

Table XII. Iteration number and average number of inner iterations for $n = 256$.

M	α	τ	IT	$\mathrm{IT}_{\mathrm{int}}$(GMRES)	IT_{iW}	$\mathrm{IT}_{\mathrm{int}}$(CG)	IT_{TW}
$5I$	48.9665	0.95	122	18	28	59	94
		0.9	150	29	52	68	98
		0.8	212	57	115	73	97
$10I$	151.7972	0.95	72	12	17	42	55
		0.9	90	17	27	49	63
		0.8	124	30	56	56	68
$15I$	629.2232	0.95	51	7	10	28	41
		0.9	68	10	16	38	52
		0.8	85	17	29	47	56
$20I$	301.5538	0.95	62	10	15	33	47
		0.9	83	15	24	43	59
		0.8	106	26	47	51	59

Table XIII. Iteration number and average number of inner iterations for $n = 1024$.

M	α	τ	IT	$\mathrm{IT}_{\mathrm{int}}$(GMRES)	IT_{iW}	$\mathrm{IT}_{\mathrm{int}}$(CG)	IT_{TW}
$15I$	2091.7	0.95	81	7	10	44	71
		0.9	118	10	18	68	100
		0.8	135	19	35	83	100
$20I$	201.3379	0.95	132	23	36	85	96
		0.9	167	38	71	86	96
		0.8	249	70	151	91	98
$25I$	217.7578	0.95	131	21	31	84	100
		0.9	163	34	63	89	100
		0.8	250	72	150	93	100
$30I$	1136.4	0.95	95	10	15	51	80
		0.9	126	15	26	73	100
		0.8	150	27	50	86	100

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Some results are listed in Tables XII and XIII, which are the numbers of outer IHNS iteration (IT), the average numbers ($\mathrm{IT_{int}}$(GMRES)) of inner GMRES(20) iteration for $\alpha I + iW$, and the average numbers ($\mathrm{IT_{int}}$(CG)) of inner CG iteration for $\alpha T + W^2$. In Tables XII and XIII, IT_{iW} denotes the iteration numbers of GMRES(20) for solving $(\alpha I + iW)z^{(k)} = r^{(k)}$, and IT_{TW} denotes the iteration numbers of CG for solving $(\alpha T + W^2)z^{(k+1/2)} = r^{(k+1/2)}$. In our numerical computations, it is easy to find the fact that the choice of the tolerance τ is important to the convergence rate of the IHNS method. According to Tables XII and XIII, the iteration numbers of the IHNS method generally decrease when the tolerance τ increases. For the convenience of comparison with the HNS iteration, in our numerical tests, we do not employ preconditioning technique in the inner iterations. We can see that the average number of inner iterations relative to the iteration numbers of outer iteration is quite small for the IHNS iteration. It is not difficult to find that the variation of τ is very small (from 0.95 to 0.8) and the change of iterations is relatively very big. This means that the IHNS method is rather sensitive to the inner stopping criterion. Therefore, finding the proper inner stopping criterion may be important for the IHNS method. With respect to this point, it needs to be studied in the future.

6. CONCLUSION

In this paper, we consider the class of complex symmetric linear systems with a Hermitian part being real symmetric and indefinite. For the iterative solution of such problems, we propose several variants of the HSS method, namely, the HNS and SHNS methods. We show that HNS (SHNS) is unconditionally convergent even if W is a real nonsingular symmetric matrix and T is a real symmetric positive definite matrix in Theorems 2.1 and 3.1. The corresponding HNS-preconditioner and the IHNS method are described. Numerical experiments are reported to illustrate the efficiency of the presented methods for an n-DOF linear system.

In our numerical experiments, we find that for the same grid, with the inertia matrix M increasing, the change of the optimal iteration parameter α is not easily described. For this reason, we may study the change of the parameter α with the size of the problem and with the quantity of the problem parameters in the future. In addition, the convergence rate of the IHNS method depends on the choices of the tolerances. Therefore, the future work needs to focus on developing the optimal tolerances as well.

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