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A parameterized SHSS iteration method for a class of complex symmetric system of linear equations[☆]

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ABSTRACT

In this paper, we present a parameterized variant of the single-step Hermitian and skew-Hermitian (SHSS) iteration method for solving a class of complex symmetric system of linear equations. We study the convergence properties of the parameterized SHSS (P-SHSS) iteration method, the choices of the parameters and the quasi-optimal parameters. Numerical experiments are given to verify the effectiveness of the P-SHSS iteration method.

1. Introduction

Many problems in scientific computing lead to a large and sparse complex symmetric linear system [1,2]

$$Ax \equiv (W + \mathrm{i}T)x = b, \tag{1.1}$$

where $\mathrm{i} = \sqrt{-1}$, $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them, e.g., W being positive definite. $b \in \mathbb{C}^n$ is a given vector and $x \in \mathbb{C}^n$ is an unknown vector.

In order to solve the system in form (1.1), an efficient splitting of the coefficient matrix A is usually required. Let $x = u+\mathrm{i}v$ and $b = p+\mathrm{i}q$, where $u, v, p, q \in \mathbb{R}^{n\times n}$, then the complex linear system (1.1) can be rewritten as 2-by-2 block real equivalent formulation

$$\begin{pmatrix} W & -T \\ T & W \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Salkuyeh et al. [3] applied the generalized successive over-relaxation (GSOR) iteration method to solve this real equivalent system. They have shown the performance of the GSOR iteration method by some numerical experiments. Then Hezari et al. [4] present a preconditioned variant of the GSOR iteration method. They also showed the effectiveness of the preconditioned GSOR method by numerical experiments.

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If the matrix A is split into its diagonal, strictly lower and upper triangular parts, then we can obtain the classical Jacobi and Gauss–Seidel iterations [5]. Based on the Hermitian and skew-Hermitian splitting (HSS) of the matrix A, i.e.,

$$A = H + S,$$

where $H = \frac{1}{2}(A + A^*) = W$, $S = \frac{1}{2}(A - A^*) = \mathrm{i}T$, Bai, Golub and Ng [6] introduced the HSS iteration method. Due to the performance and elegant mathematical properties of the HSS iteration method, a variety of considerable attentions and results appear in many papers, such as the preconditioned HSS (PHSS) iteration method [7], the modified HSS (MHSS) iteration method [8], the accelerated HSS (AHSS) iteration method [9], the generalization of HSS (GHSS) iteration method [10], the modified GHSS iteration method [11]. See [12–17] for more generalizations and comprehensive survey on the HSS iteration method.

Obviously, the HSS iteration method requires one to solve two linear subsystems with $\alpha I + H$ and $\alpha I + S$. Because the coefficient matrix $\alpha I + H$ is usually a Hermitian positive definite, then one can employ the CG method to solve the system [6]. However, the system of linear equations with coefficient matrix $\alpha I + S$ is in general not easy to obtain.

Recently, Li and Wu [18] introduced a single-step HSS (SHSS) iteration method to solve the non-Hermitian positive definite linear systems as the following algorithm.

Algorithm 1.1. The SHSS iteration method.

Given an initial guess $x^{(0)}$, for $k = 0, 1, \ldots,$ until $\{x^{(k)}\}$ converges, compute

$$(\alpha I + H)x^{(k+1)} = (\alpha I - S)x^{(k)} + b,$$

where α is a given positive constant.

It has been pointed out in [18] that the SHSS iteration method is convergent to the unique solution of the linear system for a loose restriction on the iteration parameter α. Particularly, if the smallest eigenvalue of the matrix H is greater than the largest singular value of the matrix S, then the SHSS iteration method will converge for any $\alpha > 0$. Based on the comparison between the largest singular value of S and the smallest eigenvalue of H, we will firstly multiply a parameter $\omega - \mathrm{i}$ by the two sides of the original linear equation (1.1) as

$$\mathscr{A}x := (\omega - \mathrm{i})Ax = (\omega - \mathrm{i})b := f, \tag{1.2}$$

where $\omega > 0$ is a positive constant and i is the imaginary unit. Let $\mathscr{H} = \frac{1}{2}(\mathscr{A} + \mathscr{A}^*) = \omega W + T$ and $\mathscr{S} = \frac{1}{2}(\mathscr{A} - \mathscr{A}^*) = \mathrm{i}(\omega T - W)$ be the Hermitian parts and the skew-Hermitian parts of the matrix $\mathscr{A}$, respectively. Applying the SHSS iteration method to the parameterized linear system (1.2), we will obtain the following parameterized SHSS (P-SHSS) iteration method.

Algorithm 1.2. The P-SHSS iteration method.

Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \ldots$ until $\{x^{(k)}\}$ converges, compute

$$(\alpha I + \mathscr{H})x^{(k+1)} = (\alpha I - \mathscr{S})x^{(k)} + f, \tag{1.3}$$

where α is a given positive constant.

If the parameter ω can be chosen such that the smallest eigenvalue of the matrix $\mathscr{H}$ is greater than the largest singular value of the matrix $\mathscr{S}$, then the P-SHSS iteration method would be convergent for any positive constant α. Moreover, the P-SHSS iteration method would be more efficient than the SHSS iteration method by choosing the parameter ω properly.

The outline of the paper is as follows. In Section 2, the convergence properties of the P-SHSS iteration method are analyzed, including the convergence conditions, the choices of the iterative parameter, the spectral radius of the iterative matrix and the quasi-optimal parameters. In Section 3, the numerical experiments are presented to illustrate the effectiveness of the P-SHSS iteration method. Finally, a brief conclusion is made in Section 4 to end this work.

2. Convergence analysis

In this section, we study the convergence properties of the P-SHSS iteration method. First, we reformulate the P-SHSS iteration scheme (1.3) as

$$x^{(k+1)} = M(\alpha, \omega)x^{(k)} + N(\alpha, \omega)f, \quad k = 0, 1, 2, \ldots,$$

where

$$M(\alpha, \omega) = (\alpha I + \mathscr{H})^{-1}(\alpha I - \mathscr{S}) = (\alpha I + \omega W + T)^{-1}(\alpha I - \mathrm{i}(\omega T - W)),$$
$$N(\alpha, \omega) = (\alpha I + \mathscr{H})^{-1} = (\alpha I + \omega W + T)^{-1},$$

with α and ω being given positive constants.

Denote

$$\tilde{\lambda}_{\min}(\omega) = \min_{\tilde{\lambda}_j \in sp(\omega W + T)} \{\tilde{\lambda}_j\} \quad \text{and} \quad \tilde{\sigma}_{\max}(\omega) = \max_{\mathrm{i}\tilde{\sigma}_j \in sp(\mathrm{i}(\omega T - W))} \{|\tilde{\sigma}_j|\}, \tag{2.1}$$

where $sp(X)$ represents the spectral set of the matrix X. Because $\mathscr{H}$ is a symmetric positive definite matrix, by making use of the results in [18,15], we can obtain the following lemmas concerning on the convergence properties of the P-SHSS iteration method.

Lemma 2.1 ([18,15]). *If $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them being positive definite, then the spectral radius $\rho(M(\alpha,\omega))$ is bounded by*

$$\delta(\alpha,\omega) = \frac{\sqrt{\alpha^2 + \tilde{\sigma}^2_{\max}(\omega)}}{\alpha + \tilde{\lambda}_{\min}(\omega)}.$$

Furthermore, it holds that

(i) *if $\tilde{\lambda}_{\min}(\omega) \geq \tilde{\sigma}_{\max}(\omega)$, then $\delta(\alpha,\omega) < 1$ for any $\alpha > 0$;*
(ii) *if $\tilde{\lambda}_{\min}(\omega) < \tilde{\sigma}_{\max}(\omega)$, then $\delta(\alpha,\omega) < 1$ if and only if*

$$\alpha > \frac{\tilde{\sigma}^2_{\max}(\omega) - \tilde{\lambda}^2_{\min}(\omega)}{2\tilde{\lambda}_{\min}(\omega)}.$$

Lemma 2.2 ([18,15]). *The parameter $\alpha^*(\omega)$ minimizing the upper bound of $\delta(\alpha,\omega)$ of the spectral radius $\rho(M(\alpha,\omega))$ is*

$$\alpha^*(\omega) = \arg\min_{\alpha}\left\{\frac{\sqrt{\alpha^2 + \tilde{\sigma}^2_{\max}(\omega)}}{\alpha + \tilde{\lambda}_{\min}(\omega)}\right\} = \frac{\tilde{\sigma}^2_{\max}(\omega)}{\tilde{\lambda}_{\min}(\omega)},$$

and

$$\delta(\alpha^*(\omega),\omega) = \frac{\tilde{\sigma}_{\max}(\omega)}{\sqrt{\tilde{\lambda}^2_{\min}(\omega) + \tilde{\sigma}^2_{\max}(\omega)}}.$$

From Lemma 2.1, we see if ω can be properly chosen such that $\tilde{\lambda}_{\min}(\omega) > \tilde{\sigma}_{\max}(\omega)$, then the P-SHSS iteration method will converge for any $\alpha > 0$. Denote

$$\lambda_{\min} = \min_{\lambda_j \in sp(W)}\{\lambda_j\}, \qquad \sigma_{\max} = \max_{\sigma_j \in sp(T)}\{\sigma_j\}, \qquad \lambda_{\max} = \max_{\lambda_j \in sp(W)}\{\lambda_j\}, \qquad \sigma_{\min} = \min_{\sigma_j \in sp(T)}\{\sigma_j\}. \tag{2.2}$$

The following theorem will give a way to choose ω such that the P-SHSS iteration method converges for any $\alpha > 0$.

Theorem 2.3. *Let $\tilde{\lambda}_{\min}(\omega)$, $\tilde{\sigma}_{\max}(\omega)$ be defined as in (2.1), $\lambda_{\min}$, $\lambda_{\max}$, $\sigma_{\min}$ and $\sigma_{\max}$ be defined as in (2.2). Denote*

$$\bar{\omega} = \frac{\lambda_{\min} + \lambda_{\max}}{\sigma_{\min} + \sigma_{\max}}, \qquad \omega_1 = \frac{\lambda_{\min} + \sigma_{\min}}{\sigma_{\max} - \lambda_{\min}}, \qquad \omega_2 = \frac{\lambda_{\max} - \sigma_{\min}}{\sigma_{\min} + \lambda_{\min}}.$$

The P-SHSS iteration method converges for any $\alpha > 0$ if each of the following conditions hold:

(1) *$\lambda_{\min} \geq \sigma_{\max}$ and $\omega \geq \bar{\omega}$;*
(2) *$\lambda_{\min} < \sigma_{\max}$ and $\bar{\omega} \leq \omega \leq \omega_1$;*
(3) *$\omega_2 \leq \omega < \bar{\omega}$.*

Proof. Let χ be an eigenvalue of the matrix $\omega W + T$ and u be the corresponding eigenvector, then $(\omega W + T)u = \chi u$. Because W and T are both semi-definite positive definite matrices, by simple computations, it follows

$$\chi = \omega\frac{u^*Wu}{u^*u} + \frac{u^*Tu}{u^*u} \geq \omega\lambda_{\min} + \sigma_{\min}, \quad \text{i.e., } \tilde{\lambda}_{\min}(\omega) \geq \omega\lambda_{\min} + \sigma_{\min}.$$

Let $\mathrm{i}\xi$ be an eigenvalue of the matrix $\mathrm{i}(\omega T - W)$ and v be the corresponding eigenvector, then $\mathrm{i}(\omega T - W)v = \mathrm{i}\xi v$. It follows

$$\xi = \omega\frac{v^*Tv}{v^*v} - \frac{v^*Wv}{v^*v}$$

and

$$|\xi| \leq \max\{|\omega\sigma_{\max} - \lambda_{\min}|, |\omega\sigma_{\min} - \lambda_{\max}|\} = \max\{\omega\sigma_{\max} - \lambda_{\min}, \lambda_{\max} - \omega\sigma_{\min}\}.$$

Therefore,

$$\tilde{\sigma}_{\max}(\omega) \leq \max\{\omega\sigma_{\max} - \lambda_{\min}, \lambda_{\max} - \omega\sigma_{\min}\}.$$

According to Lemma 2.1, when $\tilde{\lambda}_{\min}(\omega) \geq \tilde{\sigma}_{\max}(\omega)$, the P-SHSS iteration method will converge for any $\alpha > 0$. We will investigate the sufficient conditions for $\tilde{\lambda}_{\min}(\omega) \geq \tilde{\sigma}_{\max}(\omega)$ in two cases:

Case 1. $\omega \geq \bar{\omega}$, i.e., $\omega\sigma_{\max} - \lambda_{\min} \geq \lambda_{\max} - \omega\sigma_{\min}$;

Case 2. $0 < \omega < \bar{\omega}$, i.e., $\omega\sigma_{\max} - \lambda_{\min} < \lambda_{\max} - \omega\sigma_{\min}$.

Case 1. If $\omega \geq \bar{\omega}$, we have $\tilde{\sigma}_{\max}(\omega) \leq \omega\sigma_{\max} - \lambda_{\min}$. A sufficient condition for $\tilde{\lambda}_{\min}(\omega) \geq \tilde{\sigma}_{\max}(\omega)$ is

$$\omega\lambda_{\min} + \sigma_{\min} \geq \omega\sigma_{\max} - \lambda_{\min}.$$

Or equivalently,

$$\omega(\lambda_{\min} - \sigma_{\max}) \geq -\sigma_{\min} - \lambda_{\min}. \tag{2.3}$$

Therefore,

(1) when $\lambda_{\min} \geq \sigma_{\max}$, Eq. (2.3) holds if $\omega \geq \bar{\omega}$.
(2) when $\lambda_{\min} < \sigma_{\max}$ and $\omega \geq \bar{\omega}$, Eq. (2.3) holds if $\omega \leq \omega_1$.

Case 2. If $0 < \omega < \bar{\omega}$, we have $\tilde{\sigma}_{\max}(\omega) \leq \lambda_{\max} - \omega\sigma_{\min}$. A sufficient condition for $\tilde{\lambda}_{\min}(\omega) \geq \tilde{\sigma}_{\max}(\omega)$ is

$$\omega\lambda_{\min} + \sigma_{\min} \geq \lambda_{\max} - \omega\sigma_{\min}, \quad \text{i.e.,} \quad \omega \geq \omega_2.$$

Therefore, $\tilde{\lambda}_{\min}(\omega) \geq \tilde{\sigma}_{\max}(\omega)$ also holds under the condition (3). □

The following theorem will compare the minimal upper bound of the spectral radius of the P-SHSS iteration method with that of the spectral radius of the SHSS iteration method.

Theorem 2.4. *If*

$$\omega > \frac{\lambda_{\min}\lambda_{\max} - \sigma_{\min}\sigma_{\max}}{\lambda_{\min}(\sigma_{\min} + \sigma_{\max})},$$

then the minimal upper bound $\delta(\alpha^*(\omega), \omega)$ of the spectral radius of the P-SHSS iterative matrix is smaller than the minimal upper bound $\frac{\sigma_{\max}}{\sqrt{\lambda_{\min}^2 + \sigma_{\max}^2}}$ of the spectral radius of the SHSS iterative matrix.

Proof. Let

$$\frac{\tilde{\sigma}_{\max}(\omega)}{\sqrt{\tilde{\lambda}_{\min}^2(\omega) + \tilde{\sigma}_{\max}^2(\omega)}} < \frac{\sigma_{\max}}{\sqrt{\lambda_{\min}^2 + \sigma_{\max}^2}},$$

that is,

$$\frac{1}{\left(\frac{\tilde{\lambda}_{\min}(\omega)}{\tilde{\sigma}_{\max}(\omega)}\right)^2 + 1} < \frac{1}{\left(\frac{\lambda_{\min}}{\sigma_{\max}}\right)^2 + 1}.$$

Equivalently,

$$\frac{\tilde{\lambda}_{\min}(\omega)}{\tilde{\sigma}_{\max}(\omega)} > \frac{\lambda_{\min}}{\sigma_{\max}}.$$

It follows

$$\tilde{\lambda}_{\min}(\omega) \cdot \sigma_{\max} > \lambda_{\min} \cdot \tilde{\sigma}_{\max}(\omega). \tag{2.4}$$

According to the proof of Theorem 2.3, if

$$\omega \geq \frac{\lambda_{\min} + \lambda_{\max}}{\sigma_{\min} + \sigma_{\max}}, \tag{2.5}$$

it holds $\tilde{\sigma}_{\max}(\omega) \leq \omega\sigma_{\max} - \lambda_{\min}$. Because $\tilde{\lambda}_{\min}(\omega) \geq \omega\lambda_{\min} + \sigma_{\min}$, then if $(\omega\lambda_{\min} + \sigma_{\min}) \cdot \sigma_{\max} > \lambda_{\min} \cdot (\omega\sigma_{\max} - \lambda_{\min})$, (2.4) follows.

However, if

$$\omega < \frac{\lambda_{\min} + \lambda_{\max}}{\sigma_{\min} + \sigma_{\max}},$$

it holds $\tilde{\sigma}_{\max}(\omega) \le \lambda_{\max} - \omega\sigma_{\min}$. A sufficient condition for (2.4) is

$$(\omega\lambda_{\min} + \sigma_{\min}) \cdot \sigma_{\max} > \lambda_{\min} \cdot (\lambda_{\max} - \omega\sigma_{\min}).$$

By simple computations, we obtain

$$\omega > \frac{\lambda_{\min}\lambda_{\max} - \sigma_{\min}\sigma_{\max}}{\lambda_{\min}(\sigma_{\min} + \sigma_{\max})}.$$

Therefore, if

$$\frac{\lambda_{\min}\lambda_{\max} - \sigma_{\min}\sigma_{\max}}{\lambda_{\min}(\sigma_{\min} + \sigma_{\max})} < \omega < \frac{\lambda_{\min} + \lambda_{\max}}{\sigma_{\min} + \sigma_{\max}},$$

then (2.4) holds.

Combining with (2.5), we obtain the conclusion of this theorem. □

The following theorem gives the optimal ω^*.

Theorem 2.5. *Let $\lambda_{\min}, \lambda_{\max}, \sigma_{\min}$ and $\sigma_{\max}$ be defined as in (2.2). The quasi-optimal parameter ω^* is given by*

$$\omega^* = \bar{\omega} \left(= \frac{\lambda_{\max} + \lambda_{\min}}{\sigma_{\max} + \sigma_{\min}} \right).$$

Proof. According to Lemma 2.2 and similar analysis in [6], we need to minimize the upper bound $\delta(\alpha^*(\omega), \omega)$. Equivalently, we need to maximize the lower bound of $\frac{\tilde{\lambda}_{\min}(\omega)}{\tilde{\sigma}_{\max}(\omega)}$. According to the proof of Theorem 2.3, if

$$\omega \ge \frac{\lambda_{\min} + \lambda_{\max}}{\sigma_{\min} + \sigma_{\max}},$$

it follows,

$$\left(\frac{\tilde{\lambda}_{\min}(\omega)}{\tilde{\sigma}_{\max}(\omega)}\right)^2 \ge \left(\frac{\omega\lambda_{\min} + \sigma_{\min}}{\omega\sigma_{\max} - \lambda_{\min}}\right)^2 := \varphi(\omega).$$

If

$$0 < \omega < \frac{\lambda_{\min} + \lambda_{\max}}{\sigma_{\min} + \sigma_{\max}},$$

it follows,

$$\left(\frac{\tilde{\lambda}_{\min}(\omega)}{\tilde{\sigma}_{\max}(\omega)}\right)^2 \ge \left(\frac{\omega\lambda_{\min} + \sigma_{\min}}{\omega\sigma_{\min} - \lambda_{\max}}\right)^2 := \psi(\omega).$$

Because

$$\varphi'(\omega) = -\frac{2(\omega\lambda_{\min} + \sigma_{\min})(\lambda_{\min}^2 + \sigma_{\min}\sigma_{\max})}{(\omega\sigma_{\max} - \lambda_{\min})^3}$$

and

$$\psi'(\omega) = -\frac{2(\omega\lambda_{\min} + \sigma_{\min})(\lambda_{\min}\lambda_{\max} + \sigma_{\min}^2)}{(\omega\sigma_{\min} - \lambda_{\max})^3},$$

then if $\omega > \frac{\lambda_{\min}}{\sigma_{\max}}$, it holds $\varphi'(\omega) < 0$. If $\omega > \frac{\lambda_{\max}}{\sigma_{\min}}$, it holds $\psi'(\omega) < 0$.

By simple computation, we obtain

$$\frac{\lambda_{\max}}{\sigma_{\min}} > \frac{\lambda_{\min} + \lambda_{\max}}{\sigma_{\min} + \sigma_{\max}} > \frac{\lambda_{\min}}{\sigma_{\max}}.$$

The function $\min\{\varphi(\omega), \psi(\omega)\}$ is required to reach the maximum value.

By making use of the monotonicity properties of the two functions $\varphi(\omega)$ and $\psi(\omega)$ in $(\frac{\lambda_{\max}}{\sigma_{\min}}, \frac{\lambda_{\min}}{\sigma_{\max}})$, we can easily obtain that the maximum value of the function $\min\{\varphi(\omega), \psi(\omega)\}$ reaches at ω^*, where

$$\omega^* \equiv \arg_{\omega} \min\{\varphi(\omega), \psi(\omega)\} = \frac{\lambda_{\min} + \lambda_{\max}}{\sigma_{\min} + \sigma_{\max}}. \quad \square$$

Table 1
The experimental optimal parameters for the proposed iterative methods.

Example	Method		Grid				
			16×16	32×32	64×64	128×128	256×256
No. 1	MHSS	α^*	1.06	0.75	0.54	0.40	0.30
	SHSS	α^*	0.9	0.52	0.36	0.26	0.19
	PGSOR	α^*	0.990	0.987	0.986	0.984	0.983
		ω^*	0.657	0.624	0.602	0.590	0.583
	P-SHSS	α	0.009	0.009	0.009	0.009	0.009
		ω	0.6	0.6	0.6	0.6	0.6
No. 2	MHSS	α^*	0.04	0.007	0.003	0.0005	0.0002
	SHSS	α^*	0.04	0.01	0.002	0.0007	0.00019
	PGSOR	α^*	0.959	0.959	0.959	0.959	0.959
		ω^*	2.345	2.364	2.368	2.369	2.369
	P-SHSS	α	0.0005	0.0005	0.0005	0.0005	0.0005
		ω	3.3	3.3	3.3	3.3	3.3

3. Numerical results

In this section, we are going to test the effectiveness of the P-SHSS iteration method for solving the complex symmetric linear system (1.1). Numerical comparisons with the MHSS [8], the SHSS [18] and the PGSOR [4] iteration methods are also presented to show the advantage of the P-SHSS iteration method. We compare those methods from the point of view of the number of iterations (denoted as "IT") and CPU times (denoted as "CPU"). Our experiments are carried out in MATLAB R2013a on Intel(R) Core(TM) CPU 3.4Ghz and 8.00 GB of RAM, with machine precision 10^{-16}. According to [6,4,8], we use the sparse Cholesky factorization incorporated with the symmetric approximation minimum degree reordering. To do so, we use "symamd.m" command in the MATLAB toolbox.

In our implementations, the initial guess $x^{(0)}$ is chosen to be zero vector and the stopping criteria for all the methods are

$$\frac{\|b - Ax^{(k)}\|_2}{\|b\|_2} \leq 10^{-6},$$

where $x^{(k)}$ is the current approximation.

Example 1. Consider the linear system of the form [8,19]

$$\left[\left(K + \frac{3-\sqrt{3}}{\tau} I\right) + \mathrm{i}\left(K + \frac{3+\sqrt{3}}{\tau} I\right)\right] x = b,$$

where τ is the time step-size and K is the five-point centered difference matrix approximating the negative Laplacian operator $L = -\triangle$ with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0, 1] \times [0, 1]$ with the mesh-size $h = \frac{1}{m+1}$.

The matrix $K \in \mathbb{R}^{n \times n}$ possesses the tensor-product form

$$K = I \otimes B_m + B_m \otimes I$$

with $B_m = \frac{1}{h^2} \mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{m \times m}$. Hence K is an $n \times n$ block-tridiagonal matrix with $n = m^2$. By taking

$$W = K + \frac{3-\sqrt{3}}{\tau} I \quad \text{and} \quad T = K + \frac{3+\sqrt{3}}{\tau} I$$

the right-hand side vector $g = -f$ and f with its j-th entry $[f]_j$ being given by

$$[f]_j = \frac{(1-\mathrm{i})j}{\tau(j+1)^2}, \quad j = 1, 2, \ldots, n.$$

In our tests, we take $\tau = h$. Further, we normalize coefficient matrix and right-hand side by multiplying both by h^2.

Example 2. Consider the complex Helmholtz equation [20,13,4]

$$-\triangle u + \sigma_1 u + \mathrm{i}\sigma_2 u = f,$$

where σ_1 and σ_2 are real coefficient functions, u satisfies Dirichlet boundary conditions in $D = [0, 1] \times [0, 1]$ and $\mathrm{i} = \sqrt{-1}$. We discretize the problem with finite differences on an $m \times m$ grid with mesh size $h = 1/(m + 1)$. This leads to a system of linear equations

$$((K + \sigma_1 I) + \mathrm{i}\sigma_2 I)x = b,$$

Table 2
The numerical results for the proposed iterative methods.

Example	Method		Grid				
			16 × 16	32 × 32	64 × 64	128 × 128	256 × 256
No. 1	MHSS	IT	40	54	73	98	133
		CPU	0.006	0.02	0.16	1.32	10.32
	SHSS	IT	161	209	269	342	445
		CPU	0.01	0.04	0.27	2.38	16.49
	PGSOR	IT	4	4	5	5	5
		CPU	0.002	0.004	0.01	0.14	0.56
	P-SHSS	IT	9	10	10	10	10
		CPU	0.001	0.004	0.01	0.11	0.49
No. 2	MHSS	IT	39	41	41	41	41
		CPU	0.009	0.01	0.06	0.36	1.99
	SHSS	IT	35	31	29	25	22
		CPU	0.003	0.006	0.03	0.2	0.99
	PGSOR	IT	6	6	6	6	6
		CPU	0.001	0.006	0.02	0.17	0.64
	P-SHSS	IT	20	18	15	12	12
		CPU	0.002	0.006	0.02	0.11	0.58

Table 3
Numerical results for Example 2.

(σ_1, σ_2)	Method		Grid				
			16 × 16	32 × 32	64 × 64	128 × 128	256 × 256
$(-1, 1)$	PGSOR	α^*	0.959	0.959	0.959	0.959	0.959
		ω^*	2.345	2.345	2.345	2.345	2.345
		IT	6	6	6	6	6
		CPU	0.001	0.005	0.017	0.119	0.65
	P-SHSS	α	0.001	0.001	0.001	0.001	0.001
		ω	20	20	20	20	20
		IT	5	5	5	5	5
		CPU	0.001	0.002	0.009	0.07	0.36
$(-10, 1)$	PGSOR	α^*	0.959	0.959	0.959	0.959	0.959
		ω^*	2.5	2.5	2.5	2.5	2.5
		IT	5	5	5	5	5
		CPU	0.001	0.004	0.02	0.11	0.57
	P-SHSS	α	0.001	0.001	0.001	0.001	0.001
		ω	20	20	20	20	20
		IT	5	5	5	5	5
		CPU	0.0007	0.004	0.01	0.06	0.36

where $K = I \otimes V_m + V_m \otimes I$ is the discretization of $-\triangle$ by means of centered differences, wherein $V_m = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. The right-hand side vector b is taken to be $b = (1+\mathtt{i})A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. Furthermore, before solving the system, we normalize the coefficient matrix and the right-hand side vector by multiplying both by h^2. For the numerical tests, we set $\sigma_1 = -10$, $\sigma_2 = 10$, $\sigma_1 = -1$, $\sigma_2 = 1$ and $\sigma_1 = -10$, $\sigma_2 = 1$.

Table 1 shows the experimental optimal parameters corresponding to the MHSS iteration method [8], the PGSOR iteration method [4] and the SHSS iteration method [18] with respect to different problem sizes for both of Examples 1 and 2 with $\sigma_1 = -10$, $\sigma_2 = 10$. However, in P-SHSS iteration method of all the different dimensions, we take $\alpha = 0.009$, $\omega = 0.6$ for Example 1 and $\alpha = 0.0005$, $\omega = 3.3$ for Example 2. The results (including iterations and CPU times) listed in Table 2 are obtained from the parameters shown in Table 1.

We can see from Table 2 that, when the problem is small scale, the PGSOR method performs quite well. However, as the problem becomes larger scale, the P-SHSS iteration method needs the least CPU times.

In Table 3, we vary the values of σ_1 and σ_2 with $\sigma_1 = -1$, $\sigma_2 = 1$ and $\sigma_1 = -10$, $\sigma_2 = 1$ in Example 2. For simplicity, we fix the parameters α and ω by 0.001 and 20, respectively. It can be seen from the results in Table 3 that, the P-SHSS iteration method outperforms the PGSOR iteration method all the time both in iterations and CPU times.

Therefore, the P-SHSS iteration method is the most efficient among those methods.

4. Conclusions

In this work, we established a parameterized SHSS iteration method for solving the complex symmetric linear systems. The convergence properties are analyzed in detail. Theoretical analysis shows that, under a loose restriction on the parameter ω, the spectral radius of the iterative matrix of the P-SHSS iteration method is smaller than that of the SHSS iteration method. The quasi-optimal values of the iteration parameters for the P-SHSS iteration method are also determined. Numerical examples show that the P-SHSS iteration method is superior to the SHSS, MHSS and PGSOR iteration methods in terms of the iterations and CPU times.

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References

[1] O. Axelsson, A. Kucherov, Real valued iterative methods for solving complex symmetric linear systems, Numer. Linear Algebra Appl. 7 (2000) 197–218.
[2] Z.-Z. Bai, Rotated block triangular preconditioning based on PMHSS, Sci. China Math. 56 (2013) 2523–2538.
[3] D.K. Salkuyeh, D. Hezari, V. Edalatpour, Generalized successive overrelaxation iterative method for a class of complex symmetric linear system of equations, Int. J. Comput. Math. 92 (2015) 802–815.
[4] D. Hezari, V. Edalatpour, D.K. Salkuyeh, Preconditioned GSOR iterative method for a class of complex symmetric system of linear equations, Numer. Linear Algebra Appl. 22 (2015) 761–776.
[5] G.H. Golub, C.F. Van Loan, Matrix Computations, Johns Hopkins University, Press, Baltimore, MD, USA, 1996.
[6] Z.-Z. Bai, G.-H. Golub, M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, SIAM J. Matrix Anal. Appl. 24 (2003) 603–626.
[7] Z.-Z. Bai, G.H. Golub, J.-Y. Pan, Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems, Numer. Math. 98 (2004) 1–32.
[8] Z.-Z. Bai, M. Benzi, F. Chen, Modified HSS iteration methods for a class of complex symmetric linear systems, Computing 87 (2010) 93–111.
[9] Z.-Z. Bai, G.H. Golub, Accelerated Hermitian and skew-Hermitian splitting iteration methods for saddle-point problems, IMA J. Numer. Anal. 27 (2007) 1–23.
[10] M. Benzi, A generalization of the Hermitian and skew-Hermitian splitting iteration, SIAM J. Matrix Anal. Appl. 31 (2009) 360–374.
[11] C.-X. Li, S.-L. Wu, A modified GHSS method for non-Hermitian positive definite linear systems, Jpn. J. Ind. Appl. Math. 29 (2012) 253–268.
[12] Z.-Z. Bai, G.H. Golub, L.-Z. Lu, Block triangular and skew-Hermitian splitting methods for positive-definite linear systems, SIAM J. Sci. Comput. 26 (2005) 844–863.
[13] X. Li, A.-L. Yang, Y.-J. Wu, Lopsided PMHSS iteration method for a class of complex symmetric linear systems, Numer. Algorithms 66 (2014) 555–568.
[14] H.N. Pour, H.S. Goughery, New Hermitian and skew-Hermitian splitting methods for non-Hermitian positive-definite linear systems, Numer. Algorithms 69 (2015) 207–225.
[15] Y.-J. Wu, X. Li, J.-Y. Yuan, A non-alternating preconditioned HSS iteration method for non-Hermitian positive definite linear systems, Comput. Appl. Math. (2015) http://dx.doi.org/10.1007/s40314-015-0231-6.
[16] Z.-Z. Bai, Structured preconditioners for nonsingular matrices of block two-by-two structures, Math. Comp. 75 (2006) 791–815.
[17] Z.-Z. Bai, Block preconditioners for elliptic PDE-constrained optimization problems, Computing 91 (2011) 379–395.
[18] C.-X. Li, S.-L. Wu, A single-step HSS method for non-Hermitian positive definite linear systems, Appl. Math. Lett. 44 (2015) 26–29.
[19] Z.-Z. Bai, M. Benzi, F. Chen, On preconditioned MHSS iteration methods for complex symmetric linear systems, Numer. Algorithms 56 (2011) 297–317.
[20] D. Bertaccini, Efficient preconditioning for sequences of parametric complex symmetric linear systems, Electron. Trans. Numer. Anal. 18 (2004) 49–64.