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On Generalizations of the Theory of Consistent Orderings for Successive Over-Relaxation Methods

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Summary. A definition of consistent ordering for a matrix is generalized to include the scope of more recent definitions. The facility with which this generalized definition can be applied is exhibited.

1. Introduction

An iterative method for the solution of the matrix equation

$$Ax = b \tag{1.1}$$

generates a sequence of vectors $x^{(i)}$, $i = 1, 2, \ldots$; to each method (the Jacobi and successive over-relaxation (SOR) methods are considered here) is associated a matrix derived from the matrix A (for SOR, the matrix depends on a parameter). The sequence of vectors will converge to a solution x of (1.1) provided that the spectral radius (magnitude of the largest eigenvalue) of the associated matrix is less than unity; the smaller the spectral radius, the more rapid is the convergence, and thus, for the SOR method, it is desirable to choose the parameter so that the spectral radius is a minimum.

Thus, in choosing the SOR parameter, some knowledge of the eigenvalues of the corresponding SOR matrix is required. For matrices with certain properties, several advances have been made in this direction. Young [38] gave a theoretical analysis of the SOR method for elliptic difference problems, and this was extended to include block SOR methods by Arms, Gates, and Zondek [1]; more general matrices were considered by Varga [36], and by Broyden [3].

The various authors develop their analyses on the basis of assumptions which are embodied in their definitions of consistent ordering. Since our proposed generalization of the definition of consistent ordering can be regarded as a certain generalization of the definition given by Varga, we give this below.

We adopt a notation which includes block iterative methods. For uniformity in the definitions we assume throughout that $A = (a_{ij})$ is an $n \times n$ matrix, and that π partitions A into submatrices A_{ij}, $i, j = 1, 2, \ldots, p$, so that A_{ii}, $i = 1, 2, \ldots, p$ are square and non-singular. The block Jacobi matrix, B, is given by

$$B = L + U = I - D^{-1} A \tag{1.2}$$

where L and U are respectively strictly lower and upper triangular matrices and D is the block diagonal matrix given by the square submatrices, A_{ii},

$i = 1, 2, \ldots, p$. The block SOR matrix, $\mathscr{L}_\omega$, is given by

$$\mathscr{L}_\omega = (I - \omega L)^{-1}\{\omega U + (1-\omega) I\} \tag{1.3}$$

where ω is a parameter lying in the half-open interval (0, 2].

Definition I (VARGA [36, pp. 39, 99, 101]). (i) An $n \times n$ complex matrix A is weakly cyclic of index p ($\geqq 2$) if there exists an $n \times n$ permutation matrix P such that PAP^T is of the form

$$PAP^T = \begin{bmatrix} 0 & 0 & \ldots & 0 & A_{1p} \\ A_{21} & 0 & & 0 & 0 \\ 0 & A_{32} & & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & A_{p\,p-1} & 0 \end{bmatrix}$$

where the null diagonal submatrices are square.

(ii) If the block Jacobi matrix B for the matrix A is weakly cyclic of index $p (\geqq 2)$, then A is p-cyclic, relative to the partitioning π.

(iii) If the matrix A is p-cyclic, then A is "consistently ordered" if all the eigenvalues of the matrix $B(\alpha)$,

$$B(\alpha) = \alpha L + \alpha^{-(p-1)} U,$$

derived from the Jacobi matrix B, are independent of α, $\alpha \neq 0$.

Each of the authors mentioned above shows that there is a relationship between the eigenvalues of the Jacobi matrix, B, and those of the SOR matrix, $\mathscr{L}_\omega$, which depends on the parameter, ω, although the definitions on which the proofs are based differ, these relationships are all contained in the relationship (2.4) given below.

Basically we are interested in the dependence of the eigenvalues of the SOR matrix upon ω, and as these in turn depend upon the eigenvalues of the Jacobi matrix, Definition I appears to be the most appropriate of the definitions given, in the literature. We noticed that in the case $p=2$, the requirement of this definition that A be 2-cyclic was unnecessary. After further examination we generalized Definition I to include the class of matrices considered by BROYDEN [3], and at the same time excluded direct reference to p-cyclicity.

2. Generalized Definition

We now make the following generalization of Definition I.

Definition II. A matrix A is said to be generally consistently ordered (π, q, r) or simply GCO(π, q, r), where q and r are positive integers, if for the partitioning π of A, the diagonal square submatrices A_{ii}, $i = 1, 2, \ldots, p$, are non-singular, and the eigenvalues of

$$B(\alpha) = \alpha^r L + \alpha^{-q} U \tag{2.1}$$

are independent of α, for all $\alpha \neq 0$, where L and U are given in (1.2).

Note that if A is GCO(π, q, r), and if d is the greatest common divisor of q and r, by replacing α by $\alpha^{1/d}$ in (2.1), it follows that A is GCO$(\pi, q/d, r/d)$.

Because the eigenvalues of B and $B(\alpha)$ are identical for a matrix A which is GCO(π, q, r), it follows that

$$\det\{\beta(L+U)-\gamma I\}=\det\{\beta(\alpha^r L+\alpha^{-q} U)-\gamma I\} \tag{2.2}$$

for any choice of constants β and γ, and for all $\alpha \neq 0$.

Theorem I. If A is GCO(π, q, r) with q and r relatively prime, and μ_i is an eigenvalue of B, then $\theta\mu_i$ is also an eigenvalue of B, where θ is a $(q+r)$-th root of unity.

Proof. Since μ_i is an eigenvalue of B, for $\alpha=\theta$, (2.2) gives

$$\begin{aligned} 0 &= \det\{(L+U)-\mu_i I\} \\ &= \det\{(\theta^r L+\theta^{-q} U)-\mu_i I\} \\ &= \det\{\theta^r[L+\theta^{-(q+r)} U-\theta^{-r}\mu_i I]\} \\ &= \det\{\theta^r[B-\theta^{-r}\mu_i I]\} \end{aligned}$$

and thus $\theta^{-r}\mu_i$ is also an eigenvalue of B. Similarly for $\alpha=\theta^{-1}$ it follows that $\theta^r\mu_i$ is also an eigenvalue of B; repeating the argument it follows that $\theta^{kr}\mu_i$ is an eigenvalue for any integer k. Since q and r are relatively prime, there exist integers k and l such that $kr+l(q+r)=1$, and thus for some k,

$$\theta^{kr}=\theta^{-l(q+r)+1}=\theta,$$

and the theorem is proved.

By Definition II, the distinct eigenvalues of $B(\alpha)$ are independent of α; it can be shown by a continuity argument that the characteristic polynomial of $B(\alpha)$ is in fact independent of α. By further determining which coefficients of this polynomial are non-zero it can be seen that

$$\det\{\gamma I-B\}=\gamma^m\prod_{i=1}^{s}(\gamma^{q+r}-\mu_i^{q+r}) \tag{2.3}$$

where B has $s(q+r)$ non-zero eigenvalues and m eigenvalues equal to zero.

We now prove a generalization of VARGA's Theorem 4.3 [36, p. 106].

Theorem II. Let the matrix A be GCO(π, q, r) with q and r relatively prime. If $\omega\neq 0$, and λ is a non-zero eigenvalue of the matrix $\mathscr{L}_\omega$ as given in (1.3), and if μ satisfies

$$(\lambda+\omega-1)^{q+r}=\lambda^q\,\omega^{q+r}\,\mu^{q+r} \tag{2.4}$$

then μ is an eigenvalue of the block Jacobi matrix B given in (1.2). Conversely, if μ is an eigenvalue of B and λ satisfies (2.4), then λ is an eigenvalue $\mathscr{L}_\omega$.

Proof. The eigenvalues of $\mathscr{L}_\omega$ are the roots of its characteristic equation

$$\det\{\lambda I-\mathscr{L}_\omega\}=0.$$

Since L is strictly lower triangular, $(I-\omega L)$ is non-singular and

$$\begin{aligned} 0 &= \det\{\lambda I-\mathscr{L}_\omega\}\cdot\det\{I-\omega L\} \\ &= \det\{(\lambda+\omega-1)\,I-\lambda\,\omega\,L-\omega\,U\}. \end{aligned}$$

We set

$$\varphi(\lambda)\equiv\det\{(\lambda+\omega-1)I-\lambda\,\omega\,L-\omega\,U\},$$

and let δ be one of the $(q+r)$-th roots of λ, so that

$$\varphi(\lambda) = \det\{(\lambda+\omega-1)\, I - \omega\, \delta^q (\delta^r L + \delta^{-q} U)\}.$$

For non-zero λ and ω, (2.2) gives

$$\begin{aligned}\varphi(\lambda) &= \det\{(\lambda+\omega-1)\, I - \omega\, \delta^q B\} \\ &= (\omega\,\delta^q)^{m+s(q+r)} \cdot \det\left\{\frac{\lambda+\omega-1}{\omega\,\delta^q} I - B\right\}.\end{aligned} \tag{2.5}$$

Applying (2.3), we obtain

$$\begin{aligned}\varphi(\lambda) &= (\omega\,\delta^q)^{m+s(q+r)} \left(\frac{\lambda+\omega-1}{\omega\,\delta^q}\right)^m \prod_{i=1}^{s} \left\{\left(\frac{\lambda+\omega-1}{\omega\,\delta^q}\right)^{q+r} - \mu_i^{q+r}\right\} \\ &= (\lambda+\omega-1)^m \prod_{i=1}^{s} \{(\lambda+\omega-1)^{q+r} - \lambda^q \omega^{q+r} \mu_i^{q+r}\}.\end{aligned} \tag{2.6}$$

To prove the second part of the theorem, suppose that μ is an eigenvalue of B, and that λ satisfies (2.4). Then one of the factors of $\varphi(\lambda)$ given in (2.6) must vanish, and this implies that λ is an eigenvalue of $\mathscr{L}_\omega$.

To prove the first part of the theorem, suppose that $\omega \neq 0$, and let λ be a non-zero eigenvalue of $\mathscr{L}_\omega$ so that $\varphi(\lambda) = 0$. This implies that one factor of (2.6) must vanish. If $\mu = 0$, and μ satisfies (2.4), $(\lambda+\omega-1) = 0$, and it follows immediately from (2.5) that μ is an eigenvalue of B. If $\mu \neq 0$, and μ satisfies (2.4), then $(\lambda+\omega-1) \neq 0$. Thus

$$(\lambda+\omega-1)^{q+r} = \lambda^q \omega^{q+r} \mu_i^{q+r} \tag{2.7}$$

for some i, $1 \leqq i \leqq s$, where μ_i is non-zero. Combining (2.7) with (2.4) we obtain

$$\lambda^q \omega^{q+r} (\mu^{q+r} - \mu_i^{q+r}) = 0,$$

and since λ and ω are non-zero, $\mu^{q+r} = \mu_i^{q+r}$. Taking $(q+r)$-th roots we obtain

$$\mu = \theta\, \mu_i$$

where θ is some $(q+r)$-th root of unity. Now it follows from Theorem I that μ is an eigenvalue of B, concluding the proof.

3. Applications

We now show that a matrix with non-zero elements in only three diagonals (or three block diagonals), one being the principal diagonal, satisfies the conditions of Definition II.

Theorem III. If a matrix A with partitioning π has square non-singular block diagonal submatrices A_{ii}, $i = 1, 2, \ldots, p$, and the associated Jacobi matrix, B, consists of two block diagonals, one in L and one in U, such that q^* is the number of zero submatrices before the first non-zero submatrix in the first row, and r^* is the number of zero submatrices before the first non-zero submatrix in the first column (see Fig. 1), then A is $\mathrm{GCO}(\pi, q^*, r^*)$.

Proof. Let $\mu(\alpha)$ be an eigenvalue of $B(\alpha)$ and let a corresponding eigenvector $X^{(\alpha)}$ be partitioned by π so that

$$X^{(\alpha)} = \{X_i^{(\alpha)},\ i = 1, 2, \ldots, p\}.$$

Partitioning B similarly, we can write

$$\begin{aligned}
\alpha^{-q^*} B_{i,i+q^*} X_{i+q^*}^{(\alpha)} &= \mu(\alpha) X_i^{(\alpha)}, & i &= 1(1) r^*,\\
\alpha^{r^*} B_{i,i-r^*} X_{i-r^*}^{(\alpha)} + \alpha^{-q^*} B_{i,i+q^*} X_{i+q^*}^{(\alpha)} &= \mu(\alpha) X_i^{(\alpha)}, & i &= r^* + 1(1) p - q^*,\\
\alpha^{r^*} B_{i,i-r^*} X_{i-r^*}^{(\alpha)} &= \mu(\alpha) X_i^{(\alpha)}, & i &= p - q^* + 1(1) p.
\end{aligned}$$

Setting $Z_i^{(\alpha)} = X_i^{(\alpha)}/\alpha^i$, we obtain

$$\begin{aligned}
B_{i,i+q^*} Z_{i+q^*}^{(\alpha)} &= \mu(\alpha) Z_i^{(\alpha)}, & i &= 1(1) r^*,\\
B_{i,i-r^*} Z_{i-r^*}^{(\alpha)} + B_{i,i+q^*} Z_{i+q^*}^{(\alpha)} &= \mu(\alpha) Z_i^{(\alpha)}, & i &= r^* + 1(1) p - q^*,\\
B_{i,i-r^*} Z_{i-r^*}^{(\alpha)} &= \mu(\alpha) Z_i^{(\alpha)}, & i &= p - q^* + 1(1) p,
\end{aligned}$$

and thus $\mu(\alpha)$ is an eigenvalue of B, and the theorem is proved.

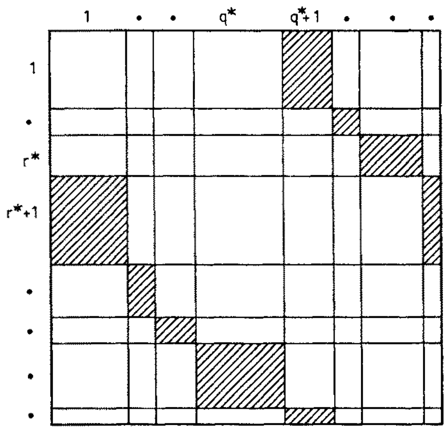

Fig. 1. The Block Matrix B. (Non-zero elements occur only in shaded blocks)

By proving a result similar to VARGA's Theorem 4.1 on the relation between the directed graph of a matrix and its cyclicity, one can show that a matrix satisfying the above conditions is a consistently ordered $(q^* + r^*)/d$ matrix where d is the greatest common divisor of q^* and r^*; thus the theorem is perhaps weak in the sense that while the conclusion allows us to apply Theorem II to A, this result could have been obtained from VARGA's work. Our derivation does however avoid the necessity of establishing the cyclicity of A.

There are certain matrices which do not have the form stated in Theorem III and are not cyclic in the sense of Definition I (ii), which are GCO (π, q, r). The following theorem is useful when the conditions of Theorem III are not satisfied.

Theorem IV. If there exists an integer k, such that the eigenvalues of

$$B(\alpha)^k = (\alpha^r L + \alpha^{-q} U)^k \tag{3.5}$$

are independent of α for $\alpha \neq 0$, then the eigenvalues of $B(\alpha)$ are independent of α. In particular if

$$B(\alpha)^k = B^k \tag{3.6}$$

then the eigenvalues of $B(\alpha)$ are independent of α.

Proof. Because the eigenvalues of $B(\alpha)^k$ are independent of α, they may be represented by the set

$$\{\eta_i \mid i = 1(1)\, m + s(q+r)\} \tag{3.7}$$

which is independent of α. Thus, for any value of α, the eigenvalues of $B(\alpha)$ are contained in

$$\{\theta\,\mu_i \mid \theta \text{ is any } k\text{-th root of unity; } \mu_i \text{ is some primitive } k\text{-th root of } \eta_i;\ i = 1(1)\, m + s(q+r)\}, \tag{3.8}$$

which is independent of α. It follows that any change in the eigenvalues of $B(\alpha)$ which accompanies a change in α, would be a discrete change. However, since the roots of an algebraic equation are continuous functions of the coefficients of the equation, the eigenvalues of a matrix must be continuous functions of the elements of the matrix, and thus any such change in the eigenvalues would be impossible. As this must be true for all changes in α, the theorem is proved.

We now wish to find q and r such that a matrix, not of the form stated in Theorem III, is GCO(π, q, r), if such numbers exist. By the remark following Definition II it is necessary only to consider values of q and r which are relatively prime. We shall further require that B have at least one non-zero eigenvalue; for by a proof similar to that of Theorem I, it can be shown that if a matrix A is GCO(π, q, r) with q and r relatively prime, and $q+r>n$, then all the eigenvalues of B are zero. Thus we need only test to determine if B is GCO(π, q, r) with q and r relatively prime, and $q+r \leqq n$, and there are two methods available. The first is to calculate the k-th power of $B(\alpha)$ for some k (in experimental work it was found convenient to set $k=q+r$); this may give a matrix independent of α, or the characteristic polynomial of the matrix may be independent of α, and in either case Theorem IV may be applied. The second method is to calculate the characteristic polynomial directly to determine whether or not it is independent of α. For large matrices, the first method may be faster if some power of $B(\alpha)$ is independent of α; however the second method will always give the required information.

For certain matrices the following heuristic method for finding q and r directly is suggested by BROYDEN'S results. If there exists a permutation matrix P such that for a matrix A, P^TAP has the form in Theorem III, select the appropriate values of q and r from P^TAP as in the statement of the theorem, and apply either of the above method to A with these values of q and r.

4. Example

We consider the matrix A given by BROYDEN [2, p. 282] which has the Jacobi matrix

$$B = \begin{bmatrix} 0 & -b & b & -b \\ -b & 0 & 0 & -b \\ b & 0 & 0 & -b \\ -b & -b & -b & 0 \end{bmatrix}$$

for the partitioning $\hat{\pi}$ with $p=4$ of A. Starting with the simplest case, $q=r=1$ (and taking larger values of q and r if necessary), we apply each of the above methods. For the first method we obtain

$$B(\alpha)^2 = \begin{bmatrix} 3b^2 & b^2 & b^2 & 0 \\ b^2 & 2b^2 & 0 & b^2 \\ b^2 & 0 & 2b^2 & -b^2 \\ 0 & b^2 & -b^2 & 3b^2 \end{bmatrix}$$

which is independent of α, and by Theorem IV, A is GCO$(\hat{\pi}, 1, 1)$. For the second method we calculate the characteristic polynomial of $B(\alpha)$, and we obtain

$$\det\{B(\alpha) - \lambda I\} = \lambda^4 - 5b^2\lambda^2 + 4b^4$$

which is independent of α, and A is GCO$(\hat{\pi}, 1, 1)$. In this example, the two methods involve about the same amount of work; however the advantages of the first method over the second (if it is applicable) are obvious for large matrices.

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