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Lopsided PMHSS iteration method for a class of complex symmetric linear systems

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Abstract Based on the preconditioned modified Hermitian and skew-Hermitian splitting (PMHSS) iteration method, we introduce a lopsided PMHSS (LPMHSS) iteration method for solving a broad class of complex symmetric linear systems. The convergence properties of the LPMHSS method are analyzed, which show that, under a loose restriction on parameter α, the iterative sequence produced by LPMHSS method is convergent to the unique solution of the linear system for any initial guess. Furthermore, we derive an upper bound for the spectral radius of the LPMHSS iteration matrix, and the quasi-optimal parameter $\alpha^{\star}$ which minimizes the above upper bound is also obtained. Both theoretical and numerical results indicate that the LPMHSS method outperforms the PMHSS method when the real part of the coefficient matrix is dominant.

Keywords Complex symmetric linear system · Positive definite · Lopsided PMHSS iteration · Spectral radius · Preconditioning · Convergence analysis

Mathematics Subject Classifications (2010) 65F10 · 65F50 · CR: G1.3

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1 Introduction

We consider the system of linear equations

$$Ax = b, \qquad A \in \mathbb{C}^{n\times n}, \quad x, b \in \mathbb{C}^n, \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$ is a complex symmetric matrix of the form

$$A = W + iT, \tag{2}$$

with $W \in \mathbb{R}^{n\times n}$ being symmetric positive definite and $T \in \mathbb{R}^{n\times n}$ being symmetric positive semidefinite. Here and in the sequel, we use $i = \sqrt{-1}$ to denote the imaginary unit. Such kind of linear systems arise in many problems in scientific computing and engineering applications. For more detailed descriptions, we refer to [1, 4, 13, 15] and the references therein.

Since the matrix $A \in \mathbb{C}^{n\times n}$ naturally possesses the Hermitian and skew-Hermitian splitting (HSS)

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^*) = W \quad \text{and} \quad S = \frac{1}{2}(A - A^*) = iT,$$

the HSS iteration method proposed by Bai et al. [8] can be used to compute the approximate solution of the complex symmetric linear system (1–2). However, a potential difficulty with the HSS iteration method is the need to solve the shifted skew-Hermitian sub-system of linear equations at each iteration step, which is as difficult as that of the original problem; see [2, 3, 7–12, 21] for more detailed descriptions about the HSS iteration method and its variants.

Recently, by making use of the special structure of the coefficient matrix $A \in \mathbb{C}^{n\times n}$, Bai et al. designed a modified HSS (MHSS) method and a preconditioned MHSS (PMHSS) method in [4] and [5], respectively, to solve the complex symmetric linear system (1–2); see also [6, 17]. The PMHSS iteration method can be described as follows:

Algorithm 1 The PMHSS iteration method

Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty} \subset \mathbb{C}^n$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha V + W)x^{(k+\frac{1}{2})} = (\alpha V - iT)x^{(k)} + b, \\ (\alpha V + T)x^{(k+1)} = (\alpha V + iW)x^{\left(k+\frac{1}{2}\right)} - ib, \end{cases} \tag{3}$$

where α is a given positive constant and $V \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix.

In particular, if we choose $V = I$, the PMHSS iteration method is reduced to the MHSS iteration method. Since matrices $\alpha V + W$ and $\alpha V + T$ are both real symmetric positive definite, the two sub-systems involved in each step of the

PMHSS iteration can be effectively solved either exactly by a sparse Cholesky factorization, or inexactly by a preconditioned conjugate gradient scheme [19]. Moreover, the PMHSS iteration method converges to the unique solution of the system of linear equations (1–2) for any positive constant α and any initial guess $x^{(0)}$ [5]. In addition, the PMHSS iteration method naturally leads to a preconditioner $\mathscr{P} = (\alpha V + W)V^{-1}(\alpha V + T)$ for the complex symmetric matrix A. We refer to $\mathscr{P}$ as the PMHSS preconditioner.

For the non-Hermitian and positive definite systems of linear equations, Li et al. [18] proposed a class of lopsided HSS (LHSS) iteration methods based on HSS scheme. Theoretical analyses in [18, 20] show that the LHSS method converges to the unique solution of the system for a wide range of the parameter α and any initial guess $x^{(0)}$. Moreover, the LHSS method outperforms HSS method when the Hermitian part of the coefficient matrix is dominant.

In this work, based on the ideas of [5] and [18], we present a new approach named as the lopsided PMHSS (LPMHSS) iteration method to solve the complex symmetric linear system of linear equations (1–2). The LPMHSS iteration method can be described as follows:

Algorithm 2 The LPMHSS iteration method

Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty} \subset \mathbb{C}^n$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} Wx^{(k+\frac{1}{2})} = -i\,Tx^{(k)} + b, \\ (\alpha V + T)x^{(k+1)} = (\alpha V + iW)x^{(k+\frac{1}{2})} - ib, \end{cases} \tag{4}$$

where α is a given positive constant and $V \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix.

Just like the PMHSS method (3), both matrices W and $\alpha V + T$ are symmetric positive definite. Hence, the two linear sub-systems in (4) can also be effectively solved either exactly by a sparse Cholesky factorization, or inexactly by a preconditioned conjugate gradient scheme. Theoretical analysis shows that the iterative sequence produced by LPMHSS iteration method converges to the unique solution of the complex symmetric linear system (1–2) for a loose restriction on the choice of α. The contraction factor of the LPMHSS iteration can be bounded by a function, which is dependent only on the choice of α, the smallest eigenvalue of matrix $V^{-1}W$ and the largest eigenvalue of matrix $V^{-1}T$.

This paper is organized as follows. In Section 2, we analyze the convergence properties, including convergence condition, spectral radius of the iterative matrix and the choices of iterative parameter etc, of the LPMHSS iteration for complex symmetric linear system (1–2). The comparison of the convergence speeds of PMHSS and LPMHSS iteration methods are implemented in Section 3. In Section 4, numerical results are presented to illustrate the effectiveness of our methods. Finally, in Section 5, we end this work with a brief conclusion.

2 Convergence analysis of the LPMHSS iteration method

In this section, we first consider the convergence properties of the LPMHSS iteration method. After straightforward derivation, we can reformulate the LPMHSS iteration scheme as

$$x^{(k+1)} = M(V;\alpha)x^{(k)} + N(V;\alpha)b, \quad k = 0, 1, 2\ldots,$$

where

$$M(V;\alpha) = -i\,(\alpha V + T)^{-1}(\alpha V + iW)W^{-1}T \tag{5}$$

and

$$N(V;\alpha) = \alpha(\alpha V + T)^{-1}VW^{-1}.$$

Here, $M(V;\alpha)$ is the iteration matrix of the LPMHSS iteration method.

Since the coefficient matrix A can be split as

$$A = B(V;\alpha) - C(V;\alpha), \tag{6}$$

where

$$B(V;\alpha) = \frac{1}{\alpha}WV^{-1}(\alpha V + T), \quad C(V;\alpha) = -\frac{i}{\alpha}(\alpha V + iW)V^{-1}T. \tag{7}$$

The LPMHSS iteration scheme can also be derived from split (6) by noticing that $M(V;\alpha) = B(V;\alpha)^{-1}C(V;\alpha)$. The matrix $B(V;\alpha)$ can be viewed as a preconditioner for the coefficient matrix $A \in \mathbb{C}^{n\times n}$. We call it as the LPMHSS preconditioner. In particular, both LPMHSS and PMHSS iteration methods lead to the same preconditioner $\mathscr{P} = \alpha W + T$ when we choose $V = W$, since the multiplicative factor $1/\alpha$ appeared in LPMHSS precoditioner has no effect on the preconditioned system.

Let $\lambda_j > 0$ and $\mu_j \geq 0$ with $j = 1, 2, \ldots, n$ being the eigenvalues of matrices $V^{-1}W$ and $V^{-1}T$, respectively. Denote $\lambda_{\min}$ the smallest eigenvalue of matrix $V^{-1}W$ and $\mu_{\max}$ the largest eigenvalue of matrix $V^{-1}T$. The following theorem gives the convergence result of the LPMHSS iteration method.

Theorem 1 *Let $A = W + iT \in \mathbb{C}^{n\times n}$, with $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ symmetric positive definite and symmetric positive semidefinite, respectively. Then, the spectral radius $\rho(M(V;\alpha))$ of the LPMHSS iteration matrix (5) satisfies $\rho(M(V;\alpha)) \leq \sigma(\alpha)$, where*

$$\sigma(\alpha) = \frac{\sqrt{\alpha^2 + \lambda_{\min}^2}}{\lambda_{\min}} \frac{\mu_{\max}}{\alpha + \mu_{\max}}. \tag{8}$$

Moreover, it holds that

(i) If $\lambda_{\min} \geq \mu_{\max}$, then $\sigma(\alpha) < 1$ for any $\alpha > 0$, which means that the LPMHSS iteration method is unconditionally convergent;

(ii) if $\lambda_{\min} < \mu_{\max}$, then $\sigma(\alpha) < 1$ if and only if

$$\alpha < \frac{2\mu_{\max}\lambda_{\min}^2}{\mu_{\max}^2 - \lambda_{\min}^2}, \tag{9}$$

which means that the LPMHSS iteration method is convergent under the condition (9).

Proof Since $V \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, we denote

$$\widetilde{W} = V^{-\frac{1}{2}} W V^{-\frac{1}{2}} \quad \text{and} \quad \widetilde{T} = V^{-\frac{1}{2}} T V^{-\frac{1}{2}}.$$

Obviously, $\widetilde{W} \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $\widetilde{T}$ is symmetric positive semidefinite. Moreover, $\widetilde{W}$ and $\widetilde{T}$ are similar to $V^{-1}W$ and $V^{-1}T$, respectively. Hence, there exist two orthogonal matrices $P,\ Q \in \mathbb{R}^{n\times n}$ such that

$$P^T \widetilde{W} P = \Lambda_w \quad \text{and} \quad Q^T \widetilde{T} Q = \Lambda_t, \tag{10}$$

where

$$\Lambda_w = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \quad \text{and} \quad \Lambda_t = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n).$$

Now, by direct computations, we have

$$\begin{aligned}
\rho(M(V;\alpha)) &= \rho((\alpha V + T)^{-1}(\alpha V + iW)W^{-1}T) \\
&= \rho\left((\alpha I + i\widetilde{W})\widetilde{W}^{-1}\widetilde{T}(\alpha I + \widetilde{T})^{-1}\right) \\
&\le \left\|(\alpha I + i\widetilde{W})\widetilde{W}^{-1}\widetilde{T}(\alpha I + \widetilde{T})^{-1}\right\|_2 \\
&\le \left\|(\alpha I + i\widetilde{W})\widetilde{W}^{-1}\right\|_2 \left\|\widetilde{T}(\alpha I + \widetilde{T})^{-1}\right\|_2 .
\end{aligned}$$

Using (10) and the orthogonal invariance of the Euclidean norm $\|\cdot\|_2$, the spectral radius $\rho(M(V;\alpha))$ can be further amplified as

$$\begin{aligned}
\rho(M(V;\alpha)) &\le \left\|(\alpha I + i\Lambda_w)\Lambda_w^{-1}\right\|_2 \left\|\Lambda_t(\alpha I + \Lambda_t)^{-1}\right\|_2 \\
&= \max_{\lambda_j \in sp(V^{-1}W)} \left|\frac{\alpha + i\lambda_j}{\lambda_j}\right| \cdot \max_{\mu_j \in sp(V^{-1}T)} \left|\frac{\mu_j}{\alpha + \mu_j}\right| \\
&= \max_{\lambda_j \in sp(V^{-1}W)} \frac{\sqrt{\alpha^2 + \lambda_j^2}}{\lambda_j} \cdot \max_{\mu_j \in sp(V^{-1}T)} \frac{\mu_j}{\alpha + \mu_j},
\end{aligned}$$

where *sp(X)* denotes the spectrum of the matrix *X*.

Since

$$\max_{\lambda_j \in sp(V^{-1}W)} \frac{\sqrt{\alpha^2 + \lambda_j^2}}{\lambda_j} = \frac{\sqrt{\alpha^2 + \lambda_{\min}^2}}{\lambda_{\min}}$$

and

$$\max_{\mu_j \in sp(V^{-1}T)} \frac{\mu_j}{\alpha + \mu_j} = \frac{\mu_{\max}}{\alpha + \mu_{\max}}.$$

The upper bound of $\rho(M(V;\alpha))$ given in (8) is obtained.

By simple calculation, $\sigma(\alpha) < 1$ is equivalent to

$$(\mu_{\max}^2 - \lambda_{\min}^2)\alpha < 2\lambda_{\min}^2 \mu_{\max}. \tag{11}$$

If $\lambda_{\min} \ge \mu_{\max}$, then (11) holds true for any $\alpha > 0$, i.e., the LPMHSS iteration converges to the unique solution of the system of linear equations (1–2); if $\lambda_{\min} < \mu_{\max}$, then (11) or $\sigma(\alpha) < 1$ follows if and only if α satisfies (9).

Therefore, in case (ii), the sufficient and necessary condition keeping the convergence of the LPMHSS iteration method is inequality (9). □

Theorem 1 gives the convergence conditions of the LPMHSS iteration method for the system of linear equations (1–2) by analyzing the upper bound $\sigma(\alpha)$ of the spectral radius of the iteration matrix $M(V;\alpha)$. Since the optimal parameter α minimizing the spectral radius $\rho(M(V;\alpha))$ is hardly obtained, we instead give the parameter $\alpha^\star$, which minimizes the upper bound $\sigma(\alpha)$ of the spectral radius $\rho(M(V;\alpha))$, in the following corollary.

Corollary 1 *Let the conditions of Theorem 1 be satisfied. Then, the parameter $\alpha^\star$ minimizing the upper bound $\sigma(\alpha)$ of the spectral radius $\rho(M(V;\alpha))$ is*

$$\alpha^\star \equiv \arg\min_{\alpha}\{\sigma(\alpha)\} = \arg\min_{\alpha}\left\{\frac{\sqrt{\alpha^2+\lambda_{\min}^2}}{\lambda_{\min}}\cdot\frac{\mu_{\max}}{\alpha+\mu_{\max}}\right\} = \frac{\lambda_{\min}^2}{\mu_{\max}}$$

and

$$\sigma(\alpha^\star) = \frac{\mu_{\max}}{\sqrt{\lambda_{\min}^2+\mu_{\max}^2}}. \qquad (12)$$

Proof Simple calculation gives

$$\sigma'(\alpha) = \frac{\mu_{\max}(\alpha\mu_{\max}-\lambda_{\min}^2)}{\lambda_{\min}(\alpha+\mu_{\max})^2\sqrt{\alpha^2+\lambda_{\min}^2}}.$$

It is obviously that $\sigma'(\alpha) > 0$ for $\alpha > \lambda_{\min}^2/\mu_{\max}$ and $\sigma'(\alpha) < 0$ for $\alpha < \lambda_{\min}^2/\mu_{\max}$. Hence, the upper bound $\sigma(\alpha)$ of the spectral radius $\rho(M(V;\alpha))$ achieves its minimum at $\alpha^\star = \lambda_{\min}^2/\mu_{\max}$. Taking $\alpha^\star$ into $\sigma(\alpha)$, the minimum value of $\sigma(\alpha)$ given in (12) is obtained. □

Remark 1 For case (ii) of Theorem 1, i.e., $\lambda_{\min} < \mu_{\max}$, we see that the $\alpha^\star$ given in Corollary 1 satisfies condition (9) since

$$\alpha^\star = \frac{\lambda_{\min}^2}{\mu_{\max}} < \frac{2\mu_{\max}\lambda_{\min}^2}{\mu_{\max}^2} < \frac{2\mu_{\max}\lambda_{\min}^2}{\mu_{\max}^2-\lambda_{\min}^2}.$$

Remark 2 The parameter $\alpha^\star$ in Corollary 1 minimizes only the upper bound $\sigma(\alpha)$ of the spectral radius of the iteration matrix. However, it is still helpful to us to choose an effective parameter α for the LPMHSS iteration method. We call $\alpha^\star$ the theoretical quasi-optimal parameter of LPMHSS iteration method.

3 Comparison of the LPMHSS and PMHSS methods

In this section, we compare the LPMHSS and PMHSS methods by analyzing their respective optimal upper bounds and give a criterion for choosing between these two methods.

We first introduce a lemma briefly reviewing the convergence analysis of the PMHSS method established in [5].

Lemma 1 *Let $A = W + iT \in \mathbb{C}^{n\times n}$, with $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ symmetric positive definite and symmetric positive semidefinite, respectively. Then, the spectral radius $\rho(L(V;\alpha))$ of the PMHSS iteration matrix $L(V;\alpha) = (\alpha V + T)^{-1}(\alpha V + iW)(\alpha V + W)^{-1}(\alpha V - iT)$ satisfies $\rho(L(V;\alpha)) \le \gamma(\alpha)$, where*

$$\gamma(\alpha) = \max_{\lambda_j \in sp(V^{-1}W)} \frac{\sqrt{\alpha^2 + \lambda_j^2}}{\alpha + \lambda_j}.$$

Thus, it holds that

$$\rho(L(V;\alpha)) \le \gamma(\alpha) < 1, \quad \forall \alpha > 0.$$

Moreover, the minimum point $\alpha_\star$ and the minimum value $\gamma(\alpha_\star)$ of the upper bound $\gamma(\alpha)$ are respectively as

$$\alpha_\star \equiv \arg\min_{\alpha}\{\gamma(\alpha)\} = \arg\min_{\alpha}\left\{\max_{\lambda_j \in SP(V^{-1}W)} \frac{\sqrt{\alpha^2 + \lambda_j^2}}{\alpha + \lambda_j}\right\} = \sqrt{\lambda_{\min}\lambda_{\max}}$$

and

$$\gamma(\alpha_\star) = \frac{\sqrt{\lambda_{\min} + \lambda_{\max}}}{\sqrt{\lambda_{\min}} + \sqrt{\lambda_{\max}}},$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and the largest eigenvalues of matrix $V^{-1}W$, respectively.

Using Corollary 1 and Lemma 1, we give the following comparison theorem for the LPMHSS and the PMHSS iteration methods.

Theorem 2 *The respective optimal upper bounds of the spectral radii of LPMHSS and PMHSS iteration matrices satisfy*

$$\sigma(\alpha^\star) \le \gamma(\alpha_\star)$$

if and only if

$$\mu_{\max} \le \sqrt{\frac{\lambda_{\min} + \lambda_{\max}}{2\sqrt{\lambda_{\min}\lambda_{\max}}}} \cdot \lambda_{\min}. \tag{13}$$

Proof From Corollary 1 and Lemma 1, inequality $\sigma(\alpha^\star) \le \gamma(\alpha_\star)$ becomes

$$\frac{\mu_{\max}}{\sqrt{\lambda_{\min}^2 + \mu_{\max}^2}} \le \frac{\sqrt{\lambda_{\min} + \lambda_{\max}}}{\sqrt{\lambda_{\min}} + \sqrt{\lambda_{\max}}},$$

which is equivalent to

$$\mu_{\max}(\sqrt{\lambda_{\min}} + \sqrt{\lambda_{\max}}) \le \sqrt{\lambda_{\min} + \lambda_{\max}}\sqrt{\lambda_{\min}^2 + \mu_{\max}^2}. \tag{14}$$

Squaring both sides of (14) and combining the coefficients of $\mu_{\max}^2$, we get

$$2\sqrt{\lambda_{\min}\lambda_{\max}} \cdot \mu_{\max}^2 \le (\lambda_{\min} + \lambda_{\max})\lambda_{\min}^2.$$

Thus, the condition (13) follows by noticing that $\lambda_{\max} \ge \lambda_{\min} > 0$. □

Remark 3 In particular, if we choose the preconditioner V such that $V = W$, then $\lambda_{\min} = \lambda_{\max} = 1$. Therefore, the condition (13) is simplified to $\mu_{\max} \le 1$. This implies that when the largest eigenvalue of the matrix $W^{-1}T$ is less than or equal to 1, we have $\sigma(\alpha^\star) \le \gamma(\alpha_\star)$.

Remark 4 Theorem 2 only compares the upper bounds on the spectral radii of the iteration matrices of LPMHSS and PMHSS methods when respective quasi-optimal values of the parameter α is employed. When inequality (13) holds, i.e., the real part of the coefficient matrix A comparing with its imaginary part is dominant, we tend to choose LPMHSS method rather than PMHSS method to solve the complex symmetric linear system (1–2) and vice verse.

4 Numerical examples

In this section, we use two different problems to illustrate the convergence results of the LPMHSS iteration method. In our implementations, the initial guess is chosen to be $x^{(0)} = \mathbf{0}$ and the iteration is terminated once the current iterate $x^{(k)}$ satisfies

$$\frac{\|b - Ax^{(k)}\|_2}{\|b\|_2} \le 10^{-6}.$$

In addition, all the linear subsystems involved in each step of the MHSS, PMHSS and LPMHSS iteration methods are solved exactly by the sparse Cholesky factorization. The preconditioners V used in PMHSS and LPMHSS methods are chosen to be $V = W$. In this case, the LPMHSS and PMHSS iteration schemes lead to the same preconditioner $\mathscr{P} = \alpha W + T$. Since the numerical results in [4, 5] show that the PMHSS iteration method outperforms the MHSS and the HSS iteration methods when they are employed as preconditioners for the GMRES method or its restarted variants [19], we just examine the efficiency of LPMHSS iteration method as a solver for solving complex symmetric linear system (1–2) by comparing the iteration numbers (denoted as IT) and CPU times (in seconds, denoted as CPU) of LPMHSS method with those of MHSS and PMHSS methods.

The iteration parameters used in MHSS and PMHSS iteration methods are the experimental optimal ones $\alpha_{\exp}$, which minimize the numbers of iteration steps. For LPMHSS method, we use both of its theoretical quasi-optimal parameters $\alpha^\star$ derived in Corollary 1 and the experimental optimal parameters $\alpha_{\exp}$.

Moreover, if the experimental optimal iteration parameters form an interval, then they are further optimized according to the least CPU time. We remark that when the right endpoints of the experimental optimal parameter intervals obtained from minimizing the iteration steps are larger than 1000, we just cut off and set them as 1000; see Tables 1 and 2. From the results in both of the two tables, we can observe an interesting and useful phenomenon, i.e., the theoretical quasi-optimal parameter $\alpha^\star$ of LPMHSS method for each case of the problems is located in its experimental optimal parameter interval. This phenomenon shows that $\alpha^\star$ derived in Corollary 1 is a good choice for the iteration parameter of LPMHSS method, although it does not minimize the spectral radius of the iteration matrix of this method.

Example 1 (See [4, 5, 13]) The complex symmetric linear system (1–2) is of the form

$$[(-\omega^2 M + K) + i(\omega C_V + C_H)]x = b,$$

where ω is the driving circular frequency, M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, respectively. We take $M = I$, $C_V = 10I$, $C_H = \mu K$ with μ a damping coefficient, and K the five-point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions on an uniform mesh $h = 1/(m+1)$ in the unit square $[0, 1] \times [0, 1]$. The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = B_m \otimes I + I \otimes B_m$, with $B_m = h^{-2} \cdot \mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. Hence, K is an $n \times n$ block-tridiagonal matrix, with $n = m^2 = 64^2$. In addition, we set $\omega = 1$ and the right-hand side vector b to be $b = (1+i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. Furthermore, we normalize coefficient matrix and right-hand side by multiplying both by h^2.

The convergence speed of the iteration method depends largely on the spectral radius of the iteration matrix. The comparisons of spectral radii of the three different iteration matrices derived by MHSS, PMHSS and LPMHSS iteration methods with different damping coefficients μ are performed in Fig. 1. From Fig. 1, we find that when μ is small (the real part of the coefficient matrix is dominant), the spectral radius of the iteration matrix of LPMHSS method is much smaller than those of MHSS and PMHSS methods, but as μ becomes large (the imaginary part is dominant), MHSS and PMHSS methods perform more and more better.

Table 1 The experimental optimal or theoretical quasi-optimal iteration parameters of MHSS, PMHSS and LPMHSS methods for Example 1

Method	$\mu = 1$	$\mu = 0.1$	$\mu = 0.01$	$\mu = 0.001$
MHSS ($\alpha_{\exp}$)	[0.195, 0.201]	[0.067, 0.071]	[0.020, 0.022]	[0.004, 0.005]
PMHSS ($\alpha_{\exp}$)	[0.909, 1.103]	[0.158, 0.905]	[0.787, 1.152]	[0.826, 1.185]
LPMHSS ($\alpha_{\exp}$)	[0.608, 0.652]	[1.160, 2.204]	[0.979, 4.911]	[1.474, 2.461]
LPMHSS ($\alpha^\star$)	0.630	1.565	1.837	1.870

Table 2 The experimental optimal or theoretical quasi-optimal iteration parameters of MHSS, PMHSS and LPMHSS methods for Example 2

Method	$\sigma_2 = 1$	$\sigma_2 = 10$	$\sigma_2 = 100$	$\sigma_2 = 1000$
MHSS (α_{exp})	[0.403, 0.410]	[0.0021, 0.0027]	[0.020, 0.030]	[0.218, 0.311]
PMHSS (α_{exp})	[0.897, 1.113]	[0.846, 1.166]	[0.766, 1.193]	[0.760, 1.088]
LPMHSS (α_{exp})	[0.396, 1000]	[1.184, 1000]	[0.870, 1.682]	[0.119, 0.120]
LPMHSS ($\alpha^{\star}$)	119.7	11.97	1.197	0.120

In Table 3, we present iteration numbers (IT) and CPU times (CPU) for MHSS, PMHSS and LPMHSS iteration methods with different damping coefficients μ. From the results in Table 3, we see that when μ is small, LPMHSS methods, no matter with the experimental optimal parameter α_{exp} or with the theoretical quasi-optimal parameter $\alpha^{\star}$, perform much better than MHSS and PMHSS methods both in iteration numbers and in CPU times. But as μ becomes large, the superiorities of LPMHSS methods disappear. In addition, the efficiencies of LPMHSS methods with iteration parameters $\alpha^{\star}$ and α_{exp} are comparable, which also means that $\alpha^{\star}$ is a good choice for the iteration parameter of LPMHSS method.

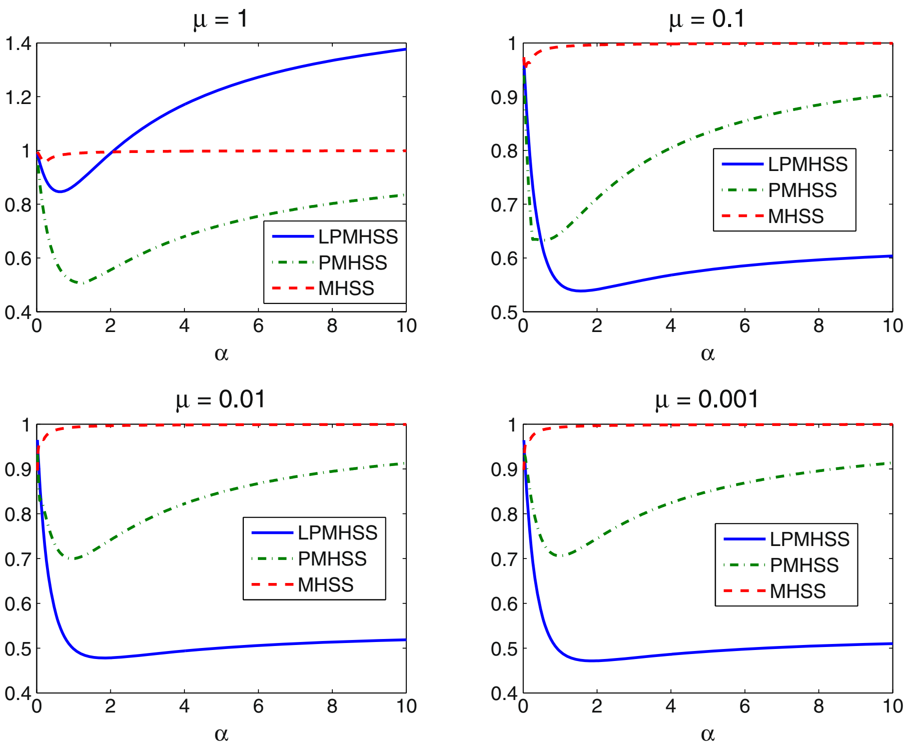

Fig. 1 The spectral radii of the iteration matrices versus α with $\mu = 1, 0.1, 0.01$ and 0.001

Table 3 The ITs and CPUs of MHSS, PMHSS and LPMHSS methods for Example 1

Method	μ	1	0.1	0.01	0.001
MHSS	$\alpha_{\exp}$	0.198	0.071	0.020	0.005
	IT	182	102	48	41
	CPU	6.5620	3.6720	1.7340	1.4690
PMHSS	$\alpha_{\exp}$	0.977	0.336	0.874	0.856
	IT	20	31	39	40
	CPU	0.6870	1.0780	1.3900	1.4370
LPMHSS	$\alpha_{\exp}$	0.630	2.054	2.307	2.137
	IT	59	16	14	13
	CPU	2.1090	0.5620	0.5000	0.4530
LPMHSS	$\alpha^{\star}$	0.630	1.565	1.837	1.870
	IT	59	16	14	13
	CPU	2.1140	0.5790	0.5010	0.4840

Example 2 (See [14, 16]) Let us consider the following complex Helmholtz equation

$$-\Delta u + \sigma_1 u + i\sigma_2 u = f,$$

where σ_1, σ_2 are real coefficient functions, and u satisfies Dirichlet boundary conditions in $D = [0, 1] \times [0, 1]$. The above equation describes the propagation of damped time-harmonic waves. We take H the five-point centered difference matrix approximating the negative Laplacian operator on an uniform mesh with mesh-size $h = 1/(m + 1)$. The matrix $H \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $H = B_m \otimes I + I \otimes B_m$, with $B_m = h^{-2} \cdot \mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. Hence, H is an $n \times n$ block-tridiagonal matrix, with $n = m^2 = 64^2$. This leads to the complex symmetric linear system (1–2) of the form

$$[(H + \sigma_1 I) + i\sigma_2 I]x = b.$$

In addition, we set $\sigma_1 = 100$ and the right-hand side vector b to be $b = (1 + i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. As before, we normalize the system by multiplying both sides by h^2.

The comparisons of spectral radii of the three different iteration matrices derived by MHSS, PMHSS and LPMHSS iteration methods with different parameter σ_2 are performed in Fig.2. From Fig. 2, we find that when σ_2 is small (the real part of the coefficient matrix is dominant), the spectral radius of the iteration matrix of LPMHSS method is much smaller than those of MHSS and PMHSS methods, but as σ_2 becomes large (the imaginary part is dominant), MHSS and PMHSS methods seems to perform better.

In Table 4, we present iteration numbers (IT) and CPU times (CPU) for MHSS, PMHSS and LPMHSS iteration methods with different parameters σ_2. From Table 4, we see that when σ_2 is small, LPMHSS methods, with the theoretical quasi-optimal parameters $\alpha^{\star}$ derived in Corollary 1 and the experimental optimal parameters $\alpha_{\exp}$, use the least iteration numbers and CPU times comparing with MHSS and PMHSS methods.

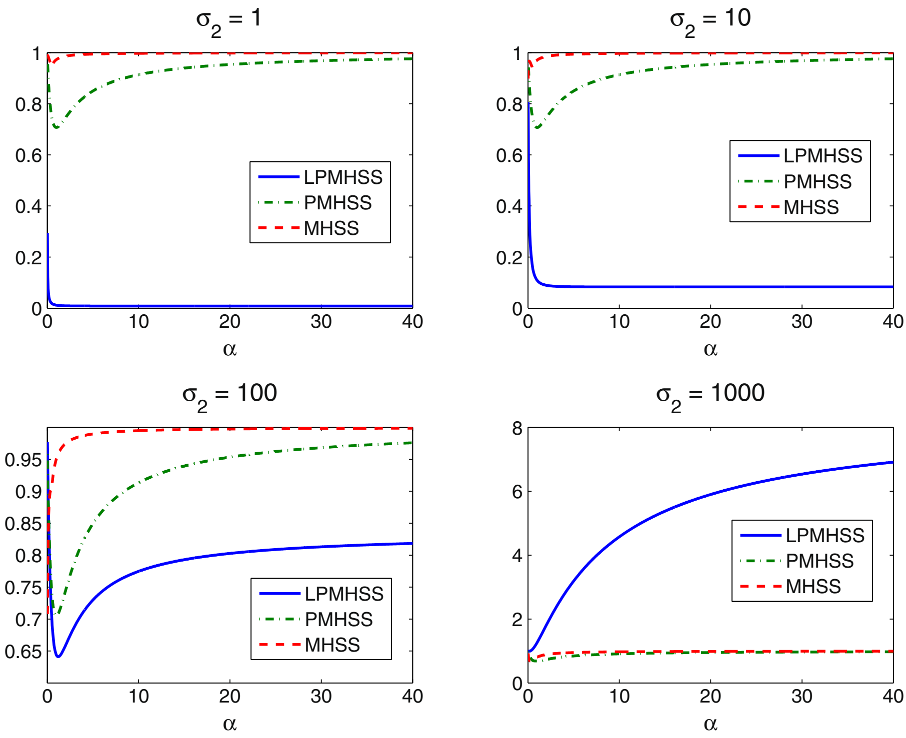

Fig. 2 The spectral radii of the iteration matrices versus α with $\sigma_2 = 1$, 10, 100 and 1000

Therefore, we tend to use LPMHSS iteration method to solve complex symmetric linear system (1–2) when the real part W is dominant comparing with the imaginary part T.

Table 4 The ITs and CPUs of MHSS, PMHSS and LPMHSS methods for Example 2

Method	σ_2	1	10	100	1000
MHSS	$\alpha_{\exp}$	0.408	0.0021	0.021	0.294
	IT	180	40	39	32
	CPU	3.3590	0.7500	0.7650	0.6400
PMHSS	$\alpha_{\exp}$	0.908	0.974	0.922	0.961
	IT	40	40	39	32
	CPU	1.4210	1.4370	1.3910	1.1560
LPMHSS	$\alpha_{\exp}$	62	12	1.360	0.120
	IT	3	5	27	1859
	CPU	0.0940	0.1720	0.9210	67.4060
LPMHSS	$\alpha^{\star}$	119.7	11.97	1.197	0.120
	IT	3	5	27	1859
	CPU	0.1250	0.1760	0.9380	68.4840

5 Conclusions

In this paper, we have introduced a lopsided PMHSS (LPMHSS) method for solving a broad class of complex symmetric linear systems. Theoretical analysis shows that for any initial guess and a wide range of parameter α, LPMHSS method converges to the unique solution of the complex linear system (1–2). We also derive an upper bound $\sigma(\alpha)$ for the spectral radius of LPMHSS iteration matrix and give the quasi-optimal parameter $\alpha^\star$ which minimizes the upper bound $\sigma(\alpha)$. In addition, both theoretical results in Theorem 2 and numerical results in Section 4 show that when the real part of the coefficient matrix is dominant, LPMHSS method performs better than MHSS and PMHSS methods. Hence, this work gives a better choice for solving the complex linear system (1–2) when the real part W of coefficient matrix A is dominant.

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