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A generalized preconditioned HSS method for non-Hermitian positive definite linear systems

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ABSTRACT

Based on the HSS (Hermitian and skew-Hermitian splitting) and preconditioned HSS methods, we will present a generalized preconditioned HSS method for the large sparse non-Hermitian positive definite linear system. Our method is essentially a two-parameter iteration which can extend the possibility to optimize the iterative process. The iterative sequence produced by our generalized preconditioned HSS method can be proven to be convergent to the unique solution of the linear system. An exact parameter region of convergence for the method is strictly proved. A minimum value for the upper bound of the iterative spectrum is derived, which is relevant to the eigensystem of the products formed by inverse preconditioner and splitting. An efficient preconditioner based on incremental unknowns is presented for the actual implementation of the new method. The optimality and efficiency are effectively testified by some comparisons with numerical results.

1. Introduction

We consider the numerical solution of a linear system which comes usually from the spatial discretization of a linear partial differential equation (see, e.g., [5,7,10]). Let the linear system be

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a nonsingular large sparse non-Hermitian and positive definite matrix, and $x, b \in \mathbb{C}^n$. Since the matrix A possesses naturally a Hermitian/skew-Hermitian splitting (HSS), i.e.,

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^*) \text{ and } S = \frac{1}{2}(A - A^*),$$

Bai et al. [1–3] proposed iterative methods called HSS and preconditioned HSS (PHSS) methods based on this particular matrix splitting. Especially, we might consider that these methods are designed for solving another preconditioned linear system $\widehat{A}\widehat{x} = \widehat{b}$ with $\widehat{A} = R^{-*}AR^{-1}$, $\widehat{x} = Rx$ and $\widehat{b} = R^{-*}b$. Where $R \in \mathbb{C}^{n\times n}$ is a prescribed nonsingular matrix, and $R^{-*} = (R^{-1})^*$ is the conjugate transpose of R^{-1}. We usually take a Hermitian positive definite matrix $P = R^*R$. In fact, the PHSS method was defined as follows (see [3]).

Method 1.1 (PHSS method). Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \ldots,$ until $\{x^{(k)}\}$ converges, compute

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$$\begin{cases} (\alpha P + H)x^{(k+\frac{1}{2})} = (\alpha P - S)x^{(k)} + b, \\ (\alpha P + S)x^{(k+1)} = (\alpha P - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{1.2}$$

where α is a given positive constant, and P is an Hermitian positive definite matrix (in particular, when we choose $P = I$, it reduces to the HSS method).

Theoretical analysis shows that both the two iterative methods converge unconditionally to the exact solution of the linear system (1.1).

With different properties of the matrices H and S, it is natural to produce different effects on the parameter in the two-steps of (1.2). Due to this consideration, we introduce two different parameters α and β in the PHSS method. This leads to the following generalized preconditioned HSS method (or, simply denoted by GPHSS method).

Method 1.2 (GPHSS method). Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \ldots$, until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha P + H)x^{(k+\frac{1}{2})} = (\alpha P - S)x^{(k)} + b, \\ (\beta P + S)x^{(k+1)} = (\beta P - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{1.3}$$

where α is a given nonnegative constant and β is a given positive constant.

The new method GPHSS is actually a two-parameter two-step iterative method. It has PHSS method as its special case with $\alpha = \beta$. And HSS method is obviously a trivial case with $\alpha = \beta$ and without preconditioner. We will argue that there exists a reasonable convergent domain of two-parameters for GPHSS method which generalizes essentially what exists for HSS or PHSS. We will also give a certain optimal upper bound for the spectral radius of iterative matrix which is located on a particular curve. This is totally a new result in this methodology. An incremental unknown (cf. [4,11]) type preconditioner and a typical model differential equation are elaborately chosen to perform the numerical computation with HSS, PHSS, GPHSS and LHSS methods.

The organization of the paper is as follows. In Section 2, the convergence of the GPHSS method is exactly considered. And for the upper bound of the spectral radius of the iteration matrix, the optimal parameters for GPHSS method are provided. In Section 3, we introduce an efficient preconditioner based on the incremental unknowns method. Numerical experiments with this preconditioner show that the GPHSS method is much more efficient than both of the HSS and PHSS methods. Finally, we give a brief concluding remark in Section 4.

2. Convergence analysis

In this section, we study the convergence rate of the GPHSS iteration. This iterative method can be generalized to the two-step splitting iterative framework, and the general convergence criterion for this two-step splitting iteration is given as follows (see [2]).

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}, A = M_i - N_i (i = 1, 2)$ be two splittings of the matrix A, and let $x^{(0)} \in \mathbb{C}^n$ be a given initial vector. If $\{x^{(k)}\}$ is a two-step iteration sequence defined by*

$$\begin{cases} M_1 x^{(k+\frac{1}{2})} = N_1 x^{(k)} + b, \\ M_2 x^{(k+1)} = N_2 x^{(k+\frac{1}{2})} + b, \quad k = 0, 1, 2, \ldots, \end{cases}$$

then

$$x^{(k+1)} = M_2^{-1} N_2 M_1^{-1} N_1 x^{(k)} + M_2^{-1}(I + N_2 M_1^{-1})b, \quad k = 0, 1, 2, \ldots$$

Moreover, if the spectral radius of the iteration matrix $M_2^{-1}N_2M_1^{-1}N_1$ is strictly less than 1, then the iterative sequence $\{x^{(k)}\}$ converges to the unique solution $x^* \in \mathbb{C}^n$ of the linear systems (1.1) for all initial vectors $x^{(0)} \in \mathbb{C}^n$.

Since S is skew-Hermitian and $P^{-1}S = R^{-1}(R^*)^{-1}S = R^{-1}(R^{*-1}SR^{-1})R$, it is clear that all eigenvalues of $P^{-1}S$ are imaginary. In particular, the jth eigenvalue of $P^{-1}S$ is of the form $ie_j (j = 1, 2, \ldots, n)$, with $i = \sqrt{-1}$. In the line of Lemma 2.1, we can obtain the convergence of the GPHSS method.

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $P \in \mathbb{C}^{n\times n}$ be an Hermitian positive definite matrix, and Λ_1 and Λ_2 be spectral sets of matrix $P^{-1}H$ and matrix $P^{-1}S$, respectively. Denote*

$$\lambda_{\max} = \max_{\lambda_k \in \Lambda_1}\{\lambda_k\}, \quad e_{\max} = \max_{ie_j \in \Lambda_2}\{|e_j|\}$$
$$\lambda_{\min} = \min_{\lambda_k \in \Lambda_1}\{\lambda_k\}, \quad e_{\min} = \min_{ie_j \in \Lambda_2}\{|e_j|\},$$

with $i = \sqrt{-1}$. Then, the GPHSS iteration (1.3) converges to the unique solution $x^* \in \mathbb{C}^n$ of the linear systems (1.1) if the parameters α and β satisfy

$$(\alpha, \beta) \in \bigcup_{i=1}^{4} \Omega_i, \tag{2.1}$$

where

$$\Omega_1 = \{(\alpha,\beta)|\alpha \leqslant \beta < \beta^*(\alpha)\},$$
$$\Omega_2 = \{(\alpha,\beta)|\beta < \min\{\alpha,\beta^*(\alpha)\}, \varphi_2(\alpha,\beta) > 0\},$$
$$\Omega_3 = \{(\alpha,\beta)|\beta^*(\alpha) \leqslant \beta < \alpha\},$$
$$\Omega_4 = \{(\alpha,\beta)|\beta \geqslant \max\{\alpha,\beta^*(\alpha)\}, \varphi_1(\alpha,\beta) > 0\},$$

with functions $\varphi_1(\alpha,\beta), \varphi_2(\alpha,\beta)$ *and* $\beta^*(\alpha)$ *denoted by*

$$\begin{aligned}\varphi_1(\alpha,\beta) &= (\beta-\alpha)\left(\lambda_{\min}^2 - e_{\max}^2\right) + 2\alpha\beta\lambda_{\min} + 2e_{\max}^2\lambda_{\min},\\ \varphi_2(\alpha,\beta) &= (\beta-\alpha)\left(\lambda_{\max}^2 - e_{\min}^2\right) + 2\alpha\beta\lambda_{\max} + 2e_{\min}^2\lambda_{\max},\\ \beta^*(\alpha) &= \frac{\alpha(\lambda_{\max}+\lambda_{\min}) + 2\lambda_{\max}\lambda_{\min}}{2\alpha + \lambda_{\max}+\lambda_{\min}} \in [\lambda_{\min},\lambda_{\max}].\end{aligned} \tag{2.2}$$

Proof. By putting

$$M_1 = \alpha P + H, \quad N_1 = \alpha P - S, \quad M_2 = \beta P + S, \text{ and } N_2 = \beta P - H$$

in Lemma 2.1 and noting that $\alpha P + H$ and $\beta P + S$ are nonsingular for any nonnegative constant α and positive constant β, one can obtain the iteration matrix of GPHSS method as follows:

$$M(\alpha,\beta) = (\beta P + S)^{-1}(\beta P - H)(\alpha P + H)^{-1}(\alpha P - S).$$

The spectral radius of the iteration matrix satisfies clearly,

$$\begin{aligned}\rho(M(\alpha,\beta)) &= \rho((\beta P - H)(\alpha P + H)^{-1}(\alpha P - S)(\beta P + S)^{-1})\\ &\leqslant \|(\beta P - H)(\alpha P + H)^{-1}\|_2 \|(\alpha P - S)(\beta P + S)^{-1}\|_2\\ &= \max_{\lambda_k \in \Lambda_1}\left|\frac{\beta-\lambda_k}{\alpha+\lambda_k}\right| \cdot \max_{ie_j\in\Lambda_2}\sqrt{\frac{\alpha^2+e_j^2}{\beta^2+e_j^2}}.\end{aligned} \tag{2.3}$$

Since $\alpha \geqslant 0$ and $\beta > 0$, it follows that

$$\max_{\lambda_k\in\Lambda_1}\left|\frac{\beta-\lambda_k}{\alpha+\lambda_k}\right| = \max\left\{\left|\frac{\beta-\lambda_{\max}}{\alpha+\lambda_{\max}}\right|, \left|\frac{\beta-\lambda_{\min}}{\alpha+\lambda_{\min}}\right|\right\}.$$

Hence using $\beta^*(\alpha)$, we have

$$\max_{\lambda_k\in\Lambda_1}\left|\frac{\beta-\lambda_k}{\alpha+\lambda_k}\right| = \begin{cases}\frac{\lambda_{\max}-\beta}{\lambda_{\max}+\alpha}, & \beta < \beta^*(\alpha),\\ \frac{\beta-\lambda_{\min}}{\alpha+\lambda_{\min}}, & \beta \geqslant \beta^*(\alpha),\end{cases} \tag{2.4}$$

and we also have

$$\max_{ie_j\in\Lambda_2}\sqrt{\frac{\alpha^2+e_j^2}{\beta^2+e_j^2}} = \begin{cases}\sqrt{\frac{\alpha^2+e_{\max}^2}{\beta^2+e_{\max}^2}}, & \alpha \leqslant \beta,\\ \sqrt{\frac{\alpha^2+e_{\min}^2}{\beta^2+e_{\min}^2}}, & \alpha > \beta,\end{cases} \tag{2.5}$$

Now, let us divide the region $D = \{(\alpha,\beta)|\alpha \geqslant 0, \beta > 0\}$ into four subregions $D = \bigcup_{i=1}^4 D_i$ (see Fig. 1 with $\alpha_0 = \sqrt{\lambda_{\max}\lambda_{\min}}$).

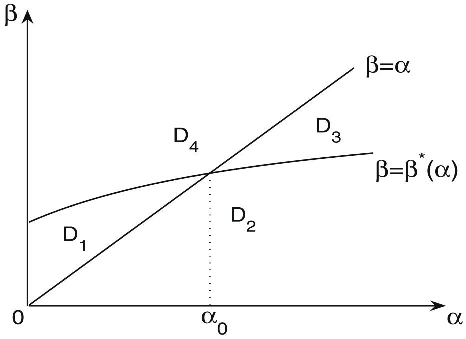

Fig. 1. The four subregions of *D*.

$$D_1 = \{(\alpha,\beta)|\alpha \leqslant \beta < \beta^*(\alpha)\}, \quad D_2 = \{(\alpha,\beta)|\beta < \min\{\alpha,\beta^*(\alpha)\}\},$$
$$D_3 = \{(\alpha,\beta)|\beta^*(\alpha) \leqslant \beta < \alpha\}, \quad D_4 = \{(\alpha,\beta)|\beta \geqslant \max\{\alpha,\beta^*(\alpha)\}\}.$$

(I) For $(\alpha,\beta) \in D_1$, we deduce easily from 2.3, 2.4 and 2.5 the following inequality

$$\rho(M(\alpha,\beta)) \leqslant \frac{\lambda_{\max}-\beta}{\lambda_{\max}+\alpha}\cdot\sqrt{\frac{\alpha^2+e_{\max}^2}{\beta^2+e_{\max}^2}} < 1.$$

(II) For $(\alpha,\beta) \in D_2$, one gets that the spectral radius $\rho(M(\alpha,\beta))$ satisfies evidently

$$\rho(M(\alpha,\beta)) \leqslant \frac{\lambda_{\max}-\beta}{\lambda_{\max}+\alpha}\cdot\sqrt{\frac{\alpha^2+e_{\min}^2}{\beta^2+e_{\min}^2}}.$$

It is clear that the right-hand side is strictly less than 1 if and only if

$$\varphi_2(\alpha,\beta) = (\beta-\alpha)\left(\lambda_{\max}^2 - e_{\min}^2\right) + 2\alpha\beta\lambda_{\max} + 2e_{\min}^2\lambda_{\max} > 0.$$

(III) For $(\alpha,\beta) \in D_3$, it follows that

$$\rho(M(\alpha,\beta)) \leqslant \frac{\beta-\lambda_{\min}}{\alpha+\lambda_{\min}}\cdot\sqrt{\frac{\alpha^2+e_{\min}^2}{\beta^2+e_{\min}^2}} < \frac{\beta}{\alpha}\cdot\frac{\alpha}{\beta} = 1.$$

(IV) For $(\alpha,\beta) \in D_4$, we know that the spectral radius $\rho(M(\alpha,\beta))$ satisfies

$$\rho(M(\alpha,\beta)) \leqslant \frac{\beta-\lambda_{\min}}{\alpha+\lambda_{\min}}\cdot\sqrt{\frac{\alpha^2+e_{\max}^2}{\beta^2+e_{\max}^2}} < 1,$$

if and only if

$$\varphi_1(\alpha,\beta) = (\beta-\alpha)\left(\lambda_{\min}^2 - e_{\max}^2\right) + 2\alpha\beta\lambda_{\min} + 2e_{\max}^2\lambda_{\min} > 0.$$

Therefore we obtain $\rho(M(\alpha,\beta)) < 1, \forall(\alpha,\beta) \in \bigcup_{j=1}^4 \Omega_j$. The proof of the theorem is completed. □

Remark 2.3. When $P = I$, the identity matrix, the GPHSS iteration method reduces to the generalized HSS (GHSS) iteration method. Obviously, the convergence property of the GPHSS method also reduces to the GHSS's.

Remark 2.4. When $\beta = \alpha$, the GPHSS method reduces to the PHSS method, which is unconditionally convergent.

Denote simply the right-hand side of (2.3) by

$$\sigma(\alpha,\beta) = \max_{\lambda_k\in\Lambda_1}\left|\frac{\beta-\lambda_k}{\alpha+\lambda_k}\right|\cdot\max_{ie_j\in\Lambda_2}\sqrt{\frac{\alpha^2+e_j^2}{\beta^2+e_j^2}}, \tag{2.6}$$

and

$$(\overline{\alpha},\overline{\beta}) = \arg\min_{\alpha,\beta}\{\sigma(\alpha,\beta)\}.$$

Apparently, $\sigma(\alpha,\beta)$ defines an upper bound of the contraction factor of the GPHSS iteration whose convergent speed depends essentially on the choice of the two-parameters α and β. Consequently, we need the properties of the function $\sigma(\alpha,\beta)$ with respect to the two-parameters.

Theorem 2.5. *The function $\sigma(\alpha,\beta)$ has its minimum on a curve $\beta = \beta^*(\alpha)$ that has been defined in (2.2). And the optimal parameters are given by*

$$(\overline{\alpha},\overline{\beta}) = \begin{cases} (\alpha_1,\beta^*(\alpha_1)), & \lambda_{\max}\lambda_{\min} \leqslant e_{\min}^2, \\ (\alpha_0,\beta^*(\alpha_0)), & e_{\min}^2 < \lambda_{\max}\lambda_{\min} < e_{\max}^2, \\ (\alpha_2,\beta^*(\alpha_2)), & \lambda_{\max}\lambda_{\min} \geqslant e_{\max}^2, \end{cases} \tag{2.7}$$

with

$$\begin{aligned} \alpha_1 &= \frac{-\left(\lambda_{\max}\lambda_{\min} - e_{\min}^2\right) + \sqrt{\left(e_{\min}^2+\lambda_{\max}^2\right)\left(e_{\min}^2+\lambda_{\min}^2\right)}}{\lambda_{\max}+\lambda_{\min}}, \\ \alpha_2 &= \frac{-\left(\lambda_{\max}\lambda_{\min} - e_{\max}^2\right) + \sqrt{\left(e_{\max}^2+\lambda_{\max}^2\right)\left(e_{\max}^2+\lambda_{\min}^2\right)}}{\lambda_{\max}+\lambda_{\min}}. \end{aligned} \tag{2.8}$$

The minimum value at the optimal parameters is

$$\sigma(\overline{\alpha},\overline{\beta}) = \begin{cases} \sigma(\alpha_1), & \lambda_{\max}\lambda_{\min} \leqslant e_{\min}^2, \\ \sigma(\alpha_0), & e_{\min}^2 < \lambda_{\max}\lambda_{\min} < e_{\max}^2, \\ \sigma(\alpha_2), & \lambda_{\max}\lambda_{\min} \geqslant e_{\max}^2, \end{cases} \tag{2.9}$$

where $\sigma(\alpha)$ will be specified in the proof process (see (2.11)).

Proof. From (2.4) and (2.5),

$$\sigma(\alpha,\beta) = \begin{cases} \frac{\lambda_{\max}-\beta}{\lambda_{\max}+\alpha} \cdot \sqrt{\frac{\alpha^2+e_{\max}^2}{\beta^2+e_{\max}^2}}, & (\alpha,\beta) \in D_1, \\ \frac{\lambda_{\max}-\beta}{\lambda_{\max}+\alpha} \cdot \sqrt{\frac{\alpha^2+e_{\min}^2}{\beta^2+e_{\min}^2}}, & (\alpha,\beta) \in D_2, \\ \frac{\beta-\lambda_{\min}}{\alpha+\lambda_{\min}} \cdot \sqrt{\frac{\alpha^2+e_{\min}^2}{\beta^2+e_{\min}^2}}, & (\alpha,\beta) \in D_3, \\ \frac{\beta-\lambda_{\min}}{\alpha+\lambda_{\min}} \cdot \sqrt{\frac{\alpha^2+e_{\max}^2}{\beta^2+e_{\max}^2}}, & (\alpha,\beta) \in D_4. \end{cases} \tag{2.10}$$

It follows that $\sigma'_\beta(\alpha,\beta) < 0$ when $(\alpha,\beta) \in D_1$ or D_2, and $\sigma'_\beta(\alpha,\beta) > 0$ with $(\alpha,\beta) \in D_3$ or D_4. The continuous function $\sigma(\alpha,\beta)$, as the upper bound of $\rho(M(\alpha,\beta))$, thus has its minimum exactly on the curve $\beta = \beta^*(\alpha)$. Now we take $\beta = \beta^*(\alpha)$ in (2.10) and denote by

$$\sigma(\alpha) := \sigma(\alpha,\beta^*(\alpha)) = \begin{cases} \frac{\beta^*(\alpha)-\lambda_{\min}}{\alpha+\lambda_{\min}} \cdot \sqrt{\frac{\alpha^2+e_{\min}^2}{\beta^*(\alpha)^2+e_{\min}^2}}, & \alpha > \alpha_0, \\ \frac{\beta^*(\alpha)-\lambda_{\min}}{\alpha+\lambda_{\min}} \cdot \sqrt{\frac{\alpha^2+e_{\max}^2}{\beta^*(\alpha)^2+e_{\max}^2}}, & \alpha \leqslant \alpha_0. \end{cases} \tag{2.11}$$

Then, the process to look for the minimum point of $\sigma(\alpha,\beta)$ reduces to the process to look for the minimum point of $\sigma(\alpha)$.

Since the calculation leads to

$$\sigma'(\alpha) := \begin{cases} c_1(\alpha)\eta_1(\alpha), & \alpha > \alpha_0, \\ c_2(\alpha)\eta_2(\alpha), & \alpha < \alpha_0, \end{cases} \tag{2.12}$$

where $c_1(\alpha)$ and $c_2(\alpha)$ are two positive functions and

$$\eta_1(\alpha) = (\lambda_{\max}+\lambda_{\min})\alpha^2 + 2(\lambda_{\max}\lambda_{\min} - e_{\min}^2)\alpha - e_{\min}^2(\lambda_{\max}+\lambda_{\min}),$$
$$\eta_2(\alpha) = (\lambda_{\max}+\lambda_{\min})\alpha^2 + 2(\lambda_{\max}\lambda_{\min} - e_{\max}^2)\alpha - e_{\max}^2(\lambda_{\max}+\lambda_{\min}).$$

We find both functions $\eta_1(\alpha)$ and $\eta_2(\alpha)$ enjoy the same property. There is a negative root and a positive root for each function.

Let the two positive roots of functions $\eta_1(\alpha)$ and $\eta_2(\alpha)$ be denoted by α_1 and α_2, respectively. Note that

$$\eta_1(\alpha_0) = \left(\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}\right)^2 (\lambda_{\max}\lambda_{\min} - e_{\min}^2),$$
$$\eta_2(\alpha_0) = \left(\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}\right)^2 (\lambda_{\max}\lambda_{\min} - e_{\max}^2),$$

we observe the following facts:

- If $\lambda_{\max}\lambda_{\min} \leqslant e_{\min}^2$, then the minimum of $\sigma(\alpha,\beta)$ is $\sigma(\alpha_1)$. Which is the value of $\sigma(\alpha,\beta)$ at the point $(\alpha_1,\beta^*(\alpha_1))$ for $\alpha_1 > \alpha_0$.
- If $e_{\min}^2 < \lambda_{\max}\lambda_{\min} < e_{\max}^2$, then the minimum of $\sigma(\alpha,\beta)$ is $\sigma(\alpha_0)$. Which is just the value of $\sigma(\alpha,\beta)$ at the point $(\alpha_0,\beta^*(\alpha_0))$.
- If $\lambda_{\max}\lambda_{\min} \geqslant e_{\max}^2$, then the minimum of $\sigma(\alpha,\beta)$ is $\sigma(\alpha_2)$. Which is the value of $\sigma(\alpha,\beta)$ at the point $(\alpha_2,\beta^*(\alpha_2))$ for $\alpha_2 < \alpha_0$. □

In Theorem 2.5, the optimal parameters $\overline{\alpha}$ and $\overline{\beta}$ minimize only the simplified upper bound $\sigma(\alpha,\beta)$ of the spectral radius. However, one usually cannot expect to minimize the spectral radii of iteration matrices with these optimal parameters.

3. Numerical example

Now we are going to perform numerical examples with a model equation. The efficiency of the GPHSS method is numerically tested. The comparisons with the HSS and PHSS methods are also made. Let us consider the following model,

$$\begin{aligned} &\nabla \cdot (-\nabla u + qu) = f, \quad \text{in } \Omega, \\ &u = 0, \qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where $q > 0$ is a constant and $\Omega = [0, 1]^2$. After using the spatial discretization in [9], we obtain the corresponding linear system denoted by

$$AU = b, \tag{3.2}$$

where A is specified to be a 225×225 matrix, and U is a vector corresponding to the nodal values of the unknown function u. In order to implement the PHSS and GPHSS methods for the linear system (3.2), we now introduce an efficient preconditioner.

3.1. Incremental unknowns preconditioner

It is well known that the incremental unknowns (IUs) method can effectively reduce the condition number of the coefficient matrix A (see [4,11]). Thus we consider it is a good strategy to choose a preconditioner based on IUs. Now, let $\overline{U}$ be the vector corresponding to the IUs and R be the transfer matrix, i.e.,

$$U = R\overline{U}.$$

Then the system (3.2) becomes

$$\overline{AU} = \overline{b},$$

with $\overline{A} = R^T AR$ and $\overline{b} = R^T b$. The preconditioned matrix P is chosen to be $P = (RR^T)^{-1}$. Due to IUs methodology, we do not need to compute P explicitly. We compute some vector Pu instead. Nevertheless, one can refer to [6] for the explicit expression of matrix P.

3.2. Spectral radius and convergence region

The spectral radius $\rho(M(\alpha, \beta^*(\alpha)))$ of the GPHSS method and the upper bound $\sigma(\alpha, \beta^*(\alpha))$ for different α are plotted in Fig. 2. From these figures, one can see that the upper bound $\sigma(\alpha, \beta^*(\alpha))$ is close to $\rho(M(\alpha, \beta^*(\alpha)))$, i.e., $\sigma(\alpha, \beta)$ is a good approximation to $\rho(M(\alpha, \beta))$ when $\beta = \beta^*(\alpha)$, especially for small q.

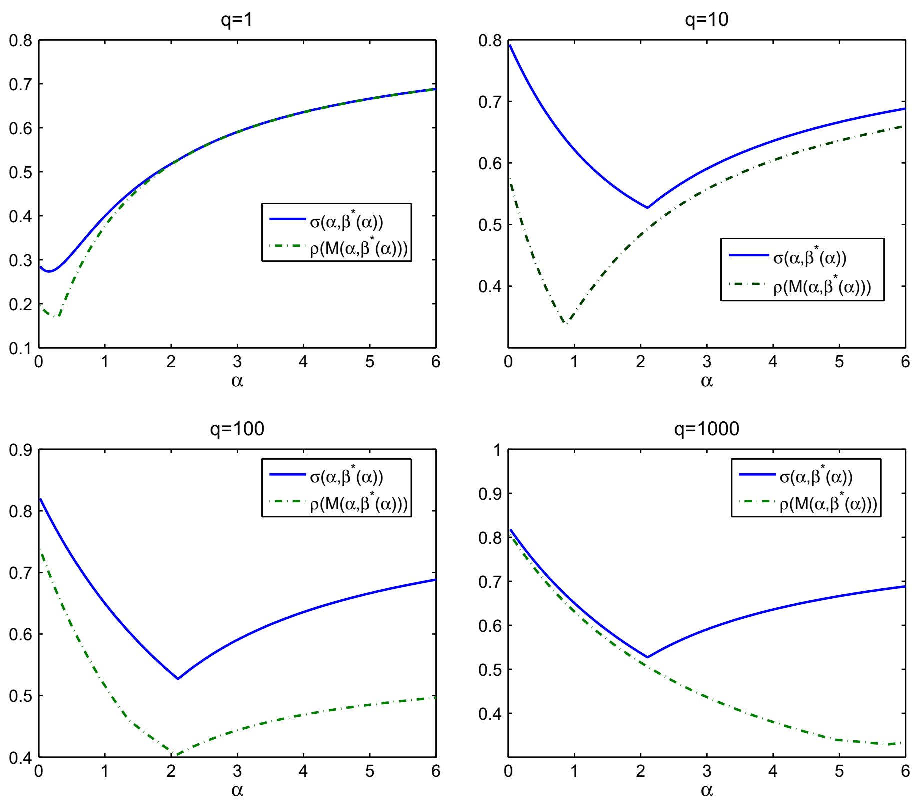

Fig. 2. The spectral radius $\rho(M(\alpha, \beta^*(\alpha)))$ and the upper bound $\sigma(\alpha, \beta^*(\alpha))$ vs. α with $q = 1, 10, 100$ and 1000.

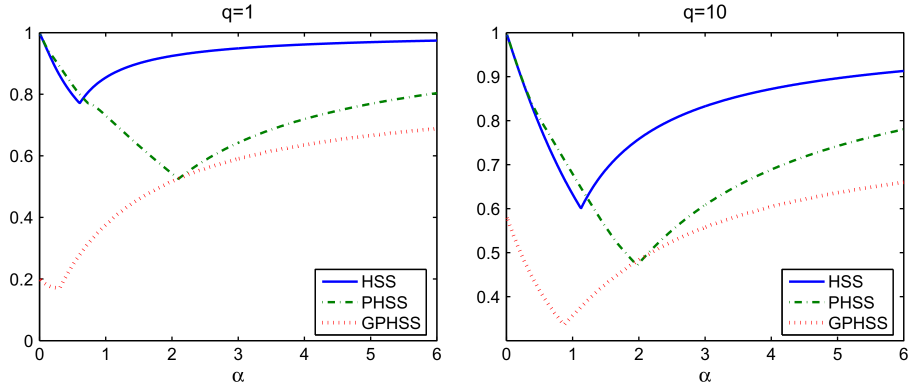

Fig. 3. The spectral radii of the iteration matrices vs. α with $q = 1$ (left) and $q = 10$ (right).

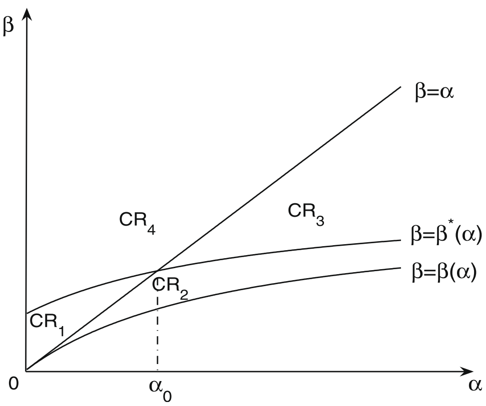

Fig. 4. The convergent region of GPHSS method.

The convergence speed of the iteration method depends largely on the spectral radius of the iteration matrix. The comparations of spectral radii of the three different iteration matrices, derived by the HSS, PHSS and GPHSS iterative methods, are performed in Fig. 3. Here, the parameters α and β of GPHSS method are only chosen on the curve line $\beta = \beta^*(\alpha)$, i.e., the figure of spectral radii of GPHSS method is the figure of $\rho(M(\alpha, \beta^*(\alpha)))$ with different α. From Fig. 3, it is easy to find that the spectral radii of the GPHSS method are much smaller than those of the HSS and PHSS methods.

By Theorem 2.2, the convergence region (CR) of GPHSS method for solving linear system (3.2) can be denoted by

$$CR = \{(\alpha, \beta) | \alpha \geqslant 0, \beta > \beta(\alpha)\} = \bigcup_{k=1}^{4} \mathrm{CR}_k,$$

with $\beta(\alpha) \approx 46.16\alpha/(46.16 + 13.59\alpha)$. See the region in Fig. 4.

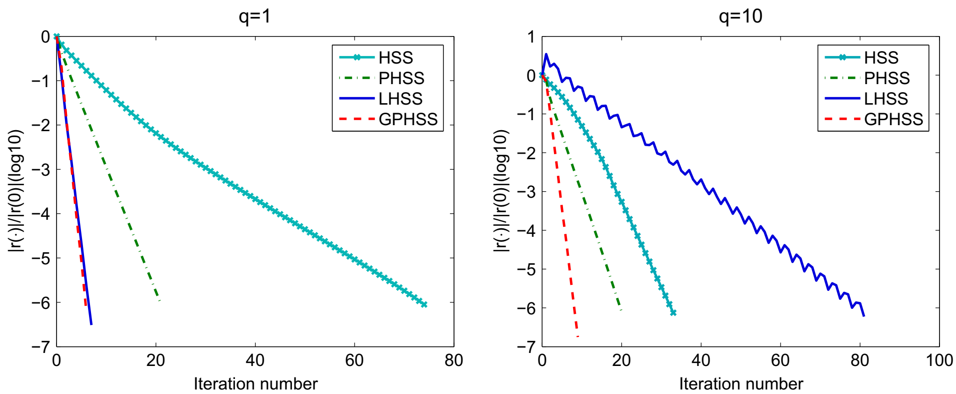

Fig. 5. The relative L^2 norm(log10) of the residual error against the iteration number with $q = 1$ (left) and $q = 10$ (right).

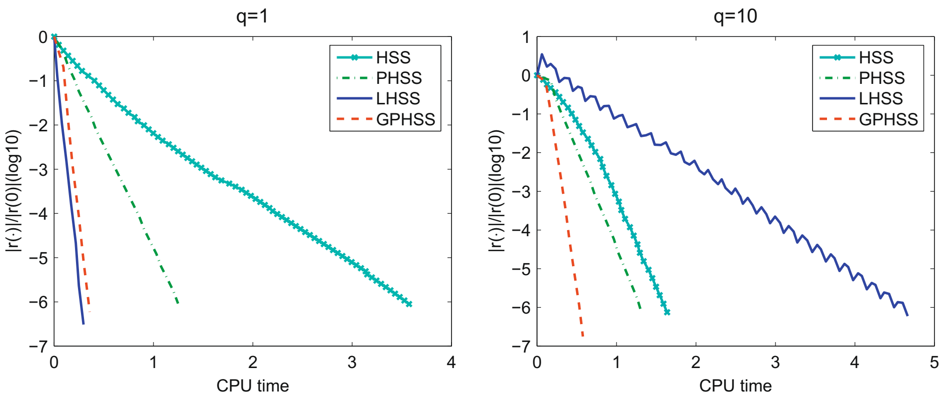

Fig. 6. The relative L^2 norm(log10) of the residual error against the CPU time with $q = 1$ (left) and $q = 10$ (right).

3.3. Comparison of the convergent speed

We performed, in this subsection, the numerical tests for HSS, PHSS, lopsided HSS (LHSS) [8] and GPHSS methods with their own numerical optimal parameters. The relative L^2 norm (log10) of the residual error is plotted against the iteration number and the CPU time in Figs. 5 and 6, respectively. From Fig. 5, one can see that the error of GPHSS method decreases faster than those of the HSS, PHSS and LHSS methods especially for $q = 10$. In Fig. 6, it follows that both of the LHSS and GPHSS methods are more efficient than HSS and PHSS when $q = 1$. For the case $q = 10$, GPHSS uses the least CPU time with a given accuracy requirement.

4. Concluding remark

As a strategy for accelerating convergence of iteration for the large sparse non-Hermitian positive definite system of linear equations, we present a two-parameter generalized preconditioned HSS method or GPHSS method. This is obviously a type of generalization of the classical HSS method because when we take $\beta = \alpha$ and do not use a preconditioner P, we shall return to the HSS method. In our work we demonstrate that the iterative series produced by GPHSS method converge to the unique solution of the system of linear equations when the parameters α and β satisfy some moderate conditions which take PHSS method as a special case. We give also initially a possible optimal upper bound on a convergent curve for the iterative spectral radius. Numerical examples with incremental unknowns preconditioner show the efficiency and effectiveness of the new GPHSS method which is typically more flexible than PHSS or HSS method.

Acknowledgements

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References

[1] Z.-Z. Bai, G.H. Golub, C.-K. Li, Convergence properties of pre-conditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite matrices, Math. Comput. 76 (2007) 287–298.
[2] Z.-Z. Bai, G.H. Golub, M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, SIAM J. Matrix Anal. Appl. 24 (2003) 603–626.
[3] Z.-Z. Bai, G.H. Golub, J.-Y. Pan, Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems, Numer. Math. 98 (2004) 1–32.
[4] M. Chen, R. Temam, Incremental Unknowns in finite differences: condition number of the matrix, SIAM J. Matrix Anal. Appl. 14 (1993) 432–455.
[5] M. Eiermann, W. Niethammer, R.S. Varga, Acceleration of relaxation methods for non-Hermitian linear systems, SIAM J. Matrix Anal. Appl. 13 (1992) 979–991.
[6] S. Garcia, Algebraic conditioning analysis of the incremental unknowns preconditioner, Appl. Math. Model. 22 (1998) 351–366.
[7] G.H. Golub, C. Van Loan, Matrix Computations, third ed., The Johns Hopkins University Press, Baltimore, 1996.
[8] L. Li, T.-Z. Huang, X.-P. Liu, Modified Hermitian and skew-Hermitian splitting methods for non-Hermitian positive-definite linear systems, Numer. Linear Algebra Appl. 14 (2007) 217–235.
[9] J.W. Thomas, Numerical Partial Differential Equations: Finite Difference Methods, Springer-Verlag, New York, 1995.
[10] C.-L. Wang, Z.-Z. Bai, Sufficient conditions for the convergent splittings of non-Hermitian positive definite matrices, Linear Algebra Appl. 330 (2001) 215–218.
[11] Y.-J. Wu, A.-L. Yang, Incremental unknowns for the heat equation with time-dependent coefficients: semi-implicit θ-schemes and their stability, J. Comput. Math. 25 (2007) 573–582.