

An Additive Variant of the Schwarz Alternating Method for the Case of Many Subregions

by

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Abstract

In recent years, there has been a revival of interest in the Schwarz alternating method. Other domain decomposition algorithms, in particular the so called iterative substructuring methods, have also been developed to solve elliptic finite element problems. In this paper, we present an additive variant of the Schwarz method for elliptic problems, which shows great promise for parallel computers. By using techniques previously developed for iterative substructuring methods, we are able to show that this method converges quite rapidly even when the region is divided into many subregions. We note that, as is the case with other fast iterative methods for many subregions, a mechanism for the global transportation of information is necessary in order to obtain fast convergence.

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1 Introduction

In this paper, we will consider linear, self adjoint, elliptic problems discretized by finite element methods. Our purpose is to develop and study an additive variant of the classical Schwarz procedure, see [16], which converges quite rapidly even for a large number of subregions and which permits the use of conjugate gradient acceleration. Adopting a term of structural engineering, we will refer to certain non-overlapping subregions as substructures.

For parallel computing systems with many processors and in the study of speed-up of algorithms on such systems, the case of many substructures is clearly of particular interest. The subproblems can then be handled conveniently by separate processors. If the work of the linear equation solver in use grows faster than linearly with the number of degrees of freedom, then fast domain decomposition methods can often also offer real benefits, even if only one processor is used. Sometimes we can also use subregions with simple geometries to reduce the overall effort of solving the problem.

In our previous work on iterative substructuring methods, we have established that a mechanism for the global transportation of information is required in order to obtain a rate of convergence that varies only very slowly with the number of degrees of freedom and subregions; see Widlund [20], [19]. In those papers, we have shown that if we only have next neighbors communication, the minimum number of iterations required grows at least as fast as $N^{1/2}$, where N is the number of substructures. The same argument applies almost verbatim here and it will therefore not be repeated. In our new algorithm for problems in the plane, we use one of the devices which has been successful in the development of iterative substructuring methods. Our main result is somewhat stronger than those discussed in [19]. The number of iterations required, to decrease the energy norm of the error by a fixed factor, is proportional to $\sqrt{(1 + \log(H/h))}$. Here H and h are the diameters of a typical substructure and an element, respectively. The hierarchical bases method due to Yserentant [21] and an iterative substructuring method due to Bramble, Pasciak and Schatz [4] are among the methods considered in Widlund [19]. The bounds for these are proportional to $(1 + \log(H/h))$. We note that, to our knowledge, a generalization of Yserentant's algorithm to three dimensional problems with fast convergence has not been found, while efficient iterative substructuring algorithms have been developed for that case; cf. Bramble, Pasciak and Schatz [5] and Dryja [7].

For self-adjoint elliptic problems, the finite element solution is the pro-

jection of the exact solution onto a finite-dimensional space. Different variants of Schwarz' algorithm, in particular the additive form considered in this paper, can also conveniently be described in terms of projections onto subspaces related to overlapping subregions which cover the region. These projections are orthogonal with respect to the symmetric bilinear form associated with the elliptic operator under consideration. In addition, we are using a special subspace which gives rise to a coarse finite element model and which provides the global transfer of information that is required.

Our point of departure has been a recent paper by P.-L. Lions [10] in which a variational frame work for the classical, multiplicative Schwarz' method is developed for continuous elliptic problems. Since it is not necessary to rely on a maximum principle, the Schwarz method can be used for a number of important elliptic problems, such as the equations of linear elasticity, Stokes' equations and the biharmonic equation. The analysis of finite element models is similarly made much simpler. In his paper, Lions gives a number of references to older results. Thus, more than 50 years ago, Sobolev [17] introduced a variational frame work in a study of Schwarz' method and the equations of linear elasticity. Early work was also done by Babuska [1], [2] and by Morgenstern [13]; see Lions [10].

The additive variant of the algorithm has been discovered independently by us, but its origin is not clear. We are quite interested in finding out about any previous work of this nature. We have found an algorithm quite similar to ours, in a paper by Nepomnyashchikh [14] , on iterative substructuring methods on regions subdivided into a fixed number of subregions. We also note that we have discovered that the so called FAC and AFAC algorithms, which are iterative refinement algorithms developed by McCormick and his coworkers, see [11] , [12], [9], can be described in terms of projections and multiplicative and additive algorithms, respectively. See Widlund [18] for a further discussion.

In Section 2, we introduce our algorithm and prove that it converges rapidly for Poisson's equation discretized by piecewise linear, continuous finite elements.

2 The Additive Algorithm and an Analysis of a Model Problem

In this paper, we consider linear, self adjoint, elliptic problems discretized by finite element methods on Lipschitz regions. In this section, we assume that Ω is a plane region, that the differential operator is the Laplacian and that we use continuous, piecewise linear finite elements. The continuous and discrete problems are of the form

$$a(u, v) = f(v), \forall v \in V,$$

and

$$a(u_h, v_h) = f(v_h), \forall v_h \in V^h, \quad (1)$$

respectively. The bilinear form is defined by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx.$$

This bilinear form defines a semi-norm $|u|_{H^1} = (a(u, u))^{1/2}$ in $H^1(\Omega)$. We assume that $\partial\Omega$, the boundary of Ω , is the union of two non-overlapping sets, Γ_D and Γ_N , on which zero Dirichlet and arbitrary Neumann boundary conditions are given. We assume that Γ_D is non empty, to avoid an unnecessary complication with a singular problem. All elements of V and its subspace V^h vanish on Γ_D . We note that we can always reduce an inhomogenous Dirichlet problem to the case considered here at the expense of solving one problem on each of the subregions which are next to Γ_D . The subregions are introduced below.

We introduce the triangulation of Ω in the following way. We first divide the region into non-overlapping substructures Ω_i , $i = 1, \dots, N$. To simplify the description and analysis of our algorithm, we confine our study to triangular substructures. In such a case, the original region must of course be a polygon. We also adopt the common assumption in finite element theory that all substructures are shape regular in the sense that the diameter H_i , divided by the radius of the largest inscribed circle in Ω_i , are uniformly bounded. All the substructures Ω_i are subdivided into elements. The elements are shape regular in the same sense as above. Since the Schwarz-type domain decomposition algorithms use overlapping subregions, we extend each substructure to a larger region Ω'_i . We assume that the distance between the boundaries $\partial\Omega_i$ and $\partial\Omega'_i$ is bounded from below by a fixed fraction of H_i , and that $\partial\Omega'_i$ does not cut through any element. We make

the same construction for the substructures that meet the boundary except that we cut off the part of Ω'_i that is outside of Ω .

We remark that some of the more intricate issues in the analysis of the Schwarz methods arise when the boundaries of the different subdomains Ω'_i intersect at one or several points; cf. the discussion in Lions [10]. This happens for example if the region is L-shaped and is partitioned into two overlapping rectangles. Tools powerful enough to treat problems of this kind are known to us. Sometimes the resulting bounds involve an extra factor $(1 + \log(H/h))$; see further Widlund [18]. In this paper, we will confine our study to the simpler case.

Our finite element space is represented as the sum of $N+1$ subspaces

$$V^h = V_0^h + V_1^h + \dots + V_N^h.$$

The first subspace V_0^h , which we also call V^H , is special. It is the space of continuous, piecewise linear functions on the coarse mesh defined by the substructures Ω_i . When convenient, we can view any element of V^H as a coarse mesh interpolant $I_H v_h$ of some element $v_h \in V^h$. The computation of the projection of an arbitrary function onto this subspace involves the solution of a standard finite element linear system of algebraic equations which is on the order of N . This coarse, global approximation of the elliptic equation can often practically be handled by a direct method such as Gaussian elimination.

The subspaces, V_i^h , which are related to interior substructures, are defined as $V^h \cap H_0^1(\Omega'_i)$. The space H_0^1 is, as usual, the subspace of H^1 functions with zero trace. It is well known that this space can be extended continuously by zero. After extending $H_0^1(\Omega'_i)$ by zero, we can regard V_i^h as a subspace of V^h . We construct subspaces related to the boundary substructures similarly. The boundary conditions of the original problem are inherited at any node of Ω'_i which falls on $\partial\Omega$. A nodal value is thus constrained to be zero if the node belongs to Γ_D , while it is free at any node on Γ_N . At all nodes of $\partial\Omega'_i$ which are in the interior of Ω we impose zero Dirichlet conditions.

The projection $P_{V_i^h} = P_i$ is defined, for all of V^h , by the unique element of V_i^h , which satisfies

$$a(P_i v_h, \phi_h) = a(v_h, \phi_h), \quad \forall \phi_h \in V_i^h. \quad (2)$$

Lions [10] has shown that the error propagation operator of the standard multiplicative variant of the Schwarz method can be written as

$$(I - P_1)(I - P_2),$$

for the case of two subregions. We can view that algorithm as an iterative method for solving

$$(P_1 + P_2 - P_1 P_2)u_h = g_h,$$

with an appropriate right hand side g_h . We note that the operator is a polynomial of degree two, and thus is not ideal for parallel computing, since two sequential steps are involved. The additive form of the Schwarz algorithm can similarly be regarded as an iterative method for solving the equation

$$Pu_h = (P_0 + P_1 + \dots + P_N)u_h = g'_h, \quad (3)$$

We must assure that this equation has the same solution as equation 1, i.e. we must use the correct right hand side. Since by equation 1, we have

$$a(u_h, \phi_h) = f(\phi_h),$$

we can construct the right-hand side g'_h by solving equation 2 for all values of i and adding the resulting vectors. We can also apply the operator P of equation 3 to any given element of V^h at the same expense by applying each projection P_i once and adding the vectors. We solve equation 3 by the standard conjugate gradient method. Much of the work, in particular that involving the individual projections, can be carried out in parallel.

It is well known that the number of steps required to decrease an appropriate norm of the error of a conjugate gradient iteration by a fixed factor is proportional to $\sqrt{\kappa}$, where κ is the condition number of P ; see e.g. Golub and Van Loan [8]. We therefore need to establish that the operator P of equation 3 is not only invertible but that satisfactory upper and lower bounds on its eigenvalues can be obtained. We will prove the following theorem.

Theorem 1 $\kappa(P)$ is bounded by $\text{const} (1 + \log(H/h))$.

Here, and elsewhere in this paper, the constants are independent of the diameters of the substructures and the individual elements. They vary with the shape of the elements and substructures. Thus in two dimensions, they depend on the minimum angle. As will become apparent in the proof, the constant in the theorem also grows if the overlap of the subregions shrinks.

The upper bound on the spectrum is obtained by bounding

$$a(Pv_h, v_h) = a(P_0v_h, v_h) + a(P_1v_h, v_h) + \dots + a(P_Nv_h, v_h),$$

from above in terms of $a(v_h, v_h)$. We first use Schwarz' inequality and the fact that P_0 is a projection to prove that the first term is bounded by $a(v_h, v_h)$. Similarly, it is easy to show that the spectrum of P is bounded from above by $(N + 1)$. Our goal, however, is to establish a bound on the condition number that grows quite slowly, and we therefore need a better upper bound.

The sum of the other terms, which are of the form

$$a(P_i v_h, v_h) = a(P_i v_h, P_i v_h),$$

can also be estimated by $\text{const } a(v_h, v_h)$. To see this, we will use the fact that $P_i v_h$ vanishes outside Ω'_i and that there is a uniform bound on the number of terms that are different from zero at any given point. It is also easy to see, from the definition of the V_i^h , that $P_i v_h$ depends only on the values of v_h in Ω'_i . Therefore

$$P_i v_h = P_i(\psi_i v_h),$$

where ψ_i is a smooth cut-off function. It has values between zero and one, is equal to one on Ω'_i and vanishes outside a neighborhood Ω''_i of Ω'_i . We can choose this new set so that the distance between its boundary and that of Ω'_i is on the order of H_i , the diameter of Ω_i . We modify this construction for boundary substructures by cutting off the part that is outside Ω . It is then easy to construct ψ_i so that its gradient is bounded by const/H_i . By using the formula for the derivative of a product, we then find that the solution of equation 2 satisfies

$$(P_i v_h, P_i v_h)_{H^1(\Omega)} = \text{const} (\|v_h\|_{H^1(\Omega''_i)}^2 + H^{-2} \|v_h\|_{L_2(\Omega''_i)}^2). \quad (4)$$

We wish to remove the L_2 - norm term from the estimate 4 and we have to consider two cases. If $\partial\Omega''_i$ does not intersect Γ_D , we first modify v_h locally by adding an arbitrary constant. This will leave the right hand side unchanged, since as is easily shown by using the definition of the projection that $P_i \text{const} = 0$. The semi-norm in the right hand side of 4 also remains unchanged. Poincaré's inequality, see e.g. Nečas [15], can now be used to estimate the second term of the right hand side by the first. If, on the other hand, $\partial\Omega''_i$ intersects Γ_D , we cannot shift by a constant and remain in the space V^h . Instead, we use an inequality due to Friedrichs, given as Theorem 1.9 in Nečas [15], to estimate the $L_2(\Omega''_i)$ - term. This inequality essentially states that, for any region $\tilde{\Omega}$ of diameter one,

$$\|u\|_{L_2(\tilde{\Omega})}^2 \leq \text{const} (\|u\|_{H^1(\tilde{\Omega})}^2 + \|u\|_{L_2(\gamma)}^2).$$

Here γ is a subset of positive measure of $\partial\tilde{\Omega}$. If we scale the variables, we find that the constant in the estimate decreases in proportion to H_i^2 , if the measure of $\partial\Omega'_i \cap \Gamma_D$ remains bounded from below by that of $\partial\Omega'_i$. This latter condition causes no difficulty, since, if necessary, we can always make Ω''_i somewhat larger. The factor H_i^2 is exactly what is required to control the $L_2(\Omega''_i)$ – term. Our proof of the upper bound for P is completed by noting that

$$\sum_{i=1}^N u_h^2_{H^1(\Omega''_i)} \leq \text{const } u_h^2_{H^1(\Omega)},$$

since the Ω''_i provides a finite cover of Ω .

We note that we have essentially used a quotient space argument which also plays an important role in the error analysis for finite element methods; cf. Ciarlet [6]. Similar arguments have been used extensively in the theory of iterative substructuring methods; see e.g. Widlund [20], [19].

We use a lemma in Lions [10] to obtain a lower bound. Since the proof of his result is quite short, we include it in this paper.

Lemma 1 (Lions) *Let $u_h = \sum_{i=0}^N u_{h,i}$, where $u_{h,i} \in V_i^h$, be a partition of an element of V^h and assume further that $\sum_{i=0}^N |u_{h,i}|_{H^1}^2 \leq C_0^2 |u_h|_{H^1}^2$, $\forall u_h \in V^h$. Then $\lambda_{\min}(P) \geq C_0^{-2}$.*

Proof: By elementary properties of symmetric projections and the representation of u_h as a sum, we find that

$$u_h^2_{H^1} = \sum_{i=0}^N (u_h, u_{h,i})_{H^1} = \sum_{i=0}^N (u_h, P_i u_{h,i})_{H^1} = \sum_{i=0}^N (P_i u_h, u_{h,i})_{H^1}$$

Therefore,

$$u_h^2_{H^1} \leq \left(\sum_{i=0}^N |P_i u_h|_{H^1}^2 \right)^{1/2} \left(\sum_{i=0}^N u_{h,i}^2_{H^1} \right)^{1/2}$$

By the assumption of the lemma

$$|u_h|_{H^1}^2 \leq C_0^2 \sum_{i=0}^N |P_i u_h|_{H^1}^2 = C_0^2 \sum_{i=0}^N (P_i u_h, u_h)_{H^1} = C_0^2 (P u_h, u_h)_{H^1}$$

and the lemma is established.

What remains is to find a suitable decomposition and to establish the bound required by Lions' lemma. We first write

$$u_h = I_H u_h + (u_h - I_H u_h) = I_H u_h + w_h,$$

where I_H is the interpolation operator related to the space $V^H = V_0^h$, which was introduced above. We use the following lemma, which plays a similar role in the theory for iterative substructuring algorithms. This result, which dates back to at least 1966, is proven in a number of papers; see e.g. Bramble [3], Bramble, Pasciak and Schatz [4] or Yserentant [21].

Lemma 2

$$\|u_h\|_{L^\infty(\Omega'_i)}^2 \leq \text{const} (1 + \log(H/h)) (|u_h|_{H^1(\Omega'_i)}^2 + H_i^{-2} \|u_h\|_{L_2(\Omega'_i)}^2).$$

It is easy to see that $|I_H u_h|_{H^1(\Omega'_i)}^2$ can be estimated by a sum of differences of functions values of u_h at the vertices of Ω'_i . As shown in Widlund [20], [19], we can use the lemma together with the same quotient space argument, as used to derive our upper bound, to establish that

$$|I_H u_h|_{H^1(\Omega'_i)}^2 \leq \text{const} (1 + \log(H/h)) |u_h|_{H^1(\Omega'_i)}^2. \quad (5)$$

The other terms in the decomposition are defined by $u_{h,i} = I_h(\theta_i w_h)$, $i = 0, \dots, N$. Here the θ_i define a partition of unity and belong to $C_0^\infty(\Omega'_i)$. We can arrange so that $\nabla \theta_i$ is bounded by const/H_i . By using the linearity of I_h , we can easily show that we obtain a correct partitioning of u_h . In order to estimate the semi-norm of $u_{h,i}$, we partition Ω'_i into elements. For each element K , we obtain

$$|u_{h,i}|_{H^1(K)}^2 \leq 2 |\bar{\theta}_i w_h|_{H^1(K)}^2 + 2 |I_h((\theta_i - \bar{\theta}_i)w_h)|_{H^1(K)}^2.$$

Here $\bar{\theta}_i$ is the average value of θ_i over K . It is easy to see, by using an inverse inequality, that

$$|I_h((\theta_i - \bar{\theta}_i)w_h)|_{H^1(K)} \leq \text{const} h^{-1} \|I_h((\theta_i - \bar{\theta}_i)w_h)\|_{L_2(K)}.$$

We can now use the fact that on K , θ_i differs from its average by at most $\text{const} h/H_i$, and, after summing over all elements of Ω'_i , we arrive at an inequality similar to 4:

$$|u_{h,i}|_{H^1(\Omega'_i)}^2 \leq \text{const} (|w_h|_{H^1(\Omega'_i)}^2 + H_i^{-2} \|w_h\|_{L_2(\Omega'_i)}^2). \quad (6)$$

At the expense of introducing a factor $(1 + \log(H/h))$, we can estimate the right hand side of 6 by the same expression with $u_h + \text{const}$ instead of w_h .

Here the constant term is arbitrary. We first note that $w_h = (I - I_H)u_h$ does not change if we add a constant to u_h . By the inequality 5, we can estimate the first term of the right hand side of 6 as required. The second term can be estimated directly by $\|w_h\|_{L^\infty(\Omega'_i)}^2$, since the area of Ω'_i is on the order of H_i^2 . This quantity can easily be estimated by $\|u_h + \text{const}\|_{L^\infty(\Omega'_i)}^2$. We now use Lemma 2, Poincaré's inequality and a finite covering argument to complete the proof of our Theorem.

References

- [1] Ivo Babuška. Über Schwarzsche Algorithmen in partielle Differentialgleichungen der mathematischen Physik. *ZAMM*, 37(7/8):243–245, 1957.
- [2] Ivo Babuška. The Schwarz algorithm in partial differential equations of mathematical physics. *Czech. Math. J.*, 83(8):328–343, 1958. In Russian.
- [3] James H. Bramble. A second order finite difference analogue of the first biharmonic boundary value problem. *Numer. Math.*, 9:236–249, 1966.
- [4] James H. Bramble, Joseph E. Pasciak, and Alfred H. Schatz. The construction of preconditioners for elliptic problems by substructuring, I. *Math. Comp.*, 47(175):103–134, 1986.
- [5] James H. Bramble, Joseph E. Pasciak, and Alfred H. Schatz. *The construction of preconditioners for elliptic problems by substructuring, IV*. Technical Report, Cornell University, 1987.
- [6] Philippe G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, 1978.
- [7] Max Dryja. A method of domain decomposition for 3-D finite element problems. In Roland Glowinski, Gene H. Golub, Gerard A. Meurant, and Jacques Périaux, editors, *Domain Decomposition Methods for Partial Differential Equations*, SIAM, Philadelphia,, 1988.
- [8] Gene H. Golub and Charles F. Van Loan. *Matrix Computations*. Johns Hopkins Univ. Press, 1983.
- [9] L. Hart and Stephen McCormick. *Asynchronous Multilevel Adaptive Methods for Solving Partial Differential Equations on Multiprocessors: Computational Analysis*. Technical Report, Comp. Math. Group, Univ. of Colorado at Denver, 1987. To appear in Multigrid Methods (the proceedings of the 1987 Cooper Mountain Multigrid Conference). Marcel Dekker, New York, 1988.
- [10] Pierre-Louis Lions. On the Schwarz alternating method. I. In Roland Glowinski, Gene H. Golub, Gerard A. Meurant, and Jacques Périaux, editors, *Domain Decomposition Methods for Partial Differential Equations*, SIAM, Philadelphia,, 1988.

- [11] Stephen McCormick. Fast adaptive composite grid (FAC) methods. In K. Böhmer and H. J. Stetter, editors, *Defect Correction Methods: Theory and Applications*, pages 115–121, Computing Supplementum 5, Springer-Verlag, Wien, 1984.
- [12] Stephen McCormick and John Thomas. The fast adaptive composite grid (FAC) method for elliptic equations. *Math. Comp.*, 46(174):439–456, 1986.
- [13] D. Morgenstern. Begründung des alternierenden Verfahrens durch Orthogonalprojektion. *ZAMM*, 36:7–8, 1956.
- [14] S. V. Nepomnyashchikh. *On the application of the method of bordering for elliptic mixed boundary value problems and on the difference norms of $W_2^{1/2}(S)$* . Technical Report 106, Computing Center of the Siberian Branch of the USSR Academy of Sciences, Novosibirsk, 1984. In Russian.
- [15] Jindřich Nečas. *Les méthodes directes en théorie des équations elliptiques*. Academia, Prague, 1967.
- [16] H.A. Schwarz. *Gesammelte Mathematische Abhandlungen*. Volume 2, Springer, Berlin, 1890. First published in *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, volume 15, 1870, pp.272–286.
- [17] S.L. Sobolev. The Schwarz algorithm in the theory of elasticity. *Dokl. Acad. N. USSR*, IV(XIII):236–238, 1936. In Russian.
- [18] Olof B. Widlund. *A Comparison of Some Domain Decomposition and Iterative Refinement Algorithms for Elliptic Finite Element Problems*. Technical Report BSC 87/4, IBM Bergen Scientific Centre, Allégaten 36, N-5007 Bergen, Norway, 1987. To appear.
- [19] Olof B. Widlund. Iterative substructuring methods: algorithms and theory for problems in the plane. In Roland Glowinski, Gene H. Golub Gerard A. Meurant, and Jacques Périaux, editors, *Domain Decomposition Methods for Partial Differential Equations*, SIAM, Philadelphia, 1988.
- [20] Olof B. Widlund. Iterative substructuring methods: the general elliptic case. In *Computational Processes and Systems*, Nauka, Moscow, 1988. Proceedings of Modern Problems in Numerical Analysis, a conference

[21] Harry Yserentant. On the multi-level splitting of finite element spaces.
Numer. Math., 49:379– 412, 1986.

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