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Efficient parameterized HSS iteration methods for complex symmetric linear systems

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ABSTRACT

Based on the new HSS (NHSS) iteration method proposed by Pour and Goughery (2015) and the efficient PSHSS iteration method by Zeng and Ma (2016), we introduce an efficient parameterized HSS (PNHSS) and a parameterized single-step HSS (PS*HSS) iteration methods for solving a class of complex symmetric linear systems. Convergence properties of the PNHSS and the PS*HSS iteration methods are studied, which show that the iterative sequences are convergent to the unique solution of the linear system for any initial guess under a loose restriction on the parameter ω. Furthermore, we derive an upper bound for the spectral radius of the PNHSS iteration matrix, and the quasi-optimal parameters α^* and ω^* which minimize the above upper bound are also considered. Both theoretical and numerical results show that the PNHSS and the PS*HSS iteration methods outperform the NHSS and the SHSS iteration methods. Little difference about the computational efficiency from the point of view of the CPU times between the PS*HSS, the PNHSS and the PSHSS iteration methods is justified by using the experimental optimal parameters. However, sometimes the PS*HSS and the PNHSS iteration methods are more efficient than the PSHSS iteration method when the experimental optimal parameters are not used.

1. Introduction and motivations

Many problems in scientific computing and engineering applications lead, to a complex symmetric system of linear equations [1–3]

$$Ax \equiv (W + iT)x = b, \tag{1.1}$$

where $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them, e.g., W being positive definite. $b \in \mathbb{C}^n$ is a given vector and $x \in \mathbb{C}^n$ is unknown. We use $i = \sqrt{-1}$ to denote the imaginary unit.

Since the matrix $A \in \mathbb{C}^{n\times n}$ possesses the Hermitian and skew-Hermitian splitting (HSS)

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^*) = W \quad \text{and} \quad S = \frac{1}{2}(A - A^*) = iT,$$

A is a non-Hermitian, but positive definite matrix. Here A^* denotes the conjugate transpose of the matrix A. To compute an approximate solution for the complex symmetric linear system (1.1), Bai et al. [4] proposed the following HSS iteration method.

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Algorithm 1. The HSS Iteration Method.

Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}$ converges, compute $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I + W)x^{\left(k+\frac{1}{2}\right)} = (\alpha I - iT)x^{(k)} + b, \\ (\alpha I + iT)x^{(k+1)} = (\alpha I - W)x^{\left(k+\frac{1}{2}\right)} + b, \end{cases}$$

where α is a given positive constant and I is the identity matrix.

A potential difficulty with the HSS iteration approach is the need to solve the shifted skew-Hermitian subsystem of linear equations with the coefficient matrix $\alpha I + iT$ in the second-half step. In some cases its solution is as difficult as that of the original problem; see [5–11] for more detailed descriptions about the HSS iteration method and its variants.

To avoid the complex arithmetic, a modified HSS (MHSS) iteration scheme is established by Bai et al. [12]. A considerable advantage of MHSS iteration method is that solving the linear subsystem with coefficient matrix $\alpha I + iT$ is avoided to be replaced by that with coefficient matrix $\alpha I + T$, which is also real and symmetric positive definite. The MHSS iteration method is given as follows:

$$\begin{cases} (\alpha I + W)x^{\left(k+\frac{1}{2}\right)} = (\alpha I - iT)x^{(k)} + b, \\ (\alpha I + T)x^{(k+1)} = (\alpha I + iW)x^{\left(k+\frac{1}{2}\right)} - ib. \end{cases}$$

To accelerate the convergence of MHSS iteration method, several variants are developed in the literature. Bai et al. [13,14] established the preconditioned MHSS (PMHSS) iteration method for complex symmetric linear systems with applications to distributed control problems. The PMHSS iteration method is given by

$$\begin{cases} (\alpha V + W)x^{\left(k+\frac{1}{2}\right)} = (\alpha V - iT)x^{(k)} + b, \\ (\alpha V + T)x^{(k+1)} = (\alpha V + iW)x^{\left(k+\frac{1}{2}\right)} - ib, \end{cases}$$

where $V \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix. Li et al. [15] developed a lopsided preconditioned MHSS iteration method for a class of complex symmetric linear systems.

On the other hand, Li et al. [16] considered asymmetric Hermitian/skew-Hermitian (AHSS) iteration. They studied the convergence properties of the AHSS method, and derived the contraction factor of the AHSS iteration method. Furthermore, they also developed the lopsided Hermitian/skew Hermitian splitting (LHSS) iteration as follows:

$$\begin{cases} Wx^{\left(k+\frac{1}{2}\right)} = -iTx^{(k)} + b, \\ (\alpha I + iT)x^{(k+1)} = (\alpha I - W)x^{\left(k+\frac{1}{2}\right)} + b. \end{cases}$$

In order to avoid the solution of the linear subsystem with coefficient matrix $\alpha I + iT$, Pour and Goughery [17] proposed the following new Hermitian and skew-Hermitian splitting (NHSS) iteration for non-Hermitian positive-definite linear systems:

$$\begin{cases} Wx^{\left(k+\frac{1}{2}\right)} = -iTx^{(k)} + b, \\ (\alpha I + W)x^{(k+1)} = (\alpha I - iT)x^{\left(k+\frac{1}{2}\right)} + b. \end{cases}$$

Recently, Li and Wu [18] introduced the following single-step HSS (SHSS) iteration method to solve the non-Hermitian positive definite linear systems:

$$(\alpha I + W)x^{(k+1)} = (\alpha I - iT)x^{(k)} + b.$$

Zeng and Ma [19] established a parameterized variant of the SHSS (PSHSS) iteration method, which is more efficient than the SHSS iteration from numerical experiments. The PSHSS iteration method is given by

$$(\alpha I + \omega W + T)x^{(k+1)} = [\alpha I - i(\omega T - W)]x^{(k)} + (\omega - i)b,$$

where α and ω are given positive constants.

Inspired by Zeng and Ma [19], we want to make improvement on the NHSS iteration method. It has been shown in [17] that the NHSS iteration method is convergent to the unique solution of the linear system with a loose restriction on the parameter α. Particularly, if the minimum eigenvalue of the matrix W is greater than the maximum singular value of the matrix iT, then the NHSS iteration method will converge for any $\alpha > 0$. Therefore, we could increase the eigenvalue of the matrix H and decrease the singular value of the matrix S by introducing a parameter to broaden the range of α so that the NHSS iteration method converges.

Multiplying a parameter $\omega - i$ by the two sides of the original linear equation (1.1) yields that

$$(\omega - i)Ax \equiv [(\omega W + T) + i(\omega T - W)]x = (\omega - i)b. \tag{1.2}$$

Let

$$\tilde{A} = (\omega - i)A, \qquad \tilde{W} = \omega W + T, \qquad \tilde{T} = \omega T - W, \qquad \tilde{b} = (\omega - i)b. \tag{1.3}$$

Then we can rewrite (1.2) as

$$\tilde{A}x \equiv (\tilde{W} + i\tilde{T})x = \tilde{b}.$$

By comparing it with the original linear equation (1.1), we can easily obtain the following variant of NHSS iteration method:

$$\begin{cases} \tilde{W}x^{\left(k+\frac{1}{2}\right)} = -i\tilde{T}x^{(k)} + \tilde{b}, \\ (\alpha I + \tilde{W})x^{(k+1)} = (\alpha I - i\tilde{T})x^{\left(k+\frac{1}{2}\right)} + \tilde{b}. \end{cases} \tag{1.4}$$

Substituting (1.3) into (1.4), we get the parameterized NHSS (PNHSS) iteration method.

Algorithm 2. The PNHSS Iteration Method.

Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}$ converges, compute $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\omega W + T)x^{\left(k+\frac{1}{2}\right)} = -i(\omega T - W)x^{(k)} + (\omega - i)b, \\ (\alpha I + \omega W + T)x^{(k+1)} = [\alpha I - i(\omega T - W)]x^{\left(k+\frac{1}{2}\right)} + (\omega - i)b, \end{cases} \tag{1.5}$$

where α and ω are given positive constants.

If the parameter ω can be chosen such that the minimum eigenvalue of the matrix $\tilde{W}$ is greater than the maximum singular value of the matrix $i\tilde{T}$, then the PNHSS iteration method would converge to the unique solution of the linear system for any positive constant α. Moreover, the PNHSS iteration method would be more efficient than the NHSS iteration method by choosing the parameter ω properly.

On the other hand, notice that the second half step of the PNHSS iteration method is just the PSHSS method, which inspires us to take out the first half step to establish the following parameterized single-step HSS (PS*HSS) iteration method. It is a variant of the single-step HSS (S*HSS) iteration method $Wx^{(k+1)} = -iTx^{(k)} + b$.

Algorithm 3. The PS*HSS Iteration Method.

Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}$ converges, compute $x^{(k+1)}$ according to the following procedure:

$$(\omega W + T)x^{(k+1)} = -i(\omega T - W)x^{(k)} + (\omega - i)b, \tag{1.6}$$

where ω is a given positive constant.

The organization of this paper is as follows. In Section 2, the convergence properties of the PNHSS iteration method are analyzed, including the convergence conditions, the choices of the iterative parameters, the spectral radius of the iterative matrix and the quasi-optimal parameters. In Section 3, the convergence properties of the PS*HSS iteration method are considered. In Section 4, some numerical examples are presented to confirm the theoretical results given in Sections 2 and 3. Finally, a brief conclusion is made in Section 5.

2. Convergence analysis about the PNHSS iteration method

First, we rewrite the PNHSS iteration scheme (1.5) as

$$x^{(k+1)} = M(\alpha, \omega)x^{(k)} + N(\alpha, \omega)(\omega - i)b, \quad k = 0, 1, 2, \ldots,$$

where

$$\begin{aligned} M(\alpha, \omega) &= -i(\alpha I + \omega W + T)^{-1}[\alpha I - i(\omega T - W)](\omega W + T)^{-1}(\omega T - W), \\ N(\alpha, \omega) &= (\alpha I + \omega W + T)^{-1} + (\alpha I + \omega W + T)^{-1}[\alpha I - i(\omega T - W)](\omega W + T)^{-1}, \end{aligned} \tag{2.1}$$

with α and ω being given positive constants.

Let

$$\tilde{\lambda}_{\min}(\omega) = \min_{\tilde{\lambda}_j \in sp(\omega W + T)} \{\tilde{\lambda}_j\} \quad \text{and} \quad \tilde{\sigma}_{\max}(\omega) = \max_{i\tilde{\sigma}_j \in sp(i(\omega T - W))} \{|\tilde{\sigma}_j|\}, \tag{2.2}$$

where $sp(X)$ is the spectral set of the matrix X. Similarly, we define

$$\lambda_{\min} = \min_{\lambda_j \in sp(W)} \{\lambda_j\}, \qquad \sigma_{\max} = \max_{\sigma_j \in sp(T)} \{\sigma_j\}, \qquad \lambda_{\max} = \max_{\lambda_j \in sp(W)} \{\lambda_j\}, \qquad \sigma_{\min} = \min_{\sigma_j \in sp(T)} \{\sigma_j\}. \tag{2.3}$$

Lemma 1. *If $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them being positive definite, then the spectral radius $\rho(M(\alpha,\omega))$ is bounded by*

$$\delta(\alpha,\omega)=\frac{\tilde{\sigma}_{\max}(\omega)\sqrt{\alpha^2+\tilde{\sigma}^2_{\max}(\omega)}}{\tilde{\lambda}_{\min}(\omega)(\alpha+\tilde{\lambda}_{\min}(\omega))}.$$

Moreover, it is easy to see that

(i) *if $\tilde{\lambda}_{\min}(\omega)\geq\tilde{\sigma}_{\max}(\omega)$, then $\delta(\alpha,\omega)<1$ i.e. the iteration converges for any $\alpha>0$;*
(ii) *if $\tilde{\lambda}_{\min}(\omega)<\tilde{\sigma}_{\max}(\omega)$ and $\tilde{\Delta}\equiv\tilde{\lambda}^4_{\min}(\omega)+\tilde{\lambda}^2_{\min}(\omega)\tilde{\sigma}^2_{\max}(\omega)-\tilde{\sigma}^4_{\max}(\omega)>0$, then $\delta(\alpha,\omega)<1$ if and only if*

$$\alpha_1\equiv\frac{\tilde{\lambda}^3_{\min}(\omega)-\tilde{\sigma}_{\max}(\omega)\sqrt{\tilde{\Delta}}}{\tilde{\sigma}^2_{\max}(\omega)-\tilde{\lambda}^2_{\min}(\omega)}<\alpha<\frac{\tilde{\lambda}^3_{\min}(\omega)+\tilde{\sigma}_{\max}(\omega)\sqrt{\tilde{\Delta}}}{\tilde{\sigma}^2_{\max}(\omega)-\tilde{\lambda}^2_{\min}(\omega)}\equiv\alpha_2;$$

(iii) *if $\tilde{\lambda}_{\min}(\omega)<\tilde{\sigma}_{\max}(\omega)$ and $\tilde{\Delta}\leq 0$, then $\delta(\alpha,\omega)\geq 1$ for any $\alpha>0$.*

Proof. According to (2.1), we have

$$\begin{aligned}\rho(M(\alpha,\omega))&\leq\|-i(\alpha I+\omega W+T)^{-1}[\alpha I-i(\omega T-W)](\omega W+T)^{-1}(\omega T-W)\|_2\\&\leq\|(\alpha I+\omega W+T)^{-1}[\alpha I-i(\omega T-W)]\|_2\|-i(\omega W+T)^{-1}(\omega T-W)\|_2\\&\leq\|(\alpha I+\omega W+T)^{-1}\|_2\|\alpha I-i(\omega T-W)\|_2\|(\omega W+T)^{-1}\|_2\|-i(\omega T-W)\|_2\\&=\Big(\max_{\tilde{\lambda}_j\in sp(\omega W+T)}\frac{1}{\alpha+\tilde{\lambda}_j}\Big)\Big(\max_{i\tilde{\sigma}_j\in sp(i(\omega T-W))}\sqrt{\alpha^2+\tilde{\sigma}_j^2}\Big)\times\frac{1}{\tilde{\lambda}_{\min}(\omega)}\times\tilde{\sigma}_{\max}(\omega)\\&=\frac{\tilde{\sigma}_{\max}(\omega)\sqrt{\alpha^2+\tilde{\sigma}^2_{\max}(\omega)}}{\tilde{\lambda}_{\min}(\omega)(\alpha+\tilde{\lambda}_{\min}(\omega))}.\end{aligned}$$

Next we only prove the result in case (ii). Since $\delta(\alpha,\omega)<1$ is equivalent to

$$(\tilde{\sigma}^2_{\max}(\omega)-\tilde{\lambda}^2_{\min}(\omega))\alpha^2-2\tilde{\lambda}^3_{\min}(\omega)\alpha+(\tilde{\sigma}^4_{\max}(\omega)-\tilde{\lambda}^4_{\min}(\omega))<0,$$

then $\delta(\alpha,\omega)<1$ if and only if $\alpha_1<\alpha<\alpha_2$ when the discriminant $\Delta=4\tilde{\sigma}^2_{\max}(\omega)\tilde{\Delta}>0$. Thus results in case (ii) and case (iii) are true. □

Mention that we have similar results in case (ii) and case (iii) for the NHSS iteration method, which may not converge if $\lambda_{\min}<\sigma_{\max}$ and $\tilde{\Delta}\equiv\lambda^4_{\min}+\lambda^2_{\min}\sigma^2_{\max}-\sigma^4_{\max}<0$. While the PNHSS iteration method converges if the parameter ω is properly chosen for solving the same linear system, see Example 1. Since $\tilde{W}=\omega W+T$ is a symmetric positive definite matrix, then by using the results in [17], we can obtain the following lemma.

Lemma 2 (*[17]*). *The optimal parameter $\alpha^*(\omega)$ minimizing the upper bound of $\delta(\alpha,\omega)$ of the spectral radius $\rho(M(\alpha,\omega))$ is*

$$\alpha^*(\omega)=arg\min_{\alpha}\left\{\frac{\tilde{\sigma}_{\max}(\omega)\sqrt{\alpha^2+\tilde{\sigma}^2_{\max}(\omega)}}{\tilde{\lambda}_{\min}(\omega)(\alpha+\tilde{\lambda}_{\min}(\omega))}\right\}=\frac{\tilde{\sigma}^2_{\max}(\omega)}{\tilde{\lambda}_{\min}(\omega)},$$

and

$$\delta(\alpha^*(\omega),\omega)=\frac{\tilde{\sigma}^2_{\max}(\omega)}{\tilde{\lambda}_{\min}(\omega)\sqrt{\tilde{\lambda}^2_{\min}(\omega)+\tilde{\sigma}^2_{\max}(\omega)}}.$$

According to Lemma 1, if the parameter ω is properly chosen such that $\tilde{\lambda}_{\min}(\omega)\geq\tilde{\sigma}_{\max}(\omega)$, then the PNHSS iteration method will converge for any $\alpha>0$. We have the following result by similar analysis as in [19].

Theorem 1 (*[19]*). *Let $\tilde{\lambda}_{\min}(\omega)$, $\tilde{\sigma}_{\max}(\omega)$ be defined as in (2.2), $\lambda_{\min}$, $\sigma_{\max}$, $\lambda_{\max}$ and $\sigma_{\min}$ be defined as in (2.3). Denote*

$$\bar{\omega}=\frac{\lambda_{\min}+\lambda_{\max}}{\sigma_{\min}+\sigma_{\max}},\qquad\omega_1=\frac{\lambda_{\min}+\sigma_{\min}}{\sigma_{\max}-\lambda_{\min}},\qquad\omega_2=\frac{\lambda_{\max}-\sigma_{\min}}{\sigma_{\min}+\lambda_{\min}}.$$

Then the PNHSS iteration method converges for any $\alpha>0$ if any of the following conditions holds:

(1) *$\lambda_{\min}\geq\sigma_{\max}$ and $\omega\geq\bar{\omega}$;*
(2) *$\lambda_{\min}<\sigma_{\max}$ and $\bar{\omega}\leq\omega\leq\omega_1$;*
(3) *$\omega_2\leq\omega<\bar{\omega}$.*

Next we consider the optimal ω^* to minimize the upper bound $\delta(\alpha^*(\omega),\omega)$.

Theorem 2. *Let $\lambda_{\min}$, $\sigma_{\max}$, $\lambda_{\max}$ and $\sigma_{\min}$ be defined as in (2.3). Then the optimal parameter ω^* is given by*

$$\omega^* = \bar{\omega} = \frac{\lambda_{\min} + \lambda_{\max}}{\sigma_{\min} + \sigma_{\max}}.$$

Proof. According to Lemma 2, we should minimize the upper bound $\delta(\alpha^*(\omega), \omega)$, which can be rewritten as

$$\delta(\alpha^*(\omega), \omega) = \left(\frac{\tilde{\lambda}_{\min}(\omega)}{\tilde{\sigma}_{\max}(\omega)} \sqrt{\frac{\tilde{\lambda}^2_{\min}(\omega)}{\tilde{\sigma}^2_{\max}(\omega)} + 1} \right)^{-1}.$$

Since it is a strictly monotone decreasing function of $\frac{\tilde{\lambda}_{\min}(\omega)}{\tilde{\sigma}_{\max}(\omega)}$, then equivalently, we need to maximize the lower bound of $\frac{\tilde{\lambda}_{\min}(\omega)}{\tilde{\sigma}_{\max}(\omega)}$. While the rest proof is the same as that of Theorem 2.5 in [19], so we omit it. □

Next, we want to compare the minimal upper bound of the spectral radius of the PNHSS iteration method with that of the spectral radius of the NHSS and the PSHSS iteration methods.

Theorem 3. *If*

$$\omega > \frac{\lambda_{\min}\lambda_{\max} - \sigma_{\min}\sigma_{\max}}{\lambda_{\min}(\sigma_{\min} + \sigma_{\max})},$$

then the minimal upper bound $\delta(\alpha^*(\omega), \omega)$ of the spectral radius of the PNHSS iterative matrix is smaller than the minimal upper bound $\delta(\alpha^*) = \frac{\sigma^2_{\max}}{\lambda_{\min}\sqrt{\lambda^2_{\min}+\sigma^2_{\max}}}$ of the spectral radius of the NHSS iterative matrix.

Proof. Let

$$\frac{\tilde{\sigma}^2_{\max}(\omega)}{\tilde{\lambda}_{\min}(\omega)\sqrt{\tilde{\lambda}^2_{\min}(\omega) + \tilde{\sigma}^2_{\max}(\omega)}} < \frac{\sigma^2_{\max}}{\lambda_{\min}\sqrt{\lambda^2_{\min} + \sigma^2_{\max}}},$$

that is

$$\left(\frac{\tilde{\lambda}_{\min}(\omega)}{\tilde{\sigma}_{\max}(\omega)} \sqrt{\frac{\tilde{\lambda}^2_{\min}(\omega)}{\tilde{\sigma}^2_{\max}(\omega)} + 1} \right)^{-1} < \left(\frac{\lambda_{\min}}{\sigma_{\max}} \sqrt{\frac{\lambda^2_{\min}}{\sigma^2_{\max}} + 1} \right)^{-1}.$$

So equivalently,

$$\frac{\tilde{\lambda}_{\min}(\omega)}{\tilde{\sigma}_{\max}(\omega)} > \frac{\lambda_{\min}}{\sigma_{\max}}.$$

Since the rest proof is the same as that of Theorem 2.4 in [19], then we omit it. □

According to the result of Lemma 2.2 and Theorem 2.5 in [19], we easily have the following result for comparison made between the PNHSS and the PSHSS iteration methods.

Theorem 4. *If $\tilde{\lambda}_{\min}(\omega) \geq \tilde{\sigma}_{\max}(\omega)$, then the minimal upper bound $\delta(\alpha^*(\omega), \omega)$ of the spectral radius of the PNHSS iterative matrix is smaller than the minimal upper bound $\frac{\tilde{\sigma}_{\max}(\omega)}{\sqrt{\tilde{\lambda}^2_{\min}(\omega)+\tilde{\sigma}^2_{\max}(\omega)}}$ of the spectral radius of the PSHSS iterative matrix. Moreover, the quasi-optimal parameters α^* and ω^* which minimize the upper bound $\delta(\alpha^*(\omega), \omega)$ are the same for the PNHSS and the PSHSS iteration methods.*

Note that under the conditions of Theorem 1, the minimal upper bound of the spectral radius of the PNHSS iterative matrix is smaller than that of the spectral radius of the PSHSS iterative matrix.

3. Convergence analysis about the PS*HSS iteration method

In this section, we simply discuss the convergence properties of the PS*HSS iteration method and derive the optimal parameter $\omega*$. First, we rewrite the PS*HSS iteration scheme (1.6) as

$$x^{(k+1)} = M(\omega)x^{(k)} + N(\omega)(\omega - i)b, \quad k = 0, 1, 2, \ldots,$$

where

$$M(\omega) = -i(\omega W + T)^{-1}(\omega T - W),$$
$$N(\omega) = (\omega W + T)^{-1},$$

with ω being given positive constant.

Lemma 3. *If $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them being positive definite, then the spectral radius $\rho(M(\omega))$ is bounded by*

$$\delta(\omega) = \frac{\tilde{\sigma}_{\max}(\omega)}{\tilde{\lambda}_{\min}(\omega)}.$$

Thus, if $\tilde{\lambda}_{\min}(\omega) > \tilde{\sigma}_{\max}(\omega)$, then $\delta(\omega) < 1$ i.e. the iteration converges.

Proof. It is easy to see that

$$\begin{aligned}\rho(M(\omega)) &= \rho(-i(\omega W + T)^{-1}(\omega T - W)) \\ &\le \| - i(\omega W + T)^{-1}(\omega T - W)\|_2 \\ &\le \|(\omega W + T)^{-1}\|_2\| - i(\omega T - W)\|_2 = \frac{\tilde{\sigma}_{\max}(\omega)}{\tilde{\lambda}_{\min}(\omega)}. \quad \square\end{aligned}$$

Mention that we have similar results for the S*HSS iteration method, which may not converge if $\lambda_{\min} < \sigma_{\max}$, see Example 1. While the PS*HSS iteration method converges if the parameter ω is properly chosen such that $\tilde{\lambda}_{\min}(\omega) > \tilde{\sigma}_{\max}(\omega)$. We have the following result by similar analysis as in [19].

Theorem 5. *Let $\tilde{\lambda}_{\min}(\omega)$, $\tilde{\sigma}_{\max}(\omega)$ be defined as in (2.2), $\lambda_{\min}$, $\sigma_{\max}$, $\lambda_{\max}$ and $\sigma_{\min}$ be defined as in (2.3). Denote*

$$\bar{\omega} = \frac{\lambda_{\min} + \lambda_{\max}}{\sigma_{\min} + \sigma_{\max}}, \qquad \omega_1 = \frac{\lambda_{\min} + \sigma_{\min}}{\sigma_{\max} - \lambda_{\min}}, \qquad \omega_2 = \frac{\lambda_{\max} - \sigma_{\min}}{\sigma_{\min} + \lambda_{\min}}.$$

*Then the PS*HSS iteration method converges if any of the following conditions holds:*

(1) $\lambda_{\min} \ge \sigma_{\max}$ *and* $\omega \ge \bar{\omega}$;
(2) $\lambda_{\min} < \sigma_{\max}$ *and* $\bar{\omega} \le \omega < \omega_1$;
(3) $\omega_2 < \omega < \bar{\omega}$.

Notice that the inequalities $\bar{\omega} \le \omega \le \omega_1$ in case (2) and $\omega_2 \le \omega < \bar{\omega}$ in case (3) of Theorem 2.3 in [19] are replaced by $\bar{\omega} \le \omega < \omega_1$ and $\omega_2 < \omega < \bar{\omega}$ respectively, since ω should be chosen such that $\tilde{\lambda}_{\min}(\omega) > \tilde{\sigma}_{\max}(\omega)$ here, rather than $\tilde{\lambda}_{\min}(\omega) \ge \tilde{\sigma}_{\max}(\omega)$.

Theorem 6. *Let $\lambda_{\min}$, $\sigma_{\max}$, $\lambda_{\max}$ and $\sigma_{\min}$ be defined as in (2.3). Then the optimal parameter ω^* for the PS*HSS iteration method is given by*

$$\omega^* = \bar{\omega} = \frac{\lambda_{\min} + \lambda_{\max}}{\sigma_{\min} + \sigma_{\max}}.$$

4. Numerical results

In this section, we will test the effectiveness of the PNHSS and the PS*HSS iteration methods for solving the complex symmetric linear system (1.1). Numerical comparisons with the MHSS [12], the PMHSS [13] using $V = W$, the NHSS [17] and the efficient PSHSS [19] iteration methods are given to show the advantage of the PNHSS and the PS*HSS iteration methods. By the way, we also test the convergence of the S*HSS iteration method, with comparison with the SHSS [18] iteration method.

We compare these methods from the point of view of the number of iterations ("IT") and CPU times ("CPU"). Our experiments are carried out in MATLAB R2013a on Intel(R) Core(TM) CPU 1.8 × 2 GHz and 4.00 GB of RAM. The CPU times is recorded by the command "tic-toc".

In our implementations, the initial guess $x^{(0)}$ is always chosen to be zero vector and the stopping criteria for all the methods are

$$\frac{\|b - Ax^{(k)}\|_2}{\|b\|_2} \le 10^{-6},$$

where $x^{(k)}$ is the current approximation.

Example 1. Consider the following linear system [12,19]

$$\left[\left(K + \frac{3-\sqrt{3}}{\tau}I\right) + i\left(K + \frac{3+\sqrt{3}}{\tau}I\right)\right]x = b,$$

where τ is the time step-size and K is the five-point centered difference matrix approximating the negative Laplacian operator $L = -\Delta$ with homogeneous Dirichlet boundary conditions, on a uniform mesh in the square $[0, 1] \times [0, 1]$ with

Table 1
The numerical results of Example 1 for different iteration methods.

Method		Grid			
		8×8	16×16	32×32	64×64
MHSS	α^*	1.7	1.15	0.8	0.54
	IT	30	39	53	73
	CPU (s)	0.0180	0.2658	10.1753	504.8936
PMHSS	α^*	1.3	1.3	1.3	1.3
	IT	21	21	21	21
	CPU (s)	0.0135	0.1594	4.1119	139.4237
SHSS	α^*	1.6	0.85	0.52	0.36
	IT	119	161	209	267
	CPU (s)	0.0351	0.6685	23.8114	987.5385
S*HSS	–	–	–	–	–
PS*HSS	$\omega*$	0.65	0.65	0.65	0.65
	IT	8	8	9	9
	CPU (s)	0.0081	0.0460	1.0547	34.0874
NHSS	–	–	–	–	–
PNHSS	α^*	0.01	0.01	0.01	0.01
	ω^*	0.65	0.65	0.65	0.65
	IT	4	4	5	5
	CPU (s)	0.0084	0.0464	1.0602	35.9105
PSHSS	α^*	0.01	0.01	0.01	0.01
	ω^*	0.65	0.65	0.65	0.65
	IT	8	8	9	9
	CPU (s)	0.0089	0.0471	1.0586	34.1689

the mesh-size $h = 1/(m+1)$. This complex symmetric system of linear equations comes from the centered difference discretizations of the R22-Padé approximations in the time integration of parabolic partial differential equations [1].

The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form

$$K = I \otimes B_m + B_m \otimes I \quad \text{with } B_m = \frac{1}{h^2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m},$$

and K is an $n \times n$ block-tridiagonal matrix with $n = m^2$. In our tests, we take

$$W = K + \frac{3-\sqrt{3}}{\tau} I, \qquad T = K + \frac{3+\sqrt{3}}{\tau} I,$$

and $\tau = h$. The right-hand side vector b with its jth entry b_j is given by

$$b_j = \frac{(1-i)j}{\tau(j+1)^2}, \quad j = 1, 2, \ldots, n.$$

Furthermore, we normalize coefficient matrix and right-hand side by multiplying both by h^2.

Example 2. The system of linear equations is of the form [12]

$$[(K - \varpi^2 M) + i(C_H + \varpi C_V)]x = b,$$

which arises in direct frequency domain analysis of an n-degree-of-freedom linear system. For more details, we refer to [3,20].

For simplicity, we consider the following linear system

$$\left[\left(K - \varpi^2 I\right) + i\left(\beta K + 10\varpi I\right)\right]x = b,$$

where ϖ, β are constants and K is the five-point centered difference matrix approximating the negative Laplacian operator $L = -\Delta$ with homogeneous Dirichlet boundary conditions, on a uniform mesh in the square $[0, 1] \times [0, 1]$ with the mesh-size $h = 1/(m+1)$. Thus, the matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form

$$K = I \otimes B_m + B_m \otimes I \quad \text{with } B_m = \frac{1}{h^2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}.$$

In our tests, we take the right-hand side vector b with its jth entry b_j is given by

$$b_j = \frac{(1+i)j}{h^2(j+1)^2}, \quad j = 1, 2, \ldots, n.$$

As before, we normalize the system by multiplying both sides by h^2.

Table 2
Numerical results of Example 2 for different iteration methods with $\varpi = \pi/4$ and $\beta = 0.02$.

Method		Grid			
		8×8	16×16	32×32	64×64
MHSS	α^*	0.2	0.1	0.06	0.03
	IT	33	37	42	61
	CPU (s)	0.0197	0.2922	8.2183	422.9038
PMHSS	α^*	0.8	0.9	0.9	0.9
	IT	33	36	37	38
	CPU (s)	0.0198	0.2551	7.5590	262.0138
SHSS	α^*	0.06	0.02	0.003	0.0005
	IT	15	14	13	13
	CPU (s)	0.0086	0.0577	1.3986	46.0612
S*HSS	IT	16	15	15	14
	CPU (s)	0.0092	0.0689	1.6876	48.7098
PS*HSS	$\omega*$	5	5	5	5
	IT	9	9	8	9
	CPU (s)	0.0071	0.0441	0.8798	32.8572
NHSS	α^*	0.0005	0.0005	0.0005	0.0005
	IT	8	8	7	7
	CPU (s)	0.0102	0.0693	1.4408	48.8727
PNHSS	α^*	0.0005	0.0005	0.0005	0.0005
	ω^*	5	5	5	5
	IT	5	5	4	5
	CPU (s)	0.0078	0.0483	0.8867	35.3436
PSHSS	α^*	0.0005	0.0005	0.0005	0.0005
	ω^*	5	5	5	5
	IT	9	9	8	9
	CPU (s)	0.0072	0.0455	0.8996	33.1284

Table 3
Numerical results of Example 2 for the PS*HSS, the PNHSS and the PSHSS iteration methods.

(ϖ, β)	Method		Grid			
			8×8	16×16	32×32	64×64
(0.2, 0.5)	PS*HSS	ω	10	10	10	10
		IT	14	15	15	15
		CPU (s)	0.0094	0.0653	1.5137	51.9204
	PNHSS	α	0.01	0.01	0.01	0.01
		ω	10	10	10	10
		IT	9	9	8	8
		CPU (s)	0.0105	0.0797	1.5889	53.8858
	PSHSS	α	0.01	0.01	0.01	0.01
		ω	10	10	10	10
		IT	18	17	16	15
		CPU (s)	0.0141	0.0756	1.7169	51.2882
(−1, 1)	PS*HSS	ω	3	3	3	3
		IT	18	20	20	20
		CPU (s)	0.0098	0.0796	2.1064	69.1587
	PNHSS	α	0.5	0.5	0.5	0.5
		ω	3	3	3	3
		IT	9	10	10	12
		CPU (s)	0.0098	0.0811	2.1102	80.2533
	PSHSS	α	0.5	0.5	0.5	0.5
		ω	3	3	3	3
		IT	17	34	104	363
		CPU (s)	0.0104	0.1508	10.7733	1238.6846

Tables 1 and 2 show the numerical results of various iteration methods with respect to different problem sizes for Examples 1 and 2 by using the experimental optimal parameters, respectively. The parameters $\omega = \pi/4$ and $\beta = 0.02$ are considered in Example 2. It can be seen that the experimental optimal parameters of the PMHSS, the NHSS, the PS*HSS, the PNHSS and the PSHSS methods are stable. The results about iterations and CPU times are listed in the tables. We can see that the CPU times of the PS*HSS, the PSHSS and the PNHSS iteration methods have little difference by using the experimental optimal parameters, and they are more efficient than the other iteration methods from the point of view of CPU times.

Table 3 shows the numerical results of the PS*HSS, the PNHSS and the PSHSS methods with respect to different problem sizes for Example 2 without using the experimental optimal parameters. We can conclude that the PS*HSS and the PNHSS iteration methods are more efficient than the PSHSS iteration method in this case.

5. Conclusions

In this work, we developed a parameterized HSS (PNHSS) iteration method based on the NHSS iteration method for solving the complex symmetric linear systems. Theoretical analysis shows that, under a loose restriction on the parameter ω, the spectral radius of the iterative matrix of the PNHSS iteration method is smaller than that of the NHSS and the PSHSS iteration methods. The quasi-optimal values of the iteration parameters for the PNHSS iteration method are also discussed. Moreover, a parameterized single-step HSS (PS*HSS) iteration method is established. Both theoretical and numerical results show that the PNHSS and the PS*HSS iteration methods are more efficient than the NHSS and the SHSS iteration methods. According to the numerical results, little difference about the CPU times between the PS*HSS, the PNHSS and the PSHSS iteration methods is justified by using the experimental optimal parameters, however, sometimes the PS*HSS and the PNHSS iteration methods outperform the PSHSS iteration method without using the experimental optimal parameters.

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