INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng 2006; **66**:250–271
Published online 7 November 2005 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/nme.1553

FETI-DP, BDDC, and block Cholesky methods

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SUMMARY

The FETI-DP and BDDC algorithms are reformulated using Block Cholesky factorizations, an approach which can provide a useful framework for the design of domain decomposition algorithms for solving symmetric positive definite linear system of equations. Instead of introducing Lagrange multipliers to enforce the coarse level, primal continuity constraints in these algorithms, a change of variables is used such that each primal constraint corresponds to an explicit degree of freedom. With the new formulation of these algorithms, a simplified proof is provided that the spectra of a pair of FETI-DP and BDDC algorithms, with the same set of primal constraints, are essentially the same. Numerical experiments for a two-dimensional Laplace's equation also confirm this result. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: domain decomposition; FETI; Neumann–Neumann; BDDC; block Cholesky; primal constraints

1. INTRODUCTION

Domain decomposition methods based on non-overlapping subdomains, for solving large symmetric positive definite linear systems arising from the discretization of elliptic partial differential equations, have been widely used and studied; see Reference [1, Chapters 4–6] and the references therein. Two of the main families of these iterative substructuring methods are the balancing Neumann–Neumann methods [2–4], and the FETI methods [5, 6]. The BDDC algorithm, introduced by Dohrmann [7, 8], represents an interesting redesign of the balancing Neumann–Neumann algorithms with the coarse, global component of a BDDC algorithm expressed in terms of a set of *primal constraints*, just as in the FETI-DP algorithms which

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Contract/grant sponsor: US Department of Energy; contract/grant numbers: DE-FG02-92ER25127, DE-FC02-01ER25482

Received 11 December 2004
Revised 12 July 2005
Accepted 17 July 2005

now have replaced the older one-level FETI methods. Fragakis and Papadrakakis [9] introduced an equivalent type of primal iterative substructuring methods, where the FETI operators are used as preconditioners, and they also found experimentally that the older balancing Neumann–Neumann and one-level FETI methods have essentially identical spectra. A Neumann–Neumann algorithm of BDDC type, inspired by the FETI-DP methods, was also developed independently by Cros [10].

All these methods are closely related to the FETI algorithms and they perform quite similarly, as shown experimentally in References [7, 9, 10]. In fact, they can be built from the same set of subprograms. Common theoretical tools were developed in Reference [11] for the study of the one-level FETI and the classical balancing Neumann–Neumann algorithms. In an important contribution to the theory, Mandel *et al.* [12] established that the preconditioned operators of a pair of BDDC and FETI-DP algorithms, with the same primal constraints, have the same non-zero eigenvalues except possibly for 0 and 1. In a very recent paper, Brenner and Sung [13] added further details. A consequence of this result is that there is no appreciable difference between the convergence rates of these algorithms when the same set of primal constraints are used. The choice of algorithm can therefore be based on other considerations. We note, in particular, that it appears easier to modify BDDC than FETI-DP when introducing additional levels; cf. References [14, 15].

The purpose of this paper is to give a simple derivation of FETI-DP and BDDC algorithms, and to give a new, short proof of the main result in Reference [12]. Throughout this paper, we will employ the language of block Cholesky elimination and our discussion can therefore also be seen as a guide to the design of domain decomposition methods using such a framework. To simplify the notation and analysis, we will make a change of the variables so that, e.g. a primal constraint related to the average of the solution over a subdomain interface edge will correspond to one of the new primal variables. The complimentary set of variables will then always have vanishing primal constraints. This approach makes our arguments much easier and it can also be the basis for computational practice as in References [16, 17, Section 6.2].

This paper is organized as follows. In Section 2, we review block Cholesky elimination for solving a symmetric, positive definite block matrix. We then consider the case of two subdomains in two dimensions in Section 3 and show that the iteration matrices of the standard Neumann–Neumann and one-level FETI methods are very closely related and that they have the same eigenvalues. A FETI-DP algorithm is also introduced in Section 3 in terms of a single primal constraint, which enforces the continuity of an edge average. In Section 4, we define, for the general case of many subdomains in either two or three dimensions, a Schur complement matrix $\widetilde{S}_\Gamma$ which represents a partially assembled system matrix obtained from the stiffness matrices of the individual subdomains. We show that all main building blocks of the FETI-DP and BDDC algorithms can be expressed in terms of $\widetilde{S}_\Gamma$, its inverse, and certain simple restriction, extension, and scaling matrices. In Section 5, we provide a new, short proof of the main result in Reference [12], which shows that the eigenvalues of a pair of FETI-DP and BDDC algorithms, with the same sets of primal constraints, are essentially the same. Finally, we provide some numerical results which indeed support the main result on the spectra of the operators and make some concluding remarks.

We note that the choice of the primal constraints is a core question for any efficient FETI-DP or BDDC algorithm. This issue will not be addressed in this paper; see instead References [16–19]. It has been established, in several cases, that the condition number of the preconditioned operator is bounded by $C(1+\log(H/h))^2$ if the primal constraints and certain

diagonal scalings are well chosen; see, e.g. References [17, 18, 20, 21]. Here, C is a constant independent of the number of subdomains as well as the size of the elements, and H/h is the maximum number of elements across any subdomain. In this paper, we will only assume that some choice of primal constraints has been made and that it satisfies the minimal requirement of positive definiteness of the partially subassembled system matrix $\widetilde{S}_\Gamma$.

For further references to domain decomposition methods of FETI and balancing type, see Reference [1], in particular Section 6.1, and the introductory parts of Sections 6.2–6.4.

2. BLOCK CHOLESKY ELIMINATION

Let us consider how to represent the inverse of a symmetric, positive definite block matrix

$$\begin{bmatrix} A & B^{\mathrm{T}} \\ B & C \end{bmatrix} \tag{1}$$

Using block Cholesky elimination, we have

$$\begin{bmatrix} A & B^{\mathrm{T}} \\ B & C \end{bmatrix} = \begin{bmatrix} I_A & \\ BA^{-1} & I_C \end{bmatrix} \begin{bmatrix} A & \\ & C - BA^{-1}B^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I_A & A^{-1}B^{\mathrm{T}} \\ & I_C \end{bmatrix}$$

where I_A and I_C are appropriate identity matrices. The matrix $S = C - BA^{-1}B^{\mathrm{T}}$ is a Schur complement, also known as the static condensation matrix, which is always positive definite. Inverting the three factors, we find that

$$\begin{aligned} \begin{bmatrix} A & B^{\mathrm{T}} \\ B & C \end{bmatrix}^{-1} &= \begin{bmatrix} I_A & -A^{-1}B^{\mathrm{T}} \\ & I_C \end{bmatrix} \begin{bmatrix} A^{-1} & \\ & S^{-1} \end{bmatrix} \begin{bmatrix} I_A & \\ -BA^{-1} & I_C \end{bmatrix} \\ &= \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \Phi S^{-1} \Phi^{\mathrm{T}} \end{aligned} \tag{2}$$

where

$$\Phi = \begin{bmatrix} -A^{-1}B^{\mathrm{T}} \\ I_C \end{bmatrix} \tag{3}$$

The second term in Equation (2) corresponds to solving the condensed problem and the first completes the retrieval of the condensed variables. The columns of the matrix Φ are known as the static modes. We can think of Φ as the extension of the columns of the identity block matrix I_C to the other, the A-block, part. We can also regard the use of Φ as a change of variables which reduces the matrix to block diagonal form.

In our applications, the A block will be a direct sum of many submatrices corresponding to the subdomains into which a given domain has been partitioned while the second diagonal block, C, will represent the global part of a preconditioner representing a select few of the subdomain interface degrees of freedom. It is clear that we should be anxious to keep the size of C and S small since factoring S can be quite expensive and does not parallelize as well as the rest.

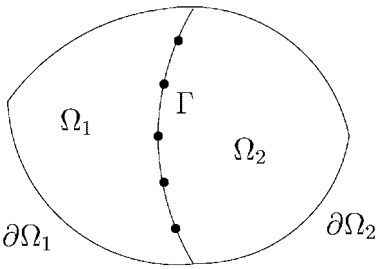

Figure 1. Partition into two non-overlapping subdomains.

3. THE TWO SUBDOMAIN

We begin by considering the two subdomain case and a domain Ω subdivided into two non-overlapping subdomains Ω_1 and Ω_2. The interface between these two open set is $\Gamma = (\partial\Omega_1 \cup \partial\Omega_2) \setminus \partial\Omega$, as shown in Figure 1; we could equally well consider a problem in three dimensions.

We introduce triangulations of the Ω_i, with common nodal points on the interface, and a finite element approximation of a second order, self-adjoint, elliptic problem such as a scalar elliptic problem or the equations of linear elasticity. We also provide standard boundary values and volume data; for simplicity, we first assume that we have a zero Dirichlet condition on all of $\partial\Omega$ and that the boundaries of both subdomains intersect that set.

We denote the stiffness matrix and the load vector for each subdomain Ω_i, with Dirichlet boundary data on $\partial\Omega_i \setminus \Gamma$ and a Neumann condition on Γ, by

$$A^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix}, \quad f^{(i)} = \begin{bmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} \end{bmatrix}, \quad i = 1, 2$$

The finite element model for the entire problem is obtained by assembling the two subproblems:

$$A = \begin{bmatrix} A_{II}^{(1)} & 0 & A_{I\Gamma}^{(1)} \\ 0 & A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma} \end{bmatrix}, \quad f = \begin{bmatrix} f_I^{(1)} \\ f_I^{(2)} \\ f_\Gamma \end{bmatrix}, \quad u = \begin{bmatrix} u_I^{(1)} \\ u_I^{(2)} \\ u_\Gamma \end{bmatrix}$$

Here, $A_{\Gamma\Gamma} = A_{\Gamma\Gamma}^{(1)} + A_{\Gamma\Gamma}^{(2)}$ and $f_\Gamma = f_\Gamma^{(1)} + f_\Gamma^{(2)}$ and they represent sums of contributions from the individual subdomains. The degrees of freedom are partitioned into those interior to Ω_1, Ω_2, and those on Γ, respectively.

After eliminating the interior unknowns of each subdomain separately, we obtain two subdomain Schur complement operators and two subdomain interface load vectors: for $i = 1, 2$,

$$S^{(i)} := A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} {A_{II}^{(i)}}^{-1} A_{I\Gamma}^{(i)} \tag{4}$$

$$g_\Gamma^{(i)} := f_\Gamma^{(i)} - A_{\Gamma I}^{(i)} {A_{II}^{(i)}}^{-1} f_I^{(i)} \tag{5}$$

The given system, $Au = f$, is reduced to

$$(S^{(1)} + S^{(2)})u_\Gamma = g_\Gamma^{(1)} + g_\Gamma^{(2)} \tag{6}$$

We note that we can compute $S^{(i)}$ times a vector by a local computation involving Ω_i only; the main effort is the solution of a Dirichlet problem and there are also some sparse matrix–vector products. Similarly, we can find ${S^{(i)}}^{-1} g_\Gamma^{(i)}$ by solving a linear system with the matrix $A^{(i)}$ and with a right-hand side $(0, g_\Gamma^{(i)})$.

3.1. A Neumann–Neumann method

In this simple case, the basic Neumann–Neumann preconditioner,

$$M^{-1} = {S^{(1)}}^{-1} + {S^{(2)}}^{-1}$$

can be used for solving the interface problem (6). Thus, instead of working with the inverse of the sum of the two Schur complements, we use the sum of the inverses. All solves are then for problems on subdomains, since, as already pointed out, the action of ${S^{(i)}}^{-1}$ on a vector $g_\Gamma^{(i)}$ can be obtained by solving a local linear system. The rate of convergence of the method depends on the eigenvalues of the generalized eigenvalue problem $S^{(1)}\phi = \lambda S^{(2)}\phi$ since the preconditioned operator, $({S^{(1)}}^{-1} + {S^{(2)}}^{-1})(S^{(1)} + S^{(2)})$, equals $2I + {S^{(1)}}^{-1}S^{(2)} + {S^{(2)}}^{-1}S^{(1)}$. It can be established that the eigenvalues of the generalized eigenvalue problem are uniformly bounded from above as well as uniformly bounded away from zero; see Reference [1, Section 1.3].

3.2. A FETI method

We now consider two local mixed Neumann–Dirichlet problems:

$$\begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} u_I^{(i)} \\ u_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} + \lambda_\Gamma^{(i)} \end{bmatrix}, \quad i = 1, 2$$

Here, $\lambda_\Gamma = \lambda_\Gamma^{(1)} = -\lambda_\Gamma^{(2)}$ is the unknown flux; once it has been found, we can obtain the solution by solving the two local problems; see Reference [1, Section 1.3.5] for more details. We obtain, with $g_\Gamma^{(i)} = f_\Gamma^{(i)} - A_{\Gamma I}^{(i)} {A_{II}^{(i)}}^{-1} f_I^{(i)}$,

$$u_\Gamma^{(i)} = {S^{(i)}}^{-1}(g_\Gamma^{(i)} + \lambda_\Gamma^{(i)})$$

Using the requirement that $u_\Gamma^{(1)} = u_\Gamma^{(2)}$, we obtain $F\lambda_\Gamma = d_\Gamma$, with $F = {S^{(1)}}^{-1} + {S^{(2)}}^{-1}$. We precondition this interface equation with $S^{(1)} + S^{(2)}$, and we find that $(S^{(1)} + S^{(2)})({S^{(1)}}^{-1} + {S^{(2)}}^{-1})$, the resulting preconditioned operator, has the same eigenvalues as the preconditioned operator of Section 3.1.

We note that we can reformulate the problem as a constrained minimization problem, minimizing the sum of the energy forms of the two subproblems subject to the constraint $u_\Gamma^{(1)} - u_\Gamma^{(2)} = 0$. We then see that λ_Γ will be a vector of Lagrange multipliers; see further the discussion in Section 3.3.

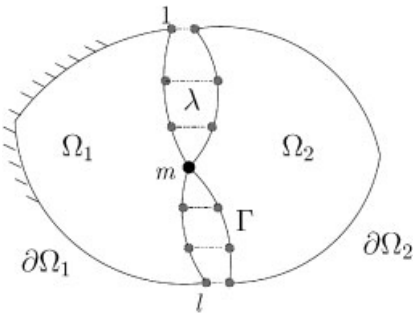

Figure 2. Partition into two subdomains, with Ω_2 floating, in the absence of a constraint.

3.3. A FETI-DP method and a change of variables

The FETI-DP methods were first introduced for two-dimensional problems by Farhat *et al.* [5]. In their algorithms, continuity of the primal variables at the subdomain vertices is maintained (by subassembly), while other continuity constraints are enforced by Lagrange multipliers but only fully so at the convergence of the algorithm. In this section, we give an example of a two-dimensional, two-subdomain case, where there is a Dirichlet boundary condition on part of $\partial\Omega_1$ and a Neumann boundary condition elsewhere as shown in Figure 2. Instead of a vertex constraint, we select the interface average as the sole primal variable; constraints of that type are necessary to obtain rapid convergence for problems in three dimensions. We note that we equally well could have treated a three-dimensional problem with a constraint expressed in terms of a face average or an average over an edge of the face, which is common to the two subdomains. We specialize to a scalar elliptic equation, which has a one-dimensional null space of constants in the case of Neumann boundary conditions. For a three-dimensional elasticity problem, six constraints would be needed on the interface to prevent the subdomain Ω_2 from floating.

We will first show how we can change variables to make the edge average degree of freedom explicit. For each subdomain Ω_i, we denote the unknowns corresponding to the nodal degrees of freedom on the interface edge Γ by $(u_1^{(i)}, \ldots, u_m^{(i)}, \ldots, u_l^{(i)})$, where the node m can be any node on the edge, and denote the other unknowns by $u_I^{(i)}$. A linear system for a subdomain can then be written as

$$A^{(i)}u^{(i)} := \begin{bmatrix} A_{II}^{(i)} & A_{1I}^{(i)\mathrm{T}} & \cdots & A_{mI}^{(i)\mathrm{T}} & \cdots & A_{lI}^{(i)\mathrm{T}} \\ A_{1I}^{(i)} & a_{11}^{(i)} & \cdots & a_{1m}^{(i)} & \cdots & a_{1l}^{(i)} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ A_{mI}^{(i)} & a_{m1}^{(i)} & \cdots & a_{mm}^{(i)} & \cdots & a_{ml}^{(i)} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ A_{lI}^{(i)} & a_{l1}^{(i)} & \cdots & a_{lm}^{(i)} & \cdots & a_{ll}^{(i)} \end{bmatrix} \begin{bmatrix} u_I^{(i)} \\ u_1^{(i)} \\ \vdots \\ u_m^{(i)} \\ \vdots \\ u_l^{(i)} \end{bmatrix} = \begin{bmatrix} f_I^{(i)} \\ f_1^{(i)} \\ \vdots \\ f_m^{(i)} \\ \vdots \\ f_l^{(i)} \end{bmatrix}$$

The interface variables of both subdomains can be changed to

$$\begin{bmatrix} u_1^{(i)} \\ \vdots \\ u_m^{(i)} \\ \vdots \\ u_l^{(i)} \end{bmatrix} = T_E \begin{bmatrix} \hat{u}_1^{(i)} \\ \vdots \\ \hat{u}_m^{(i)} \\ \vdots \\ \hat{u}_l^{(i)} \end{bmatrix} = \begin{bmatrix} 1 & & 1 & & \\ & \ddots & \vdots & & \\ -1 & \dots & 1 & \dots & -1 \\ & & \vdots & \ddots & \\ & & 1 & & 1 \end{bmatrix} \begin{bmatrix} \hat{u}_1^{(i)} \\ \vdots \\ \hat{u}_m^{(i)} \\ \vdots \\ \hat{u}_l^{(i)} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix} \hat{u}_m^{(i)} + \begin{bmatrix} \hat{u}_1^{(i)} \\ \vdots \\ -\hat{u}_1^{(i)} - \cdots - \hat{u}_{m-1}^{(i)} - \hat{u}_{m+1}^{(i)} - \hat{u}_l^{(i)} \\ \vdots \\ \hat{u}_l^{(i)} \end{bmatrix}$$

where T_E is the l by l square matrix shown above, with columns representing the new basis of the space of edge variables. The original interface nodal unknowns have now been separated into two parts. The first part corresponds to a basis function, which is constant on the edge and has a value $\hat{u}_m^{(i)}$ for the subdomain Ω_i; the second corresponds to functions with zero edge averages. Correspondingly, the transformed subdomain problem is of the form

$$T^{\mathrm{T}} \begin{bmatrix} A_{II}^{(i)} & A_{1I}^{(i)^{\mathrm{T}}} & \dots & A_{mI}^{(i)^{\mathrm{T}}} & \dots & A_{lI}^{(i)^{\mathrm{T}}} \\ A_{1I}^{(i)} & a_{11}^{(i)} & \dots & a_{1m}^{(i)} & \dots & a_{1l}^{(i)} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ A_{mI}^{(i)} & a_{m1}^{(i)} & \dots & a_{mm}^{(i)} & \dots & a_{ml}^{(i)} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ A_{lI}^{(i)} & a_{l1}^{(i)} & \dots & a_{lm}^{(i)} & \dots & a_{ll}^{(i)} \end{bmatrix} T \begin{bmatrix} u_I^{(i)} \\ \hat{u}_1^{(i)} \\ \vdots \\ \hat{u}_m^{(i)} \\ \vdots \\ \hat{u}_l^{(i)} \end{bmatrix} = T^{\mathrm{T}} \begin{bmatrix} f_I^{(i)} \\ f_1^{(i)} \\ \vdots \\ f_m^{(i)} \\ \vdots \\ f_l^{(i)} \end{bmatrix}$$

where T is a block diagonal matrix of the form

$$T = \begin{bmatrix} I & \\ & T_E \end{bmatrix}$$

The resulting matrix $T^{\mathrm{T}} A^{(i)} T$ is denser than the original stiffness matrix $A^{(i)}$ but only the blocks related to the interface degrees of freedom are affected while the block corresponding to the subdomain interior degrees of freedom, $A_{II}^{(i)}$, is not. Several different devices for maintaining sparsity are also discussed in Reference [17, Section 6.2]. We also note that the change of

basis on an individual interface edge is a local procedure, i.e. it can be carried out edge by edge, as long as the sets of variables being transformed do not contain any common degrees of freedom. We will, from now on, always assume that the subdomain variables have been changed when primal edges or faces are used. In other words, we will have explicit primal unknowns for the subdomain problem, which correspond to edge or face average finite element basis functions.

In the example shown in Figure 2, the subdomain edge average degrees of freedom, $\hat{u}_m^{(1)}$ and $\hat{u}_m^{(2)}$, are required to have a common value throughout the FETI-DP iteration. From now on, we will denote this common degree of freedom by u_Π, a notation often used for the vector of global, coarse-level, primal variables. The other interface degrees of freedom are $u_\Delta^{(1)}$ and $u_\Delta^{(2)}$, the dual displacement variables, for the neighbouring subdomains with their own values at the same interface nodes. The global matrix problem of our example can then be written as

$$\begin{bmatrix} A_{II}^{(1)} & A_{\Delta I}^{(1)^{\mathrm{T}}} & & & A_{\Pi I}^{(1)^{\mathrm{T}}} & \\ A_{\Delta I}^{(1)} & A_{\Delta\Delta}^{(1)} & & & A_{\Pi\Delta}^{(1)^{\mathrm{T}}} & B_\Delta^{(1)^{\mathrm{T}}} \\ & & A_{II}^{(2)} & A_{\Delta I}^{(2)^{\mathrm{T}}} & A_{\Pi I}^{(2)^{\mathrm{T}}} & \\ & & A_{\Delta I}^{(2)} & A_{\Delta\Delta}^{(2)} & A_{\Pi\Delta}^{(2)^{\mathrm{T}}} & B_\Delta^{(2)^{\mathrm{T}}} \\ A_{\Pi I}^{(1)} & A_{\Pi\Delta}^{(1)} & A_{\Pi I}^{(2)} & A_{\Pi\Delta}^{(2)} & A_{\Pi\Pi}^{(1)} + A_{\Pi\Pi}^{(2)} & \\ & B_\Delta^{(1)} & & B_\Delta^{(2)} & & \end{bmatrix} \begin{bmatrix} u_I^{(1)} \\ u_\Delta^{(1)} \\ u_I^{(2)} \\ u_\Delta^{(2)} \\ u_\Pi \\ \lambda \end{bmatrix} = \begin{bmatrix} f_I^{(1)} \\ f_\Delta^{(1)} \\ f_I^{(2)} \\ f_\Delta^{(2)} \\ f_\Pi^{(1)} + f_\Pi^{(2)} \\ 0 \end{bmatrix} \tag{7}$$

Here the matrices $B_\Delta^{(i)}$ have elements from the set $\{0, 1, -1\}$ and are chosen such that the $l-1$ equations of $B_\Delta^{(1)} u_\Delta^{(1)} + B_\Delta^{(2)} u_\Delta^{(2)} = 0$ are $u_k^{(1)} - u_k^{(2)} = 0$, for $k = 1, \ldots, m-1, m+1, \ldots, l$, which guarantees continuity of the dual displacement variables $u_\Delta^{(1)}$ and $u_\Delta^{(2)}$ across the interface; λ is the corresponding vector of Lagrange multipliers and has $l-1$ components in this example.

We now eliminate the local variables $u_I^{(1)}$, $u_\Delta^{(1)}$, $u_I^{(2)}$, and $u_\Delta^{(2)}$, from Equation (7) and obtain:

$$\begin{bmatrix} S_{\Pi\Pi} & \tilde{B}_{\Lambda\Pi}^{\mathrm{T}} \\ \tilde{B}_{\Lambda\Pi} & \tilde{B}_{\Lambda\Lambda} \end{bmatrix} \begin{bmatrix} u_\Pi \\ \lambda \end{bmatrix} = \begin{bmatrix} g_\Pi \\ d_\Lambda \end{bmatrix} \tag{8}$$

where

$$S_{\Pi\Pi} = \sum_{i=1}^{2} R_\Pi^{(i)^{\mathrm{T}}} \left(A_{\Pi\Pi}^{(i)} - \begin{bmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^{\mathrm{T}}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)^{\mathrm{T}}} \\ A_{\Pi\Delta}^{(i)^{\mathrm{T}}} \end{bmatrix} \right) R_\Pi^{(i)} \tag{9}$$

$$\tilde{B}_{\Lambda\Pi} = -\sum_{i=1}^{2} \begin{bmatrix} 0 & B_\Delta^{(i)} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^{\mathrm{T}}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)^{\mathrm{T}}} \\ A_{\Pi\Delta}^{(i)^{\mathrm{T}}} \end{bmatrix} R_\Pi^{(i)} \tag{10}$$

$$\tilde{B}_{\Lambda\Lambda} = -\sum_{i=1}^{2} [0 \quad B_{\Delta}^{(i)}] \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^{\mathrm{T}}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_{\Delta}^{(i)^{\mathrm{T}}} \end{bmatrix} \tag{11}$$

$$g_{\Pi} = \sum_{i=1}^{2} R_{\Pi}^{(i)^{\mathrm{T}}} \left(f_{\Pi}^{(i)} - [A_{\Pi I}^{(i)} \quad A_{\Pi\Delta}^{(i)}] \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^{\mathrm{T}}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} f_{I}^{(i)} \\ f_{\Delta}^{(i)} \end{bmatrix} \right) \tag{12}$$

$$d_{\Lambda} = -\sum_{i=1}^{2} [0 \quad B_{\Delta}^{(i)}] \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^{\mathrm{T}}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} f_{I}^{(i)} \\ f_{\Delta}^{(i)} \end{bmatrix} \tag{13}$$

and where $R_{\Pi}^{(i)}$ is the matrix, with $\{0, 1\}$ elements, which maps the global primal variable u_{Π} to its subdomain component $u_{\Pi}^{(i)}$. In this simple example, both $R_{\Pi}^{(1)}$ and $R_{\Pi}^{(2)}$ are equal to 1.

We note that the second block in the leading block diagonal part of Equation (7), corresponding to subdomain Ω_2, is invertible, since we have imposed the edge average constraint.

Equation (8) is further reduced to a linear system of equations for the Lagrange multiplier λ:

$$(\tilde{B}_{\Lambda\Lambda} - \tilde{B}_{\Lambda\Pi} S_{\Pi\Pi}^{-1} \tilde{B}_{\Lambda\Pi}^{\mathrm{T}}) \lambda = d_{\Lambda} - \tilde{B}_{\Lambda\Pi} S_{\Pi\Pi}^{-1} g_{\Pi} \tag{14}$$

A Dirichlet preconditioner is often used in FETI-DP algorithms for solving Equation (14) and it will be discussed in Section 4.1. We also typically use a conjugate gradient method to accelerate the convergence.

When designing a BDDC preconditioner, we take a different approach. We do not use any Lagrange multipliers. Instead in each iteration, we solve the positive definite, symmetric linear system obtained from (7) by dropping the rows and columns related to the constraints and the Lagrange multipliers. Solving such a linear system will typically result in a vector such that $u_{\Delta}^{(1)} \neq u_{\Delta}^{(2)}$. Continuity is then restored, in each iteration, by computing a weighted average of these interface values; details will be provided in Section 4.2.

4. MANY SUBDOMAINS

We will now consider the general case of many subdomains, in two or three dimensions. The domain Ω is decomposed into N non-overlapping polyhedral subdomains Ω_i, $i = 1, 2, \ldots, N$. The interface Γ between the subdomains is the union of the parts of the subdomain boundaries that are common to at least two subdomains. The interface of the subdomain Ω_i is defined by $\Gamma_i = \partial\Omega_i \cap \Gamma$. We denote the finite element space of continuous functions on Ω_i by $W^{(i)}$, which is decomposed into subspaces of interior and interface type,

$$W^{(i)} = W_I^{(i)} \oplus W_{\Gamma}^{(i)}$$

The associated product spaces are denoted by $W := \prod_{i=1}^{N} W^{(i)}$, $W_I := \prod_{i=1}^{N} W_I^{(i)}$, and $W_{\Gamma} := \prod_{i=1}^{N} W_{\Gamma}^{(i)}$, respectively.

The finite element solutions are continuous across the subdomain interface and we denote the corresponding subspace of W_Γ by $\hat{W}_\Gamma$; generally the functions in the space W_Γ are discontinuous across the interface. A subspace $\tilde{W}_\Gamma$, intermediate between $\hat{W}_\Gamma$ and W_Γ, is defined by

$$\tilde{W}_\Gamma = W_\Delta \oplus \hat{W}_\Pi = \left(\prod_{i=1}^{N} W_\Delta^{(i)} \right) \oplus \hat{W}_\Pi$$

where $\hat{W}_\Pi$ is the continuous, coarse-level, primal variable space. $\hat{W}_\Pi$ is typically spanned by subdomain corner nodal basis functions, and/or by interface edge and/or face basis functions with weights at the nodes of the edge or face. These basis functions correspond to the primal interface continuity constraints enforced in the FETI-DP algorithm. W_Δ is the product space of the subdomain dual displacement variable spaces $W_\Delta^{(i)}$ which consists of functions with zero values at the primal degrees of freedom. The functions in the space $\tilde{W}_\Gamma$ are only continuous at the coarse level, primal degrees of freedom and are typically discontinuous elsewhere across the subdomain interface. Here, as always, we assume that the basis has been changed so that each primal constraint corresponds to an explicit degree of freedom.

Several restriction and extension operators between these interface spaces need to be defined. $R_{\Gamma\Delta}$ and $R_{\Gamma\Pi}$ are the restriction operators from the space $\tilde{W}_\Gamma$ onto its subspaces W_Δ and $\hat{W}_\Pi$. $R_\Gamma^{(i)} : \hat{W}_\Gamma \to W_\Gamma^{(i)}$, $R_\Delta^{(i)} : W_\Delta \to W_\Delta^{(i)}$, and $R_\Pi^{(i)} : \hat{W}_\Pi \to W_\Pi^{(i)}$, correspondingly, map global interface vectors to its component on Γ_i, respectively. $R_\Gamma : \hat{W}_\Gamma \to W_\Gamma$, is the direct sum of the $R_\Gamma^{(i)}$. $\tilde{R}_\Gamma : \hat{W}_\Gamma \to \tilde{W}_\Gamma$ is the direct sum of $\hat{R}_\Pi$ and $\hat{R}_\Delta^{(i)}$; here $\hat{R}_\Pi$ and $\hat{R}_\Delta^{(i)}$ map functions in the space $\hat{W}_\Gamma$ to their components in the spaces $\hat{W}_\Pi$ and $W_\Delta^{(i)}$, respectively.

In order to define certain scaling operators, we need to introduce a positive scaling factor $\delta_i^\dagger(x)$ for any node x on the interface Γ_i of the subdomain Ω_i. In applications, these scaling factors will depend on the heat conduction coefficient and the first of the Lamé parameters for scalar elliptic problems and the equations of linear elasticity, respectively; see References [17, 18]. Here, with $\mathcal{N}_x$ the set of indices of the subregions which has x on its boundary, we will only assume that $\sum_{j\in\mathcal{N}_x} \delta_j^\dagger(x) = 1$. Given the scaling factors at the subdomain interface nodes, we can define scaled operators $R_{D,\Gamma}^{(i)}$ and $\hat{R}_{D,\Delta}^{(i)}$. We note that each row of $R_\Gamma^{(i)}$ and $\hat{R}_\Delta^{(i)}$ has only one non-zero entry which corresponds to a subdomain interface node x. Multiplying each such row with the scaling factor $\delta_i^\dagger(x)$ gives us $R_{D,\Gamma}^{(i)}$ and $\hat{R}_{D,\Delta}^{(i)}$, respectively. The scaled operators $R_{D,\Gamma}$ and $\tilde{R}_{D,\Gamma}$ are direct sums of $R_{D,\Gamma}^{(i)}$, and of $\hat{R}_\Pi$ and $\hat{R}_{D,\Delta}^{(i)}$, respectively.

4.1. FETI-DP algorithms

In a FETI-DP algorithm, a finite element space of partially subassembled variables is defined by

$$\tilde{W} = W_I \oplus \tilde{W}_\Gamma = W_I \oplus W_\Delta \oplus \hat{W}_\Pi$$

If we have primal vertex constraints only, the corresponding finite element model can be regarded as obtained by making incisions which remove the coupling between the nodes on the subdomain interface, except at the subdomain vertices as shown in Figure 3.

The global linear system of equations is a simple generalization of Equation (7) to the many subdomain case, and the reduced equation for the Lagrange multiplier λ is of the form (14)

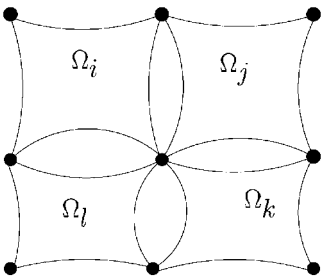

Figure 3. Decomposition of subdomains for a FETI-DP method.

where the corresponding operators and vectors are defined as in Equations (9)–(13), except for a case of many subdomains. The space of Lagrange multipliers is $V = B_\Delta W_\Delta$, where B_Δ is defined in terms of subdomain operators $B_\Delta^{(i)}$, and the continuity constraints across the interface are written as

$$B_\Delta u_\Delta = \sum_{i=1}^{N} B_\Delta^{(i)} u_\Delta^{(i)} = 0$$

Here we use all the possible continuity constraints on the interface nodes in the formulation of B_Δ and thus work with a fully redundant set of Lagrange multipliers, cf. References [6, 11, 22]. We note that the result in Theorem 1 in Section 5 applies equally well to the case of non-redundant Lagrange multipliers, cf. Remark 2 in Section 5.

We will now rewrite the FETI-DP operator in Equation (14) in terms of certain Schur complements, and also introduce a Dirichlet preconditioner.

We define a Schur complement operator $\widetilde{S}_\Gamma$ on the space $\tilde{W}_\Gamma$ by: given interface variables $w_\Gamma \in \tilde{W}_\Gamma$, determine $\widetilde{S}_\Gamma w_\Gamma \in \tilde{W}_\Gamma$, such that

$$\begin{bmatrix} A_{II}^{(1)} & A_{\Delta I}^{(1)^{\mathrm{T}}} & & & & \tilde{A}_{\Pi I}^{(1)^{\mathrm{T}}} \\ A_{\Delta I}^{(1)} & A_{\Delta\Delta}^{(1)} & & & & \tilde{A}_{\Pi\Delta}^{(1)^{\mathrm{T}}} \\ & & \ddots & & & \vdots \\ & & & A_{II}^{(N)} & A_{\Delta I}^{(N)^{\mathrm{T}}} & \tilde{A}_{\Pi I}^{(N)^{\mathrm{T}}} \\ & & & A_{\Delta I}^{(N)} & A_{\Delta\Delta}^{(N)} & \tilde{A}_{\Pi\Delta}^{(N)^{\mathrm{T}}} \\ \tilde{A}_{\Pi I}^{(1)} & \tilde{A}_{\Pi\Delta}^{(1)} & \dots & \tilde{A}_{\Pi I}^{(N)} & \tilde{A}_{\Pi\Delta}^{(N)} & \tilde{A}_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} w_I^{(1)} \\ w_\Delta^{(1)} \\ \vdots \\ w_I^{(N)} \\ w_\Delta^{(N)} \\ w_\Pi \end{bmatrix} = \begin{bmatrix} 0 \\ (\widetilde{S}_\Gamma w_\Gamma)_\Delta^{(1)} \\ \vdots \\ 0 \\ (\widetilde{S}_\Gamma w_\Gamma)_\Delta^{(N)} \\ (\widetilde{S}_\Gamma w_\Gamma)_\Pi \end{bmatrix} \tag{15}$$

Here

$$\tilde{A}_{\Pi I}^{(i)} = R_\Pi^{(i)^{\mathrm{T}}} A_{\Pi I}^{(i)}, \quad \tilde{A}_{\Pi\Delta}^{(i)} = R_\Pi^{(i)^{\mathrm{T}}} A_{\Pi\Delta}^{(i)} \quad \forall i = 1, 2, \dots, N$$

and

$$\tilde{A}_{\Pi\Pi} = \sum_{i=1}^{N} R_\Pi^{(i)^{\mathrm{T}}} A_{\Pi\Pi}^{(i)} R_\Pi^{(i)}$$

Equation (15) can be viewed as the two-by-two block matrix of Equation (1), if we take the leading block-diagonal matrix with N blocks, corresponding to the N subdomains, to be the matrix A of Equation (1), and the last diagonal block, corresponding to the coarse-level variables, to be the matrix C. Then the inverse of $\widetilde{S}_\Gamma$ can be evaluated by Cholesky elimination as in Equation (2) of Section 2. We have

$$\widetilde{S}_\Gamma^{-1} = R_{\Gamma\Delta}^{\mathrm{T}} \left(\sum_{i=1}^{N} [0 \quad R_\Delta^{(i)^{\mathrm{T}}}] \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^{\mathrm{T}}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ R_\Delta^{(i)} \end{bmatrix} \right) R_{\Gamma\Delta} + \Phi S_{\Pi\Pi}^{-1} \Phi^{\mathrm{T}} \tag{16}$$

where the first part corresponds to A^{-1} in Equation (2),

$$\Phi = R_{\Gamma\Pi}^{\mathrm{T}} - R_{\Gamma\Delta}^{\mathrm{T}} \sum_{i=1}^{N} [0 \quad R_\Delta^{(i)^{\mathrm{T}}}] \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^{\mathrm{T}}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)^{\mathrm{T}}} \\ A_{\Pi\Delta}^{(i)^{\mathrm{T}}} \end{bmatrix} R_\Pi^{(i)} \tag{17}$$

corresponds to the matrix Φ in Equation (3), and $S_{\Pi\Pi}$ is defined as in Equation (9), except for a case of many subdomains.

We define the operator B_Γ by $B_\Gamma := B_\Delta R_{\Gamma\Delta}$. From the fact that $R_{\Gamma\Delta} R_{\Gamma\Delta}^{\mathrm{T}} = I$ and $R_{\Gamma\Pi} R_{\Gamma\Delta}^{\mathrm{T}} = 0$, we have,

$$\begin{aligned} B_\Gamma \widetilde{S}_\Gamma^{-1} B_\Gamma^{\mathrm{T}} = {} & \sum_{i=1}^{N} [0 \quad B_\Delta^{(i)}] \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^{\mathrm{T}}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_\Delta^{(i)^{\mathrm{T}}} \end{bmatrix} \\ & + \left(\sum_{i=1}^{N} [0 \quad B_\Delta^{(i)}] \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^{\mathrm{T}}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)^{\mathrm{T}}} \\ A_{\Pi\Delta}^{(i)^{\mathrm{T}}} \end{bmatrix} R_\Pi^{(i)} \right) S_{\Pi\Pi}^{-1} \\ & \times \left(\sum_{i=1}^{N} R_\Pi^{(i)^{\mathrm{T}}} [A_{\Pi I}^{(i)} \quad A_{\Pi\Delta}^{(i)}] \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^{\mathrm{T}}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_\Delta^{(i)^{\mathrm{T}}} \end{bmatrix} \right) \\ = {} & -(\tilde{B}_{\Lambda\Lambda} - \tilde{B}_{\Lambda\Pi} S_{\Pi\Pi}^{-1} \tilde{B}_{\Lambda\Pi}^{\mathrm{T}}) \end{aligned}$$

We can therefore write the FETI-DP system (14) as

$$B_\Gamma \widetilde{S}_\Gamma^{-1} B_\Gamma^{\mathrm{T}} \lambda = -(d_\Lambda - \tilde{B}_{\Lambda\Pi} S_{\Pi\Pi}^{-1} g_\Pi) \tag{18}$$

We note that if there are no primal degrees of freedom, then $\tilde{S}_\Gamma$ will be a block diagonal matrix built from subdomain Schur complements and the problem operator in (18), with the inverse replaced by a pseudo-inverse, will become the one-level FETI operator.

The Dirichlet preconditioner used in the FETI-DP algorithms for solving Equation (18) is of the form $B_{D,\Gamma} \widetilde{S}_\Gamma B_{D,\Gamma}^{\mathrm{T}}$, where $B_{D,\Gamma}$ is defined by $B_{D,\Gamma} := B_{D,\Delta} R_{\Gamma\Delta}$ and where $B_{D,\Delta}$ is

constructed from subdomain operators $B^{(i)}_{D,\Delta}$ in the same way as B_Δ. Each $B^{(i)}_{D,\Delta}$ is defined as follows: each row of $B^{(i)}_\Delta$ with a non-zero entry corresponds to a Lagrange multiplier connecting the subdomain Ω_i to a neighbouring subdomain Ω_j at a point $x \in \Gamma_i \cap \Gamma_j$. Multiplying each such row of $B^{(i)}_\Delta$ with the positive scaling factor $\delta^\dagger_j(x)$ gives us $B^{(i)}_{D,\Delta}$. From the definition of $\widetilde{S}_\Gamma$ in Equation (15), we can see that multiplying $\widetilde{S}_\Gamma$ with a vector in *range* $(B^{\mathrm{T}}_{D,\Gamma})$ requires solving subdomain problems with Dirichlet boundary conditions. The preconditioned FETI-DP operator with a Dirichlet preconditioner, is therefore of the form

$$B_{D,\Gamma}\widetilde{S}_\Gamma B^{\mathrm{T}}_{D,\Gamma} B_\Gamma \widetilde{S}^{-1}_\Gamma B^{\mathrm{T}}_\Gamma \tag{19}$$

We note that a different form of the FETI-DP operator,

$$B_{D,\Delta} S_\Delta B^{\mathrm{T}}_{D,\Delta} B_\Delta \widetilde{S}^{-1} B^{\mathrm{T}}_\Delta \tag{20}$$

was introduced in References [18, 20], and used in the analysis of different FETI-DP algorithms. There S_Δ is defined as the direct sum of subdomain Schur operators $S^{(i)}_\Delta$ which are defined by: given $w^{(i)}_\Delta \in W^{(i)}_\Delta$, determine $S^{(i)}_\Delta w^{(i)}_\Delta \in W^{(i)}_\Delta$, such that

$$\begin{bmatrix} A^{(i)}_{II} & A^{(i)^{\mathrm{T}}}_{\Delta I} \\ A^{(i)}_{\Delta I} & A^{(i)}_{\Delta\Delta} \end{bmatrix} \begin{bmatrix} w^{(i)}_I \\ w^{(i)}_\Delta \end{bmatrix} = \begin{bmatrix} 0 \\ S^{(i)}_\Delta w^{(i)}_\Delta \end{bmatrix} \tag{21}$$

The operator $\widetilde{S}$ in the FETI-DP operator (20) can be defined by Equation (15), except that we use vectors in the dual displacement variable space W_Δ, and set the coarse-level part of the right-hand side of Equation (15) to zero.

From the definition of $\widetilde{S}_\Gamma$ in Equation (15), we can see that S_Δ and $\widetilde{S}^{-1}$ are the restrictions of the operators $\widetilde{S}_\Gamma$ and $\widetilde{S}^{-1}_\Gamma$, respectively, to the space W_Δ. Therefore, we have

$$S_\Delta = R_{\Gamma\Delta}\widetilde{S}_\Gamma R^{\mathrm{T}}_{\Gamma\Delta}, \quad \widetilde{S}^{-1} = R_{\Gamma\Delta}\widetilde{S}^{-1}_\Gamma R^{\mathrm{T}}_{\Gamma\Delta}$$

which shows that the two preconditioned FETI-DP operators of Equations (19) and (20) are the same. A reason for introducing the form of FETI-DP operator given in Equation (19) is that it is more convenient in the study of the connection with the BDDC algorithms. Using the definition of $S^{(i)}_\Delta$ in Equation (21), we find that $\widetilde{S}^{-1}_\Gamma$, given in Equation (16), can be written as

$$\widetilde{S}^{-1}_\Gamma = R^{\mathrm{T}}_{\Gamma\Delta} S^{-1}_\Delta R_{\Gamma\Delta} + \Phi S^{-1}_{\Pi\Pi} \Phi^{\mathrm{T}} \tag{22}$$

where S^{-1}_Δ represents the direct sum of the S^{-1}_Δ.

4.2. Neumann–Neumann methods of the same flavour: BDDC

In a Neumann–Neumann type algorithm, we solve the interface Schur complement problem: find $u_\Gamma \in \hat{W}_\Gamma$, such that

$$\hat{S} u_\Gamma = \sum_{i=1}^{N} R^{(i)^{\mathrm{T}}}_\Gamma g^{(i)}_\Gamma \tag{23}$$

with a preconditioner built from subdomain Neumann solvers, each possibly with a few constraints. In Equation (23), $g_\Gamma^{(i)}$ is the subdomain interface load vector, as in Equation (5), and $\hat{S}$ is the interface Schur complement operator, defined on the space $\hat{W}_\Gamma$.

There are different ways of representing $\hat{S}$. Let us denote by S the Schur complement operator defined on the space W_Γ, i.e. S is the direct sum of the $S^{(i)}$, the subdomain Schur complements defined on the space $W_\Gamma^{(i)}$ as in Equation (4). Then $\hat{S}$ can be regarded as the restriction of S to the space $\hat{W}$ and can therefore be written as

$$\hat{S} = R_\Gamma^{\mathrm{T}} S R_\Gamma = \sum_{i=1}^{N} (R_\Gamma^{(i)^{\mathrm{T}}} S^{(i)} R_\Gamma^{(i)})$$

$\hat{S}$ can also be written as the restriction of $\widetilde{S}_\Gamma$ to the space $\hat{W}$, and is therefore also of the form

$$\hat{S} = \tilde{R}_\Gamma^{\mathrm{T}} \widetilde{S}_\Gamma \tilde{R}_\Gamma$$

Remark 1
A one-level Neumann–Neumann preconditioner

$$M^{-1} = R_{D,\Gamma}^{\mathrm{T}} S^{-1} R_{D,\Gamma} = \sum_{i=1}^{N} R_{D,\Gamma}^{(i)^{\mathrm{T}}} S^{(i)^{-1}} R_{D,\Gamma}^{(i)}$$

can be used for solving the interface problem (23), as in Section 3.1 for the two-subdomain case. We note that the use of the scaling means that we partition the residual on the interface and then, after solving the local problems, average the values obtained on the interface; cf. also Reference [1, Section 1.3.4]. There are two problems with this approach. The $S^{(i)}$ corresponding to interior subdomains typically are singular, introducing *floating* subdomains, and there is also no global component of the preconditioner. The first problem can be overcome by adding a small positive multiple of the mass matrices to the stiffness matrices of the subdomains when constructing the preconditioner. The lack of a global component, however, will make such a preconditioner non-competitive with a rate of convergence that will necessarily deteriorate with an increasing number of subdomains; see Reference [1, Section 1.3.6]. Several successful two-level balancing Neumann–Neumann algorithm have been proposed to overcome this problem; see, e.g. References [23] and [4]. Some of these algorithms have been used extensively for large-scale problems.

The BDDC algorithm, first introduced by Dohrmann [7, 8], is a variant of the two-level Neumann–Neumann type preconditioner for solving the interface problem (23). In the BDDC preconditioner, the coarse-level problem is assembled from a special set of coarse basis functions, which are the minimum energy extension on the subdomains subject to sets of primal constraints; these coarse-level basis functions, in fact, correspond to the matrix Φ in the block Cholesky elimination in Equation (2). The primal constraints usually represent continuity at subdomain corners and/or common edge or face averages across the interface, as in a FETI-DP algorithm.

Dohrmann's BDDC preconditioner is written in the form

$$M_{\mathrm{BDDC}}^{-1} = R_{D,\Gamma}^{\mathrm{T}} (T_{\mathrm{sub}} + T_0) R_{D,\Gamma} \tag{24}$$

see Reference [12]. The coarse-level correction operator T_0 is defined by

$$T_0 = \Psi(\Psi^{\mathrm{T}} S \Psi)^{-1} \Psi^{\mathrm{T}}$$

where the coarse-level basis function matrix Ψ is of the form

$$\Psi = \begin{bmatrix} \Psi^{(1)} \\ \vdots \\ \Psi^{(N)} \end{bmatrix}$$

Each subdomain coarse-level basis function matrix $\Psi^{(i)}$ is determined by

$$\begin{bmatrix} S^{(i)} & C^{(i)^{\mathrm{T}}} \\ C^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \Psi^{(i)} \\ \Lambda^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ R_{\Pi}^{(i)} \end{bmatrix} \tag{25}$$

where $C^{(i)}$ represents the primal constraints of the subdomain Ω_i and each column of $\Lambda^{(i)}$ is a vector of Lagrange multipliers. The number of columns of each $\Psi^{(i)}$ is the same as the number of global coarse-level degrees of freedom. Only a few columns of each $\Psi^{(i)}$ are non-zero namely those supported in Ω_i. To compute a non-zero column of $\Psi^{(i)}$, a subdomain Neumann problem is solved subject to the given primal constraints, which corresponds to a non-zero column of the matrix $R_{\Pi}^{(i)}$. We can see from Equation (25), that each non-zero column of $\Psi^{(i)}$ is the minimum energy extension to the subdomain Ω_i setting one of the primal constraints equal to 1 and all others equal to 0.

The subdomain correction operator T_{sub} is defined by

$$T_{\mathrm{sub}} = \sum_{i=1}^{N} [R_{\Gamma}^{(i)^{\mathrm{T}}} \quad 0] \begin{bmatrix} S^{(i)} & C^{(i)^{\mathrm{T}}} \\ C^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} R_{\Gamma}^{(i)} \\ 0 \end{bmatrix} \tag{26}$$

which gives subdomain corrections for which all the coarse level, primal variables vanish.

If we, as we always assume, have changed the variables, then the Lagrange multipliers are no longer needed to enforce the primal continuity constraints and the BDDC preconditioner (24) can be written as

$$M_{\mathrm{BDDC}}^{-1} = R_{D,\Gamma}^{\mathrm{T}} \{ R_{\Gamma\Delta}^{\mathrm{T}} S_{\Delta}^{-1} R_{\Gamma\Delta} + \Psi(\Psi^{\mathrm{T}} S \Psi)^{-1} \Psi^{\mathrm{T}} \} R_{D,\Gamma} \tag{27}$$

where we have replaced the subdomain correction operator T_{sub}, defined in Equation (26), by $R_{\Gamma\Delta}^{\mathrm{T}} S_{\Delta}^{-1} R_{\Gamma\Delta}$, since, after changing the variables, we can enforce zero primal constraints simply by restricting the operators to the dual interface space W_{Δ}. Similarly, Equation (25) is replaced by

$$\begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^{\mathrm{T}}} & A_{\Pi I}^{(i)^{\mathrm{T}}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Pi\Delta}^{(i)^{\mathrm{T}}} \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{bmatrix} \begin{bmatrix} u_I^{(i)} \\ \Psi_{\Delta}^{(i)} \\ R_{\Pi}^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ S_{\Pi\Pi}^{(i)} R_{\Pi}^{(i)} \end{bmatrix} \tag{28}$$

where

$$\Psi^{(i)} = \begin{bmatrix} \Psi_{\Delta}^{(i)} \\ R_{\Pi}^{(i)} \end{bmatrix} = \begin{bmatrix} -[0 \quad I_{\Delta}^{(i)}] \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)\mathrm{T}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)\mathrm{T}} \\ A_{\Pi\Delta}^{(i)\mathrm{T}} \end{bmatrix} R_{\Pi}^{(i)} \\ R_{\Pi}^{(i)} \end{bmatrix} \tag{29}$$

and

$$S_{\Pi\Pi}^{(i)} = A_{\Pi\Pi}^{(i)} - [A_{\Pi I}^{(i)} \quad A_{\Pi\Delta}^{(i)}] \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)\mathrm{T}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)\mathrm{T}} \\ A_{\Pi\Delta}^{(i)\mathrm{T}} \end{bmatrix}$$

which is a subdomain contribution to the operator $S_{\Pi\Pi}$ of Equation (9). We can see, from Equations (28) and (29), that

$$\Psi^{\mathrm{T}} S \Psi = \sum_{i=1}^{N} \Psi^{(i)^{\mathrm{T}}} S^{(i)} \Psi^{(i)} = \sum_{i=1}^{N} R_{\Pi}^{(i)^{\mathrm{T}}} S_{\Pi\Pi}^{(i)} R_{\Pi}^{(i)} = S_{\Pi\Pi} \tag{30}$$

From the fact that $\Psi^{(i)}$ in Equation (29) is the same as the restriction of the matrix Φ of Equation (17) to the subdomain Ω_i, we see that the matrix Ψ in fact equals Φ, except that Ψ corresponds to interface vectors with distributed but continuous coarse-level variables and Φ corresponds to vectors with shared coarse-level degrees of freedom. From the assumption that $\sum_{j \in \mathcal{N}_x} \delta_j^{\dagger}(x) = 1$ for every $x \in \Gamma$, we have $R_{D,\Gamma}^{\mathrm{T}} \Psi = \tilde{R}_{D,\Gamma}^{\mathrm{T}} \Phi$. Therefore, from Equations (30) and (22) the BDDC preconditioner (27) can be written as

$$\begin{aligned} M_{\mathrm{BDDC}}^{-1} &= R_{D,\Gamma}^{\mathrm{T}} R_{\Gamma\Delta}^{\mathrm{T}} S_{\Delta}^{-1} R_{\Gamma\Delta} R_{D,\Gamma} + R_{D,\Gamma}^{\mathrm{T}} \Psi (\Psi^{\mathrm{T}} S \Psi)^{-1} \Psi^{\mathrm{T}} R_{D,\Gamma} \\ &= \tilde{R}_{D,\Gamma}^{\mathrm{T}} R_{\Gamma\Delta}^{\mathrm{T}} S_{\Delta}^{-1} R_{\Gamma\Delta} \tilde{R}_{D,\Gamma} + \tilde{R}_{D,\Gamma}^{\mathrm{T}} \Phi S_{\Pi\Pi}^{-1} \Phi^{\mathrm{T}} \tilde{R}_{D,\Gamma} = \tilde{R}_{D,\Gamma}^{\mathrm{T}} \widetilde{S}_{\Gamma}^{-1} \tilde{R}_{D,\Gamma} \end{aligned}$$

and the preconditioned BDDC operator is of the form

$$\tilde{R}_{D,\Gamma}^{\mathrm{T}} \widetilde{S}_{\Gamma}^{-1} \tilde{R}_{D,\Gamma} \tilde{R}_{\Gamma}^{\mathrm{T}} \widetilde{S}_{\Gamma} \tilde{R}_{\Gamma} \tag{31}$$

5. MATRIX ANALYSIS OF FETI-DP AND BDDC

We first define jump and average operators on the space $\tilde{W}_{\Gamma}$ by

$$P_D := B_{D,\Gamma}^{\mathrm{T}} B_{\Gamma}, \quad E_D := \tilde{R}_{\Gamma} \tilde{R}_{D,\Gamma}^{\mathrm{T}} \tag{32}$$

The following lemma shows that the jump and average operators are complimentary projections; cf. References [11, Lemma 5], [9, Section 2.3], [24, Section 3.1], and [12, Lemma 3].

Lemma 1

$$E_D + P_D = I; \quad E_D^2 = E_D, P_D^2 = P_D; \quad E_D P_D = P_D E_D = 0$$

Proof
Given a function $w_\Gamma \in \tilde{W}_\Gamma$, we denote its coarse-level component by w_Π and its dual subdomain components by $w_\Delta^{(i)}$, for $i = 1, 2, \ldots, N$. We have, from the definitions of E_D and $\tilde{R}_{D,\Gamma}$,

$$E_D w_\Gamma = \tilde{R}_\Gamma \tilde{R}_{D,\Gamma}^{\mathrm{T}} w_\Gamma = \tilde{R}_\Gamma \left(\hat{R}_\Pi^{\mathrm{T}} w_\Pi + \sum_{j=1}^{N} \hat{R}_{D,\Delta}^{(j)^{\mathrm{T}}} w_\Delta^{(j)} \right) \tag{33}$$

We see that the coarse-level component of $E_D w_\Gamma$ is still w_Π, and its dual subdomain components equal

$$(E_D w_\Gamma)_\Delta^{(i)}(x) = \sum_{j \in \mathcal{N}_x} \delta_j^\dagger(x) w_\Delta^{(j)}(x) \quad \forall x \in \Gamma_i \tag{34}$$

From the definition of P_D, B_Γ, and $B_{D,\Gamma}$, we have

$$P_D w_\Gamma = B_{D,\Gamma}^{\mathrm{T}} B_\Gamma w_\Gamma = R_{\Gamma\Delta}^{\mathrm{T}} B_{D,\Delta}^{\mathrm{T}} B_\Delta R_{\Gamma\Delta} w_\Gamma = R_{\Gamma\Delta}^{\mathrm{T}} \left(\sum_{j=1}^{N} B_{D,\Delta}^{(j)^{\mathrm{T}}} B_\Delta^{(j)} w_\Delta^{(j)} \right)$$

We can see that the coarse-level component of $P_D w_\Gamma$ is zero, and that

$$(P_D w_\Gamma)_\Delta^{(i)}(x) = \sum_{j \in \mathcal{N}_x} \delta_j^\dagger(x) (w_\Delta^{(i)}(x) - w_\Delta^{(j)}(x)) \quad \forall x \in \Gamma_i \tag{35}$$

By the assumption that $\sum_{j \in \mathcal{N}_x} \delta_j^\dagger(x) = 1$, for any $x \in \Gamma$, we have

$$(E_D w_\Gamma + P_D w_\Gamma)_\Delta^{(i)}(x) = \sum_{j \in \mathcal{N}_x} \delta_j^\dagger(x) (w_\Delta^{(j)}(x) + w_\Delta^{(i)}(x) - w_\Delta^{(j)}(x)) = w_\Delta^{(i)}(x)$$

We also note that the coarse-level component of $E_D w_\Gamma + P_D w_\Gamma$ equals w_Π. Therefore, for any vector $w_\Gamma \in \tilde{W}_\Gamma$, $(E_D + P_D) w_\Gamma = w_\Gamma$, i.e. $E_D + P_D = I$.

Equations (33) and (34) also show that *range* $(E_D) \subset \hat{W}$ and that for any continuous function $w_\Gamma \in \hat{W}_\Gamma$, $E_D w_\Gamma = w_\Gamma$. Therefore $E_D^2 = E_D$. Since $E_D(E_D + P_D) = E_D$, we have $E_D P_D = 0$. From Equation (35), we know that for any continuous interface function w_Γ, $P_D w_\Gamma = 0$. Therefore $P_D E_D = 0$, and since $P_D(E_D + P_D) = P_D$, we have $P_D^2 = P_D$. □

We note that the preconditioned FETI-DP operator (19) can be singular, since the matrix B_Γ^{T} is not of full column rank when redundant Lagrange multipliers are used. But we need not worry since the Lagrange multiplier λ is always restricted to *range* (B_Γ), which is orthogonal to the null space of B_Γ^{T}. We multiply the preconditioned FETI-DP operator (19) by B_Γ^{T} on the left and remove the same factor on the right to obtain

$$P_D^{\mathrm{T}} \widetilde{S}_\Gamma P_D \widetilde{S}_\Gamma^{-1} \tag{36}$$

It is easy to see that the two operators (19) and (36) have the same non-zero eigenvalues.

Remark 2
For the case of non-redundant Lagrange multipliers in the FETI-DP algorithms, the scaling in the preconditioner is defined differently, see References [5, 6, 11, 24]. But it has been proved in Reference [11, Lemma 9] that the jump operator is the same as in (32) for fully redundant

Lagrange multipliers. Therefore the operator form (36) is also valid for non-redundant Lagrange multipliers. This also shows that the result in Theorem 1 applies equally well in that case. The jump and average operators, defined in (32), are equally valid in two and three dimensions. They are inherently defined as in Equations (34) and (35), where the subdomain scaling operator $\delta_j^\dagger(x)$ is defined for the subdomain interface nodes x; see the beginning of Section 4.

We also multiply the preconditioned BDDC operator (31) by $\tilde{R}_\Gamma$ on the left and remove the same factor on the right to obtain

$$E_D \widetilde{S}_\Gamma^{-1} E_D^{\mathrm{T}} \widetilde{S}_\Gamma \tag{37}$$

which has the same non-zero eigenvalues as (31).

We will now give a short proof of the following theorem, which is the main result in Reference [12].

Theorem 1
The preconditioned FETI-DP operator and the BDDC operator, given by Equations (19) and (31), respectively, have the same eigenvalues except for eigenvalues equal to 0 or 1.

Proof
We only need to prove this result for the two operators (36) and (37). We will show that any non-zero eigenvalue, except those equal to 1, of the operator (36) is also an eigenvalue of the operator (37). The conclusion then comes from exchanging the roles of the two operators.

Let φ be an eigenvector of the operator (36) with the eigenvalue μ, and let $\psi = E_D \widetilde{S}_\Gamma^{-1} \varphi$. Then, from Lemma 1, we have,

$$\begin{aligned} E_D \widetilde{S}_\Gamma^{-1} E_D^{\mathrm{T}} \widetilde{S}_\Gamma \psi &= E_D \widetilde{S}_\Gamma^{-1} E_D^{\mathrm{T}} \widetilde{S}_\Gamma E_D \widetilde{S}_\Gamma^{-1} \varphi = E_D \widetilde{S}_\Gamma^{-1} (I - P_D^{\mathrm{T}}) \widetilde{S}_\Gamma (I - P_D) \widetilde{S}_\Gamma^{-1} \varphi \\ &= E_D \widetilde{S}_\Gamma^{-1} P_D^{\mathrm{T}} \widetilde{S}_\Gamma P_D \widetilde{S}_\Gamma^{-1} \varphi - E_D P_D \widetilde{S}_\Gamma^{-1} \varphi + E_D \widetilde{S}_\Gamma^{-1} (I - P_D^{\mathrm{T}}) \varphi \end{aligned}$$

where the first term on the right-hand side equals $\mu\psi$, since $P_D^{\mathrm{T}} \widetilde{S}_\Gamma P_D \widetilde{S}_\Gamma^{-1} \varphi = \mu\varphi$. The second term vanishes since $E_D P_D = 0$. In the third term, $(I - P_D^{\mathrm{T}})\varphi$ vanishes since $\varphi \in \mathit{range}\,(P_D^{\mathrm{T}})$. We can therefore conclude that μ is also an eigenvalue of the operator (37) if $\psi \neq 0$. If $\psi = E_D \widetilde{S}_\Gamma^{-1} \varphi = 0$, we find, from $E_D = I - P_D$, that $P_D \widetilde{S}_\Gamma^{-1} \varphi = \widetilde{S}_\Gamma^{-1} \varphi$. Therefore,

$$P_D^{\mathrm{T}} \widetilde{S}_\Gamma P_D \widetilde{S}_\Gamma^{-1} \varphi = P_D^{\mathrm{T}} \varphi = \mu\varphi$$

Since P_D^{T} is a projection, and $\varphi \in \mathit{range}\,(P_D^{\mathrm{T}})$, $P_D^{\mathrm{T}}\varphi$ does not vanish and μ must equal 1. This shows that any non-zero eigenvalue, which differs from 1, of the operator (36) is also an eigenvalue of the operator (37). □

Remark 3
Brenner and Sung [13] have recently established that any common eigenvalue, not equal to 0 or 1, have the same multiplicity. In addition, they give an example for which the eigenvalues of the FETI-DP operator all exceed 1 while the BDDC operator has an eigenvalue 1. They also give a new proof that all eigenvalues are $\geqslant 1$, cf. References [18, 20].

6. NUMERICAL EXPERIMENTS

Laplace's equation with Dirichlet boundary conditions was solved on a square domain and with bilinear finite elements. A preconditioned conjugate gradient method was used to solve the interface problems of both the BDDC and the FETI-DP algorithms, and the iterations were stopped when the relative primal residual

$$\frac{\|Au - f\|_2}{\|f\|_2} \leqslant 10^{-6}$$

Here A, u, and f are the global assembled stiffness matrix, solution vector, and load vector, respectively. In our implementation of the BDDC and FETI-DP algorithms, the subdomain variables have always been changed to create explicit degrees of freedom corresponding to the edge average constraints. In all the tables below, the condition numbers are estimated by using the smallest and largest eigenvalues obtained from the tridiagonal Lanczos matrix generated by the preconditioned conjugate gradient iterations.

Tables I and II show condition number bounds and iteration counts of the BDDC algorithm, implemented as in of (31), for varying numbers of subdomains and for different subdomain sizes. Here we have used different sets of coarse level primal variables: either only the subdomain corner variables, or only the subdomain edge average variables, or both the subdomain corner and edge average variables. The corresponding results for the BDDC algorithm formulated as in Reference [25] are also shown in parentheses. We see that both the condition number bounds

Table I. Condition number bounds (κ) and iteration counts (*iter.*) of BDDC for changing number of subdomains and $H/h = 8$.

	Corners + Edges		Edges only		Corners only	
Num. of subs	κ	*iter.*	κ	*iter.*	κ	*iter.*
4×4	1.2 (1.2)	5 (4)	1.7 (1.7)	6 (7)	2.7 (2.8)	8 (8)
8×8	1.3 (1.3)	5 (5)	1.8 (1.8)	7 (8)	3.0 (3.1)	10 (12)
12×12	1.3 (1.2)	5 (4)	1.8 (1.8)	7 (8)	3.1 (3.1)	10 (13)
16×16	1.3 (1.2)	5 (4)	1.8 (1.8)	7 (8)	3.1 (3.2)	10 (13)
20×20	1.3 (1.2)	5 (4)	1.8 (1.8)	6 (8)	3.1 (3.2)	10 (13)

Table II. Condition number bounds (κ) and iteration counts (*iter.*) of BDDC for 4×4 subdomains and changing H/h.

	Corners + Edges		Edges only		Corners only	
H/h	κ	*iter.*	κ	*iter.*	κ	*iter.*
4	1.1 (1.1)	4 (4)	1.3 (1.3)	5 (6)	2.0 (2.1)	7 (7)
8	1.2 (1.2)	5 (4)	1.7 (1.7)	6 (7)	2.7 (2.8)	8 (8)
16	1.4 (1.4)	5 (5)	2.3 (2.3)	7 (7)	3.6 (3.7)	9 (9)
32	1.7 (1.7)	6 (6)	3.0 (3.1)	8 (8)	4.6 (4.7)	10 (10)

Table III. Eigenvalue bounds of BDDC and FETI-DP algorithms for changing number of subdomains and $H/h = 8$.

	Corners + Edges				Corners only			
	$\lambda_{\min}$		$\lambda_{\max}$		$\lambda_{\min}$		$\lambda_{\max}$	
Num. of subs	BDDC	DP	BDDC	DP	BDDC	DP	BDDC	DP
4×4	1.00	1.00	1.27	1.27	1.00	1.00	2.79	2.79
8×8	1.00	1.00	1.31	1.31	1.00	1.00	3.09	3.09
12×12	1.00	1.00	1.31	1.32	1.00	1.00	3.15	3.11
16×16	1.00	1.00	1.31	1.32	1.00	1.00	3.17	3.15
20×20	1.00	1.00	1.32	1.32	1.00	1.00	3.17	3.16

Table IV. Eigenvalue bounds of BDDC and FETI-DP algorithms for 4×4 subdomains and changing H/h.

	Corners + Edges				Corners only			
	$\lambda_{\min}$		$\lambda_{\max}$		$\lambda_{\min}$		$\lambda_{\max}$	
H/h	BDDC	DP	BDDC	DP	BDDC	DP	BDDC	DP
4	1.00	1.00	1.11	1.11	1.00	1.00	2.07	2.07
8	1.00	1.00	1.27	1.27	1.00	1.00	2.79	2.79
16	1.00	1.00	1.48	1.48	1.00	1.00	3.64	3.64
32	1.00	1.00	1.73	1.73	1.00	1.00	4.64	4.64

and the iteration counts closely match for these two formulations of the BDDC algorithm. The scalability of the BDDC algorithm can also be observed.

In Tables III and IV, we compare the eigenvalue bounds of the BDDC algorithm with those of the FETI-DP algorithm, for the operators given in Equations (31) and (19), respectively. To simplify the implementation, we only consider the cases where the corner variables are always in the coarse level primal set. We see that the lower and upper eigenvalue bounds of the two algorithms match very well, for different meshes and for different sets of coarse level variables.

7. CONCLUSIONS

In this paper, the BDDC algorithm and the FETI-DP algorithm with a Dirichlet preconditioner are written in terms of a partially subassembled Schur complement, its inverse, and certain simple restriction, extension, and scaling operators. We thus show that for both the FETI-DP and the BDDC preconditioners the essential step involves solving the same partially subassembled Schur complement using block Cholesky elimination. A change of variables is used so that each coarse level continuity constraint in the BDDC and FETI-DP algorithms corresponds to an explicit degree of freedom. This new and simple approach is then used to prove that the FETI-DP and BDDC algorithms, with the same sets of primal constraints, essentially have the same eigenvalues. Since this argument is purely algebraic it is equally valid for two and three

dimensions. We believe the success of this approach has also been demonstrated by several of our friends who have quickly implemented FETI-DP and BDDC algorithms on the basis of an early version of this paper.

Some related recent work includes a study of a primal version of the FETI-DP algorithm with a lumped preconditioner, by Fragakis and Papadrakakis [26]. In new work, by the authors of this paper, cf. Reference [27], we prove that the FETI-DP algorithm with a lumped preconditioner has essentially the same eigenvalues as those of a similar BDDC algorithm; the use of inexact subdomain solvers in the BDDC algorithm is also discussed. A study of the effect of inexact solvers in the FETI-DP algorithms by Klawonn and Rheinbach [28] has also appeared recently. Extensions of BDDC algorithms to incompressible elasticity and Stokes problems are considered in References [29, 30]. Finally, quite recently, Kim *et al*. [31] have developed a BDDC method for mortar finite elements, including methods for geometrically non-conforming domain decompositions, and established a quite complete theory that parallels that for conforming finite elements.

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