XXXII. *On the Numerical Solution of Linear Simultaneous Equations by an Iterative Method.*

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1. *Introduction.*

THE numerical solution of linear simultaneous equations, by methods essentially dependent on successive elimination or by determinants, becomes laborious, if the number of unknowns is large. In addition to the actual number of calculations involved, the solution often involves differences between quantities very nearly equal. In these cases all the calculations have to be taken to a large number of significant figures in order to get a few figures in the final result. Successive approximation methods developed originally by Gauss, Seidel, and Jacobi (1) for equations in their normal form have been used to overcome these difficulties. Morris (2) has recently put forward a simple tabular method of getting the approximations and found a condition that they should converge. Successive approximation methods have also been brought into special prominence lately by Southwell (3), who used his "relaxation" process to solve many differing types of problem whose solution depends ultimately on simultaneous equations.

The methods of successive approximation have, however, suffered from the serious disadvantage that unless the equations were first put into their normal form, the approximations did not necessarily converge, whilst the labour of putting equations into their normal form is considerable, and even then the convergence may be slow.

In the present paper theorems are given which enable the solution of any system of linear simultaneous equations to be found by the method of successive approximations, without the necessity of first putting the equations into their normal form.

2. *An Exact Expression for any of the Unknowns.*

Let the equations we are to solve be

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\ldots a_{1m}x_m-b_1=0, \\ a_{21}x_1+a_{22}x_2+\ldots a_{2m}x_m-b_2=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1+a_{m2}x_2\ldots a_{mm}x_m-b_m=0. \end{array}\right\} \quad (2.1)$$

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Denote the nth approximation to the value of any unknown x_r by $x_r^{(n)}$. To solve these equations by the method of successive approximations, we assume approximations $x_2^{(0)}, x_3^{(0)}, \ldots x_m^{(0)}$ to the unknowns $x_2, x_3, \ldots x_m$. Making use of these values, we then use the first equation to find $x_1^{(1)}$, the first approximation to x_1. As the approximations will not always converge to the value of the unknowns, we shall, in future, call the numbers $x_s^{(r)}$ iterates. Using the iterates $x_1^{(1)}, x_3^{(0)}, x_4^{(0)}, \ldots x_m^{(0)}$, we now use the second equation to find $x_2^{(1)}$, the first iterate to x_2. Using the iterates $x_1^{(1)}, x_2^{(1)}, x_4^{(0)}, \ldots x_m^{(0)}$, we now use the third equation to find $x_3^{(1)}$. Continuing in this way we find the first iterates to all of the unknowns. We now repeat the whole process and find the second and then the third iterates to the unknowns and so on.

If we use finite difference notation and denote $x_r^{(n+1)}$ by $\mathrm{E}x_r^{(n)}$ the equations to determine the $(n+1)$th iterate from the nth are

$$\left.\begin{array}{l}
a_{11}\mathrm{E}x_1^{(n)}+a_{12}x_2^{(n)}+a_{13}x_3^{(n)}+ \ldots\ldots\ldots +a_{1m}x_m^{(n)}-b_1=0,\\
a_{21}\mathrm{E}x_1^{(n)}+a_{22}\mathrm{E}x_2^{(n)}+a_{23}x_3^{(n)}+ \ldots\ldots\ldots +a_{2m}x_m^{(n)}-b_2=0,\\
a_{31}\mathrm{E}x_1^{(n)}+a_{32}\mathrm{E}x_2^{(n)}+a_{33}\mathrm{E}x_3^{(n)}+ \ldots\ldots +a_{3m}x_m^{(n)}-b_3=0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
a_{m1}\mathrm{E}x_1^{(n)}+a_{m2}\mathrm{E}x_2^{(n)}+a_{m3}\mathrm{E}x_3^{(n)}+ \ldots +a_{mm}\mathrm{E}x_m^{(n)}-b_m=0.
\end{array}\right\} \quad . \quad (2.2)$$

Hence to find $x_r^{(n)}$, we have a system of m linear simultaneous difference equations to solve. Solving these equations we have

$$\begin{vmatrix}
\mathrm{E}a_{11} & a_{12} & a_{13}\ldots & a_{1m}\\
\mathrm{E}a_{21} & \mathrm{E}a_{22} & a_{23}\ldots & a_{2m}\\
\mathrm{E}a_{31} & \mathrm{E}a_{32} & \mathrm{E}a_{33}\ldots & a_{3m}\\
\cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots\\
\mathrm{E}a_{m1} & \mathrm{E}a_{m2} & \mathrm{E}a_{m3}\ldots & \mathrm{E}a_{mm}
\end{vmatrix} x_r^{(n)}$$

$$= \begin{vmatrix}
\mathrm{E}a_{11} & a_{12}\ldots & a_{1\,\overline{r-1}} & b_1 & a_{1\,\overline{r+1}}\ldots & a_{1m}\\
\mathrm{E}a_{21} & \mathrm{E}a_{22}\ldots & a_{2\,\overline{r-1}} & b_2 & a_{2\,\overline{r+1}}\ldots & a_{2m}\\
\mathrm{E}a_{31} & \mathrm{E}a_{32}\ldots & a_{3\,\overline{r-1}} & b_3 & a_{3\,\overline{r+1}}\ldots & a_{3m}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\mathrm{E}a_{m1} & \mathrm{E}a_{m2}\ldots & \mathrm{E}a_{m\,\overline{r-1}} & b_m & \mathrm{E}a_{m\,\overline{r+1}}\ldots & \mathrm{E}a_{mm}
\end{vmatrix} ,1 \quad . \quad (2.3)$$

where the first determinant operates on $x_r^{(n)}$ and the second on unity. Since the result of operating on a constant by any integral power of E is the same constant, the E's may be omitted from the second determinant.

Hence to find $x_r^{(n)}$ we have to solve a difference equation

$$(p_m \mathrm{E}^m + p_{m-1}\mathrm{E}^{m-1} + \ldots + p_1 \mathrm{E}) x_r^{(n)} = \alpha_r, \quad . \quad . \quad . \quad (2.4)$$

where p_m, p_{m-1}, $\ldots p_1$, are the coefficients of the various powers of E in the first determinant and α_r is the value of the second determinant.

The solution of a difference equation of this type is known to be [4]

$$x_r^{(n)} = \frac{\alpha_r}{p_m + p_{m-1} + \ldots + p_1} + \mathrm{A}_1^{(r)} e_1^n + \mathrm{A}_2^{(r)} e_2^n + \ldots + \mathrm{A}_{m-1}^{(r)} e_{m-1}^n. \quad (2.5)$$

where e_1, e_2, $\ldots e_{m-1}$ are the roots (supposed distinct) of

$$p_m \mathrm{E}^{m-1} + p_{m-1}\mathrm{E}^{m-2} + \ldots + p_1 = 0, \quad . \quad . \quad . \quad . \quad (2.6)$$

and $\mathrm{A}_1^{(r)}$, $\mathrm{A}_2^{(r)}$, $\ldots \mathrm{A}_{m-1}^{(r)}$ are arbitrary functions of period unity and may here be taken to be constants. We will consider the case of equal roots later.

Now $p_m + p_{m-1} + \ldots + p_1$ is the value of the first determinant when $\mathrm{E}=1$, and hence any iterate $x_r^{(n)}$ is given by

$$x_r^{(n)} = x_r + \mathrm{A}_1^{(r)} e_1^n + \mathrm{A}_2^{(r)} e_2^n + \ldots + \mathrm{A}_{m-1}^{(r)} e_{m-1}^n, \quad . \quad . \quad . \quad . \quad (2.7)$$

x_r being the value of the unknown that we are seeking. Hence we see that even if we start with approximations very close to the actual values of the unknowns, the iterates will, usually, converge if, and only if, all of the roots e_1, e_2, $\ldots e_{m-1}$ are numerically less than unity.

We can, nevertheless, find the true value of any unknown from m successive iterates to that unknown. Dropping the suffix that denotes the rth unknown, we have

$$\left.\begin{array}{l} x^{(0)} = x + \mathrm{A}_1 + \mathrm{A}_2 + \ldots + \mathrm{A}_{m-1}, \\ x^{(1)} = x + \mathrm{A}_1 e_1 + \mathrm{A}_2 e_2 + \ldots + \mathrm{A}_{m-1} e_1, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x^{(m-1)} = x + \mathrm{A}_1 e_1^{\overline{m-1}} + \mathrm{A}_2 e_2^{\overline{m-2}} + \ldots + \mathrm{A}_{m-1} e_{n-1}^{\overline{m-1}}. \end{array}\right\} \quad . \quad . \quad (2.8)$$

If some of the roots e_1, e_2, $\ldots e_m$ are not distinct, suppose that there are s_1 roots e_1, s_2 roots e_2, $\ldots$, s_t roots e_t. Then

$$s_1 + s_2 + s_3 + \ldots s_t = m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.9)$$

and some of s's may be unity.

The solution of the difference equation (2.3) is now

$$\begin{array}{l} x_r^{(n)} = x + \{\mathrm{A}_1 + (n+1)\mathrm{A}_2 + (n+1)^2 \mathrm{A}_3 + \ldots (n+1)^{\overline{s_1 - 1}} \mathrm{A} s_1\} e_1^n \\ \quad + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \quad + \{\mathrm{L}_1 + (n+1)\mathrm{L}_2 + (n+1)^2 \mathrm{L}_3 + \ldots (n+1)^{\overline{s_t - 1}} \mathrm{L} s_t\} e_t^n, \end{array} \quad . \quad (2.10)$$

and we can write down the expressions for $x_r^{(0)}$, $x_r^{(1)}$, $\ldots x_r^{(m-1)}$.

Whether the roots are distinct or not, multiply the equation for $x^{(0)}$ by p_1, the equation for $x^{(1)}$ by p_2, etc. and add the resulting equations.

This operation results in eliminating both the arbitrary constants and the roots and we deduce the following theorem.

If $x^{(0)}$, $x^{(1)}$, . . . $x^{(m-1)}$ are m successive iterates to the value of any unknown and if

$$p_m E^m + p_{m-1} E^{m-1} + \ldots + p_2 E^2 + p_1 E = 0 \quad \ldots \quad (2.11)$$

is the expansion of the determinantal equation

$$\begin{vmatrix} Ea_{11} & a_{12} & \ldots & a_{1m} \\ Ea_{21} & Ea_{22} & \ldots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ Ea_{m1} & Ea_{m2} & \ldots & a_{mm} \end{vmatrix} = 0 \quad \ldots \quad (2.12)$$

in powers of E, then the true value of the unknown is

$$\frac{p_m x^{(m-1)} + p_{m-1} x^{(m-2)} + \ldots + p_2 x^{(1)} + p_1 x^{(0)}}{p_m + p_{m-1} + \ldots + p_2 + p_1} \quad \ldots \quad (2.13)$$

It may be noted that if

$$p_m + p_{m-1} + \ldots + p_1 = 0, \quad \ldots \quad (2.14)$$

the original equations (2.1) are either inconsistent or not independent.

3. *An Example.*

A simple example will illustrate the theorem. To solve

$$x + 4y - 10z = 1, \quad \ldots \quad (3.1)$$

$$2x + 3y + 8z = 20, \quad \ldots \quad (3.2)$$

$$3x + 5y + 2z = 21, \quad \ldots \quad (3.3)$$

for which $x=3$, $y=2$, $z=1$.

If we take first approximations $y^{(0)}=0$, $z^{(0)}=0$.

(1) gives $x^{(1)} = 1 - 4y^{(0)} + 10z^{(0)}$, hence $x^{(1)} = 1$. . . (3.4)

(2) gives $3y^{(1)} = 20 - 2x^{(1)} - 8z^{(0)}$, hence $y^{(1)} = 6$. . . (3.5)

(3) gives $2z^{(1)} = 21 - 3x^{(1)} - 5y^{(1)}$, hence $z^{(1)} = -6$. . . (3.6)

(1) gives $x^{(2)} = 1 - 4y^{(1)} + 10z^{(1)}$, hence $x^{(2)} = -83$. . (3.7)

(2) gives $3y^{(2)} = 20 - 2x^{(2)} - 8z^{(1)}$, hence $y^{(2)} = 78$. . . (3.8)

(3) gives $2z^{(2)} = 21 - 3x^{(2)} - 5y^{(2)}$, hence $z^{(2)} = -60$. . (3.9)

(1) gives $x^{(3)} = 1 - 4y^{(2)} + 10z^{(2)}$, hence $x^{(3)} = -911$. . (3.10)

The equation for E is

$$\begin{vmatrix} E & 4 & -10 \\ 2E & 3E & 8 \\ 3E & 5E & 2E \end{vmatrix} = 0, \quad \ldots \quad (3.11)$$

i. e.,

$$E^3 - 11E^2 + 16E = 0. \quad \ldots \quad (3.12)$$

Hence
$$x=\{x^{(3)}-11x^{(2)}+16x^{(1)}\}/6=3. \qquad (3.13)$$
$$y=\{y^{(2)}-11y^{(1)}+16y^{(0)}\}/6=2. \qquad (3.14)$$
$$z=\{z^{(2)}-11z^{(1)}+16z^{(0)}\}/6=1. \qquad (3.15)$$

4. *Rearrangement of the Equations.*

Equation (2.12) for E presupposes that given iterates $x_1^{(n)}, x_2^{(n)}, \ldots x_m^{(n)}$ the successive iterates are found in the order $x_1^{(n+1)}, x_2^{(n+1)}, \ldots, x_m^{(n+1)}$ from the equations (2.2) whose first terms begin with $a_{11}, a_{21}, \ldots a_{m1}$, respectively. Any rearrangement of this order will, in general, alter the equation for E, and hence the roots $e_1, e_2, \ldots e_{m-1}$. The rearrangement can be made simply so as to give an equation with roots $1/e_1, 1/e_2, \ldots 1/e_m$. For, if in the determinant (2.12) we put $\mathrm{E}=1/e$, and then multiply the first column by e^2 and the other columns by e, the roots of the resulting equation will be the reciprocals of those of the original one, except for the zero root. Now make the mth column, column 2, the $(m-1)$th column 3, ... the second column, column m; then invert the determinant so that the last row becomes the first, the $(m-1)$th the second, ... etc. The resulting determinantal equation is

$$\begin{vmatrix} ea_{m1} & a_{mm} & a_{m\,\overline{m-1}} & \cdots & a_{m2} \\ ea_{\overline{m-1}\,1} & ea_{\overline{m-1}\,m} & ea_{\overline{m-1}\,\overline{m-1}} & \cdots & a_{\overline{m-1}\,2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ ea_{11} & ea_{1m} & ea_{1\,\overline{m-1}} & \cdots & a_{12} \end{vmatrix}=0, \qquad (4.1)$$

and it may be seen to be of the same form as (2.12). Hence, to get iterates involving roots the reciprocals of the earlier ones, we use the equations (2.2) in the rearranged order.

$$\begin{aligned} a_{m1}\,x_1+a_{mm}\,x_m+\ldots+a_{m2}\,x_2-b_m &=0, \\ a_{\overline{m-1}\,1}x_1+a_{\overline{m-1}\,m}x_m+\ldots+a_{\overline{m-1}\,2}x_2-b_{\overline{m-1}} &=0, \\ \cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots & \\ a_{11}\,x_1+a_{1m}\,x_m+\ldots+a_{12}\,x_2-b_1 &=0. \end{aligned} \qquad (4.2)$$

In the example given above the non-zero roots of the determinantal equation (3.11) are 1·72 and 9·27. If therefore we rearrange the equation so as to get roots the reciprocals of these, the successive iterates will converge and there is no need to use the theorem of paragraph 2 in order to get an approximation to the values of the unknowns. Rearranging the equations, we get

$$3x+2z+5y-21=0. \qquad (4.3)$$
$$2x+8z+3y-20=0. \qquad (4.4)$$
$$x-10z+4y-1=0. \qquad (4.5)$$

If we take first approximations $z^{(0)}=0$, $y^{(0)}=0$,

(3) gives $x^{(1)}=7$	(4) gives $z^{(1)}=0\cdot75$	(5) gives $y^{(1)}=0\cdot375$
(3) gives $x^{(2)}=5\cdot875$	(4) gives $z^{(2)}=0\cdot89$	(5) gives $y^{(2)}=1\cdot01$
(3) gives $x^{(3)}=4\cdot72$	(4) gives $z^{(3)}=0\cdot94$	(5) gives $y^{(3)}=1\cdot42$
(3) gives $x^{(4)}=4\cdot01$	(4) gives $z^{(4)}=0\cdot96$	(5) gives $y^{(4)}=1\cdot65$
(3) gives $x^{(5)}=3\cdot61$	(4) gives $z^{(5)}=0\cdot98$	(5) gives $y^{(5)}=1\cdot80$
(3) gives $x^{(6)}=3\cdot35$	(4) gives $z^{(6)}=0\cdot99$	(5) gives $y^{(6)}=1\cdot89$
(3) gives $x^{(7)}=3\cdot19$	(4) gives $z^{(7)}=0\cdot99$	(5) gives $y^{(7)}=1\cdot93$
(3) gives $x^{(8)}=3\cdot12$	(4) gives $z^{(8)}=1\cdot00$	(5) gives $y^{(8)}=1\cdot97$
(3) gives $x^{(9)}=3\cdot05$	(4) gives $z^{(9)}=1\cdot00$	(5) gives $y^{(9)}=1\cdot99$
(3) gives $x^{(10)}=3\cdot02$	(4) gives $z^{(10)}=1\cdot00$	(5) gives $y^{(10)}=2\cdot00$

5. *Method of Finding the Unknowns Approximately.*

In the example just given the successive iterates diverged when we found them in one order and converged when we used another order of finding them. In general, when we have m unknowns, the nth iterate to any unknown x is

$$x+A_1e_1^n+A_2e_2^n+\ldots+A_{m-1}e_{m-1}^n, \qquad (5.1)$$

in which some of the e's are less and some greater than unity. After a few iterations the values of e_r much less than unity will hardly affect the results. In that case the nth iterate will be

$$x+A_1e_1^n+A_2e_2^n+\ldots+A_pe_p^n. \qquad (5.2)$$

where $p\leqslant m$, and if the order of finding the unknowns is chosen carefully, p will often be 0, 1, 2, or 3, even with equations involving a large number of unknowns. At the stage at which the above expression begins to represent the iterate approximately, we shall change our notation and call the iterate $y^{(0)}$, and the subsequent ones $y^{(1)}$, $y^{(2)}$, etc. Hence

$$\begin{aligned} y^{(0)}&=x+B_1\;+B_2\;+\ldots+B_p,\\ y^{(1)}&=x+B_1e_1+B_2e_2+\ldots+B_pe_p,\\ y^{(2)}&=x+B_1e_1^2+B_2e_2^2+\ldots+B_pe_p^2,\\ &\cdots\cdots\cdots\cdots\\ &\cdots\cdots\cdots\cdots\\ y^{(p)}&=x+B_1e_1^p+B_2e_2^p+\ldots+B_pe_p^p. \end{aligned} \qquad (5.3)$$

Let the equation whose roots are $e_1, e_2, \ldots e_p$ be

$$q_pE^p+q_{p-1}E^{p-1}+\ldots+q_1E+q_0=0. \qquad (5.4)$$

Multiply the first equation by q_0, the second by q_1, etc., and add all resulting equations. We get

$$q_0(x-y^{(0)}+q_1(x-y^{(1)})+\ldots+q_p(x-y^{(p)})=0.$$

Similarly,

$$\left.\begin{array}{l} q_0(x-y^{(1)})+q_1(x-y^{(2)})+\ldots\ldots+q_p(x-y^{(p+1)})=0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ q_0(x-y^{(p)})+q_1(x-y^{(p+1)})+\ldots+q_p(x-y^{(2p)})=0. \end{array}\right\} \quad \ldots \quad (5.5)$$

Hence

$$\begin{vmatrix} x-y^{(0)} & x-y^{(1)} & \ldots\ldots & x-y^{(p)} \\ x-y^{(1)} & x-y^{(2)} & \ldots\ldots & x-y^{(p+1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x-y^{(p)} & x-y^{(p+1)} & \ldots & x-y^{(2p)} \end{vmatrix}=0. \quad \ldots \quad (5.6)$$

Denote $y^{(1)}-y^{(0)}$ by $\Delta y^{(0)}$, $y^{(2)}-y^{(1)}$ by $\Delta y^{(1)}$, etc., $\Delta y^{(1)}-\Delta y^{(0)}$ by $\Delta^2 y^{(0)}$, $\Delta y^{(2)}-\Delta y^{(1)}$ by $\Delta^2 y^{(1)}$ etc.

In the above determinant subtract the pth column from the $(p+1)$th, then subtract the $(p-1)$th column from the pth, then subtract the $(p-2)$th column from the $(p-1)$th, etc. Then perform an exactly similar operation on the rows. We get on solving for x and rearranging the terms

$$x=\frac{\begin{vmatrix} y^{(0)} & y^{(1)} & \ldots\ldots & y^{(p)} \\ y^{(1)} & y^{(2)} & \ldots\ldots & y^{(p+1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y^{(p)} & y^{(p+1)} & \ldots & y^{(2p)} \end{vmatrix}}{\begin{vmatrix} \Delta^2 y^{(0)} & \Delta^2 y^{(1)} & \ldots & \Delta^2 y^{(p-1)} \\ \Delta^2 y^{(1)} & \Delta^2 y^{(2)} & \ldots & \Delta^2 y^{(p)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \Delta^2 y^{(p-1)} & \Delta^2 y^{(p)} & \ldots & \Delta^2 y^{(2p-2)} \end{vmatrix}} \quad \ldots\ldots \quad (5.7)$$

Another way of writing this result is

$$x=\frac{\begin{vmatrix} y^{(0)} & \Delta y^{(0)} & \Delta^2 y^{(0)} & \ldots\ldots & \Delta^p y^{(0)} \\ \Delta y^{(0)} & & \Delta^2 y^{(0)} & \ldots\ldots & \Delta^{p+1} y^{(0)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta^p y^{(0)} & \Delta^{(p+1)} y^{(0)} & \Delta^{(p+2)} y^{(0)} & \ldots & \Delta^{(2p)} y^{(0)} \end{vmatrix}}{\begin{vmatrix} \Delta^2 y^{(0)} & \Delta^3 y^{(0)} & \ldots\ldots & \Delta^{(p+1)} y^{(0)} \\ \Delta^3 y^{(0)} & \Delta^4 y^{(0)} & \ldots\ldots & \Delta^{(p+2)} y^{(0)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \Delta^{(p+1)} y^{(0)} & \Delta^{(p+2)} y^{(0)} & \ldots & \Delta^{(2p)} y^{(0)} \end{vmatrix}}. \quad \ldots \quad (5.8)$$

The above theorems are, of course, accurate if we take all of the roots of the " E " equation into account, *i. e.*, if $p=m$, but usually it only pays to use them when we may neglect a number of the roots. To illustrate (5.8) we will apply it to the problem previously solved. In paragraph 4 iterates were found for the unknowns, the roots of the corresponding determinantal equation being 0·58 and 0·11. After a few iterations we may expect that the latter root will hardly affect the results. Hence $p=1$ and

$$x \sim \frac{\begin{vmatrix} x^{(3)} & x^{(4)} \\ x^{(4)} & x^{(4)} \end{vmatrix}}{\Delta^2 x^{(3)}} = \frac{0{\cdot}96}{0{\cdot}31} \sim 3{\cdot}10. \quad . \quad . \quad . \quad . \quad . \quad (5.9)$$

Similarly for the other unknowns.

In order to make use of the results just given it is best to arrange the equations so that r is small and then the evaluation of determinants of high order can be avoided. This is done by arranging the equations so that, as far as possible, the largest coefficient in any equation lies on the leading diagonal. If the coefficients in any equation are nearly equal, large diagonal terms can often be arranged by suitable additions and subtractions of the original equations.

6. *A Criterion showing when the Method of Approximation may be used.*

It is of importance to know when the approximate formula may be used. Fortunately a simple criterion can be found. Eliminating q_0, q_1, q_2, . . . and x from the equations (5.5), we get after a simple transformation

$$\begin{vmatrix} \Delta y^{(0)} & \Delta y^{(1)} & \Delta y^{(2)} \ldots\ldots & \Delta y^{(p)} \\ \Delta y^{(1)} & \Delta y^{(2)} & \Delta y^{(3)} \ldots\ldots & \Delta y^{(p+1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \Delta y^{(p)} & \Delta y^{(p+1)} & \Delta y^{(p+2)} \ldots & \Delta y^{(2p)} \end{vmatrix} = 0, \quad . \quad . \quad . \quad . \quad . \quad (5.10)$$

which is the required criterion. In a numerical example, the determinant will generally not vanish, but will be small compared to its constituent terms.

Some Particular Cases.

Some particular cases of the theorems of the last two paragraphs are worth noting.

If $p=1$, equation (5.7) becomes

$$x = y^{(0)} - \frac{\{\Delta y^{(0)}\}^2}{\Delta^2 y^{(0)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.11)$$

while the criterion for its use becomes

$$\frac{\Delta y^{(0)}}{\Delta y^{(1)}}=\frac{\Delta y^{(1)}}{\Delta y^{(2)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.12)$$

If the corresponding root of the determinantal equation is known, we have the relation

$$x=y^{(0)}-\frac{\Delta y^{(0)}}{e-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.13)$$

If $p=2$, equation (5.7) becomes

$$x=y^{(0)}-\frac{\{\Delta y^{(0)}\}^2}{\Delta^2 y^{(0)}}-\frac{\{\Delta y^{(0)}\Delta y^{(2)}-[\Delta y^{(1)}]^2\}^2}{\Delta^2 y^{(0)}\{\Delta^2 y^{(0)}\Delta^2 y^{(2)}-[\Delta^2 y^{(1)}]^2\}}. \quad . \quad . \quad (5.14)$$

6. *Another Method of Approximation.*

In the previous paragraph the finding of the unknown made use of the iterates to only one unknown. A method of making use of the iterates to several of the unknowns, which is usually more useful, will now be given.

In paragraph 5 it was shown that

$$x_r=\frac{q_0 y_r^{(0)}+q_1 y_r^{(1)}+\ . \ . \ . \ +q_p y_r^{(p)}}{q_0+q_1+\ . \ . \ . \ +q_p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.1)$$

Subtracting the second equation in (5.6) from the first we get

$$q_0\Delta y_r^{(0)}+q_1\Delta y_r^{(1)}+\ . \ . \ . \ +q_p\Delta y_r^{(p)}=0. \quad . \quad . \quad . \quad . \quad (6.2)$$

Now, if $x_r^{(n)}$ depends, approximately, on the roots $e_1, e_2, \ldots e_p$, only, then the iterates to other unknowns y_e, etc., will usually depend on these same roots only. Hence

$$q_0\Delta y_e^{(0)}+q_1\Delta y_e^{(1)}+\ . \ . \ \ +q_p\Delta y_e^{(n)}=0. \quad . \quad . \quad . \quad . \quad (6.3)$$

Now if we have p equations of this type, we can solve for q_0, q_1, etc., and substituting in (6.1) we can find x_r and similarly the other unknowns.

It is the first differences of the iterates which are first obtained in Morris's [2] tabular method, and the equations (6.3) can therefore be easily written down.

A criterion telling us if this theorem can be used is clearly found by eliminating $q_0, q_1, \ldots q_p$, from equations of the form (6.4). In practice, however, it is found more convenient to solve p of the equations for q_0, q_1, q_2, etc., and to find if the other equations are satisfied approximately.

7. *Application of Previous Theory to an Example.*

As an application of the foregoing theory we shall solve a set of simultaneous equations, which occurred in an aeronautical problem, and is taken from a paper by Winny [5]. The equations are given in tabular form in Table I., A. B. C. etc. being the unknowns. The elements of

TABLE I.

Coefficients of A.	B.	C.	D.	E.	F.	G.	H.	Constants.	
74·7	126·6	24·95	450	33·5	57·6	107	186	−570	(7·1)
93·1	83·0	31·1	−122·5	42·6	41·5	15·35	59·2	−367	(7·2)
62·4	−14·7	−35·1	37·2	22·75	− 5·77	−12·85	12·9	−169	(7·3)
−18·8	−15·1	−15·3	− 14·7	5·71	− 5·92	5·99	− 5·54	− 18·6	(7·4)
− 0·269	− 0·902	− 2·33	− 4·82	0·605	2·84	7·85	16·5	0	(7·5)
− 0·499	− 0·962	− 0·499	2·410	0·905	2·58	1·48	− 7·33	0	(7·6)
− 0·437	0·237	0·835	− 1·145	0·870	− 0·765	− 3·16	4·58	0	(7·7)
− 0·210	0·445	− 0·695	0·870	0·304	− 1·14	2·269	− 3·17	0	(7·8)

TABLE II.

	Coefficients of								Constants.
	A.	B.	C.	D.	E.	F.	G.	H.	
	1	—·2356	—·5625	·5962	·3646	—·0925	— ·2059	·2067	—2·7083
	·4903	1	·0044	— ·8224	·2754	·4712	1·0298	·6208	—2·9134
	—·0741	—·3778	1	—1·0037	—·0694	—·1480	·3805	—·3644	1·3975
	·1660	·2813	·0554	1	·0744	·1280	·2378	·4133	—1·2667
	—·5368	—·0146	·1614	·3357	1	—·0305	— ·8718	—·4544	0
	—·0163	—·3811	·1833	·1379	·0611	1	— ·6292	—·2037	0
	—·0687	·0689	—·3024	— ·0245	·1107	—·1490	1	·0339	0
	—·0163	—·0547	—·1412	— ·2921	·0367	·1721	·4758	1	0
Check sum (S)	—·0559	—·7134	—·6015	—1·0730	·8536	·3513	— ·5832	·2522	

the table under each letter are the coefficients of that unknown in the various equations. The resulting terms in each row are to be added, and together with the constant equated to zero, forming the eight equations (7.1), (7.2), . . . (7.8).

TABLE III.

A.	B.	C.	D.	E.	F.	G.	H.	Check sums (C).
2·7083	2·9134	−1·3975	1·2667	0	0	0	0	
	−1·3279	·2007	− ·4496	1·4538	·0441	·1861	·0441	·1513
·3736	**1·5855**	·5990	− ·4460	·0231	·6043	−·1092	·0867	1·1315
− ·3363	·0026	− **·5978**	·0331	·0965	·1096	−·1808	−·0844	−·3597
− ·2409	·3323	·4056	**·4041**	− ·1357	−·0557	·0099	·1181	·4336
− ·5242	− ·3959	·0998	− ·1070	**1·4377**	−·0878	−·1592	−·0528	−1·2271
·0568	− ·2895	·0909	− ·0786	·0187	**·6143**	·0915	−·1057	− ·2159
− ·0333	·0048	·0615	·0385	− ·1410	−·1018	−**·1617**	·0769	− ·0944
− ·0171	− ·0515	·0302	− ·0343	·0377	·0169	−·0028	**·0829**	− ·0209
− **·7214**	·3537	− ·0534	·1198	− ·3872	−·6118	−·0496	−·0118	− ·0403
− ·0102	− **·0435**	− ·0164	·0122	− ·0006	−·0166	·0030	−·0024	− ·0310
·3477	− ·0027	**·6182**	− ·0342	−·0998	−·1133	·1869	·0873	·3719
·0498	− ·0688	− ·0839	− **·0836**	·0281	·0115	−·0020	−·0244	− ·0897
·1984	·1498	− ·0378	·0405	− **·5441**	·0332	·0602	−·0200	·4643
− ·0168	·0857	− ·0269	·0233	− ·0055	−**·1819**	−·0271	·0313	·0640
·0347	− ·0050	− ·0642	− ·0401	·1470	·1061	**·1686**	·0802	·0983
− ·0041	− ·0123	·0072	− ·0082	·0090	·0040	−·0007	**·0198**	− ·0051
·5995	− ·2939	·0444	− ·0995	·3218	·0098	·0412	·0098	1·0336
− ·0347	− **·1472**	− ·0556	·0414	− ·0021	−·0561	·0101	−·0081	− ·1051
− ·1220	·0010	− **·2168**	·0120	·0350	·0397	−·0656	−·0306	− ·1305
·0182	− ·0252	− ·0307	− **·0306**	·0103	·0042	−·0007	−·0089	− ·0328
·1880	− ·1420	·0358	− ·0384	**·5155**	−·0315	−·0571	−·0189	− ·4401
·0070	− ·0359	·0113	− ·0098	·0023	**·0762**	·0114	−·0131	− ·0268
− ·0126	·0018	·0234	·0146	− ·0535	−·0386	−**·0614**	·0292	− ·0357
·0084	·0252	− ·0148	·0168	− ·0184	−·0083	·0014	−**·0406**	·0103
− **·3237**	·1587	− ·0240	·0537	− ·1738	−·0053	−·0222	−·0053	− ·0181
− ·0039	− **·0164**	− ·0062	·0046	− ·0002	−·0062	·0011	−·0009	− ·0117
− ·0029	·0000	− **·0052**	·0003	·0008	·0010	−·0016	−·0007	− ·0031
− ·0249	·0344	·0420	**·0418**	− ·0140	−·0058	·0010	·0122	·0449
·0936	·0707	− ·0178	·0191	− **·2568**	·0157	·0284	·0094	·2191
− ·0044	·0224	− ·0070	·0061	− ·0014	−**·0475**	−·0071	·0082	·0168
·0002	·0000	− ·0004	− ·0002	·0009	·0006	**·0010**	−·0005	·0006
− ·0046	− ·0139	·0082	− ·0093	·0102	·0046	−·0008	**·0224**	− ·0056
·0531	− ·0260	·0039	− ·0088	·0285	·0009	·0036	·0009	·0031
·0206	**·0876**	·0331	− ·0246	·0013	·0334	−·0060	·0048	·0626
·0349	− ·0003	**·0620**	− ·0034	− ·0100	−·0114	·0187	·0088	**·0373**
·0126			− **·0211**	·0071	·0029	−·0005	−·0062	
− ·0133				**·0366**	−·0022	−·0041	−·0013	
·0027					**·0288**	·0043	−·0050	
·0031						**·0152**	−·0072	
·0011							−**·0052**	
·0619								

Equation (1) has the coefficient of D greater than the other coefficients in the equation and will therefore be taken as the new fourth equation.

Equation (3) will be taken as the new first and (5) as the new eighth, (2)−2(4) will be taken as the new second, −(3)+3(4) as the new third, (6)+2(7)+(8) as new fifth, (6)−2(8) as new sixth, (5)+5(8) as new seventh.

Dividing each of these equations by the coefficient of the unknown in the leading diagonal term, the equationsresulting are those given in Table II.

The approximations are then found as explained by Morris [2], except that some additional calculations are made for checking purposes. The sum of the coefficients S of each unknown (*i. e.* the elements of any column), except for unity in the leading diagonal, is found and the number is placed in that column under the coefficients. The sum (C.) of the elements of any row, except for the actual iterate in that row, is also found. Then the product of S and the iterate should equal minus and check sum C. Thus the elements of each row in the calculation are checked as they are found. The results are given in Table III.

Hence the iterates to the unknowns are given by the following table :

TABLE IV.

	First.	Second.	Third.	Fourth.	Fifth.
A	2·7083	1·9869	2·5866	2·2627	2·3158
B	1·5855	1·5422	1·3948	1·3784	1·4661
C	−0·5978	0·0204	−0·01964	−0·2016	−0·1396
D	0·4041	0·3205	0·2899	0·3317	0·3106
E	1·4377	0·8936	1·4091	1·1523	1·1889
F	0·6143	0·4324	0·5086	0·4611	0·4899
G	−0·1617	0·0069	−0·0545	−0·0535	−0·0383
H	0·0829	0·1027	0·0621	0·0845	0·0793

It should be noted that we could have written down the iterates in Table IV. direct from Table II. as the iterates can be found directly with the aid of a calculating machine. This method is, however, more difficult to check.

We now take the differences of the iterates and see on how many roots of the " E " equation they depend. It is obvious that they depend on more than one, since

$$\frac{\Delta A^{(2)}}{\Delta A^{(3)}} \neq \frac{\Delta A^{(3)}}{\Delta A^{(4)}}.$$

If they depended on two, then

$$\begin{vmatrix} \Delta A^{(2)} & \Delta A^{(3)} & \Delta A^{(4)} \\ \Delta B^{(2)} & \Delta B^{(3)} & \Delta B^{(4)} \\ \Delta C^{(2)} & \Delta C^{(3)} & \Delta C^{(4)} \end{vmatrix}$$

would vanish. This is not so, but if the equations

$$\begin{array}{rrrr} 0\cdot5995\, q_0 & -0\cdot3237\, q_1 & +0\cdot0531\, q_2 & +0\cdot0619\, q_3=0 \\ 0\cdot6182\, q_0 & -0\cdot2168\, q_1 & -0\cdot0052\, q_2 & +0\cdot0620\, q_3=0 \\ -0\cdot5441\, q_0 & +0\cdot5155\, q_1 & -0\cdot2568\, q_2 & +0\cdot0366\, q_3=0 \end{array}$$

are solved, we get

$$q_0 : q_1 : q_2 \,.\, q_3 = 0\cdot0361 : 0\cdot2021 : 0\cdot3831 : 0\cdot3781.$$

The results satisfy approximately the equations

$$q_0\Delta B^{(2)}+q_1\Delta B^{(3)}+q_2\Delta B^{(4)}+q_3\Delta B^{(5)}=0. \text{ etc.}$$

We can choose any three of eight equations to find the ratios $q_0 : q_1 : q_2 : q_3$. The three equations (7.10) were selected as they have the largest coefficients and will therefore yield the most accurate values to these auxiliary unknowns.

Making use of (6.1), we find

$$\begin{array}{llll} A=2\cdot338 & B=1\cdot421 & C=-0\cdot169 & D=0\cdot315 \\ E=1\cdot209 & F=0\cdot481 & G=-0\cdot045 & H=0\cdot78. \end{array}$$

When these values are substituted into the original equations (7.1), the remainders are

$$0\cdot08,\ 0\cdot20,\ 0\cdot003,\ 0\cdot04,\ 0\cdot001,\ 0\cdot001,\ 0\cdot00001,\ 0\cdot00002.$$

Hence the solutions found are approximately correct.

If we nad evaluated the iterates to only three decimal places, found one iterate less to each of the unknowns, and solved for the ratios

$$q_0 : q_1 : q_2 : q_3,$$

we would have got

$$\begin{array}{lll} A=2\cdot33 & B=1\cdot41 & C=-0\cdot18 \\ D=0\cdot31 & E=1\cdot20 & F=0\cdot47 \\ G=-0\cdot05 & H=0\cdot08 & \end{array}$$

By evaluating the iterates to five decimal places and finding the first six iterates to the unknowns, it was found that

$$\begin{array}{lll} A=2\cdot3377 & B=1\cdot4204 & C=-0\cdot1702 \\ D=0\cdot3153 & E=1\cdot2069 & F=0\cdot4806 \\ G=-0\cdot0498 & H=0\cdot0786 & \end{array}$$

The writer is deeply indebted to Dr. W. G. Bickley for many valuable suggestions.

References.

(1) Whittaker and Robinson, The Calculus of Observations,' Section 130.
(2) J. Morris, Reports and Memoranda of the Aeronautical Research Committee No. 1711 ; and ' Journal of the Royal Aeronautical Society,' April 1935.
(3) Southwell, Proceedings of the Royal Society,' A, cli. (1935), etc.
(4) Milne-Thomson, Calculus of Finite Differences,' Cap. 8.
(5) Winny, 'Reports and Memoranda of the Aeronautical Research Committee,' **No. 1756.**