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European option pricing and hedging with both fixed and proportional transaction costs

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Abstract

In this paper we provide a systematic treatment of the utility based option pricing and hedging approach in markets with both fixed and proportional transaction costs: we extend the framework developed by Davis et al. (SIAM J. Control Optim., 31 (1993) 470) and formulate the option pricing and hedging problem. We propose and implement a numerical procedure for computing option prices and corresponding optimal hedging strategies. We present a careful analysis of the optimal hedging strategy and elaborate on important differences between the exact hedging strategy and the asymptotic hedging strategy of Whalley and Wilmott (RISK 7 (1994) 82). We provide a simulation analysis in order to compare the performance of the utility based hedging strategy against the asymptotic strategy and some other common strategies.

JEL classification: C61; G11; G13

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1. Introduction

The break-through in option valuation theory started with the publication of two seminal papers by Black and Scholes (1973) and Merton (1973). In both papers the authors introduced a continuous time model of a complete friction-free market where the price of a stock follows a geometric Brownian motion. They presented a self-financing, dynamic trading strategy consisting of a riskless security and a risky stock, which replicates the payoff of an option. Then they argued that the absence of arbitrage dictates that the option price be equal to the cost of setting up the replicating portfolio.

In the presence of transaction costs in capital markets the absence of arbitrage argument is no longer valid, since perfect hedging is impossible. Due to the infinite variation of the geometric Brownian motion, the continuous replication policy incurs an infinite amount of transaction costs over any trading interval no matter how small it might be. A variety of approaches have been suggested to deal with the problem of option pricing and hedging with transaction costs. A great deal of them are concerned with the 'financial engineering' problem of either replicating or super-replicating the option payoff. These approaches are mainly preference-free models where rehedging occurs at some discrete time intervals whether or not it is optimal in any sense. However, common sense tells us that an 'optimal' hedging policy should achieve the best possible tradeoff between the risk and the costs of replication. Recognizing the fact that risk preferences differ among individuals, the following conclusion becomes obvious: in pricing and hedging options one must consider the investor's attitude towards risk.

In modern finance it is customary to describe risk preferences by a utility function. Expected utility theory maintains that individuals behave as if they were maximizing the expectation of some utility function of the possible outcomes. Hodges and Neuberger (1989) pioneered the option pricing and hedging approach based on this theory. The key idea behind the utility based approach is the indifference argument: the writing price of an option is defined as the amount of money that makes the investor indifferent, in terms of expected utility, between trading in the market with and without writing the option. In many respects such an option price is determined in a similar manner to a certainty equivalent within the expected utility framework, which is a well grounded pricing principle in economics. The difference in the two trading strategies, with and without the option, is interpreted as 'hedging' the option.

The utility based approach proved to be probably the most successful approach to option hedging with transaction costs. Using simulation analysis, Mohamed (1994), Clewlow and Hodges (1997), and Martellini and Priaulet (2002) demonstrated that the utility based approach achieves excellent empirical performance judging against the best possible tradeoff between the risk and the costs of a hedging strategy.

Hodges and Neuberger (1989) introduced the approach with a fairly general transaction costs structure. However, they carried out computations of the optimal hedging strategies and option prices in a market with only proportional transaction costs, without really presenting the continuous time model and the numerical procedure. Davis et al. (1993) rigorously developed the model of Hodges and

Neuberger (1989) for a market with proportional transaction costs only. They showed that in this case the problem amounts to a *stochastic singular control* problem that was formulated by Davis and Norman (1990). They proved that the problem has a unique solution. They also proved the convergence of discretization schemes employed in the numerical procedure. Further contributions to the study of the utility based option pricing approach in a market with proportional transaction costs was made by Clewlow and Hodges (1997), Whalley and Wilmott (1997), Constantinides and Zariphopoulou (1999), Andersen and Damgaard (1999), and some others.

In practice, transactions often involve both fixed and proportional costs. However, very little has been done from both the theoretical and empirical sides for this transaction costs structure: Clewlow and Hodges (1997) presented the results of numerical computations of the optimal hedging strategy for a 3-period model in a market with both fixed and proportional transaction costs, without, again, really presenting the continuous time model and the numerical procedure for this case. Whalley and Wilmott (1994) provided an asymptotic analysis of the model of Hodges and Neuberger (1989) for any linear transaction costs structure, assuming that transaction costs are small. Martellini and Priaulet (2002) presented also the comparison of the performance of some simple hedging strategies in the presence of a fixed fee component. In this paper we attempt to fill this gap by providing a systematic treatment of the utility based option pricing and hedging approach in the market with both fixed and proportional transaction costs.

The introduction of fixed transaction costs in addition to proportional transaction costs makes the utility maximization/optimal portfolio selection problem more complicated. In this case, due to the presence of a fixed transaction fee irrespective of the size of transaction, the optimal control strategy is discontinuous as opposed to the case with proportional transaction costs only. Unlike the case with only proportional transaction costs where the optimal strategy is described by two free boundaries, the optimal strategy with both fixed and proportional transaction costs turns out to be characterized by four free boundaries. Moreover, the formulation of the optimal portfolio selection problem where each transaction has a fixed cost component requires application of *stochastic impulse control* theory[1] as opposed to stochastic singular control theory. Consequently, the solution procedure and numerical algorithm to compute the expected utility and optimal trading strategy with both fixed and proportional transaction costs are different from those with only proportional transaction costs. The reader is reminded that in the problem with only proportional transaction costs one uses the gradient constraints to detect the two free boundaries (see, for example, Davis et al. (1993) and Davis and Panas (1994)). In contrast, in the problem with both fixed and proportional transaction costs one needs to employ the maximum utility operator to detect two of the free boundaries and the value matching conditions to detect the other two free boundaries.

[1]For some applications of this theory to a consumption-investment problem, we refer the interested reader to, for example, Korn (1998) and Øksendal and Sulem (2002).

The paper is organized as follows. In Section 2 we generalize the framework developed by Davis et al. (1993) and formulate the option pricing and hedging problem in the market with both fixed and proportional transaction costs. In Section 3 we present the result of the asymptotic analysis provided by Whalley and Wilmott (1994) on the optimal hedging strategy in a market with both fixed and proportional transaction costs. In Section 4 we propose an original numerical procedure for computing option prices and corresponding optimal hedging strategies. The results of our numerical computations are presented in Section 5. Here we begin our presentation with a study of how the utility based option price depends on the level of the investor's risk aversion. Then we proceed to a detailed study of the optimal hedging strategy.

We consider the case of a holder of a short European call option and find that the utility based option price is always above the corresponding Black–Scholes price and is an increasing function of the option holder's risk aversion. As risk aversion decreases, the utility based option price approaches a horizontal asymptote located above the Black–Scholes price.

According to the utility based approach, the qualitative description of the optimal hedging strategy is as follows: do nothing when the hedge ratio lies within a so-called no transaction (NT) region and rehedge to the nearest target boundary inside the NT region as soon as the hedge ratio moves out of the NT region. Since hedging is widely used to reduce risk, the knowledge of the optimal hedging strategy in the presence of transaction costs is of great practical interest. With this in mind, we provide a careful analysis of the numerically computed optimal hedging strategy and find that it has two key elements: specific forms of the NT region and the region between the target boundaries, and a volatility adjustment. In particular, we focus our attention on the important differences between the optimal hedging strategies obtained using the exact numerics and the asymptotic analysis and propose the general specification of the optimal hedging strategy. To the best of the authors' knowledge, this is the first detailed study of the exact optimal hedging strategy with quantification of the sizes of the NT region and the region between the target boundaries, and the volatility adjustment.

In Section 6 we provide a simulation analysis in order to compare the performance of the numerically calculated optimal hedging strategy against the asymptotic strategy and some other common strategies. We find that the exact utility based hedging strategy outperforms all the others. The results of our simulation analysis highlight, among other things, the deficiencies of the asymptotic strategy. Section 7 summarizes the paper.

2. The formulation of the model

We consider a continuous-time economy, similar to that of Øksendal and Sulem (2002), with one risky and one risk-free asset. Let $(\Omega, \mathscr{F}, P)$ be a probability space with a given filtration $\{\mathscr{F}_t\}_{0 \leqslant t \leqslant T}$. The risk-free asset, which we will refer to as the bank account, pays a constant interest rate of $r \geqslant 0$, and, consequently, the evolution of the amount invested in the bank, $x(t)$, is given by the ordinary differential equation

$$\mathrm{d}x(t) = rx(t)\,\mathrm{d}t. \tag{1}$$

We will refer to the risky asset as the stock, and assume that the price of the stock, $S(t)$, evolves according to a geometric Brownian motion defined by

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}B(t), \tag{2}$$

where μ and σ are constants, and $B(t)$ is a one-dimensional $\mathscr{F}_t$-Brownian motion.

The investor holds $x(t)$ in the bank account and $y(t)$ shares of the stock at time t. We assume that a purchase or sale of ξ shares of the stock incurs transaction costs consisting of a sum of a fixed cost $k \geqslant 0$ (independent of the size of transaction) plus a cost $\lambda S(t)|\xi|$ proportional to the transaction ($\lambda \geqslant 0$). These costs are drawn from the bank account.

We suppose that at any time the investor can decide to transfer money from the bank account to the stock and conversely. The control of the investor is a pure *impulse control* $v = (\tau_1, \tau_2, \ldots; \xi_1, \xi_2, \ldots)$. Here $0 \leqslant \tau_1 < \tau_2 < \ldots$ are $\mathscr{F}_t$-stopping times giving the times when the investor decides to change his portfolio, and ξ_i are $\mathscr{F}_{\tau_i}$-measurable random variables giving the sizes of the transactions at these times. If such a control is applied to the system $(x(t), y(t))$, it assumes the form

$$\begin{aligned} x(\tau_{i+1}) &= x(\tau_{i+1}^-) - k - (\xi_{i+1} + \lambda|\xi_{i+1}|)S(\tau_{i+1}^-), \\ y(\tau_{i+1}) &= y(\tau_{i+1}^-) + \xi_{i+1}. \end{aligned} \tag{3}$$

Thus a positive value of ξ_{i+1} corresponds to buying shares of the stock, and conversely if ξ_{i+1} is negative.

The starting point for the utility based option pricing and hedging approach is to consider the optimal portfolio selection problem of the investor who faces transaction costs and maximizes expected utility of his terminal wealth. The investor has a finite horizon $[0, T]$ and it is assumed[2] that there are no transaction costs at terminal time T. We define the value function of the investor with no option liability at time t as

$$J_0(t, x(t), y(t), S(t)) = \max_{v \in \mathscr{A}(x,y,S)} E_t[U(x(T) + y(T)S(T))], \tag{4}$$

where $U(\cdot)$ is the investor's utility function and $\mathscr{A}(x, y, S)$ denotes the set of admissible controls available to the investor who starts at time t with an amount of x in the bank and y shares of the stock at price S.

The option contract is a cash settled European call with expiration time T, the strike K, and payoff $(S(T) - K)^+$ at expiration. Similarly to (4), the value function of the investor with option liability is defined by

$$J_w(t, x(t), y(t), S(t)) = \max_{v \in \mathscr{A}(x,y,S)} E_t[U(x(T) + y(T)S(T) - (S(T) - K)^+)]. \tag{5}$$

[2]This assumption is made for simplicity. In practice, there are two types of option settlement: either asset or cash. The type of option settlement affects, to some extent, the option price and hedging strategy.

Definition 1. The reservation write price of a European call option is defined as the compensation P such that

$$J_w(0, x + P, y, S) = J_0(0, x, y, S). \tag{6}$$

That is, the reservation write price, P, is the lowest price at which the investor is willing to sell an option, and where the investor is indifferent between the two alternatives: (i) a utility maximization problem where he trades in the riskless and risky assets only, and (ii) a utility maximization problem where the investor, in addition, sells an option at price P.

We denote the investor's optimal trading policy without option liability by $y_0(t)$. Similarly, the optimal trading policy of the investor with option liability is denoted by $y_w(t)$.

Definition 2. The option hedging strategy of the investor is defined as the difference, $y_w(t) - y_0(t)$, between the investor's trading strategies with and without option liability.

That is, option hedging is defined as the incremental trades caused by the presence of the option.

Remark 1. Hodges and Neuberger (1989) considered the special case of a 'risk-neutral' world where the stock drift is equal to the risk-free interest rate, $\mu = r$. In this case, without the option liability, the investor would choose not to invest in the stock at all. That is, $y_0(t) \equiv 0$, and $y_w(t)$ alone completely defines the option hedging strategy.

Now we turn on to the derivation of the quasi-variational inequalities satisfied by the investor's value functions.

In the framework of stochastic impulse control theory one assumes that the investor's portfolio space is divided into two disjoint regions: a *continuation region* and an *intervention region*. The intervention region is the region where it is optimal to make a transaction. We define the *intervention operator* (or the maximum utility operator) $\mathscr{M}$ by

$$\mathscr{M}J_j(t, x, y, S) = \max_{(x', y') \in \mathscr{A}(x, y, S)} J_j(t, x', y', S), \tag{7}$$

where j is 0 or w, and x' and y' are the new values of x and $y : y' = y + \xi$ and $x' = x - k - (\xi + \lambda|\xi|)S$, where ξ is the size of transaction. In other words, $\mathscr{M}J_j(t, x, y, S)$ represents the value of the strategy that consists in choosing the best transaction. The continuation region is the region where it is not optimal to rebalance the investor's portfolio. We define the continuation region D by

$$D = \{(t, x, y, S) : J_j(t, x, y, S) > \mathscr{M}J_j(t, x, y, S)\}. \tag{8}$$

Now, by giving heuristic arguments, we intend to characterize the value function and the associated optimal strategy: if for some initial point (t, x, y, S) the optimal strategy is to not transact, the utility associated with this strategy is $J_j(t, x, y, S)$. Choosing the best transaction and then following the optimal strategy gives the utility $\mathscr{M}J_j(t,x,y,S)$. The necessary condition for the optimality of the first strategy is $J_j(t,x,y,S) \geqslant \mathscr{M}J_j(t,x,y,S)$. This inequality holds with equality when it is optimal to rebalance the portfolio. Moreover, in the continuation region, the application of the dynamic programming principle gives $\mathscr{L}J_j(t,x,y,S)=0$, where the operator $\mathscr{L}$ is defined by

$$\mathscr{L}J_j(t,x,y,S) = \frac{\partial J_j}{\partial t} + rx\,\frac{\partial J_j}{\partial x} + \mu S\,\frac{\partial J_j}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 J_j}{\partial S^2}. \tag{9}$$

The subsequent theorem formalizes this intuition.

Theorem 1. *The value function J_j defined by either* (4) *or* (5) *is the unique viscosity solution of the quasi-variational Hamilton–Jacobi–Bellman inequalities* (*QVHJBI, or just QVI*)

$$\max\{\mathscr{L}J_j, \quad \mathscr{M}J_j - J_j\} = 0, \tag{10}$$

with a proper boundary condition

$$J_0(T,x,y,S) = U(x+yS),$$

$$J_w(T,x,y,S) = U(x+yS-(S-K)^+).$$

The proof of the existence of the solution can be made by following along the lines of the proof of Theorem 3.7 in Øksendal and Sulem (2002). In addition, the uniqueness of the solution can be proved in the same manner as in Theorem 3.8 (comparison of theorem with subsequent corollary) in Øksendal and Sulem (2002). The technical meaning of the existence and uniqueness of the solution is that this solution can be computed by standard discretization methods.

We further assume that the investor has the negative exponential utility function

$$U(z) = -\exp(-\gamma z), \quad \gamma > 0, \tag{11}$$

where γ is a measure of the investor's absolute risk aversion, which is independent of the investor's wealth. This choice of the utility function satisfies two very desirable properties: (i) the option hedging strategy is simple to illustrate and interpret, because it does not depend on the investor's holdings in the bank account, (ii) the computational effort needed to solve the problem is rather low. This particular choice of the utility function might seem restrictive. However, as it was conjectured by Davis et al. (1993) and shown in Andersen and Damgaard (1999), the option price is approximately invariant to the specific form of the investor's utility function, and mainly only the level of absolute risk aversion plays an important role.

It is easy to see from (1) and (3) that the amount $x(T)$ is given by

$$x(T) = \frac{x(t)}{\delta(t,T)} - \sum_{i=0}^{n} \frac{k + (\xi_i + \lambda|\xi_i|)S(\tau_i)}{\delta(\tau_i,T)}, \tag{12}$$

where $\delta(t,T)$ is the discount factor defined by

$$\delta(t,T) = \exp(-r(T-t)), \tag{13}$$

n is a random number of transactions in $[t, T]$, and $t \leqslant \tau_1 < \tau_2 < \cdots < \tau_n < T$. Therefore, taking into consideration the investor's utility function, we can write[3]

$$J_j(t, x, y, S) = \exp\left(-\gamma \frac{x}{\delta(t, T)}\right) Q_j(t, y, S), \tag{14}$$

where $Q_j(t, y, S)$ is defined by $Q_j(t, y, S) = J_j(t, 0, y, S)$. It means that the dynamics of y through time is independent of the total wealth. In other words, the choice in y is independent of x. This representation suggests transformation of (10) into the following QVI for the value function $Q_j(t, y, S)$:

$$\max\left\{\mathscr{D}Q_j, \max_{y' \in \mathscr{A}(y,S)} \exp\left(\gamma \frac{k + (y' - y + \lambda|y' - y|)S}{\delta(t, T)}\right) Q_j - Q_j\right\} = 0, \tag{15}$$

where y' is the new value of y, $\mathscr{A}(y, S)$ denotes the set of admissible controls available to the investor who starts at time t with 0 in the bank account and y shares of in the stock at price S, and the operator $\mathscr{D}$ is defined by

$$\mathscr{D}Q_j(t, y, S) = \frac{\partial Q_j}{\partial t} + \mu S \frac{\partial Q_j}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Q_j}{\partial S^2}. \tag{16}$$

This is an important simplification that reduces the dimensionality of the problem. Note that the function $Q_j(t, y, S)$ is evaluated in the three-dimensional space $[0, T] \times R \times R^+$.

In the absence of any transaction costs, the solutions for the optimal number of shares the investor would hold without and with option liability are given by (see, for example, Davis et al. (1993))

$$y_0^* = \frac{\delta(t, T)}{\gamma S} \frac{(\mu - r)}{\sigma^2}, \tag{17}$$

$$y_w^* = \frac{\delta(t, T)}{\gamma S} \frac{(\mu - r)}{\sigma^2} + \frac{\partial V}{\partial S}, \tag{18}$$

where V is the price of the option in a market with no transaction costs (that is, the Black–Scholes price) given by

$$V = SN(d_1) - K\delta(t, T)N(d_2), \tag{19}$$

where

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$

$$d_2 = d_1 - \sigma\sqrt{T - t} \tag{20}$$

and $N(\cdot)$ is the cumulative probability distribution function of a normal variable with mean 0 and variance 1. Note in particular that the definition of the option hedging strategy with transaction costs reduces to the Black–Scholes hedging

[3]See a similar discussion in Davis et al. (1993).

strategy in the absence of transaction costs:

$$y_w^* - y_0^* = \frac{\partial V}{\partial S}.$$

The numerical calculations show that in the presence of both fixed and proportional transaction costs the portfolio space is divided into three disjoint regions: *buy*, *sell* and NT, and the optimal policy is described by four boundaries. The buy and NT regions are divided by the lower NT boundary, and the sell and NT regions are divided by the upper NT boundary. If a portfolio lies in the buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the buy target boundary. Similarly, if a portfolio lies in the sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the sell target boundary. If a portfolio lies in the NT region, it is not adjusted.

For fixed values of μ, σ, r, γ, λ, and k all the NT and target boundaries are functions of the investor's horizon and the stock price and do not depend on the investor's holdings in the bank account, so that a possible description of the optimal policy may be given by

$$\begin{aligned} y &= y_u(t,S),\\ y &= y_u^*(t,S),\\ y &= y_l^*(t,S),\\ y &= y_l(t,S), \end{aligned} \tag{21}$$

where the first and the fourth equations describe the upper and the lower NT boundaries respectively, and the second and the third equations describe the sell and buy target boundaries.

If the function $Q_j(t,y,S)$ is known in the NT region, then

$$Q_j(t,y,S) = \begin{cases} \exp\left(\gamma \frac{k-(1-\lambda)S(y-y_u^*)}{\delta(t,T)}\right) Q_j(t,y_u^*,S), & \forall y(t,S) \geqslant y_u(t,S), \\ \exp\left(\gamma \frac{k+(1+\lambda)S(y_l^*-y)}{\delta(t,T)}\right) Q_j(t,y_l^*,S), & \forall y(t,S) \leqslant y_l(t,S). \end{cases} \tag{22}$$

This follows from the optimal transaction policy described above. That is, if a portfolio lies in the buy or sell region, then the investor performs the minimum transaction required to reach the closest target boundary.

Note that the reservation write price could be expressed by (follows from (6) and (14))

$$P = \frac{\delta(0,T)}{\gamma} \log\left(\frac{Q_w(0,y,S)}{Q_0(0,y,S)}\right). \tag{23}$$

Unfortunately, there are no closed-form solutions for the value functions Q_0 and Q_w. As a result, the solutions have to be obtained by numerical methods.

Remark 2. Note that a reservation write price depends, to some extent, on the investor's initial holdings in the stock. To avoid ambiguity, one usually assumes that the investor has no holdings in the stock at time 0.

3. The asymptotic analysis of Whalley and Wilmott

Recall that there are no explicit solutions for the utility based option pricing and hedging model with transaction costs. As a result, 'exact' solutions have to be obtained by numerical methods. However, the numerical methods are computationally rather hard. One of the alternatives to numerical methods is to obtain an asymptotic solution to a problem. In asymptotic analysis one studies the solution to a problem when some parameters in the problem assume large or small values.

Whalley and Wilmott (1994) provided an asymptotic analysis of the model of Hodges and Neuberger (1989) for any linear transaction costs structure, assuming that transaction costs are small. They showed that in the model with both fixed and proportional transaction costs the boundaries of the NT region, assuming $\mu = r$, are given by

$$y = \frac{\partial V}{\partial S} \pm A \tag{24}$$

and the target boundaries are given by

$$y = \frac{\partial V}{\partial S} \pm B. \tag{25}$$

To find A and B one needs to solve the following system of nonlinear equations:

$$\begin{aligned} AB(A+B) &= \frac{3\lambda\delta(t,T)\Gamma^2}{\gamma}, \\ (A-B)^3(A+B) &= \frac{12k\delta(t,T)\Gamma^2}{\gamma}, \end{aligned} \tag{26}$$

where Γ is the option gamma given by

$$\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{(T-t)}}. \tag{27}$$

Fig. 1 illustrates the Whalley and Wilmott asymptotic strategy.

4. The numerical procedure

The main objective of this section is to present numerical procedures for computing the investor's value functions and the corresponding optimal trading policies. In the very beginning it is important to note that these numerical procedures for a market with both fixed and proportional transaction costs are different from those for a market with only proportional transaction costs. The reader is reminded that the investor's value function in a market with only proportional transaction costs is characterized by variational HJB inequalities with two gradient constraints (see, for example, Davis and Norman (1990)). In addition, the investor's optimal trading policy is described by two free boundaries. Consequently, in the numerical

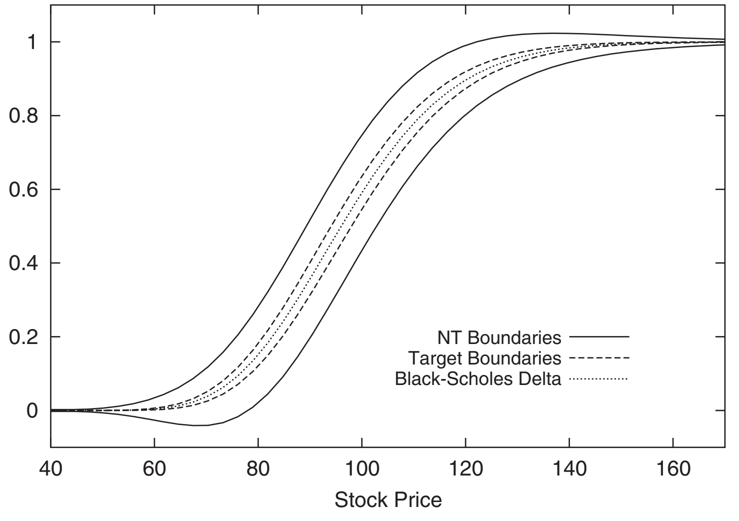

Fig. 1. The Whalley and Wilmott asymptotic strategy versus the Black–Scholes delta for the following model parameters: $\gamma = 1.0$, $\lambda = 0.01$, $k = 0.05$, $S_t = K = 100$, $\sigma = 0.25$, $\mu = r = 0.05$, and $T - t = 0.5$.

procedure one uses the two gradient constraints to detect the two free boundaries.[4] In contrast, the investor's value function in a market with both fixed and proportional transaction costs is characterized by quasi-variational HJB inequalities where one makes use of the maximum utility operator. Besides, the investor's optimal trading policy is described by four free boundaries. As a result, in the numerical procedure one needs to employ the maximum utility operator to detect two of the free boundaries and the value matching conditions to detect the other two free boundaries.

To find the solution of the continuous-time continuous-space stochastic control problem described by (15), we apply the method of the Markov chain approximation suggested by Kushner (see, for example, Kushner and Martins (1991) and Kushner and Dupuis (1992)). The basic idea involves a consistent approximation of the problem under consideration by a Markov chain, and then the solution of an appropriate optimization problem for the Markov chain model. First, according to the Markov chain approximation method, we construct discrete time approximations of the continuous time price processes presented in Section 2. Then the discrete time program is solved by using the discrete time dynamic programming algorithm (that is, the backward recursion algorithm).

Consider the partition $0 = t_0 < t_1 < \cdots < t_n = T$ of the time interval $[0, T]$ and assume that $t_i = i\Delta t$ for $i = 0, 1, \ldots, n$ where $\Delta t = T/n$. Let ε be a stochastic

[4]The discretization scheme for these variational HJB inequalities was first suggested by Davis et al. (1993). The practical implementation of this scheme was described in details in Davis and Panas (1994). Later on this scheme, with minor modifications in the practical realization, was successfully employed by Damgaard (2000), Monoyios (2004), and some others.

variable:

$$\varepsilon = \begin{cases} u & \text{with probability } p, \\ d & \text{with probability } 1-p. \end{cases}$$

We define the discrete time stochastic process of the stock as

$$S(t_{i+1}) = S(t_i)\varepsilon \tag{28}$$

and the discrete time discount factor

$$\delta(t_i, T) = \mathrm{e}^{-r(T-t_i)}. \tag{29}$$

If we choose $u = \mathrm{e}^{\sigma\sqrt{\Delta t}}$, $d = \mathrm{e}^{-\sigma\sqrt{\Delta t}}$, and $p = \frac{1}{2}\left[1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right]$, we obtain the binomial model proposed by Cox et al. (1979). An alternative choice is $u = \mathrm{e}^{(\mu-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}}$, $d = \mathrm{e}^{(\mu-\frac{1}{2}\sigma^2)\Delta t-\sigma\sqrt{\Delta t}}$, and $p = \frac{1}{2}$, which was proposed by He (1990). As n goes to infinity, the discrete time process (28) converges in distribution to its continuous counterpart (2). This is what is called the *local consistency conditions* for a Markov chain.

The following discretization scheme is proposed to find the value function $Q_j(t, y, S)$ defined by QVI (15):

$$\begin{aligned} Q_j^{\Delta t}(t_i, y, S) = \max\Bigg\{ & \max_m \exp\left(\gamma \frac{k + (1+\lambda)m\Delta y S}{\delta(t_i, T)}\right) Q_j^{\Delta t}(t_i, y + m\Delta y, S), \\ & \max_m \exp\left(\gamma \frac{k - (1-\lambda)m\Delta y S}{\delta(t_i, T)}\right) Q_j^{\Delta t}(t_i, y - m\Delta y, S), \\ & E\{Q_j^{\Delta t}(t_{i+1}, y, S\varepsilon)\}\Bigg\}, \end{aligned} \tag{30}$$

where m runs through the positive integer numbers ($m = 1, 2, \ldots$), and

$$Q_j^{\Delta t}(t_i, y + m\Delta y, S) = E\{Q_j^{\Delta t}(t_{i+1}, y + m\Delta y, S\varepsilon)\},$$

$$Q_j^{\Delta t}(t_i, y - m\Delta y, S) = E\{Q_j^{\Delta t}(t_{i+1}, y - m\Delta y, S\varepsilon)\},$$

as at time t_i we do not know yet the value function. In this case we use the known values at the next time instant, t_{i+1}. We use a binomial tree for the stock price process $S(t_i)$, and at each node of the tree we have discretized the y-space by a one-dimensional grid of size Δy. It is supposed that $\lim_{\Delta t \to 0} \Delta y \to 0$. That is, $\Delta y = h\Delta t$, where h is some constant. This scheme is a dynamic programming formulation of the discrete time problem. The solution procedure is as follows. Start at the terminal date and give the value function values by using the boundary conditions as for the continuous value function over the discrete state space. Then work backwards in time. That is, at every time instant t_i and every particular state (y, S), by knowing the value function for all the states in the next time instant, t_{i+1}, find the investor's optimal policy. This is carried out by comparing maximum attainable utilities from buying, selling, or doing nothing.

Theorem 2. *The solution* $Q_j^{\Delta t}$ *of* (30) *converges weakly to the unique continuous viscosity solution of* (15) *as* $\Delta t \to 0$.

A rigorous treatment of a proof of this type of convergence theorem is rather lengthy. We refer the reader to, for instance, Kushner and Martins (1991), Davis et al. (1993), and Davis and Panas (1994) for an example of such a proof.

So far the outputs of the discretization scheme presented above are the value function and the optimal transaction policy described as the mapping $(y, S) \mapsto (y', S)$. That is, we implicitly assumed that for every point (y, S) the algorithm finds a new point (y', S) that represents the optimal transaction. A direct implementation of such an algorithm is extremely time consuming. Below we show how the computational time can be substantially reduced by exploiting the knowledge of the form of the optimal portfolio strategy.

The practical implementation of the numerical scheme for $Q_j^{\Delta t}(t, y, S)$ is based on the qualitative knowledge of the form of the optimal trading strategy. That is, at every time t the optimal strategy is completely described by four numbers: $y_l(t, S)$ – the lower boundary of the NT region, $y_u(t, S)$ – the upper boundary of the NT region, $y_u^*(t, S)$ – the sell target boundary, and $y_l^*(t, S)$ – the buy target boundary. This qualitative knowledge of the optimal portfolio strategy can be exploited to build an efficient practical realization of a Markov chain approximation scheme. In short, the idea is to detect first the two NT and two target boundaries. The optimal number of shares y^* without transaction costs (given by either (17) or (18)) is the natural start for the search of the free boundaries. The value function $Q_j^{\Delta t}(t_i, y, S)$ inside the NT region is found assuming the NT policy. Then observe that, as in the continuous time case, if the value function $Q_j^{\Delta t}(t_i, y, S)$ is known in the NT region, then it can be calculated in the buy and sell region by using the discrete space version of (22):

$$Q_j^{\Delta t}(t_i, y, S) = \begin{cases} \exp\left(\gamma \frac{k-(1-\lambda)(y-y_u^*)S}{\delta(t_i,T)}\right) Q_j^{\Delta t}(t_i, y_u^*, S), & \forall y(t_i, S) \geqslant y_u(t_i, S), \\ \exp\left(\gamma \frac{k+(1+\lambda)(y_l^*-y)S}{\delta(t_i,T)}\right) Q_j^{\Delta t}(t_i, y_l^*, S), & \forall y(t_i, S) \leqslant y_l(t_i, S). \end{cases} \tag{31}$$

It is crucial to note that inequalities (22) hold with equalities on the points belonging to the NT boundaries. This means in particular that

$$Q_j(t, y_u, S) = \exp\left(\gamma \frac{k - (1-\lambda)S(y_u - y_u^*)}{\delta(t, T)}\right) Q_j(t, y_u^*, S), \tag{32}$$

$$Q_j(t, y_l, S) = \exp\left(\gamma \frac{k + (1+\lambda)S(y_l^* - y_l)}{\delta(t, T)}\right) Q_j(t, y_l^*, S). \tag{33}$$

Consequently, assuming we know the value function at t_{i+1}, the following sequence of steps is performed at time t_i to find the free boundaries and the value function:

Detecting the buy target boundary. Starting from y^* we take relatively big steps δy down from y^* and search for the first point $y = y^* - l\delta y$, $l = 1, 2, \ldots$, located below the NT region, where it is optimal to buy. This is done by comparing the first and the third terms in (30). The first term in (30), which is nothing else than the maximum utility operator in the buy direction of transaction, returns the point $y + m\Delta y = y_l^*$ on the buy target boundary.

Detecting the lower NT boundary. We proceed to the search of the lower NT boundary using the value matching condition (33): starting from y_l^* and going along the buy line in the opposite direction, we search for the first point where the value function is less or equal to the value on the buy target boundary. This gives us y_l.

Detecting the sell target boundary. Starting from y^* we take relatively big steps δy up from y^* and search for the first point $y = y^* + l\delta y$, located above the NT region, where it is optimal to sell. This is done by comparing the second and the third terms in (30). The second term in (30), which is nothing else than the maximum utility operator in the sell direction of transaction, returns the point $y - m\Delta y = y_u^*$ on the sell target boundary.

Detecting the upper NT boundary. We proceed to the search of the upper NT boundary using the value matching condition (32): starting from y_u^* and going along the sell line in the opposite direction, we search for the first point where the value function is less or equal to the value on the sell target boundary. This gives us y_u.

Computing the value function inside the NT region. Having determined the boundaries of the NT region, we proceed by computing and storing in memory[5] the value function inside the NT region for every grid step $y = y_l + l\Delta y,\ y \in [y_l, y_u]$. Since the optimal policy inside the NT region is to not transact, the value function is given by the third term in (30).

The increase in the speed of the above algorithm comes from the efficient implementation of the maximum utility operator given by the first and the second terms in (30), and from the efficient implementation of the search for an NT boundary. To find the maximum along the direction of transaction, we recommend first to implement a routine for *initially bracketing a maximum*, and then to implement either the classical *bracketing* algorithm or the *golden section search* algorithm to find the maximum. In the search for an NT boundary it makes sense first to take relatively big steps δy from a point on a target boundary to find the first point outside of the NT region. Apparently, the NT boundary, which we are looking for, lies in between this point and the last point inside the NT region. Then the NT boundary can be easily found by implementing a *bisection* algorithm. For some examples of efficient practical realization of the above mentioned algorithms, see, for example, Press et al. (1992).

At first sight, the procedures for computing the value functions Q_0 and Q_w are the same except for the boundary conditions. In fact, the computation of the value function without option liability is a much easier task than the computation of the value function with option liability. The trick is to note that in the problem without option liability the optimal investor's strategy is fully defined by the total amount invested in the stock, yS (see, for example, the formulation of the model in Davis and Norman (1990) or Øksendal and Sulem (2002)). This means that if we know the optimal trading strategy for some node, (t_i, S), of the binomial tree, say $y_0(t_i, S)$, then

[5]We need to keep the value function at time t_i for at least one period in order to use it for the computation of the value function at the preceding time t_{i-1}.

the optimal trading strategy for another node (t_i, S') can be found as

$$y'_0(t_i, S') = y_0(t_i, S)\frac{S}{S'}.$$

Hodges and Neuberger (1989) and Clewlow and Hodges (1997) avoided the computation of the value function Q_0 by choosing $\mu = r$. In this case the investor refrains from investing in the stock and places all his wealth in the riskless asset.

Finally, as Clewlow and Hodges (1997) pointed out, the algorithm which uses the values of the (negative) exponential utility of terminal wealth works reasonably well, but has one drawback: for high levels of the investor's risk aversion γ one cannot increase the number of periods of trading n beyond some threshold as the values of the exponential utility cause either overflow or underflow. To avoid this, they suggest the logarithmic transformation of the value function Q_j with consequent approximation of the value function by Taylor series expansion.

5. The numerical results

In this section we present the results of our numerical computations of the investor's reservation write price and the corresponding hedging strategy. In most of our calculations we used the following model parameters: the initial risky asset price $S = 100$, the strike price $K = 100$, the volatility $\sigma = 25\%$, the drift $\mu = 12\%$, and the risk-free rate of return $r = 5\%$ (all in annualized terms). The option maturity is $T = 1$ year. The proportional transaction costs $\lambda = 1\%$ and the fixed transaction fee $k = 0.05$. The discretization parameters of the Markov chain, depending on the investor's γ, are: $n \in [150, 250]$ periods of trading, and the grid size $\Delta y \in [0.5\text{e-}5, 1.0\text{e-}3]$.

We begin our presentation with the study of how the reservation write price depends on the level of the investor's absolute risk aversion γ. The results of the numerical computations are presented in Fig. 2. On the basis of studying the figure, we can make the following observations concerning the reservation write price: The reservation write price is always above the corresponding Black–Scholes price and is an increasing function of γ. Moreover, the reservation write price is always above the corresponding price in the model with no fixed transaction fee ($k = 0$). As γ increases, the discrepancy between the reservation write prices with and without a fixed transaction fee also increases. We conjecture that as $\gamma \to \infty$ the reservation write price approaches the super-replication price with transaction costs. In other words, the writing price of an option approaches the price of the underlying asset. This was rigorously proved by Bouchard et al. (2001) for the case with proportional transaction costs only.

As γ decreases, the reservation write price approaches a horizontal asymptote located above the Black–Scholes price. Here, for low values of γ, the reservation write price is virtually independent of the choice of γ. Our comparative statics analysis[6] shows that for low values of γ the reservation write price can be roughly

[6]Due to the space limitation we do not present this analysis.

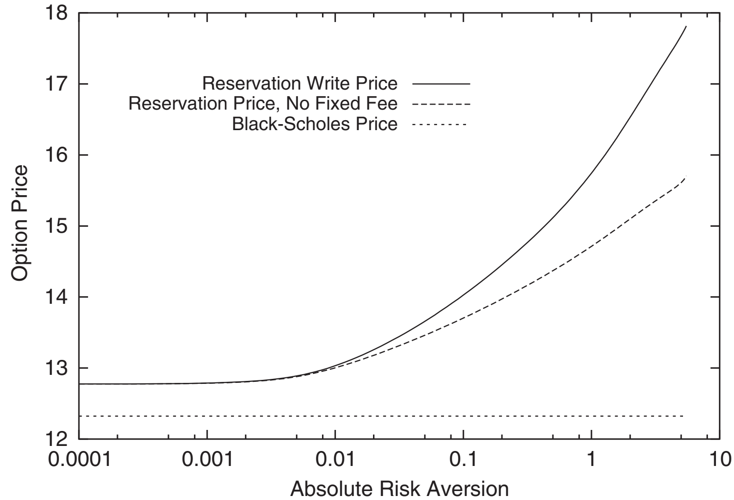

Fig. 2. Reservation write prices of the European call option versus the Black–Scholes price for different levels of the investor's absolute risk aversion γ.

approximated by

$$P = V + \lambda S \frac{\partial V}{\partial S}. \tag{34}$$

Note that the level of proportional transaction costs fully explains the magnitude of the discrepancy between the option prices with and without transaction costs. Note in addition that neither the level of the fixed transaction fee nor the drift of the risky asset (when it is appreciably higher than the risk-free rate) influences the price of the option (which seems to be counterintuitive).

Now we present the intuition behind the observed dependence of the reservation write price on the parameter γ. Eq. (18) suggests that in the absence of transaction costs the investor's overall portfolio problem can be separated into an investment problem and a hedging problem. Even though in the presence of transaction costs the investing and hedging decisions are not completely separable, still we can roughly evaluate and compare the investment and hedging demand of the investor according to Eq. (18). Recall in addition the optimal strategies for the investor without and with option liability: the optimal trading policy without options requires selling some shares of the stock when the stock price goes up and buying additional shares of the stock when the stock price falls (see Eq. (17)). The hedging of a short option position requires purchasing additional shares of the stock when the stock price increases and selling some shares of the stock when the stock price decreases. Note that the investing and hedging decisions work in the opposite directions.

As the option holder's risk aversion decreases, he invests more wealth in the risky stock. Eventually, the investment demand prevails over the hedging demand and the option holder's hedging strategy becomes 'absorbed' by the investing strategy, which means that a highly risk tolerant option holder implements mainly a so-called *static* hedge, regardless of how highly risk tolerant he is. That is why a reservation option price approaches a horizontal asymptote as γ decreases. To implement the static hedge, the option writer needs to buy some additional number of shares of the stock at the initial time and sell them when the option matures. That is why the discrepancy between the reservation write price and the Black–Scholes price (see Eq. (34)) is roughly equal to the (incremental) amount of transaction costs caused by the static hedge strategy.[7] Note that a low risk averse option writer pays the same fixed costs at the initial time regardless of the presence of the option.

However, we should point out that the reservation write price generally depends on the investor's initial holdings in the stock (see Eq. (23)). One usually assumes that the investor starts with zero holdings in the stock. This implies that the investor's initial stock inventory lies in the buy region. If for example, the investor's initial stock inventory lay in the sell region, the sign of the bias in Eq. (34) would be the opposite. That is, the investor would sell some shares of the stock at the initial time. In this case the option writer would value the option less than the Black–Scholes price, since the presence of the option would reduce the initial transaction costs, and these savings decreased the reservation write price. This phenomenon in the presence of only proportional transaction costs was closely studied in Monoyios (2004).[8]

If the investor's risk aversion is high, then the hedging demand prevails over the investment demand. This means that a larger part of the investor's holdings in the stock is devoted to hedge the risk of the option. In the presence of transaction costs the option hedging is costly. The amount of hedging transaction costs increases when the option holder's risk aversion increases (a more risk averse option holder hedges the option more often). These hedging transaction costs increase the reservation write price. That is, the writer of the option will demand a price which increases as the writer's risk aversion increases. Consequently, when transaction costs include a fixed component, the amount of hedging transaction costs is larger, which results in a higher reservation write price.

Observe in particular that when the investor's absolute risk aversion γ is rather high, then we can assume for simplicity that without option liability the investor refrains from investing in the risky asset (see Eq. (17)). Moreover, our comparative statics analysis shows that for high values of γ the reservation write price is virtually independent of the drift of the risky asset, irrespective of the presence of a fixed transaction fee. Consequently, the assumption about a highly risk averse investor is completely equivalent to the Hodges and Neuberger (1989) assumption that $\mu = r$. Under our model parameters the latter assumption is justified for an investor with $\gamma > 0.01$ (see Fig. 2). For practical applications it means that for an investor with high risk aversion we can assume for simplicity $\mu = r$, thus eliminating one variable and the evaluation of the value function Q_0.

[7]Recall our assumption that there are no transaction costs at the terminal time.

[8]Strictly speaking, Monoyios (2004) uses the definition of the option price suggested by Davis (1997). We conjecture that as $\gamma \to 0$, the reservation option price converges to that obtained by using the definition of Davis.

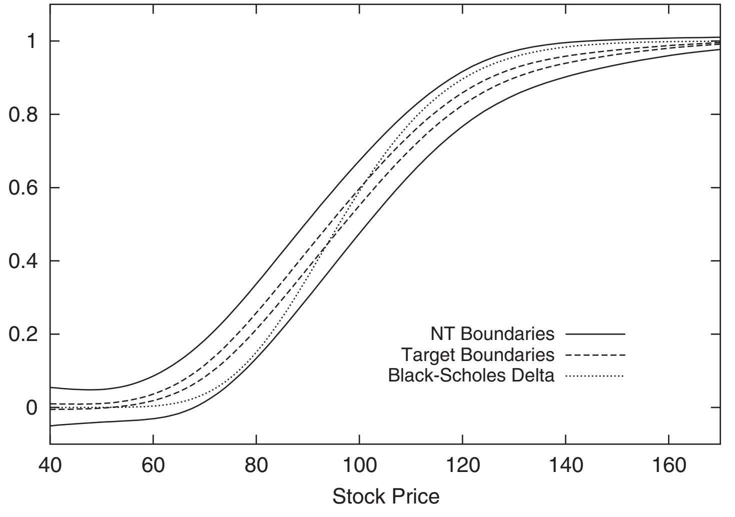

Fig. 3. Numerically calculated optimal hedging strategy versus the Black–Scholes delta for the same model parameters as in Fig. 1.

We now turn to the analysis of the optimal hedging strategy in the market with both fixed and proportional transaction costs. Assuming $\mu = r$, the numerically calculated optimal hedging strategy is depicted on Fig. 3. At first sight, the optimal hedging strategy obtained using the exact numerics seems similar to that obtained from the asymptotic analysis (see Fig. 1). However, a careful visual inspection reveals that there are some differences in the size and the form of the NT regions and the regions between the two target boundaries. Moreover, the middle of the NT region[9] of the numerically calculated optimal hedging strategy does not coincide with the Black–Scholes delta.

Let us elaborate on this more specifically. Consider first the size and the form of an NT region. The picture becomes more clear if we calculate the size of an NT region as $(y_u^w - y_l^w)S$, that is, measured as the difference in the investor's total holdings in the risky asset on the boundaries of an NT region. Fig. 4 shows the size of the numerically calculated NT region versus the size of the Whalley and Wilmott NT region. First note that the size of the NT region obtained from the asymptotic analysis is overvalued as compared to that from the exact numerics. The form of the NT region obtained from the asymptotic analysis is derived from the form of the option gamma. Recall that the option gamma is the sensitivity of the option delta to the underlying asset price. Therefore we expect to transact more in regions where the sensitivity of the delta to the stock price is high. To decrease the amount of transaction costs it makes sense to widen the NT region where the option gamma is high. However, the asymptotic strategy maintains that the size of the NT

[9]Alternatively, the middle of the region between the target boundaries.

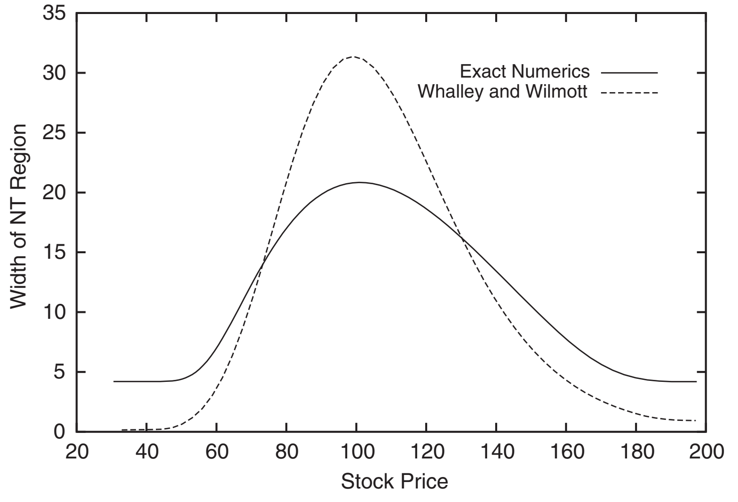

Fig. 4. The size of the numerically calculated NT region versus the size of the Whalley and Wilmott NT region for the same model parameters as in Fig. 1.

region decreases to zero as the option goes farther either out-of-the-money ($S \to 0$) or in-the-money ($S \to \infty$). On the contrary, the exact numerics show that, when the option gamma goes to zero, the size of the NT region approaches a constant value. It turns out that this constant value is actually the size of the NT region in the model without option liability. All the foregoing remarks apply to the size and the form of the region between the target boundaries as well.

Consider now the discrepancy between the middle of the NT region obtained from the exact numerics and the Black–Scholes delta. This discrepancy was first observed by Clewlow and Hodges (1997) for the model with proportional transaction costs only. They noted that this discrepancy could be conveniently described as the volatility adjustment which reflects the variance intuition of Leland (1985). For the same model, Barles and Soner (1998) performed an alternative asymptotic analysis to that of Whalley and Wilmott (1994)[10] and found that the volatility adjustment is a function of the option gamma.

Using the results of the numerical computations of the optimal hedging strategy, we can compute the volatility adjustment. That is, we compute the middle of the NT region and suppose that it is calculated according to the Black–Scholes delta with a modified volatility

$$\frac{y_u^w + y_l^w}{2} = \frac{\partial V(\sigma_m)}{\partial S} = N(d_1(\sigma_m)).$$

[10] A rigorous derivation of the asymptotic strategy for the model with only proportional transaction costs was presented in Whalley and Wilmott (1997).

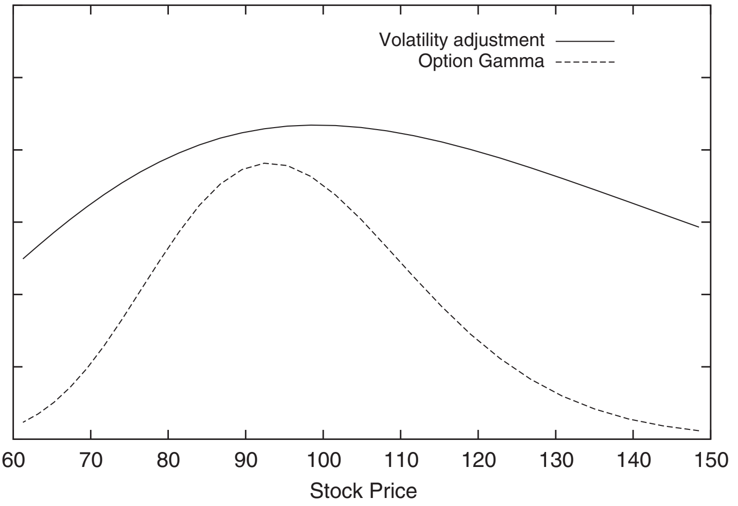

Fig. 5. The form of the optimal volatility adjustment $\sigma_m^2 - \sigma^2$, obtained using the exact numerics, versus the option gamma.

Consequently, the value of $d_1(\sigma_m)$ can be calculated as

$$d_1(\sigma_m) = N^{-1}\left(\frac{y_u^w + y_l^w}{2}\right),$$

where $N^{-1}(\cdot)$ is the inverse of the cumulative normal distribution function. From the other side, $d_1(\sigma_m)$ is defined in accordance with Eq. (20). Therefore, to find the unknown σ_m we need to solve the following quadratic equation:

$$\tfrac{1}{2}(T-t)\sigma_m^2 - \sqrt{T-t}\, d_1(\sigma_m)\sigma_m + \log\left(\frac{S}{K}\right) + r(T-t) = 0.$$

Fig. 5 depicts the form of the calculated optimal volatility adjustment versus the option gamma.

The intuition behind the volatility adjustment in the presence of transaction costs is as follows: increased volatility makes the middle of the NT region a flatter function of the stock price than the Black–Scholes delta. Consequently, the decreased sensitivity to the changes in the stock price helps to reduce the amount of transaction costs. The dependence of the volatility adjustment on the option gamma seems to be very natural, as we expect to transact more in regions where the option gamma is high.

Recall that in the asymptotic analysis, as applied to our problem, one studies the limiting behavior of the optimal hedging policy as the amount of transaction costs goes to zero. Even though such an asymptotic analysis can reveal some underlying structure of the solution, our study shows that under realistic transaction costs this method provide not quite accurate results. Consequently, the careful analysis of the numerically calculated optimal hedging policy $y_w(t)$, given $\mu = r$, suggests the following specification of the NT region (as opposed to Eq. (24)):

$$y = \frac{\partial V(\sigma_m)}{\partial S} \pm (A_w + A_0) \tag{35}$$

and the target boundaries are given by (as opposed to Eq. (25))

$$y = \frac{\partial V(\sigma_m)}{\partial S} \pm (B_w + B_0). \tag{36}$$

A_0 and B_0 are, respectively, half of the size of the NT region and half of the size of the region between the target boundaries in the model without option liability. A_w and B_w are 'additional' increases in the size of the corresponding region induced by the presence of an option. Visual observations of Figs. 4 and 5 suggest that A_w, B_w, and σ_m are functions of the option gamma.

6. The simulation analysis

In this section we compare the performance of the numerically calculated optimal hedging strategy against the Black–Scholes, Whalley and Wilmott, and two ad hoc strategies. The first ad hoc strategy is commonly known as the *delta tolerance* strategy which could be considered as a rough simplification of the utility based optimal hedging. The strategy prescribes rehedging when the hedge ratio moves outside of the prescribed tolerance from the corresponding Black–Scholes delta. More formally, the boundaries of the NT region are defined by

$$y = \frac{\partial V}{\partial S} \pm A,$$

where A is a given constant tolerance. The intuition behind this strategy is pretty obvious: the parameter A is a proxy for the measure of risk of the hedging portfolio. More risk averse hedgers would choose a low A, while more risk tolerant hedgers will accept a higher value for A. In the model with only proportional transaction costs it is optimal to rehedge to the nearest boundary of the NT region. In the presence of a fixed fee component, the rehedging should bring the hedge ratio inside the NT region. For simplicity, we assume that rehedging brings the hedge ratio to the Black–Scholes delta.

The second ad hoc strategy prescribes rehedging to the Black–Scholes delta when the percentage change in the value of the underlying asset exceeds the prescribed tolerance. We will refer to this strategy as the *asset tolerance* strategy. More formally, the series of stopping times is recursively given by

$$\tau_1 = 0, \quad \tau_{i+1} = \inf\left\{ t > \tau_i : \left| \frac{S(t) - S(\tau_i)}{S(\tau_i)} \right| > a \right\}, \quad i = 1, 2, \ldots,$$

where a is a given constant percentage. The intuition behind this strategy is similar to that of the delta tolerance strategy.

In order to compare the performance of different hedging strategies we need to choose a suitable unified risk-return framework. We adopt the mean-variance framework suggested by Clewlow and Hodges (1997). That is, we calculate the expected replication error relative to the Black–Scholes value and the standard deviation of the replication error. The reader is reminded that the mean-variance framework does not favor a priori any one hedging strategy.

The different hedging strategies were simulated and the results are presented below. The model parameters for the simulation analysis are the same as those in Section 5. The simulation proceeds as follows: at the beginning, the writer of a European call option receives the Black–Scholes value of the option and sets up a replicating portfolio. The underlying path of the stock is simulated according to the binomial tree described in Section 4. At each node of the tree a check is made to see if the option needs to be rehedged. If so, the rebalancing trade is performed and transaction costs are drawn from the bank account. Finally, at expiration, we compute the replication error, that is, the cash value of the replicating portfolio minus the due exercise payment. For the Black–Scholes strategy, we vary the rehedging interval $m\Delta t$, $m = 1, 2, \dots, \frac{n}{2}$. This means that rehedging occurs at the initial time $t = 0$, then at time $m\Delta t$, $2m\Delta t$, $3m\Delta t$, and so on. For the utility based and the asymptotic strategies we vary the risk aversion parameter $\gamma \in [0.01, 10]$. In both the delta tolerance and the asset tolerance strategies, A and a take values in $[0.01, 0.35]$. For each parameter we generate 10,000 paths.

More formally, for each possible hedging strategy i we fix a parameter α (where α is either $m\Delta t$, γ, A, or a), perform path simulations, and calculate the expectation, $E_i(\alpha)$, and the standard deviation, $\sigma_i(\alpha)$, of the replication error. Hence, for a given value of parameter α, the results of simulations can be represented by a point with $(\sigma_i(\alpha), E_i(\alpha))$ in the risk-return space. By varying the value of α we span all the possible combinations of $(\sigma_i(\alpha), E_i(\alpha))$. Thus, the obtained curve for a given strategy can be intuitively interpreted as the tradeoff between the risk and the costs of a hedging strategy. For example, an increase in γ results in a tighter NT region. Consequently, we normally expect that the risk of a hedging strategy decreases at the expense of the increase in the costs of a hedging strategy. Similarly, a decrease in γ results in a wider NT region. This reduces the costs of a hedging strategy at the expense of the increase in the risk of a hedging strategy. The reader is reminded that strategy i is considered to be better than strategy j if $\sigma_i < \sigma_j$ when $E_i = E_j$, or, alternatively, if $E_i < E_j$ when $\sigma_i = \sigma_j$.

Fig. 6 shows the results of the simulations. It is clearly seen that the utility based hedging strategy obtained using the exact numerics outperforms all the others. The comparative performance of the numerically calculated strategy and the asymptotic strategy emphasizes the importance of the two key elements of the utility based hedging strategy: optimal forms of the NT region and the region between the target boundaries, and a volatility adjustment. Once again we would like to highlight the deficiencies of the Whalley and Wilmott asymptotic strategy: in the asymptotic strategy, both deep in-the-money and out-of-the-money options, irrespective of the hedger's risk aversion, are hedged with almost zero hedging bandwidth. This is clearly suboptimal and explains why the exact strategy is better for a hedger with a low risk aversion.[11] For a hedger with high risk aversion it is very important to hedge

[11] Here we mean a value in the lower end of the risk aversion interval.

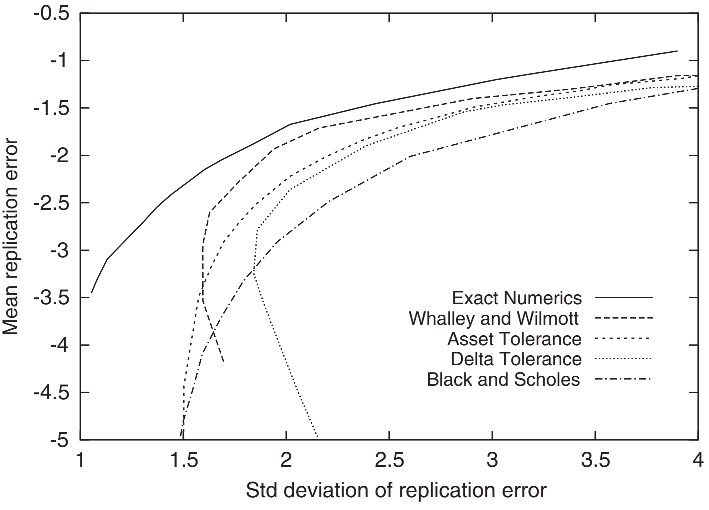

Fig. 6. Comparison of hedging strategies in the mean-variance framework.

with optimal volatility adjustment. The comparative static analysis shows that as γ increases, the size of the NT region and the region between the target boundaries decreases in order to decrease the risk of the hedging portfolio. Without volatility adjustment, the Whalley and Wilmott and the two ad hoc strategies converge to the Black–Scholes strategy. That is, in the limit, the amount of transaction costs tends to infinity. On the contrary, in the optimal strategy, as γ increases, the decrease in the size of the hedging bandwidth is largely compensated by the increase in the volatility adjustment. Roughly speaking, in the limit as $\gamma \to \infty$ the utility based strategy tends to the Leland strategy where the rehedging interval approaches zero. The reader is reminded that the latter strategy amounts to the super-replicating strategy where a single share of the stock is held to hedge an option.

7. Summary

In this paper we provided a systematic treatment of the utility based option pricing and hedging approach in the market with both fixed and proportional transaction costs: we extended the framework developed by Davis et al. (1993) and formulated the option pricing and hedging problem in a market with both fixed and proportional transaction costs. We proposed and implemented the numerical procedure for computing option prices and corresponding optimal hedging strategies.

We considered the case of a holder of a short European call option and studied how the utility based option price depends on the level of the option holder's risk aversion. We found that the utility based option price is always above the corresponding Black–Scholes price and is an increasing function of the option holder's risk aversion. As risk aversion decreases, the utility based option price approaches a horizontal asymptote located above the Black–Scholes price.

Since hedging is widely used to reduce risk, the knowledge of the optimal hedging strategy in the presence of transaction costs is of great practical interest. With this in mind, we provided a careful analysis of the numerically computed optimal hedging strategy and found that it has two distinctive features: specific forms of the NT region and the region between the target boundaries, and a volatility adjustment. In particular, we focused our attention on the important differences between the optimal hedging strategies obtained using the exact numerics and the asymptotic hedging strategy of Whalley and Wilmott (1994).

Finally, we provided a simulation analysis in order to compare the performance of the numerically calculated utility based hedging strategy against the asymptotic strategy and some other common strategies and found that the numerically calculated hedging strategy outperforms all the others. The results of our simulation analysis highlighted, in particular, the deficiencies of the asymptotic strategy.

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