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Further results on convergence of asynchronous linear iterations

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Abstract

This paper focuses on the convergence problem of asynchronous linear iterations. A stronger version of the necessity part of the classical Chazan–Miranker theorem is proved and new results for special classes of iteration matrices are also presented. © 1998 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction and terminology

1.1. Introduction

In this paper we focus on convergence criteria for linear asynchronous iterations. For a matrix $A \in \mathbb{R}^{N\times N}$, partitioned as $(A_{ij})_{m\times m}$, where $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, $n_i \geqslant 1$ and $\sum_{i=1}^{m} n_i = N$, the asynchronous linear iteration is given by

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$$x_i(k+1) = \begin{cases} \sum_{j=1}^{m} A_{ij} x_j(k - d(i,j,k)) & \text{if } i \in S(k), \\ x_i(k) & \text{otherwise,} \end{cases} \qquad k = 0, 1, 2, \ldots, \tag{1}$$

where $d(i,j,k) \geqslant 0$ are nonnegative integers, $S(k)$ are nonempty subsets of $\{1,\ldots,m\}$, the initial vectors are specified by $x(0) = x(-1) = \ldots$ Henceforth, we write the initial vector $x(0)$ to abbreviate reference to this set of equal initial vectors. We refer to the $d(i,j,k)$ as *iteration delays* and $S(k)$ as *updating sets*. Note that the iteration (1) is linear with constant coefficient matrices $A = (A_{ij})$ but it is nonstationary since the delays $d(i,j,k)$ are time-varying.

The interpretation of Eq. (1) in the modeling of block-iterative numerical methods implemented on parallel computers is as follows. Suppose we have a parallel computer consisting of m processors, assign x_i and A_{ij}, $1 \leqslant j \leqslant m$, to processor i, at iteration $k+1$, processor i receives the value of x_j from processor j for all $1 \leqslant j \leqslant m$, calculates $x_i(k+1)$ using the right-hand side of Eq. (1), where $d(i,j,k)$ represents the iteration steps that processor j needs to transfer its value of x_j to processor i at iteration $k+1$. For further discussion, see Refs. [1,2]. Eq. (1) also models a discrete-time system with time-varying delays in the interconnections [3].

Since Chazan and Miranker proposed their Chaotic Relaxation model in 1969 [4], numerous asynchronous models have been proposed and successfully applied to some practical problems, in the area of parallel and distributed computation, such as solutions of systems of linear and nonlinear equations, calculation of fixed points of nonlinear functions, optimization, eigenproblems, neural networks and some discrete problems. A comprehensive account of the western literature on asynchronism can be found in Refs. [1,2], while both Russian and Western literature are discussed in Ref. [5].

The first purpose of this paper is to classify some kinds of asynchronous iterative schemes. The second is to propose some new convergence results under this classification framework.

1.2. Terminology

The assumptions usually made in the study of linear asynchronous systems (1) can be grouped into three classes.

1.2.1. Iteration delays

Using terminology similar to that of Bru et al. [6], we say that the iteration delays $d(i,j,k)$ are *admissible* if

$$\lim_{k\to\infty} k - d(i,j,k) = \infty \quad \text{for all } i,j, \tag{2}$$

and *regulated* if there exists a nonnegative integer D such that

$$0 \leqslant d(i,j,k) \leqslant D \quad \text{for all } i,j,k. \tag{3}$$

Conditions (2) and (3) say that there is no iteration vector which will be used infinitely often. Clearly, condition (2) implies condition (3).

If for all i,j,k, the delays $d(i,j,k) = 0$, we call the iteration (1) a zero-delay iteration, otherwise it is called an iteration with delays.

If iteration delays are regulated, the system (1) with delays can always be written as a nonstationary zero-delay system in $\mathbb{R}^{N(D+1)}$ by stacking $x(k), \ldots, x(k-D)$ as one 'big' vector $X(k) \in \mathbb{R}^{N(D+1)}$, however, in this case, the coefficient matrices must, in general, be time-varying. See, for example, Ref. [7] for this kind of approach.

1.2.2. Updating sets

The updating sets $S(k)$ are called *admissible* if

$$\bigcup_{k=K}^{\infty} S(k) = \{1, \ldots, m\}, \quad \text{for any } K \tag{4}$$

and *regulated* if there exists a $K \geqslant 0$,

$$\bigcup_{k=i}^{i+K} S(k) = \{1, \ldots, m\}, \quad \text{for all } i. \tag{5}$$

Condition (4) says that every subvector should be updated infinitely often, so it is also known as a *nonstarvation* condition in the literature. Condition (5) says that every component should be updated at least once in any $K + 1$ iteration steps.

The updating sets are called *periodic* if there exists a positive integer T such that

$$S(k+T) = S(k) \quad \text{for all } k, \tag{6}$$

and aperiodic otherwise.

The *synchronous iterations* are iterations of the type (1) with no delays and the updating sets $S(k) = \{1, \ldots, m\}$ for all k. All other iterations are referred to as *asynchronous*. It is sometimes useful to specify the kind of asynchronism being considered by adding some qualifiers, e.g. zero-delay asynchronism is a very important special case that has been much studied, see, e.g., Refs. [5,8,9]. Examples of the use of this terminology can be found in Section 1.3 below.

1.2.3. Iteration matrix

We recall that the iteration matrix A in $\mathbb{R}^{N\times N}$ is partitioned as $A = (A_{ij})_{m\times m}$ where $A_{ij} \in \mathbb{R}^{n_i\times n_j}, n_i > 0$ and $\sum_{i=1}^{m} n_i = N$. The partitioning is referred to as *pointwise* if $m = N$ (i.e. $n_i = 1$ for all i). If $m = 1$, the matrix A is said to be *unpartitioned*, and for $1 < m < N$, we say that A is *block-partitioned*.

From the point of view of structure, iteration matrices can be divided into classes of nonnegative, symmetric, symmetrizable, triangular, strictly triangular, irreducible matrices, etc.

In terms of the spectral radius of $A, \rho(A)$, or the norm of $A, \|A\|$, we encounter the following kinds of conditions: $\rho(A) < 1, \rho(|A|) < 1$ where $|A| \equiv (|a_{ij}|)$, $\rho(A) = 1$, and for the block form, $\rho(H) < 1$ with $H = (h_{ij})_{m\times m}$ and $h_{ij} = \|A_{ij}\|$. Here the norms are properly chosen induced operator norms.

Matrix A is called *convergent* if $\lim_{k\to\infty} A^k$ exists. It is also called semiconvergent in some literature, e.g., Ref. [10], p. 152. In this case, we use A^∞ to denote this limit. The necessary and sufficient conditions for A to be convergent are that:

1. its spectral radius, $\rho(A)$, is less than or equal to unity, and
2. if $\rho(A) = 1$, then all the elementary divisors associated with the eigenvalue 1 of A are linear; that is rank $(I - A)^2 =$ rank $(I - A)$, and
3. if $\rho(A) = 1$ then λ is an eigenvalue of A with $|\lambda| = 1$ implies $\lambda = 1$.

1.3. Some examples

The terminology introduced above can be exemplified in the context of iterative methods for linear systems of equations. For example, if in Eq. (1), the updating set is a singleton given by $S(k) = \{i(k) := k(\text{mod } N) + 1\}$, and all the delays $d(i(k), j, k) = i(k) - 1$, then Eq. (1) represents the sequential block Jacobi iteration. If $S(k) = \{1, \ldots, m\}$, $d(i, j, k) = 0$ for all i, j, k, we have the parallel block Jacobi iteration.

If $S(k) = \{i(k) := k(\text{mod } N) + 1\}$, $d(i, j, k) = \min(k - 1, r - 1)$ for some $r, 1 \leqslant r \leqslant N$, Eq. (1) is called a periodic asynchronous scheme, see Refs. [4,11]. The classical Gauss–Seidel iteration is a periodic asynchronous scheme with $r = 1$.

If in Eq. (1), $d(i, j, k) = 0$, i.e., the zero-delay case, it is also referred as a Serial Model [2] and if there is only one element in set $S(k)$ for all k, it is called a free-steering method by Ostrowski [12]. The Gauss–Seidel method is also a special case of the free-steering method.

If the iteration delays and the updating sets are both regulated and there is no self-iteration delay, i.e., $d(k, i, i) = 0$ for all k, i, then Eq. (1) is called a *partially asynchronous linear system*. Sometimes partial asynchronism is also defined allowing self-delays, i.e. $S(k)$ and $d(i, j, k)$ are only required to be regulated [7]. Correspondingly, if iteration delays and the updating sets are only required to be admissible, the system is called *totally asynchronous*, see Ref. [1] for details.

2. Convergence results

In this section, we present new convergence results on special cases of the asynchronous linear iterations of the type (1). Some related results given earlier are also discussed.

2.1. General theorems

We first present the classical result of Chazan and Miranker [4].

Theorem 1. *Consider Eq. (1) in the pointwise case, i.e. $m = N$.*

(a) *If $\rho(|A|) < 1$, and the iteration delays and updating sets are both admissible, then for every initial vector $x(0)$, the sequence of vectors $x(k)$ determined by Eq. (1) converges to zero vector.*

(b) *If $\rho(|A|) \geqslant 1$, there exists a sequence of admissible iteration delays and a sequence of admissible updating sets such that for some initial vector $x(0), x(k)$ determined by Eq. (1) does not converge to zero.*

We prove the following stronger form of part (b) of the above theorem in which the admissible iteration delays are replaced by regulated iteration delays with a bound D of 1.

Theorem 2. *If $\rho(|A|) \geqslant 1$, there exists a linear asynchronous iteration with regulated iteration delays and iteration delay bound $D = 1$ such that $x(k)$ does not converge to zero.*

Bertsekas and Tsitsiklis [1] also gave a similar result with $D = 2$. The idea of our proof is similar to theirs, but the details of the construction of a divergent asynchronous sequence are not the same.

To prove Theorem 2, we first give a lemma which was also used in Ref. [1].

Lemma 1. *Denoting A_+ and A_- such that $A_+ + A_- = A$ and $A_+ - A_- = |A|$. Let A satisfy:*

$$\sum_{j=1}^{m} |a_{ij}| \geqslant 1, \quad \textit{for all } i. \tag{7}$$

If there exists some $k_0 \geqslant 0$ such that $x(k_0) > 0, x(k_0 + 1) < 0$, then for the asynchronous iteration:

$$x(k+1) = A_- x(k) + A_+ x(k-1), \quad k = k_0 + 1, \ldots, \tag{8}$$

with initial vectors $x(k_0)$ and $x(k_0 + 1)$, the sequence of vectors $x(k)$ does not converge to zero.

Proof. Set

$$\alpha = \min_{1 \leqslant i \leqslant N} \{ |x_i^{k_0}|, |x_i^{k_0+1}| \}.$$

By condition of lemma, $\alpha > 0$ and besides

$$x(k_0) \geqslant \alpha e > 0, \qquad x(k_0 + 1) \leqslant -\alpha e < 0 \tag{9}$$

where e is a vector with all of its components equal 1. It is easy by induction to show that for $k = 0, 1, \ldots,$ the following relations

$$x(k_0 + 2k) \geqslant \alpha e > 0, \qquad x(k_0 + 2k + 1) \leqslant -\alpha e < 0 \tag{10}$$

are valid. □

Proof of Theorem 2. We only need to prove the existence of asynchronous linear iterations not convergent to zero, under the condition $\rho(|A|) \geqslant 1$. If $1 \in \sigma(A)$ where $\sigma(A)$ is the spectrum of A, then the synchronous iteration (which is a special case of an asynchronous iteration) defined by

$$x(k+1) = Ax(k), \quad k = 0, 1, 2, \ldots$$

where $x(0)$ is an eigenvector corresponding to the eigenvalue 1 of A, determines the sequence $x(k) = x(0), \forall k$, which does not converge to the zero vector. So, without loss of generality, suppose that $1 \notin \sigma(A)$ (i.e. $I - A$ nonsingular), and also that $|A|$ is irreducible. There exists a positive eigenvector $v \in \mathbb{R}^N$ of matrix $|A|$, such that $|A|v = \rho(|A|)v$, see Ref. [10]. With renormalization of the basis elements in $\mathbb{R}^N$ we can achieve that the vector v will be the vector with all components 1, thus the condition (7) is satisfied. Since $(I - A)$ is nonsingular, there exists a vector y such that

$$(I - A)y = v.$$

This vector y is used to define the four asynchronous sequences: $r(k), s(k), t(k)$ and $u(k)$ below:

$$r(0) = y,$$

$$r(1) = Ar(0) = Ay,$$

$$r(2) = A\, r(1) + A\, r(0) = (A\, A + A\,)y,$$

$$r(3) = A\, r(2) + A\, r(1) = [A\, (A\, A + A\,) + A\, A]y,$$

$$r(k+1) = A\, r(k) + A\, r(k-1), \quad k = 3, 4, \ldots,$$

$$s(0) = y,$$

$$s(1) = As(0) = Ay,$$

$$s(2) = A\, s(1) + A\, s(0) = (A\, A + A\,)y,$$

$$s(3) = A_-s(2) + A_+s(1) = [A_-(A_+A + A_-) + A_+A]y,$$
$$s(k+1) = A_-s(k) + A_+s(k-1), \quad k = 3,4,\ldots,$$

$$t(0) = y,$$
$$t(1) = At(0) = Ay,$$
$$t(2) = A_-t(1) + A_+t(0) = (A_-A + A_+)y,$$
$$t(3) = A_+t(2) + A_-t(1) = [A_+(A_-A + A_+) + A_-A]y,$$
$$t(k+1) = A_-t(k) + A_+t(k-1), \quad k = 3,4,\ldots,$$

$$u(0) = y,$$
$$u(1) = Au(0) = Ay,$$
$$u(2) = A_+u(1) + A_-u(0) = (A_+A + A_-)y,$$
$$u(3) = A_+u(2) + A_-u(1) = [A_+(A_+A + A_-) + A_-A]y,$$
$$u(k+1) = A_-u(k) + A_+u(k-1), \quad k = 3,4,\ldots$$

For a scalar β, construct the following sequence of vectors.

$$w(k) = r(k) - s(k) - \beta t(k) + \beta u(k), \quad k = 0,1,2,\ldots \tag{11}$$

By calculation,

$$w(0) = 0,$$
$$w(1) = 0,$$
$$w(2) = (1-\beta)\rho(|A|)v,$$
$$w(3) = -\rho^2(|A|)v + (1-\beta)A_+\rho(|A|)v,$$
$$w(k+1) = A_-w(k) + A_+w(k-1), \quad k = 3,4,\ldots$$

Now choosing the value $\beta \in (0,1)$ sufficiently close to 1, we get that the components of the vector $w(2)$ are strictly positive, and the components of the vector $w(3)$ are strictly negative. Therefore by Lemma 1, the sequence of vectors $w(k), k = 0,1,\ldots,$ cannot be convergent to zero. Finally, noting that the sequence $w(k)$ is a linear combination of the sequences $r(k), s(k), t(k), u(k)$ (see Eq. (11)), we conclude that at least one of the sequences $r(k), s(k), t(k), u(k)$ does not converge to zero. □

Strikwerda [13] recently strengthened the Chazan–Miranker result in another direction: if $\rho(|A|) \geqslant 1$, there exists a linear asynchronous iteration which does not converge to 0, such that for all k, (i) there is only one element in $S(k)$, (ii) $d(i,j,k)$ have the same value for the same i,k, (iii) $D(i,j,k) \leqslant N^2$.

A generalization of part (a) of Theorem 1 to the block-partitioned case is the following theorem.

Theorem 3. *Let $H = (h_{ij})$ with $h_{ij} = \|A_{ij}\|$, where the norm is any induced operator norm. If $\rho(H) < 1$, and iteration delays and the updating sets are both admissible, then for any initial vector $x(0)$, the sequence of vectors $x(k)$ determined by Eq. (1) converges to 0.*

This theorem was proved for admissible iteration delays and updating sets by El Tarazi [14] using an induction proof and for regulated delays using a Liapunov function approach in Ref. [7]. Note that $\rho(A) \leqslant \rho(H)$ [15], p. 175, and also that in this theorem, if $m = N$, this result reduces to the sufficiency part of the Chazan–Miranker result.

2.2. Single delay and unpartitioned case

Concerning Theorem 1, Chazan and Miranker [4] commented: *Clearly weaker conditions could be sufficient to guarantee convergence of a smaller class of chaotic schemes than the full class. Any finer classification of chaotic schemes yielding successively stronger convergence results would certainly be of some interest.* From Theorem 2 we learn that even in the case of regulated iteration delays with iteration delay bound of unity, there is no convergence result better than $\rho(|A|) < 1$.

However, weaker conditions than $\rho(|A|) < 1$ may be obtained in the special case of a single delay and unpartitioned system matrix ($m = 1$) and in some other special cases. We have the following corollary to Theorem 3.

Corollary 1. *Consider the unpartitioned asynchronous linear iteration with a single iteration delay sequence, $d(k)$,*

$$x(k+1) = Ax(k - d(k)). \tag{12}$$

If the sequence of iteration delays is admissible, i.e.

$$\lim_{k\to\infty} k - d(k) = \infty,$$

and if $\rho(A) < 1$, then for any initial vector $x(0)$, the sequence $x(k)$ from Eq. (1) converges to 0.

Proof. Because $\rho(A) < 1$, there always exists a norm $\| \ \|_*$ such that $\|A\|_* < 1$. Since $m = 1$, H is a 1×1 matrix and $\rho(H) = H = \|A\|_* < 1$. □

The following proposition is the analog of Corollary 1 for convergent matrices.

Proposition 1. *If the matrix A is convergent, and the remaining conditions on the asynchronous linear iteration are the same as in Corollary 1, then for the initial vector $x(0)$, the sequence $x(k)$ converges to $A^{\infty}x(0)$.*

Proof. Each $x(k)$ can be written as $x(k) = A^{c(k)}x(0)$ with $c(k)$ is an integer such that $\lim_{k\to\infty} c(k) = \infty$. □

Remark. In this proposition, and in some results in Section 2.4 which will use this proposition, the initial vectors should be same, i.e. $x(0) = x(-1) = \dots$ Otherwise, the iteration vectors may oscillate among the neighbours of some fixed vectors $A^{\infty}x(0), A^{\infty}x(-1), \dots$

2.3. Symmetric matrix and zero-delay case

Consider the following result from [9].

Theorem 4. *Suppose that, in Eq.* (1), *A is symmetric with $\rho(A) < 1$, $d(i,j,k) = 0$, and the updating sets are admissible. Then for any initial vector $x(0)$, the sequence $x(k)$ converges to* 0.

This theorem is now generalized to the case where A is block-diagonal symmetrizable (Corollary 2 below). A matrix A is *block-diagonal symmetrizable* if $A = A_1^{-1}A_2$ with A_1 symmetric positive definite and *block diagonal*, A_2 symmetric, and A_1, A_2 conformally partitioned.

Corollary 2. *If, in Eq.* (1), *A is block-diagonal symmetrizable, and the remaining assumptions in Theorem* 4 *hold, then for any initial vector $x(0)$, the sequence $x(k)$ converges to* 0.

Proof. Let $B = A_1^{-1/2}A_2A_1^{-1/2}$ and $y(k) = A_1^{1/2}x(k)$: now, using the result of Theorem 4, we are done. □

Motivated by Lubachevsky and Mitra [16], who discussed the convergence of an asynchronous iteration for a nonnegative matrix with unity spectral radius, we consider the case in which the iteration matrix A is symmetric and $\rho(A) = 1$. Let P_1 denote the orthogonal projector on the eigenspace corresponding to the eigenvalue(s) 1, and P_0 the orthogonal projector on the orthogonal complement of this eigenspace.

Theorem 5. *Suppose that A is symmetric and*

(a) -1 *is not an eigenvalue of A,*

(b) *there is no iteration delay, i.e. $d(i,j,k) = 0$,*

(c) *the updating sets are regulated,*

then for any initial vector $x(0)$, the sequence $x(k)$ determined by (1) *satisfies:*

$$P_0x(k) \to 0. \tag{13}$$

Proof. Consider the quadratic function $F(x) = x^{\mathrm{T}}(I - A)x$ as in Ref. [9]. For any x,

$$F(x) = F(P_0x) \geqslant 0. \tag{14}$$

Denote $A(k) \equiv (B_{ij})$ as

$$B_{ij} = \begin{cases} (I)_{ij} & \text{if } i \notin S(k), \\ A_{ij} & \text{otherwise}, \end{cases}$$

where $(I)_{ij}$ is the (i,j)-th block element of the identity matrix in $\mathbb{R}^{N\times N}$. For any x, by Ref. [9], Lemma 3, p. 313

$$F(x) - F(A(k)x) = (x - A(k)x)^{\mathrm{T}}(I + A)(x - A(k)x). \tag{15}$$

Since $\rho(A) < 1$ and -1 is not an eigenvalue of A, the matrix $(I + A)$ is positive definite, so for the sequence $x(k)$, we have $F(x(k)) \geqslant F(x(k+1)) \geqslant 0$, implying that $F(x(k))$ converges to some $F^* \geqslant 0$. If $F^* = 0$, from Eq. (14), we have

$$\lim_{k\to\infty} F(P_0x(k)) = \lim_{k\to\infty} F(x(k)) = 0.$$

Since $(I - A)$ is symmetric positive definite on $P_0\mathbb{R}^N$, we have the conclusion.

Suppose $F^* > 0$. Once again, since the matrix $(I - A)$ is symmetric positive definite on $P_0\mathbb{R}^N$, $\{P_0x(k)\}_k$ is a bounded sequence. Denote $y(k) = P_0x(k)$: there exists a convergent subsequence $\{y(i(k))\}$ which converges to $y^* \neq 0, y^* \in P_0\mathbb{R}^N$. Let $F(y^*) = F^*$. For arbitrarily small $\epsilon > 0$, there exists a K_1, such that for $i(k) \geqslant K_1, F(y(i(k))) - F^* < \epsilon$ and $\|y(i(k)) - y^*\| < \epsilon$. Suppose for $k = K_1, \ldots, K_2 - 1, A(k)y^* = y^*, A(K_2)y^* \neq y^*$. If $A(K_1)y^* \neq y^*$ we let $K_2 = K_1$. We assert that $K_2 - K_1 < K$, where K is defined in Eq. (5). If this is not the case, since

$$\bigcup_{k=K_1}^{K_2-1} S(k) = \{1, \ldots m\},$$

and $A(k)y^* = y^*$ for $k = K_1, \ldots, K_2 - 1$, therefore $Ay^* = y^*$ and $F(y^*) = 0$, this contradicts $F(y^*) > 0$.

For any x, $A(k)P_1x = P_1x$ and $P_0P_1 = 0$, thus

$$\begin{aligned} \|y(K_1 + 1) - y^*\| &= \|P_0x(K_1 + 1) - y^*\| \\ &= \|P_0A(K_1)(P_1x(K_1) + y(K_1)) - y^*\| \\ &= \|P_0(A(K_1)y(K_1) - y^*)\| \leqslant \|A(K_1)(y(K_1) - y^*)\| \leqslant C\epsilon, \end{aligned}$$

where the constant $C \geqslant 1$ is independent of k because there is only a finite number of different $A(k)$. By induction, we can prove that

$$\|y(K_2) - y^*\| \leqslant C^{K_2-K_1}\epsilon \leqslant C^K\epsilon.$$

Using Eq. (15), we have

$$\begin{aligned}F(y(K_2)) - F(y(K_2+1)) &= [y(K_2+1) - y(K_2)]^{\mathrm{T}}(I+A)[y(K_2+1) - y(K_2)]\\ &\geqslant [\lambda_{\min}(I+A)]\|P_0A(K_2)y^* - y^* + (P_0A(K_2) - I)(y(K_2) - y^*)\|^2\\ &\geqslant \lambda_{\min}(I+A)\|P_0A(K_2)y^* - y^*\|^2 - C'\epsilon,\end{aligned}$$

where $\lambda_{\min}(I+A) > 0$ is the minimum eigenvalue of $I+A$ and C' is another constant independent of k. Then

$$\begin{aligned}F(y(K_2+1)) &\leqslant F(y(K_2)) - [\lambda_{\min}(I+A)]\|P_0A(K_2)y^* - y^*\|^2 + C'\epsilon\\ &\leqslant F^* + (C'+1)\epsilon - [\lambda_{\min}(I+A)]\|P_0A(K_2)y^* - y^*\|^2.\end{aligned}$$

This contradicts $F(y(K_2+1)) \geqslant F^*$ when ϵ is small enough. □

Remark. We conjecture that $\lim_{k\to\infty} x(k)$ exists.

The following proposition shows that the conjecture is true with an additional assumption.

Proposition 2. *Suppose $P_0x(k)$ has a linear rate of convergence in the sense that: there exists a fixed integer $\tilde{K} \geqslant 1$ and a constant $0 < c < 1$, such that*

$$\|P_0x(k_1)\| \leqslant c\|P_0x(k_2)\| \quad \textit{for all } k_2 \geqslant 0, k_1 \geqslant k_2 + \tilde{K},$$

where the norm $\|\ \|$ is an induced norm such that

$$|x| \leqslant |y| \Rightarrow \|x\| \leqslant \|y\|. \tag{16}$$

Then the sequence of iteration vectors $\{x(k)\}_{k=0}^{\infty}$ converges.

Proof. For any $k > 0$, from the property (16) of the norm,

$$\begin{aligned}\|x(k) - x(k+1)\| &\leqslant \|x(k) - Ax(k)\|\\ &= \|(I-A)x(k)\| = \|(I-A)P_0x(k)\|\\ &\leqslant \|I-A\|\ \|P_0x(k)\|.\end{aligned}$$

Therefore, for any $k_2 \geqslant 0, k_1 > k_2$

$$\begin{aligned}\|x(k_1) - x(k_2)\| &\leqslant \|x(k_1) - x(k_1-1)\| + \cdots + \|x(k_2+1) - x(k_2)\|\\ &\leqslant \|I-A\|(\|P_0x(k_1-1)\| + \cdots + \|P_0x(k_2)\|)\\ &\leqslant (1 + c + c^2 + \cdots)\tilde{K}\|I-A\|\ \|P_0x(k_2-\tilde{K})\|\\ &\leqslant \frac{\tilde{K}}{1-c}\|I-A\|\ \|P_0x(k_2-\tilde{K})\|.\end{aligned}$$

From Theorem 5,

$$\lim_{k_2\to\infty} \|P_0x(k_2-\tilde{K})\| = 0,$$

so applying Cauchy's theorem, we can obtain the conclusion. □

2.4. Matrix A triangular or block triangular

In this section, we consider the case in which the iteration matrix A is triangular or block triangular. All the results in this section can be simply applied to the case in which A is (block) similar (via permutation) to a triangular matrix, i.e. there exists a permutation matrix P such that $P^{\mathrm{T}}AP$ is (block) triangular. Of course, we assume that the partitioning in block triangular form is conformal with the partitioning in Eq. (1). The results may be viewed as continuous valued analogs of some results of Robert [17] on discrete data.

Theorem 6. *If A is lower block triangular and $\rho(A_{ii}) = 0$ for $i = 1, \dots, N$, the iteration delays and the updating sets are admissible, then for any initial vector $x(0), x(k)$ determined by Eq. (1) converges to 0 in a finite number of iteration steps.*

Note that the assumption that A is strictly lower triangular (cf. Ref. [17]) is unnecessary in this theorem.

Theorem 7. *Let A be lower block triangular, and $A_{ii}, i = 1, \dots, m_1$ be convergent matrices, $A_{ij} = 0$ for $i = 1, \dots, m_1, j = 1, \dots, i-1, \rho(A_{ii}) < 1, i = m_1 + 1, \dots, m$, for Eq. (1), and if the iteration delays and the updating sets are both admissible, then for any initial vector $x(0)$, the sequence of vectors $x(k)$ converges to $A^{\infty}x(0)$.*

Corollary 3. *For a block triangular matrix A with $\rho(A) < 1$ and admissible iteration delays and updating sets, for any initial vector $x(0)$, the sequence of vectors $x(k)$ converges to 0.*

These results on triangular matrices can be proved by using Corollary 1 and Proposition 1.

3. Conclusions

In this paper, we strengthened both the classical Chazan–Miranker result [4] on the convergence of asynchronous linear iterations and the proof due to Bertsekas and Tsitsiklis [1] by showing that regulated iteration delays with delay bound equal to unity are sufficient to cause nonconvergence of asynchronous iterations in the case when $\rho(|A|) \geqslant 1$. We have given some convergence results for asynchronous linear iterations. Our convergence conditions are weaker than the classical condition $\rho(|A|) < 1$, although some other assumptions on A (e.g. A is symmetric or triangular, etc.), or on the iteration delays (e.g. regulated or zero), or on updating sets are needed.

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References

[1] D.P. Bertsekas, J.N. Tsitsiklis, Parallel and Distributed Computation, Prentice-Hall, Englewood Cliffs, NJ, 1989.
[2] A. Üresin, M. Dubois, Asynchronous iterative algorithms: Models and convergence, in: L. Kronsjö, D. Shumsheruddin (Ed.), Advances in Parallel Algorithms, Ch. 10, Oxford Univ. Press, London, 1992.
[3] E. Kaszkurewicz, A. Bhaya, A delay independent robust stability condition for linear discrete-time systems, In: Proceedings of 32nd IEEE Conference on Decision and Control, vol. 4, San Antonio, TX, December 1993, pp. 3459–3460.
[4] D. Chazan, W.L. Miranker, Chaotic relaxation, Linear Algebra Appl. 2 (1969) 190–222.
[5] E.A. Asarin, V.S. Kozyakin, M.A. Krasnosel'skii, N.A. Kuznetsov, Stability Analysis of Desynchronized Discrete-Event Systems (in Russian), Nauka, Russia, 1992.
[6] R. Bru, L. Elsner, M. Neumann, Models of parallel chaotic iteration methods, Linear Algebra Appl. 103 (1988) 175–192.
[7] E. Kaszkurewicz, A. Bhaya, D.D. Šiljak, On the convergence of parallel asynchronous block-iterative computations, Linear Algebra Appl. 131 (1990) 139–160.
[8] M. Verkama, Random relaxation of fixed-point iteration, SIAM J. Sci. Comput. 17 (4) (1996) 906–912.
[9] A.F. Kleptsyn, V.S. Kozyakin, M.A. Krasnosel'skii, N.A. Kuznetsov, Effect of small synchronization errors on stability of complex systems. Part i, Automation and Remote Control 44 (7) (1983) 861–867; Part ii, Automation and Remote Control 45 (3) (1984) 309–314; Part iii, Automation and Remote Control 45 (8) (1984) 1014–1018.
[10] A. Berman, B. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Classics in Applied Mathematics, SIAM, Philadelphia, PA, 1994.
[11] F.D.P. Donnelly, Periodic chaotic relaxation, Linear Algebra Appl. 4 (1971) 117–128.
[12] A. Ostrowski, Determinanten mit überwiegender Hauptdiagonale und die absolute Konvergenz von linearen Iterationsprozessen, Comment. Math. Helv. 30, 1955.
[13] J.C. Strikwerda, A convergence theorem for chaotic asynchronous relaxation, Linear Algebra Appl. 253 (1997) 15–24.
[14] M. El Tarazi, Some convergence results for asynchronous algorithms, Numer. Math. 39 (1982) 325–340.
[15] M.A. Krasnosel'skii, Je.A. Lifshifts, A.V. Sobolev, Positive Linear Systems: The Method of Positive Operators, Heldermann, Berlin, 1989.

[16] B. Lubachevsky, D. Mitra, A chaotic asynchronous algorithm for computing the fixed point of a nonnegative matrix of unit spectral radius, J. ACM 33 (1) (1985) 130–150.
[17] F. Robert, Discrete Iterations, Springer, Berlin, 1986.