J. Soc. Indust. Appl. Math.
Vol. 3, No. 1, March, 1955
Printed in U.S.A.

ON THE NUMERICAL INTEGRATION OF $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$ BY IMPLICIT METHODS*†

JIM DOUGLAS, JR.

Introduction. The purpose of this report is to investigate how solutions of boundary value problems for the heat flow equation in two space dimensions can be obtained by solving certain difference equations in a rectangular lattice and taking the limit as the mesh size of the lattice tends to zero. All problems treated are of the form

$$
\text{(I)}\qquad
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} && && (0 \leqq x, y \leqq 1, t \geqq 0),\\
&u(x, y, 0) = f(x, y), && && \\
&u(0, y, t) = g_1(y, t), && u(1, y, t) = g_2(y, t) && (t > 0),\\
&u(x, 0, t) = h_1(x, t), && u(x, 1, t) = h_2(x, t) && (t > 0).
\end{aligned}
$$

Interest in the derivation of a numerical solution arises from two sources. First, the formal solution as given by either a quadrature using an influence function or a series resulting from the separation of variables technique (if possible) is usually difficult to evaluate. Second, numerical procedures applicable to the heat flow equation often extend in modified form to non-linear parabolic equations.

Quite a number of papers have been devoted to the study of numerical methods for the solution of the related one-dimensional equation

$$
\text{(II)}\qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}
$$

under various conditions. Most of the work has been centered around the explicit difference equation

$$
\text{(III)}\quad w(x, t + \Delta t) = w(x, t) + \frac{\Delta t}{(\Delta x)^2}[w(x - \Delta x, t) - 2w(x, t) + w(x + \Delta x, t)].
$$

It has been known [2] for some time that the solution of (III) agreeing with the initial condition of u will not approximate closely the solution of (II) except for certain special initial values [9] unless

$$
\text{(IV)}\qquad \lambda^* = \frac{\Delta t}{(\Delta x)^2} \leqq \tfrac{1}{2}.
$$

---

* Presented to the American Mathematical Society, August 31, 1954.
† Received by the editors October 18, 1954.

Under the assumption that $\lambda^* \leqq \frac{1}{2}$, various authors [5, 7] have proved that the solution of (III) converges to that of (II) in any finite region. Moreover, the dependence of the rate of convergence on the smoothness of the initial condition has been studied [7] in the special case of vanishing boundary values.

More general differential equations of the form

$$\frac{\partial u}{\partial t} = a_0(x, t) \frac{\partial^2 u}{\partial x^2} + a_1(x, t) \frac{\partial u}{\partial x} + a_2(x, t, u) \tag{V}$$

have also been studied [6] for the initial value problem on the infinite interval $-\infty < x < \infty$. Again, explicit difference formulae were used:

$$w(x, t + \Delta t) = \textstyle\sum_r c_r(x, t) w(x + r\Delta x, t) + (\Delta t) a_2(x, t, w(x, t)). \tag{VI}$$

For convergence, it was necessary to restrict λ^* to be small.

The restriction of small λ^* leads to the taking of a very large number of time steps to complete the solution of the problem. Various implicit methods have been proposed [3, 4, 8, 10] to alleviate this difficulty. Two of these difference equations, implicit in form are:

$$\begin{aligned} u(x, t + \Delta t) &- \frac{\Delta t}{(\Delta x)^2} [u(x - \Delta x, t + \Delta t) \\ &- 2u(x, t + \Delta t) + u(x + \Delta x, t + \Delta t)] = u(x, t), \end{aligned} \tag{VII}$$

$$\begin{aligned} u(x, t + \Delta t) &- \frac{\Delta t}{2(\Delta x)^2} [u(x - \Delta x, t + \Delta t) \\ &- 2u(x, t + \Delta t) + u(x + \Delta x, t + \Delta t)] \\ &= u(x, t) + \frac{\Delta t}{2(\Delta x)^2} [u(x - \Delta x, t) - 2u(x, t) + u(x + \Delta x, t)]. \end{aligned} \tag{VIII}$$

It is easy to show that either of these two difference equations is stable for any value of λ^*; however, to the author's knowledge no convergence proof has been published for either method, but the methods of this report can be used to give proofs. Thus, the number of time steps necessary to complete the problem can be comparatively small. There is, of course, the problem of solving the linear equations, but this is not troublesome if the method outlined in [1] is used.

In the two-dimensional case, the analogue of (III) is

$$\begin{aligned} u(x, y, t + \Delta t) &= u(x, y, t) \\ &+ \lambda^*[u(x + \Delta x, y, t) + u(x - \Delta x, y, t) \\ &+ u(x, y + \Delta y, t) + u(x, y - \Delta y, t) - 4u(x, y, t)] \end{aligned} \tag{IX}$$

where

$$\lambda^* = \frac{\Delta t}{(\Delta x)^2} = \frac{\Delta t}{(\Delta y)^2}. \tag{X}$$

Again it is necessary to limit the size of λ^*. Equation (IX) is stable only if $\lambda^* \leqq \frac{1}{4}$; the results in the explicit case above have obvious analogues in the two-dimensional problem. Thus, the complete solution involves a vast amount of calculation.

This report is devoted to the theoretical study of various implicit difference schemes, each stable for all λ^* for the numerical integration of (I). An accompanying report of Peaceman and Rachford [11] treats the practical aspects of the calculation in considerable detail and gives application of one method to the Dirichlet problem by treating the time step as an iteration parameter; machine time and storage requirements are included in [11].

Chapter I contains a discussion of an implicit method that has no analogue in the one space dimensional case. Chapters II and III concern the direct analogues of equations (VII) and (VIII). Each of the three procedures leads to a set of N^2 linear equations at each time step, where N is the number of internal points on the side of the mesh. The great difference between the schemes is that the first method gives equations each one of which involves three unknowns and the other two give equations involving five unknowns. Moreover, the equations in the first procedure break into N groups of N equations each, and each group may be solved directly by the method mentioned previously [1], whereas the equations of the second and third schemes must be solved by iteration.

Convergence theorems are given for all three methods. The result is that the three are essentially equivalent, as the truncation error is shown to be uniformly $O(\Delta t)$, assuming that the initial and boundary conditions are such that (I) possesses a six-times differentiable solution. No effort is made in this report to study the behavior of the rate of convergence with changing smoothness restrictions on the initial and boundary data.

In the appendix a lemma on Fourier coefficients is proved.

CHAPTER I

1. Approximation method. Consider the heat flow equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \tag{1}$$

in the semi-infinite cylinder $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, $t > 0$ with the initial condition

(1a) $$u(x, y, 0) = F(x, y)$$

and boundary conditions

(1b) $$\begin{cases} u(0, y, t) = f_1(y, t) \\ u(1, y, t) = f_2(y, t) \\ u(x, 0, t) = f_3(x, t) \\ u(x, 1, t) = f_4(x, t) \end{cases} \qquad (t > 0).$$

We shall assume that the initial and boundary conditions are such that a solution $u(x, y, t)$ exists in the *closed* region $0 \leqq x \leqq 1, 0 \leqq y \leqq 1, 0 \leqq t \leqq T$ and that u is six times continuously differentiable in the closed region.

Let

(2) $$g\left(\frac{i}{N}, \frac{j}{N}, n\Delta t\right) = g_{i,j,n}$$

where $\Delta x = \Delta y = 1/N$. Let

(3) $$\begin{cases} {}_x\Delta^2 g_{i,j,n} = (g_{i+1,j,n} - 2g_{i,j,n} + g_{i-1,j,n})/(\Delta x)^2 \\ {}_y\Delta^2 g_{i,j,n} = (g_{i,j+1,n} - 2g_{i,j,n} + g_{i,j-1,n})/(\Delta y)^2. \end{cases}$$

Replace the differential system (1) by the following finite difference approximation:

(4) $$\begin{cases} w_{i,j,0} = F_{i,j}, \\ {}_x\Delta^2 w_{i,j,2n} + {}_y\Delta^2 w_{i,j,2n+1} = (w_{i,j,2n+1} - w_{i,j,2n})/\Delta t \qquad (n \geqq 0), \\ {}_x\Delta^2 w_{i,j,2n+2} + {}_y\Delta^2 w_{i,j,2n+1} = (w_{i,j,2n+2} - w_{i,j,2n+1})/\Delta t \qquad (n \geqq 0), \\ w_{i,j,n} = u_{i,j,n} \qquad (i = 0 \text{ or } N, j = 0 \text{ or } N). \end{cases}$$

2. Difference equation for truncation error. Let us develop the difference equation for the truncation error.

(5) $$\begin{aligned} {}_x\Delta^2 u_{i,j,2n} &= \frac{\partial^2 u_{i,j,2n}}{\partial x^2} + \frac{1}{12}\frac{\partial^4 u}{\partial x^4}(\Delta x)^2 \\ &= \frac{\partial^2 u_{i,j,2n+\frac{1}{2}}}{\partial x^2} - \frac{1}{2}\frac{\partial^3 u}{\partial t\, \partial x^2}\Delta t + \frac{1}{12}\frac{\partial^4 u}{\partial x^4}(\Delta x)^2, \end{aligned}$$

where $\partial^4 u/\partial x^4$ is evaluated at $((i + \theta)\Delta x, j\Delta x, 2n\Delta t)$, $|\theta| < 1$, and $\partial^3 u/\partial t \partial x^2$ at $(i\Delta x, j\Delta x, (2n + \xi)\Delta t)$, $|\xi| < \frac{1}{2}$. Similarly,

(6) $${}_y\Delta^2 u_{i,j,2n+1} = \frac{\partial^2 u_{i,j,2n+\frac{1}{2}}}{\partial y^2} + \frac{1}{2}\frac{\partial^3 u}{\partial t\, \partial y^2}\Delta t + \frac{1}{12}\frac{\partial^4 u}{\partial y^4}(\Delta x)^2,$$

and

$$\frac{u_{i,j,2n+1} - u_{i,j,2n}}{\Delta t} = \frac{\partial u_{i,j,2n+\frac{1}{2}}}{\partial t} + \frac{1}{24}\frac{\partial^3 u}{\partial t^3}(\Delta t)^2. \tag{7}$$

As u is four times boundedly differentiable, if $(2n + 1)\Delta t \leqq T$,

$$\begin{aligned} {}_x\Delta^2 u_{i,j,2n} + {}_y\Delta^2 u_{i,j,2n+1} &= \frac{u_{i,j,2n+1} - u_{i,j,2n}}{\Delta t} + g_{i,j,2n}, \\ g_{i,j,2n} &= O(\Delta t + (\Delta x)^2). \end{aligned} \tag{8}$$

Similarly,

$$\begin{aligned} {}_x\Delta^2 u_{i,j,2n+2} + {}_y\Delta^2 u_{i,j,2n+1} &= \frac{u_{i,j,2n+2} - u_{i,j,2n+1}}{\Delta t} + g_{i,j,2n+1}, \\ g_{i,j,2n+1} &= O(\Delta t + \Delta x)^2). \end{aligned} \tag{9}$$

Let the truncation error be denoted by $v_{i,j,n}$. Then,

$$v_{i,j,n} = u_{i,j,n} - w_{i,j,n}. \tag{10}$$

Combining (4), (8), and (9),

$$\begin{cases} v_{i,j,0} = 0, \\ {}_x\Delta^2 v_{i,j,2n} + {}_y\Delta^2 v_{i,j,2n+1} = \dfrac{v_{i,j,2n+1} - v_{i,j,2n}}{\Delta t} + g_{i,j,2n} & (n \geqq 0), \\ {}_x\Delta^2 v_{i,j,2n+2} + {}_y\Delta^2 v_{i,j,2n+1} = \dfrac{v_{i,j,2n+2} - v_{i,j,2n+1}}{\Delta t} + g_{i,j,2n+1} & (n \geqq 0), \\ v_{i,j,n} = 0 \qquad (i = 0 \text{ or } N, \quad j = 0 \text{ or } N). \end{cases} \tag{11}$$

This is the basic equation for the truncation error; it is our purpose to give bounds on the size of $|v_{i,j,n}|$ in terms of the length of the time step. Since the difference equation (11) is linear, a superposition principle may be applied to simplify the analysis. Let*

$$v_n = \textstyle\sum_{k=0}^{n-1} v_n^{(k)} \tag{12}$$

where $v_n^{(k)}$ is defined as below:

$$\begin{cases} v_n^{(k)} = 0 & (n \leqq k), \\ {}_y\Delta^2 v_{2n+1}^{(2n)} - \dfrac{1}{\Delta t} v_{2n+1}^{(2n)} = g_{2n}, \\ {}_x\Delta^2 v_{2n+2}^{(2n+1)} - \dfrac{1}{\Delta t} v_{2n+2}^{(2n+1)} = g_{2n+1}, \end{cases} \tag{13}$$

* $v_n = v_{i,j,n}$.

and

$$(14)\quad \begin{cases} {}_x\Delta^2 v_{2n}^{(k)} + {}_y\Delta^2 v_{2n+1}^{(k)} = \dfrac{v_{2n+1}^{(k)} - v_{2n}^{(k)}}{\Delta t} & (2n > k), \\ {}_x\Delta^2 v_{2n+2}^{(k)} + {}_y\Delta^2 v_{2n+1}^{(k)} = \dfrac{v_{2n+2}^{(k)} - v_{2n+1}^{(k)}}{\Delta t} & (2n+1 > k). \end{cases}$$

Direct substitution shows that (12) is a solution of (11).

3. Homogeneous system. Each of the functions $v_{n-k-1}^{(k)}$ satisfies the homogeneous equation associated with (11) (or the analogue with x and y interchanged) except that the initial condition is not identically zero. Thus, the solution of the homogeneous equation with arbitrary initial condition will enable us to solve (11). Let

$$(15)\quad \begin{cases} v_{i,j,0} = f_{i,j}; \qquad f_{i,j} = 0 & (\text{if } i = 0 \text{ or } N,\ j = 0 \text{ or } N), \\ {}_x\Delta^2 v_{i,j,2n} + {}_y\Delta^2 v_{i,j,2n+1} = \dfrac{v_{i,j,2n+1} - v_{i,j,2n}}{\Delta t} & (n \geqq 0), \\ {}_x\Delta^2 v_{i,j,2n+2} + {}_y\Delta^2 v_{i,j,2n+1} = \dfrac{v_{i,j,2n+2} - v_{i,j,2n+1}}{\Delta t} & (n \geqq 0), \\ v_{i,j,n} = 0 & \text{if } i = 0 \text{ or } N,\ j = 0 \text{ or } N. \end{cases}$$

Let

$$(16)\qquad f_{i,j} = a_0 \sin \alpha x \sin \beta y,$$

and assume

$$(17)\qquad v_n = a_n \sin \alpha x \sin \beta y.$$

As

$$\Delta^2 \sin \alpha x = \left[-4 \sin^2 \frac{\alpha \Delta x}{2} \Big/ (\Delta x)^2 \right] \sin \alpha x,$$

$$(18)\quad \begin{cases} -4(\Delta x)^{-2} \sin^2 \dfrac{\alpha \Delta x}{2}\, a_{2n} - 4(\Delta y)^{-2} \sin^2 \dfrac{\beta \Delta y}{2}\, a_{2n+1} = (a_{2n+1} - a_{2n})/\Delta t \\ -4(\Delta x)^{-2} \sin^2 \dfrac{\alpha \Delta x}{2}\, a_{2n+2} - 4(\Delta y)^{-2} \sin^2 \dfrac{\beta \Delta y}{2}\, a_{2n+1} = (a_{2n+2} - a_{2n+1})/\Delta t \end{cases}$$

Thus,

$$(19)\quad \begin{cases} \dfrac{a_{2n+1}}{a_{2n}} = \dfrac{1 - 4\dfrac{\Delta t}{(\Delta x)^2} \sin^2 \dfrac{\alpha \Delta x}{2}}{1 + 4\dfrac{\Delta t}{(\Delta y)^2} \sin^2 \dfrac{\beta \Delta y}{2}} \\ \dfrac{a_{2n+2}}{a_{2n+1}} = \dfrac{1 - 4\dfrac{\Delta t}{(\Delta y)^2} \sin^2 \dfrac{\beta \Delta y}{2}}{1 + 4\dfrac{\Delta t}{(\Delta x)^2} \sin^2 \dfrac{\alpha \Delta x}{2}}. \end{cases}$$

Moreover,

$$\frac{a_{2n+2}}{a_{2n}} = \left(\frac{1 - 4\dfrac{\Delta t}{(\Delta x)^2}\sin^2\dfrac{\alpha\Delta x}{2}}{1 + 4\dfrac{\Delta t}{(\Delta x)^2}\sin^2\dfrac{\alpha\Delta x}{2}}\right)\left(\frac{1 - 4\dfrac{\Delta t}{(\Delta y)^2}\sin^2\dfrac{\beta\Delta y}{2}}{1 + 4\dfrac{\Delta t}{(\Delta y)^2}\sin^2\dfrac{\beta\Delta y}{2}}\right). \tag{20}$$

Now, for arbitrary α and β, the ratio $|a_{2n+1}/a_{2n}|$ or $|a_{2n+2}/a_{2n+1}|$ may be larger than one; however, the ratio $|a_{2n+2}/a_{2n}|$ is always less than one; i.e., stable. To prevent the intermediate ratio from becoming too large when Δt, Δx, and Δy tend to zero, set

$$4\frac{\Delta t}{(\Delta x)^2} = 4\frac{\Delta t}{(\Delta y)^2} = \lambda, \text{ a constant.} \tag{21}$$

Then, the intermediate ratios are limited by λ (assuming λ larger than one, as it will be for the desired large time step), as the numerator is no greater than the larger of one and $\lambda - 1$ in magnitude and the denominator is greater than one.

The solution with (16) as initial condition is

$$\begin{cases} v_{2n} = a_0\left(\dfrac{1 - \lambda\sin^2\dfrac{\alpha\Delta x}{2}}{1 + \lambda\sin^2\dfrac{\alpha\Delta x}{2}}\right)^n\left(\dfrac{1 - \lambda\sin^2\dfrac{\beta\Delta y}{2}}{1 + \lambda\sin^2\dfrac{\beta\Delta y}{2}}\right)^n \sin\alpha x\sin\beta y, \\ v_{2n+1} = a_0\left(\dfrac{1 - \lambda\sin^2\dfrac{\alpha\Delta x}{2}}{1 + \lambda\sin^2\dfrac{\alpha\Delta x}{2}}\right)^n\left(\dfrac{1 - \lambda\sin^2\dfrac{\beta\Delta y}{2}}{1 + \lambda\sin^2\dfrac{\beta\Delta y}{2}}\right)^n \\ \qquad\qquad \cdot\dfrac{1 - \lambda\sin^2\dfrac{\alpha\Delta x}{2}}{1 + \lambda\sin^2\dfrac{\beta\Delta x}{2}}\sin\alpha x\sin\beta y. \end{cases} \tag{22}$$

For arbitrary $f_{i,j}$ vanishing on the boundary of the unit square,

$$f_{i,j} = \textstyle\sum_{p,q=1}^{N-1} c_{pq}\sin\pi px\sin\pi qy, \tag{23}$$

where

$$c_{pq} = \frac{4}{N^2}\sum_{i,j=1}^{N-1} f_{i,j}\sin\frac{\pi pi}{N}\sin\frac{\pi qj}{N}. \tag{24}$$

Hence, the general solution of (15) is

$$(25)\quad \begin{cases} v_{2n} = \displaystyle\sum_{p,q=1}^{N=1} c_{pq} \left(\frac{1 - \lambda \sin^2 \frac{\pi p}{2N}}{1 + \lambda \sin^2 \frac{\pi p}{2N}} \right)^n \left(\frac{1 - \lambda \sin^2 \frac{\pi q}{2N}}{1 + \lambda \sin^2 \frac{\pi q}{2N}} \right)^n \sin \pi p x \sin \pi q y, \\ v_{2n+1} = \displaystyle\sum_{p,q=1}^{N-1} c_{pq} \left(\frac{1 - \lambda \sin^2 \frac{\pi p}{2N}}{1 + \lambda \sin^2 \frac{\pi p}{2N}} \right)^n \left(\frac{1 - \lambda \sin^2 \frac{\pi q}{2N}}{1 + \lambda \sin^2 \frac{\pi q}{2N}} \right)^n \\ \qquad\qquad \left(\frac{1 - \lambda \sin^2 \frac{\pi q}{2N}}{1 + \lambda \sin^2 \frac{\pi p}{2N}} \right) \sin \pi p x \sin \pi q y. \end{cases}$$

4. A convergence theorem. Equation (25) may be applied to obtain a result guaranteeing the convergence of the solution of (4) to that of (1) uniformly. Let

$$(26)\qquad v_{n+1}^{(n)} = \textstyle\sum_{p,q=1}^{N-1} c_{p,q}^{(n)} \sin \pi p x \sin \pi q y.$$

Then,*

$$(27)\quad \begin{cases} v_{2n+1}^{(2k)} = \displaystyle\sum_{p,q=1}^{N-1} c_{pq}^{(2k)} \left(\frac{1 - \lambda \sin^2 \frac{\pi p}{2N}}{1 + \lambda \sin^2 \frac{\pi p}{2N}} \right)^{n-k} \left(\frac{1 - \lambda \sin^2 \frac{\pi q}{2N}}{1 + \lambda \sin^2 \frac{\pi q}{2N}} \right)^{n-k} \\ \qquad\qquad \sin \pi p x \sin \pi q y \\ v_{2n+1}^{(2k+1)} = \displaystyle\sum_{p,q=1}^{N-1} c_{pq}^{(2k+1)} \left(\frac{1 - \lambda \sin^2 \frac{\pi p}{2N}}{1 + \lambda \sin^2 \frac{\pi q}{2N}} \right)^{n-k-1} \\ \qquad\qquad \times \left(\frac{1 - \lambda \sin^2 \frac{\pi q}{2N}}{1 + \lambda \sin^2 \frac{\pi q}{2N}} \right)^{n-k-1} \left(\frac{1 - \lambda \sin^2 \frac{\pi p}{2N}}{1 + \lambda \sin^2 \frac{\pi q}{2N}} \right) \sin \pi p x \sin \pi q y. \end{cases}$$

Thus,

---

* Only v_{2n+1} will be discussed; the argument for v_{2n} is similar.

$$
(27)\quad \left\{
\begin{aligned}
v_{2n+1} = {} & \sum_{k=0}^{n-1} \sum_{p,q=1}^{N-1} c_{pq}^{(2k)} \left(\frac{1 - \lambda \sin^2 \frac{\pi p}{2N}}{1 + \lambda \sin^2 \frac{\pi p}{2N}} \right)^{n-k} \left(\frac{1 - \lambda \sin^2 \frac{\pi q}{2N}}{1 + \lambda \sin^2 \frac{\pi q}{2N}} \right)^{n-k} \\
& \sin \pi p x \sin \pi q y \\
& + \sum_{k=0}^{n-1} \sum_{p,q=1}^{N-1} c_{pq}^{(2k+1)} \left(\frac{1 - \lambda \sin^2 \frac{\pi p}{2N}}{1 + \lambda \sin^2 \frac{\pi p}{2N}} \right)^{n-k-1} \\
& \times \left(\frac{1 - \lambda \sin^2 \frac{\pi q}{2N}}{1 + \lambda \sin^2 \frac{\pi q}{2N}} \right)^{n-k-1} \left(\frac{1 - \lambda \sin^2 \frac{\pi p}{2N}}{1 + \lambda \sin^2 \frac{\pi q}{2N}} \right) \sin \pi p x \sin \pi q y.
\end{aligned}
\right.
$$

To investigate the behavior of v_{2n+1} as N increases, it is necessary to study the properties of the Fourier coefficients and also of the ratio

$$
\frac{1 - \lambda \sin^2 \frac{\pi p}{2N}}{1 + \lambda \sin^2 \frac{\pi p}{2N}}
$$

as functions of the indices p and q. The following lemma on the Fourier coefficients is proved in the appendix:

LEMMA. *Let $f(x, y)$ be twice boundedly differentiable for $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. Let*

$$
c_{pq} = \frac{4}{N^2} \sum_{i,j=1}^{N-1} f\left(\frac{i}{N}, \frac{j}{N}\right) \sin \frac{\pi p i}{N} \sin \frac{\pi q j}{N} \quad (p, q = 1, \cdots, N-1).
$$

Then, there exists a constant $A > 0$ depending only on the magnitudes of $f(x, y)$ and its first two derivatives such that

$$
| c_{pq} | \leqq A \left(\frac{1}{pq} + \frac{1}{N^{2\epsilon}} \right)
$$

if $1 \leqq p, q \leqq N^{1-\epsilon}$.

In particular, if $f(x, y)$ contains $(\Delta t)^2$ as a factor, then A does as well. Note that there is no assumption that $f(x, y)$ vanishes on the boundary; hence, it applies to the coefficients $c_{pq}^{(k)}$ uniformly in k as u is six times continuously differentiable. Moreover, $c_{pq}^{(k)}$ contains a factor $(\Delta t)^2$. This follows from the following lemma.

LEMMA. *If* ${}_x\Delta^2 v - \rho(x)v = g(x)$, $v(0) = v(1) = 0$, and $\rho > 0$, *then*

(29) $$\| v \| \leqq \| g\rho^{-1} \|^*.$$

Proof: Let $g > 0$. First, $v < 0$; for if not, at the first positive maximum,

$$-\rho v = g - {}_x\Delta^2 v > 0,$$

which contradicts the assumption that v were positive. At a negative minimum,

$$-v = \rho^{-1}(g - {}_x\Delta^2 v) \leqq \rho^{-1} g.$$

Thus, the result follows for positive g. Similarly, for $g \leqq 0$, $v \geqq 0$ and (29) is satisfied. The proof is completed by separating g into its positive and negative parts.

We see from (13) that

(30) $${}_x\Delta^2 v_{n+1}^{(n)} - \frac{1}{\Delta t} v_{n+1}^{(n)} = g_n, \qquad v_{n+1}^{(n)}(0) = v_{n+1}^{(n)}(1) = 0,$$

where the second difference is taken in the x or y direction as n is odd or even. Hence,

(31) $$\| v_{n+1}^{(n)} \| \leqq \| g_n \| \, \Delta t.$$

As $g_n = O(\Delta t)$,

(32) $$\| v_{n+1}^{(n)} \| = O((\Delta t)^2).$$

Now, applying the lemma on Fourier coefficients,

(33) $$\begin{cases} c_{pq}^{(n)} = O\left(\left(\dfrac{1}{pq} + \dfrac{1}{N^{2\epsilon}}\right)(\Delta t)^2\right) & (0 \leqq p, q \leqq N^{1-\epsilon}), \\ c_{pq}^{(n)} = O((\Delta t)^2) & (N^{1-\epsilon} < p \text{ or } q \leqq N). \end{cases}$$

A bound for the stability ratio may be obtained as follows. Let

(34) $$\rho = \frac{1 - \lambda \sin^2 z}{1 + \lambda \sin^2 z}.$$

Then,

(35) $$| \rho | = | 1 + 2 \textstyle\sum_{k=1}^{\infty} (-1)^k (\lambda \sin^2 z)^k | < 1 - 2\lambda \sin^2 z + 2\lambda^2 \sin^4 z,$$

if $\lambda \sin^2 z < 1$. Moreover, there exists z_0 such that if $0 < z < z_0$,

(36) $$| \rho | < 1 - \lambda \sin^2 z < 1 - \lambda \frac{z^2}{4},$$

as $z/2 < \sin z < z$ for $0 < z < \pi/2$.

* $\| v \| = \max | v |$.

For $z_0 \leqq z \leqq \pi/2$, there exists $\delta > 0$ such that

$$|\rho| \leqq 1 - \delta. \tag{37}$$

A posteriori,

$$|\rho| \leqq 1 - \delta\left(\frac{2}{\pi}\right)^2 z^2. \tag{38}$$

Thus, if $\eta = \min(\delta, \lambda\pi^2/16)$,

$$|\rho| \leqq 1 - \eta\left(\frac{2z}{\pi}\right)^2 \qquad (0 \leqq x \leqq \pi/2). \tag{39}$$

For $z = \pi p/2N$,

$$|\rho| \leqq 1 - \eta\frac{p^2}{N^2}. \tag{40}$$

Hence,

$$|\rho| \leqq \begin{cases} 1 - \eta N^{-2}, & (1 \leqq p \leqq N^{1-\epsilon}) \\ 1 - \eta N^{-2\epsilon}, & (N^{1-\epsilon} \leqq p \leqq N - 1). \end{cases} \tag{41}$$

Let us consider the first sum in (28).

$$\begin{aligned} &\left| \sum_{k=0}^{n-1} \sum_{p,q=1}^{N-1} c_{pq}^{(2k)} \left(\frac{1 - \lambda \sin^2 \dfrac{\pi p}{2N}}{1 + \lambda \sin^2 \dfrac{\pi p}{2N}} \right)^{n-k} \left(\frac{1 - \lambda \sin^2 \dfrac{\pi q}{2N}}{1 + \lambda \sin^2 \dfrac{\pi q}{2N}} \right)^{n-k} \sin \pi p x \sin \pi q y \right| \\ &\leqq \sum_{k=0}^{n-1} \sum_{p,q=1}^{N^{1-\epsilon}} | c_{pq}^{(2k)} | \left(1 - \frac{\eta}{N^2}\right)^{n-k} + \sum_{k=0}^{n-1} \sum_{p \text{ or } q > N^{1-\epsilon}}^{N-1} | c_{pq}^{(2k)} | \left(1 - \frac{\eta}{N^{2\epsilon}}\right)^{n-k} \\ &< A(\Delta t)^2 \sum_{p,q=1}^{N^{1-\epsilon}} \left(\frac{1}{pq} + \frac{1}{N^{2\epsilon}}\right) \sum_{k=0}^{n-1} \left(1 - \frac{\eta}{N^2}\right)^k \\ &\qquad + A(\Delta t)^2 \sum_{p \text{ or } q > N^{1-\epsilon}} 1 \times \sum_{k=0}^{n-1} \left(1 - \frac{\eta}{N^{2\epsilon}}\right)^k \\ &< \frac{A}{\eta} (\Delta t)^2 \left\{ N^2 \sum_{p,q=1}^{N^{1-\epsilon}} \left(\frac{1}{pq} + \frac{1}{N^{2\epsilon}}\right) + N^{2\epsilon} \sum_{p \text{ or } q > N^{1-\epsilon}}^{N-1} 1 \right\}. \end{aligned}$$

Now,

$$\left\{ \sum_{p,q=1}^{N^{1-\epsilon}} \frac{1}{pq} \approx (1 - \epsilon)^2 (\log N)^2, \qquad \sum_{p,q=1}^{N^{1-\epsilon}} \frac{1}{N^{2\epsilon}} = N^{2-4\epsilon}, \sum_{p \text{ or } q > N^{1-\epsilon}}^{n-1} 1 < N^2. \right. \tag{43}$$

Thus, as $\Delta t = O(N^{-2})$,

$$(44)\quad \left| \sum_{k=0}^{n-1} \sum_{p,q=1}^{N-1} c_{pq}^{(2k)} \left(\frac{1 - \lambda \sin^2 \frac{\pi p}{2N}}{1 + \lambda \sin^2 \frac{\pi p}{2N}} \right)^{n-k} \left(\frac{1 - \lambda \sin^2 \frac{\pi q}{2N}}{1 + \lambda \sin^2 \frac{\pi q}{2N}} \right)^{n-k} \sin \pi p x \sin \pi q y \right|$$
$$= O(N^{-4\epsilon} + N^{-2}(\log N)^2 + N^{-2+2\epsilon}).$$

The second sum in (28) may be approximated in the same manner with the same result. Thus,

$$(45)\qquad \| v_{2n+1} \| = O(N^{-2+2\epsilon} + N^{-4\epsilon} + N^{-2} (\log N)^2) \quad (0 \leqq \epsilon \leqq 1).$$

The best result is obtained by choosing ϵ so that the first two exponents are equal. Thus $\epsilon = \frac{1}{3}$, and

$$(46)\qquad \| v_n \| = O(N^{-4/3}) = O((\Delta t)^{2/3})$$

uniformly in $0 \leqq x, y \leqq 1, 0 \leqq t \leqq T$. Collecting, we have the

THEOREM 1. *If, in the region $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, $0 \leqq t \leqq T$, there exists a solution $u(x, y, t)$ of the boundary value problem* (1) *having continuous derivatives of the sixth order, then the solution of the finite difference scheme* (4) *converges uniformly to $u(x, y, t)$ as $\Delta t \to 0$, $(\Delta t/(\Delta x)^2 = \lambda =$ constant), with the error being $O((\Delta t)^{2/3}) = O(\Delta x^{4/3})$ at worst.*

5. Refinement. To obtain the above result, the sums in (28) were broken into two parts; a better bound can be calculated by breaking the sums into many parts. Divide the set of index pairs (p, q) into the following subsets:

$$(47)\qquad \begin{cases} I_1 : 0 < p, q \leqq N^{1-\epsilon_1} & \\ I_j : p \text{ or } q > N^{1-\epsilon_{j-1}} & (1 \leqq p, q \leqq N^{1-\epsilon_j}, j = 2, \cdots, m), \\ I_{m+1} : p \text{ or } q > N^{1-\epsilon_m} & (1 \leqq p, q \leqq N - 1). \end{cases}$$

If $\epsilon_0 = 1$ and $\epsilon_{m+1} = 0$ are included, the general description for I_j holds for $j = 1, 2, \cdots, m + 1$. Let

$$(48)\qquad \begin{aligned} S_j = \sum_{k=0}^{n-1} \sum_{(p,q)\epsilon I_j} c_{pq}^{(2k)} &\left(\frac{1 - \lambda \sin^2 \frac{\pi p}{2N}}{1 + \lambda \sin^2 \frac{\pi p}{2N}} \right)^{n-k} \\ &\cdot \left(\frac{1 - \lambda \sin^2 \frac{\pi q}{2N}}{1 + \lambda \sin^2 \frac{\pi q}{2N}} \right)^{n-k} \sin \pi p x \sin \pi q y \end{aligned}$$

From the lemma on Fourier coefficients and (41),

$$
(49)\qquad \begin{cases} |c_{pq}^{(2k)}| < A\left(\dfrac{1}{pq} + \dfrac{1}{N^{2\epsilon_j}}\right)(\Delta t)^2, \\ \left|\left(\dfrac{1 - \lambda \sin^2 \dfrac{\pi p}{2N}}{1 + \lambda \sin^2 \dfrac{\pi p}{2N}}\right)\left(\dfrac{1 - \lambda \sin^2 \dfrac{\pi q}{2N}}{1 + \lambda \sin^2 \dfrac{\pi q}{2N}}\right)\right| < 1 - \dfrac{\eta}{N^{2\epsilon_{j-1}}}. \end{cases} \qquad (p, q) \in I_j
$$

Thus,

$$
(50)\qquad \begin{aligned} |S_j| &\leqq A(\Delta t)^2 \sum_{k=0}^{n-1} \sum_{(p,q)\in I_j} \left(\frac{1}{pq} + \frac{1}{N^{2\epsilon_j}}\right)\left(1 - \frac{\eta}{N^{2\epsilon_{j-1}}}\right)^{n-k} \\ &< A\eta^{-1}(\Delta t)^2 N^{2\epsilon_{j-1}} \sum_{(p,q)\in I_j} \left(\frac{1}{pq} + \frac{1}{N^{2\epsilon_j}}\right) \end{aligned}
$$

Now,

$$
(51)\qquad \sum_{(p,q)\in I_j} \frac{1}{pq} < 2 \sum_{p=N^{1-\epsilon_{j-1}}}^{N^{1-\epsilon_j}} \sum_{q=1}^{N^{1-\epsilon_j}}{}' \frac{1}{pq} < 3(1 - \epsilon_j)(\epsilon_{j-1} - \epsilon_j)(\log N)^2,
$$

$$
(52)\qquad \sum_{(p,q)\in I_j} \frac{1}{N^{2\epsilon_j}} = N^{-2\epsilon_j}(N^{2-2\epsilon_j} - N^{2-2\epsilon_{j-1}}).
$$

Set

$$
(53)\qquad \epsilon_j = 1 - \frac{j}{m+1} \qquad (j = 0, \cdots, m+1).
$$

Then,

$$
(54)\qquad \begin{aligned} |S_j| &\leqq A\eta^{-1}(\Delta t)^2 N^{2-(2(j-1)/m+1)} \\ &\qquad \left\{\frac{3j}{(m+1)^2}(\log N)^2 + N^{-2+(2j/m+1)}(N^{2j/m+1} - N^{(2(j-1)/m+1)})\right\} \\ &= A\eta^{-1}(\Delta t)^2\left\{\frac{3(\log N)^2 N^{2+(2/m+1)}}{(m+1)^2} jN^{-2j/m+1} + (N^{2/m+1} - 1)N^{2j/m+1}\right\}. \end{aligned}
$$

Summing,

$$
(55)\qquad \begin{aligned} \sum_{j=1}^{m+1} |S_j| &\leqq A\eta^{-1}(\Delta t)^2 \\ &\quad \cdot \left\{\frac{3(\log N)^2 N^{2+(2/m+1)}}{(m+1)^2} \sum_{j=1}^{m+1} jN^{-2j/m+1} + (N^{2/m+1} - 1)\sum_{j=1}^{m+1} N^{2j/m+1}\right\} \\ &< A\eta^{-1}(\Delta t)^2\left\{\frac{3(\log N)^2 N^2}{(m+1)^2(1 - N^{-2/(m+1)})^2} + N^{2+(2/m+1)}\right\}, \end{aligned}
$$

as

$$(56) \quad \begin{cases} \sum_{j=1}^{m+1} jx^j < \sum_{j=1}^{\infty} jx^j = \dfrac{x}{(1-x)^2} & (0 \leqq x < 1), \\ \sum_{j=1}^{m+1} y^j = y\,\dfrac{y^{m+1}-1}{y-1} & (y \neq 1). \end{cases}$$

The choice of m can be made anywhere between one and N. Take

$$(57) \qquad m + 1 = \log N.$$

Then, $N^{2/m+1} = e^2$, and

$$(58) \quad \begin{aligned} \textstyle\sum_{j=1}^{m+1} | S_j | &< A\eta^{-1}(\Delta t)^2 \,\{3(1 - e^{-2})^{-2} + e^2\}N^2 \\ &= \tfrac{1}{16}A\lambda^2\eta^{-1}\,\{3(1 - e^{-2})^{-2} + e^2\}\,N^{-2} = O(N^{-2}). \end{aligned}$$

The second sum in (28) can also be approximated as above. Thus,

$$(59) \qquad \| v_n \| = O(N^{-2}) = O(\Delta t)$$

uniformly in $0 \leqq x, y \leqq 1, 0 \leqq t \leqq T$.

THEOREM 2. *Under the same assumptions as in Theorem* 1, *the truncation error of method* (4) *is* $O(\Delta t)$ *for any* $\lambda = 4\Delta t/(\Delta x)^2$.

CHAPTER II

6. Second approximation method. Again we shall treat the boundary value problem (1). Consider as a second implicit scheme for integrating (1) the backwards difference equation:

$$(60) \quad \begin{cases} w_{i,j,0} = F_{i,j} \\ {}_x\Delta^2 w_{i,j,n+1} + {}_y\Delta^2 w_{i,j,n+1} = \dfrac{w_{i,j,n+1} - w_{i,j,n}}{\Delta t} & (n \geqq 0), \\ w_{i,j,n} = u_{i,j,n} & \text{(on boundary).} \end{cases}$$

Note that all of the Laplacian operator is approximated on the advanced time line. As a consequence, the linear equations are considerably more complex than in the first method.

7. Difference equation for truncation error. The solution u of (1) satisfies a non-homogeneous difference equation analogous to (60). It can be obtained as follows.

$$(61) \quad \begin{cases} {}_x\Delta^2 u_{i,j,n+1} + {}_y\Delta^2 u_{i,j,n+1} \\ \qquad = \dfrac{\partial^2 u_{i,j,n+1}}{\partial x^2} + \dfrac{\partial^2 u_{i,j,n+1}}{\partial y^2} + \dfrac{1}{12}\left(\dfrac{\partial^4 u}{\partial x^4} + \dfrac{\partial^4 u}{\partial y^4}\right)(\Delta x)^2 \\ \dfrac{u_{i,j,n+1} - u_{i,j,n}}{\Delta t} = \dfrac{\partial u_{i,j,n+1}}{\partial t} - \dfrac{1}{2}\dfrac{\partial^2 u}{\partial t^2}\,\Delta t, \end{cases}$$

where $\partial^4 u/\partial x^4$, $\partial^4 u/\partial y^4$, and $\partial^2 u/\partial t^2$ are evaluated at $((i + \theta_1)\Delta x, j\Delta x, (n + 1)\Delta t, (i\Delta x, (j + \theta_2)\Delta x, (n + 1)\Delta t)$, and $(i\Delta x, j\Delta x, (n + \theta_3)\Delta t)$, respectively, with $|\theta_i| < 1$. Setting

$$\frac{4\Delta t}{(\Delta x)^2} = \frac{4\Delta t}{(\Delta y)^2} = \lambda, \tag{62}$$

$$\begin{cases} {}_x\Delta^2 u_{i,j,n+1} + {}_y\Delta^2 u_{i,j,n+1} = \dfrac{u_{i,j,n+1} - u_{i,j,n}}{\Delta t} + g_{i,j,n}, \\ g_{i,j,n} = O(\Delta t), \end{cases} \tag{63}$$

assuming $\partial^4 u/\partial x^4$, $\partial^4 u/\partial y^4$, and $\partial^2 u/\partial t^2$ to be bounded in $0 \leqq x, y \leqq 1$, $0 \leqq t \leqq T$. Set

$$v_{i,j,n} = u_{i,j,n} - w_{i,j,n}. \tag{64}$$

Subtracting (60) from (63),

$$\begin{cases} v_{i,j,0} = 0 \\ {}_x\Delta^2 v_{i,j,n+1} + {}_y\Delta^2 v_{i,j,n+1} = \dfrac{v_{i,j,n+1} - v_{i,j,n}}{\Delta t} + g_{i,j,n} \\ v_{i,j,n} = 0 \qquad \text{(on boundary).} \end{cases} \tag{65}$$

A different method of attack based on [12] will be used on the truncation error equation (65) from the harmonic analysis procedure for equation (11).

8. Bound for solution of (65). The following lemma will be useful in relating the magnitude of v_{n+1} to that of v_n.

LEMMA. *If* ${}_x\Delta^2 v_{i,j} + {}_y\Delta^2 v_{i,j} - \rho(x, y)v_{i,j} = g_{i,j}$, $0 \leqq x, y \leqq 1$, $\rho > 0$, *and* $v = 0$ *on the boundary, then*

$$\max |v(x, y)| \leqq \max \left| \frac{g(x, y)}{\rho(x, y)} \right|.$$

PROOF: *Case 1.* $g \leqq 0$. First, $v \geqq 0$. Otherwise, there would exist (x, y) at which v would be at a negative minimum. Then,

$$-\rho(x, y)v(x, y) = g(x, y) - {}_x\Delta^2 v(x, y) - {}_y\Delta^2 v(x, y) \leqq 0,$$

as the second differences are positive at a minimum. But the left hand side must be positive; thus, a contradiction arises from the assumption of negative v. Therefore, $v \geqq 0$. Consider then a point (x, y) at which v has a maximum:

$$-\rho(x, y)v(x, y) = g(x, y) - {}_x\Delta^2 v(x, y) - {}_y\Delta^2 v(x, y) \geqq g(x, y),$$

as the second differences are negative. As $\rho(x, y) > 0$,

$$v(x, y) \leqq -\frac{g(x, y)}{\rho(x, y)}.$$

Thus, the conclusion holds for $g \leqq 0$.

Case 2. $g \geqq 0$. In like manner, $v \leqq 0$ and $-v(x, y) \leqq g(x, y)/\rho(x, y)$ at minimum.

Case 3. $g = g^+ + g^-$, where $g^+g^- = 0, g^+ \geqq 0, g^- \leqq 0$. Then, $v = v^+ + v^-$, and $v^+ \leqq 0$, $v^- \geqq 0$. It is easily seen that

$$\max | v(x, y) | \leqq \max \left| \frac{g(x, y)}{\rho(x, y)} \right|.$$

Rewriting (65),

$$(66) \qquad {}_x\Delta^2 v_{n+1} + {}_y\Delta^2 v_{n+1} - \frac{1}{\Delta t} v_{n+1} = -\frac{1}{\Delta t} v_n + g_n.$$

As $\Delta t > 0$ and v_n vanishes on the boundary,

$$(67) \qquad \max | v_{n+1} | \leqq \max | v_n | + \Delta t \max | g_n |.$$

Now, as $g_n = O(\Delta t)$, there exists A such that

$$(68) \qquad \max | v_{n+1} | \leqq \max | v_n | + A(\Delta t)^2.$$

As $| v_0 | = 0$,

$$(69) \qquad \max | v_n | \leqq nA(\Delta t)^2 = At(\Delta t),$$

and, consequently, v_n tends to zero uniformly in the closed region $0 \leqq x$, $y \leqq 1$, $0 \leqq t \leqq T$ as Δt tends to zero. Collecting, we have

THEOREM 3. *If there exists a solution u of* (1) *in* $0 \leqq x, y \leqq 1, 0 \leqq t \leqq T$ *having bounded u_{xxxx}, u_{yyyy}, and u_{tt}, then the solution of* (60) *converges uniformly to u in this region with the error being $O(t\Delta t)$.*

The convergence for this method then is of the same order as that of method I; it should be noted that the method of proof in this case allows the use of a weaker hypothesis. However, when both methods apply, the first method involves only $O(N^2)$ operations per time step and the second requires $O(N^3)$ by standard iteration techniques or $O(N^2 \log N)$ using an iteration technique based on the first method itself [11].

The numerical stability of method II follows from the lemma as applied to the homogeneous form of (66).

CHAPTER III

9. Third approximation method. As a third implicit method, let us consider a direct extension of the procedure (VIII) of the introduction.

$$
(70)\quad \begin{cases} w_{i,j,0} = F_{i,j} \\ {}_x\Delta^2 w_{i,j,n+1} + {}_y\Delta^2 w_{i,j,n+1} + {}_x\Delta^2 w_{i,j,n} + {}_y\Delta^2 w_{i,j,n} \\ \qquad = 2\,\dfrac{w_{i,j,n+1} - w_{i,j,n}}{\Delta t} \qquad (n \geqq 0), \\ w_{i,j,n} = u_{i,j,n} \qquad \text{(on boundary)}. \end{cases}
$$

The inclusion of the Laplacian difference operator at the old time level will reduce the elemental error due to truncation in the time direction without introducing any appreciable addition to the amount of computation for the completion of one time step.

10. Difference equation for truncation error. Assuming that the solution u is six times boundedly differentiable in $0 \leqq x, y \leqq 1, 0 \leqq t \leqq T$, it is easy to see that

$$
(71)\quad \begin{cases} {}_x\Delta^2 u_{i,j,n+1} + {}_y\Delta^2 u_{i,j,n+1} + {}_x\Delta^2 u_{i,j,n} + {}_y\Delta^2 u_{i,j,n} \\ \qquad = 2\,\dfrac{u_{i,j,n+1} - u_{i,j,n}}{\Delta t} + g_{i,j,n} \\ g_{i,j,n} = O((\Delta x)^2 + (\Delta t)^2) = O(\Delta t), \end{cases}
$$

if

$$
(72)\quad 4\,\frac{\Delta t}{(\Delta x)^2} = \lambda
$$

If we again let

$$
(73)\quad v_{i,j,n} = u_{i,j,n} - w_{i,j,n},
$$

we obtain the basic truncation error equation by subtracting (70) from (71):

$$
(74)\quad \begin{cases} v_{i,j,0} = 0 \\ {}_x\Delta^2 v_{i,j,n+1} + {}_y\Delta^2 v_{i,j,n+1} + {}_x\Delta^2 v_{i,j,n} + {}_y\Delta^2 v_{i,j,n} \\ \qquad = 2\,\dfrac{v_{i,j,n+1} - v_{i,j,n}}{\Delta t} + g_{i,j,n} \\ v_{i,j,n} = 0 \qquad \text{(on boundary)}. \end{cases}
$$

The appearance of the second differences on the known time level prevents the useful application of the method of analysis of Chapter II. We must return to the harmonic analysis procedure of Chapter I.

11. Homogeneous system. As a preliminary to the solution of (74) by means of superposition, we shall study the corresponding homogeneous equation. Let

$$
(75) \quad \begin{cases} v_{i,j,0} = f_{i,j} & (f_{i,j} = 0 \text{ on boundary}), \\ {}_x\Delta^2 v_{i,j,n+1} + {}_y\Delta^2 v_{i,j,n+1} + {}_x\Delta^2 v_{i,j,n} + {}_y\Delta^2 v_{i,j,n} = 2\,\dfrac{v_{i,j,n+1} - v_{i,j,n}}{\Delta t} & \\ v_{i,j,n} = 0 & (\text{on boundary}). \end{cases}
$$

As f_{ij} can be expanded in a finite Fourier series, it is sufficient to consider the special case:

$$
(76) \qquad f_{ij} = a_0 \sin \alpha x_i \sin \beta y_j .
$$

Assume that

$$
(77) \qquad v_{i,j,n} = a_n \sin \alpha x_i \sin \beta y_j .
$$

Substituting into (75) and using (62), we obtain

$$
-a_{n+1}\lambda\left(\sin^2\frac{\alpha\Delta x}{2} + \sin^2\frac{\beta\Delta x}{2}\right) - a_n\lambda\left(\sin^2\frac{\alpha\Delta x}{2} + \sin^2\frac{\beta\Delta x}{2}\right) = 2(a_{n+1} - a_n).
$$

Hence,

$$
(78) \qquad \frac{a_{n+1}}{a_n} = \frac{1 - \frac{\lambda}{2}\left(\sin^2\frac{\alpha\Delta x}{2} + \sin^2\frac{\beta\Delta x}{2}\right)}{1 + \frac{\lambda}{2}\left(\sin^2\frac{\alpha\Delta x}{2} + \sin^2\frac{\beta\Delta x}{2}\right)}.
$$

As $\lambda > 0$, the stability ratio is less than one in magnitude, i.e., stable. The solution of (75) under the initial condition (76) is

$$
(79) \qquad v_{i,j,n} = a_o\left[\frac{1 - \frac{\lambda}{2}\left(\sin^2\frac{\alpha\Delta x}{2} + \sin^2\frac{\beta\Delta x}{2}\right)}{1 + \frac{\lambda}{2}\left(\sin^2\frac{\alpha\Delta x}{2} + \sin^2\frac{\beta\Delta x}{2}\right)}\right]^n \sin \alpha x_i \sin \beta y_i .
$$

As

$$
(80) \qquad f_{ij} = \textstyle\sum_{p,q=1}^{N-1} c_{pq} \sin \pi p x_i \sin \pi q y_j ,
$$

$$
(81) \quad v_{i,j,n} = \sum_{p,q=1}^{N-1} c_{pq}\left[\frac{1 - \frac{\lambda}{2}\left(\sin^2\frac{\pi p\Delta x}{2} + \sin^2\frac{\pi q\Delta x}{2}\right)}{1 + \frac{\lambda}{2}\left(\sin^2\frac{\pi p\Delta x}{2} + \sin^2\frac{\pi q\Delta x}{2}\right)}\right]^n \sin \pi p x_i \sin \pi q y_i .
$$

12. Convergence theorem. To apply the results of the last section to (74), it is necessary to express v_n as a sum of functions satisfying homogeneous equations. Let

$$
(82) \qquad v_n = \textstyle\sum_{k=0}^{n-1} v_n^{(k)},
$$

where

$$
(83) \quad \begin{cases} v_m^{(k)} = 0 & (m \leqq k), \\ {}_x\Delta^2 v_{k+1}^{(k)} + {}_y\Delta^2 v_{k+1}^{(k)} - \dfrac{2}{\Delta t} v_{k+1}^{(k)} = g_k & (m = k+1) \\ {}_x\Delta^2 v_m^{(k)} + {}_y\Delta^2 v_m^{(k)} + {}_x\Delta^2 v_{m-1}^{(k)} + {}_y\Delta^2 v_{m-1}^{(k)} = 2\,\dfrac{v_m^{(k)} - v_{m-1}^{(k)}}{\Delta t} & (m \geqq k+2) \\ v_m^{(k)} = 0 & \text{(on boundary).} \end{cases}
$$

To show that v_n as defined by (82) satisfies the difference equation (74), let us substitute directly:

$$
(84) \quad \begin{aligned} &{}_x\Delta^2 v_{n+1} + {}_y\Delta^2 v_{n+1} + {}_x\Delta^2 v_n + {}_y\Delta^2 v_n - 2\,\frac{v_{n+1} - v_n}{\Delta t} \\ &\quad = ({}_x\Delta^2 + {}_y\Delta^2)\left\{\sum_{k=0}^{n} v_{n+1}^{(k)} + \sum_{k=1}^{n-1} v_n^{(k)}\right\} - \frac{2}{\Delta t}\left\{\sum_{k=1}^{n} v_{n+1}^{(k)} - \sum_{k=0}^{n-1} v_n^{(k)}\right\} \\ &\quad = {}_x\Delta^2 v_{n+1}^{(k)} + {}_y\Delta^2 v_{n+1}^{(n)} - \frac{2}{\Delta t} v_{n+1}^{(n)} \\ &\qquad + \sum_{k=0}^{n-1}\left\{({}_x\Delta^2 + {}_y\Delta^2)(v_{n+1}^{(k)} + v_n^{(k)}) - 2\,\frac{v_{n+1}^{(k)} - v_n^{(k)}}{\Delta t}\right\} = g_n, \end{aligned}
$$

by (83). Thus, (74) is satisfied. Next, let us obtain a bound for $v_{n+1}^{(n)}$; the lemma of section 8 may again be used.

$$
(85) \quad \max |v_{k+1}^{(k)}| \leqq \tfrac{1}{2}\Delta t \max |g_k| = O((\Delta t)^2).
$$

Expand $v_{k+1}^{(k)}$ in its Fourier series:

$$
(86) \quad v_{k+1}^{(k)} = \textstyle\sum_{p,q=1}^{N-1} c_{pq}^{(k)} \sin \pi p x \sin \pi q y.
$$

Then,

$$
(87) \quad v_n^{(k)} = \sum_{p,q=1}^{N-1} c_{pq}^{(k)} \left[\frac{1 - \frac{\lambda}{2}\left(\sin^2 \frac{\pi p \Delta x}{2} + \sin^2 \frac{\pi q \Delta x}{2}\right)}{1 + \frac{\lambda}{2}\left(\sin^2 \frac{\pi p \Delta x}{2} + \sin^2 \frac{\pi q \Delta x}{2}\right)}\right]^{n-k-1} \sin \pi p x \sin \pi q y.
$$

Adding,

$$
(88) \quad v_n = \sum_{k=0}^{n-1} \sum_{p,q=1}^{N-1} c_{pq}^{(k)} \left[\frac{1 - \frac{\lambda}{2}\sin^2\left(\frac{\pi p \Delta x}{2} + \sin^2 \frac{\pi q \Delta x}{2}\right)}{1 + \frac{\lambda}{2}\sin^2\left(\frac{\pi p \Delta x}{2} + \sin^2 \frac{\pi q \Delta x}{2}\right)}\right]^{n-k-1} \sin \pi p x \sin \pi q y.
$$

Under the assumption that u is six times continuously differentiable, g_k is twice boundedly differentiable; thus, the lemma of section 4 on Fourier coefficients is again applicable.

$$c_{pq}^{(k)} = \begin{cases} O((p^{-1}q^{-1} + N^{-2\epsilon})(\Delta t)^2) & (1 \leqq p, q \leqq N^{1-\epsilon}) \\ O((\Delta t)^2) & (p \text{ or } q > N^{1-\epsilon}). \end{cases} \tag{89}$$

Also,

$$\left| \frac{1 - \frac{\lambda}{2}\left(\sin^2 \frac{\pi p \Delta x}{2} + \sin^2 \frac{\pi q \Delta x}{2}\right)}{1 + \frac{\lambda}{2}\left(\sin^2 \frac{\pi p \Delta x}{2} + \sin^2 \frac{\pi q \Delta x}{2}\right)} \right| \leqq \begin{cases} 1 - \dfrac{\lambda \pi^2}{4N^2} & (1 \leqq p, q \leqq N^{1-\epsilon}) \\ 1 - \dfrac{\lambda \pi^2}{8N^{2\epsilon}} & (p \text{ or } q > N^{1-\epsilon}) \end{cases} \tag{90}$$

Hence,

$$| v_n | \leqq A(\Delta t)^2 \sum_{p,q=1}^{N^{1-\epsilon}} \left(\frac{1}{pq} + \frac{1}{N^{2\epsilon}}\right) \sum_{k=0}^{n-1} \left(1 - \frac{\lambda \pi^2}{4N^2}\right)^k + A(\Delta t)^2 \sum_{p \text{ or } q > N^{1-\epsilon}} 1 \times \sum_{k=0}^{n-1} \left(1 - \frac{\lambda \pi^2}{8N^{2\epsilon}}\right)^k.$$

$$| v_n | < A^*(N^{4-4\epsilon} + N^{2+2\epsilon})(\Delta t)^2 = A^* \lambda^2 (N^{-4\epsilon} + N^{-2+2\epsilon}) \tag{91}$$

For $\epsilon = \frac{1}{3}$,

$$| v_n | = O(N^{-4/3}) = O((\Delta t)^{2/3}). \tag{92}$$

It is obvious that the refinement procedure is identical here to that of Chapter I; hence, we may state the final result

THEOREM 4. *If there exists a solution u of* (1) *such that u is six times continuously differentiable in the region $0 \leqq x, y \leqq 1, 0 \leqq t \leqq T$, then the solution of* (70) *converges uniformly to u in this region with the truncation error being no worse than $O(\Delta t)$.*

Note that, while much more work is involved in the solution by this method, no improvement in the convergence over that of the first method was obtained. The additional work is $O(N/\log N)$.

APPENDIX

13. A lemma on Fourier coefficients. Consider a function $f(x, y)$ which is twice boundedly differentiable on the closed unit square. Its Fourier sine coefficients are given by

$$C_{pq} = 4 \int_0^1 \int_0^1 f(x, y) \sin \pi p x \sin \pi q y \, dx \, dy \quad (p, q = 1, 2, \cdots). \tag{93}$$

It is easy to see by integration by parts first with respect to x and then with respect to y that

$$C_{pq} = O\left(\frac{1}{pq}\right). \tag{94}$$

The Fourier sine coefficients of $f(x, y)$ on the finite group $\left\{\frac{i}{N}\right\}_{i=0}^{N-1} \times \left\{\frac{j}{N}\right\}_{i=0}^{N-1}$ are given by

$$c_{pq} = \frac{4}{N^2} \sum_{i,j=1}^{N-1} f_{ij} \sin \frac{\pi p i}{N} \sin \frac{\pi q i}{N} \quad (p, q = 1, \cdots, N-1). \tag{95}$$

As the sum in (95) is an approximate for the integral of (93), c_{pq} should be little different from c_{pq} for small p, q. Let

$$p, q \leqq N^{1-\epsilon}. \tag{96}$$

Then,

$$\begin{aligned} &\int_0^1 \int_0^1 f(x, y) \sin \pi p x \sin \pi q y \, dx \, dy \\ &\quad = \sum_{i,j=1}^{N-1} \int_{i-\frac{1}{2}/N}^{i+\frac{1}{2}/N} \int_{j-\frac{1}{2}/N}^{j+\frac{1}{2}/N} f(x, y) \sin \pi p x \sin \pi q y \, dx \, dy + O(N^{-1-\epsilon}), \end{aligned} \tag{97}$$

as the integrand is $O(N^{-\epsilon})$ in the bordering strip left out and its area is $O(N^{-1})$.

$$\begin{aligned} &\int_{i-\frac{1}{2}/N}^{i+\frac{1}{2}/N} \int_{j-\frac{1}{2}/N}^{j+\frac{1}{2}/N} f(x, y) \sin \pi p x \sin \pi q y \, dx \, dy \\ &\quad = \int_{i-\frac{1}{2}/N}^{i+\frac{1}{2}/N} \int_{j-\frac{1}{2}/N}^{j+\frac{1}{2}/N} f\left(\frac{i}{N}, \frac{j}{N}\right) \sin \pi p x \sin \pi q y \, dx \, dy \\ &\quad + \int_{i-\frac{1}{2}/N}^{i+\frac{1}{2}/N} \int_{j-\frac{1}{2}/N}^{j+\frac{1}{2}/N} \left[f(x, y) - f\left(\frac{i}{N}, \frac{j}{N}\right)\right] \sin \pi p x \sin \pi q y \, dx \, dy. \end{aligned} \tag{98}$$

Let us consider each of these terms separately.

$$\begin{aligned} \sin \pi p x &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n}{dx^n} \sin \pi p x\right)_{x=i/N} \left(x - \frac{i}{N}\right)^n \\ &= \sin \frac{\pi p i}{N} \sum_{n=0}^{\infty} \frac{(-1)^n (\pi p)^{2n}}{(2n)!} \left(x - \frac{i}{N}\right)^{2n} \\ &\quad + \cos \frac{\pi p i}{N} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (\pi p)^{2n+1}}{(2n+1)!} \left(x - \frac{i}{N}\right)^{2n+1}. \end{aligned} \tag{99}$$

Hence,

$$
\begin{aligned}
\sin \pi p x \sin \pi q y = {} & \sin \frac{\pi p i}{N} \sin \frac{\pi q j}{N}\left[\sum_{n=0}^{\infty} \frac{(-1)^n(\pi p)^{2n}}{(2n)!}\left(x-\frac{i}{N}\right)^{2n}\right] \\
& \times\left[\sum_{m=0}^{\infty} \frac{(-1)^m(\pi q)^{2m}}{(2m)!}\left(y-\frac{j}{N}\right)^{2m}\right]+\cos \frac{\pi p i}{N} \cos \frac{\pi q j}{N} \\
& \times\left[\sum_{n=0}^{\infty} \frac{(-1)^n(\pi p)^{2n+1}}{(2n+1)!}\left(x-\frac{i}{N}\right)^{2n+1}\right]\left[\sum_{m=0}^{\infty} \frac{(-1)^m(\pi q)^{2m+1}}{(2m+1)!}\left(y-\frac{i}{N}\right)^{2m+1}\right] \\
& +\sin \frac{\pi p i}{N} \cos \frac{\pi q j}{N}\left[\sum_{n=0}^{\infty} \frac{(-1)^n(\pi p)^{2n}}{(2n)!}\left(x-\frac{i}{N}\right)^{2n}\right] \\
& \times\left[\sum_{m=0}^{\infty} \frac{(-1)^m(\pi q)^{2m+1}}{(2m+1)!}\left(y-\frac{j}{N}\right)^{2m+1}\right]+\cos \frac{\pi p i}{N} \sin \frac{\pi q j}{N} \\
& \times\left[\sum_{n=0}^{\infty} \frac{(-1)^n(\pi p)^{2n+1}}{(2n+1)!}\left(x-\frac{i}{N}\right)^{2n+1}\right]\left[\sum_{m=0}^{\infty} \frac{(-1)^m(\pi q)^{2m}}{(2m)!}\left(y-\frac{j}{N}\right)^{2m}\right].
\end{aligned} \tag{100}
$$

Thus,

$$
\begin{aligned}
& \int_{i-\frac{1}{2}/N}^{i+\frac{1}{2}/N} \int_{j-\frac{1}{2}/N}^{j+\frac{1}{2}/N} f\left(\frac{i}{N}, \frac{j}{N}\right) \sin \pi p x \sin \pi q y \, dx \, dy \\
& \quad = f\left(\frac{i}{N}, \frac{j}{N}\right) \sin \frac{\pi p i}{N} \sin \frac{\pi q j}{N}\left[\sum_{n=0}^{\infty} \frac{(-1)^n(\pi p)^{2n}}{(2n+1)!} \frac{2}{(2N)^{2n+1}}\right] \\
& \quad \times\left[\sum_{m=0}^{\infty} \frac{(-1)^m(\pi q)^{2m}}{(2m+1)!} \frac{2}{(2N)^{2m+1}}\right]=\frac{1}{N^2} f_{ij} \sin \frac{\pi p i}{N} \sin \frac{\pi q j}{N} \\
& \quad \times\left[1+\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\pi p}{2N}\right)^{2n}\right]\left[1+\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m+1)!}\left(\frac{\pi q}{2N}\right)^{2m}\right].
\end{aligned} \tag{101}
$$

For p, q as required by (96),

$$
\begin{aligned}
\left|\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\pi p}{2N}\right)^{2n}\right| & \leqq \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}\left(\frac{\pi}{2N^{\epsilon}}\right)^{2n} \\
& =\frac{\pi^2}{4N^{2\epsilon}} \sum_{n=0}^{\infty} \frac{1}{(2n+3)!}\left(\frac{\pi}{2N^{\epsilon}}\right)^{2n}<\frac{\pi^2}{4N^{2\epsilon}} \sum_{n=0}^{\infty} \frac{1}{(2n+3)!}\left(\frac{\pi}{2}\right)^{2n}.
\end{aligned} \tag{102}
$$

Therefore,

$$
\begin{aligned}
& \int_{i-\frac{1}{2}/N}^{i+\frac{1}{2}/N} \int_{j-\frac{1}{2}/N}^{j+\frac{1}{2}/N} f\left(\frac{i}{N}, \frac{j}{N}\right) \sin \pi p x \sin \pi p y \, dx \, dy \\
& \qquad\qquad = \frac{1}{N^2} f_{ij} \sin \frac{\pi p i}{N} \sin \frac{\pi q j}{N}+O\left(N^{-2-2\epsilon}\right).
\end{aligned} \tag{103}
$$

Now,

$$f(x, y) - f\left(\frac{i}{N}, \frac{j}{N}\right) = f_x\left(\frac{i}{N}, \frac{j}{N}\right)\left(x - \frac{i}{N}\right) + f_y\left(\frac{i}{N}, \frac{j}{N}\right)\left(y - \frac{j}{N}\right)$$
$$(104) \quad + \tfrac{1}{2}f_{xx}(x', y')\left(x - \frac{i}{N}\right)^2 + f_{xy}(x'', y'')\left(x - \frac{i}{N}\right)\left(y - \frac{j}{N}\right)$$
$$+ \tfrac{1}{2}f_{yy}(x''', y''')\left(y - \frac{j}{N}\right)^2.$$

Hence,

$$\int_{i-\frac{1}{2}/N}^{i+\frac{1}{2}/N}\int_{j-\frac{1}{2}/N}^{j+\frac{1}{2}/N}\left[f(x, y) - f\left(\frac{i}{N}, \frac{j}{N}\right)\right]\sin \pi px \sin \pi qy \, dx \, dy$$
$$= f_x\left(\frac{i}{N}, \frac{j}{N}\right)\cos\frac{\pi pi}{N}\sin\frac{\pi qj}{N}\left[\sum_{n=0}^{\infty}\frac{(-1)^n(\pi p)^{2n+1}}{(2n+1)!(2n+3)}\frac{2}{(2N)^{2n+3}}\right]$$
$$(105) \quad \times\left[\sum_{m=0}^{\infty}\frac{(-1)^m(\pi q)^{2m}}{(2m+1)!}\frac{2}{(2N)^{2m+1}}\right] + f_y\left(\frac{i}{N}, \frac{j}{N}\right)\sin\frac{\pi pi}{N}\cos\frac{\pi qj}{N}$$
$$\times\left[\sum_{m=0}^{\infty}\frac{(-1)^n(\pi p)^{2n}}{(2n+1)!}\frac{2}{(2N)^{2n+1}}\right]\left[\sum_{m=0}^{\infty}\frac{(-1)^m(\pi q)^{2m+1}}{(2m+1)!(2m+3)}\frac{2}{(2N)^{2m+3}}\right]$$
$$+ O(N^{-4}) = O(N^{-3-2\epsilon} + N^{-4}).$$

Collecting,

$$(106) \quad \int_{i-\frac{1}{2}/N}^{i+\frac{1}{2}/N}\int_{j-\frac{1}{2}/N}^{j+\frac{1}{2}/N} f(x, y)\sin \pi px \sin \pi qy \, dx \, dy = \frac{1}{N^2}f_{ij}\sin\frac{\pi pi}{N}\sin\frac{\pi qj}{N} + O(N^{-2-2\epsilon}).$$

As there are $(N - 1)^2$ terms in the sum of (97),

$$(107) \quad \int_0^1\int_0^1 f(x, y)\sin \pi px \sin \pi qy \, dx \, dy = \frac{1}{N^2}\sum_{i,j,=1}^{N-1} f_{ij}\sin\frac{\pi pi}{N}\sin\frac{\pi qj}{N} + O(N^{-2\epsilon}),$$

for $0 < \epsilon < 1$; i.e.,

$$(108) \quad c_{pq} = C_{pq} + O(N^{-2\epsilon}).$$

Combining (108) and (94), we have proved the following

LEMMA. *If $f(x, y)$ is twice boundedly differentiable in $0 \leqq x, y \leqq 1$ and c_{pq} is defined by (95), then, for $p, q \leqq N^{1-\epsilon}$,*

$$c_{pq} = O\left(\frac{1}{pq} + \frac{1}{N^{2\epsilon}}\right),$$

the constant depending on $f(x, y)$ and its first two derivatives.

BIBLIOGRAPHY

1. Bruce, G. H., D. W. Peaceman, H. H. Rachford, and J. D. Rice, *Calculations of unsteady-state gas flow through porous media*, Journal of Petroleum Technology 198(1953) pp. 79–91.
2. Courant, R., K. Friedrichs, and H. Lewy, *Über die Partiellen Differenzengleichungen der Mathematischen Physik*, Math. Ann. 100(1928) pp. 32–74.
3. Crank, J., and P. Nicholson, *A practical method for numerical integration of solutions of partial differential equations of the heat-conduction type*, Proc. Camb. Phil. Soc. 43(1947) pp. 50–67.
4. Hartree, S., and J. R. Womersley, *A method for the numerical or mechanical solution of certain types of partial differential equations*, Proc. Royal Soc. London, Series A, 161(1937) pp. 353–366.
5. Hildebrand, F. B., *On the convergence of numerical solutions of the heat-flow equation*, J. Math. and Physics 31(1952) pp. 35–41.
6. John, F., *On integration of parabolic equations by difference methods*, Comm. on Pure and App. Math. 5(1952) pp. 155–211.
7. Juncosa, M. L. and D. M. Young, *On the order of convergence of solutions of a difference equation to a solution of the diffusion equation*, J. Soc. Indust. Appl. Math. 1(1953) pp. 111–135.
8. Laasonen, P., *Über eine methode zur Losung der Wärmeleitungsgleichung*, Acta Math. 81(1949) pp. 309–317.
9. Leutert, W., *On the convergence of approximate solutions of the heat equation to the exact solution*, Proc. Amer. Math. Soc. 2(1951) pp. 433–439.
10. O'Brien, G. G., M. A. Hyman, and S. Kaplan, *A study of the numerical solution of partial differential equations*, J. Math. and Physics 29(1951) pp. 223–251.
11. Peaceman, D. W., and H. H. Rachford, Jr., *The numerical solution of parabolic and elliptic differential equations*, this journal, 3(1955), pp. 28–41.
12. Rothe, E., *Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensionaler Randwertaufgaben*, Math. Ann. 102(1930) pp. 650–670.

HUMBLE OIL AND REFINING COMPANY
HOUSTON, TEXAS