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On SSOR-like preconditioners for non-Hermitian positive definite matrices

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SUMMARY

We construct, analyze, and implement SSOR-like preconditioners for non-Hermitian positive definite system of linear equations when its coefficient matrix possesses either a dominant Hermitian part or a dominant skew-Hermitian part. We derive tight bounds for eigenvalues of the preconditioned matrices and obtain convergence rates of the corresponding SSOR-like iteration methods as well as the corresponding preconditioned GMRES iteration methods. Numerical implementations show that Krylov subspace iteration methods such as GMRES, when accelerated by the SSOR-like preconditioners, are efficient solvers for these classes of non-Hermitian positive definite linear systems. Copyright © 2015 John Wiley & Sons, Ltd.

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1. INTRODUCTION

SSOR iteration method [1–3] is one of the effective linear solvers and the high-quality matrix preconditioners for Krylov subspace iteration methods [4] used to solve iteratively large sparse systems of linear equations, especially when the coefficient matrix of the linear system is Hermitian positive definite or when it is strictly or irreducibly diagonally dominant [2, 5–8]. Each iteration step of the SSOR iteration method consists of two half steps: the first is a usual (forward) SOR iteration and the second is a backward SOR iteration. The SSOR iteration method was introduced by Sheldon and constitutes a generalization of the method introduced previously by Aitken for the iteration parameter being equal to one; see [3]. Because of minimal requirement of computer storage and easiest process of computer implementation, the SSOR iteration is still the method of choice in some application areas such as computed tomography [9, 10] and magnetic resonance imaging [11, 12], where this class of iteration methods is termed as the *algebraic reconstruction technique* (ART) [13] or the *simultaneous ART* (**SART**) [14]; see also [15–19]. In addition, the SSOR iteration is often employed as preconditioners for Krylov subspace iteration methods [20–23] and smoothers for multigrid methods [24].

For a Hermitian positive definite linear system, the symmetry in the structure of an iteration scheme and the Hermitian positive definiteness of the induced splitting (or preconditioning) matrix are often important in both theory and applications, for example, in preconditioning the conjugate gradient or the minimal residual methods [1, 20], so SSOR is advantageous over SOR in this regard.

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Hence, the convergence and preconditioning properties of SSOR have been studied extensively and profoundly; see [1–3] and [25–27]. However, to our knowledge, there is little result about SSOR for the linear systems of non-Hermitian positive definite coefficient matrices; see [28–30]. The main reasons may be that for this case the SSOR iteration scheme is structurally nonsymmetric and the induced splitting (or preconditioning) matrix may lose positive definiteness even if the original coefficient matrix possesses it. Given additionally that the two-half steps in the SSOR iteration amount to a non-trivial computational cost when compared with the SOR, the SOR rather than the SSOR may be often the method of choice. Note that even for non-Hermitian positive definite linear system, the SSOR can better capture information from both lower and upper triangular parts of the original coefficient matrix than the SOR, and the corresponding SSOR splitting (or preconditioning) matrix can more accurately approximate the coefficient matrix than the SOR splitting matrix. This could make the SSOR be more effective as either a linear solver or a splitting preconditioner than the SOR in practical computations. Moreover, in both theory and applications such as in preconditioning Krylov subspace iteration methods like GMRES and BiCGSTAB [4], the structural symmetry of an iteration scheme and the positive definiteness of the induced preconditioning matrix are crucial for guaranteeing their convergence and efficiency, so SSOR may be also more attractive and deserve more concern than SOR in this regard.

In this paper, we construct and analyze SSOR-like iteration methods for solving the large sparse linear system

$$Ax = b, \quad A \in \mathbb{C}^{n \times n} \quad \text{and} \quad x, b \in \mathbb{C}^n, \tag{1}$$

when the matrix A is non-Hermitian positive definite and possesses either a strong Hermitian part H or a strong skew-Hermitian part S, where

$$H = \frac{1}{2}(A + A^*) \quad \text{and} \quad S = \frac{1}{2}(A - A^*),$$

with A^* representing the conjugate transpose of A; see [31]. These SSOR-like iteration methods are different from the classical ones, as they are constructed by sufficiently utilizing the strong dominance of the Hermitian and the skew-Hermitian parts of the coefficient matrix $A \in \mathbb{C}^{n \times n}$. Hence, they may show much better numerical property in actual applications to solving non-Hermitian positive definite linear systems of either dominant Hermitian parts or dominant skew-Hermitian parts. For the SSOR-like iteration methods, we prove theoretically their asymptotic convergence under certain conditions and discuss preconditioning properties of the corresponding SSOR-preconditioned matrices, and we numerically show advantages of the new SSOR-like iteration and the new SSOR-preconditioned GMRES methods over the classical SSOR-like iteration and the classical SSOR-preconditioned GMRES methods, respectively, when they are employed to solve non-Hermitian positive definite linear system whose coefficient matrix has either a strongly dominant Hermitian part or a strongly dominant skew-Hermitian part.

The organization of the paper is as follows. In Section 2 we introduce necessary notations, establish several basic lemmas, and review general two-step splitting iteration methods. In Sections 3 and 4, for non-Hermitian positive definite linear systems with strongly dominant Hermitian or strongly dominant skew-Hermitian parts, we construct SSOR-like iteration methods and the corresponding SSOR-like preconditioning matrices, derive tight bounds for the preconditioned matrices, and give convergence conditions for the iteration matrices. Numerical results for the non-Hermitian positive definite linear systems arising from discretizing two-dimensional linear integro-differential equations are reported in Section 5. Finally, in Section 6, we end the paper with a few concluding remarks.

2. PRELIMINARIES

First of all, we introduce a few necessary notations and establish several basic lemmas.

We represent by I the identity matrix of suitable dimension and by $\mathtt{i}$ the imaginary unit. Given a complex vector g, we use g^T to denote its transpose, g^* to denote its conjugate transpose, $|g|$ its

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absolute value, and $\|g\|_2$ its Euclidean norm. These notations can be directly carried on to complex matrices. $\Re(\lambda)$ and $\Im(\lambda)$ indicate the real and the imaginary parts of a complex number λ. For a given matrix $G \in \mathbb{C}^{n\times n}$, we use $\lambda(G)$ to represent any of its eigenvalues, $\rho(G)$ its spectral radius, and $\mathcal{H}(G)$ and $\mathcal{S}(G)$ its Hermitian and skew-Hermitian parts, respectively, that is,

$$\mathcal{H}(G) = \frac{1}{2}(G + G^*) \quad \text{and} \quad \mathcal{S}(G) = \frac{1}{2}(G - G^*).$$

For a nonsingular matrix $G \in \mathbb{C}^{n\times n}$, the following lemma describes the relationships between $\mathcal{H}(G)$ and $\mathcal{H}(G^{-1})$, as well as $\mathcal{S}(G)$ and $\mathcal{S}(G^{-1})$.

Lemma 2.1
Let $G \in \mathbb{C}^{n\times n}$ be a nonsingular matrix. Then

$$\mathcal{H}(G^{-1}) = G^{-*}\,\mathcal{H}(G)\,G^{-1} \quad \text{and} \quad \mathcal{S}(G^{-1}) = -G^{-*}\,\mathcal{S}(G)\,G^{-1}.$$

Proof
By straightforward computations, we have

$$\mathcal{H}\left(G^{-1}\right) = \frac{1}{2}\left(G^{-1} + G^{-*}\right) = \frac{1}{2}G^{-*}\left(G + G^*\right)G^{-1} = G^{-*}\,\mathcal{H}(G)\,G^{-1}$$

and

$$\mathcal{S}\left(G^{-1}\right) = \frac{1}{2}\left(G^{-1} - G^{-*}\right) = \frac{1}{2}G^{-*}\left(G^* - G\right)G^{-1} = -G^{-*}\,\mathcal{S}(G)\,G^{-1}.$$

□

Given an eigenvalue bound of a matrix $G \in \mathbb{C}^{n\times n}$, we can further determine if the matrix $I - G$ is convergent by examining conditions stated in the following lemma.

Lemma 2.2
Assume that all eigenvalues of $G \in \mathbb{C}^{n\times n}$ are enclosed in the rectangle $[\xi_{\min}, \xi_{\max}] \times [-\eta_{\max}, \eta_{\max}]$. Then $\rho(I - G) < 1$ if

$$\eta_{\max} < 1, \quad \xi_{\min} > 1 - \sqrt{1 - \eta_{\max}^2} \quad \text{and} \quad \xi_{\max} < 1 + \sqrt{1 - \eta_{\max}^2}. \tag{2}$$

Proof
Let $\mu = \xi + \mathrm{i}\eta$ be an eigenvalue of the matrix G. Then the eigenvalues of the matrix $I - G$ are of the form $\lambda = 1 - \mu$. Hence, $|\lambda| < 1$ if and only if $|1 - \mu| < 1$, or equivalently, $\xi^2 - 2\xi + \eta^2 < 0$. It follows from straightforward computations that a sufficient condition for guaranteeing the last inequality is

$$\eta_{\max} < 1 \quad \text{and} \quad 1 - \sqrt{1 - \eta_{\max}^2} < \xi < 1 + \sqrt{1 - \eta_{\max}^2},$$

which is valid if the conditions in (2) hold true. □

The generalized Bendixson theorem established in [32] describes a bound for the spectrum of the matrix $M^{-1}A$, where $A, M \in \mathbb{C}^{n\times n}$ are complex matrices with M being nonsingular.

Theorem 2.1
[32] Let $A, M \in \mathbb{C}^{n\times n}$ be two complex matrices satisfying $x^*\mathcal{H}(A)x \neq 0$ and $x^*\mathcal{H}(M)x \neq 0$ for $\forall x \in \mathbb{C}^n \setminus \{0\}$. Assume that there exist positive constants γ_l and γ_u such that

$$\gamma_l \leqslant \frac{x^*\mathcal{H}(A)x}{x^*\mathcal{H}(M)x} \leqslant \gamma_u, \quad \forall x \in \mathbb{C}^n \setminus \{0\},$$

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and nonnegative constants β_a and β_m such that

$$-\beta_a \leqslant \frac{1}{\mathrm{i}} \frac{x^*\mathcal{S}(A)x}{x^*\mathcal{H}(A)x} \leqslant \beta_a \quad \text{and} \quad -\beta_m \leqslant \frac{1}{\mathrm{i}} \frac{x^*\mathcal{S}(M)x}{x^*\mathcal{H}(M)x} \leqslant \beta_m, \quad \forall x \in \mathbb{C}^n \setminus \{0\}.$$

Then, when $\beta_a\beta_m < 1$, it holds that

$$\frac{(1-\beta_a\beta_m)\gamma_l}{1+\beta_m^2} \leqslant \Re(\lambda(M^{-1}A)) \leqslant (1+\beta_a\beta_m)\gamma_u$$

and

$$-(\beta_a+\beta_m)\gamma_u \leqslant \Im(\lambda(M^{-1}A)) \leqslant (\beta_a+\beta_m)\gamma_u.$$

Lemma 2.2 and Theorem 2.1 straightforwardly result in the following criterion for determining the convergence of an iteration matrix.

Theorem 2.2
Let $A, M \in \mathbb{C}^{n\times n}$ be two complex matrices satisfying the conditions of Theorem 2.1. Denote by $N = M - A$. Then the iteration matrix $M^{-1}N$ is convergent if $\beta_a\beta_m < 1$, $(\beta_a+\beta_m)\gamma_u < 1$ and

$$\begin{cases} \frac{(1-\beta_a\beta_m)\gamma_l}{1+\beta_m^2} > 1 - \sqrt{1-(\beta_a+\beta_m)^2\gamma_u^2}, \\ (1+\beta_a\beta_m)\gamma_u < 1 + \sqrt{1-(\beta_a+\beta_m)^2\gamma_u^2}. \end{cases}$$

In the following, we review the two-step splitting iteration method and the correspondingly induced two-step splitting iteration preconditioner for the linear system (1); see [33, 34].

Let

$$A = M_1 - N_1 = M_2 - N_2$$

be two splittings[‡] of the matrix $A \in \mathbb{C}^{n\times n}$. Then the two-step splitting iteration method for the linear system (1) can be defined as

$$\begin{cases} M_1 x^{(k+\frac{1}{2})} = N_1 x^{(k)} + b, \\ M_2 x^{(k+1)} = N_2 x^{(k+\frac{1}{2})} + b; \end{cases} \tag{3}$$

see [33, 34] for more details about convergent and computational properties.

From the first-half iterate of (3), we easily have

$$x^{(k+\frac{1}{2})} = M_1^{-1}N_1x^{(k)} + M_1^{-1}b.$$

By substituting this expression into the second-half iterate of (3), we straightforwardly obtain

$$\begin{aligned} x^{(k+1)} &= M_2^{-1}N_2\left(M_1^{-1}N_1x^{(k)} + M_1^{-1}b\right) + M_2^{-1}b \\ &= M_2^{-1}N_2M_1^{-1}N_1x^{(k)} + \left(M_2^{-1} + M_2^{-1}N_2M_1^{-1}\right)b. \end{aligned}$$

Hence, if the matrix $M_2^{-1} + M_2^{-1}N_2M_1^{-1}$ is nonsingular, by introducing matrices

$$G = M_2^{-1}N_2M_1^{-1}N_1$$

and

$$M^{-1} = \left(M_2^{-1} + M_2^{-1}N_2M_1^{-1}\right) = M_2^{-1}\left(I + N_2M_1^{-1}\right) = M_2^{-1}\left(M_1+N_2\right)M_1^{-1},$$

[‡] $A = M - N$ is called a splitting of the matrix $A \in \mathbb{C}^{n\times n}$ if $M \in \mathbb{C}^{n\times n}$ is nonsingular and $N \in \mathbb{C}^{n\times n}$.

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we can rewrite the two-step splitting iteration method (3) into the brief form

$$x^{(k+1)} = Gx^{(k)} + M^{-1}b. \tag{4}$$

Thereby, this iteration scheme is convergent if and only if its iteration matrix G is convergent, that is, $\rho(G) < 1$; see [2, 3]. By defining two matrices

$$M = M_1(M_1 + N_2)^{-1}M_2 \tag{5}$$

and

$$N = MG = M_1(M_1 + N_2)^{-1}N_2M_1^{-1}N_1 = N_2(M_1 + N_2)^{-1}N_1, \tag{6}$$

we have

$$A = M - N. \tag{7}$$

Hence, the two-step splitting iteration (4) is also induced by the matrix splitting (7) defined through the splitting matrices (5) and (6).

From (5) we see that $M \in \mathbb{C}^{n\times n}$ is nonsingular if and only if the matrix $M_1 + N_2$ is nonsingular. When $M \in \mathbb{C}^{n\times n}$ is nonsingular, it naturally defines a preconditioner for the coefficient matrix $A \in \mathbb{C}^{n\times n}$ of the linear system (1). We call this preconditioner M defined in (5) the two-step splitting iteration preconditioner of the matrix A. Evidently, because (6) and (7) yield

$$M^{-1}A = I - M^{-1}N = I - G,$$

the eigenvalue distribution of the matrix $M^{-1}A$ is completely determined by that of the iteration matrix G of the two-step splitting iteration (3) or (4), and vice versa.

Typical examples of the two-step splitting iteration method for solving the linear system (1) are the classical SSOR and the recent *Hermitian and skew-Hermitian splitting* (**HSS**) iteration methods. The former is algorithmically defined by

$$\begin{cases} \left(\frac{1}{\omega}D + L\right)x^{(k+\frac{1}{2})} = \left[\left(\frac{1}{\omega} - 1\right)D - U\right]x^{(k)} + b, \\ \left(\frac{1}{\omega}D + U\right)x^{(k+1)} = \left[\left(\frac{1}{\omega} - 1\right)D - L\right]x^{(k+\frac{1}{2})} + b, \end{cases}$$

where D, L, and U are the diagonal, the strictly lower-triangular, and the strictly upper-triangular parts of the matrix $A \in \mathbb{C}^{n\times n}$, respectively, with $\omega \in (0, 2)$ being an arbitrary real number. And the latter is algorithmically defined by

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases}$$

where H and S are the Hermitian and the skew-Hermitian parts of the matrix $A \in \mathbb{C}^{n\times n}$, with α being an arbitrary positive number. The SSOR iteration method is convergent when the matrix A is either Hermitian positive definite or strictly diagonally dominant; see [2, 3, 21, 22, 28]. Compared with SOR, SSOR requires more work per iteration and, in general, converges slower. The HSS iteration method is convergent when the matrix A is non-Hermitian positive definite; see [34]. Compared with SSOR, HSS may generally require more work per iteration because of solving two linear sub-systems with the coefficient matrices $\alpha I + H$ and $\alpha I + S$ rather than two triangular linear sub-systems as involved in SSOR. However, HSS may converge faster and can be used to solve a larger spectrum of application problems than SSOR. Because of the symmetric structure, both SSOR and HSS can be combined with the semi-iteration method or the Krylov subspace methods to produce fast and robust iteration methods for solving the linear system (1); see [3, 4]. Correspondingly, the SSOR preconditioning matrix M is given by

$$P_{\mathrm{SSOR}} = \frac{1}{\omega(2-\omega)}(D + \omega L)D^{-1}(D + \omega U) \tag{8}$$

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and the HSS preconditioning matrix M is given by

$$P_{\mathrm{HSS}} = \frac{1}{2\alpha}(\alpha I + H)(\alpha I + S).$$

For further analyses, generalizations, and applications about the SSOR and the HSS iteration methods, as well as the correspondingly induced SSOR and HSS preconditioners, we refer to [35–42].

3. MATRICES OF STRONG HERMITIAN PARTS

Assume that the coefficient matrix $A \in \mathbb{C}^{n\times n}$ of the linear system (1) is non-Hermitian positive definite and has a strongly dominant Hermitian part $H = \frac{1}{2}(A + A^*)$ over its skew-Hermitian part $S = \frac{1}{2}(A - A^*)$. Decompose H into its diagonal part D_H, strictly lower-triangular part L_H, and strictly upper-triangular part L_H^* as

$$H = D_H + L_H + L_H^*.$$

When the matrix A is of a blockwise form, the matrices H and S, as well as the matrices D_H, L_H, and L_H^* are thought of having the blockwise forms conforming to that of the matrix A, too.

For this case, we define an SSOR-like iteration method for solving the linear system (1) as follows:

$$\begin{cases} x^{(k+\frac{1}{2})} = x^{(k)} + \omega(D_H + \omega L_H)^{-1}(b - Ax^{(k)}), \\ x^{(k+1)} = x^{(k+\frac{1}{2})} + \omega\left(D_H + \omega L_H^*\right)^{-1}\left(b - Ax^{(k+\frac{1}{2})}\right), \end{cases} \tag{9}$$

or equivalently,

$$\begin{cases} \left(\frac{1}{\omega}D_H + L_H\right)x^{(k+\frac{1}{2})} = \left[\left(\frac{1}{\omega} - 1\right)D_H - L_H^* - S\right]x^{(k)} + b, \\ \left(\frac{1}{\omega}D_H + L_H^*\right)x^{(k+1)} = \left[\left(\frac{1}{\omega} - 1\right)D_H - L_H - S\right]x^{(k+\frac{1}{2})} + b. \end{cases} \tag{10}$$

Evidently, this SSOR-like iteration method belongs to the category of the two-step splitting iteration method (3), with the splitting matrices being specifically chosen as

$$\begin{aligned} A = M_1(\omega) - N_1(\omega) &= \left(\frac{1}{\omega}D_H + L_H\right) - \left[\left(\frac{1}{\omega} - 1\right)D_H - L_H^* - S\right] \\ = M_2(\omega) - N_2(\omega) &= \left(\frac{1}{\omega}D_H + L_H^*\right) - \left[\left(\frac{1}{\omega} - 1\right)D_H - L_H - S\right]. \end{aligned}$$

It naturally reduces to the classical SSOR iteration method when the matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian, that is, $S = 0$. In addition, it is a two-step generalization of the single splitting iteration methods; see [5–7, 43, 44] and the references therein.

From (5) the corresponding SSOR-like preconditioning matrix M, denoted distinctively as $P_H(\omega)$, for the matrix $A \in \mathbb{C}^{n\times n}$ is given by

$$P_H(\omega) = \left(\frac{1}{\omega}D_H + L_H\right)\left[\left(\frac{2}{\omega} - 1\right)D_H - S\right]^{-1}\left(\frac{1}{\omega}D_H + L_H^*\right). \tag{11}$$

Let

$$W_H(\omega) = \left(\frac{2}{\omega} - 1\right)D_H - S.$$

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Then we know that $W_H(\omega)$ is a positive definite matrix when $0 < \omega < 2$, as the matrices A and D_H are positive definite. Therefore, $W_H(\omega)^{-1}$ is positive definite when $0 < \omega < 2$, too, which implies that the SSOR-like preconditioning matrix $P_H(\omega) \in \mathbb{C}^{n\times n}$ is positive definite when $0 < \omega < 2$.

In order to obtain bounds for eigenvalues of the preconditioned matrix $P_H(\omega)^{-1}A$ by making use of the generalized Bendixson theorem in [32], that is, Theorem 2.1 in Section 2, we need to first derive estimates for the generalized Rayleigh quotients

$$\frac{x^*Hx}{x^*\mathcal{H}(P_H(\omega))x}, \quad \frac{1}{\mathtt{i}}\frac{x^*Sx}{x^*Hx} \quad \text{and} \quad \frac{1}{\mathtt{i}}\frac{x^*\mathcal{S}(P_H(\omega))x}{x^*\mathcal{H}(P_H(\omega))x}, \qquad \forall x \in \mathbb{C}^n \setminus \{0\},$$

in terms of the constants

$$\theta = \max_{x\neq 0}\frac{x^*D_Hx}{x^*Hx}, \quad \delta = \max_{x\neq 0}\frac{x^*\left(L_HD_H^{-1}L_H^* - \frac{1}{4}D_H\right)x}{x^*Hx} \quad \text{and} \quad \varrho = \rho(D_H^{-1}S). \tag{12}$$

Lemma 3.1
Let the constants θ, δ, and ϱ be defined as in (12). Then for $\omega \in (0,2)$ and $x \in \mathbb{C}^n \setminus \{0\}$ it holds that

$$\frac{4\tau}{\theta\tau^2 + 2\tau + 4\delta + 2} \leqslant \frac{x^*Hx}{x^*\mathcal{H}(P_H(\omega))x} \leqslant 1 + \left(\frac{\varrho}{\tau}\right)^2 \tag{13}$$

and

$$\frac{|x^*Sx|}{x^*Hx} \leqslant \theta\varrho, \qquad \frac{|x^*\mathcal{S}(P_H(\omega))x|}{x^*\mathcal{H}(P_H(\omega))x} \leqslant \frac{\varrho}{\tau}, \tag{14}$$

where $\tau = \frac{2}{\omega} - 1$.

Proof
We first verify the bounds in (13). By making use of Lemma 2.1 we have

$$\mathcal{H}(W_H(\omega)^{-1}) = \left(\frac{2}{\omega} - 1\right) W_H(\omega)^{-*}\, D_H\, W_H(\omega)^{-1}.$$

So the Hermitian part of $P_H(\omega)$ is given by

$$\mathcal{H}(P_H(\omega)) = \frac{2-\omega}{\omega}\left(\frac{1}{\omega}D_H + L_H\right) W_H(\omega)^{-*}\, D_H\, W_H(\omega)^{-1}\left(\frac{1}{\omega}D_H + L_H^*\right). \tag{15}$$

For notational simplicity we denote by

$$\widetilde{L} = \frac{1}{\omega}D_H + L_H \quad \text{and} \quad \widetilde{R} = W_H(\omega)\, D_H^{-1} W_H(\omega)^*. \tag{16}$$

Then according to (15) the Hermitian part of $P_H(\omega)$ can be briefly expressed as

$$\mathcal{H}(P_H(\omega)) = \frac{2-\omega}{\omega}\widetilde{L}W_H(\omega)^{-*}\, D_H W_H(\omega)^{-1}\widetilde{L}^* = \frac{2-\omega}{\omega}\widetilde{L}\widetilde{R}^{-1}\widetilde{L}^*. \tag{17}$$

After straightforward computations we obtain

$$\widetilde{R} = \left(\frac{2-\omega}{\omega}\right)^2 D_H + S^*D_H^{-1}S \tag{18}$$

and

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$$\widetilde{L}D_H^{-1}\widetilde{L}^* = \frac{1}{\omega}H + \left(\frac{2-\omega}{2\omega}\right)^2 D_H + \left(L_H D_H^{-1} L_H^* - \frac{1}{4}D_H\right) \tag{19}$$

$$= \frac{2-\omega}{\omega}H + F(\omega), \tag{20}$$

where

$$F(\omega) = \left[\left(1-\frac{1}{\omega}\right)D_H + L_H\right] D_H^{-1} \left[\left(1-\frac{1}{\omega}\right)D_H + L_H^*\right]$$

is a Hermitian positive semidefinite matrix; see Lemma 2.2 in [21] and Theorem 2.2 in [22].

As the matrices $\mathcal{H}(P_H(\omega))^{-1}H$ and $\frac{\omega}{2-\omega}\widetilde{R}\widetilde{L}^{-1}H\widetilde{L}^{-*}$ are similar, they have the same eigenvalue set. Hence, for an eigenvalue λ of the matrix $\mathcal{H}(P_H(\omega))^{-1}H$, there exists a nonzero vector u such that

$$\frac{\omega}{2-\omega}\widetilde{R}\widetilde{L}^{-1}H\widetilde{L}^{-*}u = \lambda u,$$

or equivalently,

$$\frac{\omega}{2-\omega}\widetilde{R}v = \lambda\widetilde{L}^*H^{-1}\widetilde{L}v, \quad \text{with} \quad v = \widetilde{L}^{-1}H\widetilde{L}^{-*}u \neq 0.$$

It then follows that

$$\lambda = \frac{\omega}{2-\omega}\,\frac{v^*\widetilde{R}v}{v^*\widetilde{L}^*H^{-1}\widetilde{L}v} = \frac{\omega}{2-\omega}\,\frac{v^*\widetilde{R}v}{v^*D_Hv}\,\frac{v^*D_Hv}{v^*\widetilde{L}^*H^{-1}\widetilde{L}v}.$$

Because H, $\widetilde{R}$, and $\widetilde{L}^*H^{-1}\widetilde{L}$ are Hermitian positive definite and $F(\omega)$ is Hermitian positive semidefinite, we further know from (18) and (20) that

$$\frac{v^*\widetilde{R}v}{v^*D_Hv} \leqslant \left(\frac{2-\omega}{\omega}\right)^2 + \varrho^2$$

and

$$\frac{v^*D_Hv}{v^*\widetilde{L}^*H^{-1}\widetilde{L}v} \leqslant \max_{w\neq 0}\frac{w^*Hw}{w^*\widetilde{L}D_H^{-1}\widetilde{L}^*w} \leqslant \frac{\omega}{2-\omega}.$$

As a result, it holds that

$$\lambda \leqslant \left(\frac{\omega}{2-\omega}\right)^2\left[\left(\frac{2-\omega}{\omega}\right)^2 + \varrho^2\right] = 1 + \left(\frac{\varrho\omega}{2-\omega}\right)^2. \tag{21}$$

Also, from (18) we know that

$$\frac{v^*\widetilde{R}v}{v^*D_Hv} \geqslant \left(\frac{2-\omega}{\omega}\right)^2.$$

As (19) results in the estimate

$$\begin{aligned}\max_{w\neq 0}\frac{w^*\widetilde{L}D_H^{-1}\widetilde{L}^*w}{w^*Hw} &\leqslant \frac{1}{\omega} + \left(\frac{2-\omega}{2\omega}\right)^2\max_{w\neq 0}\frac{w^*D_Hw}{w^*Hw} + \max_{w\neq 0}\frac{w^*\left(L_HD_H^{-1}L_H^* - \frac{1}{4}D_H\right)w}{w^*Hw}\\ &\leqslant \frac{1}{\omega} + \left(\frac{2-\omega}{2\omega}\right)^2\theta + \delta,\end{aligned}$$

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we have

$$\frac{v^* D_H v}{v^* \widetilde{L}^* H^{-1} \widetilde{L} v} \geqslant \min_{u \neq 0} \frac{u^* H u}{u^* \widetilde{L} D_H^{-1} \widetilde{L}^* u} \geqslant \left[\frac{1}{\omega} + \left(\frac{2-\omega}{2\omega} \right)^2 \theta + \delta \right]^{-1}.$$

As a result, it holds that

$$\lambda \geqslant \left[\frac{1}{\omega} + \left(\frac{2-\omega}{2\omega} \right)^2 \theta + \delta \right]^{-1} \left(\frac{2-\omega}{\omega} \right). \tag{22}$$

Therefore, the estimates (21) and (22) straightforwardly lead to the bounds in (13).

Now we turn to derive the bounds in (14). It is easily seen that

$$\frac{|x^* S x|}{x^* H x} \leqslant \max_{x \neq 0} \frac{x^* D_H x}{x^* H x} \max_{x \neq 0} \frac{|x^* S x|}{x^* D_H x} = \theta \varrho$$

for $\forall x \in \mathbb{C}^n \setminus \{0\}$. In addition, by making use of Lemma 2.1 we have

$$\mathcal{S}(W_H(\omega)^{-1}) = W_H(\omega)^{-*} \, S \, W_H(\omega)^{-1}.$$

So the skew-Hermitian part of $P_H(\omega)$ is given by

$$\mathcal{S}(P_H(\omega)) = \left(\frac{1}{\omega} D_H + L_H \right) W_H(\omega)^{-*} \, S \, W_H(\omega)^{-1} \left(\frac{1}{\omega} D_H + L_H^* \right);$$

see (11). By making use of the notation

$$\widetilde{L} = \frac{1}{\omega} D_H + L_H$$

introduced in (16), we can rewrite $\mathcal{S}(P_H(\omega))$ into the equivalent form

$$\mathcal{S}(P_H(\omega)) = \widetilde{L} W_H(\omega)^{-*} \, S \, W_H(\omega)^{-1} \widetilde{L}^*. \tag{23}$$

It then follows immediately from (17) and (23) that for any $x \in \mathbb{C}^n \setminus \{0\}$ we have

$$\begin{aligned} \frac{|x^* \mathcal{S}(P_H(\omega)) x|}{x^* \mathcal{H}(P_H(\omega)) x} &= \frac{\omega}{2-\omega} \frac{|x^* \widetilde{L} W_H(\omega)^{-*} \, S \, W_H(\omega)^{-1} \widetilde{L}^* x|}{x^* \widetilde{L} W_H(\omega)^{-*} \, D_H \, W_H(\omega)^{-1} \widetilde{L}^* x} \\ &\leqslant \frac{\omega}{2-\omega} \max_{x \neq 0} \frac{|x^* S x|}{x^* D_H x} = \frac{\omega \varrho}{2-\omega}. \end{aligned}$$

□

Based on Lemma 3.1 and Theorem 2.1, we can obtain bounds for eigenvalues of the preconditioned matrix $P_H(\omega)^{-1} A$ with respect to the SSOR-like preconditioning matrix $P_H(\omega)$ defined in (11).

Theorem 3.1

Let the matrices A and $P_H(\omega)$ be defined as in (1) and (11) and the constants θ, δ, and ϱ be defined as in (12). Then for $\omega \in (0, \frac{2}{1+\theta\varrho^2})$ it holds that

$$\frac{4\tau^2(\tau - \theta\varrho^2)}{(\theta\tau^2 + 2\tau + 4\delta + 2)(\tau^2 + \varrho^2)} \leqslant \Re(\lambda(P_H(\omega)^{-1} A)) \leqslant \frac{(\tau + \theta\varrho^2)(\tau^2 + \varrho^2)}{\tau^3}$$

and

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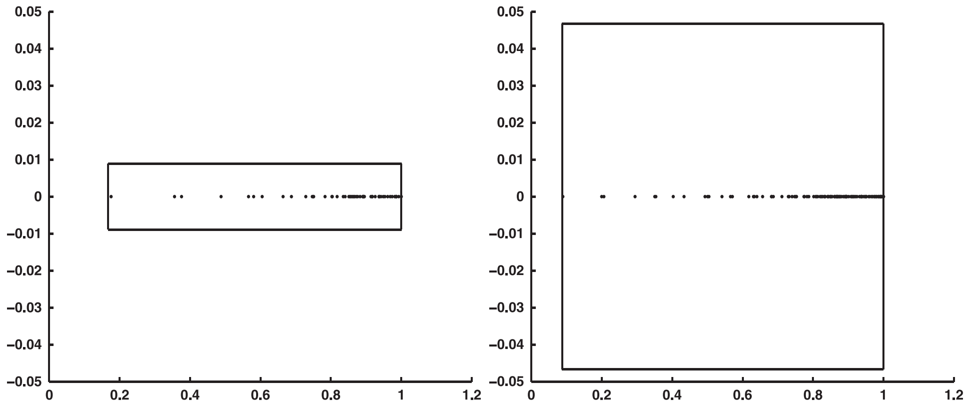

Figure 1. The exact eigenvalue distribution and the estimated eigenvalue bound for the preconditioned matrix $P_H(\omega)^{-1}A$, with A being the matrix in (41) in Example 5.1, when æ $= 10^{-4}$, $\nu = 10^{-8}$, and $\omega = 0.9999$, with $n = 8^2$ (left) and $n = 12^2$ (right). The exact eigenvalues: the dot '·', and the estimated bound: the rectangle of solid line '—'.

$$|\Im(\lambda(P_H(\omega)^{-1}A))| \leqslant \frac{\varrho(\theta\tau+1)(\tau^2+\varrho^2)}{\tau^3},$$

where $\tau = \frac{2}{\omega} - 1$.

The eigenvalue bounds given in Theorem 3.1 about the preconditioned matrix $P_H(\omega)^{-1}A$ with respect to the SSOR-like iteration method (9) or (10) are numerically validated by the linear system (41) arising from discretization of the two-dimensional linear integro-differential equation in Example 5.1; see Figure 1. From these two pictures, we observe that all eigenvalues of the preconditioned matrix $P_H(\omega)^{-1}A$ are tightly contained in a rectangle constituted by the estimated eigenvalue bounds given in Theorem 3.1, and the estimated bounds on the real parts of the eigenvalues are sharp.

According to Theorem 3.1 and Lemma 2.2 or Theorem 2.1 and Lemma 3.1, we can obtain the following convergence theory for the SSOR-like iteration method (9) or (10) for solving the linear system (1) of a strongly dominant Hermitian part.

Theorem 3.2
Let the coefficient matrix $A \in \mathbb{C}^{n\times n}$ of the linear system (1) be non-Hermitian positive definite. Then the SSOR-like iteration method (9) or (10) is convergent if the relaxation parameter $\omega \in \left(0, \frac{2}{1+\theta\varrho^2}\right)$ satisfying

$$\varrho(\theta\tau+1)\left(\tau^2+\varrho^2\right) < \tau^3$$

and

$$\begin{cases} \frac{4\tau^2(\tau-\theta\varrho^2)}{(\theta\tau^2+2\tau+4\delta+2)(\tau^2+\varrho^2)} > 1 - \sqrt{1 - \frac{\varrho^2(\theta\tau+1)^2(\tau^2+\varrho^2)^2}{\tau^6}}, \\ \frac{(\tau+\theta\varrho^2)(\tau^2+\varrho^2)}{\tau^3} < 1 + \sqrt{1 - \frac{\varrho^2(\theta\tau+1)^2(\tau^2+\varrho^2)^2}{\tau^6}}, \end{cases}$$

where $\tau = \frac{2}{\omega} - 1$ and the constants θ, δ, and ϱ are defined as in (12).

For the linear system (41) in Example 5.1, we plot the pictures of the spectral radius versus the relaxation parameter ω with respect to the iteration matrix $\mathbf{L}_H(\omega) = I - P_H(\omega)^{-1}A$ of the SSOR-like iteration scheme (9) or (10) in Figure 2. Here the *computed convergence domains* of ω are $\mathbb{I}_{ccd} = (0, 1.9980)$ for both pictures, while the *estimated convergence domains* of ω are $\mathbb{I}_{ecd} = (0.0030, 1.9845)$ for the left picture and $\mathbb{I}_{ecd} = (0.0040, 1.9840)$ for the right picture. Hence, we see that $\mathbb{I}_{ecd} \subset \mathbb{I}_{ccd}$ and the end-points of the intervals $\mathbb{I}_{ecd}$ and $\mathbb{I}_{ccd}$ respectively match very well as expected.

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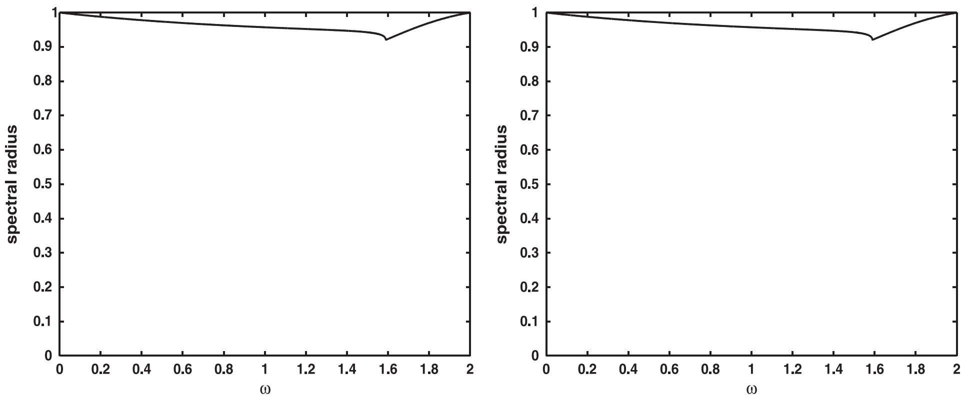

Figure 2. Pictures of the spectral radius versus the relaxation parameter with respect to the iteration matrix of the SSOR-like iteration scheme (9) for the linear system (41) in Example 5.1 when $n = 10^2$ and æ $= 10^{-6}$, with $\nu = 10^{-5}$ (left) and $\nu = 10^{-6}$ (right).

4. MATRICES OF STRONG SKEW-HERMITIAN PARTS

When the coefficient matrix $A \in \mathbb{C}^{n\times n}$ of the linear system (1) is non-Hermitian positive definite and has a strongly dominant skew-Hermitian part $S = \frac{1}{2}(A - A^*)$ over its Hermitian part $H = \frac{1}{2}(A + A^*)$, we let

$$H = D_H + B_H, \quad S = D_S + L_S + U_S \quad \text{and} \quad D = D_H + D_S,$$

where D is the diagonal part of the matrix A; D_S, L_S, and U_S are the diagonal, the strictly lower-triangular, and the strictly upper-triangular parts of the matrix S; and D_H and B_H are the diagonal and the off-diagonal parts of the matrix H, respectively. When the matrix A is of a blockwise form, the matrices H and S, as well as the matrices D_H, B_H, D_S, L_S, and U_S, are thought of having the blockwise forms conforming to that of the matrix A, too.

For this case, we define an SSOR-like iteration method for solving the linear system (1) as follows:

$$\begin{cases} x^{(k+\frac{1}{2})} = x^{(k)} + \omega[\mathrm{i} D_H + \omega(D_S + L_S)]^{-1}\left(b - Ax^{(k)}\right), \\ x^{(k+1)} = x^{(k+\frac{1}{2})} + \omega[\mathrm{i} D_H + \omega(D_S + U_S)]^{-1}\left(b - Ax^{(k+\frac{1}{2})}\right), \end{cases} \tag{24}$$

or equivalently,

$$\begin{cases} \left(\frac{\mathrm{i}}{\omega} D_H + D_S + L_S\right) x^{(k+\frac{1}{2})} = \left[\left(\frac{\mathrm{i}}{\omega} - 1\right) D_H - U_S - B_H\right] x^{(k)} + b, \\ \left(\frac{\mathrm{i}}{\omega} D_H + D_S + U_S\right) x^{(k+1)} = \left[\left(\frac{\mathrm{i}}{\omega} - 1\right) D_H - L_S - B_H\right] x^{(k+\frac{1}{2})} + b. \end{cases} \tag{25}$$

Note that the SSOR-like iteration scheme (24) or (25) is well defined if the diagonal matrix $\frac{\mathrm{i}}{\omega} D_H + D_S$ is nonsingular. Hence, throughout this section we stipulate that the diagonal matrix $\frac{\mathrm{i}}{\omega} D_H + D_S$ is nonsingular for any proper nonzero parameter ω. In fact, this statement holds true for many practical situations, for example, when one of the diagonal matrices D_H and D_S vanishes but the other is nonsingular. Evidently, this SSOR-like iteration method belongs to the category of the two-step splitting iteration method (3), with the splitting matrices being specifically chosen as

$$\begin{aligned} A = M_1(\omega) - N_1(\omega) &= \left(\frac{\mathrm{i}}{\omega} D_H + D_S + L_S\right) - \left[\left(\frac{\mathrm{i}}{\omega} - 1\right) D_H - U_S - B_H\right] \\ = M_2(\omega) - N_2(\omega) &= \left(\frac{\mathrm{i}}{\omega} D_H + D_S + U_S\right) - \left[\left(\frac{\mathrm{i}}{\omega} - 1\right) D_H - L_S - B_H\right]. \end{aligned}$$

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It is a two-step generalization of the single splitting iteration methods proposed and discussed in [43, 44].

From (5) the corresponding SSOR-like preconditioning matrix M, denoted distinctively as $P_S(\omega)$, for the matrix $A \in \mathbb{C}^{n\times n}$ is given by

$$P_S(\omega) = \left[\frac{\mathtt{i}}{\omega}D_H + (D_S + L_S)\right]\left[\left(\frac{2\mathtt{i}}{\omega}D_H + D_S\right) - H\right]^{-1}\left[\frac{\mathtt{i}}{\omega}D_H + (D_S + U_S)\right]. \quad (26)$$

Let

$$W(\omega) = \left(\frac{2\mathtt{i}}{\omega}D_H + D_S\right) - H.$$

Then we know that $W(\omega)$ is a negative definite matrix. Therefore, $W(\omega)^{-1}$ is negative definite, too, which implies that the SSOR-like preconditioning matrix $P_S(\omega) \in \mathbb{C}^{n\times n}$ is positive definite because of the facts that D_H is Hermitian positive definite, D_S is skew-Hermitian, and $U_S = -L_S^*$.

In order to obtain bounds for eigenvalues of the preconditioned matrix $P_S(\omega)^{-1}A$ by making use of the generalized Bendixson theorem in [32], that is, Theorem 2.1, we need to first derive estimates for the generalized Rayleigh quotients

$$\frac{x^*Hx}{x^*\mathcal{H}(P_S(\omega))x}, \quad \frac{1}{\mathtt{i}}\frac{x^*Sx}{x^*Hx} \quad \text{and} \quad \frac{1}{\mathtt{i}}\frac{x^*\mathcal{S}(P_S(\omega))x}{x^*\mathcal{H}(P_S(\omega))x}, \qquad \forall x \in \mathbb{C}^n \setminus \{0\},$$

in terms of the constants θ and ϱ defined as in (12), as well as the constants

$$\theta_{\min} = \min_{x\neq 0}\frac{x^*D_Hx}{x^*Hx}, \quad \phi_{\min} = \min_{x\neq 0}\frac{x^*(D_S + L_S)D_H^{-1}(D_S + L_S)^*x}{x^*Hx} \quad (27)$$

and

$$\varrho_s = \rho\left(D_H^{-1}D_S\right), \quad \phi = \max_{x\neq 0}\frac{x^*(D_S + L_S)D_H^{-1}(D_S + L_S)^*x}{x^*Hx}. \quad (28)$$

Lemma 4.1

Let the constants $\theta, \varrho, \theta_{\min}, \phi_{\min}, \varrho_s$, and ϕ be defined as in (12), (27), and (28). Then for $x \in \mathbb{C}^n\setminus\{0\}$ it holds that

$$f_{\min}(\omega) \leqslant \frac{x^*Hx}{x^*\mathcal{H}(P_S(\omega))x} \leqslant f_{\max}(\omega) \quad (29)$$

and

$$\frac{|x^*Sx|}{x^*Hx} \leqslant \theta\varrho, \qquad \frac{|x^*\mathcal{S}(P_S(\omega))x|}{x^*\mathcal{H}(P_S(\omega))x} \leqslant \theta\left(\frac{2}{\omega} + \varrho_s\right), \quad (30)$$

with $\kappa = \frac{1}{2}\theta(\varrho + \varrho_s)$ and

$$\begin{cases} f_{\min}(\omega) = \frac{1}{\theta}\left(\frac{\theta}{\omega^2} + \frac{2\kappa}{\omega} + \phi\right)^{-1}, \\ f_{\max}(\omega) = \left(\frac{1}{\theta_{\min}} + \frac{\theta(2+\varrho_s\omega)^2}{\omega^2}\right)\left(\frac{\theta_{\min}}{\omega^2} - \frac{2\kappa}{\omega} + \phi_{\min}\right)^{-1}, \end{cases} \quad (31)$$

provided either $\kappa < \sqrt{\theta_{\min}\phi_{\min}}$, or $\kappa > \sqrt{\theta_{\min}\phi_{\min}}$ and the relaxation parameter $\omega \in (0, +\infty)$ satisfies the condition

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$$\omega < \frac{\kappa}{\phi_{\min}}\left(1-\sqrt{1-\frac{\theta_{\min}\phi_{\min}}{\kappa^2}}\right) \quad \text{or} \quad \omega > \frac{\kappa}{\phi_{\min}}\left(1+\sqrt{1-\frac{\theta_{\min}\phi_{\min}}{\kappa^2}}\right).$$

Proof
We first verify the bounds in (29). By making use of Lemma 2.1 we have

$$\mathcal{H}(W(\omega)^{-1}) = -W(\omega)^{-*}\, H\, W(\omega)^{-1}.$$

So the Hermitian part of $P_S(\omega)$ is given by

$$\mathcal{H}(P_S(\omega)) = -\left[\frac{\mathrm{i}}{\omega}D_H + (D_S + L_S)\right] W(\omega)^{-*}\, H\, W(\omega)^{-1}\left[\frac{\mathrm{i}}{\omega}D_H + (D_S + U_S)\right]. \qquad (32)$$

For notational simplicity we denote by

$$\tilde{L} = \frac{\mathrm{i}}{\omega}D_H + (D_S + L_S), \quad \tilde{U} = \frac{\mathrm{i}}{\omega}D_H + (D_S + U_S) \quad \text{and} \quad \tilde{R} = W(\omega)\, H^{-1}\, W(\omega)^*. \qquad (33)$$

Note that $\tilde{U}^* = -\tilde{L}$. Then from (32) the Hermitian part of $P_S(\omega)$ can be briefly expressed as

$$\mathcal{H}(P_S(\omega)) = -\tilde{L}W(\omega)^{-*}\, H\, W(\omega)^{-1}\tilde{U} = \tilde{L}W(\omega)^{-*}\, H\, W(\omega)^{-1}\tilde{L}^* = \tilde{L}\tilde{R}^{-1}\tilde{L}^*. \qquad (34)$$

After straightforward computations we obtain

$$\tilde{R} = H - \frac{1}{\omega^2}D_M H^{-1} D_M, \quad \text{with} \quad D_M = 2\mathrm{i}D_H + \omega D_S, \qquad (35)$$

and

$$\tilde{L}D_H^{-1}\tilde{L}^* = \frac{1}{\omega^2}D_H - \frac{\mathrm{i}}{\omega}(D_S + S) + (D_S + L_S)D_H^{-1}(D_S + L_S)^*. \qquad (36)$$

As the matrices $\mathcal{H}(P_S(\omega))^{-1}H$ and $\tilde{R}\tilde{L}^{-1}H\tilde{L}^{-*}$ are similar, they have the same eigenvalue set. Hence, for an eigenvalue λ of the matrix $\mathcal{H}(P_S(\omega))^{-1}H$, there exists a nonzero vector u such that

$$\tilde{R}\tilde{L}^{-1}H\tilde{L}^{-*}u = \lambda u,$$

or equivalently,

$$\tilde{R}v = \lambda\tilde{L}^*H^{-1}\tilde{L}v, \quad \text{with} \quad v = \tilde{L}^{-1}H\tilde{L}^{-*}u \neq 0.$$

It then follows that

$$\lambda = \frac{v^*\tilde{R}v}{v^*\tilde{L}^*H^{-1}\tilde{L}v} = \frac{v^*\tilde{R}v}{v^*D_Hv}\cdot\frac{v^*D_Hv}{v^*\tilde{L}^*H^{-1}\tilde{L}v}.$$

Because H, $\tilde{R}$ and $\tilde{L}^*H^{-1}\tilde{L}$ are Hermitian positive definite, we further know from (35) that

$$\frac{v^*\tilde{R}v}{v^*D_Hv} = \frac{v^*Hv}{v^*D_Hv} + \frac{1}{\omega^2}\frac{v^*D_M^*H^{-1}D_Mv}{v^*D_Hv} \geqslant \frac{v^*Hv}{v^*D_Hv} \geqslant \frac{1}{\theta}$$

and

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$$\begin{aligned}
\frac{v^*\tilde{R}v}{v^*D_Hv} &\leqslant \max_{v\neq 0}\frac{v^*Hv}{v^*D_Hv} + \frac{1}{\omega^2}\max_{w\neq 0}\frac{w^*D_MD_H^{-1}D_M^*w}{w^*Hw}\\
&= \max_{v\neq 0}\frac{v^*Hv}{v^*D_Hv} + \frac{1}{\omega^2}\max_{w\neq 0}\frac{w^*\left(4D_H - \mathtt{i}4\omega D_S + \omega^2 D_S^*D_H^{-1}D_S\right)w}{w^*Hw}\\
&\leqslant \frac{1}{\theta_{\min}} + \frac{\theta(2+\varrho_s\omega)^2}{\omega^2}.
\end{aligned}$$

And from (36) we know that

$$\begin{aligned}
\frac{w^*\tilde{L}D_H^{-1}\tilde{L}^*w}{w^*Hw} &\geqslant \frac{1}{\omega^2}\min_{w\neq 0}\frac{w^*D_Hw}{w^*Hw} + \frac{1}{\omega}\min_{w\neq 0}\frac{w^*[-\mathtt{i}(D_S+S)]w}{w^*Hw}\\
&\quad + \min_{w\neq 0}\frac{w^*(D_S+L_S)D_H^{-1}(D_S+L_S)^*w}{w^*Hw}\\
&\geqslant \frac{\theta_{\min}}{\omega^2} - \frac{\theta(\varrho_s+\varrho)}{\omega} + \phi_{\min}
\end{aligned}$$

and

$$\begin{aligned}
\frac{w^*\tilde{L}D_H^{-1}\tilde{L}^*w}{w^*Hw} &\leqslant \frac{1}{\omega^2}\max_{w\neq 0}\frac{w^*D_Hw}{w^*Hw} + \frac{1}{\omega}\max_{w\neq 0}\frac{w^*[-\mathtt{i}(D_S+S)]w}{w^*Hw}\\
&\quad + \max_{w\neq 0}\frac{w^*(D_S+L_S)D_H^{-1}(D_S+L_S)^*w}{w^*Hw}\\
&\leqslant \frac{\theta}{\omega^2} + \frac{\theta(\varrho_s+\varrho)}{\omega} + \phi,
\end{aligned}$$

where we have applied the estimate

$$\frac{|x^*Sx|}{x^*Hx} \leqslant \theta\varrho, \quad \forall x \in \mathbb{C}^n \setminus \{0\},$$

derived in Lemma 3.1. It follows straightforwardly from

$$\min_{w\neq 0}\frac{w^*Hw}{w^*\tilde{L}D_H^{-1}\tilde{L}^*w} \leqslant \frac{v^*D_Hv}{v^*\tilde{L}^*H^{-1}\tilde{L}v} \leqslant \max_{w\neq 0}\frac{w^*Hw}{w^*\tilde{L}D_H^{-1}\tilde{L}^*w}$$

that

$$\lambda \leqslant \left(\frac{1}{\theta_{\min}} + \frac{\theta(2+\varrho_s\omega)^2}{\omega^2}\right)\left(\frac{\theta_{\min}}{\omega^2} - \frac{\theta(\varrho_s+\varrho)}{\omega} + \phi_{\min}\right)^{-1} \tag{37}$$

and

$$\lambda \geqslant \frac{1}{\theta}\left(\frac{\theta}{\omega^2} + \frac{\theta(\varrho_s+\varrho)}{\omega} + \phi\right)^{-1}. \tag{38}$$

Therefore, the estimates (37) and (38) immediately lead to the bounds in (29).

Now we turn to derive the bounds in (30). The estimate

$$\frac{|x^*Sx|}{x^*Hx} \leqslant \theta\varrho, \quad \forall x \in \mathbb{C}^n \setminus \{0\},$$

has been already derived in Lemma 3.1. By making use of Lemma 2.1 we have

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$$\mathcal{S}(W(\omega)^{-1}) = -\frac{1}{\omega} W(\omega)^{-*}\, D_M\, W(\omega)^{-1}.$$

So the skew-Hermitian part of $P_S(\omega)$ is given by

$$\mathcal{S}(P_S(\omega)) = -\frac{1}{\omega}\left[\frac{\mathtt{i}}{\omega} D_H + (D_S + L_S)\right] W(\omega)^{-*}\, D_M\, W(\omega)^{-1}\left[\frac{\mathtt{i}}{\omega} D_H + (D_S + U_S)\right];$$

see (26). By making use of the notations

$$\tilde{L} = \frac{\mathtt{i}}{\omega} D_H + (D_S + L_S) \quad \text{and} \quad \tilde{U} = \frac{\mathtt{i}}{\omega} D_H + (D_S + U_S)$$

introduced in (33), we can rewrite $\mathcal{S}(P_S(\omega))$ into the equivalent form

$$\mathcal{S}(P_S(\omega)) = -\frac{1}{\omega}\tilde{L} W(\omega)^{-*}\, D_M\, W(\omega)^{-1}\tilde{U} = \frac{1}{\omega}\tilde{L} W(\omega)^{-*}\, D_M\, W(\omega)^{-1}\tilde{L}^*. \tag{39}$$

It then follows immediately from (34) and (39) that for any $x \in \mathbb{C}^n \setminus \{0\}$ we have

$$\begin{aligned}\frac{|x^*\mathcal{S}(P_S(\omega))x|}{x^*\mathcal{H}(P_S(\omega))x} &= \frac{1}{\omega}\frac{|x^*\tilde{L} W(\omega)^{-*}\, D_M\, W(\omega)^{-1}\tilde{L}^* x|}{x^*\tilde{L} W(\omega)^{-*}\, H\, W(\omega)^{-1}\tilde{L}^* x} \leqslant \frac{1}{\omega}\max_{x\neq 0}\frac{|x^* D_M x|}{x^* H x}\\ &\leqslant \frac{1}{\omega}\left(2\frac{x^* D_H x}{x^* H x} + \omega\frac{|x^* D_S x|}{x^* H x}\right) = \theta\left(\frac{2}{\omega} + \varrho_s\right).\end{aligned}$$

□

Based on Lemma 4.1 and Theorem 2.1, we can obtain bounds for eigenvalues of the preconditioned matrix $P_S(\omega)^{-1}A$ with respect to the SSOR-like preconditioning matrix $P_S(\omega)$ defined in (26).

Theorem 4.1

Let the matrices A and $P_S(\omega)$ be defined as in (1) and (26), and the constants θ, ϱ, $\theta_{\min}$, $\phi_{\min}$, ϱ_s, and ϕ be defined as in (12), (27), and (28) satisfying $\theta^2\varrho\varrho_s < 1$. Then it holds that

$$\frac{[\omega - \theta^2\varrho(2+\varrho_s\omega)]\omega}{\omega^2 + \theta^2(2+\varrho_s\omega)^2} f_{\min}(\omega) \leqslant \Re(\lambda(P_S(\omega)^{-1}A)) \leqslant \left[1 + \theta^2\varrho\left(\varrho_s + \frac{2}{\omega}\right)\right] f_{\max}(\omega)$$

and

$$|\Im(\lambda(P_S(\omega)^{-1}A))| \leqslant \theta\left(\varrho + \varrho_s + \frac{2}{\omega}\right) f_{\max}(\omega),$$

with $f_{\min}(\omega)$ and $f_{\max}(\omega)$ being defined as in (31) and $\kappa = \frac{1}{2}\theta(\varrho + \varrho_s)$, provided that the relaxation parameter $\omega \in (0, +\infty)$ satisfies

$$\omega > \frac{2\theta^2\varrho}{1 - \theta^2\varrho\varrho_s},$$

and either $\kappa < \sqrt{\theta_{\min}\phi_{\min}}$, or $\kappa > \sqrt{\theta_{\min}\phi_{\min}}$ and

$$\omega < \frac{\kappa}{\phi_{\min}}\left(1 - \sqrt{1 - \frac{\theta_{\min}\phi_{\min}}{\kappa^2}}\right) \quad \text{or} \quad \omega > \frac{\kappa}{\phi_{\min}}\left(1 + \sqrt{1 - \frac{\theta_{\min}\phi_{\min}}{\kappa^2}}\right).$$

The eigenvalue bounds given in Theorem 4.1 about the preconditioned matrix $P_S(\omega)^{-1}A$ with respect to the SSOR-like iteration method (24) or (25) are numerically validated by the linear system (43) arising from discretization of the two-dimensional linear integro-differential equation in Example 5.2; see Figure 3. From these two pictures we observe that all eigenvalues of the preconditioned matrix $P_S(\omega)^{-1}A$ are contained in a rectangle composed of the estimated eigenvalue bounds given in Theorem 4.1. Note that these estimated bounds are not so sharp for the real as well as the imaginary parts of the eigenvalues.

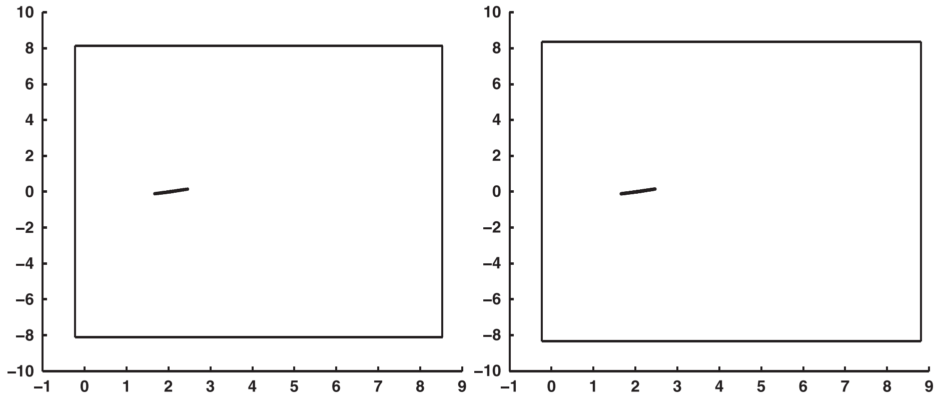

Figure 3. The exact eigenvalue distribution and the estimated eigenvalue bound for the preconditioned matrix $P_S(\omega)^{-1}A$, with A being the matrix in (43) in Example 5.2, when æ $= 10$, $\beta = 10$, and $\nu = 10^{-4}$, with $n = 8^2$ and $\omega = 1.1566e+3$ (left) and $n = 12^2$ and $\omega = 1.4782e+3$ (right). The exact eigenvalues: the dot '·', and the estimated bound: the rectangle of solid line '—'.

According to Theorem 4.1 and Lemma 2.2 or Theorem 2.1 and Lemma 4.1, we can obtain the following convergence theory for the SSOR-like iteration method (24) or (25) for solving the linear system (1) of a strongly dominant skew-Hermitian part.

Theorem 4.2
Let the coefficient matrix $A \in \mathbb{C}^{n\times n}$ of the linear system (1) be non-Hermitian positive definite, the matrix $P_S(\omega)$ be defined as in (26), and the constants θ, ϱ, $\theta_{\min}$, $\phi_{\min}$, ϱ_s, and ϕ be defined as in (12), (27), and (28) satisfying $\theta^2\varrho\varrho_s < 1$. Then the SSOR-like iteration method (24) or (25) is convergent if the relaxation parameter $\omega \in (0, +\infty)$ satisfies

$$\omega > \frac{2\theta^2\varrho}{1-\theta^2\varrho\varrho_s}, \quad \theta\left(\varrho+\varrho_s+\frac{2}{\omega}\right) f_{\max}(\omega) < 1,$$

and

$$\begin{cases} \frac{[\omega-\theta^2\varrho(2+\varrho_s\omega)]\omega}{\omega^2+\theta^2(2+\varrho_s\omega)^2} f_{\min}(\omega) > 1-\sqrt{1-\theta^2(\varrho+\varrho_s+\frac{2}{\omega})^2}, \\ \left(1+\frac{\theta^2\varrho(2+\varrho_s\omega)}{\omega}\right) f_{\max}(\omega) < 1+\sqrt{1-\theta^2(\varrho+\varrho_s+\frac{2}{\omega})^2} \end{cases}$$

when either $\kappa < \sqrt{\theta_{\min}\phi_{\min}}$, or $\kappa > \sqrt{\theta_{\min}\phi_{\min}}$ and

$$\omega < \frac{\kappa}{\phi_{\min}}\left(1-\sqrt{1-\frac{\theta_{\min}\phi_{\min}}{\kappa^2}}\right) \quad \text{or} \quad \omega > \frac{\kappa}{\phi_{\min}}\left(1+\sqrt{1-\frac{\theta_{\min}\phi_{\min}}{\kappa^2}}\right),$$

where $f_{\min}(\omega)$ and $f_{\max}(\omega)$ are defined as in (31), with $\kappa = \frac{1}{2}\theta(\varrho+\varrho_s)$.

Admittedly, this convergence result is practically not so useful because of the strict restrictions imposed on the relaxation parameter ω, which may be difficult to be examined in actual applications and may result in small convergence domain with respect to ω. These strict restrictions originate from the usage of the generalized Bendixson theorem (Theorem 2.1), which is developed especially for bounding the eigenvalues of a pair of non-Hermitian positive definite matrices of strong Hermitian parts, but not of strong skew-Hermitian parts.

For the linear system (43) in Example 5.2, we plot the pictures of the spectral radius versus the relaxation parameter ω with respect to the iteration matrix $\mathbf{L}_S(\omega) = I - P_S(\omega)^{-1}A$ of the SSOR-like iteration scheme (24) or (25) in Figure 4. Here, the *computed convergence domains* of ω are

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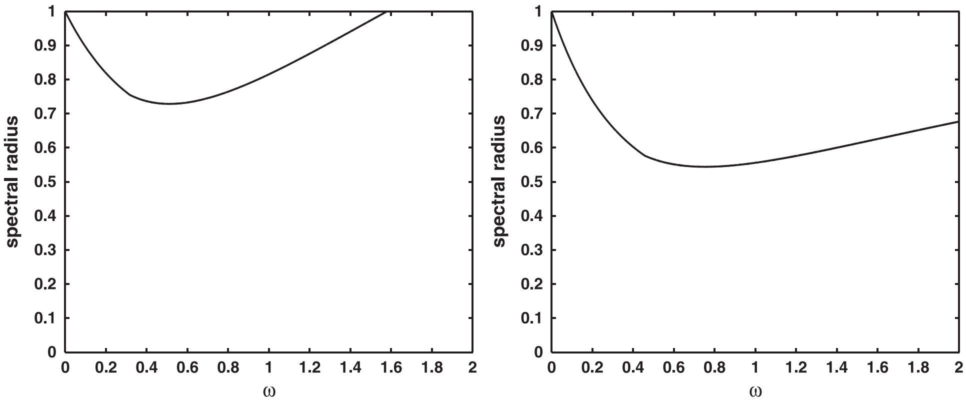

Figure 4. Pictures of the spectral radius versus the relaxation parameter with respect to the iteration matrix of the SSOR-like iteration scheme (24) for the linear system (43) in Example 5.2 when $n = 10^2$, $\beta = 15$, and $\nu = 10^{-4}$, with æ = 10 (left) and æ = 15 (right).

$\mathbb{I}_{ccd} = (0, 1.5790)$ for the left picture and $\mathbb{I}_{ccd} = (0, 2)$ for the right picture. Therefore, the SSOR-like iteration scheme (24) or (25) may converge even within a large domain of the relaxation parameter ω in practical applications.

5. NUMERICAL RESULTS

In this section, we examine the theoretical properties and the numerical behaviors of the SSOR-like preconditioning matrices $P_H(\omega)$ and $P_S(\omega)$, defined in (11) and (26), when they are employed to precondition the GMRES method [4]. Correspondingly, these two kinds of preconditioned GMRES methods are termed briefly as SSOR_H-GMRES and SSOR_S-GMRES, respectively. Besides, the GMRES preconditioned by the classical SSOR defined in (8) and by the incomplete LU (ILU) factorization are abbreviated as SSOR-GMRES and ILU-GMRES, respectively. The advantages of SSOR_H-GMRES and SSOR_S-GMRES over ILU-GMRES and SSOR-GMRES are shown by comparing their numbers of iteration steps (denoted as **IT**) and computing times in seconds (denoted as **CPU**).

The experimented linear system (1) arises from certain numerical discretizations of two-dimensional linear integro-differential equations described in the following. Here we should emphasize that these kinds of large and sparse linear systems are highly ill-conditioned, significantly unstructured, and heavily dense, so they are often problematic and challenging to compute approximate solutions of desired accuracy within satisfactory time by using either direct methods such as Gaussian elimination, or iterative methods such as SSOR.

Example 5.1 (see [45]*)*
Consider the two-dimensional linear integro-differential equation

$$-\triangle \mathrm{u} + q\left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \frac{\partial \mathrm{u}}{\partial \mathrm{y}}\right) + \gamma \int_0^{\mathrm{y}} \int_0^{\mathrm{x}} \mathcal{K}(\mathrm{x},\mathrm{y},\mathrm{s},\mathrm{t})\,\mathrm{u}(\mathrm{s},\mathrm{t})\,\mathrm{ds}\,\mathrm{dt} = \mathrm{g}(\mathrm{x},\mathrm{y}), \quad (\mathrm{x},\mathrm{y}) \in \Omega, \tag{40}$$

with the homogeneous Dirichlet boundary condition, where $\Omega = (0,1) \times (0,1)$ is the unit square, $\triangle$ is the two-dimensional Laplace operator, $\mathcal{K}(\mathrm{x},\mathrm{y},\mathrm{s},\mathrm{t})$ is the integral kernel function, $\mathrm{g}(\mathrm{x},\mathrm{y})$ is a given function, and q and γ are prescribed positive parameters.

After approximating the derivatives by the nine-point centered differences and the integral by the trapezoidal quadrature formula, with the stepsizes being both equal to $h = \frac{1}{\sqrt{m}+4}$, and normalizing the discretized coefficient matrix and right-hand side by multiplying h^2, we obtain a linear system of the form

$$(T + æ\,B + \nu K)x = b, \tag{41}$$

where $æ = qh$ and $\nu = \gamma h^4$ are positive constants,

$$T = T_1 \otimes I + I \otimes T_1 \in \mathbb{R}^{m\times m} \quad \text{and} \quad B = B_1 \otimes I + I \otimes B_1 \in \mathbb{R}^{m\times m}$$

are the two-dimensional discrete matrices corresponding to the difference and the integral operators,

$$T_1 = \frac{5}{2} I - \frac{4}{3}\left(E_1 + E_1{}^T\right) + \frac{1}{12}\left(E_2 + E_2{}^T\right) \quad \text{and} \quad B_1 = \frac{2}{3}\left(E_1 - E_1{}^T\right) - \frac{1}{12}\left(E_2 - E_2{}^T\right)$$

are the one-dimensional discrete matrices corresponding to the difference and the integral operators, with $E_1, E_2 \in \mathbb{R}^{\sqrt{m}\times\sqrt{m}}$ being given by

$$E_1 = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 0 & 0 & 1 & & & \\ & 0 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & 1 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

and $K = (k_{ij}) \in \mathbb{R}^{m\times m}$ is defined by

$$k_{ij} = \mathrm{x}_{i_1}\mathrm{y}_{i_2}\,\mathcal{K}\left(\mathrm{x}_{i_1}, \mathrm{y}_{i_2}, \mathrm{x}_{j_1}, \mathrm{y}_{j_2}\right),$$

with

$$i = \sqrt{m}(i_2 - 1) + i_1 \quad \text{and} \quad j = \sqrt{m}(j_2 - 1) + j_1, \qquad i_1, i_2, j_1, j_2 = 1, 2, \ldots, \sqrt{m}.$$

Here we choose

$$\mathcal{K}(\mathrm{x}, \mathrm{y}, \mathrm{s}, \mathrm{t}) = \frac{\mu(\sqrt{\psi} + 4)}{\mathrm{xy}}\left[\sqrt{\psi}(\mathrm{y} - \mathrm{t}) + (\mathrm{x} - \mathrm{s})\right],$$

with ψ being a positive constant, so that

$$k_{ij} = \frac{\mu(\sqrt{\psi} + 4)}{\sqrt{m} + 4}\left[\sqrt{\frac{\psi}{m}}(i - j) + \left(1 - \sqrt{\frac{\psi}{m}}\right)(i_1 - j_1)\right], \qquad i, j = 1, 2, \ldots, m.$$

For Example 5.1, the matrix T is symmetric positive definite, and the matrices B and K are skew-symmetric. Besides, both T and B are sparse, but K is dense. As a result, the linear system (41) is symmetric positive definite of a sparse symmetric part H and a dense skew-symmetric part S, with H being strongly dominant over S. So we may adopt the $\mathrm{SSOR_H}$-GMRES as the linear solver.

Example 5.2

Consider the two-dimensional linear integro-differential equation (40). Using the ansatz $\mathrm{v}(\mathrm{x}, \mathrm{y}) = \mathrm{u}(\mathrm{x}, \mathrm{y})e^{-\mathrm{i}\xi(\mathrm{x}+\mathrm{y})}$, where u and v are complex-valued functions, we can reformulate it as

$$\begin{aligned} &- \triangle \mathrm{v} + (q - \mathrm{i}\,2\xi)\left(\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \frac{\partial \mathrm{v}}{\partial \mathrm{y}}\right) + 2\xi(\mathrm{i}\,q + \xi)\mathrm{v} \\ &+ \gamma e^{-\mathrm{i}\xi(\mathrm{x}+\mathrm{y})}\int_0^{\mathrm{y}}\int_0^{\mathrm{x}} \mathcal{K}(\mathrm{x}, \mathrm{y}, \mathrm{s}, \mathrm{t})\,\mathrm{v}(\mathrm{s}, \mathrm{t})e^{\mathrm{i}\xi(\mathrm{s}+\mathrm{t})}\,\mathrm{ds}\,\mathrm{dt} = \mathrm{g}(\mathrm{x}, \mathrm{y})\mathrm{e}^{-\mathrm{i}\xi(\mathrm{x}+\mathrm{y})}, \quad (\mathrm{x}, \mathrm{y}) \in \Omega, \end{aligned} \tag{42}$$

with the homogeneous Dirichlet boundary condition, where $\Omega = (0, 1) \times (0, 1)$ is the unit square, $\triangle$ is the two-dimensional Laplace operator, $\mathcal{K}(\mathrm{x}, \mathrm{y}, \mathrm{s}, \mathrm{t})$ is the integral kernel function, $\mathrm{g}(\mathrm{x}, \mathrm{y})$ is a given function, and q and γ are prescribed positive parameters.

Then by discretizing (42) in an analogous fashion to that of equation (40) in Example 5.1, we obtain a linear system of the form

$$[T + 2\beta(\mathrm{i}\,æ + \beta)I + (æ - \mathrm{i}\,2\beta)B + \gamma K]\,x = b, \tag{43}$$

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where $æ = qh$, $\beta = \xi h$, the matrices T and B are the same as in Example 5.1, and the matrix $K = (k_{ij}) \in \mathbb{R}^{m \times m}$ is defined by

$$k_{ij} = \mathrm{x}_{i_1} \mathrm{y}_{i_2} \mathcal{K}\left(\mathrm{x}_{i_1}, \mathrm{y}_{i_2}, \mathrm{x}_{j_1}, \mathrm{y}_{j_2}\right) e^{\mathrm{i}\xi\left[\left(\mathrm{x}_{j_1}+\mathrm{y}_{j_2}\right)-\left(\mathrm{x}_{i_1}+\mathrm{y}_{i_2}\right)\right]},$$

with

$$i = \sqrt{m}(i_2 - 1) + i_1 \quad \text{and} \quad j = \sqrt{m}(j_2 - 1) + j_1, \qquad i_1, i_2, j_1, j_2 = 1, 2, \ldots, \sqrt{m}.$$

Here we choose

$$\mathcal{K}(\mathrm{x}, \mathrm{y}, \mathrm{s}, \mathrm{t}) = \frac{e^{\mathrm{i}\xi[(\mathrm{x}+\mathrm{y})-(\mathrm{s}+\mathrm{t})]}}{\mathrm{xy}} e^{-\mu|\sqrt{\psi}(\mathrm{y}-\mathrm{t})+(\mathrm{x}-\mathrm{s})|},$$

with ψ being a positive constant, so that

$$k_{ij} = e^{-\frac{\mu}{\sqrt{m}+4}\left|\sqrt{\frac{\psi}{m}}(i-j)+\left(1-\sqrt{\frac{\psi}{m}}\right)(i_1-j_1)\right|}, \qquad i, j = 1, 2, \ldots, m.$$

For Example 5.2, the matrix T is symmetric positive definite, the matrix B is skew-symmetric, and the matrix K is symmetric. Besides, both T and B are sparse, but K is dense. As a result, the linear system (43) is Hermitian positive definite of a dense Hermitian part H and a sparse skew-Hermitian part S, with S being strongly dominant over H. So we may adopt the SSOR$_\mathrm{S}$-GMRES as the linear solver.

In our implementations, the positive constants μ and ψ are set to be $\mu = 10$ and $\psi = 40$ in both examples. And in Example 5.2, the positive constant γ is chosen such that $\nu = \gamma h^4 = 10$. Moreover, the initial guess is taken to be zero and the right-hand side vector is set to be $b = h^2\mathbf{e}$, with $\mathbf{e} \in \mathbb{R}^m$ being the vector of all entries equal to 1. The iteration process is terminated once the Euclidean norm of the current residual is reduced by a factor of 10^6 from that of the initial residual, and the iteration parameters are the experimentally found optimal ones that minimize the total numbers of iteration steps of the corresponding iteration processes, which are listed in Tables II and IV with respect to Examples 5.1 and 5.2. In addition, all codes were run in MATLAB (version R2008a) in double precision, and all experiments were performed on a personal computer with 2.53 GHz central processing unit (Intel(R) Core(TM)2 Duo CPU L9400), 1.86 G memory and Windows operating system.

In Table I, we list the numbers of iteration steps and the CPU times of ILU-GMRES, SSOR-GMRES, and SSOR$_\mathrm{H}$-GMRES for Example 5.1 with respect to different values of the problem parameters q and γ as well as the problem size m, where the CPU times in seconds are shown in parentheses. The optimal iteration parameters ω_{opt} of SSOR-GMRES and SSOR$_\mathrm{H}$-GMRES used in this table are listed in Table II.

From Table I we see that for any fixed pair $(æ, \nu)$ of the constants $æ$ and ν, when the problem size m becomes large, the IT of ILU-GMRES and SSOR$_\mathrm{H}$-GMRES remains essentially constant, but that of the SSOR-GMRES is increasing. For each pair $(æ, \nu)$ and each m, both IT and CPU of SSOR$_\mathrm{H}$-GMRES are much less than those of ILU-GMRES and SSOR-GMRES. Therefore, for the linear system (41) with the coefficient matrix having a strong symmetric part, SSOR$_\mathrm{H}$-GMRES outperforms both ILU-GMRES and SSOR-GMRES in terms of both IT and CPU, and it also shows h-independent convergence property.

In Table III, we list the numbers of iteration steps and the CPU times of ILU-GMRES, SSOR-GMRES, and SSOR$_\mathrm{S}$-GMRES for Example 5.2 with respect to different values of the problem parameters q and ξ as well as the problem size m, where the CPU times in seconds are shown in parentheses. The optimal iteration parameters ω_{opt} of SSOR-GMRES and SSOR$_\mathrm{S}$-GMRES used in this table are listed in Table IV.

From Table III we see that ILU-GMRES fails to converge for most of the experimented cases except for $m = 900$ with all pairs of $(æ, \beta)$, and for $m = 1225$ with $(æ, \beta) = (10^7, 10^{-2})$ and $(10^8, 10^{-3})$, and SSOR-GMRES even fails to converge for all m when $(æ, \beta) = (10^8, 10^{-3})$. As for the convergent cases, for each pair $(æ, \beta)$ of the constants $æ$ and β and each m of the problem size, both IT and CPU of SSOR$_\mathrm{S}$-GMRES are much less than those of ILU-GMRES and

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Table I. Iteration steps and computing times of ILU-GMRES, SSOR-GMRES and $\mathrm{SSOR_H}$-GMRES for Example 5.1.

Method	(æ, ν)	m = 900	1225	1600	2025	2500
ILU-GMRES	$(10^{-4}, 10^{-4})$	26 (0.94)	26 (1.73)	25 (2.67)	25 (3.97)	25 (5.81)
	$(10^{-4}, 10^{-5})$	16 (0.88)	17 (1.76)	18 (2.54)	18 (3.83)	19 (5.64)
	$(10^{-4}, 10^{-6})$	19 (0.91)	21 (1.73)	22 (2.58)	23 (3.91)	23 (6.95)
	$(10^{-6}, 10^{-7})$	24 (1.01)	26 (1.70)	28 (2.59)	31 (3.97)	32 (5.95)
SSOR-GMRES	$(10^{-4}, 10^{-4})$	35 (0.96)	40 (1.97)	49 (4.03)	60 (7.75)	64 (15.60)
	$(10^{-4}, 10^{-5})$	26 (0.75)	33 (1.68)	39 (3.29)	47 (6.16)	61 (14.85)
	$(10^{-4}, 10^{-6})$	17 (0.56)	19 (1.06)	22 (2.10)	25 (3.64)	30 (8.02)
	$(10^{-6}, 10^{-7})$	16 (0.53)	17 (0.97)	18 (1.73)	19 (2.98)	21 (6.18)
$\mathrm{SSOR_H}$-GMRES	$(10^{-4}, 10^{-4})$	9 (0.33)	9 (0.60)	9 (1.02)	9 (1.64)	9 (3.14)
	$(10^{-4}, 10^{-5})$	11 (0.39)	12 (0.74)	12 (1.25)	12 (1.95)	12 (3.75)
	$(10^{-4}, 10^{-6})$	14 (0.45)	14 (0.80)	14 (1.39)	15 (2.35)	15 (4.41)
	$(10^{-6}, 10^{-7})$	16 (0.48)	16 (0.87)	17 (1.62)	17 (2.63)	17 (5.13)

Table II. Experimental optimal parameters ω_{opt} of SSOR-GMRES and $\mathrm{SSOR_H}$-GMRES for Example 5.1.

Method	(æ, ν)	m = 900	1225	1600	2025	2500
SSOR-GMRES	$(10^{-4}, 10^{-4})$	0.17	0.28	0.31	0.30	0.35
	$(10^{-4}, 10^{-5})$	1.49	1.53	1.62	1.63	1.42
	$(10^{-4}, 10^{-6})$	1.79	1.72	1.70	1.71	1.68
	$(10^{-6}, 10^{-7})$	1.74	1.79	1.78	1.80	1.88
$\mathrm{SSOR_H}$-GMRES	$(10^{-4}, 10^{-4})$	1.66	1.70	1.71	1.69	1.65
	$(10^{-4}, 10^{-5})$	1.77	1.74	1.78	1.82	1.84
	$(10^{-4}, 10^{-6})$	1.83	1.75	1.81	1.79	1.83
	$(10^{-6}, 10^{-7})$	1.68	1.73	1.81	1.83	1.84

Table III. Iteration steps and computing times of ILU-GMRES, SSOR-GMRES and $\mathrm{SSOR_S}$-GMRES for Example 5.2.

Method	(æ, β)	m = 900	1225	1600	2025	2500
ILU-GMRES	$(10^{6}, 10^{-1})$	846 (7.00)	—	—	—	—
	$(10^{7}, 10^{-2})$	842 (4.20)	893 (19.76)	—	—	—
	$(10^{6}, 10^{-2})$	774 (11.91)	—	—	—	—
	$(10^{8}, 10^{-3})$	714 (10.04)	945 (9.16)	—	—	—
SSOR-GMRES	$(10^{6}, 10^{-1})$	64 (4.19)	77 (8.81)	91 (17.29)	97 (29.10)	149 (64.10)
	$(10^{7}, 10^{-2})$	68 (4.41)	80 (8.95)	90 (17.07)	99 (29.58)	147 (63.24)
	$(10^{6}, 10^{-2})$	90 (5.86)	99 (11.15)	109 (20.63)	128 (38.73)	196 (83.68)
	$(10^{8}, 10^{-3})$	—	—	—	—	—
$\mathrm{SSOR_S}$-GMRES	$(10^{6}, 10^{-1})$	52 (3.39)	67 (7.66)	72 (15.56)	72 (22.03)	75 (33.72)
	$(10^{7}, 10^{-2})$	65 (4.19)	75 (8.44)	85 (16.17)	93 (27.98)	106 (47.61)
	$(10^{6}, 10^{-2})$	89 (5.70)	99 (11.12)	108 (20.50)	122 (37.09)	191 (79.89)
	$(10^{8}, 10^{-3})$	123 (7.61)	179 (19.42)	206 (37.94)	314 (91.39)	440 (188.23)

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Table IV. Experimental optimal parameters ω_{opt} of SSOR-GMRES and SSOR$_S$-GMRES for Example 5.2.

Method	(æ, β)	m 900	1225	1600	2025	2500
SSOR-GMRES	$(10^6, 10^{-1})$	0.010	0.009	0.015	0.012	0.011
	$(10^7, 10^{-2})$	0.001	0.002	0.002	0.001	0.001
	$(10^6, 10^{-2})$	0.001	0.001	0.001	0.001	0.001
	$(10^8, 10^{-3})$	—	—	—	—	—
SSOR$_S$-GMRES	$(10^6, 10^{-1})$	0.141	0.125	0.127	0.121	0.120
	$(10^7, 10^{-2})$	0.003	0.004	0.003	0.004	0.004
	$(10^6, 10^{-2})$	0.001	0.001	0.001	0.001	0.001
	$(10^8, 10^{-3})$	0.001	0.001	0.001	0.001	0.001

Table V. Iteration steps and computing times of SSOR$_H$ and SOR$_H$ for Example 5.1.

Method	(æ, ν)	m 900	1225	1600	2025	2500
SSOR$_H$	$(10^{-4}, 10^{-4})$	47 (1.45)	45 (2.58)	44 (4.29)	44 (6.83)	43 (10.16)
	$(10^{-4}, 10^{-3})$	37 (1.15)	36 (2.11)	36 (3.51)	36 (5.59)	36 (8.52)
	$(10^{-3}, 10^{-3})$	37 (1.17)	37 (2.15)	36 (3.51)	36 (5.61)	36 (8.54)
	$(10^{-5}, 10^{-4})$	48 (1.49)	48 (2.77)	48 (4.69)	48 (7.48)	48 (11.36)
SOR$_H$	$(10^{-4}, 10^{-4})$	96 (1.66)	92 (2.90)	92 (4.91)	91 (7.70)	91 (11.62)
	$(10^{-4}, 10^{-3})$	73 (1.26)	73 (2.30)	73 (3.91)	72 (6.13)	72 (9.31)
	$(10^{-3}, 10^{-3})$	73 (1.27)	75 (2.35)	75 (4.01)	74 (6.39)	74 (9.60)
	$(10^{-5}, 10^{-4})$	94 (1.61)	94 (2.96)	94 (5.02)	93 (7.92)	93 (12.15)

(or) SSOR-GMRES. Therefore, for the linear system (43) with the coefficient matrix having a strong skew-Hermitian part, SSOR$_S$-GMRES outperforms both ILU-GMRES and SSOR-GMRES in terms of both IT and CPU. However, all these three methods do not show h-independent convergence behavior as the numbers of their iteration steps are growing up when the problem size m becomes large.

In addition, for the linear system (1) we can also define the SOR-like counterparts of the SSOR-like iteration methods (9) and (24) as

$$x^{(k+1)} = x^{(k)} + \omega(D_H + \omega L_H)^{-1}\left(b - Ax^{(k)}\right) \tag{44}$$

and

$$x^{(k+1)} = x^{(k)} + \omega[\mathrm{i}D_H + \omega(D_S + L_S)]^{-1}\left(b - Ax^{(k)}\right), \tag{45}$$

correspondingly. For brevity, we denote by SOR$_H$ and SOR$_S$ the iteration schemes (44) and (45), and SSOR$_H$ and SSOR$_S$ the iteration schemes (9) and (24), respectively. It should be practically valuable to examine the numerical behaviors of these SOR-like and SSOR-like iteration methods when they are used as linear solvers.

In Table V, we list the numbers of iteration steps and the CPU times of SSOR$_H$ and SOR$_H$ for Example 5.1 with respect to different values of the problem parameters q and γ as well as the problem size m, where the CPU times in seconds are shown in parentheses. The optimal iteration parameters ω_{opt} used in this table are listed in Table VI.

From Table V we see that for any fixed pair (æ, ν) of the constants æ and ν, when the problem size m becomes large, the IT of SSOR$_H$ and SOR$_H$ remains essentially constant. And for each pair (æ, ν) and each m, the IT of SSOR$_H$ is about half of that of SOR$_H$, and the CPU of SSOR$_H$ is much less than that of SOR$_H$. Therefore, for the linear system (41) with the coefficient matrix having a

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Table VI. Experimental optimal parameters ω_{opt} of SSOR_{H} and SOR_{H} for Example 5.1.

Method	(æ, ν)	m = 900	1225	1600	2025	2500
SSOR_{H}	$(10^{-4}, 10^{-4})$	1.98	1.96	1.93	1.90	1.87
	$(10^{-4}, 10^{-3})$	1.96	1.92	1.87	1.83	1.78
	$(10^{-3}, 10^{-3})$	1.97	1.89	1.85	1.79	1.76
	$(10^{-5}, 10^{-4})$	1.96	1.89	1.81	1.74	1.68
SOR_{H}	$(10^{-4}, 10^{-4})$	1.96	1.93	1.87	1.83	1.79
	$(10^{-4}, 10^{-3})$	1.98	1.89	1.83	1.78	1.73
	$(10^{-3}, 10^{-3})$	1.98	1.85	1.77	1.71	1.65
	$(10^{-5}, 10^{-4})$	1.98	1.91	1.84	1.78	1.73

Table VII. Iteration steps and computing times of SSOR_{S} and SOR_{S} for Example 5.2.

Method	(æ, β)	m = 900	1225	1600	2025	2500
SSOR_{S}	$(10^2, 0.30)$	5 (0.78)	5 (1.44)	5 (2.55)	5 (4.14)	5 (6.37)
	$(10^4, 0.14)$	30 (4.53)	28 (8.19)	27 (14.47)	26 (21.27)	25 (31.56)
	$(10^5, 0.15)$	12 (1.85)	12 (3.47)	11 (5.44)	11 (10.00)	11 (13.90)
	$(10^2, 0.50)$	5 (0.76)	5 (1.48)	5 (2.59)	5 (4.13)	5 (6.29)
SOR_{S}	$(10^2, 0.30)$	10 (0.85)	10 (1.64)	10 (2.69)	10 (4.38)	10 (6.94)
	$(10^4, 0.14)$	61 (5.04)	56 (8.70)	54 (14.65)	52 (23.16)	51 (34.69)
	$(10^5, 0.15)$	24 (1.95)	24 (3.77)	23 (6.29)	23 (10.48)	22 (14.85)
	$(10^2, 0.50)$	10 (0.83)	10 (1.63)	10 (2.70)	11 (4.99)	11 (7.06)

Table VIII. Experimental optimal parameters ω_{opt} of SSOR_{S} and SOR_{S} for Example 5.2.

Method	(æ, β)	m = 900	1225	1600	2025	2500
SSOR_{S}	$(10^2, 0.30)$	0.21	0.18	0.19	0.18	0.15
	$(10^4, 0.14)$	0.31	0.30	0.28	0.26	0.26
	$(10^5, 0.15)$	0.24	0.23	0.26	0.25	0.24
	$(10^2, 0.50)$	0.32	0.31	0.30	0.28	0.28
SOR_{S}	$(10^2, 0.30)$	0.22	0.21	0.18	0.17	0.15
	$(10^4, 0.14)$	0.26	0.28	0.29	0.26	0.27
	$(10^5, 0.15)$	0.21	0.20	0.20	0.23	0.24
	$(10^2, 0.50)$	0.34	0.32	0.31	0.29	0.29

strong symmetric part, SSOR_{H} outperforms SOR_{H} in terms of both IT and CPU, and both methods also show h-independent convergence property.

In Table VII, we list the numbers of iteration steps and the CPU times for Example 5.2 with respect to different values of the problem parameters q and ξ as well as the problem size m, where the CPU times in seconds are shown in parentheses. The optimal iteration parameters ω_{opt} used in this table are listed in Table VIII.

From Table VII we see that for any fixed pair (æ, β) of the constants æ and β, when the problem size m becomes large, the IT of SSOR_{S} and SOR_{S} remains essentially constant. And for each pair (æ, β) and each m, the IT of SSOR_{S} is about half of that of SOR_{S}, and the CPU of SSOR_{S} is less

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than that of $\mathrm{SOR_S}$. Therefore, for the linear system (43) with the coefficient matrix having a strong skew-Hermitian part, $\mathrm{SSOR_S}$ outperforms $\mathrm{SOR_S}$ in terms of both IT and CPU, and both methods also show h-independent convergence property.

6. CONCLUSIONS AND REMARKS

For non-Hermitian positive definite matrices, the newly proposed SSOR-like preconditioning matrices are positive definite and the eigenvalues of the correspondingly preconditioned matrices possess tight bounds. Numerical experiments show that these SSOR-like preconditioners outperform the classical SSOR preconditioner and that the SSOR-like iteration methods also outperform their SOR-like counterparts in terms of both number of iteration steps and computing time, when either the Hermitian or the skew-Hermitian part of the coefficient matrix is dominant. Sufficient convergence conditions for the SSOR-like iteration methods are obtained based on the generalized Bendixson theorem stated in Theorem 2.1 and the eigenvalue bounds described in Theorems 3.1 and 4.1, but these conditions could be too rough. Hence, more precise conditions should be derived in order to verify the convergence of the SSOR-like iteration methods. Besides, the SSOR-like iteration methods could also be employed as smoothers of the multigrid methods, so their smoothing property is an important topic to be further explored and analyzed.

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