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002

A BLACK BOX GENERALIZED CONJUGATE GRADIENT SOLVER WITH INNER ITERATIONS AND VARIABLE-STEP PRECONDITIONING*

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Abstract. The generalized conjugate gradient method proposed by Axelsson is studied in the case when a variable-step preconditioning is used. This can be the case when the preconditioned system is solved approximately by an auxiliary (inner) conjugate gradient method, for instance, and the thus-obtained quasi residuals are used to construct the next search vector in the outer generalized cg-iteration method.

A monotone convergence of the method is proved and a rough convergence rate estimate is derived, provided the variable-step preconditioner (generally, a nonlinear mapping) satisfies a continuity and a coercivity assumption.

These assumptions are verified for application of the method for two-level grids and indefinite problems. This variable-step preconditioning involves, for the two-level case, the solution of the coarse grid problem and problems for the nodes on the rest of the grid—both by auxiliary (inner) iterative methods. For the indefinite problems that are considered, the special block structure of the matrix is utilized—also in an outer-inner iterative method.

For both the outer and inner iterations, parameter-free preconditioned generalized conjugate gradient methods are advocated. For indefinite problems the method used offers an alternative to the well-known Uzawa algorithm.

Key words. generalized conjugate gradient method, variable-step preconditioning, two-level method, two-grid method, indefinite problems

AMS(MOS) subject classifications. 65F10, 65N20, 65N30

1. Introduction. We consider the solution of the system of linear equations,

$$Ax = \mathbf{b} \tag{1.1}$$

by a GCG (generalized conjugate gradient) method, in the form proposed by Axelsson [1] and further developed in [2]. In general, A may be a nonsymmetric and/or indefinite matrix. A may even be a rectangular matrix, if only its column rank is complete.

The GCG method from [1] consists of the following steps.

Given a set of search directions $\{\mathbf{d}^{(s)}\}_{s=0}^{k-1}$ orthogonal with respect to $(\cdot, \cdot)_1$, one computes a new approximation $\mathbf{x}^{(k)}$, such that the quadratic functional

$$f(\mathbf{x}) = \tfrac{1}{2}(\mathbf{r}, \mathbf{r})_0 \tag{1.2}$$

is minimized over the shifted space

$$\mathbf{x}^{(0)} + \text{span}\,\{\mathbf{d}^{(s)}\}_{s=0}^{k-1},$$

where $\mathbf{x}^0$ is an initial approximation, $(\cdot, \cdot)_0$ and $(\cdot, \cdot)_1$ are inner products, and $\mathbf{r} = A\mathbf{x} - \mathbf{b}$ is the residual. Since the column rank of A is complete, there exists a unique minimizer of (1.2) on any space of vectors of dimension m, the column rank of A.

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Determining in this way the next approximation

$$\mathbf{x}^{(k)}=\mathbf{x}^{(k-1)}+\sum_{s=1}^{k}\alpha_{k-s}^{(k-1)}\mathbf{d}^{(k-s)},$$

and the next residual

$$\mathbf{r}^{(k)}=\mathbf{r}^{(k-1)}+\sum_{s=1}^{k}\alpha_{k-s}^{(k-1)}A\mathbf{d}^{(k-s)},$$

in order to accelerate the convergence one uses a preconditioning step, i.e., one computes by some procedure corresponding to a matrix B, called preconditioner to A, the vector or pseudoresidual,

$$\tilde{\mathbf{r}}^{(k)}=B\mathbf{r}^{(k)}. \tag{1.3}$$

Then the next search vector is defined by

$$\mathbf{d}^{(k)}=-\tilde{\mathbf{r}}^{(k)}+\sum_{s=1}^{k}\beta_{k-s}^{(k-1)}\mathbf{d}^{(k-s)}.$$

The coefficients $\beta_{k-s}^{(k-1)}$ are determined from the orthogonality conditions

$$(\mathbf{d}^{(k)},\mathbf{d}^{(k-s)})_1=0, \qquad s=1,2,\cdots,k.$$

As is readily seen, this approach is quite general and can be used for an arbitrary mapping B,

$$\mathbf{r}\rightarrow B[\mathbf{r}]. \tag{1.4}$$

In practice B is chosen to approximate the inverse of A, if this exists, or at any rate such that BA is sufficiently close to the identity operator. In the general case when B is a nonlinear mapping, we shall assume a certain coercivity and boundedness condition that generalizes this matrix property.

In the literature, various iterative methods with inner-outer iterations have been considered, e.g., by Axelsson [3]; Golub and Overton [12]; Bank, Welfert, and Yserentant [10]; and Verfürth [15]. However, as an outer iterative method, they used a stationary iterative method, i.e., not a conjugate gradient method.

The algebraic multilevel method considered by Axelsson and Vassilevski [5], [6] can also be seen as an inner-outer iterative method. Here the inner iterations on a given discretization level correspond to the approximate solution of the coarse-grid problem in the two-level grid context of the method, by a Chebyshev iterative method and to two problems for the nodes not lying on the coarse grid, also solved by an iterative method. However, this method is parameter-dependent, i.e., certain parameters required in the iterative method must be estimated.

In this paper we analyse the GCG method in the general case of variable-step (i.e., generally a nonlinear mapping) preconditioner $B[\cdot]$ under the following assumptions:

(i) coercivity, i.e., there exists a positive constant δ_1, such that

$$(AB[\mathbf{v}],\mathbf{v})_0\geqq\delta_1(\mathbf{v},\mathbf{v})_0, \quad \text{all } \mathbf{v}, \tag{1.5}$$

(ii) continuity, i.e., there exists a positive constant δ_2, such that

$$\|AB[\mathbf{v}]\|_0\leqq\delta_2\|\mathbf{v}\|_0, \quad \text{all } \mathbf{v}.$$

Under these assumptions we prove in § 2 that the GCG method converges monotonically and at least with a rate given by the inequality,

$$\|\mathbf{r}^{(k)}\|_0 \leqq \sqrt{1-(\delta_1/\delta_2)^2}\,\|\mathbf{r}^{(k-1)}\|_0.$$

These results are based on already-proven similar results, say, in the linear case (i.e., a fixed matrix as a preconditioner) in Axelsson [1].

In § 3 we express these conditions as algebraic conditions and verify them in § 4 in the context of the two-level nonsymmetric preconditioning method, studied in the symmetric case in Axelsson and Gustafsson [4], with corresponding variable-step preconditioners. By the theory presented in §§ 2 and 3, we thereby give a mathematical justification of the numerical experiments presented in Axelsson and Gustafsson [4], when the preconditioned conjugate gradient (PCG) method is used as an inner iterative method, to solve the systems of equations corresponding to the nodes on the finer level, not lying on the coarse grid.

In § 4 we also demonstrate the algebraic conditions for indefinite matrices on a common block form. For indefinite problems, our method offers an alternative to the Uzawa algorithm, used, for instance, in Verfürth [15] and Langer and Queck [13].

2. The generalized conjugate gradient method with variable-step preconditioning. Following an earlier presentation of Axelsson [1], the GCG method with variable-step preconditioning is defined as follows.

Let $\mathbf{x}^{(0)}$ be an initial approximation, $\mathbf{r}^{(0)} = A\mathbf{x}^{(0)} - \mathbf{b}$ the initial residual, $\tilde{\mathbf{r}}^{(0)} = B[\mathbf{r}^{(0)}]$ a corresponding pseudoresidual, and $\mathbf{d}^{(0)} = -\tilde{\mathbf{r}}^{(0)}$ an initial search vector. For $k = 1, 2, \cdots$, let

$$\{\mathbf{d}^{(s)}\}_{s=0}^{k-1}$$

be $(\cdot,\cdot)_1$-orthogonal search vectors. Then the next approximation

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + \sum_{s=1}^{k} \alpha_{k-s}^{(k-1)} \mathbf{d}^{(k-s)}$$

is determined from

$$\frac{\partial}{\partial \alpha_s^{(k-1)}} \varphi = 0, \qquad s = 0, 1, \cdots, k-1, \tag{2.1}$$

where

$$\varphi = \varphi(\alpha_0^{(k-1)}, \cdots, \alpha_{k-1}^{(k-1)}) = \tfrac{1}{2}(\mathbf{r}^{(k)}, \mathbf{r}^{(k)})_0,$$

and

$$\begin{aligned} \mathbf{r}^{(k)} &= A\mathbf{x}^{(k)} - \mathbf{b} \\ &= \mathbf{r}^{(k-1)} + \sum_{s=1}^{k} \alpha_{k-s}^{(k-1)} A\mathbf{d}^{(k-s)}. \end{aligned} \tag{2.2}$$

We have the following lemma.

LEMMA 2.1. (a) $(\mathbf{r}^{(k)}, A\mathbf{d}^{(s)})_0 = 0$, $s = 0, 1, \cdots, k-1$;

(b) $\Lambda^{(k)}\alpha^{(k)} = \gamma^{(k)}$, *where* $\Lambda^{(k)}$ *is the matrix with entries*

$$\Lambda_{l,j}^{(k)} = (A\mathbf{d}^{(k-l)}, A\mathbf{d}^{(k-j)})_0, \qquad 1 \leqq l \leqq k, 1 \leqq j \leqq k,$$

$$(\alpha^{(k)})_j = \alpha_{k-j}^{(k-1)}, \qquad j = 1, \cdots, k,$$

and

$$(\gamma^{(k)})_1 = -(\mathbf{r}^{(k-1)}, A\mathbf{d}^{(k-1)})_0, \qquad (\gamma^{(k)})_j = 0, \qquad j = 2, 3, \cdots, k.$$

Proof. (See [1]. As it is short, we present it here also.) Equations (2.1) and (2.2) give

$$0 = \frac{\partial}{\partial \alpha_s^{(k-1)}} \varphi = (\mathbf{r}^{(k)}, A\mathbf{d}^{(s)})_0 = 0, \qquad s = 0, 1, \cdots, k-1,$$

or

$$\sum_{j=1}^{k} \alpha_{k-j}^{(k-1)} (A\mathbf{d}^{(k-j)}, A\mathbf{d}^{(k-l)})_0 = -(\mathbf{r}^{(k-1)}, A\mathbf{d}^{(k-l)})_0, \qquad l = 1, \cdots, k,$$

which proves part (a) and also part (b), using an induction hypothesis. □

The inner products $(\cdot, \cdot)_0$ and $(\cdot, \cdot)_1$ can be chosen independently of each other. For any pair of inner products, $\Lambda^{(k)}$ is nonsingular. However, for practical reasons we shall here consider two special cases.

Case 1. $(\mathbf{u}, \mathbf{v})_1 = (A\mathbf{u}, A\mathbf{v})_0$.

Case 2. $(\mathbf{u}, \mathbf{v})_1 = (\mathbf{u}, \mathbf{v})_0$.

LEMMA 2.2. (a) *In Case* 1, $(\mathbf{u}, \mathbf{v})_1 = (A\mathbf{u}, A\mathbf{v})_0$, *we have that* $\Lambda^{(k)}$ *is diagonal and*

$$\begin{aligned} &\alpha_s^{(k-1)} = 0, \qquad s = 0, 1, \cdots, k-2, \\ &\alpha_{k-1}^{(k-1)} = -(\mathbf{r}^{(k-1)}, A\mathbf{d}^{(k-1)})_0 / (\mathbf{d}^{(k-1)}, \mathbf{d}^{(k-1)})_1. \end{aligned} \tag{2.3}$$

(b) *In Case* 2, $(\mathbf{u}, \mathbf{v})_1 = (\mathbf{u}, \mathbf{v})_0$, *we have that* $\Lambda^{(k)}$ *equals* $\Lambda^{(k-1)}$ *augmented with a row and a column.*

(c) $\Lambda^{(k)}$ *is symmetric and positive definite.*

Proof (see [1]). (a) In Case 1, the $(\cdot, \cdot)_1$ orthogonality of $\mathbf{d}^{(s)}$, $s = 0, 1, \cdots, k-1$, shows that

$$(A\mathbf{d}^{(k-j)}, A\mathbf{d}^{(k-l)})_0 = (\mathbf{d}^{(k-j)}, \mathbf{d}^{(k-l)})_1 = 0, \qquad j \neq l, \qquad j, l = 1, 2, \cdots, k.$$

Part (a) then follows from part (b) of Lemma 2.1. For any pair of inner products the orthogonality of $\{\mathbf{d}^{(s)}\}$ implies in particular that this vector set is linearly independent. Since A has complete column rank, the set $\{A\mathbf{d}^{(s)}\}_{s=0}^{k-1}$ is also linearly independent, so $\Lambda^{(k)}$ is nonsingular. The last parts of the statement follow by construction of $\Lambda^{(k)}$ and by the linear independence of the set $\{\mathbf{d}^{(s)}\}$. □

At the kth step of the GCG method, the new search direction is defined by

$$\mathbf{d}^{(k)} = -\tilde{\mathbf{r}}^{(k)} + \sum_{s=1}^{k} \beta_{k-s}^{(k-1)} \mathbf{d}^{(k-s)}, \tag{2.4}$$

where

$$\tilde{\mathbf{r}}^{(k)} = B[\mathbf{r}^{(k)}] \tag{2.5}$$

and the parameters $\{\beta_{k-s}^{(k-1)}\}_{s=1}^{k}$ are determined from the orthogonality conditions

$$(\mathbf{d}^{(k)}, \mathbf{d}^{(j)})_1 = 0, \qquad j = 0, 1, \cdots, k-1,$$

i.e.,

$$\beta_j^{(k-1)} = \frac{(\tilde{\mathbf{r}}^{(k)}, \mathbf{d}^{(j)})_1}{(\mathbf{d}^{(j)}, \mathbf{d}^{(j)})_1}, \qquad j = 0, 1, \cdots, k-1. \tag{2.6}$$

LEMMA 2.3. (a) $(\mathbf{r}^{(k-1)}, A\mathbf{d}^{(k-1)})_0 = -(\mathbf{r}^{(k-1)}, AB[\mathbf{r}^{(k-1)}])_0$,

(b) $\alpha_{k-1}^{(k-1)} = (\mathbf{r}^{(k-1)}, AB[\mathbf{r}^{(k-1)}])_0 \det(\Lambda^{(k-1)})/\det(\Lambda^{(k)})$,

(c) $\alpha_{k-1}^{(k-1)} > 0$ *if and only if* $(\mathbf{r}^{(k-1)}, AB[\mathbf{r}^{(k-1)}])_0 > 0$.

(Note that in Case 1, the expression in (b) can be further simplified, as shown in Lemma 2.2(a).)

Proof (see [1]). Equation (2.4) shows that

$$(\mathbf{r}^{(k-1)}, A\mathbf{d}^{(k-1)})_0 = -(\mathbf{r}^{(k-1)}, A\tilde{\mathbf{r}}^{(k-1)})_0 + \sum_{s=2}^{k} \beta_{k-s}^{(k-2)}(\mathbf{r}^{(k-1)}, A\mathbf{d}^{(k-s)})_0.$$

Part (a) follows now from Lemma 2.1(a) and (2.5). Part (b) follows from Lemma 2.2 and Cramer's rule. Since $\Lambda^{(k-1)}$ is a Gramian-type matrix, its determinant is positive, so part (c) follows directly from part (b). □

Lemmata 2.1, 2.2, and 2.3 show now that one GCG step of the algorithms in Cases 1 and 2, respectively, takes the following forms (in practice, we frequently let $(\mathbf{u}, \mathbf{v})_0 = \mathbf{u}^t\mathbf{v}$).

ALGORITHM 1. Compute $A\mathbf{d}^{(k-1)}$
Compute $(A\mathbf{d}^{(k-1)}, A\mathbf{d}^{(k-1)})_0$
Compute $(\mathbf{r}^{(k-1)}, A\mathbf{d}^{(k-1)})_0$
$\alpha_{k-1}^{(k-1)} = -(\mathbf{r}^{(k-1)}, A\mathbf{d}^{(k-1)})_0/(A\mathbf{d}^{(k-1)}, A\mathbf{d}^{(k-1)})_0$
$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + \alpha_{k-1}^{(k-1)}\mathbf{d}^{(k-1)}$
$\mathbf{r}^{(k)} = \mathbf{r}^{(k-1)} + \alpha_{k-1}^{(k-1)}A\mathbf{d}^{(k-1)}$
Compute $\tilde{\mathbf{r}}^{(k)} = B[\mathbf{r}^{(k)}]$
Compute $A\tilde{\mathbf{r}}^{(k)}$
Compute $\beta_j^{(k-1)} = (A\tilde{\mathbf{r}}^{(k)}, A\mathbf{d}^{(j)})_0/(A\mathbf{d}^{(j)}, A\mathbf{d}^{(j)})_0, \quad j = 0, \cdots, k-1$
$\mathbf{d}^{(k)} = -\tilde{\mathbf{r}}^{(k)} + \sum_{s=1}^{k} \beta_{k-s}^{(k-1)}\mathbf{d}^{(k-s)}$

ALGORITHM 2. Compute $A\mathbf{d}^{(k-1)}$
Compute $(A\mathbf{d}^{(k-1)}, A\mathbf{d}^{(k-j)})_0, \quad j = 1, \cdots, k$
Compute $(\mathbf{r}^{(k-1)}, A\mathbf{d}^{(k-1)})_0$
Solve $\Lambda^{(k)}\alpha^{(k)} = \gamma^{(k)}$
$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + \sum_{s=1}^{k} \alpha_{k-s}^{(k-1)}\mathbf{d}^{(k-s)}$
$\mathbf{r}^{(k)} = \mathbf{r}^{(k-1)} + \sum_{s=1}^{k} \alpha_{k-s}^{(k-1)}A\mathbf{d}^{(k-s)}$ (or $\mathbf{r}^{(k)} = A\mathbf{x}^{(k)} - \mathbf{b}$)
Compute $\tilde{\mathbf{r}}^{(k)} = B[\mathbf{r}^{(k)}]$
Compute $\beta_j^{(k-1)} = (\tilde{\mathbf{r}}^{(k)}, \mathbf{d}^{(j)})_0/(\mathbf{d}^{(j)}, \mathbf{d}^{(j)})_0, \quad j = 0, \cdots, k-1$
$\mathbf{d}^{(k)} = -\tilde{\mathbf{r}}^{(k)} + \sum_{s=1}^{k} \beta_{k-s}^{(k-1)}\mathbf{d}^{(k-s)}$

Note that in Algorithm 1 we need two multiplications with the matrix A, while in Algorithm 2 only one such multiplication is required. On the other hand, Algorithm 2 requires $k-1$ more inner products and $2(k-1)$ more vector updates per iteration step. Hence, if the number of iterations (k) is sufficiently small or if the cost of a matrix multiplication with A is sufficiently big, Algorithm 2 can be more efficient than Algorithm 1.

We now estimate the rate of convergence of the algorithms.

THEOREM 2.1. *Let the preconditioner* $B[\cdot]$ *satisfy the assumptions* (i) *and* (ii), *i.e.*,

(i) $(\mathbf{v}, AB[\mathbf{v}])_0 \geqq \delta_1(\mathbf{v}, \mathbf{v})_0$, *all* $\mathbf{v}$;

(ii) $\|AB[\mathbf{v}]\|_0 \leqq \delta_2\|\mathbf{v}\|_0$, *all* $\mathbf{v}$, *for some positive constants* δ_1, δ_2. *Then the variable-step* GCG *method converges monotonically and the following convergence rate estimate is valid*:

$$\|\mathbf{r}^{(k)}\|_0 \leqq \sqrt{1-(\delta_1/\delta_2)^2}\,\|\mathbf{r}^{(k-1)}\|_0, \qquad k = 1, 2, \cdots.$$

Proof (see Theorem 2.2 in [1]). Lemma 2.1 shows that

$$(\mathbf{r}^{(k)},\mathbf{r}^{(k)})_0=(\mathbf{r}^{(k)},\mathbf{r}^{(k-1)}+\sum_{j=1}^{k}\alpha_{k-j}^{(k-1)}A\mathbf{d}^{(k-j)})_0=(\mathbf{r}^{(k)},\mathbf{r}^{(k-1)})_0$$

$$=(\mathbf{r}^{(k-1)}+\sum_{j=1}^{k}\alpha_{k-j}^{(k-1)}A\mathbf{d}^{(k-j)},\mathbf{r}^{(k-1)})_0$$

$$=(\mathbf{r}^{(k-1)},\mathbf{r}^{(k-1)})_0+\alpha_{k-1}^{(k-1)}(\mathbf{r}^{(k-1)},A\mathbf{d}^{(k-1)})_0.$$

Hence Lemma 2.3 shows that

$$(\mathbf{r}^{(k)},\mathbf{r}^{(k)})_0=(\mathbf{r}^{(k-1)},\mathbf{r}^{(k-1)})_0-(\mathbf{r}^{(k-1)},AB[\mathbf{r}^{(k-1)}])_0^2\det(\Lambda^{(k-1)})/\det(\Lambda^{(k)}).$$

It is shown in [1] that

$$\det(\Lambda^{(k)})/\det(\Lambda^{(k-1)})=\min_{\mathbf{g}\in W_{k-2}}\|AB[\mathbf{r}^{(k-1)}]-\mathbf{g}\|_0^2 \tag{2.7}$$

where W_{k-2} is the vectorspace spanned by $\{A\mathbf{d}^{(s)}\}_{s=0}^{k-2}$. Simply letting $\mathbf{g}=0$ in (2.7), we get the upperbound

$$(\mathbf{r}^{(k)},\mathbf{r}^{(k)})_0\leqq(\mathbf{r}^{(k-1)},\mathbf{r}^{(k-1)})_0-((\mathbf{r}^{(k-1)},AB[\mathbf{r}^{(k-1)}])_0^2/\|AB[\mathbf{r}^{(k-1)}]\|_0)^2.$$

Assumptions (i) and (ii) now show Theorem 2.1. □

Remark 2.1. Note that Theorem 2.1 also holds for a variable-step preconditioner, i.e., the preconditioner can change from one step to the next. In fact, it is readily seen that the rate of convergence estimate in Theorem 2.1 can be derived even for the steepest descent algorithm where $\alpha_s^{(k-1)}=0$, $s=0,1,\cdots,k-2$, and $\beta_s^{(k-1)}=0$, $s=0,1,\cdots,k-1$.

Remark 2.2. If $\mathbf{d}^{(k)}=0$, for some k, then it follows, by (2.4), that $\tilde{\mathbf{r}}^{(k)}$ is a linear combination of $\{\mathbf{d}^{(k-s)}\}_{s=1}^{k}$. Then

$$(\mathbf{r}^{(k)},A\tilde{\mathbf{r}}^{(k)})_0=\sum_{s=1}^{k}\beta_{k-s}^{(k-1)}(\mathbf{r}^{(k)},A\mathbf{d}^{(k-s)})_0=0,$$

by Lemma 2.1. By the coercivity assumption (i), we then have,

$$0=(\mathbf{r}^{(k)},A\tilde{\mathbf{r}}^{(k)})_0=(\mathbf{r}^{(k)},AB[\mathbf{r}^{(k)}])_0\geqq\delta_1\|\mathbf{r}^{(k)}\|_0^2.$$

Hence $\mathbf{r}^{(k)}=0$, i.e., the problem has been solved.
Thus we proved the following result.

THEOREM 2.2. *If the preconditioner* $B[\cdot]$ *satisfies the coercivity assumption* (i), *then the* (*variable-step*) *preconditioned* GCG *method with this preconditioner cannot fail.*

Note finally that even if A is indefinite, for instance, there can still exist a mapping $B[\cdot]$ for which the coercivity and boundedness assumptions hold.

Remark 2.3. When applying the GCG method there is a simple way to automatically determine if the preconditioner is sufficiently accurate. We simply check the sign of $(\mathbf{r}^{(k-1)},AB[\mathbf{r}^{(k-1)}])_0$, which by Lemma 2.3(a) equals $-(\mathbf{r}^{(k-1)},A\mathbf{d}^{(k-1)})_0$. Equivalently, we can check the sign of $\alpha_{k-1}^{(k-1)}$. If this is negative, we restart the algorithm without updating the approximation at the last step and compute in the following iterations a more accurate preconditioner B by making the inner iteration parameters ε_1, ε_2 (see § 3) smaller, or by simply performing more inner iterations. This corresponds to one form of a variable-step preconditioning and makes the algorithm a "black box" solver.

3. Verification of coercivity and continuity assumptions. We consider here two important types of problems where matrices on a two-by-two block form naturally arise. Hence, consider a matrix A, partitioned as

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{3.1}$$

A can be symmetric or nonsymmetric, but in the case when A is indefinite we assume here that A is symmetric.

We want to solve

$$A\mathbf{x}=\mathbf{b}$$

or

$$\begin{aligned} A_{11}\mathbf{x}_1+A_{12}\mathbf{x}_2&=\mathbf{b}_1 \\ A_{21}\mathbf{x}_1+A_{22}\mathbf{x}_2&=\mathbf{b}_2. \end{aligned} \tag{3.2}$$

We shall assume that A_{11} is invertible (in fact even with a positive-definite symmetric part, $\frac{1}{2}(A_{11} + A_{11}^T)$). However, in the case where $A_{22} = 0$ and A_{11} is singular or indefinite (a case occurring frequently in constrained optimization problems), we consider the equivalent system (that is, with the same solution),

$$\begin{aligned} \left(A_{11}+\frac{1}{\varepsilon}A_{12}A_{12}^T\right)\mathbf{x}_1+A_{12}\mathbf{x}_2&=\mathbf{b}_1+\frac{1}{\varepsilon}A_{12}\mathbf{b}_2, \\ A_{12}^T\mathbf{x}_1&=\mathbf{b}_2, \end{aligned}$$

and we assume then that, for some $\varepsilon > 0$, $A_{11} + (1/\varepsilon)A_{12}^TA_{12}$ has a positive-definite symmetric part. Hence, we must assume that A_{11} is positive definite on the nullspace of A_{12}^T. Therefore, we might as well assume that A_{11} is positive definite from the onset.

We assume also that

$$S=A_{22}-A_{21}A_{11}^{-1}A_{12}$$

is definite (positive or negative) and that A_{22} is positive definite if S is positive definite and A_{22} is negative semidefinite if S is negative definite.

This means that A_{22} is definite (positive or negative) on the nullspace of A_{12}.

Next we consider the following exact block-form of the inverse of A (see, for an earlier derivation, Banachiewicz [8]), which is readily derived by inverting the block matrix factorization of A,

$$A=\begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

and hence

$$\begin{aligned} A^{-1}&=\begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}\begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \\ &=\begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}\begin{bmatrix} A_{11}^{-1} & 0 \\ -S^{-1}A_{21}A_{11}^{-1} & S^{-1} \end{bmatrix}. \end{aligned}$$

Note that the application of this form to compute $A^{-1}\mathbf{v}$ for any block vector

$$\mathbf{v}=\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{bmatrix},$$

involves the following steps.

ALGORITHM 3.
(1) $\mathbf{w}_1 = A_{11}^{-1}\mathbf{v}_1$;
(2) $\mathbf{w}_2 = -A_{21}\mathbf{w}_1 + \mathbf{v}_2$;
(3) $\mathbf{x}_2 = S^{-1}\mathbf{w}_2$;
(4) $\mathbf{y}_1 = A_{12}\mathbf{x}_2$;
(5) $\mathbf{z}_1 = A_{11}^{-1}\mathbf{y}_1$;
(6) $\mathbf{x}_1 = \mathbf{w}_1 - \mathbf{z}_1$.

Then

$$A^{-1}\mathbf{v}=\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}.$$

The preconditioner, approximating A^{-1}, is now defined as follows. Every occurrence of the inverse of A_{11} (i.e., steps (1) and (5) above) is replaced by an (inner) iterative method to solve the corresponding systems with A_{11}, i.e., $A_{11}\mathbf{w}_1 = \mathbf{v}_1$ and $A_{11}\mathbf{z}_1 = \mathbf{y}_1$, above. Likewise, the occurrence of S^{-1} in (3) is replaced by an (inner) iterative method, i.e., to solve $S\mathbf{x}_2 = \mathbf{w}_2$ approximately. In all cases we iterate until the iteration error is sufficiently small.

In order to define preconditioner B on each (outer iteration) step we need (in general, nonlinear) mappings

$$\mathbf{v}_1 \to B_{11}[\mathbf{v}_1], \qquad \mathbf{v}_2 \to C[\mathbf{v}_2]$$

such that

$$\|A_{11}B_{11}[\mathbf{v}_1]-\mathbf{v}_1\|_0 \leqq \varepsilon_1\|\mathbf{v}_1\|_0, \quad \text{all } \mathbf{v}_1 \tag{3.3a}$$

$$\|SC[\mathbf{v}_2]-\mathbf{v}_2\|_0 \leqq \varepsilon_2\|\mathbf{v}_2\|_0, \quad \text{all } \mathbf{v}_2, \tag{3.3b}$$

and ε_1, ε_2 are sufficiently small positive numbers.

The application of the (variable-step) preconditioner $B = B[\cdot]$ involves, therefore, the following steps.

ALGORITHM 4.
(1) $\mathbf{w}_1 = B_{11}[\mathbf{v}_1]$;
(2) $\mathbf{w}_2 = -A_{21}\mathbf{w}_1 + \mathbf{v}_2$;
(3) $\mathbf{x}_2 = C[\mathbf{w}_2]$;
(4) $\mathbf{y}_1 = A_{12}\mathbf{x}_2$;
(5) $\mathbf{z}_1 = B_{11}[\mathbf{y}_1]$;
(6) $\mathbf{x}_1 = \mathbf{w}_1 - \mathbf{z}_1$.

Then

$$B[\mathbf{v}]=\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}.$$

We shall now estimate the deviation of $AB[\mathbf{v}]$ from $\mathbf{v}$. Note first, then, that Algorithm 4 shows that

$$\begin{aligned} AB[\mathbf{v}] &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11}(\mathbf{w}_1 - \mathbf{z}_1) + A_{12}C[\mathbf{w}_2] \\ A_{21}(\mathbf{w}_1 - \mathbf{z}_1) + A_{22}C[\mathbf{w}_2] \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} + \begin{bmatrix} A_{11}(\mathbf{w}_1 - B_{11}[\mathbf{y}_1]) - \mathbf{v}_1 + A_{12}C[\mathbf{w}_2] \\ A_{21}(\mathbf{w}_1 - B_{11}[\mathbf{y}_1]) + A_{22}C[\mathbf{w}_2] - \mathbf{v}_2 \end{bmatrix} \qquad (3.4) \\ &= \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} + \begin{bmatrix} (A_{11}\mathbf{w}_1 - \mathbf{v}_1) - (A_{11}\mathbf{z}_1 - \mathbf{y}_1) \\ A_{22}C[\mathbf{w}_2] - \mathbf{w}_2 - A_{21}B_{11}[\mathbf{y}_1] \end{bmatrix}. \end{aligned}$$

Therefore, since $C[\mathbf{w}_2] = \mathbf{x}_2$ and $B_{11}[\mathbf{y}_1] = \mathbf{z}_1$, the application of the preconditioner on each (outer iteration) step can be realized as Algorithm 4′.

ALGORITHM 4′. Given sufficiently small positive numbers ε_1, ε_2, we iterate in steps (1) and (5) with some method until the iterations $\mathbf{w}_1$, $\mathbf{z}_1$ satisfy

$$\|A_{11}\mathbf{w}_1 - \mathbf{v}_1\|_0 \leqq \varepsilon_1 \|\mathbf{v}_1\|_0, \qquad \|A_{11}\mathbf{z}_1 - \mathbf{y}_1\|_0 \leqq \varepsilon_1 \|\mathbf{y}_1\|_0, \tag{3.5a}$$

and in step (3) until the iteration $\mathbf{x}_2$ satisfies

$$\|A_{22}\mathbf{x}_2 - A_{21}\mathbf{z}_1 - \mathbf{w}_2\|_0 \leqq \varepsilon_2 \|\mathbf{w}_2\|_0 \tag{3.5b}$$

where

$$\mathbf{w}_2 = \mathbf{v}_2 - A_{21}\mathbf{w}_1.$$

Since (3.5b) involves the computation of $\mathbf{z}_1$ in step (5), the test (3.5b) is actually performed after step (5). This means that we may have to repeat steps (4) and (5) if (3.5b) fails to be satisfied. Hence, in practice, it can be advisable to choose a certain (fixed) number of iterations in step (3) and test on the sign of $\alpha_{k-1}^{(k-1)}$, instead, as was already mentioned in Remark 2.2. If the sign test is violated, we repeat Algorithm 4 with smaller values of ε_1, ε_2.

To continue the estimate of $AB[\mathbf{v}] - \mathbf{v}$ in (3.4), note first that by (3.5a),

$$\|A_{11}B_{11}[\mathbf{v}_1] - \mathbf{v}_1\|_0 = \|A_{11}\mathbf{w}_1 - \mathbf{v}_1\|_0 \leqq \varepsilon_1 \|\mathbf{v}_1\|_0, \tag{3.6}$$

and hence that

$$\begin{aligned} \|\mathbf{w}_2\|_0 &\leqq \|\mathbf{v}_2\|_0 + \|A_{21}A_{11}^{-1}A_{11}\mathbf{w}_0\|_0 \leqq \|\mathbf{v}_2\|_0 + \|A_{21}A_{11}^{-1}\|_0 \|A_{11}B_{11}[\mathbf{v}_1]\|_0 \\ &\leqq \|\mathbf{v}_2\|_0 + \|A_{21}A_{11}^{-1}\|_0 (1+\varepsilon_1) \|\mathbf{v}_1\|_0. \end{aligned} \tag{3.7}$$

Next note that

$$\begin{aligned} &\|SC[\mathbf{w}_2] - \mathbf{w}_2\|_0 = \|S\mathbf{x}_2 - \mathbf{w}_0\|_0 \\ &= \|A_{22}\mathbf{x}_2 - A_{21}A_{11}^{-1}A_{12}\mathbf{x}_2 - \mathbf{w}_2\|_0 \\ &= \|A_{22}\mathbf{x}_2 - A_{21}B_{11}[\mathbf{y}_1] - \mathbf{w}_2 + A_{21}(B_{11}[\mathbf{y}_1] - A_{11}^{-1}A_{12}\mathbf{x}_2)\|_0 \\ &\times \|A_{22}\mathbf{x}_2 - A_{21}\mathbf{z}_1 - \mathbf{w}_2 + A_{21}A_{11}^{-1}[A_{11}\mathbf{z}_1 - \mathbf{y}_1]\|_0 \\ &\leqq \varepsilon_2 \|\mathbf{w}_2\|_0 + \|A_{21}A_{11}^{-1}\|_0 \varepsilon_1 \|\mathbf{y}_1\|_0 \quad (\text{by}\,(3.5\text{b}), (3.5\text{a})). \end{aligned}$$

Further,

$$\|\mathbf{y}_1\|_0 = \|A_{12}\mathbf{x}_2\|_0 = \|A_{12}S^{-1}SC[\mathbf{w}_2]\| \leqq \|A_{12}S^{-1}\|_0 \|SC[\mathbf{w}_2]\|_0. \tag{3.8}$$

Hence

$$\|SC[\mathbf{w}_2]-\mathbf{w}_2\|_0 \leqq \varepsilon_2\|\mathbf{w}_2\|_0+\varepsilon_1\|A_{21}A_{11}^{-1}\|_0\|A_{12}S^{-1}\|_0[\|SC[\mathbf{w}_2]-\mathbf{w}_2\|_0+\|\mathbf{w}_2\|_0],$$

so, if ε_1 is sufficiently small,

$$\|SC[\mathbf{w}_2]-\mathbf{w}_2\|_0 \leqq (\varepsilon_2+\varepsilon_1\|A_{21}A_{11}^{-1}\|_0\|A_{12}S^{-1}\|_0)\|\mathbf{w}_2\|/(1-\varepsilon_1\|A_{21}A_{11}^{-1}\|_0\|A_{12}S^{-1}\|_0)$$

and

(3.9)
$$\|SC[\mathbf{w}_2]\|_0 \leqq \|SC[\mathbf{w}_2]-\mathbf{w}_2\|_0+\|\mathbf{w}_2\|_0 \leqq (1+\varepsilon_2)\|\mathbf{w}_2\|_0/(1-\varepsilon_1\|A_{21}A_{11}^{-1}\|_0\|A_{12}S^{-1}\|_0).$$

Therefore, by (3.8) and (3.9)

(3.10)
$$\|A_{11}\mathbf{z}_1-\mathbf{y}_1\|_0 \leqq \varepsilon_1\|\mathbf{y}_1\|_0 \leqq \varepsilon_1\|A_{12}S^{-1}\|_0(1+\varepsilon_2)\|\mathbf{w}_2\|_0/(1-\varepsilon_1\|A_{21}A_{11}^{-1}\|_0\|A_{12}S^{-1}\|_0).$$

Finally (3.4), (3.5a), and (3.10) show that

$$\begin{aligned}\|AB[\mathbf{v}]-\mathbf{v}\|_0^2 &\leqq (\|A_{11}\mathbf{w}_1-\mathbf{v}_1\|_0+\|A_{11}\mathbf{z}_1-\mathbf{y}_1\|_0)^2+\|A_{22}\mathbf{x}_2-A_{21}\mathbf{z}_1-\mathbf{w}_2\|_0^2\\ &\leqq [\varepsilon_1\|\mathbf{v}_1\|_0+\varepsilon_1\|A_{12}S^{-1}\|_0(1+\varepsilon_2)\|\mathbf{w}_2\|_0/(1-\varepsilon_1\|A_{21}A_{11}^{-1}\|_0\|A_{12}S^{-1}\|_0)]^2\\ &\quad+\varepsilon_2^2\|\mathbf{w}_2\|_0^2,\end{aligned}$$

so by (3.7)

$$\|AB[\mathbf{v}]-\mathbf{v}\|_0^2 \leqq (\varepsilon_1+\varepsilon_2)^2 C_1(\sigma_1,\sigma_2)\|\mathbf{v}\|_0^2, \tag{3.11}$$

where $\|\mathbf{v}\|_0^2=\|\mathbf{v}_1\|_0^2+\|\mathbf{v}_2\|_0^2$, $C_1=C_1(\sigma_1,\sigma_2)$,

$$\begin{aligned}C_1(\sigma_1,\sigma_2)&=[1+\sigma_1\sigma_2(1+\varepsilon_2)(1+\varepsilon_1)/(1-\varepsilon_1\sigma_1\sigma_2)]^2+2\sigma_2(1+\varepsilon_2)^2\\ &\quad+[1+\sigma_1(1+\varepsilon_2)/(1-\varepsilon_1\sigma_1\sigma_2)]^2\end{aligned}$$

and

$$\sigma_1=\|A_{12}S^{-1}\|_0, \qquad \sigma_2=\|A_{21}A_{11}^{-1}\|_0.$$

We summarize the result in the following theorem.

THEOREM 3.1. *Let the norms* $\|\cdot\|_0$ *in the vectorspaces for* $\mathbf{v}_1$, $\mathbf{v}_2$, *respectively, be such that*

$$\sigma_1=\|A_{12}S^{-1}\|_0, \qquad \sigma_2=\|A_{21}A_{11}^{-1}\|_0$$

are bounded uniformly with respect to the problem parameter. Then for ε_1, ε_2 *sufficiently small, the mapping* $B[\cdot]$ *defined by Algorithm* 4, *with* $B_{11}[\cdot]$ *and* $C[\cdot]$ *satisfying* (3.3a,b), *is coercive and bounded; that is,*

$$(\mathbf{v},AB[\mathbf{v}])_0 \geqq [1-C_1^{1/2}(\varepsilon_1+\varepsilon_2)]\|\mathbf{v}\|_0^2, \quad \textit{all } \mathbf{v},$$

where $C_1=C_1(\sigma_1,\sigma_2)$ *is a function of* σ_1, σ_2, *bounded for all bounded values of* σ_1, σ_2, *and*

$$\|AB[\mathbf{v}]\|_0 \leqq [1+C_1^{1/2}(\varepsilon_1+\varepsilon_2)]\|\mathbf{v}\|_0, \quad \textit{all } \mathbf{v},$$

respectively.

Proof. Equation (3.10) shows that

$$|\|AB[\mathbf{v}]\|_0-\|\mathbf{v}\|_0| \leqq \|AB[\mathbf{v}]-\mathbf{v}\|_0 \leqq C_1^{1/2}(\varepsilon_1+\varepsilon_2)\|\mathbf{v}\|_0.$$

Hence

$$[1-C_1^{1/2}(\varepsilon_1+\varepsilon_2)]\,\|\mathbf{v}\|_0 \leqq \|AB[\mathbf{v}]\|_0 \leqq [1+C_1^{1/2}(\varepsilon_1+\varepsilon_2)]\,\|\mathbf{v}\|_0. \tag{3.12}$$

Further, (3.11) shows that

$$\|\mathbf{v}\|_0^2 - 2(\mathbf{v},AB[\mathbf{v}])_0 + \|AB[\mathbf{v}]\|_0^2 \leqq C_1(\varepsilon_1+\varepsilon_2)^2\|\mathbf{v}\|_0^2.$$

Hence

$$2(\mathbf{v},AB[\mathbf{v}])_0 \geqq [1-C_1(\varepsilon_1+\varepsilon_2)^2]\,\|\mathbf{v}\|_0^2 + \|AB[\mathbf{v}]\|_0^2,$$

and this, together with the left-hand side part of (3.12), show that

$$2(\mathbf{v},AB[\mathbf{v}])_0 \geqq [2-2C_1^{1/2}(\varepsilon_1+\varepsilon_2)]\,\|\mathbf{v}\|_0^2. \qquad \square$$

In the next section we make the corresponding choice of norms $\|\mathbf{v}_1\|_0$, $\|\mathbf{v}_2\|_0$ in two particular and important applications: the two-level multilevel method for nonselfadjoint elliptic problems and a mixed finite element discretization of the Stokes problem or of the second-order elliptic equation.

4. Applications for the two-level multigrid method for nonselfadjoint elliptic problems and for problems arising in mixed finite element solution of elliptic equations.

PROBLEM 4.1. *Consider the following boundary value problem,*

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \mathbf{v}\cdot\nabla u + bu = f(x), \qquad x\in\Omega\subset\mathbb{R}^2,$$

$u = 0$ *on* $\Gamma = \partial\Omega$.

Here Ω is a polygonal domain and the matrix $[a_{ij}(x)]_{i,j=1}^2$ is assumed to be symmetric and uniformly positive definite on $x \in \bar{\Omega}$.

The form

$$a(u,w) = \int_\Omega \sum a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial w}{\partial x_j}\,dx + \int_\Omega [(\mathbf{v}\cdot\nabla u)w + buw]\,dx$$

is assumed to be H^1-coercive, that is,

$$a(u,u) = \int_\Omega \sum a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx + \int_\Omega [b-\tfrac{1}{2}\operatorname{div}\mathbf{v}]u^2\,dx$$

$$\geqq c_0|u|_{1,\Omega}^2 + b_0|u|_{0,\Omega}^2.$$

for some $c_0 > 0$, $b_0 \geqq 0$.

To satisfy this, it suffices to have

$$b(x) - \tfrac{1}{2}\operatorname{div}\mathbf{v}(x) \geqq b_0 \geqq 0.$$

The standard variational formulation of this problem is:

Find $u \in H_0^1(\Omega)$ such that

$$a(u,\phi) = (f,\phi), \qquad \text{all } \phi\in H_0^1(\Omega).$$

Since the form $a(\cdot,\cdot)$ is H^1-coercive, this problem has a unique solution $u \in H_0^1(\Omega)$ for any right-hand side function $f \in L_2(\Omega)$.

Also, we shall need the bilinear form $\mathring{a}(\cdot,\cdot)$, the symmetric part of $a(\cdot,\cdot)$, defined by

$$\mathring{a}(u,\phi)=\int_\Omega \sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_j}\,dx+\int_\Omega [b-\tfrac{1}{2}\operatorname{div}\mathbf{v}]u\phi\,dx. \tag{4.1}$$

Consider now a finite element space V split up into two spaces V_1, V_2 such that

$$V=V_1+V_2,\qquad V_1\cap V_2=\{0\}.$$

The following are examples of such partitionings.

Example 4.1. Let τ_2 be a triangulation of Ω, consisting of a set of nonoverlapping triangles. Let

$$V_2=\operatorname{span}\{\phi_i^{(2)}\}_{i=1}^{n_2},$$

where n_2 is the number of vertices in τ_2 not lying on Γ_D and where $\phi_i^{(2)}$ is piecewise linear on the triangles in τ_2,

$$\phi_i^{(2)}(x_j^{(2)})=\delta_{i,j},$$

and $x_j^{(2)}$ runs over all vertices of the triangles in τ_2. By a refining procedure, e.g., by bisection or by pairwise connecting the center points of the edges of the triangles (see Fig. 4.1), we get a finer triangulation τ_1. Then V_1 is defined by

$$V_1=\operatorname{span}\{\phi_i^{(1)}\}_{i=1}^{n_1},$$

where $\phi_i^{(1)}$ forms a nodal basis in V_1 and are piecewise linear on the triangles in τ_2 and vanish on the vertices of the triangles in τ_1 (except on the ith).

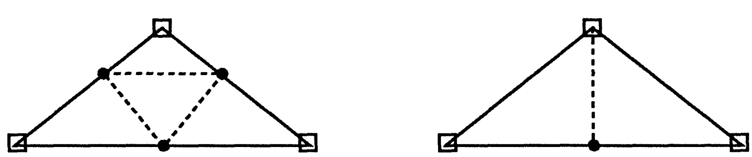

FIG. 4.1

Example 4.2. Let τ_2 be a triangulation of Ω, as in Example 4.1; let V_2 be defined in the same way; and let V_1 be the set of continuous functions, which are piecewise polynomials of degree p in each triangle, vanishing on the vertices of the triangles in τ_2 and spanning the complete monomials up to degree p, except 1, x_1, and x_2. Here p is a fixed integer greater than 1.

Example 4.3. Assume that Ω can be divided into a set of rectangular elements. To the vertices of the elements we associate piecewise bilinear functions, which span the space V_2. V_1 is spanned by the corresponding serendipity piecewise polynomials of degree less than or equal to p, which vanish on the vertices of the elements (see Fig. 4.2).

In all these examples we have two disjoint sets of nodes N_2, N_1, such that

$$V_1=\{\phi\in V \text{ and } \phi(x_j^{(2)})=0, x_j^{(2)}\in N_2\}.$$

Using this block ordering of the nodes, namely first ordering nodes in N_1 and then in N_2, we get the following two-by-two block form of the stiffness matrix A:

$$A=\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}, \tag{4.2}$$

where

$$A_{11}=\{a(\phi_j^{(1)},\phi_i^{(1)})\}_{x_i,x_j\in N_1},$$

$$A_{21}=\{a(\phi_j^{(1)},\phi_i^{(2)})\}_{x_j\in N_1,x_i\in N_2},\qquad A_{12}=\{a(\phi_j^{(2)},\phi_i^{(1)})\}_{x_i\in N_1,x_j\in N_2},$$

and

$$A_{22}=\{a(\phi_j^{(2)},\phi_i^{(2)})\}_{x_i,x_j\in N_2}.$$

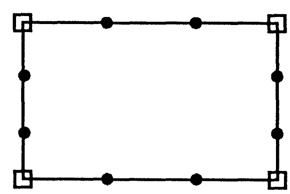

FIG. 4.2

The symmetric part of A, $\mathring{A}=\frac{1}{2}(A+A^T)$, is obtained from the bilinear form $\mathring{a}(\cdot,\cdot)$ defined by (4.1); that is,

$$\mathring{A}=\begin{bmatrix}\mathring{A}_{11} & \mathring{A}_{12}\\ \mathring{A}_{21} & \mathring{A}_{22}\end{bmatrix} \tag{4.3}$$

with

$$\mathring{A}_{11}=\{\mathring{a}(\phi_j^{(1)},\phi_i^{(1)})\}_{x_i,x_j\in N_1},$$

$$\mathring{A}_{12}^T=\mathring{A}_{21}=\{\mathring{a}(\phi_j^{(1)},\phi_i^{(2)})\}_{x_j\in N_1,x_i\in N_2},$$

and

$$\mathring{A}_{22}=\{\mathring{a}(\phi_j^{(2)},\phi_i^{(2)})\}_{x_i,x_j\in N_2}.$$

The following strengthened Cauchy inequality, proved in Bank and Dupont [9] and Axelsson and Gustafsson [4], will be used later:

There exists a constant $\gamma\in(0,1)$, independent of the mesh parameter (but dependent on p), such that

$$\mathbf{v}_1^t\mathring{A}_{12}\mathbf{v}_2\leqq\gamma\{\mathbf{v}^t\mathring{A}_{11}\mathbf{v}\}^{1/2}\{\mathbf{v}_2^t\mathring{A}_{22}\mathbf{v}_2\}^{1/2}\quad\text{for all } \mathbf{v}_1,\mathbf{v}_2. \tag{4.4}$$

We shall also use the following relations, valid for any s.p.d. (symmetric, positive-definite) stiffness matrix partitioned into the block form (4.3).

LEMMA 4.1. *Let* $\mathring{S}=\mathring{A}_{22}-\mathring{A}_{21}\mathring{A}_{11}^{-1}\mathring{A}_{12}$. *Then*

(a) *The condition number of* $\mathring{A}_{11}$ *is bounded above by a number independent of the mesh parameter*;

(b) $1-\gamma^2\leqq\mathbf{v}_2^t\mathring{S}\mathbf{v}_2/\mathbf{v}_2^t\mathring{A}_{22}\mathbf{v}_2\leqq 1$, *for all* $\mathbf{v}_2$, *where* γ *is the constant in* (4.4).

These results have been proved in Axelsson and Gustafsson [4].

Since the bilinear form $a(\cdot,\cdot)$ is bounded on $H_0^1\times H_0^1$, one can easily verify the following estimate.

LEMMA 4.2. *There exists a constant* $\gamma_2\geqq 1$, *such that*

$$\mathbf{v}^tA\mathbf{w}\leqq\gamma_2(\mathbf{v}^t\mathring{A}\mathbf{v})^{1/2}(\mathbf{w}^t\mathring{A}\mathbf{w})^{1/2}\quad\textit{for all } \mathbf{v},\mathbf{w}.$$

COROLLARY 4.1. *Consider the Schur complements*

$$S = A_{22} - A_{21} A_{11}^{-1} A_{12}, \qquad \mathring{S} = \mathring{A}_{22} - \mathring{A}_{21} \mathring{A}_{11}^{-1} \mathring{A}_{12},$$

of A *and* $\mathring{A}$, *partitioned into block forms* (4.2) *and* (4.3), *respectively. Then the following inequality is valid*:

$$\mathbf{v}_2^t S \mathbf{w}_2 \leqq \gamma_2^2 \{\mathbf{v}_2^t \mathring{S} \mathbf{v}_2\}^{1/2} \{\mathbf{w}_2^t \mathring{S} \mathbf{w}_2\}^{1/2} \quad \textit{for all } \mathbf{v}, \mathbf{w},$$

and

$$\mathbf{v}_2^t S \mathbf{v}_2 \geqq \mathbf{v}_2^t \mathring{S} \mathbf{v}_2 \quad \textit{for all } \mathbf{v}_2.$$

Proof. (See also Ewing, Lazarov, Pasciak, and Vassilevski [11] and Axelsson and Vassilevski [7]. Since the proof is short, we present it here for completeness.)

Given $\mathbf{v}_2$, $\mathbf{w}_2$, choose $\mathbf{v}_1$ arbitrary and $\mathbf{w}_1$, so that

$$A\mathbf{w} = \begin{bmatrix} 0 \\ S\mathbf{w}_2 \end{bmatrix},$$

that is,

$$A_{11}\mathbf{w}_1 + A_{12}\mathbf{w}_2 = 0.$$

Then, by Lemma 4.2, we have

$$\begin{aligned} \mathbf{v}_2^t S \mathbf{w}_2 = \mathbf{v}^t A \mathbf{w} &\leqq \gamma_2 (\mathbf{v}^t \mathring{A} \mathbf{v})^{1/2} (\mathbf{w}^t \mathring{A} \mathbf{w})^{1/2} \\ &= \gamma_2 (\mathbf{v}^t \mathring{A} \mathbf{v})^{1/2} (\mathbf{w}_2^t S \mathbf{w}_2)^{1/2}. \end{aligned} \tag{4.5}$$

If we choose $\mathbf{v}_2 = \mathbf{w}_2$ above, then

$$\mathbf{w}_2^t S \mathbf{w}_2 \leqq \gamma_2^2 \mathbf{v}^t \mathring{A} \mathbf{v}$$

and hence

$$\{\mathbf{w}_2^t S \mathbf{w}_2\}^{1/2} \leqq \gamma_2 \left\{ \inf_{\mathbf{v}_1} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{w}_2 \end{bmatrix}^t \mathring{A} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{w}_2 \end{bmatrix} \right\}^{1/2} = \gamma_2 (\mathbf{w}_2^t \mathring{S} \mathbf{w}_2)^{1/2}.$$

Inserting the last inequality into (4.5), we get

$$\begin{aligned} \mathbf{v}_2^t S \mathbf{w}_2 &\leqq \gamma_2^2 \left\{ \inf_{\mathbf{v}_1} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}^t \mathring{A} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \right\}^{1/2} \{\mathbf{w}_2^t \mathring{S} \mathbf{w}_2\}^{1/2} \\ &= \gamma_2^2 \{\mathbf{v}_2^t \mathring{S} \mathbf{v}_2\}^{1/2} \{\mathbf{w}_2^t \mathring{S} \mathbf{w}_2\}^{1/2}. \end{aligned}$$

The last inequality follows from

$$\mathbf{v}^t A \mathbf{v} = \mathbf{v}^t \mathring{A} \mathbf{v} \geqq \mathbf{v}_2^t \mathring{S} \mathbf{v}_2 \quad \text{for all } \mathbf{v}_1,$$

and hence for $\mathbf{v}_1$, such that $A_{11}\mathbf{v}_1 + A_{12}\mathbf{v}_2 = 0$, that is,

$$A\mathbf{v} = \begin{bmatrix} 0 \\ S\mathbf{v}_2 \end{bmatrix},$$

we get

$$\mathbf{v}^t A \mathbf{v} = \mathbf{v}_2^t S \mathbf{v}_2 \geqq \mathbf{v}_2^t \mathring{S} \mathbf{v}_2. \qquad \square$$

COROLLARY 4.2. *Assume that the eigenvalues of* $\mathring{A}_{11}$ *are contained in the interval* $[\alpha_1, \alpha_2]$, *which is defined in Lemma* 4.1, *and hence where* $\alpha_1 > 0$. *Then the eigenvalues* *of A_{11} are contained in the fixed segment of a disc in the right-half complex plane*

$$\{z \in C; \mathrm{Re}\, z \geqq \alpha_1, |z| \leqq \gamma_2\alpha_2\}.$$

Proof. By Lemma 4.2, we have

$$\begin{aligned}\mathbf{v}_1^t A_{11}\mathbf{w}_1 &\leqq \gamma_2\{\mathbf{v}_1^t \mathring{A}_{11}\mathbf{v}_1\}^{1/2}\{\mathbf{w}_1^t \mathring{A}_{11}\mathbf{w}_1\}^{1/2}\\ &\leqq \gamma_2\alpha_2\|\mathbf{v}_1\|\,\|\mathbf{w}_1\|.\end{aligned}$$

Hence the eigenvalues of A_{11} satisfy

$$|\lambda(A_{11})| \leqq \gamma_2\alpha_2$$

and

$$\mathrm{Re}\,\lambda(A_{11}) \geqq \lambda_{\min}\left[\tfrac{1}{2}(A_{11}+A_{11}^T)\right] \geqq \alpha_1. \qquad \square$$

Corollary 4.2 shows that A_{11} is well conditioned and hence, we can solve the system with A_{11} (occurring in Algorithm 3) using inner iterations with a generalized conjugate gradient method in a number of iterations independent of the mesh parameter and to any desired relative accuracy (see Axelsson [1], for instance). In practice, one will use a preconditioned form of the generalized conjugate gradient method, with a preconditioner, such as the diagonal part, or an incomplete factorization of A_{11}. Further, Lemma 4.1(b) and Corollary 4.1 show that we can solve the system with S occurring in Algorithm 3 with $\mathring{A}_{22}$ (a coarse-grid symmetric and positive-definite stiffness matrix) as a spectrally equivalent preconditioner in a preconditioned generalized conjugate gradient method. Since $\mathring{A}_{22}$ corresponds to a stiffness matrix on the coarse mesh (τ_2) we can expect to be able to solve systems with $\mathring{A}_{22}$ with much less cost than A. In the symmetric case, this was discussed in Axelsson and Gustafsson [4]. Alternatively, we can solve $\mathring{A}_{22}$ using a recursive factorization with two-by-two block matrix splittings. This has been analysed in Axelsson and Vassilevski [5], [6] and shall now be discussed further in the present context.

In order to apply the theory in § 3, we need to define the norms $\|\mathbf{v}_1\|_0$ and $\|\mathbf{v}_2\|_0$ and estimate the corresponding numbers

$$\sigma_1 = \|A_{12}S_2^{-1}\|_0, \qquad \sigma_2 = \|A_{21}A_{11}^{-1}\|_0.$$

We choose here

$$\|\mathbf{v}_1\|_0 = \{\mathbf{v}_1^t \mathring{A}_{11}^{-1}\mathbf{v}_1\}^{1/2}, \qquad \|\mathbf{v}_2\|_0 = \{\mathbf{v}_2^t \mathring{S}^{-1}\mathbf{v}_2\}^{1/2}. \tag{4.6}$$

For practical purposes, however, one must choose

$$\|\mathbf{v}_1\|_0 = \{\mathbf{v}_1^t \mathring{B}_{11}\mathbf{v}_1\}^{1/2}, \qquad \|\mathbf{v}_2\|_0 = \{\mathbf{v}_2^t \mathring{A}_{22}^{-1}\mathbf{v}_2\}^{1/2}, \tag{4.6$'$}$$

with $\mathring{B}_{11}$ s.p.d. and spectrally equivalent to $\mathring{A}_{11}^{-1}$ (such as $\mathring{B}_{11}^{-1} = \mathrm{diag}\,(\mathring{A}_{11})$), which give uniformly equivalent norms to the previous ones. To simplify the presentation, we consider here only the choice (4.6).

By the definition of σ_1, we have

$$\begin{aligned}\sigma_1^2 &= \sup_{\mathbf{v}_2}\left(\frac{\|A_{12}S^{-1}\mathbf{v}_2\|_0}{\|\mathbf{v}_2\|_0}\right)^2\\ &= \sup_{\mathbf{v}_2}\frac{(A_{12}S^{-1}\mathbf{v}_2)^T\mathring{A}_{11}^{-1}A_{12}S^{-1}\mathbf{v}_2}{\mathbf{v}_2^t\mathring{S}^{-1}\mathbf{v}_2}\\ &= \sup_{\mathbf{v}_2}\frac{\mathbf{v}_2^t S^{-T}A_{12}^T\mathring{A}_{11}^{-1}A_{12}S^{-1}\mathbf{v}_2}{\mathbf{v}_2^t\mathring{S}^{-1}\mathbf{v}_2}\end{aligned}$$

$$= \sup_{\mathbf{v}_2} \frac{\mathbf{v}_2^t[\mathring{S}^{1/2}S^{-T}\mathring{S}^{1/2}][\mathring{S}^{-1/2}A_{12}^T\mathring{A}_{11}^{-1/2}]R_1R_2\mathbf{v}_2}{\mathbf{v}_2^t\mathbf{v}_2}$$

$$= \sup_{\mathbf{v}_2} \frac{\mathbf{v}_2^t(R_1R_2)^TR_1R_2\mathbf{v}_2}{\mathbf{v}_2^t\mathbf{v}_2} \leqq \|R_1\|^2\|R_2\|^2,$$

where $R_1 = \mathring{A}_{11}^{-1/2}A_{12}\mathring{S}^{-1/2}$, $R_2 = \mathring{S}^{1/2}S^{-1}\mathring{S}^{1/2}$. Hence

$$\sigma_1 = \|R_1\|\,\|R_2\|.$$

Note that

$$\|R_1\| = \sup_{\mathbf{v}_1,\mathbf{v}_2} \frac{|\mathbf{v}_1^tA_{12}\mathbf{v}_2|}{\{\mathbf{v}_1^t\mathring{A}_{11}\mathbf{v}_1\}^{1/2}\{\mathbf{v}_2^t\mathring{S}\mathbf{v}_2\}^{1/2}},$$

so by Lemma 4.1,

$$\|R_1\| \leqq \gamma_2\{\mathbf{v}_2^t\mathring{A}_{22}\mathbf{v}_2/\mathbf{v}_2^t\mathring{S}\mathbf{v}_2\}^{1/2} \leqq \gamma_2/(1-\gamma^2)^{1/2}.$$

Further, Corollary 4.1 shows that

$$\|R_2\| \leqq 1. \tag{4.7}$$

Hence

$$\sigma_1 \leqq \gamma_2/(1-\gamma^2)^{1/2}.$$

Also,

$$\sigma_2^2 = \|A_{21}A_{11}^{-1}\|_0^2$$

$$= \sup_{\mathbf{v}_1}\left(\frac{\|A_{21}A_{11}^{-1}\mathbf{v}_1\|_0}{\|\mathbf{v}_1\|_0}\right)^2$$

$$= \sup_{\mathbf{v}_1} \frac{(A_{21}A_{11}^{-1}\mathbf{v}_1)^T\mathring{S}^{-1}A_{21}A_{11}^{-1}\mathbf{v}_1}{\mathbf{v}_1^t\mathring{A}_{11}^{-1}\mathbf{v}_1}$$

$$= \sup_{\mathbf{v}_1} \frac{\mathbf{v}_1^tG_2^TG_2\mathbf{v}_1}{\mathbf{v}_1^t\mathbf{v}_1} = \|G_2\|^2$$

where

$$G_2 = \mathring{S}^{-1/2}A_{21}A_{11}^{-1}\mathring{A}_{11}^{1/2}.$$

Hence

$$\sigma_2 = \sup_{\mathbf{v}_2,\mathbf{w}_1} \frac{\mathbf{v}_2^t\mathring{S}^{-1/2}A_{21}A_{11}^{-1}\mathring{A}_{11}^{1/2}\mathbf{w}_1}{\|\mathbf{v}_2\|\,\|\mathbf{w}_1\|} \leqq \gamma_2 \sup_{\mathbf{v}_2,\mathbf{w}_1} \frac{\{\mathbf{v}_2^t\mathring{S}^{-1/2}\mathring{A}_{22}\mathring{S}^{-1/2}\mathbf{v}_2\}^{1/2}\{\mathbf{w}_1^tG_1^TG_1\mathbf{w}_1\}^{1/2}}{\|\mathbf{v}_2\|\,\|\mathbf{w}_1\|},$$

where $G_1 = \mathring{A}_{11}^{1/2}A_{11}^{-1}\mathring{A}_{11}^{1/2}$, so by Lemma 4.2,

$$\sigma_2 \leqq \gamma_2(1-\gamma^2)^{-1/2}\|G_1\|,$$

and by the construction of $\mathring{A}_{11}$, $\|G_1\| \leqq 1$, so

$$\sigma_2 \leqq \gamma_2/(1-\gamma^2)^{1/2}. \tag{4.8}$$

We summarize the result in the following theorem.

THEOREM 4.1. *Let the norms* $\|\mathbf{v}_1\|_0$, $\|\mathbf{v}_2\|_0$ *be defined by* (4.6) (*or by* (4.6′)). *Then for* ε_1, ε_2 *sufficiently small, the mapping* $B[\cdot]$ *defined by Algorithm* 4, *with* $B_{11}[\cdot]$ *and* $C[\cdot]$ *satisfying* (3.3), *is an optimal order variable-step preconditioning.* □

PROBLEM 4.2. *Consider the following saddle point problem*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

with A_{11} symmetric, positive definite, $A_{21} = A_{12}^T$ and A_{22} negative semidefinite. In certain applications, such as mixed finite element discretizations of second-order elliptic problems, A_{22} is, in fact, zero.

In this case the Schur complement, $-S$,

$$-S = A_{22} - A_{21} A_{11}^{-1} A_{12}$$

is negative definite.

In order to apply the two-by-two block variable step (with inner iterations) preconditioner B, we need to specify $B_{11}[\mathbf{w}_1]$, a mapping which approximates the solution of

$$A_{11}\mathbf{v}_1 = \mathbf{w}_1,$$

and a mapping $C[\mathbf{w}_2]$, which approximates the solution of

$$S\mathbf{v}_2 = \mathbf{w}_2.$$

Note that, hence, $-C[\cdot]$ will approximate $-S^{-1}$.

If A is derived by a finite element approximation of the Stokes problem, then $B_{11}[\mathbf{w}_1]$ can, for example, be the approximation of $A_{11}^{-1}\mathbf{w}_1$, which one gets when applying ν steps of an optimal order preconditioned conjugate gradient method, for instance, for the solution of the Poisson equation based on a multigrid or on the algebraic multilevel preconditioner in Axelsson and Vassilevski [5], [6]. Let us denote the corresponding optimal (symmetric and positive-definite) preconditioning matrix in this inner iterative method by $\mathring{A}_{11}$.

In the case when A is derived by a mixed finite element discretization of second-order elliptic equations, $\mathring{A}_{11}$ will be, say, a lumped mass matrix, or more generally a (modified) incomplete factorization of A_{11}, since cond $(A_{11}) = O(1)$.

Since the actions of the Schur complement S are not generally available, first use an approximation of S by

$$\hat{S} = A_{22} - A_{21}\hat{A}_{11}^{-1}A_{12} \tag{4.9}$$

where $\hat{A}_{11}^{-1}$ is generally a more accurate approximation to A_{11}^{-1} than is $\mathring{A}_{11}^{-1}$. $\hat{A}_{11}^{-1}$ is obtained by a fixed number of steps of an optimal stationary (inner) iterative method.

Finally, let D be an optimal order preconditioner to S. For the Stokes problem, D can, for example, be the unity matrix (see Langer and Queck [13], Verfürth [15]). For the mixed finite element approximation of second-order elliptic problems, the (best) choice of D is not clear, as it can depend on the discretization used. However, for the lowest order Raviart–Thomas finite element spaces discretization (see, for example, [14]), D can be chosen as a multigrid step applied to the corresponding equation obtained after elimination of the velocity.

Then $C[\mathbf{w}_2]$ corresponds to the approximations obtained by a fixed step preconditioned conjugate gradient method with D as a preconditioner applied to solve the system $\hat{S}\mathbf{v}_2 = \mathbf{w}_2$ (see (4.9)).

The corresponding norms are chosen as

$$\|\mathbf{v}_1\|_0 = \{\mathbf{v}_1^T \mathring{A}_{11}^{-1} \mathbf{v}_1\}^{1/2}, \qquad \|\mathbf{v}_2\|_0 = \{\mathbf{v}_2^T D^{-1} \mathbf{v}_2\}^{1/2}. \tag{4.10'}$$

However, in order to somewhat simplify the analysis, we shall assume that

$$\|\mathbf{v}_1\|_0=\{\mathbf{v}_1^T A_{11}^{-1}\mathbf{v}_1\}^{1/2}, \qquad \|\mathbf{v}_2\|_0=\{\mathbf{v}_2^T S^{-1}\mathbf{v}_2\}^{1/2}. \tag{4.10}$$

Finally, it remains to estimate

$$\sigma_1=\|A_{12}S^{-1}\|_0, \qquad \sigma_2=\|A_{21}A_{11}^{-1}\|_0.$$

We have

$$\begin{aligned}\sigma_1^2&=\sup_{\mathbf{v}_2}\left(\frac{\|A_{12}S^{-1}\mathbf{v}_2\|_0}{\|\mathbf{v}_2\|_0}\right)^2\\ &=\sup_{\mathbf{v}_2}\frac{(A_{12}S^{-1}\mathbf{v}_2)^T A_{11}^{-1}(A_{12}S^{-1}\mathbf{v}_2)}{\mathbf{v}_2^t S^{-1}\mathbf{v}_2}\\ &\leqq\sup_{\mathbf{v}_2}\frac{\mathbf{v}_2^t S^{-1/2}A_{21}A_{11}^{-1}A_{12}S^{-1/2}\mathbf{v}_2}{\mathbf{v}_2^t\mathbf{v}_2}\\ &\leqq\sup_{\mathbf{v}_2}\frac{\mathbf{v}_2^t S^{-1/2}(A_{21}A_{11}^{-1}A_{12}-A_{22})S^{-1/2}\mathbf{v}_2}{\mathbf{v}_2^t\mathbf{v}_2}\\ &\leqq 1.\end{aligned}$$

Similarly,

$$\begin{aligned}\sigma_2^2&=\sup_{\mathbf{v}_1}\left(\frac{\|A_{21}A_{11}^{-1}\mathbf{v}_1\|_0}{\|\mathbf{v}_1\|_0}\right)^2\\ &=\sup_{\mathbf{v}_1}\frac{\mathbf{v}_1^t A_{11}^{-1}A_{12}S^{-1}A_{21}A_{11}^{-1}\mathbf{v}_1}{\mathbf{v}_1^t A_{11}^{-1}\mathbf{v}_1}\\ &=\sup_{\mathbf{v}_1}\frac{\mathbf{v}_1^t A_{11}^{-1/2}A_{12}S^{-1}A_{21}A_{11}^{-1/2}\mathbf{v}_1}{\mathbf{v}_1^t\mathbf{v}_1}.\end{aligned}$$

We have

$$\begin{aligned}\mathbf{w}_1^t A_{12}\mathbf{w}_2&=(A_{11}^{1/2}\mathbf{w}_1)^T(A_{11}^{-1/2}A_{12}\mathbf{w}_2)\\ &\leqq\|A_{11}^{1/2}\mathbf{w}_1\|\,(\mathbf{w}_2^t A_{21}A_{11}^{-1}A_{12}\mathbf{w}_2)^{1/2}\\ &\leqq\|A_{11}^{1/2}\mathbf{w}_1\|\,\|S^{1/2}\mathbf{w}_2\|.\end{aligned}$$

Hence, with $\mathbf{v}_1=A_{11}^{1/2}\mathbf{w}_1$, $\mathbf{v}_2=S^{1/2}\mathbf{w}_2$, we find

$$\mathbf{v}_1^t A_{11}^{-1/2}A_{12}S^{-1/2}\mathbf{v}_2\leqq\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|,$$

that is,

$$\sigma_2=\|S^{-1/2}A_{21}A_{11}^{-1/2}\|\leqq 1.$$

Here the norm is the standard Euclidean norm, $\|\mathbf{v}\|=\{\sum v_i^2\}^{1/2}$.

Remark 4.1. Note that

$$\frac{|(A_{21}\mathbf{x},\mathbf{y})|}{(A_{11}\mathbf{x},\mathbf{x})^{1/2}(S\mathbf{y},\mathbf{y})^{1/2}}=\frac{|(S^{-1/2}A_{21}A_{11}^{-1/2}\mathbf{x},\mathbf{y})|}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\leqq\sigma_2\leqq 1 \quad \text{for all } \mathbf{x},\mathbf{y}.$$

This shows the upper bound in the well-known Babuska–Brezzi condition. Note also that

$$\sup_{\mathbf{x}\neq\mathbf{0}} \frac{(A_{21}\mathbf{x},\mathbf{y})}{(A_{11}\mathbf{x},\mathbf{x})^{1/2}(S\mathbf{y},\mathbf{y})^{1/2}} = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{(\mathbf{x}, A_{11}^{-1/2}A_{12}S^{-1/2}\mathbf{y})}{\|\mathbf{x}\|\,\|\mathbf{y}\|} = \frac{\|A_{11}^{-1/2}A_{12}S^{-1/2}\mathbf{y}\|}{\|\mathbf{y}\|}.$$

Hence

$$\inf_{\mathbf{y}\neq 0}\sup_{\mathbf{x}\neq 0} \frac{(A_{21}\mathbf{x},\mathbf{y})}{(A_{11}\mathbf{x},\mathbf{x})^{1/2}(S\mathbf{y},\mathbf{y})^{1/2}} = \mu_1,$$

where μ_1^2 is the smallest eigenvalue of the matrix

$$S^{-1/2}A_{21}A_{11}^{-1}A_{12}S^{-1/2} = (A_{11}^{-1/2}A_{12}S^{-1/2})^T(A_{11}^{-1/2}A_{12}S^{-1/2}).$$

This is positive (i.e., the Babuska–Brezzi inf-sup condition is satisfied) if and only if A_{12} has a complete column rank. The algebraic formulation of the Babuska–Brezzi condition can be found earlier in Bank, Welfert, and Yserentant [10].

The result for Problem 4.2 is summarized in the following theorem.

THEOREM 4.2. *Let the norms* $\|\mathbf{v}_1\|_0$, $\|\mathbf{v}_2\|_0$ *be defined by* (4.10′). *Then the mapping* $B[\cdot]$ *defined by Algorithm* 4, *with* ε_1, ε_2 *sufficiently small and* $B_{11}[\cdot]$, $C[\cdot]$ *defined accordingly as above for Problem* 4.2, *gives an optimal variable-step preconditioner.*

5. Conclusion. We have derived a general framework for a parameter-free variable step preconditioned generalized conjugate gradient method, which is applied for solving two-by-two block matrix problems arising, for example, in two-level nonsymmetric problems, as well as for indefinite saddle-point problems. For them, the general coercivity and boundedness properties of the variable-step preconditioner have been verified. The method can be implemented as a black box solver for any problem satisfying the coercivity and boundedness assumptions.

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