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OPTIMAL DISCRETE TRANSMISSION CONDITIONS FOR A NONOVERLAPPING DOMAIN DECOMPOSITION METHOD FOR THE HELMHOLTZ EQUATION*

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Abstract. This paper is dedicated to recent developments of a two-Lagrange multipliers domain decomposition method for the Helmholtz equation [C. Farhat et al., *Comput. Methods Appl. Mech. Engrg.*, 184 (2000), pp. 213–240; M. J. Gander, F. Magoulès, and F. Nataf, *SIAM J. Sci. Comput.*, 24 (2002), pp. 38–60] involving an additional augmented operator along the interface between the subdomains. Most methods for optimizing the augmented interface operator are based on the discretization of approximations of the continuous transparent operator [B. Després, *Proceedings of the Second International Conference on Mathematical and Numerical Aspects of Wave Propagation*, R. Kleinman et al., eds., SIAM, Philadelphia, 1993, pp. 197–206; J.-D. Benamou and B. Després, *J. Comput. Phys.*, 136 (1997), pp. 68–82; P. Chevalier and F. Nataf, *Domain Decomposition Methods* 10, AMS, Providence, RI, 1998, pp. 400–407; M. J. Gander, *Proceedings of the* 12*th International Conference on Domain Decomposition Methods*, (Chiba, Japan), ddm.org, 2000, pp. 247–254; M. J. Gander, F. Magoulès, and F. Nataf, *SIAM J. Sci. Comput.*, 24 (2002), pp. 38–60]. At the discrete level, the optimal operator can be proved to be equal to the Schur complement of the outer domain. This Schur complement can be directly approximated using purely algebraic techniques like sparse approximate inverse methods or incomplete factorization. The main advantage of such an algebraic approach is that it is much easier to implement in existing code without any information on the geometry of the interface and the finite element formulation used. Convergence results and parallel efficiency of several algebraic optimization techniques of an interface operator for acoustic analysis applications will be presented.

Key words. domain decomposition methods, algebraic approximation, transmission conditions, transparent operator

AMS subject classifications. 65F10, 65N22

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1. Introduction.

The simulation of time-harmonic acoustic waves in a homogeneous medium Ω can be formulated as the following Helmholtz boundary value problem:

$$\begin{aligned} -\nabla^2 u - \omega^2 u &= f && \text{in } \Omega, \\ u &= g_1 && \text{on } \Gamma_D, \\ \nabla u \cdot \nu &= g_2 && \text{on } \Gamma_N, \\ \nabla u \cdot \nu + \gamma u &= g_3 && \text{on } \Gamma_R, \end{aligned} \tag{1.1}$$

where ω is the wave number associated with the problem, f the field of given external forces, g_1 the prescribed Dirichlet data on the boundary Γ_D, g_2 the prescribed Neumann data on boundary Γ_N, and g_3 the prescribed Robin data on boundary Γ_R.

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Finally, ν denotes the unit outward normal on the boundary $\Gamma_N \cup \Gamma_R$. The last equation of system (1.1) is the Robin boundary condition, which models the absorbing material properties involving the complex coefficient γ.

The finite element discretization of the above boundary value problem leads to the following system of complex equations:

$$\begin{aligned} Zx &= b, \\ \text{where} \quad Z &= K - \omega^2 M + \gamma M_R. \end{aligned} \tag{1.2}$$

Matrices Z, K, and M are the so-called impedance, stiffness, and mass matrices of the problem, and b is the right-hand side vector. Matrix M_R is induced by the Robin boundary condition and is nonzero only at the degrees of freedom lying on the boundary Γ_R.

High-frequency acoustic problems call for fine meshes and therefore lead to large-scale systems of equations. For such problems, solving (1.2) by a direct method entails memory and CPU requirements that rapidly overwhelm even the largest resources that are currently available. For this reason, a significant amount of research continues to be invested in the development of Krylov-subspace, multigrid, and domain decomposition (DD) based iterative algorithms for the solution of the matrix problem (1.2) [2], [24], [22], [19]. However, for high frequencies and therefore large values of the wave number ω, Z is usually indefinite, which implies serious challenges for analysis, implementation, and performance of iterative solvers.

In this paper, we present a DD method with Lagrange multipliers for the solution of Helmholtz problems and show that an optimal variant can be derived, which leads to scalable convergence properties. This DD method uses two-Lagrange multipliers fields and two augmented operators along the interface and is presented in detail in section 2. This method is essentially an optimal extension to indefinite or complex problems of the augmented version in [6] and [8] of the finite element tearing and interconnecting (FETI) method [11]. Here, we show in section 3 the theoretical optimal choice of the augmented operator in order to have a full scalability upon the number of subdomains, and in section 4 we present some algebraic approximations of the augmented operator. In section 5, we apply the DD method to the parallel solution of three-dimensional acoustic model problems and then to an industrial problem of an air-conditioned tube. We report interesting performance results on a 16-processor Origin 3400 that highlight the efficiency of the DD method for the solution of acoustic problems. Finally, we conclude this paper in section 6.

2. Algebraic formulation of DD methods. The main motivation of nonoverlapping DD methods is to determine an algorithm able to solve the linear system (1.2) by the use of a nonoverlapping splitting of the initial domain into small subdomains and by an uncoupled formulation of the associated linear systems.

2.1. General presentation. Consider a splitting of the domain as in Figure 2.1 and note by subscripts i and p the degrees of freedom located inside subdomain $\Omega^{(s)}$ and on interface Γ_p. Then, the contribution of subdomain $\Omega^{(s)}$, $s = 1, 2$, to the impedance matrix and the right-hand side can be written as (see [11], [23])

$$Z^{(s)} = \begin{bmatrix} Z_{ii}^{(s)} & Z_{ip}^{(s)} \\ Z_{pi}^{(s)} & Z_{pp}^{(s)} \end{bmatrix}, \qquad b^{(s)} = \begin{bmatrix} b_i^{(s)} \\ b_p^{(s)} \end{bmatrix}.$$

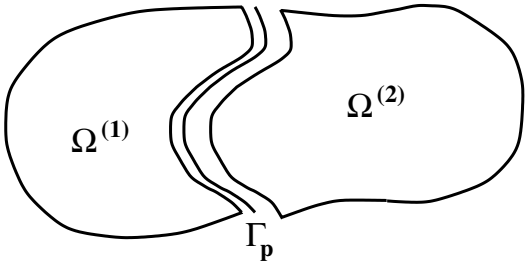

FIG. 2.1. *Nonoverlapping domain splitting.*

The global problem is a block system obtained by assembling a local contribution of each subdomain:

$$\begin{bmatrix} Z_{ii}^{(1)} & 0 & Z_{ip}^{(1)} \\ 0 & Z_{ii}^{(2)} & Z_{ip}^{(2)} \\ Z_{pi}^{(1)} & Z_{pi}^{(2)} & Z_{pp} \end{bmatrix} \begin{bmatrix} x_i^{(1)} \\ x_i^{(2)} \\ x_p \end{bmatrix} = \begin{bmatrix} b_i^{(1)} \\ b_i^{(2)} \\ b_p \end{bmatrix}. \tag{2.1}$$

The matrices $Z_{pp}^{(1)}$ and $Z_{pp}^{(2)}$ represent the interaction matrices between the nodes on the interface obtained by integration on $\Omega^{(1)}$ and $\Omega^{(2)}$. Block Z_{pp} is the sum of these two blocks. In the same way, the term $b_p = b_p^{(1)} + b_p^{(2)}$ is obtained by local integration of the right-hand side over each subdomain and summation on the interface. With this notation, the problem (2.1) is equivalent to the following coupled problems:

$$\begin{bmatrix} Z_{ii}^{(1)} & Z_{ip}^{(1)} \\ Z_{pi}^{(1)} & Z_{pp}^{(1)} \end{bmatrix} \begin{bmatrix} x_i^{(1)} \\ x_p \end{bmatrix} = \begin{bmatrix} b_i^{(1)} \\ b_p - Z_{pi}^{(2)} x_i^{(2)} - Z_{pp}^{(2)} x_p \end{bmatrix}, \tag{2.2}$$

$$\begin{bmatrix} Z_{ii}^{(2)} & Z_{ip}^{(2)} \\ Z_{pi}^{(2)} & Z_{pp}^{(2)} \end{bmatrix} \begin{bmatrix} x_i^{(2)} \\ x_p \end{bmatrix} = \begin{bmatrix} b_i^{(2)} \\ b_p - Z_{pi}^{(1)} x_i^{(1)} - Z_{pp}^{(1)} x_p \end{bmatrix}, \tag{2.3}$$

where $Z_{pp} = Z_{pp}^{(1)} + Z_{pp}^{(2)}$. We introduce the following theorem.

THEOREM 2.1 (existence and unicity). *Under a splitting of the form* $Z_{pp} = Z_{pp}^{(1)} + Z_{pp}^{(2)}$ *and* $b_p = b_p^{(1)} + b_p^{(2)}$, *for all matrices* $A^{(1)}$, $A^{(2)}$ *there is one and only one associated value* $\lambda^{(1)}$, $\lambda^{(2)}$, *such as the following coupled problems:*

$$\begin{bmatrix} Z_{ii}^{(1)} & Z_{ip}^{(1)} \\ Z_{pi}^{(1)} & Z_{pp}^{(1)} + A^{(1)} \end{bmatrix} \begin{bmatrix} x_i^{(1)} \\ x_p^{(1)} \end{bmatrix} = \begin{bmatrix} b_i^{(1)} \\ b_p^{(1)} + \lambda^{(1)} \end{bmatrix}, \tag{2.4}$$

$$\begin{bmatrix} Z_{ii}^{(2)} & Z_{ip}^{(2)} \\ Z_{pi}^{(2)} & Z_{pp}^{(2)} + A^{(2)} \end{bmatrix} \begin{bmatrix} x_i^{(2)} \\ x_p^{(2)} \end{bmatrix} = \begin{bmatrix} b_i^{(2)} \\ b_p^{(2)} + \lambda^{(2)} \end{bmatrix}, \tag{2.5}$$

$$x_p^{(1)} - x_p^{(2)} = 0, \tag{2.6}$$

$$\lambda^{(1)} + \lambda^{(2)} - A^{(1)} x_p^{(1)} - A^{(2)} x_p^{(2)} = 0 \tag{2.7}$$

are equivalent to the problem (2.1).

Proof. The admissibility condition (2.6) derives from the relation $x_p^{(1)} = x_p^{(2)} = x_p$. If $x_p^{(1)} = x_p^{(2)} = x_p$, the first rows of local systems (2.4) and (2.5) are the same as the two first rows of global system (2.1), and adding the last rows of local systems (2.4)

and (2.5) gives

$$Z_{pi}^{(1)}x_i^{(1)} + Z_{pi}^{(2)}x_i^{(2)} + Z_{pp}\ x_p - b_p = \lambda^{(1)} + \lambda^{(2)} - A^{(1)}x_p^{(1)} - A^{(2)}x_p^{(2)}.$$

So, the last equation of global system (2.1) is satisfied only if

$$\lambda^{(1)} + \lambda^{(2)} - A^{(1)}x_p^{(1)} - A^{(2)}x_p^{(2)} = 0.$$

Conversely, if $x_p^{(1)}$, $x_p^{(2)}$, and x_p are derived from global system (2.1), then local systems (2.4) and (2.5) define $\lambda^{(1)}$ and $\lambda^{(2)}$ in a unique way. ∎

It follows from Theorem 2.1 that the coupled problems (2.4)–(2.7) are equivalent to the coupled problems (2.2)–(2.3). The two relations (2.6) and (2.7) represent the admissibility constraint and the equilibrium constraint ensured by the interface variables $x_p^{(1)}$, $\lambda^{(1)}$ and $x_p^{(2)}$, $\lambda^{(2)}$ in order to have the equivalence between these coupled problems. Such relations on the interface are usually called discrete transmission conditions in the literature.

2.2. DD algorithms. In the previous theorem, the matrices $A^{(1)}$ and $A^{(2)}$ correspond to augmented matrices (along the interface Γ_p) with no particular properties. Upon the value of these matrices, different nonoverlapping DD methods can be derived. In the following paragraphs, such methods are described in a new homogeneous formulation.

2.2.1. The case of null augmented matrices. The first case of interest corresponds to the basic case where no augmented matrices are added on the interface, which means $A^{(1)} = A^{(2)} = 0$. With this choice, the coupled problems to solve reduce to

$$\begin{bmatrix} Z_{ii}^{(1)} & Z_{ip}^{(1)} \\ Z_{pi}^{(1)} & Z_{pp}^{(1)} \end{bmatrix} \begin{bmatrix} x_i^{(1)} \\ x_p^{(1)} \end{bmatrix} = \begin{bmatrix} b_i^{(1)} \\ b_p^{(1)} + \lambda^{(1)} \end{bmatrix}, \tag{2.8}$$

$$\begin{bmatrix} Z_{ii}^{(2)} & Z_{ip}^{(2)} \\ Z_{pi}^{(2)} & Z_{pp}^{(2)} \end{bmatrix} \begin{bmatrix} x_i^{(2)} \\ x_p^{(2)} \end{bmatrix} = \begin{bmatrix} b_i^{(2)} \\ b_p^{(2)} + \lambda^{(2)} \end{bmatrix}, \tag{2.9}$$

$$x_p^{(1)} - x_p^{(2)} = 0, \tag{2.10}$$

$$\lambda^{(1)} + \lambda^{(2)} = 0. \tag{2.11}$$

One way to solve this problem, called the dual Schur complement method, is to suppose that the relation (2.11) is ensured, that is $\lambda^{(1)} = -\lambda^{(2)} = \lambda$, with λ an arbitrary known value. If the matrix $Z_{ii}^{(s)}$ is nonsingular and if the local problems in each subdomain are well-posed, a direct relation between $x_p^{(s)}$ and λ can be obtained from (2.8) and (2.9):

$$x_p^{(1)} = [S^{(1)}]^{-1}(c_p^{(1)} + \lambda),$$
$$x_p^{(2)} = [S^{(2)}]^{-1}(c_p^{(2)} - \lambda),$$

where $S^{(s)} = Z_{pp}^{(s)} - Z_{pi}^{(s)}[Z_{ii}^{(s)}]^{-1}Z_{ip}^{(s)}$ is the Schur complement and $c_p^{(s)} = b_p^{(s)} - Z_{pi}^{(s)}[Z_{ii}^{(s)}]^{-1}b_i^{(s)}$ is the condensed right-hand side in subdomain $\Omega^{(s)}$, $s = 1, 2$. The substitution of $x_p^{(1)}$ and $x_p^{(2)}$ in (2.10) leads to the following linear system:

$$([S^{(1)}]^{-1} + [S^{(2)}]^{-1})\lambda = -[S^{(1)}]^{-1}c_p^{(1)} + [S^{(2)}]^{-1}c_p^{(2)}.$$

The solution of this linear system leads to the unique, convenient value of λ. Then, the solution of (2.8) and (2.9) gives the value of $x_i^{(1)}$, $x_p^{(1)}$ and $x_i^{(2)}$, $x_p^{(2)}$. The reader is referred to [10], [11], [9] for more details on this method.

Opposite to the dual Schur complement method is the primal Schur complement method [18], [23], which is based on the hypothesis (2.10), that is, $x_p^{(1)} = x_p^{(2)} = x_p$, with x_p an arbitrary known value. If the matrix $Z_{ii}^{(s)}$ is nonsingular and if the local problems in each subdomain are well-posed, the elimination of $x_i^{(1)}$ and $x_i^{(2)}$ in favor of x_p in (2.8) and (2.9) leads to

$$\begin{aligned} S^{(1)}x_p &= c_p^{(1)} + \lambda^{(1)}, \\ S^{(2)}x_p &= c_p^{(2)} + \lambda^{(2)}, \end{aligned}$$

and after substitution in (2.11), we obtain the following linear system:

$$(S^{(1)} + S^{(2)})x_p = c_p^{(1)} + c_p^{(2)}.$$

Here again, the solution of this linear system leads to the unique, correct value of x_p, and the first line of (2.8) and (2.9),

$$\begin{aligned} x_i^{(1)} &= [Z_{ii}^{(1)}]^{-1}(b_i^{(1)} - Z_{ip}^{(1)}x_p^{(1)}), \\ x_i^{(2)} &= [Z_{ii}^{(2)}]^{-1}(b_i^{(2)} - Z_{ip}^{(2)}x_p^{(2)}), \end{aligned}$$

gives, respectively, the value of $x_i^{(1)}$ and $x_i^{(2)}$.

2.2.2. The case of opposite augmented matrices. The second case of interest corresponds to the case where $A^{(1)} + A^{(2)} = 0$, which means $A^{(1)} = -A^{(2)} = A$. With this choice the two relations (2.6) and (2.7) become

$$\begin{aligned} x_p^{(1)} - x_p^{(2)} &= 0, \\ \lambda^{(1)} + \lambda^{(2)} - A(x_p^{(1)} - x_p^{(2)}) &= 0, \end{aligned}$$

equivalent to the relations

$$\begin{aligned} x_p^{(1)} - x_p^{(2)} &= 0, \\ \lambda^{(1)} + \lambda^{(2)} &= 0. \end{aligned}$$

Here again, different DD methods can be derived. If the primal Schur complement method leads to the same linear system as in the previous subsection when $A^{(1)} = A^{(2)} = 0$, this is not the case for the dual Schur complement method. Indeed, following the same steps as in the dual Schur complement method, and if we suppose that the local problems are well-posed, the linear system becomes

$$F\lambda = d$$

with the corresponding matrix and right-hand side

$$\begin{aligned} F &= [S^{(1)} + A]^{-1} + [S^{(2)} - A]^{-1}, \\ d &= -[S^{(1)} + A]^{-1}c_p^{(1)} + [S^{(2)} - A]^{-1}c_p^{(2)}. \end{aligned}$$

More details on this method called the FETI-H method can be found in [7], [8], [20].

2.2.3. The case of general augmented matrices. The third case corresponds to the case where the matrices $A^{(1)}$ and $A^{(2)}$ are not linked together. Furthermore, if no restrictions are imposed on the interface variables, the expression of $x_p^{(1)}$ and $x_p^{(2)}$ can be obtained from (2.4) and (2.5):

$$x_p^{(1)} = [S^{(1)} + A^{(1)}]^{-1}(c_p^{(1)} + \lambda^{(1)}),$$
$$x_p^{(2)} = [S^{(2)} + A^{(2)}]^{-1}(c_p^{(2)} + \lambda^{(2)}).$$

The only restriction on the choice of the matrices $A^{(1)}$ and $A^{(2)}$ is to ensure that both matrices $[S^{(1)}+A^{(1)}]$ and $[S^{(2)}+A^{(2)}]$ are nonsingular. In order to have the equivalence with the global problem, the two constraints (2.6) and (2.7) must be ensured at the same time. After substitution of $x_p^{(1)}$ and $x_p^{(2)}$ in (2.6) and (2.7) the following linear system is obtained:

$$\begin{bmatrix} [S^{(1)} + A^{(1)}]^{-1} & -[S^{(2)} + A^{(2)}]^{-1} \\ I - A^{(1)}[S^{(1)} + A^{(1)}]^{-1} & I - A^{(2)}[S^{(2)} + A^{(2)}]^{-1} \end{bmatrix} \begin{bmatrix} \lambda^{(1)} \\ \lambda^{(2)} \end{bmatrix}$$
$$= \begin{bmatrix} -[S^{(1)} + A^{(1)}]^{-1}c_p^{(1)} + [S^{(2)} + A^{(2)}]^{-1}c_p^{(2)} \\ A^{(1)}[S^{(1)} + A^{(1)}]^{-1}c_p^{(1)} + A^{(2)}[S^{(2)} + A^{(2)}]^{-1}c_p^{(2)} \end{bmatrix}. \tag{2.12}$$

Because two constraints on the interface are ensured at the same time, the size of this unusual linear system is two times bigger than the size of the linear systems issued from the classical DD algorithms presented before. Of course, the previous matrix is not computed explicitly, and the linear system (2.12) is solved through an iterative method. The block form of the linear system (2.12) is only shown to analyze the iterative algorithm.

2.3. Discrete transmission conditions as local preconditioner. In this section, the case of general augmented matrices in Theorem 2.1 is considered, and both matrices $[S^{(1)} + A^{(1)}]$ and $[S^{(2)} + A^{(2)}]$ are assumed nonsingular. Rather than ensure relations (2.6) and (2.7) on the interface, it may be more interesting to consider another relation, such as

$$C^{(1)}\ (2.6) + (2.7) = 0,$$
$$-C^{(2)}\ (2.6) + (2.7) = 0,$$

which become equivalent to the initial relations as soon as the two matrices $C^{(1)}$ and $C^{(2)}$ satisfy the property $C^{(1)}+C^{(2)}$ invertible. Following the same steps as in section 2.2.3, the matrix and the right-hand side of the linear system take the block form

$$\begin{bmatrix} I - (A^{(1)} - C^{(1)})[S^{(1)} + A^{(1)}]^{-1} & I - (A^{(2)} + C^{(1)})[S^{(2)} + A^{(2)}]^{-1} \\ I - (A^{(1)} + C^{(2)})[S^{(1)} + A^{(1)}]^{-1} & I - (A^{(2)} - C^{(2)})[S^{(2)} + A^{(2)}]^{-1} \end{bmatrix},$$
$$\begin{bmatrix} (A^{(1)} - C^{(1)})[S^{(1)} + A^{(1)}]^{-1}c_p^{(1)} + (A^{(2)} + C^{(1)})[S^{(2)} + A^{(2)}]^{-1}c_p^{(2)} \\ (A^{(1)} + C^{(2)})[S^{(1)} + A^{(1)}]^{-1}c_p^{(1)} + (A^{(2)} - C^{(2)})[S^{(2)} + A^{(2)}]^{-1}c_p^{(2)} \end{bmatrix}.$$

This manipulation on the interface relations simply corresponds to a left multiplication of the linear system (2.12) by the following preconditioner:

$$\begin{bmatrix} C^{(1)} & I \\ -C^{(2)} & I \end{bmatrix}.$$

Different choices can be considered for the matrices $C^{(1)}$ and $C^{(2)}$, but a natural choice consists of $C^{(1)} = A^{(1)}$ and $C^{(2)} = A^{(2)}$ if the condition $A^{(1)} + A^{(2)}$ invertible is assumed. Indeed, with this choice, the constraints on the interface become

$$\lambda^{(1)} + \lambda^{(2)} - (A^{(1)} + A^{(2)})x_p^{(1)} = 0,$$
$$\lambda^{(1)} + \lambda^{(2)} - (A^{(1)} + A^{(2)})x_p^{(2)} = 0,$$

and the diagonal block of the matrix of the linear system reduces to the identity block

$$\begin{bmatrix} I & I - (A^{(1)} + A^{(2)})[S^{(2)} + A^{(2)}]^{-1} \\ I - (A^{(1)} + A^{(2)})[S^{(1)} + A^{(1)}]^{-1} & I \end{bmatrix} \begin{bmatrix} \lambda^{(1)} \\ \lambda^{(2)} \end{bmatrix}$$
$$= \begin{bmatrix} (A^{(1)} + A^{(2)})[S^{(2)} + A^{(2)}]^{-1} c_p^{(2)} \\ (A^{(1)} + A^{(2)})[S^{(1)} + A^{(1)}]^{-1} c_p^{(1)} \end{bmatrix}. \tag{2.13}$$

This method is usually called the two-Lagrange multipliers method in the literature and is usually solved with an iterative procedure (see [20], [8], [13]). It is interesting to notice that in [13], the authors show that this linear system can be obtained through the fixed point solution of the additive Schwarz algorithm without overlap.

This new algebraic approach clearly shows that nonoverlapping DD methods can immediately be induced from Theorem 2.1. In addition, this analysis shows that transmission conditions can be interpreted as a simple local preconditioner of the linear system condensed on the interface. Thus, the determination of good transmission conditions is reduced to the analysis of this preconditioner.

After the presentation of this last preconditioned method, we are naturally interested in the choice of the augmented matrices $A^{(s)}$, $s = 1, 2$. Indeed, these matrices, used in the transmission conditions, may lead to better convergence behavior of the iterative algorithm by reducing the number of iterations. This is the meaning of the optimal convergence of the method.

3. Optimal discrete transmission conditions.

3.1. Preliminaries. It is shown in [14], [4] that the best choice for the continuous operators $\mathcal{A}^{(s)}$ associated to the matrices $A^{(s)}$, $s = 1, 2$, corresponds to the continuous Steklov–Poincaré operators, which are not partial differential operators. Different techniques of approximation have been analyzed in recent years in the context of the additive Schwarz method with no overlap. These techniques, based on two-dimensional Fourier analysis of the Steklov–Poincaré operators in a half-space, consist of approximating these operators with partial differential operators involving a tangential derivative as $(\mathcal{A}^{(s)} = \alpha^{(s)} + \beta^{(s)}\partial^2_{\tau^2})$ with $\alpha^{(s)}$, $\beta^{(s)}$ two coefficients and τ the unit tangential vector.

The discretization leads to the discrete operator $A^{(s)} = \alpha^{(s)} M_\Gamma + \beta^{(s)} K_\Gamma$, where M_Γ and K_Γ are, respectively, a surface mass matrix and a surface stiffness matrix defined on the interface of the subdomain. The definition of these augmented matrices ensures that no additional cost of the algorithm is required at each iteration, since the bandwidth of the local matrices (2.4)–(2.5) does not change, if the matrix $A^{(s)}$, $s = 1, 2$, is added or not along the interface.

The first results presented in [5], [1] for the Helmholtz equation use a simple zeroth order Taylor development, i.e., a choice of $\alpha^{(s)} = i\omega$ and $\beta^{(s)} = 0$, $s = 1, 2$, where $i = \sqrt{-1}$ denotes the imaginary complex number. Then in [16], [17] for convection diffusion equations, in [3] for the Maxwell equation, in [20] for the

Helmholtz equation, and in [12] for the Laplace equation, the techniques were based on optimization procedures. The coefficients are chosen in such a way that they minimize the convergence rate of the Jacobi algorithm involved in the additive Schwarz method. The key points of these techniques are both an analysis performed on the continuous problem and very robust behavior when applied on the discrete problem; see [13].

In [16], a variation has been performed in order to extend the previous work directly on the discrete problem obtained from a finite difference scheme. The expression of the associated coefficients $\alpha_h^{(s)}$, $\beta_h^{(s)}$ was close from the coefficients $\alpha^{(s)}$, $\beta^{(s)}$ obtained on the continuous problem, and thus the number of iterations does not change. The major drawback of this last approach is the dependence of the coefficients $\alpha_h^{(s)}$, $\beta_h^{(s)}$ upon the numerical scheme.

As we can see, in such an approximation there is a strong link between the optimal coefficients, the considered equations, and the discretization scheme. Moreover, up to now there has been no theory concerning the optimization of such coefficients in the case of a nonhomogeneous medium. Another drawback of such an approach is the major intrusion inside the existing code, which requires the knowledge of the geometry of the interface in order to compute the interface matrices M_Γ and K_Γ.

In the following, a new analysis is performed directly on the discrete problem and we show that the optimal convergence of a two-Lagrange multipliers algorithm is obtained with a choice of the augmented matrices equal to the complete outer Schur complement, that is, the discrete Steklov–Poincaré operator. The extension to the case of a one-way splitting is also performed. Then, a new approach based on an algebraic approximation of the discrete Steklov–Poincaré operator helps to avoid any additional cost. Some properties of this new method are analyzed on a model problem, and an application on an industrial acoustic problem shows the efficiency of this new method.

3.2. Two-domain splitting. Consider a two-domain splitting of the domain as represented in Figure 2.1. The global system takes the following block form:

$$\begin{bmatrix} Z_{ii}^{(1)} & 0 & Z_{ip}^{(1)} \\ 0 & Z_{ii}^{(2)} & Z_{ip}^{(2)} \\ Z_{pi}^{(1)} & Z_{pi}^{(2)} & Z_{pp} \end{bmatrix} \begin{bmatrix} x_i^{(1)} \\ x_i^{(2)} \\ x_p \end{bmatrix} = \begin{bmatrix} b_i^{(1)} \\ b_i^{(2)} \\ b_p \end{bmatrix} \tag{3.1}$$

with $Z_{pp} = Z_{pp}^{(1)} + Z_{pp}^{(2)}$ and $b_p = b_p^{(1)} + b_p^{(2)}$. The elimination of the unknown $x_i^{(q)}$, $q = 1, 2$, $q \neq s$ in (3.1), leads to

$$\begin{bmatrix} Z_{ii}^{(s)} & Z_{ip}^{(s)} \\ Z_{pi}^{(s)} & Z_{pp}^{(s)} + S^{(q)} \end{bmatrix} \begin{bmatrix} x_i^{(s)} \\ x_p \end{bmatrix} = \begin{bmatrix} b_i^{(s)} \\ b_p^{(s)} + c_p^{(q)} \end{bmatrix}, \tag{3.2}$$

where $S^{(q)}$ and $c_p^{(q)}$ denote the condensed matrix and right-hand side,

$$\begin{aligned} S^{(q)} &= Z_{pp}^{(q)} - Z_{pi}^{(q)} [Z_{ii}^{(q)}]^{-1} Z_{ip}^{(q)}, \\ c_p^{(q)} &= b_p^{(q)} - Z_{pi}^{(q)} [Z_{ii}^{(q)}]^{-1} b_i^{(q)}, \end{aligned}$$

introduced in the previous sections. Equation (3.2) suggests that the optimal augmented matrix to add to the local impedance matrix on the interface is $S^{(q)}$, since then system (3.2) is very similar to the augmented local problem:

$$\begin{bmatrix} Z_{ii}^{(s)} & Z_{ip}^{(s)} \\ Z_{pi}^{(s)} & Z_{pp}^{(s)} + A^{(s)} \end{bmatrix} \begin{bmatrix} x_i^{(s)} \\ x_p^{(s)} \end{bmatrix} = \begin{bmatrix} b_i^{(s)} \\ b_p^{(s)} + \lambda^{(s)} \end{bmatrix}.$$

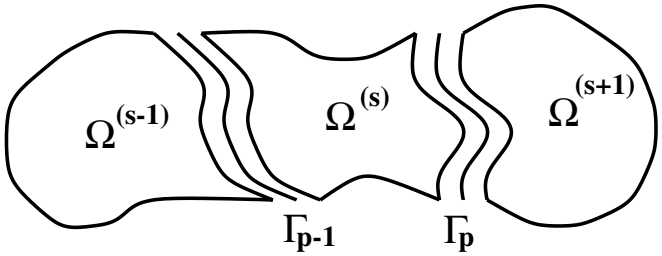

FIG. 3.1. *One-way domain splitting without cross-points.*

THEOREM 3.1. *In a case of a two-domain splitting, the simple (Jacobi) iterative algorithm for two-Lagrange multipliers with augmented matrix equal to complete outer Schur complement defined as in* (3.2) *converges in at most one iteration.*

Proof. Choosing the augmented local matrix $A^{(s)} = S^{(q)}$, $s = 1, 2$, $q = 1, 2$, $s \neq q$, makes the matrix of condensed interface system (2.13) equal to identity. ∎

3.3. One-way splitting. Consider a one-way splitting of the domain as in Figure 3.1 and note by subscripts i, $p-1$, and p the degrees of freedom located inside subdomain $\Omega^{(s)}$, on left interface Γ_{p-1} and right interface Γ_p. Then, the contribution of subdomain $\Omega^{(s)}$ to the impedance matrix and the right-hand side can be written as

$$Z^{(s)} = \begin{bmatrix} Z^{(s)}_{ii} & Z^{(s)}_{ip-1} & Z^{(s)}_{ip} \\ Z^{(s)}_{p-1i} & Z^{(s)}_{p-1p-1} & 0 \\ Z^{(s)}_{pi} & 0 & Z^{(s)}_{pp} \end{bmatrix}, \qquad b^{(s)} = \begin{bmatrix} b^{(s)}_{i} \\ b^{(s)}_{p-1} \\ b^{(s)}_{p} \end{bmatrix}.$$

The global system of equations can be reduced on the interfaces by elimination of inner degrees of freedom. The contribution of subdomain $\Omega^{(s)}$ to the condensed matrix and the right-hand side is as follows:

$$\begin{bmatrix} S^{(s)}_{p-1p-1} & S^{(s)}_{p-1p} \\ S^{(s)}_{pp-1} & S^{(s)}_{pp} \end{bmatrix} = \begin{bmatrix} Z^{(s)}_{p-1p-1} - Z^{(s)}_{p-1i}[Z^{(s)}_{ii}]^{-1} Z^{(s)}_{ip-1} & -Z^{(s)}_{p-1i}[Z^{(s)}_{ii}]^{-1} Z^{(s)}_{ip} \\ -Z^{(s)}_{pi}[Z^{(s)}_{ii}]^{-1} Z^{(s)}_{ip-1} & Z^{(s)}_{pp} - Z^{(s)}_{pi}[Z^{(s)}_{ii}]^{-1} Z^{(s)}_{ip} \end{bmatrix},$$

$$\begin{bmatrix} c^{(s)}_{p-1} \\ c^{(s)}_{p} \end{bmatrix} = \begin{bmatrix} b^{(s)}_{p-1} & - & Z^{(s)}_{p-1i} & [Z^{(s)}_{ii}]^{-1} & b^{(s)}_{i} \\ b^{(s)}_{p} & - & Z^{(s)}_{pi} & [Z^{(s)}_{ii}]^{-1} & b^{(s)}_{i} \end{bmatrix}.$$

The global condensed problem on interfaces is a block 3-diagonal system obtained by assembling a local contribution of each subdomain:

$$\begin{bmatrix} \dots & \dots & 0 & 0 \\ S^{(s-1)}_{p-1p-2} & S^{(s-1)}_{p-1p-1} + S^{(s)}_{p-1p-1} & S^{(s)}_{p-1p} & 0 \\ 0 & S^{(s)}_{pp-1} & S^{(s)}_{pp} + S^{(s+1)}_{pp} & S^{(s+1)}_{pp+1} \\ 0 & 0 & \dots & \dots \end{bmatrix} \begin{bmatrix} & \dots & \\ c^{(s-1)}_{p-1} & + & c^{(s)}_{p-1} \\ c^{(s)}_{p} & + & c^{(s+1)}_{p} \\ & \dots & \end{bmatrix}. \tag{3.3}$$

If the system (3.3) is factorized by successive condensation of the matrix and the right-hand side starting from both ends up to the block associated with the subdomain $\Omega^{(s)}$, the following final condensed problem on one subdomain is obtained:

$$\begin{bmatrix} S^{-}_{p-1p-1} + S^{(s)}_{p-1p-1} & S^{(s)}_{p-1p} \\ S^{(s)}_{pp-1} & S^{(s)}_{pp} + S^{+}_{pp} \end{bmatrix} \begin{bmatrix} x_{p-1} \\ x_p \end{bmatrix} = \begin{bmatrix} c^{-}_{p-1} & + & c^{(s)}_{p-1} \\ c^{(s)}_{p} & + & c^{+}_{p} \end{bmatrix}. \tag{3.4}$$

The condensed right and left blocks and right-hand sides of system (3.4) that are denoted with plus and minus superscripts are defined by the following recurrence relations:

$$
\begin{aligned}
S^{+}_{p-1p-1} &= S^{(s)}_{p-1p-1} - S^{(s)}_{p-1p}\ [S^{(s)}_{pp} + S^{+}_{pp}]^{-1} S^{(s)}_{pp-1},\\
S^{-}_{pp} &= S^{(s)}_{pp} - S^{(s)}_{pp-1}\ [S^{-}_{p-1p-1} + S^{(s)}_{p-1p-1}]^{-1} S^{(s)}_{p-1p},\\
c^{+}_{p-1} &= c^{(s)}_{p-1} - S^{(s)}_{p-1p}\ [S^{(s)}_{pp} + S^{+}_{pp}]^{-1} [c^{(s)}_{p} + c^{+}_{p}],\\
c^{-}_{p} &= c^{(s)}_{p} - S^{(s)}_{pp-1}\ [S^{-}_{p-1p-1} + S^{(s)}_{p-1p-1}]^{-1} [c^{-}_{p-1} + c^{(s)}_{p-1}].
\end{aligned}
\tag{3.5}
$$

Equation (3.4) suggests that the optimal augmented matrix to add to the local impedance matrix $Z^{(s)}$ on the left or right interface is, respectively, S^{-}_{p-1p-1} and S^{+}_{pp} since, if $\Omega^{(s)}$ is the only subdomain with a nonzero right-hand side, then $c^{-}_{p-1} = 0$ and $c^{+}_{p} = 0$, and system (3.4) is exactly the condensation of the augmented local problem:

$$
\begin{bmatrix} Z^{(s)}_{ii} & Z^{(s)}_{ip-1} & Z^{(s)}_{ip} \\ Z^{(s)}_{p-1i} & Z^{(s)}_{p-1p-1} + S^{-}_{p-1p-1} & 0 \\ Z^{(s)}_{pi} & 0 & Z^{(s)}_{pp} + S^{+}_{pp} \end{bmatrix}
\begin{bmatrix} x^{(s)}_{i} \\ x^{(s)}_{p-1} \\ x^{(s)}_{p} \end{bmatrix} =
\begin{bmatrix} b^{(s)}_{i} \\ b^{(s)}_{p-1} \\ b^{(s)}_{p} \end{bmatrix}.
\tag{3.6}
$$

THEOREM 3.2. *In the case of a one-way splitting, the simple (Jacobi) iteration algorithm for two-Lagrange multipliers with an augmented matrix equal to the complete outer Schur complement defined as in* (3.6) *converges in at most the number of subdomains minus* 1 *iteration.*

Proof. If $\Omega^{(s)}$ is the only subdomain with a nonzero right-hand side, (3.4) and (3.6) mean that the first iteration with null initial Lagrange multipliers gives an exact solution in $\Omega^{(s)}$ and zero in the other subdomains.

Since λ and x are zero everywhere except on Γ_{p-1} and Γ_p, the initial gradient is nonzero on adjacent interfaces only:

$$
\begin{aligned}
g^{(s-1)}_{p-1} &= \lambda^{(s-1)}_{p-1} + \lambda^{(s)}_{p-1} - (S^{-}_{p-1p-1} + S^{+}_{p-1p-1})\ \ x^{(s)}_{p-1}\\
&= -(S^{-}_{p-1p-1} + S^{+}_{p-1p-1})\ \ x^{(s)}_{p-1},\\
g^{(s+1)}_{p} &= \lambda^{(s+1)}_{p} + \lambda^{(s)}_{p} - (\ S^{-}_{pp} + S^{+}_{pp}\ )\ \ x^{(s)}_{p}\\
&= -(\ S^{-}_{pp} + S^{+}_{pp}\ )\ \ x^{(s)}_{p}.
\end{aligned}
$$

By condensation of (3.4), the initial solution on interface Γ_{p-1} satisfies

$$
\begin{aligned}
&(S^{-}_{p-1p-1} + S^{(s)}_{p-1p-1} - S^{(s)}_{p-1p}[S^{(s)}_{pp} + S^{+}_{pp}]^{-1} S^{(s)}_{pp-1}) x^{(s)}_{p-1}\\
&\qquad = c^{(s)}_{p-1} - S^{(s)}_{p-1p}[S^{(s)}_{pp} + S^{+}_{pp}]^{-1} c^{(s)}_{p}.
\end{aligned}
\tag{3.7}
$$

Under the assumption that the right-hand side is nonzero in $\Omega^{(s)}$ only, $c^{-}_{p-1} = 0$ and $c^{+}_{p} = 0$. So, from the definition of the condensed matrices and the right-hand sides of (3.5), it derives from (3.7) that

$$
(\ S^{-}_{p-1p-1} + S^{+}_{p-1p-1}\ )\ x^{(s)}_{p-1} \ = \ c^{+}_{p-1}
$$

and so

$$
g^{(s-1)}_{p-1} = -\ (\ S^{-}_{p-1p-1} + S^{+}_{p-1p-1}\ )\ \ x^{(s)}_{p-1} = -c^{+}_{p-1}.
$$

A similar result is obtained for $g_p^{(s+1)}$.

The Jacobi algorithm on the condensed interface problem consists of updating λ by $\lambda - g$. So, at the second iteration, both subdomains $s-1$ and $s+1$ will have their complete condensed right-hand side, as well as subdomain s for which $\lambda_{p-1}^{(s)}$ and $\lambda_p^{(s)}$ will remain unchanged and equal to zero. After the second iteration, the solution in the three subdomains will be the exact restriction of the solution of the global problem.

It is easy to see now that, at iteration 2, the situation on interface Γ_{p-2} between subdomains $\Omega^{(s-2)}$ and $\Omega^{(s-1)}$ is exactly the same as that at iteration 1 on interface Γ_{p-1} between subdomains $\Omega^{(s-1)}$ and $\Omega^{(s)}$. So, an exact condensed right-hand side will be passed to subdomain $\Omega^{(s-2)}$ when updating λ at iteration 2.

On the other hand, if λ is such that in two neighboring subdomains $\Omega^{(s-1)}$ and $\Omega^{(s)}$ with interface Γ_{p-1}, the local condensed right-hand sides are complete, then $x_{p-1}^{(s-1)} = x_{p-1}^{(s)} = x_{p-1}$. Condensation on interface Γ_{p-1} of (3.4) gives

$$(S_{p-1p-1}^{-} + S_{p-1p-1}^{+})x_{p-1} = c_{p-1}^{-} + c_{p-1}^{+}.$$

So, if $\lambda_{p-1}^{(s-1)} + \lambda_{p-1}^{(s)} = c_{p-1}^{-} + c_{p-1}^{+}$, then

$$g_{p-1}^{(s-1)} = g_{p-1}^{(s)} = \lambda_{p-1}^{(s-1)} + \lambda_{p-1}^{(s)} - (S_{p-1p-1}^{-} + S_{p-1p-1}^{+})x_{p-1} = 0.$$

This is exactly the situation between $\Omega^{(s-1)}$ and $\Omega^{(s)}$ as well as between $\Omega^{(s)}$ and $\Omega^{(s+1)}$ at iteration 2. This means that the gradient will be zero on all the interfaces of these subdomains at iteration 2.

In the same way, it can be proved by recurrence that each Jacobi iteration will propagate the complete condensed right-hand side one subdomain further on the left and on the right while leaving the values of λ unmodified in all subdomains, where the condensed right-hand side is already complete.

So, the Jacobi method will converge in at most (number of subdomains minus 1) iterations if the initial right-hand side is nonzero in only one subdomain. As any general right-hand side can be decomposed in the sum of right-hand sides that are nonzero in one subdomain only, and since the Jacobi procedure is additive, the same result holds for any case. ∎

4. Approximation of optimal discrete transmission conditions. In the previous section, we have shown that the best choice for the operators $A^{(s)}$ ($s = 1 \ldots$ number of subdomains) in the case of a one-way splitting DD into several subdomains is the complete outer Schur complement matrix. Indeed, this choice leads to the convergence of the Jacobi iterative algorithm in at most (number of subdomains minus 1) iterations. Anyway, such a choice cannot be done in practice since the computational cost of the complete outer Schur complement matrix is too expensive and because this matrix is a dense matrix. Thus, the solution we investigate now is to approximate this complete outer Schur complement with different matrices through an algebraic analysis.

4.1. Neighbor Schur complement matrix. If we would like to reduce the computational cost and the exchange of data, the first idea consists of approximating the complete outer Schur complement with the Schur complement of the neighboring subdomains only. Unfortunately, this choice leads to a matrix $A^{(s)}$, which is still dense, and its addition inside $Z^{(s)}$ increase the bandwidth of the latest, which implies a high additional cost of the algorithm at each iteration.

4.2. Sparse approximate inverse matrix. Rather than consider the exact neighboring Schur complement (which is a dense matrix), we consider its approximation with a sparse matrix.

One of the most popular methods to approximate a dense matrix is the sparse approximate inverse (SPAI) method; see [15]. It consists of approximating the inverse of an $N \times N$ matrix A by a sparse matrix M, which minimizes the Frobenius norm of the difference: $AM - I$. As we look for a sparse matrix M, we choose NS as the number of nonzero terms in a column of the approximate inverse M. Then, the columns of M are computed by solving N least-squares problems:

$$\min \|AM - I\|_F^2 = \sqrt{\sum_{k=1}^{N} \min \|Am^k - e^k\|},$$

where m^k and e^k represent, respectively, the kth column of M and I. It leads to N QR-factorizations of rectangular systems $N \times NS$ extracted from the $N \times N$ matrix A. Of course, NS should be a compromise between the desired precision of the inverse and the size of the QR-systems.

We choose here to approximate by a sparse matrix $[Z_{ii}^{(s)}]^{-1} Z_{ip}$. It consists of minimizing the norm of $\|Z_{ii}^{(s)} M - Z_{ip}^{(s)}\|_F^2$. Matrix M is an $N_i \times N_p$ matrix (N_i corresponds to the number of strictly interior vertices and N_p to the number of interface vertices). Among the N_i unknowns in each column of M, we choose to keep only N_p nonzero terms. Indeed, M is multiplied by $Z_{pi}^{(s)}$ to calculate the Schur complement matrix, and nonzero terms are those corresponding to the neighbors of the nodes on the interface. The number of neighbor nodes in a first approximation (with a plane interface) is the same as the number of nodes on the interface. Then, writing $Z_{ii}^{(s)} M^k = {Z_{ip}^{(s)}}^k$ gives nonzero terms on lines corresponding to the neighbors of the neighbors of the nodes on the interface. We denote this number by N_{pp}, and we have to solve N_p QR-systems of size $N_{pp} \times N_p$, which is a reasonable cost (see [15]). We see immediately that the number of columns chosen in the SPAI approximation will have a strong influence on the quality of the approximation. On the other hand, the more columns are chosen, the more computing time is required for the method.

Compared to the previous subsection, this method reduces the bandwidth of $A^{(s)}$ and then the computational cost of the algorithm at each iteration. The most important computational cost is of course located in the preliminary computation of the Schur complement matrix!

4.3. Lumped matrix. This approach is based on the same idea as the lumped preconditioner for the classical FETI method [11] and consists of approximating the Schur complement matrix $Z_{pp}^{(s)} - Z_{pi}^{(s)} [Z_{ii}^{(s)}]^{-1} Z_{ip}^{(s)}$ by the matrix $Z_{pp}^{(s)}$. With this choice, the interface block of the augmented matrix corresponds to the block associated to the degrees of freedom located on the interface of the global assembled matrix: $Z_{pp} = Z_{pp}^{(s)} + Z_{pp}^{(s+1)}$. The practical interest of this remark is fundamental. With this approach, it is not necessary, as in the classical DD method, to know how to assemble the local matrices in each subdomain, since it is possible to apply the partitioning directly on the assembly matrix. Such a methodology definitely increases the "black box" implementation inside an existing code. Later on, if we would like to improve the approximation, we just need to build a sparse approximation of $-Z_{pi}^{(s)} [Z_{ii}^{(s)}]^{-1} Z_{ip}^{(s)}$, which can be done by using the global assembly matrix too.

TABLE 5.1
Number of iterations for different augmented matrices and different numbers of subdomains for the model problem.

Number of subdomains N_s	Exact Schur complement		Approximate Schur complement			
	Complete outer Schur complement	Neighbor Schur complement	Sparse inverse	Lumped	Taylor	Optimized
2	1	2	**11**	**13**	13	14
4	3	8	**14**	**19**	65	42
8	7	17	**19**	**25**	85	50
16	15	33	**33**	**35**	107	65

5. Numerical experiments. Three sets of numerical experiments are presented. The first set corresponds to a model problem and the results illustrate the convergence properties of the proposed DD method. The second numerical experiment comes from industry and consists of analyzing the noise levels in the interior of an air-conditioned tube. The third numerical experiment consists of an acoustic scattering problem, where an additional two-level approach is used to prove the full scalability of the proposed DD method.

5.1. Model problem. A three-dimensional cavity Ω is considered. Homogeneous Dirichlet boundary conditions are imposed all around the cube except on the left and right sides, where radiation conditions of Robin type are imposed. A uniform quadrilateral mesh is used for the discretization, such as the mesh size equal to $h = \pi/(5\omega)$, where $\omega = 2\pi$ is the wave number considered. For all the experiments, we consider each subdomain as a unit cube, thus adding more subdomains in the one-way splitting will increase the geometry of the global domain.

Table 5.1 shows the number of iterations needed for different number of subdomains for both exact Schur complement and approximate Schur complement. To solve the linear system condensed on the interface (2.13), the ORTHODIR Krylov method is used. The convergence criterion is evaluated on the relative error of the global problem (1.2):

$$\|Zx - b\|_{L_2} \leq 10^{-9} \|b\|_{L_2}. \tag{5.1}$$

The number of iterations obtained with the complete outer Schur complement confirms the result expected from the theoretical analysis, since the number of iterations is always lower than the number of subdomains minus 1. The neighbor Schur complement is presented for information purpose only since it is a dense matrix and leads to an expensive computational cost. The approximate Schur complement consists, on the one hand, of the sparse inverse, on the other hand, of the lumped matrix introduced in this paper and the Taylor and optimized transmission conditions from Fourier analysis; see [13]. It is fundamental to notice that the only difference in cost of the approximate Schur complement matrices (lumped, Taylor, optimized) consists of different entries in the interface matrices, without enlarging the bandwidth of the matrices. Opposite, the sparse inverse approximation increases the bandwidth of the local impedance matrix and leads to additional computational cost at each iteration. In this example, we can see that a sparse inverse approximation gives better results than the lumped approximate as expected, since more coefficients are used in this matrix. In addition, we can see that the lumped approximation leads to better convergence than the Taylor or optimized transmission conditions. This method works

FIG. 5.1. *Decomposition of the air-conditioned tube into four subdomains.*

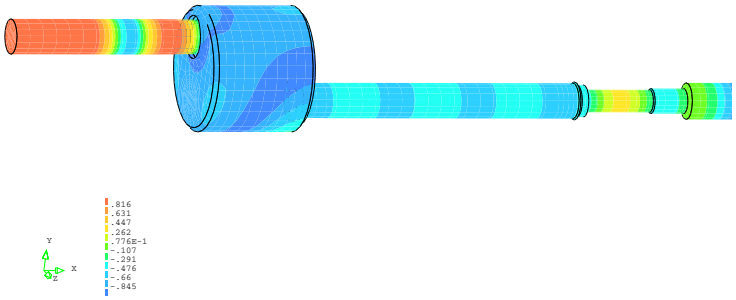

FIG. 5.2. *Acoustic field in the noise-reducer of the air-conditioned tube.*

well compared to the others, even on a small number of processors. When the optimized one needs 50 iterations, the lumped method asks only for half!

Of course, because no global preconditioners are used, a dependence upon the number of subdomains occurs, but this dependence is less important with the lumped approximation than with Taylor or optimized transmission conditions. Nevertheless the method is not optimal with respect to the number of subdomains, but when using a preconditioner (see the numerical experiment dealing with the acoustic scattering problem) it becomes scalable!

5.2. Noise level in an air-conditioned tube. The noise level distribution in an air-conditioned tube is now analyzed. Some vibrations stem from the friction of the fluid along the boundary of the air-conditioned tube. This example is representative for a large class of industrial problems where one tries to determine the acoustic response in the interior of a cavity caused by some friction of fluid. A three-dimensional simulation of the air-conditioned tube is performed. A total number of 2500 degrees of freedom is used for the mesh and Figure 5.1 shows the decomposition of the initial domain into four subdomains. The problem is characterized by a frequency F of 2500 Hz in the air-conditioned tube of length equal to 1.6365 meters and diameter equal to 0.045 meters. The computations are performed in parallel on an SGI-3400 with 16 processors. An ORTHODIR Krylov method is used for the solution of (2.13) and the convergence criterion is always given by (5.1).

Here again the number of iterations obtained with the complete outer Schur complement is always lower than the number of subdomains minus 1. We must keep in mind that the neighbor Schur complement, which gives very good results, is a dense matrix! For the approximation of the Schur complement method, we employ only the sparse inverse and lumped approximations, because we have already shown that these methods are superior to their variants (Taylor and optimized). Results reported in Table 5.2 show that a sparse inverse approximation gives better results than the lumped approximations as already observed for the model problem, and Figure 5.2 shows the acoustic field obtained in the noise-reducer of the air-conditioned tube.

TABLE 5.2
Number of iterations for different augmented matrices and different numbers of subdomains for the air-conditioned tube.

Number of subdomains N_s	Exact Schur complement		Approximate Schur complement	
	Complete outer Schur complement	Neighbor Schur complement	Sparse inverse	Lumped
2	1	1	**12**	**10**
4	3	4	**27**	**30**
6	5	8	**41**	**52**
8	7	12	**56**	**77**

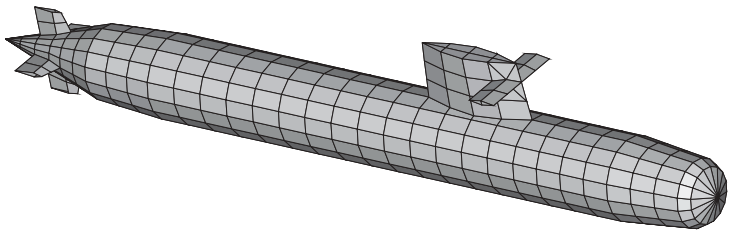

FIG. 5.3. *Shape of the submarine.*

5.3. Acoustic scattering problem. Here, we consider the three-dimensional solution of an acoustic scattering problem where the obstacle has the shape of a submarine of length 76.00 m, height 9.25 m, and diameter 7.50 m; see Figure 5.3. The characteristics of the surrounding domain are the water characteristics, i.e., a density equal to 1000 kg/m^3 and a sound celerity equal to $c = 1500$ m/s. The analysis consists of determining the acoustic scattering field issued from the vibrations of the shell of the submarine which are generated by an incident wave. The computations are performed in the following two steps:

(i) A coupled fluid-structure computation is performed first, where the fluid is modeled with boundary elements and the structure is modeled with shell-type finite elements. An incident planar wave is sent on the submarine. The solution of the coupled problem gives the acoustic pressure for the fluid and the displacement value for the structure.

(ii) Then an acoustic problem is solved. The ocean around the submarine is modeled with finite and infinite elements. The submarine is surrounded by an ellipsoid and only the volume between the submarine and the ellipsoid is meshed with finite elements. Infinite elements are located on the surface of the ellipsoid. The criteria of six elements per wavelength is ensured over the entire mesh. We discretize the domain by $162,000$ $\mathbb{P}_1$ finite elements, $11,500$ infinite elements of order 3, $11,500$ grid points; see [21] for more details. Using the displacement of the structure issued from the first step as boundary conditions for this acoustic problem, the acoustic

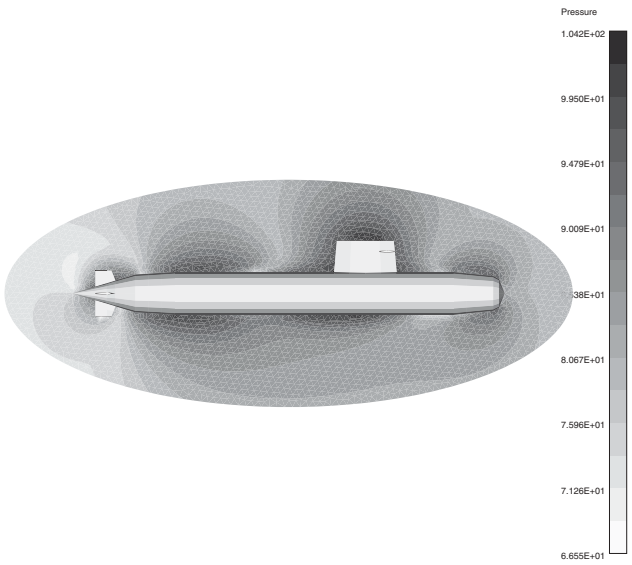

FIG. 5.4. *Two-dimensional cut of the acoustic field around the submarine.*

field can be obtained for different frequencies. This acoustic problem is solved with the nonoverlapping DD method described in this paper. Figure 5.4 shows a two-dimensional cut of the acoustic scattering field for a frequency equal to 10 Hz.

We partition the corresponding mesh into 2, 4, 8 and 16 subdomains as presented in Figure 5.5 such that the infinite elements are located inside the same subdomain; notice that the mesh partitioning does not contain any cross points, and thus the Theorem 3.2 is still valid. For the simulation, we employ only the lumped approximation for solving the resulting systems of equations, because we have already shown that this method is superior to its variant (Taylor, optimized), and present a lower computational cost compared to the sparse inverse approximate. A two-level projected GMRES algorithm is used for the solution of the interface problem (2.13).

The two-level projector considered here is the one we have introduced in [8] for Taylor approximation, which leads to scalability on the number of subdomains. Since the lumped method already converges faster than the Taylor approximation method, it is no wonder that when applying an additional coarse space preconditioner, these methods become fully scalable too. In [8] the coarse space was built with a set of N_θ planar waves. The performance results of the nonoverlapping DD method applied to the solution of the acoustic scattering problem with a two-level approach are reported in Table 5.3. These results confirm the numerical scalability of the lumped approximation with respect to subdomains.

The reader can observe in [8] that the cost of forming the second-level interface problem sometimes can be a significant percentage of the total solution cost, and that the value of N_θ has a great importance for the convergence. For a fixed value of N_θ the cost of forming the projector is affected by N_s in two different ways. For a fixed mesh size h, increasing the number of subdomains decreases the size of the subdomain problems and therefore has a tendency to reduce the CPU time needed for forming the projector. However, increasing the number of subdomains also increases the number

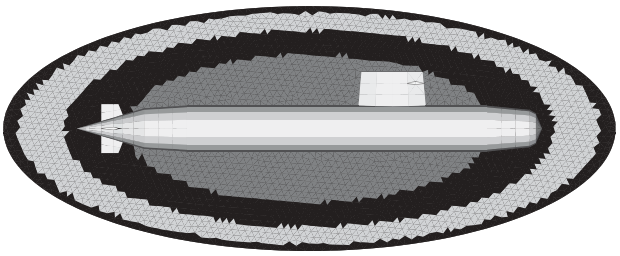

FIG. 5.5. *Two-dimensional cut of the decomposition into four subdomains.*

TABLE 5.3
Number of iterations for the lumped approximate and different numbers of subdomains for the acoustic scattering problem.

Number of subdomains N_s	Coarse space parameter N_θ	Number of iterations
2	32	**13**
4	32	**14**
8	32	**9**
16	32	**8**

of subdomain problems to be solved, and therefore also has a tendency to increase the CPU time required for forming the second-level problem matrix.

Since N_θ must be big enough to ensure the scalability, the construction of the projector represents an important computational cost. An alternative will be to build a different coarse space involving another function, an issue under current investigation.

6. Conclusions. A general algebraic presentation of a DD method with two-Lagrange multipliers and additional augmented operator along the interface has been introduced. Optimal transmission conditions have been derived from this algebraic analysis. Since the optimal augmented operator in a subdomain is the Schur complement of the outer domain, it is not possible to compute it in practice. Promising results have been obtained using simple approximation techniques for this Schur complement. The key issue to improve the method presented in this paper lies in the design of good sparse approximation methods.

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