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On the solution of a class of complex symmetric linear systems

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A R T I C L E I N F O

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We present an iterative method for solving the complex symmetric system $(W + iT)x = b$, where $T \in \mathbb{R}^{n\times n}$ is a positive definite matrix and $W \in \mathbb{R}^{n\times n}$ is indefinite. Convergence of the method is investigated. The induced preconditioner is applied to accelerate the convergence rate of the GMRES(ℓ) method and the numerical results are compared with those of the Hermitian normal splitting (HNS) preconditioner.

1. Introduction

We consider the system of linear equations

$$Ax \equiv (W + iT)x = b, \tag{1}$$

where $W, T \in \mathbb{R}^{n\times n}$, $b \in \mathbb{C}^n$ and $i = \sqrt{-1}$. We assume that the matrices W and T are symmetric. Systems of the form (1) arise in a variety of scientific and engineering applications. Practical background of these kinds of the problems can be found in [1–7] and references therein.

Several iterative methods have been presented to solve the system (1) in the literature (for example see [8–10]). In [8], Bai et al. proposed the Hermitian and skew-Hermitian splitting (HSS) method to solve non-Hermitian positive definite system of linear equations. The HSS iteration method with the Hermitian and skew-Hermitian parts of the matrix A, which are defined by $H = (A+A^H)/2 = W$ and $S = (A-A^H)/2 = iT$, can be directly applied to solve the system (1). However, a modified version of the HSS (MHSS) method was presented by Bai et al. in [1] to solve the system (1) which can be summarized as following.

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The MHSS method. Let $x^{(0)} \in \mathbb{C}^n$ be an initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}_{k=1}^{\infty}$ converges, compute the next iterate $x^{(k+1)}$ via the following procedure:

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - iT)x^{(k)} + b, \\ (\alpha I + T)x^{(k+1)} = (\alpha I + iW)x^{(k+\frac{1}{2})} - ib, \end{cases} \tag{2}$$

where α is a given positive constant and I is the identity matrix.

When both of the matrices W and T are symmetric positive semidefinite with at least one of them (e.g., W) being positive definite, the theoretical analysis shows that the MHSS method converges unconditionally to the unique solution of (1) (see [1]). In each iterate of the MHSS method two subsystems with the coefficient matrices $\alpha I + W$ and $\alpha I + T$ should be solved. Since both of these matrices are symmetric positive definite, the corresponding systems can be solved exactly using the Cholesky factorization or inexactly using the conjugate gradient (CG) method or its preconditioned version, PCG.

When W is symmetric indefinite matrix, then $\alpha I + W$ may be indefinite or singular. In this case, the MHSS iteration method may fail to converge. In [11], Bai designed the skew-normal splitting (SNS) to solve non-Hermitian positive definite systems. The SNS method for solving (1) can be described as following.

The SNS method. Let $x^{(0)} \in \mathbb{C}^n$ be an initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty}$ converges, compute the next iterate $x^{(k+1)}$ via:

$$\begin{cases} (\alpha I - iT)x^{(k+\frac{1}{2})} = (\alpha W - T^2)x^{(k)} - iTb, \\ (\alpha W + T^2)x^{(k+1)} = (\alpha I + iT)x^{(k+\frac{1}{2})} - iTb, \end{cases} \tag{3}$$

where α is a given positive constant.

Bai proved that when W is positive definite and T is invertible, then the SNS method is unconditionally convergent (see [11]).

Let the matrix W be symmetric indefinite. In this case, the matrix W^2 is symmetric positive definite. Having this in mind, Wu in [12] multiplies both sides of the system (1) to get the following system

$$WAx = (W^2 + iWT)x = Wb,$$

which can be equivalently written in the forms

$$\begin{cases} (\alpha I + iW)Tx = (\alpha T - W^2)x + Wb, \\ (\alpha T + W^2)x = (\alpha I - iW)Tx + Wb. \end{cases}$$

This results in the Hermitian normal splitting (HNS) method which can be summarized as following (see [12]).

The HNS method. Let $x^{(0)} \in \mathbb{C}^n$ be an initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty}$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I + iW)x^{(k+\frac{1}{2})} = (\alpha T - W^2)x^{(k)} + Wb, \\ (\alpha T + W^2)x^{(k+1)} = (\alpha I - iW)x^{(k+\frac{1}{2})} + Wb, \end{cases} \tag{4}$$

where α is a given positive constant.

It is easy to see that the application of the SNS method, proposed by Bai in [11], to the system $-iAx = -ib$ results in the HNS iteration method. In [12], the authors proved that if W is symmetric indefinite and T is symmetric positive definite, then the HNS iteration is unconditionally convergent. Computing $x^{(k+\frac{1}{2})}$ from the first equation in (4) and substituting in the second equation yields

$$x^{(k+1)} = M_\alpha x^{(k)} + N_\alpha Wb,$$

where $M_\alpha = (\alpha T + W^2)^{-1}(\alpha I - iW)(\alpha I + iW)^{-1}(\alpha T - W^2)$ and $N_\alpha = 2\alpha(\alpha T + W^2)^{-1}(\alpha I + iW)^{-1}$. In addition, if we define

$$B_\alpha = \frac{1}{2\alpha}(\alpha I + iW)(\alpha T + W^2) \quad \text{and} \quad C_\alpha = \frac{1}{2\alpha}(\alpha I - iW)(\alpha T - W^2),$$

then $WA = B_\alpha - C_\alpha$ and $M_\alpha = B_\alpha^{-1}C_\alpha$. Using these equations, we have $M_\alpha = I - B_\alpha^{-1}WA$. From the convergence of the HNS method we conclude that the eigenvalues of the matrix $B_\alpha^{-1}WA$ are located in a circle centered at $(1, 0)$ with radius 1. Therefore, a Krylov-subspace method like GMRES for solving $B_\alpha^{-1}WAx = B_\alpha^{-1}b$ often has rapid convergence rate (see [13,14]).

In this paper, we present a modification of the SNS iteration method, called MSNS, for solving (1) which unconditionally converges to the solution of (1). Then, the induced preconditioner is numerically compared with HNS preconditioner.

2. The MSNS iteration method

Let W be a symmetric indefinite matrix and the matrix T be symmetric positive definite. To derive the method, similar to the SNS iteration method (see [11]) we multiply both sides of (1) by iT to obtain

$$iTAx = (iTW - T^2)x = iTb. \tag{5}$$

Then, we consider the following two equivalent forms of Eq. (5):

$$(\alpha I + T)iWx = (i\alpha W + T^2)x + iTb,$$

and

$$(i\alpha W - T^2)x = (\alpha I - T)iWx + iTb.$$

The MSNS iteration method is obtained by alternating these two splittings. It is noted that the term iWx does not bring about extra work and for an initial guess $x^{(0)}$ the method can be described as follows.

The MSNS method. Let $x^{(0)} \in \mathbb{C}^n$ be an initial guess. For $k = 0, 1, 2, \ldots,$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty}$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I + T)x^{(k+\frac{1}{2})} = (i\alpha W + T^2)x^{(k)} + iTb, \\ (i\alpha W - T^2)x^{(k+1)} = (\alpha I - T)x^{(k+\frac{1}{2})} + iTb, \end{cases} \tag{6}$$

where α is a given positive constant.

Eliminating the intermediate vector $x^{(k+\frac{1}{2})}$ in Eq. (6), gives the one-step iteration form of MSNS as

$$x^{(k+1)} = P_\alpha x^{(k)} + Q_\alpha Tb,$$

where

$$P_\alpha = (i\alpha W - T^2)^{-1}(\alpha I - T)(\alpha I + T)^{-1}(i\alpha W + T^2),$$

is iteration matrix of the method and $Q_\alpha = 2\alpha i(i\alpha W - T^2)^{-1}(\alpha I + T)^{-1}$. It is not difficult to see that if we define

$$E_\alpha = \frac{1}{2\alpha i}(\alpha I + T)(i\alpha W - T^2) \quad \text{and} \quad F_\alpha = \frac{1}{2\alpha i}(\alpha I - T)(i\alpha W + T^2),$$

then $TA = E_\alpha - F_\alpha$ and $P_\alpha = E_\alpha^{-1}F_\alpha$. Therefore, it is easy to see that the MSNS iteration method can be induced by the matrix splitting $TA = E_\alpha - F_\alpha$. The next theorem is stated for the convergence of the method.

Theorem 1. *Let W be a real symmetric indefinite matrix, T be a real symmetric positive definite matrix and $\alpha > 0$. Then, the spectral radius $\rho(P_\alpha)$ of the MSNS iteration matrix is bounded by*

$$\eta(\alpha) \equiv \max_{\mu_j \in \sigma(T)} \left| \frac{\alpha - \mu_j}{\alpha + \mu_j} \right|, \tag{7}$$

where $\sigma(T)$ is the spectrum of the matrix T. Therefore, it follows from (7) that $\rho(P_\alpha) \leq \eta(\alpha) < 1$ for all $\alpha > 0$. That is, the MSNS iteration method converges unconditionally to the unique solution of the system (1).

Proof. Clearly

$$\begin{aligned}
\rho(P_\alpha) &= \rho\left((\alpha I - T)(\alpha I + T)^{-1}(i\alpha W + T^2)(i\alpha W - T^2)^{-1}\right) \\
&= \rho\left((\alpha I - T)(\alpha I + T)^{-1} T (i\alpha T^{-1} W T^{-1} + I) T T^{-1} (i\alpha T^{-1} W T^{-1} - I)^{-1} T^{-1}\right) \\
&= \rho\left(T^{-1}(\alpha I - T)(\alpha I + T)^{-1} T (i\alpha T^{-1} W T^{-1} + I)(i\alpha T^{-1} W T^{-1} - I)^{-1}\right) \\
&= \rho\left((\alpha T^{-1} - I)(\alpha T^{-1} + I)^{-1}(i\alpha T^{-1} W T^{-1} + I)(i\alpha T^{-1} W T^{-1} - I)^{-1}\right) \\
&\leq \|(\alpha T^{-1} - I)(\alpha T^{-1} + I)^{-1}(i\alpha T^{-1} W T^{-1} + I)(i\alpha T^{-1} W T^{-1} - I)^{-1}\|_2 \\
&\leq \|(\alpha T^{-1} - I)(\alpha T^{-1} + I)^{-1}\|_2 \|(i\alpha T^{-1} W T^{-1} + I)(i\alpha T^{-1} W T^{-1} - I)^{-1}\|_2.
\end{aligned}$$

Let $U_\alpha = (i\alpha T^{-1} W T^{-1} + I)(i\alpha T^{-1} W T^{-1} - I)^{-1}$. Since the matrix $iT^{-1} W T^{-1}$ is skew-Hermitian, we conclude that the matrix U_α is a Cayley transform and as a result is unitary (see [11,15]). Therefore, the matrix U_α is unitary and we have $\|U_\alpha\|_2 = 1$. On the other hand, the matrix $V_\alpha = (\alpha T^{-1} - I)(\alpha T^{-1} + I)^{-1}$ is symmetric and we deduce that

$$\|V_\alpha\|_2 = \rho\left((\alpha T^{-1} - I)(\alpha T^{-1} + I)^{-1}\right) = \max_{\mu_j \in \sigma(T)} \left| \frac{\frac{\alpha}{\mu_j} - 1}{\frac{\alpha}{\mu_j} + 1} \right| = \max_{\mu_j \in \sigma(T)} \left| \frac{\alpha - \mu_j}{\alpha + \mu_j} \right|.$$

Summarizing the above results we see that

$$\rho(P_\alpha) \leq \|U_\alpha\|_2 \|V_\alpha\|_2 = \max_{\mu_j \in \sigma(T)} \left| \frac{\alpha - \mu_j}{\alpha + \mu_j} \right| < 1,$$

which completes the proof. □

Corollary 2. *Let W be a symmetric indefinite matrix and T be a symmetric positive definite. Let also $\mu_{\max}$ and $\mu_{\min}$ be the largest and smallest eigenvalues of T, respectively. Then,*

$$\alpha^* = \arg\min_{\alpha} \max_{\mu_j \in \sigma(T)} \left| \frac{\alpha - \mu_j}{\alpha + \mu_j} \right| = \sqrt{\mu_{\min} \mu_{\max}},$$

and

$$\eta(\alpha^*) = \frac{\sqrt{\kappa(T)} - 1}{\sqrt{\kappa(T)} + 1},$$

where $\kappa(T)$ is the spectral condition number of T.

Proof. The proof is similar to Corollary 2.3 in [8] and is omitted here. □

Similar to the simplified SNS method we can obtain the simplified version of the MSNS (hereafter is denoted by SMSNS). To do so, the linear system (1) is rewritten in the following two equivalent forms:

$$(iT+\frac{i}{\alpha}T^2)x=(-W+\frac{i}{\alpha}T^2)x+b,$$
$$(W+\frac{i}{\alpha}T^2)x=(-iT+\frac{i}{\alpha}T^2)x+b.$$

Simple computation reveals that

$$\begin{cases}(\alpha I+T)Tx=(i\alpha W+T^2)x-i\alpha b,\\ (i\alpha W-T^2)x=(\alpha I+T)Tx+i\alpha b.\end{cases} \tag{8}$$

From (8), the SMSNS iteration method can be written as

$$\begin{cases}(\alpha I+T)x^{(k+\frac{1}{2})}=(i\alpha W+T^2)x^{(k)}-i\alpha b,\\ (i\alpha W-T^2)x^{(k+1)}=(\alpha I+T)x^{(k+\frac{1}{2})}+i\alpha b.\end{cases}$$

3. Numerical experiments

In this section, the numerical results of the MSNS iteration method are compared with those of the HNS iteration method. The Hermitian positive definite and non-Hermitian subsystems are solved by the Cholesky and LU factorizations, respectively. A zero vector is always used as an initial guess and the iteration is stopped as the residual norm is reduced by a factor of 10^5. Also, the numerical results of GMRES(ℓ) in conjunction with the preconditioners B_α and E_α for solving, respectively, the systems $WAx=Wb$ and $TAx=Tb$, are presented. In the implementation of the preconditioners the sub-systems with symmetric positive definite coefficient matrices are solved by the conjugate gradient (CG) method and the other subsystems are solved by the GMRES(10) method. For all the iterations methods (inner and outer), the value of restart is set to 10 and the maximum number of iterations is set to 500. The outer iteration is stopped as soon as the residual norm of the preconditioned system is reduced by a factor of 10^5 and the inner iterations are terminated as soon as the residual norm is reduced by a factor of 10^3. We always use a zero vector as an initial guess. In the tables, a dagger (†) shows that the method has not converged in 2000 iterations.

Now, we consider the complex symmetric linear system of equations (see [1,2])

$$[(-\omega^2M+K)+i(\omega C_V+C_H)]x=b, \tag{9}$$

where M and K are the inertia and stiffness matrices, respectively; C_V and C_H are the viscous and hysteretic damping matrices, respectively; and ω is the driving circular frequency. In our numerical experiments, we set $C_H=\mu K$ with $\mu=0.02$ and K the five-point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0,1]\times[0,1]$ with the mesh size $h=\frac{1}{m+1}$. In this case, we have $K=I\otimes V_m+V_m\otimes I\in\mathbb{R}^{n\times n}$ with $V_m=-h^2\mathrm{tridiag}(-1,2,-1)\in\mathbb{R}^{m\times m}$. Hence, the total number of variables is $n=m^2$. In addition, the right-hand side vector b is to be adjusted such that $b=(1+i)Ae$ where $e=(1,1,\dots,1)^T$.

In Table 1–3 we present the numerical results. To see that the matrices T and W are, respectively, SPD and symmetric indefinite, we have presented the smallest eigenvalues of these matrices in tables. In all the tables the value of ω is set to 4π. Numerical results of the problem for $C_V=0.7M$, $C_V=0.8M$ and $C_V=0.9M$ with different matrices M are presented in Tables 1–3, respectively. In the tables the optimal value of the parameter α (α_{opt}), the CPU time (in seconds) for the convergence (CPU), the number of iterations (Iters) and $R_k=\|r_k\|_2/\|b\|_2$, where r_k is the residual vector of the computed solution. Let "$It=(i_1,i_2)$" be the number of restarts returned by the GMRES(ℓ) method. In this case, the value of Iters of GMRES(ℓ)

Table 1
Numerical results for $n = 1024$ and $C_V = 0.7M$.

Method	M	I	$\frac{6}{5}I$	$\frac{7}{5}I$	$\frac{8}{5}I$	$\frac{9}{5}I$
	$\lambda_{\min}(T)$	0.0084	0.0101	0.0117	0.0133	0.0149
	$\lambda_{\min}(W)$	−0.1269	−0.1559	−0.1849	−0.2139	−0.2429
MSNS	α_{opt}	0.03	0.034	0.036	0.038	0.04
	Iter	20	18	17	16	15
	CPU	0.04	0.04	0.03	0.03	0.03
	R_k	6.85e−06	8.47e−06	7.52e−06	7.74e−06	8.75e−06
HNS	α_{opt}	3.2	2.1	3.97	—	3.62
	Iter	408	605	312	†	321
	CPU	0.36	0.51	0.28	—	0.28
	R_k	9.93e−06	9.95e−06	9.94e−06	—	9.87e−06
MSNS-GMRES(10)	α_{opt}	0.0035	0.0046	0.0055	0.0078	0.0079
	Iters	7	7	7	7	7
	CPU	0.55	0.50	0.48	0.50	0.44
	R_k	1.83e−03	2.24e−03	2.05e−03	1.69e−03	1.84e−03
HNS-GMRES(10)	α_{opt}	0.11	0.2	0.33	0.3	0.37
	Iters	28	28	29	39	34
	CPU	1.00	0.88	0.76	0.95	0.84
	R_k	4.13e−03	2.90e−03	2.49e−03	2.24e−03	1.57e−03

Table 2
Numerical results for $n = 1024$ and $C_V = 0.8M$.

Method	M	I	$\frac{6}{5}I$	$\frac{7}{5}I$	$\frac{8}{5}I$	$\frac{9}{5}I$
	$\lambda_{\min}(T)$	0.0096	0.0114	0.0133	0.0151	0.0170
	$\lambda_{\min}(W)$	−0.1269	−0.1559	−0.1849	−0.2139	−0.2429
MSNS	α_{opt}	0.033	0.036	0.038	0.041	0.044
	Iter	18	17	16	15	14
	CPU	0.04	0.04	0.03	0.03	0.03
	R_k	9.55e−06	7.83e−06	7.62e−06	7.79e−06	9.05e−06
HNS	α_{opt}	3	1.97	3.7	—	3.4
	Iter	427	636	326	†	336
	CPU	0.37	0.64	0.29	—	0.31
	R_k	9.96e−06	9.98e−06	9.93e−06	—	9.97e−06
MSNS-GMRES(10)	α_{opt}	0.0037	0.005	0.0062	0.0087	0.013
	Iters	7	7	7	7	6
	CPU	0.49	0.46	0.44	0.47	0.50
	R_k	1.71e−03	1.78e−03	2.63e−03	1.93e−03	1.31e−03
HNS-GMRES(10)	α_{opt}	0.07	0.22	0.3	0.44	0.25
	Iters	27	28	29	55	35
	CPU	1.01	0.86	0.77	1.19	0.95
	R_k	7.05e−03	2.37e−03	2.46e−03	1.86e−03	1.83e−03

presented in the tables are computed via $Iters = \ell \times (i_1 - 1) + i_2$. As we see, the MSNS method outperforms the HNS method as both an iterative solver and as a preconditioner for GMRES. Another observation which can be posed here that the optimal value of the parameter α in the MSNS method is rather small. This can be explained as follows. In the second half-step of the MSNS method a linear system with the coefficient matrix $T^2 - i\alpha W$ should be solved. Since the matrix T^2 is SPD, for small values of α this term dominates over the term $i\alpha W$. On the other hand, the value of α cannot be chosen very small, because it affects the condition number of the coefficient matrix of the system of the first half-step of the MSNS method.

4. Conclusion

We have presented a modified version of the skew-normal splitting (SNS) called MSNS for solving the complex symmetric system of linear equations $(W + iT)x = b$, where W is indefinite and T is positive definite. Convergence of the method has been investigated. Our numerical results show that the MSNS method outperforms the HNS method, both as an iterative solver and as a preconditioner.

Table 3
Numerical results for $n = 1024$ and $C_V = 0.9M$.

Method	M	I	$\frac{6}{5}I$	$\frac{7}{5}I$	$\frac{8}{5}I$	$\frac{9}{5}I$
	$\lambda_{\min}(T)$	0.0107	0.0128	0.0149	0.0170	0.0191
	$\lambda_{\min}(W)$	−0.1269	−0.1559	−0.1849	−0.2139	−0.2429
MSNS	α_{opt}	0.035	0.038	0.041	0.044	0.047
	Iter	17	16	15	14	14
	CPU	0.04	0.03	0.03	0.03	0.03
	R_k	9.70e−06	8.25e−06	8.04e−06	8.81e−06	5.21e−06
HNS	α_{opt}	2.81	1.85	3.5	—	3.24
	Iter	446	666	340	†	351
	CPU	0.38	0.56	0.29	—	0.30
	R_k	9.96e−06	9.96e−06	9.82e−06	—	9.79e−06
MSNS-GMRES(10)	α_{opt}	0.004	0.0056	0.0076	0.0096	0.015
	Iters	7	7	7	7	6
	CPU	0.44	0.44	0.44	0.45	0.49
	R_k	1.95e−03	1.43e−03	1.76e−03	2.26e−03	1.29e−03
HNS-GMRES(10)	α_{opt}	0.11	0.23	0.3	0.44	0.37
	IT	28	28	29	55	35
	CPU	0.99	0.86	0.77	1.19	0.86
	R_k	4.25e−03	2.01e−03	2.66e−03	1.69e−03	1.01e−03

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