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OPTIMIZED SCHWARZ METHODS*

MARTIN J. GANDER†

Abstract. Optimized Schwarz methods are a new class of Schwarz methods with greatly enhanced convergence properties. They converge uniformly faster than classical Schwarz methods and their convergence rates dare asymptotically much better than the convergence rates of classical Schwarz methods if the overlap is of the order of the mesh parameter, which is often the case in practical applications. They achieve this performance by using new transmission conditions between subdomains which greatly enhance the information exchange between subdomains and are motivated by the physics of the underlying problem. We analyze in this paper these new methods for symmetric positive definite problems and show their relation to other modern domain decomposition methods like the new Finite Element Tearing and Interconnect (FETI) variants.

Key words. optimized Schwarz methods, optimized transmission conditions, domain decomposition, parallel preconditioning

AMS subject classifications. 65N55, 65F10, 65N22.

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1. Introduction. The convergence properties of the classical Schwarz methods are well understood for a wide variety of problems; see, for example, the books [37], [35] or the survey articles [4], [40], [41] and references therein. Over the last decade, people have looked at different transmission conditions for the classical Schwarz method. There were three main motivations: the first motivation for different transmission conditions came from the nonoverlapping variant of the Schwarz method proposed by Lions, since without overlap the classical Schwarz method does not converge. Lions proposed to use Robin conditions to obtain a convergent algorithm in [31]. At the end in his paper, we find the following remark: "First of all, it is possible to replace the constants in the Robin conditions by two proportional functions on the interface, or even by local or nonlocal operators." Lions then gives a simple example in one dimension and shows that the optimal choice for the parameters in the Robin transmission conditions of the algorithm are constants in that case. In higher dimensions, however, the optimal choice involves a nonlocal transmission operator, as was shown for a two-dimensional convection diffusion problem by Charton, Nataf, and Rogier in [5], where a parabolic factorization of the operator was used to derive the optimal transmission conditions. Since nonlocal operators are not convenient to implement and costly ("ils se prêtent peu au calcul numérique" [5]), the authors propose for the convection dominated convection-diffusion problem to expand the symbols of the nonlocal operators in the small viscosity parameter to obtain local approximations. A different approximation using a Taylor expansion in the frequency parameter to obtain local transmission conditions for the convection diffusion equation is proposed in [33]; see also [34] and [32]. For symmetric coercive problems, a formulation of the nonoverlapping Schwarz method with Robin transmission conditions which avoids the explicit use of normal derivatives was introduced independently in [7], and convergence of the resulting algorithm was proved using energy estimates. A first optimization of the

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http://www.siam.org/journals/sinum/44-2/42540.html
†Section de Mathématiques, Université de Genève, 2-4 rue du Lièvre, CP 64, CH-1211 Genève, Switzlerland (Martin.Gander@math.unige.ch).

transmission conditions for the performance of the algorithm was done by Japhet in [26] for convection diffusion problems, where one coefficient in a second order transmission condition was optimized, which led to the first optimized Schwarz method in this context. This approach was further developped and refined in [29], [27], and [28] for convection diffusion problems.

The second motivation for changing the transmission conditions came from acoustics. For problems of Helmholtz type, the classical Schwarz algorithm is not convergent, even with overlap. Després therefore proposed in [8] to use radiation conditions instead for the Helmholtz equation and proved convergence of a nonoverlapping variant of the Schwarz algorithm with these transmission conditions using energy estimates; see also [9]. The radiation conditions used were again Robin conditions, and the same conditions were also used in an overlapping context in [3]. Higher order local transmission conditions for the Helmholtz equation were introduced in [6] and a first attempt was made to optimize the free parameter in the transmission conditions for the performance of the algorithm, leading to the first optimized Schwarz method without overlap for the Helmholtz equation. The optimization problem for the Robin transmission conditions was then completely solved for this case in [22] and a simple strategy to optimize the second order transmission conditions was also presented. For a complete optimization of the second order transmission conditions for Helmholtz problems, see [14] for the case without overlap and [21] for the case with overlap.

The third motivation was that the convergence rate of the classical Schwarz method is rather slow and very much dependent on the size of the overlap. In a short note on nonlinear problems [24], Hagstrom, Tewarson, and Jazcilevich introduced Robin transmission conditions between subdomains and suggested, "Indeed, we advocate the use of nonlocal conditions." Later and independently, Tang introduced in [39] the generalized Schwarz alternating method, which uses a weighted average of Dirichlet and Neumann conditions at the interfaces, which is equivalent to a Robin condition. Numerically, optimal values for the weighting parameter were determined, and it was shown that a good choice of the parameter leads to a significant speedup of the algorithm. The main difficulty remaining in this approach is the determination of these parameters on the interfaces, like for successive overrelaxation (SOR) methods. Even stronger coupling was proposed in [38], where the authors introduced the overdetermined Schwarz algorithm, which enforces the coupling not only on the interfaces but also in the overlap itself, in so-called artificial boundary layers, and the relaxation parameter is now a function depending on space, as proposed earlier by Lions [31]. But the link with absorbing boundary conditions was only made later in [10], where an overlapping version of the Schwarz algorithm for Laplace's equation was analyzed with Robin and second order transmission conditions and a first attempt was made to determine asymptotically optimal parameters. In the waveform relaxation community, a link made with Schwarz methods in [23] opened up the way for better transmission conditions in the Schwarz waveform relaxation algorithms; see [19]. This led to optimized Schwarz algorithms for evolution problems, where one can clearly see that the optimal transmission conditions are absorbing boundary conditions. For the case of the wave equation with discontinuous coefficients, a nonoverlapping optimized Schwarz method is introduced and analyzed in detail at both the continuous and the discrete level in [20]. For the heat equation, see [17].

Optimized Schwarz methods have several key features:

1. They converge necessarily faster than classical Schwarz methods, at the same cost per iteration.

2. There are simple optimization procedures to determine the best parameters to be used in the transmission conditions, sometimes even closed formulas, depending on the problem solved.
3. Classical Schwarz implementations need only a small change in the implementation, in the information exchange routine, to benefit from the additional performance.
4. Optimized Schwarz methods can be used with or without overlap.

We present here a complete analysis of optimized Schwarz methods for a symmetric positive definite model problem and analyze in detail the optimization problems and the asymptotic performance of different approximations to the optimal transmission conditions. We restrict our analysis to the simple case of two subdomains, because optimized Schwarz method are greatly enhancing the local coupling between subdomains. Once optimized coupling conditions are found, they can be used in the general context of many subdomains, as we show with numerical examples at the end (see also [22]). As for classical Schwarz methods, a coarse grid is necessary as soon as many subdomains are used, if a convergence rate independent of the number of subdomains is desired, but we do not consider this issue here.

2. The classical Schwarz algorithm for a model problem. We use throughout the paper the model problem

$$\mathcal{L}(u) = (\eta - \Delta)(u) = f \quad \text{on } \Omega = \mathbb{R}^2,\ \eta > 0, \tag{2.1}$$

where we require the solution to decay at infinity. To introduce the ideas behind optimized Schwarz methods, we start by analyzing a parallel variant of the classical alternating Schwarz method introduced by Lions [30], applied to the model problem (2.1). We decompose the domain Ω into the two overlapping subdomains

$$\Omega_1 = (-\infty, L) \times \mathbb{R}, \quad \Omega_2 = (0, \infty) \times \mathbb{R}. \tag{2.2}$$

The Jacobi–Schwarz method for the two subdomains and the model problem is then given by

$$\begin{array}{llll} (\eta - \Delta)u_1^n = f & \text{in } \Omega_1, & (\eta - \Delta)u_2^n = f & \text{in } \Omega_2, \\ u_1^n(L, y) = u_2^{n-1}(L, y), & y \in \mathbb{R}, & u_2^n(0, y) = u_1^{n-1}(0, y), & y \in \mathbb{R}, \end{array} \tag{2.3}$$

and we require the iterates to decay at infinity. By linearity it suffices to consider only the case $f = 0$ and analyze convergence to the zero solution. Our analysis is based on the Fourier transform,

$$\hat{f}(k) = \mathcal{F}(f) := \int_{-\infty}^{\infty} e^{-ikx} f(x)dx, \quad f(x) = \mathcal{F}^{-1}(\hat{f}) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{f}(k)dk, \quad k \in \mathbb{R}. \tag{2.4}$$

Taking a Fourier transform of the Schwarz algorithm (2.3) in the y direction, and using the property of the Fourier transform that derivatives in y become multiplications by ik, we obtain

$$\begin{array}{ll} (\eta + k^2 - \partial_{xx})\hat{u}_1^n = 0, \quad x < L,\ k \in \mathbb{R}, & (\eta + k^2 - \partial_{xx})\hat{u}_2^n = 0, \quad x > 0,\ k \in \mathbb{R}, \\ \hat{u}_1^n(L, k) = \hat{u}_2^{n-1}(L, k),\ k \in \mathbb{R}, & \hat{u}_2^n(0, k) = \hat{u}_1^{n-1}(0, k),\ k \in \mathbb{R}. \end{array} \tag{2.5}$$

Hence subdomain solutions in the Fourier transformed domain are of the form

$$\hat{u}_j^n(x, k) = A_j(k)e^{\lambda_1(k)x} + B_j(k)e^{\lambda_2(k)x}, \quad j = 1, 2, \tag{2.6}$$

where the $\lambda_j(k)$, $j = 1, 2$ satisfy the characteristic equation $\eta - \lambda_j^2 + k^2 = 0$ and hence $\lambda_1(k) = \sqrt{k^2+\eta}$ and $\lambda_2(k) = -\sqrt{k^2+\eta}$. By the condition on the iterates at infinity, we obtain for the subdomain solutions

$$\hat{u}_1^n(x,k) = \hat{u}_2^{n-1}(L,k)e^{\sqrt{k^2+\eta}\,(x-L)}, \qquad \hat{u}_2^n(x,k) = \hat{u}_1^{n-1}(0,k)e^{-\sqrt{k^2+\eta}\,x}.$$

Inserting these solutions into algorithm (2.5), we obtain by induction

$$\hat{u}_1^{2n}(0,k) = \rho_{cla}^n \hat{u}_1^0(0,k), \qquad \hat{u}_2^{2n}(L,k) = \rho_{cla}^n \hat{u}_2^0(L,k), \tag{2.7}$$

where the convergence factor $\rho_{cla}(k, \eta, L)$ of the classical Schwarz algorithm is given by

$$\rho_{cla} = \rho_{cla}(k, L, \eta) := e^{-2\sqrt{k^2+\eta}L} < 1 \quad \forall k \in \mathbb{R}. \tag{2.8}$$

Note that we have chosen here to define the convergence factor over two iterations, which would correspond to one iteration of the Gauss–Seidel–Schwarz method in this two-subdomain case. From (2.8), we see that the iterates converge to zero on the line $x = 0$ and $x = L$. Since with zero boundary conditions the solution vanishes identically, we have shown that the classical Schwarz method converges for all frequencies, provided $\eta > 0$. The convergence factor depends on the problem parameter η, the size of the overlap L, and the frequency parameter k: the top curve in Figure 4.1 shows the dependence of ρ_{cla} on k for an overlap $L = \frac{1}{100}$ and $\eta = 1$. One can see that the Schwarz algorithm is a smoother; it damps high frequencies effectively, whereas for low frequencies the convergence factor is close to one and hence the algorithm is very slow.

3. The optimal Schwarz algorithm.

We now introduce the key modification in the classical Schwarz method: new transmission conditions between the subdomains. The new algorithm is given by

$$\begin{array}{ll} (\eta - \Delta)u_1^n = f \quad \text{in } \Omega_1, & (\eta - \Delta)u_2^n = f \quad \text{in } \Omega_2, \\ (\partial_x + \mathcal{S}_1)(u_1^n)(L,\cdot) = (\partial_x + \mathcal{S}_1)(u_2^{n-1})(L,\cdot), & (\partial_x + \mathcal{S}_2)(u_2^n)(0,\cdot) = (\partial_x + \mathcal{S}_2)(u_1^{n-1})(0,\cdot), \end{array} \tag{3.1}$$

where $\mathcal{S}_j$, $j = 1, 2$, are linear operators along the interface in the y direction which we will determine in what follows to get the best possible performance of the new Schwarz algorithm. As for the classical Schwarz method, taking a Fourier transform in the y direction for $f = 0$, we obtain

$$\begin{array}{c} \eta\hat{u}_1^n - \partial_{xx}\hat{u}_1^n + k^2\hat{u}_1^n = 0, \quad x < L, \ k \in \mathbb{R}, \\ (\partial_x + \sigma_1(k))(\hat{u}_1^n)(L,k) = (\partial_x + \sigma_1(k))(\hat{u}_2^{n-1})(L,k), \quad k \in \mathbb{R}, \end{array} \tag{3.2}$$

where $\sigma_1(k)$ denotes the symbol of the operator $\mathcal{S}_1$, and

$$\begin{array}{c} \eta\hat{u}_2^n - \partial_{xx}\hat{u}_2^n + k^2\hat{u}_2^n = 0, \quad x > 0, \ k \in \mathbb{R}, \\ (\partial_x + \sigma_2(k))(\hat{u}_2^n)(0,k) = (\partial_x + \sigma_2(k))(\hat{u}_1^{n-1})(0,k), \quad k \in \mathbb{R}, \end{array} \tag{3.3}$$

where $\sigma_2(k)$ is the symbol of $\mathcal{S}_2$. The solutions on the subdomains are again of the form (2.6), and using the condition on the iterates at infinity, the transmission conditions, and the fact that

$$\frac{\partial \hat{u}_1^n}{\partial x} = \sqrt{k^2+\eta}\,\hat{u}_1^n, \qquad \frac{\partial \hat{u}_2^n}{\partial x} = -\sqrt{k^2+\eta}\,\hat{u}_2^n,$$

we find the subdomain solution in Fourier space to be

$$\hat{u}_1^n(x,k) = \frac{\sigma_1(k) - \sqrt{k^2+\eta}}{\sigma_1(k) + \sqrt{k^2+\eta}} e^{\sqrt{k^2+\eta}(x-L)} \hat{u}_2^{n-1}(L,k),$$
$$\hat{u}_2^n(x,k) = \frac{\sigma_2(k) + \sqrt{k^2+\eta}}{\sigma_2(k) - \sqrt{k^2+\eta}} e^{-\sqrt{k^2+\eta}\,x} \hat{u}_1^{n-1}(0,k).$$

Inserting these solutions into algorithm (3.1), we obtain by induction

$$\hat{u}_1^{2n}(0,k) = \rho_{opt}^n \hat{u}_1^0(0,k), \qquad \hat{u}_2^{2n}(L,k) = \rho_{opt}^n \hat{u}_2^0(L,k), \tag{3.4}$$

where the new convergence factor ρ_{opt} is given by

$$\rho_{opt} = \rho_{opt}(k,L,\eta,\sigma_1,\sigma_2) := \frac{\sigma_1(k) - \sqrt{k^2+\eta}}{\sigma_1(k) + \sqrt{k^2+\eta}} \cdot \frac{\sigma_2(k) + \sqrt{k^2+\eta}}{\sigma_2(k) - \sqrt{k^2+\eta}} e^{-2\sqrt{k^2+\eta}L}. \tag{3.5}$$

The only difference between the new convergence factor ρ_{opt} and the one of the classical Schwarz method, ρ_{cla} given in (2.8), is the factor in front of the exponential. But this factor has a tremendous influence on the performance of the method: choosing for the symbols

$$\sigma_1(k) := \sqrt{k^2+\eta}, \quad \sigma_2(k) := -\sqrt{k^2+\eta}, \tag{3.6}$$

the new convergence factor vanishes identically, $\rho_{opt} \equiv 0$, and the algorithm converges in two iterations, independently of the initial guess, the overlap L, and the problem parameter η. This is an optimal result, since the solution in one subdomain depends on the forcing function f in the other subdomain and hence a first solve is necessary to incorporate the influence of f into the subdomain solution, then one information exchange is performed to give this information to the neighboring subdomain and a second solve on the subdomains incorporates this information into the new subdomain solution. Convergence in less than two steps is not possible. One can also see from (3.6) that the optimal choice depends on the problem. The optimal convergence result for two subdomains in two iterations can be generalized to $J > 2$ subdomains and convergence in J iterations (see, for example, [33] or [16]), provided the subdomains are arranged in a sequence. In addition, with this choice of σ_j, the exponential factor in the convergence factor becomes irrelevant and we can have Schwarz methods without overlap.

To use the optimal choice of σ_j in practice, we need to back-transform the transmission conditions involving σ_1 and σ_2 from the Fourier domain into the physical domain to obtain the transmission operators $\mathcal{S}_1$ and $\mathcal{S}_2$. Hence we need

$$\mathcal{S}_1(u_1^n) = \mathcal{F}_k^{-1}(\sigma_1 \hat{u}_1^n), \qquad \mathcal{S}_2(u_2^n) = \mathcal{F}_k^{-1}(\sigma_2 \hat{u}_2^n), \tag{3.7}$$

and thus for the optimal choice of σ_j we have to evaluate a convolution in each step of the algorithm, because the σ_j contain a square root and thus the optimal $\mathcal{S}_j$ are nonlocal operators, as advocated in [24]. If the symbols σ_j were, however, polynomials in ik, then the operators $\mathcal{S}_j$ would consist of derivatives in y and thus be local operators. We will therefore approximate the optimal choice of σ_j by polynomials in the following sections, which leads to the new class of optimized Schwarz methods.

4. Optimized Schwarz algorithms. We approximate the symbols of the optimal transmission conditions found in (3.6) by polynomial symbols in ik which corresponds to local approximations. We choose polynomials of degree two here,

$$\sigma_1^{app}(k) = p_1 + q_1 k^2, \quad \sigma_2^{app}(k) = -p_2 - q_2 k^2. \tag{4.1}$$

Note that we do not consider a first order term, because the operator of the underlying problem is self-adjoint. Higher order approximations would be possible as well, as long as the subdomain problems remain well posed. With the approximation (4.1), the convergence factor of the optimized Schwarz algorithm becomes

$$\rho = \rho(k, L, \eta, p_1, p_2, q_1, q_2) = \frac{\sqrt{k^2+\eta} - p_1 - q_1 k^2}{\sqrt{k^2+\eta} + p_1 + q_1 k^2} \cdot \frac{\sqrt{k^2+\eta} - p_2 - q_2 k^2}{\sqrt{k^2+\eta} + p_2 + q_2 k^2} e^{-2\sqrt{k^2+\eta}L}. \tag{4.2}$$

THEOREM 4.1. *The optimized Schwarz method* (3.1) *with transmission conditions defined by the symbols* (4.1) *converges for* $p_j > 0$, $q_j \geq 0$, $j = 1, 2$, *faster than the classical Schwarz method* (2.3), $|\rho_{opt}(k)| < |\rho_{cla}(k)|$ *for all* k.

Proof. The only difference between ρ_{cla} in (2.8) and ρ_{opt} in (4.2) is the additional factor in front of the exponential, which satisfies for $p_j > 0$ and $q_j \geq 0$

$$\left| \frac{\sqrt{k^2+\eta} - p_1 - q_1 k^2}{\sqrt{k^2+\eta} + p_1 + q_1 k^2} \cdot \frac{\sqrt{k^2+\eta} - p_2 - q_2 k^2}{\sqrt{k^2+\eta} + p_2 + q_2 k^2} \right| < 1 \quad \forall k,$$

and hence $|\rho(k)| < |\rho_{cla}(k)|$ for all k. ∎

The goal of optimized Schwarz methods is now to choose the free parameters $p_j, q_j \geq 0$ for $j = 1, 2$ to further improve the performance of the method.

4.1. Low-frequency approximations. As we have seen, the classical Schwarz method is effective, due to the overlap, for high frequencies but ineffective for low frequencies. The low frequencies can, however, be treated in the new Schwarz algorithm with the transmission conditions: expanding the symbols $\sigma_j(k)$ of the optimal operators $\mathcal{S}_j$ in a Taylor series, we find

$$\sigma_1(k) = \sqrt{\eta} + \frac{1}{2\sqrt{\eta}} k^2 + O(k^4), \qquad \sigma_2(k) = -\sqrt{\eta} - \frac{1}{2\sqrt{\eta}} k^2 + O(k^4), \tag{4.3}$$

and hence a second order Taylor approximation would lead to the values $p_1 = p_2 = \sqrt{\eta}$, $q_1 = q_2 = \frac{1}{2\sqrt{\eta}}$, whereas a zeroth order approximation could be obtained by setting $q_1 = q_2 = 0$ for the same values of p_j. The corresponding optimized Schwarz methods have the convergence factors

$$\begin{aligned} \rho_{T0}(k, L, \eta) &= \left(\frac{\sqrt{k^2+\eta} - \sqrt{\eta}}{\sqrt{k^2+\eta} + \sqrt{\eta}} \right)^2 e^{-2\sqrt{k^2+\eta}L}, \\ \rho_{T2}(k, L, \eta) &= \left(\frac{\sqrt{k^2+\eta} - \sqrt{\eta} - \frac{1}{2\sqrt{\eta}} k^2}{\sqrt{k^2+\eta} + \sqrt{\eta} + \frac{1}{2\sqrt{\eta}} k^2} \right)^2 e^{-2\sqrt{k^2+\eta}L}, \end{aligned} \tag{4.4}$$

where we used the index $T0$ to denote a Taylor approximation of order zero and $T2$ to denote a Taylor approximation of order two of the optimal symbol in the transmission condition. Figure 4.1 shows on the left the convergence factors obtained with this

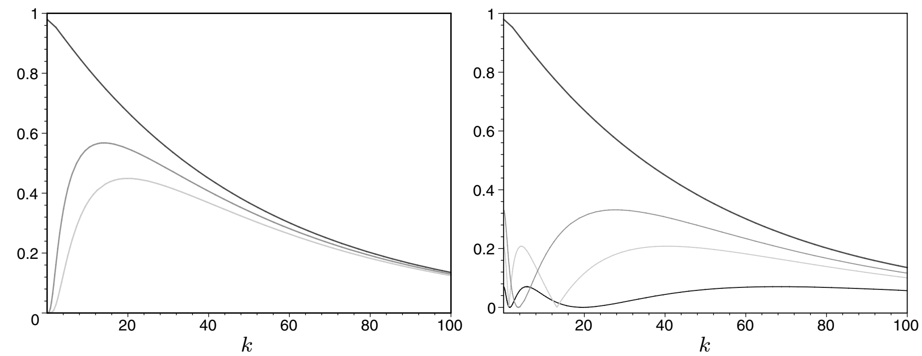


FIG. 4.1. *Convergence factor ρ_{cla} of the classical Schwarz method (top curve) as a function of k, compared on the left to ρ_{T0} (middle curve) and ρ_{T2} (bottom curve) of the optimized Schwarz methods with zeroth and second order transmission conditions, respectively, obtained by Taylor expansion, and on the right compared to the $OO0$ and $OO2$ Schwarz methods, and the optimized Schwarz method with two-sided optimized Robin transmission conditions, which lies in between $OO0$ and $OO2$.*

choice of transmission conditions for the model problem with two subdomains, overlap $L = \frac{1}{100}$ and problem parameter $\eta = 1$, together with the classical convergence factor ρ_{cla}. First one can clearly see that the optimized Schwarz methods are uniformly better than the classical Schwarz method; in particular the low-frequency behavior is greatly improved. The maximum of the convergence factor of classical Schwarz is about 0.980, whereas the maximum of the convergence factor with zeroth order Taylor condition is 0.568 and the maximum with second order Taylor condition is 0.449 in this example. Hence the classical Schwarz method needs about 28 iterations to obtain the contraction factor of one iteration of the optimized Schwarz method with zeroth order Taylor conditions, and about 40 iterations are needed to obtain the contraction of one iteration of the optimized Schwarz method with second order transmission conditions from Taylor expansion.

As we mentioned earlier, the classical Schwarz method does not converge without overlap: for $L = 0$ we obtain $\rho_{cla}(k, 0, \eta) = 1$ and hence convergence is lost for all modes. Optimized Schwarz methods, however, can be used without overlap, and nonoverlapping Schwarz methods can be of great interest, if the physical properties in the subdomains differ, for example, when there are jumps in the coefficients of the equation as in [20] or the nature of the equations changes, like in the case of coupling of hyperbolic and parabolic problems; see, for example, [18] and references therein. If we set $L = 0$ in the convergence factor (4.4) of the optimized Schwarz method, the exponential term becomes one, but the factor in front remains unchanged, and thus $\rho_{T0}(k, 0, \eta) < 1$ and $\rho_{T2}(k, 0, \eta) < 1$ for all k. In a numerical implementation there is a maximum frequency which can be represented on a grid with grid spacing h. An estimate for this maximum frequency is $k_{\max} = \frac{\pi}{h}$. Hence the slowest convergence for the optimized Schwarz method without overlap and Taylor transmission conditions is obtained for the highest frequency: the method is a rougher as opposed to the smoother the classical Schwarz method is.

In practice, even when using the Schwarz method with overlap, the overlap is often only a few grid cells wide, and thus $L = O(h)$. In that case the convergence factor of the classical Schwarz method deteriorates as well as one refines the mesh and h goes to zero and we have the following comparison theorem.

THEOREM 4.2. *The optimized Schwarz methods with Taylor transmission conditions and overlap $L = h$ have an asymptotically superior performance than the* *classical Schwarz method with the same overlap. As h goes to zero, we have*

$$\max_{|k|\leq\frac{\pi}{h}} |\rho_{cla}(k,h,\eta)| = 1-2\sqrt{\eta}h+O(h^2), \tag{4.5}$$

$$\max_{|k|\leq\frac{\pi}{h}} |\rho_{T0}(k,h,\eta)| = 1-4\sqrt{2}\eta^{\frac{1}{4}}\sqrt{h}+O(h), \tag{4.6}$$

$$\max_{|k|\leq\frac{\pi}{h}} |\rho_{T2}(k,h,\eta)| = 1-8\eta^{\frac{1}{4}}\sqrt{h}+O(h). \tag{4.7}$$

Without overlap, the optimized Schwarz methods with Taylor transmission conditions are asymptotically comparable to the classical Schwarz method with overlap $L = h$. As h goes to zero, we have

$$\max_{|k|\leq\frac{\pi}{h}} |\rho_{T0}(k,0,\eta)| = 1-4\frac{\sqrt{\eta}}{\pi}h+O(h^2), \tag{4.8}$$

$$\max_{|k|\leq\frac{\pi}{h}} |\rho_{T2}(k,0,\eta)| = 1-8\frac{\sqrt{\eta}}{\pi}h+O(h^2). \tag{4.9}$$

Proof. For the second result it suffices to expand the convergence factors (4.4) for $L = 0$ at $k = k_{\max} = \frac{\pi}{h}$ for h small. Similarly for the classical Schwarz method one expands the convergence factor (2.8) with $L = h$ for h small at $k = 0$. For the optimized Schwarz methods with Taylor transmission conditions and overlap $L = h$, the convergence factors (4.4) attain their maximum in the interior, at

$$\bar{k}_{T0} = \frac{\sqrt{2}\eta^{\frac{1}{4}}}{\sqrt{L}} \quad \text{and} \quad \bar{k}_{T2} = \frac{2\eta^{\frac{1}{4}}}{\sqrt{L}},$$

respectively, as a direct computation shows. Hence with overlap $L = h$, these maxima are in the range of the computational frequencies, since they are smaller than $k_{\max} = \frac{\pi}{h}$ and thus relevant for the convergence factor. Expanding the corresponding convergence factor at these maxima for $L = h$ as h goes to zero leads to the results (4.8) and (4.9). ∎

Hence already for Taylor expansions of the optimal symbols in the transmission conditions the asymptotic performance of the new Schwarz method is better than the one of the classical Schwarz method when the overlap is of the order of the mesh parameter, which is often the case in applications. One can, however, also see that increasing the order of the Taylor approximation does not increase the asymptotic performance further—there is only an initial gain from h to $\sqrt{h}$. This changes with the approach described in the next subsection.

4.2. Uniformly optimized approximations. We now develop an even better choice for the transmission conditions: one can choose the parameters p_j and q_j to optimize the performance of the new Schwarz method, which means minimizing the convergence factor over all frequencies relevant to the problem. For the zeroth order transmission condition we have the min-max problem

$$\min_{p_j\geq 0}\left(\max_{k_{\min}\leq k\leq k_{\max}} \left|\frac{\sqrt{\eta+k^2}-p_1}{\sqrt{\eta+k^2}+p_1}\right|\left|\frac{\sqrt{\eta+k^2}-p_2}{\sqrt{\eta+k^2}+p_2}\right| e^{-2\sqrt{\eta+k^2}L}\right), \tag{4.10}$$

and for the second order optimized Schwarz method the min-max problem is

$$\min_{p_j,q_j\geq 0}\left(\max_{k_{\min}\leq k\leq k_{\max}} \left|\frac{\sqrt{\eta+k^2}-p_1-q_1k^2}{\sqrt{\eta+k^2}+p_1+q_1k^2}\right|\left|\frac{\sqrt{\eta+k^2}-p_2-q_2k^2}{\sqrt{\eta+k^2}+p_2+q_2k^2}\right| e^{-2\sqrt{\eta+k^2}L}\right), \tag{4.11}$$

where we have also introduced a lower bound $k_{\min}$ on the frequency range. This is useful for bounded subdomains: if, for example, the subdomains Ω_j were strips of width 1 in the y direction with homogeneous Dirichlet boundary conditions, then the lowest possible frequency on those domains would be $k_{\min} = \pi$, as one can see from a sine expansion. More general, if one uses a coarse grid, which is necessary as soon as one has many subdomains for good performance on elliptic problems, then the highest frequency representable on the coarse grid would be an estimate for $k_{\min}$, since the subdomain iteration does not need to be effective on the coarse grid frequencies.

Since the optimal transmission conditions (3.6) are of the same size with opposite signs, we first analyze the simpler optimization problems when the approximation of the optimal transmission conditions is also of the same size with opposite signs, which means $p_1 = p_2 = p$ and $q_1 = q_2 = q$. In subsection 4.3 we will analyze how much is lost in performance due to this simplifying assumption.

4.2.1. Zeroth order optimized transmission conditions. Using the same zeroth order transmission condition on both sides of the interface, $p_1 = p_2 = p$ and $q_1 = q_2 = 0$, the expression (4.2) of the convergence factor simplifies to

$$\rho_{OO0}(k, L, \eta, p) := \left(\frac{\sqrt{k^2+\eta} - p}{\sqrt{k^2+\eta} + p} \right)^2 e^{-2\sqrt{k^2+\eta}L}. \tag{4.12}$$

To determine the optimal parameter p of the associated optimized Schwarz method (which we call OO0 for "Optimized of Order 0"), we have to solve the min-max problem

$$\min_{p\geq 0}\left(\max_{k_{\min}\leq k\leq k_{\max}} |\rho_{OO0}(k, L, \eta, p)| \right) = \min_{p\geq 0}\left(\max_{k_{\min}\leq k\leq k_{\max}} \left(\frac{\sqrt{\eta+k^2} - p}{\sqrt{\eta+k^2} + p} \right)^2 e^{-2\sqrt{\eta+k^2}L} \right). \tag{4.13}$$

The following Lemma will be needed for several of the results on the min-max problems that arise in the optimization of the new Schwarz methods.

LEMMA 4.3. *Let $f(x,\gamma)$ be a continuously differentiable function, $f : [a,b] \times [c,d] \mapsto \mathbb{R}$, with a unique interior maximum in x at $x^*(\gamma) \in (a,b)$ for each $\gamma \in [c,d]$, $\frac{\partial f}{\partial x}(x^*(\gamma),\gamma) = 0$, and assume that $x^*(\gamma)$ is differentiable and $\frac{\partial f}{\partial \gamma} < 0$ for $x \in [a,b]$, $\gamma \in [c,d]$. Then*

$$\frac{df}{d\gamma}(x^*(\gamma),\gamma) < 0 \quad \forall \gamma \in [c,d].$$

Proof. Since $\frac{\partial f}{\partial x}(x^*(\gamma),\gamma) = 0$ for all $\gamma \in [c,d]$, we have

$$\frac{df}{d\gamma}(x^*(\gamma),\gamma) = \frac{\partial f}{\partial \gamma}(x^*(\gamma),\gamma) + \frac{\partial f}{\partial x}(x^*(\gamma),\gamma)\frac{\partial x^*}{\partial \gamma}(\gamma) = \frac{\partial f}{\partial \gamma}(x^*(\gamma),\gamma) < 0$$

by assumption on the partial derivative with respect to γ. ∎

THEOREM 4.4 (optimal Robin parameter). *For $L > 0$ and $k_{\max} = \infty$, the solution p^* of the min-max problem* (4.13) *is given by the unique root of the equation*

$$\rho_{OO0}(k_{\min}, L, \eta, p^*) = \rho_{OO0}(\bar{k}(p^*), L, \eta, p^*), \qquad \bar{k}(L,\eta,p) = \frac{\sqrt{L(2p + L(p^2-\eta))}}{L}. \tag{4.14}$$

For $L = 0$ and $k_{\max}$ finite, the optimal parameter p^* is given by

$$p^* = ((k_{\min}^2 + \eta)(k_{\max}^2 + \eta))^{\frac{1}{4}}. \tag{4.15}$$

Proof. The key idea is to use a transformation: the partial derivative of ρ_{OO0} with respect to p is

$$\frac{\partial \rho_{OO0}}{\partial p} = 4\frac{(p - \sqrt{k^2+\eta})\sqrt{k^2+\eta}e^{-2\sqrt{k^2+\eta}L}}{(\sqrt{k^2+\eta}+p)^3},$$

which shows that as long as $p < \sqrt{k_{\min}^2 + \eta}$, increasing p decreases ρ_{OO0} for all $k \in [k_{\min}, \infty)$. Hence one can restrict the range for p in the min-max problem to $p \geq \sqrt{k_{\min}^2 + \eta}$, the solution cannot lie outside this range. This implies that for the new range of p, ρ_{OO0} has a unique zero in $[k_{\min}, \infty)$, namely, at $k = k_1 = \sqrt{p^2 - \eta}$. We can thus transform the min-max problem into a new, equivalent one in the parameter k_1. Defining the function

$$R(k, L, \eta, k_1) := \frac{(\sqrt{k^2+\eta} - \sqrt{k_1^2+\eta})}{\sqrt{k^2+\eta} + \sqrt{k_1^2+\eta}}e^{-\sqrt{k^2+\eta}L}, \tag{4.16}$$

which is negative for $k \in [k_{\min}, k_1)$ and positive for $k > k_1$, the new min-max problem which is equivalent to (4.13) is

$$\min_{k_1 \geq k_{\min}} \left(\max_{k_{\min} \leq k \leq k_{\max}} |R(k, L, \eta, k_1)| \right).$$

Now in the case of overlap, $L > 0$, the derivative with respect to k,

$$\frac{\partial R}{\partial k} = \frac{ke^{-\sqrt{k^2+\eta}L}(2\sqrt{k_1^2+\eta} - Lk^2 + Lk_1^2)}{(\sqrt{k^2+\eta} + \sqrt{k_1^2+\eta})^2\sqrt{k^2+\eta}}$$

shows that the function has a maximum at

$$\bar{k} = \bar{k}(k_1) = \sqrt{\frac{2\sqrt{k_1^2+\eta}}{L} + k_1^2} > k_1.$$

Hence the maximum in the min-max problem can be attained either at $k = k_{\min}$ or at $k = \bar{k}$. Since

$$\frac{\partial R}{\partial k_1} = -2\frac{k_1 e^{-\sqrt{k^2+\eta}L}\sqrt{k^2+\eta}}{(\sqrt{k^2+\eta} + \sqrt{k_1^2+\eta})^2\sqrt{k_1^2+\eta}} < 0, \tag{4.17}$$

the function R decreases monotonically with k_1. For $k_1 = k_{\min}$ we have $0 = |R(k_{\min}, L, \eta, k_{\min})| < R(\bar{k}, L, \eta, k_{\min})$ and for k_1 large, we have $|R(k_{\min}, L, \eta, k_1)| > R(\bar{k}, L, \eta, k_1)$, since in the limit as k_1 goes to infinity, $R(\bar{k}(k_1), L, \eta, k_1)$ goes to zero. By continuity there exists at least one k_1^* such that $-R(k_{\min}, L, \eta, k_1^*) = R(\bar{k}, L, \eta, k_1^*)$. Using now that R decreases monotonically in k_1, we have that $|R(k_{\min}, L, \eta, k_1)|$ increases monotonically with k_1 and by Lemma 4.3 that $R(\bar{k}(k_1), L, \eta, k_1)$ decreases monotonically with k_1. Hence k_1^* is unique and therefore the unique solution of the min-max problem. Back-transforming to the p variable gives the first result of the theorem.

In the case without overlap, $L = 0$, the function R has no interior maximum, hence the maximum can be attained only on the boundary at either $k = k_{\min}$ or at $k = k_{\max}$. Since the sign of the derivative (4.17) remains the same for $L = 0$, the function R decreases monotonically with k_1. For $k_1 = k_{\min}$ we have $0 = |R(k_{\min}, 0, \eta, k_{\min})| < R(k_{\max}, 0, \eta, k_{\min})$ and for $k_1 = k_{\max}$, we have $|R(k_{\min}, L, \eta, k_{\max})| > R(k_{\max}, L, \eta, k_{\max}) = 0$. By continuity there exists at least one k_1^* such that

$$-R(k_{\min}, 0, \eta, k_1^*) = R(k_{\max}, 0, \eta, k_1^*) \tag{4.18}$$

and since R decreases monotonically in k_1, we have that $|R(k_{\min}, L, \eta, k_1)|$ increases monotonically with k_1 and $R(k_{\max}, L, \eta, k_1)$ decreases monotonically with k_1. Hence k_1^* is unique and thus the unique solution of the min-max problem. Solving (4.18) and back-transforming the result to the p variable leads then to the second result of the theorem. □

Figure 4.1 shows on the right the convergence factors obtained with the optimized Robin transmission condition for the model problem with overlap $L = \frac{1}{100}$ and $\eta = 1$, comparing the classical Schwarz method and the OO0 Schwarz method. The maximum of the convergence factor of the OO0 Schwarz method is 0.332, which means that about 55 iterations of the classical Schwarz method with convergence factor 0.980 are needed to attain the performance of the OO0 Schwarz method.

THEOREM 4.5 (Robin asymptotics). *The asymptotic performance of the new Schwarz method with optimized Robin transmission conditions and overlap $L = h$, as h goes to zero, is given by*

$$\max_{k_{\min} \le |k| \le \frac{\pi}{h}} |\rho_{OO0}(k, h, \eta, p^*)| = 1 - 4 \cdot 2^{\frac{1}{6}} (k_{\min}^2 + \eta)^{\frac{1}{6}} h^{\frac{1}{3}} + O(h^{\frac{2}{3}}). \tag{4.19}$$

The asymptotic performance without overlap, $L = 0$, is given by

$$\max_{k_{\min} \le |k| \le \frac{\pi}{h}} |\rho_{OO0}(k, 0, \eta, p^*)| = 1 - 4 \frac{(k_{\min}^2 + \eta)^{\frac{1}{4}}}{\sqrt{\pi}} \sqrt{h} + O(h). \tag{4.20}$$

Proof. For the first result, we need to find an asymptotic expansion for the optimal parameter p^* for small h from (4.14). We make the ansatz $p^* = Ch^\alpha$ for $\alpha < 0$, since we know from Theorem 4.4 that the optimal parameter is growing when h diminishes. Inserting this ansatz into (4.14) satisfied by p^* and expanding for small h, we find the leading order terms in the equation to be $4C\sqrt{k_{\min}^2 + \eta}\, h^\alpha - 4\sqrt{2} C^{\frac{5}{2}} h^{\frac{5}{2}\alpha + \frac{1}{2}}$. Since (4.14) holds for all h, this expression must vanish and hence both the exponents and the coefficients must match, which leads to

$$\alpha = -\frac{1}{3}, \quad C = \frac{(4(k_{\min}^2 + \eta))^{\frac{1}{3}}}{2}$$

and hence the optimal parameter p^* behaves asymptotically like

$$p^* = \frac{(4(k_{\min}^2 + \eta))^{\frac{1}{3}}}{2} h^{-\frac{1}{3}}. \tag{4.21}$$

With this asymptotic behavior of p^*, the interior maximum $\bar{k}$ behaves asymptotically like

$$\bar{k} = (4(k_{\min}^2 + \eta^2)^{\frac{1}{6}} h^{-\frac{2}{3}}, \tag{4.22}$$

which is less than $k_{\max} = \frac{\pi}{h}$ for h small and hence the optimal result given for $k_{\max} = \infty$ in (4.14) is indeed asymptotically the relevant one on the bounded frequency range $|k| \leq k_{\max} = \frac{\pi}{h}$ for $L = O(h)$. Now inserting the asymptotic value of the optimal parameter p^* from (4.21) into the convergence factor (4.12) and expanding at $k = k_{\min}$ leads to (4.19).

For the second result where $L = 0$, the optimal parameter p^* is known in closed form from (4.15) and hence it suffices to insert this p^* into the convergence factor (4.12), to set $k_{\max} = \frac{\pi}{h}$, and to expand the result at $k = k_{\min}$ in a series for small h to find (4.20). □

4.2.2. Second order optimized transmission conditions. Using the same second order transmission condition on both sides of the interface, $p_1 = p_2 = p$ and $q_1 = q_2 = q$, the expression (4.2) of the convergence factor simplifies to

$$\rho_{OO2}(k, L, \eta, p, q) := \left(\frac{\sqrt{k^2+\eta} - p - qk^2}{\sqrt{k^2+\eta} + p + qk^2} \right)^2 e^{-2\sqrt{k^2+\eta}L}. \tag{4.23}$$

To determine the optimal parameters p and q for the associated Schwarz method (which we call OO2 for "Optimized of Order 2," a term introduced in [25]), we have to solve the min-max problem

$$\begin{aligned} &\min_{p,q\geq 0} \left(\max_{k_{\min}\leq k\leq k_{\max}} |\rho_{OO2}(k, L, \eta, p, q)| \right) \\ &= \min_{p,q\geq 0} \left(\max_{k_{\min}\leq k\leq k_{\max}} \left(\frac{\sqrt{\eta+k^2} - p - qk^2}{\sqrt{\eta+k^2} + p + qk^2} \right)^2 e^{-2\sqrt{\eta+k^2}L} \right). \end{aligned} \tag{4.24}$$

We need a second technical lemma for the analysis of the optimal parameters.

LEMMA 4.6. *Let $R_1(k_1, k_2)$ and $R_2(k_1, k_2)$ be two continuously differentiable functions, $R_j : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, $j = 1, 2$, such that the partial derivatives satisfy*

$$\frac{\partial R_1}{\partial k_1} < 0, \quad \frac{\partial R_1}{\partial k_2} < 0, \quad \frac{\partial R_2}{\partial k_1} < 0, \quad \frac{\partial R_2}{\partial k_2} > 0 \tag{4.25}$$

and assume that there exists a unique differentiable $k_1^*(k_2)$ such that

$$R_1(k_1^*(k_2), k_2) + R_2(k_1^*(k_2), k_2) = 0. \tag{4.26}$$

Then we must have

$$\frac{dR_2}{dk_2}(k_1^*(k_2), k_2) > 0. \tag{4.27}$$

Proof. Using implicit differentiation of (4.26), we find

$$\frac{dk_1^*}{dk_2}(k_2) = -\frac{\frac{\partial R_1}{\partial k_2}(k_1^*(k_2), k_2) + \frac{\partial R_2}{\partial k_2}(k_1^*(k_2), k_2)}{\frac{\partial R_1}{\partial k_1}(k_1^*(k_2), k_2) + \frac{\partial R_2}{\partial k_1}(k_1^*(k_2), k_2)}$$

and inserting this result into the total derivative, we obtain

$$\begin{aligned} \frac{dR_2}{dk_2}(k_1^*(k_2), k_2) &= \frac{\partial R_2}{\partial k_1}(k_1^*(k_2), k_2)\frac{dk_1^*}{dk_2}(k_2) + \frac{\partial R_2}{\partial k_2}(k_1^*(k_2), k_2) \\ &= \frac{-\frac{\partial R_2}{\partial k_1}(k_1^*(k_2), k_2)\frac{\partial R_1}{\partial k_2}(k_1^*(k_2), k_2) + \frac{\partial R_2}{\partial k_1}(k_1^*(k_2), k_2)\frac{\partial R_1}{\partial k_1}(k_1^*(k_2), k_2)}{\frac{\partial R_1}{\partial k_1}(k_1^*(k_2), k_2) + \frac{\partial R_2}{\partial k_1}(k_1^*(k_2), k_2)} \\ &> 0 \end{aligned}$$

using the assumption on the signs of the partial derivatives. □

THEOREM 4.7 (optimal second order parameters). *For $L > 0$ and $k_{\max} = \infty$, the solution p^*, q^* of the min-max problem* (4.24) *is given by the unique root of the system of equations*

$$\rho_{OO2}(k_{\min}, L, \eta, p^*, q^*) = \rho_{OO2}(\bar{k}_1, L, \eta, p^*, q^*) = \rho_{OO2}(\bar{k}_2, L, \eta, p^*, q^*), \tag{4.28}$$

where the locations of the maxima $\bar{k}_1$ and $\bar{k}_2$ are given by

$$\bar{k}_{1,2}(L, \eta, p, q) = \frac{1}{q}\sqrt{\frac{L + 2q - 2Lpq \mp \sqrt{L^2 + 4Lq - 4L^2pq + 4q^2 - 16Lpq^2 + 16Lq^3\eta + 4L^2q^2\eta}}{2L}}. \tag{4.29}$$

For $L = 0$ and $k_{\max}$ finite, the optimal parameters p^* and q^* are given by

$$\begin{aligned} p^* &= \frac{k_{\max}^2\sqrt{k_{\min}^2+\eta} - k_{\min}^2\sqrt{k_{\max}^2+\eta}}{\sqrt{2(k_{\max}^2-k_{\min}^2)}\left(\left(\sqrt{k_{\max}^2+\eta}-\sqrt{k_{\min}^2+\eta}\right)\left((k_{\max}^2+\eta)\sqrt{k_{\min}^2+\eta}-(k_{\min}^2+\eta)\sqrt{k_{\max}^2+\eta}\right)\right)^{\frac{1}{4}}}, \\ q^* &= \frac{\left(\sqrt{k_{\max}^2+\eta}-\sqrt{k_{\min}^2+\eta}\right)^{\frac{3}{4}}}{\sqrt{2(k_{\max}^2-k_{\min}^2)}\left((k_{\max}^2+\eta)\sqrt{k_{\min}^2+\eta}-(k_{\min}^2+\eta)\sqrt{k_{\max}^2+\eta}\right)^{\frac{1}{4}}}. \end{aligned} \tag{4.30}$$

Proof. The argument is again based on a transformation: the partial derivatives of ρ_{OO2} with respect to p and q are

$$\frac{\partial \rho_{OO2}}{\partial p} = 4\sqrt{k^2+\eta}\frac{p + qk^2 - \sqrt{k^2+\eta}}{(p + qk^2 + \sqrt{k^2+\eta})^3}e^{-2\sqrt{k^2+\eta}L}, \quad \frac{\partial \rho_{OO2}}{\partial q} = k^2\frac{\partial \rho_{OO2}}{\partial p}, \tag{4.31}$$

and hence ρ_{OO2} is monotonically decreasing when p and q are decreased for all $k > k_{\min}$ as long as $p + qk^2 > \sqrt{k^2+\eta}$. This implies that at the solution of the min-max problem ρ_{OO2} must have at least one zero $k_1 > k_{\min}$. Then instead of using the parameter p, we can use equivalently the parameter k_1 in the min-max problem by setting $p := \sqrt{k_1+\eta} - qk_1^2$, which leads to the new form of the convergence factor

$$\rho'_{OO2} = \frac{(\sqrt{k_1^2+\eta} - \sqrt{k^2+\eta} + q(k^2 - k_1^2))^2}{(\sqrt{k^2+\eta} + \sqrt{k_1^2+\eta} + q(k^2 - k_1^2))^2}e^{-2\sqrt{k^2+\eta}L},$$

which has now necessarily a zero at $k_1 > k_{\min}$. If we suppose that k_1 is the only zero at the optimum, we reach again a contradiction, because a partial derivative with respect to q gives

$$\frac{\partial \rho'_{OO2}}{\partial q} = 4\sqrt{k^2+\eta}(k^2 - k_1^2)\frac{\sqrt{k_1^2+\eta} - \sqrt{k^2+\eta} + q(k^2 - k_1^2)}{(\sqrt{k^2+\eta} + \sqrt{k_1^2+\eta} + q(k^2 - k_1^2))^3}e^{-2\sqrt{k^2+\eta}L},$$

where the denominator is positive, since $\sqrt{k_1+\eta} - qk_1^2 = p \geq 0$, and the numerator changes sign only at $k = k_1$ by assumption, which together with the factor $(k^2 - k_1^2)$ in front makes the sign of the derivative negative for all $q > 0$ as long as there is only one zero at k_1. Thus increasing q the convergence factor ρ'_{OO2} can be decreased for all $k > k_{\min}$ as long as there is no second zero. Hence at the optimum, ρ'_{OO2} must have a second zero, without loss of generality at $k_2 \geq k_1 > k_{\min}$. Thus we can use the parameter k_2 instead of q, which leads to the change of variables

$$p = \frac{\sqrt{k_1^2+\eta}k_2^2 - k_1^2\sqrt{k_2^2+\eta}}{k_2^2 - k_1^2}, \quad q = \frac{\sqrt{k_2^2+\eta} - \sqrt{k_1^2+\eta}}{k_2^2 - k_1^2} \tag{4.32}$$

and the new min-max problem, which is equivalent to (4.24), is

$$\min_{k_{\min}<k_1\leq k_2}\left(\max_{k_{\min}\leq k\leq k_{\max}}|R(k,L,\eta,k_1,k_2)|\right), \tag{4.33}$$

with the new function $R(k,L,\eta,k_1,k_2)$, representing the square root of the convergence factor, given by

$$R(k,L,\eta,k_1,k_2) = \frac{\sqrt{k^2+\eta}(k_2^2-k_1^2)-(\sqrt{k_1^2+\eta}k_2^2-\sqrt{k_2^2+\eta}k_1^2)-(\sqrt{k_2^2+\eta}-\sqrt{k_1^2+\eta})k^2}{\sqrt{k^2+\eta}(k_2^2-k_1^2)+(\sqrt{k_1^2+\eta}k_2^2-\sqrt{k_2^2+\eta}k_1^2)+(\sqrt{k_2^2+\eta}-\sqrt{k_1^2+\eta})k^2}e^{-\sqrt{k^2+\eta}L}. \tag{4.34}$$

Taking the partial derivatives with respect to k_1 and k_2, we find

$$\frac{\partial R}{\partial k_1} = \frac{2k_1(k^2-k_2^2)\sqrt{k^2+\eta}\left(\sqrt{k_2^2+\eta}-\sqrt{k_1^2+\eta}\right)^2}{\sqrt{k_1^2+\eta}\left(\sqrt{k^2+\eta}(k_1^2-k_2^2)+\sqrt{k_1^2+\eta}(k^2-k_2^2)+(k_1^2-k^2)\sqrt{k_2^2+\eta}\right)^2}e^{-\sqrt{k^2+\eta}L}, \tag{4.35}$$

$$\frac{\partial R}{\partial k_2} = \frac{2k_2(k^2-k_1^2)\sqrt{k^2+\eta}\left(\sqrt{k_2^2+\eta}-\sqrt{k_1^2+\eta}\right)^2}{\sqrt{k_2^2+\eta}\left(\sqrt{k^2+\eta}(k_1^2-k_2^2)+\sqrt{k_1^2+\eta}(k^2-k_2^2)+(k_1^2-k^2)\sqrt{k_2^2+\eta}\right)^2}e^{-\sqrt{k^2+\eta}L}, \tag{4.36}$$

which shows that for $k<k_2$, the function R is decreasing when k_1 is increasing and for $k>k_2$ it is increasing with k_1. Similarly for $k<k_1$, the function R is decreasing when k_2 is increasing, and for $k>k_1$ it is increasing with k_2.

Now for $L>0$, $R=(-1+O(\frac{1}{k}))e^{-Lk}$ as k goes to infinity. Hence the maximum in the min-max problem can be attained, by continuity of R and knowing that there are two zeros at $k_1,k_2>k_{\min}$, either at $k_{\min}$, where R is negative, or at $k=\bar{k}_1$ given in (4.29), where R has a maximum, $k_1\leq\bar{k}_1\leq k_2$, or at $k=\bar{k}_2$ given in (4.29), where R has a negative minimum, $k_2\leq\bar{k}_2$. To show that the solution of the min-max problem is indeed when the three are balanced, we first note that for any fixed k_2, there exists a unique $k_1^*=k_1^*(k_2)\in[k_{\min},k_2]$ such that $|R(k_{\min},L,\eta,k_1^*(k_2),k_2)|=R(\bar{k}_1,L,\eta,k_1^*(k_2),k_2)$, because of continuity and $0=|R(k_{\min},L,\eta,k_{\min},k_2)|<R(\bar{k}_1,L,\eta,k_{\min},k_2)$ and $|R(k_{\min},L,\eta,k_2,k_2)|>R(\bar{k}_1,L,\eta,k_2,k_2)=R(k_2,L,\eta,k_2,k_2)=0$, and $|R(k_{\min},L,\eta,k_1,k_2)|$ is monotonically increasing with k_1 by (4.35) and $R(\bar{k}_1,L,\eta,k_1,k_2)$ is monotonically decreasing in k_1 by (4.35) and Lemma 4.3. Hence denoting by $R_1(k_1,k_2):=R(k_{\min},L,\eta,k_1,k_2)$ and $R_2(k_1,k_2):=R(\bar{k}_1,L,\eta,k_1,k_2)$ Lemma 4.6 applies and therefore $|R(k_{\min},L,\eta,k_1^*(k_2),k_2)|=R(\bar{k}_1,L,\eta,k_1^*(k_2),k_2)$ is monotonically increasing with k_2. Now for $k_2=k_{\min}$, we have $k_1^*(k_{\min})=k_{\min}$ and thus $0=|R(k_{\min},L,\eta,k_{\min},k_{\min})|<|R(\bar{k}_2,L,\eta,k_{\min},k_{\min})|$ and for large k_2 we have $|R(k_{\min},L,\eta,k_1^*(k_2),k_2)|>|R(\bar{k}_2,L,\eta,k_1^*(k_2),k_2)|$ (since the right-hand term goes to zero in the limit). Therefore by continuity, Lemma 4.6 for $|R(k_{\min},L,\eta,k_1^*(k_2),k_2)|$ and Lemma 4.3 for $|R(\bar{k}_2,L,\eta,k_1^*(k_2),k_2)|$ (note that $(k_1^*)'\geq 0$), there exists a unique k_2^* where these two expressions are equal, $|R(k_{\min},L,\eta,$

$k_1^*(k_2^*), k_2^*)| = |R(\bar{k}_2, L, \eta, k_1^*(k_2^*), k_2^*)|$, which is the unique solution of the min-max problem. Back-transforming to the p and q variables using (4.32) we obtain the equations for the solution of the min-max problem given in (4.28).

In the case without overlap, $L = 0$, the function R behaves for large k like $-1+O(\frac{1}{k})$ and hence the maximum in the min-max problem for $L = 0$ can be attained either at $k_{\min}$, $k_{\max}$, or at the interior maximum $\bar{k}_1$ which satisfies $k_1 \leq \bar{k}_1 \leq k_2$ and is given in the p and q variables by

$$\bar{k}_1 = \frac{\sqrt{q(p - 2q\eta)}}{q}.$$

The same argument used for the case $L > 0$ is still valid, and hence there exists a unique solution p^*, q^* of the min-max problem which is characterized by the system of equations

$$\rho_{OO2}(k_{\min}, 0, \eta, p^*, q^*) = \rho_{OO2}(\bar{k}_1, 0, \eta, p^*, q^*) = \rho_{OO2}(k_{\max}, 0, \eta, p^*, q^*). \tag{4.37}$$

This system can be solved in closed form by first solving $\rho_{OO2}(k_{\min}, 0, \eta, p, q^*) = \rho_{OO2}(k_{\max}, 0, \eta, p, q^*)$ for $q^* = q^*(p)$, which leads to

$$q^*(p) = \frac{p(\sqrt{k_{\max}^2 + \eta} - \sqrt{k_{\min}^2 + \eta})}{\sqrt{k_{\min}^2 + \eta}k_{\max}^2 - k_{\min}^2\sqrt{k_{\max}^2 + \eta}}.$$

Inserting this solution into the remaining equation $\rho_{OO2}(k_{\min}, 0, \eta, p^*, q^*(p^*)) = \rho_{OO2}(\bar{k}_1, L, \eta, p^*, q^*(p^*))$ and solving for p^* leads to the closed form solution (4.30) of the min-max problem for $L = 0$. ∎

Figure 4.1 shows on the right the convergence factor obtained with the second order optimized transmission conditions for our model problem with overlap $L = \frac{1}{100}$ and $\eta = 1$, comparing it to the convergence factor of the classical Schwarz method. The maximum of the convergence factor of the new Schwarz method with optimized second order transmission conditions is 0.0704, which means that about 131 iterations of the classical Schwarz method with convergence factor 0.980 are needed to attain the performance of the second order optimized Schwarz method.

THEOREM 4.8 (OO2 asymptotics). *The asymptotic performance of the new Schwarz method with optimized second order transmission conditions and overlap $L = h$, as h goes to zero, is given by*

$$\max_{k_{\min} \leq k \leq \frac{\pi}{h}} |\rho_{OO2}(k, h, \eta, p^*, q^*)| = 1 - 4 \cdot 2^{\frac{3}{5}}(k_{\min}^2 + \eta)^{\frac{1}{10}} h^{\frac{1}{5}} + O(h^{\frac{2}{5}}). \tag{4.38}$$

The asymptotic performance without overlap, $L = 0$, is for h small given by

$$\max_{k_{\min} \leq k \leq \frac{\pi}{h}} |\rho_{OO2}(k, 0, \eta, p^*, q^*)| = 1 - 4\frac{\sqrt{2}(k_{\min}^2 + \eta)^{\frac{1}{8}}}{\pi^{\frac{1}{4}}} h^{\frac{1}{4}} + O(h^{\frac{1}{2}}). \tag{4.39}$$

Proof. To obtain the first result, we need to solve the nonlinear equations (4.28) asymptotically in h for the optimal parameters p^* and q^*. We make the ansatz $p = C_1 h^\alpha$ and $q = C_2 h^\beta$, insert this together with $L = h$ into the nonlinear equations (4.28), and expand for small h. The search for the lowest order terms is simplified by the knowledge that $\alpha < 0$ and $\beta > 0$ since p is growing when h is decaying and q is diminishing with h. Expanding for h small, we find from the equation $\rho_{OO2}(k_{\min}, h, \eta, p^*, q^*) = \rho_{OO2}(\bar{k}_1, h, \eta, p^*, q^*)$ the leading order terms

$$-4\sqrt{2}C_1 h^\alpha \sqrt{h} + 8C_2 h^\beta \sqrt{C_1 h^\alpha} C_1 h^\alpha$$

and from the equation $\rho_{OO2}(k_{\min}, h, \eta, p^*, q^*) = \rho_{OO2}(k_2, h, \eta, p^*, q^*)$ the leading order terms

$$-4\sqrt{2}C_1^2 h^{2\alpha}\sqrt{h} + 4\sqrt{k_{\min}^2 + \eta}\sqrt{C_2 h^\beta}C_1 h^\alpha.$$

Since the equations hold at the optimum, the leading order terms must match, which leads to a system of equations for the unknown exponents α and β,

$$\frac{3}{2}\alpha + \beta = \alpha + \frac{1}{2}, \quad 2\alpha + \frac{1}{2} = \alpha + \frac{\beta}{2},$$

whose solution is $\alpha = -\frac{1}{5}$ and $\beta = \frac{3}{5}$, and a system of equations for the constants C_1 and C_2, whose solution is

$$C_1 = 2^{-\frac{3}{5}}(k_{\min}^2 + \eta)^{\frac{2}{5}}, \quad C_2 = (2(k_{\min}^2 + \eta))^{-\frac{1}{5}}.$$

Hence asymptotically the optimal parameters p^* and q^* are

$$p^* = 2^{-\frac{3}{5}}(k_{\min}^2 + \eta)^{\frac{2}{5}}h^{-\frac{1}{5}}, \quad q^* = (2(k_{\min}^2 + \eta))^{-\frac{1}{5}}h^{\frac{3}{5}}. \tag{4.40}$$

To see that the min-max solution given in (4.38) on the infinite frequency domain $k \in [k_{\min}, \infty)$ is really the relevant one asymptotically on the bounded frequency domain $|k| < k_{\max} = \frac{\pi}{h}$, we must have that the second maximum $\bar{k}_2$ given in (4.29) satisfies asymptotically $\bar{k}_2 \leq k_{\max}$. Inserting the asymptotic expressions of p^* and q^* from (4.40) into the expression of $\bar{k}_2$ in (4.29), setting $L = h$ and expanding for h small, we find

$$\bar{k}_2 = \frac{2^{\frac{3}{5}}(k_{\min}^2 + \eta)^{\frac{1}{10}}}{h^{\frac{4}{5}}} + O(h^{-\frac{2}{5}}) \tag{4.41}$$

and hence indeed asymptotically $\bar{k}_2 \leq k_{\max} = \frac{\pi}{h}$. Inserting now the asymptotically optimal parameters p^* and q^* from (4.40) into the convergence factor ρ_{OO2} and expanding as h goes to zero, we obtain the result (4.38).

For the second result without overlap, we have the closed formulas (4.30) for the optimal parameters p^* and q^*. It suffices therefore to insert them into the convergence factor and to expand it in h for $k_{\max} = \frac{\pi}{h}$ at $k = k_{\min}$ to find the result (4.39). ∎

4.3. A two-sided optimized Robin transmission condition. We now investigate how the simplifying assumption $p_1 = p_2$ and $q_1 = q_2$ in the min-max problem (4.11) affects the performance of the optimized Schwarz methods. We do this only for the case of Robin transmission conditions to illustrate the change. We thus have $q_1 = q_2 = 0$ and the optimization problem (4.10).

THEOREM 4.9 (optimal two-sided Robin conditions). *If there is overlap, $L > 0$, then the optimal two-sided Robin parameters are given by*

$$p_1^* = \frac{1 - \sqrt{1 + 4\eta(q^*)^2 - 4p^*q^*}}{2q^*}, \quad p_2^* = \frac{1 + \sqrt{1 + 4\eta(q^*)^2 - 4p^*q^*}}{2q^*}, \tag{4.42}$$

where p^* and q^* are solutions of (4.28) *with L replaced by $2L$. If there is no overlap, $L = 0$, then the optimal two-sided Robin parameters are* (4.42), *where p^* and q^* are given by* (4.30).

Proof. Multiplying the two factors in the optimization problem (4.10), we obtain the optimization problem

$$\min_{p_j\geq 0}\left(\max_{k_{\min}<k<k_{\max}}\left|\frac{\sqrt{\eta+k^2}-\frac{\eta+p_1p_2}{p_1+p_2}-\frac{k^2}{p_1+p_2}}{\sqrt{\eta+k^2}+\frac{\eta+p_1p_2}{p_1+p_2}+\frac{k^2}{p_1+p_2}}\right|e^{-2\sqrt{\eta+k^2}L}\right) \tag{4.43}$$

and hence in the new parameters

$$p=\frac{\eta+p_1p_2}{p_1+p_2},\quad q=\frac{1}{p_1+p_2}, \tag{4.44}$$

this optimization problem is equivalent to the optimization problem (4.24) provided L is replaced by $2L$. The solution for this problem is given for $L>0$ in (4.28) and for $L=0$ in (4.30). Back-transforming these results using (4.44) concludes the proof. ∎

The preceding theorem shows that one can generate the performance of higher order transmission conditions using lower order transmission conditions which are not equal on both sides. In the case without overlap, one needs to perform two iterations of the two-sided optimized Robin transmission algorithm to attain an error reduction equivalent to the one from one iteration of the optimized second order transmission conditions algorithm. With overlap, two iterations of the algorithm with optimized two-sided Robin transmission conditions is even a bit better than one iteration of the algorithm with second order transmission conditions, since the overlap has been effective twice.

Figure 4.1 shows on the right the convergence factors obtained with the two-sided optimized Robin conditions for our model problem with overlap $L=\frac{1}{100}$ and $\eta=1$, comparing it to the convergence factor of the classical and the optimized zeroth and second order Schwarz methods. The maximum of the convergence factor of the new Schwarz method with two-sided optimized Robin conditions is 0.208, which means that about 78 iterations of the classical Schwarz method with convergence factor 0.980 are needed to attain the performance of the two-sided optimized Robin Schwarz method.

COROLLARY 4.10. *The asymptotic performance of the two-sided optimized Schwarz method with $L=h$ is*

$$\max_{k_{\min}\leq k\leq\frac{\pi}{h}}|\rho(k,h,\eta,p_1^*,p_2^*)|=1-2\cdot 2^{\frac{4}{5}}(k_{\min}^2+\eta)^{\frac{1}{10}}h^{\frac{1}{5}}+O(h^{\frac{2}{5}}). \tag{4.45}$$

Without overlap, $L=0$, the asymptotic performance is given by

$$\max_{k_{\min}\leq k\leq\frac{\pi}{h}}|\rho(k,0,\eta,p_1^*,p_2^*)|=1-2\frac{\sqrt{2}(k_{\min}^2+\eta)^{\frac{1}{8}}}{\pi^{\frac{1}{4}}}h^{\frac{1}{4}}+O(h^{\frac{1}{2}}). \tag{4.46}$$

Hence asymptotically, the second order optimized algorithm and the two-sided optimized Robin algorithm are equivalent: one can get the same asymptotic performance from Robin transmission conditions that one gets from second order transmission conditions, provided one uses different parameters in the two transmission conditions.

The idea of not using the same parameters on each side can be generalized by not using the same parameter in each iteration: one uses a sequence of transmission conditions with Robin parameters p_i, $i=1,2,\ldots,I$, where I is a number of parameters chosen and one cycles through the transmission conditions from 1 to I in the

Schwarz iteration. This adds more degrees of freedom in the optimization problem and leads to Schwarz algorithms that have an arbitrarily weak dependence of the convergence factor on h, even without overlap (see [15]), but at the cost of having to solve subdomain problems with varying transmission conditions per interation.

5. Optimized Schwarz methods compared to Schur and FETI methods. We now investigate what the relation is between optimized Schwarz methods, which can be used without overlap, to other domain decomposition methods without overlap, like the Schur methods and FETI (Finite Element Tearing and Interconnect [13]). To this end we will address two questions:

1. What conditions can one impose to couple subdomain problems ?
2. Which of these conditions are good to build efficient domain decomposition algorithms ?

Although the ideas in this section hold for general second order elliptic problems, we will use our self-adjoint coercive model problem (2.1) to fix ideas.

5.1. Classical coupling conditions between subdomains. There are two classical ways to couple subdomain problems. For the first one, one uses an overlapping decomposition of Ω, say, $\Omega_1 = (-\infty, L)$ and $\Omega_2 = (0, \infty)$ for $L > 0$, and the coupled subproblems are given by

$$\begin{array}{llll} \mathcal{L}(u_1) = f & \text{in } \Omega_1, & \mathcal{L}(u_2) = f & \text{in } \Omega_2, \\ u_1(L, y) = u_2(L, y), & y \in \mathbb{R}, & u_2(0, y) = u_1(0, y), & y \in \mathbb{R}. \end{array} \tag{5.1}$$

Note that we do not introduce an algorithm to find the solution of the coupled subproblems here; we only define coupled subdomain problems which are equivalent to the original problem. The equivalence can be seen in this case, for example, by studying the associated Schwarz algorithm.

For the second approach, one uses subdomains without overlap, for example, $\Omega_1 = (-\infty, 0)$ and $\Omega_2 = (0, \infty)$, and the coupled subdomain problems are

$$\begin{array}{llll} \mathcal{L}(u_1) = f & \text{in } \Omega_1, & \mathcal{L}(u_2) = f & \text{in } \Omega_2, \\ u_1(0, y) = u_2(0, y), & y \in \mathbb{R}, & \partial_x u_2(0, y) = \partial_x u_1(0, y), & y \in \mathbb{R}. \end{array} \tag{5.2}$$

Note the key difference: in the decomposition without overlap, both the solution values as well as the normal derivatives are imposed to agree on the interface for this second order problem, whereas in the approach with overlap, only solution values are imposed to agree, but at two different locations, which implies the agreement of normal derivatives.

The classical algorithm to find a solution for the case of an overlapping decomposition is the one given by Schwarz in [36],

$$\begin{array}{llll} \mathcal{L}(u_1^n) = f & \text{in } \Omega_1, & \mathcal{L}(u_2^n) = f & \text{in } \Omega_2, \\ u_1^n(L, y) = u_2^{n-1}(L, y), & y \in \mathbb{R}, & u_2^n(0, y) = u_1^{n-1}(0, y), & y \in \mathbb{R}, \end{array} \tag{5.3}$$

and we have derived the linear convergence factor of this algorithm in (2.8).

Can a similar iterative method be used for the nonoverlapping decomposition? This would lead, for example, to

$$\begin{array}{llll} \mathcal{L}(u_1^n) = f & \text{in } \Omega_1, & \mathcal{L}(u_2^n) = f & \text{in } \Omega_2, \\ u_1^n(0, y) = u_2^{n-1}(0, y), & y \in \mathbb{R}, & \partial_x u_2^n(0, y) = \partial_x u_1^{n-1}(0, y), & y \in \mathbb{R}. \end{array} \tag{5.4}$$

In general not, because this algorithm does not converge, as one can see with Fourier analysis. Setting for the convergence analysis $f = 0$ by linearity and taking a Fourier transform in y with parameter k of (5.4) leads to the transformed iterates

$$\hat{u}_1^n(x,k) = \hat{u}_2^{n-1}(0,k)e^{\sqrt{\eta+k^2}x}, \quad \hat{u}_2^n(x,k) = -\hat{u}_1^{n-1}(0,k)e^{-\sqrt{\eta+k^2}x}.$$

Thus inserting $\hat{u}_2^{n-1}(0,k)$ from the second equation into the first one and evaluating at $x=0$, we find

$$\hat{u}_1^n(0,k) = -\hat{u}_1^{n-2}(0,k) \quad \text{and similarly} \quad \hat{u}_2^n(0,k) = -\hat{u}_2^{n-2}(0,k).$$

Hence the convergence factor of this algorithm is $\rho = -1$ and thus it does not converge.

A first remedy consists of introducing relaxation parameters γ_j, $j=1,2$, which leads to the transmission conditions

$$\begin{aligned} u_1^n(0,y) &= \gamma_1 u_2^{n-1}(0,y) + (1-\gamma_1)u_1^{n-1}(0,y), \\ \partial_x u_2^n(0,y) &= \gamma_2 \partial_x u_1^{n-1}(0,y) + (1-\gamma_2)\partial_x u_2^{n-1}(0,y), \end{aligned} \tag{5.5}$$

for which convergence results have been established. In [1] we find that for the so-called Dirichlet–Neumann method, $\gamma_2 = 1$, there exist γ_1 for which the algorithm converges, and in [35] we find that for the Neumann–Dirichlet method, $\gamma_1 = 1$, there exist γ_2 for which the algorithm converges. For our model problem, we find for the interface system in the Fourier domain

$$\begin{pmatrix} \hat{u}_1^n(0,k) \\ \partial_x \hat{u}_2^n(0,k) \end{pmatrix} = \begin{bmatrix} 1-\gamma_1 & \frac{-\gamma_1}{\sqrt{\eta+k^2}} \\ \gamma_2\sqrt{\eta+k^2} & 1-\gamma_2 \end{bmatrix} \begin{pmatrix} \hat{u}_1^{n-1}(0,k) \\ \partial_x \hat{u}_2^{n-1}(0,k) \end{pmatrix}. \tag{5.6}$$

The asymptotic convergence factor of this matrix iteration is governed by the spectral radius of the 2×2 matrix, which is given by the larger eigenvalue in modulus,

$$\rho = \left| 1 - \frac{1}{2}(\gamma_1+\gamma_2) + \frac{1}{2}\sqrt{(\gamma_1-\gamma_2)^2 - 4\gamma_1\gamma_2} \right|. \tag{5.7}$$

Note that ρ is independent of the frequency parameter k, which implies that the convergence factor is independent of the mesh parameter h if the algorithm is discretized. In the case of the Dirichlet–Neumann algorithm, where $\gamma_2 = 1$, the asymptotic convergence factor for our model problem is

$$\rho = \frac{1}{2}\left| 1-\gamma_1 + \sqrt{\gamma_1^2 - 6\gamma_1 + 1} \right|,$$

which is less than 1 for $0 < \gamma_1 < 1$. The optimal value which minimizes the convergence factor is $\gamma_1 = 3 - 2\sqrt{2} \approx 0.1716$, for which the convergence factor becomes $\rho \approx 0.4142$. The same results we find by the symmetry of the parameters γ_i also in the case of the Neumann–Dirichlet algorithm, where $\gamma_1 = 1$. But one could also use both relaxation parameters simultaneously to minimize the convergence factor. With both parameters, we can achieve that both eigenvalues vanish simultaneously by setting the term under the square root and the one outside of the square root in (5.7) equal to zero. We find that for the choice

$$\gamma_1 = 1 \pm \frac{1}{\sqrt{2}}, \quad \gamma_2 = 1 \mp \frac{1}{\sqrt{2}},$$

the spectral radius vanishes identically, $\rho \equiv 0$. Hence this method will converge in at most two iterations for any initial guess. (The matrix is not normal; otherwise convergence would be in one iteration, which we know is not possible.)

In a Gauss–Seidel version of this iteration, subdomain Ω_2 would use directly the newest values at the interface from subdomain Ω_1. In that case the relaxed interface iteration can be found after a short calculation to be

$$\left(\begin{array}{c}\hat{u}_1^n(0,k)\\ \partial_x\hat{u}_2^n(0,k)\end{array}\right)=\left[\begin{array}{cc}1-\gamma_1 & \frac{-\gamma_1}{\sqrt{\eta+k^2}}\\ \gamma_2(1-\gamma_1)\sqrt{\eta+k^2} & 1-\gamma_2-\gamma_1\gamma_2\end{array}\right]\left(\begin{array}{c}\hat{u}_1^{n-1}(0,k)\\ \partial_x\hat{u}_2^{n-1}(0,k)\end{array}\right). \tag{5.8}$$

As before the asymptotic convergence of this matrix iteration is governed by the spectral radius of the 2×2 matrix and the term depending on the frequency parameter k cancels; the convergence factor is independent of k. In this case, however, both the Dirichlet–Neumann and the Neumann–Dirichlet algorithm can achieve already a convergence factor $\rho = 0$; one parameter suffices. The optimal choice is $\gamma_1 = \frac{1}{2}$ for the Dirichlet–Neumann case, where $\gamma_2 = 1$, and $\gamma_2 = \frac{1}{2}$ for the Neumann–Dirichlet case, where $\gamma_1 = 1$, results found already in [1] and [35]. Unfortunately all these results depend strongly on the symmetry in the problem; otherwise the two symbols depending on the frequency parameter k and containing the square root would not cancel. Hence for a more general situation with uneven domains or variable coefficients, convergence in two steps will not be possible with this approach. The optimal Schwarz method using the exact Dirichlet-to-Neumann map, however, does still converge in two iterations also in these more general cases.

A second remedy, and this is really the classical approach for subdomain problems coupled without overlap, consists of avoiding an iteration first. One keeps the coupled problem and introduces a name for the quantities at the interface,

$$\begin{array}{llll}\mathcal{L}(u_1)=f & \text{in } \Omega_1, & \mathcal{L}(u_2)=f & \text{in } \Omega_2,\\ u_1(0,y)=u_2(0,y) & =:\ \lambda(y), & \partial_x u_2(0,y)=\partial_x u_1(0,y) & =:\ \lambda_x(y).\end{array} \tag{5.9}$$

The primal Schur method then works as follows: supposing that $\lambda(y)$ is known, one computes $u_1(x,y,\lambda)$ and $u_2(x,y,\lambda)$ and then sets

$$\partial_x u_1(0,y,f,\lambda)-\partial_x u_2(0,y,f,\lambda)=0,$$

which is a linear equation to determine the interface function λ. Solving this linear problem with a Krylov method requires at each step two subdomain solves with Dirichlet conditions,

$$\mathcal{A}_p\lambda := \partial_x u_1(0,y,0,\lambda)-\partial_x u_2(0,y,0,\lambda) = -\partial_x u_1(0,y,f,0)+\partial_x u_2(0,y,f,0) =: b_p. \tag{5.10}$$

To learn more about the conditioning of the primal Schur complement system $\mathcal{A}_p\lambda = b_p$, we take a Fourier transform of $\mathcal{A}_p\lambda$ to find the symbol of $\mathcal{A}_p$,

$$\hat{\mathcal{A}}_p\hat{\lambda} = \hat{v}_x(0,y,0,\hat{\lambda})-\hat{w}_x(0,y,0,\hat{\lambda}) = 2\sqrt{\eta+k^2}\,\hat{\lambda}. \tag{5.11}$$

This symbol is symmetric in k and hence the condition number of the corresponding operator can be estimated using the ratio of the symbol at the maximum and minimum frequencies occurring in a given computation. Estimating the minimum frequency by 0 and the maximum frequency by $k_{\max} = \frac{\pi}{h}$ as before, where h is the mesh parameter, we find the asymptotic condition number for h small to be

$$\mathcal{K}(\mathcal{A}_p) = \frac{\pi}{\sqrt{\eta}h} + O(h). \tag{5.12}$$

Note that the original operator $(\eta - \Delta)u = f$ had a condition number estimate of $O(\frac{1}{h^2})$ and thus the primal Schur method improves the condition number by a square root. On the negative side the matrix vector product is now more expensive, since it involves subdomain solves.

The dual Schur method, which became famous under the name FETI, is similar, although the key feature of a natural coarse space cannot be seen in this simple setting: supposing that λ_x is known, we compute $u_1(x, y, f, \lambda_x)$ and $u_2(x, y, f, \lambda_x)$ and then set

$$u_1(0, y, f, \lambda_x) - u_2(0, y, f, \lambda_x) = 0,$$

which is now a linear equation for λ_x. Solving this linear problem with a Krylov method requires at each step two subdomain solves with Neumann conditions,

$$\mathcal{A}_d\lambda_x := u_1(0, y, 0, \lambda_x) - u_2(0, y, 0, \lambda_x) = -u_1(0, y, f, 0) + u_2(0, y, f, 0) =: b_d. \tag{5.13}$$

The Fourier transform of the dual Schur complement system $\mathcal{A}_d\lambda_x = b_d$ leads to

$$\hat{\mathcal{A}}_d\hat{\lambda}_x = \hat{v}(0, y, 0, \hat{\lambda}_x) - \hat{w}(0, y, 0, \hat{\lambda}_x) = \frac{2}{\sqrt{\eta + k^2}}\hat{\lambda}_x, \tag{5.14}$$

which shows that the operator $\mathcal{A}_d$ has the symbol $\frac{2}{\sqrt{\eta+k^2}}$. This symbol is also symmetric in k and as in the case of the primal Schur complement, we find the condition number for h small to be

$$\mathcal{K}(\mathcal{A}_d) = \frac{\pi}{\sqrt{\eta}h} + O(h). \tag{5.15}$$

Now note that the dual Schur complement with the symbol $\frac{2}{\sqrt{\eta+k^2}}$ is the inverse of the primal Schur complement that had the symbol $2\sqrt{\eta + k^2}$, up to the constant 4, and hence one is the ideal preconditioner for the other. This led to the famous Neumann–Neumann preconditioner for the primal Schur complement, with condition number independent of the mesh parameter [2]. Similarly, one could use a Dirichlet–Dirichlet preconditioner for the dual Schur complement or FETI to obtain a mesh independent domain decomposition method.

But why should one give preference to either the Dirichlet or the Neumann condition when formulating a Schur method? And why should we impose the same type of interface conditions on each subdomain? In the recent FETI-DP method [11], for some parts of the interfaces continuity of the dual variables is imposed, and for other parts continuity of the primal variables. One could go a step further and first assume that both λ and λ_x are known, then solve for $u_1(x, y, f, \lambda)$ and $u_2(x, y, f, \lambda_x)$, for example, and set

$$\begin{aligned} u_1(0, y, f, \lambda) - u_2(0, y, f, \lambda_x) &= 0, \\ \partial_x u_1(0, y, f, \lambda) - \partial_x u_2(0, y, f, \lambda_x) &= 0, \end{aligned}$$

which is now a two-field formulation for the two unknown fields, λ and λ_x. Solving this linear problem with a Krylov method requires at each step one subdomain solve with Dirichlet and one with Neumann conditions,

$$\begin{aligned} \mathcal{A}_{pd}\begin{pmatrix} \lambda \\ \lambda_x \end{pmatrix} &:= \begin{bmatrix} 1 & -u_2(0, y, 0, \cdot) \\ -\partial_x u_1(0, y, 0, \cdot) & 1 \end{bmatrix}\begin{pmatrix} \lambda \\ \lambda_x \end{pmatrix} \\ &= \begin{pmatrix} u_2(0, y, f, 0) \\ \partial_x u_1(0, y, f, 0) \end{pmatrix} =: b_{pd}. \end{aligned} \tag{5.16}$$

Taking a Fourier transform of the operator $\mathcal{A}_{pd}$, we find

$$\hat{\mathcal{A}}_{pd} = \begin{bmatrix} 1 & \frac{1}{\sqrt{\eta+k^2}} \\ -\sqrt{\eta+k^2} & 1 \end{bmatrix}, \tag{5.17}$$

which is precisely the matrix to which we have applied a Richardson iteration trying simply to relax the interface conditions in (5.4), an iteration which did not converge. By applying a Krylov method to solve the problem directly, however, it would converge in two steps, since the eigenvalues are independent of k, there are only two distinct points in the spectrum.

We can also write the coupled subdomain problems with overlap in substructured form. If we give the unknown functions at the interfaces the names $\lambda_0(y)$ and $\lambda_L(y)$, we get

$$\begin{array}{ll} \mathcal{L}(u_1) = f \text{ in } \Omega_1, & \mathcal{L}(u_2) = f \text{ in } \Omega_2, \\ u_1(L,y) = u_2(L,y) \ =: \ \lambda_L(y) & u_2(0,y) = u_1(0,y) \ =: \ \lambda_0(y). \end{array} \tag{5.18}$$

If we assume that both λ_L and λ_0 are known, then we can compute $u_1(x,y,f,\lambda_L)$ and $u_2(x,y,f,\lambda_0)$ and then set

$$\begin{array}{r} u_2(0,y,f,\lambda_0) - u_1(0,y,f,\lambda_L) = 0, \\ -u_2(L,y,f,\lambda_0) + u_1(L,y,f,\lambda_L) = 0, \end{array}$$

which is a linear system of equations for the unknowns λ_0 and λ_L. Solving this linear problem with a Krylov method requires at each step two subdomain solves with Dirichlet conditions,

$$\mathcal{A}_s \begin{pmatrix} \lambda_0 \\ \lambda_L \end{pmatrix} := \begin{bmatrix} 1 & -u_1(0,y,0,\cdot) \\ -u_2(L,y,0,\cdot) & 1 \end{bmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_L \end{pmatrix} = \begin{pmatrix} u_1(0,y,f,0) \\ u_2(L,y,f,0) \end{pmatrix} =: b_s. \tag{5.19}$$

In Fourier the symbol of the operator $\mathcal{A}_s$ is given by

$$\hat{\mathcal{A}}_s = \begin{bmatrix} 1 & -e^{-\sqrt{\eta+k^2}L} \\ -e^{-\sqrt{\eta+k^2}L} & 1 \end{bmatrix}, \tag{5.20}$$

and we see that the operator is symmetric in this case. If one applies a Richardson iteration to this operator, one recovers the classical Schwarz method for which we have seen that it converges independently of the discretization parameter. The eigenvalues in Fourier are $1 \pm e^{-\sqrt{\eta+k^2}L}$, which shows that the eigenvalues are clustering for large k around 1, a very desirable property when a Krylov method is used to solve the corresponding linear system. The condition number of this symmetric operator can be estimated by the ratio of the largest and smallest eigenvalue,

$$\mathcal{K}(\mathcal{A}_s) = \frac{1+e^{-\sqrt{\eta}L}}{1-e^{-\sqrt{\eta}L}}, \tag{5.21}$$

and it is independent of the mesh parameter h, as long as the overlap L is independent of h. For an overlap which depends on h, $L = h$, we have for h small

$$\mathcal{K}(\mathcal{A}_s) = \frac{2}{\sqrt{\eta}h} + O(h) \tag{5.22}$$

as for the primal and dual Schur methods.

5.2. Coupling conditions optimized for the computation. Optimized Schwarz methods bring the overlapping and nonoverlapping strategies together. They do not use either Dirichlet or Neumann conditions, and they work with or without overlap. The fundamental idea is that the coupled problem can be written with any set of conditions that implies the classical coupling conditions. The coupled problems

$$\begin{aligned} \mathcal{L}(u_1) &= f \quad \text{in } \Omega_1, & \mathcal{L}(u_2) &= f \quad \text{in } \Omega_2, \\ (\partial_x + \mathcal{S}_1)(u_1)(L) &= (\partial_x + \mathcal{S}_1)(u_2)(L), & (\partial_x + \mathcal{S}_2)(u_2)(0) &= (\partial_x + \mathcal{S}_1)(u_1)(0), \end{aligned} \tag{5.23}$$

are equivalent to the original, unpartitioned problem, as long as the choice of $\mathcal{S}_j$, $j = 1, 2$, leads to well-posed subdomain problems and implies, for $L > 0$, $u_1(0) = u_2(0)$ and $u_1(L) = u_2(L)$, and for $L = 0$, $u_1(0) = u_2(0)$ and $\partial_x u_1(0) = \partial_x u_2(0)$. To write this system in substructured form, we assume again that the interface functions $\lambda_1(y)$ and $\lambda_2(y)$ are known,

$$\begin{aligned} (\partial_x + \mathcal{S}_1)(u_1)(L, y) = (\partial_x + \mathcal{S}_1)(u_2)(L, y) =: \lambda_1(y), \\ (\partial_x + \mathcal{S}_2)(u_2)(0, y) = (\partial_x + \mathcal{S}_1)(u_1)(0, y) =: \lambda_2(y), \end{aligned}$$

solve the subdomain problems, and then set

$$\begin{aligned} -(\partial_x + \mathcal{S}_1)(u_2(0, y, f, \lambda_2)) + \lambda_1 = 0, \\ \lambda_2 - (\partial_x + \mathcal{S}_2)(u_1(L, y, f, \lambda_1)) = 0. \end{aligned}$$

This is again a linear system to be solved for λ_1 and λ_2. Using a Krylov method, at each iteration two problems with the new transmission conditions need to be solved,

$$\begin{aligned} \mathcal{A}\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} &:= \begin{bmatrix} 1 & -(\partial_x + \mathcal{S}_1)(u_2(0, y, 0, \cdot)) \\ -(\partial_x + \mathcal{S}_2)(u_1(L, y, 0, \cdot)) & 1 \end{bmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_L \end{pmatrix} \\ &= \begin{pmatrix} (\partial_x + \mathcal{S}_1)(u_2(0, y, f, 0)) \\ (\partial_x + \mathcal{S}_2)(u_1(L, y, f, 0)) \end{pmatrix} =: b. \end{aligned} \tag{5.24}$$

In the Fourier domain, the symbol of the operator $\mathcal{A}$ becomes for our model problem

$$\hat{\mathcal{A}} = \begin{bmatrix} 1 & -\frac{\sqrt{\eta+k^2}-\sigma_1(k)}{\sqrt{\eta+k^2}-\sigma_2(k)} e^{-\sqrt{\eta+k^2}L} \\ -\frac{\sqrt{\eta+k^2}+\sigma_2(k)}{\sqrt{\eta+k^2}+\sigma_1(k)} e^{-\sqrt{\eta+k^2}L} & 1 \end{bmatrix}. \tag{5.25}$$

For well-posedness of the subdomain problems, we need that $\mathcal{S}_1$ is a positive operator and $\mathcal{S}_2$ a negative one, as one can also see from the denominators in the symbol of the operator $\mathcal{A}$. The iterative optimized Schwarz method is obtained when a Richardson iteration is applied to this system, and we have seen that this iteration converges in two steps, if $= \sigma_2 = -\sigma_1 = \sqrt{\eta + k^2}$, or very fast, if the symbols approximate this choice. If we choose $\sigma_2 = -\sigma_1 > 0$, then the operator becomes symmetric and its condition number equals one for the optimal choice, or it can be made small choosing good approximations. This is the heart of the optimized Schwarz methods: the optimal choice always exists, it is the Dirichlet-to-Neumann map, and good approximations lead to the optimized Schwarz methods with superior performance. The FETI methods have also started to incorporate these ideas; see, for example, the variant FETI-H presented in [12], where the authors state, "The modified Lagrangian formulation presented here can be related to alternative transmission conditions for the subdomain interfaces." FETI-H constructs (5.25) using optimized Robin conditions.

TABLE 6.1

Number of iterations of the classical Schwarz method compared to the different optimized Schwarz methods with fixed small overlap of the size $L = \frac{1}{50}$ between subdomains.

	Classical	Taylor 0	Taylor 2	Optimized 0	Two-sided optimized 0	Optimized 2
h	Schwarz as an iterative solver					
1/50	65	16	11	7	6	4
1/100	77	17	12	7	6	4
1/200	86	16	11	7	6	4
1/400	91	16	12	7	6	4
1/800	93	16	11	7	6	4
	Schwarz use as a preconditioner					
1/50	11	8	7	5	5	3
1/100	12	8	7	5	5	3
1/200	13	8	7	5	5	3
1/400	13	8	7	5	5	3
1/800	13	8	7	5	5	3

6. Numerical experiments. We perform numerical experiments for our model problem on the unit square, $\Omega = (0,1) \times (0,1)$,

$$
\begin{aligned}
(\eta - \Delta)(u) &= f \qquad \text{in } \Omega,\\
u &= 0 \qquad \text{on } \partial\Omega.
\end{aligned} \tag{6.1}
$$

We decompose the unit square Ω into two subdomains $\Omega_1 = (0,\beta) \times (0,1)$ and $\Omega_2 = (\alpha,1) \times (0,1)$, where $0 < \alpha \le \beta < 1$ and hence the overlap is $L = \beta - \alpha$. Note that we explicitly allow $\alpha = \beta$ such that the method does not have any overlap, $L = 0$. We use a finite difference discretization with the classical five-point discretization for the Laplacian and a uniform mesh with mesh parameter h.

6.1. Overlapping optimized Schwarz methods. Classically the overlap in the Schwarz method is held constant as the mesh is refined to obtain mesh independent convergence factors for the method. The same is true for optimized Schwarz methods because of Theorem 4.1, as iteration counts to reach an error reduction of $1e{-}6$ show in Table 6.1 for a fixed overlap $L = \beta - \alpha = \frac{1}{50}$. We simulate directly the error equations, $f = 0$, and use a random initial guess so that all the frequency components are present. The results show clearly how important transmission conditions are for this algorithm. Note also that while the Krylov method has a big impact on the classical Schwarz method, for the second order optimized Schwarz method the acceleration with the Krylov method does not reduce the iteration count significantly. This situation is well known for multigrid methods, which do not need Krylov acceleration either when applied to a Poisson problem. The Krylov acceleration is then used to improve the performance of the method on more complex problems.

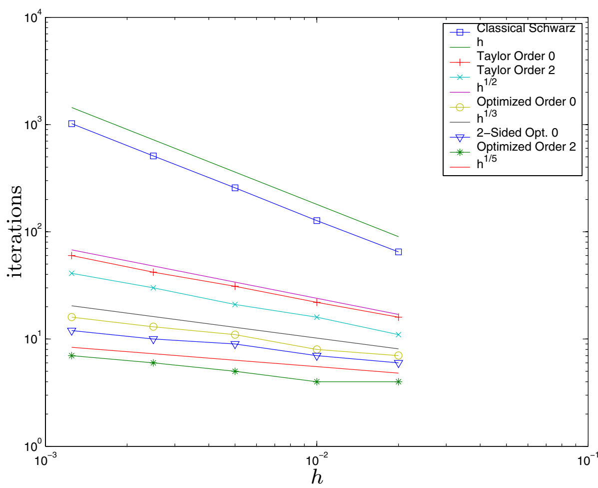
In practical computations, one can often not afford many mesh cells to overlap, so the overlap depends on the mesh parameter h. In the following experiments we choose therefore the overlap $L = \beta - \alpha = h$. Table 6.2 shows the iteration counts for this case. It is interesting to note that the second order optimized Schwarz method without Krylov acceleration is already six times faster than classical Schwarz with Krylov acceleration at high resolution.

In Figure 6.1, we show the number of iterations on a log-log plot so they can be compared to the theoretical asymptotic results. On the top, the Schwarz methods are used as iterative solvers and the numerical results show the asymptotic behavior predicted by the theory. On the bottom, the Schwarz methods are used as preconditioners. This improves the asymptotic performance by a square root, as one can

TABLE 6.2

Number of iterations of the classical Schwarz method compared to the different optimized Schwarz methods with overlap $L = h$ between subdomains.

	Classical	Taylor 0	Taylor 2	Optimized 0	Two-sided optimized 0	Optimized 2
h	Schwarz as an iterative solver					
1/50	65	16	11	7	6	4
1/100	127	22	16	8	7	4
1/200	257	31	21	11	9	5
1/400	510	42	30	13	10	6
1/800	1020	60	41	16	12	7
	Schwarz use as a preconditioner					
1/50	11	8	7	5	5	3
1/100	16	9	8	6	6	4
1/200	21	11	9	6	6	4
1/400	31	13	11	7	7	4
1/800	42	16	13	8	8	5



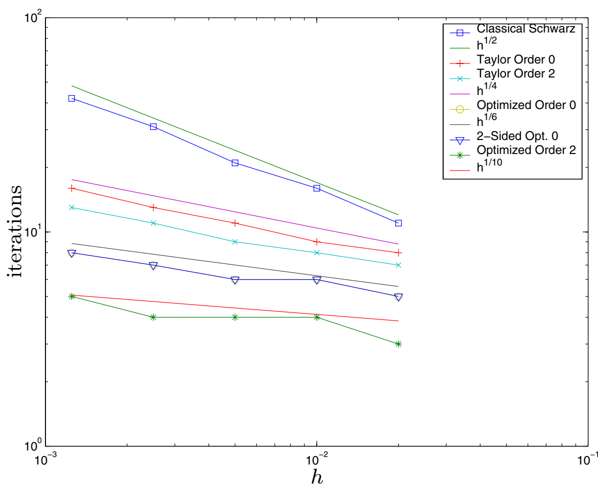


FIG. 6.1. *Number of iterations required by the classical and the optimized Schwarz methods, with overlap $L = h$. On the top the methods are used as iterative solvers, and on the bottom they are used as preconditioners for a Krylov method.*

show in ideal situations, since a square root is taken off the condition number of the preconditioned system. This is also visible in our numerical results.

We now investigate how well the continuous analysis predicts the optimal parameters to be used in the numerical setting. To this end we vary the parameter p in the Robin transmission conditions for a fixed problem of mesh size $h = \frac{1}{100}$ and count for each value of p the number of iterations to reach a residual of $1e-6$. The results for both optimized Schwarz used as an iterative solver and as a preconditioner are shown in Figure 6.2 on the top. The analysis predicts very well the optimal parameter, and when the method is used as a preconditioner, the area where the optimum is attained is widened considerably, which shows that the optimized Schwarz method is robust with respect to the optimal parameter. Similar results hold for the second order optimized Schwarz method, as one can see in Figure 6.2 in the middle when the method is used iteratively and on the bottom when used as a preconditioner.

6.2. Nonoverlapping optimized Schwarz methods. Nonoverlapping Schwarz methods are of interest if the physical properties vary from subdomain to subdomain and one has formulated the subdomain decomposition motivated by this fact; see, for example, [20]. They also facilitate the construction of nonmatching grids per subdomain and the formulation of algorithms in that case. We illustrate the performance of the optimized Schwarz methods without overlap, $\alpha = \beta$ or $L = 0$, by choosing for the mesh parameter diminishing values and counting the number of iterations the methods take to reduce the error by a factor $1e-6$. Table 6.3 shows the performance of the different optimized Schwarz methods in that case. Note that the classical Schwarz method is not shown because classical Schwarz does not converge without overlap. Comparing with the performance of the methods with overlap h, one can see that the number of iterations is by a factor 1.5–1.7 higher for the second order optimized Schwarz method, whereas the cost per subdomain is only slightly higher for the method with overlap; there are m more variables in one subdomain for matrices of size m^2. Hence a physical motivation must outweigh the increased cost of a nonoverlapping Schwarz method.

In Figure 6.3 we show the number of iterations on a log-log plot so they can be compared to the theoretical asymptotic factors.

On the left the methods are used as iterative solvers and one can see that again the numerical results show the asymptotic behavior predicted by the analysis. On the right the results are shown when the Schwarz methods are used as preconditioners, and one can see again that Krylov acceleration improves the performance by about a square root.

We finally show in Figure 6.4 how well the analysis predicts the optimization parameters in the nonoverlapping case.

6.3. An application. We now show how a nonoverlapping optimized Schwarz method can be used to compute the temperature distribution in our apartment on Durocher in Montreal. In Figure 6.5, we show on top the floor plan of our apartment with a finite element discretization and a decomposition into the different rooms: on the left is the living room, connected to the kitchen and with a long hallway to the bathroom and bedroom on the right. Insulated walls are shown in blue, the windows on top are shown in black, where we assume -20 degrees Celsius for a regular Montreal winter day, and the doors at the bottom and on the right are also shown in black. They lead to a heated public hallway, at about 15 degrees Celsius. The interfaces are shown in red, and we introduced curved interfaces and nonrectangular domains, so that the Fourier analysis presented in this paper cannot be applied any more. In

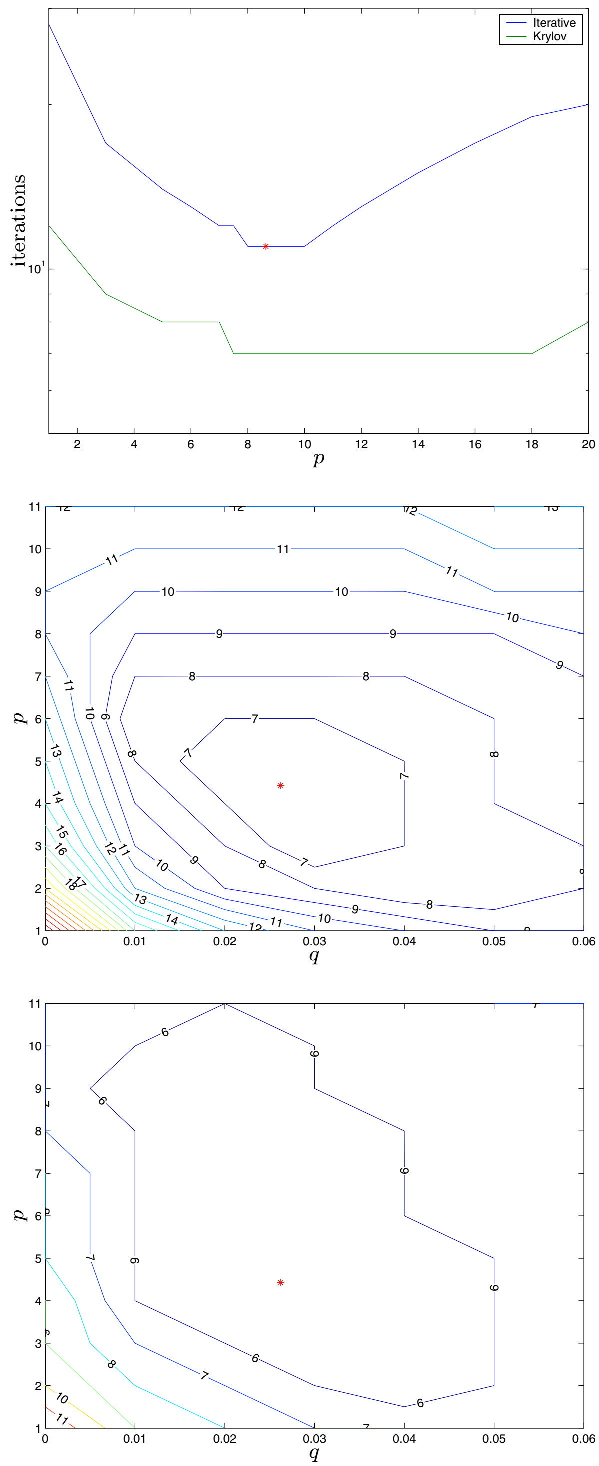
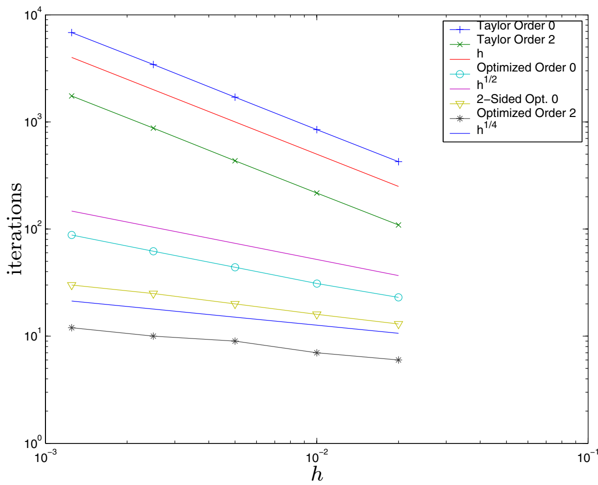


FIG. 6.2. *Optimal parameter (*) found by the analytical optimization compared to the performance of other values of the parameters: on the top for the Robin case, in the middle for the second order case used iteratively, and on the bottom used as a preconditioner.*

TABLE 6.3
Number of iterations of different optimized Schwarz methods without overlap between subdomains.

	Taylor 0	Taylor 2	Optimized 0	Two-sided optimized 0	Optimized 2
h	Optimized Schwarz as an iterative solver				
1/50	425	109	23	13	6
1/100	847	217	31	16	7
1/200	1702	434	44	20	9
1/400	3432	875	62	25	10
1/800	6824	1746	88	30	12
	Optimized Schwarz as a preconditioner				
1/50	21	15	9	8	5
1/100	28	20	11	10	5
1/200	35	26	13	11	6
1/400	46	34	15	12	6
1/800	59	45	18	13	7



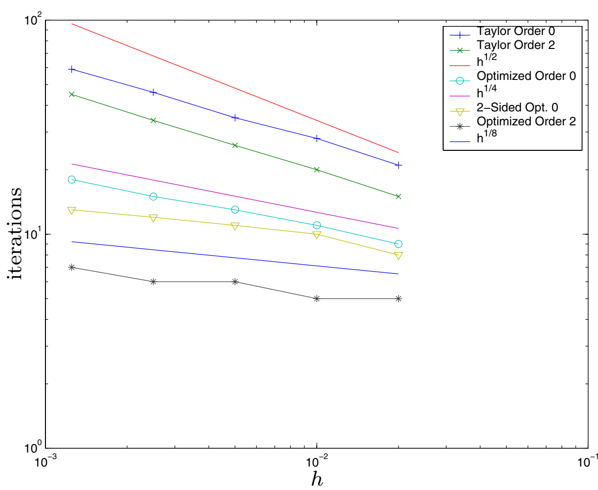


FIG. 6.3. *Asymptotic number of iterations required by the nonoverlapping optimized Schwarz methods: on the top the methods are used as iterative solvers, and on the bottom they are used as preconditioners for a Krylov method.*

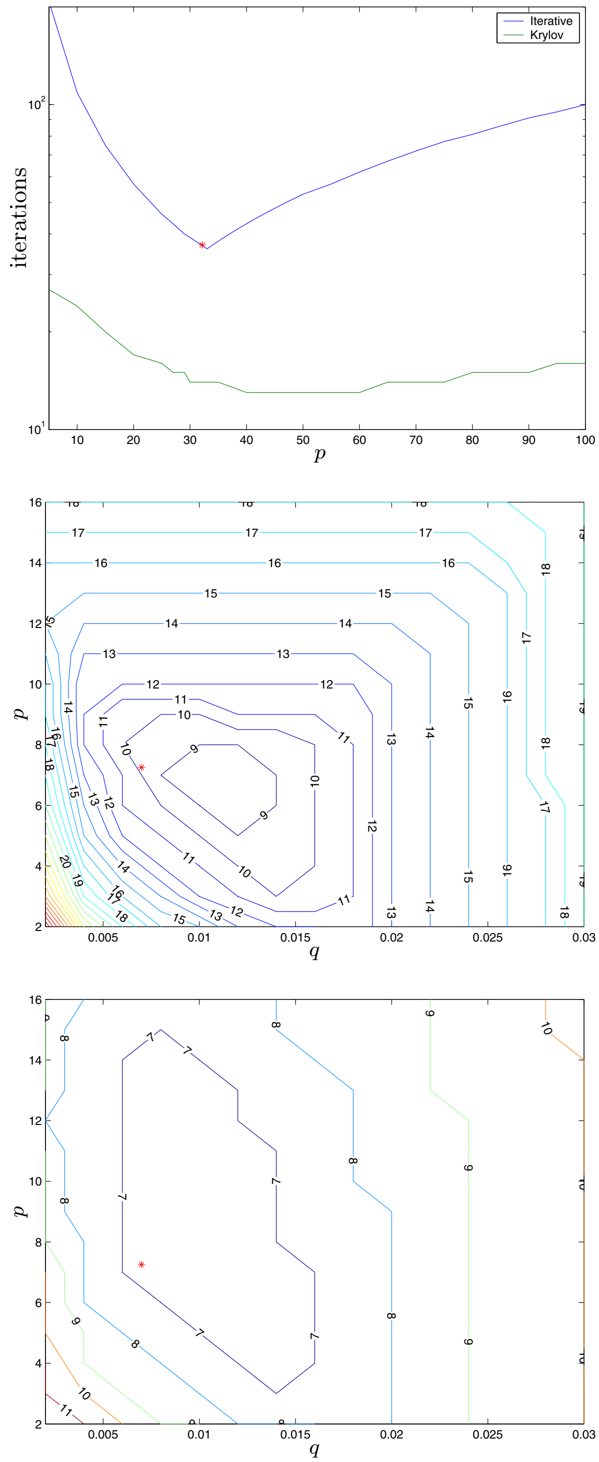


FIG. 6.4. *Comparison of the optimal parameter (*) found in the analysis with the numerical performance of other values of the parameters: for the Robin case on the top, the second order case used iteratively in the middle, and as a preconditioner on the bottom.*

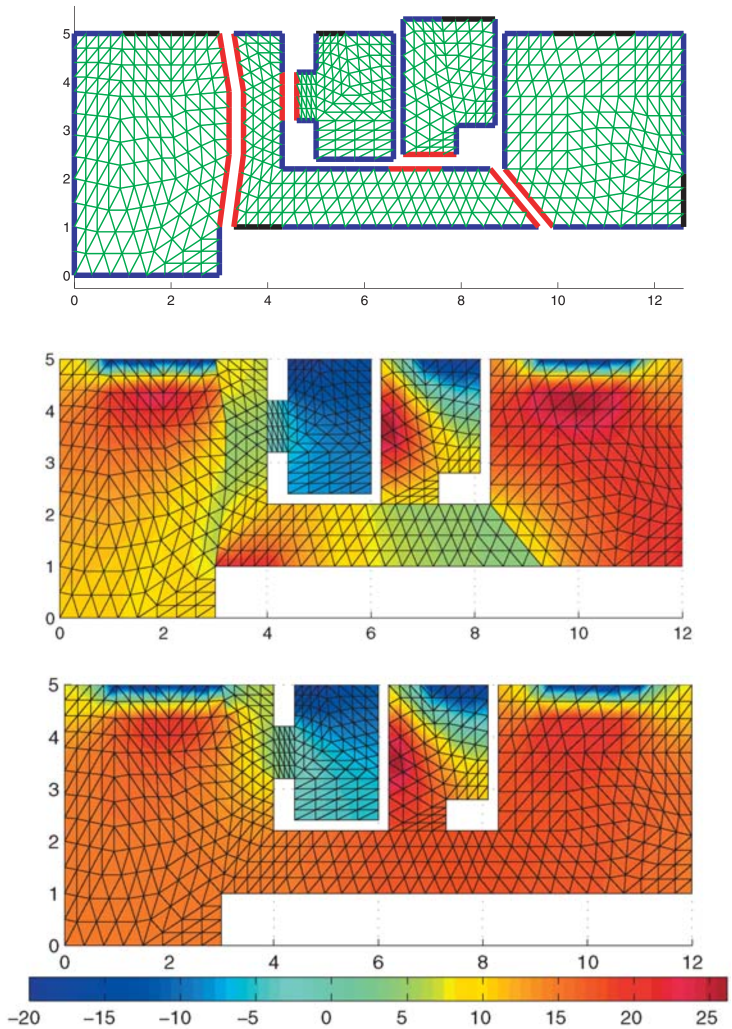


Fig. 6.5. *On top the decomposition of a two-dimensional cross section of an apartment in Montreal, in the middle the first iteration, and at the bottom the final temperature distribution computed in winter with an optimized Schwarz method.*

the middle in Figure 6.5 we show the first iteration of the optimized Schwarz method with Robin transmission conditions, where one can clearly see the isolated effect of the heaters and warm doors in each subdomain: the iterate is discontinuous. In Figure 6.5 at the bottom we show the final result of the simulation, which is now continuous. The method took 25 iterations to converge to a relative residual of $1e-3$ in the iterative case and 12 iterations when used as a preconditioner, using the optimal parameter $p^* = 2.7207$ from the two-subdomain theory. Refining once more, the method took 32 iterations in the iterative case and 13 in the preconditioned case, with the optimal parameter $p^* = 3.8576$ from the two-subdomain theory. The ratio in the iterative case is $32/25 = 1.28 \approx 2^{1/3} = 1.26$, as predicted by the two-subdomain theory for the simple two-subdomain case with straight interfaces, and in the preconditioned case, the ratio is $13/12 = 1.08 \approx 2^{1/6} = 1.12$. This shows that although the Fourier analysis cannot be applied in the more general case, the results predicted by the theory for the two-subdomain case are also observed in more practical situations.

The results of this simulation were interesting to us: one can see that while the heaters in the living room on the left and the bedroom on the right are well placed to block the cold from the windows, the heater on the left wall in the bathroom is not effective to keep the room warm, a fact we strongly felt in winter. Also, the kitchen is not heated and stays cold, except when cooking and baking.

7. Conclusion. We introduced the reader to a new class of Schwarz methods, the optimized Schwarz methods. The algorithm is the same as for the classical Schwarz method and it can be used either iteratively or as a preconditioner. The difference is a new type of transmission conditions between subdomains, instead of the classical Dirichlet condition. We analyzed for a symmetric positive definite model problem and two subdomains the influence of the transmission conditions on the convergence factor of the Schwarz algorithm. We showed both analytically and numerically that the optimized Schwarz methods have a greatly improved performance compared to the classical Schwarz method. The number of iterations required to achieve a certain accuracy is by a factor smaller, often more than an order of magnitude. This performance is achieved without an increased cost for the subdomain solves, since the same type of matrix problem has to be solved in the subdomains, and the new subdomain matrices have the same bandwidth as the original ones. We also proved that the optimized Schwarz methods are always faster than the classical Schwarz method and since their implementation is not more difficult than the implementation of a classical Schwarz method, they represent a very attractive alternative. We finally showed in numerical experiments that the results derived for the simple two-subdomain configuration with a straight interface also apply in more complicated situations in practice, where Fourier analysis cannot be applied any more.

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