Numerical Linear Algebra with Applications, Vol. 3(2), 113–124 (1996)

# Two-stage Multisplitting Methods with Overlapping Blocks

Mark T. Jones

*Department of Computer Science, The University of Tennessee, Knoxville, Tennessee, 37996-1301, USA*

and

Daniel B. Szyld

*Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122-2585, USA*

Parallel two-stage multisplitting methods with overlap for the solution of linear systems of algebraic equations are studied. It is shown that, under certain hypotheses, the method with overlap is asymptotically faster than that without overlap. Experiments illustrating this phenomenon are presented.

KEY WORDS iterative methods; linear systems; multisplittings; overlap; parallel algorithms.

## 1. Introduction

Consider the iterative solution of linear systems of the form $Ax = b$ on parallel computers, where $A \in \mathbb{R}^{n\times n}$ is non-singular. O'Leary and White [1] introduced the notion of a multisplitting of $A$ as a collection of $n \times n$ matrices $(M_\ell, N_\ell, E_\ell)$, $\ell = 1, \cdots, L$, where $A = M_\ell - N_\ell$, $M_\ell$ are non-singular, $E_\ell$ are non-negative diagonal, and $\sum_{\ell=1}^{L} E_\ell = I$. The iterative method associated with this multisplitting is the following.

**Algorithm 1.1. (Multisplitting)** *Given the initial vector* $x_0$.

For $i = 0, 1, \cdots$, until convergence.

CCC 1070–5325/96/020113–12
©1996 by John Wiley & Sons, Ltd.

*Received 14 March 1994*
*Accepted 3 March 1995*

$$\begin{aligned} &\text{For } \ell = 1 \text{ to } L \\ &\qquad M_\ell y_\ell = N_\ell x_i + b \\ &x_{i+1} = \sum_{\ell=1}^{L} E_\ell y_\ell \end{aligned} \tag{1.1}$$

The strength of Algorithm 1.1 is that the linear systems (1.1) can be solved in parallel by different processors. Several authors have studied this algorithm, e.g. Frommer and Mayer [2], Neumann and Plemmons [3], and White [4,5]. Often the splittings $A = M_\ell - N_\ell$ are all the same, i.e., $M_\ell = M$, $N_\ell = N$, $\ell = 1, \cdots, L$, and the weighting matrices are such that the multisplitting algorithm is a rendition of classical block iterative methods, such as block-Jacobi; see, e.g., Varga [6], Young [7] and Definition 2.3 (a) in the next section. Thus, Algorithm 1.1 can be interpreted as an extension and parallel generalization of the block-Jacobi method; see e.g., [8]. In fact, the common application of the multisplitting algorithm is when the splittings $A = M_\ell - N_\ell$ correspond to $M_\ell$ being a diagonal block of $A$; see e.g. Definition 2.3 (b) in the next section.

We can write directly $x_{i+1} = Hx_i + c$, where $H = \sum_{\ell=1}^{L} E_\ell M_\ell^{-1} N_\ell$ is then the iteration matrix of the multisplitting algorithm, and $c$ is an appropriate vector. In the case $M_\ell = M$, $\ell = 1, \cdots, L$, as is the case of block-Jacobi, we have $H = M^{-1}N$.

In many practical applications, the linear systems (1.1) represent $(d-1)$-dimensional grids of a $d$-dimensional model. In those cases, and in others, the solution of these linear systems by direct methods implies a certain amount of fill. This fill produces excessive storage requirements either local to each processor, or in a global memory, as well as the associated increased computational cost. Thus, it is natural to consider the solution of the linear systems (1.1) by iterative methods. When the linear systems (1.1) are solved iteratively, one obtains the two-stage multisplitting algorithm [9]. It is based on a collection of $n \times n$ matrices $(M_\ell, B_\ell, C_\ell, N_\ell, E_\ell)$, $\ell = 1, \cdots, L$, where $A = M_\ell - N_\ell$, $M_\ell = B_\ell - C_\ell$, $M_\ell$ and $B_\ell$ are non-singular, $E_\ell$ are non-negative diagonal, and $\sum_{\ell=1}^{L} E_\ell = I$.

**Algorithm 1.2.** **(Two-stage Multisplitting)** *Given the initial vector $x_0$, and the fixed number $p$ of inner iterations.*

$$\begin{aligned} &\text{For } i = 0, 1, \cdots, \text{ until convergence.} \\ &\qquad \text{For } \ell = 1 \text{ to } L \\ &\qquad\qquad y_{\ell,0} = x_i \\ &\qquad\qquad \text{For } j = 0 \text{ to } p-1 \\ &\qquad\qquad\qquad B_\ell y_{\ell,j+1} = C_\ell y_{\ell,j} + N_\ell x_i + b \\ &\qquad x_{i+1} = \sum_{\ell=1}^{L} E_\ell y_{\ell,p} \end{aligned}$$

Since the number of inner iterations $p$ is considered fixed, we can write directly $x_{i+1} =$

$H_p x_i + c$, where

$$H_p = \sum_{\ell=1}^{L} E_\ell (B_\ell^{-1} C_\ell)^p + \sum_{\ell=1}^{L} E_\ell \sum_{j=0}^{p-1} (B_\ell^{-1} C_\ell)^j B_\ell^{-1} N_\ell \tag{1.2}$$

and $c$ is an appropriate vector; cf. [9–11].

When the group of grid points represented by the matrices $(M_\ell, N_\ell, E_\ell)$ overlap with those of other adjacent groups of points in the grid, it has been observed that convergence is improved; see e.g. [5,9]. This overlap may occur either by (i) assigning different weights to the contributions of each group, i.e., by having entries with values different than zero or one in $E_\ell$, or (ii) by having the matrices $M_\ell$ capture the contributions of larger groups of nodes, while the weighting matrices $E_\ell$ have entries zero or one in a smaller set of variables; see further Definition 2.3 (b). Frommer and Pohl [12] provide a proof for the observed behavior in case (ii) for Algorithm 1.1. Namely, they show that the method with this kind of overlap converges asymptotically faster than the method without overlap. Examples in [12] imply that similar results for weights different than zero or one do not hold in general. We concentrate our work on case (ii).

In this paper we first provide a slight generalization of the comparison theorem in [12], then prove a similar result for the Two-stage Multisplitting Algorithm 1.2, and illustrate our results with numerical experiments.

## 2. Notation, Preliminaries and First Comparison Result

We include in this section definitions and results collected from the literature, and present the generalization to the comparison theorem of Frommer and Pohl [12].

**Definition 2.1.** *[6] A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is an M-matrix if it is non-singular, $a_{ij} \leq 0$ for $i \neq j$, and $A^{-1} \geq O$.*

**Lemma 2.1.** *[6,13] Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an M-matrix and denote its diagonal part by $D = diag(a_{11}, \cdots, a_{nn})$. Then,*

*(a) $a_{ii} > 0$ for $i = 1, \cdots, n$, and in particular $D$ is non-singular.*
*(b) Any matrix $B$ satisfying $A \leq B \leq D$ is an M-matrix.*
*(c) If $C \in \mathbb{R}^{n\times n}$ is another M-matrix satisfying $A \leq C$, then $C^{-1} \leq A^{-1}$.*

**Definition 2.2.** *The splitting $A = M - N$ is called*
*(a) regular if $M^{-1} \geq O$ and $N \geq O$ [6,14],*
*(b) left weak regular if $M^{-1} \geq O$ and $M^{-1}N \geq O$ [13,15],*
*(c) weak regular if it is left weak regular and $NM^{-1} \geq O$, and*
*(d) weak if $M^{-1}N \geq O$ [16,17].*

We point out that often in the literature, what we call a left weak regular splitting is known as weak regular; see, e.g. Berman and Plemmons [18], but here, as in [11,13,15], we need to make the distinction between left weak regular splitting and weak regular splittings.

**Lemma 2.2.** *[6,18], Let $A = M - N$ be a left weak regular splitting. Then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq O$.*

**Lemma 2.3.** *[11] Given a non-singular matrix $A \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{n\times n}$ such that $(I-H)^{-1}$ exists, there exists a unique pair of matrices $F$, $G$, such that $H = F^{-1}G$ and $A = F - G$, where $F$ is non-singular. The matrices are $F = A(I-H)^{-1}$ and $G = F - A$.*

In the context of this lemma, we say that $H$ *induces* the splitting $A = F - G$.

**Lemma 2.4.** *[16] Let $A = M - N = \tilde{M} - \tilde{N}$ be weak splittings with $H = M^{-1}N$ and $\tilde{H} = \tilde{M}^{-1}\tilde{N}$ such that $\rho(H) < 1$ and $\rho(\tilde{H}) < 1$. Let $x \geq 0$, $z \geq 0$ be such that $Hx = \rho(H)x$ and $\tilde{H}z = \rho(\tilde{H})z$. If either*

$$Nx \geq 0 \tag{2.1}$$

*or $\tilde{N}z \geq 0$ with $z > 0$, and if $M^{-1} \geq \tilde{M}^{-1}$, then $\rho(H) \leq \rho(\tilde{H})$.*

We remark that in Lemma 2.4, the matrix $A$ need not have a non-negative inverse, since the splittings are not necessarily left weak regular, cf. Lemma 2.2. Lemma 2.4 can be applied in particular to the case of two left weak regular splittings and the added hypothesis needed is (2.1), for example. The following simple lemma can be used to test this additional hypothesis.

**Lemma 2.5.** *Let $A = M - N$ be a weak splitting with $H = M^{-1}N$ and $\rho = \rho(H) < 1$. Let $x$ be the Frobenius eigenvector of $H$, i.e., $Hx = \rho x \geq 0$. Then $Nx \geq 0$ if and only if $Mx \geq 0$ if and only if $Ax \geq 0$.*

*Proof* The lemma follows from the equalities $Nx = \rho Mx$ and $Ax = M(I-H)x = M(1-\rho)x$. ∎

**Definition 2.3.** *[12] Let $S_1, \cdots, S_L$ be a partition of $\{1, \cdots, n\}$, i.e., the sets $S_\ell$ are pairwise disjoint non-empty subsets of $\{1, \cdots, n\}$, so that $\cup_{\ell=1}^{L} S_\ell = \{1, \cdots, n\}$. Moreover, let $S_\ell \subset T_\ell \subset \{1, \cdots, n\}$, $\ell = 1, \cdots, L$, with at least one $\ell \in \{1, \cdots, L\}$, for which $S_\ell \neq T_\ell$.*

*(a) The multisplitting $(M_\ell, N_\ell, E_\ell)$, $\ell = 1, \cdots, L$ of $A \in \mathbb{R}^{n\times n}$, where*

$$\begin{cases} M_\ell = ((M_\ell)_{ij}) \ \text{ with } (M_\ell)_{ij} = \begin{cases} a_{ij} & \text{if } i \in S_\ell \text{ and } j \in S_\ell, \\ a_{ii} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \\ N_\ell = M_\ell - A, \\ E_\ell = ((E_\ell)_{ij}) \ \text{ with } (E_\ell)_{ij} = \begin{cases} 1 & \text{if } i = j \in S_\ell, \\ 0 & \text{otherwise,} \end{cases} \end{cases} \tag{2.2}$$

*is called a (non-overlapping) block-Jacobi splitting of $A$.*

*(b) Any multisplitting $(\tilde{M}_\ell, \tilde{N}_\ell, \tilde{E}_\ell)$, $\ell = 1, \cdots, L$ of $A \in \mathbb{R}^{n\times n}$ where*

$$\begin{cases} \tilde{M}_\ell = ((\tilde{M}_\ell)_{ij}) \ \text{ with } (\tilde{M}_\ell)_{ij} = \begin{cases} a_{ij} & \text{if } i \in T_\ell \text{ and } j \in T_\ell \\ a_{ii} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \\ \tilde{N}_\ell = \tilde{M}_\ell - A, \\ \tilde{E}_\ell = ((\tilde{E}_\ell)_{ij}) \ \text{ with } (\tilde{E}_\ell)_{ii} = 0 \text{ if } i \notin T_\ell \end{cases} \tag{2.3}$$

*is called an overlapping block-Jacobi multisplitting.*

We remark that the condition $S_\ell \subset T_\ell$, with at least one of the inclusions being strict, means that $T_1, \cdots, T_L$ is not a partition of $\{1, \cdots, n\}$. The non-empty intersection of pairs of sets $T_\ell$ and $T_{\ell'}$, say, is indeed the overlap. Furthermore, the entries $(\tilde{E}_\ell)_{ii}$ in the diagonal matrix defined in (2.3) cannot all have value 1, for $i \in T_\ell$, for otherwise the multisplitting condition $\sum_{\ell=1}^{L} \tilde{E}_\ell = I$ would not hold. In fact, one can choose to use the diagonal weighting matrices $E_\ell$ of (2.2) corresponding to the non-overlapping sets, while using the larger (overlapping) matrices defined in (2.3). This is the usage studied by Frommer and Pohl [12] and the case we study in this paper, e.g., in Theorem 2.1.

Note also that due to the structure of the matrices in (2.2), the iteration matrix of the block-Jacobi splitting,

$$H = \sum_{\ell=1}^{L} E_\ell M_\ell^{-1} N_\ell \tag{2.4}$$

can also be expressed as the iteration matrix $H = M^{-1}N$ corresponding to the splitting $A = M - N$, where

$$M = (M_{ij}) \text{ with } M_{ij} = \begin{cases} a_{ij} & \text{if } i, j \in S_\ell \text{ for some } \ell \in \{1, \cdots, L\} \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

cf. Lemma 2.3, as well as [12], and the discussions in Section 1 and in [9, Section 3].

**Theorem 2.1.** *Let $A^{-1} \geq 0$. Let $(M_\ell, N_\ell, E_\ell)$ and $(\tilde{M}_\ell, \tilde{N}_\ell, E_\ell)$ denote the non-overlapping block-Jacobi splitting and the overlapping block-Jacobi multisplitting of A, respectively, as defined in Definition 2.3 with the same non-overlapping weighting matrices $E_\ell$. Let H and $\tilde{H}$ be the corresponding iteration matrices, i.e., H defined as in (2.4) and*

$$\tilde{H} = \sum_{\ell=1}^{L} E_\ell \tilde{M}_\ell^{-1} \tilde{N}_\ell \tag{2.6}$$

*Let the induced splittings be weak and convergent, i.e., $H \geq 0$, $\tilde{H} \geq 0$, $\rho(H) < 1$, $\rho(\tilde{H}) < 1$, and $Mx \geq 0$, where M is defined in (2.5) and x is such that $Hx = \rho(H)x$. Furthermore, assume that $M_\ell^{-1} \leq \tilde{M}_\ell^{-1}$, $\ell = 1, \cdots, L$. Then, $\rho(H) \geq \rho(\tilde{H})$.*

Frommer and Pohl [12] proved the preceding comparison result as well as a similar one for the case of $A$ being an $M$-matrix, using the stronger hypotheses that $A = M_\ell - N_\ell$ are regular and the splittings $A = \tilde{M}_\ell - \tilde{N}_\ell$ are left weak regular. Their proof is based on a lemma in [19]. The proof of Theorem 2.1 follows in a similar way using Lemmas 2.4 and 2.5. In the next section we extend Theorem 2.1 to the Two-stage Multisplitting Algorithm 1.2 using the hypotheses that all splittings are weak regular and $Nx \geq 0$, where $A = M - N$, and $M^{-1}Nx = \rho(M^{-1}N)x$, which we show are satisfied.

Theorem 2.1 says that the method with overlap (with non-overlapping weighting matrices) is asymptotically faster than the one without overlap. It is natural then to ask if increasing the amount of overlap decreases the asymptotic convergence rate. Frommer and Pohl [12] already observed that in their experience, this is the case. In Table 1 we report the spectral radius for iteration matrices of the form (2.6) corresponding to different amounts of overlap in the overlapping block-Jacobi multisplitting (2.3). The $25 \times 25$ matrix corresponds to a nine-point discretization of the Laplacian on a square periodic grid, i.e., the matrix $A$

Table 1. Spectral radius for the iteration matrix of overlapping block-Jacobi for different amounts of overlap

| elements in $(T_1, T_2)$ | (10,15) | (15,15) | (15,20) | (20,20) |
|---|---|---|---|---|
| overlap (elements in $T_1 \cap T_2$) | 0 | 5 | 10 | 15 |
| $\rho(\tilde{H})$ | 0.7145 | 0.5164 | 0.3276 | 0.2068 |

has nine non-zero diagonals and a typical row is $[00\cdots 0aba000b1b000aba0\cdots 0]$, where $a = -0.05$ and $b = -0.2$. There are two splittings, i.e., $L = 2$, and $E_1$ has the first 10 diagonal entries equal to one, and zero otherwise. The fact that the spectral radius decreases as the overlap increases can be readily appreciated.

It is still an open question to prove or disprove this observation. Frommer and Pohl [12] showed that the splittings induced by $H$ and $\tilde{H}$ are left weak regular; see also [1,9]. Thus, they pointed out, the lemma in [19] cannot be used to study this question. As it turns out, the additional hypothesis (2.1) is not satisfied by the matrices corresponding to non-empty overlap studied in Table 1, and therefore Lemma 2.4 cannot be used to study this question either.

It should be pointed out that Theorem 2.1 is not inconsistent with the results by Elsner, Neumann and Vemmer [20] where non-overlapping block-Jacobi methods are considered. In particular, in Example 4 of [20] the hypotheses of Theorem 2.1 are not satisfied.

## 3. More Comparison of Rates of Convergence

We first show the convergence of a two-stage non-overlapping block-Jacobi multisplitting for a matrix with a non-negative inverse.

**Lemma 3.1.** *Let $A^{-1} \geq O$. Let $(M_\ell, N_\ell, E_\ell)$ be a non-overlapping block-Jacobi splitting of $A$. Let the splittings $A = M_\ell - N_\ell$ be regular and the splittings $M_\ell = B_\ell - C_\ell$ be weak regular and defined in such a way that if $(M_\ell)_{ij} = O$, then $(B_\ell)_{ij} = O$. Let $p$ be the number of inner iterations of the two-stage multisplitting $(M_\ell, B_\ell, C_\ell, N_\ell, E_\ell)$. Then, the induced splitting $A = F_p - G_p$ is a weak regular splitting, and thus convergent. Furthermore, $G_p x \geq 0$, where $H_p x = \rho(H_p)x$, $H_p = F_p^{-1} G_p$.*

*Proof* From the hypothesis that if $(M_\ell)_{ij} = O$, then $(B_\ell)_{ij} = O$, it follows that if $A = M - N$ is defined as in (2.5), we can write $M = B - C$, where $B_{ij} = (B_\ell)_{ij}$ if $i, j \in S_\ell$ for some $\ell \in \{1, \cdots, L\}$ and zero otherwise. Therefore, as in the remark after Definition 2.3, we can rewrite $H_p$ defined in (1.2) as

$$H_p = (B^{-1}C)^p + \sum_{j=0}^{p-1} (B^{-1}C)^j B^{-1} N \tag{3.1}$$

Moreover, $A = M - N$ is a regular splitting and $M = B - C$ is a weak regular splitting. Let $U = B^{-1}C \geq O$. From Lemma 2.3, after some simple matrix algebra we have $A = F_p - G_p$

where

$$F_p^{-1} = \sum_{j=0}^{p-1} U^j B^{-1} = (I - U^p)(I - U)^{-1} B^{-1} = (I - U^p) M^{-1} \geq O \tag{3.2}$$

and

$$G_p = F_p U^p + N = M(I - U^p)^{-1} U^p + N = M U^p (I - U^p)^{-1} + N \tag{3.3}$$

The last equality follows from the fact that $U^p$ and $(I - U^p)^{-1}$ commute. From (3.1), it is easy to see that $H_p = F_p^{-1} G_p \geq O$. Since

$$M = B - C = (I - CB^{-1})B \tag{3.4}$$

we have

$$G_p F_p^{-1} = M(B^{-1}C)^p M^{-1} + N F_p^{-1} = (CB^{-1})^p + N F_p^{-1} \geq O$$

Thus, $A = F_p - G_p$ is a weak regular splitting, and thus by Lemma 2.2 $\rho(H_p) < 1$. Let $\rho = \rho(H_p)$. Let $V = U^p \geq O$. Since $H_p = V + F_p^{-1} N$, then $H_p \geq V$, and thus

$$\rho(V) \leq \rho \tag{3.5}$$

which implies that $(\rho I - V)$ is non-singular and its inverse is non-negative. Let $H_p x = \rho x \geq O$. Then,

$$\begin{aligned}
\rho F_p x &= G_p x \\
\rho \left( M(I - V)^{-1} \right) x &= M(I - V)^{-1} V x + N x \\
\rho (I - V)^{-1} x &= (I - V)^{-1} V x + M^{-1} N x \\
(I - V)^{-1} (\rho I - V) x &= M^{-1} N x \\
(I - V)^{-1} x &= (\rho I - V)^{-1} M^{-1} N x
\end{aligned} \tag{3.6}$$

Consider now $G_p x$ using the form (3.3) and replace the identity (3.6) to obtain

$$G_p x = MV(\rho I - V)^{-1} M^{-1} N x + N x$$

It suffices to show now that

$$M(B^{-1}C)^p \left( \rho I - (B^{-1}C)^p \right)^{-1} M^{-1} \geq O$$

Using (3.4) and some algebra and matrix commutations, this matrix is equal to

$$(CB^{-1})^p \left( \rho I - (CB^{-1})^p \right)^{-1}$$

which is non-negative since $M = B - C$ is a weak regular splitting and $\rho(CB^{-1})^p = \rho(B^{-1}C)^p < \rho$ from (3.5). ■

The tools used in the second part of the proof of Lemma 3.1 are those found in [11, Theorems 4.2 and 5.1].

**Theorem 3.1.** *Let $A^{-1} \geq O$. Let $(M_\ell, N_\ell, E_\ell)$ and $(\tilde{M}_\ell, \tilde{N}_\ell, E_\ell)$ denote the non-overlapping block-Jacobi splitting and the overlapping block-Jacobi multisplitting of A, respectively, as defined in Definition 2.3 with the same non-overlapping weighting matrices $E_\ell$. Let the splittings $A = M_\ell - N_\ell$ and $A = \tilde{M}_\ell - \tilde{N}_\ell$ be regular. Let the splittings $M_\ell = B_\ell - C_\ell$ and $\tilde{M}_\ell = \tilde{B}_\ell - \tilde{C}_\ell$ be weak regular, and defined in such a way that if $(M_\ell)_{ij} = O$, then $(B_\ell)_{ij} = O$. In addition, let $\tilde{B}_\ell^{-1} \geq B_\ell^{-1} \geq O$, and $\tilde{B}_\ell^{-1}\tilde{C}_\ell \geq B_\ell^{-1}C_\ell \geq O$. Let $H_p$ and $\tilde{H}_p$ be the iteration matrices of the two-stage multisplitting methods defined by $(M_\ell, B_\ell, C_\ell, N_\ell, E_\ell)$ and $(\tilde{M}_\ell, \tilde{B}_\ell, \tilde{C}_\ell, \tilde{N}_\ell, E_\ell)$, with p inner iterations, respectively. Then,*

$$\rho(H_p) \geq \rho(\tilde{H}_p) \tag{3.7}$$

*Proof* Let $A = F_p - G_p = \tilde{F}_p - \tilde{G}_p$ be such that $H_p = F_p^{-1}G_p$ and $\tilde{H}_p = \tilde{F}_p^{-1}\tilde{G}_p$, cf. Lemma 2.3. Let $A = M - N$ and $M = B - C$ as in the proof of Lemma 3.1. Also, as in (3.2) we have that

$$F_p^{-1} = \sum_{j=0}^{p-1}(B^{-1}C)^j B^{-1} = \sum_{\ell=1}^{L} E_\ell \sum_{j=0}^{p-1}(B_\ell^{-1}C_\ell)^j B_\ell^{-1}$$

$$\leq \sum_{\ell=1}^{L} E_\ell \sum_{j=0}^{p-1}(\tilde{B}_\ell^{-1}\tilde{C}_\ell)^j \tilde{B}_\ell^{-1} = \tilde{F}_p^{-1}$$

If $H_p$ is irreducible, and $x$ is such that $H_p x = \rho(H_p)x$, then $x > 0$ [6], and (3.7) follows from Lemmas 3.1 and 2.4. If $H_p$ is reducible, consider $A - \varepsilon E$, where $E$ is a matrix with all ones and $\varepsilon > 0$. Consider further the regular splittings $A - \varepsilon E = M_\ell - (N_\ell + \varepsilon E) = \tilde{M}_\ell - (\tilde{N}_\ell + \varepsilon E)$. Then, from (1.2) or (3.1) it follows that corresponding matrix $H_p(\varepsilon) > 0$ and in particular it is irreducible. Thus, the inequality $\rho(H_p(\varepsilon)) \leq \rho(\tilde{H}_p(\varepsilon))$ follows for all $\varepsilon > 0$ and the theorem follows by taking the limit $\varepsilon \to 0$. ∎

The following theorem applies to block-Jacobi algorithms for $M$-matrices. This situation appears often in applications and the hypotheses are sometimes easier to check than Theorem 3.1.

**Theorem 3.2.** *Let A be an M-matrix and D be its diagonal. Let $(M_\ell, N_\ell, E_\ell)$ and $(\tilde{M}_\ell, \tilde{N}_\ell, E_\ell)$ denote the non-overlapping block-Jacobi splitting and the overlapping block-Jacobi multisplitting of A, respectively, as defined in Theorem 3.1 with the same non-overlapping weighting matrices $E_l$. Let the splittings $M_\ell = B_\ell - C_\ell$ and $\tilde{M}_\ell = \tilde{B}_\ell - \tilde{C}_\ell$ be defined in such a way that $M_\ell \leq B_\ell \leq D$ and $\tilde{M}_\ell \leq \tilde{B}_\ell \leq D$. In addition, let the overlap be such that $Q_\ell = M_\ell - \tilde{M}_\ell \geq B_\ell - \tilde{B}_\ell = R_\ell \geq O$. Let $H_p$ and $\tilde{H}_p$ be the iteration matrices of the two-stage multisplitting methods defined by $(M_\ell, B_\ell, C_\ell, N_\ell, E_\ell)$ and $(\tilde{M}_\ell, \tilde{B}_\ell, \tilde{C}_\ell, \tilde{N}_\ell, E_\ell)$, with p inner iterations, respectively. Then, $\rho(H_p) \geq \rho(\tilde{H}_p)$.*

*Proof* By the definition of the block-Jacobi algorithm the matrices $M_\ell$ and $\tilde{M}_\ell$, lie between $A$ and $D$ and thus, by Lemma 2.1, they are $M$-matrices. Since $A \leq M_\ell \leq D$, then $N_\ell \geq 0$. Similarly, $\tilde{N}_\ell \geq 0$. Therefore, the splittings $A = M_\ell - N_\ell = \tilde{M}_\ell - \tilde{N}_\ell$ are regular splittings. Using the same arguments, $B_\ell$ and $\tilde{B}_\ell$ are $M$-matrices and the splittings $M_\ell = B_\ell - C_\ell$

and $\tilde{M}_\ell = \tilde{B}_\ell - \tilde{C}_\ell$ are also regular splittings. Given that $B_\ell \geq \tilde{B}_\ell$ are $M$-matrices, then $\tilde{B}_\ell^{-1} \geq B_\ell^{-1} \geq O$. Also, $\tilde{B}_\ell^{-1}\tilde{C}_\ell \geq B_\ell^{-1}C_\ell \geq O$, because

$$\begin{aligned} M_\ell - \tilde{M}_\ell &= Q_\ell \\ (B_\ell - \tilde{B}_\ell) - (C_\ell - \tilde{C}_\ell) &= Q_\ell \\ \tilde{C}_\ell - C_\ell &= Q_\ell - R_\ell \geq O \end{aligned}$$

The conditions of Theorem 3.1 are met and the theorem follows directly. ∎

## 4. Numerical Experiments

The two-stage methods with $p > 1$ and overlap do have a significant advantage on parallel architectures with non-uniform memory access times: the inner iterations can be accomplished with references exclusively to memory that is 'local' to that processor. Examples of architectures that fall into this class are the Intel iPSC/860, Intel DELTA, and the IBM SP/1; on such computers, a non-local reference implies the sending/receiving of messages. The two-stage methods can gain an advantage from this data locality only if the number of outer iterations necessary to converge to a solution decreases as $p$ increases. Similarly, the use of overlap is only beneficial if the number of outer iterations necessary for convergence is smaller with overlap than without. In [9] it is shown that asymptotically, the two-stage method, where $p > 1$ is slower than the standard block algorithm where $p = 1$. However, it is also shown experimentally in [9] that this 'extra' work is offset by the gain in computational efficiency afforded by the improved locality of operations; thus, in certain situations, $p > 1$ is more efficient than $p = 1$. In Theorem 3.1, it is shown that asymptotically, the two-stage method with overlap is faster than without overlap. If the extra work required by the use of overlap is offset by a reduction in the number of outer iterations, then the use of overlap will reduce the computation time. Therefore, there should be some 'optimal' number of inner iterations, $p$, and size of overlap, $q$, which would depend on the architecture and the matrix in question.

To briefly demonstrate the practical advantages of using $p > 1$ inner iterations with overlap on parallel processors with non-uniform memory access times, experiments were performed on the Intel DELTA parallel computer. To run the experiments, a parallel block iterative code was written with the block-Jacobi method as the outer iteration and the point Gauss–Seidel method as the inner iteration. In the case of overlap, the unknowns associated with the overlap were ordered first in the point Gauss–Seidel inner iteration. This choice is based on the experience reported in [5] and [9]. A $4 \times 4$ array of 16 processors of the DELTA was used to perform the experiment. Both the number of inner iterations, $p$, and the size of the overlap, $q$, were varied to determine the optimal values for a particular problem. After each outer iteration, the residual is computed and the algorithm is stopped if the norm of the residual is less than $10^{-3}$.

A matrix was generated using the standard nine-point stencil to discretize Laplace's equation on a $64 \times 64$ grid. Every element of the right-hand side was identical and chosen such that the norm of the right-hand side was one. To minimize communication, each of the 16 processors was assigned a $16 \times 16$ subgrid. The processors were arranged in a square grid and numbered in a 'natural' ordering; i.e., processor 0 received the lower left subgrid and processor 15 received the upper right subgrid. The grid points, and corresponding unknowns,

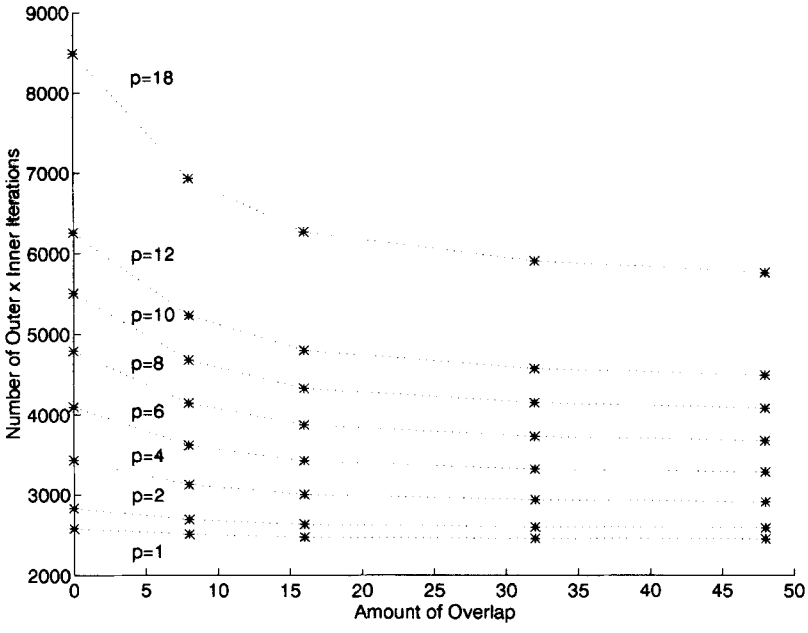


Figure 1. The total number of inner iterations performed as a function of the amount of overlap for different numbers of inner iterations

were numbered such that all the grid points on a processor were numbered consecutively, and in a natural ordering on that subgrid, and such that all of processor 0's unknowns were numbered first, then all of processor 1's, etc. When $q \neq 0$, a processor would overlap with the $q$ unknowns ordered before and the $q$ unknowns ordered after its unknowns (where such unknowns exist). On a $4 \times 4$ grid, using 4 processors such a scheme yields the following numbering:

$$\begin{array}{cccc} 9 & 11 & 13 & 15 \\ 8 & 10 & 12 & 14 \\ 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 \end{array}$$

where processor 0 has unknowns 0–3, processor 1 has 4–7, etc. With $q = 2$ processor 1 would have overlap unknowns 2, 3, 8, and 9.

An examination of the results in Figure 1 shows that, as expected from the theoretical results, the number of outer iterations decreases as $p$ increases, but the total number of inner iterations increases. In addition, as predicted, the number of outer iterations is smaller with overlap than without. Moreover, in agreement with Table 1, the number of outer iterations decreases as the overlap, $q$, increases. As mentioned in Section 2, whether this is true in general is an open question.

Figure 2 shows that the cost of the extra work per outer iteration for certain values of $p$ and $q$ is offset by the benefit of locality and the reduction in the number of outer iterations. In particular, the lowest solution time among those plotted occurs at $p = 6$ and $q = 16$.

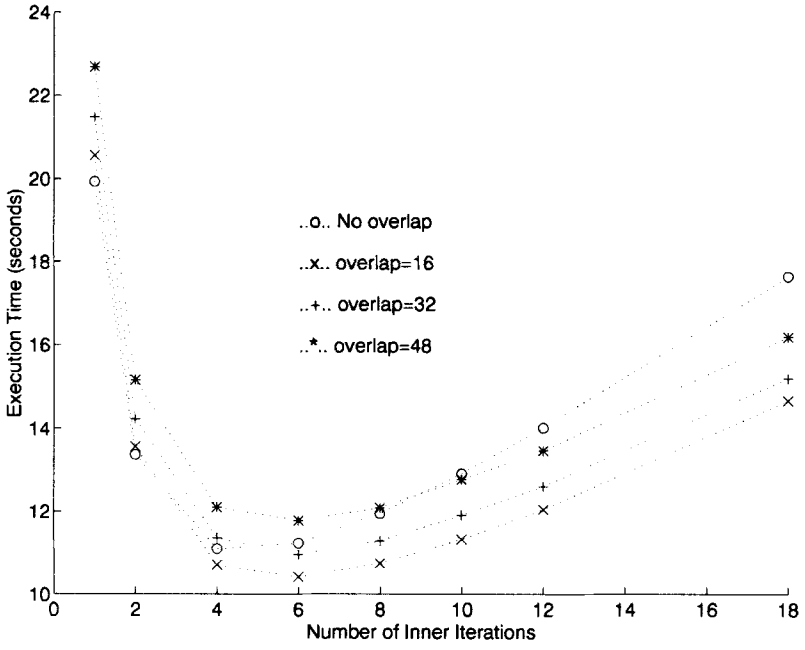


Figure 2. The total execution time as a function of the number of inner iterations for different amounts of overlap

## Acknowledgements

The authors thank the Mathematics and Computer Science Division of Argonne National Laboratory for the use of the Intel DELTA computer. They also thank Andreas Frommer, and an anonymous referee for their comments and suggestions. The work of the second author was supported by the National Science Foundation grant DMS-9201728.

## REFERENCES

1. D. P. O'Leary and R. E. White. Multi-splittings of matrices and parallel solution of linear systems. *SIAM Journal on Algebraic and Discrete Methods*, 6, 630–640, 1985.
2. A. Frommer and G. Mayer. Convergence of relaxed parallel multisplitting methods. *Linear Algebra and its Applications*, 119, 141–152, 1989.
3. M. Neumann and R. J. Plemmons. Convergence of parallel multisplitting methods for $M$-matrices. *Linear Algebra and its Applications*, 88–89, 559–574, 1987.
4. R. E. White. Multisplittings and parallel iterative methods. *Computer Methods in Applied Mechanics and Engineering*, 64, 567–577, 1987.
5. R. E. White. Multisplitting of a symmetric positive definite matrix. *SIAM Journal on Matrix Analysis and Applications*, 11, 69–82, 1990.
6. R. S. Varga. *Matrix Iterative Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey, 1962.
7. D. M. Young. *Iterative Solution of Large Linear Systems*. Academic Press, New York, 1971.
8. R. Bru, V. Migallón, J. Penadés and D. B. Szyld. Parallel, synchronous and asynchronous two-stage multisplitting methods. *Electronic Transactions on Numerical Analysis*, 3, 24–38, 1995.

9. D. B. Szyld and M. T. Jones. Two-stage and multisplitting methods for the parallel solution of linear systems. *SIAM Journal on Matrix Analysis and Applications*, 13, 671–679, 1992.
10. A. Frommer and D. B. Szyld. $H$-splittings and two-stage iterative methods. *Numerische Mathematik*, 63, 345–356, 1992.
11. P. J. Lanzkron, D. J. Rose and D. B. Szyld. Convergence of nested classical iterative methods for linear systems. *Numerische Mathematik*, 58, 685–702, 1991.
12. A. Frommer and B. Pohl. A comparison result for multisplittings based on overlapping blocks and its application to waveform relaxation methods. *Numerical Linear Algebra with Applications*, 2, 335–346, 1995.
13. J. M. Ortega and W. C. Rheinboldt. *Iterative Solution of Nonlinear Equations in Several Variables*. Academic Press, New York and London, 1970.
14. R. S. Varga. Factorization and normalized iterative methods. In R. E. Langer, editor, *Boundary Problems in Differential Equations*, pages 121–142, The University of Wisconsin Press. Madison, 1960.
15. J. J. Moré. Global convergence of Newton–Gauss–Seidel methods. *SIAM Journal on Numerical Analysis*, 8, 325–336, 1971.
16. I. Marek and D. B. Szyld. Comparison theorems for weak splittings of bounded operators. *Numerische Mathematik*, 58, 387–397, 1990.
17. I. Marek and D. B. Szyld. Splittings of $M$-operators: irreducibility and the index of the iteration operator. *Numerical Functional Analysis and Optimization*, 11, 529–553, 1990.
18. A. Berman and R. J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Academic Press, New York, third edition, 1979. Reprinted by SIAM, Philadelphia, 1994.
19. L. Elsner. Comparisons of weak regular splittings and multisplitting methods. *Numerische Mathematik*, 56, 283–289, 1989.
20. L. Elsner, M. Neumann and B. Vemmer. The effect of the number of processors on the convergence of the parallel block Jacobi method. *Linear Algebra and its Applications*, 154–156, 311–330, 1991.