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Convergence analysis of two-stage waveform relaxation method for the initial value problems

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Abstract

We present a class of two-stage waveform relaxation methods for solving the initial value problems of ordinary differential equations. By making use of the Laplace transform we discuss the convergence of these methods, and derive sufficient conditions for guaranteeing their convergence when the system matrices are specifically H-matrices. Also we discuss the monotonicity of the convergence sequence. Further we compare the sequences generated by two different inner splittings of the same initial value problem.

Keywords: Waveform relaxation method; Two-stage iteration; Convergence

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1. Introduction

Consider the following initial value problem of ordinary differential equations with a given initial value x_0:

$$\dot{x}(t) + Ax(t) = f(t), \quad x(0) = x_0, \tag{1}$$

where A is an $n \times n$ complex matrix, and $f(t)$ is a vector-valued function and is assumed to be continuous.

Basically, the traditional numerical methods for iteratively solving the problem (1) first use a feasible discretization formula to approximate the differential operator, resulting in a series of algebraic linear systems, and then, compute approximate solutions of such algebraic linear systems by either a direct or an iterative algorithm, obtaining an approximate solution of the original problem (1).

In 1987, Nevanlinna [5] studied the waveform relaxation method, also called the dynamic iteration method, for the initial value problem of ordinary differential equations (1). In fact, the waveform relaxation method is a continuous time method iterating directly on functions. Hence, it can save computing time and computer storage. The waveform relaxation method has also been applied to solve BVM problems of ODEs with circulant-based preconditioners. See for instance [4].

Let $A = M-N$ be a splitting of the matrix A. Then the waveform relaxation method can be described as follows:

$$\begin{cases} \dot{x}^{(k)}(t) + Mx^{(k)}(t) = Nx^{(k-1)}(t) + f(t), \\ \\ x^{(k)}(0) = x_0, \end{cases} \qquad k = 1, 2, \ldots, \tag{2}$$

with $x^{(0)}(t)$ a given initial function satisfying $x^{(0)}(0) = x_0$.

In actual applications, we often need not solve $x^{(k)}(t)$ in (2) exactly, but solve it iteratively by a prescribed iteration method. This naturally results in an inexact variant of the waveform relaxation method (2) for solving the problem (1). If a waveform relaxation method resulting from the splitting $M = F-G$ of the matrix M, with F being nonsingular, is employed to compute the current iterate $x^{(k)}(t)$ in (2), then we particularly obtain a two-stage waveform relaxation method for the problem (1).

Historically, the two-stage iterative method was first proposed for solving systems of linear equations by Nichols in 1973, see [7] and also [8] for instance. Then, it has been further developed and studied by many authors, e.g., Golub and Overton [3], Migallón and Penadés [6] and Bai and Wang [2], etc. Bai also discussed a class of two-stage iterative methods for the systems of weakly nonlinear equations in [1].

By developing the idea of designing the two-stage iterative method for the system of linear equations to the ordinary differential equation, in this paper we establish a class of Two-Stage Waveform Relaxation (TSWR) method for solving the initial value problem of ordinary differential equation (1). By making use of the Laplace transform, we derive a neat iteration scheme of the TSWR method in the frequency domain, and then prove its convergence when the outer and the inner splittings $A = M-N$ and $M = F-G$ are both convergent splittings. In particular, when the matrix A is an H-matrix which will be defined in Section 4 or see [13], we give sufficient conditions for guaranteeing the convergence of the TSWR method.

In Section 5, the convergent sequence's monotonicity is discussed. It will be showed that the monotonicity is completely dependent on the initial function. But no matter what initial function is chosen, the sequence will convergent to the same value. Especially for the different inner splittings, not only does the monotonicity of the two sequences are guaranteed respectively, also the relation of the values size is kept.

2. The two-stage waveform relaxation method

Denote by $g(t) := Nx^{(k-1)}(t) + f(t)$ and $y(t) := x^{(k)}(t)$. Then at each iteration step of the waveform relaxation method (2) we need to solve an initial value problem of ordinary differential equation of the form

$$\dot{y}(t) + My(t) = g(t), \quad y(0) = x_0.$$

If the problem (3) is solved by another waveform relaxation iteration induced by the splitting $M = F-G$, i.e.,

$$\begin{cases} \dot{y}^{(\ell)}(t) + Fy^{(\ell)}(t) = Gy^{(\ell-1)}(t) + g(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ell = 1, 2, \ldots, m, \\ y^{(\ell)}(0) = x_0, \end{cases} \tag{3}$$

where $y^{(0)}(t) := x^{(k-1)}(t)$ and m is a given number of inner iteration steps, then we obtain the following two-stage waveform relaxation method for the initial value problem of ordinary differential equation (1).

Method 2.1 (THE TSWR METHOD). *Given a starting function $x^{(0)}(t)$. Suppose that we have obtained the iteration functions $\{x^{(\ell)}(t)\}_{\ell=1}^{k-1}$, then the next iteration function $x^{(k)}(t)$ is defined by $x^{(k)}(t) := y^{(k,m_k)}(t)$, with $\{y^{(k,\ell)}(t)\}_{\ell=1}^{m_k}$ being successively computed through solving the initial value problems of ordinary differential equations (3), i.e.,*

$$\begin{cases} \dot{y}^{(k,\ell)}(t) + Fy^{(k,\ell)}(t) = Gy^{(k,\ell-1)}(t) + Nx^{(k-1)}(t) + f(t), \\ y^{(k,\ell)}(0) = x_0, \end{cases} \tag{4}$$

where $y^{(k,0)}(t) := x^{(k-1)}(t)$ and m_k is the number of inner iteration steps within the kth outer iteration.

The TSWR method is called stationary if the number of inner iteration steps is fixed, i.e., $m_k \equiv (k = 1, 2, \ldots)$. Otherwise, it is called nonstationary. In particular, when $G = 0$, the TSWR method reduces to the waveform relaxation method (2).

3. Convergence analysis

Let $\mathscr{L}$ be the operator of the Laplace transform, and denote by

$$\mathscr{X}(z) = \mathscr{L}[x(t)], \quad \mathscr{Y}(z) = \mathscr{L}[y(t)] \quad \text{and} \quad \mathscr{F}(z) = \mathscr{L}[f(t)].$$

Then according to the definition of the TSWR method we know that

$$\mathscr{X}^{(k)}(z) = \mathscr{Y}^{(k,m_k)}(z) \quad \text{and} \quad \mathscr{Y}^{(k,0)}(z) = \mathscr{X}^{(k-1)}(z).$$

Because the Laplace transform has the property

$$\mathscr{L}[\dot{x}(t)] = z\mathscr{X}(z) - \mathscr{X}(0),$$

after applying the operator $\mathscr{L}$ to both sides of (4) we obtain

$$\mathscr{L}[\dot{y}^{(k,\ell)}(t) + Fy^{(k,\ell)}(t)] = \mathscr{L}[Gy^{(k,\ell-1)}(t) + Nx^{(k-1)}(t) + f(t)].$$

Hence, it holds that

$$z\mathscr{Y}^{(k,\ell)}(z) + F\mathscr{Y}^{(k,\ell)}(z) = G\mathscr{Y}^{(k,\ell-1)}(z) + \mathscr{N}\mathscr{Y}^{(k,0)}(z) + \mathscr{F}(z) + \mathscr{Y}^{(k,\ell)}(0),$$

or in the neat form,

$$(zI + F)\mathscr{Y}^{(k,\ell)}(z) = G\mathscr{Y}^{(k,\ell-1)}(z) + \mathscr{N}\mathscr{X}^{(k-1)}(z) + \mathscr{F}(z) + x_0.$$

Here we have used the facts that

$$\mathscr{Y}^{(k,0)}(z) = \mathscr{X}^{(k-1)}(z) \quad \text{and} \quad \mathscr{Y}^{(k,\ell)}(0) = x_0.$$

If we further assume that the matrix $(zI + F)$ is nonsingular, then

$$\begin{aligned}
\mathscr{Y}^{(k,\ell)}(z) &= (zI+F)^{-1}G\mathscr{Y}^{(k,\ell-1)}(z) + (zI+F)^{-1}N\mathscr{X}^{(k-1)}(z) \\
&\quad + (zI+F)^{-1}(\mathscr{F}(z) + x_0) \\
&= (zI+F)^{-1}G[G\mathscr{Y}^{(k,\ell-2)}(z) + N\mathscr{X}^{(k-1)}(z) \\
&\quad + (\mathscr{F}(z) + x_0)] + (zI+F)^{-1}N\mathscr{X}^{(k-1)}(z) \\
&\quad + (zI+F)^{-1}(\mathscr{F}(z) + x_0) \\
&= \ldots = \left\{ [(zI+F)^{-1}G]^{\ell} + \sum_{i=0}^{\ell-1} [(zI+F)^{-1}G]^{i}(zI+F)^{-1}N \right\} \mathscr{X}^{(k-1)}(z) \\
&\quad + \sum_{i=0}^{\ell-1} [(zI+F)^{-1}G]^{i}(zI+F)^{-1}(\mathscr{F}(z) + x_0).
\end{aligned}$$

Now, by taking $\ell = m_k$, we obtain an explicit expression of the TSWR method as follows:

$$\mathscr{X}^{(k)}(z) = T_{m_k}\mathscr{X}^{(k-1)}(z) + \sum_{i=0}^{m_k-1}[(zI+F)^{-1}G]^i(zI+F)^{-1}(\mathscr{F}(z)+x_0), \tag{5}$$

where

$$T_{m_k} = [(zI+F)^{-1}G]^{m_k} + \sum_{i=0}^{m_k-1}[(zI+F)^{-1}G]^i(zI+F)^{-1}N \tag{6}$$

is the iteration matrix of the TSWR method.

Let $x^*(t)$ be the exact solution of the initial value problem of ordinary differential equation (1), and $e^{(k)}(t) := x^{(k)}(t) - x^*(t)$ be the error at the kth iteration step of the TSWR method. Denote by

$$\mathscr{X}^*(z) = \mathscr{L}[x^*(t)] \quad \text{and} \quad \mathscr{E}^{(k)}(z) = \mathscr{L}[e^{(k)}(t)].$$

Because $x^*(t)$ satisfies

$$\dot{x}^*(t) + Fx^*(t) = Gy^*(t) + Nx^*(t) + f(t), \quad x^*(0) = x_0,$$

by applying the Laplace transform operator $\mathscr{L}$ to both sides of the above equality and after direct computations we obtain

$$\mathscr{X}^*(z) = T_{m_k}\mathscr{X}^*(z) + \sum_{i=0}^{m_k-1}[(zI+F)^{-1}G]^i(zI+F)^{-1}(\mathscr{F}(z)+x_0).$$

After subtracting this equality from (5) we get $\mathscr{E}^{(k)}(z) = T_{m_k}\mathscr{E}^{(k-1)}(z)$, $(k = 1,2,\ldots)$. It then follows from successive recurrence that

$$\mathscr{E}^{(k)}(z) = T_{m_k} \cdot T_{m_{k-1}} \cdots T_{m_0} \cdot \mathscr{E}^{(0)}(z).$$

Therefore, to prove the convergence of the TSWR method, we only need to demonstrate that $\lim_{k\to\infty}\mathscr{E}^{(k)}(z) = O$ holds for any complex z such that $\Re(z) \geqslant 0$, where O denotes the zero matrix and $\Re(z)$ the real part of the complex number z. Obviously, a sufficient condition for guaranteeing the validity of this fact is that $\lim_{k\to\infty}\sup_{\Re(z)\geqslant 0}\|\mathscr{E}^{(k)}(z)\| = 0$ holds for certain matrix norm $\|\cdot\|$.

Theorem 3.1. *Let $A = M-N$ and $M = F-G$ be splittings of the matrices A and M, respectively, and both M and F be positive real. Assume that there exists a matrix norm $\|\cdot\|$ such that*

$$\Theta := \sup_{\Re(z)\geqslant 0}\|(zI+M)^{-1}N\| < 1,$$

and the splitting $M = F-G$ is a convergent splitting, i.e., the spectral radius of the matrix $(zI + F)^{-1}G$ is uniformly less than 1 for all complex number z satisfying $\Re(z) \geqslant 0$. Denote by

$$\theta := \sup_{\Re(z) \geqslant 0} \rho((zI + F)^{-1}G)) < 1.$$

If the sequence of the inner iteration steps satisfies $\lim_{k\to\infty} m_k = \infty$, *then*

(*a*) *the sequence* $\{x^{(k)}(t)\}_{k=0}^{\infty}$ *generated by the TSWR method is convergent*;
(*b*) *the* R_1 *convergence factor of the sequence* $\{x^{(k)}(t)\}_{k=0}^{\infty}$ *is at most* Θ.

Proof. Evidently, the positive real property of the matrices M and F implies that both $(zI + M)$ and $(zI + F)$ are nonsingular for all complex number z such that $\Re(z) \geqslant 0$.

Because

$$\theta = \sup_{\Re(z) \geqslant 0} \rho((zI + F)^{-1}G)) < 1,$$

we know that

$$\sup_{\Re(z) \geqslant 0} \rho(((zI + F)^{-1}G)^p) = \left(\sup_{\Re(z) \geqslant 0} \rho((zI + F)^{-1}G)\right)^p$$
$$= \theta^p \to 0, \quad \text{as } p \to +\infty.$$

Therefore, for any $\varepsilon > 0$, there exists a positive integer p_0 such that for all $p \geqslant P_0$ there holds

$$\sup_{\Re(z) \geqslant 0} \|((zI + F)^{-1}G)^p\| < \varepsilon.$$

It then follows from $\lim_{k\to\infty} m_k = \infty$ that there is a positive integer k_0 such that when $k \geqslant k_0$ there holds $m_k > m_{k_0} \geqslant p_0$ and

$$\sup_{\Re(z) \geqslant 0} \|((zI + F)^{-1}G)^{m_k}\| < \varepsilon \text{ (for all } k \geqslant k_0).$$

From (6) we know that

$$\begin{aligned} \|T_{m_k}\| &= \left\| [(zI + F)^{-1}G]^{m_k} + \sum_{i=0}^{m_k-1} [(zI + F)^{-1}G]^i (zI + F)^{-1}N \right\| \\ &\leqslant \|(zI + F)^{-1}G]^{m_k}\| + \|\{I - [(zI + F)^{-1}G]^{m_k}\}[I - (zI + F)^{-1}G]^{-1} \\ &\times (zI + F)^{-1}N\| \leqslant \|(zI + F)^{-1}G]^{m_k}\| + \|I - [(zI + F)^{-1}G]^{m_k}\| \\ &\times \|(zI + M)^{-1}N\| \leqslant \varepsilon + (1 + \varepsilon)\Theta := \phi(\varepsilon) \end{aligned}$$

holds for all $k \geqslant k_0$. Let ε be small enough such that $\varepsilon \leqslant \frac{1-\Theta}{1+\Theta}$. Then we obtain $\phi(\varepsilon) < 1$. Hence,

$$\begin{aligned}\|\mathscr{E}^{(k)}(z)\| &= \|T_{m_k}\cdot T_{m_{k-1}}\cdots T_{m_0}\cdot \mathscr{E}^{(0)}(z)\|\\ &\leqslant \left(\prod_{i=k_0}^{k}\|T_{m_i}\|\right)\cdot\left(\prod_{i=0}^{k_0-1}\|T_{m_i}\|\right)\cdot\|\mathscr{E}^{(0)}(z)\|\\ &\leqslant \phi(\varepsilon)^{k-k_0+1}\cdot\left(\prod_{i=0}^{k_0-1}\|T_{m_i}\|\right)\cdot\|\mathscr{E}^{(0)}(z)\| \leqslant \phi(\varepsilon)^{k-k_0+1}c_{k_0}\cdot\|\mathscr{E}^{(0)}(z)\|,\end{aligned}$$

where $c_{k_0} = (\prod_{i=0}^{k_0-1}\|T_{m_i}\|)$ is a constant. This estimate immediately implies that

$$\lim_{k\to\infty}\sup_{\Re(z)\geqslant 0}\|\mathscr{E}^{(k)}(z)\| = 0.$$

Therefore, the sequence $\{\mathscr{E}^{(k)}(z)\}_{k=0}^{\infty}$, or the sequence $\{x^{(k)}(t)\}_{k=0}^{\infty}$ generated by the TSWR method, is convergent. This proves the conclusion (a).

Moreover, it obviously holds that

$$\lim_{k\to\infty}\sup_{\Re(z)\geqslant 0}\|\mathscr{E}^{(k)}(z)\|^{\frac{1}{k}} \leqslant \lim_{k\to\infty}(\phi(\varepsilon)^{k-k_0+1}c_{k_0})^{\frac{1}{k}}\cdot\lim_{k\to\infty}\sup_{\Re(z)\geqslant 0}\|\mathscr{E}^{(0)}(z)\|^{\frac{1}{k}} = \phi(\varepsilon).$$

By considering that ε is arbitrary, we immediately know that

$$\lim_{k\to\infty}\sup_{\Re(z)\geqslant 0}\|\mathscr{E}^{(k)}(z)\|^{\frac{1}{k}} \leqslant \Theta,$$

which shows that the R_1 convergence factor of the sequence $\{\mathscr{E}^{(k)}(z)\}_{k=0}^{\infty}$, or the sequence $\{x^{(k)}(t)\}_{k=0}^{\infty}$ generated by the TSWR method, is at most Θ. This proves the conclusion (b). □

It is clear from Theorem 3.1(b) that the global convergence rate of the TSWR method is, at least, as fast as the waveform relaxation method (2) denned only by its outer iteration, provided the sequence $\{m_k\}_{k=0}^{\infty}$ of the inner iteration steps tends to infinity.

Otherwise, if $m_k(k = 0,1,2,\ldots)$ is sufficient large as k is large enough, we can also demonstrate the convergence of the TSWR method.

Theorem 3.2. *Let $A = M-N$ and $M = F-G$ be splittings of the matrices A and M, respectively, and both M and F be positive real. Assume*

$$\Theta := \sup_{\Re(z)\geqslant 0}\|(zI+M)^{-1}N\| < 1.$$

Let $\tilde{k}$ be a positive integer such that for all $k \geqslant \tilde{k}$, it holds that

$$\|[(zI+F)^{-1}G]^k\| \leqslant \delta < \frac{1-\Theta}{1+\Theta}.$$

If the sequence $\{m_k\}_{k=0}^{\infty}$ *of the inner iteration steps satisfy*, $\lim_{k\to\infty} m_k > \tilde{k}$, *then*

(*a*) *the sequence* $\{x^{(k)}(t)\}_{k=0}^{\infty}$ *generated by the TSWR method is convergent*;
(*b*) *the* R_1 *convergence factor of the sequence* $\{x^{(k)}(t)\}_{k=0}^{\infty}$ *is at most* $\delta + (1+\delta)\Theta < 1$.

Proof. It is analogous to the proof of Theorem 3.1, and is thus omitted. □

4. Sufficient convergence conditions

An $n \times n$ real matrix $A = (a_{ij})$ is said to be nonnegative, denoted by $A \geqslant O$, if all of its entries are nonnegative. The absolute value of the matrix A is defined as $|A| = (|a_{ij}|)$. Obviously, it holds that $|A| \geqslant O$ and $|AB| \leqslant |A||B|$, where B is a matrix having compatible size with A. We use $\rho(A)$ to denote the spectral radius of the matrix A. From [9,10,12] we know that $\rho(A) \leqslant \rho(B)$ if $|A| \leqslant B$.

The matrix A is called an L-matrix if $a_{ii} \geqslant 0 (i = 1, 2, \ldots, n)$ and $a_{ij} \leqslant 0 (i \neq j, i, j = 1, 2, \ldots, n)$. It is called an M-matrix if A is a nonsingular L-matrix satisfying $A^{-1} \geqslant O$. The comparison matrix of A is defined as $\langle A \rangle = (\langle a_{ij} \rangle)$, where $\langle a_{ij} \rangle = |a_{ij}|$ if $i = j$, and $\langle a_{ij} \rangle = -|a_{ij}|$ if $i \neq j$, A is said to be an H-matrix if its comparison matrix $\langle A \rangle$is an M-matrix. From [9,10] we know that if A is an M-matrix and B is an L-matrix satisfying $A \leqslant B$, then B is also an M-matrix. Moreover, if A is an H-matrix, then $|A^{-1}| \leqslant \langle A \rangle^{-1}$.

Let $A = M - N$ be a splitting of the matrix A. Then it is called a regular splitting if $M^{-1} \geqslant O$ and $N \geqslant O$, a weak regular splitting if $M^{-1} \geqslant O$ and $M^{-1}N \geqslant 0$, an M-splitting if M is an M-matrix and $N \geqslant O$, an H-splitting if $\langle M \rangle - |N|$ is an M-matrix, and an H-compatible splitting if $\langle A \rangle = \langle M \rangle - |N|$. If the splitting $A = M - N$ is an H-splitting, then both A and M are H-matrices and it holds that

$$\rho(M^{-1}N) \leqslant \rho(\langle M \rangle^{-1}|N|) < 1.$$

In addition, if the splitting $A = M - N$ is an H-compatible splitting and A is an H-matrix, then it is an H-splitting and thus convergent. See for instance [1,2,6,9–11].

Theorem 4.1. *Let* $A = M - N$ *be an* H-*splitting, and* $M = F - G$ *be an* H-*compatible splitting. Assume that all diagonal entries of the matrix* F *are nonnegative. Then, the TSWR method is convergent for any sequence* $\{m_k\}_{k=1}^{\infty}$ *of inner iteration steps.*

Proof. Because $A = M - N$ is an H-splitting and $M = F - G$ is an H-compatible splitting, we know that M is an H-matrix. Thus,

$$\rho(M^{-1}N) \leqslant \rho(|M^{-1}N|) \leqslant \rho(|M^{-1}||N|) \leqslant \rho(\langle M \rangle^{-1}|N|) < 1$$

and

$$\rho(F^{-1}G) \leqslant \rho(|F^{-1}G|) \leqslant \rho(|F^{-1}||G|) \leqslant \rho(\langle F\rangle^{-1}|G|) < 1.$$

Let $F = D_F - B_F$, with $D_F = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ being the diagonal matrix of F, and denote $D = zI + D_F$. Then we have $D_F \geqslant 0$ and $|D_F| \leqslant |D|$. Noticing that $|D|^{-1}|B_F| \leqslant |D_F|^{-1}|B_F|$ and F is an H-matrix, we know that the matrix $zI + F$ is nonsingular and satisfies

$$\begin{aligned}|(zI+F)^{-1}| &= |(D-B_F)^{-1}| = \left|\sum_{i=0}^{\infty}(D^{-1}B_F)^i D^{-1}\right| \leqslant \sum_{i=0}^{\infty}(|D|^{-1}|B_F|)^i|D|^{-1} \\ &\leqslant \sum_{i=0}^{\infty}(|D_F|^{-1}|B_F|)^i|D_F|^{-1} = (|D_F|-|B_F|)^{-1} = \langle F\rangle^{-1}\end{aligned}$$

for any complex number z such that $\Re(z) \geqslant 0$.

Now, from (6) we obtain

$$\begin{aligned}|T_{m_k}| &= \left|[(zI+F)^{-1}G]^{m_k} + \sum_{i=0}^{m_k-1}[(zI+F)^{-1}G]^i(zI+F)^{-1}N\right| \\ &\leqslant (\langle F\rangle^{-1}|G|)^{m_k} + \sum_{i=0}^{m_k-1}(\langle F\rangle^{-1}|G|)^{m_k}\langle F\rangle^{-1}|N|.\end{aligned}$$

By the assumptions of this theorem we know that $\langle M\rangle - |N|$ is a regular splitting of a monotone matrix, say $\overline{A}$, and $\langle M\rangle = \langle F\rangle - |G|$ is also a regular splitting. Obviously, the matrix $\overline{T}_{m_k} := |T_{m_k}|$ can be considered as the iteration matrix at the kth iteration step of the nonstationary two-stage iteration method for the linear system $\overline{A}\bar{x} = \bar{b}$ with respect to the outer splitting $\overline{A} = \langle M\rangle - |N|$ and the inner splitting $\langle M\rangle = \langle F\rangle - |G|$. Therefore, there exists a maximum norm, say $\|\cdot\|_{\max}$, and a constant $\vartheta \in [0,1)$ such that $\|\overline{T}_{m_k}\|_{\max} \leqslant \vartheta < 1$. See [1,2,6]. This shows the convergence of the TSWR method for any sequence $\{m_k\}_{k=1}^{\infty}$ of inner iteration steps. □

5. Monotonicity and comparison theory

Theorem 5.1. *Let $A = M - N$ and $M = F - G$ be two splittings, matrix $N \geqslant O$, $G \geqslant O$ where, O is the zero matrix. Assume that $\{x_1^p\}, \{x_2^p\}$ are two sequences that generated by the TSWR method and starting from the functions $\{x_1^0\}$ and $\{x_2^0\}$ respectively, if $\dot{x}_1^0 + Ax_1^0 \leqslant f, \dot{x}_2^0 + Ax_2^0 \geqslant f$, and $x_1^0 \leqslant x_2^0$, then*

(*a*) $x_1^p \leqslant x_1^{p+1} \leqslant x_2^{p+1} \leqslant x_2^p, p = 0, 1, 2 \ldots;$

(*b*) $\lim_{p\to\infty} x_1^p = x_1^* = x_2^* = \lim_{p\to\infty} x_2^p$ *and* $x_1^* = x_2^*$ *is the unique solution of the initial value problem,* (*c*) *For any*, $z_0 \in \mathscr{R}^n$ *satisfying* $x_1^0 \leqslant z^0 \leqslant x_2^0$, *the sequence*, $\{z^p\}$ *starting from* z^0 *and generated by the TSWR method, satisfies* $x_1^p \leqslant z^p \leqslant x_2^p$ *then* $\lim_{p\to\infty} z^p = x_1^* = x_2^*$.

Proof. (a) Firstly, we compare x_1^p and x_1^{p+1}. By the definition of the TSWR method, we know that $y_1^0 = x_1^0$. After the first step iteration, we get

$$\dot{y}_1^1 + Fy_1^1 = Gy_1^0 + Ny_1^0 + f.$$

By the assumption $A = F - G - N$, we get

$$\dot{y}_1^1 - \dot{y}_1^0 + F(y_1^1 - y_1^0) = f - Ay_1^0 - \dot{y}_1^0.$$

Let we have $h_1 = y_1^1 - y_1^0$, since $\dot{x}_1^0 + Ax_1^0 \geqslant f$ we have

$$\dot{h}_1 + Fh_1 = f - Ay_1^0 - \dot{y}_1^0 \geqslant O,$$

where O is the zero vector. By simple proof, we can get $h_1 \geqslant O$, i.e. $y_1^1 \geqslant y_1^0$. And $N \geqslant O$, $G \geqslant O$, thus $(G+N)y_1^0 \leqslant (G+N)y_1^1$. By the induction principle we get $y_1^m \geqslant y_1^{m-1} \geqslant \cdots \geqslant y_1^0$ with $\dot{y}_1^m + Ay_1^m \leqslant f$. As $y_1^m = x_1^1$, we have $x_1^0 \leqslant x_1^1$. By the induction principle again, we have $x_1^p \leqslant x_1^{p+1}, p = 0, 1, \ldots$. Secondly, by the similar process, we can prove $x_2^p \geqslant x_2^{p+1}, p = 0, 1, \ldots$.

Lastly, we compare x_1^p and x_2^p.

As $p = 0$, the result is appearance. As $p = 1$, since

$$\dot{y}_1^{l+1} + Fy_1^{l+1} = Gy_1^l + Nx_1^0 + f$$

and

$$\dot{y}_2^{l+1} + Fy_2^{l+1} = Gy_2^l + Nx_2^0 + f$$

is satisfied, computing the difference between them, we get

$$\dot{y}_1^{l+1} - \dot{y}_2^{l+1} + F(y_1^{l+1} + y_2^{l+1}) = G(y_1^l - y_2^l) + N(x_1^0 - x_2^0).$$

By the former proof and the assumption, we have $N(x_1^0 - x_2^0) \leqslant N(y_1^{l+1} + y_2^{l+1})$ and $G(y_1^l - y_2^l) \leqslant G(y_1^{l+1} - y_2^{l+1})$, hence $(\dot{y}_1^{l+1}) - \dot{y}_2^{l+1} + A(y_1^{l+1} - y_2^{l+1}) \leqslant O$, further we get $y_1^{l+1} \leqslant y_2^{l+1}$. After the mth step iteration, we have $x_1^1 = y_1^m \leqslant y_2^m = x_2^1$. By the induction principle, we have $x_1^p \leqslant x_2^p$.

(b) By the result (a), we get $\lim_{p\to\infty} x^p = x_1^* \leqslant x_2^* = \lim_{p\to\infty} y^p$. Since x_1^* and x_2^* are both the solutions of the initial value problem, by the uniqueness of it, we get $x_1^* = x_2^*$.

(c) By the similar proof with (a), we can get the result. □

Theorem 5.2. *Let* $A = M - N$ *be a splitting and* $N \geqslant O$. *Let* $M = F_i - G_i, (i = 1,2)$ *be two different splittings and hold* $G_i \geqslant O, (i = 1,2)$. *Assume that* $x_1^0 = x_2^0$ *are the starting functions, sequences* $\{x_1^p\}$ *and* $\{x_2^p\}$ *are generated by*

$$\dot{y}_1^{l+1} + F_1 y_1^{l+1} = G_1 y_1^l + Nx_1^k + f$$

and

$$\dot{y}_2^{l+1} + F_2 y_2^{l+1} = G_2 y_2^l + N x_2^k + f$$

respectively. If $G_2 \geqslant G_1$, *then*

(*a*) *If* $\dot{x}_1^0 + A x_1^0 \leqslant f$ *or* $\dot{x}_2^0 + A x_2^0 \leqslant f$, *then* $x_1^p \geqslant x_2^p$, $p = 0, 1, 2, \ldots$;
(*b*) *If* $\dot{x}_1^0 + A x_1^0 \geqslant f$ *or* $\dot{x}_2^0 + A x_2^0 \geqslant f$, *then* $x_1^p \leqslant x_2^p$, $p = 0, 1, 2, \ldots$.

Proof. (a) As $p = 0$, by the assumption, the result is appearance. As $p = 1$, by the definition of the TSWR method, we get $y_1^0 = x_1^0 = x_2^0 = y_2^0$. Then $\dot{y}_1^{l+1} + F_1 y_1^{l+1} = G_1 y_1^l + N y_1^0 + f$ and $\dot{y}_2^{l+1} + F_2 y_2^{l+1} = G_2 y_2^l + N y_2^0 + f$. Computing the difference between the two formulas, we have

$$\dot{y}_1^{l+1} - \dot{y}_2^{l+1} + F_1 y_1^{l+1} - F_2 y_2^{l+1} = G_1 y_1^l - G_2 y_2^l.$$

Let $z^{l+1} = y_1^{l+1} - y_2^{l+1}$, since $F_1 - G_1 = F_2 - G_2$, we get

$$\dot{z}_1^{l+1} + F_2 z^{l+1} + G_1(y_1^{l+1} - y_1^l) = G_2(y_1^{l+1} - y_2^l).$$

Further we have

$$\dot{z}_1^{l+1} + F_2 z^{l+1} = G_2 z^l + (G_2 - G_1)(y_1^{l+1} - y_1^l).$$

Let $u^l = (G_2 - G_1)(y_1^{l+1} - y_1^l)$. As $\dot{y}_1^0 + A y_1^0 \leqslant f$, by the monotone convergence theory, we have $y_1^{l+1} \geqslant y_1^l$. By the assumption $G_2 \geqslant G_1$, thus $u^l \geqslant O$. As has been proved in [11], we know that the solution satisfies $z^l \geqslant O, l = 0, 1, \ldots, m$. Thus $x_1^1 \geqslant x_1^2$. By the induction principle, we can get $x_1^p \geqslant x_2^p, (p = 0, 1, 2, \ldots)$.

(b) By the similar proof, we can get the second result. □

6. Conclusion

In summary, the TSWR method is convergent for any initial function on condition that the matrix splitting is a convergent splitting. We have particularly proved that it is true for H-splitting. Further when the initial function satisfies the differential inequality $\dot{x} + Ax - f \leqslant O$ or ($\geqslant O$), the TSWR method also guarantees the monotonicity. Additionally, when we perform different inner splittings, the sequences generated by the TSWR method will satisfy the monotonicity and the comparison theory contemporarily.

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