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Contractive mappings with maximum norms: Comparison of constants of contraction and application to asynchronous iterations

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Abstract

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In this paper, we give two extensions of Stein-Rosenberg's theorem. The first, which we name the general result, is an abstract nonlinear extension and can be described as follows: Given a first fixed point mapping on a Banach product space, we define a more implicit second fixed point mapping, possibly after a redecomposition of our product space, and we get that the new mapping has a constant of contraction lower or equal to the constant of contraction of the initial mapping. This result allows an efficient use of El Tarazi's theorem [5], about the convergence of asynchronous iterations. The second extension is close to the linear case and permits us to compare the constants of contraction using strict inequality. We give two applications of these results: the first, in a context near from the one studied by D.J. Evans and W. Deren, about a diagonal monotone perturbation of linear problems [6]. The second is a short example in a totally different framework about the formulation of asynchronous waveform relaxation for a system of ordinary differential equations with initial conditions. For another point of view concerning Stein-Roseinberg's theorem and asynchronous algorithms, see [7] and [10].

Keywords. Asynchronous iteration; diagonal monotone perturbation; Stein-Rosenberg's theorem; differential algebraic problems.

0. Introduction

In this paper, we give two extensions of Stein-Rosenberg's theorem.
The first, which we name the general result, is an abstract nonlinear extension and can be described as follows: given a first fixed point mapping on a Banach product space, we define a more implicit second fixed point mapping, possibly after a redecomposition of our product space, and we get that the new mapping has a constant of contraction lower or equal to the constant of contraction of the initial mapping.

This result allows an efficient use of El Tarazi's theorem, about the convergence of asynchronous iterations. This use can be described as follows: If we are, for example, in the

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mework of the assumptions of El Tarazi's theorem for a by point decomposition of the ite dimensional problem, then our theorem implies that we are also in the same kind of mework for a more implicit, by block decomposition of the same fixed point problem.

Moreover, it is worthwhile to note that this abstract form of Stein-Rosenberg's theorem, e El Tarazi's theorem, deals with maximum norms which in application are often related by way of the maximum principle, with order structures; this is the usual framework of in-Rosenberg's theorem.

The second extension is close to the linear case and permits us to compare the constants of ntraction using strict inequality. We apply these points of view to two kinds of situations: First, in a context like the one studied by Evans and Deren [6] about a diagonal monotone perturbation of linear problems (in short: DMPL problems), associated to an M-matrix, which thanks to Evans and Deren's study, allows to include asynchronous Schwarz alternating methods. Besides the fact that our proofs of convergence are quite different from those of Evans and Deren, our statement includes here multivalued mappings (this is interesting for the case of constrained problems), and a somewhat different parameter: namely the spectral radius (or approximate spectral radius in reducible matrices) of the Jacobi associated matrix, instead of the norms used by these authors. Moreover, it appears that our general result has something to do with the classical Stein-Rosenberg's theorem: both allow an inequality:

- between constants of contraction for our theorem,
- between spectral radii for Stein-Rosenberg's theorem, the corresponding inequality being strict in this case, by the way of Perron-Frobenius theorem. The second extension leads us to take advantage of the matricial aspect of the DMPL problems, to improve the interval of convergence of the relaxation parameter and also the constant of contraction of the corresponding fixed point mapping.

Second, we give a short example in a totally different framework about the formulation of asynchronous waveform relaxation for a system of ordinary differential equations with initial conditions. In Bahi's thesis and a forthcoming paper (by M. Bahi, E. Griepentrog and J.C. Miellou), the same point of view is emphasized in order to study the waveform relaxation method for classes of differential algebraic problems.

This paper is organized in the following way: In Section 1 we give some backgrounds about nchronous iterations. Section 2 is dedicated to the statement and the proof of the general ult. In Section 3, we apply this result to the parallel treatment associated to a block composition of the nonlinear algebraic DMPL problems:

$$b - Ax \in M(x).$$

Section 4, we use the classical Stein-Rosenberg theorem in order to prove the second ension in the context of DMPL problems, which allows us to derive a best contraction nstant and convergence condition on the relaxation parameter of our fixed point mapping. Section 5, we give the second application concerning asynchronous waveform relaxation for linary differential equations.

Background: Asynchronous algorithms

Consider a Banach product space $E = \prod_{i=1}^{\alpha} E_i$ and a fixed point mapping G defined on E. roduce the following symbols: $J = \{J(p)\}_{p \in \mathbb{N}}$ is a sequence of nonempty subsets of $\ldots, \alpha\}$. $S = \{(s_1(p), \ldots, s_\alpha(p))\}_{p \in \mathbb{N}}$ is a sequence of $\mathbb{N}^\alpha$ such that:

$\forall i \in \{1, \ldots, \alpha\}$, the subset $\{p \in \mathbb{N},\ i \in J(p)\}$ is infinite.

$\forall i \in \{1, \ldots, \alpha\}$, $\forall p \in \mathbb{N}$, $s_i(p) \leqslant p$.

$\forall i \in \{1, \ldots, \alpha\}$, $\lim_{p \to \infty} s_i(p) = \infty$.

The asynchronous (or synchronous) algorithm associated to G and noted (G, u^0, J, S) is defined as follows:

$$u_i^{p+1} = \begin{cases} G_i(\dots, u_k^{s_k(p)}, \dots) & \text{if } i \in J(p) \\ u_i^p & \text{if } i \notin J(p). \end{cases} \tag{1.1}$$

The following statement is useful in order to study the behaviour of asynchronous iterations.

Theorem 1 [5]. *Let G be a mapping from $D(G) \subset E$ in E an suppose that*:
(a) $D(G) \subset \prod_{i=1}^{\alpha} D_i(G)$
(b) $\exists u^* \in D(G)$, *such that* $u^* = G(u^*)$
(c) $\forall u \in D(G)$, $]|G(u) - G(u^*)|[\leqslant \beta]|u - u^*|[$
with $0 < \beta < 1$, *then every asynchronous algorithm correspondents to G with a starting point $u^0 \in D(G)$, converges to the fixed point of G, u^*.*

Proof. See [5]. □

2. An abstract nonlinear extension of Stein-Rosenberg's theorem

Let E_i, $i \in \{1, \dots, n\}$ be Banach spaces equiped with the norms:

$$\| \dots \|_i. \tag{2.1}$$

Consider $E = \prod_{i=1}^{n} E_i$ equiped with the norm:

$$\| \dots \| = \max_{1 \leqslant i \leqslant n} \| \dots \|_i. \tag{2.2}$$

Consider a mapping:

$$T : E = \prod_{i=1}^{n} E_i \longrightarrow E = \prod_{i=1}^{n} E_i. \tag{2.3}$$

$$u \longrightarrow v = T(u).$$

Suppose that: $\forall$a $(\gamma_1, \dots, \gamma_n) \in \mathbb{R}_+^{*n}$ such that T is contractive in the norm:

$$]| \dots |[= \max_{1 \leqslant i \leqslant n} \frac{\| \dots \|_i}{\gamma_i}. \tag{2.4}$$

Definition 1. Let $I_k (k \in \{1, \dots, \alpha\}$, $\alpha \leqslant n)$ be subsets of $\mathbb{N}$ such that:

(a) $\quad I_k \cap I_j = \emptyset, \forall k \neq j$

(b) $\quad \bigcup_{k=1}^{\alpha} I_k = \{1, \dots, n\}.$

Define:

$$\mathbb{E}_k = \prod_{i \in I_k} E_i. \tag{2.5}$$

Thus, we can consider E as the product:

$$E = \prod_{k=1}^{\alpha} \mathbb{E}_k. \tag{2.6}$$

ɔw, let $U^k \in \mathbb{E}_k$, we have:

$$(U^k)_i = u_i, \quad \forall i \in I_k. \tag{2.7}$$

:t J_k be subsets of $\{1, \ldots, \alpha\}$, and

$$\mathbb{I}_k = \bigcup_{l \in J_k} I_l.$$

ɪr $U \in E = \prod_{i=1}^{\alpha} \mathbb{E}_i$ and $v \in E = \prod_{i=1}^{n} E_i$, we define the vector $\sigma_k(U, v)$ as follows:

$$\sigma_k(U, v) = (w_1, \ldots, w_n), \quad \text{where:}$$

$$\begin{cases} w_j = u_j & \text{if } j \in \mathbb{I}_k \\ w_j = v_j & \text{if } j \notin \mathbb{I}_k. \end{cases} \tag{2.8}$$

tablishment of the mapping $\mathbb{T}$. We consider the following mapping:

$$\mathbb{T} : E = \prod_{i=1}^{n} E_i \longrightarrow E = \prod_{i=1}^{\alpha} \mathbb{E}_i.$$

$$(v_1, \ldots, v_n) \longrightarrow (U^1, \ldots, U^\alpha).$$

ɔh that:

$$T_i(\sigma_k(U, v)) = (U^k)_i. \tag{2.9}$$

mark 1.

$$(\mathbb{T}_k(v))_i = T_i(\sigma_k(\mathbb{T}(v), v)), \tag{2.10}$$

$$(\mathbb{T}_k(u^*))_i = (u^*)_i \Leftrightarrow T_i(u^*) = u_i^*, \tag{2.11}$$

if T and $\mathbb{T}$ are contractive then they have the same fixed point.

eorem 2 (General result). *If T is a contraction mapping in the norm:*

$$]|\ldots|[= \max_{1 \leqslant i \leqslant n} \frac{\|\ldots\|_i}{\gamma_i}, \quad \gamma_i > 0, \quad \forall i \in \{1, \ldots, n\}, \tag{2.12}$$

n $\mathbb{T}$ is also contractive in the norm:

$$]\|\ldots\|[= \max_{1 \leqslant l \leqslant \alpha} \left(\max_{i \in \mathbb{I}_l} \frac{\|\ldots\|_i}{\gamma_i} \right). \tag{2.13}$$

ɪreover, if ν and ν' are the respective contraction constants of T and $\mathbb{T}$ then:

$$\nu \geqslant \nu'. \tag{2.14}$$

ɔof.

$$]|T(w) - T(v)|[\leqslant \nu]|w - v|[. \tag{2.15}$$

ɔm (2.10) we have:

$$(\mathbb{T}_k(v))_i = T_i(\sigma_k(\mathbb{T}(v), v)), \tag{2.16(a)}$$

$$(\mathbb{T}_k(w))_i = T_i(\sigma_k(\mathbb{T}(w), w)). \tag{2.16(b)}$$

Then:

$$\frac{\|(\mathbb{T}_k(v))_i - (\mathbb{T}_k(w))_i\|_i}{\gamma_i}$$
$$\leqslant \nu \max\left[\max_{J\in \mathbb{J}_k} \frac{\|(\mathbb{T}_k(v))_j - (\mathbb{T}_k(w))_j\|_j}{\gamma_j};\ \max_{1\leqslant l\leqslant\alpha}\ \max_{j\in\mathbb{J}_l} \frac{\|v_j - w_j\|_j}{\gamma_j}\right].$$

Denote:

$$\rho_k = \max_{j\in\mathbb{J}_k} \frac{\|(\mathbb{T}_k(v))_j - (\mathbb{T}_k(w))_j\|_j}{\gamma_j} \tag{2.17}$$

then

$$\rho_k \leqslant \nu\left(\max\left(\rho_k;\ \max_{1\leqslant l\leqslant\alpha}\ \max_{j\in\mathbb{J}_l} \frac{\|v_j - w_j\|_j}{\gamma_j}\right)\right).$$

Therefore, either

$$\rho_k \leqslant \nu\rho_k, \quad \text{so } \rho_k = 0, \tag{2.18}$$

or

$$\rho_k \leqslant \max_{1\leqslant l\leqslant\alpha}\ \max_{j\in\mathbb{J}_l} \frac{\|v_j - w_j\|_j}{\gamma_j}. \tag{2.19}$$

Thus,

$$\max_{1\leqslant k\leqslant\alpha}\ \max_{j\in\mathbb{J}_k} \frac{\|(\mathbb{T}_k(v))_j - (\mathbb{T}_k(w))_j\|_j}{\gamma_j} \leqslant \nu \max_{1\leqslant l\leqslant\alpha}\ \max_{j\in\mathbb{J}_l} \frac{\|v_j - w_j\|_j}{\gamma_j},$$

that is

$$]\|\mathbb{T}(v) - \mathbb{T}(w)\|[\leqslant \nu]\|v - w\|[. \quad \square$$

Remark 2. Particular case. If $\mathbb{J}_k = I_k(J_k = \{k\})$, the mapping $\mathbb{T}_k$ corresponds to the block Jacobi mapping which is the usual basic fixed point mapping which allows to generate block asynchronous iterations. We may use the previous theorem associated to this choice of $\mathbb{J}_k$ in sections 3 and 5. In section 3.5 we present another kind of choice of $\mathbb{J}_k$ in order to modelize mixed Gauss-Seidel or Jacobi asynchronous iterations.

3. The first application of the general result

In this section, we are interested in asynchronous parallel algorithms for solving nonlinear algebraic problems:

$$b - Ax \in M(x), \quad x \in D(M), \tag{3.1}$$

where:

$$A = a_{ij} \tag{3.2}$$

is an irreducible M matrix,

$$M(x) = (M_1(x_1), \ldots, M_n(x_n))^{\mathrm{T}} \tag{3.3}$$

is a diagonal, monotone, maximal and possibly multi-valued operator defined on $D(M)$. We ill subdivise problem (3.1) into α subproblems which will be treated by α processors. First e define a fixed point mapping whose fixed point coincides with the solution of (3.1).

1. The by point fixed point mapping T

We consider the following fixed point mapping:

$$T: D(M) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$x \longrightarrow y$$

ch that:

$$\begin{cases} \lambda_i(y_i) + a_{ii} y_i = -\sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j + b_i \\ \lambda_i(y_i) \in M_i(y_i). \end{cases} \tag{3.4}$$

mark 3.
) T is well defined, indeed: M_i is monotone maximal so y_i exists and is unique.
) If x^* is the fixed point of T then x^* is the unique solution of (3.1), indeed:

$$\begin{cases} \lambda_i(x_i^*) + a_{ii} x_i^* = -\sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^* + b_i \\ \lambda_i(x_i^*) \in M_i(x_i^*) \end{cases}$$

that

$$0 \in Ax^* + M(x^*) - b.$$

oposition 3. *There exists a vector $e > 0$ (all the components of e are > 0), such that T is ntractive with respect to the norm:*

$$]|\ldots|[= \max_{1 \leqslant i \leqslant n} \frac{\|\ldots\|_i}{e_i}.$$

oof. We are here in a particular case of more general frameworks such as in El Tarazi [5], ellou and Spiteri [8]; but we detail the proof for an easy reading: Let $y = T(x)$ and $= T(\tilde{x})$, then:

$$\lambda_i(y_i) + a_{ii} y_i = -\sum_{\substack{j=1 \\ j \neq 1}}^{n} a_{ij} x_j + b_i$$

d

$$\lambda_i(\tilde{y}_i) + a_{ii} \tilde{y}_i = -\sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} \tilde{x}_j + b_i$$

n

$$\lambda_i(y_i) - \lambda_i(\tilde{y}_i) + a_{ii}(y_i - \tilde{y}_i) = -\sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}(\tilde{x}_j - x_j). \tag{3.5}$$

Therefore:

$$(y_i-\tilde{y}_i)[\lambda_i(y_i)-\lambda_i(\tilde{y}_i)+a_{ii}(y_i-\tilde{y}_i)]=\left[-\sum_{\substack{j=1\\j\neq i}}^{n}a_{ij}(\tilde{x}_j-x_j)\right](y_i-\tilde{y}_i).$$

From (3.3) we have:

$$(y_i-\tilde{y}_i)(\lambda_i(y_i)-\lambda_i(\tilde{y}_i))\geqslant 0.$$

Then

$$(y_i-\tilde{y}_i)a_{ii}(y_i-\tilde{y}_i)\leqslant\left[-\sum_{\substack{j=1\\j\neq i}}^{n}a_{ij}(\tilde{x}_j-x_j)\right](y_i-\tilde{y}_i)\tag{3.6}$$

so

$$(y_i-\tilde{y}_i)\leqslant-\frac{\sum_{\substack{j=1\\j\neq i}}^{n}a_{ij}(x_j-\tilde{x}_j)}{a_{ii}}.\tag{3.7}$$

Let $\|\dots\|_i$ be a norm on $\mathbb{R}$. From the Perron-Frobenius theorem, there exists $e>0$ such that:

$$J(e)\leqslant\rho(J).e,$$

where J is the Jacobi matrix of A and ρ(J) its spectral radius. Therefore, as in [9]:

$$\|y_i-\tilde{y}_i\|_i\leqslant-\frac{\sum_{\substack{j=1\\j\neq i}}^{n}a_{ij}e_j}{a_{ii}}\frac{\|x_j-\tilde{x}_j\|_j}{e_j},$$

so

$$\|y_i-\tilde{y}_i\|_i\leqslant\rho(J)e_i\frac{\|x_j-\tilde{x}_j\|_j}{e_j}.\tag{3.8}$$

Therefore

$$\frac{\|y_i-\tilde{y}_i\|_i}{e_i}\leqslant\rho(J)\frac{\|x_j-\tilde{x}_j\|_j}{e_j},\tag{3.9}$$

so that

$$]|y-\tilde{y}|[\leqslant\rho(J)]|x-\tilde{x}|[.$$

Since A is an M matrix $\rho(J)<1$, T is contractive on $\mathbb{R}^n$ endowed with the norm $]|\dots|[$. □

Remark 4. If A is reducible it is sufficient to replace $\rho(J)$ by $\rho(J)+\epsilon$, where ϵ is an arbitrary positive real number.

3.2. Decomposition of (3.1)

Let $(n_1,\dots,n_\alpha)\in\mathbb{N}_+^\alpha$, such that:

$$\sum_{i=1}^{\alpha}n_i=n,\tag{3.10}$$

e decompose a vector $x \in \mathbb{R}^n$ into α blocks X_i of n_i components:

$$x = X = (X_1, \ldots, X_\alpha)^T. \tag{3.11}$$

r an $n \times n$ matrix $A = a_{ij}$, we denote:

$$\mathcal{A} = A_{ij} \text{ the correspondent } \alpha \times \alpha \text{ matrix.} \tag{3.12}$$

e also denote:

$$M(x) = \mathbb{M}(X) = (\mathbb{M}_1(X_1), \ldots, \mathbb{M}_\alpha(X_\alpha))^T \tag{3.13}$$

d:

$$b = B = (B_1, \ldots, B_\alpha)^T. \tag{3.14}$$

1) can be written after decomposition:

$$\begin{cases} 0 = \Lambda_i(X_i) + \sum_{j=1}^{\alpha} A_{ij} X_j - B_i \\ \Lambda_i(X_i) \in \mathbb{M}_i(X_i), \end{cases} \tag{3.15}$$

ere

$$\lambda(x) = \Lambda(x) = (\Lambda_1(x_1), \ldots, \Lambda_\alpha(x_\alpha))^T. \tag{3.16}$$

mark 5.
This decomposition is general, and it is worthwhile to note that the Schwartz alternating procedure can be viewed as a block asynchronous (or synchronous) relaxation algorithm applied to a problem of the kind of (3.1), but set on a space of greater dimension, see [6] for the explication of the correspondent extended matrices and problems, and for asynchronous forms of the Schwartz alternating method.
For a related point of view about redecomposition techniques, see Comte [4].

. The more implicit fixed point mapping $\mathbb{T}$

We consider the following fixed point mapping:

$$\mathbb{T} : \mathbb{R}^\alpha \longrightarrow D(\mathbb{M}) \cap \mathbb{R}^\alpha$$
$$X \longrightarrow Y$$

ch that

$$\begin{cases} \Lambda_i(Y_i) + A_{ii} Y_i = -\sum_{\substack{j=1 \\ j \neq i}}^{\alpha} A_{ij} X_j + B_i \\ \Lambda_i(Y_i) \in \mathbb{M}_i(Y_i). \end{cases} \tag{3.17}$$

mark 6. The same remark as in Remark 3 can be made.

oposition 4. *$\mathbb{T}$ is contractive, moreover its contraction constant is lower or equal to the ntraction constant of T.*

oof. We have only to use the particular case (Remark 2) of the general result applied to T fined in (3.4) and $\mathbb{T}$ defined by (3.17). □

Theorem 5. *Every asynchronous algorithm associated to* $\mathbb{T}$ *with a starting point* $u^0 \in D(\mathbb{M})$, *converges to the unique solution of the problem* (3.1).

Proof. Since $\mathbb{T}$ is contractive in the norm:

$$\max_{1 \leqslant l \leqslant \alpha} \max_{J \in \mathbb{I}_l} \frac{\| \ \|_j}{\gamma_j} = \max_{1 \leqslant l \leqslant \alpha} \frac{\| \ \|_1}{\gamma_l}, \quad \text{where:}$$

$$\frac{\| \ \|_l}{\gamma_l} = \max_{j \in \mathbb{I}_l} \frac{\| \ \|_j}{\gamma_j},$$

it is sufficient to apply Theorem 1. □

3.4. The relaxed fixed point mappings

3.4.1. The relaxed fixed point mapping T_ω

Consider the mapping:

$$T_\omega : \mathbb{R}^n \longrightarrow \mathbb{R}^n \cap D(M)$$
$$x \longrightarrow y$$

such that

$$y = (1 - \omega)x + \omega T(x) \tag{3.18}$$

where T is defined in (3.4).
Then:

Theorem 6. *If* $\omega \in]0, 2/1 + e(J)[$ *where* J *is the Jacobi matrix of the matrix* A, *then every asynchronous algorithm correspondent to* T_ω *converges to the unique solution of* (3.1).

Proof. We will prove that T_ω is contractive. Let $y = T_\omega(x)$ and $\tilde{y} = T_\omega(\tilde{x})$, then:

$$\begin{aligned}]| y - \tilde{y} |[&=]|(1-\omega)x + \omega T(x) - (1-\omega)\tilde{x} - \omega T(\tilde{x}) |[\\ &=]|(1-\omega)(x - \tilde{x}) + \omega(T(x) - T(\tilde{x})) |[\\ &\leqslant |1 - \omega|]| x - \tilde{x} |[+ |\omega| \rho(J)]| x - \tilde{x} |[. \end{aligned}$$

It is obvious that if $0 < \omega < 2/1 + e(J)$, then:

$$|1 - \omega| + |\omega| \rho(J) < 1,$$

so T_ω is contractive. Now it is sufficient to apply Theorem 1 to prove Theorem 6. □

3.4.2. The relaxed fixed point mapping $\mathbb{T}_\omega$

Consider the mapping $\mathbb{T}_\omega$ defined as follows:

$$\mathbb{T}_\omega : \mathbb{R}^\alpha \longrightarrow \mathbb{R}^\alpha \cap D(\mathbb{M})$$
$$X \longrightarrow Y$$

such that

$$Y = (1 - \omega)X + \omega \mathbb{T}(X). \tag{3.19}$$

Theorem 7. *If* $\omega \in]0, 2/1 + e(J)[$ *where* J *is the Jacobi matrix of the matrix* A, *then every asynchronous algorithm correspondent to* $\mathbb{T}_\omega$ *converges to the unique solution of* (3.1).

oof. It is sufficient to apply the general result applied to T_ω defined by (3.18), and $\mathbb{T}_\omega$ fined by (3.19). □

. Mixed Gauss-Seidel asynchronous iterations

Let us assume a by point decomposition of each diagonal block A_{ii} of the form:

$$A_{ii} = D_i - L_i - U_i,$$

ere with the usual notation D_i (resp L_i; U_i) is denoted the by point diagonal part of A_{ii} sp triangular lower; triangular upper part of A_{ii}). Let us define the following fixed point ıpping: Given $X = (X_1, \ldots, X_\alpha)$ and $Y = (Y_1, \ldots, Y_\alpha)$, $\mathbb{T}_{G,S}(X) = Y$ satisfies

$$A_i(Y_i) + (D_i - L_i)Y_i = U_i X_i - \sum_{j \neq i} A_{ij} X_j + B_i.$$

t us introduce the subsets of indices: $I_i' = \{l_i, \ldots, l_{i+1} - 1\}$ defining the ith block A_{ii}. $\in \{1, \ldots, n\}$, $\exists i$ such that $l \in I_i'$. Then we have: $\mathbb{I}_l = \{k,\ l_i \leqslant k \leqslant l\}$, which by the way of (2.8) ɔws to define $\sigma_1(Y, X)$, and so the fixed point mapping $\mathbb{T}_{G.S}$ satisfies the condition of eorem 2, and so we have:

oposition 8. *$\mathbb{T}_{G.S}$ is contractive with a constant of contraction lower or equal to the one of the point mapping T defined in* (3.4).

mark 7. In fact the kind of result given by the previous statement can also be obtained by ısidering the fact that all the asynchronous algorithms associated to the fixed point ıpping $\mathbb{T}_{G.S}$, can be included in asynchronous methods associated to the by point fixed point ıpping T.

A second nonlinear extension of Stein-Roseinberg's theorem closer to the linear case

We are interested in the present section only in the case of strict inequalities between ıstants of contraction of fixed point mappings, because the case of non-strict inequalities is ʼays solved by our previous Theorem 2. So we have to assume for the matrices involved in r problems and statement the irreducibility property. We have to use the following theorem e [3]):

eorem 9. *Let A be an M-matrix. Let $A = B_1 - C_1$ and $A = B_2 - C_2$ be two regular splittings A with*:

$$C_2 < C_1 \quad (C_2 \leqslant C_1 \text{ and } C_2 \neq C_1) \tag{4.1}$$

d

$$B_1^{-1} C_1 \text{ irreducible}.$$

n

$$\rho(B_2^{-1} C_2) < \rho(B_1^{-1} C_1).$$

oof. See [3], p. 183, and [11] p. 57. □

ɪw let us consider that

$$A = B - C \tag{4.2}$$

is a regular decomposition of the M-matrix A, in which B is also an M-matrix, such that $B^{-1}C$ is irreducible. Let us introduce the fixed point mapping T defined by

$$T(x) = y \text{ such that}$$
$$M(y) + By = Cx + b, \tag{4.3}$$

where M is defined in (3.3). Let us introduce the norm

$$]|u|[= \max_l \frac{|u|_l}{e_l}, \tag{4.4}$$

where e is the eigenvector associated to the spectral radius of $B^{-1}C$, that is to say by Perron-Frobenius theorem:

$$B^{-1}Ce = \rho(B^{-1}C)e = \rho e. \tag{4.5}$$

Then we have:

Proposition 10.

$$]|T(v) - T(v')|[\leqslant \rho]|v - v'|[.$$

Proof. We have

$$\lambda(y) + By = Cx + b$$
$$\lambda(y') + By' = Cx' + b$$

so

$$\lambda(y) - \lambda(y') + B(y - y') = C(x - x').$$

Then

$$(\lambda_l(y_l) - \lambda_l(y_l'))(y_l - y_l') + \sum_k b_{lk}(y_k - y_k')(y_l - y_l') = \sum_k c_{lk}(x_k - x_k')(y_l - y_l').$$

Since $b_k \leqslant 0$ for $l \neq k$, and $b_{ll} > 0$, we deduce (like in the proof of Proposition 3):

$$B|y - y'| \leqslant C|x - x'|.$$

Therefore,

$$|y - y'| \leqslant B^{-1}C|x - x'| \quad (B^{-1}C \geqslant 0).$$

Then with the use of The Perron-Frobenius theorem, we deduce:

$$]|y - y'|[\leqslant \rho(B^{-1}C)]|x - x'|[. \quad \square$$

Let us consider now two decompositions of A:

$$A = B_1 - C_1 \quad \text{and} \quad A = B_2 - C_2,$$

both satisfying the assumptions of Theorem 9, except the irreducibility which has to be satisfied only by $B_1^{-1}C_1$. Then if T_1, $]|\cdot|[_1$, ρ_1 and T_2, $]|\cdot|[_2$, ρ_2 are the fixed point mappings, the norms and the constants of contraction, corresponding by (4.3), (4.4), (4.5), to the respective decompositions: $A = B_1 - C_1$; and $B = B_2 - C_2$, then:

Corollary 11 (nonlinear variant of Stein-Rosenberg's theorem). *The constants of contraction of T_1 and T_2 can be compared and we have*:

$$\rho_2 < \rho_1.$$

Proof. Results both from Theorem 9 and Proposition 10. $\square$

position 12. *Under the assumptions of Theorem 9, suppose that A is irreducible in the blem (3.15), then: if $\omega \in]0, 2/(1+\rho(J'))[$ where J' is the Jacobi matrix correspondent to the ular block decomposition of the matrix A, then every asynchronous algorithm correspondent $\mathcal{T}_\omega$ converges to the unique solution of (4.1). Moreover:*

$$\rho(J') < \rho(J),$$

re J is the by point Jacobi matrix associated to A.

of. It is sufficient to take:

$$B_1 = \mathrm{diag}(\ldots, a_{ii}, \ldots);\ C_1 = B_1 - A$$

l

$$B_2 = \mathrm{diag}(\ldots, A_{ii}, \ldots);\ C_2 = B_2 - A.$$

being irreducible, J is also irreducible, so we are in the framework of our previous ducibility assumptions which ensure us of strict inequality. □

nark 8. All the previous results can also be applied to the case in which we replace the umption 'A is an M-matrix' by 'A is an H-matrix with positive diagonal elements', and in ch we consider that: $A = B - C$, where B is obtained by cancelling extradiagonal elements A (this is the case in each one of the examples associated both to block decompositions or ed Gauss-Seidel blocks decompositions considered above).

he second application of the general result

Consider an ordinary differential equation with initial condition:

$$\begin{cases} y'(t) = f(y(t), t) \\ y(t_0) = y_0, \end{cases} \tag{5.1}$$

re $y(t) = (y_1(t), \ldots, y_n(t))^{\mathrm{T}}$; $f = (f_1, \ldots, f_n)^{\mathrm{T}}$; $y_0 = (y_{01}, \ldots, y_{0n})^{\mathrm{T}}$. and $t \in [t_0, t_0 + h]$ (h). Consider $|x|_\infty = \max_i |x_i|$, the maximum norm on $\mathbb{R}^n$. Suppose that f satisfies, for the e of simplicity, a global Lipschitz condition:

$$|f(y, t) - f(\tilde{y}, t)|_\infty \leqslant L|y - \tilde{y}|_\infty,$$

re L does not depend on t. It is a classical result that for h sufficiently small the problem) has a unique solution, and that the following fixed point mapping:

$$T(y) = x \text{ such that:}$$

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = f_i(y_i(t), \ldots, y_{i-1}(t), x_i(t), y_{i+1}(t), \ldots, y_n(t), t) \\ x_i(t_0) = y_{0i} \end{cases} \tag{5.2}$$

ontractant and converges to the unique solution of (5.1). Now, we can write problem (5.1) r decomposition:

$$\begin{cases} Y'(t) = F(Y(t), t) \\ Y(t_0) = Y_0, \end{cases} \tag{5.3}$$

re $Y(t) = (Y_1(t), \ldots, Y_\alpha(t))^{\mathrm{T}}$; $F = (F_1, \ldots, F_\alpha)^{\mathrm{T}}$; $Y_0 = (Y_{01}, \ldots, Y_{0\alpha})^{\mathrm{T}}$, and $t \in [t_0, t_0 + h]$(h

> 0). Y_i, F_i and Y_{0i} have n_i components, and $\sum_{i=1}^{\alpha} n_i = n$. Let us consider the norm on $C^n([t_0, t_0 + h])$:

$$\| x \|_\infty = \max_{1 \leqslant i \leqslant n} \max_{t_0 \leqslant t \leqslant t_0 + h} | x_i(t) |.$$

Consider the following mapping: $\mathbb{T}(Y) = X$ such that:

$$\begin{cases} \dfrac{dX_i(t)}{dt} = f_i(Y_1(t), \ldots, Y_{i-1}(t), X_i(t), Y_{i+1}(t), \ldots, Y_\alpha(t), t) \\ X_i(t_0) = Y_{0i}, \end{cases} \tag{5.4}$$

then we have:

Theorem 13. *Every asynchronous algorithm associated to $\mathbb{T}$ with a starting point Y_0, converges to the unique solution of the problem* (5.1).

Proof. It is a classical result that for h sufficiently small T is contractive with respect to the norm $\| \ldots \|_\infty$. We deduce by the redecomposition theorem that $\mathbb{T}$ is also contractive with respect to the same norm, and then it is sufficient to apply Theorem 1. □

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