Generalized Multisplitting Asynchronous Iteration

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ABSTRACT

We give a generalized asynchronous iteration, based on the multisplitting of linear or nonlinear operators, to solve systems of equations. The condition for convergence can be imposed only on the components which actually enter the computation. The results contain synchronous multisplitting iterations as a special case.

1. MULTISPLITTING AND MULTISPLITTING ASYNCHRONOUS ITERATION

Let $A \in \mathscr{R}^{N \times N}$, nonsingular. Then (M_l, N_l), $l = 1, \ldots, L$, is called a multisplitting of A (see [7]) if $A = M_l - N_l$, M_l nonsingular, $l = 1, \ldots, L$. Let E_l, $l = 1, \ldots, L$ be L nonnegative, nonzero, diagonal matrices, $\sum_{l=1}^{L} E_l = I$. Then the (*parallel linear*) *multisplitting iterative method* for solving the linear system of equations

$$Ax = b \tag{1.1}$$

is

$$x_0 = \text{initial guess of } x^* = A^{-1}b,$$

$$x^{k+1} = \sum_{l=1}^{p} E_l M_l^{-1} (N_l x^k + b) \tag{1.2}$$

Assume that the multiprocessor system consists of $L + 1$ processors. Then its parallel implementation can be divided into two steps:

1. proc(l), $l = 1, \ldots, L$: input x^k, the current approximation vector; compute

$$y^{k,l} = E_l M_l^{-1}(N_l x^k + b). \tag{1.3}$$

$y^{k,l}$ are the input variables of proc(0).

2. proc(0): input $y^{k,l}$, $l = 1, \ldots, L$; compute

$$x^{k+1} = \sum_{l=1}^{L} y^{k,l}. \tag{1.4}$$

x^{k+1} is the next input of Step 1.

In step 1, $y^{k,l}$, $l = 1, \ldots, L$, can be computed concurrently on L processors. But, as is easily seen, step 1 and step 2 must be executed sequentially. In other words, when proc(0) [note that proc(0) can be any one of proc(1), $\ldots$, proc(L), but we can express the algorithm conveniently and clearly if proc(0) is used] is executing (1.4), the other L processors must be in the waiting state until proc(0) has finished its task (1.4). Besides, if the times proc(l) used to compute (1.3) are not equal, then some processor(s) will be required to wait for synchronization.

To make full use of the efficiency of a multiprocessor system, Bru, Elsner, and Neumann [1] proposed two chaotic multisplitting iterative models to decrease the waiting time. The first one is synchronous; we omit it here. The second one is *multisplitting asynchronous iteration* (MAI):

proc(l), $l = 1, \cdots, L$:
REPEAT UNTIL convergence DO

1. Input z, the newest approximation to the solution residing in proc(0).
2. Compute

$$y = E_l T_l(z),$$
$$T_l : \mathscr{R}^N \to \mathscr{R}^N, \qquad T_l(z) = M_l^{-1}(N_l z + b). \tag{1.5}$$

3. Output y to proc(0)

proc(0):   initial x. REPEAT UNTIL convergence DO
wait until some processor, say proc(l), sends its local approximation (its value is y); input y; compute

$$x = (I - E_l)x + y. \tag{1.6}$$

Note that z needn't be an approximation solution x. In this model, it is assumed that no two processors output their y's at the same instant; otherwise, some queueing priorities will have to be introduced into the algorithm. With this strategy, we can see that if two or more, say c, processors output their y's concurrently, then there must be one processor that waits $C - 1$ iteration time steps such as (1.6). Maximum efficiency in exploiting the resources of the multiprocessor system has not yet been attained.

Our idea to improve this model is: if more than one processor outputs y's concurrently, proc(0) inputs all these outputs concurrently and uses them to update the old approximation. Mathematically, for the initial vector x^0,

$$x^{k+1} = \left(I - \sum_{l \in S_k} E_l\right) x^k + \sum_{l \in S_k} E_l M_l^{-1}\left(N_l z^{k,l} + b\right),$$

(1.7)

$$\left(z^{k,l}\right)_n = \left(x^{k-d_{k,l,n}}\right)_n,$$

where $S_k \subseteq \{1, \ldots, L\}$, and $(z^k)_n$ is the nth component of the vector z^k. Obviously, synchronous multisplitting iteration (1.2) and asynchronous multisplitting iteration (1.5)–(1.6) are special cases.

In the next section, we prove a convergence theorem on a more generalized asynchronous iteration based on the above-mentioned multisplitting. In Section 3, we give some applications of the result in Section 2.

2.   CONVERGENCE OF GENERALIZED MULTISPLITTING ASYNCHRONOUS ITERATION

In this section, we consider the generalized multisplitting asynchronous iteration (GMAI):

$$x^0 = \text{initial iterate},$$

$$x^{k+1} = \left(I - \sum_{l \in S_k} E_l\right) x^k + \sum_{l \in S_k} E_l T_{k,l}(z^{k,l}),$$

(2.1)

$$T_{k,l} : \mathscr{R}^N \to \mathscr{R}^N,$$

$$z^{k,l}: \quad \left(z^{k,l}\right)_n = \left(x^{k-d_{k,l,n}}\right)_n.$$

(2.2)

Some basic restrictions on the parameters herein are:

n: $1 \leqslant n \leqslant N$.
S_k: subset of $\{1, \ldots, L\}$, nonempty, $0 \leqslant k < \infty$.
E_l: nonnegative, nonzero, diagonal, $1 \leqslant l \leqslant L$, $\sum_{l=1}^{L} E_l$ nonsingular.
$d_{k,l,n}$: $0 \leqslant d_{k,l,n} \leqslant k$ for all $k \geqslant 0$, $l \in S_k$, $1 \leqslant n \leqslant N$.

We'll use the monotone norm to express and prove the convergence theorem. For x, $y \in \mathscr{R}^N$, the expressions $x \leqslant y$, $x \ll y$ mean $(x)_n \leqslant (y)_n$, $(x)_n < (y)_n$ for all $1 \leqslant n \leqslant N$ respectively. If $x \gg 0$ or $x \geqslant 0$, we refer to x as a positive vector or a nonnegative vector respectively. For a fixed positive vector v, the monotone norm $\| \cdot \|_v$ is defined as

$$\|x\|_v = \max_{1 \leqslant n \leqslant N} \frac{|(x)_n|}{(v)_n}.$$

The monotone norm plays an important role in the field of nonnegative matrices; see [2]. Two simple but frequently used properties of monotone norms are

$$\|x\|_v \leqslant \gamma \iff |x| \leqslant \gamma v,$$

$$\|B\|_v = \|Bv\|_v \qquad \text{for} \quad B \geqslant 0,$$

where $\|B\|_v$ is the induced operator norm of the matrix B. In the following, we'll denote $\| \cdot \|_v$ by $\| \cdot \|$.

It will be convenient to introduce some special notation:

$$I_l \equiv E_l E_l^+ \qquad (E_l^+ \text{ is the Moore-Penrose inverse of } E_l),$$

$$E^k \equiv \sum_{l \in S_k} E_l,$$

$$I^k \equiv E^k (E^k)^+.$$

THEOREM 1. *Suppose $x^* \in \mathscr{R}^N$. Suppose $\forall l \in S_k$ $\exists \delta > 0$, $0 < \beta < 1$, independent of k, l, such that for all z, $\|z - x^*\| \leqslant \delta$,*

$$\|T_{k,l}(z) - x^*\| \leqslant \beta \|z - x^*\| \qquad (2.3)$$

or

$$\|I_l[T_{k,l}(z) - x^*]\| \leqslant \beta \|I_l(z - x^*)\|. \qquad (2.4)$$

Suppose E_l, $l = 1, \cdots, L$, satisfy

$$\sum_{l=1}^{L} E_l < \frac{2}{1+\beta} I; \qquad (2.5)$$

$\{S_k\}_k$ *satisfies* $\forall K$ ($K \geqslant 0$, *integer*)

$$\bigcup_{k=0}^{\infty} S_{K+k} = \{1, \cdots, L\}; \qquad (2.6)$$

and $\{d_{k,l,n}\}_{k,l,n}$ *satisfies*

$$\lim_{k \to \infty} (k - d_{k,l,n}) = \infty. \qquad (2.7)$$

Then $\forall x^0$ *such that* $\|x^0 - x^*\| \leqslant \delta$, *the sequence* $\{x^k\}_k$ *defined by* (2.1)–(2.2) *satisfies*

$$\lim_{k \to \infty} x^k = x^*.$$

To prove the above theorem, we prove two lemmas first.

LEMMA 2.   *$T_{k,l}$ satisfies* (2.3) *or* (2.4). *E_l satisfies* (2.5). *Denote*

$$\gamma_k \equiv \max\left\{\|x^k - x^*\|, \max_{l \in S_k}\|z^{k,l} - x^*\|\right\}. \qquad (2.8)$$

If $\gamma_k \leqslant \delta$, *then there exists a* β_1, $0 \leqslant \beta_1 < 1$, *independent of k, such that*

$$|I^k(x^{k+1} - x^*)| \leqslant \beta_1 \gamma_k I^k v, \qquad (2.9)$$

$$(I - I^k)|x^{k+1} - x^*| = (I - I^k)|x^k - x^*|. \qquad (2.10)$$

Proof.

$$x^{k+1} - x^* = (I - E^k)(x^k - x^*) + \sum_{l \in S_k} E_l\big[T_{k,l}(z^{k,l}) - x^*\big].$$

Then

$$(I - I^k)|x^{k+1} - x^*|$$

$$= \left| (I - I^k - E^k + E^k)(x^k - x^*) + \sum_{l \in S_k} (E_l - E_l)\left[T_{k,l}(z^{k,l}) - x^*\right] \right|$$

$$= (I - I^k)|x^k - x^*|.$$

In the case of (2.3), using $I^k = (I^k)^2$, we have

$$I^k|x^{k+1} - x^*|$$

$$\leqslant I^k \left(|I^k - E^k| \cdot |x^k - x^*| + \sum_{l \in S_k} E_l |T_{k,l}(z^{k,l}) - x^*| \right)$$

$$\leqslant I^k \left(|I^k - E^k|\gamma_k v + \sum_{l \in S_k} E_l \beta \gamma_k v \right)$$

$$= I^k(|I^k - E^k| + \beta E^k)v\gamma_k.$$

In the case of (2.4),

$$I^k|x^{k+1} - x^*| \qquad \left(\text{use } E_l = E_l I_l\right)$$

$$\leqslant I^k \left(|I^k - E^k| \cdot |x^k - x^*| + \sum_{l \in S_k} E_l |I_l\left[T_{k,l}(z^{k,l}) - x^*\right]| \right)$$

$$\leqslant I^k \left(|I^k - E^k|\gamma_k v + \sum_{l \in S_k} E_l \beta \gamma_k v \right)$$

$$= I^k \left(|I^k - E^k|\gamma_k v + \sum_{l \in S_k} E_l I_l \beta v \gamma_k \right).$$

We always have

$$I^k|x^{k+1} - x^*| \leqslant I^k(|I^k - E^k| + \beta E^k)v\gamma_k. \qquad (2.11)$$

Denote

$$e_{\min} \equiv \min\{(E_l)_{nn} | 1 \leqslant l \leqslant L, 1 \leqslant n \leqslant N, (E_l)_{nn} \neq 0\},$$

$$e_{\max} \equiv \max\left\{ \left(\sum_{l=1}^{L} E_l \right)_{nn} \middle| 1 \leqslant n \leqslant N \right\}$$

and

$$\beta_2 \equiv \begin{cases} 0, & e_{\min} > 1, \\ 1 - (1 - \beta) e_{\min}, & e_{\min} \leqslant 1; \end{cases}$$

$$\beta_3 \equiv \begin{cases} 0, & e_{\max} < 1, \\ (1 + \beta) e_{\max} - 1, & e_{\max} \geqslant 1. \end{cases}$$

From $\beta < 1$ and (2.5), we have $0 \leqslant \beta_2 < 1$, $0 \leqslant \beta_3 < 1$. Let $\beta_1 = \max\{\beta_2, \beta_3\}$; then $\beta_1 < 1$, and

$$|I^k - E^k| + \beta E^k \leqslant \beta_1 I. \tag{2.12}$$

Combining (2.11) and (2.12), the inequality (2.9) follows immediately. ∎

LEMMA 3.   *If there exist K_1, K_2, $K_1 \leqslant K_2$, such that*

$$\bigcup_{k=K_1}^{K_2} S_k = \{1, \ldots, L\}, \tag{2.13}$$

then

$$\prod_{k=K_1}^{K_2} (I - I^k) = 0. \tag{2.14}$$

Proof.   It is not difficult to prove that $\sum_{k=K_1}^{K_2} I^k$ nonsingular. Using $I^k(I - I^k) = 0$, we have

$$\sum_{i=K_1}^{K_2} I^i \cdot \prod_{k=K_1}^{K_2} (I - I^k)$$

$$= \sum_{i=K_1}^{K_2} \left(I^i(I - I^i) \prod_{\substack{K_1 \leqslant k \leqslant K_2 \\ k \neq i}} (I - I^k) \right) = 0.$$

Hence (2.14) holds.   ∎

Proof of Theorem 1. Because of (2.6) and (2.7), we can define two integer sequences $\{K_m\}_{m \geqslant 0}$ and $\{K_{m+1/2}\}_{m \geqslant 0}$ as follows:

$K_0 = 0$;

$K_{m+1/2}$ is the smallest integer K such that $k - d_{k,l,n} > K_m$ holds for all $k \geqslant K$;

K_{m+1} is the smallest integer K greater than $K_{m+1/2}$ such that

$$\bigcup_{k=K_{m+1/2}}^{K} S_k = \{1, \ldots, L\}. \tag{2.15}$$

There is no difficulty in proving

$$\lim_{m \to \infty} K_m = \infty.$$

Now we prove

$$\|x^k - x^*\| \leqslant \beta_1^m \gamma_0 \qquad \forall m \geqslant 0, \ k > K_m \tag{2.16}$$

by induction on m, where β_1 and γ_k are defined in Lemma 2 and its proof. If (2.16) holds, this theorem is proved

Step 1. We prove (2.16) holds for $m = 0$, i.e.,

$$\|x^k - x^*\| \leqslant \gamma_0 \qquad \forall k > 0. \tag{2.17}$$

If $k = 1$, (2.17) holds by Lemma 2. Suppose $k \leqslant i$; then (2.17) holds, i.e.,

$$\|x^k - x^*\| \leqslant \gamma_0 \qquad \text{for} \quad 1 \leqslant k \leqslant i. \tag{2.18}$$

From (2.18) and the definition of γ_i, we have

$$\gamma_i \leqslant \gamma_0.$$

Using Lemma 2, we get

$$|x^{i+1} - x^*\| = (I - I^i)|x^{i+1} - x^*| + I^i|x^{i+1} - x^*|$$

$$\leqslant (I - I^i)|x^i - x^*| + I^i \gamma_i \beta v$$

$$\leqslant (I - I^i)\gamma_i v + I^i \gamma_i \beta v$$

$$\leqslant \gamma_i v \leqslant \gamma_0 v.$$

Thus (2.17) is proved.

Step 2. Suppose (2.16) holds for m.

Step 3. Let $m := m + 1$; we have

$$k - d_{k,l,n} > K_m \qquad \forall k \geqslant K_{m+1/2}.$$

By step 2, we have

$$\gamma_k \leqslant \beta_1^m \gamma_0 \qquad \forall k \geqslant K_{m+1/2}. \tag{2.19}$$

Using induction on k, we prove: for any $k \geqslant K_{m+1/2}$

$$\left(1 - \prod_{r=K_{m+1/2}}^{k} (I - I^r)\right)|x^{k+1} - x^*|$$

$$\leqslant \left(I - \prod_{r=K_{m+1/2}}^{k} (I - I^r)\right)\beta_1^{m+1}\gamma_0 v. \tag{2.20}$$

For $k = K_{m+1/2}$, (2.20) holds by Lemma 2. Suppose (2.20) holds for $k = i$. Then for $k = i + 1$, by Lemma 2 and (2.19), we have

$$I^{i+1}|x^{i+2} - x^*| \leqslant I^{i+1}\gamma_{i+1}\beta_1 v$$

$$\leqslant I^{i+1}v \cdot \gamma_0 \beta_1^{m+1},$$

$$(I - I^{i+1})|x^{i+2} - x^*| = (I - I^{i+1})|x^{i+1} - x^*|.$$

In addition,

$$I - \prod_{r=K_{m+1/2}}^{i+1} (I - I^r) = I^{i+1} + (I - I^{i+1}) - \prod_{r=K_{m+1/2}}^{i+1} (I - I^r)$$

$$= I^{i+1} + (I - I^{i+1})\left(I - \prod_{r=K_{m+1/2}}^{i} (I - I^r)\right).$$

So

$$\left(I - \prod_{r=K_{m+1/2}}^{i+1} (I - I^r)\right)|x^{i+2} - x^*|$$

$$\leqslant I^{i+1}v \cdot \gamma_0 \beta_1^{m+1}$$

$$+ (I - I^{i+1})\left(I - \prod_{r=K_{m+1/2}}^{i} (I - I^r)\right)|x^{i+1} - x^*|$$

$$\leqslant I^{i+1}v \cdot \gamma_0 \beta_1^{m+1}$$

$$+ (I - I^{i+1})\left(I - \prod_{r=K_{m+1/2}}^{i} (I - I^r)\right)v \cdot \gamma_0 \beta_1^{m+1}$$

$$= \left(I - \prod_{r=K_{m+1/2}}^{i+1} (I - I^r)\right)v \cdot \gamma_0 \beta_1^{m+1}.$$

Thus, (2.20) is proved.

By the definition of K_{m+1}, from Lemma 3 we have, for any $k \geqslant K_{m+1}$,

$$\prod_{r=K_{m+1/2}}^{k} (I - I^r) = 0. \tag{2.21}$$

Substituting this equation into (2.20), we have

$$|x^{k+1} - x^*| \leqslant \beta_1^{m+1}\gamma_0 v \qquad \text{for any} \quad k \geqslant K_{m+1}. \tag{2.22}$$

This is equivalent to

$$\|x^{k+1} - x^*\| \leqslant \beta_1^{m+1}\gamma_0 \qquad \text{for any} \quad k \geqslant K_{m+1}. \tag{2.23}$$

We have completed the proof of (2.16).   ∎

We end this section with some comments on the conditions of Theorem 1.

REMARK 1.   It is clear that neither of the conditions (2.3) and (2.4) is included in the other. Note that the condition (2.4) is imposed only on those components of the operator $T_{k,l}$ which actually enter the computation. Consider teh following synchronous iteration (1.2) (a special case of the asynchronous iteration):

$$A = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & & \\ & & 2 & -1 \\ & & -1 & 2 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} -1 & 0 & & \\ -1 & 1 & & \\ & & 2 & 0 \\ & & -1 & 2 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 2 & 0 & & \\ -1 & 2 & & \\ & & 1 & 0 \\ & & -1 & 1 \end{pmatrix},$$

$$E_1 = \mathrm{diag}(0,0,1,1), \qquad E_2 = \mathrm{diag}(1,1,0,0).$$

In this example, $\rho(M_1^{-1}N_1) = \rho(M_2^{-1}N_2) = 1$; this means the convergence condition doesn't hold, and therefore convergence can't be obtained if one uses the traditional multisplitting result as in [7]. But, if we choose $v = (1,1,1,1)^T$, then the condition (2.4), i.e.

$$\| I_l [T_l(z) - x^*] \| \leqslant \beta \| I_l(z - x^*) \|$$

holds for $l = 1,2$ with $\beta = 0.5$, so the multisplitting iteration converges. This result seems to be new in the literature on synchronous or asynchronous multisplitting iteration.

REMARK 2.   In numerous papers about the multisplitting iteration, the particular E_l, $l = 1,\cdots, L$, satisfy $\sum_{l=1}^{L} E_l = I$. Then a relaxed multisplitting iteration, i.e. $\sum_{l=1}^{L} E_l = \omega I$, becomes relevant; e.g., [5, 3, 4]. In Theorem 1, we don't require that $\sum_{l=1}^{L} E_l$ have the same diagonal elements, and E_l can play a relaxing-factor role.

REMARK 3.   In some cases, E_l may vary with k, i.e., $E_{k,l}$. Using a method similar to that in Theorem 1, it is not difficult to obtain a similar condition on $E_{k,l}$ for convergence.

REMARK 4.   In [1], if (2.6) and (2.7) are valid, then $\{S_k\}_k$ and $\{d_{k,l,n}\}_{k,l,n}$ are called admissible. Consider the following stronger conditions: there exist $K \geqslant 0$ and $D \geqslant 0$ such that

$$\bigcup_{i=k}^{k+K} S_i = \{1, \cdots, L\} \qquad \forall k \geqslant 0 \tag{2.24}$$

and

$$d_{k,l,n} \leqslant D \qquad \forall k, l, n. \tag{2.25}$$

If (2.24) and (2.25) are valid, $\{S_k\}_k$ and $\{d_{k,l,n}\}_{k,l,n}$ are called regulated. We have proved the convergence under more weaker assumptions.

3.   SOME APPLICATIONS TO SOLVERS OF SYSTEMS OF EQUATIONS

In this section, we give some special versions of GMAI in some special cases, then obtain the convergence theorems by applying Theorem 1. We will use following asynchronous iterative scheme:

$$x^{k+1} = \left(I - \sum_{l \subset S_k} E_l\right) x^k + \sum_{l \subset S_k} E_l(T_l)^{\mu_{k,l}}(z^{k,l}), \qquad \mu_{k,l} \geqslant 1, \tag{3.1}$$

where

$$(T_l)^{\mu}(x) = (T_l)^{\mu-1}(T_l(x)).$$

With different T_l, we can handle different problems.

3.1. Linear Cases
To solve linear systems of equations

$$Ax = b, \tag{3.2}$$

where $A^{-1} \geqslant 0$, we let

$$T_l(x) = M_l^{-1}(N_l x + b),$$

and $A = M_l - N_l$, $l = 1, \cdots, L$, are L weakly regular splittings of A, i.e., $M_l^{-1} \geqslant 0$, $M_l^{-1} N_l \geqslant 0$. We know $\|M_l^{-1} N_l\|_v < 1$, where v is any positive vector such that $Av \gg 0$; see [2].

THEOREM 4.   If $A^{-1} \geqslant 0$, if $\mu_{k,l} \geqslant 1$ for $k \geqslant 1, l \in S_k$, and if (2.6) and (2.7) hold, then for any $x^0 \in \mathscr{R}^N$,

$$\lim_{k \to \infty} x^k = x^* = A^{-1}b.$$

The results in [1] are special cases of this.

3.2. Nonlinear Cases
Consider a nonlinear system of equations

$$F(x) = 0, \qquad F : \mathscr{R}^N \to \mathscr{R}^N. \tag{3.3}$$

Let x^* be its zero point, F be continuously differentiable in some neighborhood of x^*, and $F'(x^*)$ be nonsingular. We use the concept of *nonlinear multisplitting* of F which was introduced by Frommer in [5]. Let $\tilde{F}_l : \mathscr{R}^N \times \mathscr{R}^N \to \mathscr{R}^N$ satisfy $\forall x \in \mathscr{R}^N$, $\tilde{F}_l(x, x) = F(x)$. Suppose $\partial_2 \tilde{F}_l(x, y)$, the Jacobian of $\tilde{F}_l$ with respect to the second variable y, is nonsingular. Then we call F_l, $l = 1, \cdots, L$, a nonlinear multisplitting of F.

Suppose for all x, $\tilde{F}_l(x, y) = 0$ is uniquely solvable on y, which means there exists

$$T_l : \mathscr{R}^N \to \mathscr{R}^N, \qquad \tilde{F}_l(x, T_l(x)) = 0. \tag{3.4}$$

In practice, for a fixed x, $T_l(x)$ can't be solved exactly, and only can be approximated iteratively. An iteration similar to Newtonian iteration (to avoid introducing too much notation, we denote it by T_l also) is

$$T_l(x) = x - \left[\partial_2 \tilde{F}_l(x, x) \right]^{-1} \tilde{F}_l(x, x)$$

$$= x - \left[\partial_2 \tilde{F}_l(x, x) \right]^{-1} F(x). \tag{3.5}$$

As $\tilde{F}_l(x, x) = F(x)$, we have

$$F'(x) = \partial_2 \tilde{F}_l(x, x) - \left[-\partial_1 \tilde{F}_l(x, x) \right].$$

Denote

$$M_l(x) \equiv \partial_2 \tilde{F}_l(x, x), \qquad N_l(x) \equiv -\partial_1 \tilde{F}_l(x, x);$$

then

$$F'(x^*) = M_l(x^*) - N_l(x^*).$$

LEMMA 5. *Suppose* $F'(x^*) = M_l(x^*) - N_l(x^*)$, $l = 1, \cdots, L$, *are* L *weakly regular splittings and* $[F'(x^*)]^{-1} \geqslant 0$. *Let* T_l *be defined by* (3.4) *or* (3.5). *Then there exists* $\delta > 0$, $q < 1$, *such that for any* x *with* $\|x - x^*\| \leqslant \delta$,

$$\|T_l(x) - x^*\| \leqslant q \|x - x^*\|.$$

THEOREM 6. *Let* $\tilde{F}_l$, $l = 1, \cdots, L$, *satisfy the condition in Lemma 5;* T_l *be defined by* (3.4) *or* (3.5); E_l, $l = 1, \cdots, L$, *be* L *nonnegative diagonal matrices satisfying* (2.5); *and* q *be given by Lemma 5. For* $\{x^k\}_k$ *defined by* (3.1), *if* $\{d_{k,l,n}\}$ *and* $\{S_k\}$ *satisfy* (2.6) *and* (2.7), *and* $\mu_{k,l} \geqslant 1$, *then for any* x^0 *such that* $\|x^0 - x^*\| \leqslant \delta$, *we have*

$$\lim_{k \to \infty} x^k = x^*.$$

The synchronous case (i.e., $\sum_{l=1}^{L} E_l = \omega I$, $d_{k,l,n} = 0$, $S_k = \{1, \cdots, L\}$) of Theorem 6 is the main result of Frommer in [5].

Now we consider a special nonlinear function F:

$$F(x) = Ax + \Phi(x) - b, \qquad (3.6)$$

where A is an M-matrix and Φ is a continuous diagonal isotone mapping; see [6] or [3] for details. Such nonlinear systems may arise in the discretization of semilinear elliptic differential equations. Suppose (M_l, N_l), $l = 1, \cdots, L$, are L weakly regular (linear) splittings of A, and M_l are upper (or lower) triangular matrices. Let

$$\tilde{F}_l(x, y) = M_l y + \Phi(y) - (N_l x + b), \qquad l = 1, \cdots, L, \quad (3.7)$$

be L nonlinear splittings of F. Then

$$T_l(x) = (M_l + \Phi)^{-1}(N_l x + b).\qquad(3.8)$$

For any $x,\ y \in \mathscr{R}^n$, using a method similar to the one in [6], we can prove

$$|(M_l + \Phi)^{-1}(x) - (M_l + \Phi)^{-1}(y)| \leqslant M_l^{-1}|x - y|;$$

therefore

$$|T_l(x) - T_l(y)| \leqslant M_l^{-1}|N_l x + b - N_l y - b|$$

$$\leqslant M_l^{-1}N_l|x - y|.$$

Then there exists a positive constant q $(q < 1)$ such that

$$\|T_l(x) - T_l(y)\| \leqslant q\|x - y\|.$$

Note that the radius of the contracting region of T_l is $\delta = \infty$. So we can easily obtain the following global convergence theorem.

THEOREM 7.   *For the nonlinear system* (3.6), *the nonlinear multisplitting of F is given by* (3.7). *If the iterative parameters in* (3.7) *and* (3.87) *satisfy the conditions in Theorem 6, then for any* $x^0 \in \mathscr{R}^N$, *the sequence* $\{x^k\}_k$ *produced by* (3.1) *and* (3.8) *satisfies*

$$\lim_{k \to \infty} x^k = x^*,$$

where x^* *is the unique solution of* (3.6).

This result generalizes the results of [3].

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