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# ON THE CONVERGENCE AND STABILITY OF THE EPSILON ALGORITHM*

P. WYNN†

**1. Introduction.** The $\epsilon$-algorithm is a device for accelerating the convergence of slowly convergent sequences or determining a limit for divergent sequences; considerable numerical experience in its use has been gained.

It has been observed that there is often a considerable degree of regularity in the sequences produced, in particular that certain monotonic sequences are transformed into sequences of quantities with fixed sign and that certain oscillating sequences are transformed into oscillating sequences. Again, in some circumstances the transformed sequences produced by means of the $\epsilon$-algorithm converge very much more rapidly than do the original sequences; in others they do not. Lastly we remark that the $\epsilon$-algorithm is a recursive process involving repeated subtraction and division, and as such one would expect it to be numerically unstable: in certain circumstances it is, but in others it is quite remarkably stable.

From the viewpoint of mathematical analysis the $\epsilon$-algorithm is a transformation of the partial sums of a series into its Padé quotients or, equivalently, a process by means of which a series may be transformed into the convergents of its associated and corresponding continued fractions. For this reason the various quantities produced by means of the $\epsilon$-algorithm may very simply be expressed in terms of the successive orthogonal polynomials and associated orthogonal polynomials which are derived by using the coefficients of the given series as a sequence of moments.

Using the well established theory of associated and corresponding continued fractions it is possible, for a large class of examples, to establish the convergence of the derived sequences and even to be able to state that in certain circumstances they are monotonic or oscillating. But although we can decide upon qualitative questions of this nature, the theory of orthogonal polynomials and associated orthogonal polynomials does not at the present time provide us with sufficient results to enable us to assess the order of magnitude of the quantities produced by means of the $\epsilon$-algorithm, and thus we are in no position to make any general a priori pronouncements concerning the stability of the process.

However, in a few simple cases it transpires that everything that need be said concerning the sequences produced by the $\epsilon$-algorithm can in fact be said; moreover a study of these cases throws into sharp contrast the types of behavior exhibited by the $\epsilon$-algorithm in far more general cases.

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† Mathematics Research Center, University of Wisconsin, Madison, Wisconsin.

In this paper we shall consider, with regard to the nature of the derived sequences and their convergence, the behavior produced by application of the $\epsilon$-algorithm in these cases, and we shall discuss the stability of the process that produced these sequences.

**2. The $\epsilon$-algorithm** [1]. The fundamental relationships of the $\epsilon$-algorithm are

$$(1) \qquad (\epsilon_{s+1}^{(m)} - \epsilon_{s-1}^{(m+1)})(\epsilon_s^{(m+1)} - \epsilon_s^{(m)}) = 1;$$

they relate quantities $\epsilon_s^{(m)}$ which may be placed in a two-dimensional array in which the suffix $s$ indicates a column number and the superscript $m$ a diagonal (see Fig. 1).

$$\begin{array}{ccccc}
 & \epsilon_0^{(0)} & & & \\
\epsilon_{-1}^{(1)} & & \epsilon_1^{(0} & & \\
 & \epsilon_0^{(1)} & & \epsilon_2^{(0)} & \\
\epsilon_{-1}^{(2)} & & \epsilon_1^{(1)} & & \epsilon_3^{(0} \\
 & \epsilon_0^{(2)} & & \epsilon_2^{(1)} & \\
\epsilon_{-1}^{(3)} & & \epsilon_1^{(2)} & & \vdots \\
 & \epsilon_0^{(3)} & & \vdots & \\
\vdots & & \vdots & & \\
 & \vdots & & &
\end{array}$$

FIG. 1

The quantities occurring in relationships (1) are to be found at the vertices of a lozenge in this array as shown in Fig. 2.

$$\begin{array}{ccc}
 & \epsilon_s^{(m)} & \\
\epsilon_{s-1}^{(m+1)} & & \epsilon_{s+1}^{(m)} \\
 & \epsilon_s^{(m+1)} &
\end{array}$$

FIG. 2

In the applications which we shall have consistently in mind in this paper, relationships (1) are rearranged to give

$$(2) \qquad \epsilon_{s+1}^{(m)} = \epsilon_{s-1}^{(m+1)} + (\epsilon_s^{(m+1)} - \epsilon_s^{(m)})^{-1},$$

a formula which expresses the extreme right-hand member of the lozenge in Fig. 2 in terms of the remaining three. The initial conditions are taken to be

$$(3) \qquad \epsilon_{-1}^{(m)} = 0, \qquad m = 1, 2, \cdots,$$

$$(4) \qquad \epsilon_0^{(m)} = S_m, \qquad m = 0, 1, \cdots$$

where the quantities $S_m$, $m = 0, 1, \cdots$, are members of a slowly convergent or divergent sequences. Equations (3) and (4) denote the insertion of values in the first two columns of Fig. 1. The rest of the $\epsilon$-array (or part of it) is then constructed from left to right, column by column, by systematic use of (2).

The process which we have just described is the so-called forward application of the $\epsilon$-algorithm (i.e., in which the column $\{\epsilon_{-1}^{(m)}\}$ is filled with zeros, and the column $\{\epsilon_0^{(m)}\}$ is taken up by the sequence $\{S_m\}$ to be transformed). In order to clarify some of the terms used in the following text we introduce the following.

DEFINITION 1. If quantities $\epsilon_s^{(m)}$ are constructed by the use of relationships (2) in conjunction with (3) and (4), we shall then say that the $\epsilon$-algorithm has been applied to the sequences $\{S_m\}$, or alternatively that the $\epsilon$-algorithm has been applied to the initial values $S_m$, $m = 0, 1, \cdots$.

Resort to this linguistic artifice enables us to dispense with repeated reference to the initial conditions relating to the sequence $\{\epsilon_{-1}^{(m)}\}$ which in all the examples of this paper is taken to be identically zero.

It is often found that the quantities $\epsilon_{2s}^{(m)}$ (i.e., the quantities lying in the even order columns of Fig. 1) tend more rapidly to the formal limit of the sequence $\{S_m\}$ than does this sequence itself: indeed it is this property that enables the $\epsilon$-algorithm to be used as a convergence accelerating transformation.

The theoretical significance of the $\epsilon$-algorithm may in part be elucidated by describing its connection with the Padé table [2].

Given a power series $\sum_{s=0}^{\infty} c_s x^s$, it is formally possible to construct a certain double sequence $P_{i,j}$ $(i = 0, 1, \cdots; j = 0, 1, \cdots)$ of rational functions of $x$. The function $P_{i,j}$ is the quotient of two polynomials, the numerator of the $j$th degree, the denominator of the $i$th degree: this quotient is characterised by the property that its series expansion in ascending powers of $x$ agrees with the series $\sum_{s=0}^{\infty} c_s x^s$ as far as the term $c_{i+j}x^{i+j}$. Specifically,

$$(5) \qquad P_{i,j} \equiv \frac{\sum_{s=0}^{j} p_{i,j,s}\, x^s}{\sum_{s=0}^{i} p'_{i,j,s}\, x^s} = \sum_{s=0}^{i+j} c_s\, x^s + \sum_{s=i+j+1}^{\infty} d_s\, x^s,$$

where in general $d_s \neq c_s$, $s = i + j + 1, i + j + 2, \cdots$.

The connection between the $\epsilon$-algorithm and the Padé table is given by the following theorem.

THEOREM 1. *If the $\epsilon$-algorithm is applied to the initial values*

$$(6) \qquad \epsilon_0^{(m)} = \sum_{s=0}^{m} c_s x^s, \qquad m = 0, 1, \cdots,$$

*then*

$$(7) \qquad \epsilon_{2r}^{(m)} = P_{r,m+r}.$$

The functions $P_{i,j}$ are usually arranged in a two-dimensional array (the Padé table) in which the first suffix $i$ indicates a row number and the second suffix $j$ a column number. For the sake of completeness and to avoid confusion, we point out that the quantities of the even order $\epsilon$-array occur in the transpose of the Padé table. The quantities $\epsilon_{2r}^{(m)}$, $m = 0, 1, \cdots$, lie in a column of the $\epsilon$-array; the quantities $P_{r,m+r}$, $m = 0, 1, \cdots$, lie in a row in the Padé table.

As was mentioned in the Introduction there is a further strong connection between the $\epsilon$-algorithm and the theory of associated and corresponding continued fractions [3], but since this part of the theory of the $\epsilon$-algorithm will not be exploited in the following text, we shall not describe it here.

For the purposes of later work we now give certain determinantal formulae which express the quantities $\epsilon_s^{(m)}$ in terms of the members of the sequence $\{S_r\}$.

Hankel determinants of the form

$$\begin{vmatrix} f_m & f_{m+1} & \cdots & f_{m+k-1} \\ f_{m+1} & f_{m+2} & \cdots & f_{m+k} \\ \vdots & \vdots & & \vdots \\ f_{m+k-1} & f_{m+k} & \cdots & f_{m+2k-2} \end{vmatrix} \tag{8}$$

are frequently encountered in the theory of continued fractions: we shall denote (8) by the symbol

$$H_k^{(m)}\{f_r\}, \quad m = 0, 1, \cdots, \quad k = 1, 2, \cdots, \tag{9}$$

and stipulate that

$$H_0^{(m)}\{f_r\} = 1, \qquad m = 0, 1, \cdots. \tag{10}$$

We also introduce the difference operator $\Delta_r$ defined by

$$\begin{aligned} &\Delta_r f(p, q, \cdots, r, \cdots, s, t) \\ &\qquad = f(p, q, \cdots, r + 1, \cdots, s, t) - f(p, q, \cdots, r, \cdots, s, t). \end{aligned} \tag{11}$$

THEOREM 2. *If the $\epsilon$-algorithm is applied to the sequence $\{S_m\}$, then the quantities $\epsilon_s^{(m)}$ produced are given by*

$$\left.\begin{aligned} \epsilon_{2n}^{(m)} &= \frac{H_{n+1}^{(m)}\{S_r\}}{H_n^{(m)}\{\Delta_r^2 S\}}, &(12)\\ \epsilon_{2n+1}^{(m)} &= \frac{H_n^{(m)}\{\Delta_r^3 S_r\}}{H_{n+1}^{(m)}\{\Delta_r S_r\}}, &(13) \end{aligned}\right\} \quad m, n = 0, 1, \cdots.$$

We conclude this expository section on the $\epsilon$-algorithm by remarking that the relationships of the $\epsilon$-algorithm are not those of a linear operator. If the $\epsilon$-algorithm is applied to the sequence $\{A_m\}$ to produce quantities $\epsilon_s^{(m)}$, to the sequence $\{B_m\}$ to produce quantities $\epsilon_s^{(m)\prime}$, and to the sequence $\{A_m + B_m\}$ to produce quantities $\epsilon_s^{(m)\prime\prime}$, then in general it is certainly not true that, for example, $\epsilon_{2r}^{(m)\prime\prime} = \epsilon_{2r}^{(m)} + \epsilon_{2r}^{(m)\prime}$. Indeed it does not in general seem to be possible to derive a simple relationship between $\epsilon_{2r}^{(m)}$, $\epsilon_{2r}^{(m)\prime}$, and $\epsilon_{2r}^{(m)\prime\prime}$. But in one special case a result in this direction can be given. It is contained in the following.

THEOREM 3. *If the relationships of the $\epsilon$-algorithm are applied to the initial values*

$$\epsilon_0^{(m)} = S_m, \qquad m = 0, 1, \cdots, \tag{14}$$

*to produce quantities $\epsilon_s^{(m)}$, and to the initial values*

$$\epsilon_0^{(m)\prime} = A + S_m, \qquad m = 0, 1, \cdots, \tag{15}$$

*to produce quantities $\epsilon_s^{(m)\prime}$, then*

$$\left.\begin{aligned} &\epsilon_{2n}^{(m)\prime} = A + \epsilon_{2n}^{(m)}, &&(16)\\ &\epsilon_{2n+1}^{(m)\prime} = \epsilon_{2n+1}^{(m)\prime}, &&(17)\end{aligned}\right\} \qquad m, n = 0, 1, \cdots.$$

This theorem can often be utilised to simplify the presentation of certain results in the theory of the $\epsilon$-algorithm: a result which is derived with regard to a sequence (14) which possesses certain properties can immediately be extended to the consideration of a sequence of the form (15) which may not possess the same properties.

## 3. The behavior of the derived sequences.

**3.1. Monotonicity.** The results which we shall derive concerning the general behavior of the sequences produced by means of the $\epsilon$-algorithm are based on the theory of totally monotone sequences.

DEFINITION 2. If

$$S_r \geqq 0, \qquad r = 0, 1, \cdots, \tag{18}$$

and

$$(-1)^p \Delta_r^p S_r \geqq 0, \qquad r = 0, 1, \cdots, \quad p = 1, 2, \cdots, \tag{19}$$

then the sequence $\{S_r\}$ is said to be *totally monotone.*

Following from this definition we have the following trivial result.

LEMMA 1. *If $\{S_r\}$ is a totally monotone sequence then so are the sequences* $\{(-1)^k \Delta_r^k S_r\}$, $k = 0, 1, \cdots$.

The significance of totally monotone sequences in the present context arises from their influence on the behavior of Hankel determinants; indeed we have the fundamental:

LEMMA 2. *If the sequence $\{S_r\}$ is totally monotone, then*

$$H_k^{(m)}\{S_r\} \geqq 0. \tag{20}$$

Using Lemmas 1 and 2 in conjunction with the determinantal expressions (12) and (13) of Theorem 2 we obtain:

THEOREM 4. *If the relationships of the $\epsilon$-algorithm are applied to a sequence $\{S_r\}$ which is totally monotone, then the quantities produced satisfy the inequalities*

$$\left.\begin{aligned} &\epsilon_{2n}^{(m)} \geqq 0, && (21)\\ &\epsilon_{2n+1}^{(m)} \leqq 0, && (22)\end{aligned}\right\} \quad m, n = 0, 1, \cdots .$$

We have of course the following corollary.

COROLLARY. *If the sequence $\{S_r\}$ to which the $\epsilon$-algorithm is applied is such that the sequence $\{-S_r\}$ is totally monotone, then*

$$\left.\begin{aligned} &\epsilon_{2n}^{(m)} \leqq 0, && (23)\\ &\epsilon_{2n+1}^{(m)} \geqq 0, && (24)\end{aligned}\right\} \quad m, n = 0, 1, \cdots .$$

Another way of stating Theorem 4 and its corollary is to say that if the $\epsilon$-algorithm is applied to a totally monotone sequence then the even order columns are consistently positive and the odd order columns consistently negative, while if the initial conditions form the negative of a totally monotone sequence then the opposite is true.

By way of introducing an application of the above theorem we remark that

$$\Delta_r^{\,p} \frac{k!}{(r + \mu)\,(r + \mu + 1)\,\cdots\,(r + \mu + k)} = \frac{(k + p)!}{(r + \mu)\,(r + \mu + 1)\,\cdots\,(r + \mu + k + p)}, \tag{25}$$

$$r + \mu \neq 0, -1, \cdots, -k - p.$$

Thus, using Theorem 4, we have:

THEOREM 5. *If the relationships of the $\epsilon$-algorithm are applied to the initial values*

$$\epsilon_0^{(m)} = \sum_{s=0}^{\infty} b_s \frac{s!}{(m + \mu)\,(m + \mu + 1)\,\cdots\,(m + \mu + s)}, \tag{26}$$

*where the Newton series on the right-hand side of* (26) *either terminates or converges for $m = 0$, and furthermore,*

$$b_s \geqq 0, \qquad s = 0, 1, \cdots, \tag{27}$$

$$\mu > 0, \tag{28}$$

*then*

$$\left.\begin{aligned} &(29) && \epsilon_{2n}^{(m)} \geqq 0, \\ &(30) && \epsilon_{2n+1}^{(m)} \leqq 0, \end{aligned}\right\} \qquad m, n = 0, 1, \cdots .$$

A further special case of Theorem 4 follows from the observation that

$$(31) \qquad \Delta_r^p \lambda^r = (\lambda - 1)^p \lambda^r, \qquad r = 0, 1, \cdots, \quad p = 1, 2, \cdots .$$

Thus we have:

THEOREM 6. *If the relationships of the* $\epsilon$*-algorithm are applied to the initial values*

$$(32) \qquad \epsilon_0^{(m)} = \sum_{s=0}^{\infty} b_s \lambda_s^m, \qquad m = 0, 1, \cdots,$$

*where the Dirichlet series on the right-hand side of* (32) *either terminates or converges for* $m = 0$ *and furthermore*

$$(33) \qquad b_s \geqq 0, \qquad s = 0, 1, \cdots,$$

*and*

$$(34) \qquad 0 \leqq \lambda_s < 1, \qquad s = 1, 2, \cdots,$$

*then*

$$\left.\begin{aligned} &(35) && \epsilon_{2n}^{(m)} \geqq 0, \\ &(36) && \epsilon_{2n+1}^{(m)} \leqq 0, \end{aligned}\right\} \qquad m, n = 0, 1, \cdots .$$

**3.2. Oscillation.** In order to facilitate our discussion of the transformation of oscillating sequences, we introduce:

DEFINITION 3. If the sequence $\{(-1)^r S_r\}$ is totally monotone, then the sequence $\{S_r\}$ is said to be *totally oscillating*.

The analogue of Lemma 1 for totally oscillating sequences is of course:

LEMMA 3. *If* $\{S_r\}$ *is a totally oscillating sequence, then so are the sequences* $\{(-1)^k \Delta_r^k S_r\}$, $k = 0, 1, \cdots$.

Again corresponding to Lemma 2, we have:

LEMMA 4. *If the sequence* $\{S_r\}$ *is totally oscillating, then*

$$(37) \qquad (-1)^m H_k^{(m)}\{S_r\} \geqq 0.$$

Finally, in analogy to Theorem 4, we have:

THEOREM 7. *If the relationships of the* $\epsilon$*-algorithm are applied to a sequence* $\{S_r\}$ *which is totally oscillating, then the quantities produced satisfy the inequalities*

$$\left.\begin{aligned} &(38) && (-1)^m \epsilon_{2n}^{(m)} \geqq 0, \\ &(39) && (-1)^m \epsilon_{2n+1}^{(m)} \leqq 0, \end{aligned}\right\} \qquad m, n = 0, 1, \cdots .$$

Again this theorem has a corollary similar to that following Theorem 4. In other words if the $\epsilon$-algorithm is applied to totally oscillating sequences then each column of the $\epsilon$-array is occupied by a sequence of numbers alternating in sign.

We shall have occasion to consider sequences of slightly greater generality than those of the preceding theorem and therefore introduce the following:

DEFINITION 4. If the sequence $\{S_r\}$ is totally oscillating then the sequence $\{A + S_r\}$, where $A$ is some finite constant, is said to be *substantially totally oscillating.*

Theorem 7 and Theorem 3 in conjunction then give the next result.

THEOREM 8. *If the $\epsilon$-algorithm is applied to a substantially totally oscillating sequence then for fixed $n$ the quantities $\epsilon_{2n}^{(m)}$, $m = 0, 1, \cdots$, form an oscillating sequence, and the quantities $\epsilon_{2n+1}^{(m)}$, $m = 0, 1, \cdots$, alternate in sign.*

Using the result of (25) we have:

THEOREM 9. *If the relationships of the $\epsilon$-algorithm are applied to the initial values*

$$\epsilon_0^{(m)} = A + (-1)^m \sum_{s=0}^{\infty} b_s \frac{s!}{(m+\mu)(m+\mu+1)\cdots(m+\mu+s)}, \tag{40}$$

*where the series on the right-hand side of (40) either terminates or converges for $m = 0$, and furthermore*

$$b_s \geqq 0, \qquad s = 0, 1, \cdots, \tag{41}$$

$$\mu > 0, \tag{42}$$

*then for fixed $n$, $\{\epsilon_{2n}^{(m)}\}$ is an oscillating sequence, and $\{\epsilon_{2n+1}^{(m)}\}$ alternates in sign.*

Finally, from (31), we have:

THEOREM 10. *If the relationships of the $\epsilon$-algorithm are applied to the initial values*

$$\epsilon_0^{(m)} = A + (-1)^m \sum_{s=0}^{\infty} b_s \lambda_s^m, \tag{43}$$

*where the series on the right-hand side of (43) either terminates or converges for $m = 0$, and furthermore*

$$b_s \geqq 0, \qquad s = 0, 1, \cdots, \tag{44}$$

*and*

$$0 < \lambda_s < 1, \qquad s = 0, 1, \cdots, \tag{45}$$

*then for fixed* $n$, $\{\epsilon_{2n}^{(m)}\}$ *is an oscillating sequence and* $\{\epsilon_{2n+1}^{(m)}\}$ *alternates in sign.*

Although we have established that the derived sequences are either one signed or oscillating, we have not as yet said anything concerning their convergence and the order of magnitude of the constituent terms: this is the concern of the following section.

**4. Convergence and order of magnitude.** In this section we shall very much be concerned with sequences whose latter members may be asymptotically represented by series of the form

$$g_m \sim \sum_{s=1}^{\infty} a_s(\mu + m)^{-s}, \qquad \mu > 0, \tag{46}$$

or

$$h_m \sim \sum_{s=1}^{\infty} a_s \lambda_s^{\,m}, \qquad 1 > |\lambda_1| > |\lambda_2| > \cdots > |\lambda_s| > \cdots. \tag{47}$$

The sign $\sim$ of asymptotic representation refers to the manner in which the coefficients $\{a_s\}$ are obtained from the behavior of the function $g_m$ or $h_m$, as for example in the sense of Poincaré. Its meaning will also be that often accorded to it in numerical analysis, in the sense that, for example,

$$g_m \sim \frac{a_1}{(\mu + m)} \tag{48}$$

and

$$h_m \sim a_1 \lambda_1^{\,m} \tag{49}$$

mean that given any arbitrarily small positive quantity $\epsilon$, an integer $m_0$ can be found such that the inequalities of the form

$$\left| \frac{g_m - \dfrac{a_1}{(\mu + m)}}{g_m} \right| < \epsilon, \qquad m = m_0, m_0 + 1, \cdots, \tag{50}$$

or

$$\left| \frac{h_m - a_1 \lambda_1^{\,m}}{h_m} \right| < \epsilon, \qquad m = m_0, m_0 + 1, \cdots, \tag{51}$$

are satisfied. It will be recalled, furthermore, that such asymptotic series can be added, subtracted, multiplied and divided to yield further series of the same kind.

We shall now proceed to derive asymptotic estimates for the quantities $\epsilon_s^{(m)}$ produced by application of the $\epsilon$-algorithm to the sequences $\{S_m\}$

whose members have the asymptotic representations

$$\left.\begin{aligned} &(52)\quad S_m \sim A + \sum_{s=1}^{\infty} a_s(\mu + m)^{-s}, \\ &(53)\quad S_m \sim A + (-1)^m \sum_{s=1}^{\infty} a_s(\mu + m)^{-s}, \end{aligned}\right\} \qquad \mu > 0,$$

$$\left.\begin{aligned} &(54)\quad S_m \sim A + \sum_{s=1}^{\infty} a_s \lambda_s^{\,m}, \\ &(55)\quad S_m \sim A + (-1)^m \sum_{s=1}^{\infty} a_s \lambda_s^{\,m}, \end{aligned}\right\} \quad 1 > \lambda_1 > \cdots > \lambda_s > \cdots > 0.$$

In order to simplify the working we shall in all cases confine our attention in the first instance to the substitution $A = 0$, and later generalise the results obtained by means of application of Theorem 3.

### 4.1. Newton series.

4.1.1. *Monotonic sequences.* We now consider in detail the case in which

$$(56)\qquad S_m \sim \sum_{s=1}^{\infty} a_s(\mu + m)^{-s}, \qquad a_1 \neq 0.$$

The two preliminary results upon which we base the assault are these: firstly that by using a special case of (25) we have

$$(57)\qquad \Delta_m^{\,p} S_m \sim \frac{p!\, a_1}{(\mu + m)^{p+1}}\,;$$

and secondly that from trivial manipulations upon the determinantal formulae of Theorem 2 we have

$$(58)\qquad \epsilon_{2n}^{(m)} = \frac{H_{n+1}^{(0)}\{\Delta_m^{\,r} S_m\}}{H_n^{(2)}\{\Delta_m^{\,r} S_m\}}$$

and

$$(59)\qquad \epsilon_{2n+1}^{(m)} = \frac{H_n^{(3)}\{\Delta_m^{\,r} S_m\}}{H_{n+1}^{(1)}\{\Delta_m^{\,r} S_m\}}\,.$$

Our asymptotic estimates of $\epsilon_s^{(m)}$ will then follow by substituting expressions of the form (57) in formulae (58) and (59), and determining the orders of magnitude of the numerators and denominators of these expressions.

We illustrate the process by which this is done by considering in detail the numerator of (58). We write this out in full and multiply rows and columns by various powers of $(\mu + m)$ which we append to the formula thus:

$$\begin{array}{c|cccc|}
 & (\mu + m) & (\mu + m)^2 & \cdots & (\mu + m)^{n+1} \\
 & \frac{a_1\, 0!}{(\mu + m)} & \frac{a_1\, 1!}{(\mu + m)^2} & \cdots & \frac{a_1\, n!}{(\mu + m)^{n+1}} \\
(\mu + m) & \frac{a_1\, 1!}{(\mu + m)^2} & \frac{a_1\, 2!}{(\mu + m)^3} & \cdots & \frac{a_1(n + 1)!}{(\mu + m)^{n+2}} \\
\vdots & \vdots & \vdots & & \vdots \\
(\mu + m)^n & \frac{a_1\, n!}{(\mu + m)^{n+1}} & \frac{a_1(n + 1)!}{(\mu + m)^{n+2}} & \cdots & \frac{a_1(2n)!}{(\mu + m)^{2n+1}}
\end{array}\; . \tag{60}$$

In this way we show that in the case under consideration

$$(\mu + m)^{(n+1)^2} H^{(0)}_{n+1}\{\Delta_m^{\;r} S_m\} \sim a_1^{\;n+1} H^{(0)}_{n+1}\{r!\}. \tag{61}$$

Proceeding in a similar manner we show finally that

$$\epsilon_{2n}^{(m)} \sim \frac{H^{(0)}_{n+1}\{r!\}}{H_n^{(2)}\{r!\}}\, a_1(\mu + m)^{-1}, \tag{62}$$

$$\epsilon_{2n+1}^{(m)} \sim -\frac{H_n^{(3)}\{r!\}(\mu + m)^2}{H^{(1)}_{n+1}\{r!\}\; a_1}\, . \tag{63}$$

With regard to the Hankel determinants occurring in these formulae it can be shown, either by recourse to the $q - d$ algorithm [4] or by direct appeal to the fundamental recursion satisfied by Hankel determinants [5], namely,

$$\{H_k^{(n)}\}^2 - H_k^{(n-1)} H_k^{(n+1)} + H_{k+1}^{(n-1)} H_{k-1}^{(n+1)} = 0, \tag{64}$$

that

$$\frac{H^{(0)}_{n+1}\{r!\}}{H_n^{(2)}\{r!\}} = \frac{1}{(n + 1)}, \tag{65}$$

$$\frac{H_n^{(3)}\{r!\}}{H_n^{(1)}\{r!\}} = \frac{(n + 1)(n + 2)}{2}\, . \tag{66}$$

Thus we have obtained:

THEOREM 11. *If the* $\epsilon$*-algorithm is applied to the sequence* $\{S_m\}$ *whose members have the asymptotic representation*

$$S_m \sim A + \sum_{s=1}^{\infty} a_s(\mu + m)^{-s}, \qquad a_1 \neq 0, \tag{67}$$

*then for fixed* $n$,

$$\epsilon_{2n}^{(m)} \sim A + \frac{a_1}{(n + 1)(\mu + m)}, \tag{68}$$

$$\epsilon_{2n+1}^{(m)} \sim \frac{(n+1)(n+2)(\mu+m)^2}{2a_1}. \tag{69}$$

The general results of this theorem are strikingly vindicated by the following result, which is easily proved by induction:

THEOREM 12. *If the relationships of the $\epsilon$-algorithm are applied to the initial values*

$$\epsilon_0^{(m)} = A + \frac{a_1}{(\mu+m)}, \quad a_1 \neq 0, \quad m = 0, 1, \cdots, \tag{70}$$

*then*

$$\left.\begin{aligned} &(71) \quad \epsilon_{2n}^{(m)} = A + \frac{a_1}{(n+1)(\mu+m+n)}, \\ &(72) \quad \epsilon_{2n+1}^{(m)} = \frac{-(n+1)(n+2)(\mu+m+n)(\mu+m+n+1)}{2a_1}, \end{aligned}\right\}$$

$$m, n = 0, 1, \cdots.$$

To conclude this section on the transformation of the monotone sequences under consideration we remark on the one hand that, bearing in mind the classes of sequences being considered, convergent sequences are transformed into convergent sequences in the sense that both $\{S_m\}$ and $\{\epsilon_{2n}^{(m)}\}$ tend to the same limit as $m$ tends to infinity, but on the other that the rate of convergence has hardly been increased.

4.1.2. *Oscillating sequences.* We now turn to oscillating sequences of the form

$$S_m \sim (-1)^m \sum_{s=1}^{\infty} a_s(\mu+m)^{-s}, \qquad a_1 \neq 0. \tag{73}$$

The result corresponding to (57) for such sequences is

$$\Delta_m^{\,p}(-1)^m S_m \sim \frac{a_1\, p!}{(\mu+m)^{p+1}}, \tag{74}$$

and in order to use it we shall have to express the quantities $\epsilon_s^{(m)}$ in terms of the differences, not of $\{S_m\}$ but of $\{(-1)^m S_m\}$. We have, it will be recalled, in this case,

$$\epsilon_{2n}^{(m)} = \frac{H_{n+1}^{(m)}\{S_r\}}{H_n^{(m)}\{\Delta_r^2 S_r\}}. \tag{75}$$

Now, as is easily verified, we have for Hankel determinants the simple relationship

$$H_{n+1}^{(m)}\{S_r\} = (-1)^m H_{n+1}^{(m)}\{(-1)^r S_r\}. \tag{76}$$

The estimate (74) can be inserted into the right-hand side of (76), the order of magnitude of the numerator of (75) can be determined just as in the case of (58) and indeed, apart from a factor of $(-1)^m$, we obtain the same estimate.

The denominator is not dismissed so lightly; we rewrite it firstly as

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ \Delta_m S_m & \Delta_m^2 S_m & \Delta_m^2 S_{m+1} & \cdots & \Delta_m^2 S_{m+n-1} \\ \Delta_m S_{m+1} & \Delta_m^2 S_{m+1} & \Delta_m^2 S_{m+2} & \cdots & \Delta_m^2 S_{m+n} \\ \vdots & \vdots & \vdots & & \vdots \\ \Delta_m S_{m+n-1} & \Delta_m^2 S_{m+n-1} & \Delta_m^2 S_{m+n} & \cdots & \Delta_m^2 S_{m+2n-2} \end{vmatrix} \tag{77}$$
$$= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta_m S_m & \Delta_m S_{m+1} & \cdots & \Delta_m S_{m+n} \\ \Delta_m S_{m+1} & \Delta_m S_{m+2} & \cdots & \Delta_m S_{m+n+1} \\ \vdots & \vdots & & \vdots \\ \Delta_m S_{m+n-1} & \Delta_m S_{m+n} & \cdots & \Delta_m S_{m+2n-1} \end{vmatrix},$$

and this, by use of a similar artifice and subsequent row and column multiplication by appropriate powers at $-1$, can be reduced to

$$\begin{vmatrix} 0 & (-1)^m & (-1)^{m+1} & \cdots & (-1)^{m+n} \\ 1 & (-1)^m S_m & (-1)^{m+1} S_{m+1} & \cdots & (-1)^{m+n} S_{m+n} \\ -1 & (-1)^{m+1} S_{m+1} & (-1)^{m+2} S_{m+2} & \cdots & (-1)^{m+n+1} S_{m+n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^n & (-1)^{m+n} S_{m+n} & (-1)^{m+n+1} S_{m+n+1} & \cdots & (-1)^{m+2n} S_{m+2n} \end{vmatrix} \tag{78}$$

or

$$\begin{vmatrix} 0 & (-1)^m & (-1)^m(-2) & \cdots & (-1)^m(-2)^n \\ 1 & (-1)^m S_m & \Delta_m(-1)^m S_m & \cdots & \Delta_m^n(-1)^m S_m \\ -2 & \Delta_m(-1)^m S_m & \Delta_m^2(-1)^m S_m & \cdots & \Delta_m^{n+1}(-1)^m S_m \\ \vdots & \vdots & \vdots & & \vdots \\ (-2)^n & \Delta_m^n(-1)^m S_m & \Delta_m^{n+1}(-1)^m S_m & \cdots & \Delta_m^{2n}(-1)^m S_m \end{vmatrix}. \tag{79}$$

The entries in the first row and column arise from the fact that

$$\Delta_r^p(-1)^r = (-1)^r(-2)^p. \tag{80}$$

In this way we show that

$$(\mu + m)^{n^2} H_n^{(m)}\{\Delta_r^2 S_r\} \sim a_1^n 2^{2n} H_n^{(0)}\{r!\}. \tag{81}$$

Before proceeding further it is important to point out that there is a great deal of difference between the processes of obtaining asymptotic estimates for the quantities derived from the transformation of the monotonic and oscillating sequences under consideration, apart, that is, from the technical difficulties involved. In the case of the monotonic sequence $\{S_r\}$ of (56) the estimate of the denominator given by a process similar to that involved in the construction of (60) yields for this sequence

$$H_n^{(m)}\{\Delta_r^2 S_r\} = O((m+\mu)^{-n(n+2)}). \tag{82}$$

The presence of the first row and column in the determinant (79) means, however, that for the oscillating sequence $\{S_r\}$ of (73),

$$H_n^{(m)}\{\Delta_r^2 S_r\} = O((m+\mu)^{-n^2}). \tag{83}$$

These differences lead in turn to the quite distinct behavior of the $\epsilon$-algorithm in the two cases. For in the monotonic case

$$\epsilon_{2n}^{(m)} = O((\mu+m)^{-1}), \tag{84}$$

and in the oscillating case

$$\epsilon_{2n}^{(m)} = O((\mu+m)^{-2n-1}). \tag{85}$$

Asymptotic estimates may also be obtained for the numerator and denominator in the expression

$$\epsilon_{2n+1}^{(m)} = \frac{H_n^{(m)}\{\Delta_r^3 S_r\}}{H_{n+1}^{(m)}\{\Delta_r\, S_r\}}. \tag{86}$$

The numerator evolves to the form

$$\begin{vmatrix} 0 & 0 & (-1)^m & \cdots & (-)^{m+n} \\ 0 & 0 & (-1)^m S_m & \cdots & (-1)^{m+n} S_{m+n} \\ -1 & -1 & (-1)^{m+1} S_{m+1} & \cdots & (-1)^{m+n+1} S_{m+n+1} \\ 1 & 2 & (-1)^{m+2} S_{m+2} & \cdots & (-1)^{m+n+2} S_{m+n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n+2} & (-1)^{n+1}(n+1) & (-1)^{m+n+1} S_{m+n+1} & \cdots & (-1)^{m+2n+1} S_{m+2n+1} \end{vmatrix}. \tag{87}$$

When forming the difference of rows and columns of this determinant, we have not only to take (80) into account, but also use the result that if

$$f_s = (-1)^s s, \tag{88}$$

then

$$\Delta_s^{2p} f_s = (-1)^s (p + s) 2^{2p}, \tag{89}$$

$$\Delta_s^{2p+1} f_s = (-1)^{s+1} (2p + 2s + 1) 2^{2p}. \tag{90}$$

We arrive finally at the result

$$(\mu + m)^{-2n-1} \epsilon_{2n+1}^{(m)} \sim \frac{(-1)^{m+1} 2^{2n-1} H_n^{(0)}\{r!\}}{a_1 H_{n+1}^{(0)}\{r!\}}. \tag{91}$$

Again the ratios of Hankel determinants of factorial functions can be simplified and we obtain:

THEOREM 13. *If the $\epsilon$-algorithm is applied to the sequence $\{S_m\}$ whose members have the asymptotic representation*

$$S_m \sim A + (-1)^m \sum_{s=1}^{\infty} a_s (\mu + m)^{-s}, \qquad a_1 \neq 0, \tag{92}$$

*then for fixed $n$,*

$$\epsilon_{2n}^{(m)} \sim A + \frac{(-1)^m (n!)^2 a_1}{2^{2n} (\mu + m)^{2n+1}}, \tag{93}$$

$$\epsilon_{2n+1}^{(m)} \sim \frac{(-1)^{m+1} 2^{2n-1} (\mu + m)^{2n+1}}{(n!)^2 a_1}. \tag{94}$$

The general estimates of this theorem are in part confirmed by the following result, which can be verified after some manipulation.

THEOREM 14. *If the relationships of the $\epsilon$-algorithm are applied to the initial values*

$$\epsilon_0^{(m)} = A + \frac{(-1)^m a_1}{(\mu + m)}, \qquad a_1 \neq 0, \tag{95}$$

*and we adopt the notation*

$$m_r = \mu + m + r, \qquad m_r' = \mu + m + r + \tfrac{1}{2}, \tag{96}$$

*then successively*

$$\epsilon_0^{(m)} = A + \frac{(-1)^m a_1}{m_0}, \tag{97}$$

$$\epsilon_1^{(m)} = (-1)^{m+1} \frac{m_0'^2 - \frac{1}{4}}{2 m_0' a_1}, \tag{98}$$

$$\epsilon_2^{(m)} = A + (-1)^m \frac{a_1}{m_1 (2 m_1^2 - 1)}, \tag{99}$$

$$\epsilon_3^{(m)} = (-1)^{m+1} \frac{m_1' (m_1'^2 - \frac{1}{4})(m_1'^2 - \frac{7}{4})}{(m_1'^2 + \frac{1}{4}) a_1}, \tag{100}$$

$$\epsilon_4^{(m)} = A + (-1)^m \frac{a_1}{4m_2(m_2^4 - m_2^2 + \frac{3}{4})}, \tag{101}$$

$$\epsilon_5^{(m)} = (-1)^{m+1} \frac{2(m_2'^2 - \frac{1}{4})(m_2'^6 - \frac{17}{4}m_2'^4 + \frac{79}{16}m_2'^2 + \frac{81}{64})}{m_2'(m_2'^2 + \frac{5}{4})a_1}, \tag{102}$$

$$\epsilon_6^{(m)} = A + (-1)^m \frac{9}{16} \frac{a_1}{m_3(m_3^6 - \frac{1}{2}m_3^4 + 4m_3^2 - \frac{9}{4})}. \tag{103}$$

We have not obtained a general expression for the quantities $\epsilon_s^{(m)}$ in this case, but it can easily be proved by induction that $\epsilon_{2r}^{(m)}$ is, apart from a factor of $m_r^{-1}$, a function of $m_r^2$ alone, and that correspondingly $\epsilon_{2r+1}^{(m)}$ is, apart from a factor of $m_r'^{-1}$, a function of $m_r'^2$ alone.

**4.2. Dirichlet series.** The principal algebraic result [1] which has so far been derived in the theory of the $\epsilon$-algorithm is as follows:

THEOREM 15. *If the $\epsilon$-algorithm is applied to the sequence $\{S_m\}$, where a homogeneous linear recursion of the form*

$$\sum_{r=0}^{n} b_r S_{m+r} = A', \qquad m = 0, 1, \cdots, \tag{104}$$

*prevails among the members of $\{S_m\}$, then*

$$\epsilon_{2n}^{(m)} = A, \qquad m = 0, 1, \cdots, \tag{105}$$

*where*

$$A' = \left(\sum_{r=0}^{n} b_r\right) A. \tag{106}$$

This result includes the case in which $S_m$ is represented by a terminating Dirichlet series

$$S_m = A + \sum_{s=1}^{n} a_s \lambda_s^m, \quad m = 0, 1, \cdots, \quad |\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|, \tag{107}$$

for the quantities $S_m$ ($m = 0, 1, \cdots$) then satisfy a linear recursion of the form (104) in which the quantities $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the roots of the characteristic equation

$$\sum_{r=0}^{n} b_r \lambda^r = 0. \tag{108}$$

Although we have shown that in this case the $2n$th column of the $\epsilon$-array contains the quantity $A$ identically, we have not as yet examined the way in which the quantities in the columns of lesser order converge to $A$, if indeed they do. It is to this problem and to its extension concerning the representation of the initial values by nonterminating Dirichlet series that we now address ourselves.

The manipulations involved in obtaining asymptotic estimates for the quantities $\epsilon_s^{(m)}$ produced by application of the $\epsilon$-algorithm to the initial values

$$(109) \qquad S_m \sim A + \sum_{s=1}^{\infty} a_s \lambda_s^m, \qquad |\lambda_1| > |\lambda_2| > \cdots,$$

are of a vastly simpler kind than those of the preceding two sections. They are based on the fundamental result for Hankel determinants stated in Lemma 5 and on a trivial consequence of Lemma 5, formulated as Lemma 6.

LEMMA 5. *If*

$$(110) \qquad S_r \sim \sum_{s=1}^{\infty} a_s \lambda_s^r, \qquad |\lambda_1| > |\lambda_2| > \cdots$$

*then, in the notation of* (8),

$$(111) \quad H_n^{(m)}\{S_r\} \sim a_1 a_2 \cdots a_n V\{\lambda_1, \lambda_2, \cdots, \lambda_n\}^2 \lambda_1^m \lambda_2^m \cdots \lambda_n^m,$$

*where* $V\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ *is the Vandermonde determinant constructed from the quantities* $\lambda_1, \lambda_2, \cdots, \lambda_n$.

LEMMA 6. *If* $S_r$ *is asymptotically represented by* (109), *then*

$$(112) \quad H_n^{(m)}\{\Delta^p S_r\} \sim (\lambda_1 - 1)^p (\lambda_2 - 1)^p \cdots (\lambda_n - 1)^p H_n^{(m)}\{S_r\}.$$

These asymptotic estimates will be substituted into the formulae

$$(113) \qquad \epsilon_{2n}^{(m)} = \frac{H_{n+1}^{(m)}\{S_r\}}{H_n^{(m)}\{\Delta_r^2 S_r\}},$$

$$(114) \qquad \epsilon_{2n+1}^{(m)} = \frac{H_n^{(m)}\{\Delta_r^3 S_r\}}{H_{n+1}^{(m)}\{\Delta_r S_r\}}.$$

4.2.1. *Monotonic sequences.* We immediately have:

THEOREM 16. *If the* $\epsilon$*-algorithm is applied to the sequence* $\{S_m\}$ *whose members have the asymptotic representation*

$$(115) \qquad S_m \sim A + \sum_{s=1}^{\infty} a_s \lambda_s^m, \qquad 1 > \lambda_1 > \lambda_2 > \cdots > 0,$$

*then for fixed n,*

$$(116) \quad \epsilon_{2n}^{(m)} \sim A + \frac{a_{n+1}\{(\lambda_{n+1} - \lambda_1)(\lambda_{n+1} - \lambda_2) \cdots (\lambda_{n+1} - \lambda_n)\}^2 \lambda_{n+1}^m}{\{(1 - \lambda_1)(1 - \lambda_2) \cdots (1 - \lambda_n)\}^2},$$

$$(117) \quad \epsilon_{2n+1}^{(m)} \sim \frac{\{(1 - \lambda_1)(1 - \lambda_2) \cdots (1 - \lambda_n)\}^2}{a_{n+1}\{(\lambda_{n+1} - \lambda_1)(\lambda_{n+1} - \lambda_2) \cdots (\lambda_{n+1} - \lambda_n)\}^2 (\lambda_{n+1} - 1)\lambda_{n+1}^m}.$$

4.2.2. *Oscillating sequences.* The case in which $\{S_m\}$ is asymptotically represented by the formula

$$(118) \qquad S_m \sim A + (-1)^m \sum_{s=1}^{\infty} a_s \lambda_s^{\,m}$$

is of course summarily dismissed by changing the sign of the $\lambda_s$ in the preceding analysis; we have:

THEOREM 17. *If the $\epsilon$-algorithm is applied to the sequence $\{S_m\}$ whose members have the asymptotic representation*

$$(119) \qquad S_m \sim A + (-1)^m \sum_{s=1}^{\infty} a_s \lambda_s^{\,m}, \qquad 1 > \lambda_1 > \lambda_2 > \cdots > 0,$$

*then for fixed $n$,*

$$(120) \qquad \epsilon_{2n}^{(m)} \sim A + \frac{a_{n+1}\{(\lambda_{n+1} - \lambda_1)(\lambda_{n+1} - \lambda_2) \cdots (\lambda_{n+1} - \lambda_n)\}^2(-\lambda_{n+1})^m}{\{(1 + \lambda_1)(1 + \lambda_2) \cdots (1 + \lambda_n)\}^2},$$

$$(121) \qquad \epsilon_{2n+1}^{(m)} \sim \frac{\{(1 + \lambda_1)(1 + \lambda_2) \cdots (1 + \lambda_n)\}^2}{a_{n+1}\{(\lambda_{n+1} - \lambda_1)(\lambda_{n+1} - \lambda_2) \cdots (\lambda_{n+1} - \lambda_n)\}^2(1 + \lambda_{n+1})(-\lambda_{n+1})^m}.$$

With regard to the convergence of the derived sequences $\epsilon_{2n}^{(m)}$, $m = 0, 1, \cdots$, in the two cases it will be seen that the way in which each approaches a limit is like that of a constant multiplied by an exponential term in $\lambda_{n+1}$ : it is true that the constant in the case of an oscillating initial sequence is smaller than that relating to the monotonic sequence (see (116) and (120) and the inequalities contained in (115), (119)) but this is hardly a matter of great consequence.

In both cases the $\epsilon$-algorithm has again proved to be a convergence preserving transformation. As to whether the derived sequences converge much more rapidly than the initial sequence depends upon the ratios $(\lambda_{n+1}/\lambda_1)$, $n = 2, 3, \cdots$ : if this is small the improvement in convergence will be very marked; if not, the $\epsilon$-algorithm will hardly have affected the rate of convergence.

**5. Analysis of stability.** Error analyses of recursive processes can be approached in various ways. Making strong assumptions concerning the way in which the error is introduced it is sometimes possible to obtain very sharp estimates of the error, and indeed in this way to devise a process for correcting the quantities being produced. This has been done for the $\epsilon$-algorithm in [6]. Again, by making certain assumptions concerning the way in which the error is introduced, it is occasionally possible to utilise the theory of the algorithmic process involved to diagnose the conditions under which the numerical computations are unstable. This occurs for example in the well-known case of three-term linear recursions: it has been carried out for the $\epsilon$-algorithm, assuming that the initial values are functions of a parameter and are computed in a certain way, in [7]. Lastly it may not be possible to exploit the theory of the algorithm in any way and it is then necessary, by adopting a linearised perturbation technique, to content ourself with a computational process which yields an upper bound for the error associated with the various quantities being computed: this has been carried out for the $\epsilon$-algorithm in [8].

In the present inquiry we have been able to derive estimates for the order of magnitude of the quantities being computed. Using these estimates we are able to obtain approximate values for the factors which magnify or diminish the small errors which are introduced at each stage of the calculation. We shall, that is, give a stability analysis for the cases being considered.

Although the $\epsilon$-array is built up from left to right by means of a single relationship which is used consistently, it is nevertheless true, as we have established, that the entries in the odd and even order columns behave in quite different ways. For this reason we shall consider in our analysis not only what happens as we proceed from one column to the next, but also the further effect upon the column next but one to that in which a disturbance is introduced.

Let us suppose that from an accurate value of $\epsilon_{s-1}^{(m)}$, accurate values of $\epsilon_s^{(m)}$, $\epsilon_s^{(m-1)}$, and so on may be computed: and that by introducing a relative error of $\delta_{s-1}^{(m)}$ in $\epsilon_{s-1}^{(m)}$, relative errors of $\delta_s^{(m)}$ in $\epsilon_s^{(m)}$, $\delta_s^{(m-1)}$ in $\epsilon_s^{(m-1)}$, and so on arise. We obtain a constellation of values in the $\epsilon$-array as shown in Fig. 3.

If the relative errors are all small, then we have the following relationships:

$$(122)\qquad \delta_s^{(m-1)} = -\frac{\epsilon_{s-1}^{(m)}}{\epsilon_s^{(m-1)}\left(\epsilon_{s-1}^{(m)} - \epsilon_{s-1}^{(m-1)}\right)^2}\,\delta_{s-1}^{(m)},$$

$$(123)\qquad \delta_s^{(m)} = \frac{\epsilon_{s-1}^{(m)}}{\epsilon_s^{(m)}\left(\epsilon_{s-1}^{(m+1)} - \epsilon_{s-1}^{(m)}\right)^2}\,\delta_{s-1}^{(m)},$$

$$(124)\qquad \delta_{s+1}^{(m-2)} = \frac{\epsilon_{s-1}^{(m)}}{\epsilon_{s+1}^{(m-2)}\left(\epsilon_{s-1}^{(m)} - \epsilon_{s-1}^{(m-1)}\right)^2\left(\epsilon_s^{(m-1)} - \epsilon_s^{(m-2)}\right)^2}\,\delta_{s-1}^{(m)},$$

$$(125)\qquad \begin{aligned}\delta_{s+1}^{(m-1)} = \frac{\epsilon_{s-1}^{(m)}}{\epsilon_{s+1}^{(m-1)}}\Bigg[1 - \Bigg\{&\frac{1}{\left(\epsilon_{s-1}^{(m+1)} - \epsilon_{s-1}^{(m)}\right)^2}\\ &+ \frac{1}{\left(\epsilon_{s-1}^{(m)} - \epsilon_{s-1}^{(m-1)}\right)^2}\Bigg\}\frac{1}{\left(\epsilon_s^{(m)} - \epsilon_s^{(m-1)}\right)^2}\Bigg]\,\delta_{s-1}^{(m)},\end{aligned}$$

$$(126)\qquad \delta_{s+1}^{(m)} = -\frac{\epsilon_{s-1}^{(m)}}{\epsilon_{s+1}^{(m)}\left(\epsilon_{s-1}^{(m+1)}\right) - \left(\epsilon_{s-1}^{(m)}\right)^2\left(\epsilon_s^{(m+1)} - \epsilon_s^{(m)}\right)^2}\,\delta_{s-1}^{(m)}.$$

$$\begin{array}{cccc}
 & & \epsilon_s^{(m-2)} & \\
 & \epsilon_{s-1}^{(m-1} & & \epsilon_{s+1}^{(m-2)}(1+\delta_{s+1}^{(m-2)}) \\
\epsilon_{s-2}^{(m)} & & \epsilon_s^{(m-1)}(1+\delta_s^{(m-1)}) & \\
 & \epsilon_{s-1}^{(m)}(1+\delta_{s-1}^{(m)}) & & \epsilon_{s+1}^{(m-1)}(1+\delta_{s+1}^{(m-1)}) \\
\epsilon_{s-2}^{(m+1)} & & \epsilon_s^{(m)}(1+\delta_s^{(m)}) & \\
 & \epsilon_{s-1}^{(m+1)} & & \epsilon_{s+1}^{(m)}(1+\delta_{s+1}^{(m)}) \\
 & & \epsilon_s^{(m+1)} &
\end{array}$$

FIG. 3

In order to examine the propagation of absolute error we shall again consider a constellation of $\epsilon$-quantities similar to that in Fig. 3. Resulting from an absolute error of $a_{s-1}^{(m)}$ in $\epsilon_{s-1}^{(m)}$, absolute errors of $a_s^{(m)}$ in $\epsilon_s^{(m)}$, $a_s^{(m-1)}$ in $\epsilon_s^{(m-1)}$, and so on are introduced. Again if these errors are small then we have

$$a_s^{(m-1)} = -\frac{a_{s-1}^{(m)}}{(\epsilon_{s-1}^{(m)} - \epsilon_{s-1}^{(m-1)})^2}, \tag{127}$$

$$a_s^{(m)} = \frac{a_{s-1}^{(m)}}{(\epsilon_{s-1}^{(m+1)} - \epsilon_{s-1}^{(m)})^2}, \tag{128}$$

$$a_{s+1}^{(m-2)} = \frac{a_{s-1}^{(m)}}{(\epsilon_{s-1}^{(m)} - \epsilon_{s-1}^{(m-1)})^2(\epsilon_s^{(m-1)} - \epsilon_s^{(m-2)})^2}, \tag{129}$$

$$\begin{aligned} a_{s+1}^{(m-1)} = \Bigg[1 - \Bigg\{&\frac{1}{(\epsilon_{s-1}^{(m+1)} - \epsilon_{s-1}^{(m)})^2} \\ &+ \frac{1}{(\epsilon_{s-1}^{(m)} - \epsilon_{s-1}^{(m-1)})^2}\Bigg\} \frac{1}{(\epsilon_s^{(m)} - \epsilon_s^{(m-1)})^2}\Bigg] a_{s-1}^{(m)}, \end{aligned} \tag{130}$$

$$a_{s+1}^{(m)} = -\frac{a_{s-1}^{(m)}}{(\epsilon_{s-1}^{(m+1)} - \epsilon_{s-1}^{(m)})^2(\epsilon_s^{(m+1)} - \epsilon_s^{(m)})^2}. \tag{131}$$

These are the amplification formulae upon which our stability analysis will be based.

### 5.1. Newton series.

5.1.1. *Monotonic sequences.* We now consider the stability of the computations that result from the application of the $\epsilon$-algorithm to the sequence $\{S_m\}$, where

$$S_m \sim A + \sum_{s=1}^{\infty} a_s(\mu + m)^{-s}, \qquad a_1 \neq 0. \tag{132}$$

We recall from Theorem 11 that then

$$\epsilon_{2n}^{(m)} \sim A + \frac{a_1}{(n+1)(\mu + m)}, \tag{133}$$

$$\epsilon_{2n+1}^{(m)} \sim -\frac{(n+1)(n+2)(\mu+m)^2}{2a_1}, \tag{134}$$

and we remark that it can be shown that

$$\epsilon_{2n}^{(m+1)} - \epsilon_{2n}^{(m)} \sim -\frac{a_1}{(n+1)(\mu+m)^2}, \tag{135}$$

$$\epsilon_{2n+1}^{(m+1)} - \epsilon_{2n+1}^{(m)} \sim -\frac{(n+1)(n+2)(\mu+m)}{a_1} \tag{136}$$

(the last two formulae will be used in conjunction with (122)–(131)).

In the following analysis we shall make the further distinction, concerning the magnitudes of the two terms on the right-hand side of (127), between the following two cases: that in which the constant $A$ is the larger, and that in which the opposite is true. The former case will be that which is the most frequently encountered, since in practical numerical analysis we seldom deal with sequences whose limit is exactly or very nearly zero.

To begin, we put $s - 1 = 2n$ in (122)–(131). Then by substituting (134) and (135) in (122), we find that when $A$ is large,

$$\delta_{2n+1}^{(m-1)} \sim \frac{2A(n+1)(\mu+m)^2}{(n+2)a_1}\,\delta_{2n}^{(m)}, \tag{137}$$

and that when $A$ is small,

$$\delta_{2n+1}^{(m-1)} \sim \frac{2(\mu+m)}{(n+2)}\,\delta_{2n}^{(m)}. \tag{138}$$

The formulae expressing $\delta_{2n+1}^{(m)}$ in terms of $\delta_{2n}^{(m)}$ are very much the same.

Before we draw any conclusions from these two formulae let us write down a few more. We have, when $A$ is large,

$$\delta_{2n+2}^{(m)} \sim \frac{(\mu+m)^2}{(n+2)^2}\,\delta_{2n}^{(m)}, \tag{139}$$

and when $A$ is small,

$$\delta_{2n+2}^{(m)} \sim \frac{(\mu+m)^2}{(n+1)(n+2)}\,\delta_{2n}^{(m)}. \tag{140}$$

The formulae expressing $\delta_{2n+2}^{(m-2)}$ in terms of $\delta_{2n}^{(m)}$ are again very similar.

Finally, to complete the formulae following from the substitution $s - 1 = 2n$, we have when $A$ is large,

$$\delta_{2n+2}^{(m-1)} \sim \left\{1 - \frac{2(\mu+m)^2}{(n+2)^2}\right\}\delta_{2n}^{(m)}, \tag{141}$$

and when $A$ is small,

$$\delta_{2n+2}^{(m-1)} \sim \frac{(n+2)}{(n+1)}\left\{1 - \frac{2(\mu+m)^2}{(n+2)^2}\right\} \delta_{2n}^{(m)}. \tag{142}$$

When $s - 1 = 2n - 1$, we derive when $A$ is large,

$$\delta_{2n}^{(m-1)} \sim -\frac{a_1}{2n(n+1)A}\, \delta_{2n-1}^{(m)}, \tag{143}$$

and when $A$ is small,

$$\delta_{2n}^{(m-1)} \sim -\frac{(\mu+m)}{2n}\, \delta_{2n-1}^{(m)}. \tag{144}$$

We remark firstly that the formulae expressing $\delta_{2n}^{(m)}$ in terms of $\delta_{2n-1}^{(m)}$ are very similar to (143) and (144) and that we could derive further formulae similar to (139)–(142): but in fact we already have at our disposal a sufficiency of results to enable us to deduce exactly what is happening.

We verse our conclusions in terms of a description of the computations that carry us from one even order column to the next. In both cases ($A$ being large or small) the relative error is increased by a factor of the order of magnitude of $(\mu + m)^2$. When $A$ is large this increase takes place effectively in the progression from the even order column to its neighbor ((137) contains a term in $(\mu + m)^2$, (143) does not; (139) and (140) are still restricted to containing terms in $(\mu + m)^2$). When $A$ is small the increase takes place in two stages: as we progress from one column to the next a magnification by a term in $(\mu + m)$ occurs (see (138), (144) in conjunction with (140) and (142)). Since the whole analysis has depended on $(\mu + m)$ being large we see that the growth of error is catastrophic.

An analysis of the propagation of absolute error can be given but this only serves to confirm us in our conclusions.

5.1.2. *Oscillating sequences.* We now consider application of the $\epsilon$-algorithm to the sequence $\{S_m\}$, where

$$S_m \sim A + (-1)^m \sum_{s=1}^{\infty} a_s(\mu+m)^{-s}, \qquad a_1 \neq 0. \tag{145}$$

We recall from Theorem 14 that then

$$\epsilon_{2n}^{(m)} \sim A + \frac{(-1)^m (n!)^2 a_1}{2^{2n}(\mu+m)^{2n+1}}, \tag{146}$$

$$\epsilon_{2n+1}^{(m)} \sim \frac{(-1)^{m+1} 2^{2n-1}(\mu+m)^{2n+1}}{(n!)^2 a_1}. \tag{147}$$

Firstly we derive propagation formulae similar to (137)–(142). We have for large $A$,

$$\delta_{2n+1}^{(m-1)} \sim \frac{(-1)^{m+1}2^{2n-1}(\mu + m)^{2n+1}A}{(n!)^2 a_1}\delta_{2n}^{(m)}, \tag{148}$$

and for small $A$,

$$\delta_{2n+1}^{(m-1)} \sim \tfrac{1}{2}\delta_{2n}^{(m)}, \tag{149}$$

and two similar formulae expressing $\delta_{2n+1}^{(m)}$ in terms of $\delta_{2n}^{(m)}$. Further we have for large $A$,

$$\delta_{2n+2}^{(m-2)} \sim \tfrac{1}{4}\delta_{2n}^{(m)}, \tag{150}$$

and for small $A$,

$$\delta_{2n+2}^{(m-2)} \sim \frac{4(\mu + m)^2}{(n + 1)^2}\delta_{2n}^{(m)}, \tag{151}$$

with two similar formulae expressing $\delta_{2n+2}^{(m)}$ in terms of $\delta_{2n}^{(m)}$. To conclude the formulae derived by writing $s - 1 = 2n$ in the set (122)–(126), we have for large $A$,

$$\delta_{2n+2}^{(m-1)} \sim \tfrac{1}{2}\delta_{2n}^{(m)}, \tag{152}$$

and for small $A$,

$$\delta_{2n+2}^{(m-1)} \sim \frac{2(\mu + m)^2}{(n + 1)^2}\delta_{2n}^{(m)} . \tag{153}$$

We can derive a similar set of formulae by substituting $s - 1 = 2n - 1$ in (122)–(126). The first of these yields for large $A$,

$$\delta_{2n}^{(m-1)} \sim -\frac{a_1}{2^{2n-1}((n - 1)!)^2(\mu + m)^{2n-1}A}\delta_{2n-1}^{(m)}, \tag{154}$$

and when $A$ is small,

$$\delta_{2n}^{(m-1)} \sim \frac{2(\mu + m)^2}{((n!)\,(n - 1)!)^2}\delta_{2n-1}^{(m)} ; \tag{155}$$

two similar formulae express $\delta_{2n}^{(m)}$ in terms of $\delta_{2n-1}^{(m)}$.

In contrast to our examination of the error propagation that takes place during the transformation of monotonic sequences, we shall find it meaningful to consider the propagation of absolute error. Here it transpires that the same formulae apply whether the constant $A$ is large or small: we have successively

$$a_{2n+1}^{(m-1)} \sim \frac{2^{4n-2}(\mu + m)^{4n+2}}{(n!)^4 a_1^2} a_{2n}^{(m)} \tag{156}$$

and a similar formula for $a_{2n+1}^{(m)}$;

$$a_{2n+2}^{(m-2)} \sim \tfrac{1}{4} a_{2n}^{(m)}, \tag{157}$$

with a similar formula for $a_{2n+2}^{(m)}$;

$$a_{2n+2}^{(m-1)} \sim \tfrac{1}{2} a_{2n}^{(m)}; \tag{158}$$

and finally

$$a_{2n}^{(m-1)} \sim \frac{((n-1)!)^4 a_1^2}{2^{4n-4}(\mu+m)^{4n-2}} a_{2n-1}^{(m)} \tag{159}$$

and a similar formula expressing $a_{2n}^{(m)}$ in terms of $a_{2n-1}^{(m)}$.

We have now assembled sufficient formulae to be able adequately to comment upon the propagation of error that takes place during the transformation of the oscillating sequences of this section: we shall first consider the case in which $A$ is large, and this as we have remarked is that which occurs the more frequently in practical numerical analysis.

From (148) we see that considerable loss of accuracy takes place in progress from an even order column to an odd order column, but (150) and (152) indicate that during the computations which extend the $\epsilon$-array from one even order column to the next, the relative error is if anything slightly attenuated. Thus in this case we can say that the even order columns are well determined while the odd order columns are not. But in view of the fact that principal interest attaches to the even order columns, and the odd order columns contain only auxiliary quantities of minor interest in themselves, this situation entirely favors our purposes.

When $A$ is small we see from (149) that relatively little loss of accuracy takes place during the transition from an even order column to one of odd order, but (from (151) and (153)) it would appear that the relative accuracy of the quantities being produced is diminished by a factor of the order of $(\mu + m)^2$ during progress from one even order column to the next.

The difference between the two cases is perhaps best understood by considering (157) and (158), which indicate that in both cases (when $A$ is large and small) the absolute error is slightly attentuated as the construction of the $\epsilon$-array progresses from one even order column to the next. When $A$ is large, and all the entries in the even columns of the $\epsilon$-array are approximately equal to $A$, this also means that the relative error is slightly attenuated. But when $A$ is small there is indeed a loss of relative error, not due to any instability in the computations, but simply because the successive estimates of the limit of the sequence being transformed become steadily smaller and smaller (and indeed by a factor of $(\mu + m)^2$, the factor of amplification of the relative error). This is of course entirely natural and in no way due to malfunctioning of the acceleration apparatus: if we wish to compute the limit of a sequence to a certain number of decimal places and the limit ultimately turns out to be very small, then we must initially work with a large number of decimal places. But this, as we have already remarked, is an anomalous case, and in general the computations resulting from the transformation of the oscillating sequences under consideration are entirely stable.

### 5.2. Dirichlet series.

Although our analyses of the transformation of sequences whose members can be expressed with the help of Newton series and Dirichlet series have been based on entirely different function-theoretic considerations, it transpires that with regard to stability the conclusions we reach are very much the same. Perhaps the contrast between the two cases is not as marked, but it is again true that accuracy can be lost in the transformation of monotone sequences while application of the $\epsilon$-algorithm to the oscillating sequences about to be considered is unconditionally stable.

Since the conclusion is the same as before and the method of error analysis has already been outlined we content ourselves with fewer formulae.

5.2.1. *Monotonic sequences.* We first consider application of the $\epsilon$-algorithm to the sequence $\{S_m\}$ which is asymptotically represented by the formulae

$$S_m \sim A + \sum_{s=1}^{\infty} a_s\lambda_s^{\,m}, \quad 1 > \lambda_1 > \lambda_2 > \cdots > 0. \tag{160}$$

We recall from Theorem 16 that we then have

$$\epsilon_{2n}^{(m)} \sim \frac{a_{n+1}\{(\lambda_{n+1}-\lambda_1)(\lambda_{n+1}-\lambda_2)\cdots(\lambda_{n+1}-\lambda_n)\}^2\lambda_{n+1}^m}{\{(1-\lambda_1)(1-\lambda_2)\cdots(1-\lambda_n)\}^2}, \tag{161}$$

$$\epsilon_{2n+1}^{(m)} \sim \frac{\{(1-\lambda_1)(1-\lambda_2)\cdots(1-\lambda_n)\}^2}{a_{n+1}\{(\lambda_{n+1}-\lambda_1)(\lambda_{n+1}-\lambda_2)\cdots(\lambda_{n+1}-\lambda_n)\}^2(\lambda_{n+1}-1)\lambda_{n+1}^m}. \tag{162}$$

In the notation of Fig. 3 we then have when $A$ is small,

$$\delta_{2n+1}^{(m-1)} \sim \frac{\lambda_{n+1}}{1-\lambda_{n+1}}\,\delta_{2n}^{(m)}, \tag{163}$$

and when $A$ is large,

$$\delta_{2n+1}^{(m-1)} \sim \frac{A\lambda_{n+1}\{(1-\lambda_1)(1-\lambda_2)\cdots(1-\lambda_n)\}^2}{(\lambda_{n+1}-1)\{(\lambda_{n+1}-\lambda_1)(\lambda_{n+1}-\lambda_2)\cdots(\lambda_{n+1}-\lambda_n)\}^2a_{n+1}\lambda_{n+1}^m}\,\delta_{2n}^{(m)}. \tag{164}$$

When $A$ is small,

$$\delta_{2n+2}^{(m)} \sim -\frac{a_{n+1}\{(\lambda_{n+1}-\lambda_1)(\lambda_{n+1}-\lambda_2)\cdots(\lambda_{n+1}-\lambda_n)\}^2}{a_{n+2}\{(\lambda_{n+2}-\lambda_1)(\lambda_{n+2}-\lambda_2)\cdots(\lambda_{n+2}-\lambda_n)(\lambda_{n+2}-\lambda_{n+1})\}^2\lambda_{n+2}^m}, \tag{165}$$

and when $A$ is large,

$$\delta^{(m)}_{2n+2} \sim -\left(\frac{\lambda_{n+1}}{1-\lambda_{n+1}}\right)^2 \delta^{(m)}_{2n}. \tag{166}$$

With regard to the propagation of absolute error, we have for both cases in which $A$ is large and small

$$a^{(m-1)}_{2n+1} \sim \frac{\{(1-\lambda_1)(1-\lambda_2)\cdots(1-\lambda_n)\}^4 a^{(m)}_{2n}}{(1-\lambda_{n+1})^2\{(\lambda_{n+1}-\lambda_1)(\lambda_{n+1}-\lambda_2)\cdots(\lambda_{n+1}-\lambda_n)\}^4 a^2_{n+1}\lambda^{2m-2}_{n+1}}, \tag{167}$$

$$a^{(m)}_{2n+2} \sim -\left(\frac{\lambda_{n+1}}{1-\lambda_{n+1}}\right)^2 a^{(m)}_{2n}. \tag{168}$$

Of these formulae, (168) is perhaps the most critical for an understanding of the numerical stability of the process being examined. If the value of $\lambda_{n+1}$ is very close to unity, then clearly the absolute error is considerably increased during the transition from one even order column to the next. If $A$ is large the relative error is increased by the same factor (see (166)): if $A$ is small then there is a further magnification of the relative error, due not only to the growth of the absolute error, but also to the decrease in the size of the estimate of the limit of the sequence being transformed.

It is of interest to point out that if $\lambda_{n+1} \leqq \frac{1}{2}$, then the magnitude of the coefficient of $\delta^{(m)}_{2n}$ in (168) is less than unity, and the $\epsilon$-algorithm is in this case stable, although we are transforming a monotone sequence. This is in contrast to the transformation of monotone sequences whose members can be approximated with the help of Newton series.

5.2.2. *Oscillating sequences.* We now consider the transformation of the sequence $\{S_m\}$ whose members are asymptotically represented by the formula

$$S_m \sim A + (-1)^m \sum_{s=1}^{\infty} a_s\lambda_s{}^m, \qquad 1 > \lambda_1 > \lambda_2 > \cdots > 0. \tag{169}$$

The error amplification formulae are derived merely by reversing the signs attached to the variables $\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_{n+1}, \cdots$ in (163)–(168). In particular we have for both large and small values of $A$,

$$a^{(m)}_{2n+2} \sim \left(\frac{\lambda_{n+1}}{1+\lambda_{n+1}}\right)^2 a^{(m)}_{2n}. \tag{170}$$

In view of the inequalities contained in (169) we see that in any case the absolute error is attenuated as we pass from one even order column to the next: the degree to which the relative error is also attenuated depends upon the estimate of the limit of the sequence being transformed. However, whether this estimate be large or small, the $\epsilon$-algorithm is unconditionally stable.

**6. Conclusion.** In this paper the behavior of the $\epsilon$-algorithm has been submitted to a rigorous mathematical analysis. We have proved that in certain circumstances the $\epsilon$-algorithm is a regular (i.e., convergence preserving) transformation: we have shown that certain types of slowly convergent monotonic sequences are transformed into slowly convergent sequences of single signed terms, that certain slowly convergent oscillating sequences are transformed into rapidly convergent oscillating sequences and in both cases have given asymptotic estimates for the quantities produced.

The results have been used in an analysis of stability. We have shown in particular that transformation of the monotonic sequences considered necessitates the repeated subtraction of approximately equal quantities, and that this in turn induces the sort of rocketing instability in which the rounding errors jump up and take complete charge of the computations after a few steps. On the other hand we have seen that transformations of the oscillating sequences considered ensures the consistent subtraction of quantities having opposite sign: no loss of figures due to cancellation takes place and the computations turn out to be completely stable.

To conclude we give some numerical examples which fully illustrate the two types of convergence and stability behavior that have been examined. All examples have been computed with the aid of a floating point arithmetic system which uses a mantissa of 36 binary places (approximately equal to 10.8 decimal places). The errors which arise in the computation of each quantity $\epsilon_s^{(m)}$ are due firstly to the rounding off error involved in this system of arithmetic and secondly to any error that may have been propagated from previous computations.

In the first example the $\epsilon$-algorithm has been applied to the monotonic initial values

$$\epsilon_0^{(m)} = 1.0 + (10.0 + m)^{-1}, \qquad m = 0, 1, \cdots ; \tag{171}$$

the even order columns of the resulting $\epsilon$-array are displayed in Table 1.

TABLE 1

| $m$ | $s$ | | | | |
|---|---|---|---|---|---|
| | 0 | 2 | 4 | 6 | 8 |
| 0 | 1.1 | | | | |
| 1 | 1.09090 90909 | 1.04545 45454 | | | |
| 2 | 1.08333 33333 | 1.04166 66667 | | | |
| 3 | 1.07692 30769 | 1.03846 15385 | 1.02564 10262 | 1.01923 07642 | |
| 4 | 1.07142 85714 | 1.03571 42887 | 1.02380 95249 | 1.01785 70772 | 1.01428 72742 |
| 5 | 1.06666 66667 | 1.03333 33334 | 1.C2222 22189 | 1.01666 67388 | |
| 6 | 1.06520 00000 | 1.03125 00000 | 1.02083 33314 | | |
| 7 | 1.05882 35294 | 1.02941 17646 | | | |
| 8 | 1.05555 55556 | | | | |

It will be observed that all the numbers contained in Table 1 are greater than unity (in accordance with Theorem 4) and that the increase in the rate of convergence of the derived sequences is virtually negligible (confirming the estimates of Theorem 11 and indeed the exact result of Theorem 12). With regard to stability we remark that it follows from Theorem 12 that the value of $\epsilon_8^{(0)}$ should be

$$1.0 + (5 \times 14)^{-1} = 1.01428\ 57143; \tag{172}$$

but the computed value of $\epsilon_8^{(0)}$ (i.e., the entry in the last column of Table 1) is clearly in error by 0.00000 15601. This indicates quite strikingly enough the instability of the whole computation.

By way of contrast the $\epsilon$-algorithm has been applied to the oscillating initial conditions

$$\epsilon_0^{(m)} = 1.0 + (-1)^m(10.0 + m)^{-1}, \qquad m = 0, 1, \cdots ; \tag{173}$$

the even order columns of the resulting $\epsilon$-array are given in Table 2.

Here the sequences of numbers contained in the various columns are oscillating (see Theorem 7), the convergence of the derived sequences is very much more rapid than that of the original sequence (in accordance with Theorem 13), and lastly there is indeed very little evidence of any instability.

The numerical results contained in Tables 3 and 4 relate to the transformation of the sequences

$$S_m = 1.0 + \sum_{s=1}^{\infty} \left(\frac{1}{s+1}\right)^2 \left(\frac{1}{s+1}\right)^m, \qquad m = 0, 1, \cdots , \tag{174}$$

and

$$S_m = 1.0 + (-1)^m \sum_{s=1}^{\infty} \left(\frac{1}{s+1}\right)^2 \left(\frac{1}{s+1}\right)^m, \qquad m = 0, 1, \cdots , \tag{175}$$

respectively.

TABLE 2

| $m$ | $s$ | | | | |
|---|---|---|---|---|---|
| | 0 | 2 | 4 | 6 | 8 |
| 0 | 1.1 | | | | |
| 1 | 0.90909 09091 | 1.00018 86081 | | | |
| 2 | 1.08333 33333 | 0.99985 48199 | 1.00000 10117 | | |
| 3 | 0.92307 69231 | 1.00011 41292 | 0.99999 93227 | 1.00000 00090 | |
| 4 | 1.07142 85714 | 0.99990 86591 | 1.00000 04672 | 0.99999 99946 | 1.00000 00001 |
| 5 | 0.93333 33333 | 1.00007 42390 | 0.99999 96693 | 1.00000 00033 | |
| 6 | 1.06250 00000 | 0.99993 88454 | 1.00000 02393 | | |
| 7 | 0.94117 64705 | 1.00005 09736 | | | |
| 8 | 1.05555 55556 | | | | |

Again it will be observed that the results of Theorems 4 and 7, relating to the sign and oscillation of the derived sequences, are confirmed.

The asymptotic estimates of Theorems 16 and 17 are also substantiated: the degree of agreement between the theoretical and numerical results decreases as, in the notation of the theorems referred to, the suffix of $\lambda_n$ increases. The estimates provided by these theorems depend upon the relative difference of $\lambda_n$ and $\lambda_{n+1}$, being sharper when this relative difference is larger. However, as one can observe by inspection of the series (171) and (175), this relative difference decreases markedly as $n$ increases, and hence the disparity between the theoretical and numerical results also increases.

The convergence of the derived sequences obtained from the oscillating sequence (175) is better than that obtained from the monotonic sequence (174) but, as was stated at the end of §4.2.2, the difference in the rate of convergence is slight and not, in the event, in any way comparable to the difference exhibited in Tables 1 and 2.

**7. Acknowledgment.** The numerical computations contained in this paper have been carried out on the CDC 3600 at Madison with the aid of

TABLE 3

| $m$ | $s$ | | | | |
|---|---|---|---|---|---|
| | 0 | 2 | 4 | 6 | 8 |
| 0 | 1.64493 40668 | | | | |
| 1 | 1.20205 69031 | 1.03795 85726 | | | |
| 2 | 1.08232 32337 | 1.00920 64796 | 1.00137 07376 | | |
| 3 | 1.03692 77551 | 1.00248 25995 | 1.00023 54818 | 1.00002 70088 | |
| 4 | 1.01734 30619 | 1.00071 17681 | 1.00004 43733 | 1.00000 35149 | 1.00000 03268 |
| 5 | 1.00834 92773 | 1.00021 25026 | 1.00000 89237 | 1.00000 05006 | |
| 6 | 1.00407 73562 | 1.00006 52737 | 1.00000 18823 | | |
| 7 | 1.00200 83928 | 1.00002 04666 | | | |
| 8 | 1.00099 45751 | | | | |

TABLE 4

| $m$ | $s$ | | | | |
|---|---|---|---|---|---|
| | 0 | 2 | 4 | 6 | 8 |
| 0 | 1.64493 40668 | | | | |
| 1 | 0.79794 30968 | 1.01084 17704 | | | |
| 2 | 1.08232 32337 | 0.99830 44096 | 1.00011 55057 | | |
| 3 | 0.96307 22448 | 1.00036 92783 | 0.99998 68317 | 1.00000 08606 | |
| 4 | 1.01734 30619 | 0.99990 57281 | 1.00000 19547 | 0.99999 99209 | 1.00000 00047 |
| 5 | 0.99165 07226 | 1.00002 63230 | 0.99999 96609 | 1.00000 00089 | |
| 6 | 1.00407 73561 | 0.99999 22324 | 1.00000 00649 | | |
| 7 | 0.99799 16071 | 1.00000 23760 | | | |
| 8 | 1.00099 45751 | | | | |

an ALGOL compiler constructed under the direction of A. A. Grau and L. L. Bumgarner. The author is grateful to C. Pfluger for help in running the necessary programs.

**8. Appendix. The smoothing effect of the $\epsilon$-algorithm.** In an earlier section the stability of the $\epsilon$-algorithm was discussed: the inquiry was based on an examination of the amplification of error which occurs during the continuation from one point in the $\epsilon$-array to those points immediately to the right.

Certain of the formulae derived are sufficiently simple to allow some conclusions to be drawn concerning the global propagation of error: we have in mind the results (148), (150), (152), (154) and (156)–(159). Subject to the assumptions which led to these formulae we see that a (relative or absolute) error source of strength $\delta$ in the quantity $\epsilon_{2n-2}^{(m)}$ leads to the pattern (shown in Fig. 4) of error in two adjacent columns of the even order $\epsilon$-array. (The quantities in the second column represent the errors in the quantities $\epsilon_{2n}^{(m-2)}$, $\epsilon_{2n}^{(m-1)}$, and $\epsilon_{2n}^{(m)}$, respectively.)

$$\begin{array}{cc} & \frac{1}{4}\delta \\ \delta & \frac{1}{2}\delta \\ & \frac{1}{4}\delta \end{array}$$

FIG. 4

The quantities in the second column may now be regarded as three sources of error acting in conjunction: they lead in turn to the distribution in Fig. 5. The process may be continued at will. The manner in which binomial coefficients and powers of two occur in the expressions describing the error transmitted to an even order column of general suffix is now clear. Indeed the reader will immediately be put in mind of the way in which errors build up during the formation of a difference table. The only distinctions in the mechanisms of error propagation here are that during the passage from an even order column to one of odd order the error is multiplied by a factor of $k(\mu + m)^{2n+1}$ (where $k$ is independent of $\mu$ and $m$), and conversely, during passage from an odd to an even order column the errors are divided by a factor of $4k(\mu + m)^{2n+1}$ (see (148), (154), (156) and (159)).

Concerning the consequences of introducing an error in one of the initial

$$\begin{array}{ccc} & & \frac{1}{16}\delta \\ & \frac{1}{4}\delta & \frac{1}{4}\delta \\ \delta & \frac{1}{2}\delta & \frac{3}{8}\delta \\ & \frac{1}{4}\delta & \frac{1}{4}\delta \\ & & \frac{1}{16}\delta \end{array}$$

FIG. 5

$$\begin{array}{cc}
 & \frac{1}{4}\delta \\
\delta & \frac{3}{4}\delta \\
\delta & \delta \\
\delta & \delta \\
\vdots & \vdots \\
\delta & \delta \\
\delta & \delta \\
\delta & \frac{3}{4}\delta \\
 & \frac{1}{4}\delta
\end{array}$$

FIG. 6

values we have the following:

THEOREM 18. *An error occurring in the initial value* $\epsilon_0^{(m')}$ *affects the quantities* $\epsilon_{2s}^{(m'')}$ $(m'' = m' - 2s, m' - 2s + 1, \cdots, m'; m'' \geqq 0)$ *lying in the column with suffix* $2s$ *in the even order* $\epsilon$*-array. If the initial values are given by*

$$\epsilon_0^{(m)} \sim A + (-1)^m \sum_{s=1}^{\infty} a_s(\mu + m)^{-s}, \qquad a_1 \neq 0, \tag{176}$$

(*where* $A$ *is large*) *and the quantity* $\epsilon_0^{(m')}$ *is vitiated by a* (*relative or absolute*) *error of* $\delta$ *then the resulting errors in the even column* $\epsilon$*-array are given by*

$$e_{2s}^{(m'')} = \left\{\binom{2s}{m' - m''}\Big/2^{2s}\right\}\delta, \quad m'' = m' - 2s, m' - 2s + 1, \cdots, m', \quad m'' \geqq 0, \tag{177}$$

*where* $e_{2s}^{(m'')}$ *is the* (*relative or absolute*) *error in the quantity* $\epsilon_{2s}^{(m'')}$.

Since the coefficient of $\delta$ in (176) tends to zero as $s$ increases, it can be seen that the original error is smoothed out during the course of the computation. This takes place, we emphasize, if one initial value is in error and the rest are correct.

If the initial values are consistently in error (we assume for simplicity that in each case the error is $\delta$) then the column of errors is substantially translated according to the scheme in Fig. 6. If all the initial values are vitiated by the same error then this error is transmitted to every estimate of the transformed limit: indeed in these circumstances the $\epsilon$-algorithm thinks that the error is an integral part of the sequence to be transformed, and indeed it may be pardoned for so doing.

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