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ON THE CONVERGENCE OF TWO-STAGE ITERATIVE PROCESSES FOR SOLVING LINEAR EQUATIONS*

NANCY K. NICHOLS†

Abstract. This paper considers two-stage iterative processes for solving the linear system $Af = b$. The outer iteration is defined by $Mf^{k+1} = Nf^k + b$, where M is a nonsingular matrix such that $M - N = A$. At each stage f^{k+1} is computed approximately using an inner iteration process to solve $Mv = Nf^k + b$ for v. At the kth outer iteration, p_k inner iterations are performed. It is shown that this procedure converges if $p_k \geqq P$ for some P provided that the inner iteration is convergent and that the outer process would converge if f^{k+1} were determined exactly at every step. Convergence is also proved under more specialized conditions, and for the procedure where $p_k = p$ for all k, an estimate for p is obtained which optimizes the convergence rate. Examples are given for systems arising from the numerical solution of elliptic partial differential equations and numerical results are presented.

1. Introduction. Various two-stage iterative processes have recently been developed for solving certain classes of linear equations

$$Af = b \tag{1}$$

which arise from the discretization of elliptic boundary value problems. These methods are particularly useful for solving difference approximations to simultaneous partial differential equations (Wachspress [8], Smith [7], and Nichols [5]). Such procedures are also of interest since they can frequently be extended to the solution of nonlinear equations (Douglas [1], Gunn [4], D'Yakonov [3], Dupont [2]). Convergence of these processes has been proved for specific problems and estimates have been established for the number of operations required to reduce the norm of the initial error in the iteration by a given factor.

In this paper we give results on the convergence of general two-stage iterative techniques under very broad conditions. We assume only that the inner process converges and that the outer process converges when the inner equations are solved exactly at every step. We also prove convergence when more specialized conditions are satisfied and obtain, in addition, estimates for the number of inner steps required for *optimal* convergence rates. Applications to various two-stage processes are discussed.

2. Definitions. We consider iterative methods of the form

$$Mf^{k+1} = Nf^k + b, \tag{2}$$

where $A = M - N$ is a splitting of the nonsingular matrix A. Method (2) is called the outer iteration and the iteration matrix of (2) is given by

$$C = M^{-1}N. \tag{3}$$

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† Department of Mathematics, University of Dundee, Dundee, Scotland. Now at Department of Mathematics, University of Reading, Reading, England. The research for this paper was completed while the author was recipient of a National Cash Register fellowship at the University of Dundee.

At each step of procedure (2) we must solve the "inner equations"

$$Mv = d, \qquad d = Nf^k + b. \tag{4}$$

When equations (4) are solved exactly for each k, then method (2) converges if $\rho(C)$, the spectral radius of C, is less than unity.

With two-stage processes, we solve (4) by another iteration procedure (called the inner iteration), and we find only an approximate solution of (4) at each step of the outer iteration. We let T_p be the error operator of the inner process and then T_p satisfies

$$v - v^p = T_p(v - v^0), \tag{5}$$

where v^m is the mth iterate in the solution of (4). If $T_p \to 0$ as $p \to \infty$, then the inner process is convergent.

We let $\{g^k\}$ be the iterates in the approximate two-stage procedure. At the kth step of the outer process we use p_k inner iterations, so that $g^{k+1} = v^{p_k}$, and we start the inner process with $v^0 = g^k$. From (5) we obtain

$$v - g^{k+1} = T_{p_k}(v - g^k), \tag{6}$$

where

$$v = M^{-1}(Ng^k + b), \tag{7}$$

and then g^{k+1} actually satisfies

$$g^{k+1} = C_{p_k}g^k + (I - C_{p_k})A^{-1}b, \tag{8}$$

where

$$C_{p_k} = C + T_{p_k}(I - C) = I - (I - T_{p_k})(I - C). \tag{9}$$

Thus (8) is a nonstationary iterative procedure and defines the overall iteration of our two-stage process.

3. Convergence. We now show that procedure (8) is convergent providing the inner and outer iteration methods are both convergent.

THEOREM 1. *Given that* $\rho(C) < 1$ *and* $T_m \to 0$ *as* $m \to \infty$, *then there exist* $P < \infty$ *and* $K < \infty$, *such that if* $p_k \geqq P$ *for all* $k \geqq K$, *then the iteration method* (8) *converges and* $g^k \to f = A^{-1}b$, *the solution of equations* (1).

Proof. Since $\rho(C) < 1$, there exists a matrix norm $\|\cdot\|_*$, subordinate to some vector norm, such that

$$\|C\|_* < 1. \tag{10}$$

(See, for example, Ortega and Rheinboldt [6].) Since $T_m \to 0$ as $m \to \infty$, we also have $\|T_m\|_* \to 0$ as $m \to \infty$, and therefore we can make $\|T_m\|_*$ arbitrarily small by choosing m large enough.

From (9) we find that

$$\|C_{p_k}\|_* \leqq \|C\|_* + \|T_{p_k}\|_* \cdot \|I - C\|_*. \tag{11}$$

We define P to be the minimum number such that

$$\|T_m\|_* < (1 - \|C\|_*)/\|I - C\|_* \quad \text{for all} \quad m \geqq P. \tag{12}$$

Then $p_k \geqq P$ implies

$$\|C_{p_k}\|_* < \|C\|_* + Q(1 - \|C\|_*)/Q = 1, \qquad Q = \|I - C\|_*. \tag{13}$$

We observe that

$$\begin{aligned} \left\| \prod_{k=0}^{\nu} C_{p_k} \right\|_* \leqq \prod_{0}^{\nu} \|C_{p_k}\|_* &= \left(\prod_{0}^{K} \|C_{p_k}\|_* \right) \left(\prod_{K+1}^{\nu} \|C_{p_k}\|_* \right) \\ &= q \left(\prod_{K+1}^{\nu} \|C_{pk}\|_* \right), \end{aligned} \tag{14}$$

and if $p_k \geqq P$ for all $k > K$ then

$$\prod_{K+1}^{\nu} \|C_{p_k}\|_* \to 0 \quad \text{as} \quad \nu \to \infty. \tag{15}$$

Hence $\|\prod_0^\nu C_{p_k}\|_* \to 0$ as $\nu \to \infty$, and process (8) converges. Moreover g^k converges to the fixed point of (8) which is just $f = A^{-1}b$, the solution of equations (1).

If we keep $p_k = p$ constant throughout the iteration, then process (8) is stationary with $C_{p_k} = C_p$, and we have the following corollary.

COROLLARY. *Given the conditions of Theorem 1 where $p_k = p$ for all k, if $p \geqq P$, then $\rho(C_p) < 1$ and iteration process (8) converges to the solution of (1).*

Proof. Since $p \geqq P$ we have

$$\rho(C_p) \leqq \|C_p\|_* \leqq \|C\|_* + Q \cdot \|T_p\|_* < 1. \tag{16}$$

Now let us consider convergence under more specialized conditions. Assume $p_k = p$ is constant, so that procedure (8) is stationary.

THEOREM 2. *Let the eigenvalues $\lambda_i(C)$ of C and the eigenvalues $\lambda_i(T_p)$ of T_p lie in the ranges*

$$-1 < a \leqq \lambda_i(C) \leqq b < 1, \qquad -1 < \xi_1 \leqq \lambda_i(T_p) \leqq \xi_2 < 1. \tag{17}$$

Also let one of the following conditions hold:

(i) *T_p and C are real symmetric matrices;*

(ii) *there exists matrix S such that $C' = SCS^{-1}$ and $T'_p = ST_pS^{-1}$ are both real and symmetric;*

(iii) *there exists matrix S such that $C' = SCS^{-1}$ is symmetric and T_p is symmetric and commutes with S.*

Then

$$\rho(C_p) \leqq \max\{|a + \xi_1(1 - a)|, |b + \xi_2(1 - b)|\}, \tag{18}$$

and iteration process (8) converges to the solution of equations (1) provided

$$-\xi_1 < (1 + a)/(1 - a). \tag{19}$$

Proof. From (9) we have

$$C_p = I - (I - T_p)(I - C) \tag{20}$$

and therefore, $\lambda_i(C_p)$, the eigenvalues of C_p, are bounded by

$$1 - \lambda_{\max}\{(I - T_p)(I - C)\} \leqq \lambda_i(C_p) \leqq 1 - \lambda_{\min}\{(I - T_p)(I - C)\}. \tag{21}$$

When (i) holds, T_p and C are symmetric so that

$$\begin{aligned} \lambda_{\min}\{(I - T_p)(I - C)\} &\geqq \lambda_{\min}(I - T_p) \cdot \lambda_{\min}(I - C), \\ \lambda_{\max}\{(I - T_p)(I - C)\} &\leqq \lambda_{\max}(I - T_p) \cdot \lambda_{\max}(I - C). \end{aligned} \tag{22}$$

Since $\lambda_i(C)$, $\lambda_i(T_p)$ lie in the interval $(-1, 1)$ we find

$$\begin{aligned} 0 \leqq 1 - \xi_2 \leqq \lambda_i(I - T_p) \leqq 1 - \xi_1, \\ 0 \leqq 1 - b \leqq \lambda_i(I - C) \leqq 1 - a, \end{aligned} \tag{23}$$

and therefore,

$$\rho(C_p) = \max |\lambda_i(C_p)| \leqq \max \{|1 - (1 - \xi_1)(1 - a)|, |1 - (1 - \xi_2)(1 - b)|\}, \tag{24}$$

which is equivalent to (18). When (ii) holds, then

$$C'_p = SC_pS^{-1} = I - (I - T'_p)(I - C') \tag{25}$$

is similar to C_p, and since C' and T'_p are both symmetric and similar to C and T_p, it follows from the first part that

$$\rho(C_p) = \rho(C'_p) \leqq \max \{|a + \xi_1(1 - a)|, |b + \xi_2(1 - b)|\}. \tag{26}$$

Finally, when (iii) holds, T_p commutes with S so that $T'_p = T_p$, and therefore

$$C'_p = I - (I - T_p)(I - C'). \tag{27}$$

Since T_p and C' are symmetric, and C' is similar to C, it follows that $\rho(C_p) = \rho(C'_p)$ again satisfies (18).

Now, from the conditions of the theorem, we find

$$|b + \xi_2(1 - b)| = |1 - (1 - b)(1 - \xi_2)| < 1, \tag{28}$$

and if

$$-\xi_1 < (1 + a)/(1 - a), \tag{29}$$

then we also have

$$|a + \xi_1(1 - a)| = |1 - (1 - a)(1 - \xi_1)| < 1, \tag{30}$$

and hence $\rho(C_p) < 1$. Therefore iteration process (8) converges, provided (19) is satisfied, and the theorem is proved.

From this theorem we derive the following corollaries.

COROLLARY 2.1. *Let the conditions of Theorem* 2 *hold. Then*

$$\rho(C_p) \leqq \rho(C) + \rho(T_p)(1 + \rho(C)) \tag{31}$$

and process (8) *converges if p is chosen so that*

$$\rho(T_p) < (1 - \rho(C))/(1 + \rho(C)). \tag{32}$$

Proof. Clearly

$$|\lambda_i(C)| \leqq \rho(C) < 1, \qquad |\lambda_i(T_p)| \leqq \rho(T_p) < 1. \tag{33}$$

We therefore let

$$b = -a = \rho(C), \qquad \xi_2 = -\xi_1 = \rho(T_p), \tag{34}$$

and substituting into (18) and (19), we obtain (31) and (32) directly.

COROLLARY 2.2. *Let the conditions of Theorem 2 hold. In addition, let either C or T_p have only nonnegative eigenvalues. Then*

$$\rho(C_p) \leqq \rho(C) + \rho(T_p)(1 - \rho(C)) < 1, \tag{35}$$

and process (8) *is guaranteed to converge.*

Proof. If the matrix C has nonnegative eigenvalues, then we choose

$$a = 0, \qquad b = \rho(C), \quad \text{and} \quad \xi_2 = -\xi_1 = \rho(T_p). \tag{36}$$

Otherwise we let

$$b = -a = \rho(C) \quad \text{and} \quad \xi_1 = 0, \qquad \xi_2 = \rho(T_p). \tag{37}$$

Substituting into (18) we obtain (35) immediately, and the corollary is proved.

In Theorem 1 we required that $p \geqq P$, where P is the minimum number such that (12) holds. We observe that condition (19) of Theorem 2 is a slightly weaker condition for convergence and, moreover, that (18) gives a more accurate bound on $\rho(C_p)$ than does (16) under the given conditions.

4. Examples and applications. We now consider applications of these theorems to some two-stage iterative methods which have already been developed. We begin with methods investigated by Gunn [4], D'Yakonov [3], and Dupont [2] for solving linear equations of the form

$$Lf = g, \tag{38}$$

where L is a positive definite matrix operator derived from the finite difference approximation of an elliptic partial differential equation over a bounded region. These methods all satisfy conditions of the types required in Theorem 2.

In the first scheme (Gunn [4], D'Yakonov [3]) the outer iteration matrix is

$$C = (I - \omega A^{-1}L), \tag{39}$$

where A is symmetric and positive definite. The eigenvalues of $A^{-1}L$ are bounded by

$$0 < \mu_0 \leqq \lambda_i(A^{-1}L) \leqq \mu^0 \tag{40}$$

and therefore the eigenvalues of C lie in the range

$$1 - \omega\mu^0 \leqq \lambda_i(C) \leqq 1 - \omega\mu_0. \tag{41}$$

Clearly,

$$\rho(C) < 1 \quad \text{if} \quad 0 < \omega < 2/\rho(A^{-1}L). \tag{42}$$

The inner iteration is given by $T_p = \Lambda$, where Λ is symmetric and commutes with A, and $\rho(\Lambda) = \xi < 1$. For the problem in Gunn [4], Λ is the iteration error operator for one cycle of length p of the Peaceman–Rachford process and p is chosen to make ξ less than unity. Since A is symmetric and positive definite, we may choose $S = A^{1/2}$, and then condition (iii) of Theorem 2 is satisfied. Therefore,

$$\rho(C_p) \leqq \max\{|1 - (1 - \xi)\omega\mu_0|, |1 - (1 + \xi)\omega\mu^0|\} = Q \tag{43}$$

and iteration process (8) converges provided

$$\omega < 2/(1 + \xi)\mu^0. \tag{44}$$

If the parameter ω is defined to be

$$\omega = 2/[(1 + \xi)\mu^0 + (1 - \xi)\mu_0], \tag{45}$$

as in Gunn [4], then Q is minimized and

$$Q = [(1 + \xi)\mu^0 - (1 - \xi)\mu_0]/[(1 + \xi)\mu^0 + (1 - \xi)\mu_0] < 1 \tag{46}$$

and the iteration is guaranteed to converge.

In the second scheme (Dupont [2], D'Yakonov [3]) the outer iteration matrix (39) is again used, but the inner iteration is defined by

$$T_p = \prod_{i=1}^{p} (I - \beta_i(A + B)^{-1}A), \tag{47}$$

where p and $\{\beta_i\}$ are chosen so that $\rho(T_p) = \xi < 1$. (For this to hold, it is sufficient that $0 < \beta_i < 2/\rho((A + B)^{-1}A)$ for all $i \leqq p$.) The matrices L, A and $A + B$ are all symmetric and positive definite. Therefore if we choose $S = A^{1/2}$, then the iteration satisfies condition (ii) of Theorem 2 and converges, provided the parameter ω satisfies (44).

Other two-stage methods which involve a single inner iteration are given by Wachspress [8] for solving eigenvalue problems. These methods use power iteration to determine the fundamental mode of the inverse of a given positive definite matrix M. To avoid explicit computation of M^{-1}, an inner iteration is used at each step of the outer process to solve equations of form (4). Matrix M and the error operator of the inner iteration, T_p, are defined so that the procedures satisfy condition (i) of Theorem 2 and are convergent by Corollary 2.2.

Two-stage processes for solving approximations to simultaneous differential equations are given by Smith [7] and Nichols [5]. These methods all involve at least two inner iterations, so that the form of the error operator T_p is more complicated than in other cases. We consider, for example, the method of Smith for solving the coupled-equation form of the biharmonic problem. The outer iteration matrix for this process is

$$\begin{aligned} C &= \begin{bmatrix} L & 0 \\ -c^{-1}h^2\omega I & L \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2ch^{-2}M \\ 0 & (1 - \omega)L \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2ch^{-2}L^{-1}M \\ 0 & (1 - \omega)I - 2\omega L^{-2}M \end{bmatrix}, \end{aligned} \tag{48}$$

where L is symmetric and positive definite. Each of the two sets of inner matrix equations is solved by an iterative process with error operator $\mathscr{T}_p$, where $\mathscr{T}_p$ is symmetric and commutes with L, and $\mathscr{T}_m \to 0$ as $m \to \infty$. The overall error operator T_p of the complete two-stage procedure is then

$$T_p = \begin{bmatrix} \mathscr{T}_p & 0 \\ \omega c^{-1}h^2(I - \mathscr{T}_p)L^{-1}\mathscr{T}_p & \mathscr{T}_p \end{bmatrix}, \tag{49}$$

and we also have $T_m \to 0$ as $m \to \infty$. Smith shows that $\rho(C) < 1$ and therefore, the method satisfies the conditions of Theorem 1 and is convergent.

Other applications of the theory to coupled equations can be found in Nichols [5]. One example for solving a biharmonic problem uses the Jacobi method for the outer iteration procedure. Matrix C is given by

$$C = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -B \\ -B & 0 \end{bmatrix} = \begin{bmatrix} 0 & -A_1^{-1}B \\ -A_2^{-1}B & 0 \end{bmatrix}, \tag{50}$$

where A_1, A_2 are two-cyclic, consistently ordered, symmetric and positive definite with constant diagonal elements, and B is symmetric. When the Jacobi process is also used to solve the two sets of inner equations, then the error operator is

$$T_p = \begin{bmatrix} H_1^p & 0 \\ 0 & H_2^p \end{bmatrix}, \tag{51}$$

where H_i is symmetric and commutes with A_i, $i = 1, 2$. Under the given conditions, it is known that the Jacobi procedures converge [5], so that $\rho(C) < 1$, $\rho(H_i) < 1$, $i = 1, 2$, and hence if

$$\mu = \max_i \{\rho(H_i)\}, \tag{52}$$

then

$$\rho(T_p) = \mu^p < 1 \quad \text{for all} \quad p \geqq 1. \tag{53}$$

If we let

$$S = \begin{bmatrix} A_1^{1/2} & 0 \\ 0 & A_2^{1/2} \end{bmatrix}, \tag{54}$$

then condition (iii) of Theorem 2 is satisfied, and if we choose

$$p > (\log \mu)^{-1} \log \{(1 - \rho(C))/(1 + \rho(C))\}, \tag{55}$$

then $\rho(T_p) = \mu^p$ satisfies (32) and by Corollary 2.1 the overall iteration is convergent.

For the same problem, when the SOR method is used for the outer (or inner) iteration, then the conditions of Theorem 2 are not satisfied, but we can still prove convergence using Theorem 1 and arguments similar to those applied to Smith's problem. Matrix C is of the form

$$C = \begin{bmatrix} A_1 & 0 \\ \omega B & A_2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} (1-\omega)A_1 & -\omega B \\ 0 & (1-\omega)A_2 \end{bmatrix}, \tag{56}$$

and the error operator becomes

$$T_p = \begin{bmatrix} H_1^p & 0 \\ -\omega(I - H_2^p)A_2^{-1}BH_1^p & H_2^p \end{bmatrix}, \tag{57}$$

where H_i are the inner iteration matrices, and $\rho(H_i) < 1$, $i = 1, 2$. Under the given conditions $\rho(C) < 1$, and since $H_i^p \to 0$ as $p \to \infty$, we also have $T_p \to 0$ as $p \to \infty$, and the process converges by Theorem 1.

5. Estimates of work. When the number of inner iterations $p_k = p$ is kept constant, and $\rho(C_p) < 1$, then the asymptotic rate of convergence of the two-stage process (8) is inversely proportional to

$$k = -p[\log \rho(C_p)]^{-1}. \tag{58}$$

If we know γ_p such that $\rho(C_p) \leqq \gamma_p < 1$, then

$$k' = -p(\log \gamma_p)^{-1} \geqq k \tag{59}$$

is an upper bound for k and we can choose p to minimize k'. We note that minimizing k' is equivalent to minimizing an upper bound on $[\rho(C_p)]^{1/p}$, the "average spectral radius" of C_p over p inner iterations.

Now suppose that for sufficiently large p, we can find an upper bound for $\rho(C_p)$ of the form

$$\rho(C_p) \leqq \gamma_p = \alpha + \beta\mu^p < 1, \tag{60}$$

where $0 < \alpha < 1, 0 < \mu < 1$, and $\beta > 0$. Clearly $\gamma_p < 1$ when

$$p > P = (-\log \mu)^{-1}(-\log ((1 - \alpha)/\beta)). \tag{61}$$

The problem is then to find

$$\min_{p>P} k' = \min_{p>P} \{-p[\log (\alpha + \beta\mu^p)]^{-1}\}. \tag{62}$$

If we write $\delta = \mu^p$, then

$$p = (-\log \mu)^{-1}(-\log \delta) \tag{63}$$

and

$$\min_{p>P} k' = \min_{\delta < \mu^P = (1-\alpha)/\beta} \{(-\log \mu)^{-1}[-\log (\alpha + \beta\delta)]^{-1}(-\log \delta)\}. \tag{64}$$

If δ is sufficiently small, then

$$\begin{aligned} -\log (\alpha + \beta\delta) &= -\log \alpha - \log (1 + \beta\delta/\alpha) \\ &\doteq -\log \alpha - \beta\delta/\alpha \\ &\doteq \beta/\alpha(-(\alpha \log \alpha)/\beta - \delta). \end{aligned} \tag{65}$$

Therefore the problem is to find $\delta < (1 - \alpha)/\beta$ such that

$$[-(\alpha \log \alpha)/\beta - \delta]^{-1}(-\log \delta) = \text{minimum}. \tag{66}$$

The solution to this problem is given by Smith [7], and we obtain the following theorem.

THEOREM 3. *If $p = p_k$ for all k, and if $\rho(C_p)$ is bounded as in* (60), *then a lower bound on the asymptotic rate of convergence of iteration process* (8) *is given by the inverse of k', where*

$$k' = -p(\log \gamma_p)^{-1} = -p[\log (\alpha + \beta\mu^p)]^{-1}. \tag{67}$$

The minimum of k' with respect to p is given approximately by

$$\min_{p>P} k' \doteq k'(p_0) \doteq (-\log \mu)^{-1}\eta^{-1}[1 - \log \eta](\alpha/\beta) \tag{68}$$

and

$$p_0 = (\log \mu)^{-1} \log \{\eta(1 - \log \eta)^{-1}\}, \qquad \eta = -(\alpha \log \alpha)/\beta. \tag{69}$$

For the general process (8) we find from (16) that an upper bound for $\rho(C_p)$ is given by

$$\rho(C_p) \leqq \gamma_p = \|C\|_* + \|T_p\|_*(1 + \|C\|_*). \tag{70}$$

The norm $\|\cdot\|_*$ is defined by

$$\|A\|_* = \|(JD)^{-1}A(JD)\|_1, \tag{71}$$

where $\|\cdot\|_1$ is the usual l_1-norm,

$$D = \operatorname{diag}\{1, \varepsilon, \cdots, \varepsilon^{n-1}\}, \quad 0 < \varepsilon < 1 - \rho(C), \tag{72}$$

and J is a matrix which transforms C into Jordan canonical form. The matrix $\hat{C} = D^{-1}J^{-1}CJD$ is identical to the Jordan form of C except that each off-diagonal element equals ε. (See Ortega and Rheinboldt [6].) Therefore,

$$\|C\|_* = \|\hat{C}\|_1 \leqq \rho(C) + \varepsilon < 1. \tag{73}$$

Furthermore, for T_p we have

$$\begin{aligned} \|T_p\|_* &\leqq \|D\|_1 \cdot \|D^{-1}\|_1 \cdot \|J\|_1 \cdot \|J^{-1}\|_1 \cdot \|T_p\|_1 \\ &\leqq \varepsilon^{-(n-1)}\|J\|_1 \cdot \|J^{-1}\|_1 \cdot \|T_p\|_1 \\ &= \varepsilon^{-(n-1)}K\|T_p\|_1. \end{aligned} \tag{74}$$

Now suppose there is a constant $\mu < 1$ such that

$$\|T_p\|_1 \leqq a\mu^p \quad \text{for all} \quad p \quad \text{and} \quad 0 < a < \infty. \tag{75}$$

(For example, when T_p is of the form $T_p = H^p$, as frequently occurs, then $\|T_p\|_1 \leqq \|H\|_1^p$ and if $\|H\|_1 < 1$, we can choose $\mu = \|H\|_1$, $a = 1$.) Substituting from (75) and (73) into (70) we find

$$\gamma_p \leqq \rho(C) + \varepsilon + \varepsilon^{-(n-1)}Ka\mu^p(1 + \rho(C) + \varepsilon) \tag{76}$$

and we have the following corollary.

COROLLARY 3.1. *Let the conditions of Theorem* 1 *hold. If there exists* $\mu < 1$ *such that* $\|T_p\|_1 \leqq a\mu^p$, *for all p, then the minimum of k′ with respect to p is given by* (68) *and obtained at* $p = p_0$ *given by* (69) *with*

$$\alpha = \rho(C) + \varepsilon, \qquad \beta = \varepsilon^{-(n-1)} \cdot Ka \cdot (1 + \rho(C) + \varepsilon). \tag{77}$$

Unfortunately, in practice we cannot usually compute p_0 as given by Corollary 3.1, since the constant $K = \|J\|_1 \cdot \|J^{-1}\|_1$ is rarely known. However, when the special conditions of Theorem 2 are satisfied, then Theorem 3 does give useful estimates for p_0.

Suppose there exists μ such that $\rho(T_p) \leqq \mu^p < 1$ and let $\rho = \rho(C) < 1$. Then from (15) we have

$$\rho(C_p) \leqq \gamma_p = \rho + \mu^p(1 + \rho) \tag{78}$$

and we obtain the following.

COROLLARY 3.2. *Let the conditions of Theorem* 2 *be satisfied. Let* $\rho(T_p) \leqq \mu^p$, *where* $\mu < 1$ *is known. Then the minimum of k′ with respect to p is given by* (68) *and* *is obtained at* $p = p_0$ *given by* (69) *with*

$$a = \rho(C) \quad \text{and} \quad \beta = (1 + \rho(C)). \tag{79}$$

For example, we consider the method of Nichols [5] for solving coupled equations using the Jacobi method for both the inner and outer iterations. This procedure satisfies the conditions of Theorem 2, as shown in § 4. If we define μ as in (52), then $\rho(T_p) \leqq \mu^p$, and we may use Corollary 3.2 to find optimal values for the number of inner iterations p. In Table 1 we give some computations for values of ρ and μ obtained in practice. The table shows P, the minimum number of inner iterations required for convergence, and p_0, the number of inner iterations needed for optimal convergence.

TABLE 1

$\rho(C)$	μ	P	p_0
0.15	0.72	1	7
0.34	0.94	12	36
0.50	0.988	91	196

For the same problem, when the SOR iteration is used, Theorem 2 is not satisfied, and we have not yet been able to find sufficiently accurate upper bounds for $\rho(C_p)$ of the form required to obtain good estimates for p_0. Further research on this problem is still needed. In practice we frequently allow p_k to vary from one outer step to the next, rather than keeping $p_k = p$ constant. Application of this technique is given in Nichols [5], but a theoretical analysis to determine the optimal choices for the p_k is still required.

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