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A new iteration method for a class of complex symmetric linear systems☆

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ABSTRACT

In this paper, a new iteration method is proposed for solving the complex symmetric linear systems. In theory, we show that the convergence factor or the upper bound of the spectral radius of the iteration matrix of the new method are smaller than that of the PMHSS method proposed by Bai et al. (2011). Moreover, we analyze and compare the parameter-free versions and the spectrum distributions of the preconditioned matrix of the new method and the PMHSS method. Finally, we present some numerical experiments on a few model problems to illustrate the theoretical results and show the effectiveness of our new method.

1. Introduction

In this paper, we consider the following system of linear equations:

$$Ax = (W + iT)x = b, \quad A \in C^{n\times n} \text{ and } x, b \in C^n, \tag{1}$$

where A is a nonsingular complex symmetric matrix, W; T are real, symmetric and positive semidefinite matrices. Here, we do not impose the restriction that at least one of W and T, e.g., W, is positive definite, as in [1].

The system such as (1) is important and arises in a variety of scientific and engineering applications. For example, diffuse optical tomography [2], FFT-based solution of certain time-dependent PDEs [3], molecular scattering [4], lattice quantum chromodynamics [5], quantum chemistry, and eddy current problem [6]. For more examples, we refer to [7–22] and the references therein.

To solve efficiently the complex symmetric linear system (1), one approach is to deal with some $2n \times 2n$ equivalent real formulations to avoid solving the complex linear system [7,23,24], and the other is to tackle the $n \times n$ linear system (1) directly, such as conjugate orthogonal conjugate gradient, complex symmetric (CSYM), and quasi-minimal Residual; see [25–27]. Hezari et al. in [18] proposed a preconditioned variant of the generalized successive overrelaxation (GSOR) iterative method for solving linear system (1), they also studied conditions under which the spectral radius of the iteration matrix of the preconditioned GSOR method is smaller than that of the GSOR method [19] and determined the optimal values of iteration parameters.

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Moreover, based on the Hermitian and skew-Hermitian parts of the coefficient matrix $A \in C^{n\times n}$: $A = H + S$ with $H = \frac{1}{2}(A+A^*) = W$ and $S = \frac{1}{2}(A-A^*) = \mathrm{i}T$, Bai et al. proposed the HSS (Hermitian and skew-Hermitian splitting) iterative method [28] for solving non-Hermitian linear system as well as its preconditioned variant such as PHSS method [29]. To avoid solving the shifted skew-Hermitian sub-system of linear equations at each iteration step for the HSS method, Bai et al. skillfully designed a modified HSS (MHSS) method [8] and its preconditioned variant: the PMHSS iteration method [1].

In this paper, we propose a novel iteration method for solving the complex symmetric linear system (1). We establish the convergence theory for the new iteration method under the condition that both W and T are symmetric positive semidefinite without imposing the restriction that at least one of W and T is positive definite. We show that the optimal convergence factor and the upper bound of the spectral radius of the iteration matrix of our new method are smaller than those of the PMHSS method. We also analyze and compare the parameter-free versions of the new method and the PMHSS method. By theoretic analysis and numerical experiments, we show that the new method can outperform the PMHSS method.

The organization of the remaining part of this paper is as follows. In Section 2, we establish the new iteration method for the complex symmetric linear system (1) and study the convergence property of this new method. In Section 3, we compare our new method with the PMHSS method. Numerical results are given in Section 4 to show the effectiveness of this new iteration method. Finally, in Section 5, we propose some conclusions and remarks to end the paper.

The following notations will be used throughout this paper. We denote the identity matrix with proper dimension by I. For a vector v, we denote the l_2 norm of v by $\|v\|_2$. And for a matrix B, we denote the transpose and the inverse of B by B^T and B^{-1}, respectively. Moreover, $\rho(B)$ denotes the spectral radius of B and null(B) represents the null space of B. When A and B are symmetric, $A \preceq B$ means that $B - A$ is positive semidefinite.

2. New iterative method

In this section, to solve complex symmetric linear system (1), we propose a so-called combination method of real part and imaginary part, which will be simply called the CRI method.

Now, we establish the CRI iteration method. The system of linear equations (1) can be written as the following two equivalent forms:

$$\begin{aligned}&(\alpha T + W)x = (\alpha - \mathrm{i})Tx + b,\\ &(\alpha W + T)x = (\alpha + \mathrm{i})Wx - \mathrm{i}b\end{aligned} \tag{2}$$

for any $\alpha \neq 0$. From (2), the CRI method is derived as follows.

The CRI *iterative method.* Given an initial vector $x^{(0)} \in C^n$. For $k = 0, 1, 2, \ldots$, until the sequence of iterates $\{x^{(k)}\}_0^\infty$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases}(\alpha T + W)x^{(k+\frac{1}{2})} = (\alpha - \mathrm{i})Tx^{(k)} + b,\\ (\alpha W + T)x^{(k+1)} = (\alpha + \mathrm{i})Wx^{(k+\frac{1}{2})} - \mathrm{i}b,\end{cases} \tag{3}$$

where $\alpha > 0$ is a given constant.

The CRI iteration scheme (3) can be reformulated into the following standard form

$$x^{(k+1)} = \mathcal{T}(\alpha)x^{(k)} + \mathcal{M}(\alpha)^{-1}b. \tag{4}$$

Here,

$$\mathcal{T}(\alpha) = (\alpha^2 + 1)(\alpha W + T)^{-1}W(\alpha T + W)^{-1}T$$

and

$$\mathcal{M}(\alpha)^{-1} = \alpha(\alpha W + T)^{-1}(W - \mathrm{i}T)(\alpha T + W)^{-1}.$$

Obviously, we see that $\mathcal{T}(\alpha)$ is the iteration matrix of the CRI iteration (3) or (4). Therefore, the CRI iteration (3) is convergent if and only if $\rho(\mathcal{T}(\alpha)) < 1$; see [30–32].

Furthermore, if let

$$\mathcal{N}(\alpha) = \mathcal{M}(\alpha) - A = \frac{\alpha^2 + 1}{\alpha}(\alpha T + W)(W - \mathrm{i}T)^{-1}W(\alpha T + W)^{-1}T,$$

then

$$A = \mathcal{M}(\alpha) - \mathcal{N}(\alpha) \tag{5}$$

defines a splitting of the coefficient matrix of the complex symmetric linear system (1), and the CRI iteration method (3) can also be induced by the matrix splitting (5). Obviously, we see that $\mathcal{T}(\alpha) = \mathcal{M}(\alpha)^{-1}\mathcal{N}(\alpha)$ is the iteration matrix of the CRI iteration method (3). It follows that the splitting matrix $\mathcal{M}(\alpha)$ can be used as a preconditioner for the complex symmetric matrix $A \in C^{n\times n}$.

Now, we discuss the convergence property of the CRI iteration. We will show that the CRI method converges to the unique solution x_* of the complex symmetric linear system (1) for arbitrary initial guess and $\alpha > 0$. To do it, we first present two important lemmas which are useful for the discussion.

Lemma 2.1. *Let $W \in R^{n\times n}$ and $T \in R^{n\times n}$ be symmetric positive semidefinite matrices, and $M = W + T$. Then* $\text{null}(W)\bigcap \text{null}(T) = \text{null}(M)$.

Proof. If $z \in \text{null}(W)\bigcap \text{null}(T)$, then $Wz = 0,\ Tz = 0$ and so $Mz = 0$ or $z \in \text{null}(M)$. Conversely, if $0 \neq z \in \text{null}(M)$, then $(W + T)z = 0$. So, we have $z^T(W + T)z = z^TWz + z^TTz = 0$. Since W and T are symmetric positive semidefinite, we have both $z^TWz = 0$ and $z^TTz = 0$ and so $Wz = Tz = 0$ or $z \in \text{null}(W)\bigcap \text{null}(T)$. The proof is completed. □

Corollary 2.1. *Let $W \in R^{n\times n}$ and $T \in R^{n\times n}$ be symmetric positive semidefinite matrices. Let $A = W + iT$ and $M = W + T$. If A is nonsingular, then $M = W + T$ is a symmetric positive definite matrix.*

Proof. It is obvious that $A = W + iT$ is nonsingular if and only if $\text{null}(W)\bigcap \text{null}(T) = \{0\}$. Under the assumption of the corollary, by Lemma 2.1, $\text{null}(W)\bigcap \text{null}(T) = \text{null}(M)$. Thus, A is nonsingular if and only if $\text{null}(M) = \{0\}$. On the other hand, M is positive semidefinite since both W and T are positive semidefinite. So $\text{null}(M) = \{0\}$ means that M is positive definite. The proof is completed. □

Lemma 2.2. *Let $W \in R^{n\times n}$ and $T \in R^{n\times n}$ be symmetric positive semidefinite matrices satisfying* $\text{null}(W)\bigcap \text{null}(T) = \{0\}$. *Then, there exists a nonsingular matrix $P \in R^{n\times n}$ such that*

$$W = P^T\Lambda P \text{ and } T = P^T\widetilde{\Lambda}P. \tag{6}$$

Here, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\widetilde{\Lambda} = \text{diag}(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_n)$ and $\widetilde{\lambda}_i$, $\widetilde{\lambda}_i$ satisfy

$$\lambda_i + \widetilde{\lambda}_i = 1,\ \lambda_i \geq 0,\ \widetilde{\lambda}_i \geq 0\ (i = 1, 2, \ldots, n). \tag{7}$$

Proof. It follows from the result in Lemma 2.1 that $M = W + T$ is a symmetric positive definite matrix. So, there exists a nonsingular matrix V such that $V^TMV = I$. Note V^TWV is a symmetric positive semidefinite matrix and $V^TWV \preceq I$, there exists an orthogonal matrix Q such that

$$V^TWV = Q^T\Lambda Q = Q^T\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)Q,$$

with $\lambda_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$. Let $\widetilde{\Lambda} = I - \Lambda$, then (7) holds true and

$$V^TTV = V^T(M - W)V = Q^T(I - \Lambda)Q = Q^T\widetilde{\Lambda}Q = Q^T\text{diag}(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_n)Q.$$

Let $P = QV^{-1}$, then (6) holds true. The proof is completed. □

Now, we apply the lemma to analyze the convergence property of the CRI method.

Theorem 2.1. *Let $A = W + iT \in C^{n\times n}$ be nonsingular, $W \in R^{n\times n}$ and $T \in R^{n\times n}$ symmetric positive semidefinite matrices and α a positive constant. Then the spectral radius of the* CRI *method satisfies*

$$\rho(\mathcal{T}(\alpha)) \leq \sigma(\alpha) \equiv \frac{\alpha^2 + 1}{(\alpha + 1)^2} < 1,\ \forall\alpha > 0. \tag{8}$$

Therefore, the CRI *method converges to the unique solution x_* of the complex symmetric linear system (1) for any initial guess.*

Proof. Since W, T are positive semidefinite matrices satisfying $\text{null}(W)\bigcap \text{null}(T) = \{0\}$, it follows from Lemma 2.2 that there exists a nonsingular matrix P such that (6) and (7) hold true. So, we have

$$\begin{aligned}\mathcal{T}(\alpha) &= (\alpha^2 + 1)(\alpha W + T)^{-1}W(\alpha T + W)^{-1}T\\ &= (\alpha^2 + 1)P^{-1}(\alpha\Lambda + \widetilde{\Lambda})^{-1}\Lambda(\alpha\widetilde{\Lambda} + \Lambda)^{-1}\widetilde{\Lambda}P.\end{aligned}$$

This implies that the iteration matrix $\mathcal{T}(\alpha)$ is similar to

$$\widetilde{\mathcal{T}(\alpha)} \equiv (\alpha^2 + 1)(\alpha\Lambda + \widetilde{\Lambda})^{-1}\Lambda(\alpha\widetilde{\Lambda} + \Lambda)^{-1}\widetilde{\Lambda}.$$

Then, by making use of (7), we can obtain

$$\begin{aligned}\rho(\mathcal{T}(\alpha)) &= \rho(\widetilde{\mathcal{T}(\alpha)})\\ &= \max_{1\leq i\leq n}\left\{\frac{(\alpha^2 + 1)\lambda_i\widetilde{\lambda}_i}{(\alpha\lambda_i + \widetilde{\lambda}_i)(\alpha\widetilde{\lambda}_i + \lambda_i)}\right\}\\ &= \max_{1\leq i\leq n}\left\{\frac{(\alpha^2 + 1)\lambda_i\widetilde{\lambda}_i}{(\alpha^2 + 1)\lambda_i\widetilde{\lambda}_i + \alpha(\lambda_i^2 + \widetilde{\lambda}_i^2)}\right\}\\ &\leq \max_{1\leq i\leq n}\left\{\frac{(\alpha^2 + 1)\lambda_i\widetilde{\lambda}_i}{(\alpha^2 + 1)\lambda_i\widetilde{\lambda}_i + 2\alpha\lambda_i\widetilde{\lambda}_i}\right\}\\ &= \frac{\alpha^2 + 1}{(\alpha + 1)^2} < 1.\end{aligned}$$

Therefore, the CRI iteration converges to the unique solution of the complex symmetric linear system (1). The proof is completed. □

3. Comparison the CRI with the PMHSS

In this section, we compare the CRI method with the PMHSS method proposed by Bai et al. in [1].

The PMHSS *method.* Let $x \in C^n$ be an arbitrary initial guess and α be a given positive constant. For $k = 0, 1, 2, \ldots$, until the sequence of iterates $\{x^{(k)}\}_0^\infty$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha V + W)x^{(k+\frac{1}{2})} = (\alpha V - \mathrm{i}T)x^{(k)} + b, \\ (\alpha V + T)x^{(k+1)} = (\alpha V + \mathrm{i}W)x^{(k+\frac{1}{2})} - \mathrm{i}b, \end{cases}$$

where $V \in C^{n\times n}$ is a prescribed symmetric positive definite matrix and is the preconditioned matrix of the PMHSS method.

Bai et al. showed that the PMHSS iteration converges to the unique solution of the complex symmetric linear system (1) for any initial guess, and its convergence rate is bounded by

$$\widetilde{\sigma}(\alpha) \equiv \max_{\widetilde{\mu}_i \in sp(V^{-1}W)} \frac{\sqrt{\alpha^2 + \widetilde{\mu}_i^2}}{\alpha + \widetilde{\mu}_i} \max_{\widetilde{\lambda}_i \in sp(V^{-1}T)} \frac{\sqrt{\alpha^2 + \widetilde{\lambda}_i^2}}{\alpha + \widetilde{\lambda}_i} \leq \max_{\widetilde{\mu}_i \in sp(V^{-1}W)} \frac{\sqrt{\alpha^2 + \widetilde{\mu}_i^2}}{\alpha + \widetilde{\mu}_i} < 1.$$

Moreover, when W is positive definite, V is taken as W and in the case it holds that $\widetilde{\sigma}(\alpha) \leq \frac{\sqrt{\alpha^2+1}}{\alpha+1} < 1$.

Corollary 3.1. *Let $A = W + iT \in C^{n\times n}$, with $W \in R^{n\times n}$ and $T \in R^{n\times n}$ being symmetric positive semidefinite matrices satisfying* $\mathrm{null}(W) \bigcap \mathrm{null}(T) = \{0\}$, *and let α be a positive constant. Then, the upper bound of the spectral radius of the* CRI *method is smaller than that of the* PMHSS *method.*

Proof. From the theoretical results given in [1], we know that the upper bound of the spectral radius of the PMHSS method is $\widetilde{\sigma}(\alpha) \equiv \frac{\sqrt{\alpha^2+1}}{\alpha+1}$, which satisfies

$$1 > \widetilde{\sigma}(\alpha) = \sqrt{\sigma(\alpha)} > \sigma(\alpha) > 0, \ \forall \alpha > 0.$$

Thus, the result holds true. □

Remark 3.1. From the proof of this corollary, we see that the upper bound of the spectral radius of the CRI method is exactly the square root of that of the PMHSS method.

We know that wherever in the PMHSS method or in the CRI method, computing the optimal parameter α^* is a time consuming work. To reduce the computational cost, in actual implementations of the PMHSS iteration, the parameter α can simply be taken as 1, see [1]. This skillful approach will result in the parameter-free methods and is effective and efficient. So, in the following, we also take the parameter $\alpha = 1$ to produce a parameter-free CRI iteration method. In fact, the parameter $\alpha = 1$ is also the optimal iteration parameter α^* which minimize the upper bound of the spectral radius of the CRI method ($\sigma(\alpha)$) as well as the upper bound of the spectral radius of the PMHSS method ($\widetilde{\sigma}(\alpha)$) (Here, we just discuss the case that the preconditioned matrix V in the PMHSS iteration is taken as W).

From [1], we know that the iteration matrix of the PMHSS iteration is

$$\mathcal{L}(\alpha) = \frac{\alpha + i}{\alpha + 1}(\alpha W + T)^{-1}(\alpha W - iT).$$

So by using Lemma 2.1, we can get the following results which are important for the PMHSS iteration.

Lemma 3.1. *Let $A = W + iT \in C^{n\times n}$, with $W \in R^{n\times n}$ and $T \in R^{n\times n}$ being symmetric positive semidefinite matrices satisfying* $\mathrm{null}(W) \bigcap \mathrm{null}(T) = \{0\}$, *and let $\alpha = 1$. Then, the spectral radius of the* PMHSS *method satisfies*

$$\frac{1}{2} \leq \rho(\mathcal{L}(1)) \leq \frac{\sqrt{2}}{2}$$

for $V = W$. (Here, V is the preconditioned matrix for PMHSS *method and it is always taken as W in actual implementations.)*

Proof. When $\alpha = 1$, applying Lemma 2.1, we have

$$\begin{aligned} \mathcal{L}(1) &= \frac{1+i}{2}(W + T)^{-1}(W - iT) \\ &= \frac{1+i}{2}P^{-1}(\Lambda + \widetilde{\Lambda})^{-1}(\Lambda - i\widetilde{\Lambda})P. \end{aligned}$$

That is, $\mathcal{L}(1)$ is similar to the matrix $\widetilde{\mathcal{L}} = \frac{1+i}{2}(\Lambda + \widetilde{\Lambda})^{-1}(\Lambda - i\widetilde{\Lambda})$. So, we have

$$\begin{aligned}\rho(\mathcal{L}(1)) &= \left|\frac{1+i}{2}\right|\rho((\Lambda+\widetilde{\Lambda})^{-1}(\Lambda - i\widetilde{\Lambda})) \\ &= \frac{\sqrt{2}}{2}\max_{1\le i\le n}\left\{\left|\frac{\lambda_i - i\widetilde{\lambda}_i}{\lambda_i+\widetilde{\lambda}_i}\right|\right\} \\ &= \frac{\sqrt{2}}{2}\max_{1\le i\le n}\left\{\frac{\sqrt{\lambda_i^2+\widetilde{\lambda}_i^2}}{\lambda_i+\widetilde{\lambda}_i}\right\} \\ &= \frac{\sqrt{2}}{2}\max_{1\le i\le n}\left\{\sqrt{\lambda_i^2+\widetilde{\lambda}_i^2}\right\}.\end{aligned}$$

Note that $\lambda_i + \widetilde{\lambda}_i = 1$, $\lambda_i \ge 0$ and $\widetilde{\lambda}_i \ge 0$ for $i = 1, 2, \ldots, n$, we have $\frac{1}{2} \le \rho(\mathcal{L}(1)) \le \frac{\sqrt{2}}{2}$. This completes the proof. □

Remark 3.2. Let $A = W + \mathrm{i}T \in C^{n\times n}$, with $W \in R^{n\times n}$ and $T \in R^{n\times n}$ being symmetric positive semidefinite matrices satisfying $\mathrm{null}(W)\bigcap\mathrm{null}(T) = \{0\}$, and let $\alpha = 1$. It follows from Lemma 3.1 that the eigenvalues of the PMHSS preconditioned matrix are clustered within the complex circle centered at 1 with out radius $\frac{\sqrt{2}}{2}$ and inner radius $\frac{1}{2}$. Here, the PMHSS preconditioned matrix is $\widehat{\mathcal{M}}^{-1}(\alpha)A$, where $\widehat{M}(\alpha)$ is the PMHSS preconditioner $\frac{(\alpha+1)(1+\mathrm{i})}{2\alpha}$ (see [1]).

By making use of Theorem 2.1 and Lemma 3.1, we can get an important result as follows:

Theorem 3.1. *Let $A = W + iT \in C^{n\times n}$, with $W \in R^{n\times n}$ and $T \in R^{n\times n}$ being symmetric positive semidefinite matrices satisfying* $\mathrm{null}(W)\bigcap\mathrm{null}(T) = \{0\}$, *and let $\alpha = 1$. Then, the convergence factor and the upper bound of the spectral radius of the iteration matrix for the* CRI *method are smaller than those of the* PMHSS *method.*

Proof. Combine Theorem 2.1 and Lemma 3.1, we have

$$0 \le \rho(\mathcal{T}(1)) \le \frac{1}{2} \le \rho(\mathcal{L}(1)) \le \frac{\sqrt{2}}{2}.$$

Thus, the result is set up directly. The proof is completed. □

We can see from Theorem 3.1 that the CRI method may converge much faster than the PMHSS method for solving complex symmetric linear systems when the iteration parameters α are taken as 1. In fact, when the CRI iterative method is in its best case, $\rho(\mathcal{T}(1))$ can be 0 if in Lemma 2.2, λ_i and $\widetilde{\lambda}_i$ satisfy

$$\lambda_i\widetilde{\lambda}_i = 0\ (i = 1, 2, \ldots, n). \tag{9}$$

In this case, the CRI method needs just one step to converge. However, the spectral radius of the iteration matrix of PMHSS method becomes $\rho(\mathcal{L}(1)) = \frac{\sqrt{2}}{2}$ which is the worst case for the PMHSS iteration (because $\frac{1}{2} \le \rho(\mathcal{L}(1)) \le \frac{\sqrt{2}}{2}$). For example, consider the following complex system of linear equations (1) with

$$W = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \text{ and } T = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

For this case, we can take $P = I$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2) = \mathrm{diag}(1, 0)$ and $\widetilde{\Lambda} = \mathrm{diag}(\widetilde{\lambda}_1, \widetilde{\lambda}_2) = \mathrm{diag}(0, 1)$. Then, the iteration sequence for the CRI method is

$$x^{(0)} = \begin{pmatrix}0\\0\end{pmatrix},\ x^{(\frac{1}{2})} = \begin{pmatrix}1\\1\end{pmatrix},\ x^{(1)} = \begin{pmatrix}1\\-i\end{pmatrix},\ x^{(2)} = x^{(1)},$$

since $\begin{pmatrix}1\\-i\end{pmatrix}$ is just the unique solution for this system. But, the PMHSS method needs more than 40 iteration steps to converge.

On the other hand, when $\lambda_i = \widetilde{\lambda}_i = \frac{1}{2}$ $(i = 1, 2, \ldots, n)$, the CRI method is in its worst case. In this case, $\rho(\mathcal{T}(1)) = \frac{1}{2} = \rho(\mathcal{L}(1))$.

In the remaining part of this section, we study the special property of the CRI preconditioned matrix for $\alpha = 1$. As the CRI preconditioner is

$$\mathcal{M}(1) = (T + W)(W - \mathrm{i}T)^{-1}(W + T),$$

the corresponding CRI preconditioned matrix is $\mathcal{M}(1)^{-1}A$. In fact, a cluster spectrum of the preconditioned matrix is a welcome property, because it enables the efficient use of the Krylov subspace methods. The spectral distribution of $\mathcal{M}(1)^{-1}A$ is described in the following theorem.

Theorem 3.2. *Let $A = W + iT \in C^{n\times n}$, with $W \in R^{n\times n}$ and $T \in R^{n\times n}$ being symmetric positive semidefinite matrices satisfying* $\text{null}(W)\bigcap\text{null}(T) = \{0\}$, *and let $\alpha = 1$. Then, all the eigenvalues of $\lambda(\mathcal{M}(1)^{-1}A)$ satisfies*

$$|\lambda(\mathcal{M}^{-1}(1)A) - 1| \leq \frac{1}{2}. \tag{10}$$

That is, the eigenvalues of the preconditioned matrix $\mathcal{M}(1)^{-1}A$ are clustered within the complex disk centered at 1 with radius $\frac{1}{2}$.

Proof. Note $\mathcal{T}(1) = \mathcal{M}^{-1}(1)\mathcal{N}(1)$ and $A = \mathcal{M}(1) - \mathcal{N}(1)$, we have

$$\mathcal{M}^{-1}(1)A = I - \mathcal{M}^{-1}(1)\mathcal{N}(1) = I - \mathcal{T}(1).$$

So, $\lambda(\mathcal{M}^{-1}(1)A) = 1 - \lambda(\mathcal{T}(1))$. From the results in Theorem 2.1, we can see $|\lambda(\mathcal{T}(1))| \leq \frac{1}{2}$. Therefore, we have $|\lambda(\mathcal{M}^{-1}(1)A) - 1| \leq \frac{1}{2}$. The proof is completed. □

From Theorem 3.2, the following corollary holds true.

Corollary 3.2. *Let $A = W + iT \in C^{n\times n}$, with $W \in R^{n\times n}$ and $T \in R^{n\times n}$ being symmetric positive semidefinite matrices satisfying* $\text{null}(W)\bigcap\text{null}(T) = \{0\}$, *and let $\alpha = 1$. Then, the spectral distribution of the* CRI *preconditioned matrix is more cluster than that of the* PMHSS *preconditioned matrix.*

Proof. Combine Remark 3.2 and Theorem 3.2, this result holds true directly. □

4. Numerical experiments

In this section, we perform three numerical examples to illustrate the theoretical results and show the effectiveness of the CRI iteration method for solving the complex symmetric linear system (1) in terms of both iteration count (denoted as IT) and computing time (in seconds, denoted as CPU), and the norm of the residual (denoted as "RES") defined by

$$\text{RES} = \|b - Ax^{(k)}\|_2.$$

In actual computations, the iteration schemes are started from the zero vector and terminated if the current iterations satisfy ERR$\leq 10^{-6}$ or the number of the prescribed iteration steps $k = 400$ are exceeded, where

$$\text{ERR} = \frac{\|b - Ax^{(k)}\|_2}{\|b\|_2}.$$

All experiments are performed in MATLAB (R2010b) with machine precision 10^{-16}, and all experiments are implemented on a personal computer with 2.20 GHz central processing unit, 2.00 G memory and Win7 operating system. The CRI method is compared with the PMHSS method and GMRES(10) method. Moreover, we also examine the their numerical behavior as preconditioners for the GMRES(10) method. The iteration parameters α used in both PMHSS and CRI iteration methods as well as the corresponding PMHSS and CRI preconditioners in our experiments are taken as $\alpha = 1$.

Example 4.1 (*See [1,8,10]*)**.** The system of linear equations (1) is of the form

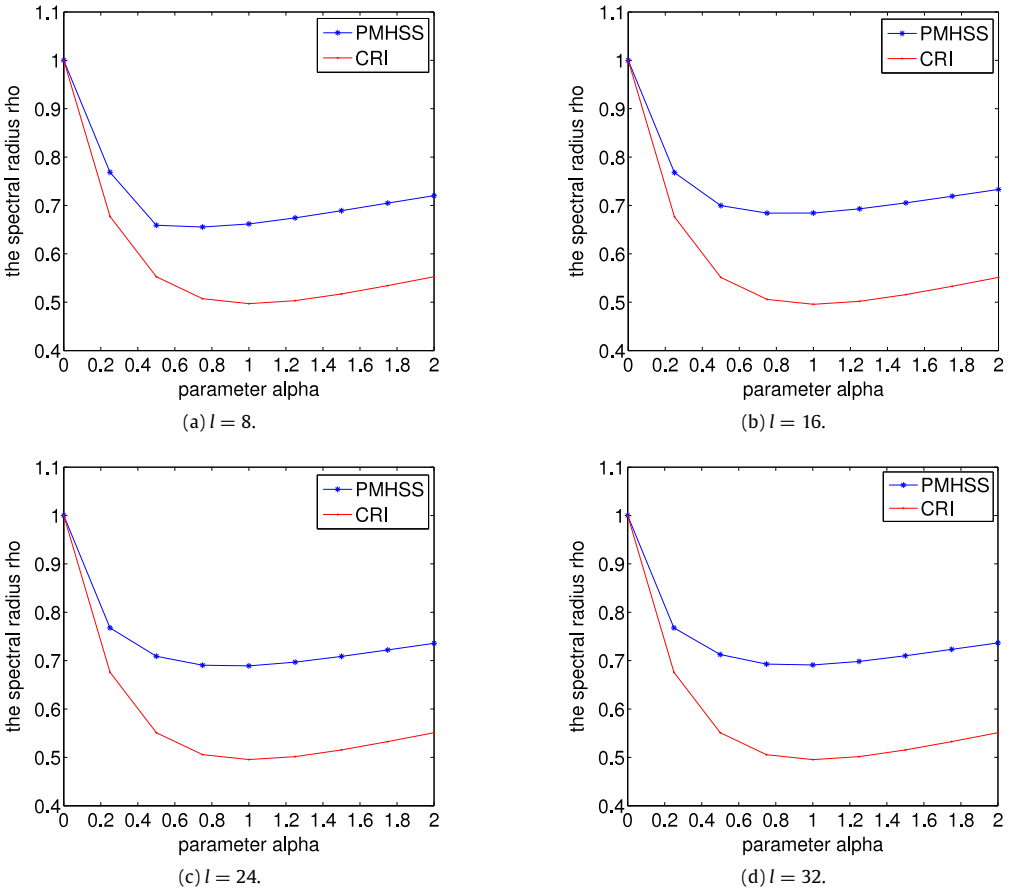
$$[(-\omega^2 M + K) + \text{i}(\omega C_V + C_H)]x = b, \tag{11}$$

where M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, respectively, and ω is the driving circular frequency. We take $C_H = \mu K$ with μ a damping coefficient, $M = I$, $C_V = 10I$, and K the five-point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0, 1]\times[0, 1]$ with the mesh-size $h = \frac{1}{m+1}$. The matrix $K \in R^{n\times n}$ possesses the tensor-product form $K = I \otimes B_m + B_m \otimes I$, with $B_m = h^{-2}\text{tridiag}(-1, 2, -1) \in R^{m\times m}$. Hence, K is an $n \times n$ block-tridiagonal matrix, with $n = m^2$. In addition, we set $\omega = \pi, \mu = 0.02$, and the right-hand side vector b to be $b = (1 + \text{i})A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. As before, we normalize the system by multiplying both sides through by h^2. In fact, this complex symmetric system of linear equations arises in direct frequency domain analysis of an n-degree-of-freedom (n-DOF) linear system. For more details, we refer to [7,33].

The numerical results about IT, CPU and RES for PMHSS, CRI, GMRES(10) and preconditioned GMRES(10) methods for Example 4.1 are presented in Table 1. As we can see from Table 1, we can see that the CRI method considerably outperforms PMHSS, GMRES(10) and preconditioned GMRES(10) methods, because it needs less iteration counts and CPU times to reach the stopping criterion. Moreover, we depict the spectral radius of the PMHSS iteration matrix and CRI iteration matrix for Example 4.1 with different l and different parameters α in Fig. 1. From this figure, we see that the spectral radius of the CRI iteration matrix is always smaller than that of the PMHSS iteration matrix. This illustrates that our CRI method converges faster than the PMHSS method, which is in accordance with the result in Theorem 2.1.

Table 1
Numerical results of different methods for Example 4.1.

Method	$m \times m$	8×8	16×16	24×24	32×32	48×48
PMHSS	IT	29	34	36	37	37
	CPU	0.0318	0.0497	0.2208	0.7244	3.8407
	RES	1.04e−7	3.79e−8	1.14e−8	9.13e−9	8.15e−9
CRI	IT	15	14	13	13	12
	CPU	0.0032	0.0475	0.1660	0.5478	2.8824
	RES	1.01e−7	2.84e−8	2.13e−8	9.13e−9	6.86e−9
GMRES	IT	5(8)	13(10)	27(10)	35(10)	18(9)
	CPU	0.0095	0.0512	0.6291	2.5140	6.2078
	RES	7.69e−7	9.95e−7	7.91e−7	8.13e−7	9.92e−7
PMHSS-GMRES(10)	IT	2(2)	2(2)	2(4)	2(4)	2(4)
	CPU	0.0066	0.0026	0.2291	1.3641	3.2098
	RES	1.21e−7	8.23e−7	7.16e−8	9.90e−8	2.39e−7
CRI-GMRES(10)	IT	1(6)	1(6)	1(7)	1(7)	1(7)
	CPU	0.0045	0.0014	0.0099	1.0980	1.0205
	RES	9.18e−8	8.70e−7	1.03e−7	1.64e−7	4.82e−7



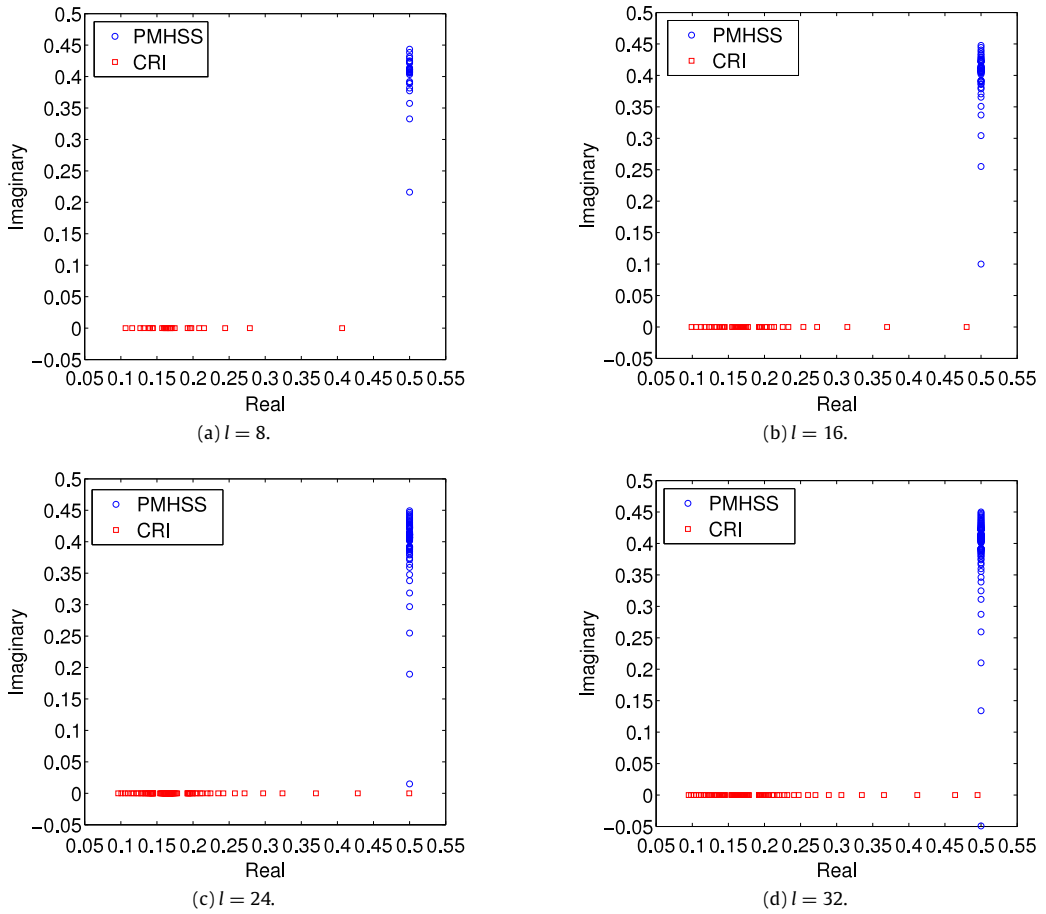
(a) $l = 8$. (b) $l = 16$. (c) $l = 24$. (d) $l = 32$.

Fig. 1. The spectral radius of the PMHSS iteration matrix and CRI iteration matrix for Example 4.1 with different l.

Example 4.2 (*See [8,18]*). The system of linear equation (1) is of the form $(W + \mathrm{i}T)x = b$, with

$$T = I \otimes V + V \otimes I \text{ and } W = 10(I \otimes V_c + V_c \otimes I) + 9(e_1 e_l^\top + e_l e_1^\top) \otimes I,$$

where $V = \text{tridiag}(-1, 2, -1) \in R^{l\times l}$, $V_c = V - e_1 e_l^\top - e_l e_1^\top \in R^{l\times l}$, e_1 and e_l are the first and the last unit vectors in R^l, respectively. We take the right-hand side vector b to be $b = (1 + \mathrm{i})A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1.



(a) $l = 8$. (b) $l = 16$. (c) $l = 24$. (d) $l = 32$.

Fig. 2. The eigenvalue distributions of the PMHSS iteration matrix and CRI iteration matrix for Example 4.2 with different l.

Here T and W correspond to the five-point centered difference matrices approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions and periodic boundary conditions, respectively, on a uniform mesh in the unit square $[0, 1] \times [0, 1]$ with the mesh-size $h = \frac{1}{l+1}$. This problem is an artificially constructed one, but it is quite challenging for iterative solvers.

The numerical results about IT, CPU and RES for PMHSS, CRI, GMRES(10) and preconditioned GMRES(10) methods for Example 4.2 are presented in Table 2. We can see from this table that the CRI method considerably outperforms PMHSS, GMRES(10) and preconditioned GMRES(10) methods not only in terms of iteration counts but also in terms of CPU times. In Fig. 2, we depict the eigenvalue distributions of the PMHSS iteration matrix and CRI iteration matrix for Example 4.2 with different l. From Fig. 2, we see that the eigenvalue distributions of the CRI iteration matrix is always more cluster around (0,0) than that of the PMHSS iteration matrix. Moreover, the spectral radius of the CRI iteration matrix is smaller than $\frac{1}{2}$, which is in accordance with the result in Theorem 3.1.

Example 4.3 (*See [8,18]*)**.** In the last test, we consider the complex Helmholtz equation

$$-\Delta u + \sigma_1 u + i\sigma_2 u = f, \tag{12}$$

where σ_1 and σ_2 are real coefficient functions, u satisfies Dirichlet boundary conditions in $D = [0, 1] \times [0, 1]$ and $i = \sqrt{-1}$. We discretize the problem with finite differences on a $l \times l$ grid with mesh size $h = 1/(l+1)$. This leads to a system of linear equations

$$[(K + \sigma_1 I) + i\sigma_2 I] = b,$$

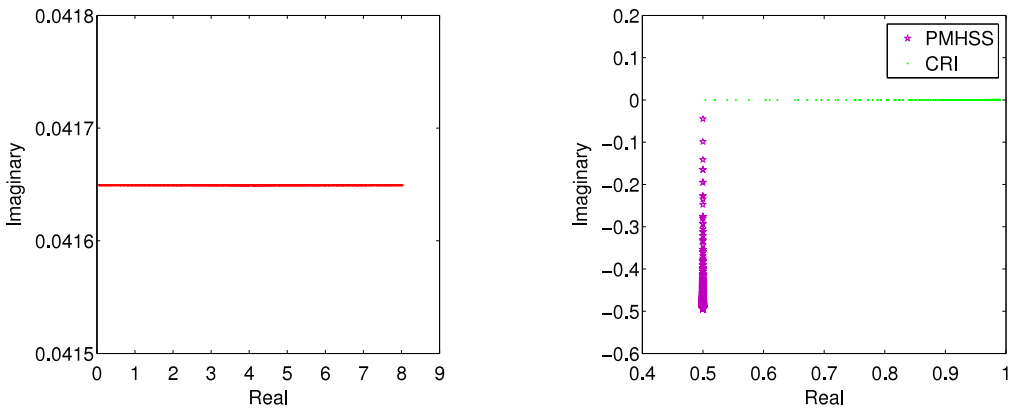
where $K = I \otimes V_l + V_l \otimes I$ is the discretization of $-\Delta$ by means of centered differences, wherein $V_l = h^{-2}\text{tridiag}(-1, 2, -1) \in R^{l\times l}$. The right-hand side vector b is taken as $b = (1 + i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. Furthermore, before solving the system we normalize the coefficient matrix and the right-hand side vector by multiplying both by h^2. For the numerical tests, we set $\sigma_1 = \sigma_2 = 100$.

Table 2
Numerical results of different methods for Example 4.2.

Method	$m \times m$	8×8	16×16	24×24	32×32	48×48
PMHSS	IT	32	32	32	32	32
	CPU	0.0079	0.0414	0.2234	0.7053	3.7473
	RES	3.41e−5	4.82e−5	5.87e−5	6.74e−5	8.20e−5
CRI	IT	15	17	17	17	16
	CPU	0.0063	0.0395	0.2118	0.6732	3.4675
	RES	2.75e−5	6.12e−5	5.59e−5	9.54e−5	8.04e−5
GMRES	IT	8(8)	22(3)	37(8)	–	–
	CPU	0.0198	0.0924	0.8187	–	–
	RES	8.77e−7	9.85e−7	9.83e−7	–	–
PMHSS-GMRES(10)	IT	1(10)	1(8)	1(10)	1(10)	1(10)
	CPU	0.0071	0.0753	0.6299	2.8407	9.0998
	RES	2.63e−8	6.60e−7	1.07e−8	3.45e−7	9.04e−7
CRI-GMRES(10)	IT	1(4)	1(5)	1(5)	1(5)	1(5)
	CPU	0.0044	0.0503	0.3888	1.3485	5.9975
	RES	8.05e−8	6.03e−8	3.42e−7	8.76e−7	5.89e−7

Table 3
Numerical results of different methods for Example 4.3.

Method	$m \times m$	8×8	16×16	24×24	32×32	48×48
PMHSS	IT	24	31	34	36	38
	CPU	0.0284	0.0462	0.2154	0.7248	3.8601
	RES	2.31e−7	4.65e−8	2.68e−8	1.55e−8	7.53e−9
CRI	IT	20	19	19	18	18
	CPU	0.0074	0.0391	0.2080	0.4771	2.3363
	RES	1.96e−7	5.94e−8	1.87e−8	1.64e−9	5.03e−9
GMRES	IT	3(2)	5(8)	8(10)	12(8)	12(15)
	CPU	0.0065	0.0191	0.0199	0.8505	10.9824
	RES	8.29e−7	9.16e−7	8.31e−7	9.42e−7	7.50-7
PMHSS-GMRES(10)	IT	1(10)	2(3)	2(4)	2(4)	2(4)
	CPU	0.0095	0.0753	0.0949	0.8765	9.4578
	RES	8.29e−7	8.18e−7	1.07e−8	8.96e−7	6.04e−7
CRI-GMRES(10)	IT	1(5)	1(7)	1(7)	1(7)	1(7)
	CPU	0.0076	0.0503	0.0698	0.6007	4.9998
	RES	6.53e−7	2.90e−7	9.13e−7	2.61e−7	5.08e−7



(a) No preconditioning. (b) Preconditioned matrices.

Fig. 3. The eigenvalue distributions of the original saddle matrix and preconditioned matrices (i.e., the PMHSS preconditioned matrix $\widehat{\mathcal{M}}^{-1}(1)A$ and the CRI preconditioned matrix $\mathcal{M}^{-1}(1)A$) for Example 4.3 with different $l = 48$.

The numerical results about IT, CPU and RES for PMHSS, CRI, GMRES(10) and preconditioned GMRES(10) methods for Example 4.3 are presented in Table 3. We can see from this table that the CRI method considerably outperforms PMHSS, GMRES(10) and preconditioned GMRES(10) methods. In Fig. 3, we depict the eigenvalue distributions of the original saddle point matrix and preconditioned matrices for Example 4.3 with $l = 48$. From Fig. 3, we see that the eigenvalue distributions of the CRI preconditioned matrix is more clustered than that of the PMHSS preconditioned matrix.

5. Conclusions

In this paper, we have proposed a new efficient iteration method (CRI method) for a class of complex symmetric linear systems. We presented some analysis for the CRI iteration method, which includes the convergence theory of the CRI method as well as it is used as a preconditioner. In theory, we showed that the convergence factor and the upper bound of the spectral radius of the iteration matrix of our CRI method are much smaller than those of the PMHSS method. We also gave several numerical experiments to show that the CRI method can outperform the PMHSS method.

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