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# $H$-Splittings and two-stage iterative methods*

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**Summary.** Convergence of two-stage iterative methods for the solution of linear systems is studied. Convergence of the non-stationary method is shown if the number of inner iterations becomes sufficiently large. The $R_1$-factor of the two-stage method is related to the spectral radius of the iteration matrix of the outer splitting. Convergence is further studied for splittings of $H$-matrices. These matrices are not necessarily monotone. Conditions on the splittings are given so that the two-stage method is convergent for any number of inner iterations.

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## 1. Introduction

Consider iterative methods for the solution of linear systems of equations of the form

$$Ax = b\,, \tag{1}$$

where $A$ is a square nonsingular matrix, and $x$ and $b$ are vectors. In this paper we study methods of the form

$$Mx_{k+1} = Nx_k + b, \quad k = 0, \ldots, \tag{2}$$

where $x_0$ is an initial approximation to the solution of (1), and $A = M - N$. A representation $A = M - N$ is called a *splitting* of $A$ when $M$ is nonsingular. It is well known that the method (2) converges for any initial vector $x_0$ if and only if the spectral radius $\rho(S) < 1$, where $S := M^{-1}N$ is called the iteration matrix; see further Sect. 2. We say that a matrix $S$ is *zero-convergent* if $\rho(S) < 1$. We say that a splitting $A = M - N$ is *convergent* if $\rho(M^{-1}N) < 1$. As is well-known, this is

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equivalent to saying that $A$ is nonsingular and $\lim_{k\to\infty} x_k = A^{-1}b$ for any sequence generated by (2). The methods we study can be seen either as extensions of the classical iterative methods [37, 41], or as preconditioners for conjugate gradient-type methods [3, 35].

Two-stage methods, also called inner/outer methods, consist of solving the linear system in (2) iteratively; i.e., let the splitting $M = F - G$ and perform, say, $s$ "inner" iterations. Thus the resulting method is

$$(3) \qquad x_{k+1} = (F^{-1}G)^s x_k + \sum_{j=0}^{s-1} (F^{-1}G)^j F^{-1}(Nx_k + b), \quad k = 0, \ldots .$$

Distinction should be made between the *stationary* two-stage method, in which the number of inner iterations $s$ stays fixed in each of the outer steps, and *non-stationary* two-stage methods, in which the number of inner iterations $s(k)$ changes with $k$, the outer iteration index[1]; see e.g. Moré [18] for a similar situation.

Stationary two-stage methods have been studied by Nichols [24], who showed that the iteration matrix corresponding to (3) is

$$(4) \qquad \begin{aligned} T_s &= I - (I - (F^{-1}G)^s)(I - M^{-1}N) \\ &= (F^{-1}G)^s + (I - (F^{-1}G)^s)M^{-1}N , \end{aligned}$$

where $I$ denotes the identity matrix. Hence, convergence of (3) is equivalent to requiring $\rho(T_s) < 1$ and Nichols showed that if $\rho(M^{-1}N) < 1$ and $\rho(F^{-1}G) < 1$, then $\rho(T_s) < 1$ for large enough $s$; see also Wachspress [40].

For the non-stationary case, let $x^* := A^{-1}b$ and consider the errors $e_k := x_k - x^*$. An easy calculation then shows $e_{k+1} = T_{s(k)} e_k$, $k = 0, 1, \ldots$ and thus

$$(5) \qquad e_{k+1} = H_k e_0, \quad k = 0, 1, \ldots, \quad \text{where } H_k = T_{s(k)} \cdot T_{s(k-1)} \cdots T_{s(0)} .$$

Hence, convergence for any initial guess $x_0$ is equivalent to $\lim_{k\to\infty} H_k = O$, where $O$ denotes the zero matrix. Since $\rho(T_{s(k)}) < 1$, $k = 0, 1, \ldots$ does not necessarily imply $\lim_{k\to\infty} H_k = O$, see e.g. [12] and [30], the convergence of the non-stationary two-stage method needs tools other than the spectral radius for its analysis.

Golub and Overton studied the convergence of two-stage methods for the case of outer Richardson or Chebyshev methods [10, 11]. More recently, Lanzkron, Rose and Szyld [14], gave conditions on the outer and inner splittings, so that both the stationary and non-stationary two-stage methods converge for any number of inner iterations; see also [34]. They also showed a monotonicity result, namely that under the same conditions, $\rho(T_s) \leqq \rho(T_r)$ if $s \geqq r$. The conditions given are for regular and weak regular splittings (see Sect. 3 for definitions), and their results only apply to monotone matrices, i.e. for matrices $A$ such that $A^{-1}$ is nonnegative; see [6] and Theorem 3.4(a).

In this paper, we study further stationary and non-stationary two-stage methods and answer several open questions.

In Sect. 2 we show that for any convergent outer and inner splitting, the non-stationary two-stage method converges at least as fast as the outer iteration (2) if

$$(6) \qquad \lim_{k\to\infty} s(k) = \infty.$$

[1] Non-stationary two-stage methods are called dynamic two-stage methods in [14].

In particular, there is convergence for any sequence $s(k)$, $k = 1, 2, \ldots$ satisfying (6). We also analyze the case where (6) does not hold but $s(k)$ becomes sufficiently large.

We introduce $H$-splittings and $H$-compatible splittings of $H$-matrices in Sect. 3, we study them, and present the background needed in the theorems of the following section.

In Sect. 4 we give alternative proofs of the convergence results in [14], and use these proofs to extend the convergence results to splittings of $H$-matrices. These matrices are not necessarily monotone. Thus, we extend the convergence results to an important class of non-monotone matrices. We give examples to show that our theorems are best possible in the sense that their principal hypotheses cannot be weakened. We also give an example that shows that a monotonicity result similar to that in [14] does not hold in the $H$-matrix case.

## 2. Non-stationary two-stage iterative methods

In this section we give two convergence results which apply to any two-stage method, requiring only that the inner and outer splittings are both convergent and that 'enough' inner iterations are performed for each outer iteration. Our theorems may be regarded as extensions of results in Nichols [24]. For convenience, let us explicitly restate (3) for the non-stationary case as

$$x_{k+1} = T_{s(k)} x_k + \sum_{j=0}^{s(k)-1} (F^{-1}G)^j F^{-1} b, \quad k = 0, 1, \ldots, \tag{7}$$

where the number of inner iterations $s(k)$ is allowed to depend on $k$ and $T_{s(k)}$ is given by (4). We show in this section that — in a sense to be defined later — the speed of convergence of the outer splitting is conserved in (7) if $s(k)$ becomes arbitrarily large as $k$ increases. Although this kind of asymptotic result may be of less importance in practice, it deserves attention since it shows what one can reasonably expect on the convergence of (7) without imposing additional conditions upon the inner and outer splittings.

The following definition can be found e.g. in [26].

**Definition 2.1.** Let $\{x_k\}_{k=0}^{\infty}$ be a sequence in $\mathbb{R}^n$ such that $\lim_{k\to\infty} x_k = x^*$. Then

$$\sigma(\{x_k\}_{k=0}^{\infty}) := \limsup_{k\to\infty} \|x_k - x^*\|^{1/k}$$

is the $R_1$-factor of the sequence $\{x_k\}_{k=0}^{\infty}$. This factor is independent of the choice of the norm $\|\cdot\|$.

If $S \in \mathbb{R}^{n\times n}$ is zero-convergent, and $\{x_k\}_{k=0}^{\infty}$ is generated by the stationary method

$$x_{k+1} = Sx_k + d,$$

then $\lim_{k\to\infty} x_k = (I - S)^{-1} d$, and $\sigma(\{x_k\}_{k=0}^{\infty}) = \rho(S)$; see e.g. [25, 37].

In order to examine the $R_1$-factor of sequences generated by the non-stationary method (7), we need the following two well known auxiliary results [25, 37].

**Lemma 2.2.** *Let $S \in \mathbb{R}^{n\times n}$. Then $\lim_{p\to\infty} S^p = O$ if and only if $\rho(S) < 1$.*

**Lemma 2.3.** *Given any $\varepsilon > 0$ there exists an operator norm $\|\cdot\|$ such that*

$$\rho(M^{-1}N) \leqq \|M^{-1}N\| \leqq \rho(M^{-1}N) + \varepsilon .$$

We are now able to prove the following

**Theorem 2.4.** *Let $A = M - N$ and $M = F - G$ be convergent splittings. Assume that $\lim_{k\to\infty} s(k) = \infty$ and let $\{x_k\}_{k=0}^{\infty}$ be the sequence generated by (7) with a given initial guess $x_0$. Then*

(*i*) $\lim_{k\to\infty} x_k = x^* \, (= A^{-1}b)$,

(*ii*) $\sigma(\{x_k\}_{k=0}^{\infty}) \leqq \rho(M^{-1}N) < 1$.

*Proof.* Let $\rho_0 := \rho(M^{-1}N)$ and $\rho_1 := \rho(F^{-1}G)$. By hypothesis, $\rho_0 < 1$ and $\rho_1 < 1$. The errors $e_k = x_k - x^*$ satisfy (5). Let $\varepsilon > 0$ and chose a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that the induced operator norm satisfies $\rho_0 \leqq \|M^{-1}N\| \leqq \rho_0 + \varepsilon$. This is possible by Lemma 2.3. Since $\rho_1 < 1$ we have $\lim_{p\to\infty}(F^{-1}G)^p = O$ (cf. Lemma 2.2) which implies that there is $p_0 \in \mathbb{N}$ such that $\|(F^{-1}G)^p\| \leqq \varepsilon$ for $p > p_0$. Since $\lim_{k\to\infty} s(k) = \infty$ there is $k_0 \in \mathbb{N}$ such that

$$\|(F^{-1}G)^{s(k)}\| \leqq \varepsilon \quad \text{for } k > k_0 .$$

For $k > k_0$ we thus have

$$\begin{aligned} \|T_{s(k)}\| &= \|(F^{-1}G)^{s(k)} + [I - (F^{-1}G)^{s(k)}]M^{-1}N\| \\ &\leqq \|(F^{-1}G)^{s(k)}\| + [1 + \|(F^{-1}G)^{s(k)}\|]\cdot\|M^{-1}N\| \\ &\leqq \varepsilon + (1+\varepsilon)(\rho_0 + \varepsilon) \\ &= \rho(\varepsilon) , \end{aligned}$$

with $\rho(\varepsilon) := \varepsilon + (1+\varepsilon)(\rho_0 + \varepsilon)$. If $\varepsilon > 0$ is chosen small enough, then $\rho(\varepsilon) < 1$. We then obtain

$$\|H_k\| \leqq \left(\prod_{i=0}^{k_0} \|T_{s(i)}\|\right)\cdot\rho(\varepsilon)^{k-k_0}, \quad k > k_0 \tag{8}$$

and thus $\lim_{k\to\infty} H_k = O$ which proves (i). Moreover, (8) yields

$$\|e_{k+1}\| \leqq \left(\prod_{i=0}^{k_0} \|T_{s(i)}\|\cdot\rho(\varepsilon)^{-k_0-1}\right)\cdot\rho(\varepsilon)^{k+1}\cdot\|e_0\|, \quad k > k_0 ,$$

from which

$$\limsup_{k\to\infty} \|e_k\|^{1/k} \leqq \rho(\varepsilon) ,$$

and thus (ii) follow since $\lim_{\varepsilon\to 0}\rho(\varepsilon) = \rho_0$. $\square$

This result shows that a non-stationary two-level method is (asymptotically) at least as fast as the iterative method induced by the only outer splitting, provided (6) holds. This is quite in contrast to an analogous situation occurring for super-linearly convergent two-level iterative methods for nonlinear systems. In this case, rather stringent additional assumptions have to be imposed to achieve the same *order* of convergence as the 'true' outer method [7, 33]. For example, Sherman [33] discusses a case for which the number of inner iterations should at least be doubled from step to step.

If (6) does not hold, the non-stationary two-stage method still converges, provided $s(k)$ is sufficiently large. The speed of convergence may then be slower than that of the outer method. We state this fact in an additional theorem, the proof of which follows almost verbatim that of Theorem 2.4 and is therefore omitted.

**Theorem 2.5.** *Let $A = M - N$ and $M = F - G$ be convergent splittings. Let $\|\cdot\|$ be any operator norm such that $\|M^{-1}N\| < 1$. Let $\tilde{s} \in \mathbb{N}$ be such that (see Lemma 2.2)*

$$\|(F^{-1}G)^s\| \leqq q < \frac{1 - \|M^{-1}N\|}{1 + \|M^{-1}N\|} \quad \text{for all } s \geqq \tilde{s}\,.$$

*Assume that $\liminf_{k\to\infty} s(k) > \tilde{s}$ and let $\{x_k\}_{k=0}^{\infty}$ be the sequence generated by (7) with a given initial guess $x_0$. Then*

*(i)* $\lim_{k\to\infty} x_k = x^*$ $(=A^{-1}b)$,

*(ii)* $\sigma(\{x_k\}_{k=0}^{\infty}) \leqq q + (1+q)\|M^{-1}N\| < 1$.

## 3. Splittings of *H*-matrices

We start the section with some basic definitions and then discuss basic properties of $M$- and $H$-matrices. We introduce $H$-splittings and $H$-compatible splittings, review others and investigate the relationship between these different splittings and their convergence.

We say that a vector $x$ is nonnegative (positive), denoted $x \geqq 0$ $(x > 0)$, if all its entries are nonnegative (positive). Similarly, a matrix $B$ is said to be nonnegative, denoted $B \geqq O$, if all its entries are nonnegative or, equivalently, if it leaves invariant the set of all nonnegative vectors. We compare two matrices $A \geqq B$, when $A - B \geqq O$, and two vectors $x \geqq y$ $(x > y)$ when $x - y \geqq 0$ $(x - y > 0)$. Given a matrix $A = (a_{ij})$, we define the matrix $|A| = (|a_{ij}|)$. It follows that $|A| \geqq O$ and that $|AB| \leqq |A|\,|B|$ for any two matrices $A$ and $B$ of compatible size.

Let $Z^{n\times n}$ denote the set of all real $n\times n$-matrices which have all non-positive off-diagonal entries. A nonsingular matrix $A \in Z^{n\times n}$ is called $M$-matrix if $A^{-1} \geqq O$.[2]

For any matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, we define its comparison matrix $\langle A\rangle = (\alpha_{ij})$ by[3]

$$\alpha_{ii} = |a_{ii}|, \qquad \alpha_{ij} = -\,|a_{ij}|, \quad i \neq j\,.$$

Following Ostrowski [27, 28], $A$ is said to be an $H$-matrix if $\langle A\rangle$ is an $M$-matrix. Of course, $M$-matrices are special $H$-matrices. In addition, as was already noted in [28], important classes of matrices such as strictly or irreducibly diagonally dominant matrices are contained in the class of all $H$-matrices. Actually, an

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[2] Some authors, e.g. Berman and Plemmons [5] distinguish between singular and nonsingular $M$-matrices. Our definition is equivalent to having $A = \beta I - B$ with $B \geqq O$ and $\beta > \rho(B)$, whereas for singular $M$-matrices one has $\beta = \rho(B)$ [5]. Here we follow Fan [8], Varga [37] and others [9] and include in the definition of $M$-matrix the nonsingularity of the matrix.

[3] Sometimes (see Berman and Plemmons [5] and Varga and his co-authors [2, 22, 38, 39]) the comparison matrix is denoted by $\mathcal{M}(A)$. We follow the notation in Neumaier [19–21], Mayer [17], and Frommer and Mayer [9].

$H$-matrix $A = (a_{ij})$ may be equivalently characterized by being generalized diagonally dominant, i.e.

$$|a_{ii}|u_i > \sum_{j \neq i} |a_{ij}|u_j, \quad i = 1, \ldots, n$$

for some vector $u = (u_1, \ldots, u_n)^T > 0$. This characterization follows directly from a result by Fan [8], which states that $A \in Z^{n \times n}$ is an $M$-matrix if and only if there is a positive vector $u$ such that $Au > 0$; see also Varga [38] and the references given therein.

$H$-matrices were studied by many authors in connection to iterative solutions of linear systems; see e.g. Alefeld [1], Beauwens [4], Frommer and Mayer [9], Krishna [13], Neumaier and Varga [22], Neumann [23], Ostrowski [28] and Robert [31]. $H$-matrices are always nonsingular [27] but, in contrast to $M$-matrices, $H$-matrices need not be monotone.

*Example 3.1.* Consider the matrix

$$A = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}, \ \langle A \rangle \text{ is an } M\text{-matrix, while } A^{-1} = \frac{1}{17}\begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix} \ngeq O .$$

We end this introduction to $M$- and $H$-matrices with an auxiliary lemma which we use later.

**Lemma 3.2.** *Let $A, B \in \mathbb{R}^{n \times n}$.*
*(a) If $A$ is an $M$-matrix, $B \in Z^{n \times n}$ and $A \leqq B$, then $B$ is an $M$-matrix.*
*(b) If $A$ is an $H$-matrix, then $|A^{-1}| \leqq \langle A \rangle^{-1}$.*
*(c) If $|A| \leqq B$ then $\rho(A) \leqq \rho(B)$.*

*Proof.* (a) and (c) can be found, e.g. in [26], 2.4.10 and 2.4.9, respectively. Part (b) goes back to Ostrowski [27] ; see also [9] or [19]. □

We collect old and new definitions of special splittings in the following

**Definition 3.3.** The splitting $A = M - N$ is called
(a) regular if $M^{-1} \geqq O$ and $N \geqq O$ [36, 37] ,
(b) weak regular[4] if $M^{-1} \geqq O$ and $M^{-1}N \geqq O$ [5, 26] ,
(c) $M$-splitting if $M$ is an $M$-matrix and $N \geqq O$ [16, 32] ,
(d) $H$-splitting if $\langle M \rangle - |N|$ is an $M$-matrix, and
(e) $H$-compatible splitting if $\langle A \rangle = \langle M \rangle - |N|$.

$H$-splittings were used in the context of interval matrices by Neumaier [19–21] and Mayer [17], where they are called strong splittings[5]. As we show in

[4] Ortega and Rheinboldt [26], Moré [18], and others, call this a left regular splitting, while reserving the name weak regular for a splitting with the additional hypothesis $NM^{-1} \geqq O$, and this is also how it is used in the proof of the convergence results in [14]. We choose here the most commonly used form; see e.g. Berman and Plemmons [5].
[5] $H$-splittings can in particular be weak splittings [15, 16], or weak regular splittings, and thus the name strong seems less appropriate in $\mathbb{R}^{n \times n}$.

Theorem 3.4, if $A = M - N$ is an $H$-splitting, $M$ is an $H$-matrix, and so is $A$. $H$-compatible splittings are used implicitly in Neumaier [19, Lemma 8] and in Frommer and Mayer [9, Theorem 4.1], but it appears that they have not been explicitly studied in the literature. The following theorem summarizes relations between different splittings and results on their convergence properties.

**Theorem 3.4.** *Let $A = M - N$ be a splitting.*

(*a*) *If the splitting is regular or weak regular, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geqq O$.*

(*b*) *If the splitting is an $M$-splitting, then $\rho(M^{-1}N) < 1$ if and only if $A$ is an $M$-matrix.*

(*c*) *If the splitting is an $H$-splitting, then $A$ and $M$ are $H$-matrices and $\rho(M^{-1}N) \leqq \rho(\langle M\rangle^{-1}|N|) < 1$.*

(*d*) *If the splitting is an $M$-splitting, then it is a regular splitting.*

(*e*) *If the splitting is an $M$-splitting and $A$ is an $M$-matrix, then it is an $H$-splitting and also an $H$-compatible splitting.*

(*f*) *If the splitting is an $H$-compatible splitting and $A$ is an $H$-matrix, then it is an $H$-splitting and thus convergent.*

*Proof.* (a) can be found, e.g. in [5, 37]. (d) follows from the definitions. To prove (b) assume first that $\rho(M^{-1}N) < 1$. From (a) we then have $A^{-1} \geqq O$, whereas from the definition of an $M$-splitting, $A \in Z^{n\times n}$. Hence, $A$ is an $M$-matrix. On the other hand, if $A$ is an $M$-matrix, we conclude that $\rho(M^{-1}N) < 1$ directly from (a). The first part of (c) was shown in [17, 19]. The inequality for the spectral radii follows by repeated application of Lemma 3.2(b) and (c) to obtain

$$\rho(M^{-1}N) \leqq \rho(|M^{-1}N|) \leqq \rho(|M^{-1}||N|) \leqq \rho(\langle M\rangle^{-1}|N|),$$

and from the fact that $\langle M\rangle - |N|$ is an $M$-splitting, whence by (d) and (a) $\rho(\langle M\rangle^{-1}|N|) < 1$. To show (e) just note that for an $M$-splitting of an $M$-matrix we have $A = \langle A\rangle = \langle M\rangle - |N| = M - N$. Finally, if we have an $H$-compatible splitting and $A$ is an $H$-matrix we see from $\langle M\rangle - |N| = \langle A\rangle$ that $\langle M\rangle - |N|$ is an $M$-matrix. This proves (f). □

The following example shows that an $H$-splitting of an $H$-matrix is not necessarily an $H$-compatible splitting.

*Example 3.5.* Consider the $H$-matrix $A$ of Example 3.1, and the splitting $A = M - N$ given by

$$M = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Then

$$\langle A\rangle \neq \langle M\rangle - |N| = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

which is an $M$-matrix.

As an immediate corollary to Theorem 3.4(f) we have that the (point) Jacobi as well as the Gauss-Seidel method (see e.g. [37]) applied to an $H$-matrix are both convergent. Indeed, since in these methods $M$ is the diagonal or lower triangular part of $A$, respectively, we have $H$-compatible splittings in both cases.

We mention in passing that SOR, SSOR and similar classical relaxation methods applied to an $H$-matrix are convergent if the relaxation parameter $\omega$ is positive and less than $2/(1+\rho(|J|))$, where $J$ is the iteration matrix of the Jacobi method; see e.g. Alefeld and Varga [2], Frommer and Mayer [9], Neumaier and Varga [22], or Varga [38].

## 4. Convergence results

In this section we show new proofs of the convergence theorems in [14] and present similar convergence results for splittings of $H$-matrices.

We begin with two lemmas on bounds for the spectral radius. The proof of the first can be found in [15, 16, 29] or [37], while that of the second in [30].

**Lemma 4.1.** *Let $T \geqq O$. If there exist $x > 0$ and a scalar $\alpha > 0$ such that $Tx \leqq \alpha x$, then $\rho(T) \leqq \alpha$. Moreover, if $Tx < \alpha x$, then $\rho(T) < \alpha$.*

**Lemma 4.2.** *Let $T_1, T_2, \ldots, T_q, \ldots$ be a sequence of nonnegative matrices in $\mathbb{R}^{n\times n}$. If there exist a real number $0 \leqq \theta < 1$, and a vector $x > 0$ in $\mathbb{R}^n$, such that*

$$T_s x \leqq \theta x, \ s = 1, 2, \ldots,$$

*then $\rho(H_q) \leqq \theta^q < 1$, where $H_q = T_q \ldots T_2 \cdot T_1$, and therefore $\lim_{q\to\infty} H_q = O$.*

**Theorem 4.3.** *Let $A = M - N$ be a convergent regular splitting, and let $M = F - G$ be a convergent weak regular splitting. Then, the stationary two-stage method* (3) *is convergent for any number $s \geqq 1$ of inner iterations.*

*Proof.* From the hypothesis on the splittings it follows that

$$T_s = (F^{-1}G)^s + \sum_{j=0}^{s-1} (F^{-1}G)^j F^{-1} N \geqq 0 ,$$

cf. (3) and (4). Consider any fixed vector $e > 0$ (e.g. with all components equal to 1), and $x = A^{-1}e$. Since by Theorem 3.4(a), $A^{-1} \geqq O$ and no row of $A^{-1}$ can have all null entries, then $x > 0$. Also, by the same arguments $F^{-1}e > 0$. Since $(I - M^{-1}N)x = M^{-1}Ax = M^{-1}e$, and $M^{-1} = (I - F^{-1}G)^{-1}F^{-1} = \sum_{j=0}^{\infty}(F^{-1}G)^j F^{-1}$, we have from (4) that

$$T_s x = x - \sum_{j=0}^{s-1} (F^{-1}G)^j F^{-1} e = x - F^{-1}e - \sum_{j=1}^{s-1} (F^{-1}G)^j F^{-1} e .$$

Moreover, since $T_s x \geqq 0$ and $x - F^{-1}e < x$, there exists $0 \leqq \theta < 1$ such that $x - F^{-1}e \leqq \theta x$. Thus $T_s x \leqq \theta x$, and by Lemma 4.1 $\rho(T_s) \leqq \theta < 1$. These bounds are independent of $s$. $\square$

We point out that in our hypothesis, the condition $GF^{-1} \geqq O$ is not needed; cf. [14].

**Theorem 4.4.** *Let $A = M - N$ be a convergent regular splitting, and let $M = F - G$ be a convergent weak regular splitting. Then, the non-stationary two-stage method* (7) *is convergent for any sequence $s(k) \geqq 1$, $k = 1, 2, \ldots$ of inner iterations.*

*Proof.* From the proof of Theorem 4.3 it follows that there exist a vector $x > 0$ and a scalar $0 \leqq \theta < 1$ such that $T_{s(k)}x \leqq \theta x$, $k = 1, 2, \ldots$ Therefore by Lemma 4.2 $\lim_{q \to \infty} H_q = O$. □

We turn to the case where $A$ is an $H$-matrix, and therefore not necessarily a monotone matrix. In the following theorems, the fact that $A$ is an $H$-matrix follows from Theorem 3.4(c).

**Theorem 4.5.** *Let $A = M - N$ be an $H$-splitting and let $M = F - G$ be an $H$-compatible splitting. Then, the stationary two-stage method* (3) *is convergent for any number $s \geqq 1$ of inner iterations.*

*Proof.* By Theorem 3.4(f) and (c), $F$ is an $H$-matrix. We use Lemma 3.2(b) to obtain the following bound.

$$
\begin{aligned}
|T_s| &\leqq (|F^{-1}||G|)^s + \sum_{j=0}^{s-1} (|F|^{-1}|G|)^j |F|^{-1}|N| \\
&\leqq (\langle F \rangle^{-1}|G|)^s + \sum_{j=0}^{s-1} (\langle F \rangle^{-1}|G|)^j \langle F \rangle^{-1}|N| \, .
\end{aligned} \tag{9}
$$

Let us denote by $\tilde{T}_s$ the matrix in (9). This is the iteration matrix of a stationary two-stage method for the matrix $\langle M \rangle - |N|$ with the regular splittings $\langle M \rangle - |N|$ and $\langle M \rangle = \langle F \rangle - |G|$, and with $s$ inner iterations. Thus, by Theorem 4.3 $\rho(\tilde{T}_s) < 1$, and by Lemma 3.2(c) $\rho(T_s) \leqq \rho(\tilde{T}_s) < 1$. □

**Theorem 4.6.** *Let $A = M - N$ be an $H$-splitting and let $M = F - G$ be an $H$-compatible splitting. Then, the non-stationary two-stage method* (7) *is convergent for any sequence $s(k) \geqq 1$, $k = 1, 2, \ldots$ of inner iterations.*

*Proof.* Let $H_q$ be defined as in (5). We use the bound (9) to obtain

$$
|H_q| \leqq |T_{s(q)}| \ldots |T_{s(2)}| \cdot |T_{s(1)}| \leqq \tilde{T}_{s(q)} \ldots \tilde{T}_{s(2)} \cdot \tilde{T}_{s(1)} \, . \tag{10}
$$

Let us denote by $\tilde{H}_q$ the matrix on the right hand side. As in the proof of Theorem 4.5, $\tilde{H}_q$ is the matrix corresponding to $q$ steps of a non-stationary two-stage method for the matrix $\langle M \rangle - |N|$ with the regular splittings $\langle M \rangle - |N|$ and $\langle M \rangle = \langle F \rangle - |G|$. Thus, by Theorem 4.4, $\lim_{q \to \infty} \tilde{H}_q = O$, and therefore by (10) $\lim_{k \to \infty} H_q = O$ as well. □

The following example shows that the hypothesis of Theorems 4.5 and 4.6 cannot be weakened. Namely, if the inner splitting is an $H$-splitting, but not $H$-compatible, there might not be convergence.

*Example 4.7.* Consider the $H$-matrix $A$ of Example 3.1, the $H$-splitting $A = M - N$ of Example 3.5, and the $H$-splitting $M = F - G$ given by $F = 2I$ and

$G = -I$. It is not an $H$-compatible splitting since $\langle F \rangle - |G| = I \neq M$. A simple calculation shows that

$$T_1 = \frac{1}{2}\begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad \rho(T_1) = \sqrt{\frac{5}{4}} > 1 .$$

It is interesting to contrast the hypotheses of Theorems 4.3 and 4.4 with those of Theorems 4.5 and 4.6. In the first case, the conditions on the outer splitting are more stringent than those on the inner splitting. The opposite is the case in the latter theorems. It is natural then to ask if Theorem 4.5 holds if the outer splitting is $H$-compatible and the inner is merely an $H$-splitting. The following example shows that this is not the case.

*Example 4.8.* Consider the $H$-compatible splitting of the $H$-matrix $A = M - N$ ($A$ is actually an $M$-matrix) given by

$$M = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

Consider further the $H$-splitting $M = F - G$ given in Example 4.7. A simple calculation shows that

$$T_1 = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \rho(T_1) = 1 .$$

We end the paper with an example that shows that a monotonicity result of the kind shown in [14] does not hold for splittings fulfilling the hypothesis of Theorem 4.5.

*Example 4.9.* Consider the $H$-matrix $A$ of Example 3.1, the $H$-splitting $A = M - N$ of Example 3.5, and the $H$-compatible splitting $M = F - G$ given by $F = 4I$ and $G = I$. One can then compute

$$T_1 = \frac{1}{4}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T_2 = -\frac{1}{16}\begin{bmatrix} 4 & -5 \\ 5 & 4 \end{bmatrix}, \quad \rho(T_1) = \frac{1}{4} < \rho(T_2) = \frac{\sqrt{41}}{16} .$$

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