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Efficient preconditioned NHSS iteration methods for solving complex symmetric linear systems

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ABSTRACT

Based on the new HSS (NHSS) iteration method introduced by Pour and Goughery (2015), we propose a preconditioned variant of NHSS (P*NHSS) and an efficient parameterized P*NHSS (PPNHSS) iteration methods for solving a class of complex symmetric linear systems. The convergence properties of the P*NHSS and the PPNHSS iteration methods show that the iterative sequences are convergent to the unique solution of the linear system for any initial guess when the parameters are properly chosen. Moreover, we discuss the quasi-optimal parameters which minimize the upper bounds for the spectral radius of the iteration matrices. Numerical results show that the PPNHSS iteration method is superior to several iteration methods whether the experimental optimal parameters are used or not.

1. Introduction and motivations

In this paper, we consider to solve the complex symmetric system of linear equations

$$Ax \equiv (W + iT)x = b, \tag{1.1}$$

where $i = \sqrt{-1}$ denotes the imaginary unit, and $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them being positive definite. Throughout the paper, without loss of generality, we assume that W is symmetric positive definite. The right-hand side vector $b \in \mathbb{C}^n$ is given and $x \in \mathbb{C}^n$ is unknown. In scientific computing and engineering applications, many problems depend on solving this kind of complex symmetric linear systems, we refer to [1–5] and the references therein.

In order to use iteration methods for solving the complex symmetric linear system (1.1), an efficient matrix splitting of the coefficient matrix A is necessary. A commonly used Hermitian and skew-Hermitian splitting (HSS) is given by

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^*) = W \quad \text{and} \quad S = \frac{1}{2}(A - A^*) = iT.$$

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It is obvious that the coefficient matrix A is non-Hermitian but positive definite. Here A^* is the conjugate transpose of the matrix A. Now we can straightforwardly solve the system (1.1) by the following HSS iteration method [6] with the scheme

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - iT)x^{(k)} + b, \\ (\alpha I + iT)x^{(k+1)} = (\alpha I - W)x^{(k+\frac{1}{2})} + b, \end{cases}$$

where α is a given positive constant and I is the identity matrix.

It is apparent that the matrix $\alpha I + iT$ is complex and non-Hermitian, which becomes a potential difficulty with the HSS iteration method for solving the shifted skew-Hermitian subsystem. See [7–13] for more details about the HSS iteration method and its variants.

To avoid the complex arithmetic, a modified HSS (MHSS) iteration method is introduced by Bai et al. [2]. To improve the speed of convergence, Bai et al. [14,15] further proposed the preconditioned MHSS (PMHSS) iteration method and applied it to the distributed control problems. The PMHSS iteration method is given by

$$\begin{cases} (\alpha V + W)x^{(k+\frac{1}{2})} = (\alpha V - iT)x^{(k)} + b, \\ (\alpha V + T)x^{(k+1)} = (\alpha V + iW)x^{(k+\frac{1}{2})} - ib, \end{cases}$$

where $V \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix, and the MHSS iteration method is just a special case when $V = I$.

Furthermore, Li et al. [16] developed a lopsided PMHSS iteration method for a class of complex symmetric linear systems when the matrix W is dominant. Li et al. [17] considered an asymmetric HSS iteration method, and developed a lopsided HSS iteration as follows:

$$\begin{cases} Wx^{(k+\frac{1}{2})} = -iTx^{(k)} + b, \\ (\alpha I + iT)x^{(k+1)} = (\alpha I - W)x^{(k+\frac{1}{2})} + b. \end{cases}$$

To avoid solving the linear subsystem with coefficient matrix $\alpha I + iT$ and accelerate the rate of convergence, Pour and Goughery [18] proposed the following new HSS (NHSS) iteration for non-Hermitian positive-definite linear systems:

$$\begin{cases} Wx^{(k+\frac{1}{2})} = -iTx^{(k)} + b, \\ (\alpha I + W)x^{(k+1)} = (\alpha I - iT)x^{(k+\frac{1}{2})} + b. \end{cases}$$

Recently, Li and Wu [19] proposed the following single-step HSS (SHSS) iteration method to solve the non-Hermitian positive definite linear systems:

$$(\alpha I + W)x^{(k+1)} = (\alpha I - iT)x^{(k)} + b.$$

Zeng and Ma [20] introduced a parameterized variant of the SHSS (PSHSS) iteration method, which is much more efficient than the SHSS iteration from their numerical experiments. The PSHSS iteration method is given by

$$(\alpha I + \omega W + T)x^{(k+1)} = [\alpha I - i(\omega T - W)]x^{(k)} + (\omega - i)b,$$

where α and ω are given positive constants. Based on the NHSS and the PSHSS iteration methods, Xiao and Yin [21] developed a parameterized variant of the NHSS (PNHSS) iteration method which is given by

$$\begin{cases} (\omega W + T)x^{(k+\frac{1}{2})} = -i(\omega T - W)x^{(k)} + (\omega - i)b, \\ (\alpha I + \omega W + T)x^{(k+1)} = [\alpha I - i(\omega T - W)]x^{(k+\frac{1}{2})} + (\omega - i)b. \end{cases}$$

Mention that the HSS-like iteration methods and their variants for solving systems of linear equations have been extended in a large number of literatures, we refer to [22–39]. Recently, a regularized HSS [40] and a simplified HSS [41] iteration methods are developed for solving saddle point problems.

On the other hand, let $x = u + iv$ and $b = p + iq$ with $u, v, p, q \in \mathbb{R}^n$. Then the complex linear system (1.1) is reduced to a 2-by-2 block real equivalent formulation

$$\begin{bmatrix} W & -T \\ T & W \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{1.2}$$

where u, v are unknown real vectors. For solving the real system (1.2), Salkuyeh et al. [42] proposed the following generalized successive overrelaxation (GSOR) iterative method

$$\begin{cases} Wu^{(k+1)} = (1-\alpha)Wu^{(k)} + \alpha Tv^{(k)} + \alpha p, \\ Wv^{(k+1)} = -\alpha Tu^{(k+1)} + (1-\alpha)Wv^{(k)} + \alpha q, \end{cases}$$

where $u^{(0)}$ and $v^{(0)}$ are the initial approximations of u and v, respectively.

In order to accelerate the rate of convergence, Hezari et al. [43] constructed a preconditioned variant of the GSOR (PGSOR) iterative method by multiplying both sides of the real equivalent system (1.2) by a nonsingular matrix $\mathcal{P}$

$$\mathcal{P}\begin{bmatrix} W & -T \\ T & W \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \mathcal{P}\begin{bmatrix} p \\ q \end{bmatrix}. \tag{1.3}$$

In particular, they mainly analyze the convergence properties of the PGSOR method with the preconditioner

$$\mathcal{P} = \begin{bmatrix} \omega I & I \\ -I & \omega I \end{bmatrix},$$

and solved the real equivalent system (1.3) by the following scheme

$$\begin{cases} (\omega W + T)u^{(k+1)} = (1-\alpha)(\omega W + T)u^{(k)} + \alpha(\omega T - W)v^{(k)} + \alpha(\omega p + q), \\ (\omega W + T)v^{(k+1)} = -\alpha(\omega T - W)u^{(k+1)} + (1-\alpha)(\omega W + T)v^{(k)} + \alpha(\omega q - p), \end{cases}$$

where α and ω are given positive constants.

In this paper, we want to further generalize the NHSS and the PNHSS iteration methods. Notice that the second-half step of the NHSS iteration method is just the first-half step of the MHSS iteration method, which inspires us to replace the second-half step of the NHSS iteration method by the first-half step of the PMHSS iteration method and establish the following preconditioned variant of NHSS (P*NHSS) iteration method.

Algorithm 1. The P*NHSS Iteration Method.

Given an arbitrary initial guess $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \ldots$ until $\{x^{(k)}\}$ converges, compute $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} Wx^{(k+\frac{1}{2})} = -iTx^{(k)} + b, \\ (\alpha V + W)x^{(k+1)} = (\alpha V - iT)x^{(k+\frac{1}{2})} + b. \end{cases} \tag{1.4}$$

where α is a given positive constant and $V \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix.

It is obvious that the NHSS iteration method is a special case of $V = I$. Next, multiplying a parameter $\omega - i$ by the two sides of the original linear equation (1.1), we obtain

$$(\omega - i)Ax \equiv [(\omega W + T) + i(\omega T - W)]x = (\omega - i)b. \tag{1.5}$$

Define

$$\tilde{A} = (\omega - i)A, \quad \tilde{W} = \omega W + T, \quad \tilde{T} = \omega T - W, \quad \tilde{b} = (\omega - i)b, \tag{1.6}$$

then we can rewrite (1.5) as

$$\tilde{A}x \equiv (\tilde{W} + i\tilde{T})x = \tilde{b}.$$

By comparing it with the original linear equation (1.1), we easily develop the following variant of P*NHSS iteration method:

$$\begin{cases} \tilde{W}x^{(k+\frac{1}{2})} = -i\tilde{T}x^{(k)} + \tilde{b}, \\ (\alpha V + \tilde{W})x^{(k+1)} = (\alpha V - i\tilde{T})x^{(k+\frac{1}{2})} + \tilde{b}. \end{cases} \tag{1.7}$$

Substituting (1.6) into (1.7), we get the parameterized variant of the P*NHSS (PPNHSS) iteration method.

Algorithm 2. The PPNHSS Iteration Method.

Given an arbitrary initial guess $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \ldots$ until $\{x^{(k)}\}$ converges, compute $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\omega W + T)x^{(k+\frac{1}{2})} = -i(\omega T - W)x^{(k)} + (\omega - i)b, \\ (\alpha V + \omega W + T)x^{(k+1)} = [\alpha V - i(\omega T - W)]x^{(k+\frac{1}{2})} + (\omega - i)b. \end{cases} \tag{1.8}$$

where α, ω are given positive constants, and $V \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix.

Thus the PNHSS iteration method is a special case of $V = I$. In this paper, we are going to study how to choose the proper parameters such that the P*NHSS iteration method is superior to the NHSS iteration method, or the PPNHSS iteration method is more efficient than the PNHSS iteration method.

The paper is organized as follows. In Section 2, the convergence properties of the P*NHSS iteration method are considered, including the convergence conditions, the spectral radius of the iterative matrix and the quasi-optimal parameter. In Section 3, the convergence properties of the PPNHSS iteration method are discussed. In Section 4, some numerical examples are given to evaluate the effectiveness of the P*NHSS and PPNHSS iteration methods by comparing with several other iteration methods. Finally, a brief conclusion is made in Section 5.

2. Convergence analysis about the P*NHSS iteration method

First, we rewrite the P*NHSS iteration scheme (1.4) as

$$x^{(k+1)} = M^V(\alpha)x^{(k)} + N^V(\alpha)b, \quad k = 0, 1, 2, \ldots,$$

where

$$\begin{aligned} M^V(\alpha) &= (\alpha V + W)^{-1}(\alpha V - iT)W^{-1}(-iT) \\ &= (\alpha I + V^{-1}W)^{-1}(\alpha I - iV^{-1}T)(V^{-1}W)^{-1}(-iV^{-1}T) \\ &= [V^{-\frac{1}{2}}(\alpha I + V^{-\frac{1}{2}}WV^{-\frac{1}{2}})V^{\frac{1}{2}}]^{-1}[V^{-\frac{1}{2}}(\alpha I - iV^{-\frac{1}{2}}TV^{-\frac{1}{2}})V^{\frac{1}{2}}] \\ &\quad \times [V^{-\frac{1}{2}}(V^{-\frac{1}{2}}WV^{-\frac{1}{2}})V^{\frac{1}{2}}]^{-1}[-iV^{-\frac{1}{2}}(V^{-\frac{1}{2}}TV^{-\frac{1}{2}})V^{\frac{1}{2}}], \\ &= V^{-\frac{1}{2}}(\alpha I + V^{-\frac{1}{2}}WV^{-\frac{1}{2}})^{-1}(\alpha I - iV^{-\frac{1}{2}}TV^{-\frac{1}{2}})(V^{-\frac{1}{2}}WV^{-\frac{1}{2}})^{-1}(-iV^{-\frac{1}{2}}TV^{-\frac{1}{2}})V^{\frac{1}{2}}, \\ N^V(\alpha) &= (\alpha V + W)^{-1}(\alpha V - iT)W^{-1} + (\alpha V + W)^{-1}, \end{aligned} \tag{2.1}$$

with $V \in \mathbb{R}^{n\times n}$ being a prescribed symmetric positive definite matrix.

For analysis of the convergence properties of the P*NHSS iteration method, we define

$$\tilde{\lambda}^V_{min} = \min_{\tilde{\lambda}_j \in sp(V^{-\frac{1}{2}}WV^{-\frac{1}{2}})} \{\tilde{\lambda}_j\} \quad \text{and} \quad \tilde{\sigma}^V_{max} = \max_{\tilde{\sigma}_j \in sp(V^{-\frac{1}{2}}TV^{-\frac{1}{2}})} \{\tilde{\sigma}_j\}, \tag{2.2}$$

where $sp(X)$ is the spectral set of the matrix X.

Theorem 1. *If $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them being positive definite, then the spectral radius $\rho(M^V(\alpha))$ is bounded by*

$$\delta^V(\alpha) = \frac{\tilde{\sigma}^V_{max}\sqrt{\alpha^2 + (\tilde{\sigma}^V_{max})^2}}{\tilde{\lambda}^V_{min}(\alpha + \tilde{\lambda}^V_{min})}. \tag{2.3}$$

Moreover,

(i) *if $\tilde{\lambda}^V_{min} \geq \tilde{\sigma}^V_{max}$, then $\delta^V(\alpha) < 1$ i.e. the iteration converges for any $\alpha > 0$;*

(ii) *if $\tilde{\lambda}^V_{min} < \tilde{\sigma}^V_{max}$ and $\tilde{\Delta} \equiv (\tilde{\lambda}^V_{min})^4 + (\tilde{\lambda}^V_{min})^2(\tilde{\sigma}^V_{max})^2 - (\tilde{\sigma}^V_{max})^4 > 0$, then $\delta^V(\alpha) < 1$ if and only if*

$$\alpha_1 \equiv \frac{(\tilde{\lambda}^V_{min})^3 - \tilde{\sigma}^V_{max}\sqrt{\tilde{\Delta}}}{(\tilde{\sigma}^V_{max})^2 - (\tilde{\lambda}^V_{min})^2} < \alpha < \frac{(\tilde{\lambda}^V_{min})^3 + \tilde{\sigma}^V_{max}\sqrt{\tilde{\Delta}}}{(\tilde{\sigma}^V_{max})^2 - (\tilde{\lambda}^V_{min})^2} \equiv \alpha_2;$$

(iii) *if $\tilde{\lambda}^V_{min} < \tilde{\sigma}^V_{max}$ and $\tilde{\Delta} \leq 0$, then $\delta^V(\alpha) \geq 1$ for any $\alpha > 0$.*

Proof. According to (2.1), it holds that

$$\begin{aligned} \rho(M^V(\alpha)) &= \rho((\alpha I + V^{-\frac{1}{2}}WV^{-\frac{1}{2}})^{-1}(\alpha I - iV^{-\frac{1}{2}}TV^{-\frac{1}{2}})(V^{-\frac{1}{2}}WV^{-\frac{1}{2}})^{-1}(-iV^{-\frac{1}{2}}TV^{-\frac{1}{2}})) \\ &\leq \|(\alpha I + V^{-\frac{1}{2}}WV^{-\frac{1}{2}})^{-1}(\alpha I - iV^{-\frac{1}{2}}TV^{-\frac{1}{2}})(V^{-\frac{1}{2}}WV^{-\frac{1}{2}})^{-1}(-iV^{-\frac{1}{2}}TV^{-\frac{1}{2}})\|_2 \\ &\leq \|(\alpha I + V^{-\frac{1}{2}}WV^{-\frac{1}{2}})^{-1}(\alpha I - iV^{-\frac{1}{2}}TV^{-\frac{1}{2}})\|_2\|(V^{-\frac{1}{2}}WV^{-\frac{1}{2}})^{-1}(-iV^{-\frac{1}{2}}TV^{-\frac{1}{2}})\|_2 \\ &\leq \|(\alpha I + V^{-\frac{1}{2}}WV^{-\frac{1}{2}})^{-1}\|_2\|\alpha I - iV^{-\frac{1}{2}}TV^{-\frac{1}{2}}\|_2\|(V^{-\frac{1}{2}}WV^{-\frac{1}{2}})^{-1}\|_2\|iV^{-\frac{1}{2}}TV^{-\frac{1}{2}}\|_2 \\ &= \Big(\max_{\tilde{\lambda}_j \in sp(V^{-\frac{1}{2}}WV^{-\frac{1}{2}})} \frac{1}{\alpha + \tilde{\lambda}_j}\Big)\Big(\max_{\tilde{\sigma}_j \in sp(V^{-\frac{1}{2}}TV^{-\frac{1}{2}})} \sqrt{\alpha^2 + \tilde{\sigma}_j^2}\Big) \times \frac{1}{\tilde{\lambda}^V_{min}} \times \tilde{\sigma}^V_{max} \\ &= \frac{\tilde{\sigma}^V_{max}\sqrt{\alpha^2 + (\tilde{\sigma}^V_{max})^2}}{\tilde{\lambda}^V_{min}(\alpha + \tilde{\lambda}^V_{min})}. \end{aligned}$$

Obviously, $\tilde{\lambda}^V_{min} \geq \tilde{\sigma}^V_{max}$ will lead to $\delta^V(\alpha) < 1$. When $\tilde{\lambda}^V_{min} < \tilde{\sigma}^V_{max}$, since $\delta^V(\alpha) < 1$ is equivalent to

$$[(\tilde{\sigma}^V_{max})^2 - (\tilde{\lambda}^V_{min})^2]\alpha^2 - 2(\tilde{\lambda}^V_{min})^3\alpha + [(\tilde{\sigma}^V_{max})^4 - (\tilde{\lambda}^V_{min})^4] < 0,$$

then $\delta^V(\alpha) < 1$ if and only if $\alpha_1 < \alpha < \alpha_2$ and the discriminant $\Delta = 4\tilde{\sigma}^2_{max}(\omega)\tilde{\Delta} > 0$. Hence the results in case (ii) and case (iii) are true. □

Corollary 1. *Under the conditions of Theorem 1, the optimal parameter α_*^V minimizing the upper bound $\delta^V(\alpha)$ of the spectral radius $\rho(M^V(\alpha))$ is*

$$\alpha_*^V = arg\min_{\alpha}\left\{\frac{\tilde{\sigma}_{max}^V\sqrt{\alpha^2+(\tilde{\sigma}_{max}^V)^2}}{\tilde{\lambda}_{min}^V(\alpha+\tilde{\lambda}_{min}^V)}\right\} = \frac{(\tilde{\sigma}_{max}^V)^2}{\tilde{\lambda}_{min}^V}, \tag{2.4}$$

and

$$\delta^V(\alpha_*^V) = \frac{(\tilde{\sigma}_{max}^V)^2}{\tilde{\lambda}_{min}^V\sqrt{(\tilde{\lambda}_{min}^V)^2+(\tilde{\sigma}_{max}^V)^2}}. \tag{2.5}$$

Proof. By simple calculations, we get

$$\delta^V(\alpha)' = \frac{\tilde{\sigma}_{max}^V}{\tilde{\lambda}_{min}^V}\frac{\alpha\tilde{\lambda}_{min}^V-(\tilde{\sigma}_{max}^V)^2}{(\alpha+\tilde{\lambda}_{min}^V)^2\sqrt{\alpha^2+(\tilde{\sigma}_{max}^V)^2}},$$

which implies that the upper bound $\delta^V(\alpha)$ of the spectral radius $\rho(M^V(\alpha))$ achieves its minimum at $\alpha_*^V = (\tilde{\sigma}_{max}^V)^2/\tilde{\lambda}_{min}^V$, i.e., (2.4) holds true. Thus, we obtain the minimum value $\delta^V(\alpha_*^V)$ by substituting α_*^V into (2.3). □

Moreover, when $\tilde{\lambda}_{min}^V \geq \tilde{\sigma}_{max}^V$, if the optimal parameter α_*^V is chosen, then it holds that

$$\rho(M^V(\alpha_*^V)) \leq \delta^V(\alpha_*^V) \leq \frac{\sqrt{2}}{2}.$$

Remark 1. According to (2.5), theoretically the optimal symmetric positive definite matrix V_* minimizing $\delta^V(\alpha_*^V)$ is given by

$$V_* = \underset{V}{argmin}\left\{\frac{(\tilde{\sigma}_{max}^V)^2}{\tilde{\lambda}_{min}^V\sqrt{(\tilde{\lambda}_{min}^V)^2+(\tilde{\sigma}_{max}^V)^2}}\right\} = \underset{V}{argmin}\left\{\frac{\tilde{\sigma}_{max}^V}{\tilde{\lambda}_{min}^V}\right\},$$

since $f(x) \equiv \frac{x^2}{\sqrt{1+x^2}}$ is a strictly monotonic increasing function. Practically we usually choose the positive definite matrix W as the matrix V. If both W and T are positive definite, then the better matrix V can be chosen in terms of

$$\frac{\tilde{\sigma}_{max}^V}{\tilde{\lambda}_{min}^V} = \min\left\{\frac{\tilde{\sigma}_{max}^W}{\tilde{\lambda}_{min}^W}, \frac{\tilde{\sigma}_{max}^T}{\tilde{\lambda}_{min}^T}\right\} = \min\left\{\tilde{\sigma}_{max}^W, \frac{1}{\tilde{\lambda}_{min}^T}\right\}.$$

In particular, if we take $V = W$, then the P*NHSS iteration scheme (1.4) becomes

$$\begin{cases} Wx^{(k+\frac{1}{2})} = -iTx^{(k)} + b, \\ (\alpha+1)Wx^{(k+1)} = (\alpha W - iT)x^{(k+\frac{1}{2})} + b. \end{cases} \tag{2.6}$$

and

$$M^W(\alpha) = \frac{1}{\alpha+1}(\alpha I - iW^{-1}T)(-iW^{-1}T), \quad N^W(\alpha) = W^{-1} - \frac{1}{\alpha+1}iW^{-1}TW^{-1}.$$

Since $\tilde{\lambda}_{min}^W = 1$, the right hand sides of (2.3)–(2.5) can be simplified, and we obtain the following results directly.

Proposition 1. *If we take $V = W$ for the P*NHSS iteration method, then the spectral radius $\rho(M^W(\alpha))$ is bounded by*

$$\delta^W(\alpha) = \frac{\tilde{\sigma}_{max}^W\sqrt{\alpha^2+(\tilde{\sigma}_{max}^W)^2}}{\alpha+1}. \tag{2.7}$$

Thus, if $\tilde{\sigma}_{max}^W \leq 1$, then $\delta^W(\alpha) < 1$ i.e. the iteration converges for any $\alpha > 0$.

Remark 2. If $\tilde{\sigma}_{max}^W > 1$, then $\delta^W(\alpha) < 1$ is equivalent to

$$(\tilde{\sigma}_{max}^W)^2[\alpha^2+(\tilde{\sigma}_{max}^W)^2] - (\alpha+1)^2 < 0,$$

which holds true if and only if

$$\tilde{\Delta} \equiv -(\tilde{\sigma}_{max}^W)^4 + (\tilde{\sigma}_{max}^W)^2 + 1 > 0, \quad \text{i.e.} \quad 1 < (\tilde{\sigma}_{max}^W)^2 < \frac{1+\sqrt{5}}{2},$$

and

$$\frac{1-\tilde{\sigma}_{max}^{W}\sqrt{\tilde{\Delta}}}{(\tilde{\sigma}_{max}^{W})^2-1}<\alpha<\frac{1+\tilde{\sigma}_{max}^{W}\sqrt{\tilde{\Delta}}}{(\tilde{\sigma}_{max}^{W})^2-1}.$$

Therefore, if $(\tilde{\sigma}_{max}^{W})^2\geq\frac{1+\sqrt{5}}{2}$, then the P*NHSS iteration method may not converge for any $\alpha>0$.

Furthermore, according to Corollary 1, the optimal parameter α_*^W minimizing the upper bound $\delta^W(\alpha)$ of the spectral radius $\rho(M^W(\alpha))$ is

$$\alpha_*^W=arg\min_{\alpha}\left\{\frac{\tilde{\sigma}_{max}^{W}\sqrt{\alpha^2+(\tilde{\sigma}_{max}^{W})^2}}{\alpha+1}\right\}=(\tilde{\sigma}_{max}^{W})^2, \tag{2.8}$$

and the minimal value is

$$\delta^W(\alpha_*^W)=\frac{(\tilde{\sigma}_{max}^{W})^2}{\sqrt{1+(\tilde{\sigma}_{max}^{W})^2}}. \tag{2.9}$$

Since $\delta^W(\alpha_*^W)$ is strictly monotonic increasing with respect to $\tilde{\sigma}_{max}^{W}$, then the smaller the $\tilde{\sigma}_{max}^{W}$ is, the faster the P*NHSS iteration method converges for $V=W$.

Remark 3. When $V=I$, the P*NHSS iteration method (1.4) is reduced to the NHSS iteration [18], and the minimal upper bound $\delta^I(\alpha)$ of the spectral radius $\rho(M^I(\alpha))$ for the NHSS iteration is given by

$$\delta^I(\alpha_*^I)=\frac{(\tilde{\sigma}_{max}^{I})^2}{\tilde{\lambda}_{min}^{I}\sqrt{(\tilde{\lambda}_{min}^{I})^2+(\tilde{\sigma}_{max}^{I})^2}}=\frac{\left(\frac{\tilde{\sigma}_{max}^{I}}{\tilde{\lambda}_{min}^{I}}\right)^2}{\sqrt{1+\left(\frac{\tilde{\sigma}_{max}^{I}}{\tilde{\lambda}_{min}^{I}}\right)^2}}, \tag{2.10}$$

where $\tilde{\lambda}_{min}^{I}$ is the smallest eigenvalue of matrix W and $\tilde{\sigma}_{max}^{I}$ is the largest eigenvalue of matrix T. While from (2.9) and (2.10), using the fact that

$$\tilde{\sigma}_{max}^{W}=\|W^{-1/2}TW^{-1/2}\|_2\leq(\|W^{-1/2}\|_2)^2\|T\|_2=\|W^{-1}\|_2\|T\|_2=\frac{\tilde{\sigma}_{max}^{I}}{\tilde{\lambda}_{min}^{I}},$$

we obtain

$$\delta^W(\alpha_*^W)\leq\delta^I(\alpha_*^I),$$

since $f(x)\equiv\frac{x^2}{\sqrt{1+x^2}}$ is a strictly monotonic increasing function. Therefore, the P*NHSS iteration method using $V=W$ would be more efficient than the NHSS iteration when the optimal parameters are used.

3. Convergence analysis about the PPNHSS iteration method

In this section, we discuss the convergence properties of the PPNHSS iteration method and derive the optimal parameters α_*^V, ω_*^V. First, we rewrite the PPNHSS iteration scheme (1.8) as

$$x^{(k+1)}=M^V(\alpha,\omega)x^{(k)}+N^V(\alpha,\omega)(\omega-i)b,\quad k=0,1,2,\ldots,$$

where

$$\begin{aligned}M^V(\alpha,\omega)&=(\alpha V+\omega W+T)^{-1}[\alpha V-i(\omega T-W)](\omega W+T)^{-1}[-i(\omega T-W)]\\&=[\alpha I+V^{-1}(\omega W+T)]^{-1}[\alpha I-iV^{-1}(\omega T-W)]\\&\quad\times[V^{-1}(\omega W+T)]^{-1}[-iV^{-1}(\omega T-W)]\end{aligned} \tag{3.1}$$

and

$$N^V(\alpha,\omega)=(\alpha V+\omega W+T)^{-1}[\alpha V-i(\omega T-W)](\omega W+T)^{-1}+(\alpha V+\omega W+T)^{-1},$$

with α,ω being given positive constants.

Let

$$\hat{\lambda}_{min}^{V}(\omega)=\min_{\hat{\lambda}_j\in sp(V^{-\frac{1}{2}}(\omega W+T)V^{-\frac{1}{2}})}\{\hat{\lambda}_j\}\quad\text{and}\quad\hat{\sigma}_{max}^{V}(\omega)=\max_{\hat{\sigma}_j\in sp(V^{-\frac{1}{2}}(\omega T-W)V^{-\frac{1}{2}})}\{|\hat{\sigma}_j|\}. \tag{3.2}$$

Since $\tilde{W}=\omega W+T$ is a symmetric positive definite matrix, then according to the convergence analysis about the P*NHSS iteration method given in Section 2, we can obtain the following results.

Theorem 2. *If $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them being positive definite, then the spectral radius $\rho(M^V(\alpha,\omega))$ is bounded by*

$$\delta^V(\alpha,\omega) = \frac{\hat{\sigma}^V_{max}(\omega)\sqrt{\alpha^2 + (\hat{\sigma}^V_{max}(\omega))^2}}{\hat{\lambda}^V_{min}(\omega)(\alpha + \hat{\lambda}^V_{min}(\omega))}. \tag{3.3}$$

Moreover, it is easy to see that

(i) *if $\hat{\lambda}^V_{min}(\omega) \geq \hat{\sigma}^V_{max}(\omega)$, then $\delta^V(\alpha,\omega) < 1$ i.e. the iteration converges for any $\alpha > 0$;*

(ii) *if $\hat{\lambda}^V_{min}(\omega) < \hat{\sigma}^V_{max}(\omega)$ and $\tilde{\Delta} \equiv (\hat{\lambda}^V_{min}(\omega))^4 + (\hat{\lambda}^V_{min}(\omega))^2(\hat{\sigma}^V_{max}(\omega))^2 - (\hat{\sigma}^V_{max}(\omega))^4 > 0$, then $\delta^V(\alpha,\omega) < 1$ if and only if*

$$\alpha_1 \equiv \frac{(\hat{\lambda}^V_{min}(\omega))^3 - \hat{\sigma}^V_{max}(\omega)\sqrt{\tilde{\Delta}}}{(\hat{\sigma}^V_{max}(\omega))^2 - (\hat{\lambda}^V_{min}(\omega))^2} < \alpha < \frac{(\hat{\lambda}^V_{min}(\omega))^3 + \hat{\sigma}^V_{max}(\omega)\sqrt{\tilde{\Delta}}}{(\hat{\sigma}^V_{max}(\omega))^2 - (\hat{\lambda}^V_{min}(\omega))^2} \equiv \alpha_2;$$

(iii) *if $\hat{\lambda}^V_{min}(\omega) < \hat{\sigma}^V_{max}(\omega)$ and $\tilde{\Delta} \leq 0$, then $\delta^V(\alpha,\omega) \geq 1$ for any $\alpha > 0$.*

Corollary 2. *Under the conditions of Theorem 2, the optimal parameter $\alpha^V_*(\omega)$ minimizing the upper bound $\delta^V(\alpha,\omega)$ of the spectral radius $\rho(M^V(\alpha,\omega))$ is*

$$\alpha^V_*(\omega) = \underset{\alpha}{argmin}\left\{\frac{\hat{\sigma}^V_{max}(\omega)\sqrt{\alpha^2 + (\hat{\sigma}^V_{max}(\omega))^2}}{\hat{\lambda}^V_{min}(\omega)(\alpha + \hat{\lambda}^V_{min}(\omega))}\right\} = \frac{(\hat{\sigma}^V_{max}(\omega))^2}{\hat{\lambda}^V_{min}(\omega)}, \tag{3.4}$$

and

$$\delta^V(\alpha^V_*(\omega),\omega) = \frac{(\hat{\sigma}^V_{max}(\omega))^2}{\hat{\lambda}^V_{min}(\omega)\sqrt{(\hat{\sigma}^V_{max}(\omega))^2 + (\hat{\lambda}^V_{min}(\omega))^2}}. \tag{3.5}$$

Moreover, when $\hat{\lambda}^V_{min}(\omega) \geq \hat{\sigma}^V_{max}(\omega)$, if we take the optimal parameter $\alpha^V_*(\omega)$, then it holds that

$$\rho(M^V(\alpha^V_*(\omega),\omega)) \leq \delta^V(\alpha^V_*(\omega),\omega) \leq \frac{\sqrt{2}}{2}.$$

Next define

$$\tilde{\lambda}^V_{max} = \max_{\tilde{\lambda}_j \in sp(V^{-\frac{1}{2}}WV^{-\frac{1}{2}})}\{\tilde{\lambda}_j\}, \qquad \tilde{\sigma}^V_{min} = \min_{\tilde{\sigma}_j \in sp(V^{-\frac{1}{2}}TV^{-\frac{1}{2}})}\{|\tilde{\sigma}_j|\}. \tag{3.6}$$

According to Theorem 2, the PPNHSS iteration method converges for any $\alpha > 0$ if $\hat{\lambda}^V_{min}(\omega) \geq \hat{\sigma}^V_{max}(\omega)$. Therefore, when $\tilde{\lambda}^V_{min} < \tilde{\sigma}^V_{max}$ and the P*NHSS iteration method does not converge, we can choose a proper ω satisfying $\hat{\lambda}^V_{min}(\omega) \geq \hat{\sigma}^V_{max}(\omega)$ so that the PPNHSS iteration method converges. In addition, since

$$\omega(V^{-\frac{1}{2}}WV^{-\frac{1}{2}}) + (V^{-\frac{1}{2}}TV^{-\frac{1}{2}}) = V^{-\frac{1}{2}}(\omega W + T)V^{-\frac{1}{2}},$$

and

$$\omega(V^{-\frac{1}{2}}TV^{-\frac{1}{2}}) - (V^{-\frac{1}{2}}WV^{-\frac{1}{2}}) = V^{-\frac{1}{2}}(\omega T - W)V^{-\frac{1}{2}},$$

we have the following results by similar analysis as in [20].

Theorem 3. *Let $\hat{\lambda}^V_{min}(\omega)$ and $\hat{\sigma}^V_{max}(\omega)$ be defined as in (3.2), $\tilde{\lambda}^V_{min}$ and $\tilde{\sigma}^V_{max}$, $\tilde{\lambda}^V_{max}$ and $\tilde{\sigma}^V_{min}$ be defined as in (2.2) and (3.6) respectively. Denote*

$$\bar{\omega}^V = \frac{\tilde{\lambda}^V_{min} + \tilde{\lambda}^V_{max}}{\tilde{\sigma}^V_{min} + \tilde{\sigma}^V_{max}}, \quad \omega^V_1 = \frac{\tilde{\lambda}^V_{min} + \tilde{\sigma}^V_{min}}{\tilde{\sigma}^V_{max} - \tilde{\lambda}^V_{min}}, \quad \omega^V_2 = \frac{\tilde{\lambda}^V_{max} - \tilde{\sigma}^V_{min}}{\tilde{\sigma}^V_{min} + \tilde{\lambda}^V_{min}}.$$

Then $\hat{\lambda}^V_{min}(\omega) \geq \hat{\sigma}^V_{max}(\omega)$ if any of the following conditions holds:

(1) *$\tilde{\lambda}^V_{min} \geq \tilde{\sigma}^V_{max}$ and $\omega \geq \bar{\omega}^V$;*

(2) *$\tilde{\lambda}^V_{min} < \tilde{\sigma}^V_{max}$ and $\bar{\omega}^V \leq \omega \leq \omega^V_1$;*

(3) *$\omega^V_2 \leq \omega < \bar{\omega}^V$.*

Moreover, the quasi-optimal parameter ω^V_* satisfying $\hat{\lambda}^V_{min}(\omega^V_*) \geq \hat{\sigma}^V_{max}(\omega^V_*)$ for the PPNHSS iteration method is given by

$$\omega^V_* = \bar{\omega}^V = \frac{\tilde{\lambda}^V_{min} + \tilde{\lambda}^V_{max}}{\tilde{\sigma}^V_{min} + \tilde{\sigma}^V_{max}}. \tag{3.7}$$

Remark 4. When $V = I$, the PPNHSS iteration is reduced to the PNHSS iteration [21]. According to Corollary 2, by similar analysis as in Remark 3, for comparing the PPNHSS iteration using $V = \tilde{W} = \omega W + T$ and the PNHSS iteration, we have

$$\delta^{\tilde{W}}(\alpha_*^{\tilde{W}}(\omega), \omega) \leq \delta^{I}(\alpha_*^{I}(\omega), \omega).$$

Therefore, the PPNHSS iteration using $V = \omega W + T$ would be more efficient than the PNHSS iteration when the optimal parameters $\alpha_*^{\tilde{W}}(\omega)$ and $\alpha_*^{I}(\omega)$ are used for the same ω.

Next, for simplicity, we also consider the special case of $V = W$, and the PPNHSS iteration scheme (1.8) becomes

$$\begin{cases} (\omega W + T)x^{(k+\frac{1}{2})} = -i(\omega T - W)x^{(k)} + (\omega - i)b, \\ [(\alpha + \omega)W + T]x^{(k+1)} = [\alpha W - i(\omega T - W)]x^{(k+\frac{1}{2})} + (\omega - i)b, \end{cases} \tag{3.8}$$

which can be rewritten as

$$x^{(k+1)} = M^{W}(\alpha, \omega)x^{k} + N^{W}(\alpha, \omega)(\omega - i)b,$$

where

$$\begin{aligned} M^{W}(\alpha, \omega) &= [(\alpha + \omega)W + T]^{-1}[\alpha W - i(\omega T - W)](\omega W + T)^{-1}[-i(\omega T - W)] \\ &= [(\alpha + \omega)I + W^{-1}T]^{-1}[\alpha I - i(\omega W^{-1}T - I)] \\ &\quad \times (\omega I + W^{-1}T)^{-1}[-i(\omega W^{-1}T - I)], \\ &= W^{-\frac{1}{2}}[(\alpha + \omega)I + W^{-\frac{1}{2}}TW^{-\frac{1}{2}}]^{-1}[\alpha I - i(\omega W^{-\frac{1}{2}}TW^{-\frac{1}{2}} - I)] \\ &\quad \times (\omega I + W^{-\frac{1}{2}}TW^{-\frac{1}{2}})^{-1}[-i(\omega W^{-\frac{1}{2}}TW^{-\frac{1}{2}} - I)]W^{\frac{1}{2}}, \\ N^{W}(\alpha, \omega) &= [(\alpha + \omega)W + T]^{-1}[\alpha W - i(\omega T - W)](\omega W + T)^{-1} + [(\alpha + \omega)W + T]^{-1}. \end{aligned} \tag{3.9}$$

Proposition 2. *If we take $V = W$ for the PPNHSS iteration method, then the spectral radius $\rho(M^{W}(\alpha, \omega))$ is bounded by*

$$\delta^{W}(\alpha, \omega) = \frac{\sqrt{\alpha^2 + \omega^2(\tilde{\sigma}_{max}^{W})^2 + 1}}{\alpha + \omega + \tilde{\sigma}_{min}^{W}} \frac{\sqrt{\omega^2(\tilde{\sigma}_{max}^{W})^2 + 1}}{\omega + \tilde{\sigma}_{min}^{W}}. \tag{3.10}$$

Moreover, the PPNHSS iteration using $V = W$ converges if any of the following conditions holds:
(1) *$\omega = 1$ for any $\alpha > 0$;*
(2) *ω satisfying $(\omega + \tilde{\sigma}_{min}^{W})^2 \geq \omega^2(\tilde{\sigma}_{max}^{W})^2 + 1$ for any $\alpha > 0$.*

Proof. We only prove case (1). According to (3.9), it is easy to see that

$$\rho(M^{W}(\alpha, \omega)) \leq \max_{\tilde{\sigma}_j \in sp(W^{-\frac{1}{2}}TW^{-\frac{1}{2}})} \frac{\sqrt{\alpha^2 + (\omega\tilde{\sigma}_j - 1)^2}}{\alpha + \omega + \tilde{\sigma}_j} \cdot \max_{\tilde{\sigma}_j \in sp(W^{-\frac{1}{2}}TW^{-\frac{1}{2}})} \frac{|\omega\tilde{\sigma}_j - 1|}{\omega + \tilde{\sigma}_j} \leq \delta^{W}(\alpha, \omega). \tag{3.11}$$

Furthermore, when $\omega = 1$ for any $\alpha > 0$, we have

$$\frac{\sqrt{\alpha^2 + (\omega\tilde{\sigma}_j - 1)^2}}{\alpha + \omega + \tilde{\sigma}_j} < 1 \quad \text{and} \quad \frac{|\omega\tilde{\sigma}_j - 1|}{\omega + \tilde{\sigma}_j} < 1$$

for any $\tilde{\sigma}_j \in sp(W^{-\frac{1}{2}}TW^{-\frac{1}{2}})$, which leads to $\rho(M^{W}(\alpha, \omega)) < 1$. □

Similar as Corollary 1, we have the following results.

Corollary 3. *Under the conditions of Proposition 2, the quasi-optimal parameter $\alpha_*^{W}(\omega)$ minimizing the upper bound $\delta^{W}(\alpha, \omega)$ of the spectral radius $\rho(M^{W}(\alpha, \omega))$ is*

$$\alpha_*^{W}(\omega) = \underset{\alpha}{argmin} \left\{ \frac{\sqrt{\alpha^2 + \omega^2(\tilde{\sigma}_{max}^{W})^2 + 1}}{\alpha + \omega + \tilde{\sigma}_{min}^{W}} \right\} = \frac{\omega^2(\tilde{\sigma}_{max}^{W})^2 + 1}{\omega + \tilde{\sigma}_{min}^{W}}, \tag{3.12}$$

and

$$\delta^{W}(\alpha_*^{W}(\omega), \omega) = \frac{\omega^2(\tilde{\sigma}_{max}^{W})^2 + 1}{(\omega + \tilde{\sigma}_{min}^{W})\sqrt{\omega^2(\tilde{\sigma}_{max}^{W})^2 + 1 + (\omega + \tilde{\sigma}_{min}^{W})^2}}. \tag{3.13}$$

Note that if the parameter ω is chosen such that $(\omega + \tilde{\sigma}_{min}^{W})^2 \geq \omega^2(\tilde{\sigma}_{max}^{W})^2 + 1$, then we obtain $\delta^{W}(\alpha_*^{W}(\omega), \omega) \leq \frac{1}{\sqrt{2}}$. On the other hand, according to the definitions in (2.2), (3.2) and (3.6), we have

$$\tilde{\lambda}_{min}^{W} = \tilde{\lambda}_{max}^{W} = 1, \quad \hat{\lambda}_{min}^{W}(\omega) = \omega + \tilde{\sigma}_{min}^{W},$$

and

$$\hat{\sigma}^{W}_{max}(\omega) = \max_{\tilde{\sigma}_j \in sp(W^{-\frac{1}{2}} T W^{-\frac{1}{2}})} \{|\omega\tilde{\sigma}_j - 1|\} \leq \sqrt{\omega^2(\tilde{\sigma}^{W}_{max})^2 + 1}.$$

Thus, the results of Theorem 2–3 can be simplified.

4. Numerical results

In this section, we want to test the effectiveness of the P*NHSS and the PPNHSS iteration methods for solving the complex symmetric linear system (1.1). Numerical comparisons with the PMHSS [14], the PGSOR [43], the PSHSS [20], the NHSS [18] and the PNHSS [21] iteration methods are also given to show the advantage of the P*NHSS and the PPNHSS iteration methods.

We compare these methods in terms of the number of iterations (denoted as IT) and the CPU times (denoted as CPU). Our experiments are carried out in MATLAB R2013a on Intel(R) Core(TM) CPU 1.8 × 2 Ghz and 4.00 GB of RAM. In each step of iteration for all these methods, we compute $x^{(k+1)}$ in the system $Bx^{(k+1)} = C^{(k)}$ by using $x^{(k+1)} = B \setminus C^{(k)}$ for the sparse matrices B and $C^{(k)}$. The CPU time is recorded by the command "tic-toc".

In our experiments, we always take the prescribed symmetric positive definite matrix $V = W$ for the PMHSS, the P*NHSS and the PPNHSS iteration methods. The initial value $x^{(0)} = u^{(0)} + iv^{(0)}$ is always chosen to be zero vector and the stopping criteria for all the methods are

$$\frac{\|b - Ax^{(k)}\|_2}{\|b\|_2} \leq 10^{-6},$$

where $x^{(k)} = u^{(k)} + iv^{(k)}$ is the current approximation.

Example 1 (*See [2,9]*)**.** Consider a system of linear equations in the form of $(W + iT)x = b$. The matrix $T \in \mathbb{R}^{n\times n}$ possesses the tensor-product form

$$T = I \otimes B_m + B_m \otimes I \quad \text{with} \quad B_m = tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m},$$

which implies that T is an $n \times n$ block-tridiagonal matrix with $n = m^2$. The matrix $W \in \mathbb{R}^{n\times n}$ is defined by

$$W = 10(I \otimes B_c + B_c \otimes I) + 9(e_1 e_m^T + e_m e_1^T) \otimes I \quad \text{with} \quad B_c = B_m - e_1 e_m^T - e_m e_1^T \in \mathbb{R}^{m\times m},$$

where e_1 and e_m are the first and mth unit basis vectors in $\mathbb{R}^m$, respectively. Actually, W is the five-point centered difference matrix approximating the negative Laplacian operator $L = -\Delta$ with periodic boundary conditions, on a uniform mesh in the square $[0, 1] \times [0, 1]$ with the mesh-size $h = 1/(m + 1)$. We choose the vector $b = (1 + i)A\mathbf{1}$, where $\mathbf{1}$ is a special vector of all entries equal to 1.

Example 2. Consider the following complex Helmholtz equation [5,16,20,43]

$$-\Delta u + \sigma_1 u + i\sigma_2 u = f,$$

where σ_1, σ_2 are real coefficient functions, and u satisfies Dirichlet boundary conditions in the square $D = [0, 1] \times [0, 1]$. We discretize this complex Helmholtz equation with finite differences on an $m \times m$ grid with mesh size $h = 1/(m + 1)$. Therefore, we obtain a system of linear equations

$$[(K + \sigma_1 I) + i\sigma_2 I]x = b,$$

where the matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form

$$K = I \otimes B_m + B_m \otimes I \quad \text{with} \quad B_m = \frac{1}{h^2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}.$$

In fact, K is the five-point centered difference matrix approximating the negative Laplacian operator $L = -\Delta$. In our tests, we take the right-hand side vector $b = (1 + i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. In addition, we normalize the system by multiplying both sides by h^2.

Example 3. Consider the following system of linear equations [2]

$$[(K - \varpi^2 M) + i(C_H + \varpi C_V)]x = b,$$

which comes from direct frequency domain analysis of an n-degree-of-freedom linear system. Here M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, respectively, and ϖ is the driving circular frequency. For more details, we refer to [4,44].

For simplicity, we only consider the linear system

$$\left[\left(K - \varpi^2 I\right) + i\left(\beta K + 10\varpi I\right)\right]x = b,$$

Table 1
The numerical results of Example 1 for different iteration methods.

Method		Grid				
		8 × 8	16 × 16	32 × 32	64 × 64	125 × 125
PMHSS	α_*	0.7	0.7	0.7	0.7	0.7
	IT	33	33	34	34	34
	CPU(s)	0.0284	0.1088	0.4439	1.9964	9.9445
PGSOR	α_*	0.99	0.99	0.99	0.99	0.99
	ω_*	10	10	10	10	10
	IT	4	4	4	4	4
	CPU(s)	0.0096	0.0220	0.0706	0.2879	1.3527
PSHSS	α_*	0.01	0.01	0.01	0.01	0.01
	ω_*	10	10	10	10	10
	IT	4	3	3	4	4
	CPU(s)	0.0074	0.0146	0.0491	0.3216	2.4573
NHSS	α_*	0.01	0.01	0.01	0.01	0.01
	IT	4	4	4	4	4
	CPU(s)	0.0082	0.0211	0.0825	0.4253 s	2.9891
PNHSS	α_*	0.01	0.01	0.01	0.01	0.01
	ω_*	10	10	10	10	10
	IT	2	2	2	2	2
	CPU(s)	0.0074	0.0166	0.0587	0.3161	2.4323
P*NHSS	α_*	0.01	0.01	0.01	0.01	0.01
	IT	4	4	4	4	4
	CPU(s)	0.0078	0.0211	0.0764	0.3155	1.4261
PPNHSS	α_*	0.01	0.01	0.01	0.01	0.01
	ω_*	10	10	10	10	10
	IT	2	2	2	2	2
	CPU(s)	0.0073	0.0159	0.0495	0.1985	0.8475

where ϖ, β are given real constants. The matrix $K \in \mathbb{R}^{n\times n}$ is the same as in Example 2, which possesses the tensor-product form

$$K = I \otimes B_m + B_m \otimes I \quad \text{with} \quad B_m = \frac{1}{h^2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}.$$

In our tests, the right-hand side vector b with its j-th entry b_j is given by

$$b_j = \frac{(1+i)j}{h^2(j+1)^2}, \quad j = 1, 2, \ldots, n.$$

Moreover, we normalize the system by multiplying both sides by h^2.

Tables 1–3 show the numerical results of various iteration methods with respect to different problem sizes for Examples 1–3 by using the experimental optimal parameters α_* and ω_*, respectively. The parameters $(\sigma_1, \sigma_2) = (-1, 1)$ and $(\varpi, \beta) = (0.2, 0.5)$ are considered in Examples 2 and 3, respectively. The results about iterations and CPU times are listed in the tables. It can be seen that the experimental optimal parameters of all the considered methods are stable, which will certainly save us much time and labor to find the optimal parameters, particularly for large linear systems. By comparison, we find that the P*NHSS method has a little advantage over the NHSS method from the point of view of iterations and CPU times, because of the introduction of prescribed symmetric positive definite matrix $V = W$. Similarly, the PPNHSS iteration method is a little more efficient than the PNHSS iteration method from the point of view of CPU times, because it is a combination of the P*NHSS and the PNHSS methods, and it inherits the merits of both. According to the results of Tables 1–3, we can conclude that the PMHSS and the NHSS iteration methods are less efficient than the other methods when the experimental optimal parameters are used.

In order to make further comparisons among the efficient PGSOR, the PSHSS, the PNHSS, the P*NHSS, and the PPNHSS methods, Tables 4–5 show the numerical results of these methods with respect to different problem sizes for Example 3 without using the experimental optimal parameters. For the sake of fairness, the non-optimal parameter α is always fixed, because the distances between 0.5 and the experimental optimal parameters α_* are almost the same for all these iteration methods. From Tables 4–5, it is easy to see that the PPNHSS method is the most efficient from the point of view of iterations and CPU times.

Overall, the numerical results confirm that the PPNHSS iteration method is very effective whether the experimental optimal parameters are used or not.

5. Conclusions

In this work, we introduce a preconditioned variant of NHSS (P*NHSS) and an efficient parameterized P*NHSS (PPNHSS) iteration methods for solving a class of complex symmetric linear systems. Theoretical analysis shows that, under a loose

Table 2
Numerical results of Example 2 for different iteration methods when $(\sigma_1, \sigma_2) = (-1, 1)$.

Method		Grid				
		8×8	16×16	32×32	64×64	125×125
PMHSS	α_*	0.9	0.9	0.9	0.9	0.9
	IT	40	40	40	40	41
	CPU(s)	0.0152	0.0406	0.1436	0.7213	4.1168
PGSOR	α_*	0.985	0.985	0.985	0.985	0.985
	ω_*	11	11	11	11	11
	IT	4	4	4	4	4
	CPU(s)	0.0073	0.0150	0.0396	0.1968	0.8523
PSHSS	α_*	0.01	0.01	0.01	0.01	0.01
	ω_*	11	11	11	11	11
	IT	6	6	6	6	11
	CPU(s)	0.0063	0.0121	0.0371	0.2494	2.4829
NHSS	α_*	0.01	0.01	0.01	0.01	0.01
	IT	3	3	3	3	3
	CPU(s)	0.0063	0.0120	0.0371	0.2406	2.2263
PNHSS	α_*	0.01	0.01	0.01	0.01	0.01
	ω_*	11	11	11	11	11
	IT	3	3	3	3	3
	CPU(s)	0.0062	0.0118	0.0369	0.2385	2.2022
P*NHSS	α_*	0.01	0.01	0.01	0.01	0.01
	IT	3	2	2	2	2
	CPU(s)	0.0062	0.0104	0.0258	0.1213	0.5576
PPNHSS	α_*	0.01	0.01	0.01	0.01	0.01
	ω_*	11	11	11	11	11
	IT	3	3	3	3	3
	CPU(s)	0.0061	0.0116	0.0299	0.1424	0.6183

Table 3
Numerical results of Example 3 for different iteration methods when $(\varpi, \beta) = (0.2, 0.5)$.

Method		Grid				
		8×8	16×16	32×32	64×64	125×125
PMHSS	α_*	0.75	0.75	0.75	0.75	0.75
	IT	21	21	21	21	21
	CPU(s)	0.0111	0.0236	0.0822	0.4416	2.3874
PGSOR	α_*	0.99	0.99	0.99	0.99	0.99
	ω_*	2	2	2	2	2
	IT	4	4	4	4	4
	CPU(s)	0.0068	0.0147	0.0351	0.1548	0.7202
PSHSS	α_*	0.001	0.001	0.001	0.001	0.001
	ω_*	2	2	2	2	2
	IT	6	5	5	5	8
	CPU(s)	0.0068	0.0116	0.0353	0.2245	2.3122
NHSS	α_*	0.1	0.001	0.001	0.001	0.001
	IT	12	13	12	11	11
	CPU(s)	0.0084	0.0187	0.0633	0.3542	2.9779
PNHSS	α_*	0.001	0.001	0.001	0.001	0.001
	ω_*	2	2	2	2	2
	IT	3	3	3	3	3
	CPU(s)	0.0066	0.0143	0.0362	0.2372	2.0907
P*NHSS	α_*	0.15	0.1	0.1	0.1	0.1
	IT	12	12	11	11	10
	CPU(s)	0.0077	0.0179	0.0524	0.2724	1.3206
PPNHSS	α_*	0.001	0.001	0.001	0.001	0.001
	ω_*	2	2	2	2	2
	IT	3	3	3	3	3
	CPU(s)	0.0059	0.0121	0.0307	0.1465	0.6695

restriction on the parameters α and ω, the iterative sequences are convergent to the unique solution of the linear system for any initial guess. Furthermore, the quasi-optimal values of the iteration parameters for the P*NHSS and the PPNHSS iteration methods are also discussed. Numerical results show that the PPNHSS iteration method outperforms several other iteration methods from the point of view of iterations and CPU times whether the experimental optimal parameters are used or not.

Table 4
Numerical results of Example 3 for different iteration methods with the fixed parameters $\alpha = 0.5$ and $\omega = 10$ when $(\varpi, \beta) = (0.2, 0.5)$.

Method		Grid				
		8×8	16×16	32×32	64×64	125×125
PGSOR	IT	20	20	20	20	20
	CPU(s)	0.0118	0.0252	0.0812	0.4025	2.1629
PSHSS	IT	16	18	39	120	404
	CPU(s)	0.0084	0.0198	0.0882	1.1928	21.1043
PNHSS	IT	9	9	12	14	14
	CPU(s)	0.0076	0.0172	0.0637	0.4143	3.2886
PPNHSS	IT	9	9	8	8	8
	CPU(s)	0.0073	0.0169	0.0479	0.2146	1.2148
P*NHSS	IT	12	11	11	10	10
	CPU(s)	0.0089	0.0187	0.0552	0.2407	1.3216

Table 5
Numerical results of Example 3 for different iteration methods with the fixed parameters $\alpha = 0.5$ and $\omega = 0.5$ when $(\varpi, \beta) = (0.2, 0.5)$.

Method		Grid				
		8×8	16×16	32×32	64×64	125×125
PGSOR	IT	20	20	20	20	20
	CPU(s)	0.0127	0.0264	0.0821	0.4042	2.2376
PSHSS	IT	36	93	306	1092	3828
	CPU(s)	0.0104	0.0479	0.4973	9.1620	180.2613
PNHSS	IT	19	22	27	32	34
	CPU(s)	0.0112	0.0261	0.1075	0.6798	5.0277
PPNHSS	IT	17	17	18	18	18
	CPU(s)	0.0101	0.0230	0.0778	0.3734	2.1164

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