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# SCHWARZ ANALYSIS OF ITERATIVE SUBSTRUCTURING ALGORITHMS FOR ELLIPTIC PROBLEMS IN THREE DIMENSIONS*

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*Dedicated to Seymour Parter on the occasion of his 65th birthday.*

**Abstract.** Domain decomposition methods provide powerful preconditioners for the iterative solution of the large systems of algebraic equations that arise in finite element or finite difference approximations of partial differential equations. The preconditioners are constructed from exact or approximate solvers for the same partial differential equation restricted to a set of subregions into which the given region has been divided. In addition, the preconditioner is often augmented by a coarse, second-level approximation that provides additional, global exchange of information that can enhance the rate of convergence considerably. The iterative substructuring methods, based on decompositions of the region into nonoverlapping subregions, form one of the main families of such algorithms.

Many domain decomposition algorithms can conveniently be described and analyzed as Schwarz methods. These algorithms are fully defined in terms of a set of subspaces and auxiliary bilinear forms. A general theoretical framework has previously been developed. In this paper, these techniques are used in an analysis of iterative substructuring methods for elliptic problems in three dimensions. A special emphasis is placed on the difficult problem of designing good coarse models and obtaining robust methods for which the rate of convergence is insensitive to large variations in the coefficients of the differential equation.

Domain decomposition algorithms can conveniently be built from modules that represent local and global components of the preconditioner. In this paper, a number of such possibilities are explored, and it is demonstrated how a great variety of fast algorithms can be designed and analyzed.

**Key words.** domain decomposition, finite elements, iterative substructuring, Schwarz methods

**AMS subject classifications.** 65N22, 65N30, 65N55

## 1. Introduction.

Domain decomposition algorithms are preconditioned iterative methods where the preconditioners are constructed from exact or approximate solvers for the given partial differential equation restricted to subregions, also called *substructures,* into which the given region is subdivided or from which it originally is assembled. Each of these subregions can naturally be associated with a set of nodes and a finite element subspace.

All algorithms of this kind known to us, which have satisfactory convergence properties for the case of many subregions, have one feature in common. In addition to subspaces and subproblems directly related to individual or small groups of adjacent

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substructures, there is a *global, coarse subspace.* Only a few global degrees of freedom per subregion are associated with this special subspace. As demonstrated in Widlund [62], using only simple arguments, the absence of such a subspace always results in slow convergence. This effect is also clearly evident in numerical experiments; cf., e.g., Smith [59]. We note that it is also quite natural to include additional levels; cf., e.g. Dryja, Sarkis, and Widlund [22], Dryja and Widlund [26], Xu [65], and Zhang [66]. However, in this paper, we will focus exclusively on two-level algorithms.

The design, analysis, and implementation of the coarse space problem pose the most challenging technical problems in work of this kind. In this paper, we demonstrate that it is profitable to view any coarse space as the range of an interpolation operator, often of a quite unconventional type, and that many questions in the analysis reduce to providing an estimate of the norm of this operator. In the study of the local components of the preconditioners, we can draw on the extensive knowledge of substructuring methods for a few subdomains; cf., e.g., Bjørstad and Widlund [3].

Throughout, we regard our methods as *Schwarz methods,* generalizations of the alternating method of Schwarz [54] discovered more than 120 years ago. Historically, Schwarz methods have primarily been associated with a division of the region into overlapping subregions. In recent years, research on this classical method and its additive variants has been quite active; cf., e.g., Dryja and Widlund [24], [25], Widlund [63], Matsokin and Nepomnyaschikh [45], and Nepomnyaschikh [46], [47]. It has been known for about five years that the *iterative substructuring* methods, based on a decomposition into nonoverlapping subregions, also fit well into a common Schwarz framework; see Dryja and Widlund [24]. This will be our point of view in this paper.

The idea behind the Schwarz methods is straightforward; the solution space $V$ is divided into subspaces $V_i$ and the solution in $V$ of the given problem is determined in an iteration by projecting the current error onto these subspaces. We can use projections $P_i$, which are orthogonal with respect to the bilinear form $a(\cdot,\cdot)$ naturally associated with the elliptic problem, or operators $T_i$ defined in terms of alternative bilinear forms $b_i(\cdot,\cdot)$ defined on $V_i \times V_i$. A particular choice of the subspaces and bilinear forms provides a complete mathematical description of a Schwarz algorithm. For recent work in which such a framework is developed and used, see Bramble, Pasciak, Wang, and Xu [8], Cai [10], Cai and Widlund [12], [13], Dryja and Widlund [24], [25], [27]–[30], Lions [36], Mathew [43], [44], Nepomnyaschikh [46], Pavarino [48], [49], Pavarino and Widlund [50], [51], Sarkis [53], Smith [56]–[59], Widlund [63], Xu [65], and Zhang [66]–[68]. In §2, we will demonstrate that rapid convergence of the iterative methods occurs if and only if all $u \in V$ can be decomposed into components in $V_i$, i.e., $u = \sum_i u_i, u_i \in V_i$, in such a way that $\sum b_i(u_i, u_i)$ can be bounded uniformly by a relatively small multiple of $a(u,u)$.

In this paper, we use the abstract Schwarz theory to develop a unified method for the design and analysis of a variety of fast iterative substructuring methods for problems in three dimensions. These methods form one of the major families of domain decomposition algorithms. For these methods, the communication of information between neighboring subdomains is confined to the exchange of values of the variables directly associated with the interfaces.

We both reexamine old algorithms, using the common framework, and also introduce new methods and estimates of their rates of convergence; to the best of our knowledge Algorithms 6.2, 6.7, and 6.10 and Theorems 6.2, 6.7, and 6.10 are new. Among the new methods are several preconditioners with condition numbers on the order of $(1+\log(H/h))$. These bounds are also independent of jumps in the coefficients

across subdomain boundaries.

The global coarse problem, of any two-level Schwarz method, is completely defined by selecting the subspace $V_0$ and the associated bilinear form $b_0(\cdot,\cdot)$. It would appear that a natural candidate for $V_0$ would be $V^H$, the space of continuous, piecewise linear functions using the substructures as elements. This approach is successful in the case of two dimensions, but for the three-dimensional problems considered in this paper quite unsatisfactory algorithms can result; see Bramble, Pasciak, and Schatz [6], [7], Smith [56], [57], and §6 for a discussion. In certain cases when the decomposition of the functions into subspaces is unique (i.e., when $V$ is a direct sum of the subspaces $V_i$), we necessarily obtain a poor bound on $b_0(u_0,u_0)$ and, as a consequence, a poor convergence rate. However, by introducing sufficient overlap between the local subspaces, rapidly convergent methods can be designed; see Dryja and Widlund [23], [24], [28], Smith [58], Widlund [63], and §6. A problem still remains for these algorithms, which use the $V^H$ space; it is not known if bounds for the condition number can be obtained that are independent of jumps in the coefficients of the differential operator. We note that recent results by Dryja, Sarkis, and Widlund [22] can be used to obtain polylogarithmic bounds on the condition number for certain algorithms that use this coarse space, for problems with coefficients that are *quasi-monotone.* Indeed, Theorem 6.3 will be valid for any problems with coefficients of this class.

An element of the space $V^H$ is defined completely by its values at the substructure vertices, with the values elsewhere obtained by linear interpolation; we therefore call such an algorithm *vertex based.* The alternative coarse spaces, considered in this paper, can also conveniently be characterized in terms of an interpolation and/or extension process. Some of them are defined by the values at the nodes shared by more than two subdomains, i.e., by the values on the *wire baskets* of the substructures; we call such algorithms *wire basket based.* These spaces can also be extended straightforwardly to more complicated substructures, which are not necessarily conventional large elements. We note that the first algorithms of this class were introduced in an important paper of Bramble, Pasciak, and Schatz [7]. Others can be called *face based*; the values on the different faces of the substructures are essential in determining the values of the interpolant. There are also many opportunities to create hybrid algorithms. We note that, recently, several of the algorithms of this paper have been modified for the case of higher-order spectral finite elements; cf. Pavarino and Widlund [50], [51].

As we have already noted, it is crucial to have a satisfactory, almost uniform bound on the energy of the coarse space interpolant. For rapid convergence, the coarse space interpolating operator should also reproduce the null space of the given elliptic operator; see Mandel [38], [39] or Smith [56]. For the case of scalar elliptic problems considered here, the null space contains only constants; for the three-dimensional linear elasticity operator it is the six-dimensional space of rigid body motions. Several examples of iterative substructuring algorithms, which satisfy both these requirements for problems in three dimensions, are given in the last section.

In this paper, we focus on scalar, self-adjoint, second-order elliptic problems, including those with large variations in the coefficients. The basic analysis is carried out for problems without a zero-order term; it is quite easy to extend the results to more general self-adjoint positive definite problems. We are also confident that much of the theory can be carried over to systems of elliptic equations, such as those of linear elasticity.

This paper is organized as follows. In the next section, we summarize an abstract theory for the Schwarz methods that has been developed in earlier work; see, in particular, Dryja and Widlund [29] for a recent overview of the theory. In §3, we introduce the elliptic problems and the finite element methods. We also introduce matrix notations, which are quite important in any discussion of the implementation of the algorithms. When analyzing the algorithms, we will work almost exclusively with the bilinear forms $b_i(\cdot,\cdot)$, but in an implementation, the matrix representations of the operators play the major role; we believe that both points of view are essential for a complete description of a Schwarz algorithm. In §4, we develop and collect the technical tools needed in the analysis of iterative substructuring algorithms in three dimensions. In §5, we discuss how various local solvers can be designed. Finally, in the last section, a variety of coarse solvers are introduced and the resulting algorithms are analyzed.

**2. Abstract theory for Schwarz methods.** A Schwarz algorithm defines an iterative method for the solution of linear systems of algebraic equations arising in the discretization of partial differential equations. The solution space is decomposed into subspaces, and the approximation of the solution is updated by using corrections obtained by projecting the error onto these subspaces. In practice, the basic iterative schemes are normally accelerated by a Krylov space method, the conjugate gradient method for symmetric problems, or, for instance, GMRES for problems with non-symmetric operators; cf. Hestenes [34] and Saad and Schultz [52], respectively. For a recent survey on Krylov space methods, see Freund, Golub, and Nachtigal [31].

**2.1. Additive and multiplicative Schwarz methods.** In this subsection, we outline an abstract convergence theory for the Schwarz methods. We have written on this topic before, most recently in Dryja and Widlund [29], where detailed proofs can be found.

Consider the following abstract variational problem: Find $u^* \in V$ such that

$$a(u^*, v) = f(v) \quad \forall v \in V.$$

The bilinear form $a(\cdot,\cdot)$ is symmetric, positive definite. We assume that there is a decomposition of the space $V$,

$$V = V_0 + V_1 + \cdots + V_N,$$

and that we are willing and can afford to solve problems of the following form: Given an inner product $b_i(\cdot,\cdot)$ defined on $V_i \times V_i$, and an element $w \in V$, find $T_i w$ such that

$$b_i(T_i w, v) = a(w, v) \quad \forall v \in V_i.$$

We note that when $b_i(\cdot,\cdot) = a(\cdot,\cdot)$ then $T_i w$ is the projection of $w$ onto $V_i$ that is orthogonal with respect to the energy inner product $a(\cdot,\cdot)$. We will generally refer to the $T_i$ as approximate projections.

If $u$ is an approximation to the solution $u^*$, then the approximate projection of the error, $u^* - u$, onto the subspace $V_i$ can be calculated by using the fact that

$$\begin{aligned} b_i(T_i(u^* - u), v) &= a(u^* - u, v) \quad \forall v \in V_i, \\ &= f(v) - a(u, v) \quad \forall v \in V_i. \end{aligned}$$

Thus, we can approximately project the error onto the subspaces, without knowing the true solution.

Several simple iterative methods can be built using the operators $T_i$. (Without limiting the generality of the methods, we assume that we are starting from a zero initial approximation.) The first method is the *multiplicative Schwarz method*:

$u^0 \leftarrow 0$
For $i = 0$ until convergence,
  $w \leftarrow u^i$
  For $j = 0$ to $N$,
    $w \leftarrow w + T_j(u^* - w)$
  End $j$
  $u^{i+1} \leftarrow w$
End $i$

We note that we can regard the algorithm as a simple iterative method for solving the equation

$$T_{\text{ms}} u^\star = g_{\text{ms}}$$

where the operator $T_{\text{ms}}$ satisfies

$$T_{\text{ms}} = I - (I - T_N) \dots (I - T_0).$$

This generally nonsymmetric operator equation can be solved with GMRES or a similar iterative method.

Since we are interested in using the conjugate gradient method, we will also consider the *symmetrized multiplicative Schwarz method*:

$u^0 \leftarrow 0$
For $i = 0$ until convergence,
  $w \leftarrow u^i$
  For $j = 0$ to $N$,
    $w \leftarrow w + T_j(u^* - w)$
  End $j$
  For $j = N$ to 0,
    $w \leftarrow w + T_j(u^* - w)$
  End $j$
  $u^{i+1} \leftarrow w$
End $i$

If we use this scheme to define a preconditioner for the conjugate gradient method, then the preconditioned operator is given by

$$\begin{aligned} T_{\text{sms}} &= I - (I - T_0) \cdots (I - T_{N-1})(I - T_N)(I - T_N)(I - T_{N-1}) \dots (I - T_0) \\ &= I - (I - T_{\text{ms}})^T (I - T_{\text{ms}}) = T_{\text{ms}} + T_{\text{ms}}^T - T_{\text{ms}}^T T_{\text{ms}}. \end{aligned}$$

We can simplify the algorithm by removing one of the factors $(I - T_N)$. If the exact projection $P_N$ onto $V_N$ is used, the algorithm remains exactly the same. In the general case, we can still obtain a, somewhat weaker, bound on the rate of convergence of the resulting algorithm by interpreting it as a multiplicative Schwarz method using the spaces

$$V_0, V_1, \dots, V_{N-1}, V_N, V_{N-1}, \dots, V_1, V_0.$$

The second main iterative scheme is the *additive Schwarz method*:

$u^0 \leftarrow 0$
For $i = 0$ until convergence,
    $u^{i+1} \leftarrow u^i + \tau \sum_j T_j(u^* - u^i)$
End $i$

Here $\tau$ is a scalar parameter chosen to ensure a good rate of convergence.

If we use this method to define a preconditioner for the conjugate gradient method, then the preconditioned operator is

$$T_a = \sum T_i.$$

We note that there are other interesting algorithms, based on the $T_i$, besides the multiplicative and additive Schwarz methods. Thus, with a balancing parameter $\gamma > 0$, Cai [11] advocates the use of the polynomial

$$\gamma T_0 + I - (I - T_N) \cdots (I - T_1).$$

This choice makes it possible to take advantage of the intrinsically more rapid convergence of a multiplicative method, while solving the special coarse problem at the same time as the local problems. In this way, one or several processors can work on the coarse problem while the rest of the processors are assigned to the local problems. We note that in the standard multiplicative algorithms, there is a potential bottleneck with many processors idly waiting for the solution of the coarse problem.

Still another interesting possibility is to replace $T_{\text{sms}}$ by the polynomial

$$T_{\text{ms}} + T_{\text{ms}}^T.$$

This operator, which is symmetric, has a larger smallest eigenvalue than $T_{\text{sms}}$ and an upper spectral bound of 4, while the largest eigenvalue of $T_{\text{sms}}$ is bounded by 2. In comparison with the symmetric multiplicative Schwarz method, this new algorithm involves only about half as many fractional steps per iteration if different processors can be assigned to the two parts of the operator.

There is a remarkably simple formula for $T_a^{-1}$ (cf. Zhang [66], [67]), which plays an important role in the understanding and systematic development of the theory.

LEMMA 2.1.

$$a(T_a^{-1}u, u) = \min_{\substack{u_i \in V_i \\ \sum u_i = u}} \sum b_i(u_i, u_i). \tag{1}$$

We note that if $b_0(u_0, u_0)$ is very large in comparison with $a(u, u)$ for some $u$ then there must exist a quite small eigenvalue of $T_a$, and the convergence of the conjugate gradient method can then suffer.

The abstract convergence theory centers around three parameters that measure the interactions of the subspaces $V_i$ and the bilinear forms $b_i(\cdot, \cdot)$, and their suitability in the construction of preconditioners.

We first consider the partitioning of the elements of $V$ and the first parameter.

• *Let $C_0$ be the minimum constant such that for all $u \in V$ there exists a representation $u = \sum u_i, u_i \in V_i$, with*

$$\sum_i b_i(u_i, u_i) \leq C_0^2 a(u, u). \tag{2}$$

We note that it is sometimes natural to make a distinction between the case when the decomposition is a direct sum, i.e., when each element of $u \in V$ is always uniquely represented by components in the $V_i$, and the case where there is some freedom in the choice of the decomposition of $u$.

The second parameter is given in terms of strengthened Cauchy–Schwarz inequalities. Here the angles between the different subspaces are measured. The space $V_0$, normally a global coarse space that intersects all the other spaces, is not included in these bounds.

- *Let $\mathcal{E}$ be the matrix of strengthened Cauchy–Schwarz coefficients, defined by*

$$|a(v_i, v_j)| \leq \epsilon_{ij} a(v_i, v_i)^{1/2} a(v_j, v_j)^{1/2} \quad \forall v_i \in V_i, \quad \forall v_j \in V_j, \quad i, j = 1, \ldots N, \tag{3}$$

*and let $\rho(\mathcal{E})$ be its spectral radius.*

The third parameter provides a bound on the norm of the operators $T_i$.

- *Let $\omega$ be the minimum constant such that*

$$a(u, u) \leq \omega b_i(u, u) \quad \forall u \in V_i, \quad i = 0, \ldots, N. \tag{4}$$

It is easy to see that we can choose $\omega = \max \|T_i\|_a$. We note that it is always possible to scale $b_i(\cdot, \cdot)$ so that $\omega \in [1, 2)$. Such a scaling will, of course, also affect the value of $C_0$.

Basic convergence results for additive and multiplicative variants of the Schwarz method can now be given. Results for the additive form are due primarily to Dryja and Widlund [23], [25] and Nepomnyaschikh [46], while the result for multiplicative Schwarz methods is a variant of results of Bramble, Pasciak, Wang, and Xu [8] and Xu [65]; see also Lions [36] for early work on the case of two subspaces. A proof of Theorem 2.2 is given in Dryja and Widlund [29], and a proof of Theorem 2.1 can also be derived directly from the results of the same paper.

THEOREM 2.1. *The abstract symmetric multiplicative Schwarz method satisfies*

$$\kappa(T_{\text{sms}}) \leq \frac{(1 + 2\hat{\omega}^2 \rho(\mathcal{E})^2) C_0^2}{2 - \hat{\omega}}.$$

*Here $\hat{\omega} = \max(1, \omega)$.*

THEOREM 2.2. *The abstract additive Schwarz method satisfies*

$$\kappa(T_a) \leq \omega(\rho(\mathcal{E}) + 1) C_0^2.$$

*In particular, $1/C_0^2$ is a sharp lower bound on the smallest eigenvalue of $T_a = \sum T_i$, and $\omega(\rho(\mathcal{E}) + 1)$ an upper bound on the largest eigenvalue.*

**2.2. Local analysis.** An example of the problems considered in detail in this paper is provided by

$$a(u, v) = \sum_j \int_{\Omega_j} \rho_j \nabla u \cdot \nabla v \, dx, \tag{5}$$

where $\rho_j > 0$ is a constant in $\Omega_j$, but with possibly large jumps between subdomains. We will develop our theory for the piecewise constant case, but all our results are equally valid for the case when the coefficients vary moderately in each subdomain. When $\rho_j \equiv 1$ $\forall j$, we have the special case of Poisson's equation. In order to be successful with problems that have large variations in the coefficients, it is important

to be able to carry out a local analysis. This can sometimes be done in a Schwarz framework:

Let $V^{(j)}$ be the restriction of the functions in the solution space $V$ to the subdomain $\Omega_j$. Decompose $V^{(j)}$ into subspaces $V_i^{(j)}$ and introduce bilinear forms $b_i^{(j)}(\cdot,\cdot)$ on $V_i^{(j)} \times V_i^{(j)}$, where the bilinear forms

$$b_i(u,v) = \sum_j b_i^{(j)}(u,v)$$

are obtained by subassembly, just as the bilinear form $a(\cdot,\cdot)$ can be obtained from

$$a^{(j)}(u,v) = \int_{\Omega_j} \rho_j \nabla u \cdot \nabla v \, dx;$$

see our further discussion in §3. The case when the space $V^{(j)}$ is a direct sum of the local subspaces $V_i^{(j)}$ has been considered by Mandel [38], [39]. He showed that if, for each $\Omega_j$, one of the local subspaces contains the null space of $a^{(j)}(\cdot,\cdot)$, then bounds on the condition number of the global preconditioned problem can be obtained from bounds for individual subdomains. The local bounds can be obtained by using the techniques outlined earlier in this section.

We formulate a related result that is also useful when the local subspaces do not form a direct sum decomposition.

LEMMA 2.2. *Assume that there exists constants $C_0^{(j)}$ so that for all $u \in V$ there exists a representation $u = \sum_i u_i, u_i \in V_i$, such that*

$$\sum_i b_i^{(j)}(u_i,u_i) \le C_0^{(j)2} a^{(j)}(u,u) \quad \forall j;$$

*then the $C_0$ of (2) is given by $C_0 = \max_j C_0^{(j)}$.*

*Proof.* This result follows immediately by summing over the substructures. □

In the case of piecewise constant coefficients, the construction of $b_i^{(j)}(\cdot,\cdot)$ is straightforward. We simply construct appropriate bilinear forms for the Laplacian and then scale them by multiplying by $\rho_j$. The bounds needed for the lemma follow immediately.

We note that an algorithm due to Smith [57] (Algorithm 6.4) and the standard iterative substructuring method (Algorithm 6.1) can be analyzed using substructure by substructure estimates. The overlapping additive Schwarz algorithms of Dryja and Widlund [23]–[25], [30] and Widlund [63] and another algorithm due to Smith [58] (Algorithm 6.3), on the other hand, are Schwarz methods for which we have been unable to perform an analysis using only local estimates.

**3. The elliptic problem and its discretization.** In this paper, we will consider only scalar, second-order, self-adjoint, coercive, bilinear forms $a(u,v)$ on $\Omega \subset R^3$, a Lipschitz region of diameter 1; in fact, to simplify matters, we assume, without limiting the generality of our theory, that the region is polyhedral. We impose a homogeneous Dirichlet condition on $\Gamma_0 \subset \partial\Omega$ and Neumann boundary conditions on $\Gamma_1 = \partial\Omega \setminus \Gamma_0$. We denote the subspace of $H^1(\Omega)$ with zero trace on $\Gamma_0$ by $H^1_{\Gamma_0}(\Omega)$. We assume that the set $\Gamma_0$ is of nonzero measure and that the underlying elliptic operator has no zero-order terms. The variational problem is then: Find $u^* \in H^1_{\Gamma_0}(\Omega)$ such that

$$a(u^*,v) = (f,v) \quad \forall v \in H^1_{\Gamma_0}(\Omega).$$

An example of such a problem, (5), which will serve as our model problem, has already been introduced in the preceding section.

The Sobolev space $H^1(\Omega)$ is closely related to our family of elliptic problems. This space is defined by the seminorm

$$|u|^2_{H^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla u \, dx \tag{6}$$

and the norm

$$\|u\|^2_{H^1(\Omega)} = |u|^2_{H^1(\Omega)} + \|u\|^2_{L_2(\Omega)}.$$

In the case of a region of diameter $H$, such as a substructure $\Omega_j$, we use a norm with different relative weights,

$$\|u\|^2_{H^1(\Omega_j)} = |u|^2_{H^1(\Omega_j)} + \frac{1}{H^2}\|u\|^2_{L_2(\Omega_j)}. \tag{7}$$

We introduce a discretization, which satisfies the usual rules for finite element triangulations such as shape regularity of the elements; cf. Ciarlet [16]. Let $V^h(\Omega)$ be the space of continuous, piecewise linear functions on this triangulation, which vanish on $\Gamma_0$. For the construction of the preconditioner, we assume that the set of elements is partitioned into subsets forming disjoint substructures $\Omega_j$. For many of the algorithms considered in this paper, the shapes of the substructures can be quite arbitrary. However, to simplify the analysis, we restrict our attention to the case where the $\Omega_i$ are shape-regular finite elements with a characteristic diameter $H$. We denote the *interface* between the subdomains by $\Gamma = \cup\partial\Omega_i \setminus \Gamma_0$. We also assume that the $\Gamma_0$ is the union of the closures of faces of some, or all, of the substructures.

The discrete problem is then of the form: Find $u^h \in V^h(\Omega)$ such that

$$a(u^h, v^h) = (f, v^h) \quad \forall\, v^h \in V^h(\Omega). \tag{8}$$

If we expand $u^h$ in the standard nodal basis, $u^h = \sum_k u_k \phi_k$, the variational problem (8) can be written as the linear system

$$K\underline{u} = \underline{f}.$$

The elements of the stiffness matrix $K$ are given by

$$K_{ij} = a(\phi_i, \phi_j),$$

and those of the right-hand side $f$ by

$$f_i = (f, \phi_i).$$

The local contributions to the stiffness matrix and the right-hand side can be formed one subdomain at a time. The stiffness matrix is then obtained by *subassembly* of these parts. We order the nodes interior to the subdomains first, followed by those on the interface $\Gamma$. All the matrices and vectors are expanded by zeros, giving them each the same dimension as the global stiffness matrix and the vector of unknowns. We can then write the linear system as

$$\begin{pmatrix} K_{II} & K_{IB} \\ K_{IB}^T & K_{BB} \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{u}_B \end{pmatrix} = \sum_j \begin{pmatrix} K_{II}^{(j)} & K_{IB}^{(j)} \\ K_{IB}^{(j)^T} & K_{BB}^{(j)} \end{pmatrix} \begin{pmatrix} \underline{u}_I^{(j)} \\ \underline{u}_B^{(j)} \end{pmatrix} = \sum_j \begin{pmatrix} \underline{f}_I^{(j)} \\ \underline{f}_B^{(j)} \end{pmatrix}.$$

Thus, to multiply $K$ by the vector $\underline{u}$, we first restrict the vectors $\underline{u}_I$ and $\underline{u}_B$ to the substructures, then multiply them by the stiffness matrices of the individual substructures, and, finally, obtain the product $K\underline{u}$ by padding with zeros and adding the resulting vectors.

In most discussions of Schwarz methods, there are technically two spaces: the space of finite element functions $V^h$, and the space of coefficients of the finite element functions. We will denote functions in $V^h$ by $u^h$, and the coefficient vectors of the finite element functions by $\underline{u}$.

In a first step of many iterative substructuring algorithms, the unknowns in the interior of the subdomains are eliminated. In this step, the Schur complements, with respect to the variables associated with the boundaries of the individual substructures, are calculated. The resulting linear system can be written as

$$\begin{pmatrix} K_{II} & K_{IB} \\ 0 & S \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{u}_B \end{pmatrix} = \sum_j \begin{pmatrix} K_{II}^{(j)} & K_{IB}^{(j)} \\ 0 & S_{BB}^{(j)} \end{pmatrix} \begin{pmatrix} \underline{u}_I^{(j)} \\ \underline{u}_B^{(j)} \end{pmatrix}$$
$$= \sum_j \begin{pmatrix} \underline{f}_I^{(j)} \\ \underline{f}_B^{(j)} - K_{IB}^{(j)^T} K_{II}^{(j)^{-1}} \underline{f}_I^{(j)} \end{pmatrix},$$

where

$$S^{(j)} = S_{BB}^{(j)} = K_{BB}^{(j)} - K_{IB}^{(j)^T} K_{II}^{(j)^{-1}} K_{IB}^{(j)},$$

and the reduced system is given by

$$S\underline{u}_B = \tilde{\underline{f}}_B.$$

Thus, the matrix $S$ is obtained from the $S^{(j)}$ by subassembly. In practice, the matrix $S$ is often not formed explicitly, since this is a potentially expensive operation. Instead, a sparse representation of the $K_{IB}^{(j)}$ and the sparse, triangular factors of the $K_{II}^{(j)}$ are stored, and the action of $S$ on a vector is calculated as needed.

The space of *discrete harmonic functions*, $\tilde{V}^h \subset V^h$, is an important subspace, which is directly related to the Schur complements and to the values at the nodes on $\Gamma$. These functions satisfy the linear relation $K_{II}\underline{u}_I + K_{IB}\underline{u}_B = 0$. It is easy to see that they are completely defined by their values on the interfaces, and that they are orthogonal, in the $a(\cdot,\cdot)$ inner product, to the spaces $V^h \cap H_0^1(\Omega_j)$. In the analysis to be given, the important inner product is the one induced by $S$; we will define our preconditioners with respect to the inner product $s(u^h, v^h) = \underline{u}_B^T S \underline{v}_B$, where $u^h$, $v^h \in \tilde{V}^h$, are discrete harmonic functions. We note that it is an elementary algebraic result that

$$s(u^h, u^h) = \min_{v^h|_\Gamma = u^h} a(v^h, v^h).$$

Thus the discrete harmonic extension is the extension that minimizes the energy.

We need to introduce notation related to certain geometrical objects, since the iterative substructuring algorithms are based on subspaces directly related to the substructures, faces, etc. Let $\Omega_{ij}$ be the union of two substructures, $\Omega_i$ and $\Omega_j$, which share a common face, and denote that face by $F^k$. Let $E^\ell$ represent an edge, $V^m$ a vertex of a substructure, and $W^j$ the wire basket of the subdomain $\Omega_j$; see Fig. 1. We note that a face in the interior of the region $\Omega$ is common to exactly

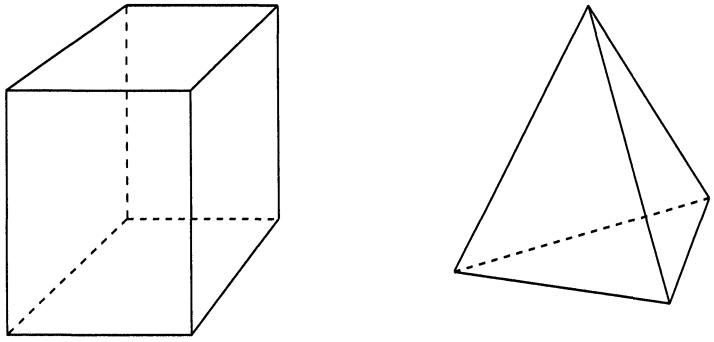

FIG. 1. *Faces, edges, and wire baskets.*

two substructures, an edge is shared by more than two, and a vertex is common to still more substructures. All the substructures, faces, and edges are regarded as open sets. The sets of nodes in $\Omega_j, F^k$, $E^\ell$, and $W^j$ are denoted by $\Omega_{i,h}, F_h^k, E_h^\ell$, and $W_h^j$, respectively.

The matrix $S$ can be represented as a block matrix with a block for each face, edge, and vertex. We often combine all the edge and all the vertex blocks of $\Omega_i$ into single blocks. We can also merge them all into a single block corresponding to the wire basket. We then obtain

$$S^{(j)} = \begin{pmatrix} S_{FF}^{(j)} & S_{FE}^{(j)} & S_{FV}^{(j)} \\ S_{FE}^{(j)^T} & S_{EE}^{(j)} & S_{EV}^{(j)} \\ S_{FV}^{(j)^T} & S_{EV}^{(j)^T} & S_{VV}^{(j)} \end{pmatrix} \tag{9}$$

and

$$S^{(j)} = \begin{pmatrix} S_{FF}^{(j)} & S_{FW}^{(j)} \\ S_{FW}^{(j)^T} & S_{WW}^{(j)} \end{pmatrix}, \tag{10}$$

respectively. Here $S_{FF}^{(j)}$ is constructed from the blocks that correspond to the individual faces, and to pairs of faces, of $\Omega_j$, etc. We will use both block structures in the description of different algorithms, as appropriate.

All of the algorithms considered in this paper can be formulated by using inexact interior solvers. We explain briefly how this can be done. The exact inverse $K^{-1}$ can be written as

$$K^{-1} = \begin{pmatrix} I & -K_{II}^{-1}K_{IB} \\ 0 & I \end{pmatrix} \begin{pmatrix} K_{II}^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -K_{BI}K_{II}^{-1} & I \end{pmatrix}.$$

If we have a good preconditioner for $S$, $B_S^{-1}$, and a good preconditioner for $K_{II}$ (i.e., an approximate solver $B_I^{-1}$ for the interior problems), we can create a preconditioner for $K$ of the form

$$B_K^{-1} = \begin{pmatrix} I & -B_I^{-1}K_{IB} \\ 0 & I \end{pmatrix} \begin{pmatrix} B_I^{-1} & 0 \\ 0 & B_S^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -K_{BI}B_I^{-1} & I \end{pmatrix}.$$

We note that an application of $B_K^{-1}$ to a vector need only involve $B_I^{-1}$ twice and $B_S^{-1}$ once. It is also possible to use different approximate interior solvers in the three factors of $B_K^{-1}$ and to construct nonsymmetric preconditioners of a similar form.

In the analysis presented in this paper, we will always require exact interior subdomain solvers. Progress has been made in analyzing algorithms that use approximate interior solvers; cf. Börgers [4] and Haase, Langer, and Meyer [33]. Since it is important first to fully understand the case when exact interior solvers are used, we will focus on that case. We can then exclusively work with the space of discrete harmonic functions $\tilde{V}^h$ and the bilinear form $s(\cdot,\cdot)$. Numerical experiments (cf. Börgers [4], Haase, Langer, and Meyer [33], Skogen [55], and Smith [59]) indicate that a good rate of convergence can be maintained when one multigrid $V$-cycle is used, instead of an exact solver, to solve the interior problems.

**4. Technical tools.** A number of auxiliary results are needed for the Schwarz analysis of the iterative substructuring algorithms. The relevant norms and seminorms have been introduced in the preceding section; some of them contain a large multiple of the $L_2$ norm (cf. (7)).

The first lemma illustrates the limitations of the interpolation operator $I^H : V^h \to V^H$. $I^H u^h$ is the result of piecewise linear interpolation of the finite element function $u^h$ onto the coarse space $V^H$. The lemma follows easily from the inequality

$$\|u^h\|^2_{L_\infty(\Omega_j)} \leq C(1/h)\|u^h\|^2_{H^1(\Omega_j)}$$

(cf. Lemma 2.3 of Bramble and Xu [9]), and by using Poincaré's inequality. The given bounds are sharp.

LEMMA 4.1. *In three dimensions,*

$$||u^h - I^H u^h||^2_{L^2(\Omega_j)} \leq C(H/h)H^2|u^h|^2_{H^1(\Omega_j)}$$

*and*

$$|I^H u^h|^2_{H^1(\Omega_j)} \leq C(H/h)|u^h|^2_{H^1(\Omega_j)}.$$

The next lemma concerns an operator for which the bounds are much improved. We note that the norms are now given in terms of the entire region $\Omega$. In fact, it is not possible to provide the same estimates for the $H^1$ and $L^2$ norms, weighted by the values $\rho_i$ of the coefficient of the elliptic problem, if we require that the constants in the estimates be independent of the $\rho_i$; cf. Xu [64]. For a proof of Lemma 4.2 and a general discussion, see Bramble and Xu [9].

LEMMA 4.2. *Let $Q^H u^h$ be the $L^2$ projection of the finite element function $u^h$ onto the coarse space $V^H$. Then, in three dimensions,*

$$||u^h - Q^H u^h||^2_{L^2(\Omega)} \leq CH^2|u^h|^2_{H^1(\Omega)}$$

*and*

$$|Q^H u^h|^2_{H^1(\Omega)} \leq C|u^h|^2_{H^1(\Omega)}.$$

Results very similar to those of the next lemma can be found in Bramble, Pasciak, and Schatz [7], Bramble and Xu [9], Dryja [21], and Dryja and Widlund [29].

LEMMA 4.3. *Let $\bar{u}^h_{W^j}$ be the average value of $u^h$ on $W^j_h$, the nodes of the wire basket of the subdomain $\Omega_j$. Then*

$$||u^h||^2_{L^2(W^j)} \leq C(1+\log(H/h))||u^h||^2_{H^1(\Omega_j)}$$

*and*

$$||u^h - \bar{u}^h_{W^j}||^2_{L^2(W^j)} \leq C(1+\log(H/h))|u^h|^2_{H^1(\Omega_j)}.$$

*Similar bounds also hold for an individual substructure edge.*

When we estimate the parameter $C_0^2$, introduced in the abstract convergence theory, we must demonstrate that all functions in the finite element space can be decomposed into components in the subspaces in such a way that the sum of the resulting energies are uniformly, or almost uniformly, bounded with respect to the parameters $h$, $H$, etc. The main technique for deriving such decompositions is the use of suitable partitions of unity. In the next two lemmas, we explicitly construct such a partition.

LEMMA 4.4. *Let $\theta_{F^k}$ be the finite element function that is equal to 1 on $F^k_h$, where $F^k$ is the face common to $\Omega_i$ and $\Omega_j$, equal to 0 zero on $(\partial\Omega_{i,h} \cup \partial\Omega_{j,h}) \setminus F^k_h$, and discrete harmonic in $\Omega_i$ and $\Omega_j$. Then*

$$|\theta_{F^k}|^2_{H^1(\Omega_i)} \leq C(1+\log(H/h))H$$

*and*

$$||\theta_{F^k}||^2_{L^2(\Omega_i)} \leq CH^3.$$

*The same bounds also hold for the other subregion $\Omega_j$.*

*Proof.* We begin with a proof of the first inequality. We prove this result by constructing a function $\vartheta_{F^k}$, with the same boundary values as $\theta_{F^k}$, for which we can establish this bound. The proof is then completed by noting that a discrete harmonic function has at least as small an energy as any other function with the same boundary values. We consider in detail only the case of a tetrahedral substructure. The four functions, which correspond to the four faces of the tetrahedron, also form a partition of unity at all nodes of the closure of the substructure except those on the wire basket; this property will be used in the proof of Lemma 4.5.

We will first construct four functions, which form a partition of unity, but which are not finite element functions. The four finite element functions $\vartheta_{F^k}$ are then obtained by piecewise linear interpolation. It is easy to verify that these new functions also define a partition of unity, and that the bound on the gradient, previously obtained, is preserved. We will use the same notation for these two sets of functions.

We divide the substructure into four tetrahedra by connecting its centroid $C$, by line segments, to the four vertices of the tetrahedron. Similarly, we divide each triangular face of the substructure into three triangles by extending the bisectors of the three vertices of the triangle until they meet. We denote the resulting points on the faces by $C_k$; see Fig. 2. By connecting the $C_k$ with $C$, we obtain the wire baskets of twelve tetrahedra.

We construct the function $\vartheta_{F^k}$, associated with the face $F^k$, as follows: At $C$ the value is 1/4. We interpolate linearly between the value 1/4 and 1 or 0, whichever is appropriate, along the line segments connecting $C$ to the $C_k$. The values elsewhere are constant on the intersection of any plane, through the unique substructure edge that

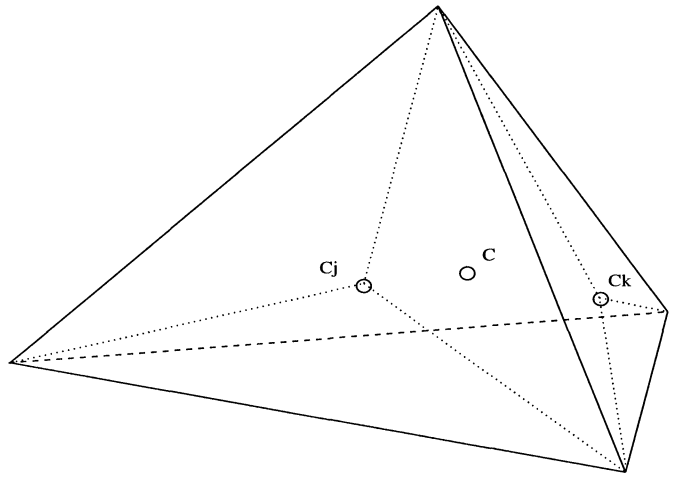


FIG. 2. *Construction of the partition function in a tetrahedron.*

belongs to a specific subtetrahedron, and that same subtetrahedron. This constant value is determined by the value, already known, at the point on the appropriate line segment, which is one of the edges of the same subtetrahedron. Finally, we modify the function by changing its values in the elements that have at least one vertex on an edge of the substructure. We make the function zero on the wire basket and continuous, by piecewise linear interpolation, using the previously constructed values at the nodes that are not on, but next to, the edge. We also replace the function elsewhere by its piecewise linear interpolant.

We return for a moment to consider the function prior to the interpolation. The values on any two planes associated with two different substructure edges, which intersect at a point on the appropriate line segment, are the same. The partition functions are therefore continuous across the boundaries of the subtetrahedra. Explicit formulas for the gradient and estimates thereof can, at least in principle, be given. The most important observation is that $|\nabla \vartheta_{F^k}| \leq C/r$, where $r$ is the distance to the nearest edge of the original tetrahedron.

It is also easy to show that $\{\vartheta_{F^k}\}$ form a partition of unity on the special line segments, and everywhere else, except in the special elements next to the edges of the original substructure.

To complete the proof of the first inequality, we return to the finite element functions and first note that the contribution to the energy from the union of the elements with at least one vertex on an edge of the substructure can be bounded from above by $CH$. This follows by considering their combined volume, the fact that $\vartheta_{F^k}$ vanishes at the edges, and the estimate $|\nabla \vartheta_{F^k}| \leq C/h$.

To estimate the contribution to the energy from the rest of the substructure, we consider one subtetrahedron at a time and introduce cylindrical coordinates using the appropriate substructure edge as the $z$-axis. The bound now follows from the bound on the gradient and elementary considerations. (We note that a similar argument, in a somewhat more complicated situation, is given in the proof of the next lemma.)

We now turn to the proof of the second inequality. To avoid irrelevant scaling factors, we consider the special case of $H = 1$, and we also denote the region by $\Omega$. In this case, we only have to prove that $\|\theta_{F^k}\|_{L^2(\Omega)}$ is bounded. We introduce an

auxiliary function $u$ by solving

$$-\triangle u = \theta_{F^k}, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega.$$

Since $\Omega$ is convex, a standard regularity result shows that

$$|u|_{H^2(\Omega)} \leq C\|\theta_{F^k}\|_{L^2(\Omega)}.$$

By Green's formula,

$$\|\theta_{F^k}\|^2_{L^2(\Omega)} = \int_\Omega \nabla u \cdot \nabla \theta_{F^k} dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} \theta_{F^k} ds.$$

Since $\theta_{F^k}$ is discrete harmonic, we find that

$$\int_\Omega \nabla u \cdot \nabla \theta_{F^k} dx = \int_\Omega \nabla(u - w^h) \nabla \theta_{F^k} dx \quad \forall w^h \in V^h(\Omega) \cap H^1_0(\Omega).$$

The right-hand side can be estimated from above by

$$\begin{aligned} \inf_{w^h \in V^h \cap H^1_0(\Omega)} |u - w^h|_{H^1(\Omega)} |\theta_{F^k}|_{H^1(\Omega)} &\leq Ch|u|_{H^2(\Omega)} |\theta_{F^k}|_{H^1(\Omega)} \\ &\leq Ch\|\theta_{F^k}\|_{L^2(\Omega)} |\theta_{F^k}|_{H^1(\Omega)}. \end{aligned}$$

Here we have used a standard error bound and the regularity result. By using the bound for $|\theta_{F^k}|_{H^1(\Omega)}$, we see that the first term originating from the Green's formula is $o(\|\theta_{F^k}\|_{L^2(\Omega)})$.

For the second term, the line integral, we use Schwarz's inequality, a standard trace theorem, and the regularity result. We obtain

$$\left| \int_{\partial\Omega} \frac{\partial u}{\partial n} \theta_{F^k} ds \right| \leq C\|\theta_{F^k}\|_{L^2(\Omega)} \|\theta_{F^k}\|_{L^2(\partial\Omega)}.$$

The argument can now easily be concluded by observing that $\|\theta_{F^k}\|_{L^2(\partial\Omega)} \leq C$. □

The following lemma is an extension to three dimensions of a result of Dryja and Widlund [24]. The present approach makes it possible to prove nontrivial bounds for iterative substructuring algorithms without the use of an extension theorem; cf. Widlund [61]. Here, we can always work in subspaces of the original finite element spaces, and we never need to use trace and extension theorems. One advantage of this approach is that the constants obtained in the estimates can be calculated explicitly from geometric information. When working with the trace and extension theorems it is more difficult to determine the exact relationship between the geometry (e.g., the aspect ratios of the substructures) and the constants of the bounds.

LEMMA 4.5. *Let $\vartheta_{F^k}(x)$ be the functions introduced in the proof of Lemma 4.4, let $F^k$ be a face of the substructure $\Omega_j$, and let $I^h$ denote the interpolation operator associated with the finite element space $V^h$. Then,*

$$\sum_k I^h(\vartheta_{F^k} u^h)(x) = u^h(x) \quad \forall x \in \overline{\Omega}_{j,h} \setminus W^j_h, \forall x \in \overline{\Omega_i} \setminus W_i$$

*and*

$$|I^h(\vartheta_{F^k} u^h)|^2_{H^1(\Omega_j)} \leq C(1 + \log(H/h))^2 ||u^h||^2_{H^1(\Omega_j)}.$$

*Proof.* We provide a proof only for the case of a tetrahedral substructure. The first formula follows immediately from the proof of the previous lemma. To prove the other, we first consider the contributions to the energy from the elements that touch the wire basket. By definition, $\vartheta_{F^k}$ vanishes on the wire basket. It is then easy to show that the energy contributed from this small neighborhood of the wire basket can be bounded by $h\sum_x |u^h(x)|^2$, where the sum is taken over all the nodal points that are within a mesh width of the wire basket. As in Lemma 4.3, this sum can be bounded by $C(1+\log(H/h))||u^h||^2_{H^1(\Omega_j)}$.

By using elementary considerations, we obtain

$$|I^h(\vartheta_{F^k}u^h)|^2_{H^1(\Omega_j)} \le 2\sum_{K\subset\Omega_j} |\bar{\vartheta}_{F^k}u^h|^2_{H^1(K)} + 2\sum_{K\subset\Omega_j} |I^h(\vartheta_{F^k}-\bar{\vartheta}_{F^k})u^h|^2_{H^1(K)}.$$

Here $0<\bar{\vartheta}_{F^k}<1$ is the average of $\vartheta_{F^k}$ over the element $K$. When we estimate these sums, we can ignore the elements that touch the wire basket, since they have already been accounted for.

The bound for the first term is trivial, but that of the second term is more complicated. We first use an inverse inequality and obtain

$$|I^h((\vartheta_{F^k}-\bar{\vartheta}_{F^k})u^h)|^2_{H^1(K)} \le Ch^{-2}||I^h((\vartheta_{F^k}-\bar{\vartheta}_{F^k})u^h)||^2_{L^2(K)}.$$

By using the bound on the gradient of $\vartheta_{F^k}$, we can bound $\vartheta_{F^k}-\bar{\vartheta}_{F^k}$ by $Ch/r$, where $r$ is the distance to the wire basket. Hence,

$$\sum_K |I^h((\vartheta_{F^k}-\bar{\vartheta}_{F^k})u^h)|^2_{H^1(K)} \le C\sum_K r^{-2}||u^h||^2_{L^2(K)}.$$

We partition the elements of $\Omega_i$ into groups, in accordance with the closest edge of $\Omega_i$; the exact rule for the assignment of the elements that are halfway between is of no importance. For each edge of the wire basket, we use a local cylindrical coordinate system with the $z$ axis coinciding with the edge, and the radial direction, $r$, normal to the edge. In cylindrical coordinates, we estimate the sum by an integral

$$\sum_{K\subset\Omega_j} r^{-2}||u^h||^2_{L^2(K)} \le C\int_{r=h}^{H}\int_\theta\int_z (u^h)^2\frac{r}{r^2}drd\theta dz.$$

The integral with respect to $z$ can be bounded by using Lemma 4.3. We obtain

$$\sum_{K\subset\Omega_j} r^{-2}||u^h||^2_{L^2(K)} \le C(1+\log(H/h))||u^h||^2_{H^1(\Omega_j)}\int_{r=h}^{H} r^{-1}dr$$

and thus

$$\sum_{K\subset\Omega_j} |I^h(\vartheta_{F^k}-\bar{\vartheta}_{F^k})u^h|^2_{H^1(K)} \le C(1+\log(H/h))^2||u^h||^2_{H^1(\Omega_j)}. \qquad \square$$

We also need some trivial results.

LEMMA 4.6. *Let* $\bar{u}^h_{F^k}$, $\bar{u}^h_{\partial\Omega_k}$, $\bar{u}^h_{\partial F^k}$, $\bar{u}^h_{E^k}$, *and* $\bar{u}^h_{W^k}$ *be the average of the nodal values of* $u^h$ *on* $F^k$, $\partial\Omega_k$, $\partial F^k$, $E^k$, *and* $W^k$, *respectively. Then,*

$$(\bar{u}^h_{F^k})^2 \le C\frac{1}{H^2}||u^h||^2_{L^2(F^k)},$$

$$(\bar{u}^h_{\partial\Omega_k})^2 \leq C\frac{1}{H^2}||u^h||^2_{L^2(\partial\Omega_k)},$$

$$(\bar{u}^h_{\partial F^k})^2 \leq C\frac{1}{H}||u^h||^2_{L^2(\partial F^k)},$$

$$(\bar{u}^h_{E^k})^2 \leq C\frac{1}{H}||u^h||^2_{L^2(E^k)},$$

$$(\bar{u}^h_{W^k})^2 \leq C\frac{1}{H}||u^h||^2_{L^2(W^k)}.$$

The proofs are direct consequences of the Schwarz inequality.

LEMMA 4.7. *Let $u^h$ be zero on the faces of $\Omega_j$ and discrete harmonic in $\Omega_j$. Then*

$$|u^h|^2_{H^1(\Omega_j)} \leq C||u^h||^2_{L^2(W^j)}.$$

This result follows by estimating the energy norm of the zero extension of the boundary values and by noting that the harmonic extension has a smaller energy.

We note that we will use both $h\|\underline{u}\|^2_{l^2(E^i)}\|$ and $\|u^h\|^2_{L^2(E^i)}$. While the first expression is appropriate when defining bilinear forms on a subspace related to the edge $E^i$, the two are, for all theoretical purposes, interchangeable since the mass matrix related to the second expression is uniformly well conditioned.

**5. Local solvers.** Iterative substructuring algorithms with good convergence properties are constructed from two types of components: many local solvers and a coarse grid solver. In this section, we describe two basic methods of constructing the local solvers.

The first approach can essentially be viewed as a classical splitting of the Schur complement matrix. For simplicity, we write down the preconditioners only for the additive algorithms; similar, but more complicated, formulas can be given for the multiplicative Schwarz methods.

We first recall that the Schur complement for the entire problem is obtained through subassembly of the matrices given in (9). This results in the formula

$$S = \begin{pmatrix} S_{FF} & S_{FE} & S_{FV} \\ S^T_{FE} & S_{EE} & S_{EV} \\ S^T_{FV} & S^T_{EV} & S_{VV} \end{pmatrix}. \tag{11}$$

As in the classical theory for iterative methods (cf. Varga [60]) a preconditioner for $S$ can be obtained by a splitting, i.e., by dropping certain blocks or elements. Here we eliminate not only the off-diagonal blocks of (11) but also the sub-blocks representing the coupling between all pairs of faces, edges, and vertices. The resulting preconditioner has the form

$$B^{-1} = \begin{pmatrix} \bar{S}^{-1}_{FF} & 0 & 0 \\ 0 & \bar{S}^{-1}_{EE} & 0 \\ 0 & 0 & \bar{S}^{-1}_{VV} \end{pmatrix}.$$

The matrix $\bar{S}_{FF}$ is block diagonal with a block for each face, $\bar{S}_{EE}$ has a block for each edge, and $\bar{S}_{VV}$ is diagonal. This is a block-Jacobi preconditioner. We note that each block corresponds to a set of adjacent variables on the interface $\Gamma$.

We need to introduce some additional notation. Let $S_{F^iF^i}$ be the submatrix of $S$ associated with the face $F^i$, and let $S_{E^iE^i}$ be that of the edge $E^i$. Similarly, $S_{V^iV^i}$ is the diagonal element of $S$ associated with the vertex $V^i$. Let $R_{F^i}$ be the rectangular restriction matrix which returns only the components of a global vector associated with the face $F^i$. Similar restriction matrices, $R_{E^i}$ and $R_{V^i}$, are introduced for the edges and individual vertices, respectively. We note that, for instance, $S_{F^iF^i} = R_{F^i}SR_{F^i}^T$.

The preconditioner $B^{-1}$ can now be rewritten as

$$B^{-1} = \sum R_{F^i}^T S_{F^iF^i}^{-1} R_{F^i} + \sum R_{E^i}^T S_{E^iE^i}^{-1} R_{E^i} + \sum R_{V^i}^T S_{V^iV^i}^{-1} R_{V^i},$$

and we also find that

$$B^{-1}S = \sum R_{F^i}^T S_{F^iF^i}^{-1} R_{F^i}S + \sum R_{E^i}^T S_{E^iE^i}^{-1} R_{E^i}S + \sum R_{V^i}^T S_{V^iV^i}^{-1} R_{V^i}S.$$

This preconditioned matrix is the same as that obtained from an additive Schwarz method with the spaces $\tilde{V}_{F^i}^h = \{u \in \tilde{V}^h | u(x) = 0, \forall x \in \Gamma_h \setminus F_h^i\}$, $\tilde{V}_{E^i}^h = \{u \in \tilde{V}^h | u(x) = 0, \forall x \in \Gamma_h \setminus E_h^i\}$, and $\tilde{V}_{V^i}^h = \{u \in \tilde{V}^h | u(x) = 0, \forall x \in \Gamma_h \setminus V^i\}$.

To decrease the cost and to avoid computing the elements of the Schur complements, we make some further simplifications. We note that the matrices $S_{E^iE^i}$ are quite well conditioned; it follows from Lemmas 4.3 and 4.7 that their condition numbers are $O(1 + \log(H/h))$. We therefore replace $S_{E^iE^i}^{-1}$, in the preconditioner, by $1/(h\rho_{E^i})I$. Here $I$ is an identity matrix and $\rho_{E^i} = \sum_{\partial\Omega_j \cap E^i \neq \emptyset} \rho_j$. In the Schwarz framework, this corresponds to replacing the bilinear form $s(\cdot,\cdot)$ on the spaces $\tilde{V}_{E^i}$ by $b_{E^i}(u^h, u^h) = h\rho_{E^i}||\underline{u}||_{l^2(E^i)}^2$. We can also replace the diagonal element $S_{V^iV^i}$ by $h\rho_{V^i} = h\sum_{\partial\Omega_j \cap V^i \neq \emptyset} \rho_j$. The modified preconditioner can then be written as

$$B^{-1} = \sum R_{F^i}^T S_{F^iF^i}^{-1} R_{F^i} + \sum \frac{1}{h\rho_{E^i}} R_{E^i}^T R_{E^i} + \sum \frac{1}{h\rho_{V^i}} R_{V^i}^T R_{V^i}.$$

We note that the second and third sums could be combined into one. One of the terms of the sum will then correspond to the wire basket $W^{(i)}$ and

$$B^{-1} = \sum R_{F^i}^T S_{F^iF^i}^{-1} R_{F^i} + \frac{1}{h} R_W^T (D_W^\rho)^{-1} R_W. \tag{12}$$

Here the elements of the diagonal matrix $D_W^\rho$ equal $\rho_{E^i}$ and $\rho_{V^i}$ for the components corresponding to edge and vertex nodes, respectively.

We should also provide a relatively inexpensive algorithm for calculating the action of each $S_{F^kF^k}^{-1}$. We do so by solving a linear system associated with the two domains $\Omega_i$ and $\Omega_j$ that share the face $F^k$. Let $K^{(ij)}$ denote the submatrix of $K$ associated with $\Omega_{ij} = \Omega_i \cup \Omega_j \cup F^k$. Then

$$S_{F^kF^k}^{-1} = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} K_{II}^{(ij)} & K_{IB}^{(ij)} \\ K_{IB}^{(ij)^T} & K_{BB}^{(ij)} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{13}$$

Here the subscripts $I$ and $B$ represent the nodes of $\Omega_i \cup \Omega_j$ and $F^k$, respectively. Hence the action of $S_{F^kF^k}^{-1}$ can be calculated by solving a homogeneous Dirichlet problem on $\Omega_{ij}$ with a right-hand side that differs from zero only on $F_h^k$. In this construction, we could also replace $\Omega_{ij}$ by any shape-regular region that contains the face $F^k$

in its interior. We stress that the solution of the local problems never requires the explicit construction of elements of $S$. Instead, in each iteration, independent Dirichlet boundary value problems are solved for regions enclosing the individual faces.

$S_{F^k F^k}$ can also be replaced by the $J$ operator introduced in Dryja [20], or another of many other preconditioners that are known to be effective for problems on the union of two substructures; cf. Bjørstad and Widlund [3].

In the splittings just considered, we eliminate the coupling between all pairs of faces. In our second main approach, we attempt to maintain this coupling. To keep the problems local, we instead eliminate the coupling between neighboring subdomains working with the full Schur complements of the individual substructures.

The preconditioner is given by

$$B^{-1} = \left( \sum_i \Lambda_i^\dagger S^{(i)^\dagger} \Lambda_i^\dagger \right),$$

where the $\Lambda_i$ are diagonal matrices with nonzero elements only for the components of $\partial\Omega_{i,h}$. The diagonal element of $\Lambda_i$, which corresponds to $x \in \partial\Omega_i$, is given by

$$\lambda_i(x) = \sum_j \rho_j^{1/2}, \tag{14}$$

where the sum is taken over all $j$ such that $x \in \partial\Omega_j \cap \partial\Omega_i$. We work with pseudoinverses, since the Neumann problems for interior substructures are singular and all the elements of the diagonal matrix $\Lambda_i$ are set to zero for $x \in \Gamma_h \setminus \partial\Omega_i$.

The following formula has much in common with (13). However, the subscripts $I$ and $B$ now represent the nodes of $\Omega_i$ and $\partial\Omega_i$, respectively:

$$S^{(i)^\dagger} = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} K_{II}^{(i)} & K_{IB}^{(i)} \\ K_{IB}^{(i)^T} & K_{BB}^{(i)} \end{pmatrix}^\dagger \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

Again, there is no need to compute the Schur complements. Instead, in each iteration step, we can solve a Neumann problem for each subdomain in order to calculate the action of $S^{(i)^\dagger}$ on a vector.

This second approach, with a different scaling, originates in the work of Bourgat, Glowinski, Le Tallec, and Vidrascu [5]. Their algorithms have been modified, extended, and analyzed by Cowsar, Mandel, and Wheeler [17], De Roeck [18], De Roeck and Le Tallec [19], Dryja and Widlund [25], [27], [29], Le Tallec, De Roeck, and Vidrascu [35], Mandel [40], [41], and Mandel and Brezina [42].

For these methods, the Schwarz subspaces are given by

$$\tilde{V}_i^h = \{u^h \in \tilde{V}^h | u^h(x) = 0, \ \forall x \in \Gamma_h \setminus \partial\Omega_{i,h}\}, \tag{15}$$

and the bilinear forms on these subspaces can be given by

$$b_i(u^h, v^h) = s^{(i)}(I^h(\lambda_i u^h), I^h(\lambda_i v^h)). \tag{16}$$

For an interior subdomain, or a boundary substructure that does not touch the Dirichlet part of the boundary, $\Gamma_0$, the local Neumann problem is singular. There are several ways of dealing with this. Instead of working with the pseudoinverses of the Schur complements, which can be computationally expensive, we can solve a

Neumann problem for a different elliptic operator. This is the main approach taken in Dryja and Widlund [29]. The local bilinear form

$$b_i(u^h, v^h) = \hat{s}^{(i)}(I^h(\lambda_i u^h), I^h(\lambda_i v^h)) \tag{17}$$

is used, where $\hat{s}^{(i)}(\cdot,\cdot)$ is the Schur complement of the bilinear form

$$\hat{a}^{(i)}(u,v) = \int_{\Omega_i} \nabla u \cdot \nabla v dx + \frac{1}{H_i^2} \int_{\Omega_i} uv dx. \tag{18}$$

We will refer to this as the *standard Neumann–Neumann local solver.* We note that Dryja and Widlund [29] contains a detailed discussion on the choice of bilinear forms for the boundary substructures that touch $\Gamma_0$ only at a point or along an edge. The former are treated as if they were interior substructures (i.e., (17) is used), and the latter in the same way as a substructure that shares an entire face with $\Gamma_0$ (i.e., (16) is used).

An alternative approach to avoiding singular problems is to impose zero Dirichlet boundary conditions on the wire basket while Neumann boundary conditions are maintained on the faces. In this case the local subspace associated with the individual substructures is given by

$$\tilde{V}_{i,\text{mix}}^h = \{u^h \in \tilde{V}_i^h | u^h(x) = 0,\ \forall x \in W_h^i\}. \tag{19}$$

We will refer to these as the *mixed* Neumann–Neumann subspaces. We can use the bilinear forms given by (16). We refer to the resulting problem as a mixed Neumann–Neumann local solver. When no coarse problem is used, this preconditioner must be augmented by terms related to the wire basket. We can, for instance, add the operator

$$\sum \frac{1}{h} R_W^T (D_W^\rho)^{-1} R_W$$

previously introduced; see also the last section of Dryja and Widlund [29] for further details.

**6. Coarse grid algorithms and condition numbers.** In addition to the local solvers discussed in the preceding section, any successful domain decomposition preconditioner must also contain a global space component. We can either add a coarse solver to a preconditioner based only on local solvers or replace part of the preconditioner. In this section, we will discuss a large number of coarse spaces. The first of them is based on the space $V^H$ of continuous, piecewise linear functions using the substructures as elements. Conceptually this is clearly the simplest, but as will be shown, it can be inadequate in three dimensions, basically because of Lemma 4.1. In the remaining subsections, we discuss wire basket-based and face-based coarse problems.

**6.1. Vertex-based methods.** To incorporate a global component of the preconditioner, we first represent $S$ in a partially hierarchical basis. The face and edge nodal basis functions are not changed, but those associated with the vertices are replaced by piecewise linear functions on the coarse triangulation. The basis change from the partial hierarchical to the nodal basis is represented by

$$\begin{pmatrix} I & 0 & R_F^T \\ 0 & I & R_E^T \\ 0 & 0 & I \end{pmatrix},$$

where the operators $R_F^T$ and $R_E^T$ represent coarse space linear interpolation from the values on the vertices to the faces and edges, respectively.

The Schur complement can be rewritten as

$$S = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -R_F & -R_E & I \end{pmatrix} \begin{pmatrix} S_{FF} & S_{FE} & \tilde{S}_{FV} \\ S_{FE}^T & S_{EE} & \tilde{S}_{EV} \\ \tilde{S}_{FV}^T & \tilde{S}_{EV}^T & \tilde{S}_{VV} \end{pmatrix} \begin{pmatrix} I & 0 & -R_F^T \\ 0 & I & -R_E^T \\ 0 & 0 & I \end{pmatrix}. \tag{20}$$

As in the preceding section, we now drop the coupling between the faces, edges, and vertices, but we keep those between the vertices. We obtain

$$B = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -R_F & -R_E & I \end{pmatrix} \begin{pmatrix} \bar{S}_{FF} & 0 & 0 \\ 0 & \bar{S}_{EE} & 0 \\ 0 & 0 & \tilde{S}_{VV} \end{pmatrix} \begin{pmatrix} I & 0 & -R_F^T \\ 0 & I & -R_E^T \\ 0 & 0 & I \end{pmatrix}.$$

The preconditioner can now be written as

$$B^{-1} = \begin{pmatrix} I & 0 & R_F^T \\ 0 & I & R_E^T \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \bar{S}_{FF}^{-1} & 0 & 0 \\ 0 & \bar{S}_{EE}^{-1} & 0 \\ 0 & 0 & \tilde{S}_{VV}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ R_F & R_E & I \end{pmatrix}$$

and

$$B^{-1}S = \sum R_{F^i}^T S_{F^iF^i}^{-1} R_{F^i} S + \sum R_{E^i}^T S_{E^iE^i}^{-1} R_{E^i} S + R_H^T \tilde{S}_{VV}^{-1} R_H S,$$

where $R_H = (R_F\, R_E\, I)$. Thus, we obtain an additive Schwarz preconditioner with the same face and edge spaces as before, but with a coarse space, $\tilde{V}^H$, in place of the set of individual, local vertex spaces. In the case of piecewise linear finite elements, the matrix $\tilde{S}_{VV}$ is equal to $K_H$, the stiffness matrix obtained by treating the substructures as elements. We can therefore replace the two last terms in the preconditioner and obtain

$$B^{-1} = \sum R_{F^i}^T S_{F^iF^i}^{-1} R_{F^i} + \sum \frac{1}{h\rho_{E^i}} R_{E^i}^T R_{E^i} + R_H^T K_H^{-1} R_H.$$

As before, there is no need to form the matrix $S$ explicitly.

ALGORITHM 6.1. *Use a Schwarz method with the subspaces $\tilde{V}^H$, $\tilde{V}_{F^i}^h$, and $\tilde{V}_{E^j}^h$. For all the $\tilde{V}_{E^j}^h$ spaces, use the bilinear forms associated with $h\rho_{E^i}||\underline{u}||_{l^2(E^j)}^2$.*

THEOREM 6.1. *Algorithm* 6.1 *satisfies the three assumptions with*

$$C_0^2 \leq C(H/h)(1+\log(H/h))^2, \quad \rho(\mathcal{E}) \leq C, \quad \omega \leq C.$$

*The constants are independent of the jumps in the coefficient $\rho_i$.*

*Proof.* We estimate the first parameter, $C_0$. We note that we are only going to work with discrete harmonic functions for which $s(u^h, u^h) = a(u^h, u^h)$. Let $u_0^h = I^H u^h$. We use Lemma 4.1 and find, by adding over the substructures, that

$$a(u_0^h, u_0^h) \leq C \sum_i \rho_i |I^H u^h|_{H^1(\Omega_i)}^2 \leq C \frac{H}{h} a(u^h, u^h).$$

We next bound the energy for the parts of the decomposition of the function $u^h$ that are associated with the faces. This requires the use of Lemmas 4.1 and 4.5. Let

$w^h = u^h - I^H u^h$, and let $u^h_{F^k} = \mathcal{H}(\vartheta_{F^k} w^h)$. Here $\mathcal{H}v^h$ denotes the discrete harmonic extension of the function $v^h$ given on the interface $\Gamma$. Then,

$$\begin{aligned} a(u^h_{F^k}, u^h_{F^k}) &\le C(\rho_i ||u^h_{F^k}||^2_{H^1(\Omega_i)} + \rho_j ||u^h_{F^k}||^2_{H^1(\Omega_j)}) \\ &\le C(1 + \log(H/h))^2 (\rho_i ||w^h||^2_{H^1(\Omega_i)} + \rho_j ||w^h||^2_{H^1(\Omega_j)}) \\ &\le C\frac{H}{h}(1 + \log(H/h))^2 (\rho_i |u^h|^2_{H^1(\Omega_i)} + \rho_j |u^h|^2_{H^1(\Omega_j)}). \end{aligned}$$

Therefore,

$$\sum_k a(u^h_{F^k}, u^h_{F^k}) \le C\frac{H}{h}(1 + \log(H/h))^2 a(u^h, u^h).$$

Let $u^h_{E^i}$ be the restriction of $(u^h - I^H u^h)$ to $E^i$. Then, by Lemmas 4.1 and 4.3,

$$\begin{aligned} \sum_i \rho_{E^i} ||u^h_{E^i}||^2_{L^2(E^i)} &\le C(1 + \log(H/h)) \sum_i \sum_{\partial\Omega_j \cap E^i \neq \emptyset} \rho_j ||u^h - I^H u^h||^2_{H^1(\Omega_j)} \\ &\le C\frac{H}{h}(1 + \log(H/h)) \sum_j \rho_j |u^h|^2_{H^1(\Omega_j)} \\ &= C\frac{H}{h}(1 + \log(H/h)) a(u^h, u^h). \end{aligned}$$

To obtain a bound for $\omega$, we have only to consider the edge spaces. The constant upper bound follows directly from Lemma 4.7.

It is easy to obtain a bound on $\rho(\mathcal{E})$. We simply note that the subdomains associated with the local subspaces form an overlapping cover of the domain, and that every point in the domain is covered by a finite, uniformly bounded number of such subregions. The subregions can be grouped into sets, with elements that do not overlap, and the subspaces related to these sets can be merged. The value of $N$ is then reduced to a constant, and a uniform upper bound for $\rho(\mathcal{E})$ is obtained. This argument is valid for all of the proofs in this section and will not be repeated. □

It is clear that the $H/h$ term is directly attributable to the large energy of the coarse mesh interpolant. In the proof given above, we must use $I^H u^h$ because all functions in the other subspaces vanish at the vertices. In the next algorithm, we add the one-dimensional spaces associated with each vertex and its standard nodal basis function. After doing so, we obtain a much stronger result, but the bounds are no longer independent of the variation of the coefficient of (5) across the interface $\Gamma$. The additive Schwarz preconditioner is now given by

$$B^{-1} = \sum R^T_{F^i} S^{-1}_{F^i F^i} R_{F^i} + \sum \frac{1}{h\rho_{E^i}} R^T_{E^i} R_{E^i} + R^T_H K^{-1}_H R_H + \sum \frac{1}{h\rho_{V^i}} R^T_{V^i} R_{V^i}.$$

We note that, as in (12), we can combine the edge and vertex spaces into a single wire basket space, $\tilde{V}^h_W$, with a corresponding restriction operator, $R_W$. We obtain

$$B^{-1} = \sum R^T_{F^i F^i} S^{-1}_{F^i} R_{F^i} + R^T_H K^{-1}_H R_H + \frac{1}{h} R^T_W (D^\rho_W)^{-1} R_W.$$

Here $D^\rho_W$ is the global diagonal matrix constructed from the weights $\rho_{E^i}$ and $\rho_{V^i}$ as in formula (12).

ALGORITHM 6.2. *Use a Schwarz method with the subspaces* $\tilde{V}^H$, $\tilde{V}^h_{F^i}$, *and* $\tilde{V}^h_W$. *In addition, on the spaces* $\tilde{V}^h_W$, *use the bilinear form given by* $h\underline{u}^T D^\rho_W \underline{u}$.

THEOREM 6.2. *Algorithm* 6.2 *satisfies the three assumptions with*

$$C_0^2 \leq C(1+\log(H/h))^2, \qquad \rho(\mathcal{E}) \leq C, \qquad \omega \leq C.$$

*Here we cannot guarantee that the estimate of* $C_0^2$ *is independent of the jumps in the coefficients of* (5).

*Proof.* The proof is almost identical to that given above, except that we use $u_0^h = Q^H u^h$ and Lemma 4.2 rather than $I^H u^h$ and Lemma 4.1. □

We can increase the overlap between the subspaces and obtain methods with condition numbers that are uniformly bounded and independent of $H$ and $h$. Such a method was given in Smith [58]. This algorithm, known as the vertex space or Copper Mountain algorithm, has much in common with the original additive Schwarz method of Dryja and Widlund [24]; cf. also Dryja and Widlund [28] and Nepomnyaschikh [46].

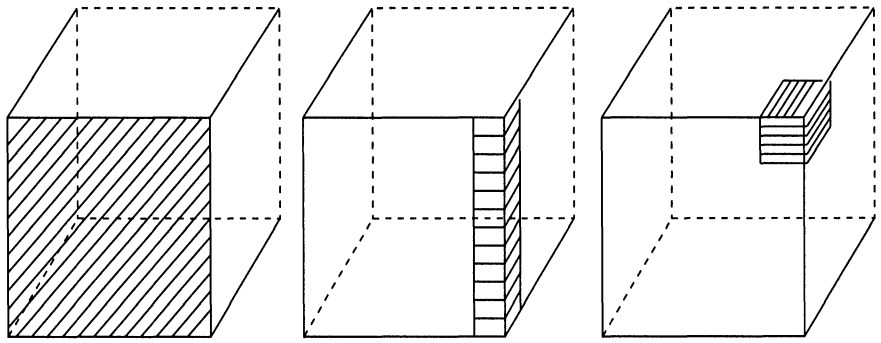

FIG. 3. *Face, edge, and vertex spaces.*

To define this algorithm, we first define edge spaces associated with a set $\Gamma_{E^j}$ that includes all parts of the faces adjacent to the edge $E^j$ that are within a distance $cH$ from the edge; see Fig. 3. We also define the vertex region $\Gamma_{V^j}$ as the part of $\Gamma$ that is at a distance less than $cH$ from the $j$th vertex of the substructure. The space related to this set is

$$\tilde{V}^h_{\Gamma^{V^j}} = \{u^h \in \tilde{V}^h | u^h(x) = 0, \forall x \in \Gamma_h \setminus \Gamma_{V^j}\},$$

with a similar definition for $\tilde{V}^h_{\Gamma^{E^j}}$. For this algorithm, we first use exact projections. Therefore, the algorithm is completely defined by its subspaces.

ALGORITHM 6.3. *Use a Schwarz method with the subspaces given by* $\tilde{V}^H$, $\tilde{V}^h_{F^i}$, $\tilde{V}^h_{\Gamma^{E^j}}$, *and* $\tilde{V}^h_{\Gamma^{V^j}}$.

In [58], Smith proved the following result.

THEOREM 6.3. *Algorithm* 6.3 *satisfies the three assumptions with*

$$C_0^2 \leq C, \qquad \rho(\mathcal{E}) \leq C, \qquad \omega = 1.$$

*Here we cannot guarantee that the estimate of* $C_0^2$ *is independent of the jumps in the coefficient of* (5).

Using the definitions given above, we find that

$$B^{-1} = R_H^T K_H^{-1} R_H + \sum_i R_{F^i}^T S_{F^i F^i}^{-1} R_{F^i} + \sum_j R_{\Gamma E^j}^T S_{\Gamma E^j \Gamma E^j}^{-1} R_{\Gamma E^j} + \sum_k R_{\Gamma V^k}^T S_{\Gamma V^k \Gamma V^k}^{-1} R_{\Gamma V^k}.$$

We note that the first term essentially involves solving a system associated with a block of $S$, represented in the partial hierarchical basis, while the other terms involve systems given by blocks of $S$ in the usual nodal basis. In practical implementations, the $S_{F^i F^i}$, $S_{\Gamma E^j \Gamma E^j}$, and $S_{\Gamma V^k \Gamma V^k}$ need not be formed explicitly. Instead we can solve problems such as (13). Another approach to cutting costs is to use probing to obtain approximations of the blocks of the Schur complement; cf., e.g., Chan and Mathew [14] and Chan, Mathew, and Shao [15].

In the analysis given in [58], Smith considered only the case when the overlap was generous, i.e., on the order of $H$. However, numerical experiments in two dimensions suggest that good convergence can also be obtained with minimal overlap. Thus motivated, Dryja and Widlund [28] showed that if the overlap is uniformly on the order of $\delta$, then Algorithm 6.3 satisfies

$$C_0^2 \leq C(1 + \log(H/\delta))^2.$$

In the same paper, they also demonstrated that for the standard overlapping Schwarz method with small overlap,

$$C_0^2 \leq C(1 + H/\delta).$$

Numerical experiments (cf. Bjørstad et al. [1], [2] and Gropp and Smith [32]) confirm that the rate of convergence of this algorithm is very satisfactory.

**6.2. Wire basket-based algorithms.** We now consider another class of coarse problems based on averages and the wire basket. Methods of this class use a different approach to overcome the difficulties associated with the piecewise linear interpolation over the coarse triangulation, which led to the poor result of Theorem 6.1 or to estimates that are not known to be valid uniformly for all values of the coefficient of (5). Instead, we now essentially interpolate using averages of $u^h$ over the wire basket. These algorithms work extremely well for problems with large jumps in the coefficients $\rho_i$; cf. Smith [59]. We note that Bramble, Pasciak, and Schatz [7] pioneered the use of similar ideas. Here, we begin by describing a method introduced in Smith [56]; cf. also [57].

For the wire basket-based methods, we work with the block matrix (10) rather than (9). Let $T^T$ be the operator that maps the values on the wire basket onto the faces by assigning, to each node on a face, the average value of the nodal values on the boundary of the face. This represents an alternative change of basis of the space. $S$ can now be written as

$$S = \begin{pmatrix} I & 0 \\ -T & I \end{pmatrix} \begin{pmatrix} S_{FF} & \bar{S}_{FW} \\ \bar{S}_{FW}^T & \bar{S}_{WW} \end{pmatrix} \begin{pmatrix} I & -T^T \\ 0 & I \end{pmatrix}.$$

We note the similarity with (20), but we are now using piecewise constant interpolation onto the faces rather than piecewise linear interpolation onto the faces and edges. We

proceed as in the preceding subsection and drop the coupling between pairs of faces, and the faces and the wire basket. We obtain

$$B^{-1} = \begin{pmatrix} I & T^T \\ 0 & I \end{pmatrix} \begin{pmatrix} \bar{S}_{FF}^{-1} & 0 \\ 0 & \bar{S}_{WW}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ T & I \end{pmatrix}$$

and

$$B^{-1}S = R_0^T \bar{S}_{WW}^{-1} R_0 S + \sum_i R_{F^i}^T S_{F^i F^i}^{-1} R_{F^i} S,$$

where $R_0 = (T\, I)$. This is also an additive Schwarz scheme that uses the same face spaces $\tilde{V}_{F^i}^h$ as the vertex-based algorithms. The coarse space $\tilde{V}_W^h$ can conveniently be defined as the range of an interpolation operator $I_W^h : \tilde{V}^h \to \tilde{V}_W^h$, defined by

$$I_W^h u^h = \sum_{x_k \in W_h} u^h(x_k)\varphi_k + \sum_k \bar{u}_{\partial F^k}^h \theta_{F^k}.$$

Here, $\varphi_k$ is the discrete harmonic extension of the standard nodal basis functions $\phi_k$. The resulting finite element function is continuous across all substructure boundaries. Therefore, $\tilde{V}_W^h$ is a conforming subspace of $\tilde{V}^h$.

We use the bilinear form given by

$$b_0^W(u^h, u^h) = (1 + \log(H/h))h \sum_i \rho_i \min_{\bar{\omega}_i} ||\underline{u} - \bar{\omega}_i \underline{z}^{(i)}|^2_{l^2(W^i)}$$

for this subspace. Here all the components of the vector $\underline{z}^{(i)}$ are equal to one. The introduction of this bilinear form corresponds to replacing $\bar{S}_{WW}$ by a matrix that, locally on each substructure, is a simple rank-one perturbation of a multiple of the identity matrix. To solve the corresponding linear system, we can use a fast technique suggested by Mandel [37]; also cf. Smith [56], [57].

Letting $B^{(i)} = h(1 + \log(H/h))\rho_i I$, we rewrite the problem as

$$\min_{\underline{u}} \sum_i \min_{\bar{\omega}_i} \frac{1}{2}(\underline{u}^{(i)} - \bar{\omega}_i \underline{z}^{(i)})^T B^{(i)}(\underline{u}^{(i)} - \bar{\omega}_i \underline{z}^{(i)}) - \underline{u}^T \underline{r}.$$

We then take derivatives with respect to $\bar{\omega}_i$ and $\underline{u}$ and obtain the linear system

$$\underline{z}^{(i)^T} B^{(i)}(\underline{u}^{(i)} - \underline{z}^{(i)}\bar{\omega}_i) = 0 \quad \forall i,$$

$$B\underline{u} - \sum_i B^{(i)} \underline{z}^{(i)} \bar{\omega}_i = \underline{r}. \tag{21}$$

Here, $B$ is the diagonal matrix obtained by subassembling the $B^{(i)}$. We then eliminate $\underline{u}$ and get the following system for the $\bar{\omega}_i$ :

$$(\underline{z}^{(i)^T} B^{(i)} \underline{z}^{(i)})\bar{\omega}_i - \underline{z}^{(i)^T} B^{(i)} B^{-1} \sum_j B^{(j)} \underline{z}^{(j)} \bar{\omega}_j = \underline{z}^{(i)^T} B^{(i)} B^{-1} \underline{r}.$$

Once the $\bar{\omega}_i$ are known, $\underline{u}$ can be found by solving (21).

ALGORITHM 6.4. *Use a Schwarz method with the subspaces given by $\tilde{V}_W^h$ and $\tilde{V}_{F^k}^h$ and the bilinear form given by $b_0^W(u^h, u^h)$ for the space $\tilde{V}_W^h$.*

THEOREM 6.4. *Algorithm* 6.4 *satisfies the three assumptions with*

$$C_0^2 \le C(1+\log(H/h))^2, \quad \rho(\mathcal{E}) \le C, \quad \omega \le C.$$

*The constants in the bounds are independent not only of the mesh size and the number of substructures, but also of the values* $\rho_i$ *of the coefficient of* (5).

*Proof.* We begin by estimating $\omega$. For the spaces $\tilde{V}_{F^k}^h$, $\omega = 1$, trivially, since we use exact projections. Let $u_0^h \in \tilde{V}_W^h$ and let $u_0^{(i),h}$ be the restriction of $u_0^h$ to $\bar{\Omega}_i$. Let $w_0^{(i),h} = u_0^{(i),h} - \bar{u}_0^{(i),h}$, where $\bar{u}_0^{(i),h}$ is the average of $u_0^{(i),h}$ over the wire basket. We split $w_0^{(i),h}$ into two parts, $w_0^{(i),h} = \sum \bar{w}_{0,F^k}^h \theta_{F^k} + w_W^{(i),h}$. The first has constant values on the faces, while the second vanishes there. Then, using Lemmas 4.4, 4.6, and 4.7, we obtain

$$\begin{aligned}
a(u_0^h, u_0^h) &= \sum_i \rho_i |u_0^{(i),h}|^2_{H^1(\Omega_i)} \\
&= \sum_i \rho_i |w_0^{(i),h}|^2_{H^1(\Omega_i)} \\
&\le C \sum_i \rho_i \left( \sum_k (\bar{w}_{0,F^k}^h)^2 |\theta_{F^k}|^2_{H^1(\Omega_i)} + |w_W^{(i),h}|^2_{H^1(\Omega_i)} \right) \\
&\le C(1+\log(H/h)) \sum_i \rho_i ||w_0^{(i),h}||^2_{L^2(W^i)} \\
&\le C b_0^W(u_0^h, u_0^h).
\end{aligned}$$

We now estimate $C_0^2$. We bound $b_0^W(u_0^h, u_0^h)$, using the interpolation operator and Lemma 4.3, by

$$\begin{aligned}
b_0^W(u_0^h, u_0^h) &\le (1+\log(H/h)) \sum_i \rho_i ||u_0^h - \bar{u}_0^h||^2_{L^2(W^i)} \\
&\le C(1+\log(H/h))^2 \sum_i \rho_i |u^h|^2_{H^1(\Omega_i)} \\
&= C(1+\log(H/h))^2 a(u^h, u^h).
\end{aligned}$$

Let $u_{F^k}^h = \mathcal{H}(\vartheta_{F^k}(u^h - u_0^h)) - \bar{u}_{\partial F^k}^h \theta_{F^k}$, where $\bar{u}_{\partial F^k}^h$ is defined in Lemma 4.6 and $u_0^h$ is the finite element function that equals $u^h$ on the wire basket nodes, vanishes on the faces, and is discrete harmonic in the interior of the subdomains.

Noting that the values of $u_0^h$ are irrelevant since $\vartheta_{F^k}$ vanishes on the wire basket, we then find that

$$\begin{aligned}
a(u_{F^k}^h, u_{F^k}^h) &\le C\{\rho_i \big( |I^h(\vartheta_{F^k}(u^h - u_0^h))|^2_{H^1(\Omega_i)} + (\bar{u}_{\partial F^k}^h)^2 |\theta_{F^k}|^2_{H^1(\Omega_i)} \big) \\
&\qquad + \rho_j \big( |I^h(\vartheta_{F^k}(u^h - u_0^h))|^2_{H^1(\Omega_j)} + (\bar{u}_{\partial F^k}^h)^2 |\theta_{F^k}|^2_{H^1(\Omega_j)} \big) \} \\
&\le C(1+\log(H/h))^2 \big( \rho_i |u^h|^2_{H^1(\Omega_i)} + \rho_j |u^h|^2_{H^1(\Omega_j)} \big).
\end{aligned}$$

Here, we use Lemmas 4.3–4.6. The full $H^1$ norm on the right-hand side can be reduced to the seminorm by noting that $u_{F^k}^h$ is invariant under the addition of a constant to $u^h$. We then sum over the subregions to obtain the necessary bound. □

We next consider two Neumann–Neumann algorithms.

ALGORITHM 6.5. *Use a Schwarz method with the subspaces given by* $\tilde{V}_W^h$ *and* $\tilde{V}_i^h$. *The bilinear form for the global space is given by* $b_0^W(u^h, u^h)$, *and those for the local spaces by the bilinear forms are given by* (16) *and* (17), *as described in* §5.

ALGORITHM 6.6. *Use a Schwarz method obtained from Algorithm* 6.5 *by replacing the local spaces* $\tilde{V}_i^h$ *by* $\tilde{V}_{i,\mathrm{mix}}^h$. *For all these local subspaces, the bilinear forms are given by* (16).

THEOREM 6.5. *Algorithms* 6.5 *and* 6.6 *satisfy the three assumptions with*

$$C_0^2 \leq C(1+\log(H/h))^2, \quad \rho(\mathcal{E}) \leq C, \quad \omega \leq C(1+\log(H/h))^2.$$

*The constants in the bounds are independent not only of the mesh size and the number of substructures, but also of the values* $\rho_i$ *of the coefficient of* (5).

A proof of these results can be given using the techniques of Dryja and Widlund [27]. Details will not be provided since these results are of relatively limited interest.

We conclude this subsection by discussing two earlier wire basket-based algorithms due to Bramble, Pasciak, and Schatz [7]. Their work has influenced much of the later work in the field. One of their coarse spaces is given in terms of the averages of the nodal values over the substructure boundaries $\partial\Omega_i$. The other space is defined by extending the wire basket values as a two-dimensional discrete harmonic function onto the faces, and then as a discrete harmonic function into the interiors of the subdomains. For both methods, Bramble, Pasciak, and Schatz proved the following; cf. [7].

THEOREM 6.6. *The condition number of the preconditioned problem is bounded by* $C(1+\log(H/h))^2$, *where the constant is independent not only of the mesh size and the number of substructures, but also of the values* $\rho_i$ *of the coefficient of* (5).

**6.3. Face-based algorithms.** We know from the previous subsection that the vertex space method, Algorithm 6.3, has a condition number that is independent of the parameters $h$ and $H$, but that this bound might not be independent of the variations of the coefficients across the interface $\Gamma$. We could explore the possibility of replacing the coarse space $\tilde{V}^H$ by the wire basket space $\tilde{V}_W^h$ as in Algorithm 6.4 and the use of the bilinear form $b_0^W(\cdot,\cdot)$. The local spaces could be chosen as in Algorithm 6.3. This leads to an algorithm for which we can prove the same type of bounds as in Theorem 6.4, i.e., the condition number is bounded by $C(1+\log(H/h))^2$. We can also show that a bound of $C(1+\log(H/h))$ holds if we allow the constant to depend on the variation of the $\rho_i$.

However, we have recently discovered two alternative coarse spaces for which it is possible to derive bounds on the condition number that are independent of the values of $\rho_i$ and that are linear in $(1+\log(H/h))$. The main ideas behind the first of these new algorithms is to expand the coarse space by allowing an additional degree of freedom for each face, rather than specifying the values on the face in terms of values on all or part of the wire basket. Later in this subsection, we will explore two more spaces which have in common the fact that the average values over the faces, or entire substructure boundaries, are important in the interpolation formulas that define the coarse space component and the coarse space as a whole.

The first coarse space of this kind, $\tilde{V}_M^h$, can be viewed as the range of the following interpolation operator:

$$I_M^h u^h(x) = \sum_{x_k \in W_h} u^h(x_k)\varphi_k(x) + \sum_{F^k \subset \Gamma} \bar{u}_{F^k}^h \theta_{F^k}(x).$$

The bilinear form is given by

$$b_0^M(u^h,u^h) = \sum_i \min_{\bar{\omega}_i} \rho_i \left\{ h||\underline{u} - \bar{\omega}_i \underline{z}^{(i)}||_{l^2(W^i)}^2 + H(1+\log(H/h)) \sum_{F^k \subset \partial\Omega_i} (\bar{u}_{F^k}^h - \bar{\omega}_i)^2 \right\}.$$

ALGORITHM 6.7. *Use a Schwarz method with the subspaces given by* $\tilde{V}_M^h$, $\tilde{V}_{F^i}^h$, $\tilde{V}_{\Gamma^{Ej}}^h$, *and* $\tilde{V}_{\Gamma^{Vj}}^h$ *and the bilinear form just given by* $b_0^M(u^h, u^h)$.

THEOREM 6.7. *Algorithm* 6.7 *satisfies the three assumptions with*

$$C_0^2 \leq C(1 + \log(H/h)), \quad \rho(\mathcal{E}) \leq C, \quad \omega \leq C.$$

*The constants in the bounds are independent of the values* $\rho_i$ *of the coefficient of* (5).

*Proof.* The proof of the first assumption is almost identical to that given for Theorem 6.3 in Smith [58] except that we now use $u_0 = I_M^h u^h$. Instead of Lemma 4.2, we use the following estimates:

$$\|u^h - I_M^h u^h\|_{L^2(\Omega_i)}^2 \leq CH^2 |u^h|_{H^1(\Omega_i)}^2 \tag{22}$$

and

$$|I_M^h u^h|_{H^1(\Omega_i)}^2 \leq C(1 + \log(H/h))|u^h|_{H^1(\Omega_i)}^2. \tag{23}$$

Inequality (22) follows from Lemmas 4.4 and 4.6.

The second inequality, (23), is established by using Lemmas 4.3, 4.4, 4.6, 4.7, and Poincaré's inequality.

The estimate

$$b_0^M(u_0, u_0) \leq C(1 + \log(H/h))a(u^h, u^h)$$

follows from Lemmas 4.3, 4.4, 4.6, and Poincaré's inequality.

Finally, we use Lemma 4.4 and an inverse inequality to establish that

$$a(u^h, u^h) \leq C b_0^M(u^h, u^h) \quad \forall u^h \in \tilde{V}_M^h. \qquad \square$$

We again consider two algorithms based on Neumann–Neumann solvers.

ALGORITHM 6.8. *Use a Schwarz method obtained from Algorithm* 6.5 *by replacing the coarse space* $\tilde{V}_W^h$ *by* $\tilde{V}_M^h$ *and the bilinear form* $b_0^W(\cdot,\cdot)$ *by* $b_0^M(\cdot,\cdot)$.

THEOREM 6.8. *Algorithm* 6.8 *satisfies the three assumptions with*

$$C_0^2 \leq C, \quad \rho(\mathcal{E}) \leq C, \quad \omega \leq C(1 + \log(H/h))^2.$$

*The constants in the bounds are independent not only of the mesh size and the number of substructures, but also of the values* $\rho_i$ *of the coefficient of* (5).

Proofs of this and the next result are given in Dryja and Widlund [29].

ALGORITHM 6.9. *Use a Schwarz method obtained from Algorithm* 6.6 *by replacing the coarse space* $\tilde{V}_W^h$ *by* $\tilde{V}_M^h$ *and the bilinear form* $b_0^W(\cdot,\cdot)$ *by* $b_0^M(\cdot,\cdot)$.

THEOREM 6.9. *Algorithm* 6.9 *satisfies the three assumptions with*

$$C_0^2 \leq C(1 + \log(H/h)), \quad \rho(\mathcal{E}) \leq C, \quad \omega \leq C(1 + \log(H/h))^2.$$

*The constants in the bounds are independent not only of the mesh size and the number of substructures, but also of the values* $\rho_i$ *of the coefficient of* (5).

We can decrease the dimension of the global space just considered. Rather than using the coarse subspace, involving all of the nodes on the edges, only one degree of freedom per edge, an average value, can be used. The resulting space, denoted by $\tilde{V}_B^h$, is the range of the interpolation operator

$$I_B^h u^h(x) = \sum_{V^k \in \Gamma} u^h(V^k)\varphi_k(x) + \sum_{E^i \subset W} \bar{u}_{E^i}^h \theta_{E^i}(x) + \sum_{F^k \subset \Gamma} \bar{u}_{F^k}^h \theta_{F^k}(x).$$

Here $\bar{u}^h_{E^i}$ is the average of the values of $u^h$ on $E^i_h$, and $\theta_{E^i}$ is the discrete harmonic function which equals 1 on that set and vanishes elsewhere on $\Gamma_h$. We define the bilinear form for this space by

$$b_0^B(u^h,u^h)=\sum_i \rho_i \min_{\bar{\omega}_i}\Big\{h\sum_{V^k\in\partial\Omega_i}(u^h(V^k)-\bar{\omega}_i)^2 +H\sum_{E^i\subset\partial\Omega_i}(\bar{u}^h_{E^i}-\bar{\omega}_i)^2+H(1+\log(H/h))\sum_{F^k\subset\partial\Omega_i}(\bar{u}^h_{F^k}-\bar{\omega}_i)^2\Big\}.$$

ALGORITHM 6.10. *Use a Schwarz method with the subspaces given by* $\tilde{V}^h_B$, $\tilde{V}^h_{F^i}$, $\tilde{V}^h_{\Gamma^{Ej}}$, *and* $\tilde{V}^h_{\Gamma^{Vj}}$ *and the bilinear form associated with* $b_0^B(u^h,u^h)$.

THEOREM 6.10. *Algorithm* 6.10 *satisfies the three assumptions with*

$$C_0^2\le C(1+\log(H/h)),\quad \rho(\mathcal{E})\le C,\quad \omega\le C.$$

*The constants in the bounds are independent not only of the mesh size and the number of substructures, but also of the values* $\rho_i$ *of the coefficient of* (5).

*Proof.* Except for an upper bound on $b_0^B(u^h,u^h)$, the estimate of $C_0^2$ is almost the same as in the proofs of Theorems 6.3 and 6.7. We use the interpolant $I^h_B u^h$ to define the coarse space component $u_0$. Using similar techniques as before, we can establish the estimates

$$\|u^h-I^h_Bu^h\|^2_{L^2(\Omega_i)}\le H^2(1+\log(H/h))|u^h|^2_{H^1(\Omega_i)}$$

and

$$|I^h_Bu^h|^2_{H^1(\Omega_i)}\le C(1+\log(H/h))|u^h|^2_{H^1(\Omega_i)}.$$

The upper bound

$$b_0^B(u_0,u_0)\le C(1+\log(H/h))a(u^h,u^h)$$

now follows from these inequalities, and that of Poincaré, choosing $\bar{\omega}_i=\bar{u}^h_{W^i}$.

Finally, we use the same bounds and an inverse inequality to establish that $a(u^h,u^h)\le Cb_0^B(u^h,u^h)$. ∎

We can also use the Neumann–Neumann solvers for the local components of the preconditioner.

ALGORITHM 6.11. *Use a Schwarz method obtained from Algorithm* 6.5 *by replacing the coarse space* $\tilde{V}^h_W$ *by* $\tilde{V}^h_B$ *and the bilinear form* $b_0^W(\cdot,\cdot)$ *by* $b_0^B(\cdot,\cdot)$.

THEOREM 6.11. *Algorithm* 6.11 *satisfies the three assumptions with*

$$C_0^2\le C,\quad \rho(\mathcal{E})\le C,\quad \omega\le C(1+\log(H/h))^2.$$

*The constants in the bounds are independent not only of the mesh size and the number of substructures, but also of the values* $\rho_i$ *of the coefficient of* (5).

Proofs of this and the next result are given in Dryja and Widlund [29].

ALGORITHM 6.12. *Use a Schwarz method obtained from Algorithm* 6.6 *by replacing the coarse space* $\tilde{V}^h_W$ *by* $\tilde{V}^h_B$ *and the bilinear form* $b_0^W(\cdot,\cdot)$ *by* $b_0^B(\cdot,\cdot)$.

THEOREM 6.12. *Algorithm* 6.12 *satisfies the three assumptions with*

$$C_0^2\le C(1+\log(H/h)),\quad \rho(\mathcal{E})\le C,\quad \omega\le C(1+\log(H/h))^2.$$

*The constants in the bounds are independent not only of the mesh size and the number of substructures, but also of the values $\rho_i$ of the coefficient of* (5).

Our last coarse space, which has much in common with one developed and analyzed recently by Mandel and Brezina [42], has only one degree of freedom per substructure. This space, $V_C^h$, is defined as the range of the interpolation operator

$$I_C^h u^h(x) = \sum_i \bar{u}^h_{\partial\Omega_i} \rho_i^{1/2} \lambda_i^\dagger(x).$$

The summation is over all substructures that do not intersect $\Gamma_0$ in more than one or a few points. $\lambda_i^\dagger(x)$ is the pseudoinverse of the function defined in formula (14) except that it is also set to zero at single points of $\Gamma_0$ if $\partial\Omega_i$ intersects $\Gamma_0$ in just one or a few points. $\bar{u}^h_{\partial\Omega_i}$ is the average value of $u^h$ over the set $\partial\Omega_{i,h}$.

The only essential difference between this coarse space and the one introduced in [42] is that the exponent of $\rho_i$ is $1/2$ rather than 1. In fact, any exponent greater than or equal to $1/2$ may be used; see Dryja, Sarkis, and Widlund [22]. We also note that our careful treatment of the boundary substructures that do not share a face with the boundary $\partial\Omega$ allows us to obtain good bounds without imposing extra restrictions on the intersection of the boundaries of the individual substructures and that of the original region.

The bilinear form is chosen as

$$b_0^C(u^h, v^h) = (1 + \log(H/h)) \sum_i s^{(i)}(u^h, v^h).$$

ALGORITHM 6.13. *Use a Schwarz method with the global space $V_C^h$, the bilinear form $b_0^C(\cdot,\cdot)$, and the local subspaces $\tilde{V}_i^h$, choosing the bilinear forms as in formulas* (16) *and* (17) *according to the rule given in* §5.

THEOREM 6.13. *Algorithm* 6.13 *satisfies the three assumptions with*

$$C_0^2 \leq C, \quad \rho(\mathcal{E}) \leq C, \quad \omega \leq C(1 + \log(H/h))^2.$$

*The constants in the bounds are independent not only of the mesh size and the number of substructures, but also of the values $\rho_i$ of the coefficients of* (5).

A proof of this result is given in Dryja and Widlund [29].

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