Computing 7, 17—24 (1971)

A Note on the ε-Algorithm[1]

By

J. B. McLeod, Madison and Oxford

(Received May 12, 1969)

Summary — Zusammenfassung

A Note on the ε-Algorithm. This paper contains the proof of a fundamental algebraic result in the theory of the vector ε-algorithm. The relationships of this algorithm involve the addition, subtraction and inversion of vectors of complex numbers: the first two operations are defined by component-wise addition and subtraction; the inverse of the vector $z = (z_1, \ldots, z_N)$ is taken to be

$$z^{-1} = \frac{(\bar{z}_1, \ldots, \bar{z}_N)}{\sum\limits_{i=1}^{N} |z_i|^2}$$

where the bar denotes a complex conjugate. It is proved that if vectors $\varepsilon_s^{(m)}$ can be constructed from the initial values $\varepsilon_{-1}^{(m)} = 0$, $(m = 1, 2, \ldots)$, $\varepsilon_0^{(m)} = s_m$, $(m = 0, 1, \ldots)$ by means of the relationships $\varepsilon_{s+1}^{(m)} = \varepsilon_{s-1}^{(m+1)} + (\varepsilon_s^{(m+1)} - \varepsilon_s^{(m)})^{-1}$, $(m, s = 0, 1, \ldots)$; and if the recursion relations $\sum\limits_{i=0}^{n} \beta_i s_{m+i} = \left(\sum\limits_{i=0}^{n} \beta_i\right) a$, $(m = 0, 1, \ldots)$ hold for the initial values, where the coefficients β_i $(i = 0, 1, \ldots, n)$ are real and $\beta_n \neq 0$, then for $m = 0, 1, \ldots$, $\varepsilon_{2n}^{(m)} = a$, if $\sum\limits_{i=0}^{n} \beta_i \neq 0$, and $\varepsilon_{2n}^{(m)} = 0$, if $\sum\limits_{i=0}^{n} \beta_i = 0$.

Bemerkung zum ε-Algorithmus. Diese Arbeit beinhaltet ein fundamentales algebraisches Ergebnis der Theorie des vektoriellen ε-Algorithmus. Als Verknüpfungen dieses Algorithmus werden verwendet die Addition, die Subtraktion und der inverse Vektor mit komplexen Komponenten. Die ersten beiden Operationen sind definiert durch komponentenweise Addition beziehungsweise Subtraktion. Sei $z = (z_1, \ldots, z_N)$ ein vorgegebener Vektor, so soll der inverse Vektor auf folgende Weise gebildet werden

$$z^{-1} = \frac{(\bar{z}_1, \ldots, \bar{z}_N)}{\sum\limits_{i=1}^{N} |z_i|^2},$$

wobei der Querstrich die konjugiert komplexe Zahl bedeutet. Unter der Voraussetzung, daß der Vektor $\varepsilon_s^{(m)}$ aus den Anfangsbedingungen $\varepsilon_{-1}^{(m)} = 0$, $(m = 1, 2, \ldots)$,

[1] This work was sponsored by the Mathematics Research Center, United States Army, Madison, Wis., under Contract No. DA-31-124-ARO-D-462.

$\varepsilon_0^{(m)} = s_m$, $(m = 0, 1, \ldots)$ mittels der Beziehungen $\varepsilon_{s+1}^{(m)} = \varepsilon_{s-1}^{(m+1)} + (\varepsilon_s^{(m+1)} - \varepsilon_s^{(m)})^{-1}$, $(m, s = 0, 1, \ldots)$ gebildet werden kann und unter der weiteren Voraussetzung, daß die Rekursionsformel $\sum_{i=0}^{n} \beta_i s_{m+1} = \left(\sum_{i=0}^{n} \beta_i\right) a$, $(m = 0, 1, \ldots)$ auch für die Anfangsbedingungen gilt, wobei die Koeffizienten β_i $(i = 0, 1, \ldots, n)$ reell und ungleich Null sein sollen, wird für $m = 0, 1, \ldots$ bewiesen, daß die Beziehungen $\varepsilon_{2n}^{(m)} = a$ gilt für $\sum_{i=0}^{n} \beta_i \neq 0$ und $\varepsilon_{2n}^{(m)} = 0$ gilt, wenn $\sum_{i=0}^{n} \beta_i = 0$.

1.

Let us display an array of mathematical entities in the form

$$
\begin{array}{ccccccc}
\varepsilon_{-1}^{(0)} & & & & & & \\
 & \varepsilon_{0}^{(0)} & & & & & \\
\varepsilon_{-1}^{(1)} & & \varepsilon_{1}^{(0)} & & & & \\
 & \varepsilon_{0}^{(1)} & & \ddots & & & \\
\varepsilon_{-1}^{(2)} & & \varepsilon_{1}^{(1)} & & \varepsilon_{s}^{(0)} & & \\
 & \varepsilon_{0}^{(2)} & & \ddots & & \varepsilon_{s+1}^{(0)} & \\
\varepsilon_{-1}^{(3)} & \cdot & \varepsilon_{1}^{(2)} & & \varepsilon_{s}^{(1)} & & \ddots \\
\cdot & \cdot & \cdot & \ddots & & \varepsilon_{s+1}^{(1)} & \\
\cdot & \cdot & \cdot & & \varepsilon_{s}^{(2)} & & \ddots \\
\cdot & & \cdot & & \cdot & \varepsilon_{s+1}^{(2)} & \\
 & & & & \cdot & \cdot & \ddots \\
 & & & & \cdot & \cdot & \\
 & & & & & \cdot &
\end{array}
$$

so that the superscript (m) denotes a diagonal and the subscript s denotes a column. In the ε-algorithm, the elements in the array are related by the equations

$$\varepsilon_{s+1}^{(m)} - \varepsilon_{s-1}^{(m+1)} = \{\varepsilon_s^{(m+1)} - \varepsilon_s^{(m)}\}^{-1}, \ (m = 0, 1, 2, \ldots; \ s = 0, 1, 2, \ldots), \tag{1.1}$$

which imply, of course, some definition for the operations of subtraction and taking the multiplicative inverse. Graphically, the rule (1.1) means that if we take four $\varepsilon_s^{(m)}$ arranged in a "lozenge",

$$
\begin{array}{ccc}
 & \varepsilon_s^{(m)} & \\
\varepsilon_{s-1}^{(m+1)} & & \varepsilon_{s+1}^{(m)} \\
 & \varepsilon_s^{(m+1)} &
\end{array}
$$

then the member most to the right is obtained by adding the member most to the left to the inverse of the difference of the two in the middle. The generation of an array in this manner has important applications in numerical analysis [1].

The simplest way of ensuring the existence of the requisite operations would be to suppose that the $\varepsilon_s^{(m)}$ are real of complex numbers, although the possibility $\varepsilon_s^{(m+1)} - \varepsilon_s^{(m)} = 0$ would have to be excluded. In this case, a fundamental result is the following, proved in § 5 of [2]:

Theorem 1. *Let the complex numbers s_m satisfy the relations*

$$\sum_{i=0}^{n} \beta_i \, s_{m+i} = \left(\sum_{i=0}^{n} \beta_i \right) a, \quad (m = 0, 1, \ldots)$$

where β_i $(i = 0, 1, \ldots, n)$ and a are fixed complex numbers and (without loss of generality) $\beta_n \neq 0$.

If then $\varepsilon_{-1}^{(m)} = 0$, $(m = 0, 1, 2, \ldots)$ and $\varepsilon_0^{(m)} = s_m$, $(m = 0, 1, 2, \ldots)$, we have for all $m = 0, 1, 2, \ldots$ that

$$\varepsilon_{2n}^{(m)} = a, \quad \text{if } \sum_{i=0}^{n} \beta_i \neq 0,$$

$$\varepsilon_{2n}^{(m)} = 0, \quad \text{if } \sum_{i=0}^{n} \beta_i = 0,$$

provided that at no point in the development of the array does

$$\varepsilon_s^{(m+1)} - \varepsilon_s^{(m)} = 0.$$

Perhaps it should be remarked at this point that the requirement $\varepsilon_s^{(m+1)} - \varepsilon_s^{(m)} \neq 0$ can be avoided by suitably redefining the array, as in [3]. The point is that though $\varepsilon_s^{(m+1)} - \varepsilon_s^{(m)} = 0$ leaves one element of the array undefined, the subsequent elements of the array may, if we consider the limiting process as $\varepsilon_s^{(m+1)} - \varepsilon_s^{(m)} \to 0$, tend to well-defined finite limits. If, when $\varepsilon_s^{(m+1)} - \varepsilon_s^{(m)} = 0$, we define the subsequent terms in the array by these limiting values, we have obtained an extension of the definition of the array for which, by continuity, Theorem 1 must still hold good. Similar remarks about extending the definition of the array will be possible at various other points in this paper, but we will not bother to make them. We will assume that the definition of the array is valid in its original form.

Reference [2] actually proves Theorem 1 under wider conditions than those we have so far stated.

Theorem 2. *If s_m, a are elements of an associative division algebra over the complex numbers, and the β_i are complex numbers, then the conclusion of Theorem 1 continues to hold good.*

It may be desirable to state the precise definition of an associative division algebra over the field of complex numbers, since the assumptions stated in [2] are not in fact sufficient to ensure the validity of the subsequent analysis. (For example, distributivity, which is essential, is not assumed.) The reader will readily verify that, with the definition given below, the analysis of [2] is valid. It is also worth remarking that we have restricted the field over which the algebra is taken to be the field of complex numbers, which again is not done in [2]; but some restriction on the field is certainly necessary, since there is a tacit assumption in Theorem 4. 5 of [2] that the field is algebraically closed; and though the result could be generalized, the restriction to complex numbers covers all interesting cases.

Let $\alpha, \beta, \gamma, \ldots$ be complex numbers. A system A consisting of elements $a, b, c, \ldots$ is said to be an associative division algebra over the complex numbers if there are defined for the elements of A the operations of addition of two elements $(a + b)$, multiplication of two elements $(a\,b)$ and multiplication of an element by a complex number ($\alpha\,a$ or $a\,\alpha$), with the following properties.

(i) If $a, b \in A$ and α is a complex number, then

$$a + b, \quad a\,b, \quad \alpha\,a, \quad a\,\alpha \in A.$$

(ii) Addition is commutative and associative, i. e.

$$a + b = b + a, \qquad (a + b) + c = a + (b + c).$$

(iii) Multiplication by a complex number satisfies

$$\alpha\,a = a\,\alpha, \qquad (\alpha\,\beta)\,a = \alpha\,(\beta\,a), \qquad (\alpha\,a)\,(\beta\,b) = (\alpha\,\beta)\,(a\,b),$$
$$(\alpha + \beta)\,a = \alpha\,a + \beta\,a, \qquad \alpha\,(a + b) = \alpha\,a + \alpha\,b,$$
$$1\,a = a.$$

(iv) Multiplication is associative and distributive with respect to addition, i. e.

$$(a\,b)\,c = a\,(b\,c),$$
$$(a + b)\,c = a\,c + b\,c, \qquad a\,(b + c) = a\,b + a\,c.$$

Multiplication is not necessarily commutative.

(v) There is a zero element $0 \in A$ such that

$$0 + a = a + 0 = a,$$
$$0\,a = 0.$$

The last equation is to hold whether 0 on the left-hand side is the zero of the complex numbers or the zero of the algebra. In fact, the context always makes it quite clear which zero is required.

(vi) There is a unit element $e \in A$ such that

$$e\,a = a\,e = a.$$

(vii) If $a\ (\neq 0) \in A$, then a has a multiplicative inverse a^{-1} such that

$$a\,a^{-1} = a^{-1}\,a = e.$$

The question has been raised by Professor P. WYNN whether the result of *Theorem 2* can be extended to vectors over the complex numbers. To do this, we have to define the inverse of the vector

$$z = (z_1, \ldots, z_N),$$

and this we do by writing

$$z^{-1} = \frac{(\bar{z}_1, \ldots, \bar{z}_N)}{\sum\limits_{i=1}^{N} |z_i|^2},$$

where the bar denotes complex conjugate. With this definition, we can set up an ε-array of vectors, and the object of this paper is to prove

Theorem 3. *With the above definition for the inverse of a vector, the conclusion of Theorem 1 holds good if s_m, a are N-dimensional vectors over the complex numbers and the β_i are real numbers.*

The restriction of the β_i to be real is to be noted. It arises in the proof because a correspondence which we wish to set up between the vectors and what is in effect an associative division algebra fails to hold except when the β_i are real. No counter-example to the theorem is known, however, even when the β_i are complex, and it remains an open question whether or not the theorem is true in this case. The proof of *Theorem 3* is given in §§ 2, 3.

I am grateful to Professor WYNN for bringing this problem to my notice, and also for directing my attention to his work on continued fractions on which the solution is based.

2.

To prove *Theorem 3*, we note first that we can extend *Theorem 2* to the case where s_m, a are $M \times M$ matrices (M any integer) over the complex numbers, provided, of course, that as always the matrices which have to be inverted in the course of forming the ε-array are indeed invertible. The difficulty in carrying through this extension is that the $M \times M$ matrices do not form an associative division algebra because of the absence of an inverse for singular matrices, and this affects not only the ε-array itself, where the difficulty is overcome by assuming that the matrices occurring are invertible, but also the proof, in the course of which it is regularly assumed that elements of the division algebra, now matrices, are invertible.

However, given any specific relation of the form

$$\sum_{i=0}^{n} \beta_i s_{m+i} = \left(\sum_{i=0}^{n} \beta_i\right) a, \quad (m = 0, 1, 2, \ldots), \tag{2.1}$$

if we wish to prove, using the proof of *Theorem 2*, that any finite number of elements of the form $\varepsilon_{2n}^{(m)}$ equal a or 0 (n being fixed by (2.1) and m

varying), then it is easy to verify by a casual reading of the proof of *Theorem 2* that only a finite number of matrices need to be inverted. These matrices depend on the entries in the matrices $s_0, s_1, \ldots, s_{n-1}$, the matrices $s_n, s_{n+1}, \ldots$ being fixed then by (2.1). The matrices which we require to invert are indeed invertible unless certain determinants vanish, and that these determinants should not vanish is equivalent to demanding that certain polynomials in the entries in $s_0, s_1, \ldots, s_{n-1}$ should not vanish. The proof of *Theorem 2* therefore extends to $M \times M$ matrices provided that the matrices $s_0, s_1, \ldots, s_{n-1}$ are such that a finite number of polynomials in their entries do not vanish.

We now remark that none of these polynomials vanishes identically. We see this by setting $s_0 - a, s_1 - a, \ldots, s_{n-1} - a$ equal to scalar multiples of the unit matrix I. Then the whole of the proof of *Theorem 2* is carried through in terms of scalar multiples of I, which implies that we are in a division algebra, that the proof of *Theorem 2* holds, and that the various determinants which appear do not vanish.

To complete the extension of *Theorem 2* to $M \times M$ matrices, we have merely to deal with the case in which $s_0, s_1, \ldots, s_{n-1}$ are such that one or more of the polynomials vanishes (without, of course, the elements in the ε-array becoming meaningless.) This we do by continuity. For if one of the polynomials vanishes, we adjust an entry in one of $s_0, s_1, \ldots, s_{n-1}$ slightly so as to render the polynomial non-zero. Then we know that $\varepsilon_{2n}^{(m)} = a$ or 0, and so by continuity this must continue to hold even when the polynomial vanishes.

3.

In view of this extension of *Theorem 2*, it will be sufficient, in order to prove *Theorem 3*, if we can find an integer M and a $1 - 1$ correspondence between the N-dimensional vectors and a subset of the $M \times M$ matrices over the complex numbers which maintains linearity (with real scalars) and division, i. e. if $z \leftrightarrow A_z$, then

$$\alpha z_1 + \beta z_2 \leftrightarrow \alpha A_{z_1} + \beta A_{z_2},$$
$$z^{-1} \leftrightarrow (A_z)^{-1},$$

α, β being any two real numbers. For *Theorem 3* then becomes a particular case of *Theorem 2* (extended to matrices) by virtue of the correspondence $z \leftrightarrow A_z$. (Real scalars are sufficient because the β_i are restricted to be real and in the formation of the array the only linear operations are those of addition and subtraction.)

To set up the correspondence try

$$A_z = x_1 B_1 + y_1 C_1 + x_2 B_2 + y_2 C_2 + \ldots + x_N B_N + y_N C_N,$$

where $z = (z_1, \ldots, z_N) = (x_1 + i y_1, \ldots, x_N + i y_N)$ and $B_1, C_1, \ldots, B_N$, C_N are fixed $M \times M$ matrices which are linearly independent but which will be more precisely specified later. This correspondence is certainly

1 — 1 between the vectors and matrices of the form

$$\sum_{i=1}^{N} (x_i B_i + y_i C_i),$$

and it preserves linearity with real scalars.

It only remains to insist that the correspondence maintains division, i. e. we want

$$A_z A_{z^{-1}} = I,$$

where I is the unit $M \times M$ matrix. The condition for this is that

$$\begin{aligned}(x_1 B_1 + y_1 C_1 + \dots + x_N B_N + y_N C_N)\times \\ (x_1 B_1 - y_1 C_1 + \dots + x_N B_N - y_N C_N) = \\ = (x_1^2 + y_1^2 + \dots + x_N^2 + y_N^2)\, I,\end{aligned}$$

and comparing coefficients on both sides, we have

$$\begin{aligned}
&B_r^2 = - C_r^2 = I && (r = 1, \dots, N),\\
&B_r C_s - C_s B_r = 0 && (r, s = 1, \dots, N),\\
&B_r B_s + B_s B_r = 0 && (r, s = 1, \dots, N;\ r \neq s),\\
&C_r C_s + C_s C_r = 0 && (r, s = 1, \dots, N;\ r \neq s).
\end{aligned}$$

If we have a set of $2N + 1$ $M \times M$ matrices A_r such that

$$\begin{aligned}
&A_r^2 = I && (r = 1, \dots, 2N + 1),\\
&A_r A_s + A_s A_r = 0 && (r, s = 1, \dots, 2N + 1;\ r \neq s),
\end{aligned}$$

then it is easily verified that if we set

$$\begin{aligned}
&B_r = A_{r+1} && (r = 1, \dots, N),\\
&C_r = A_1 A_{N+r+1} && (r = 1, \dots, N),
\end{aligned}$$

then the equations above for B_r, C_r are satisfied, so that the problem reduces to finding the set A_r.[2]

The matrices A_r engender a Clifford algebra ([5], [6]).

We can find a set A_r as follows. We take

$$A_1 = \begin{pmatrix} I & O \\ O & -I \end{pmatrix},$$

the partitioning being into four squares of equal size. (This implies that M is divisible by 2.) Certainly, $A_1^2 = I$.

[2] I originally proved *Theorem 3* for the case of real vectors, in which case the correspondence $z \leftrightarrow z_1 A_1 + \dots + z_N A_N$ preserves linearity and division, as can be easily checked, and there is then no need to introduce the matrices B_r, C_r. I am grateful to Professor Wynn (cf. [7]) for pointing out to me that the introduction of B_r, C_r makes possible the extension to complex vectors.

The equation

$$A_1 A_s + A_s A_1 = 0$$

is certainly satisfied for $s > 1$ if A_s has the form

$$\begin{pmatrix} O & X_s \\ X_s & O \end{pmatrix},$$

with the same partitioning as before. Moreover, if $X_s^2 = I$, then $A_s^2 = I$, so that in particular, if we set

$$X_2 = \begin{pmatrix} I & O \\ O & -I \end{pmatrix},$$

making X_2, as it were, the same as A_1 but half the size, then $A_2^2 = I$. (Note that the partitioning of X_2 implies that M must be divisible by 4.)

The procedure can be repeated. For $t > 2$, we set

$$X_t = \begin{pmatrix} O & Y_t \\ Y_t & O \end{pmatrix},$$

with

$$Y_t = \begin{pmatrix} I & O \\ O & -I \end{pmatrix},$$

all the partitionings being into four equal squares, so that M must be divisible by 8. Then for $t > 2$

$$A_2 A_t + A_t A_2 = 0,$$

and it is also clear that, if $Y_t^2 = I$, then $X_t^2 = I$, giving $A_t^2 = I$. In particular, $A_3^2 = I$.

If $M = 2^{2N+1}$, we can define in this way all the A_r, and it should generally be remarked that the distribution of the zero entries in the A_r ensures that the matrices $B_1, C_1, \ldots, B_N, C_N$ are linearly independent.

References

[1] Wynn, P.: On a device for computing the $e_m(S_n)$ transformation. M. T. A. C. **10**, 91–96.

[2] Wynn, P.: Continued fractions whose coefficients obey a non-commutative law of multiplication. Arch. Rat. Mech. Anal. **12**, 273–312 (1963).

[3] Wynn, P.: Singular rules for certain non-linear algorithms. Bit **3**, 175–195 (1963).

[4] Clifford, W. K.: Applications of Grassmann's extensive algebra. Amer. Jour. Math. **1**, 350–358 (1878).

[5] Brauer, R., and H. Weyl: Spinors in n dimensions. Amer. J. Math. **57**, 425–449 (1935).

[6] Chevalley, C.: The algebraic theory of spinors, New York. 1954.

[7] Wynn, P.: Complex number and other extensions to the Clifford algebra with an application to the theory of continued fractions, Math. Research Center, Univ. of Wisconsin, Report 646.

J. B. McLeod
Wadham College
Oxford, Great Britain