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Asynchronous multisplitting methods for nonlinear fixed point problems

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Our aim is to present for nonlinear problems asynchronous multisplitting algorithms including both the basic situation of O'Leary and White and the discrete analogue of Schwarz's alternating method and its multisubdomain extensions and moreover their two-stage counterparts. The analysis of these methods is based on El Tarazi's convergence theorem for asynchronous iterations and leads to a good level of asynchronism in each of the considered situations.

Keywords: asynchronous algorithms, multisplitting methods, nonlinear problems

1. Introduction

The term "multisplitting methods" now covers a wide class of iterative methods in order to get the solution of linear or nonlinear systems of equations having their origin in the pioneer work of O'Leary and White [14]. The first attempt to give asynchronous variants (more precisely chaotic in the sense of Chazan and Miranker) of this method goes back to Bru et al. [4]. Since this first approach, several points of view have been developed about the theme of asynchronous multisplitting methods, see [5,9,12] on the one hand, and [2,6], on the other. In the algorithms considered here we do not take into account multiacceleration parameters studied in the last two quoted papers.

With the above quoted exception, the aim of our paper is to give an algorithmic formulation and the corresponding mathematical analysis for such discrete overlapping subdomain techniques, also including the most basic one, namely the ancient Schwarz "alternating" technique, a process also considered here in the context of the discrete analogue of multisubdomain decomposition methods. We present also the two-stage counterparts of the considered methods. This aim is achieved through:

– the introduction of suitably extended fixed point mappings associated with a collection of fixed point mappings that we call a formal multisplitting;
– the use of El Tarazi's theorem in order to study the behavior of general asynchronous iterations applied to all such fixed point mappings, including the ones associated with the corresponding two-stage methods;
– the introduction of notions of s-regular splitting and strongly regular splitting well suited for nonlinear problems and slightly more particular than the standard regular splitting when we consider the linear case associated with an M-matrix;
– the introduction of a collection of weighting matrices which are a generalization of the original ones proposed by O'Leary and White.

It is worthwhile noting that asynchronous fixed point methods are not only a family of algorithms suitable for asynchronous computations on multiprocessors, but also a general framework in order to formulate general iteration methods associated with a fixed point mapping on a product space, including the most standard ones such as the successive approximation method (linear or nonlinear Jacobi method) and linear or nonlinear Gauss–Seidel method among many others. In this perspective we are able to particularize our general formulation of asynchronous multisplitting methods to the situations of O'Leary and White multisplittings and Schwarz alternating methods and the corresponding two-stage algorithms.

It is of course worth noting that we are concerned with a fixed point methodology which is efficient when the corresponding fixed point mapping is contractive with a good constant of contraction. Otherwise, it is certainly preferable to use other kinds of subdomain methods: multilevel ones, preconditioned conjugate gradient techniques and nonsymmetric variants for which we refer to [11,19].

It is not within the scope of the present paper to propose a study of the constants of contraction of the introduced fixed point mappings. In the context of overlapping subdomain methods associated with partial differential equations, it can nevertheless be said that good contraction properties for such fixed point mappings depend on:

– the boundary condition (Dirichlet ones are better),
– the kind of elliptic operator (some diffusion convection operators are good candidates),
– the amount of overlapping and the number of subdomains.

In circumstances when these fixed point methods converge in few iterations, they can appear to be simpler than others and moreover, in a parallel computing framework, they do not need synchronization. This last aspect implies weaker delays due to interprocessor communications and to load balancing defect.

The rest of the paper is organized as follows.

Section 2 is devoted to a succinct background about asynchronous algorithms and El Tarazi's theorem, see [8].

In section 3 we first give a general definition of formal fixed point multisplittings associated with a nonlinear problem. Then we introduce an extended fixed point mapping with the help of general weighting matrices which generalize those usually used in the case of multisplitting techniques. We then give a brief comment on the use of the level of generality of these weighting matrices, with respect to the description of various algorithmic models (general asynchronous "O'Leary and White", "Schwarz" algorithms and mainly their two-stage counterparts). After such a justification of our general framework, we present a contraction property of the extended fixed point mapping with respect to a weighted maximum norm, which with the help of El Tarazi's theorem leads us to the convergence property for associated asynchronous iterations. Then we introduce the notion of s-regular splitting of a suitably contractive mapping.

In section 4, we are interested in overlapping multisplitting in the context of nonlinear fixed point mappings, which appears to be a particular case of multi s-regular splittings. Our formulation gives rise to asynchronous O'Leary and White and Schwarz multisplitting methods for nonlinear problems. We formulate the corresponding convergence results, we describe some specific properties related to the use of weighting matrices and we obtain a unification of these two approaches in the case of the exact solution of subproblems. The section ends by an illustration of our results for the case of so called pseudolinear problems.

In section 5 we first introduce and characterize the notion of strongly regular splitting through an underrelaxed s-regular splitting fixed point mapping. This allows us to deal with the nonlinear perturbation of a class of H-matrices. Then our strongly regular splitting can be associated with a simple variant of H-compatible splittings defined in [5]. The rest of the section is devoted to a study of two-stage multisplitting algorithms for pseudolinear problems with the help of our generalization of weighting matrices. These algorithms are considered in the context of both Schwarz discrete multisubdomain, and multisplitting O'Leary and White algorithms. The asynchronism formalism is of great help here to describe some two-stage algorithms, even if they are not asynchronous algorithms in a computer science point of view. Moreover, the two kinds of methods (O'Leary and White on the one hand and Schwarz on the other) can now be closely associated during the execution of each outer iteration of a considered algorithm. Moreover, in linear situations we get similar results in the context of weakly regular splittings.

Our presentation covers such previously known methods (with the exception of multi acceleration parameters techniques of [2,6]), with weaker synchronization constraints, that is to say, it permits taking a better advantage of asynchronism.

2. Preliminaries. Asynchronous iterations, contraction with respect to a weighted maximum norm, some basic definitions

Asynchronous algorithms are used in the mathematical modelling of the parallel treatment of problems taking into account interaction processes. Let B_i, $i \in \{1, \dots, \alpha\}$, be Banach spaces equipped with the norms $\| \cdot \|_i$. Consider $B = \prod_{i=1}^{\alpha} B_i$ equipped with the norm

$$| \ |_{\infty,\gamma} = \max_{1 \leqslant l \leqslant \alpha} \frac{\| \ \|_l}{\gamma_l}, \quad \gamma_l > 0,\ l \in \{1, \ldots, \alpha\}.$$

Define

$J = \{J(p)\}_{p \in \mathbb{N}}$ a sequence of nonempty subsets of $\{1, \ldots, \alpha\}$,

$S = \{\overline{\rho}_1(p), \ldots, \overline{\rho}_\alpha(p)\}_{p \in \mathbb{N}}$ a sequence of $\mathbb{N}^\alpha$ such that:

(h1) $\forall i \in \{1, \ldots, \alpha\}$, the subset $\{p \in \mathbb{N},\ i \in J(p)\}$ is infinite,

(h2) $\forall i \in \{1, \ldots, \alpha\}$, $\forall p \in \mathbb{N}$, $\overline{\rho}_i(p) \leqslant p$,

(h3) $\forall i \in \{1, \ldots, \alpha\}$, $\lim_{p \to \infty} \overline{\rho}_i(p) = \infty$.

The asynchronous algorithm associated with the operator T and denoted (T, u^0, J, S) is defined by, see [8],

$$\begin{cases} u^0 = \left(u^{0,1}, \ldots, u^{0,\alpha}\right), \\ u^{l,p+1} = \begin{cases} T_l\left(u^{1,\overline{\rho}_1(p)}, \ldots, u^{\alpha,\overline{\rho}_\alpha(p)}\right) & \text{if } l \in J(p), \\ u^{l,p} & \text{if } l \notin J(p), \end{cases} \\ \quad p = 1, 2, \ldots,\ l = 1, \ldots, \alpha, \end{cases} \tag{2.1}$$

where $J(p)$ is the set of components to be updated at step p, $p - \overline{\rho}_l(p)$ is the delay due to the lth processor when it computes the lth block at the pth iteration.

If we take $\overline{\rho}_i(p) = p$ then (2.1) describes synchronous algorithms.

If we take

$$\begin{cases} \overline{\rho}_i(p) = p & \forall p = 1, 2, \ldots,\ i = 1, 2, \ldots, \\ J(p) = \{1, 2, \ldots, \alpha\} & \forall p = 1, 2, \ldots, \end{cases}$$

then (2.1) describes the algorithm of Jacobi.

If we take

$$\begin{cases} \overline{\rho}_i(p) = p & \forall p = 1, 2, \ldots,\ i = 1, 2, \ldots, \\ J(p) = 1 + p \pmod{\alpha} & \forall p = 1, 2, \ldots, \end{cases}$$

then (2.1) describes the algorithm of Gauss–Seidel.

Theorem 2.1. Let T be a mapping from $D(T) \subset B$ in B and suppose that:

(a) $D(T) = \prod_{i=1}^{\alpha} D_i(T)$, where $D_i(T)$ are closed and convex,

(b) $T(D(T)) \subset D(T)$,

(c) $\exists u^* \in D(T)$, such that $u^* = T(u^*)$,

(d) $\forall u \in D(T)$, $|T(u) - u^*|_{\infty,\gamma} \leqslant \beta |u - u^*|_{\infty,\gamma}$ with $0 < \beta < 1$.

Then every asynchronous algorithm (T, u^0, J, S) associated with T with a starting point $u^0 \in D(T)$, converges to the fixed point u^* of T.

Proof. See [8]. □

Definition 2.2. A mapping T which satisfies condition (d) of theorem 2.1 will be called $|\ |_{\infty,\gamma}$ contractive with respect to u^*.

If T is contractive in the norm $|\ |_{\infty,\gamma}$ on $D(T)$ we say that it is $|\ |_{\infty,\gamma}$ contractive on $D(T)$.

Definition 2.3. We say that a vector x is nonnegative (positive), denoted $x \geqslant 0$ ($x > 0$), if all its entries are nonnegative (positive). A matrix B is said to be nonnegative, denoted $B \geqslant 0$, if all its entries are nonnegative. We compare two matrices $A \geqslant B$, when $A - B \geqslant 0$, and two vectors $x \geqslant y$ ($x > y$) when $x - y \geqslant 0$ ($x - y > 0$).

$L(\mathbb{R}^n)$ denotes the space of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Definition 2.4. Let $A \in L(\mathbb{R}^n)$. The representation $A = M - N$ is called a splitting if M is nonsingular. It is called a convergent splitting if $\rho(M^{-1}N) < 1$. A splitting $A = M - N$ is called:

(a) regular if $M^{-1} \geqslant 0$ and $N \geqslant 0$,

(b) weakly regular if $M^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$.

Definition 2.5. A maximal monotone graph Λ is a subset of $\mathbb{R}^2$ such that if (x, y) and (x', y') belong to Λ then the product $(y - y')(x - x')$ is positive or equal to zero. Moreover, it is impossible to add any point to the graph without losing the above monotonicity property.

The domain $D(\Lambda)$, the range $R(\Lambda)$ and $\Lambda(x)$ are defined by

$$\begin{cases} D(\Lambda) = \{x \in \mathbb{R},\ \exists y \in \mathbb{R} \text{ with } (x, y) \in \Lambda\}, \\ R(\Lambda) = \{y \in \mathbb{R},\ \exists x \in \mathbb{R} \text{ with } (x, y) \in \Lambda\}, \\ \Lambda(x) = \{y \in \mathbb{R},\ (x, y) \in \Lambda\} \quad \text{for } x \in D(\Lambda). \end{cases}$$

3. The general framework of nonlinear asynchronous multisplitting methods

3.1. The general formal multisplitting

We consider problems in the form

$$0 \in H(x^*), \quad x^* \in D \subset \mathbb{R}^n, \tag{3.1}$$

where H is a nonlinear and possibly multivalued operator defined on a closed set D, where

(h4) $D = \prod_{i=1}^{n} d_i$ and $d_i \subset \mathbb{R}$ are closed and convex.

We also consider m mappings F_l on D such that

$$F_l(D) \subset D. \tag{3.2}$$

We suppose that for $l \in \{1, \dots, m\}$, F_l is $|\ |_{\infty,\gamma}$ contractive with respect to x^* with the constant ν_l so

$$\begin{cases} |x|_{\infty,\gamma} = \max\limits_{1 \leqslant i \leqslant n} \dfrac{|x_i|}{\gamma_i}, \\ \gamma_i > 0, \end{cases} \tag{3.3}$$

where $|x_i|$ is the absolute value of x_i in $\mathbb{R}$,

$$\big|F_l(x) - x^*\big|_{\infty,\gamma} \leqslant \nu_l |x - x^*|_{\infty,\gamma}. \tag{3.4}$$

We also suppose that for $l \in \{1, \dots, m\}$

$$F_l(x^*) = x^*. \tag{3.5}$$

Definition 3.1. We call formal multisplitting associated with (3.1) the collection of fixed point problems

$$x - F_l(x) = 0, \quad l \in \{1, \dots, m\}.$$

3.2. The extended fixed point mapping associated with the formal multisplitting of nonlinear multivalued problems

We use the following notations: x^l, x^k, $l, k \in \{1, \dots, m\}$, denote vectors of $\mathbb{R}^n$ the components of which are x_i^l, x_j^k, $i, j \in \{1, \dots, n\}$.

Define the fixed point mapping

$$\begin{cases} T_1 : (\mathbb{R}^n)^m \to (\mathbb{R}^n)^m, \\ X = (x^1, \dots, x^m) \to Y = (y^1, \dots, y^m) \end{cases}$$

such that for $l \in \{1, \dots, m\}$

$$\begin{cases} y^l = F_l(z^l), \\ z^l = \sum_{k=1}^m E_{lk}(X) x^k, \end{cases} \tag{3.6}$$

where $E_{lk}(X)$ are weighting matrices satisfying

$$\begin{cases} E_{lk}(X) \text{ are diagonal matrices}, \\ E_{lk}(X) \geqslant 0, \\ \sum_{k=1}^m E_{lk}(X) = I_n \text{ (identity matrix in } L(\mathbb{R}^n)), \quad \forall l \in \{1, \dots, m\}. \end{cases} \tag{3.7}$$

Since $F_l(D) \subset D$ and by the use of assumption (h4) we have

$$T_1(U) \subset U, \tag{3.8}$$

where $U = \prod_{i=1}^m D$.

3.3. Comments with respect to the use of our generalization of weighting matrices

In the above definition of the extended fixed point mapping, especially about the introduction of the family of matrices $E_{lk}(X)$, it is worthwhile to note that our formulation allows:

- To take $E_{lk}(X) = E_k$ in order to obtain O'Leary and White multisplitting algorithms, see section 4.2.
- To define $E_{lk}(X) = E_{lk}$ depending on the index l in order to give a presentation of either the Schwarz alternating method or general Schwarz multisplitting methods, see section 4.3. Nevertheless see section 4.3.3 for the reduction of these situations to the O'Leary and White case.
- To take also $E_{lk}(X) = E_{lk}$ as is needed in the case of two-stage Schwarz methods, see remark 5.19 of section 5.4.
- To take $E_{lk}(X)$ depending on both the index l and on the element X of $(\mathbb{R}^n)^m$, the value of which must be the current iterates X^p in order to describe two-stage multisplitting methods, see section 5.5 and also section 5.6 in which O'Leary and White and Schwarz approaches are mixed.

Moreover, and particularly when card(Im(T)) $< \infty$, if we have to deal with the same value of x for several iterates X^p, we can consider that the family $E_{lk}(X)$ is a selection of a multivalued mapping $x \rightarrow \widetilde{E}_{kl}(X)$ such that each such selection satisfies condition (3.7). It is easy to see that in such a situation the following proposition 3.2 and corollary 3.3 and their proofs remain valid.

3.4. Contraction property in a weighted maximum norm. Convergence of asynchronous algorithms corresponding to T_1

Under assumptions (h4), (3.2), (3.4) and (3.7), T_1 satisfies the following proposition.

Proposition 3.2. Denote $X^* = (x^*, \ldots, x^*)$, where x^* is the solution of (3.1), then T_1 is $|\ |_{\infty,\gamma}$ contractive with respect to X^*; $|\ |_{\infty,\gamma}$ is defined by

$$|X|_{\infty,\gamma} = \max_{1\leqslant k\leqslant m} \max_{1\leqslant i\leqslant n} \frac{|(x^k)_i|}{\gamma_i}. \tag{3.9}$$

Its constant of contraction is

$$\nu = \max_{1\leqslant l\leqslant m} \nu_l \tag{3.10}$$

and X^* is the fixed point of T_1.

Proof. Let $Y = T_1(X)$, by (3.6) we have

$$\left|y^l - x^*\right|_{\infty,\gamma} = \left|F_l\left(\sum_{k=1}^{m} E_{lk}(X)x^k\right) - x^*\right|_{\infty,\gamma}$$

and

$$\frac{\left|\left(\sum_{k=1}^{m} E_{lk}(X)(x^k - x^*)\right)_i\right|}{\gamma_i} = \frac{\left|\sum_{k=1}^{m}\sum_{j=1}^{n}(E_{lk}(X))_{ij}(x^k - x^*)_j\right|}{\gamma_i}.$$

Since the weighting matrices $E_{lk}(X)$ are diagonal, we have

$$\left|\sum_{j=1}^{n}\left(E_{lk}(X)\right)_{ij}\left(x^k - x^*\right)_j\right| = \left|\left(E_{lk}(X)\right)_{ii}\left(x^k - x^*\right)_i\right|.$$

Condition (3.7) gives

$$\left|\sum_{k=1}^{m}\left(E_{lk}(X)\right)_{ii}\left(x^k - x^*\right)_i\right| \leqslant \sum_{k=1}^{m}\left(E_{lk}(X)\right)_{ii}\max_{1\leqslant k\leqslant m}\left|\left(x^k - x^*\right)_i\right|,$$

so that

$$\max_{1\leqslant i\leqslant n}\frac{\left|\left(F_l\left(\sum_{k=1}^{m} E_{lk}(X)x^k\right) - x^*\right)_i\right|}{\gamma_i} \leqslant \nu_l \max_{1\leqslant i\leqslant n}\max_{1\leqslant k\leqslant m}\frac{|(x^k - x^*)_i|}{\gamma_i}.$$

Consequently

$$\max_{1\leqslant l\leqslant m}\max_{1\leqslant i\leqslant n}\frac{|(y^l - x^*)_i|}{\gamma_i} \leqslant \max_{1\leqslant l\leqslant m}\left(\nu_l \max_{1\leqslant i\leqslant n}\frac{|(x^l - x^*)_i|}{\gamma_i}\right),$$

so that by (3.9) and (3.10) we have

$$|Y - X^*|_{\infty,\gamma} \leqslant \nu|X - X^*|_{\infty,\gamma}.$$

Since $\sum_{k=1}^{m} E_{lk}(X) = I_n$ and $x^* = F_l(x^*)$ we have $T_1(X^*) = X^*$. □

Corollary 3.3. Under the assumptions of proposition 3.2, any asynchronous algorithm (T_1, X^0, J, S) corresponding to T_1 and starting with $X^0 \in U$ converges to the solution of (3.1).

Proof. Condition (3.8) implies that T_1 has a unique fixed point. Theorem 2.1 and proposition 3.2 end the proof. □

3.5. *S-regular splittings associated with a $|\ |_{\infty,\gamma}$ contractive fixed point mapping G*

Suppose that problem (3.1) can be written equivalently in the form

$$x^* = G(x^*), \quad x^* \in D \subset \mathbb{R}^n, \tag{3.11}$$

where D satisfies assumption (h4), and G is $|\ |_{\infty,\gamma}$ contractive:

$$\bigl|G(x) - G(y)\bigr|_{\infty,\gamma} \leqslant \nu|x - y|_{\infty,\gamma}, \quad 0 < \nu < 1.$$

Suppose that

$$G(D) \subset D. \tag{3.12}$$

Definition 3.4. For $i, j \in \{1, \ldots, n\}$ let us introduce an array β of weights $\beta_{ij} \in [0, 1]$. For $i \in \{1, \ldots, n\}$ we define the vector

$$\begin{cases} \sigma_i^{\beta}(v, u) = (w_1, \ldots, w_n) \quad \text{such that} \\ w_j = \beta_{ij} v_j + (1 - \beta_{ij}) u_j. \end{cases} \tag{3.13}$$

Definition 3.5. Define the implicit fixed point mapping F_β by

$$\forall x \in D,\ \forall i \in \{1, \ldots, n\}, \quad F_{\beta,i}(x) = G_i\bigl(\sigma_i^{\beta}\bigl(F_\beta(x), x\bigr)\bigr). \tag{3.14}$$

F_β will be called an s-regular splitting associated with the $|\ |_{\infty,\gamma}$ contractive fixed point mapping G.

Remark 3.6. The implicit mapping F_β is well defined. Indeed for every fixed x the mapping $\widetilde{G}$ defined by

$$\widetilde{G}(y) = z^{\beta} \Leftrightarrow z_i^{\beta} = G_i\bigl(\sigma_i^{\beta}(y, x)\bigr)$$

is $|\ |_{\infty,\gamma}$ contractive:

$$\begin{aligned} \frac{|\tilde{z}_i^{\beta} - z_i^{\beta}|}{\gamma_i} &= \frac{|G_i(\sigma_i^{\beta}(\tilde{y}, x)) - G_i(\sigma_i^{\beta}(y, x))|}{\gamma_i} \leqslant \nu\bigl|\sigma_i^{\beta}(\tilde{y}, x) - \sigma_i^{\beta}(y, x)\bigr|_{\infty,\gamma} \\ &\leqslant \nu\left(\max_{1\leqslant j\leqslant n} \frac{|\beta_{ij}(\tilde{y}_j - y_j)|}{\gamma_j}\right) \leqslant \nu \max_{1\leqslant j\leqslant n} \frac{|\tilde{y}_j - y_j|}{\gamma_j}. \end{aligned}$$

Hence

$$\bigl|\tilde{z}^{\beta} - z^{\beta}\bigr|_{\infty,\gamma} \leqslant \nu|\tilde{y} - y|_{\infty,\gamma},$$

so for every fixed x, $\widetilde{G}$ has a unique fixed point $z^{\beta}(x)$ ($\widetilde{G}$ maps D into itself by (3.12)):

$$z^{\beta}(x) = \widetilde{G}\bigl(z^{\beta}(x)\bigr) \Leftrightarrow z_i^{\beta}(x) = G_i\bigl(\sigma_i^{\beta}\bigl(z^{\beta}(x), x\bigr)\bigr).$$

Put $F_\beta(x) = z^{\beta}(x)$.

Under assumption (h4), (3.11) and (3.12) we have:

Proposition 3.7. The fixed point mapping F_β is $|\ |_{\infty,\gamma}$ contractive, its constant is less than or equal to the constant of G and the fixed point of F_β is x^*.

Proof.

$$\begin{aligned}\frac{|F_{\beta,i}(x) - F_{\beta,i}(y)|}{\gamma_i} &= \frac{|G_i(\sigma_i^\beta(F_\beta(x),x)) - G_i(\sigma_i^\beta(F_\beta(y),y))|}{\gamma_i} \\ &\leqslant \nu \left|\sigma_i^\beta\big(F_\beta(x),x\big) - \sigma_i^\beta\big(F_\beta(y),y\big)\right|_{\infty,\gamma} \\ &\leqslant \nu \max_{1\leqslant j\leqslant n} \frac{|\beta_{ij}(F_{\beta,j}(x) - F_{\beta,j}(y)) + (1-\beta_{ij})(x_j - y_j)|}{\gamma_j} \\ &\leqslant \nu \max\left(\max_{1\leqslant j\leqslant n} \frac{|F_{\beta,j}(x) - F_{\beta,j}(y)|}{\gamma_j}, \max_{1\leqslant j\leqslant n} \frac{|x_j - y_j|}{\gamma_j}\right),\end{aligned}$$

so either

$$\forall i \in \{1,\dots,n\}, \quad \frac{|F_{\beta,i}(x) - F_{\beta,i}(y)|}{\gamma_i} \leqslant \nu \max_{1\leqslant j\leqslant n} \frac{|F_{\beta,j}(x) - F_{\beta,j}(y)|}{\gamma_j},$$

which implies that $|F_\beta(x) - F_\beta(y)|_{\infty,\gamma} = 0$, or

$$\frac{|F_{\beta,i}(x) - F_{\beta,i}(y)|}{\gamma_i} \leqslant \nu \max_{1\leqslant j\leqslant n} \frac{|x_j - y_j|}{\gamma_j},$$

so

$$\left|F_\beta(x) - F_\beta(y)\right|_{\infty,\gamma} \leqslant \nu |x - y|_{\infty,\gamma},$$

which implies that F_β is contractive and that its constant is less than or equal to ν.

Moreover, by (3.12) F and G have a unique fixed point. Let

$$F_{\beta,i}(x) = x_i,$$

so equivalently

$$x_i = G_i\big(\sigma_i^\beta(x,x)\big),$$

so

$$x = G(x).$$

Hence F_β and G have the same fixed point: x^*. □

Let us consider a collection of s-regular splittings: to each $l \in \{1,\dots,m\}$ is associated an array β^l .

Corollary 3.8. Under the assumptions of proposition 3.7, if we replace F_l by F_{β^l} in the definition of T_1, then any asynchronous algorithm (T_1, X^0, J, S) corresponding to T_1 and starting with $X^0 \in U$ converges to the solution of (3.11).

4. O'Leary and White and Schwarz asynchronous multisplitting methods for nonlinear problems

In this section, we do not need the dependence on X of the weighting matrices $E_{lk}(X)$, so we consider only the cases, where $E_{lk}(X) = E_k$ and where $E_{lk}(X) = E_{lk}$.

4.1. *Overlapping block multisplitting of nonlinear fixed point problems*

We consider problems in the form (3.11). Let I_l, $l \in \{1, \ldots, m\}$, be subsets of $\{1, \ldots, n\}$ and I_l^C their complements:

$$I_l \cup I_l^C = \{1, \ldots, n\}, \quad \forall l \in \{1, \ldots, m\}. \tag{4.1}$$

Consider the particular case of weights

$$\begin{cases} \beta_{ij}^l = 1 & \text{if } (i, j \in I_l) \text{ or } \big(i, j \in I_l^C\big), \\ \beta_{ij}^l = 0 & \text{elsewhere,} \end{cases}$$

then the vector $\sigma_i^\beta(u, v)$, which we now denote $\sigma_i^l(u, v)$, can be written in the form

$$\begin{aligned} &\sigma_i^l(v, u) = (w_1, \ldots, w_n) \quad \text{such that} \\ &\quad \begin{cases} w_j = v_j & \text{if } (i, j \in I_l) \text{ or } \big(i, j \in I_l^C\big), \\ w_j = u_j & \text{elsewhere.} \end{cases} \end{aligned} \tag{4.2}$$

Definition 4.1. Following definition 3.5, define the mapping F_l by

$$\forall x \in D,\ \forall i \in \{1, \ldots, n\}, \quad F_{l,i}(x) = G_i\big(\sigma_i^l\big(F_l(x), x\big)\big). \tag{4.3}$$

We call this splitting the block (I_l, I_l^C) splitting.

Remark 4.2. In such a case we usually take

$$(E_{lk})_{ij} = 0 \quad \text{or} \quad (E_k)_{ij} = 0 \quad \text{for } j \in I_k^C. \tag{4.4}$$

If we except particular problems which admit a natural block decomposition structure suitable for block iterative algorithms, the previous condition (4.4) is very important especially for overlapping block decomposition techniques because in the evaluation of F_l, for any k we never have to use any component from I_k^C, so in our block (I_l, I_l^C) splitting the solution of a diagonal block subproblem associated with any I_l^C never has to be computed.

Under assumption (h4), (3.11) and (3.12) we have

Corollary 4.3. For $l \in \{1, \ldots, m\}$, F_l is $|\ |_{\infty,\gamma}$ contractive, its constant is less than or equal to the constant of G and the fixed point of F_l is x^*.

Proof. This is a particular case of proposition 3.7. □

4.2. Extended fixed point mapping associated with O'Leary and White multisplitting

Take the diagonal positive matrices $E_{lk}(X)$ depending only on k

$$E_{lk}(X) = E_k$$

and satisfying

$$\begin{cases} \sum_{k=1}^{m} E_k = I_n, \\ (E_k)_{ii} = 0, \quad \forall i \notin I_k. \end{cases} \tag{4.5}$$

With the notations of section 2, where we take $\alpha = m$, $B_l = \mathbb{R}^n$, the asynchronous iterations corresponding to O'Leary and White multisplitting are defined by the fixed point mapping

$$T_1^{\mathrm{OW}}\bigl(x^1, \ldots, x^m\bigr) = \bigl(y^1, \ldots, y^m\bigr) \quad \text{such that}$$

$$\begin{cases} y^l = F_l(z), \\ z = \sum_{k=1}^{m} E_k x^k, \end{cases} \tag{4.6}$$

where for $l \in \{1, \ldots, m\}$, F_l is defined by (4.3).

As a consequence of corollary 4.3 and theorem 2.1 we have

Corollary 4.4. Any asynchronous algorithm $(T_1^{\mathrm{OW}}, X^0, J, S)$ corresponding to T_1^{OW} and starting with $X^0 \in U$ converges to the solution of (3.11).

4.3. Fixed point mappings associated with the discrete analogue of Schwarz's alternating method and its multisubdomain generalization

4.3.1. Discrete analogue of the Schwarz alternating method

Let us first consider the case $l = 2$. Suppose $I_1 \cap I_2 \neq \varnothing$, so we have an overlap between the 1st and the 2nd subdomains. Consider the matrices E_{lk} such that

$$\begin{aligned} (E_{11})_{ii} &= \begin{cases} 1 & \forall i \in I_1, \\ 0 & \forall i \notin I_1, \end{cases} & (E_{12})_{ii} &= \begin{cases} 0 & \forall i \in I_1, \\ 1 & \forall i \notin I_1, \end{cases} \\ (E_{21})_{ii} &= \begin{cases} 1 & \forall i \notin I_2, \\ 0 & \forall i \in I_2, \end{cases} & (E_{22})_{ii} &= \begin{cases} 0 & \forall i \notin I_2, \\ 1 & \forall i \in I_2. \end{cases} \end{aligned} \tag{4.7}$$

Define the fixed point mapping

$$T_1^S\bigl(x^1, x^2\bigr) = \bigl(y^1, y^2\bigr) \quad \text{such that for } l = 1, 2$$

$$\begin{cases} y^l = F_l(z^l), \\ z^l = \sum_{k=1}^{2} E_{lk} x^k, \end{cases} \tag{4.8}$$

where for $l \in \{1, 2\}$, F_l is defined by (4.3). Then the additive discrete analogue of the Schwarz alternating method corresponds to the successive approximation method applied to T_1^S, and the multiplicative discrete analogue of the Schwarz alternating method corresponds to the block nonlinear Gauss–Seidel method applied to T_1^S. Of course it is not necessary to use as above weighting matrices in order to express such algorithms; for this and the use of such methods as preconditioners of Krylov space methods, we refer to [11,19].

4.3.2. Discrete analogue of the multisubdomain Schwarz method

We introduce the weighting matrices E_k satisfying (4.5) and the matrices E_{lk} such that for $l \in \{1, \dots, m\}$

$$(E_{ll})_{ii} = \begin{cases} 1 & \text{if } i \in I_l, \\ 0 & \text{if } i \notin I_l, \end{cases} \qquad (E_{lk})_{ii} = \begin{cases} 0 & \text{if } i \in I_l, \\ (E_k)_{ii} & \text{if } i \notin I_l, \end{cases} \tag{4.9}$$

the asynchronous iterations corresponding to the discrete analogue of the multisubdomain Schwarz method are defined by the fixed point mapping T_1^{MS}

$$T_1^{\mathrm{MS}}\bigl(x^1, \dots, x^m\bigr) = \bigl(y^1, \dots, y^m\bigr) \quad \text{such that}$$
$$\begin{cases} y^l = F_l(z^l), \\ z^l = \sum_{k=1}^m E_{lk} x^k, \end{cases} \tag{4.10}$$

where E_{lk} are defined by (4.9) and F_l are defined by (4.3).

T_1 being T_1^{OW} or T_1^S or T_1^{MS}, we have the following corollary.

Corollary 4.5. Any asynchronous algorithm (T_1, X^0, J, S) corresponding to T_1 and starting with $X^0 \in U$ converges to the solution of (3.11).

4.3.3. Some complements and unification of the formulation of the algorithms

(i) In the situation of the two subdomains Schwarz alternating method, one evaluation of the corresponding fixed point mapping corresponds to one step of the additive Schwarz alternating method, and to a block Jacobi-like iteration, while one step of the multiplicative Schwarz alternating method corresponds to one step of a block Gauss–Seidel-like algorithm.

Moreover, in the case of the two subdomains Schwarz mapping, the general successive approximation algorithm associated with the formulation of asynchronous iterations cannot give rise to other iteration than either one Gauss–Seidel-like algorithm or two independent ones. In this case, each one never communicates any intermediate result to the other. This is related here with the following property of such a fixed point mapping; namely: let us consider (4.1)–(4.3), let us take in (4.3)

$$y^l = F_l(x),$$

so that, by taking into account remark 4.2, it can be rewritten as

$$y_i^l = G_i\bigl(\sigma_i^l\bigl(y^l, x\bigr)\bigr) \quad \text{with } i \in I_l,$$

where

$$\bigl(\sigma_i^l\bigl(y^l, x\bigr)\bigr)_j = \begin{cases} y_j^l, & \forall j \in I_l, \\ x_j, & \forall j \notin I_l. \end{cases}$$

Let us come back to the definition of T_1^S in (4.8): we have in this case to deal with

$$y_i^l = G_i\big(\sigma_i^l\big(y^l, z^l\big)\big), \tag{4.11}$$

where

$$z^l = \sum_{k=1}^{2} E_{lk} x^k.$$

If we take into account the property of matrices E_{lk} to be diagonal, we see that

$$\sigma_i^l\big(y^l, z^l\big) = \sigma_i^l\big(y^l, E_{lk} x^k\big) \quad \text{with } k \neq l, \text{ and } k, l \in \{1, 2\}. \tag{4.12}$$

(ii) In the situation of multisubdomain Schwarz-kind methods, (4.12) takes the form

$$\sigma_i^l\big(y^l, z^l\big) = \sigma_i^l\left(y^l, \sum_{k=1, k\neq l}^{m} E_{lk} x^k\right). \tag{4.13}$$

Proposition 4.6. T_1^{MS} and T_1^{OW} can be identified as being the same fixed point mapping.

Proof. It is a simple consequence of the construction (4.9) of E_{lk} and of (4.13). □

Remark 4.7. It is worthwhile to note that the distinction between O'Leary and White and Schwarz algorithms in introducing the matrices E_{lk} remains necessary in order to deal with corresponding two-stage algorithms studied in section 5.

4.4. *The example of linear problems perturbed by multivalued diagonal monotone operators*

4.4.1. *Convergence of the general asynchronous multisplitting methods*

We give here an application of the general framework to linear problems perturbed by multivalued diagonal monotone operators, in short: pseudolinear problems. Let

(h5) $A = (a_{ij}) \in L(\mathbb{R}^n)$ be an M-matrix (see [3]), and

(h6) Λ a diagonal maximal monotone operator,

and consider the pseudolinear problem

$$b - Ax \in \Lambda(x). \tag{4.14}$$

Let $D = \mathrm{diag}(\ldots, a_{ii}, \ldots)$ be the diagonal part of A and

$$B = -A + D.$$

Remark that $B \geqslant 0$ since A is an M-matrix.

$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} = \overbrace{\begin{pmatrix} \begin{array}{|c|c|c|} \hline \tilde{A}_l & 0 & \tilde{A}_l \\ \hline 0 & \hat{A}_l & 0 \\ \hline \tilde{A}_l & 0 & \tilde{A}_l \\ \hline \end{array} \end{pmatrix}}^{M_l} - \overbrace{\begin{pmatrix} \begin{array}{|c|c|c|} \hline 0 & & 0 \\ \hline & 0 & \\ \hline 0 & & 0 \\ \hline \end{array} \end{pmatrix}}^{N_l}$$

Figure 1.

Let us define in the following way a fixed point mapping G associated with the problem (4.14) which we choose to be the nonlinear Jacobi mapping, namely

$$\forall x \in \mathbb{R}^n, \quad y = G(x) \Leftrightarrow b + Bx - Dy \in \Lambda(y). \tag{4.15}$$

That is to say, each component $y_i \in \mathbb{R}$ of y satisfies

$$b_i - (Bx)_i - a_{ii} y_i \in \Lambda_i(y_i). \tag{4.16}$$

Equation (4.16) admits a unique solution by the strict positivity of a_{ii} and the maximal monotonicity of Λ_i.

Consider for $l \in \{1, \dots, m\}$ subsets $I_l \subset \{1, \dots, n\}$, and the family of operators σ_i^l satisfying respectively (4.1) and (4.2). Let us use the following notations:

$$\begin{aligned} &\big(\widehat{A}_l\big)_{ij} = a_{ij} \quad \text{for } i, j \in I_l, \quad \big(\widetilde{A}_l\big)_{ij} = a_{ij} \quad \text{for } i, j \in I_l^C, \quad \text{and} \\ &M_l = \operatorname{diag}\big(\widehat{A}_l, \widetilde{A}_l\big), \end{aligned}$$

then we have the splitting of A described by figure 1. It is worthwhile to remark that if we renumber the indices in order that after this renumbering I_l becomes the first card(I_l) numbers of $\{1, \dots, n\}$, then the matrix $M_l = \operatorname{diag}(\widehat{A}_l, \widetilde{A}_l)$ appears to be a block diagonal matrix in the usual sense.

The following proposition will be useful in the sequel.

Proposition 4.8. G is $|\,|_{\infty,\gamma}$ contractive, where γ is obtained from the Perron–Frobenius theory for nonnegative matrices. The constant of contraction of G is either $\rho(D^{-1}B)$ if $D^{-1}B$ is irreducible or $\rho(D^{-1}B) + \varepsilon$ (for any $\varepsilon > 0$) if $D^{-1}B$ is reducible.

Proof. See, for example, [1]. □

By definition 4.1 we are able to introduce the corresponding family of fixed point mappings F_l which satisfy condition (4.3) with respect to σ_i^l and G defined by (4.15).

Under assumptions (h4)–(h6) we have the following proposition.

Proposition 4.9. For $l \in \{1, \dots, m\}$, F_l is $|\,|_{\infty,\gamma}$ contractive, its constant is less than or equal to the constant of G and the fixed point of F_l is x^*, the solution of (4.14).

Proof. This is a direct application of proposition 3.7. □

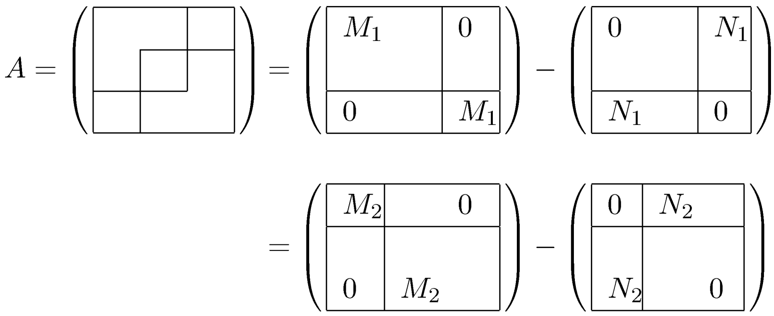

Figure 2.

Consider the fixed point mapping

$$T_2\big(x^1,\dots,x^m\big) = \big(y^1,\dots,y^m\big) \quad \text{such that}$$
$$\begin{cases} y^l = F_l(z^l), \\ z^l = \sum_{k=1}^m E_{lk}(X)x^k. \end{cases} \tag{4.17}$$

Corollary 4.10. Any asynchronous algorithm (T_2, X^0, J, S) corresponding to T_2 and starting with $X^0 \in U$ converges to the solution of (4.14).

Proof. This is a direct application of theorem 2.1 and proposition 4.9. □

4.4.2. Discrete analogue of the Schwarz alternating method

Let

$$A = M_1 - N_1 = M_2 - N_2.$$

Suppose $I_1 \cap I_2 \neq \varnothing$, so we have an overlap as shown in figure 2. As in section 4.3.1, consider the matrices E_{lk} satisfying (4.7) and consider the fixed point mapping $T_2^{\mathrm{OW}}(x^1,x^2) = T_2^S(x^1,x^2) = (y^1,y^2)$ such that for $l = 1,2$

$$\begin{cases} y^l = F_l(z^l), \\ z^l = \sum_{k=1}^2 E_{lk}x^k. \end{cases}$$

4.4.3. Discrete analogue of the multisubdomain methods

Let

$$A = M_i - N_i, \quad i \in \{1,\dots,m\}.$$

Suppose for some $i,j \in \{1,\dots,m\}$, $I_i \cap I_j \neq \varnothing$, so we have an overlap between the ith and the jth subdomains.

The asynchronous iterations corresponding to the discrete analogue of the multisubdomain Schwarz method are defined by the fixed point mapping

$$T_2^{\mathrm{MS}}\big(x^1,\dots,x^m\big) = T_2^{\mathrm{OW}}\big(x^1,\dots,x^m\big) = \big(y^1,\dots,y^m\big) \quad \text{(by proposition 4.6)}$$

such that

$$\begin{cases} y^l = F_l(z^l), \\ z^l = \sum_{k=1}^{m} E_{lk} x^k, \end{cases}$$

where E_{lk} are defined by (4.9). T_2 being $T_2^{\mathrm{MS}} = T_2^{\mathrm{OW}}$, we have the following corollary.

Corollary 4.11. Any asynchronous algorithm (T_2, X^0, J, S) converges to the solution of (4.14).

5. Two-stage multisplitting algorithms for pseudolinear problems

5.1. Strongly regular splitting associated with an s-regular splitting of an off-diagonal fixed point mapping

Definition 5.1. A set $\Omega = \{\omega_1, \ldots, \omega_n\}$ of parameters ω_i is called a set of underrelaxation parameters if

$$\forall i \in \{1, \ldots, n\}, \quad \omega_i \in]0, 1].$$

Definition 5.2. The mapping F^{Ω} defined by

$$F_i^{\Omega}(v) = \omega_i F_i(v) + (1 - \omega_i) v_i \tag{5.1}$$

is called an underrelaxation mapping associated with F.

Proposition 5.3. If F is $|\ |_{\infty,\gamma}$ contractive on $D(F)$ then any associated underrelaxation fixed point mapping is also $|\ |_{\infty,\gamma}$ contractive on $D(F)$.

Proof. Let ν be the constant of contraction of F, then

$$\begin{aligned}
\frac{|F_i^{\Omega}(v) - F_i^{\Omega}(w)|}{\gamma_i} &= \frac{|\omega_i(F_i(v) - F_i(w)) + (1 - \omega_i)(v_i - w_i)|}{\gamma_i} \\
&\leqslant \omega_i \frac{|F_i(v) - F_i(w)|}{\gamma_i} + (1 - \omega_i) \frac{|v_i - w_i|}{\gamma_i} \\
&\leqslant \omega_i \nu \frac{|v_i - w_i|}{\gamma_i} + (1 - \omega_i) \frac{|v_i - w_i|}{\gamma_i} \\
&\leqslant \bigl(\omega_i \nu + (1 - \omega_i)\bigr) \frac{|v_i - w_i|}{\gamma_i}.
\end{aligned}$$

We have

$$\nu^{\Omega} = \max_{1 \leqslant i \leqslant n} \bigl(\omega_i \nu + (1 - \omega_i)\bigr) = \max_{1 \leqslant i \leqslant n} \bigl(1 - (1 - \nu)\omega_i\bigr) < 1,$$

so

$$\left|F_i^{\Omega}(v) - F_i^{\Omega}(w)\right|_{\infty,\gamma} \leqslant \nu^{\Omega}|v-w|_{\infty,\gamma}, \quad \nu^{\Omega} < 1. \qquad \square$$

Corollary 5.4. Any underrelaxed fixed point mapping associated with an s-regular splitting of a $|\ |_{\infty,\gamma}$ contractive fixed point mapping is $|\ |_{\infty,\gamma}$ contractive.

Proof. By the result of proposition 3.7, any fixed point mapping associated with an s-regular splitting of a $|\ |_{\infty,\gamma}$ contractive fixed point mapping is $|\ |_{\infty,\gamma}$ contractive. The corollary is then proved by proposition 5.3. □

Definition 5.5. Let us call a fixed point mapping G off-diagonal if for all $i \in \{1,\dots,n\}$, $G_i(\dots,x_j,\dots)$ does not depend on x_i.

Proposition 5.6. Let us consider an s-regular splitting associated with the off-diagonal $|\ |_{\infty,\gamma}$ contractive mapping G:

$$w = F^{\beta}(v).$$

Then there exists a $|\ |_{\infty,\gamma}$ contractive underrelaxed mapping $(F^{\beta})^{\Omega}(v)$ defined by

$$\left(F^{\beta}\right)^{\Omega}(v) = u \Leftrightarrow u_i = \omega_i F_i^{\beta}(v) + (1-\omega_i)v_i \tag{5.2}$$

which satisfies

$$\alpha_{ii}(u_i) + (1-\alpha_{ii})v_i = G_i\left(\dots,\alpha_{ij}u_j + (1-\alpha_{ij})v_j,\dots\right),$$

where

$$\begin{cases} \alpha_{ij} \in [0,1], \\ \alpha_{ii} > 1. \end{cases}$$

Proof. w satisfies

$$w_i = G_i\left(\sigma_i^{\beta}(w,v)\right), \quad \forall i \in \{1,\dots,n\}. \tag{5.3}$$

By proposition 3.7, F^{β} is $|\ |_{\infty,\gamma}$ contractive, and by corollary 5.4, $(F^{\beta})^{\Omega}$ is also $|\ |_{\infty,\gamma}$ contractive.

Equation (5.3) can be written as

$$w_i = G_i\left(\dots,\beta_{ij}w_j + (1-\beta_{ij})v_j,\dots\right).$$

By (5.2) we have

$$w_i = \frac{1}{\omega_i}u_i + \left(1 - \frac{1}{\omega_i}\right)v_i,$$

so

$$\frac{1}{\omega_i}u_i + \left(1-\frac{1}{\omega_i}\right)v_i = G_i\left(\dots,\beta_{ij}\left(\frac{1}{\omega_j}u_j + \left(1-\frac{1}{\omega_j}\right)v_j\right) + (1-\beta_{ij})v_j,\dots\right)$$

or

$$\frac{1}{\omega_i}u_i + \left(1 - \frac{1}{\omega_i}\right)v_i = G_i\left(\dots, \frac{\beta_{ij}}{\omega_j}u_j + \left(1 - \frac{\beta_{ij}}{\omega_j}\right)v_j, \dots\right).$$

Then we can choose ω_i, $i \in \{1, \dots, n\}$, such that

$$\begin{cases} \alpha_{ii} = \dfrac{1}{\omega_i} > 1, \\ \alpha_{ij} = \dfrac{\beta_{ij}}{\omega_j} \in [0,1], \end{cases}$$

so

$$\alpha_{ii}u_i + (1 - \alpha_{ii})v_i = G_i\big(\dots, \alpha_{ij}u_j + (1 - \alpha_{ij})v_j, \dots\big). \qquad \square$$

Definition 5.7. A strongly regular splitting of an operator $H(x) = x - G(x)$ associated with a $|\ |_{\infty,\gamma}$ contractive off-diagonal fixed point mapping G is defined by

(i) an s-regular splitting of G defined by the vector σ_i^α of definition 3.4,

(ii) a set of parameters $\alpha_{ii} \geqslant 1$ ($i \in \{1, \dots, n\}$),

(iii) a mapping $\overline{F}_\alpha(u) = v$ defined by

$$\begin{aligned} &\alpha_{ii}v_i + (1 - \alpha_{ii})u_i = G_i\big(\sigma_i^\alpha(v,u)\big) \\ &\quad \Leftrightarrow \alpha_{ii}v_i + (1 - \alpha_{ii})u_i = G_i\big(\dots, \alpha_{ij}v_j + (1 - \alpha_{ij})u_j, \dots\big). \end{aligned} \tag{5.4}$$

Remark 5.8. Since G is off-diagonal we do not need $\alpha_{ii} \in [0,1]$ in $\sigma_i^\alpha(v,u)$.

Proposition 5.9. If G is $|\ |_{\infty,\gamma}$ contractive and off-diagonal then the mapping $\overline{F}_\alpha$, the strongly regular splitting of $H(x) = x - G(x)$ defined by (5.4), is an underrelaxed $|\ |_{\infty,\gamma}$ contractive fixed point mapping associated with an s-regular splitting of G.

Proof. $\alpha_{ii} > 1$ allows to take

$$\omega_i = \frac{1}{\alpha_{ii}} \in]0,1].$$

Then (5.4) can be written as

$$\frac{1}{\omega_i}v_i + \left(1 - \frac{1}{\omega_i}\right)u_i = G_i\big(\dots, \alpha_{ij}v_j + (1 - \alpha_{ij})u_j, \dots\big).$$

Let

$$w_i = \frac{1}{\omega_i}v_i + \left(1 - \frac{1}{\omega_i}\right)u_i \quad \text{for } i \in \{1, \dots, n\}, \tag{5.5}$$

so

$$v_i = \omega_i w_i + (1 - \omega_i)u_i.$$

Then

$$w_i = G_i\big(\dots, \alpha_{ij}\big(\omega_j w_j + (1-\omega_j)u_j\big) + (1-\alpha_{ij})u_j, \dots\big),$$
$$w_i = G_i\big(\dots, \alpha_{ij}\omega_j w_j + (1-\alpha_{ij}\omega_j)u_j, \dots\big).$$

Denote

$$\beta_{ij} = \alpha_{ij}\omega_j \in [0,1],$$

thus

$$w_i = G_i\big(\dots, \beta_{ij} w_j + (1-\beta_{ij})u_j, \dots\big),$$

so by (5.5) we have

$$w_i = \omega_i G_i\big(\dots, \beta_{ij} w_j + (1-\beta_{ij})u_j, \dots\big) + (1-\omega_i)u_i,$$

which is, by corollary 5.4, a $|\ |_{\infty,\gamma}$ contractive underrelaxed fixed point mapping associated with an s-regular splitting of G. □

5.2. *Application to pseudolinear problems*

Let us associate to the real matrix A the matrix $\langle A\rangle_+$ the coefficients $\overline{a}_{ij}$ of which satisfy

$$\overline{a}_{ii} = a_{ii} \quad \text{and} \quad \overline{a}_{ij} = -|a_{ij}| \quad \text{if } i \neq j.$$

Definition 5.10. A will be said to be an H_+-matrix if the matrix $\langle A\rangle_+$ is an M-matrix.

Definition 5.11. A splitting $A = M - N$ will be called H_+ compatible if

$$\langle A\rangle_+ = \langle M\rangle_+ - |N|,$$

where, if we denote $N = (n_{ij})$, then $|N| = (|n_{ij}|)$.

Consider again the problem of section 4.4: $b - Ax \in \Lambda(x)$. We introduce the mapping G defined by

$$v = G(u) \Leftrightarrow \begin{cases} \delta_i(v_i) \in \Lambda(v_i), \\ \delta_i(v_i) + a_{ii}v_i = -\sum_{j\neq i}^n a_{ij}u_j + b_i. \end{cases} \tag{5.6}$$

Proposition 5.12. It is equivalent to consider a strongly regular splitting of $v - G(v)$ or to consider an H_+ compatible splitting of the H_+-matrix A.

Proof. Consider a strongly regular splitting of $v - G(v)$

$$\delta_i\big(\alpha_{ii}v_i + (1-\alpha_{ii})u_i\big) + a_{ii}\big(\alpha_{ii}v_i + (1-\alpha_{ii})u_i\big)$$
$$= -\sum_{j\neq i}^n a_{ij}\big(\alpha_{ij}v_j + (1-\alpha_{ij})u_j\big) + b_i,$$

$$\delta\big(\alpha_{ii}v_i + (1-\alpha_{ii})u_i\big) \in \Lambda\big(\alpha_{ii}v_i + (1-\alpha_{ii})u_i\big).$$

Let

$$\overline{m}_{ij} = \alpha_{ij}a_{ij}, \qquad \overline{n}_{ij} = (\alpha_{ij}-1)a_{ij}.$$

Since the splitting is strongly regular we have

$$\alpha_{ii} \geqslant 1 \quad \text{and} \quad \alpha_{ij} \in [0,1],$$

thus

$$a_{ii} = \overline{m}_{ii} - \overline{n}_{ii} \quad \text{with} \quad \overline{m}_{ii} \geqslant a_{ii} \text{ and } \overline{n}_{ii} \geqslant 0. \tag{5.7}$$

For $i \neq j$ we have

$$\begin{aligned} |a_{ij}| &= \big|\big(\alpha_{ij} + (1-\alpha_{ij})\big)a_{ij}\big| = \big(\alpha_{ij} + (1-\alpha_{ij})\big)|a_{ij}| \\ &= |\alpha_{ij}a_{ij}| + |(1-\alpha_{ij})a_{ij}| = |\overline{m}_{ij}| + |\overline{n}_{ij}|. \end{aligned} \tag{5.8}$$

If we denote

$$\overline{M} = (\overline{m}_{ij}), \qquad \overline{N} = (\overline{n}_{ij}),$$

then by (5.7) and (5.8) we get that the splitting

$$A = \overline{M} - \overline{N} \tag{5.9}$$

is H_+ compatible.

Conversely, consider an H_+ compatible splitting (5.9) of A, then

$$\begin{cases} a_{ii} = \overline{m}_{ii} - |\overline{n}_{ii}|, \\ a_{ii} = \overline{m}_{ii} - \overline{n}_{ii}, \end{cases} \tag{5.10}$$

which implies

$$\overline{n}_{ii} = |\overline{n}_{ii}| \geqslant 0 \quad \text{and} \quad \overline{m}_{ii} = a_{ii} + \overline{n}_{ii} \geqslant a_{ii}.$$

Let us write

$$\alpha_{ii} = \frac{\overline{m}_{ii}}{a_{ii}} \geqslant 1, \tag{5.11}$$

then

$$\begin{cases} \overline{m}_{ii} = \alpha_{ii}a_{ii}, \\ \overline{n}_{ii} = (\alpha_{ii}-1)a_{ii}. \end{cases}$$

For $i \neq j$

$$\begin{cases} a_{ij} = \overline{m}_{ij} - \overline{n}_{ij}, \\ |a_{ij}| = |\overline{m}_{ij}| + |\overline{n}_{ij}|, \end{cases}$$

so

$$|a_{ij}| = |\overline{m}_{ij}| + |a_{ij} - \overline{m}_{ij}|,$$

which implies that $\overline{m}_{ij}$ and $a_{ij} - \overline{m}_{ij}$ have the same sign, thus

$$\begin{cases} \overline{m}_{ij} \geqslant 0 \Rightarrow a_{ij} \geqslant \overline{m}_{ij} \geqslant 0 \Rightarrow 0 \leqslant \frac{\overline{m}_{ij}}{a_{ij}} \leqslant 1, \\ \overline{m}_{ij} \leqslant 0 \Rightarrow a_{ij} \leqslant \overline{m}_{ij} \leqslant 0 \Rightarrow 0 \leqslant \frac{\overline{m}_{ij}}{a_{ij}} \leqslant 1; \end{cases}$$

then put

$$\alpha_{ij} = \frac{\overline{m}_{ij}}{a_{ij}} \in [0,1], \tag{5.12}$$

so we can consider a splitting of $u - G(u)$

$$\begin{gathered} \delta_i\big(\alpha_{ii}v_i + (1-\alpha_{ii})u_i\big) + a_{ii}\big(\alpha_{ii}v_i + (1-\alpha_{ii})u_i\big) \\ = -\sum_{j\neq i}^{n} a_{ij}\big(\alpha_{ij}v_j + (1-\alpha_{ij})u_j\big) + b_i, \\ \delta\big(\alpha_{ii}v_i + (1-\alpha_{ii})u_i\big) \in \Lambda\big(\alpha_{ii}v_i + (1-\alpha_{ii})u_i\big), \end{gathered}$$

which is strongly regular by (5.10) and (5.12). □

Now consider the problem (4.14), where we make the hypothesis

(h7) A is an H_+-matrix.

Suppose that we want to solve iteratively the subproblem corresponding to $\widehat{A}_l$, so we consider the splittings

$$\widehat{A}_l = \widehat{M}_l - \widehat{N}_l. \tag{5.13}$$

Suppose that

(h8) the splitting (5.13) is H_+ compatible.

Put

$$\overline{M}_l = \operatorname{diag}\big(\widehat{M}_l, \widetilde{A}_l\big), \qquad \overline{N}_l = N_l + \operatorname{diag}\big(\widehat{N}_l, 0\big).$$

As described by figure 3 for $l \in \{1,\dots,m\}$, we have the H_+ compatible splittings

$$A = \overline{M}_l - \overline{N}_l, \tag{5.14}$$

where $\overline{M}_l = (\overline{m}_{ij})_{1\leqslant i,j\leqslant n}$ and $\overline{N}_l = (\overline{n}_{ij})_{1\leqslant i,j\leqslant n}$.

$$\overbrace{\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix}}^{A} = \overbrace{\begin{pmatrix} \tilde{A}_l & 0 & \tilde{A}_l \\ 0 & \hat{M}_l & 0 \\ \tilde{A}_l & 0 & \tilde{A}_l \end{pmatrix}}^{\overline{M}_l} - \overbrace{\begin{pmatrix} 0 & & 0 \\ & \hat{N}_l & \\ 0 & & 0 \end{pmatrix}}^{\overline{N}_l}$$

Figure 3.

Consider the splittings

$$\overline{F}_l(x) = y \tag{5.15}$$

defined by

$$\begin{aligned} &\delta_i\big(\alpha_{ii}y_i + (1-\alpha_{ii})x_i\big) + a_{ii}\big(\alpha_{ii}y_i + (1-\alpha_{ii})x_i\big) \\ &\quad = -\sum_{j\neq i}^{n} a_{ij}\big(\alpha_{ij}y_j + (1-\alpha_{ij})x_j\big) + b_i, \\ &\quad \delta\big(\alpha_{ii}y_i + (1-\alpha_{ii})x_i\big) \in \Lambda\big(\alpha_{ii}y_i + (1-\alpha_{ii})x_i\big). \end{aligned}$$

Proposition 5.13. Under the assumptions (h4) and (h6)–(h8), $\overline{F}_l$ is $|\ |_{\infty,\gamma}$ contractive.

Proof. Proposition 5.9 implies that $\overline{F}_l$ is a strongly regular splitting which is $|\ |_{\infty,\gamma}$ contractive by proposition 5.6. □

Corollary 5.14. Under the assumptions of proposition 5.13 any asynchronous algorithm corresponding to the two-stage multisplitting method defined by (3.6), where F_l is replaced by $\overline{F}_l$ defined by (5.15), converges to the solution of (4.14).

5.3. *Linear problems*

Consider the linear problem

$$Ax = b. \tag{5.16}$$

Suppose that A is split as in figure 1 and (5.13). Assume that

(h9) the splittings of figure 1 are regular;

(h10) the splittings (5.13) are weakly regular.

The splittings (5.14) $A = \overline{M}_l - \overline{N}_l$ are then weakly regular. Consider the splittings defined by $\overline{F}_l$ such that

$$\overline{F}_l(x) = y \Leftrightarrow \overline{M}_l(y) = \overline{N}_l(x) + b. \tag{5.17}$$

We need the following proposition:

Proposition 5.15. If we have a weakly regular splitting of a matrix with inverse $A^{-1} \geqslant 0$,

$$A = \overline{M}_l - \overline{N}_l,$$

then there exists $\gamma \in \mathbb{R}^n$ the components of which are strictly positive and $\nu_l \in\ ]0,1[$ such that

$$\forall l \in \{1,\ldots,m\}, \quad \overline{M}_l^{-1}\overline{N}_l\gamma < \nu_l\gamma. \tag{5.18}$$

Proof. (See also theorem 2.2 of [9] in a different technical framework.) Let us consider a positive vector v and the linear system

$$A\gamma = v.$$

Since $A^{-1} \geqslant 0$ and $v > 0$ then $\gamma > 0$. Let us write the weakly regular splittings of matrices M_l:

$$M_l = \overline{M}_l - \mathcal{N}_l,$$

so we have

$$A = M_l - N_l = \overline{M}_l - \mathcal{N}_l - N_l$$

and so

$$A\gamma = \overline{M}_l\gamma - \mathcal{N}_l\gamma - N_l\gamma = v.$$

From the last equation and assumption (h10) we have

$$\gamma - \overline{M}_l^{-1}\mathcal{N}_l\gamma - \overline{M}_l^{-1}N_l\gamma = \overline{M}_l^{-1}v > 0,$$

and, moreover,

$$\overline{M}_l^{-1}\mathcal{N}_l \geqslant 0.$$

From assumption (h9), $N_l \geqslant 0$ and so $\overline{M}_l^{-1}N_l \geqslant 0$, then

$$\overline{M}_l^{-1}\overline{N}_l = \overline{M}_l^{-1}(\mathcal{N}_l + N_l) \geqslant 0.$$

So vectors γ, v satisfy

$$\gamma - \overline{M}_l^{-1}\overline{N}_l\gamma = \overline{M}_l^{-1}v > 0,$$

which imply

$$\overline{M}_l^{-1}\overline{N}_l\gamma < \gamma. \qquad \square$$

Proposition 5.16. Under hypotheses (h9) and (h10) the mapping $\overline{F}_l$ defined by (5.17) is contractive and its constant is ν_l defined by (5.18).

Proof. Take any $z^l = \overline{F}_l(y)$ and $\tilde{z}^l = \overline{F}_l(\tilde{y})$, so

$$\begin{cases} \overline{M}_l y = \overline{N}_l z^l + b, \\ \overline{M}_l \tilde{y} = \overline{N}_l \tilde{z}^l + b, \end{cases}$$

therefore

$$\overline{M}_l(\tilde{y} - y) = \overline{N}_l\big(\tilde{z}^l - z^l\big)$$

and

$$(\tilde{y} - y) = \overline{M}_l^{-1}\overline{N}_l\big(\tilde{z}^l - z^l\big),$$

$$|\tilde{y} - y| \leqslant \overline{M}_l^{-1} \overline{N}_l \left|\tilde{z}^l - z^l\right|,$$

thus by proposition 1 of [13] we get that

$$|\tilde{y} - y|_{\infty,\gamma} \leqslant \nu_l \left|\tilde{z}^l - z^l\right|_{\infty,\gamma}. \qquad \square$$

As a direct consequence of proposition 3.2 and theorem 2.1 we have the following corollary.

Corollary 5.17. Under assumptions (h4), (h9) and (h10) any asynchronous algorithm corresponding to the two-stage multisplitting method defined by (3.6), where F_l is replaced by $\overline{F}_l$ defined by (5.17), converges to the solution of (5.16).

Remark 5.18. This corollary presents some novelty with respect to theorem 4.3 of [5]: that is to say, it allows more freedom with respect to asynchronism. The corresponding comments quoted in the pseudolinear case (see remark 5.21 and also section 5.6.2) are valid.

Moreover, in the linear case when A is an H-matrix we can also use the results of the pseudolinear case by premultiplying by the sign of the corresponding diagonal element of both parts of each equation of the linear system to be solved, and so deal with the situations covered by theorem 4.4 of [5] with the same increase in asynchronism freedom as quoted above. For a consideration of theorems 4.3 and 4.4 of [5] with respect to relaxation parameters, see remark 5.24 below, which includes also overrelaxation ability.

5.4. General two-stage Schwarz algorithms

We are placed under the hypothesis of proposition 5.13 (respectively proposition 5.16).

We associate with the multisplitting (5.15) (respectively (5.17)) the fixed point mapping

$$\begin{cases} \overline{T}^{\mathrm{MS}} : \left(\mathbb{R}^n\right)^m \to \left(\mathbb{R}^n\right)^m, \\ X = \left(x^1, \ldots, x^m\right) \to Y = \left(y^1, \ldots, y^m\right) \end{cases}$$

such that for all $l \in \{1, \ldots, m\}$

$$\begin{cases} y^l = \overline{F}_l(z), \\ z = \sum_{k=1}^m E_{lk} x^k, \end{cases} \tag{5.19}$$

where E_{lk} are the weighting matrices satisfying (4.9).

Remark 5.19. In the two stage Schwarz method the computation on the lth subdomain corresponds to the solution of a block diagonal subproblem, the matrix of which is $\widehat{A}_l = \widehat{M}_l - \widehat{N}_l$.

The inner iteration involves the above splitting of $\widehat{A}_l$ with, at each resolution of a linear system with matrix $\widehat{M}_l$, we need two kinds of data:

(i) Data issued from the progress of outer iterations, which are of the same kind as in the case of the exact Schwarz algorithm. The same kind means with the same weighting of the data issued from other subproblems.

(ii) Data issued from the progress of the inner iterations. In the Schwarz method it seems natural to use data issued solely from the previous inner iteration, and not a weighting of data partly issued from other subproblems.

This means that we must modify the weighting matrices' indices proposed by O'Leary and White in order that the diagonal coefficients of matrix E_l, the indices of which belong to I_l, be equal to one. The consequence is that we must introduce a second subscript so that E_l becomes E_{ll} and for $k \neq l$ and $I_k \cap I_l \neq \varnothing$ we must replace E_k by E_{lk} where the diagonal coefficient of E_k corresponding to an index which belongs to $I_k \cap I_l$ is put to zero in E_{lk}.

In the two-subdomain two-level Schwarz alternating method, the general formalism of asynchronous iterations does not now reduce to one or two independent Gauss–Seidel-like algorithms as quoted in the case of the standard Schwarz two-subdomain fixed point mapping in section 4.3.3(i).

Following section 4 we obtain

Corollary 5.20. Under the assumptions of proposition 5.13 (respectively proposition 5.16), any asynchronous algorithm $(\overline{T}^{\mathrm{MS}}, X^0, J, S)$ corresponding to the two-stage Schwarz multisplitting method converges to the solution of (4.14) (respectively (5.16)).

First particular situation: Outer two-stage Gauss–Seidel iterations

In order to make the presentation and notations more clear, let us at first consider the Gauss–Seidel algorithm applied to the fixed point mapping T_2^{MS} introduced in section 4.4.3, that is to say: we choose the general asynchronous formulation

$$\begin{cases} J(p) = 1 + p \pmod m, \\ \overline{\rho}_j(p) = p. \end{cases}$$

A whole standard Gauss–Seidel iteration step (the qth, say), corresponds to m consecutive steps of the above iteration with indices p such that

$$qm \leqslant p < (q+1)m.$$

To solve iteratively the lth subproblem, let us now consider $s(q,l)$ inner iterations at the qth outer Gauss–Seidel iteration. In order to formulate the corresponding two-stage Gauss–Seidel algorithms, let us introduce the following mapping:

$$\begin{cases} \overline{p}_l : (q+1) \to \overline{p}_l(q+1) = \overline{p}_m(q) + \sum_{r=0}^{l-1} s(q,r), \\ \overline{p}_l(0) = 0 \ \forall l \in \{1,\dots,m\}, \quad s(q,0) = 0 \ \forall q = 0,1,\dots . \end{cases} \tag{5.20}$$

Then the outer two-stage Gauss–Seidel algorithm can be expressed by asynchronous iterations associated with $\overline{T}^{\mathrm{MS}}$ defined by (5.19) with $\overline{\rho}_j(p)$ and $J(p)$ satisfying

$$\begin{aligned}&\overline{\rho}_j(p) = p,\\ &J(p) = \{l\} \Leftrightarrow \overline{p}_l(q+1) \leqslant p < \overline{p}_l(q+1) + s(q,l).\end{aligned} \tag{5.21}$$

Second particular situation: Outer two-stage Jacobi iterations

The outer two-stage Jacobi algorithm can be described by asynchronous iterations associated with $\overline{T}^{\mathrm{MS}}$ defined by (5.19) with $J(p)$ satisfying (5.21) and $\overline{\rho}_k(p)$ satisfying

$$\text{for } p = \overline{p}_l(q+1) \quad \begin{cases} \overline{\rho}_k(p) \in [\overline{p}_{k+1}(q), \overline{p}_1(q+1)] & \text{if } k \leqslant l,\\ \overline{\rho}_k(p) \in [\overline{p}_{k+1}(q), p] & \text{if } k > l,\end{cases} \tag{5.22}$$

and

$$\text{for } \overline{p}_l(q+1) < p < \overline{p}_{l+1}(q+1) \quad \begin{cases} \overline{\rho}_l(p) = p,\\ \overline{\rho}_k(p) \in [\overline{p}_{k+1}(q), \overline{p}_1(q+1)] & \text{if } k < l,\\ \overline{\rho}_k(p) \in [\overline{p}_{k+1}(q), p] & \text{if } k > l.\end{cases} \tag{5.23}$$

Remark 5.21. (5.23) is a constraint on $\overline{\rho}_k(p)$. It expresses that any local inner iteration cannot take advantage of the progress of any other local inner iteration corresponding to the same qth outer iteration.

Bibliographical comment

The practical situations considered by Rodrigue et al. in [16–18] correspond to choosing

$$s(l,q) = s \quad \forall l \in \{1,\dots,m\},\ \forall q = 0,1,\dots .$$

5.5. *Two-stage O'Leary and White algorithms*

Denote $[u]_{I_l}$ the vector of $\mathbb{R}^{\mathrm{card}(I_l)}$ such that

$$\forall j \in I_l, \quad \big([u]_{I_l}\big)_j = u_j.$$

Let $\overline{T}$ denote the fixed point mapping which describes the two-stage asynchronous multisplitting algorithms; using the splitting (5.14) for the linear part, $\overline{T}$ can be written as: $\forall l \in \{1,\dots,m\}$,

$$\begin{aligned}&\overline{T}_l\big(x^1,\dots,x^m\big) = y^l = \big(\hat{y}^l, \tilde{y}^l\big) \quad \text{such that}\\ &\delta\big(\hat{y}^l\big) + \widehat{M}_l\hat{y}^l = \left[\overline{N}_l\left(\sum_{k=1}^m E_{lk}(X)x^k\right) + b\right]_{I_l},\\ &\delta\big(\tilde{y}^l\big) + \widetilde{A}_l\tilde{y}^l = \left[N_l\left(\sum_{k=1}^m E_{lk}(X)x^k\right) + b\right]_{I_l^C}.\end{aligned} \tag{5.24}$$

Denote $X^p = (x^{1,p}, \ldots, x^{l,p}, \ldots, x^{m,p})$, the asynchronous algorithms corresponding to (5.24) are

- if $l \in J(p)$, then

$$\begin{aligned} \delta\big[x^{l,p+1}\big]_{I_l} + \widehat{M}_l\big[x^{l,p+1}\big]_{I_l} &= \widehat{N}_l\big[x^{l,\overline{\rho}_l(p)}\big]_{I_l} + \left[N_l\left(\sum_{k=1}^{m} E_{lk}\big(X^p\big)x^{k,\overline{\rho}_k(p)}\right) + b\right]_{I_l}, \\ \delta\big[x^{l,p+1}\big]_{I_l^C} + \widetilde{A}_l\big[x^{l,p+1}\big]_{I_l^C} &= \left[N_l\left(\sum_{k=1}^{m} E_{lk}\big(X^p\big)x^{k,\overline{\rho}_k(p)}\right) + b\right]_{I_l^C}, \end{aligned} \tag{5.25}$$

- if $l \notin J(p)$, then $x^{l,p+1} = x^{l,p}$.

Now in order to get synchronous and asynchronous algorithms defined by Bru et al. in [5,20], we restrict ourselves to linear problems and we denote $R_l = \widehat{M}_l^{-1}\widehat{N}_l$. Due to the comments of section 3.3 we choose $E_{lk}(X^p)$ such that

$$\text{for } \overline{p}_l(q+1) < p < \overline{p}_{l+1}(q+1), \quad \begin{cases} \big(E_{ll}(X^p)\big)_{ii} = \begin{cases} 1 & \text{if } i \in I_l, \\ 0 & \text{if } i \notin I_l, \end{cases} \\ \big(E_{lk}(X^p)\big)_{ii} = \begin{cases} 0 & \text{if } i \in I_l, \\ (E_k)_{ii} & \text{if } i \notin I_l, \end{cases} \end{cases} \tag{5.26}$$

and

$$\text{for } p = \overline{p}_l(q+1), \quad E_{lk}\big(X^p\big) = E_k. \tag{5.27}$$

5.5.1. Synchronous two-stage multisplitting algorithms

To get the synchronous two-stage algorithms defined in [5], we consider (5.25) with the above $E_{lk}(X^p)$, and $J(p)$ and $\overline{\rho}_k(p)$ defined respectively by (5.21), (5.22) and (5.23), so we obtain

$$x^{q+1} = \sum_{l=1}^{m} E_l\left(R_l^{s(l,q)}\big[x^q\big]_{I_l} + \sum_{r=0}^{s(l,q)-1} R_l^r \widehat{M}_l^{-1}\big[N_l x^q + b\big]_{I_l}\right).$$

Remark 5.22. For $p = \overline{p}_1(q+1)$ we have $\overline{\rho}_k(p) \in \{\overline{p}_2(q), \ldots, \overline{p}_m(q)\}$ since $J(p) = \{1\}$ and $p = \overline{p}_1(q+1) = \overline{p}_m(q)$.

5.5.2. Outer asynchronous multisplitting two-stage algorithms

If we choose $J(p)$ satisfying (5.21) and $\overline{\rho}_k(p)$ such that

$$\overline{\rho}_k(p) = \overline{p}_l\big(\rho_k(q)\big)$$

with $\rho_k(q)$ satisfying conditions (h2) and (h3) of section 2, then we get the outer asynchronous two-stage algorithms defined in [5]:

$$x^{l,q+1} = \begin{cases} x^{l,q} \quad \forall l \notin J(q), \\ E_l\left(R_l^{s(l,q)} x^{l,\rho_l(q)} + \displaystyle\sum_{r=0}^{s(l,q)-1} R_l^r \widehat{M}_l^{-1}\left[N_l \sum_{k=1}^{m} x^{k,\rho_k(q)} + b\right]_{I_l}\right). \end{cases}$$

5.5.3. Totally asynchronous multisplitting algorithms

To get the totally asynchronous algorithms defined in [5], take $J(p)$ satisfying (5.21) and put

$$\begin{cases} \overline{\rho}_k(p) = \overline{p}_l(\rho_k(q,r)), \\ r = p - \overline{p}_l(q). \end{cases}$$

Then we obtain

$$x^{l,q+1} = \begin{cases} x^{l,q} \quad \forall l \notin J(q), \\ E_l\left(R_l^{s(l,q)} x^{l,\rho_l(q)} + \displaystyle\sum_{r=0}^{s(l,q)-1} R_l^r \widehat{M}_l^{-1}\left[N_l \sum_{k=1}^{m} x^{k,\rho_k(q,r)} + b\right]_{I_l}\right). \end{cases}$$

Remark 5.23. About the above O'Leary and White-kind algorithms: in the inner current iteration, the update of components which belong to the lth subdomain depends partly on the use of previous values of these components. These previous values are issued

– either exclusively from the inside of the corresponding subdomain, and we have to take $E_{ll}(X^p)$ (and $E_{lk}(X^p)$, which correspond to the outside data) defined by (5.26),
– or partly from several subdomains, and we have to take the $E_{lk}(X^p)$ satisfying (5.27).

5.6. Some complement about asynchronism and mixing the O'Leary and White and Schwarz approaches

5.6.1. About asynchronism

In the formulation of Bru et al. in [5] $\rho_k(q)$ satisfy condition (h2), that is to say,

$$\rho_k(q) \leqslant q,$$

which in the corresponding expression of $\overline{\rho}_k(p)$ corresponds to the constraint (5.23), which has not necessarily to be satisfied in the formulation of asynchronous iterations applied to the mapping $\overline{T}$.

5.6.2. About mixing the O'Leary and White and Schwarz approaches

Moreover, in this general framework, at each iteration indexed by p, we are able to choose either O'Leary and White's or Schwarz's weighting matrices. So we can have these two points of view in the same iterative algorithm and even in the same outer qth loop of such algorithms.

Remark 5.24. For all contractive fixed point mappings described in this paper, we can introduce in a standard way a relaxation parameter ω satisfying

$$0 < \omega < \frac{2}{1+\nu},$$

where ν is the constant of contraction of the considered fixed point mapping, and we can prove that the asynchronous algorithms corresponding to the relaxed fixed point mapping converge.

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