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Preconditioned GSOR iterative method for a class of complex symmetric system of linear equations

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SUMMARY

In this paper, we present a preconditioned variant of the generalized successive overrelaxation (GSOR) iterative method for solving a broad class of complex symmetric linear systems. We study conditions under which the spectral radius of the iteration matrix of the preconditioned GSOR method is smaller than that of the GSOR method and determine the optimal values of iteration parameters. Numerical experiments are given to verify the validity of the presented theoretical results and the effectiveness of the preconditioned GSOR method. Copyright © 2015 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Consider the system of linear equations of the form

$$Au \equiv (W + iT)u = b, \quad u, b \in \mathbb{C}^n, \tag{1}$$

where $i = \sqrt{-1}$ and $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semidefinite matrices with at least one of them, for example, W, being positive definite. We assume that $T \neq 0$, which implies that A is non-Hermitian. Such systems arise in many problems such as quantum mechanics [1], diffuse optical tomography [2], structural dynamics [3], FFT-based solution of certain time-dependent PDEs [4] and molecular scattering [5]. For more examples and additional references, the reader is referred to [6].

The Hermitian/skew-Hermitian (HS) splitting of the matrix A is given by

$$A = H + S, \tag{2}$$

where

$$H = \frac{1}{2}(A + A^H) = W \quad \text{and} \quad S = \frac{1}{2}(A - A^H) = iT,$$

with A^H being the conjugate transpose of A. Based on the HS splitting (2), the HSS iteration method [7] can be straightforwardly applied to solve (1). Bai *et al.* [8] recently proposed the use of the following modified Hermitian and skew-Hermitian splitting (MHSS) method, which is more efficient than the HSS iteration method for solving the complex symmetric linear system (1):

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The MHSS iteration method. Given an initial guess $u^{(0)} \in \mathbb{C}^n$ and positive constant α for $k = 0, 1, 2 \ldots$ until $\{u^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + W)u^{(k+\frac{1}{2})} = (\alpha I - iT)u^{(k)} + b, \\ (\alpha I + T)u^{(k+1)} = (\alpha I + iW)u^{(k+\frac{1}{2})} - ib, \end{cases} \tag{3}$$

where I is the identity matrix.

In [8], Bai and coworkers proved that the MHSS iterative method is convergent for any positive constant α. Obviously, both of the matrices $\alpha I + W$ and $\alpha I + T$ are symmetric positive definite. Therefore, the two sub-systems involved in each step of the MHSS iteration can be solved effectively by using the Cholesky factorization of the matrices $\alpha I + W$ and $\alpha I + T$. Moreover, to solve both of the sub-systems in the inexact variant of the MHSS method, one can use a preconditioned conjugate gradient method. This is different from the HSS iteration method, in which a shifted skew-Hermitian linear sub-system with coefficient matrix $\alpha I + iT$ needs to be solved at the second half-step of every iteration. More recently, Bai *et al.* [9] proposed a preconditioned variant of the MHSS (PMHSS) for solving a class of complex symmetric systems of linear equations. It is necessary to mention that a potential difficulty with the HSS and MHSS iteration methods is the need to use complex arithmetic. Moreover, Axelsson *et al.* [10] have presented a comparison of iterative methods to solve the complex symmetric linear system of equation (1).

Letting $u = x + iy$ and $b = p + iq$ where $x, y, p, q \in \mathbb{R}^n$, the complex linear system (1) can be rewritten as 2-by-2 block real equivalent formulation

$$\mathcal{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{4}$$

where

$$\mathcal{A} = \begin{bmatrix} W & -T \\ T & W \end{bmatrix}.$$

This linear system can be formally regarded as a special case of the generalized saddle point problem [11, 12]. Recently, more efficient preconditioners for the real formulation (4) have been proposed [6, 13–15]. Rather than solving the original complex linear system (1), Salkuyeh *et al.* [16] solved the real equivalent system (4) by the generalized successive overrelaxation (GSOR) iterative method. By some numerical experiments, they have shown that the performance of the GSOR method is much more better than the MHSS method.

In order to solve the linear system (4) by the GSOR method, the matrix $\mathcal{A}$ is split as

$$\mathcal{A} = \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -T & 0 \end{bmatrix} - \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}.$$

So, for $0 \neq \alpha \in \mathbb{R}$, the GSOR method can be constructed as follows:

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix} = \mathcal{G}_\alpha \begin{bmatrix} x^k \\ y^k \end{bmatrix} + C_\alpha \begin{bmatrix} p \\ q \end{bmatrix}, \tag{5}$$

where

$$\begin{aligned} \mathcal{G}_\alpha &= \begin{bmatrix} W & 0 \\ \alpha T & W \end{bmatrix}^{-1} \left((1-\alpha) \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} + \alpha \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \right), \\ &= \begin{bmatrix} I & 0 \\ \alpha S & I \end{bmatrix}^{-1} \begin{bmatrix} (1-\alpha)I & \alpha S \\ 0 & (1-\alpha) \end{bmatrix}, \end{aligned}$$

is the iteration matrix, wherein $S = W^{-1}T$, and

$$C_\alpha = \alpha \begin{bmatrix} W & 0 \\ \alpha T & W \end{bmatrix}^{-1}.$$

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It is easy to see that (5) is equivalent to

$$\begin{cases} Wx^{(k+1)} = (1-\alpha)Wx^{(k)} + \alpha Ty^{(k)} + \alpha p, \\ Wy^{(k+1)} = -\alpha Tx^{(k+1)} + (1-\alpha)Wy^{(k)} + \alpha q, \end{cases} \tag{6}$$

where $x^{(0)}$ and $y^{(0)}$ are the initial approximations for x and y, respectively. Salkuyeh *et al.* [16] proved that, under certain condition on α, the GSOR method is convergent and determined the optimal value of the iteration parameter α and the corresponding optimal convergence factor. In the GSOR method, two sub-systems with the coefficient matrix W should be solved, which can be done by the Cholesky factorization or inexactly by the CG algorithm. Moreover, the right-hand side of the sub-systems are real. Therefore, the solution of the system can be obtained by the real version of the algorithms.

By decreasing the spectral radius of the iteration matrix $\mathcal{G}_\alpha$, the convergence rate of the GSOR method can been improved. For this purpose, an effective method is to transform the linear system (4) into the preconditioned form

$$\mathcal{P}\mathcal{A}\begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{P}\begin{bmatrix} p \\ q \end{bmatrix},$$

where $\mathcal{P} \in \mathbb{R}^{2n\times 2n}$ is nonsingular. The basic GSOR iterative method corresponding to the preconditioned system will be referred to as the preconditioned GSOR iterative method (PGSOR).

In this paper, we are going to consider the PGSOR method with the following preconditioner

$$\mathcal{P}_{\rightarrow} = \begin{bmatrix} \omega I & I \\ -I & \omega I \end{bmatrix},$$

where $0 < \omega \in \mathbb{R}$. We study condition on ω under which the spectral radius of the iterative matrix of the PGSOR method is smaller than that of the GSOR method and determine the optimal values of α and ω. Moreover, we propose the approximation values for α and ω such that the performance of the corresponding PGSOR method is close to that of the optimal parameters. Finally, numerical examples are presented to verify the theoretical results and the effectiveness of the PGSOR method. It is noteworthy that the motivation of choosing this preconditioner stems from [6] in which Benzi *et al.* have presented some examples of preconditioners for the real formulation (4) of the system (1).

Throughout this paper, for a square matrix X, $\sigma(X)$ and $\rho(X)$ denote for the spectrum and the spectral radius of X, respectively. For a vector $z \in \mathbb{C}^n$, $\|z\|_2$ denotes for the Euclidean norm of z.

The paper is organized as follows. In Section 2, we discuss the convergence of the PGSOR iterative method. Section 3 is devoted to some numerical experiments to confirm the theoretical results given in Section 2. Finally, in Section 4, some concluding remarks are given.

2. THE PRECONDITION GENERALIZED SUCCESSIVE OVERRELAXATION ITERATIVE METHOD

In this section, we express the PGSOR iterative method and its convergence properties. Before presenting this method, we review the established theorems in [16] that relate to the convergence properties of the GSOR method.

Theorem 2.1
Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric, respectively. Then, the GSOR method to solve Eqn (4) is convergent if and only if

$$0 < \alpha < \frac{2}{1+\rho(S)},$$

where $S = W^{-1}T$.

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Theorem 2.2
Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric, respectively. Then, the optimal value of the parameter α for the GSOR iterative method (5) is given by

$$\alpha^* = \frac{2}{1+\sqrt{1+\rho(S)^2}}, \tag{7}$$

and the corresponding optimal convergence factor of the method is given by

$$\rho(\mathcal{G}_{\alpha^*}) = 1-\alpha^* = 1-\frac{2}{1+\sqrt{1+\rho(S)^2}}, \tag{8}$$

where $S = W^{-1}T$.

Corollary 2.1
Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semi-definite, respectively. Then, the GSOR method is convergent if and only if

$$0 < \alpha < \frac{2}{1+\mu_{\max}(S)},$$

where $\mu_{\max}(S)$ is largest eigenvalue of $S = W^{-1}T$. Moreover, the optimal value of iteration parameter α and corresponding optimal convergence factor can be computed as follows:

$$\alpha^* = \frac{2}{1+\sqrt{1+\mu_{\max}(S)^2}} \quad \text{and} \quad \rho(\mathcal{G}_{\alpha^*}) = 1-\alpha^* = 1-\frac{2}{1+\sqrt{1+\mu_{\max}(S)^2}}. \tag{9}$$

Remark 2.1
When the matrices W and T are symmetric positive definite and symmetric positive semi-definite, respectively, if $\mu_{\max}(S) = 0$ ($S = W^{-1}T$), then according to Eqn(9), we have $\alpha^* = 1$ and $\rho(\mathcal{G}_{\alpha^*}) = 0$. This means that the method would have the highest speed of convergence. Hence, hereafter, we assume $\mu_{\max}(S) \neq 0$.

In order to speed up the convergence rate of the GSOR method, it is known that the spectral radius of the iterative matrix $\mathcal{G}_{\oslash}$ must be decreased. According to Theorem 2, the spectral radius of the iteration matrix $\mathcal{G}_{\oslash}$ tends to zero as the spectral radius of S approaches to zero. For decreasing the spectral radius of S, an effective method is to precondition the linear system (4). We apply $\mathcal{P}_\omega$ to the system (4) to obtain the preconditioned linear system

$$\tilde{\mathcal{A}}_\omega \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}, \tag{10}$$

with

$$\tilde{\mathcal{A}}_\omega = \begin{bmatrix} \tilde{W}_\omega & -\tilde{T}_\omega \\ \tilde{T}_\omega & \tilde{W}_\omega \end{bmatrix} = \begin{bmatrix} \omega W + T & -(\omega T - W) \\ \omega T - W & \omega W + T \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix} = \begin{bmatrix} \omega p + q \\ \omega q - p \end{bmatrix}.$$

Remark 2.2
Since W and T are symmetric positive definite and symmetric positive semi-definite, respectively, we can easily conclude that $\tilde{W}_\omega = \omega W + T$ and $\tilde{T}_\omega = \omega T - W$ are symmetric positive definite and symmetric, respectively.

By splitting the coefficient matrix $\tilde{\mathcal{A}}_\omega$ as

$$\tilde{\mathcal{A}}_\omega = \begin{bmatrix} \tilde{W}_\omega & 0 \\ 0 & \tilde{W}_\omega \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -\tilde{T}_\omega & 0 \end{bmatrix} - \begin{bmatrix} 0 & \tilde{T}_\omega \\ 0 & 0 \end{bmatrix},$$

the GSOR method for solving (10) is given by

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix} = \tilde{\mathcal{G}}_\alpha(\omega) \begin{bmatrix} x^k \\ y^k \end{bmatrix} + \tilde{C}_\alpha(\omega) \begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}, \tag{11}$$

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where

$$\tilde{\mathcal{G}}_\alpha(\omega) = \begin{bmatrix} I & 0 \\ \alpha\tilde{S}_\omega & I \end{bmatrix}^{-1} \begin{bmatrix} (1-\alpha)I & \alpha\tilde{S}_\omega \\ 0 & (1-\alpha)I \end{bmatrix},$$

is iteration matrix, wherein $\tilde{S}_\omega = \tilde{W}_\omega^{-1}\tilde{T}_\omega$ and

$$\tilde{C}_\alpha(\omega) = \alpha \begin{bmatrix} \tilde{W} & 0 \\ \alpha\tilde{T} & \tilde{W} \end{bmatrix}^{-1}.$$

It is easy to see that (11) is equivalent to

$$\begin{cases} \tilde{W}_\omega x^{(k+1)} = (1-\alpha)\tilde{W}_\omega x^{(k)} + \alpha\tilde{T}_\omega y^{(k)} + \alpha\tilde{p}, \\ \tilde{W}_\omega y^{(k+1)} = -\alpha\tilde{T}_\omega x^{(k+1)} + (1-\alpha)\tilde{W}_\omega y^{(k)} + \alpha\tilde{q}, \end{cases} \tag{12}$$

where $x^{(0)}$ and $y^{(0)}$ are the initial approximations for x and y, respectively. In the PGSOR method, two sub-systems with coefficient matrix $\tilde{W}_\omega = \omega W + T$ should be solved, which can be done by the Cholesky factorization or inexactly by the CG algorithm. Often , $\omega W + T$ is better conditioned than W itself.

Remark 2.3
The PGSOR method is equivalent to apply the GSOR method for real equivalent formation of the new complex system that is obtained by multiplying the complex number $(\omega - i)$ through both sides of the complex system (1).

Lemma 2.1 (see[16]*)*
Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric, respectively. Then, the eigenvalues of the matrix $S = W^{-1}T$ are all real. If T be a symmetric positive semi-definite matrix, then, the eigenvalues of S are all nonnegative.

Lemma 2.2
Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semi-definite, respectively. Also let ω be a positive constant, $\tilde{W}_\omega = \omega W + T$ and $\tilde{T}_\omega = \omega T - W$. If λ is an eigenvalue of $\tilde{S}_\omega = \tilde{W}_\omega^{-1}\tilde{T}_\omega$, then, there is an eigenvalue μ of $S = W^{-1}T$ that satisfies

$$\lambda = \frac{\omega\mu - 1}{\omega + \mu}. \tag{13}$$

Moreover

$$\rho(\tilde{S}_\omega) = \max\left\{\frac{1-\omega\mu_{\min}}{\omega+\mu_{\min}}, \frac{\omega\mu_{\max}-1}{\omega+\mu_{\max}}\right\}, \tag{14}$$

where $\mu_{\min}$ and $\mu_{\max}$ are the smallest and largest eigenvalues of S, respectively.

Proof
Let (λ, x) be an eigenpair of $\tilde{S}_\omega$. Then $\tilde{S}_\omega x = \lambda x$, which is equivalent to

$$(\omega T - W)x = \lambda(\omega W + T)x.$$

Multiplying both sides of the earlier equation by W^{-1}, we obtain

$$(\omega - \lambda)Sx = (\omega\lambda + 1)x.$$

Evidently, we have $\lambda \neq \omega$. Then, from latter equation we get

$$Sx = \frac{\omega\lambda + 1}{\omega - \lambda}x,$$

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and hence, there is an eigenvalue μ of $S = W^{-1}T$ such that

$$\mu = \frac{\omega\lambda + 1}{\omega - \lambda},$$

and from this we can easily obtain (13).

The second part is a consequence of the fact that

$$h(\mu) = \frac{\omega\mu - 1}{\omega + \mu},$$

is an increasing function with respect to variable μ. □

Since $\tilde{W}_\omega$ and $\tilde{T}_\omega$ are, respectively, symmetric positive definite and symmetric matrices, analogously to Theorems 2.1 and 2.2, we can prove that the PGSOR iteration is convergent if and only if

$$0 < \alpha < \frac{2}{1 + \rho(\tilde{S}_\omega)},$$

and the optimal value of the iteration parameter α for the PGSOR iterative method (11) is given by

$$\alpha^* = \frac{2}{1 + \sqrt{1 + \rho(\tilde{S}_\omega)^2}}, \tag{15}$$

and moreover, the corresponding optimal convergence factor of the method is given by

$$\rho(\mathcal{G}_{\alpha^*}(\omega)) = 1 - \alpha^* = 1 - \frac{2}{1 + \sqrt{1 + \rho(\tilde{S}_\omega)^2}}, \tag{16}$$

where $\tilde{S}_\omega = (\omega W + T)^{-1}(\omega T - W)$.

Note that in the aforementioned relations, for the value of $\rho(\tilde{S}_\omega)$, one may use Eqn (14).

The main objective of applying $\mathcal{P}_\omega$ is to decrease the spectral radius of the iteration matrix. This work is done by decreasing the spectral radius of $\tilde{S}_\omega$ in comparison to that of S, then, from (8) and (16), we will get $\rho\left(\tilde{\mathcal{G}}_{\tilde{\alpha}^*}(\omega)\right) < \rho\left(\mathcal{G}_{\alpha^*}\right)$. The necessary and sufficient conditions to achieve these desired results are formally stated in the following lemma and theorem.

Lemma 2.3

Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semi-definite, respectively. Also let ω be a positive constant, $\tilde{W}_\omega = \omega W + T$ and $\tilde{T}_\omega = \omega T - W$. Then, the spectral radius of $\tilde{S}_\omega = \tilde{W}_\omega^{-1}\tilde{T}_\omega$ is smaller than that of $S = W^{-1}T$ if and only if

$$\max\left\{0, \frac{1 - \mu_{\min}\mu_{\max}}{\mu_{\min} + \mu_{\max}}\right\} < \omega \tag{17}$$

where $\mu_{\min}$ and $\mu_{\max}$ are the smallest and largest eigenvalues of S, respectively.

Proof

Suppose λ is an arbitrary eigenvalue of $\tilde{S}$. Using Lemma 2.2, there is an eigenvalue μ of S such that $\lambda = \frac{\omega\mu - 1}{\omega + \mu}$. Moreover, from Lemma 2.1, we saw that the eigenvalues of S are nonnegative. Then, the spectral radius of $\tilde{S}_\omega$ is smaller than that of S if and only if, for every eigenvalue μ of S, the following inequality holds

$$-\mu_{\max} < \frac{\omega\mu - 1}{\omega + \mu} < \mu_{\max}. \tag{18}$$

It is easy to see that the right inequality of (18) holds if and only if $0 < \omega$ and the left inequality of (18) holds if and only if

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$$\frac{1-\mu\mu_{\max}}{\mu+\mu_{\max}} < \omega \quad \forall \mu. \tag{19}$$

Define

$$f(\mu) = \frac{1-\mu\mu_{\max}}{\mu+\mu_{\max}}.$$

Because $f(\mu)$ is a decreasing function, then, the inequality (19) holds if and only if

$$\frac{1-\mu_{\min}\mu_{\max}}{\mu_{\min}+\mu_{\max}} < \omega,$$

which completes the proof. □

Theorem 2.3
Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semi-definite, respectively, and ω be a positive constant. Let $\mathcal{G}_{\alpha^*}$ and $\tilde{\mathcal{G}}_{\tilde{\alpha}^*}(\omega)$ be the iteration matrices of the GSOR and PGSOR methods, respectively. Then, $\rho\left(\tilde{\mathcal{G}}_{\tilde{\alpha}^*}(\omega)\right) < \rho\left(\mathcal{G}_{\alpha^*}\right)$ if and only if

$$\max\left\{0, \frac{1-\mu_{\min}\mu_{\max}}{\mu_{\min}+\mu_{\max}}\right\} < \omega, \tag{20}$$

where $\mu_{\min}$ and $\mu_{\max}$ are the smallest and largest eigenvalues of $S = W^{-1}T$, respectively.

Proof
From (8) and (16), we can find that $\rho\left(\tilde{\mathcal{G}}_{\tilde{\alpha}^*}(\omega)\right) < \rho\left(\mathcal{G}_{\alpha^*}\right)$ if and only if $\rho(\tilde{S}_\omega) < \rho(S)$, where $\tilde{S}_\omega = (\omega W + T)^{-1}(\omega T - W)$, and by the Lemma 2.3, the latter inequality holds if and only if ω satisfies (20). □

In the previous theorem, for every ω satisfying (20), we saw that the spectral radius of the PGSOR method is smaller than that of the GSOR method. Now, in the next theorem, we obtain the optimal value of the parameter ω, which minimizes the spectral radius of the iterative matrix of the PGSOR method, that is

$$\rho\left(\tilde{\mathcal{G}}_{\alpha^*}(\omega^*)\right) = \min_{\omega} \rho\left(\tilde{\mathcal{G}}_{\alpha^*}(\omega)\right).$$

Lemma 2.4
Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semi-definite, respectively, $\tilde{W}_\omega = \omega W + T$ and $\tilde{T}_\omega = \omega T - W$. Moreover, let $\mu_{\min}$ and $\mu_{\max}$ be the smallest and largest eigenvalues of $S = W^{-1}T$, respectively, and ω be a positive parameter. Then, the optimal value of the parameter ω which minimizes the spectral radius $\rho(\tilde{S}_\omega)$ of the matrix $\tilde{S}_\omega = \tilde{W}_\omega^{-1}\tilde{T}_\omega$ is given by

$$\omega^* = \frac{1-\mu_{\min}\mu_{\max}+\sqrt{\left(1+\mu_{\min}^2\right)\left(1+\mu_{\max}^2\right)}}{\mu_{\min}+\mu_{\max}}. \tag{21}$$

Proof
By using Lemma 2.1, it is known that the eigenvalues μ of S are nonnegative. Then, according to Lemma 2.2, we can write

$$\rho(\tilde{S}_\omega) = \max_{\mu\in\sigma(S)} \frac{|\omega\mu - 1|}{\omega+\mu}.$$

Note that

$$g(\mu) = \frac{\omega\mu - 1}{\omega+\mu},$$

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is an increasing function with respect to variable μ. Now, if $\mu_{\max} \leqslant \frac{1}{\omega}$, then for all $\mu \in \sigma(S)$, we have $\omega\mu - 1 \leqslant 0$. Hence

$$h(\omega) \equiv \max_{\mu\in\mu(S)} \frac{|\omega\mu - 1|}{\omega + \mu} = \frac{1 - \omega\mu_{\min}}{\omega + \mu_{\min}} \quad \text{if} \quad \mu_{\max} \leqslant \frac{1}{\omega}. \tag{22}$$

If $\frac{1}{\omega} \leqslant \mu_{\max}$, then $0 \leqslant \omega\mu_{\max} - 1$. First, we suppose $\mu_{\min} \neq 0$ and consider the following two cases:

Case I: If $\mu_{\min} \leqslant \frac{1}{\omega}$, we have $\omega\mu_{\min} - 1 \leqslant 0$, which implies that

$$h(\omega) = \max\left\{\frac{1 - \omega\mu_{\min}}{\omega + \mu_{\min}}, \frac{\omega\mu_{\max} - 1}{\omega + \mu_{\max}}\right\} \quad \text{if} \quad \mu_{\min} \leqslant \frac{1}{\omega} \leqslant \mu_{\max}. \tag{23}$$

Case II: If $\frac{1}{\omega} \leqslant \mu_{\min}$, then $0 \leqslant \omega\mu_{\min} - 1$, which implies that

$$h(\omega) = \frac{\omega\mu_{\max} - 1}{\omega + \mu_{\max}} \quad \text{if} \quad \frac{1}{\omega} \leqslant \mu_{\min}. \tag{24}$$

From (22), (23) and (24), we can obtain the following result:

$$h(\omega) = \begin{cases} \dfrac{1 - \omega\mu_{\min}}{\omega + \mu_{\min}}, & \text{for} \quad \omega \leqslant \frac{1}{\mu_{\max}}, \\ \max\left\{\dfrac{1 - \omega\mu_{\min}}{\omega + \mu_{\min}}, \dfrac{\omega\mu_{\max} - 1}{\omega + \mu_{\max}}\right\}, & \text{for} \quad \frac{1}{\mu_{\max}} \leqslant \omega \leqslant \frac{1}{\mu_{\min}}, \\ \dfrac{\omega\mu_{\max} - 1}{\omega + \mu_{\max}}, & \text{for} \quad \frac{1}{\mu_{\min}} \leqslant \omega. \end{cases} \tag{25}$$

Define

$$f_1(\omega) = \frac{1 - \omega\mu_{\min}}{\omega + \mu_{\min}} \quad \text{and} \quad f_2(\omega) = \frac{\omega\mu_{\max} - 1}{\omega + \mu_{\max}}.$$

The functions $f_1(\omega)$ and $f_2(\omega)$ are decreasing and increasing, respectively. From this fact, we can conclude that if ω^* be the minimum point of $h(\omega)$, then it belongs to interval $\left(\frac{1}{\mu_{\max}}, \frac{1}{\mu_{\min}}\right)$ and must satisfy

$$\frac{1 - \omega\mu_{\min}}{\omega + \mu_{\min}} = \frac{\omega\mu_{\max} - 1}{\omega + \mu_{\max}}.$$

By simplifying the aforementioned equation, we get

$$\omega^2(\mu_{\min} + \mu_{\max}) + 2\omega(\mu_{\min}\mu_{\max} - 1) - (\mu_{\min} + \mu_{\max}) = 0. \tag{26}$$

The roots of Eqn (26) are

$$\omega_{\pm} = \frac{1 - \mu_{\min}\mu_{\max} \pm \sqrt{\left(1 + \mu_{\min}^2\right)\left(1 + \mu_{\max}^2\right)}}{\mu_{\min} + \mu_{\max}}.$$

It is easy to observe that $\omega_- < 0$, then

$$\omega^* = \arg\min_{\omega} h(\omega) = \frac{1 - \mu_{\min}\mu_{\max} + \sqrt{\left(1 + \mu_{\min}^2\right)\left(1 + \mu_{\max}^2\right)}}{\mu_{\min} + \mu_{\max}}. \tag{27}$$

Now, suppose $\mu_{\min} = 0$. Then, from (22) and (23), we can deduce

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$$h(\omega) = \begin{cases} \dfrac{1}{\omega}, & \text{for } \omega \leqslant \frac{1}{\mu_{\max}}, \\ \max\left\{\dfrac{1}{\omega}, \dfrac{\omega\mu_{\max} - 1}{\omega + \mu_{\max}}\right\}, & \text{for } \frac{1}{\mu_{\max}} \leqslant \omega. \end{cases} \tag{28}$$

In this case, it is easy to see that ω^* is obtained by setting $\mu_{\min} = 0$ in (27). □

Theorem 2.4
Let $W, T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semi-definite, respectively. Then, the optimal values of the parameters α and ω for the PGSOR iterative method (11) is given by

$$\omega^* = \frac{1 - \mu_{\min}\mu_{\max} + \sqrt{\left(1 + \mu_{\min}^2\right)\left(1 + \mu_{\max}^2\right)}}{\mu_{\min} + \mu_{\max}} \quad \text{and} \quad \alpha^* = \frac{2}{1 + \sqrt{1 + \xi^2}}, \tag{29}$$

and the corresponding optimal convergence factor of the method is given by

$$\rho\left(\tilde{\mathcal{G}}_{\alpha^*}(\omega^*)\right) = 1 - \alpha^* = 1 - \frac{2}{1 + \sqrt{1 + \xi^2}}, \tag{30}$$

where

$$\xi \equiv \rho\left(\tilde{S}_{\omega^*}\right) = \frac{1 - \omega^*\mu_{\min}}{\omega^* + \mu_{\min}} \left(= \frac{\omega^*\mu_{\max} - 1}{\omega^* + \mu_{\max}}\right),$$

wherein $\mu_{\min}$ and $\mu_{\max}$ are the smallest and largest eigenvalues of $S = W^{-1}T$, respectively, and $\tilde{S}_{\omega^*} = (\omega^* W + T)^{-1}(\omega^* T - W)$.

Proof
According to (15) and (16), for every positive parameter ω, we have

$$\alpha^* = \frac{2}{1 + \sqrt{1 + \rho(\tilde{S}_\omega)^2}} \quad \text{and} \quad \rho\left(\tilde{\mathcal{G}}_{\alpha^*}(\omega)\right) = 1 - \alpha^* = 1 - \frac{2}{1 + \sqrt{1 + \rho(\tilde{S}_\omega)^2}},$$

where $\tilde{S}_\omega = (\omega W + T)^{-1}(\omega T - W)$. Noticing that $\rho\left(\tilde{\mathcal{G}}_{\alpha^*}(\omega)\right)$ is an increasing function with respect to $\rho(\tilde{S}_\omega)$, it is minimized when $\rho(\tilde{S}_\omega)$ is minimum. From Lemma 2.4, we know that $\rho(\tilde{S}_\omega)$ is minimized by

$$\omega^* = \frac{1 - \mu_{\min}\mu_{\max} + \sqrt{\left(1 + \mu_{\min}^2\right)\left(1 + \mu_{\max}^2\right)}}{\mu_{\min} + \mu_{\max}}.$$

In the proof of Lemma 2.4, for the previous ω^*, we have seen that

$$\frac{1 - \omega^*\mu_{\min}}{\omega^* + \mu_{\min}} = \frac{\omega^*\mu_{\max} - 1}{\omega^* + \mu_{\max}}.$$

Then, by using Lemma 2.2, and replacing $\rho\left(\tilde{S}_{\omega^*}\right)$ by ξ, the proof is completed. □

Now, we can obtain an upper bound for the spectral radius of $\tilde{\mathcal{G}}_{\alpha^*}(\omega^*)$ that is formally stated in the following Corollary.

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Corollary 2.2
Let the conditions of Theorem 2.4 hold. Then

$$\rho\big(\tilde{\mathcal{G}}_{\alpha^*}(\omega^*)\big) < \frac{\sqrt{2}-1}{\sqrt{2}+1} \approx 0.172, \tag{31}$$

and

$$\alpha^* \in \left(\frac{2}{1+\sqrt{2}}, 1\right) \approx (0.828, 1).$$

Proof
According to Eqns (29) and (30), the proof will be completed if we show the following inequality:

$$\xi = \rho\left(\tilde{S}_{\omega^*}\right) = \frac{1-\omega^*\mu_{\min}}{\omega^*+\mu_{\min}} < 1. \tag{32}$$

Substituting ω^* defined in Eqn (29) in the aforementioned inequality and simplifying, yields

$$\mu_{\max} + \mu_{\min}^2\mu_{\max} - \mu_{\min}\sqrt{\left(1+\mu_{\min}^2\right)\left(1+\mu_{\max}^2\right)} < 1 + \sqrt{\left(1+\mu_{\min}^2\right)\left(1+\mu_{\max}^2\right)} + \mu_{\min}^2,$$

and therefore

$$\left(1+\mu_{\min}^2\right)(\mu_{\max}-1) < (1+\mu_{\min})\sqrt{\left(1+\mu_{\min}^2\right)\left(1+\mu_{\max}^2\right)}. \tag{33}$$

Now, if $\mu_{\max} \leqslant 1$, then the inequality (33) holds, and so (32) is true. On the other hand, if $\mu_{\max} > 1$, from Eqn (33) we get

$$\left(1+\mu_{\min}^2\right)(\mu_{\max}-1)^2 < (1+\mu_{\min})^2\left(1+\mu_{\max}^2\right),$$

which is always true, and hence (32) holds. □

Note that the upper bound (31) for the spectral radius of $\tilde{\mathcal{G}}_{\alpha^*}(\omega^*)$ is a constant independent of both data and size of the problem.

From the proof of Corollary 2.2, we can see that all of the eigenvalues of $\tilde{S}_{\omega^*}$ are clustered in the interval $(-1, 1)$. Hence, when the spectral radius of S is large, from (8) and (31), we can find that the spectral radius of $\mathcal{G}_{\alpha^*}$ is close to 1, whereas, the spectral radius $\tilde{\mathcal{G}}_{\alpha^*}(\omega^*)$ is smaller than 0.172. This demonstrates the superiority of the PGSOR method over the GSOR method. But, when the spectral radius of S is smaller than 1, it is expected that the performance of PGSOR method is similar to that of the GSOR method.

Because it may turn out to be difficult to find the optimal values of the parameters α and ω, we propose to use the values of ω and α such that the performance of the corresponding PGSOR method is close to that of the optimal parameters. To do so, we consider the following two cases:

Case I: If $\mu_{\min} = 0$ (the smallest eigenvalue of $S = W^{-1}T$ is zero, e.g., when T is $spsd$) then, with assumption $\omega = 1$, from (28), we have $h(\omega) = \rho(\tilde{S}_\omega) = 1$ and this means that all of eigenvalues of $\tilde{S}_\omega$ are clustered in the interval $[-1, 1]$, and on the other hand, from (15) and (16), we have $\alpha^* = \frac{2}{1+\sqrt{2}} \approx 0.828$ and $\rho\left(\tilde{\mathcal{G}}_{\alpha^*}(\omega)\right) = \frac{\sqrt{2}-1}{\sqrt{2}+1} \approx 0.172$, respectively.

Case II: If $\mu_{\min} \neq 0$ then, with assumption $\omega = 1$, from (25), it is easy to see that $h(\omega) = \rho(\tilde{S}_\omega) < 1$, hence, all of eigenvalues of $\tilde{S}_\omega$ are clustered in the interval $(-1, 1)$. So, from (15), we have $\alpha^* \in \left(\frac{2}{1+\sqrt{2}}, 1\right) \approx (0.828, 1)$. When $\mu_{\min}$ is very small and close to zero or $\mu_{\max}$ (the largest eigenvalue of $S = W^{-1}T$) is rather large, from (25), it is easy to find that $h(\omega) = \rho(\tilde{S}_\omega) \approx 1$, therefore, from (15), it can be expected that $\alpha^* \approx \frac{2}{1+\sqrt{2}} \approx 0.828$.

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Therefore, for a broad class of problems, we can consider $\omega = 1$ and $\alpha = 0.828$.

It is noteworthy that, if there exist real numbers β and δ such that both matrices $\hat{W} := \beta W + \delta T$ and $\hat{T} := \beta T - \delta W$ are symmetric positive semidefinite with at least one of them positive definite, we can first multiply both sides of (1) by the complex number $\beta - i\delta$ to get the equivalent system

$$(\hat{W} + i\hat{T})x = \hat{b} \quad \text{with} \quad \hat{b} := (\beta - i\delta)b,$$

and then employ the PGSOR iteration method to the equivalent real system that is obtained from the aforementioned system.

3. NUMERICAL EXPERIMENTS

In this section, we use three test problems from [8] and an example of [4] to illustrate the effectiveness of the PGSOR iteration method for solving the equivalent real system (4). We also compare the performance of the PGSOR method with the HSS, MHSS and GSOR methods, in terms of both iteration count (denoted by IT) and CPU time (in seconds, denoted by CPU). The HSS and MHSS iterations are employed to solve the complex system (1) and the two other methods to solve the equivalent real system (4). The two half-steps comprising each iteration of the HSS method are computed by the Cholesky factorization and LU decomposition of the coefficient matrices. In each iteration of the MHSS, GSOR and PGSOR iteration methods, we use the Cholesky factorization of the coefficient matrices to solve the sub-systems. The CPU times reported are the sum of the CPU times for the convergence of the method and the CPU times for computing the Cholesky factorization and LU decomposition. It is necessary to mention that to solve symmetric positive definite system of linear equations, we have used the sparse Cholesky factorization incorporated with the symmetric approximate minimum degree reordering. To do so, we have used the `symamd` command of MATLAB Version 7.

All the numerical experiments were computed in double precision using some MATLAB codes on a 64-bit 1.73 GHz intel Q740 core i7 processor and 4 GB RAM running Windows 7. We use a null vector as an initial guess, and the stopping criterion

$$\frac{\|b - Au^{(k)}\|_2}{\|b\|_2} < 10^{-6},$$

is always used where $u^{(k)} = x^{(k)} + iy^{(k)}$.

Example 3.1 (see [8]*)*
Consider the linear system of equations

$$\left[\left(K + \frac{3-\sqrt{3}}{\tau}I\right) + i\left(K + \frac{3+\sqrt{3}}{\tau}I\right)\right]x = b,$$

where τ is the time step-size and K is the five-point centered difference matrix approximating the negative Laplacian operator $L = -\Delta$ with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0, 1] \times [0, 1]$ with the mesh-size $h = 1/(m + 1)$. The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$, with $V_m = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. Hence, K is an $n \times n$ block-tridiagonal matrix, with $n = m^2$. We take

$$W = K + \frac{3-\sqrt{3}}{\tau}I \quad \text{and} \quad T = K + \frac{3+\sqrt{3}}{\tau}I,$$

and the right-hand side vector b with its jth entry b_j being given by

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$$b_j = \frac{(1-i)j}{\tau(j+1)^2}, \quad j = 1, 2, \ldots, n.$$

In our tests, we take $\tau = h$. Furthermore, we normalize coefficient matrix and right-hand side by multiplying both by h^2.

Example 3.2 (see [8]*)*
Consider the linear system of equations

$$\left[(-\omega^2 M + K) + i(\omega C_V + C_H)\right] x = b,$$

Table I. The optimal parameters α^* and ω^* for the HSS, MHSS, GSOR and PGSOR methods.

			Grid				
Example	Method		16×16	32×32	64×64	128×128	256×256
No. 1	HSS	α^*	0.81	0.55	0.37	0.28	0.20
	MHSS	α^*	1.06	0.75	0.54	0.40	0.30
	GSOR	α^*	0.550	0.495	0.457	0.432	0.428
	PGSOR	α^*	0.990	0.987	0.986	0.984	0.983
		ω^*	0.657	0.624	0.602	0.590	0.583
No. 2	HSS	α^*	0.42	0.23	0.12	0.07	0.04
	MHSS	α^*	0.21	0.08	0.04	0.02	0.01
	GSOR	α^*	0.455	0.455	0.455	0.455	0.455
	PGSOR	α^*	0.898	0.896	0.895	0.895	0.895
		ω^*	1.309	1.323	1.328	1.330	1.330
No. 3	HSS	α^*	4.41	2.71	1.61	0.93	0.53
	MHSS	α^*	1.61	1.01	0.53	0.26	0.13
	GSOR	α^*	0.908	0.776	0.566	0.353	0.199
	PGSOR	α^*	0.982	0.956	0.918	0.885	0.864
		ω^*	3.001	1.980	1.437	1.181	1.063
No. 4	HSS	α^*	0.45	0.25	0.13	0.07	0.04
	MHSS	α^*	0.04	0.007	0.003	0.0005	0.0002
	GSOR	α^*	0.820	0.821	0.822	0.822	0.822
	PGSOR	α^*	0.959	0.959	0.959	0.959	0.959
		ω^*	2.345	2.364	2.368	2.369	2.369

Note: HSS, Hermitian/skew-Hermitian splitting; MHSS, modified Hermitian/skew-Hermitian splitting; GSOR, generalized successive overrelaxation; PGSOR, preconditioned GSOR.

Table II. Numerical results for Example 3.1.

Method	$m \times m$	16×16	32×32	64×64	128×128	256×256
HSS	IT	44	65	97	136	191
	CPU	0.07	0.31	2.28	18.97	187.79
MHSS	IT	40	54	73	98	133
	CPU	0.08	0.30	1.53	8.50	52.57
GSOR	IT	19	22	24	26	27
	CPU	0.04	0.05	0.15	0.64	2.88
PGSOR	IT	4	4	5	5	5
	CPU	0.03	0.04	0.07	0.23	1.01

Note: HSS, Hermitian/skew-Hermitian splitting; MHSS, modified Hermitian/skew-Hermitian splitting; GSOR, generalized successive overrelaxation; PGSOR, preconditioned GSOR.

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where M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, respectively, and ω is the driving circular frequency. We take $C_H = \mu K$ with μ a damping coefficient, $M = I$, $C_V = 10I$, and K the five-point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0, 1] \times [0, 1]$ with the mesh-size $h = 1/(m + 1)$.

Table III. Numerical results for Example 3.2.

Method	$m \times m$	16×16	32×32	64×64	128×128	256×256
HSS	IT	86	153	284	540	1084
	CPU	0.11	1.15	5.77	61.86	692.11
MHSS	IT	34	38	50	81	139
	CPU	0.08	0.22	1.08	7.24	54.88
GSOR	IT	26	24	24	23	23
	CPU	0.04	0.06	0.16	0.60	2.54
PGSOR	IT	8	7	8	8	8
	CPU	0.03	0.04	0.08	0.29	1.26

Table IV. Numerical results for Example 3.3.

Method	$m \times m$	16×16	32×32	64×64	128×128	256×256
HSS	IT	84	137	223	390	746
	CPU	0.11	0.60	4.77	47.10	556.16
MHSS	IT	53	76	130	246	468
	CPU	0.11	0.43	2.84	22.34	194.36
GSOR	IT	7	11	20	35	71
	CPU	0.03	0.05	0.17	1.05	8.69
PGSOR	IT	5	6	7	8	8
	CPU	0.03	0.04	0.09	0.38	1.67

Note: HSS, Hermitian/skew-Hermitian splitting; MHSS, modified Hermitian/skew-Hermitian splitting; GSOR, generalized successive overrelaxation; PGSOR, preconditioned GSOR.

Table V. Numerical results for Example 3.4.

Method	$m \times m$	16×16	32×32	64×64	128×128	256×256
HSS	IT	81	147	271	502	981
	CPU	0.03	0.11	0.85	6.30	53.40
MHSS	IT	39	41	41	41	41
	CPU	0.03	0.07	0.24	1.11	4.51
GSOR	IT	9	9	9	9	9
	CPU	0.02	0.02	0.04	0.16	0.71
PGSOR	IT	6	6	6	6	6
	CPU	0.01	0.02	0.03	0.15	0.56

Note: HSS, Hermitian/skew-Hermitian splitting; MHSS, modified Hermitian/skew-Hermitian splitting; GSOR, generalized successive overrelaxation; PGSOR, preconditioned GSOR.

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The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$, with $V_m = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. Hence, K is an $n \times n$ block-tridiagonal matrix, with $n = m^2$. In addition, we set $\omega = \pi$, $\mu = 0.02$, and the right-hand side vector b to be $b = (1 + i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. As before, we normalize the system by multiplying both sides through by h^2.

Example 3.3 (see [8]*)*
Consider the linear system of equations $(W + iT)x = b$, with

$$T = I \otimes V + V \otimes I \quad \text{and} \quad W = 10(I \otimes V_c + V_c \otimes I) + 9\left(e_1 e_m^T + e_m e_1^T\right) \otimes I,$$

where $V = \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$, $V_c = V - e_1 e_m^T - e_m e_1^T \in \mathbb{R}^{m\times m}$ and e_1 and e_m are the first and last unit vectors in $\mathbb{R}^m$, respectively. We take the right-hand side vector b to be $b = (1 + i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1.

Here T and W correspond to the five-point centered difference matrices approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions and periodic boundary conditions, respectively, on a uniform mesh in the unit square $[0, 1] \times [0, 1]$ with the mesh-size $h = 1/(m + 1)$.

Example 3.4 (see [4, 17]*)*
We consider the complex Helmholtz equation

$$-\triangle u + \sigma_1 u + i\sigma_2 u = f,$$

where σ_1 and σ_2 are real coefficient functions, u satisfies Dirichlet boundary conditions in $D = [0, 1] \times [0, 1]$ and $i = \sqrt{-1}$. We discretize the problem with finite differences on a $m \times m$ grid with mesh size $h = 1/(m + 1)$. This leads to a system of linear equations

$$((K + \sigma_1 I) + i\sigma_2 I)\, x = b,$$

where $K = I \otimes V_m + V_m \otimes I$ is the discretization of $-\triangle$ by means of centered differences, wherein $V_m = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. The right-hand side vector b is taken to be $b = (1 + i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. Furthermore, before solving the system, we normalize the coefficient matrix and the right-hand side vector by multiplying both by h^2. For the numerical tests, we set $\sigma_1 = -10$ and $\sigma_2 = 10$.

For all the examples, the optimal values of the parameters α and ω (denoted by α^* and ω^*) are listed in Table I for different values of m, that the former is used in the HSS, MHSS, GSOR and

Table VI. Spectral radius of the iterative matrices of GSOR and PGSOR.

		Grid				
Example		16×16	32×32	64×64	128×128	256×256
No. 1	$\rho(\mathcal{G}_{\alpha^*})$	0.450	0.505	0.543	0.568	0.572
	$\rho\left(\tilde{\mathcal{G}}_{\alpha^*}(\omega^*)\right)$	0.010	0.013	0.014	0.016	0.017
No. 2	$\rho(\mathcal{G}_{\alpha^*})$	0.545	0.545	0.545	0.545	0.545
	$\rho\left(\tilde{\mathcal{G}}_{\alpha^*}(\omega^*)\right)$	0.102	0.104	0.105	0.105	0.105
No. 3	$\rho(\mathcal{G}_{\alpha^*})$	0.092	0.224	0.434	0.647	0.801
	$\rho\left(\tilde{\mathcal{G}}_{\alpha^*}(\omega^*)\right)$	0.018	0.044	0.082	0.115	0.136
No. 4	$\rho(\mathcal{G}_{\alpha^*})$	0.180	0.179	0.178	0.178	0.178
	$\rho\left(\tilde{\mathcal{G}}_{\alpha^*}(\omega^*)\right)$	0.027	0.041	0.041	0.041	0.041

Note: GSOR, generalized successive overrelaxation; PGSOR, preconditioned GSOR.

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PGSOR iterative methods and the latter is only used in the PGSOR iterative method. The experimentally found optimal parameters α^* for the HSS and MHSS are the ones resulting in the least numbers of iterations for the two methods for each of the numerical examples and those are presented in [8] (expect for Example 3.4). The value of α^* for the GSOR method is obtained from (9) and the values of α^* and ω^* for the PGSOR method are obtained from (24) in which the largest eigenvalue of matrix S ($\mu_{\max}(S)$) has been estimated by a few iterations of the power method, and also if T is symmetric positive definite, then the smallest eigenvalue of matrix S ($\mu_{\min}(S)$) can be estimated by a few iterations of the inverse power method and if T is symmetric positive semi-definite, then $\mu_{\min}(S) = 0$.

From Table I, as we expected for all the examples, the value of α^* in the PGSOR method belongs to the interval $(0.828, 1)$, independently of the data and problem size (in fact, this confirms Corollary 2.2) and decreases as the mesh-size h decreases. But note that the rate of decrease for α^* in the PGSOR method decreases as the mesh-size h decreases, and for large values of m, the value of α^* for Examples 3.2 and 3.4 becomes approximately constant and is approximately equal to 0.895 and 0.959, respectively. Moreover, for Examples 3.1 and 3.3, the optimal parameter ω^* decreases as the mesh-size h decreases, whereas for Examples 3.2 and 3.4 it increases. Note that, for all the examples, the rate of change of ω^* decreases as the mesh-size h decreases.

In Tables II–V, we have reported numerical results for Examples 3.1–3.4. These tables present IT and CPU for the HSS, MHSS, GSOR and PGSOR methods. As seen, the PGSOR method outperforms GSOR and behaves much better than MHSS and HSS, especially when problem size increases. The cause of such performance is easily predictable when we observe the spectral radius of the iteration matrices of the GSOR and PGSOR methods for the four examples (Table VI) that is obtained from (8) and (30), respectively. We see that the spectral radius of the iteration matrix of the PGSOR method is smaller than that of the GSOR method when the optimal parameters are employed. For the numerical results of Example 3.3 reported in Table VI, we see that the spectral radius of the iteration matrix of the GSOR method grows rapidly with problem size, while that of

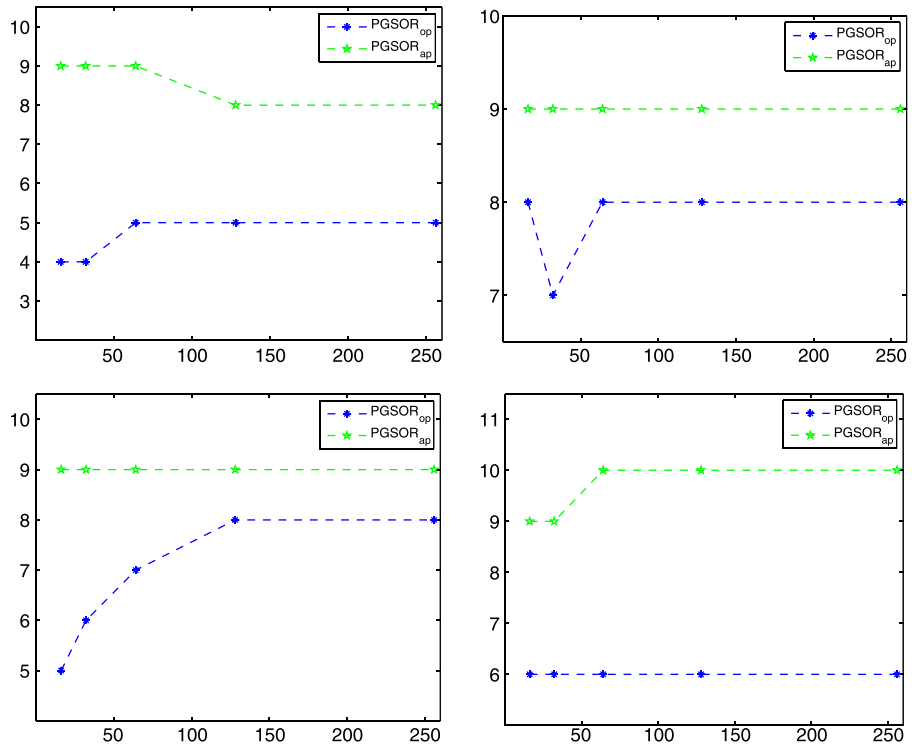

Figure 1. Demonstration of IT versus m for the PGSOR method with optimal value of the parameters (α, ω) in Table I and approximation parameters $\alpha = 0.828$ and $\omega = 1$; *top-left*: Example 3.1, *top-right*: Example 3.2, *down-left*: Example 3.3, *down-right*: Example 3.4.

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the PGSOR method grows very slowly and is very smaller than that of the GSOR method when the problem size is large.

In Figure 1, we have compared the number of iterations for the convergence of the PGSOR method in conjunction with optimal values of (α, ω) (denoted by PGSOR_{op}) and with $(\alpha, \omega) = (0.828, 1)$ (denoted by PGSOR_{ap}) for all the examples. As seen, $(\alpha, \omega) = (0.828, 1)$ can be considered as a reasonable approximation of the optimal value of (α, ω).

4. CONCLUDING REMARKS

In this paper, we have presented a preconditioned variant of the generalized successive overrelaxation (GSOR) iterative method to solve the equivalent real formulation of complex linear system (1), where W is symmetric positive definite and T is symmetric positive semi-definite. Convergence properties of the method have been also investigated. Some numerical results have been presented to show the effectiveness of the method. Our numerical examples show that our method is quite suitable for such problems. Moreover, the presented numerical experiments show that the PGSOR method is superior to GSOR, MHSS and HSS in terms of the iterations and CPU times.

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