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ON THE CONVERGENCE OF THE GAVER–STEHFEST ALGORITHM*

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Abstract. The Gaver–Stehfest algorithm for numerical inversion of Laplace transform was developed in the late 1960s. Due to its simplicity and good performance it is becoming increasingly more popular in such diverse areas as geophysics, operations research and economics, financial and actuarial mathematics, computational physics, and chemistry. Despite the large number of applications and numerical studies, this method has never been rigorously investigated. In particular, it is not known whether the Gaver–Stehfest approximations converge or what the rate of convergence is. In this paper we answer the first of these two questions: We prove that the Gaver–Stehfest approximations converge for functions of bounded variation and functions satisfying an analogue of Dini criterion.

Key words. Gaver–Stehfest algorithm, inverse Laplace transform, Lambert W-function, generating functions, Dini criterion, Jordan criterion

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1. Introduction and main results.

In this paper we are concerned with the classical problem of numerical inversion of Laplace transform. More specifically, assume that $f : (0,\infty) \mapsto \mathbb{R}$ is a locally integrable function, such that its Laplace transform

$$F(z) := \int_0^\infty e^{-zx} f(x)\mathrm{d}x$$

is finite for all $z > 0$. The problem consists in recovering the original function $f(x)$ given that we know $F(z)$. This problem has numerous applications, and it has attracted a lot of attention from researchers over the last 50 years. (See [4] for an up-to-date exposition of this area.)

More specifically, we are interested in the Gaver–Stehfest algorithm, which aims to approximate $f(x)$ by a sequence of functions

$$f_n(x) := \ln(2)x^{-1} \sum_{k=1}^{2n} a_k(n) F\left(k\ln(2)x^{-1}\right), \quad n \geq 1,\ x > 0, \tag{1.1}$$

where the coefficients are defined as follows:

$$a_k(n) := \frac{(-1)^{n+k}}{n!} \sum_{j=[(k+1)/2]}^{\min(k,n)} j^{n+1} \binom{n}{j} \binom{2j}{j} \binom{j}{k-j}, \quad n \geq 1,\ 1 \leq k \leq 2n.$$

In order to demonstrate the intuition behind these rather complicated formulas, let us explain where they come from. In 1966 Gaver [9] introduced the following sequence of approximations:

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$$\tilde{f}_k(x) := \ln(2)x^{-1}\frac{(2k)!}{k!(k-1)!}\sum_{i=0}^{k}\binom{k}{i}(-1)^i F((k+i)\ln(2)x^{-1}), \tag{1.2}$$

where $k \geq 1$ and $x > 0$. Applying the binomial theorem, the above expression can be rewritten in an equivalent integral form

$$\tilde{f}_k(x) = \int_0^\infty p_k(u) f\left(\frac{xu}{\ln(2)}\right) \mathrm{d}u, \tag{1.3}$$

where

$$p_k(u) := \frac{(2k)!}{k!(k-1)!}(1-e^{-u})^k e^{-ku}, \quad k \geq 1,\ u \geq 0.$$

The function $p_k(u)$ is positive and its integral over $[0,\infty)$ is equal to one, therefore we can think of it as the density function of a positive random variable U_k. One can check by direct calculations that $\mathbb{E}[U_k] = \ln(2) + O(k^{-1})$ and $\mathrm{Var}(U_k) = O(k^{-1})$ as $k \to +\infty$. This shows that the random variables $\{U_k\}_{k\geq 1}$ converge in distribution to $\ln(2)$; therefore for any continuous function $f(x)$ and any $x > 0$ we have

$$\tilde{f}_k(x) = \mathbb{E}\left[f\left(\frac{xU_k}{\ln(2)}\right)\right] \to f(x), \quad k \to \infty.$$

Gaver [9] also proved that as $k \to +\infty$

$$\tilde{f}_k(x) = f(x) + \frac{\beta_1(x)}{k} + \frac{\beta_2(x)}{k^2} + \cdots + \frac{\beta_{m-1}(x)}{k^{m-1}} + O\left(k^{-m+\frac{1}{2}}\right), \tag{1.4}$$

under the assumptions $f \in C^{2m}(\mathbb{R}^+)$ and $f^{(j)}(x) \in L_\infty(\mathbb{R}^+)$ for all $j = 0, 1, \ldots, 2m$. Formula (1.4) shows that $\tilde{f}_k(x)$ converge to $f(x)$ rather slowly, but it also suggests using a convergence acceleration technique. This is exactly what was done by Stehfest [19, 20] in 1970. He defined a new approximation,

$$f_n(x) = \sum_{k=1}^{n} c_k(n)\tilde{f}_k(x), \tag{1.5}$$

where the coefficients $c_k(n)$ are chosen so that the asymptotic terms $\beta_j(x)k^{-j}$, $j = 1, 2, \ldots, n-1$, in (1.4) are eliminated:

$$\sum_{k=1}^{n} c_k(n)k^{-j} = \begin{cases} 1, & j = 0, \\ 0, & j = 1, 2, \ldots, n-1. \end{cases} \tag{1.6}$$

It turns out that the above conditions are satisfied by the sequence

$$c_k(n) := (-1)^{n+k}\frac{k^n}{k!(n-k)!}, \quad n \geq 1, 1 \leq k \leq n.$$

Combining the above formula with (1.2) and (1.5) gives us the Gaver–Stehfest approximations in the form (1.1).

The Gaver–Stehfest algorithm has a number of desirable properties: (i) the approximations $f_n(x)$ are linear in values of $F(z)$; (ii) it requires the values of $F(z)$ for real z and does not need any complex numbers; (iii) the coefficients $a_k(n)$ can be easily computed; (iv) the Gaver–Stehfest approximations are exact for constant functions, that is, if $f(x) \equiv c$, then $f_n(x) \equiv c$ for all $n \geq 1$. This algorithm was studied in [2, 7, 11, 16, 21], where it was demonstrated numerically that $f_n(x)$ converge very quickly to $f(x)$ for many examples of initial functions $f(x)$ (provided that $f(x)$ is nonoscillating). Another universally accepted fact is that this algorithm requires high-precision arithmetic for its implementation (which is rather obvious, since the coefficients $a_k(n)$ are growing very rapidly and have alternating signs).

Over the last 40 years the Gaver–Stehfest algorithm has been applied to solving various problems in geophysics [14], probability [1, 15], actuarial mathematics [3] and mathematical finance [18], chemistry [17], and economics [12]. This is just a small sample, and a quick search on the Internet produces hundreds of papers with references to the Gaver–Stehfest algorithm.

Despite all this popularity, there has not yet been a single rigorous investigation of this algorithm. In particular, it is not known what sufficient conditions on f will ensure convergence of $f_n(x)$ or what the rate of convergence would be. Stehfest [19] writes that "theoretically $f_n(x)$ become the more accurate the greater n," and Proposition 8.2 in Abate and Whitt [1] states that for any $k > 0$ we have $f_n(x) - f(x) = o(n^{-k})$ as $n \to +\infty$; however, both these statements are not very precise and lack rigorous proof. The authors seem to assume that the derivation of the Gaver–Stehfest approximations via formulas (1.2), (1.4), (1.5), and (1.6) automatically gives the proof of convergence. This is not the case. First, validity of the asymptotic expansion (1.4) for all $m \geq 1$ would require a very restrictive assumption on the function f. (It has to be an infinitely differentiable function plus some additional assumptions.) The second issue is that even if the asymptotic expansion (1.4) is valid for all $m \geq 1$, it is still not clear that $f_n(x) \to f(x)$, since the coefficients $c_k(n)$ grow very rapidly and have alternating signs. Therefore, it is not at all obvious that $f_n(x)$ will converge to $f(x)$, given (1.5), (1.6), and the fact that $\tilde{f}_n(x) \to f(x)$.

In this paper we present the first rigorous investigation of the Gaver–Stehfest algorithm. We derive two sufficient conditions on the function f, which ensure convergence of $f_n(x)$. Our main results are presented in the next theorem.

THEOREM 1.1. *Assume that $f : (0,\infty) \mapsto \mathbb{R}$ is a locally integrable function such that the Laplace transform $F(z) = \int_0^{\infty} e^{-zx} f(x) \mathrm{d}x$ exists for all $z > 0$ and that $f_n(x)$ are defined by* (1.1).

(i) *The convergence of $f_n(x)$ depends only on the values of the function f in the neighborhood of x.*

(ii) *Assume that for some $c \in \mathbb{R}$ and some $\epsilon \in (0, 1/4)$*

$$\int_0^{\epsilon} \left| f(-x \log_2(\tfrac{1}{2} + v)) + f(-x \log_2(\tfrac{1}{2} - v)) - 2c \right| v^{-1} \mathrm{d}v < \infty. \tag{1.7}$$

Then $f_n(x) \to c$ as $n \to +\infty$.

(iii) *Assume that the function f has bounded variation in the neighborhood of x. Then $f_n(x) \to (f(x+0) + f(x-0))/2$ as $n \to +\infty$.*

Our sufficient conditions are very similar to the corresponding results from the theory of convergence of Fourier series. Item (i) has its counterpart in [13, Theorem 4.1.1], item (ii) should be compared with the Dini criterion [13, Theorem 4.1.3(i)]

$$\int_0^{\epsilon} \left| f(x+v) + f(x-v) - 2c \right| v^{-1} \mathrm{d}v < \infty, \tag{1.8}$$

and item (iii) is exactly the same as the Jordan criterion [13, Theorem 4.1.3(ii)]. Theorem 1.1(ii) also provides the following useful corollary.

COROLLARY 1.2. *Under the assumptions of Theorem* 1.1*, if*

$$f(x+v) - f(x) = O(|v|^{\alpha})$$

for some $\alpha > 0$ *and all* v *in some neighborhood of* x*, then* $f_n(x) \to f(x)$ *as* $n \to +\infty$*.*

We do not address the second important problem related to the rate of convergence of $f_n(x)$. We hope that the methods developed in this paper may be useful for solving this problem, and we leave it for future work.

2. Proof of Theorem 1.1. Let us review some properties of the Lambert W-function (see [6]), which will be used in the proof of Theorem 1.1. We will be interested in the principal branch of the Lambert W-function, which will be denoted by $W(z)$: It is an analytic function in the neighborhood of $z = 0$ and it satisfies

$$W(z)\exp(W(z)) = z.$$

Let us investigate the range and domain of $W(z)$ in more detail. The function $w = x + \mathrm{i}y \mapsto we^w$ takes real values only if $y = 0$ or $x = -y\cot(y)$ (see section 4 in [6]). In particular, the open set

$$\mathcal{A} := \{w = x + \mathrm{i}y, \quad x > -y\cot(y), \quad -\pi < y < \pi\}$$

is mapped by $z = we^w$ onto the cut complex plane $\mathbb{C}\backslash(-\infty, -1/e]$; see Figure 2.1. The function $w \mapsto we^w$ is one-to-one on the set $\mathcal{A}$, and $W(z)$ is defined as the inverse of this function. Therefore, $W(z)$ is analytic in $\mathbb{C}\backslash(-\infty, -1/e]$ and maps this set onto $\mathcal{A}$. It is known that $W(z)$ has the explicit Taylor series at $z = 0$ (see formula (3.1) in [6])

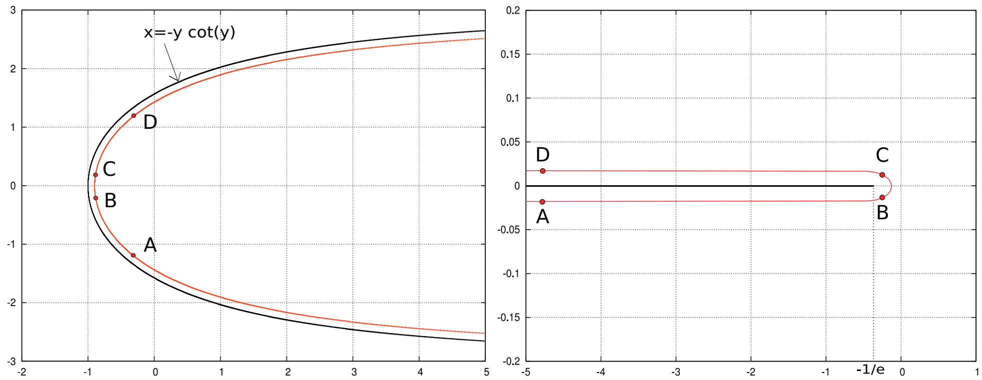
$$W(z) = \sum_{n\ge1} (-n)^{n-1}\frac{z^n}{n!}, \quad |z| < 1/e, \tag{2.1}$$

and has a branching singularity at $z = -1/e$,

$$W(z) = \sum_{n\ge0} \mu_n p^n = -1 + p - \frac{p^2}{3} + \frac{11}{72}p^3 - \frac{43}{540}p^4 + \frac{769}{17280}p^5 + \cdots, \tag{2.2}$$

where $p = p(z) := \sqrt{2(1+ez)}$ and the series converges for $|p| < \sqrt{2}$ (see formula (4.22) in [6]).

We will extend function $W(z)$ to the interval $\{z \in \mathbb{R},\ z < -1/e\}$ so that $W(z)$ is continuous in the upper half-plane $\mathrm{Im}(z) \ge 0$. Thus $W(z)$ maps the interval $\{z \in \mathbb{R},\ z < -1/e\}$ onto the curve $x = -y\cot(y)$, $y \in (0, \pi)$ (which is the upper half of the black curve in Figure 2.1(a)). The function $W(z)$ is smooth on $\{z \in \mathbb{R},\ z < -1/e\}$, and it is clear that $|W(z)|$ and $\mathrm{Im}(W(z))$ are decreasing functions of z. Assuming that z is real and $z < -1/e$, the equation $we^w = z$ has infinitely many solutions, which correspond to different branches of the Lambert W-function. However, only two of these solutions (given by $w = W(z)$ and $w = \overline{W(z)}$) satisfy $|\mathrm{Im}(w)| < \pi$. These solutions necessarily lie on the black curve on Figure 2.1(a), and they will play a very important role in our investigation.



(a) $w = x + \mathrm{i}y$ (b) $z = we^w$

FIG. 2.1. *Domain* (b) *and range* (a) *of the principal branch of the Lambert W-function.*

Our main tool used in the proof Theorem 1.1 is the following sequence of polynomials:

$$q_n(v) := \sum_{k=1}^{n} \frac{k^{n+1}(\frac{1}{2})_k}{(n-k)!(k!)^2}(-1)^{n+k}v^k, \quad n \ge 1. \tag{2.3}$$

Recall that $(a)_k := a(a+1)\dots(a+k-1)$ denotes the Pochhammer symbol. Combining (1.3) and (1.5) we obtain an integral representation

$$f_n(x) = \int_0^\infty q_n\left(4e^{-u}(1-e^{-u})\right) f\left(\frac{xu}{\ln(2)}\right) \mathrm{d}u, \tag{2.4}$$

which explains why these polynomials are important for Gaver–Stehfest approximations. Our plan for proving Theorem 1.1 is to establish uniform asymptotics for $q_n(v)$, $0 \le v \le 1$, as $n \to \infty$ and then use (2.4) to study convergence of $f_n(x)$.

Let us also define

$$G(z) := \sum_{n \ge 1} \frac{(\frac{1}{2})_n}{(n!)^2}(-1)^n n^{n+1} z^n. \tag{2.5}$$

Using Stirling's asymptotic formula for the Gamma function one can check that the above series converges for $|z| < 1/e$.

PROPOSITION 2.1. *For* $0 \le v \le 1$ *and* $|t| < 1/(2e)$

$$G\left(vte^t\right) = \sum_{n \ge 1} q_n(v)(-1)^n t^n. \tag{2.6}$$

Proof. Using (2.3), the trivial estimates $(\frac{1}{2})_k < (1)_k = k!$ and $k^{n+1} < n^{n+1}$, and the binomial theorem, we obtain an upper bound

$$|q_n(v)| \le n^{n+1} \sum_{k=1}^{n} \frac{1}{(n-k)!(k!)} < \frac{n^{n+1}2^n}{n!} \quad \text{for all } \ 0 \le v \le 1,$$

which implies that the series in the left-hand side of (2.6) converges absolutely for $|t| < 1/(2e)$. We combine (2.3) and (2.6), interchange the order of summation, and obtain

$$\begin{aligned}\sum_{n\ge1} q_n(v)(-1)^n t^n &= \sum_{n\ge1}\sum_{k=1}^{n} \frac{k^{n+1}(\frac{1}{2})_k}{(n-k)!(k!)^2}(-1)^k v^k t^n\\ &= \sum_{k\ge1} \frac{(\frac{1}{2})_k}{(k!)^2}(-1)^k k^{k+1}(vt)^k \sum_{n\ge k}\frac{k^{n-k}t^{n-k}}{(n-k)!}\\ &= \sum_{k\ge1} \frac{(\frac{1}{2})_k}{(k!)^2}(-1)^k k^{k+1}(vte^t)^k = G(vte^t). \qquad \square\end{aligned}$$

It is well known that the asymptotic behavior of the coefficients of the power series is closely related with the asymptotic behavior of the generating function at its dominant singularities. Our next goal is to obtain analytic continuation of the function $G(z)$ and to study its asymptotic expansion at the dominant singularity.

PROPOSITION 2.2. *There exists a function $g(z)$, which is analytic in $\mathbb{C}\backslash(-\infty,-1/e]$ and continuous in the half-plane $\mathrm{Im}(z)\ge 0$, such that*

$$G(z) = \frac{1}{\sqrt{2}\pi}\left[(1+ez)^{-1} + \frac{5}{24}\ln(1+ez)\right] + g(z). \tag{2.7}$$

Proof. Let us denote $H(z) := -\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 W(z)$. From (2.1) we find that

$$H(z) = \sum_{n\ge1}\frac{n^{n+1}}{n!}(-1)^n z^n, \quad |z|<1/e.$$

Using the identity (see formula 3.621.3 in [10])

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} (\sin(t))^{2n}\mathrm{d}t = \frac{(\frac{1}{2})_n}{n!}$$

and formula (2.5) we obtain the integral representation

$$G(z) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} H(z\sin(t)^2)\mathrm{d}t, \quad |z|<1/e. \tag{2.8}$$

The function $W(z)$ is analytic in the cut plane $\mathbb{C}\backslash(-\infty,-1/e]$; therefore $H(z)$ is analytic in the same domain. Applying the Leibniz rule to (2.8) we see that $G(z)$ is also analytic in $\mathbb{C}\backslash(-\infty,-1/e]$.

From (2.2) and the above definition of $H(z)$ we obtain

$$H(z) = \sum_{n\ge-3} c_n p(z)^n = p(z)^{-3} - \frac{11}{24}p(z)^{-1} - \frac{4}{135} - \frac{p(z)}{1152} + \cdots, \tag{2.9}$$

where $p(z) := \sqrt{2(1+ez)}$ and the above series converges for $|p(z)|<\sqrt{2}$. We subtract the dominant asymptotic terms from $H(z)$ and define the two functions

$$\begin{aligned} h(z) &:= H(z) - p(z)^{-3} + \frac{11}{24}p(z)^{-1} = \sum_{n\ge0} c_n p(z)^n, \\ g_1(z) &:= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} h(z\sin(t)^2)\mathrm{d}t. \end{aligned} \tag{2.10}$$

It is clear that $h(z)$ is analytic in $\mathbb{C}\backslash(-\infty,-1/e]$ and continuous in the upper half-plane $\mathrm{Im}(z)\ge 0$ and therefore $g_1(z)$ satisfies the same properties.

Next, we use formula 3.681.1 in [10] and formula (14) on p. 110 in [8] and conclude that for all $z\in\mathbb{C}\backslash(-\infty,-1/e]$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(1+ez\sin(t)^2)^{-\frac{3}{2}}\mathrm{d}t = {}_2F_1(\tfrac{3}{2},\tfrac{1}{2};1;-ez) = \frac{2}{\pi(1+ez)}-\frac{1}{2\pi}\ln(1+ez)+g_2(z), \tag{2.11}$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(1+ez\sin(t)^2)^{-\frac{1}{2}}\mathrm{d}t = {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;-ez) = -\frac{1}{\pi}\ln(1+ez)+g_3(z), \tag{2.12}$$

where ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function and both functions $g_2(z)$ and $g_3(z)$ are analytic in $\mathbb{C}\backslash(-\infty,-1/e]$ and continuous in the half-plane $\mathrm{Im}(z)\ge 0$. Combining (2.8), (2.10), (2.11), and (2.12) we see that

$$\begin{aligned} G(z) &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} H(z\sin(t)^2)\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} p(z\sin(t)^2)^{-3}\mathrm{d}t - \frac{11}{24}\times\frac{2}{\pi}\int_0^{\frac{\pi}{2}} p(z\sin(t)^2)^{-1}\mathrm{d}t + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} h(z\sin(t)^2)\mathrm{d}t \\ &= \frac{1}{\sqrt{2}\pi(1+ez)} + \frac{5}{24\sqrt{2}\pi}\ln(1+ez) + \left[g_1(z)+g_2(z)+g_3(z)\right], \end{aligned}$$

which is equivalent to (2.7). ∎

LEMMA 2.3. *For any $\epsilon\in(0,1)$ there exist constants $b=b(\epsilon)>1$ and $C=C(\epsilon)>0$ such that $|q_n(v)|<Cb^{-n}v$ for all $v\in[0,1-\epsilon]$.*

Proof. The function te^t maps the unit circle $|t|<1$ into some open domain which is a subset of $\mathbb{C}\backslash(-\infty,-1/e]$ (see Figure 2.1). Using this fact combined with the results of Proposition 2.2, we see that the function $t\mapsto G(te^t)$ is analytic for $|t|<1$; therefore there exists $R>1$ such that $t\mapsto G((1-\epsilon)te^t)$ is analytic for $|t|<R$. It is clear that for all $v\in[0,1-\epsilon]$ the function $t\mapsto G'(vte^t)te^t$ is also analytic for $|t|<R$.

We differentiate both sides of (2.6) with respect to v and find

$$q_n'(v)=\frac{(-1)^n}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(G'(vte^t)te^t\right)\bigg|_{t=0}\right].$$

We set $b=(1+R)/2$ and use Cauchy estimates (see [5, Theorem 2.14])

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(G'(vte^t)te^t\right)\bigg|_{t=0}\right|\le n!b^{-n}M_v,$$

where $M_v:=\max\{|G'(vte^t)te^t| \,:\, |t|=b\}$. Applying the maximum modulus principle shows that for all $v\in[0,1-\epsilon]$

$$\begin{aligned} M_v &\le \max\{|G'(vte^t)| \,:\, |t|=b, 0\le v\le 1-\epsilon\}\times\max\{|te^t| \,:\, |t|=b\} \\ &= \max\{|G'((1-\epsilon)te^t)| \,:\, |t|=b\}\times be^b = C. \end{aligned}$$

So far we have established that there exist $b > 1$ and $C > 0$ such that $|q_n'(v)| < Cb^{-n}$ for all $v \in [0, 1-\epsilon]$. Since $q_n(0) = 0$, we conclude that $|q_n(v)| \le \int_0^v |q_n'(v)| \mathrm{d}v < Cb^{-n} v$ for all $v \in [0, 1-\epsilon]$. $\square$

From now on, we will use the notation $w = w(v) := W(-1/(ev))$ where $v \in (0, 1]$ and W is principal branch of the Lambert W-function. In other words, $z = w$ is the unique solution of the equation $1 + vze^{1+z} = 0$ in the strip $0 \le \mathrm{Im}(z) < \pi$. (See the discussion preceding formula (2.3).) Note that $w(1) = -1$ and both $\mathrm{Im}(w(v))$ and $|w(v)|$ are decreasing functions of $v \in (0, 1]$ (see Figure 2.1).

PROPOSITION 2.4. *As $n \to +\infty$ we have uniformly for all $v \in [1/2, 1)$*

$$q_n(v) = (-1)^n \frac{\sqrt{2}}{\pi} \mathrm{Re}\left[\frac{w^{-n}}{(1+w)}\right] + O(w^{-n}). \tag{2.13}$$

Proof. As noted in the proof of Lemma 2.3, for any $v \in [0, 1]$ the function $z \mapsto G(vze^z)$ is analytic in the circle $|z| < 1$. Therefore, using the Cauchy integral formula and (2.6) we find that for all $v \in [1/2, 1)$

$$q_n(v) = \frac{(-1)^n}{2\pi \mathrm{i}} \int_{U_{1/2}} G(vze^z) z^{-n-1} \mathrm{d}z, \tag{2.14}$$

where the contour of integration U_r denotes the circle of radius r and center at $z = 0$, winding counterclockwise around the origin.

Let $g(z)$ be the function defined by (2.7). From (2.7) and (2.14) we find

$$q_n(v) = (-1)^n \left[\frac{1}{\sqrt{2\pi}} q_{n,1}(v) + \frac{5}{24\sqrt{2\pi}} q_{n,2}(v) + q_{n,3}(v)\right], \tag{2.15}$$

where we have defined

$$\begin{aligned}
q_{n,1}(v) &:= \frac{1}{2\pi \mathrm{i}} \int_{U_{1/2}} \frac{z^{-n-1}}{1 + vze^{1+z}} \mathrm{d}z, \\
q_{n,2}(v) &:= \frac{1}{2\pi \mathrm{i}} \int_{U_{1/2}} \ln(1 + vze^{1+z}) z^{-n-1} \mathrm{d}z, \\
q_{n,3}(v) &:= \frac{1}{2\pi \mathrm{i}} \int_{U_{1/2}} g(vze^z) z^{-n-1} \mathrm{d}z.
\end{aligned}$$

Our first goal is to prove that for all $v \in [1/2, 1)$ we have

$$q_{n,1}(v) = 2\mathrm{Re}\left[\frac{w^{-n}}{(1+w)}\right] + O(w^{-n}), \quad n \to +\infty, \tag{2.16}$$

and that the implied constant in $O(w^{-n})$ does not depend on v. One can check numerically that $|w(1/2)| = |W(-2/e)| \approx 1.2508 < 2$. In particular, for all $v \in [1/2, 1)$ we have $1 < |w(v)| < 2$ and the equation $1 + vze^{1+z} = 0$ has exactly two solutions in the circle U_2, given by $z = w$ and $z = \bar{w}$, and no other solutions in the circle U_π. This shows that for $v \in [1/2, 1)$ the meromorphic function $z \mapsto z^{-n-1}/(1 + vze^{1+z})$ has two simple poles at $z = w$ and $z = \bar{w}$ and no other singularities in the circle U_π, while if $v = 1$ we have a double pole at $z = w = \bar{w} = -1$. We assume that $v \in [1/2, 1)$ and compute the residues of $z^{-n-1}/(1 + vze^{1+z})$ at $z = w$ and $z = \bar{w}$, and applying Cauchy's residue theorem we obtain

$$\frac{1}{2\pi \mathrm{i}} \int_{U_{1/2}} \frac{z^{-n-1}}{1 + vze^{1+z}} \mathrm{d}z = 2\mathrm{Re}\left[\frac{w^{-n}}{(1+w)}\right] + \frac{1}{2\pi \mathrm{i}} \int_{U_3} \frac{z^{-n-1}}{1 + vze^{1+z}} \mathrm{d}z.$$

The integral in the right-hand side of the above formula is estimated as follows:

$$\left|\frac{1}{2\pi\mathrm{i}}\int_{U_\pi}\frac{z^{-n-1}}{1+vze^{1+z}}\mathrm{d}z\right| \le \frac{3^{-n}}{C_1} = O(w^{-n}),$$

where $C_1 := \min\{|1+vze^{1+z}| \ : \ 1/2 \le v < 1, |z| = 3\}$. Note that C_1 is strictly positive, since we have shown above that for $v \in [1/2, 1]$ the function $z \mapsto 1 + vze^{1+z}$ has no roots in the domain $2 \le |z| \le \pi$.

Our second goal is to prove that for all $v \in [1/2, 1)$ we have

$$q_{n,2}(v) = O(w^{-n}), \quad n \to +\infty. \tag{2.17}$$

We isolate the dominant singularities at $z = w$ and $z = \bar{w}$ and obtain

$$\begin{aligned} q_n^{(2)}(v) = \frac{1}{2\pi\mathrm{i}}\int_{U_{1/2}} z^{-n-1}\ln(1-z/w)\mathrm{d}z + \frac{1}{2\pi\mathrm{i}}\int_{U_{1/2}} z^{-n-1}\ln(1-z/\bar{w})\mathrm{d}z \\ + \frac{1}{2\pi\mathrm{i}}\int_{U_{1/2}} z^{-n-1}\ln\left[\frac{1+vze^{1+z}}{(1-z/w)(1-z/\bar{w})}\right]\mathrm{d}z. \end{aligned}$$

The first and second integrals can be evaluated explicitly to $(-n^{-1}w^{-n})$ and $(-n^{-1}\bar{w}^{-n})$, respectively; thus they contribute $O(w^{-n})$. The integrand in the third integral is analytic in the circle $|z| \le \pi$, and using the same technique as used above for estimating $q_{n,1}(v)$, we find that for all $v \in [1/2, 1)$ the third integral is bounded from above by $O(3^{-n}) = O(w^{-n})$.

Finally, for any $v \in [1/2, 1]$ the function $z \mapsto g(vze^z)$ is analytic in the circle $U_{|w|}$ and continuous on the boundary of this circle. (Note that $g(z)$ is continuous at $z = -1/e$ due to Proposition 2.2.) Therefore

$$q_{n,3}(v) = \frac{1}{2\pi\mathrm{i}}\int_{U_{1/2}} g(vze^z)z^{-n-1}\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\int_{U_{|w|}} g(vze^z)z^{-n-1}\mathrm{d}z$$

and we obtain the estimate

$$|q_{n,3}(v)| = \left|\frac{1}{2\pi\mathrm{i}}\int_{U_{|w|}} g(vze^z)z^{-n-1}\mathrm{d}z\right| \le |w|^{-n}C_2,$$

where $C_2 := \max\{|g(vze^z)| \ : \ 1/2 \le v \le 1, z = |w|e^{2\pi\mathrm{i}t}, \ 0 \le t \le \pi\}$. Since $w(v)$ is continuous for $v \in [1/2, 1]$ and $g(z)$ is continuous in the upper half-plane $\mathrm{Im}(z) \ge 0$, we see that C_2 is finite and we obtain

$$q_{n,3}(v) = O(w^{-n}), \quad n \to +\infty, \tag{2.18}$$

for all $v \in [1/2, 1)$.

Combining (2.15), (2.16), (2.17), and (2.18) gives us (2.13). □

Remark 1. With some extra work one can improve the results of Propositions 2.2 and 2.4 in the following way:

(i) The function $G(z)$ admits an asymptotic expansion at $z = -1/e$ of the form

$$G(z) \approx \frac{1}{\sqrt{2\pi}}\left[(1+ez)^{-1} + \sum_{n\ge 0}(1+ez)^n(a_n\ln(1+ez) + b_n)\right],$$

where $a_0 = 5/24$, $a_1 = 25/1152$, and $\{a_n\}_{n\ge 2}$ are computable rational numbers.

(ii) As $n \to +\infty$ we have uniformly in $v \in [1/2, 1)$

$$q_n(v) = (-1)^n \frac{\sqrt{2}}{\pi} \mathrm{Re}\left[\frac{w^{-n}}{(1+w)} - \frac{5}{24n} w^{-n} + \frac{25}{1152n^2}(1+w)w^{-n}\right] + O(n^{-3} w^{-n}),$$

where $w = W(-1/(ev))$. The above formula remains valid as $v \to 1^-$, in which case we obtain

$$q_n(1) = \frac{\sqrt{2}}{\pi}\left[n + \frac{1}{3} - \frac{5}{24n}\right] + O(n^{-3}).$$

For $v \in [0, 1/2)$ we define

$$\xi(v) = w(1-4v^2) = W(-1/(e(1-4v^2))), \quad \alpha(v) = \mathrm{Im}\left[\xi(v)\right].$$

These two function will play an important role in the proof of Theorem 1.1. We summarize some of their properties in the next lemma.

Lemma 2.5.

(i) *The function $|\xi(v)|$ is smooth and strictly increasing for $v \in [0, 1/2)$. It is analytic in the neighborhood of $v = 0$ and satisfies*

$$\xi(v) = -1 + 2\sqrt{2}\mathrm{i}v + \frac{8}{3}v^2 + \frac{14\sqrt{2}}{9}\mathrm{i}v^3 + O(v^4), \quad v \to 0^+. \tag{2.19}$$

(ii) *The function $\alpha(v)$ is smooth and strictly increasing for $v \in [0, 1/2)$. It is analytic in the neighborhood of $v = 0$ and satisfies*

$$\alpha(v) = 2\sqrt{2}v + \frac{14\sqrt{2}}{9}v^3 + O(v^5), \quad v \to 0^+. \tag{2.20}$$

(iii) *For any function $h \in L_1(0, 1/2)$ we have*

$$\lim_{n \to +\infty} \int_0^{\frac{1}{2}} h(v)\xi(v)^{-n} \mathrm{d}v = 0.$$

Proof. Items (i) and (ii) follow easily from properties of the Lambert W-function. In particular the series expansion (2.19) follows from (2.2), while (2.20) is a simple corollary of (2.19). Item (iii) is obvious, given that $|\xi(0)| = 1$ and $|\xi(v)|$ is a strictly increasing function. ∎

Proposition 2.6. *Under the assumptions of Theorem 1.1, $f_n(x) \to c$ as $n \to \infty$ if and only if for any $\epsilon \in (0, 1/4)$*

$$\int_0^{\epsilon} \frac{\sin(n\alpha(v))}{\alpha(v)|\xi(v)|^n}\left[f\left(-x\log_2\left(\tfrac{1}{2}+v\right)\right) + f\left(-x\log_2\left(\tfrac{1}{2}-v\right)\right) - 2c\right]\mathrm{d}v \to 0 \tag{2.21}$$

as $n \to +\infty$.

Proof. Assume that $\epsilon \in (0,1/4)$ and define $u_0 := -\ln(\frac{1}{2}+\epsilon)$ and $u_1 := -\ln(\frac{1}{2}-\epsilon)$. Note that $0 < u_0 < \ln(2) < u_1 < \infty$. The function $u \mapsto 4e^{-u}(1-e^{-u})$ is strictly increasing for $0 < u < \ln(2)$ and strictly decreasing for $\ln(2) < u$, and its value at $u = \ln(2)$ is one. We conclude that there exists $\delta = \delta(\epsilon) > 0$ such that for all $u \in [0,u_0] \cup [u_1,\infty)$ we have $0 < 4e^{-u}(1-e^{-u}) < 1-\delta$. Using Lemma 2.3 we find that there exist $b > 1$ and $C > 0$ such that

$$|q_n\left(4e^{-u}(1-e^{-u})\right)| < 4Cb^{-n}e^{-u}(1-e^{-u}) < 4Cb^{-n}e^{-u}$$

for all $u \in [0,u_0]\cup[u_1,\infty)$. This fact and our assumptions on the function f guarantee that

$$\lim_{n\to+\infty} \int_{[0,u_0]\cup[u_1,\infty)} q_n\left(4e^{-u}(1-e^{-u})\right) f\left(\frac{xu}{\ln(2)}\right) \mathrm{d}u = 0.$$

Thus we have established the following result:

$$f_n(x) = \int_{u_0}^{u_1} q_n\left(4e^{-u}(1-e^{-u})\right) f\left(\frac{xu}{\ln(2)}\right) \mathrm{d}u + o(1), \quad n \to +\infty.$$

As we pointed out in the introduction, Gaver–Stehfest approximations are exact for constant functions: if $f(x) \equiv C$, then $f_n(x) \equiv C$ for all $n \ge 1$. This result and the above formula imply

$$f_n(x) - c = \int_{u_0}^{u_1} q_n\left(4e^{-u}(1-e^{-u})\right) \left[f\left(\frac{xu}{\ln(2)}\right) - c\right] \mathrm{d}u + o(1) \tag{2.22}$$

as $n \to +\infty$.

Our next goal is to simplify the integral in (2.22). We separate this integral into two parts: the integral over $[u_0,\ln(2)]$ and the integral over $[\ln(2),u_1]$. For the first (resp., second) integral we change the variable of integration $u = -\ln(\frac{1}{2}+v)$ (resp., $u = -\ln(\frac{1}{2}-v)$) and after combining the two parts we obtain

(2.23)
$$f_n(x) - c = \int_0^{\epsilon} q_n(1-4v^2)\left[\frac{f(-x\log_2(\frac{1}{2}+v)) - c}{\frac{1}{2}+v} + \frac{f(-x\log_2(\frac{1}{2}-v)) - c}{\frac{1}{2}-v}\right] \mathrm{d}v + o(1)$$

as $n \to +\infty$.

For our next step, we will need the result

$$q_n(1-4v^2) = \frac{\sqrt{2}}{\pi}|\xi(v)|^{-n}\frac{\sin(n\alpha(v))}{\alpha(v)} + O(\xi(v)^{-n}) \tag{2.24}$$

uniformly for all $v \in (0,1/4]$. Note that the Lambert W-function satisfies $W(z) = z\exp(-W(z))$; thus for $z < -1/e$ we have $\arg(W(z)) = \pi - \mathrm{Im}[W(z)]$, which implies $\arg(\xi(v)) = \pi - \alpha(v)$. This gives us

$$\xi(v)^{-n} = |\xi(v)|^{-n}e^{-\mathrm{i}n(\pi-\alpha(v))} = (-1)^n|\xi(v)|^{-n}\left(\cos(n\alpha(v)) + \mathrm{i}\times\sin(n\alpha(v))\right).$$

Combining the above formula with the identity

$$\frac{1}{1+\xi} = \frac{1+\bar{\xi}}{|1+\xi|^2} = \frac{1+\mathrm{Re}(\xi)}{|1+\xi|^2} - \mathrm{i}\frac{\mathrm{Im}(\xi)}{|1+\xi|^2}$$

we obtain

$$\mathrm{Re}\left[\frac{\xi(v)^{-n}}{1+\xi(v)}\right] = |\xi(v)|^{-n}\left[\cos(n\alpha(v))\frac{1+\mathrm{Re}(\xi(v))}{|1+\xi(v)|^2} + \sin(n\alpha(v))\frac{\alpha(v)}{|1+\xi(v)|^2}\right].$$

Formulas (2.19) and (2.20) show that uniformly for all $v \in (0, 1/4]$

$$\frac{1+\mathrm{Re}(\xi(v))}{|1+\xi(v)|^2} = O(1), \quad \frac{\alpha(v)}{|1+\xi(v)|^2} = \frac{1}{\alpha(v)} + O(1);$$

thus

$$\mathrm{Re}\left[\frac{\xi(v)^{-n}}{1+\xi(v)}\right] = (-1)^n|\xi(v)|^{-n}\frac{\sin(n\alpha(v))}{\alpha(v)} + O(\xi(v)^{-n}),$$

which together with (2.13) implies (2.24).

Next, we define

$$\begin{aligned}(2.25)\qquad g(x,v,c) &:= f\left(-x\log_2\left(\tfrac{1}{2}+v\right)\right) + f\left(-x\log_2\left(\tfrac{1}{2}-v\right)\right) - 2c,\\ \tilde{g}(x,v,c) &:= -\frac{f\left(-x\log_2\left(\frac{1}{2}+v\right)\right)-c}{\frac{1}{2}+v} + \frac{f\left(-x\log_2\left(\frac{1}{2}-v\right)\right)-c}{\frac{1}{2}-v}.\end{aligned}$$

It is easy to check that

$$\left[\frac{f(-x\log_2(\frac{1}{2}+v))-c}{\frac{1}{2}+v} + \frac{f(-x\log_2(\frac{1}{2}-v))-c}{\frac{1}{2}-v}\right] \equiv 2g(x,v,c) + 2v\tilde{g}(x,v,c);$$

therefore

$$\begin{aligned}(2.26)\qquad f_n(x) - c = 2\int_0^{\epsilon} q_n(1-4v^2)g(x,v,c)\mathrm{d}v\\ +2\int_0^{\epsilon} q_n(1-4v^2)v\tilde{g}(x,v,c)\mathrm{d}v + o(1), \quad n\to+\infty.\end{aligned}$$

We use (2.24) and find that

$$\begin{aligned}&\int_0^{\epsilon} q_n(1-4v^2)v\tilde{g}(x,v,c)\mathrm{d}v\\ &\quad= \frac{\sqrt{2}}{\pi}\int_0^{\epsilon}\left[|\xi(v)|^{-n}\sin(n\alpha(v))\frac{v}{\alpha(v)} + vO(\xi(v)^{-n})\right]\tilde{g}(x,v,c)\mathrm{d}v \to 0\end{aligned}$$

as $n\to+\infty$, due to Lemma 2.5(iii). (Note that the function $v\mapsto\tilde{g}(x,v,c)$ is integrable and the function $v/\alpha(v)$ is bounded.) This fact combined with formula (2.26) give us (2.21). ∎

Proof of Theorem 1.1(i) *and* (ii). Theorem 1.1(i) follows directly from Proposition 2.6. Let us prove Theorem 1.1(ii). Formulas (2.21) and (2.25) give us

$$\begin{aligned}f_n(x) - c &= 2\int_0^{\epsilon}|\xi(v)|^{-n}\frac{\sin(n\alpha(v))}{\alpha(v)}g(x,v,c)\mathrm{d}v + o(1)\\ &= 2\int_0^{\epsilon}|\xi(v)|^{-n}\left[\sin(n\alpha(v))\times\frac{v}{\alpha(v)}\times\frac{g(x,v,c)}{v}\right]\mathrm{d}v + o(1).\end{aligned}$$

The functions $\sin(n\alpha(v))$ and $v/\alpha(v)$ are bounded and the function $v \mapsto g(x,v,c)/v$ is integrable over $[0,\epsilon)$; thus applying Lemma 2.5(iii) we see that the integral in the right-hand side of the above formula converges to zero as $n \to +\infty$, and therefore $f_n(x) \to c$ as $n \to +\infty$. ∎

Proof of Theorem 1.1(iii). Assume that for some $\delta > 0$ the function f has bounded variation on the interval $[x-\delta, x+\delta]$. We will need the following simple property of functions of bounded variation:

If $f(y)$ has bounded variation for $y \in [a,b]$ and $g(x)$ is monotone for $x \in [c,d]$ and satisfies $g([c,d]) \subseteq [a,b]$, then $f(g(x))$ has bounded variation for $x \in [c,d]$.

Using the above property we see that there exists $\epsilon_1 \in (0,1/4)$ such that both functions $v \mapsto f(-x\log_2(\frac{1}{2}-v))$ and $v \mapsto f(-x\log_2(\frac{1}{2}+v))$ have bounded variation in the interval $v \in [0,\epsilon_1]$.

Formula (2.20) shows that there exists $\epsilon_2 \in (0,1/4)$ such that $\alpha''(v) > 0$ for all $v \in (0,\epsilon_2)$. We set $\epsilon_3 = \min(\epsilon_1,\epsilon_2)$ and take $\epsilon < \epsilon_3$ to be a small positive number (to be specified later). We rewrite formula (2.21), where ϵ is chosen in the above way and $c = \frac{1}{2}\left(f(x+0)+f(x-0)\right)$:

(2.27)
$$f_n(x) - c = \int_0^{\epsilon} |\xi(v)|^{-n} \frac{\sin(n\alpha(v))}{\alpha(v)} \left[f(-x\log_2(\tfrac{1}{2}-v)) - f(x+0)\right] \mathrm{d}v + \int_0^{\epsilon} |\xi(v)|^{-n} \frac{\sin(n\alpha(v))}{\alpha(v)} \left[f(-x\log_2(\tfrac{1}{2}+v)) - f(x-0)\right] \mathrm{d}v + o(1)$$

as $n \to +\infty$. Since the two functions $v \mapsto f(-x\log_2(\frac{1}{2}-v))$ and $v \mapsto f(-x\log_2(\frac{1}{2}+v))$ have bounded variation for $v \in [0,\epsilon_3]$, there exist four increasing functions $h_i(v)$, $1 \le i \le 4$, satisfying $h_i(0+) = 0$, such that $f(-x\log_2(\frac{1}{2}-v)) - f(x+0) = h_1(v) - h_2(v)$ and $f(-x\log_2(\frac{1}{2}+v)) - f(x-0) = h_3(v) - h_4(v)$ for all $v \in [0,\epsilon_3]$. We rewrite (2.27) in the form

$$f_n(x) - c = J_{n,1} - J_{n,2} + J_{n,3} - J_{n,4} + o(1), \quad n \to +\infty, \tag{2.28}$$

where

$$J_{n,i} := \int_0^{\epsilon} |\xi(v)|^{-n} \frac{\sin(n\alpha(v))}{\alpha(v)} h_i(v)\mathrm{d}v. \tag{2.29}$$

Since h_1 is positive and increasing, we apply the mean value theorem and conclude that there exists $\theta_1 \in (0,\epsilon)$ such that

$$J_{n,1} = \int_0^{\epsilon} |\xi(v)|^{-n} \frac{\sin(n\alpha(v))}{\alpha(v)} h_1(v)\mathrm{d}v = h_1(\epsilon) \int_{\theta_1}^{\epsilon} |\xi(v)|^{-n} \frac{\sin(n\alpha(v))}{\alpha(v)}\mathrm{d}v. \tag{2.30}$$

Recall that $\alpha''(0) > 0$ for $v \in [0,\epsilon_3]$ and $\alpha'(0) = 2\sqrt{2}$ (see (2.20)), which shows that $\alpha'(v)$ is positive and increasing for $v \in [0,\epsilon_3]$, which in turn implies that the function $|\xi(v)|^{-n}/\alpha'(v)$ is positive and decreasing for $v \in [0,\epsilon_3]$. Applying the mean-value theorem again, we find that there exists some $\theta_2 \in (\theta_1,\epsilon)$ such that

$$\begin{aligned} \int_{\theta_1}^{\epsilon} |\xi(v)|^{-n} \frac{\sin(n\alpha(v))}{\alpha(v)}\mathrm{d}v &= \int_{\theta_1}^{\epsilon} \left[\frac{|\xi(v)|^{-n}}{\alpha'(v)}\right] \times \left[\frac{\sin(n\alpha(v))}{\alpha(v)}\alpha'(v)\right] \mathrm{d}v \\ &= \frac{|\xi(\theta_1)|^{-n}}{\alpha'(\theta_1)} \int_{\theta_1}^{\theta_2} \frac{\sin(n\alpha(v))}{\alpha(v)}\alpha'(v)\mathrm{d}v = \frac{|\xi(\theta_1)|^{-n}}{\alpha'(\theta_1)} \int_{n\alpha(\theta_1)}^{n\alpha(\theta_2)} \frac{\sin(u)}{u}\mathrm{d}u \\ &= \frac{|\xi(\theta_1)|^{-n}}{\alpha'(\theta_1)} \left[\mathrm{Si}(n\alpha(\theta_1)) - \mathrm{Si}(n\alpha(\theta_2))\right], \end{aligned} \tag{2.31}$$

where $\mathrm{Si}(y) := \int_0^y \sin(u)u^{-1}\mathrm{d}u$ is the sine-integral function. Since $|\xi(\theta_1)| > 1$ and $\alpha'(\theta_1) > 2\sqrt{2}$ and $|\mathrm{Si}(x)| < \pi$ for all $x \geq 0$, we conclude from (2.29), (2.30), and (2.31) that

$$|J_{n,1}| < \frac{\pi}{\sqrt{2}} h_1(\epsilon).$$

In the same way we can obtain corresponding estimates for $J_{n,i}$, $i = 2, 3, 4$; therefore (2.28) gives us

$$\limsup_{n\to+\infty} |f_n(x) - c| < \frac{\pi}{\sqrt{2}}(h_1(\epsilon) + h_2(\epsilon) + h_3(\epsilon) + h_4(\epsilon)).$$

Since $h_i(0+) = 0$, the right-hand side can be made arbitrarily small provided that ϵ is small enough. This shows that $f_n(x) \to c$ as $n \to +\infty$. $\square$

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