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On the convergence of additive and multiplicative splitting iterations for systems of linear equations[☆]

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Abstract

We study convergence conditions for the additive and the multiplicative splitting iteration methods, i.e., two generalizations of the additive and the multiplicative Schwarz iterations, for Hermitian and non-Hermitian systems of linear equations, under an algebraic setting. Theoretical analyses show that when the coefficient and the splitting matrices are Hermitian, or non-Hermitian but diagonalizable, satisfying mild conditions, both additive and multiplicative splitting iteration methods are convergent, even if the coefficient matrix is indefinite. © 2003 Elsevier Science B.V. All rights reserved.

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1. Introduction

The matrix splitting methods play an important role for solving large sparse system of linear equations

$$Ax = b, \quad A \in \mathbb{C}^{n \times n} \text{ nonsingular, and } x, b \in \mathbb{C}^n, \quad (1)$$

as either solvers or preconditioners, in both theoretical studies and practical applications. Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{C}^{n \times n}$, i.e., $M \in \mathbb{C}^{n \times n}$ is nonsingular and $N \in \mathbb{C}^{n \times n}$. Then a

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typical splitting iteration induced by the matrix splitting $A = M - N$ has the form

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \quad k = 0, 1, 2, \dots, \quad (2)$$

where $x^{(0)} \in \mathbb{C}^n$ is a given starting vector. Research results about convergence properties and numerical behaviours of iteration (2) are comprehensively and systematically summarized in [1,3,4,7].

In this paper, we study convergence conditions for the additive and the multiplicative splitting iteration methods, i.e., two generalizations of iteration (2), for the system of linear equations (1), under an algebraic setting. Theoretical analyses show that when the coefficient and the splitting matrices are Hermitian, or non-Hermitian but diagonalizable, satisfying mild conditions, both additive and multiplicative splitting iteration methods are convergent, even if the coefficient matrix is indefinite. These results not only extend existing convergence theory for iteration (2) for Hermitian positive definite linear systems [1], but also yield new ones for iteration (2) as well as the additive and the multiplicative Schwarz iterations.

2. Additive/multiplicative splitting iterations

Let $A = M_i - N_i$ ($i = 1, 2$) be two splittings of the matrix $A \in \mathbb{C}^{n \times n}$. The additive splitting iteration (ASI-) method for solving the system of linear equations (1) is defined as follows:

Method 2.1 (ASI-METHOD). Given a starting vector $x^{(0)} \in \mathbb{C}^n$. For $k = 0, 1, 2, \dots$ until $\{x^{(k)}\}$ convergence, compute

$$\begin{aligned} u^{(k+1)} &= M_1^{-1}N_1x^{(k)} + M_1^{-1}b, \\ v^{(k+1)} &= M_2^{-1}N_2x^{(k)} + M_2^{-1}b, \quad k = 0, 1, 2, \dots, \\ x^{(k+1)} &= \omega u^{(k+1)} + (1 - \omega)v^{(k+1)}, \end{aligned}$$

where $\omega \in \mathbb{R}^1$ is a relaxation factor.

If we introduce matrices

$$H_{\text{asi}}(\omega) = \omega M_1^{-1}N_1 + (1 - \omega)M_2^{-1}N_2, \quad G_{\text{asi}}(\omega) = \omega M_1^{-1} + (1 - \omega)M_2^{-1},$$

then the ASI-method can be equivalently written in the form

$$x^{(k+1)} = H_{\text{asi}}(\omega)x^{(k)} + G_{\text{asi}}(\omega)b, \quad k = 0, 1, 2, \dots \quad (3)$$

Clearly, the ASI-method is a special case of the matrix multisplitting method [2], and it is a general form of the additive Schwarz [5] as well as the domain decomposition methods [6].

The multiplicative splitting iteration (MSI-) method for solving the system of linear equations (1) is defined as follows:

Method 2.2 (MSI-METHOD). Given a starting vector $x^{(0)} \in \mathbb{C}^n$. For $k = 0, 1, 2, \dots$ until $\{x^{(k)}\}$ convergence, compute

$$\begin{aligned} u^{(k+1)} &= M_1^{-1}N_1x^{(k)} + M_1^{-1}b, \\ x^{(k+1)} &= M_2^{-1}N_2u^{(k+1)} + M_2^{-1}b, \quad k = 0, 1, 2, \dots \end{aligned}$$

If we introduce matrices

$$H_{\text{msi}} = M_2^{-1}N_2M_1^{-1}N_1, \quad G_{\text{msi}} = M_2^{-1}N_2M_1^{-1} + M_2^{-1},$$

then the MSI-method can be equivalently written in the form

$$x^{(k+1)} = H_{\text{msi}}x^{(k)} + G_{\text{msi}}b, \quad k = 0, 1, 2, \dots \quad (4)$$

Obviously, the MSI-method presents a general description for the multiplicative Schwarz method [5], and it covers many classical iterations such as the alternating direction implicit (ADI-) method [1].

Moreover, we can unify the typical splitting iteration (2), as well as the ASI and the MSI-methods to define the following general splitting iteration (GSI-) method.

Method 2.3 (GSI-METHOD). Given a starting vector $x^{(0)} \in \mathbb{C}^n$. For $k = 0, 1, 2, \dots$ until $\{x^{(k)}\}$ convergence, compute

$$\begin{aligned} u^{(k+1)} &= M_1^{-1}N_1x^{(k)} + M_1^{-1}b, \\ v^{(k+1)} &= M_2^{-1}N_2w^{(k)} + M_2^{-1}b, \quad w^{(k)} \in \{x^{(k)}, u^{(k+1)}\}, \quad k = 0, 1, 2, \dots, \\ x^{(k+1)} &= \omega u^{(k+1)} + (1 - \omega)v^{(k+1)}, \end{aligned}$$

where $\omega \in \mathbb{R}^1$ is a relaxation factor.

In the GSI-method, choosing $w^{(k)} = x^{(k)} (k = 0, 1, 2, \dots)$ gives the ASI-method, and choosing $\omega = 0$ and $w^{(k)} = u^{(k+1)} (k = 0, 1, 2, \dots)$ gives the MSI-method. In addition, when $\omega \neq 0$ and $w^{(k)} = u^{(k+1)} (k = 0, 1, 2, \dots)$, the GSI-method yields a relaxed variant of the MSI-method. Besides, many new methods can be obtained through different choices of the parameter ω , or different switching rules of the intermediate iterate $w^{(k)}$ for $k = 0, 1, 2, \dots$

Analogously, if we define matrices

$$\begin{aligned} H_{\text{gsi}}(\gamma, \omega) &= \gamma(1 - \omega)M_2^{-1}N_2M_1^{-1}N_1 + \omega M_1^{-1}N_1 + (1 - \gamma)(1 - \omega)M_2^{-1}N_2, \\ G_{\text{gsi}}(\gamma, \omega) &= \gamma(1 - \omega)M_2^{-1}N_2M_1^{-1} + \omega M_1^{-1} + (1 - \omega)M_2^{-1}, \end{aligned}$$

then the GSI-method can be equivalently expressed in the form

$$x^{(k+1)} = H_{\text{gsi}}(\gamma, \omega)x^{(k)} + G_{\text{gsi}}(\gamma, \omega)b, \quad k = 0, 1, 2, \dots, \quad (5)$$

where $\gamma=1, 0$ correspond to $w^{(k)}=u^{(k+1)}, x^{(k)}$, respectively, which may switch with the iterate index k .

Evidently, iterations (3)–(5) are consistent with the original system of linear equations (1) if the matrices $G_{\xi}(\cdot)$ ($\xi = \text{asi, msi, gsi}$) are nonsingular, and they are convergent if the spectral radii of the iteration matrices $H_{\xi}(\cdot)$ ($\xi = \text{asi, msi, gsi}$) are less than one, i.e., $\rho(H_{\xi}(\cdot)) < 1$ ($\xi = \text{asi, msi, gsi}$).

3. Convergence theories for Hermitian matrix

In this section, we discuss convergence of iterations (3)–(5) under assumptions such as Hermitian and/or positive definiteness, or commutativity.

3.1. Hermitian case

Let $B = (b_{ij}) \in \mathbb{C}^{n \times n}$. We use $\bar{B} = (\bar{b}_{ij})$, $B^T = (b_{ji})$ and $B^* = (\bar{B})^T$ to denote the complex conjugate, the transpose and the Hermitian transpose of the matrix B , respectively. The matrix B is said to be positive definite (positive semidefinite), denoted by $B \succ O$ ($B \succeq O$), in \mathbb{C}^n if its quadratic form (Bx, x) is real and positive (nonnegative) for all nonzero $x \in \mathbb{C}^n$, where O denotes the zero matrix. For $B, C \in \mathbb{C}^{n \times n}$, we define $B \succ C$ ($B \succeq C$) if $B - C \succ O$ ($B - C \succeq O$). In particular, if the matrices B and C are Hermitian positive definite, then $B \succeq C$ ($B \succ C$) implies $B^{-1} \preceq C^{-1}$ ($B^{-1} \prec C^{-1}$), and there exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ such that $PBP^* = I$ and $PCP^* = D$, where I is the identity matrix, and D a diagonal matrix with all diagonal entries being positive. More generally, we have the following result.

Lemma 3.1. *Let $B \in \mathbb{C}^{n \times n}$ be a Hermitian positive definite matrix and $C \in \mathbb{C}^{n \times n}$ be a Hermitian matrix. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ such that $PBP^* = I$ and $PCP^* = D$, with D a diagonal matrix.*

Proof. Because $B \in \mathbb{C}^{n \times n}$ is Hermitian positive definite, there exists a unitary matrix $Q_B \in \mathbb{C}^{n \times n}$ such that $Q_B B Q_B^* = A_B$, where Q_B satisfies $Q_B Q_B^* = I$ and A_B is a diagonal matrix with all diagonal entries being positive. Letting $\tilde{Q}_B = A_B^{-1/2} Q_B$ and $\tilde{C} = \tilde{Q}_B C \tilde{Q}_B^*$, we have $\tilde{Q}_B B \tilde{Q}_B^* = I$ and $\tilde{C}^* = \tilde{C}$. Therefore, there exists another unitary matrix $Q_{\tilde{C}} \in \mathbb{C}^{n \times n}$ such that $Q_{\tilde{C}} \tilde{C} Q_{\tilde{C}}^* = A_{\tilde{C}}$, where $Q_{\tilde{C}}$ satisfies $Q_{\tilde{C}} Q_{\tilde{C}}^* = I$ and $A_{\tilde{C}} \in \mathbb{R}^{n \times n}$ is a diagonal matrix. If we define $P = Q_{\tilde{C}} \tilde{Q}_B = Q_{\tilde{C}} A_B^{-1/2} Q_B$, then we immediately get $PBP^* = I$ and $PCP^* = D$, with $D = A_{\tilde{C}}$. \square

We call the matrix P in Lemma 3.1 as the simultaneously diagonalizing matrix with respect to the matrices B and C . Moreover, to specify the dependence of the matrix P upon the matrices B and C , and to identify the mapping relationships between the transformed matrices I and D and the original matrices B and C , respectively, we may represent the simultaneously diagonalizing matrix P by $P_{B/C}$, i.e., $P_{B/C} B P_{B/C}^* = I$ and $P_{B/C} C P_{B/C}^* = D$.

Theorem 3.1. *Let $A \in \mathbb{C}^{n \times n}$ be Hermitian and nonsingular, and $A = M_i - N_i$ ($i=1, 2$) be two splittings such that M_i ($i=1, 2$) are Hermitian. Then*

- (a) *the matrix $G_{\text{asi}}(\omega)$ is Hermitian, and it is nonsingular if and only if¹ the matrix $R_{\text{asi}}(\omega) = (1 - \omega)M_1 + \omega M_2$ is nonsingular;*
- (b) *the matrix G_{msi} is Hermitian, if either $M_1^{-1}N_2M_2^{-1} = M_2^{-1}N_2M_1^{-1}$ or $M_1A^{-1}M_2 = M_2A^{-1}M_1$ holds, and it is nonsingular iff the matrix $R_{\text{msi}} = M_1 + M_2 - A$ is nonsingular;*
- (c) *the matrix $G_{\text{gsi}}(\gamma, \omega)$ is Hermitian, if either $M_1^{-1}N_2M_2^{-1} = M_2^{-1}N_2M_1^{-1}$ or $M_1A^{-1}M_2 = M_2A^{-1}M_1$ holds, and it is nonsingular iff the matrix $R_{\text{gsi}}(\gamma, \omega) = (1 - \omega)M_1 + (\omega + \gamma(1 - \omega))M_2 - \gamma(1 - \omega)A$ is nonsingular.*

Proof. It is obvious that $G_{\text{asi}}(\omega)$ is a Hermitian matrix.

¹ We abbreviate “if and only if” by “iff” throughout this paper.

In addition, because

$$G_{\text{asi}}(\omega) = \omega M_1^{-1} + (1 - \omega)M_2^{-1} = M_1^{-1}R_{\text{asi}}(\omega)M_2^{-1},$$

we know that $G_{\text{asi}}(\omega)$ is nonsingular iff $R_{\text{asi}}(\omega)$ is nonsingular.

It follows directly from $M_1^{-1}N_2M_2^{-1} = M_2^{-1}N_2M_1^{-1}$ that the matrices G_{msi} and $G_{\text{gsi}}(\gamma, \omega)$ are Hermitian. Moreover, because

$$M_1^{-1}N_2M_2^{-1} = M_1^{-1}(M_2 - A)M_2^{-1} = M_1^{-1} - M_1^{-1}AM_2^{-1}$$

and

$$M_2^{-1}N_2M_1^{-1} = M_2^{-1}(M_2 - A)M_1^{-1} = M_2^{-1} - M_2^{-1}AM_1^{-1}$$

and because $M_2A^{-1}M_1 = M_1A^{-1}M_2$ is equivalent to $M_1^{-1}AM_2^{-1} = M_2^{-1}AM_1^{-1}$, the condition $M_1A^{-1}M_2 = M_2A^{-1}M_1$ implies that both matrices G_{msi} and $G_{\text{gsi}}(\gamma, \omega)$ are Hermitian, too.

Finally, noticing that

$$\begin{aligned} G_{\text{msi}} &= M_2^{-1}N_2M_1^{-1} + M_2^{-1} = M_2^{-1}(N_2 + M_1)M_1^{-1} \\ &= M_2^{-1}(M_1 + M_2 - A)M_1^{-1} = M_2^{-1}R_{\text{msi}}M_1^{-1} \end{aligned}$$

and

$$\begin{aligned} G_{\text{gsi}}(\gamma, \omega) &= \gamma(1 - \omega)M_2^{-1}N_2M_1^{-1} + \omega M_1^{-1} + (1 - \omega)M_2^{-1} \\ &= M_2^{-1}[\gamma(1 - \omega)N_2 + (1 - \omega)M_1 + \omega M_2]M_1^{-1} \\ &= M_2^{-1}[\gamma(1 - \omega)(M_2 - A) + (1 - \omega)M_1 + \omega M_2]M_1^{-1} \\ &= M_2^{-1}R_{\text{gsi}}(\gamma, \omega)M_1^{-1}, \end{aligned}$$

we know that the matrices G_{msi} and $G_{\text{gsi}}(\gamma, \omega)$ are nonsingular iff the matrices R_{msi} and $R_{\text{gsi}}(\gamma, \omega)$ are nonsingular, respectively. \square

The proof of Theorem 3.1 shows that $M_1^{-1}N_2M_2^{-1} = M_2^{-1}N_2M_1^{-1}$ and $M_2A^{-1}M_1 = M_1A^{-1}M_2$ are equivalent under the conditions of the theorem. In addition, Theorem 3.1 guarantees that the ASI, the MSI and the GSI methods are consistent with the original system of linear equations (1) iff the matrices R_{asi} , R_{msi} and $R_{\text{gsi}}(\gamma, \omega)$ are nonsingular, respectively.

Moreover, we can establish stronger properties for the matrices G_{asi} , G_{msi} and $G_{\text{gsi}}(\gamma, \omega)$ in the following. To this end, we use $\text{diag}(\cdot)$ and $\text{Diag}(\cdot)$ to represent a pointwise and a blockwise diagonal matrix, respectively. For a real pointwise diagonal matrix $A = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, we define $\text{sign}(A) = \text{diag}(\text{sign}(\lambda_1), \text{sign}(\lambda_2), \dots, \text{sign}(\lambda_n))$, where for $i = 1, 2, \dots, n$, $\text{sign}(\lambda_i)$ denotes the sign of the real λ_i .

Theorem 3.2. *Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian and nonsingular matrix, and $A = M_i - N_i$ ($i = 1, 2$) be two splittings such that M_i ($i = 1, 2$) are Hermitian. Then:*

(a) *When $M_1 \in \mathbb{C}^{n \times n}$ is positive definite, it holds that*

(a₁) *the matrix $G_{\text{asi}}(\omega)$ is Hermitian positive definite iff*

$$(1 - \omega)\text{sign}(D) + \omega|D| \succ O;$$

(a₂) the matrix G_{msi} is Hermitian positive definite iff $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ and

$$\text{sign}(D) + |D| - \text{sign}(D) P_{M_1/M_2} A P_{M_1/M_2}^* \succ O;$$

(a₃) the matrix $G_{\text{gsi}}(\gamma, \omega)$ is Hermitian positive definite iff $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ and

$$(1 - \omega) \text{sign}(D) + (\omega + \gamma(1 - \omega)) |D| - \gamma(1 - \omega) \text{sign}(D) P_{M_1/M_2} A P_{M_1/M_2}^* \succ O,$$

where $D = P_{M_1/M_2} M_2 P_{M_1/M_2}^*$, and P_{M_1/M_2} is the simultaneously diagonalizing matrix with respect to the matrices M_1 and M_2 .

(b) When $M_2 \in \mathbb{C}^{n \times n}$ is positive definite, it holds that

(b₁) the matrix $G_{\text{asi}}(\omega)$ is Hermitian positive definite iff

$$(1 - \omega) |D| + \omega \text{sign}(D) \succ O;$$

(b₂) the matrix G_{msi} is Hermitian positive definite iff $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ and

$$|D| + \text{sign}(D) - P_{M_2/M_1} A P_{M_2/M_1}^* \text{sign}(D) \succ O;$$

(b₃) the matrix $G_{\text{gsi}}(\gamma, \omega)$ is Hermitian positive definite iff $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ and

$$(1 - \omega) |D| + (\omega + \gamma(1 - \omega)) \text{sign}(D) - \gamma(1 - \omega) P_{M_2/M_1} A P_{M_2/M_1}^* \text{sign}(D) \succ O,$$

where $D = P_{M_2/M_1} M_1 P_{M_2/M_1}^*$, and P_{M_2/M_1} is the simultaneously diagonalizing matrix with respect to the matrices M_1 and M_2 .

Proof. Because $G_{\text{asi}}(\omega) = G_{\text{gsi}}(0, \omega)$ and $G_{\text{msi}} = G_{\text{gsi}}(1, 0)$, we only need to investigate the Hermitian positive definiteness of the matrix $G_{\text{gsi}}(\gamma, \omega)$.

We first demonstrate the validity of (a). By Lemma 3.1 we know that there exists a simultaneously diagonalizing matrix $P = P_{M_1/M_2} \in \mathbb{C}^{n \times n}$ such that $P M_1 P^* = I$ and $P M_2 P^* = D$. Here, without loss of generality, we assume that the real diagonal matrix D has the block form

$$\begin{bmatrix} d_1 I_{n_1} & & & \\ & d_2 I_{n_2} & & \\ & & \ddots & \\ & & & d_r I_{n_r} \end{bmatrix} \quad (6)$$

with $d_i \neq d_j$ for $i \neq j$, $i, j = 1, 2, \dots, r$, and n_i ($i = 1, 2, \dots, r$) being positive integers satisfying $\sum_{i=1}^r n_i = n$. Let $\tilde{A} = P A P^* = (\tilde{A}_{ij})$ be blocked in the same fashion as the matrix D . By straightforward computation we have

$$M_1^{-1} A M_2^{-1} = P^* \tilde{A} D^{-1} P \quad \text{and} \quad M_2^{-1} A M_1^{-1} = P^* D^{-1} \tilde{A} P.$$

Therefore, the condition $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ immediately yields $\tilde{A} D^{-1} = D^{-1} \tilde{A}$, which obviously implies that

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & & & \\ & \tilde{A}_{22} & & \\ & & \ddots & \\ & & & \tilde{A}_{rr} \end{bmatrix}, \quad \tilde{A}_{ii} \in \mathbb{C}^{n_i \times n_i}, \quad i = 1, 2, \dots, r. \quad (7)$$

Because

$$\begin{aligned} PR_{\text{gsi}}(\gamma, \omega)P^* &= P[(1 - \omega)M_1 + (\omega + \gamma(1 - \omega))M_2 - \gamma(1 - \omega)A]P^* \\ &= (1 - \omega)I + (\omega + \gamma(1 - \omega))D - \gamma(1 - \omega)\tilde{A}, \end{aligned}$$

we obtain

$$\begin{aligned} G_{\text{gsi}}(\gamma, \omega) &= M_2^{-1}R_{\text{gsi}}(\gamma, \omega)M_1^{-1} \\ &= (P^*D^{-1}P)R_{\text{gsi}}(\gamma, \omega)(P^*P) \\ &= P^*D^{-1}[(1 - \omega)I + (\omega + \gamma(1 - \omega))D - \gamma(1 - \omega)\tilde{A}]P \\ &= P^*[(1 - \omega)D^{-1} + (\omega + \gamma(1 - \omega))I - \gamma(1 - \omega)D^{-1}\tilde{A}]P. \end{aligned}$$

Let

$$\tilde{S} = (1 - \omega)D^{-1} + (\omega + \gamma(1 - \omega))I - \gamma(1 - \omega)D^{-1}\tilde{A}.$$

Then we see that $G_{\text{gsi}}(\gamma, \omega)$ is Hermitian positive definite iff all diagonal blocks

$$\tilde{S}_{ii} = \frac{1 - \omega}{d_i}I_{n_i} + (\omega + \gamma(1 - \omega))I_{n_i} - \frac{\gamma(1 - \omega)}{d_i}\tilde{A}_{ii}, \quad i = 1, 2, \dots, r$$

of the matrix \tilde{S} are Hermitian positive definite, or in other words,

$$\begin{aligned} S_{ii} &\succ O \quad \text{for } d_i > 0, \\ S_{ii} &\prec O \quad \text{for } d_i < 0, \end{aligned} \quad i = 1, 2, \dots, r,$$

where

$$S_{ii} = (1 - \omega)I_{n_i} + (\omega + \gamma(1 - \omega))d_iI_{n_i} - \gamma(1 - \omega)\tilde{A}_{ii}, \quad i = 1, 2, \dots, r.$$

If $\tilde{\mu}_j^{(i)} (j = 1, 2, \dots, n_i)$ are eigenvalues of the matrix \tilde{A}_{ii} , then it straightforwardly follows from the above derivation that $G_{\text{gsi}}(\gamma, \omega)$ is Hermitian positive definite iff

$$\begin{aligned} 1 - \omega + (\omega + \gamma(1 - \omega))d_i - \gamma(1 - \omega)\tilde{\mu}_j^{(i)} &> 0 \quad \text{for } d_i > 0, \\ 1 - \omega + (\omega + \gamma(1 - \omega))d_i - \gamma(1 - \omega)\tilde{\mu}_j^{(i)} &< 0 \quad \text{for } d_i < 0, \\ j &= 1, 2, \dots, n_i, \quad i = 1, 2, \dots, r, \end{aligned}$$

or equivalently,

$$\begin{aligned} (1 - \omega)\text{sign}(d_i) + (\omega + \gamma(1 - \omega))|d_i| - \gamma(1 - \omega)\text{sign}(d_i)\tilde{\mu}_j^{(i)} &> 0, \\ j &= 1, 2, \dots, n_i, \quad i = 1, 2, \dots, r, \end{aligned} \tag{8}$$

where

$$\text{sign}(d_i) = \begin{cases} 1 & \text{for } d_i > 0, \\ -1 & \text{for } d_i < 0, \end{cases} \quad i = 1, 2, \dots, r.$$

By rewriting (8) in matrix form, we obtain

$$(1 - \omega)\text{sign}(d_i)I_{n_i} + (\omega + \gamma(1 - \omega))|d_i|I_{n_i} - \gamma(1 - \omega)\text{sign}(d_i)\tilde{A}_{ii} \succ O, \quad i = 1, 2, \dots, r,$$

and with additional operations, we get

$$(1 - \omega) \operatorname{sign}(D) + (\omega + \gamma(1 - \omega))|D| - \gamma(1 - \omega) \operatorname{sign}(D)\tilde{A} \succ O.$$

Now, by specializing the parameters γ and ω , and applying Theorem 3.1, we immediately obtain the conclusions (a₁)–(a₃).

The demonstrations of (b₁)–(b₃) can be completed in an analogous way. Therefore, it is omitted here. \square

Following a similar analysis, we can immediately obtain a duality of Theorem 3.2.

Theorem 3.3. *Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian and nonsingular matrix, and $A = M_i - N_i$ ($i = 1, 2$) be two splittings such that M_i ($i = 1, 2$) are Hermitian. Then:*

(a) *When $M_1 \in \mathbb{C}^{n \times n}$ is positive definite, it holds that*

(a₁) *the matrix $G_{\text{asi}}(\omega)$ is Hermitian negative definite iff*

$$(1 - \omega) \operatorname{sign}(D) + \omega|D| \prec O;$$

(a₂) *the matrix G_{msi} is Hermitian negative definite iff $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ and*

$$\operatorname{sign}(D) + |D| - \operatorname{sign}(D) P_{M_1/M_2} A P_{M_1/M_2}^* \prec O;$$

(a₃) *the matrix $G_{\text{gsi}}(\gamma, \omega)$ is Hermitian negative definite iff $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ and*

$$(1 - \omega) \operatorname{sign}(D) + (\omega + \gamma(1 - \omega))|D| - \gamma(1 - \omega) \operatorname{sign}(D) P_{M_1/M_2} A P_{M_1/M_2}^* \prec O,$$

where $D = P_{M_1/M_2} M_2 P_{M_1/M_2}^$, and P_{M_1/M_2} is the simultaneously diagonalizing matrix with respect to the matrices M_1 and M_2 .*

(b) *When $M_2 \in \mathbb{C}^{n \times n}$ is positive definite, it holds that*

(b₁) *the matrix $G_{\text{asi}}(\omega)$ is Hermitian negative definite iff*

$$(1 - \omega)|D| + \omega \operatorname{sign}(D) \prec O;$$

(b₂) *the matrix G_{msi} is Hermitian negative definite iff $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ and*

$$|D| + \operatorname{sign}(D) - P_{M_2/M_1} A P_{M_2/M_1}^* \operatorname{sign}(D) \prec O;$$

(b₃) *the matrix $G_{\text{gsi}}(\gamma, \omega)$ is Hermitian negative definite iff $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ and*

$$(1 - \omega)|D| + (\omega + \gamma(1 - \omega)) \operatorname{sign}(D) - \gamma(1 - \omega) P_{M_2/M_1} A P_{M_2/M_1}^* \operatorname{sign}(D) \prec O,$$

where $D = P_{M_2/M_1} M_1 P_{M_2/M_1}^$, and P_{M_2/M_1} is the simultaneously diagonalizing matrix with respect to the matrices M_1 and M_2 .*

The following theorem gives general criterions for examining the convergence of the ASI, the MSI and the GSI methods.

Theorem 3.4. Let $A = M_i - N_i$ ($i = 1, 2$) be two splittings of the matrix $A \in \mathbb{C}^{n \times n}$. Then it holds that

$$H_\xi(\cdot) = I - G_\xi(\cdot)A, \quad \xi = \text{asi, msi, gsi.}$$

Moreover, for $\xi = \text{asi, msi, gsi}$, if all eigenvalues of the matrix $(G_\xi(\cdot)A)$ are reals, then $\rho(H_\xi(\cdot)) < 1$ iff $0 \prec G_\xi(\cdot)A \prec 2I$.

Proof. We only need to consider the case $\xi = \text{gsi}$, as the other two cases can be got by special choices of the parameters γ and ω .

By direct computation, we have

$$\begin{aligned} G_{\text{gsi}}(\gamma, \omega)A &= [\gamma(1 - \omega)M_2^{-1}N_2M_1^{-1} + \omega M_1^{-1} + (1 - \omega)M_2^{-1}]A \\ &= \gamma(1 - \omega)M_2^{-1}N_2(I - M_1^{-1}N_1) + \omega(I - M_1^{-1}N_1) + (1 - \omega)(I - M_2^{-1}N_2) \\ &= -\gamma(1 - \omega)M_2^{-1}N_2M_1^{-1}N_1 - (1 - \gamma)(1 - \omega)M_2^{-1}N_2 - \omega M_1^{-1}N_1 + I \\ &= -H_{\text{gsi}}(\gamma, \omega) + I. \end{aligned}$$

That is to say, $H_{\text{gsi}}(\gamma, \omega) = I - G_{\text{gsi}}(\gamma, \omega)A$.

Because all eigenvalues of the matrix $(G_{\text{gsi}}(\gamma, \omega)A)$, and hence, those of the matrix $H_{\text{gsi}}(\gamma, \omega)$, are reals, we know that $\rho(H_{\text{gsi}}(\gamma, \omega)) < 1$ iff all eigenvalues of the matrix $(G_{\text{gsi}}(\gamma, \omega)A)$ are located in the interval $(0, 2)$, or in other words, $0 \prec G_{\text{gsi}}(\gamma, \omega)A \prec 2I$. \square

We remark that the eigenvalues of the matrix $(G_\xi(\cdot)A)$ may be complex even if both $G_\xi(\cdot)$ and A are Hermitian. For example, for the matrices

$$G = \begin{bmatrix} g_1 & g_0 \\ g_0 & -g_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & a_0 \\ a_0 & 0 \end{bmatrix}, \quad a_0 > 0, \quad g_i > 0, \quad i = 0, 1, 2,$$

the two eigenvalues of the matrix (GA) are $(a_0g_0 \pm a_0\sqrt{-g_1g_2})$, which are obviously complex. However, when both of the matrices $G_\xi(\cdot)$ and A are Hermitian, and at least, one of them is definite, we immediately know that all eigenvalues of the matrix $(G_\xi(\cdot)A)$ must be reals.

To derive convergence conditions for the ASI, the MSI and the GSI methods, according to Theorem 3.4 we need to investigate spectral distribution of the matrix $W_\xi(\cdot) = 2I - G_\xi(\cdot)A$, for $\xi = \text{asi, msi, gsi}$.

Theorem 3.5. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian and nonsingular matrix, and $A = M_i - N_i$ ($i = 1, 2$) be two splittings such that M_i ($i = 1, 2$) are Hermitian. Then:

(a) When $M_1 \in \mathbb{C}^{n \times n}$ is positive definite, it holds that

(a₁) the matrix $W_{\text{asi}}(\omega)$ has real spectrum and is positive definite iff

$$2|D| - [(1 - \omega)\text{sign}(D) + \omega|D|]P_{M_1/M_2}AP_{M_1/M_2}^* \succ O;$$

(a₂) the matrix W_{msi} has real spectrum and is positive definite iff $M_1A^{-1}M_2 = M_2A^{-1}M_1$ and

$$2|D| - [\text{sign}(D) + |D| - \text{sign}(D)P_{M_1/M_2}AP_{M_1/M_2}^*]P_{M_1/M_2}AP_{M_1/M_2}^* \succ O;$$

(a₃) the matrix $W_{\text{gsi}}(\gamma, \omega)$ has real spectrum and is positive definite iff $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ and

$$2|D| - [(1 - \omega) \text{sign}(D) + (\omega + \gamma(1 - \omega))|D|$$

$$- \gamma(1 - \omega) \text{sign}(D) P_{M_1/M_2} A P_{M_1/M_2}^* P_{M_1/M_2} A P_{M_1/M_2}^* \succ O,$$

where $D = P_{M_1/M_2} M_2 P_{M_1/M_2}^*$, and P_{M_1/M_2} is the simultaneously diagonalizing matrix with respect to the matrices M_1 and M_2 .

(b) When $M_2 \in \mathbb{C}^{n \times n}$ is positive definite, it holds that

(b₁) the matrix $W_{\text{asi}}(\omega)$ has real spectrum and is positive definite iff

$$2|D| - [(1 - \omega)|D| + \omega \text{sign}(D)] P_{M_2/M_1} A P_{M_2/M_1}^* \succ O;$$

(b₂) the matrix W_{msi} has real spectrum and is positive definite iff $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ and

$$2|D| - [|D| + \text{sign}(D) - P_{M_2/M_1} A P_{M_2/M_1}^* \text{sign}(D)] P_{M_2/M_1} A P_{M_2/M_1}^* \succ O;$$

(b₃) the matrix $W_{\text{gsi}}(\gamma, \omega)$ has real spectrum and is positive definite iff $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ and

$$2|D| - [(1 - \omega)|D| + (\omega + \gamma(1 - \omega)) \text{sign}(D)$$

$$- \gamma(1 - \omega) P_{M_2/M_1} A P_{M_2/M_1}^* \text{sign}(D)] P_{M_2/M_1} A P_{M_2/M_1}^* \succ O,$$

where $D = P_{M_2/M_1} M_1 P_{M_2/M_1}^*$, and P_{M_2/M_1} is the simultaneously diagonalizing matrix with respect to the matrices M_1 and M_2 .

Proof. Because of $W_{\text{asi}}(\omega) = W_{\text{gsi}}(0, \omega)$ and $W_{\text{msi}} = W_{\text{gsi}}(1, 0)$, we only need to investigate the positive definiteness of the matrix $W_{\text{gsi}}(\gamma, \omega)$.

We first demonstrate the validity of (a). The proof of Theorem 3.2 has shown that for the simultaneously diagonalizing matrix $P = P_{M_1/M_2} \in \mathbb{C}^{n \times n}$, it holds that

$$D = \text{Diag}(d_1 I_{n_1}, d_2 I_{n_2}, \dots, d_r I_{n_r}),$$

$$\tilde{A} = \text{Diag}(\tilde{A}_{11}, \tilde{A}_{22}, \dots, \tilde{A}_{rr})$$

and

$$G_{\text{gsi}}(\gamma, \omega) = P^* [(1 - \omega) D^{-1} + (\omega + \gamma(1 - \omega)) I - \gamma(1 - \omega) D^{-1} \tilde{A}] P,$$

where n_i ($i = 1, 2, \dots, r$) are positive integers satisfying $\sum_{i=1}^r n_i = n$, and d_i ($i = 1, 2, \dots, r$) are real constants satisfying $d_i \neq d_j$ if $i \neq j$, and \tilde{A}_{ii} ($i = 1, 2, \dots, r$) are $n_i \times n_i$ ($i = 1, 2, \dots, r$) Hermitian matrices.

Through straightforward manipulations we have

$$\begin{aligned} (P^*)^{-1} W_{\text{gsi}}(\gamma, \omega) P^* &= (P^*)^{-1} (2I - G_{\text{gsi}}(\gamma, \omega) A) P^* \\ &= 2I - [(1 - \omega) D^{-1} + (\omega + \gamma(1 - \omega)) I - \gamma(1 - \omega) D^{-1} \tilde{A}] \tilde{A}. \end{aligned}$$

Let

$$\tilde{T} = 2I - [(1 - \omega) D^{-1} + (\omega + \gamma(1 - \omega)) I - \gamma(1 - \omega) D^{-1} \tilde{A}] \tilde{A}.$$

Then we observe that $W_{\text{gsi}}(\gamma, \omega)$ has real spectrum, and it is positive definite iff all diagonal blocks

$$\tilde{T}_{ii} = 2I_{n_i} - \left[\frac{1-\omega}{d_i} I_{n_i} + (\omega + \gamma(1-\omega))I_{n_i} - \frac{\gamma(1-\omega)}{d_i} \tilde{A}_{ii} \right] \tilde{A}_{ii}, \quad i = 1, 2, \dots, r,$$

of the matrix \tilde{T} are Hermitian positive definite.

If $\tilde{\mu}_j^{(i)} (j = 1, 2, \dots, n_i)$ are eigenvalues of the matrix \tilde{A}_{ii} , then it straightforwardly follows from the above derivation that $W_{\text{gsi}}(\gamma, \omega)$ is positive definite iff

$$2|d_i| - [(1-\omega)\text{sign}(d_i) + (\omega + \gamma(1-\omega))|d_i| - \gamma(1-\omega)\text{sign}(d_i)\tilde{\mu}_j^{(i)}]\tilde{\mu}_j^{(i)} > 0, \\ j = 1, 2, \dots, n_i, \quad i = 1, 2, \dots, r. \quad (9)$$

By rewriting (9) in matrix form, we obtain

$$2|d_i|I_{n_i} - [(1-\omega)\text{sign}(d_i)I_{n_i} + (\omega + \gamma(1-\omega))|d_i|I_{n_i} - \gamma(1-\omega)\text{sign}(d_i)\tilde{A}_{ii}]\tilde{A}_{ii} \succ O, \\ i = 1, 2, \dots, r,$$

and with additional operations, we get

$$2|D| - [(1-\omega)\text{sign}(D) + (\omega + \gamma(1-\omega))|D| - \gamma(1-\omega)\text{sign}(D)\tilde{A}]\tilde{A} \succ O.$$

Now, by specializing the parameters γ and ω , we immediately obtain the conclusions (a₁)–(a₃).

The demonstrations of (b₁)–(b₃) can be completed in an analogous way. Therefore, it is omitted here. \square

Now, by considering Theorems 3.2–3.5 together, we can get the following convergence properties about the ASI, the MSI and the GSI methods, respectively.

Theorem 3.6 (ASI CONVERGENCE THEOREM). *Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian and nonsingular matrix, and $A = M_i - N_i$ ($i = 1, 2$) be two splittings such that M_i ($i = 1, 2$) are Hermitian. Then:*

(a) *When $M_1 \in \mathbb{C}^{n \times n}$ is positive definite, the ASI-method is convergent iff either of the following two conditions is satisfied:*

(a₁) *$A \succ O$ and*

$$(1-\omega)\text{sign}(D) + \omega|D| \succ O, \\ 2|D| - [(1-\omega)\text{sign}(D) + \omega|D|]P_{M_1/M_2}AP_{M_1/M_2}^* \succ O;$$

(a₂) *$A \prec O$ and*

$$(1-\omega)\text{sign}(D) + \omega|D| \prec O, \\ 2|D| - [(1-\omega)\text{sign}(D) + \omega|D|]P_{M_1/M_2}AP_{M_1/M_2}^* \succ O,$$

where $D = P_{M_1/M_2}M_2P_{M_1/M_2}^*$, and P_{M_1/M_2} is the simultaneously diagonalizing matrix with respect to the matrices M_1 and M_2 .

(b) When $M_2 \in \mathbb{C}^{n \times n}$ is positive definite, the ASI-method is convergent iff either of the following two conditions is satisfied:

(b₁) $A \succ O$ and

$$(1 - \omega)|D| + \omega \operatorname{sign}(D) \succ O,$$

$$2|D| - [(1 - \omega)|D| + \omega \operatorname{sign}(D)]P_{M_2/M_1}AP_{M_2/M_1}^* \succ O;$$

(b₂) $A \prec O$ and

$$(1 - \omega)|D| + \omega \operatorname{sign}(D) \prec O,$$

$$2|D| - [(1 - \omega)|D| + \omega \operatorname{sign}(D)]P_{M_2/M_1}AP_{M_2/M_1}^* \succ O,$$

where $D = P_{M_2/M_1}M_1P_{M_2/M_1}^*$, and P_{M_2/M_1} is the simultaneously diagonalizing matrix with respect to the matrices M_1 and M_2 .

Theorem 3.7 (MSI CONVERGENCE THEOREM). Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian and nonsingular matrix, and $A = M_i - N_i$ ($i = 1, 2$) be two splittings such that M_i ($i = 1, 2$) are Hermitian and $M_1A^{-1}M_2 = M_2A^{-1}M_1$. Then:

(a) When $M_1 \in \mathbb{C}^{n \times n}$ is positive definite, the MSI-method is convergent iff either of the following two conditions is satisfied:

(a₁) $A \succ O$ and

$$\operatorname{sign}(D) + |D| - \operatorname{sign}(D)P_{M_1/M_2}AP_{M_1/M_2}^* \succ O,$$

$$2|D| - [\operatorname{sign}(D) + |D| - \operatorname{sign}(D)P_{M_1/M_2}AP_{M_1/M_2}^*]P_{M_1/M_2}AP_{M_1/M_2}^* \succ O;$$

(a₂) $A \prec O$ and

$$\operatorname{sign}(D) + |D| - \operatorname{sign}(D)P_{M_1/M_2}AP_{M_1/M_2}^* \prec O,$$

$$2|D| - [\operatorname{sign}(D) + |D| - \operatorname{sign}(D)P_{M_1/M_2}AP_{M_1/M_2}^*]P_{M_1/M_2}AP_{M_1/M_2}^* \succ O,$$

where $D = P_{M_1/M_2}M_2P_{M_1/M_2}^*$, and P_{M_1/M_2} is the simultaneously diagonalizing matrix with respect to the matrices M_1 and M_2 .

(b) When $M_2 \in \mathbb{C}^{n \times n}$ is positive definite, the MSI-method is convergent iff either of the following two conditions is satisfied:

(b₁) $A \succ O$ and

$$|D| + \operatorname{sign}(D) - P_{M_2/M_1}AP_{M_2/M_1}^* \operatorname{sign}(D) \succ O,$$

$$2|D| - [|D| + \operatorname{sign}(D) - P_{M_2/M_1}AP_{M_2/M_1}^* \operatorname{sign}(D)]P_{M_2/M_1}AP_{M_2/M_1}^* \succ O;$$

(b₂) $A \prec O$ and

$$|D| + \operatorname{sign}(D) - P_{M_2/M_1}AP_{M_2/M_1}^* \operatorname{sign}(D) \prec O,$$

$$2|D| - [|D| + \operatorname{sign}(D) - P_{M_2/M_1}AP_{M_2/M_1}^* \operatorname{sign}(D)]P_{M_2/M_1}AP_{M_2/M_1}^* \succ O,$$

where $D = P_{M_2/M_1} M_1 P_{M_2/M_1}^*$, and P_{M_2/M_1} is the simultaneously diagonalizing matrix with respect to the matrices M_1 and M_2 .

Theorem 3.8 (GSI CONVERGENCE THEOREM). *Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian and nonsingular matrix, and $A = M_i - N_i$ ($i = 1, 2$) be two splittings such that M_i ($i = 1, 2$) are Hermitian and $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$. Then:*

(a) *When $M_1 \in \mathbb{C}^{n \times n}$ is positive definite, the GSI-method is convergent iff either of the following two conditions is satisfied:*

(a₁) $A \succ O$ and

$$(1 - \omega) \operatorname{sign}(D) + (\omega + \gamma(1 - \omega))|D| - \gamma(1 - \omega) \operatorname{sign}(D) P_{M_1/M_2} A P_{M_1/M_2}^* \succ O,$$

$$2|D| - [(1 - \omega) \operatorname{sign}(D) + (\omega + \gamma(1 - \omega))|D| - \gamma(1 - \omega) \operatorname{sign}(D) P_{M_1/M_2} A P_{M_1/M_2}^*] P_{M_1/M_2} A P_{M_1/M_2}^* \succ O;$$

(a₂) $A \prec O$ and

$$(1 - \omega) \operatorname{sign}(D) + (\omega + \gamma(1 - \omega))|D| - \gamma(1 - \omega) \operatorname{sign}(D) P_{M_1/M_2} A P_{M_1/M_2}^* \prec O,$$

$$2|D| - [(1 - \omega) \operatorname{sign}(D) + (\omega + \gamma(1 - \omega))|D| - \gamma(1 - \omega) \operatorname{sign}(D) P_{M_1/M_2} A P_{M_1/M_2}^*] P_{M_1/M_2} A P_{M_1/M_2}^* \prec O,$$

where $D = P_{M_1/M_2} M_2 P_{M_1/M_2}^*$, and P_{M_1/M_2} is the simultaneously diagonalizing matrix with respect to the matrices M_1 and M_2 .

(b) *When $M_2 \in \mathbb{C}^{n \times n}$ is positive definite, the GSI-method is convergent iff either of the following two conditions is satisfied:*

(b₁) $A \succ O$ and

$$(1 - \omega)|D| + (\omega + \gamma(1 - \omega)) \operatorname{sign}(D) - \gamma(1 - \omega) P_{M_2/M_1} A P_{M_2/M_1}^* \operatorname{sign}(D) \succ O,$$

$$2|D| - [(1 - \omega)|D| + (\omega + \gamma(1 - \omega)) \operatorname{sign}(D) - \gamma(1 - \omega) P_{M_2/M_1} A P_{M_2/M_1}^* \operatorname{sign}(D)] P_{M_2/M_1} A P_{M_2/M_1}^* \succ O;$$

(b₂) $A \prec O$ and

$$(1 - \omega)|D| + (\omega + \gamma(1 - \omega)) \operatorname{sign}(D) - \gamma(1 - \omega) P_{M_2/M_1} A P_{M_2/M_1}^* \operatorname{sign}(D) \prec O,$$

$$2|D| - [(1 - \omega)|D| + (\omega + \gamma(1 - \omega)) \operatorname{sign}(D) - \gamma(1 - \omega) P_{M_2/M_1} A P_{M_2/M_1}^* \operatorname{sign}(D)] P_{M_2/M_1} A P_{M_2/M_1}^* \prec O,$$

where $D = P_{M_2/M_1} M_1 P_{M_2/M_1}^*$, and P_{M_2/M_1} is the simultaneously diagonalizing matrix with respect to the matrices M_1 and M_2 .

3.2. Commutative case

Given a matrix $B \in \mathbb{C}^{n \times n}$, let its Jordan form be $J = \text{Diag}(J_1, J_2, \dots, J_r)$, i.e., there exists a nonsingular matrix $X \in \mathbb{C}^{n \times n}$ such that $XBX^{-1} = J$, where

$$J_i = \begin{bmatrix} \lambda_1^{(i)} & 1 & & & \\ & \lambda_2^{(i)} & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_{n_i-1}^{(i)} & 1 \\ & & & & \lambda_{n_i}^{(i)} \end{bmatrix} \in \mathbb{C}^{n_i \times n_i}, \quad i = 1, 2, \dots, r$$

and n_i ($i = 1, 2, \dots, r$) are positive integers satisfying $\sum_{i=1}^r n_i = n$. Then we define the sign matrix, $\text{sign}(B)$, of the matrix B by $\text{sign}(B) = \text{Diag}(\hat{I}_1, \hat{I}_2, \dots, \hat{I}_r)$, where

$$\hat{I}_i = \text{diag}(\text{sign}(\text{Re}(\lambda_1^{(i)})), \text{sign}(\text{Re}(\lambda_2^{(i)})), \dots, \text{sign}(\text{Re}(\lambda_{n_i}^{(i)}))) \in \mathbb{R}^{n_i \times n_i}$$

and $\text{Re}(\cdot)$ denotes the real part of a complex. The following lemma is useful for our discussion in this subsection.

Lemma 3.2. *Let $B, C \in \mathbb{C}^{n \times n}$ be two Hermitian matrices. Then $BC = CB$ iff B and C have a common set of orthonormal eigenvectors.*

Theorem 3.9. *Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian and nonsingular matrix, and $A = M_i - N_i$ ($i = 1, 2$) be two splittings such that M_i ($i = 1, 2$) are Hermitian and $M_1 M_2 = M_2 M_1$. Then*

(a) *the matrix $G_{\text{asi}}(\omega)$ is Hermitian positive definite iff*

$$\text{sign}(M_1 M_2)[(1 - \omega)M_1 + \omega M_2] \succ O;$$

(b) *the matrix G_{msi} is Hermitian positive definite iff $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ and*

$$\text{sign}(M_1 M_2)[M_1 + M_2 - A] \succ O;$$

(c) *the matrix $G_{\text{gsi}}(\gamma, \omega)$ is Hermitian positive definite iff $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ and*

$$\text{sign}(M_1 M_2)[(1 - \omega)M_1 + (\omega + \gamma(1 - \omega))M_2 - \gamma(1 - \omega)A] \succ O.$$

Moreover, when the orderings in the above inequalities are reversed, we get necessary and sufficient conditions for the matrices $G_{\xi}(\cdot)$ ($\xi = \text{asi}, \text{msi}, \text{gsi}$) being Hermitian negative definite, respectively.

Proof. In light of Theorem 3.1 we know that $G_{\text{asi}}(\omega)$, G_{msi} and $G_{\text{gsi}}(\gamma, \omega)$ are all Hermitian matrices. Noticing that $G_{\text{asi}}(\omega) = G_{\text{gsi}}(0, \omega)$ and $G_{\text{msi}} = G_{\text{gsi}}(1, 0)$, in the following we only investigate the positive definiteness of the matrix $G_{\text{gsi}}(\gamma, \omega)$.

By Lemma 3.2 we know that there exists a unitary matrix $Q \in \mathbb{C}^{n \times n}$ and two diagonal matrices $D_1, D_2 \in \mathbb{R}^{n \times n}$ such that $QM_i Q^* = D_i$ ($i = 1, 2$). Let $D = D_2 D_1^{-1}$, and without loss of generality

we assume that D is of the block form

$$D = \text{Diag}(d_1 I_{n_1}, d_2 I_{n_2}, \dots, d_r I_{n_r})$$

with $d_i \neq d_j$ for $i \neq j$, $i, j = 1, 2, \dots, r$, and n_i ($i = 1, 2, \dots, r$) positive integers satisfying $\sum_{i=1}^r n_i = n$. Partition the matrix $\tilde{A} = Q A Q^* = (\tilde{A}_{ij})$ in the same fashion as the matrix D . Then we have $\tilde{A} D = D \tilde{A}$ by equivalent transformation of the condition $M_1^{-1} A M_2^{-1} = M_2^{-1} A M_1^{-1}$. It follows from this equality and the diagonal structure of the matrix D that \tilde{A} must be a block diagonal matrix of the form

$$\tilde{A} = \text{Diag}(\tilde{A}_{11}, \tilde{A}_{22}, \dots, \tilde{A}_{rr}), \quad \tilde{A}_{ii} \in \mathbb{C}^{n_i \times n_i}, \quad i = 1, 2, \dots, r.$$

Obviously, each block \tilde{A}_{ii} ($i = 1, 2, \dots, r$) is Hermitian.

Because

$$\begin{aligned} Q R_{\text{gsi}}(\gamma, \omega) Q^* &= Q[(1 - \omega)M_1 + (\omega + \gamma(1 - \omega))M_2 - \gamma(1 - \omega)A]Q^* \\ &= (1 - \omega)D_1 + (\omega + \gamma(1 - \omega))D_2 - \gamma(1 - \omega)\tilde{A}, \end{aligned}$$

we can obtain

$$\begin{aligned} G_{\text{gsi}}(\gamma, \omega) &= M_2^{-1} R_{\text{gsi}}(\gamma, \omega) M_1^{-1} \\ &= (Q^* D_2^{-1} Q) R_{\text{gsi}}(\gamma, \omega) (Q^* D_1^{-1} Q) \\ &= Q^* D_2^{-1} [(1 - \omega)D_1 + (\omega + \gamma(1 - \omega))D_2 - \gamma(1 - \omega)\tilde{A}] D_1^{-1} Q \\ &= Q^* D_2^{-1} [(1 - \omega)D_2 + (\omega + \gamma(1 - \omega))D_2 D - \gamma(1 - \omega)\tilde{A} D] D_2^{-1} Q. \end{aligned}$$

Let

$$\tilde{S} = (1 - \omega)D_2 + (\omega + \gamma(1 - \omega))D_2 D - \gamma(1 - \omega)\tilde{A} D$$

and

$$D_i = \text{Diag}(D_{11}^{(i)}, D_{22}^{(i)}, \dots, D_{rr}^{(i)}), \quad i = 1, 2,$$

where for $i = 1, 2$ and $j = 1, 2, \dots, r$, $D_{jj}^{(i)}$ is a diagonal matrix of size $n_j \times n_j$. It then follows from $D = D_2 D_1^{-1}$ that $D_{ii}^{(2)} = d_i D_{ii}^{(1)}$. Therefore, the i th diagonal block \tilde{S}_{ii} of the block diagonal matrix \tilde{S} can be written as

$$\tilde{S}_{ii} = d_i [(1 - \omega)D_{ii}^{(1)} + (\omega + \gamma(1 - \omega))D_{ii}^{(2)} - \gamma(1 - \omega)\tilde{A}_{ii}].$$

The above derivation immediately shows that $G_{\text{gsi}}(\gamma, \omega)$ is positive definite iff

$$S_{ii} = \text{sign}(d_i) [(1 - \omega)D_{ii}^{(1)} + (\omega + \gamma(1 - \omega))D_{ii}^{(2)} - \gamma(1 - \omega)\tilde{A}_{ii}], \quad i = 1, 2, \dots, r$$

are positive definite, or in other words,

$$S = \text{sign}(D) [(1 - \omega)D_1 + (\omega + \gamma(1 - \omega))D_2 - \gamma(1 - \omega)\tilde{A}] \succ O.$$

Considering that

$$\begin{aligned} Q^* S Q &= Q^* \text{sign}(D) [(1 - \omega)D_1 + (\omega + \gamma(1 - \omega))D_2 - \gamma(1 - \omega)\tilde{A}] Q \\ &= \text{sign}(M_1 M_2) [(1 - \omega)M_1 + (\omega + \gamma(1 - \omega))M_2 - \gamma(1 - \omega)A], \end{aligned}$$

we know that $G_{\text{gsi}}(\gamma, \omega)$ is positive definite if and only if

$$\text{sign}(M_1 M_2)[(1 - \omega)M_1 + (\omega + \gamma(1 - \omega))M_2 - \gamma(1 - \omega)A] \succ O.$$

Analogously, we know that $G_{\text{gsi}}(\gamma, \omega)$ is negative definite if and only if

$$\text{sign}(M_1 M_2)[(1 - \omega)M_1 + (\omega + \gamma(1 - \omega))M_2 - \gamma(1 - \omega)A] \prec O. \quad \square$$

Theorem 3.10. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian and nonsingular matrix, and $A = M_i - N_i$ ($i = 1, 2$) be two splittings such that M_i ($i = 1, 2$) are Hermitian and $M_1 M_2 = M_2 M_1$. Let $W_\xi(\cdot) = 2I - G_\xi(\cdot)A$, $\xi = \text{asi}, \text{msi}$ and gsi . Then:

(a) the matrix $W_{\text{asi}}(\omega)$ has real spectrum iff $AM_2 = M_2A$, and it is positive definite iff

$$2I - [(1 - \omega)M_1 + \omega M_2]M_2^{-1}AM_1^{-1} \succ O;$$

(b) the matrix W_{msi} has real spectrum iff $AM_2 = M_2A$, and it is positive definite iff

$$2I - [M_1 + M_2 - A]M_2^{-1}AM_1^{-1} \succ O;$$

(c) the matrix $W_{\text{gsi}}(\gamma, \omega)$ has real spectrum iff $AM_2 = M_2A$, and it is positive definite iff

$$2I - [(1 - \omega)M_1 + (\omega + \gamma(1 - \omega))M_2 - \gamma(1 - \omega)A]M_2^{-1}AM_1^{-1} \succ O.$$

Proof. Noticing that $W_{\text{asi}}(\omega) = W_{\text{gsi}}(0, \omega)$ and $W_{\text{msi}} = W_{\text{gsi}}(1, 0)$, we only need to investigate the positive definiteness of the matrix $W_{\text{gsi}}(\gamma, \omega)$ in the following. From the proof of Theorem 3.9 we know that there exists an orthonormal matrix $Q \in \mathbb{C}^{n \times n}$ and two diagonal matrices $D_1, D_2 \in \mathbb{R}^{n \times n}$ such that $QM_iQ^* = D_i$ ($i = 1, 2$) and $\tilde{A}D = D\tilde{A}$, with $D = D_2D_1^{-1}$, $\tilde{A} = QAQ^*$, and

$$D = \text{Diag}(d_1I_{n_1}, d_2I_{n_2}, \dots, d_rI_{n_r}),$$

$$\tilde{A} = \text{Diag}(\tilde{A}_{11}, \tilde{A}_{22}, \dots, \tilde{A}_{rr}),$$

where $d_i \neq d_j$ for $i \neq j$, $i, j = 1, 2, \dots, r$; $\tilde{A}_{ii} \in \mathbb{C}^{n_i \times n_i}$ for $i = 1, 2, \dots, r$; and n_i ($i = 1, 2, \dots, r$) being positive integers satisfying $\sum_{i=1}^r n_i = n$. In addition, it holds that

$$G_{\text{gsi}}(\gamma, \omega) = Q^*D_2^{-1}[(1 - \omega)D_2 + (\omega + \gamma(1 - \omega))D_2D - \gamma(1 - \omega)\tilde{A}D]D_2^{-1}Q$$

and

$$D_i = \text{Diag}(D_{11}^{(i)}, D_{22}^{(i)}, \dots, D_{rr}^{(i)}), \quad i = 1, 2,$$

where for $i = 1, 2$ and $j = 1, 2, \dots, r$, $D_{jj}^{(i)}$ is a diagonal matrix of size $n_j \times n_j$.

Because

$$\begin{aligned} QW_{\text{gsi}}(\gamma, \omega)Q^* &= Q(2I - G_{\text{gsi}}(\gamma, \omega)A)Q^* \\ &= 2I - D_2^{-1}[(1 - \omega)D_2 + (\omega + \gamma(1 - \omega))D_2D - \gamma(1 - \omega)\tilde{A}D]D_2^{-1}\tilde{A} \\ &= 2I - [(1 - \omega)I + (\omega + \gamma(1 - \omega))D - \gamma(1 - \omega)DD_2^{-1}\tilde{A}]D_2^{-1}\tilde{A}, \end{aligned}$$

we see that the matrix $W_{\text{gsi}}(\gamma, \omega)$ has real spectrum iff each of the submatrices $D_{ii}^{(2)-1}\tilde{A}_{ii}$ ($i = 1, 2, \dots, r$) has real spectrum, and it is positive definite iff all diagonal blocks

$$\tilde{T}_{ii} = 2I_{n_i}[(1 - \omega)I_{n_i} + (\omega + \gamma(1 - \omega))d_iI_{n_i} - \gamma(1 - \omega)d_iD_{ii}^{(2)-1}\tilde{A}_{ii}]D_{ii}^{(2)-1}\tilde{A}_{ii}, \quad i = 1, 2, \dots, r$$

are positive definite. Furthermore, we know from Lemma 3.2 that $W_{\text{gsi}}(\gamma, \omega)$ has real spectrum iff for $i = 1, 2, \dots, r$, $D_{ii}^{(2)}$ and \tilde{A}_{ii} has common set of orthonormal eigenvectors, which is equivalent to $AM_2 = M_2A$; and that it is positive definite iff

$$2I - [(1 - \omega)I + (\omega + \gamma(1 - \omega))M_2M_1^{-1} - \gamma(1 - \omega)M_1^{-1}A]M_2^{-1}A \succ O,$$

or equivalently,

$$2I - [(1 - \omega)M_1 + (\omega + \gamma(1 - \omega))M_2 - \gamma(1 - \omega)A]M_2^{-1}AM_1^{-1} \succ O. \quad \square$$

A technical combination of Theorems 3.9 and 3.10 straightforwardly results in the following convergence theories for the ASI, the MSI, and the GSI methods, respectively.

Theorem 3.11. *Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian and nonsingular matrix, and $A = M_i - N_i$ ($i = 1, 2$) be two splittings such that M_i ($i = 1, 2$) are Hermitian, $M_1M_2 = M_2M_1$ and $AM_2 = M_2A$. Then:*

(a) *If $A \succ O$, then*

(a₁) *the ASI-method is convergent iff*

$$\begin{aligned} \text{sign}(M_1M_2)[(1 - \omega)M_1 + \omega M_2] &\succ O, \\ 2I - [(1 - \omega)M_1 + \omega M_2]M_2^{-1}AM_1^{-1} &\succ O; \end{aligned}$$

(a₂) *the MSI-method is convergent iff*

$$\begin{aligned} \text{sign}(M_1M_2)[M_1 + M_2 - A] &\succ O, \\ 2I - [M_1 + M_2 - A]M_2^{-1}AM_1^{-1} &\succ O; \end{aligned}$$

(a₃) *the GSI-method is convergent iff*

$$\begin{aligned} \text{sign}(M_1M_2)[(1 - \omega)M_1 + (\omega + \gamma(1 - \omega))M_2 - \gamma(1 - \omega)A] &\succ O, \\ 2I - [(1 - \omega)M_1 + (\omega + \gamma(1 - \omega))M_2 - \gamma(1 - \omega)A]M_2^{-1}AM_1^{-1} &\succ O. \end{aligned}$$

(b) *If $A \prec O$, then*

(b₁) *the ASI-method is convergent iff*

$$\begin{aligned} \text{sign}(M_1M_2)[(1 - \omega)M_1 + \omega M_2] &\prec O, \\ 2I - [(1 - \omega)M_1 + \omega M_2]M_2^{-1}AM_1^{-1} &\succ O; \end{aligned}$$

(b₂) *the MSI-method is convergent iff*

$$\begin{aligned} \text{sign}(M_1M_2)[M_1 + M_2 - A] &\prec O, \\ 2I - [M_1 + M_2 - A]M_2^{-1}AM_1^{-1} &\succ O; \end{aligned}$$

(b₃) the GSI-method is convergent iff

$$\begin{aligned} \text{sign}(M_1 M_2)[(1 - \omega)M_1 + (\omega + \gamma(1 - \omega))M_2 - \gamma(1 - \omega)A] &\prec O, \\ 2I - [(1 - \omega)M_1 + (\omega + \gamma(1 - \omega))M_2 - \gamma(1 - \omega)A]M_2^{-1}AM_1^{-1} &\succ O. \end{aligned}$$

4. Convergence theories for non-Hermitian matrix

In this section, we investigate convergence of iterations (2.1)–(2.3) under assumptions such as nonsingularity and diagonalizability.

Theorem 4.1. Let $A \in \mathbb{C}^{n \times n}$ be a nonsingular matrix, and $A = M_i - N_i$ ($i = 1, 2$) be two splittings such that $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$. Represent by $\rho_1 = \rho(M_1^{-1} N_1)$ and $\rho_2 = \rho(M_2^{-1} N_2)$. Then:

(a) the ASI-method is convergent, with the convergent factor being at least $\sigma_{\text{asi}}(\omega) = |\omega|\rho_1 + |1 - \omega|\rho_2$, if either of the following conditions is satisfied:

- (a₁) $1 < \omega < \frac{1 + \rho_2}{\rho_1 + \rho_2}$;
- (a₂) $\rho_2 < 1$, and $\omega < \frac{\rho_2 - 1}{\rho_2 + 1}$;
- (a₃) $\rho_2 < \min\{1, \rho_1\}$, and $0 \leq \omega < \min\left\{1, \frac{1 - \rho_2}{\rho_1 - \rho_2}\right\}$;
- (a₄) $\rho_1 \leq 1 < \rho_2$, and $\frac{\rho_2 - 1}{\rho_2 - \rho_1} < \omega \leq 1$.

(b) the MSI-method is convergent, with the convergent factor being at least $\sigma_{\text{msi}} = \rho_1 \rho_2$, if $\rho_1 \rho_2 < 1$.

(c) the GSI-method is convergent, with the convergent factor being at least $\sigma_{\text{gsi}}(\gamma, \omega) = \sigma_{\text{asi}}(\omega) + [|\gamma|\rho_1 + |1 - \gamma| - 1]|1 - \omega|\rho_2$, if either of the following conditions is satisfied:

- (c₁) $\sigma_{\text{asi}}(\omega) < 1$, $\rho_1 < 1$, and $0 \leq \gamma \leq 1$;
- (c₂) $\sigma_{\text{asi}}(\omega) < 1$, $\rho_1 > 1$, and $0 \leq \gamma \leq \min\left\{1, \frac{1 - \sigma_{\text{asi}}(\omega)}{(\rho_1 - 1)\rho_2|1 - \omega|}\right\}$;
- (c₃) $\sigma_{\text{asi}}(\omega) < 1 - (1 + \rho_1)\rho_2|1 - \omega|$, and $1 < \gamma < \frac{2 + \eta(\omega)}{1 + \rho_1}$, with $\eta(\omega) = \frac{1 - \sigma_{\text{asi}}(\omega)}{|1 - \omega|\rho_2}$;
- (c₄) $\sigma_{\text{asi}}(\omega) < 1$, and $-\frac{\eta(\omega)}{1 + \rho_1} < \gamma < 0$;
- (c₅) $\frac{\eta(\omega)}{\rho_1 - 1} < \gamma < \frac{2 + \eta(\omega)}{\rho_1 + 1}$.

Proof. We only need to demonstrate (c), because (a) and (b) are its straightforward corollaries when the parameters γ and ω are suitably specified.

Because $M_1 A^{-1} M_2 = M_2 A^{-1} M_1$ is equivalent to that the two matrices $(M_1 A^{-1})$ and $(M_2 A^{-1})$ are commutative, according to Lemma 3.2 we know that $(M_1 A^{-1})$ and $(M_2 A^{-1})$ have a common set of orthonormal eigenvectors. That is to say, there exists a unitary matrix $Q \in \mathbb{C}^{n \times n}$ and two diagonal matrices $A_i = \text{diag}(\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_n^{(i)}) \in \mathbb{C}^{n \times n}$, $i = 1, 2$, such that $Q M_i^{-1} A Q^* = A_i$, $i = 1, 2$. Noticing that

$$\begin{aligned} H_{\text{gsi}}(\gamma, \omega) &= \gamma(1 - \omega) M_2^{-1} N_2 M_1^{-1} N_1 + \omega M_1^{-1} N_1 + (1 - \gamma)(1 - \omega) M_2^{-1} N_2 \\ &= \gamma(1 - \omega) M_2^{-1} (M_2 - A) M_1^{-1} (M_1 - A) \\ &\quad + \omega M_1^{-1} (M_1 - A) + (1 - \gamma)(1 - \omega) M_2^{-1} (M_2 - A) \\ &= Q^* [\gamma(1 - \omega)(I - A_2)(I - A_1) + \omega(I - A_1) + (1 - \gamma)(1 - \omega)(I - A_2)] Q, \end{aligned}$$

we have

$$\begin{aligned} \rho(H_{\text{gsi}}(\gamma, \omega)) &\leq \max_{1 \leq i \leq n} |\gamma(1 - \omega)(1 - \lambda_i^{(1)})(1 - \lambda_i^{(2)}) \\ &\quad + \omega(1 - \lambda_i^{(1)}) + (1 - \gamma)(1 - \omega)(1 - \lambda_i^{(2)})| \\ &\leq |\gamma| \|1 - \omega\| \max_{1 \leq i \leq n} |1 - \lambda_i^{(1)}| \max_{1 \leq i \leq n} |1 - \lambda_i^{(2)}| \\ &\quad + |\omega| \max_{1 \leq i \leq n} |1 - \lambda_i^{(1)}| + |1 - \gamma| \|1 - \omega\| \max_{1 \leq i \leq n} |1 - \lambda_i^{(2)}| \\ &= |\gamma| \|1 - \omega\| \rho(I - M_1^{-1} A) \rho(I - M_2^{-1} A) \\ &\quad + |\omega| \rho(I - M_1^{-1} A) + |1 - \gamma| \|1 - \omega\| \rho(I - M_2^{-1} A) \\ &= |\gamma| \|1 - \omega\| \rho(M_1^{-1} N_1) \rho(M_2^{-1} N_2) \\ &\quad + |\omega| \rho(M_1^{-1} N_1) + |1 - \gamma| \|1 - \omega\| \rho(M_2^{-1} N_2) \\ &= |\gamma| \|1 - \omega\| \rho_1 \rho_2 + |\omega| \rho_1 + |1 - \gamma| \|1 - \omega\| \rho_2 \\ &= \sigma_{\text{gsi}}(\gamma, \omega). \end{aligned}$$

Through straightforward computations we can easily obtain that $\sigma_{\text{gsi}}(\gamma, \omega) < 1$ under either of the conditions (c₁)–(c₅). \square

5. Concluding remarks

Criteria for examining the convergence of additive and multiplicative splitting iteration methods are established for both Hermitian and non-Hermitian systems of linear equations from the viewpoint of matrix analysis. The new results show that these iterations can converge for a wider range of matrix classes other than the Hermitian positive definite one. Moreover, they present new theories for studying preconditioning properties of these class of splitting iterations. This is quite different from the existing convergence theories, which are based on the background of numerical partial

differential equations and are restricted to the case of self-adjoint and elliptic problems. Therefore, this work presents novel general convergence properties and analysis means for the additive and the multiplicative splitting iteration methods.

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