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Optimal Convergence for Overlapping and Non-Overlapping Schwarz Waveform Relaxation

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INTRODUCTION

We are interested in solving time dependent problems of parabolic and hyperbolic type using domain decomposition techniques. Contrary to the classical approach where one discretizes time to obtain a sequence of steady problems to which the domain decomposition algorithms are applied (see [Cai91, Meu91, Cai94] for parabolic and [BGT97, WCK98] for hyperbolic problems), we formulate algorithms directly for the original problem without discretization. We decompose the spatial domain into subdomains and solve iteratively time dependent problems on subdomains, exchanging information at the boundary. Thus the algorithm is defined as in the classical Schwarz case, but like in waveform relaxation, time dependent subproblems are solved, which explains the name of these methods. In [Gan96, Gan97b, GS98] and [GK97] the overlapping version of such an algorithm has been studied for different types of parabolic problems. We investigate the algorithm applied to two new problems in this paper, the wave equation

$$\mathcal{L}_1(u) := u_{tt} - c^2 u_{xx} = f(x,t), \quad c > 0$$

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and the linear convection reaction diffusion equation

$$\mathcal{L}_2(u) := u_t - \nu u_{xx} - a u_x - bu = f(x,t), \quad \nu > 0,\ a,b \in \mathbb{R}$$

on $\mathbb{R} \times [0,T]$ with appropriate initial conditions. Without loss of generality we assume for the convection reaction diffusion equation $a > 0$. We first analyze the convergence behavior of the overlapping Schwarz waveform relaxation algorithm applied to the above problems. We then show that the Dirichlet conditions at the artificial interfaces inhibit the information exchange between subdomains and therefore slow down the convergence of the algorithms. Using ideas introduced in [Hal86] and [NRdS95] we derive optimal transmission conditions for the convergence of the algorithms. These transmission conditions coincide with the absorbing boundary conditions studied in great detail to truncate computational domains in [EM77] for hyperbolic problems and in [Hal86] for convection diffusion problems. They lead to non-overlapping Schwarz waveform relaxation algorithms which converge in a finite number of steps, identical to the number of subdomains. In general however the exact absorbing boundary conditions are not available or expensive to compute. Similar to the approach for stationary problems in [NR95] and [Jap96] and for control problems in [Ben97] we approximate the exact absorbing boundary conditions locally. We optimize the convergence rate including an overlap in the optimization if desired. Numerical experiments show that the convergence rates are improved by orders of magnitudes.

OVERLAPPING SCHWARZ WAVEFORM RELAXATION

We decompose the spatial domain $\mathbb{R}$ into two overlapping subdomains $(-\infty, L]$ and $[0,\infty)$. By linearity it suffices to analyze the overlapping Schwarz waveform relaxation algorithm for the homogeneous problems with zero initial conditions,

$$\begin{array}{rcll} \mathcal{L}_i(v^{k+1}) & = & 0, & x \in (-\infty, L) \\ v^{k+1} & = & w^k, & x = L \\ \mathcal{L}_i(w^{k+1}) & = & 0, & x \in (0,\infty) \\ w^{k+1} & = & v^k, & x = 0 \end{array} \tag{1}$$

for $i = 1,2$ and prove convergence to zero. Existence and uniqueness of the iterates is easily ensured by classical methods.

Theorem 1 *For the wave equation, $i = 1$ in (1), the algorithm converges in a finite number of iterations, $v^{2k+1} \equiv w^{2k+1} \equiv 0$ as soon as*

$$k \geq \frac{Tc}{2L}.$$

Proof Applying the Laplace transform with parameter $s \in \mathbb{C}$, $\Re(s) > 0$, we find the transformed solutions to be

$$\hat{v}^{k+1}(x,s) = \hat{w}^k(L,s)e^{s(x-L)/c} \tag{2}$$

$$\hat{w}^{k+1}(x,s) = \hat{v}^k(0,s)e^{-sx/c} \tag{3}$$

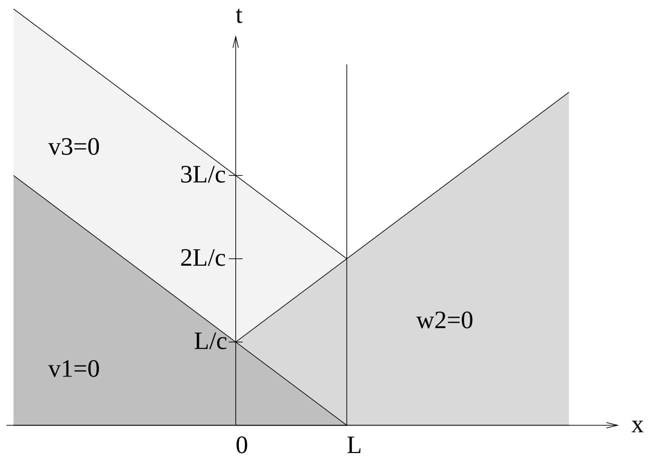

Figure 1 Regions where the iterates of the overlapping Schwarz waveform relaxation algorithm for the wave equation vanish due to the finite speed of propagation.

Evaluating (3) at iteration step k for $x = L$ and inserting it into (2), evaluated at $x = 0$, we find

$$\hat{v}^{k+1}(0, s) = e^{-2sL/c}\hat{v}^{k-1}(0, s).$$

Defining the convergence rate $\rho := e^{-2sL/c}$ we find by induction

$$\hat{v}^{2k}(0, s) = \rho^k \hat{v}^0(0, s). \tag{4}$$

A similar result holds for $\hat{w}^{2k}(L, s)$ and thus the iteration converges for all frequencies with $\Re(s) > 0$. To obtain the convergence result for bounded time intervals, we back-transform (4). Since

$$e^{-2kLs/c} = \int_0^\infty e^{-st}\delta(t - 2kL/c)dt$$

we find on using the convolution theorem of the Laplace transform

$$v^{2k}(0, t) = \int_0^t \delta(t - \tau - 2kL/c)v^0(0, \tau)d\tau = v^0(0, t - 2kL/c).$$

A similar result holds for $w^{2k}(L, t)$ and hence if $k > \frac{Tc}{2L}$ the transmission conditions imposed are identically zero and thus the next step leads to convergence. ■

Figure 1 shows intuitively why the overlapping Schwarz waveform relaxation algorithm for the wave equation converges in a finite number of steps. It is due to the finite speed of propagation: the iterates are identically zero before the arrival of the first disturbance from the artificial interfaces.

Theorem 2 *For the convection reaction diffusion equation, $i = 2$ in (1), the asymptotic convergence rate is superlinear and governed by the diffusion parameter ν,*

$$\frac{||v^{2k}(0, \cdot) + w^{2k}(L, \cdot)||_T}{||v^0(0, \cdot) + w^0(L, \cdot)||_T} \leq C erfc(\frac{kL}{\sqrt{\nu T}}).$$

with the constant $C = \max(1, e^{(b-a^2/4\nu)T})$.

Proof We take again a Laplace transform in time with parameter $s \in \mathbb{C}$, $\Re(s) > b$ and find the transformed solutions to be

$$\hat{v}^{k+1}(x,s) = \hat{w}^k(L,s)e^{\frac{-a+\sqrt{a^2+4\nu(s-b)}}{2\nu}(x-L)} \tag{5}$$
$$\hat{w}^{k+1}(x,s) = \hat{v}^k(0,s)e^{\frac{-a-\sqrt{a^2+4\nu(s-b)}}{2\nu}x}. \tag{6}$$

Evaluating (6) at $x = L$ for iteration index k, inserting it into (5) and evaluating at $x = 0$ we find

$$\hat{v}^{k+1}(0,s) = e^{-2\sqrt{\frac{a^2}{4\nu^2}+\frac{s-b}{\nu}}L}\hat{v}^{k-1}(0,s).$$

Defining the convergence rate $\rho := e^{-2\sqrt{\frac{a^2}{4\nu^2}+\frac{s-b}{\nu}}L}$ we find by induction

$$\hat{v}^{2k}(0,s) = \rho^k \hat{v}^0(0,s). \tag{7}$$

A similar result holds for $\hat{w}^{2k}(L,s)$ and thus the additive Schwarz method converges for all frequencies $\Re(s) > b$. To obtain the desired convergence result for bounded time, we back-transform (7) on noting that [AS64]

$$e^{-x\sqrt{s+q}} = \int_0^\infty e^{-st}K(x,t)e^{-qt}dt$$

where the kernel K is given by

$$K(x,t) = \frac{x}{2\sqrt{\pi}t^{3/2}}e^{-\frac{x^2}{4t}}.$$

We find on using the convolution theorem for the Laplace transform

$$v^{2k}(0,t) = \int_0^t K_x(\frac{2kL}{\sqrt{\nu}}, t-\tau)e^{-(a^2/4\nu-b)(t-\tau)}v^0(0,\tau)d\tau.$$

Taking the supremum in time on a bounded time interval $0 < t < T$,

$$||v^{2k}(0,\cdot)||_T := \sup_{0<t<T}|v^{2k}(0,t)|$$

and estimating the exponential with $\max(1, e^{(b-a^2/4\nu)T})$ we get

$$||v^{2k}(0,\cdot)||_T \le \max(1, e^{(b-a^2/4\nu)T})\int_0^T K_x(\frac{2kL}{\sqrt{\nu}}, T-\tau)d\tau||v^0(0,\cdot)||_T.$$

Now applying the variable transform $y = kL/\sqrt{\nu(t-\tau)}$ in the integration leads to

$$||v^{2k}(0,\cdot)||_T \le \max(1, e^{(b-a^2/4\nu)T})\mathrm{erfc}(\frac{kL}{\sqrt{\nu T}})||v^0(0,\cdot)||_T.$$

By a similar argument for the second subdomain, the result follows. ∎

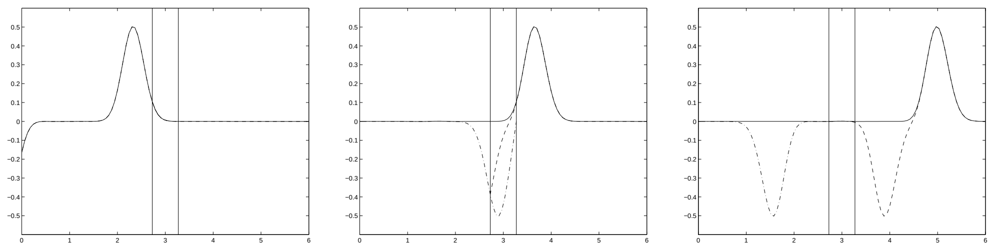

Figure 2 Snapshots of $v^1(x,t)$ (dash-dot) and $w^2(x,t)$ (dashed) in the wave equation case, together with the exact solution (solid) showing the erroneous reflections caused by the Dirichlet transmission conditions.

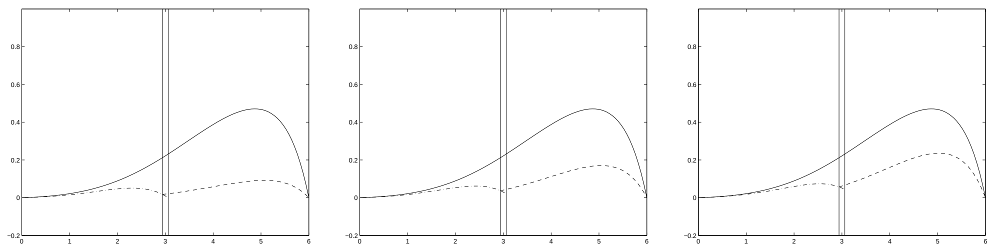

Figure 3 Iterates $v^k(x,T)$ (dash-dot) and $w^{k+1}(x,T)$ (dashed) at the end of the time interval for $k = 1, 3, 5$ in the convection reaction diffusion case together with the exact solution (solid) showing how the Dirichlet transmission conditions inhibit the information transport.

Both results differ from the classical linear convergence of the overlapping Schwarz method for elliptic problems. The convergence in a finite number of steps in the wave equation case and the superlinear convergence in the convection reaction diffusion case depend both on the time interval under consideration. For the wave equation it is evident that the convergence rate does not depend on the number of subdomains, whereas for the convection reaction diffusion equation one can show that the convergence rate depends only lower order on the number of subdomains [Gan97a]. This shows that coarse grid preconditioners are not necessary in this case.

In both cases however the Dirichlet transmission conditions at the interfaces are responsible for slow convergence, as one can see in Figure 2 where wrong reflected waves are created and in Figure 3 where the convection and diffusion of the information across the interface is inhibited. We are thus looking for a remedy of this by investigating the transmission conditions in the next section.

NON-OVERLAPPING SCHWARZ WAVEFORM RELAXATION

We are using the same algorithm as before, but with different transmission conditions, namely

$$\begin{array}{rcll} \mathcal{L}_i(v^{k+1}) & = & 0, & x \in (-\infty, L) \\ v_x^{k+1} + \Lambda_v(v^{k+1}) & = & w_x^k + \Lambda_v(w^k), & x = L \\ \mathcal{L}_i(w^{k+1}) & = & 0, & x \in (0, \infty) \\ w_x^{k+1} + \Lambda_w(w^{k+1}) & = & v_x^k + \Lambda_w(v^k), & x = 0, \end{array} \tag{8}$$

where Λ_v and Λ_w are linear operators acting along the boundary in time. Well-posedness is ensured in the case $i = 1$ by [EM77] and in the case $i = 2$ by [Hal86].

Theorem 3 *For the wave equation, $i = 1$ in (8), the algorithm converges in two iterations, independently of the size of the overlap, if*

$$\Lambda_v = \frac{1}{c}\partial_t, \quad \Lambda_w = -\frac{1}{c}\partial_t.$$

Proof Taking a Laplace transform of the above equations for $i = 1$ with parameter s, $\Re(s) > 0$, we find

$$\begin{array}{rcl} s^2\hat{v}^{k+1} & = & c^2\hat{v}_{xx}^{k+1}, \\ \hat{v}_x^{k+1}(L,s) + \lambda_v(s)(\hat{v}^{k+1}(L,s)) & = & \hat{w}_x^k(L,s) + \lambda_v(s)(\hat{w}^k(L,s)), \\ s^2\hat{w}^{k+1} & = & c^2\hat{w}_{xx}^{k+1}, \\ \hat{w}_x^{k+1}(0,s) + \lambda_w(s)(\hat{w}^{k+1}(0,s)) & = & \hat{v}_x^k(0,s) + \lambda_w(s)(\hat{v}^k(0,s)). \end{array}$$

Solving for $\hat{w}$ at iteration k for $x = L$ and inserting into the solution for $\hat{v}^{k+1}$ one obtains after evaluating at $x = 0$

$$\hat{v}^{k+1}(0,s) = \frac{-\frac{s}{c} + \lambda_v}{\frac{s}{c} + \lambda_v} \cdot \frac{\frac{s}{c} + \lambda_w}{-\frac{s}{c} + \lambda_w} e^{-2\frac{s}{c}L}\hat{v}^{k-1}(0,s).$$

A similar result holds for $\hat{w}^{k+1}(L,s)$. Thus choosing $\lambda_v = \frac{s}{c}$ and $\lambda_w = -\frac{s}{c}$ the iteration converges in two steps, $v^2 \equiv w^2 \equiv 0$, and the factor $e^{-2\frac{s}{c}L}$ stemming from the overlap becomes irrelevant. Back-transforming this choice, the result follows. ■

Theorem 4 *For the convection reaction diffusion equation, $i = 2$ in (8), the above algorithm converges in two iterations independently of the size of the overlap, if the operators Λ_v and Λ_w have the corresponding symbols*

$$\lambda_v = \frac{a}{2\nu} + \frac{\sqrt{a^2 + 4\nu(s-b)}}{2\nu}, \quad \lambda_w = \frac{a}{2\nu} - \frac{\sqrt{a^2 + 4\nu(s-b)}}{2\nu}.$$

Proof Using the Laplace transform as before with parameter $s \in \mathbb{C}$, $\Re(s) > b$, we find

$$\begin{array}{rcl} s\hat{v}^{k+1} & = & \nu\hat{v}_{xx}^{k+1} + a\hat{v}_x^{k+1} + b\hat{v}^{k+1}, \\ \hat{v}_x^{k+1}(L,s) + \lambda_v(s)(\hat{v}^{k+1}(L,s)) & = & \hat{w}_x^k(L,s) + \lambda_v(s)(\hat{w}^k(L,s)), \\ s\hat{w}^{k+1} & = & \nu\hat{w}_{xx}^{k+1} + a\hat{w}_x^{k+1} + b\hat{w}^{k+1}, \\ \hat{w}_x^{k+1}(0,s) + \lambda_w(s)(\hat{w}^{k+1}(0,s)) & = & \hat{v}_x^k(0,s) + \lambda_w(s)(\hat{v}^k(0,s)) \end{array}$$

Solving for $\hat{w}$ at iteration k for $x = L$ and inserting into the solution $\hat{v}^{k+1}$ one obtains after evaluating at $x = 0$

$$\hat{v}^{k+1}(0,s) = \frac{\lambda_2 + \lambda_v}{\lambda_1 + \lambda_v} \cdot \frac{\lambda_1 + \lambda_w}{\lambda_2 + \lambda_w} e^{(\lambda_2 - \lambda_1)L} \hat{v}^{k-1}(0,s)$$

where λ_{12} denote the characteristic roots,

$$\lambda_{12} = -\frac{a}{2\nu} \pm \frac{\sqrt{a^2 + 4\nu(s-b)}}{2\nu}.$$

A similar result holds for $\hat{w}^{k+1}(L,s)$. Thus by choosing $\lambda_v = -\lambda_2$ and $\lambda_w = -\lambda_1$ the algorithm converges in two steps, $v^2 \equiv w^2 \equiv 0$, independently of the overlap. ■

Note that in this case however, the symbols lead to nonlocal operators in time, which are more expensive to implement in an algorithm than the local ones found for the wave equation. It is therefore of interest to approximate the nonlocal operators by local ones, whose symbols are polynomials. We propose here four different approximations: Taylor approximations of zeroth and first order

$$\Lambda_{vw} = \frac{a \pm \sqrt{a^2 - 4\nu b}}{2\nu}, \quad \Lambda_{vw} = \frac{a \pm \sqrt{a^2 - 4\nu b}}{2\nu} \pm \frac{1}{\sqrt{a^2 - 4\nu b}} \partial t$$

and optimized constant and first order polynomials

$$\Lambda_{vw} = \frac{a \pm p}{2\nu} \qquad , \quad \Lambda_{vw} = \frac{a \pm \sqrt{a^2 - 4\nu b}}{2\nu} \pm \frac{q}{2\nu} \partial t$$

where we optimize the convergence rate using p in

$$\min_{p>0} \left(\max_{\Re(s)>0} \left| \frac{(p - \sqrt{a^2 + 4\nu(s-b)})^2}{(p + \sqrt{a^2 + 4\nu(s-b)})^2} e^{-\frac{L}{\nu}\sqrt{a^2+4\nu(s-b)}} \right| \right)$$

and q in

$$\min_{q>0} \left(\max_{\Re(s)>0} \left| \frac{(qs + \sqrt{a^2 - 4\nu b} - \sqrt{a^2 + 4\nu(s-b)})^2}{(qs + \sqrt{a^2 - 4\nu b} + \sqrt{a^2 + 4\nu(s-b)})^2} e^{-\frac{L}{\nu}\sqrt{a^2+4\nu(s-b)}} \right| \right).$$

The optimization is performed numerically to obtain the convergence results in the next section.

NUMERICAL RESULTS

We show numerical results for the parabolic problem to test the effectiveness of the approximately absorbing transmission conditions. We consider the model problem

$$u_t = u_{xx} - 2u_x + \frac{1}{2}u \quad 0 < x < 6, \ 0 < t < T = 2$$

with given data

$$u(0,t) = 0, \quad u(6,t) = 0, \quad u(x,0) = e^{-3(\frac{3}{2}-x)^2}$$

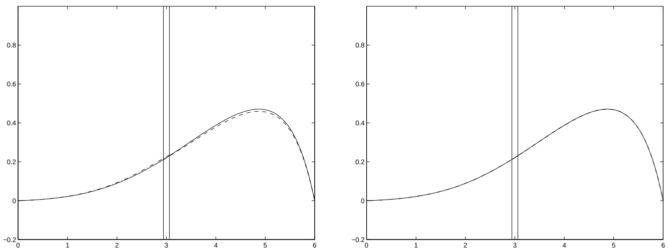

Figure 4 Iterates $v^k(x,T)$ (dash-dot) and $w^{k+1}(x,T)$ (dashed) at the end of the time interval for $k=1,3$ in the convection reaction diffusion case together with the exact solution (solid) using optimized first order transmission conditions.

and to compare the performance with the Dirichlet case, we employ an overlap of 2%. Figure 4 shows the iterates $v^1(x,T)$ and $w^2(x,T)$ on the left and $v^3(x,T)$ and $w^4(x,T)$ on the right at the end of the time interval with the optimized first order transmission conditions and should be compared with the results in Figure 3. Clearly the information is now convected and diffused across the artificial interface with the new transmission conditions. Figure 5 shows the performance of the same algorithm when the transmission conditions are changed from Dirichlet to the new approximately absorbing transmission conditions.

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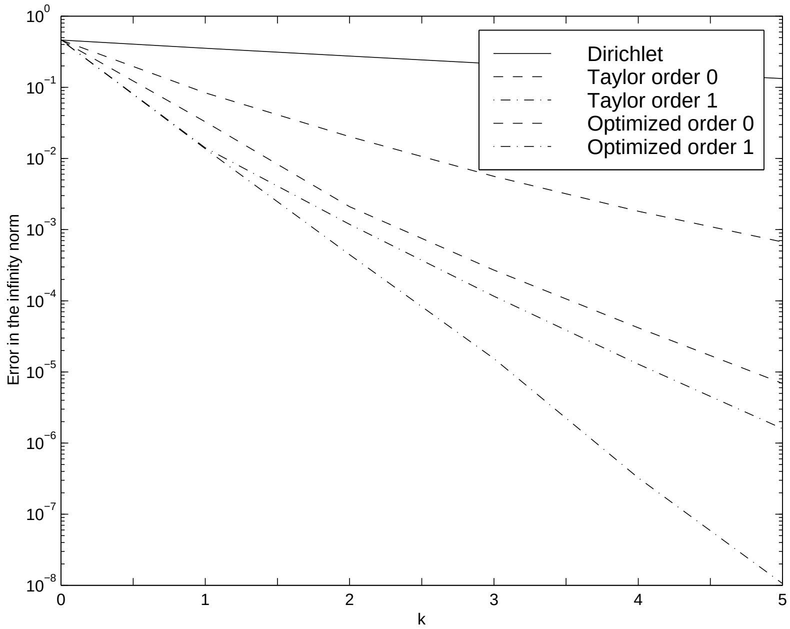

Figure 5 Convergence rates of the classical Schwarz with Dirichlet transmission conditions compared to the same algorithm with the new transmission conditions.

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