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On the convergence of nonstationary multisplitting two-stage iteration methods for hermitian positive definite linear systems

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Abstract

Convergence properties of the nonstationary multisplitting two-stage iteration methods for solving large sparse system of linear equations are further studied when the coefficient matrices are hermitian positive definite matrices.

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Keywords: Nonstationary two-stage iteration; Matrix multisplitting; Convergence theory; Hermitian positive definite matrix

1. Introduction

The matrix multisplitting iteration is an efficient method for obtaining an approximate solution of the large sparse system of linear equations

$$Ax = b, \quad A \in \mathbb{C}^{n\times n} \text{ nonsingular}, \quad x, b \in \mathbb{C}^n \tag{1}$$

on a multiprocessor system. Here, $\mathbb{C}^n$ denotes the n-dimensional complex vector space and $\mathbb{C}^{n\times n}$ the $n \times n$ complex matrix space. By applying the inner/outer iteration technique to this multisplitting

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method, Szyld and Jones [17] proposed a multisplitting two-stage iteration method. Then, this multisplitting two-stage technique was further extended to asynchronous situations (see [10,13]) and to system of mildly nonlinear equations (see [1,4–7]). The asymptotic convergence properties for H-matrix class and the monotone convergence properties for monotone matrix class were established for these methods in a variety of literature. Besides the above-mentioned works, we refer the readers to [9,3,8,18], and references therein.

In [2], the convergence theory and the comparison theorem of the sequential two-stage iteration method for solving the system of linear equations (1) were established when the coefficient matrix $A \in \mathbb{C}^{n\times n}$ is a hermitian positive definite matrix. However, these results were not developed to the multisplitting two-stage iteration method yet, due to the insufficiency of the P-regular splitting for guaranteeing the convergence of the iteration sequence. It is then natural for us to ask if this multisplitting two-stage iteration method converges for the hermitian positive definite matrix class, and how the splitting matrices influence its asymptotic convergence rate. This paper tries to give definite answers for these two interesting questions.

The contents of this paper are arranged as follows. In Section 2, we briefly restate the nonstationary multisplitting two-stage iteration method, and present its relaxed variant. Section 3 lists some necessary notations, concepts and lemmas. Finally, in Section 4, we study the convergence properties of these two nonstationary multisplitting two-stage iteration methods, and investigate their asymptotic convergence rates for the hermitian positive definite matrix class.

2. The nonstationary multisplitting two-stage iteration methods

Let $A=M_i-N_i(i=1,2,\dots,\alpha)$ be α splittings of the matrix $A\in\mathbb{C}^{n\times n}$, namely, $M_i,N_i\in\mathbb{C}^{n\times n}$ and $\det(M_i)\neq 0$, $i=1,2,\dots,\alpha$. Let $M_i=F_i-G_i(i=1,2,\dots,\alpha)$ be, respectively, splittings of the matrices $M_i(i=1,2,\dots,\alpha)$, and $E_i(i=1,2,\dots,\alpha)$ be α nonnegative diagonal matrices satisfying $\sum_{i=1}^{\alpha}E_i=I$ (the $n\times n$ identity matrix). Then we call the collection of triples $(M_i,N_i,E_i)(i=1,2,\dots,\alpha)$ a multisplitting and the collection of quintuples $(M_i:F_i,G_i;N_i;E_i)(i=1,2,\dots,\alpha)$ a two-stage multisplitting, of the matrix $A\in\mathbb{C}^{n\times n}$, respectively.

The nonstationary multisplitting two-stage iteration method for solving the system of linear equations (1) can be described as follows:

Method 2.1 (*Nonstationary Multisplitting Two-stage Iteration Method*). Given an initial vector $x_0\in\mathbb{C}^n$ and α positive integer sequences $\{s(i,k)\}_{k=1}^{\infty}(i=1,2,\dots,\alpha)$. For $k=1,2,\dots,$ until $\{x_k\}$ convergence, compute

$$x_k=\sum_{i=1}^{\alpha}E_ix_{i,s(i,k)},$$

where $x_{i,s(i,k)}$ is recursively obtained by the following formula:

$$F_ix_{i,j}=G_ix_{i,j-1}+N_ix_{k-1}+b,\quad j=1,2,\dots,s(i,k),$$

with the starting vector $x_{i,0}=x_{k-1}$.

When $s(i,k)=s(i=1,2,\cdots,\alpha,k=1,2,\dots)$, Method 2.1 becomes the stationary multisplitting two-stage iteration method studied in [17,9], respectively; when $N_i=0$, it reduces to the non-stationary multisplitting iteration method discussed in [12], and when $s(i,k) = 1 (i = 1,2,\dots,\alpha, k = 1,2,\dots)$ and $N_i = 0$, it recovers the matrix multisplitting iteration method initiated in [16]. Moreover, when $\alpha = 1$, it recovers the two-stage iteration method studied in [14,2].

By introducing relaxation factors to each of the inner iterations, we can straightforwardly get the following relaxed variant of Method 2.1.

Method 2.2 (*Relaxed Nonstationary Multisplitting Two-stage Iteration Method*). Given an initial vector $x_0 \in \mathbb{C}^n$ and α positive integer sequences $\{s(i,k)\}_{k=1}^{\infty} (i = 1,2,\dots,\alpha)$. For $k = 1,2,\dots,$ until $\{x_k\}$ convergence, compute

$$x_k = \sum_{i=1}^{\alpha} E_i x_{i,s(i,k)},$$

where $x_{i,s(i,k)}$ is recursively obtained by the following formula:

$$F_i x_{i,j} = \omega_i (G_i x_{i,j-1} + N_i x_{k-1} + b) + (1 - \omega_i) F_i x_{i,j-1},$$

with the starting vector $x_{i,0} = x_{k-1}$. Here, $\omega_i (i = 1,2,\dots,\alpha)$ are relaxation factors.

Evidently, when $\omega_i = 1 (i = 1,2,\dots,\alpha)$, Method 2.2 automatically becomes Method 2.1. Moreover, by suitably choosing the relaxation factors $\omega_i (i = 1,2,\dots,\alpha)$, the convergence property of Method 2.2 can be considerably improved.

3. Concepts and lemmas

Let $A \in \mathbb{C}^{n\times n}$ be a $n \times n$ complex matrix. We call A hermitian positive definite (semi-definite) and denote it by $A > O (A \geqslant O)$ if for all $x \in \mathbb{C}^n$ and $x \neq 0$, it holds that $x^H A x > 0$ $(x^H A x \geqslant 0)$. Here, O denotes the zero matrix, and $(\cdot)^H$ denotes the conjugate transpose of either a vector or a matrix. For $A, B \in \mathbb{C}^{n\times n}$, we use $A > B$ $(A \geqslant B)$ to represent $(A - B) > O ((A - B) \geqslant O)$. The spectral radius of the matrix A is denoted by $\rho(A)$, and $\|\cdot\|$ always refers to the $\|\cdot\|_2$.

Let $A \in \mathbb{C}^{n\times n}$ and $A = M - N$ be a splitting. Then this splitting is called a hermitian P-regular splitting if $M > O$ and $N \geqslant O$; and a P-regular splitting if $M^H + N > O$.

In the following, we list several lemmas which are necessary for the subsequent discussion.

Lemma 3.1 (Berman and Plemmons [11]). *Let $A \in \mathbb{C}^{n\times n}$ be a hermitian matrix, and $A = M - N$ a P-regular splitting. Then $\|A^{1/2} M^{-1} N A^{-1/2}\| < 1$ if and only if $A \in \mathbb{C}^{n\times n}$ is a positive definite matrix.*

Lemma 3.2 (Nabben [15]). *Let $A \in \mathbb{C}^{n\times n}$ be a hermitian positive definite matrix, and $(M_i, N_i, E_i)(i = 1,2,\dots,\alpha)$ be a multisplitting of it. Assume that each of the splittings $A = M_i - N_i (i = 1,2,\dots,\alpha)$ is P-regular, and the weighting matrices satisfy $E_i = \beta_i I$, where β_i is a scalar, $i = 1,2,\dots,\alpha$. Let $G = \sum_{i=1}^{\alpha} E_i M_i^{-1}$ and $H = \sum_{i=1}^{\alpha} E_i M_i^{-1} N_i$. Then the matrix G is nonsingular, and the splitting $A = G^{-1} - (G^{-1} H)$ is P-regular. Furthermore, this splitting is a hermitian P-regular splitting if $A = M_i - N_i (i = 1,2,\dots,\alpha)$ are all hermitian P-regular splittings.*

Lemma 3.3. *Let $T_1, T_2, \ldots, T_\alpha$ be $n \times n$ complex matrices, satisfying $\|T_i\| < 1 (i = 1, 2, \ldots, \alpha)$, and $E_i (i = 1, 2, \ldots, \alpha)$ be the weighting matrices satisfying $E_i = \beta_i I (i = 1, 2, \ldots, \alpha)$. Then*

$$\left\| \sum_{i=1}^{\alpha} E_i T_i \right\| \leqslant \sum_{i=1}^{\alpha} \beta_i \|T_i\| < 1.$$

Lemma 3.4. *Let $\{T(k)\}_{k=1}^{\infty}$ be a sequence of $n \times n$ complex matrices. If there exists a nonnegative constant $\theta \in [0, 1)$ such that $\|T(k)\| \leqslant \theta (k = 1, 2, \ldots)$, then $\lim_{k\to\infty} T(k)T(k-1) \cdots T(1) = O$.*

4. Convergence theories

We first prove the convergence of Methods 2.1 and 2.2 when the coefficient matrix $A \in \mathbb{C}^{n\times n}$ is hermitian positive definite. For this purpose, we introduce the following matrices, which will be used throughout the subsequent discussions.

$$\begin{aligned}
R(s,k) &= \sum_{i=1}^{\alpha} E_i R_i(s,k), \quad & R_i(s,k) &= \sum_{j=0}^{s(i,k)-1} (F_i^{-1} G_i)^j F_i^{-1}, \\
T(s,k) &= \sum_{i=1}^{\alpha} E_i T_i(s,k), \quad & T_i(s,k) &= (F_i^{-1} G_i)^{s(i,k)} + \sum_{j=0}^{s(i,k)-1} (F_i^{-1} G_i)^j F_i^{-1} N_i, \\
R(s) &= \sum_{i=1}^{\alpha} E_i R_i(s), & R_i(s) &= \sum_{j=0}^{s-1} (F_i^{-1} G_i)^j F_i^{-1}, \\
T(s) &= \sum_{i=1}^{\alpha} E_i T_i(s), & T_i(s) &= (F_i^{-1} G_i)^s + \sum_{j=0}^{s-1} (F_i^{-1} G_i)^j F_i^{-1} N_i, \\
P_i(s) &= (F_i^{-1} G_i)^s, & P_i(s,k) &= (F_i^{-1} G_i)^{s(i,k)},
\end{aligned}$$

where $A = M_i - N_i (i = 1, 2, \ldots, \alpha)$ and $M_i = F_i - G_i (i = 1, 2, \ldots, \alpha)$ are splittings of the matrix $A \in \mathbb{C}^{n\times n}$ and $M_i \in \mathbb{C}^{n\times n} (i = 1, 2, \ldots, \alpha)$, respectively, $E_i (i = 1, 2, \ldots, \alpha)$ are weighting matrices, and $s(i,k) (i = 1, 2, \ldots, \alpha, k = 1, 2, \ldots)$ and s are positive integers.

After straightforward operations we get the following relations about the above matrices:

$$\begin{aligned}
T(s,k) &= I - R(s,k)A, & T_i(s,k) &= I - R_i(s,k)A, \\
T(s) &= I - R(s)A, & T_i(s) &= I - R_i(s)A, \\
R_i(s,k) &= (I - P_i(s,k))M_i^{-1}, \quad & R_i(s) &= (I - P_i(s))M_i^{-1}.
\end{aligned}$$

Assume that $R_i(s,k)$, $R(s,k)$ and $R(s)$ are nonsingular matrices, and define

$$\begin{aligned}
B_i(s,k) &= R_i(s,k)^{-1}, \quad & C_i(s,k) &= R_i(s,k)^{-1} T_i(s,k), \\
B(s,k) &= R(s,k)^{-1}, & C(s,k) &= R(s,k)^{-1} T(s,k), \\
B(s) &= R(s)^{-1}, & C(s) &= R(s)^{-1} T(s).
\end{aligned}$$

Then we have

$$\begin{aligned} A &= B_i(s,k) - C_i(s,k) \\ &= B(s,k) - C(s,k) \\ &= B(s) - C(s). \end{aligned}$$

Furthermore, we introduce constants

$$\rho_i^{(1)} = \rho(A^{-1}N_i), \qquad \rho_i^{(2)} = \rho(F_i^{-1}G_i)^2 = \rho(M_i^{1/2}(F_i^{-1}G_i)M_i^{-1/2})^2.$$

Now, we are ready to establish convergence theorems for Methods 2.1 and 2.2.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ be a hermitian positive definite matrix, and $(M_i : F_i, G_i; N_i; E_i)(i = 1,2,\ldots,\alpha)$ be a two-stage multisplitting of the matrix A. Assume that $A = M_i - N_i (i = 1,2,\ldots,\alpha)$ are hermitian P-regular splittings, $M_i = F_i - G_i (i = 1,2,\ldots,\alpha)$ are P-regular splittings, and $E_i = \beta_i I (i = 1,2,\ldots,\alpha)$ the weighting matrices. Then for any given initial vector x_0 and any positive integer sequences of inner iteration numbers $\{s(i,k)\}_{k=1}^{\infty} (i = 1,2,\ldots,\alpha)$,*

(i) *Method* 2.1 *converges to the unique solution x_* of the system of linear equations* (1);
(ii) *Method* 2.2 *converges to the unique solution x_* of the system of linear equations* (1), *provided the relaxation factors $\omega_i (i = 1,2,\ldots,\alpha)$ satisfy*

$$\omega_i > 0, \quad (2-\omega_i)M_i + G_i^H + G_i > O, \quad i = 1,2,\ldots,\alpha. \tag{2}$$

Proof. Method 2.1 can be briefly expressed in the following matrix-vector form:

$$x_{k+1} = T(s,k)x_k + R(s,k)b, \quad k = 1,2,\ldots. \tag{3}$$

We assert that $R(s,k)$ are nonsingular matrices, and hence, $A = B(s,k) - C(s,k)$ exist. Moreover, these splittings are P-regular splittings.

In fact, because

$$R_i(s,k) = (I - P_i(s,k))M_i^{-1}, \qquad T_i(s,k) = I - (I - P_i(s,k))M_i^{-1}A,$$

we know that $R_i(s,k)$ is nonsingular, i.e., $B_i(s,k)$ exists, if $\rho(P_i(s,k)) < 1$; and that $A = B_i(s,k) - C_i(s,k)$ is P-regular, iff the matrix $S_i(s,k) \equiv B_i(s,k)^H + C_i(s,k) > O$.

The assumption that $A = M_i - N_i$ is hermitian P-regular implies that $M_i > O$. Therefore, by Lemma 3.1 and the assumption that $M_i = F_i - G_i$ is P-regular, we see that there exists a nonnegative constant $\theta \in [0,1)$ such that

$$\rho_i^{(2)} = \|M_i^{1/2}(F_i^{-1}G_i)M_i^{-1/2}\|^2 \leqslant \theta^2.$$

Since

$$\begin{aligned} \|M_i^{1/2}P_i(s,k)M_i^{-1/2}\| &= \|M_i^{1/2}(F_i^{-1}G_i)^{s(i,k)}M_i^{-1/2}\| = \|(M_i^{1/2}(F_i^{-1}G_i)M_i^{-1/2})^{s(i,k)}\| \\ &\leqslant \|M_i^{1/2}(F_i^{-1}G_i)M_i^{-1/2}\|^{s(i,k)} \leqslant \|M_i^{1/2}(F_i^{-1}G_i)M_i^{-1/2}\| \leqslant \theta < 1, \end{aligned} \tag{4}$$

we immediately get

$$\rho(P_i(s,k)) = \rho(M_i^{1/2}P_i(s,k)M_i^{-1/2}) \leqslant \|M_i^{1/2}P_i(s,k)M_i^{-1/2}\| \leqslant \theta < 1.$$

Hence, $R_i(s,k)$ is a nonsingular matrix and $B_i(s,k)$ exists.

By direct manipulations we have

$$\begin{aligned}
\bar{S}_i(s,k) &= (I - P_i(s,k))^H S_i(s,k)(I - P_i(s,k)) \\
&= (I - P_i(s,k))^H (B_i(s,k)^H + C_i(s,k))(I - P_i(s,k)) \\
&= (I - P_i(s,k))^H (R_i(s,k)^{-H} + R_i(s,k)^{-1}T_i(s,k))(I - P_i(s,k)) \\
&= (R_i(s,k)^{-1}(I - P_i(s,k)))^H (I - P_i(s,k)) \\
&\quad +(I - P_i(s,k))^H (R_i(s,k)^{-1}(I - P_i(s,k))) \\
&\quad -(I - P_i(s,k))^H A(I - P_i(s,k)) \\
&= M_i^H(I - P_i(s,k)) + (I - P_i(s,k))^H M_i \\
&\quad -(I - P_i(s,k))^H A(I - P_i(s,k)) \\
&= 2M_i - M_i^H P_i(s,k) - P_i(s,k)^H M_i \\
&\quad -(I - P_i(s,k))^H (M_i - N_i)(I - P_i(s,k)) \\
&= (M_i + N_i) - N_i P_i(s,k) - P_i(s,k)^H N_i - P_i(s,k)^H M_i P_i(s,k) \\
&\quad +P_i(s,k)^H N_i P_i(s,k) \\
&= M_i - P_i(s,k)^H M_i P_i(s,k) + (I - P_i(s,k))^H N_i (I - P_i(s,k)).
\end{aligned}$$

Again, that $A = M_i - N_i$ is a hermitian P-regular splitting shows that $M_i > O$ and $N_i \geqslant O$. By making use of this fact we obtain

$$\begin{aligned}
\bar{S}_i(s,k) &\geqslant M_i - P_i(s,k)^H M_i P_i(s,k) \\
&= M_i^{1/2}[I - (M_i^{-1/2}P_i(s,k)^H M_i^{1/2})(M_i^{1/2}P_i(s,k)M_i^{-1/2})]M_i^{1/2} \\
&= M_i^{1/2}[I - (M_i^{1/2}P_i(s,k)M_i^{-1/2})^2]M_i^{1/2}.
\end{aligned}$$

Noticing (4) we immediately know that $\bar{S}_i(s,k) > O$, and therefore, $S_i(s,k) > O$ holds, independently of the iterate indices k, i and $s(i,k)$. Hence, $A = B_i(s,k) - C_i(s,k)$ is P-regular independently of the iterate indices k, i and $s(i,k)$, too.

By making use of Lemma 3.2, we get that the matrix $R(s,k)$ is nonsingular, and the splitting $A = B(s,k) - C(s,k)$ is P-regular, independently of the iterate indices k, i and $s(i,k)$. Again, by Lemma 3.1 we obtain that

$$\begin{aligned}
\rho(T(s,k)) = \rho(B(s,k)^{-1}C(s,k)) &= \rho(A^{1/2}(B(s,k)^{-1}C(s,k))A^{-1/2}) \\
&\leqslant \|A^{1/2}(B(s,k)^{-1}C(s,k))A^{-1/2}\| < 1
\end{aligned}$$

holds independently of the iterate indices k, i and $s(i,k)$. Therefore, Method 2.1 converges to the unique solution x_* of the system of linear equations (1).

Obviously, Method 2.2 is equivalent to Method 2.1 applied to the two-stage multisplitting $(M_i : F_i(\omega_i), G_i(\omega_i); N_i; E_i)(i = 1,2,\dots,\alpha)$ of the matrix A, where $F_i(\omega_i) = (1/\omega_i)F_i$ and $G_i(\omega_i) = (1/\omega_i - 1)F_i + G_i$. Under the assumptions we can easily verify that $M_i = F_i(\omega_i) - G_i(\omega_i)(i=1,2,\dots,\alpha)$ are P-regular splittings. Therefore, Method 2.1 converges to the unique solution x_* of the system of linear equations (1), too. $\square$

We remark that in Theorem 4.1, if $G_i^H + G_i \geqslant O$ hold for all $i = 1,2,\dots,\alpha$, then the restriction (2) is satisfied, provided $0 < \omega_i < 2$, $i = 1,2,\dots,\alpha$.

The following theorem describes the asymptotic convergence speeds of both Methods 2.1 and 2.2.

Theorem 4.2. *Let the conditions of Theorem 4.1 be satisfied. Then*

(i) *the iterative sequence $\{x_k\}$ generated by Method 2.1 converges to x_* with the R_1-factor being at least σ_1, where*

$$\sigma_1 = \sum_{i=1}^{\alpha} \beta_i \left(\frac{\rho_i^{(1)} + \rho_i^{(2)}}{1 + \rho_i^{(1)}} \right)^{1/2};$$

(ii) *the iterative sequence $\{x_k\}$ generated by Method 2.2 converges to x_* with the R_1-factor being at least σ_2, where*

$$\sigma_2 = \sum_{i=1}^{\alpha} \beta_i \left(\frac{\rho_i^{(1)} + \|(1-\omega_i)I + \omega_i M_i^{1/2}(F_i^{-1}G_i)M_i^{-1/2}\|^2}{1 + \rho_i^{(1)}} \right)^{1/2},$$

provided the relaxation factors $\omega_i (i = 1,2,\dots,\alpha)$ satisfy (2); or with the R_1-factor being at least $\bar{\sigma}_2$, where

$$\bar{\sigma}_2 = \sum_{i=1}^{\alpha} \beta_i \left(\frac{\rho_i^{(1)} + \left(|1-\omega_i| + \omega_i \sqrt{\rho_i^{(2)}} \right)^2}{1 + \rho_i^{(1)}} \right)^{1/2},$$

provided the relaxation factors $\omega_i (i = 1,2,\dots,\alpha)$ satisfy

$$0 < \omega_i < \frac{2}{1 + \sqrt{\rho_i^{(2)}}}, \quad i = 1,2,\dots,\alpha. \tag{5}$$

Proof. Let $\varepsilon_k = x_k - x_*$ be the error of the kth out iterate. Then, it follows from (3) that

$$\varepsilon_k = T(s,k)T(s,k-1)\cdots T(s,1)\varepsilon_0. \tag{6}$$

Because $A = M_i - N_i$ being a hermitian P-regular splitting implies that $M_i > O$ and $N_i \geqslant O$, we have

$$\begin{aligned} A - T_i(s,k)^H A T_i(s,k) &= A - (I - (I - P_i(s,k))M_i^{-1}A)^H A (I - (I - P_i(s,k))M_i^{-1}A) \\ &= AM_i^{-1}(I - P_i(s,k))^H A + A(I - P_i(s,k))M_i^{-1}A \end{aligned}$$

$$
\begin{aligned}
&-AM_i^{-1}(I - P_i(s,k))^H A(I - P_i(s,k))M_i^{-1}A \\
&= AM_i^{-1}[(I - P_i(s,k))^H M_i + M_i(I - P_i(s,k)) \\
&\quad -(I - P_i(s,k))^H A(I - P_i(s,k))]M_i^{-1}A \\
&= AM_i^{-1}[M_i - P_i(s,k)^H M_i P_i(s,k) \\
&\quad +(I - P_i(s,k))^H N_i(I - P_i(s,k))]M_i^{-1}A \\
&\geqslant AM_i^{-1}(M_i - P_i(s,k)^H M_i P_i(s,k))M_i^{-1}A.
\end{aligned}
$$

This estimate is equivalent to

$$
\begin{aligned}
&I - A^{-1/2}T_i(s,k)^H AT_i(s,k)A^{-1/2} \\
&\quad \geqslant A^{1/2}M_i^{-1/2}(I - M_i^{-1/2}P_i(s,k)^H M_i P_i(s,k)M_i^{-1/2})M_i^{-1/2}A^{1/2},
\end{aligned}
$$

which directly leads to

$$
\begin{aligned}
&(I - A^{-1/2}T_i(s,k)^H AT_i(s,k)A^{-1/2})^{-1} \\
&\quad \leqslant A^{-1/2}M_i^{1/2}(I - M_i^{-1/2}P_i(s,k)^H M_i P_i(s,k)M_i^{-1/2})^{-1}M_i^{1/2}A^{-1/2}.
\end{aligned}
$$

By straightforward computations we can further obtain the estimate

$$
\frac{1}{1 - \|A^{1/2}T_i(s,k)A^{-1/2}\|^2} \leqslant \frac{1 + \|A^{-1/2}N_iA^{-1/2}\|}{1 - \|M_i^{1/2}P_i(s,k)M_i^{-1/2}\|^2}. \tag{7}
$$

Hence, it holds that

$$
\|A^{1/2}T_i(s,k)A^{-1/2}\|^2 \leqslant \frac{\rho_i^{(1)} + \rho_i^{(2)}}{1 + \rho_i^{(1)}}.
$$

Here, we have used the estimate

$$
\|M_i^{1/2}P_i(s,k)M_i^{-1/2}\| \leqslant \|M_i^{1/2}(F_i^{-1}G_i)M_i^{-1/2}\|,
$$

which comes from (4). Now, by making use of Lemma 3.3, we know that

$$
\|A^{1/2}T(s,k)A^{-1/2}\| \leqslant \sum_{i=1}^{\alpha} \beta_i \left(\frac{\rho_i^{(1)} + \rho_i^{(2)}}{1 + \rho_i^{(1)}} \right)^{1/2} = \sigma_1
$$

holds. By taking norms on both sides of (6), we have

$$
\begin{aligned}
\|\varepsilon_k\| &= \|T(s,k)T(s,k-1)\cdots T(s,1)\varepsilon_0\| \\
&\leqslant \|A^{-1/2}\|\|A^{1/2}T(s,k)T(s,k-1)\cdots T(s,1)A^{-1/2}\|\|A^{1/2}\varepsilon_0\| \\
&\leqslant \|A^{-1/2}\|\prod_{j=1}^{k}\|A^{1/2}T(s,j)A^{-1/2}\|\|A^{1/2}\varepsilon_0\| \\
&\leqslant \sigma_1^k\|A^{-1/2}\|\|A^{1/2}\varepsilon_0\|.
\end{aligned}
$$

Therefore,

$$\overline{\lim_{k\to\infty}} \|\varepsilon_k\|^{1/k} \leqslant \sigma_1.$$

This shows that the R_1-factor of the iterative sequence generated by Method 2.1 is at least σ_1.

We now turn to demonstrate (ii). Considering that the splittings $A = M_i - N_i (i = 1, 2, \dots, \alpha)$ are hermitian P-regular and $M_i = F_i(\omega_i) - G_i(\omega_i) (i = 1, 2, \dots, \alpha)$ are P-regular, and that the relaxation factors $\omega_i (i = 1, 2, \dots, \alpha)$ satisfy (2), we know that

$$\|M_i^{1/2}((1-\omega_i)I + \omega_i F_i^{-1} G_i) M_i^{-1/2}\| < 1, \quad i = 1, 2, \dots, \alpha$$

hold from Lemma 3.1, and

$$|1-\omega_i| + \omega_i \sqrt{\rho_i^{(2)}} < 1, \quad i = 1, 2, \dots, \alpha$$

hold from (5).

Because

$$\begin{aligned}
\|M_i^{1/2}(F_i(\omega_i)^{-1}G_i(\omega_i))^{s(i,k)}M_i^{-1/2}\| &= \|(M_i^{1/2}(F_i(\omega_i)^{-1}G_i(\omega_i))M_i^{-1/2})^{s(i,k)}\| \\
&\leqslant \|M_i^{1/2}(F_i(\omega_i)^{-1}G_i(\omega_i))M_i^{-1/2}\|^{s(i,k)} \\
&\leqslant \|M_i^{1/2}(F_i(\omega_i)^{-1}G_i(\omega_i))M_i^{-1/2}\| \\
&= \|M_i^{1/2}((1-\omega_i)I + \omega_i F_i^{-1}G_i)M_i^{-1/2}\| \\
&\leqslant |1-\omega_i| + \omega_i \|M_i^{1/2}(F_i^{-1}G_i)M_i^{-1/2}\| \\
&= |1-\omega_i| + \omega_i \sqrt{\rho_i^{(2)}},
\end{aligned}$$

by making use of inequality (7) we immediately obtain

$$\begin{aligned}
\|A^{1/2}T_i(s,k)A^{-1/2}\|^2 &\leqslant \frac{\rho_i^{(1)} + \left(\|(1-\omega_i)I + \omega_i M_i^{1/2}(F_i^{-1}G_i)M_i^{-1/2}\|\right)^2}{1+\rho_i^{(1)}} \\
&\leqslant \frac{\rho_i^{(1)} + \left(|1-\omega_i| + \omega_i\sqrt{\rho_i^{(2)}}\right)^2}{1+\rho_i^{(1)}}.
\end{aligned}$$

Again, Lemma 3.3 straightforwardly yields

$$\begin{aligned}
\|A^{1/2}T(s,k)A^{-1/2}\| &\leqslant \sum_{i=1}^{\alpha} \beta_i \left(\frac{\rho_i^{(1)} + \left(\|(1-\omega_i)I + \omega_i M_i^{1/2}(F_i^{-1}G_i)M_i^{-1/2}\|\right)^2}{1+\rho_i^{(1)}} \right)^{1/2} = \sigma_2 \\
&\leqslant \sum_{i=1}^{\alpha} \beta_i \left(\frac{\rho_i^{(1)} + \left(|1-\omega_i| + \omega_i\sqrt{\rho_i^{(2)}}\right)^2}{1+\rho_i^{(1)}} \right)^{1/2} = \bar{\sigma}_2.
\end{aligned}$$

Now, similar to the proof of (i), we know that the R_1-factor of the iterative sequence generated by Method 2.2 is at least either σ_2, or $\bar{\sigma}_2$, depending upon different restrictions on the relaxation factors $\omega_i (i = 1, 2, \ldots, \alpha)$. $\square$

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