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An algebraic theory for primal and dual substructuring methods by constraints

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Abstract

FETI and BDD are two widely used substructuring methods for the solution of large sparse systems of linear algebraic equations arising from discretization of elliptic boundary value problems. The two most advanced variants of these methods are the FETI-DP and the BDDC methods, whose formulation does not require any information beyond the algebraic system of equations in a substructure form. We formulate the FETI-DP and the BDDC methods in a common framework as methods based on general constraints between the substructures, and provide a simplified algebraic convergence theory. The basic implementation blocks including transfer operators are common to both methods. It is shown that commonly used properties of the transfer operators in fact determine the operators uniquely. Identical algebraic condition number bounds for both methods are given in terms of a single inequality, and, under natural additional assumptions, it is proved that the eigenvalues of the preconditioned problems are the same. The algebraic bounds imply the usual polylogarithmic bounds for finite elements, independent of coefficient jumps between substructures. Computational experiments confirm the theory.

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1. Introduction

FETI (Finite Element Tearing and Interconnecting) and BDD (Balancing Domain Decomposition) are two widely used substructuring methods for the solution of large sparse systems of linear algebraic equations arising from discretization of elliptic boundary value problems. The main difference between those methods is that FETI is a dual method, which enforces the equality of the degrees of freedom on substructure interfaces by Lagrange multipliers, while BDD is a primal method, which operates on the common interface degrees of freedom. The original versions of these methods require information about the nullspace of the problem in the absence of essential boundary conditions, i.e., rigid body modes in the case of elasticity. The two most advanced variants of these methods are the FETI-DP (FETI Dual–Primal) and the BDDC (BDD based on Constraints) methods, whose formulation does not require any information beyond the algebraic system of equations in a substructure form.

The FETI-DP method was proposed in [6] as a further development of the FETI method [4], suitable for 4th order problems in 2D. The FETI-DP method was modified to achieve better performance in 3D in [5,12]. Convergence theory for the FETI method was given in [18,20], modified to cover the FETI-DP method in [19], and extended to the FETI-DP method for 3D problems with coefficient jumps in [12].

The BDDC method was first proposed in [2]. It can be understood as a further development of the BDD method [15,13] and it belongs to the class of additive Schwarz methods of Neumann–Neumann type [3]. Convergence bounds for the BDDC method were provided in [17]. The main advantages of the BDDC method over the original BDD method are:

(1) the coarse problem has the same form as the original problem,
(2) the coarse problem is sparser than in the BDD method, and
(3) the algorithmic framework allows for the coarse problem to be solved only approximately.

Other constructions of a sparser coarse space in Neumann–Neumann methods were given in [3].

Because of the apparent duality between FETI and BDD methods, their relation has been a subject of interest, but first specific results have emerged only recently. Identities between the transfer operators in FETI and BDD were given in [23] and generalized and complemented by further important relations in [10]; these identities allow one to reduce the analysis of both methods to a common core. In [7], it was observed that such identities hold between many versions of FETI and BDD, and a part of this paper expands on some of those identities. An important new development is the reformulation and analysis of the FETI-DP method as a Schwarz method [1].

In this paper, we investigate the connections between the most advanced variants of FETI and BDD class methods, namely the FETI-DP and the BDDC methods. We identify the necessary algebraic properties of the transfer operators, which we expect to be useful in future work, and we show that in several important cases known from the literature, these properties determine the transfer operators uniquely. We present a new, mathematically equivalent formulation of FETI-DP with a positive definite coarse problem, which builds on the approach in [12]. We show that the condition number bound for both FETI-DP and BDDC can be reduced to the same single inequality, and we prove that, under natural additional assumptions, the eigenvalues of the preconditioned system are the same except possibly for zeros and ones. The principal advantages of BDDC and the new formulation of FETI-DP is that, unlike the original formulation in [5], only the solution of symmetric positive definite coarse problems is required, which allows the use of the Cholesky algorithm, which is much more robust and efficient than methods for indefinite sparse systems. In addition, the implementation of BDDC is simpler, and the additive Schwarz setting of BDDC allows for a straightforward extension to a multilevel algorithm.

The paper is organized as follows. In Section 2, we introduce the notation and define the components for both methods. Section 3 is concerned with the formulation of the FETI-DP method in our setting. Section 4 contains a brief formulation of the BDDC method, based on the same components as FETI-DP. Section 5 establishes properties of the spaces and transfer operators. In Section 6, we restate the convergence theory from [19] so that it applies to our abstract formulation of FETI-DP, and reduce the condition number bound to verifying a single inequality based on the problem matrices. In Section 7, we give a bound on the condition number of BDDC, which is the same as for FETI-DP. We then refer to existing literature for the usual polylogarithmic bounds. Section 8 contains the proof of equality of the eigenvalues of the preconditioned FETI-DP and BDDC operators. Computational experiments are in Section 9.

2. Problem setting and common components

This paper is concerned with the iterative solution of the problem

$$\min_{w\in\widehat{W}} \frac{1}{2} w^{\mathrm{T}} S w - w^{\mathrm{T}} g, \tag{1}$$

where

$$\widehat{W} = \operatorname{range} R,$$
$$S = \begin{bmatrix} S_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & S_N \end{bmatrix}, \qquad g = \begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix}, \qquad w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, \qquad R = \begin{bmatrix} R_1 \\ \vdots \\ R_N \end{bmatrix},$$

S is a symmetric, positive semidefinite block diagonal matrix, and R is a zero-one matrix of full column rank such that

$$R_i R_i^{\mathrm{T}} = I.$$

Here and below, I is the identity matrix of appropriate dimension. The problem (1) is an abstract formulation of a substructuring setting, in which (1) is minimization of energy in a structure, S_i is the Schur complement of the stiffness matrix for substructure i after elimination of interior degrees of freedom, g_i is the substructure load vector, w_i is the vector of the substructure degrees of freedom, and R_i maps (restricts) global degrees of freedom to substructure degrees of freedom.

We consider matrices to be operators between the appropriate Euclidean spaces,

$$\begin{aligned} &R_i : U \to W_i, \qquad R : U \to W = W_1 \times \cdots \times W_N, \\ &S_i : W_i \to W_i, \qquad S : W \to W. \end{aligned} \tag{2}$$

In view of the substructuring setting, the space W_i will be called the space of degrees of freedom on the interfaces of substructure i, the space U will be called the space of the global interface degrees of freedom, and $\widehat{W}$ will be called the space of degrees of freedom continuous across the substructure interfaces. In the following we will freely use the substructuring terminology.

Let $B : W \to \Lambda$ be a matrix such that the constraint $Bu = 0$ enforces the continuity across the substructure interfaces, i.e.,

$$\operatorname{range} R = \operatorname{null} B. \tag{3}$$

Here, Λ is the space of the values of the constraints, that is, jumps between the substructures. Each global degree of freedom on the interfaces between the substructures gives rise to one or more rows of B. With the block structure given by (2), write B in the block form

$$B = [B_1, \ldots, B_N], \qquad B_i : W_i \to \Lambda.$$

The primal approach, taken by BDD methods, is to enforce the condition $w \in \widehat{W}$ by making $w = Ru$ for some u; the dual approach, taken by FETI methods, is to enforce $w \in \widehat{W}$ by the constraint $Bw = 0$.

Throughout the iterations, additional linearly dependent constraints are imposed by both FETI-DP and BDDC methods to speed up the convergence of the iterations. In FETI-DP, the additional constraints are enforced directly throughout the iteration, while in BDDC, they are used to build subspace corrections. Let n_c be the number of the additional constraints. In the case of BDDC, the additional constraints are written by means of the matrix

$$Q_P : \mathbb{R}^{n_c} \to U.$$

During the iterations, the BDDC method makes use of subspace correction with respect to the coarse degrees of freedom u_c, defined by

$$u_c = Q_P^{\mathrm{T}} u.$$

Let n_{ci} be the number the coarse degrees of freedom that can be nonzero for substructure i, let $R_{ci} : \mathbb{R}^{n_c} \to \mathbb{R}^{n_{ci}}$ be a zero-one matrix that selects those degrees of freedom, and define

$$C_i = R_{ci} Q_P^{\mathrm{T}} R_i^{\mathrm{T}}, \qquad R_c = \begin{bmatrix} R_{c1} \\ \vdots \\ R_{cN} \end{bmatrix}, \qquad C = \begin{bmatrix} C_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & C_N \end{bmatrix}.$$

We assume that R_c has full column rank. The coarse degrees of freedom are, e.g., values of degrees of freedom at substructure corners, or weighted averages of degrees of freedom over substructure edges or faces. Let Φ be a block matrix whose columns form a basis for the coarse degrees of freedom, i.e.,

$$\Phi = \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_N \end{bmatrix}, \qquad C\Phi = R_c. \tag{4}$$

Note that Φ_i will typically have many zero columns; its nonzero columns correspond to basis functions whose support intersects substructure i. We will be particularly interested in coarse basis functions defined by minimizing the energy of the coarse basis functions, which is the same as minimizing the diagonal terms of $\Phi^{\mathrm{T}} S \Phi$. We denote such energy minimizing basis functions by Ψ. The columns of Ψ are the solutions of the saddle-point problem with multiple right sides,

$$\begin{bmatrix} S & C^{\mathrm{T}} \\ C & 0 \end{bmatrix} \begin{bmatrix} \Psi \\ \Lambda \end{bmatrix} = \begin{bmatrix} 0 \\ R_c \end{bmatrix}. \tag{5}$$

Let $\widetilde{W}$ be the subspace of W consisting of vectors with the generalized degrees of freedom continuous across substructure interfaces:

$$\widetilde{W} = \{w \in W\colon\ \exists u_c\colon\ Cw = R_c u_c\} = \{w \in W\colon\ Cw \in \operatorname{range} R_c\}, \tag{6}$$

and let $\widetilde{W}_\Delta$ be the space of vectors with all coarse degrees of freedom equal to zero, called the *dual space*

$$\widetilde{W}_\Delta = \operatorname{null} C,$$

and denote

$$\widetilde{W}_\Pi = \operatorname{range} \Phi,$$

called the *primal space*. In Lemma 8 below, we will show that

$$\widetilde{W} = \widetilde{W}_\Pi \oplus \widetilde{W}_\Delta. \tag{7}$$

We also assume that

$$S \text{ is positive definite on } \widetilde{W}. \tag{8}$$

Another important ingredient of BDDC is the primary weight matrices $D_{Pi} : W_i \to W_i$ (usually diagonal), and the block diagonal matrix

$$D_P = \begin{bmatrix} D_{P1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & D_{PN} \end{bmatrix}.$$

We assume that the primary weight matrices form a decomposition of unity,

$$R^{\mathrm{T}} D_P R = I. \tag{9}$$

The purpose of the primary weight matrices is to average a vector of degrees of freedom $w \in W$ on the interfaces between the substructures to obtain a global vector $u \in U$,

$$u = R^{\mathrm{T}} D_P w.$$

For formulating the FETI-DP method, we will also need a matrix B_D such that

$$B_D^{\mathrm{T}} B + R R^{\mathrm{T}} D_P = I. \tag{10}$$

The matrix B_D^{T} plays the role of a generalized inverse of B. For further details on the operators D_P and B_D, see Section 5 below.

3. The FETI-DP method

As in other FETI methods, Lagrange multipliers are introduced into (1) to enforce the condition $Bw = 0$. Then (1) is equivalent to

$$\min_{w \in W} \sup_{\lambda \in \Lambda} \mathcal{L}(w, \lambda), \tag{11}$$

where $\mathcal{L}(w, \lambda)$ is the Lagrangian,

$$\mathcal{L}(w, \lambda) = \frac{1}{2} w^{\mathrm{T}} S w - w^{\mathrm{T}} g + w^{\mathrm{T}} B^{\mathrm{T}} \lambda. \tag{12}$$

Because $\sup_{\lambda\in\Lambda}\mathcal{L}(w,\lambda)<+\infty$ if and only if $w\in\widehat{W}$, and $\widehat{W}\subset\widetilde{W}\subset W$, it holds that

$$\min_{w\in W}\sup_{\lambda\in\Lambda}\mathcal{L}(w,\lambda)=\min_{w\in\widetilde{W}}\sup_{\lambda\in\Lambda}\mathcal{L}(w,\lambda),$$

so (11) is equivalent to the dual problem

$$\max_{\lambda\in\Lambda}\mathcal{F}(\lambda), \tag{13}$$

where

$$\mathcal{F}(\lambda)=\min_{w\in\widetilde{W}}\mathcal{L}(w,\lambda), \tag{14}$$

is the dual functional.

With the decomposition (7), the dual functional can be written in terms of nested minimization as

$$\mathcal{F}(\lambda)=\min_{w\in\widetilde{W}}\mathcal{L}(w,\lambda)=\min_{w_\Pi\in\widetilde{W}_\Pi}\ \min_{w_\Delta\in\widetilde{W}_\Delta}\mathcal{L}(w_\Pi+w_\Delta,\lambda). \tag{15}$$

Remark 1. The dual functional $\mathcal{F}(\lambda)$ does not depend on the choice of $\widetilde{W}_\Pi$, as long as (7) holds.

Now we introduce Lagrange multipliers μ to enforce the zero coarse degrees of freedom in $\widetilde{W}_\Delta$ and replace in (15) the minimization over $\widetilde{W}_\Delta$ by the minimization over $\widetilde{W}$,

$$\mathcal{F}(\lambda)=\min_{w_\Pi\in\widetilde{W}_\Pi}\ \min_{w_\Delta\in\widetilde{W}}\ \sup_{\mu}\mathcal{L}(w_\Pi+w_\Delta,\lambda)+w_\Delta^{\mathrm{T}}C^{\mathrm{T}}\mu,$$

because

$$\sup_{\mu}\mathcal{L}(w_\Pi+w_\Delta,\lambda)+w_\Delta^{\mathrm{T}}C^{\mathrm{T}}\mu=\begin{cases}\mathcal{L}(w_\Pi+w_\Delta,\lambda) & \text{if } w_\Delta\in\widetilde{W}_\Delta,\\ +\infty & \text{otherwise.}\end{cases}$$

Consequently, the dual linear system (13) is equivalent to the stationary conditions for the augmented Lagrangian $\mathcal{L}(w_\Pi+w_\Delta,\lambda)+w_\Delta^{\mathrm{T}}C^{\mathrm{T}}\mu$. Writing $w_\Pi=\Phi u_c$ and differentiating with respect to w_Δ, μ, u_c, and λ, we get the linear system, equivalent to (13),

$$\begin{cases}Sw_\Delta+C^{\mathrm{T}}\mu+S\Phi u_c+B^{\mathrm{T}}\lambda=g,\\ Cw_\Delta=0,\\ \Phi^{\mathrm{T}}Sw_\Delta+\Phi^{\mathrm{T}}S\Phi u_c+\Phi^{\mathrm{T}}B^{\mathrm{T}}\lambda=\Phi^{\mathrm{T}}g,\\ Bw_\Delta+B\Phi u_c=0.\end{cases} \tag{16}$$

Eliminating w_Δ and μ from the first two equations, the third equation yields the coarse problem

$$S_cu_c=r_c, \tag{17}$$

where

$$S_c=\Phi^{\mathrm{T}}S\Phi-\begin{bmatrix}S\Phi\\0\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}S & C^{\mathrm{T}}\\ C & 0\end{bmatrix}^{-1}\begin{bmatrix}S\Phi\\0\end{bmatrix}, \tag{18}$$

$$r_c=\Phi^{\mathrm{T}}\left(I-\begin{bmatrix}S\\0\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}S & C^{\mathrm{T}}\\ C & 0\end{bmatrix}^{-1}\begin{bmatrix}I\\0\end{bmatrix}\right)\left(g-B^{\mathrm{T}}\lambda\right).$$

The matrix S_c is symmetric and positive definite because S is positive on range $\Phi\subset\widetilde{W}$ by assumption, cf. (8).

The first two equations in (16) are independent local systems equations in the substructures

$$\begin{cases} S_i w_{\Delta i} + C_i^{\mathrm{T}} \mu_i = g_i - S_i \Phi_i u_c - B_i^{\mathrm{T}} \lambda, \\ C_i w_{\Delta i} = 0, \end{cases} \qquad i = 1, \ldots, N. \tag{19}$$

Remark 2. For computational efficiency, the entries of $w_{\Delta i}$ that are constrained by the condition $C_i w_{\Delta i} = 0$ to be zero can be set to zero upfront. If there are enough corners in the substructure decomposition so that S_i is positive definite on the resulting subspace, then the solution of (19) can be reduced by eliminating $w_{\Delta i}$ to solving two symmetric positive definite systems, one sparse and one small and dense, cf. [2].

Finally, the dual system operator is evaluated by computing u_c from (17), w_Δ from (19), and substituting into the last equation in (16),

$$\frac{\partial \mathcal{F}(\lambda)}{\partial \lambda} = B w_\Delta + B \Phi u_c \equiv d - F\lambda, \tag{20}$$

for suitable F and d. The FETI-DP method is preconditioned conjugate gradients for the dual linear system

$$F\lambda = d,$$

with the preconditioner defined by

$$M = B_D S B_D^{\mathrm{T}}. \tag{21}$$

The calculation of the coarse matrix can be further simplified.

Theorem 3. *The coarse matrix S_c, given by* (18), *satisfies $S_c = \Psi^{\mathrm{T}} S \Psi$, where Ψ are coarse functions defined by energy minimization* (5). *In particular, S_c does not depend on the particular choice of the coarse basis functions Φ.*

Proof. Note that Ψ and Φ can be expressed as

$$\Psi = C^{\mathrm{T}} \left(C C^{\mathrm{T}}\right)^{-1} R_c + C_\perp A_\Psi, \qquad \Phi = C^{\mathrm{T}} \left(C C^{\mathrm{T}}\right)^{-1} R_c + C_\perp A_\Phi,$$

where range $C_\perp = \operatorname{null} C$. It follows that

$$\Psi = \Phi + C_\perp A_\delta, \tag{22}$$

where $A_\delta = A_\Psi - A_\Phi$. Substituting (22) into (5) and noting that $C\Phi = R_c$, cf. (4), leads to

$$\begin{bmatrix} S & C^{\mathrm{T}} \\ C & 0 \end{bmatrix} \begin{bmatrix} C_\perp A_\delta \\ \Lambda \end{bmatrix} = \begin{bmatrix} -S\Phi \\ 0 \end{bmatrix}. \tag{23}$$

Multiplying the first equation in (23) by $A_\delta^{\mathrm{T}} C_\perp^{\mathrm{T}}$ and noting that $C C_\perp = 0$ gives

$$A_\delta^{\mathrm{T}} C_\perp^{\mathrm{T}} S C_\perp A_\delta = -A_\delta^{\mathrm{T}} C_\perp^{\mathrm{T}} S \Phi. \tag{24}$$

Solving (23) for $C_\perp A_\delta$ yields

$$C_\perp A_\delta = [\, I \quad 0 \,] \begin{bmatrix} S & C^{\mathrm{T}} \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} -S\Phi \\ 0 \end{bmatrix}. \tag{25}$$

Substituting (22), (24), and (25) into $\Psi^{\mathrm{T}} S\Psi$ gives

$$\Psi^{\mathrm{T}} S\Psi = \Phi^{\mathrm{T}} S\Phi - \begin{bmatrix} S\Phi \\ 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} S & C^{\mathrm{T}} \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} S\Phi \\ 0 \end{bmatrix},$$

which is the same as (18). □

For an approximate solution λ, the corresponding solution of the assembled system is obtained by averaging the primal solution,

$$u = R^{\mathrm{T}} D_P^{\mathrm{T}} w, \quad w = w_\Delta + \Phi u_c.$$

The norm of the residual of the assembled system

$$r = R^{\mathrm{T}} g - R^{\mathrm{T}} S R u = R^{\mathrm{T}} g - R^{\mathrm{T}} S R R^{\mathrm{T}} D_P^{\mathrm{T}} w, \tag{26}$$

is used as the stopping criterion of the iterations. It can be efficiently obtained during the evaluation of the preconditioned residual as follows.

Theorem 4. *The primal residual r of the averaged primal solution can be computed together with the preconditioned dual residual ξ as*

$$r = R^{\mathrm{T}} v, \qquad \xi = B_D v, \quad \text{where } v = S B_D^{\mathrm{T}}(d - F\lambda). \tag{27}$$

In other words, the primal residual is obtained by assembling the output of the multiplication by the Schur complement in the preconditioner.

Proof. The dual problem is

$$d - F\lambda = Bw = 0, \tag{28}$$

where w is determined by minimizing $\mathcal{L}(w, \lambda)$ over $w \in \widetilde{W}$ for a given λ, cf. (14), (12) and (20). Hence, w satisfies

$$w \in \widetilde{W}, \quad y^{\mathrm{T}}\big(Sw + B^{\mathrm{T}}\lambda\big) = y^{\mathrm{T}} g, \quad \forall y \in \widetilde{W},$$

and using the facts that range $R \subset \widetilde{W}$ and $BR = 0$, it follows that $R^{\mathrm{T}} S w = R^{\mathrm{T}} g$. Substituting for $R^{\mathrm{T}} g$ into (26) and using (10) yields

$$r = R^{\mathrm{T}} S\big(I - R R^{\mathrm{T}} D_P^{\mathrm{T}}\big) w = R^{\mathrm{T}} S B_D^{\mathrm{T}} B w.$$

The preconditioned dual residual computed in PCG is

$$\zeta = M(d - F\lambda) = -B_D S B_D^{\mathrm{T}} B w,$$

from (21) and (28). □

The procedure (27) for calculating the primal residual is a generalization of a physics based argument from [22]. Theorem 4 and its proof applies also to the original FETI method; the only change in the proof is that we need to replace $\widetilde{W}$ by W.

Remark 5. The original formulation of FETI-DP in [5] can be written in this context as follows. The space W is decomposed

$$W = W_c \oplus W_r, \tag{29}$$

where the space W_c consists of functions that are continuous across interfaces and have all degrees of freedom equal to zero except at substructure corners, and the space W_r consists of functions with corner degrees of freedom equal to zero. A dual weight matrix Q_D is introduced such that

$$\widetilde{W} = \operatorname{null} Q_D^{\mathrm{T}} B. \tag{30}$$

Using new multipliers μ_i to enforce continuity of constraint values over substructure interfaces, $Q_D^{\mathrm{T}} B w = 0$, $w \in W_r$, the dual functional (14) is written as

$$\begin{aligned}
\mathcal{F}(\lambda) &= \min_{w\in\widetilde{W}} \mathcal{L}(w,\lambda) = \min_{w\in W} \max_{\mu} \mathcal{L}(w,\lambda) + w^{\mathrm{T}} B^{\mathrm{T}} Q_D \mu \\
&= \min_{w_c\in W_c} \min_{w_r\in W_r} \max_{\mu} \mathcal{L}(w_r + w_c, \lambda) + (w_r + w_c)^{\mathrm{T}} B^{\mathrm{T}} Q_D \mu \\
&= \min_{w_c\in W_c} \max_{\mu} \min_{w_r\in W_r} \mathcal{L}(w_r + w_c, \lambda) + w_r^{\mathrm{T}} B Q_D \mu,
\end{aligned}$$

because $B w_c = 0$. Note that here some of the constraints $Q_D^{\mathrm{T}} B w_r = 0$ may always be satisfied for $w_r \in W_r$, such as continuity at corners, but this only means that the associated multipliers μ are always zero. A linear system equivalent to the dual problem (13) is obtained from the stationary conditions for the augmented Lagrangian $\mathcal{L}(w_r + w_c, \lambda) + w_r^{\mathrm{T}} B Q_D \mu$. Taking first the derivative with respect to w_r gives the independent substructure problems, then the derivatives with respect to w_c and μ give the coarse problem, and finally the derivative with respect to λ, with substitutions for w_r, w_c, and μ, gives the dual operator $-F\lambda + d$, which is the same as in (20). However, the coarse problem from [5] is indefinite if more than just corners are used for the coarse space (called the augmented coarse space in [5]), because it is obtained from a combined minimization and maximization problem.

Remark 6. The main difference between the setting here and in [5,12] is that we choose to incorporate all constraints into Q_D (or C_i), not just the continuity of averages across faces or edges. This is a matter of notational convenience and implementation only, with no effect on the mathematical method.

4. The BDDC method

The system operator of the BDDC method is the assembled Schur complement

$$A = R^{\mathrm{T}} S R.$$

The preconditioner P is defined by

$$Pr = R^{\mathrm{T}} D_P (\Psi u_c + z), \tag{31}$$

where

$$\Psi^{\mathrm{T}} S \Psi u_c = \Psi^{\mathrm{T}} D_P^{\mathrm{T}} R r, \tag{32}$$

and

$$Sz + C^{\mathrm{T}} \mu = D_P^{\mathrm{T}} R r, \qquad Cz = 0. \tag{33}$$

Recall that the coarse basis functions Ψ are defined by energy minimization, cf. (5). The coarse system matrix $\Psi^{\mathrm{T}} S \Psi$ in (32) is the same as in the present formulation of FETI-DP, cf. Theorem 3. The matrices of the independent local problems in the substructures (33) are also the same as in FETI-DP, cf. (19), and no indefinite problems need to be solved in practice, cf. Remark 2. For a presentation of the BDDC preconditioner as an additive Schwarz method, see [17]. We will need another form of the preconditioned operator, for theoretical purposes.

Lemma 7. *The preconditioned operator PA satisfies, for any $u \in U$*

$$PAu = R^{\mathrm{T}} D_P w, \tag{34}$$

where w is defined by

$$w \in \widetilde{W}, \quad \langle Sw, v\rangle = \left\langle Au, R^{\mathrm{T}} D_P v\right\rangle \quad \forall v \in \widetilde{W}. \tag{35}$$

Proof. Denote in (31)

$$w = w_\Pi + w_\Delta, \quad w_\Pi = \Psi u_c,\ w_\Delta = z,$$

and let $r = Au$ and $\widetilde{W}_\Pi = \operatorname{range} \Psi$. Then from (32),

$$w_\Pi \in \widetilde{W}_\Pi, \quad \langle S w_\Pi, v_\Pi\rangle = \left\langle Au, R^{\mathrm{T}} D_P v_\Pi\right\rangle \quad \forall v_\Pi \in \widetilde{W}_\Pi, \tag{36}$$

and from (33),

$$w_\Delta \in \widetilde{W}_\Delta, \quad \langle S w_\Delta, v_\Delta\rangle = \left\langle Au, R^{\mathrm{T}} D_P v_\Delta\right\rangle \quad \forall v_\Delta \in \widetilde{W}_\Delta. \tag{37}$$

Adding (36) and (37) and using the fact that $\widetilde{W}_\Pi$ and $\widetilde{W}_\Delta$ are S-orthogonal by Lemma 8 gives the result. □

5. Properties of spaces and transfer operators

This section contains details on properties of the spaces and operators in the problem setting and formulation of the methods.

Lemma 8. *The primal space and the dual space form a decomposition of $\widetilde{W}$,*

$$\widetilde{W} = \widetilde{W}_\Pi \oplus \widetilde{W}_\Delta. \tag{38}$$

If the primal space is spanned by energy minimal functions, $\widetilde{W}_\Pi = \operatorname{range} \Psi$, then the decomposition is S-orthogonal:

$$\langle S w_\Pi, w_\Delta\rangle = 0 \quad \forall w_\Pi \in \widetilde{W}_\Pi,\ w_\Delta \in \widetilde{W}_\Delta. \tag{39}$$

Proof. Let $w \in \widetilde{W}_\Pi \cap \widetilde{W}_\Delta$. Then $w = \Phi u_c$ and $Cw = 0$. From (4), $C\Phi u_c = R_c u_c = 0$, so $u_c = 0$ because R_c has full column rank, and $w = \Phi u_c = 0$. Let $w \in \widetilde{W}$. From the definition of $\widetilde{W}$, cf. (6), $Cw = R_c u_c$ for some u_c. Consequently, $w = w_\Pi + w_\Delta$ where $w_\Pi = \Phi u_c \in \widetilde{W}_\Pi$, and $w_\Delta \in \widetilde{W}_\Delta$ because

$$C w_\Delta = C(w - \Phi u_c) = Cw - R_c u_c = 0.$$

If $\widetilde{W}_{\Pi} = \operatorname{range} \Psi$, then it follows from the first equation in (5) that

$$S\widetilde{W}_{\Pi} = \operatorname{range} S\Psi \subset \operatorname{range} C_{\perp}^{\mathrm{T}} \operatorname{null} C = \widetilde{W}_{\Delta},$$

which proves (39). □

The decomposition (38) was introduced in [12], where it was assumed that the primal space is in $\widehat{W}$. Here we will only need continuous coarse basis functions for theoretical purposes; we will assume that there exists a matrix $\widehat{\Phi}$ of coarse basis functions that are continuous across the substructure interfaces,

$$\exists \widehat{\Phi}\colon\ B\widehat{\Phi} = 0, \qquad C\widehat{\Phi} = R_c, \tag{40}$$

and denote

$$\widehat{W}_{\Pi} = \operatorname{range} \widehat{\Phi}.$$

The property (40) is needed only for Lemma 21 and Theorem 22, and $\widehat{\Phi}$ is not used in any algorithm. In fact, we find it convenient computationally to allow energy optimal coarse basis functions, which are discontinuous across substructure interfaces. By Lemma 8, it also holds that

$$\widetilde{W} = \widehat{W}_{\Pi} \oplus \widetilde{W}_{\Delta}. \tag{41}$$

As shown in the next lemma, the property (40) will be satisfied when each constraint (column of Q_P) involves only degrees of freedom from a single corner, edge, or a face of the substructures, which is the case in existing methods in the literature (though not always in implementations). Such sets of degrees of freedom were called globs in [16] and they are denoted by $\mathcal{G}_j$ below. The set $\mathcal{D}_i$ below is the set of the global degrees of freedom on substructure i.

Lemma 9. *Suppose that there are sets $\mathcal{D}_i$ and $\mathcal{G}_j$ such that $\{1, \ldots, m\} = \bigcup_{i=1}^{N} \mathcal{D}_i$, the sets $\mathcal{G}_j$ are mutually disjoint, each set $\mathcal{D}_i$ is the union of some of the sets $\mathcal{G}_j$,*

$$\left(R_i R_i^{\mathrm{T}}\right)_{k\ell} = \begin{cases} 1 & \text{if } k = \ell \in \mathcal{D}_i, \\ 0 & \text{otherwise}, \end{cases} \tag{42}$$

and for each column k of Q_P there is a unique set $\mathcal{G}_{s(k)}$ such that the column has nonzeros only in the rows with indices in the set $\mathcal{G}_{s(k)}$,

$$j \notin \mathcal{G}_{s(k)} \implies (Q_p)_{jk} = 0. \tag{43}$$

Then (40) *is satisfied.*

Proof. Let $\widehat{\Phi} = RQ_P(Q_P^{\mathrm{T}} Q_P)^{-1}$. Then $B\widehat{\Phi} = 0$ and

$$C_i \widehat{\Phi}_i = R_{ci} Q_P^{\mathrm{T}} R_i^{\mathrm{T}} R_i Q_P \left(Q_P^{\mathrm{T}} Q_P\right)^{-1}.$$

Since the sets of nonzeros of the columns of Q_P are mutually disjoint from (43), $Q_P^{\mathrm{T}} Q_P$ is diagonal. Fix i. Let $\mathcal{K}_i = \{k\colon\ \mathcal{G}_{s(k)} \subset \mathcal{D}_i\}$; this is the set of the indices of the columns of Q_P associated with the substructure i. Using (42), we get that

$$\left(Q_P^{\mathrm{T}} R_i^{\mathrm{T}} R_i Q_P \left(Q_P^{\mathrm{T}} Q_P\right)^{-1}\right)_{\mathcal{K}_i, \mathcal{K}_i} = I,$$

and because only the columns of R_{ci} with indices in the set $\mathcal{K}_i$ can be nonzero, we have that $C_i \widehat{\Phi}_i = R_{ci}$. □

We will now study in an abstract setting the properties of the matrices B, R, D_P, and B_D that will be needed in our convergence analysis. Some of the arguments in our proofs are abstract adaptations of the ideas from [10].

Lemma 10. *Let B_D be a matrix of the same size as B. Then for the statements below,* (44) *implies* (45)–(49), *and* (47)–(50) *imply* (44):

$$B_D^{\mathrm{T}}B + RR^{\mathrm{T}}D_P = I, \tag{44}$$
$$BB_D^{\mathrm{T}}B = B, \tag{45}$$
$$BB_D^{\mathrm{T}} \quad \textit{is a projection}, \tag{46}$$
$$B_D^{\mathrm{T}}B \quad \textit{is a projection}, \tag{47}$$
$$R^{\mathrm{T}}D_P B_D^{\mathrm{T}}B = 0, \tag{48}$$
$$B_D^{\mathrm{T}}Be = 0 \implies Be = 0, \tag{49}$$
$$Be = 0, \qquad e^{\mathrm{T}}D_P e = 0 \implies e = 0. \tag{50}$$

Proof. Multiplying (44) by B from the left and using (3) gives (45). Multiplying (45) by B_D^{T} from the right gives (46) and by B_D^{T} from the left gives (47). To show that (44) implies (48), multiply (44) by $B_D^{\mathrm{T}}B$ from the right and use (47) and R having full column rank. Finally, (49) follows from (45). Now assume that (48), (49) hold. We proceed similarly as in the proof of [10, Lemma 4.3]. Let $X = B_D^{\mathrm{T}}B + RR^{\mathrm{T}}D_P - I$ and note that

$$R^{\mathrm{T}}D_P X = \underbrace{R^{\mathrm{T}}D_P B_D^{\mathrm{T}}B}_{=0} + \underbrace{R^{\mathrm{T}}D_P R}_{=I} R^{\mathrm{T}}D_P - R^{\mathrm{T}}D_P = 0, \tag{51}$$

using (48) and (9), and

$$B_D^{\mathrm{T}}BX = \underbrace{B_D^{\mathrm{T}}BB_D^{\mathrm{T}}B}_{=B_D^{\mathrm{T}}B} + B_D^{\mathrm{T}}\underbrace{BR}_{=0}R^{\mathrm{T}}D_P - B_D^{\mathrm{T}}B = 0,$$

from (47) and (3). Let $e \in \operatorname{range} X$. From (49), $Be = 0$, so $e \in \operatorname{range} R$ by (3), $e^{\mathrm{T}}D_P e = 0$ from (51), and $e = 0$ follows by (50). □

In Theorems 14 (uniqueness of B_D) and 26 (equality of BDDC and FETI-DP eigenvalues), we will also need the following property (53), which is somewhat stronger than (48).

Lemma 11. *If* (44) *holds, then the following statements are equivalent*:

$$B_D^{\mathrm{T}}BB_D^{\mathrm{T}} = B_D^{\mathrm{T}}, \tag{52}$$
$$R^{\mathrm{T}}D_P B_D^{\mathrm{T}} = 0. \tag{53}$$

Proof. Multiplication of (44) by B_D^{T} from the right gives

$$R\left(R^{\mathrm{T}}D_P B_D^{\mathrm{T}}\right) = B_D^{\mathrm{T}} - B_D^{\mathrm{T}}BB_D^{\mathrm{T}},$$

and R has full column rank. This shows that (52) and (53) are equivalent. □

We can now identify a small and easy to verify set of assumptions that give all the needed properties of the operator B_D.

Theorem 12. *Assume that D_P is symmetric positive definite. Then* (45), (53),

$$BB_D^{\mathrm{T}}B = B, \qquad R^{\mathrm{T}}D_P B_D^{\mathrm{T}} = 0,$$

imply (44), (52),

$$B_D^{\mathrm{T}}B + RR^{\mathrm{T}}D_P = I, \qquad B_D^{\mathrm{T}}BB_D^{\mathrm{T}} = B_D^{\mathrm{T}},$$

and the rest of the properties in Lemmas 10 *and* 11.

Proof. Use Lemmas 10 and 11, noting that (53) implies (48) and (45) multiplied by B_D^{T} from the right gives (47). □

Remark 13. The property (53) means that the dual weights are compatible with the primal weights in such a way that when B_D^{T} makes values on the interfaces into jumps, then the weighted averaging by $R^{\mathrm{T}}D_P$ will make the jumps into zero. The advantage of assuming (45) and (53) is that they involve B_D only once.

Theorem 14. *If D_P is symmetric positive definite, then the properties* (44) *and* (52)

$$B_D^{\mathrm{T}}B + RR^{\mathrm{T}}D_P = I, \qquad B_D^{\mathrm{T}}BB_D^{\mathrm{T}} = B_D^{\mathrm{T}},$$

imply that the matrix B_D^{T} is the Moore–Penrose pseudoinverse of B with respect to the inner product $\langle D_P u, v\rangle$ on W and an inner product on Λ such that the projection BB_D^{T} is orthogonal. In particular, in the case when $BB_D^{\mathrm{T}} = I$ (that is, for nonredundant Lagrange multipliers), *the properties* (44) *and* (52) *determine B_D uniquely.*

Proof. As noted before, (45) follows from (44). We have

$$\big\langle D_P\big(RR^{\mathrm{T}}D_P u\big), v\big\rangle = \big\langle D_P u, RR^{\mathrm{T}}D_P v\big\rangle,$$

so $RR^{\mathrm{T}}D_P$ is selfadjoint in the inner product $\langle D_P u, v\rangle$. From (44), $B_D^{\mathrm{T}}B = I - RR^{\mathrm{T}}D_P$, so $B_D^{\mathrm{T}}B$ is also selfadjoint. Therefore, from (45), and (52), and from the facts that $B_D^{\mathrm{T}}B$ and BB_D^{T} are selfadjoint, it follows that B_D is the Moore–Penrose pseudoinverse of B, which is unique [21]. □

As an example, consider a construction of B_D as

$$B_D = [D_{D1}B_1, \ldots, D_{DN}B_N], \tag{54}$$

where $D_{Di} : \Lambda \to \Lambda$. We first specify the matrices R and B. For a global degree of freedom α, let u^α be its value in a global vector u, let $G(\alpha)$ denote the set of indices of substructures that share this degree of freedom, and let $\ell_i(\alpha)$ be the local number of this degree of freedom in substructure i; the entry of w_i that corresponds to u^α is then $(w_i)_{\ell_i(\alpha)}$; however, to simply notation, we will write w_i^α instead. The matrix R is constructed so that

$$w = Ru \iff w_i^\alpha = u^\alpha \quad \forall\alpha\forall i \in G(\alpha). \tag{55}$$

Each row of B enforces a continuity constraint for a degree of freedom u^α between substructures i and j,

$$(Bw)_{(\alpha,i,j)} = w_i^\alpha - w_j^\alpha = 0. \tag{56}$$

The rows of B are indexed by the triples (α, i, j) in some index set $\mathcal{C}$. Assume that

$$(\alpha, i, j) \in \mathcal{C} \implies (\alpha, j, i) \notin \mathcal{C}. \tag{57}$$

This means that the same constraint (56) cannot be present in both directions, and trivial constraints with $i = j$ are not allowed as well. However, other redundant constraints are possible. Denote

$$\mathcal{S} = \{(\alpha, i, j)\colon (\alpha, i, j) \in \mathcal{C} \text{ or } (\alpha, j, i) \in \mathcal{C}\}.$$

That is, $(\alpha, i, j) \in \mathcal{S}$ if the condition $Bw = 0$ constrains the degree of freedom α to have the same value in substructures i and j.

We now specify the weight matrices D_P and D_{Dk}. Let the primal weight matrix D_P be diagonal with positive diagonal entries, denoted by d_i^α, so that

$$(D_P w)_i^\alpha = d_i^\alpha w_i^\alpha, \quad d_i^\alpha > 0 \ \forall i \in G(\alpha). \tag{58}$$

As is well known, the decomposition of unity property (9) becomes

$$\sum_{i \in G(\alpha)} d_i^\alpha = 1 \quad \forall \alpha. \tag{59}$$

Let the dual weight matrices D_{Dk} in (54) be also diagonal, and denote the diagonal entry of D_{Di} in the row (α, i, j) by d_{ij}^α. In this notation, (54) becomes

$$(B_D w)_{(\alpha,i,j)} = d_{ij}^\alpha w_i^\alpha - d_{ji}^\alpha w_j^\alpha, \quad (\alpha, i, j) \in \mathcal{C}. \tag{60}$$

Lemma 15. *It holds that $BB_D^{\mathrm{T}}B = B$ if and only if*

$$\forall (\alpha, i, j) \in \mathcal{C}\colon\ d_{ji}^\alpha + \sum_{k\colon (\alpha,i,k) \in \mathcal{S}} d_{ik}^\alpha = 1, \tag{61}$$

$$\forall \alpha \forall k \in G(\alpha)\ \exists \delta_k^\alpha\ \forall i, (\alpha, i, k) \in \mathcal{S}\colon\ d_{ik}^\alpha = \delta_k^\alpha. \tag{62}$$

Proof. By a direct computation,

$$\begin{aligned}
\left(B_D^{\mathrm{T}}\lambda\right)_i^\alpha &= \sum_{j\colon (\alpha,i,j)\in\mathcal{C}} d_{ij}^\alpha \lambda_{(\alpha,i,j)} - \sum_{j\colon (\alpha,j,i)\in\mathcal{C}} d_{ij}^\alpha \lambda_{(\alpha,j,i)}, \\
\left(B_D^{\mathrm{T}}Bw\right)_i^\alpha &= \sum_{j\colon (\alpha,i,j)\in\mathcal{C}} d_{ij}^\alpha \left(w_i^\alpha - w_j^\alpha\right) - \sum_{j\colon (\alpha,j,i)\in\mathcal{C}} d_{ij}^\alpha \left(w_j^\alpha - w_i^\alpha\right) = \sum_{k\colon (\alpha,i,k)\in\mathcal{S}} d_{ik}^\alpha \left(w_i^\alpha - w_k^\alpha\right).
\end{aligned}$$

Now the equation $Bw = BB_D^{\mathrm{T}}Bw$ becomes, for $(\alpha, i, j) \in \mathcal{C}$,

$$\begin{aligned}
w_i^\alpha - w_j^\alpha &= \left[B_D^{\mathrm{T}}Bw\right]_i^\alpha - \left[B_D^{\mathrm{T}}Bw\right]_j^\alpha \\
&= \sum_{k\colon (\alpha,i,k)\in\mathcal{S}} d_{ik}^\alpha \left(w_i^\alpha - w_k^\alpha\right) - \sum_{k\colon (\alpha,j,k)\in\mathcal{S}} d_{jk}^\alpha \left(w_j^\alpha - w_k^\alpha\right).
\end{aligned}$$

Comparing the coefficients of w_i^α gives (61), and comparing the coefficients of w_j^α gives (61) again with i and j switched. Fix α. If k is such that $(\alpha, i, k) \in \mathcal{S}$ and $(\alpha, j, k) \in \mathcal{S}$, then the comparison of the coefficients of w_k^α gives that $d_{ik}^\alpha = d_{jk}^\alpha$. The graph with the nodes $G(\alpha)$ and the edges $\{(i,j)\colon (\alpha,i,j)\in\mathcal{S}\}$ is connected, otherwise (56) could not constrain all w_i^α, $i\in G(\alpha)$, to the same value. Hence, d_{ik}^α have the same value for all i such that $(\alpha,i,k)\in\mathcal{S}$, and denoting the value by δ_k^α gives (62). □

Lemma 16. *It holds that $R^{\mathrm T} D_P B_D^{\mathrm T} = 0$ if and only if $d_{ij}^\alpha d_i^\alpha = d_{ji}^\alpha d_j^\alpha$ for all $(\alpha,i,j)\in\mathcal{S}$.*

Proof. From (55), (58), and (60),

$$(B_D D_P R u)_{(\alpha,i,j)} = d_{ij}^\alpha d_i^\alpha u^\alpha - d_{ji}^\alpha d_j^\alpha u^\alpha = \left(d_{ij}^\alpha d_i^\alpha - d_{ji}^\alpha d_j^\alpha\right) u^\alpha,$$

and $R^{\mathrm T} D_P B_D^{\mathrm T} = 0$ if and only if the coefficients at all u^α are zero. □

Theorem 17. *If the matrices R, B, and B_D are constructed according to* (54)–(60), *then the conditions* (44) *and* (52),

$$B_D^{\mathrm T} B + R R^{\mathrm T} D_P = I, \qquad B_D^{\mathrm T} B B_D^{\mathrm T} = B_D^{\mathrm T},$$

determine the weights in B_D uniquely as

$$d_{ij}^\alpha = d_j^\alpha \quad \forall(\alpha,i,j)\in\mathcal{S}, \tag{63}$$

and all properties in Lemmas 10 *and* 11 *are satisfied.*

Proof. From Lemma 16,

$$\frac{d_{ij}^\alpha}{d_j^\alpha} = \frac{d_{ji}^\alpha}{d_i^\alpha} \quad \forall(\alpha,i,j)\in\mathcal{S}.$$

From (62), the common value of the fractions does not depend on i and j, so $d_{ij}^\alpha = c^\alpha d_j^\alpha$ for some c^α. Substituting into (61) and using (59) and (57) gives $c^\alpha = 1$. □

Remark 18. The recipe (63) is the same as the construction of B_D in [5,10]. The recipe given in [12] can be viewed as a special case of the construction given here. The identity (44), $B_D^{\mathrm T} B + R R^{\mathrm T} D_P = I$, was first formulated and proved in [23] for D_P by counting, and in [10] for general weights in the setting (54)–(60). The identity (45), $B = B B_D^{\mathrm T} B$, was first stated and proved in [10]. Also in [10], the construction $B_D^{\mathrm T} = D_P^{-1} B^{\mathrm T} (B D_P^{-1} B^{\mathrm T})^{-1}$ was used in the case of nonredundant multipliers (i.e., linearly independent rows of B). From Theorem 14, this is the same B_D as in (54). It was observed in [7] that, in many variants of FETI and BDD methods, (44) and (47) hold. A construction of [22], called mechanically consistent, also satisfies the conditions (45) and (44). The role of the identity (52), $B_D^{\mathrm T} B B_D^{\mathrm T} = B_D^{\mathrm T}$, in the substructuring context appears to be new.

In the rest of this paper, it will be assumed that (44) holds, thus all of the statements (44)–(49) hold. We need a few more auxiliary results.

Lemma 19. *It holds that*

$$B_D^{\mathrm T} B \widetilde{W} \subset \widetilde{W}. \tag{64}$$

Proof. From range $R = \widehat{W} \subset \widetilde{W}$, we have

$$B_D^{\mathrm{T}} B \widetilde{W} = \left(I - RR^{\mathrm{T}} D_P\right) \widetilde{W} \subset \widetilde{W} + \operatorname{range} R = \widetilde{W},$$

which is (64). □

Lemma 20. *If* $w \in \widetilde{W}_\Delta$ *and* $\|B_D^{\mathrm{T}} B w\|_S = 0$ *then* $Bw = 0$.

Proof. Let $w \in \widetilde{W}_\Delta$ and $\|B_D^{\mathrm{T}} B w\|_S = 0$. Because $B_D^{\mathrm{T}} B w \in \widetilde{W}$ by (64) and S is positive definite on $\widetilde{W}$, it holds that $B_D^{\mathrm{T}} B w = 0$, and $Bw = 0$ follows from (49). □

6. The FETI-DP estimate

The purpose of our analysis is to reduce the estimate of the condition number of FETI-DP to the single inequality (68). Throughout this section, we assume that the primal functions are continuous across the substructure interfaces, $\widetilde{W}_\Pi \subset \widehat{W}$. Since the FETI-DP preconditioned operator does not depend on the specific choice of $\widetilde{W}_\Pi$, cf. Remark 1, this does not restrict the generality of the analysis. Define the space of jumps on the subdomain interfaces and its dual by

$$V = B\widetilde{W}_\Delta, \qquad V' = \left\{\lambda:\ B^{\mathrm{T}}\lambda \in \widetilde{W}_\Delta\right\},$$

equipped with the norms

$$\|\zeta\|_V = \left\|B_D^{\mathrm{T}}\zeta\right\|_S, \tag{65}$$

and

$$\|\lambda\|_{V'} = \max_{\substack{\zeta \in V \\ \zeta \neq 0}} \frac{|\zeta^{\mathrm{T}}\lambda|}{\|\zeta\|_V} = \max_{\substack{v_\Delta \in \widetilde{W}_\Delta \\ Bv_\Delta \neq 0}} \frac{|v_\Delta^{\mathrm{T}} B^{\mathrm{T}} \lambda|}{\|B_D^{\mathrm{T}} B v_\Delta\|_S}. \tag{66}$$

From Lemma 20, $\|B_D^{\mathrm{T}}\zeta\|_S = 0$, $\zeta \in B\widetilde{W}_\Delta$ implies $\zeta = 0$, so (65) indeed defines a norm, and V' is a space of representants for the factorspace $\Lambda/(B\widetilde{W}_\Delta)^\perp$, so (66) also defines a norm. A form of the operator F useful for analysis is given by the following restatement of [19, Lemma 4.2] in the present context (the “corner degrees of freedom” u_c in [19] become “primal” quantities w_Π, and the “remaining degrees of freedom” u_r from [19] become the “dual” quantities w_Δ).

Lemma 21. *Let* F *be the dual operator defined by* (20)*. Then*

$$\lambda^{\mathrm{T}} F \lambda = \lambda^{\mathrm{T}} B \widetilde{S}^{-1} B^{\mathrm{T}} \lambda \quad \forall \lambda \in V',$$

where $\widetilde{S} : \widetilde{W}_\Delta \to \widetilde{W}_\Delta$ *is the symmetric operator defined by the quadratic form*

$$\forall w \in \widetilde{W}_\Delta:\ w_\Delta^{\mathrm{T}} \widetilde{S} w_\Delta = \min_{w_\Pi \in \widehat{W}_\Pi} (w_\Delta + w_\Pi)^{\mathrm{T}} S (w_\Delta + w_\Pi). \tag{67}$$

Proof. From (41) and $Bw_\Pi = 0$ for $w_\Pi \in \widehat{W}_\Pi$, it follows that

$$\begin{aligned}
\mathcal{F}(\lambda) &= \min_{w\in\widetilde{W}} \mathcal{L}(w,\lambda) \\
&= \min_{w_\Delta\in\widetilde{W}_\Delta} \min_{w_\Pi\in\widehat{W}_\Pi} \frac{1}{2}(w_\Pi+w_\Delta)^{\mathrm{T}} S(w_\Pi+w_\Delta) - (w_\Pi+w_\Delta)^{\mathrm{T}}\big(g-B^{\mathrm{T}}\lambda\big) \\
&= \min_{w_\Delta\in\widetilde{W}_\Delta} \min_{w_\Pi\in\widehat{W}_\Pi} \left(\frac{1}{2}(w_\Pi+w_\Delta)^{\mathrm{T}} S(w_\Pi+w_\Delta) - w_\Pi^{\mathrm{T}} g\right) - w_\Delta^{\mathrm{T}}\big(g-B^{\mathrm{T}}\lambda\big) \\
&= \min_{w_\Delta\in\widetilde{W}_\Delta} \frac{1}{2} w_\Delta^{\mathrm{T}}\widetilde{S} w_\Delta + w_\Delta^{\mathrm{T}} B^{\mathrm{T}}\lambda - w_\Delta^{\mathrm{T}} e \\
&= -\frac{1}{2}\big(B^{\mathrm{T}}\lambda - e\big)^{\mathrm{T}}\widetilde{S}^{-1}\big(B^{\mathrm{T}}\lambda - e\big),
\end{aligned}$$

for some e. Consequently, the quadratic term in $\mathcal{F}(\lambda)$ is $-\frac{1}{2}\lambda^{\mathrm{T}} B\widetilde{S}^{-1}B^{\mathrm{T}}\lambda$, and a comparison with (20) concludes the proof. □

It follows from Remark 1 that F is independent of the particular choices of $\widetilde{W}_\Pi$ and $\widehat{W}_\Pi$. The following theorem provides a bound on the condition number. It is an abstract restatement of the arguments in [12, 19]. We will assume that

$$\big\|B_D^{\mathrm{T}} B u\big\|_S^2 \leqslant \omega \|u\|_S^2 \quad \forall u\in\widetilde{W}. \tag{68}$$

Theorem 22. *The condition number of FETI-DP is bounded by*

$$\kappa_{\text{FETI-DP}} = \frac{\lambda_{\max}(MF)}{\lambda_{\min}(MF)} \leqslant \omega.$$

Proof. From Lemma 21 and (66), we have $F: V' \to V$ and

$$\lambda^{\mathrm{T}} F\lambda = \big\|B^{\mathrm{T}}\lambda\big\|_{\widetilde{S}^{-1}}^2 = \max_{\substack{v_\Delta\neq 0\\ v_\Delta\in\widetilde{W}_\Delta}} \frac{|v_\Delta^{\mathrm{T}} B^{\mathrm{T}}\lambda|^2}{\|v_\Delta\|_{\widetilde{S}}^2}. \tag{69}$$

A lower bound on F follows in turn from the substitution $v_\Delta = B_D^{\mathrm{T}} B w_\Delta$, $w_\Delta\in\widetilde{W}_\Delta$, in (69) using the property $B_D^{\mathrm{T}} B\widetilde{W}\subset\widetilde{W}$ from (64), minimizing over a subset, the property $BB_D^{\mathrm{T}}B = B$ from (45), and Lemma 21:

$$\begin{aligned}
\lambda^{\mathrm{T}} F\lambda &= \max_{\substack{v_\Delta\neq 0\\ v_\Delta\in\widetilde{W}_\Delta}} \frac{|v_\Delta^{\mathrm{T}} B^{\mathrm{T}}\lambda|^2}{\|v_\Delta\|_{\widetilde{S}}^2} \geqslant \max_{\substack{B_D^{\mathrm{T}} B w_\Delta\neq 0\\ w_\Delta\in\widetilde{W}_\Delta}} \frac{|w_\Delta^{\mathrm{T}} B^{\mathrm{T}} B_D B^{\mathrm{T}}\lambda|^2}{\|B_D^{\mathrm{T}} B w_\Delta\|_{\widetilde{S}}^2} \\
&= \max_{\substack{B_D^{\mathrm{T}} B w_\Delta\neq 0\\ w_\Delta\in\widetilde{W}_\Delta}} \frac{|w_\Delta^{\mathrm{T}} B^{\mathrm{T}}\lambda|^2}{\|B_D^{\mathrm{T}} B w_\Delta\|_{\widetilde{S}}^2} \geqslant \max_{\substack{B_D^{\mathrm{T}} B w_\Delta\neq 0\\ w_\Delta\in\widetilde{W}_\Delta}} \frac{|w_\Delta^{\mathrm{T}} B^{\mathrm{T}}\lambda|^2}{\|B_D^{\mathrm{T}} B w_\Delta\|_{S}^2} = \|\lambda\|_{V'}^2.
\end{aligned}$$

To establish an upper bound, let $v_\Delta\in\widetilde{W}_\Delta$. From (67), $\|v_\Delta\|_{\widetilde{S}}^2 = \|v_\Delta - v_\Pi\|_S^2$ for some $v_\Pi\in\widehat{W}_\Pi\subset\widetilde{W}$. Then $v_\Delta - v_\Pi\in\widetilde{W}$, and, using (68) and $Bv_\Pi = 0$,

$$\omega\|v_\Delta\|_{\widetilde{S}}^2 = \omega\|v_\Delta - v_\Pi\|_S^2 \geqslant \big\|B_D^{\mathrm{T}} B(v_\Delta - v_\Pi)\big\|_S^2 = \big\|B_D^{\mathrm{T}} B v_\Delta\big\|_S^2.$$

Thus,

$$\lambda^{\mathrm{T}} F \lambda = \max_{\substack{v_\Delta \neq 0 \\ v_\Delta \in \widetilde{W}_\Delta}} \frac{|v_\Delta^{\mathrm{T}} B^{\mathrm{T}} \lambda|^2}{\|v_\Delta\|_{\widetilde{S}}^2} \leqslant \omega \max_{\substack{B v_\Delta \neq 0 \\ v_\Delta \in \widetilde{W}_\Delta}} \frac{|v_\Delta^{\mathrm{T}} B^{\mathrm{T}} \lambda|^2}{\|B_D^{\mathrm{T}} B v_\Delta\|_{\widetilde{S}}^2} = \omega \|\lambda\|_{V'}^2.$$

From the definition of $\|\zeta\|_V$ and the definition of the preconditioner M, cf. (21), we have $M : V \to V'$ and $\zeta^{\mathrm{T}} M \zeta = \|\zeta\|_V^2$. Consequently,

$$c_1 \|\lambda\|_{V'}^2 \leqslant \langle \lambda, F\lambda \rangle \leqslant c_2 \|\lambda\|_{V'}^2 \quad \forall \lambda \in V',$$
$$c_3 \|v\|_V^2 \leqslant \langle v, Mv \rangle \leqslant c_4 \|v\|_V^2 \quad \forall v \in V,$$

with $c_1 = 1$, $c_2 = \omega$, $c_3 = c_4 = 1$, and, using [18, Lemma 3.1], if follows that $\kappa_{\text{FETI-DP}} \leqslant \frac{c_2 c_4}{c_1 c_3} = \omega$. $\square$

Remark 23. Because the preconditioned FETI-DP operator works in a factorspace of Λ, its computational realization, which is a linear operator on Λ, will in general have zero eigenvalues. This is not a problem, the iterations will simply run in the factorspace.

Remark 24. The condition number bound in Theorem 22 does not depend on the selection of the primal space $\widetilde{W}_\Pi$.

7. The BDDC estimate

Theorem 25. *The condition number BDDC method is bounded by*

$$\kappa_{\mathrm{BDDC}} = \frac{\lambda_{\max}(PA)}{\lambda_{\min}(PA)} \leqslant \omega.$$

Proof. The proof follows the ideas from [10]. The main difference is that here the coarse component of BDDC lends itself to much the same treatment as the local components, which allows for a significant simplification, while the analysis in [10] had to be done on the complement of the coarse space of the original BDD method.

We will find bounds on $\langle Au, PAu \rangle$ in terms of $\langle Au, u \rangle$. For the lower bound, first set in (35) $v = Ru$, note that $R^{\mathrm{T}} D_P R = I$, and use the Cauchy inequality to get

$$\langle Au, u \rangle = \left\langle Au, R^{\mathrm{T}} D_P R u \right\rangle = \langle Sw, Ru \rangle \leqslant \langle Sw, w \rangle^{1/2} \langle SRu, Ru \rangle^{1/2}. \tag{70}$$

Setting in (35) $v = w$, and using (34) gives

$$\langle Sw, w \rangle = \left\langle Au, R^{\mathrm{T}} D_P w \right\rangle = \langle Au, PAu \rangle. \tag{71}$$

From the definition of A,

$$\langle SRu, Ru \rangle = \left\langle R^{\mathrm{T}} SRu, u \right\rangle = \langle Au, u \rangle. \tag{72}$$

Substituting (71) and (72) into (70) gives

$$\langle Au, u \rangle \leqslant \langle Au, PAu \rangle^{1/2} \langle Au, u \rangle^{1/2},$$

and, consequently,

$$\langle Au, u \rangle \leqslant \langle Au, PAu \rangle,$$

which implies $\lambda_{\min}(PA) \geqslant 1$.

For the upper bound, use (34) and (71) to get

$$\langle APAu, PAu\rangle = \left\langle R^{\mathrm{T}}SRPAu, PAu\right\rangle = \langle SRPAu, RPAu\rangle$$
$$= \left\|RR^{\mathrm{T}}D_P w\right\|_S^2 \leqslant \left\|RR^{\mathrm{T}}D_P\right\|_S^2 \|w\|_S^2 = \left\|RR^{\mathrm{T}}D_P\right\|_S^2 \langle Au, PAu\rangle,$$

where $\|RR^{\mathrm{T}}D_P\|_S$ is the norm of $RR^{\mathrm{T}}D_P$ as an operator on $\widetilde{W}$. Since $RR^{\mathrm{T}}D_P$ is a projection and the norm of a nontrivial projection in an inner product space depends only on the angle between its range and its nullspace [8], it holds that

$$\left\|RR^{\mathrm{T}}D_P\right\|_S = \left\|I - RR^{\mathrm{T}}D_P\right\|_S = \left\|B_D^{\mathrm{T}}B\right\|_S.$$

From the assumption (68), it follows that

$$\langle APAu, PAu\rangle \leqslant \omega\langle Au, PAu\rangle.$$

Using this and the Cauchy inequality gives

$$\langle Au, PAu\rangle \leqslant \langle Au, u\rangle^{1/2}\langle APAu, PAu\rangle^{1/2} \leqslant \langle Au, u\rangle^{1/2}\omega^{1/2}\langle Au, PAu\rangle^{1/2},$$

and, consequently,

$$\langle Au, PAu\rangle \leqslant \omega\langle Au, u\rangle,$$

which implies $\lambda_{\max}(PA) \leqslant \omega$. □

So, the bounds on the condition numbers of BDDC and FETI-DP are the same. The next theorem shows that there is an even closer correspondence between the methods. The proof is somewhat tedious, so it is postponed until the next section.

Theorem 26. *The eigenvalues of the preconditioned operators of the FETI-DP and the BDDC methods are the same except possibly for the eigenvalues equal to zero and one.*

We refer to the literature for estimates of the constant ω: In [19], it was proved under the assumptions usual in substructuring methods that

$$\omega \leqslant \mathrm{const}\left(1 + \log^2\frac{H}{h}\right),$$

where H is the characteristic size of a substructure, and h is the element size, for both second and fourth order problems in 2D, and constraints on values at substructure corners. In [12], this estimate was extended to problems in 3D and for coefficient jumps between substructures by including constraints based on averages over substructure edges. However, the logarithmic estimates are a subtle matter, particularly for elasticity [9]. For selecting relatively few coarse degrees of freedom in elasticity so that the logarithmic bounds still apply, see [11]. For selecting relatively few coarse degrees of freedom in elasticity and still guarantee that the coarse problem is nonsingular, see [14].

8. Proof of equality of eigenvalues of FETI-DP and BDDC

In this section, we assume that the coarse basis functions in FETI-DP are defined by energy minimization, that is, $\Phi = \Psi$. Because the preconditioned operator of FETI-DP is invariant to the selection of coarse basis functions, this does not cause any loss of generality.

Here is an explicit form of the preconditioned operators.

Lemma 27. *The preconditioned operators of FETI-DP and BDDC are,*

$$MF = B_D S B_D^{\mathrm{T}} B H B^{\mathrm{T}}, \tag{73}$$

$$PA = R^{\mathrm{T}} D_P H D_P^{\mathrm{T}} R R^{\mathrm{T}} S R, \tag{74}$$

respectively, where

$$H = \begin{bmatrix} I \\ 0 \\ \Psi^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} S & C^{\mathrm{T}} & 0 \\ C & 0 & 0 \\ 0 & 0 & \Psi^{\mathrm{T}} S \Psi \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \\ \Psi^{\mathrm{T}} \end{bmatrix}.$$

Proof. From the definition of the BDDC preconditioner (31), it follows that

$$P = R^{\mathrm{T}} D_P H D_P^{\mathrm{T}} R,$$

which gives the BDDC preconditioned operator (74).

An explicit expression for the FETI-DP operator: $\lambda \mapsto F\lambda$ is obtained by setting $g = 0$ in the first three equations of (16), eliminating w_Δ, and u_c from the first three equations and substituting in the fourth, which gives

$$F = B \overline{H} B^{\mathrm{T}},$$

where

$$\overline{H} = \begin{bmatrix} I \\ 0 \\ \Psi^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} S & C^{\mathrm{T}} & S\Psi \\ C & 0 & 0 \\ \Psi^{\mathrm{T}} S & 0 & \Psi^{\mathrm{T}} S \Psi \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \\ \Psi^{\mathrm{T}} \end{bmatrix}.$$

The FETI-DP preconditioner is $M = B_D S B_D^{\mathrm{T}}$, so to finish the proof of (73), it is enough to show that $H = \overline{H}$.

Let $w = Hz$ and $\bar{w} = \overline{H} z$. Then

$$w = x + \Psi \mu, \qquad \bar{w} = \bar{x} + \Psi \bar{\mu},$$

where

$$\begin{bmatrix} S & C^{\mathrm{T}} & 0 \\ C & 0 & 0 \\ 0 & 0 & \Psi^{\mathrm{T}} S \Psi \end{bmatrix} \begin{bmatrix} x \\ \zeta \\ \mu \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ \Psi^{\mathrm{T}} z \end{bmatrix}, \tag{75}$$

$$\begin{bmatrix} S & C^{\mathrm{T}} & S\Psi \\ C & 0 & 0 \\ \Psi^{\mathrm{T}} S & 0 & \Psi^{\mathrm{T}} S \Psi \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{\zeta} \\ \bar{\mu} \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ \Psi^{\mathrm{T}} z \end{bmatrix}. \tag{76}$$

From the second equation in (76), it follows that $C\bar{x} = 0$. From the definition of Ψ by energy minimization,

$$S\Psi = -C^{\mathrm{T}} \Lambda, \tag{77}$$

which gives by taking the transpose

$$\Psi^{\mathrm{T}} S \bar{x} = -\Lambda^{\mathrm{T}} \underbrace{C\bar{x}}_{=0} = 0.$$

Consequently, from the last equations in (75) and in (76),

$$\bar{\mu} = \left(\Psi^{\mathrm{T}} S \Psi\right)^{-1}\left(\Psi^{\mathrm{T}} z - \Psi^{\mathrm{T}} S \bar{x}\right) = \left(\Psi^{\mathrm{T}} S \Psi\right)^{-1} \Psi z = \mu.$$

Using (77) again, we have $S\Psi\bar{\mu} = -C^{\mathrm{T}}\Lambda\bar{\mu}$, and we get from the first row of (76),

$$S\bar{x} + C^{\mathrm{T}}\left(\bar{\zeta} - \Lambda\bar{\mu}\right) = z.$$

Consequently, $x = \bar{x}$ and $\zeta = \bar{\zeta} - \Lambda\bar{\mu}$ satisfy the first two equations in (75), which determine the value of x. □

We need some properties of H.

Lemma 28. *It holds that*

$$HS \text{ is an } S\text{-orthogonal projection onto } \widetilde{W}, \tag{78}$$

$$\operatorname{range} H = \widetilde{W}, \tag{79}$$

$$HSB_D^{\mathrm{T}}BH = B_D^{\mathrm{T}}BH, \tag{80}$$

$$HSR = R, \tag{81}$$

and

$$B_D SHB^{\mathrm{T}} \quad \text{is a projection,} \tag{82}$$

$$HD_P^{\mathrm{T}}RR^{\mathrm{T}}S \quad \text{is a projection.} \tag{83}$$

Proof. Note that the S-orthogonal projection onto $\widetilde{W}$ is well defined because S was assumed to be positive definite on $\widetilde{W}$. To prove (78), consider $w = Hz$, which is the same as

$$w = x + \Psi\mu, \qquad \begin{bmatrix} S & C^{\mathrm{T}} & 0 \\ C & 0 & 0 \\ 0 & 0 & \Psi^{\mathrm{T}} S \Psi \end{bmatrix} \begin{bmatrix} x \\ \zeta \\ \mu \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ \Psi^{\mathrm{T}} z \end{bmatrix}. \tag{84}$$

Clearly, x is determined only from the first two equations, and $x \in \ker C$, hence $w = x + \Psi\mu \in \ker C \oplus \operatorname{range}\Psi = \widetilde{W}$. This shows that

$$\operatorname{range} H \subset \widetilde{W}.$$

Now let in (84) $z = Sy$, that is, $w = HSy$, which gives

$$\left.\begin{array}{l} Sx + C^{\mathrm{T}}\zeta = Sy, \\ Cx = 0, \end{array}\right\} \iff \left\{\begin{array}{l} S(x - y) \in \operatorname{range} C^{\mathrm{T}} = (\ker C)^{\perp}, \\ x \in \ker C. \end{array}\right.$$

Consequently, x is the S-orthogonal projection of y on $\ker C = W_{\Delta}$. Clearly, $\Psi\mu$ is the S-orthogonal projection of z onto $\operatorname{range}\Psi$, because

$$\Psi\mu = \Psi\left(\Psi^{\mathrm{T}} S \Psi\right)^{-1} \Psi^{\mathrm{T}} S y.$$

Because $\ker C$ and $\operatorname{range}\Psi$ are S-orthogonal by (39), $x + \Psi\mu$ is the S-orthogonal projection of y onto $\ker C \oplus \operatorname{range}\Psi = \widetilde{W}$, which proves (78). This also implies that $\operatorname{range} H \supset \widetilde{W}$, which completes the proof of (79). To prove (80), note that

$$\operatorname{range} B_D^{\mathrm{T}} B H = B_D^{\mathrm{T}} B \operatorname{range} H = B_D^{\mathrm{T}} B \widetilde{W} \subset \widetilde{W}$$

from (64) and (75), and HS restricted to $\widetilde{W}$ is the identity. Eq. (81) follows from the facts that range $R = \widehat{W} \subset \widetilde{W}$ and HS restricted to $\widetilde{W}$ is the identity. To prove (82), consider that

$$B_D \underbrace{SHB^{\mathrm{T}}B_DSH}_{SH} B^{\mathrm{T}} = B_DSHB^{\mathrm{T}},$$

using the facts that range $SH = \widetilde{W}$, $B^{\mathrm{T}}B_D\widetilde{W} \subset \widetilde{W}$, and SH is a projection onto $\widetilde{W}$. Finally, to prove (83), compute

$$SRR^{\mathrm{T}}D_P \underbrace{HS}_{=I \text{ on range } R} RR^{\mathrm{T}}D_PH = SRR^{\mathrm{T}}D_PRR^{\mathrm{T}}D_PH = SRR^{\mathrm{T}}D_PH$$

using the facts that HS is a projection onto $\widetilde{W} \supset \widehat{W} = \operatorname{range} R$ and $RR^{\mathrm{T}}D_P$ is a projection. □

The proof that the eigenvalues of the preconditioned operators are the same follows by mapping the eigenvectors.

Lemma 29. *Let $T = R^{\mathrm{T}}D_PHB^{\mathrm{T}}$ and $Q = MFB_DSR$. Then*

$$PAT = TMF, \tag{85}$$
$$QPA = MFQ, \tag{86}$$
$$MFu = \lambda u, \quad u \neq 0,\ \lambda \neq 0, 1 \implies Tu \neq 0, \tag{87}$$
$$PAv = \lambda v, \quad v \neq 0,\ \lambda \neq 1 \implies Qv \neq 0. \tag{88}$$

Consequently, for $\lambda \neq 0, 1$, the matrices T and Q from Lemma 29 give a correspondence of eigenvectors of the preconditioned operators MF and PA:

$$MFu = \lambda u, \quad u \neq 0 \implies TMFu = PATu = \lambda Tu, \quad Tu \neq 0 \implies PAv = \lambda v, \quad v \neq 0,$$

and

$$PAv = \lambda v, \quad v \neq 0 \implies QPAv = MFQv = \lambda Qv, \quad Qv \neq 0 \implies MFu = \lambda u, \quad u \neq 0,$$

so Theorem 26 holds.

Proof. To prove (85), compute, using $RR^{\mathrm{T}}D_P = I - B_D^{\mathrm{T}}B$,

$$TMF = \underbrace{R^{\mathrm{T}}D_PHB^{\mathrm{T}}}_{T} \underbrace{B_DSB_D^{\mathrm{T}}BHB^{\mathrm{T}}}_{MF} = R^{\mathrm{T}}D_PH\big(B_D^{\mathrm{T}}B\big)^{\mathrm{T}}S\big(B_D^{\mathrm{T}}B\big)HB^{\mathrm{T}},$$

and

$$PAT = \underbrace{R^{\mathrm{T}}D_PHD_P^{\mathrm{T}}RR^{\mathrm{T}}SR}_{PA} \underbrace{R^{\mathrm{T}}D_PHB^{\mathrm{T}}}_{T} = R^{\mathrm{T}}D_PH\big(I - B_D^{\mathrm{T}}B\big)^{\mathrm{T}}S\big(I - B_D^{\mathrm{T}}B\big)HB^{\mathrm{T}},$$

so

$$\begin{aligned} PAT = {} & \underbrace{R^{\mathrm{T}}D_PH\big(B_D^{\mathrm{T}}B\big)^{\mathrm{T}}BS\big(B_D^{\mathrm{T}}B\big)HB^{\mathrm{T}}}_{TMF} \\ & - R^{\mathrm{T}}D_PHS\big(B_D^{\mathrm{T}}B\big)HB^{\mathrm{T}} && (89) \\ & + R^{\mathrm{T}}D_PH\big(I - B_D^{\mathrm{T}}B\big)^{\mathrm{T}}SHB^{\mathrm{T}}. && (90) \end{aligned}$$

We will show that the last two terms vanish. To show that (89) vanishes, note that from (80),

$$R^{\mathrm{T}} D_P H S B_D^{\mathrm{T}} B H B^{\mathrm{T}} = R^{\mathrm{T}} D_P B_D^{\mathrm{T}} B H B^{\mathrm{T}},$$

where $R^{\mathrm{T}} D_P B_D^{\mathrm{T}} B = 0$ by (48). To show that the term (90) vanishes, consider

$$\begin{aligned}&\bigl(R^{\mathrm{T}} D_P H \bigl(I - B_D^{\mathrm{T}} B\bigr)^{\mathrm{T}} S H B^{\mathrm{T}}\bigr)^{\mathrm{T}} \\ &\quad = B H S \bigl(I - B_D^{\mathrm{T}} B\bigr) H D_P^{\mathrm{T}} R = \underbrace{B\bigl(I - B_D^{\mathrm{T}} B\bigr)}_{=0} H D_P^{\mathrm{T}} R,\end{aligned}$$

because $B_D^{\mathrm{T}} B$ preserves $\widetilde{W} = \operatorname{range} H$, HS is a projection on $\widetilde{W}$, and $B B_D^{\mathrm{T}} B = B$.

To prove (86), compute

$$\begin{aligned}MFQ &= MFMFB_D SR \\ &= B_D S\bigl(B_D^{\mathrm{T}} B\bigr) H \bigl(B_D^{\mathrm{T}} B\bigr)^{\mathrm{T}} S \bigl(B_D^{\mathrm{T}} B\bigr) H \bigl(B_D^{\mathrm{T}} B\bigr)^{\mathrm{T}} SR,\end{aligned}$$

and

$$\begin{aligned}QPA &= MFB_D SRPA \\ &= B_D S B_D^{\mathrm{T}} B H B^{\mathrm{T}} B_D S R R^{\mathrm{T}} D_P H D_P^{\mathrm{T}} R R^{\mathrm{T}} S R \\ &= \underbrace{B_D S B_D^{\mathrm{T}} B H B^{\mathrm{T}}}_{MF} B_D S R \underbrace{R^{\mathrm{T}} D_P H D_P^{\mathrm{T}} R R^{\mathrm{T}} S R}_{PA} \\ &= B_D S\bigl(B_D^{\mathrm{T}} B\bigr) H \bigl(B_D^{\mathrm{T}} B\bigr)^{\mathrm{T}} S \bigl(I - B_D^{\mathrm{T}} B\bigr) H \bigl(I - B_D^{\mathrm{T}} B\bigr)^{\mathrm{T}} SR,\end{aligned}$$

so

$$\begin{aligned}QPA = {}&\underbrace{B_D S\bigl(B_D^{\mathrm{T}} B\bigr) H \bigl(B_D^{\mathrm{T}} B\bigr)^{\mathrm{T}} S \bigl(B_D^{\mathrm{T}} B\bigr) H \bigl(B_D^{\mathrm{T}} B\bigr)^{\mathrm{T}} SR}_{MFQ} \\ &- B_D S\bigl(B_D^{\mathrm{T}} B\bigr) H \bigl(B_D^{\mathrm{T}} B\bigr)^{\mathrm{T}} S B_D^{\mathrm{T}} \underbrace{BHSR}_{=0} && (91) \\ &+ B_D S\bigl(B_D^{\mathrm{T}} B\bigr) \underbrace{H \bigl(B_D^{\mathrm{T}} B\bigr)^{\mathrm{T}} S H \bigl(I - B_D^{\mathrm{T}} B\bigr)^{\mathrm{T}}}_{=0} SR. && (92)\end{aligned}$$

We need to show that the last two terms vanish. First, from $BR = 0$ and $HSR = R$ follows $BHSR = 0$, so the term (91) is zero. To show that (92) vanishes, consider

$$\bigl(I - B_D^{\mathrm{T}} B\bigr) H S B_D^{\mathrm{T}} B H = \bigl(I - B_D^{\mathrm{T}} B\bigr) B_D^{\mathrm{T}} B H = 0,$$

because $\operatorname{range} H = \widetilde{W}$, $B_D^{\mathrm{T}} B \widetilde{W} \subset \widetilde{W}$, HS is projection onto $\widetilde{W}$ and thus the identity on $\widetilde{W}$, and $B_D^{\mathrm{T}} B$ is a projection.

To prove (87), let $MFu = \lambda u$, where $u \neq 0$ and $\lambda \neq 0, 1$. Recall that $MF = B_D S B_D^{\mathrm{T}} B H B^{\mathrm{T}}$ and $T = R^{\mathrm{T}} D_P H B^{\mathrm{T}}$. Now suppose that $Tu = 0$. Then $R R^{\mathrm{T}} D_P H B^{\mathrm{T}} u = 0$. Now since $B_D^{\mathrm{T}} B + R R^{\mathrm{T}} D_P = I$, it follows that $B_D^{\mathrm{T}} B H B^{\mathrm{T}} u = H B^{\mathrm{T}} u$ and thus

$$\underbrace{B_D S B_D^{\mathrm{T}} B H B^{\mathrm{T}}}_{MF} u = B_D S H B^{\mathrm{T}} u, \qquad MFu = B_D S H B^{\mathrm{T}} u, \qquad \lambda u = B_D S H B^{\mathrm{T}} u.$$

This is a contradiction because $B_D S H B^{\mathrm{T}}$ is a projection by (82).

To prove (88): The expressions for Qv and $(I - PA)v$ have common parts. Using $HSR = R$ and $B_D^{\mathrm{T}} B + RR^{\mathrm{T}} D_P = I$, we have

$$\begin{aligned} Qv &= B_D S B_D^{\mathrm{T}} B H B^{\mathrm{T}} B_D S R v \\ &= B_D S B_D^{\mathrm{T}} B H \bigl(I - D_P^{\mathrm{T}} R R^{\mathrm{T}}\bigr) S R v \\ &= B_D S B_D^{\mathrm{T}} B \bigl(I - H D_P^{\mathrm{T}} R R^{\mathrm{T}} S\bigr) R v. \end{aligned} \tag{93}$$

On the other hand, from the decomposition of unity $R^{\mathrm{T}} D_P R = I$,

$$(I - PA)v = \bigl(I - R^{\mathrm{T}} D_P H D_P^{\mathrm{T}} R R^{\mathrm{T}} S R\bigr) v = R^{\mathrm{T}} D_P \bigl(I - H D_P^{\mathrm{T}} R R^{\mathrm{T}} S\bigr) R v. \tag{94}$$

Suppose that $PAv = \lambda v$, $\lambda \neq 1$, and $Qv = 0$. We need to show that $v = 0$. If $(I - HD_P^{\mathrm{T}} RR^{\mathrm{T}} S)Rv = 0$, then $(I - PA)v = 0$ from (94), and it follows that $v = 0$. It remains to consider the case $(I - HD_P^{\mathrm{T}} RR^{\mathrm{T}} S)Rv \neq 0$. Because $\operatorname{range} R = \ker B = (\operatorname{range} B^{\mathrm{T}})^{\perp}$, we have

$$\bigl(I - H D_P^{\mathrm{T}} R R^{\mathrm{T}} S\bigr) R v = R q_1 + B^{\mathrm{T}} q_2. \tag{95}$$

Since $\operatorname{range} H \subset \widetilde{W}$ and $\operatorname{range} R = \widehat{W} \subset \widetilde{W}$, multiplying (95) by B from the left and using Lemma 20 and $BR = 0$ gives $BB^{\mathrm{T}} q_2 = 0$, hence $B^{\mathrm{T}} q_2 = 0$. Then

$$\bigl(I - H D_P^{\mathrm{T}} R R^{\mathrm{T}} S\bigr) R v = R q_1. \tag{96}$$

Substitution of (96) into (94) and use of the decomposition of unity $R^{\mathrm{T}} D_P R = I$ leads to

$$q_1 = (1 - \lambda) v. \tag{97}$$

Substitution of (97) into (96) yields

$$\bigl(I - H D_P^{\mathrm{T}} R R^{\mathrm{T}} S\bigr) R v = R (1 - \lambda) v. \tag{98}$$

Since $HD_P^{\mathrm{T}} RR^{\mathrm{T}} S$ is a projection by (83), premultiplication of (98) by $(I - HD_P^{\mathrm{T}} RR^{\mathrm{T}} S)$ leads to

$$\bigl(I - H D_P^{\mathrm{T}} R R^{\mathrm{T}} S\bigr) R v = (1 - \lambda) \bigl(I - H D_P^{\mathrm{T}} R R^{\mathrm{T}} S\bigr) R v, \tag{99}$$

and it follows that $(I - HD_P^{\mathrm{T}} RR^{\mathrm{T}} S)Rv = 0$ since $\lambda \neq 0$, hence $(I - PA)v = 0$ so $v = 0$. □

9. Computational experiments

Initial comparisons of the FETI-DP and BDDC methods were made in [2]. However, the comparisons were influenced by the fact that numerical results were based on a specific implementation of FETI-DP found in a large-scale, parallel, structural dynamics code. As a result, it was not possible to duplicate all components of the methods such as weights and constraints. To obtain a closer comparison, we programmed both methods in Matlab to use identical components. For convenience, the coarse space functions in FETI-DP were selected to be the same as the energy minimal coarse space functions of BDDC [2]; as explained in Section 3, this has no effect on the method. Constraints were defined by averages of each displacement field for each glob. The primal weights were defined in the standard manner, with the diagonal entry of D_{Pi} equal to the ratio of the corresponding diagonal entry of K_i and the sum of all corresponding diagonal entries from all substructures that share the same global degree of freedom. We note that the choice of weights and constraints used here is somewhat different than that in [2].

Table 1
Comparisons of BDDC and FETI-DP. Here, ndof denotes the number of degrees of freedom in the model, nsub is the number of substructures, niter is the number of iterations, and cond is the condition number estimate

Name	ndof	nsub	BDDC		FETI-DP	
			niter	cond	niter	cond
square4 × 4Hh4	544	16	11	2.1	10	2.1
square4 × 4Hh8	2112	16	13	3.1	12	3.1
square4 × 4Hh16	8320	16	15	4.4	14	4.4
square4 × 4Hh32	33024	16	17	6.0	16	5.9
square4 × 4Hh64	131584	16	20	7.7	18	7.6
square4 × 4Hh4-jagged	544	16	27	9.3	27	9.1
square4 × 4Hh8-jagged	2112	16	44	22.0	45	21.6
square4 × 4Hh16-jagged	8320	16	49	33.8	50	33.2
square4 × 4Hh32-jagged	33024	16	54	58.0	52	57.0
square4 × 4Hh64-jagged	131584	16	61	107	59	105
square4 × 4−1e−4	1200	16	11	2.9	10	2.9
square4 × 4−1e−2	1200	16	11	2.9	10	2.9
square4 × 4−1e0	1200	16	10	2.7	9	2.6
square4 × 4−1e2	1200	16	10	2.2	9	2.2
square4 × 4−1e4	1200	16	11	2.2	10	2.2
cube4 × 4 × 4−1e−4	45000	64	14	2.8	13	2.8
cube4 × 4 × 4−1e−2	45000	64	14	2.8	13	2.8
cube4 × 4 × 4−1e0	45000	64	12	2.6	12	2.6
cube4 × 4 × 4−1e2	45000	64	12	2.3	11	2.3
cube4 × 4 × 4−1e4	45000	64	13	2.3	11	2.2

Four sets of comparisons are shown in Table 1. The iterations were stopped based on the norm of the relative primal residual,

$$\frac{\|r\|}{\|g\|} \leqslant 10^{-8}, \quad r = R^{\mathrm{T}} g - R^{\mathrm{T}} S R u. \tag{100}$$

The condition number was estimated from the extremal eigenvalues of the tridiagonal Lanczos matrix generated by preconditioned conjugate gradients.

Results with the square4 × 4Hhn designation are for plane stress analysis of a unit square with all degrees of freedom fixed on one side. The square is decomposed into 16 square substructures each containing n^2 elements. Thus, the substructure to element length ratio H/h equals n. The elastic modulus E and Poisson's ratio ν equal 1 and 0.3, respectively, throughout the entire domain. Similar results for jagged mesh decompositions (see Fig. 1) bear the square4 × 4Hhn-jagged designation. Results with the cube4 × 4 × 4−1ep designation are for 3D elasticity analysis of a unit cube with all degrees of freedom fixed on one side. The cube is decomposed into 64 cube substructures each with $H/h = 4$. For these problems, $E = 1$ and $\nu = 0.3$ everywhere except in a centered cube region of length $1/2$ where $E = 10^p$. Here the substructure boundaries are aligned with material property jumps and the theory of this study holds. Corresponding problems for a unit square decomposed into 16 square substructures with $H/h = 6$ and aligned material property jumps are designated by square4 × 4−1ep.

Numerical results indicate nearly identical performance of FETI-DP and BDDC in terms of iterations and condition numbers for the problems considered. The condition number of both methods grows with decreasing mesh size very slowly for straight mesh decompositions, as predicted by the theory, but much

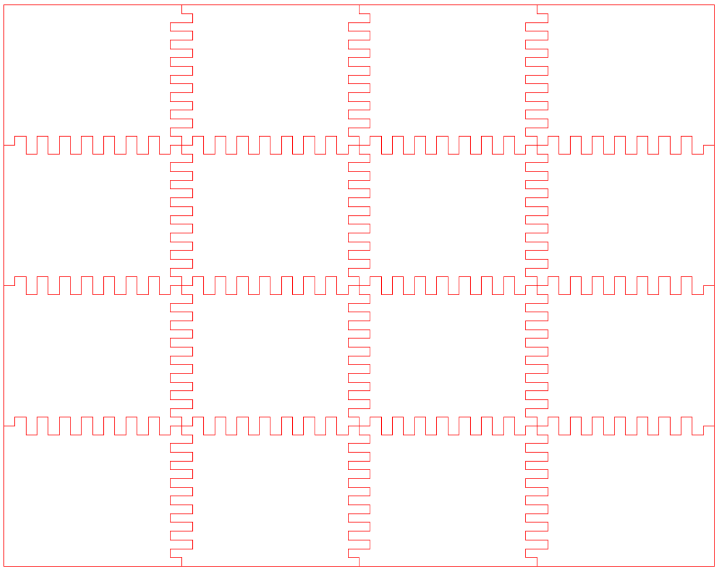

Fig. 1. Mesh used for square4 $\times$ 4Hh 16-jagged results.

faster for jagged mesh decompositions; this will be studied theoretically elsewhere. Finally, the square4 $\times$ 4$-$1ep and cube4 $\times$ 4 $\times$ 4$-$1ep results are consistent with the theoretical result that condition numbers are bounded independently of the magnitude of material property jumps that are aligned with substructure boundaries.

We have also computed the eigenvalues of the preconditioned operator both for BDDC and FETI-DP for several small problems where the calculation was feasible. We have found that the eigenvalues coincide except for rounding errors and different multiplicities of the eigenvalues equal to zero and one, which confirms Theorem 26.

References

[1] S.C. Brenner, Analysis of two-dimensional FETI-DP preconditioners by the standard additive Schwarz framework, Electronic Trans. Numer. Anal. 16 (2003) 165–185.

[2] C.R. Dohrmann, A preconditioner for substructuring based on constrained energy minimization, SIAM J. Sci. Comput. 25 (1) (2003) 246–258.

[3] M. Dryja, O.B. Widlund, Schwarz methods of Neumann–Neumann type for three-dimensional elliptic finite element problems, Comm. Pure Appl. Math. 48 (2) (1995) 121–155.

[4] C. Farhat, F.X. Roux, Implicit parallel processing in structural mechanics, Comput. Mech. Adv. 2 (1994) 1–124.

[5] C. Farhat, M. Lesoinne, K. Pierson, A scalable dual-primal domain decomposition method, Numer. Linear Algebra Appl. 7 (2000) 687–714.

[6] C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, D. Rixen, FETI-DP: A dual-primal unified FETI method. I. A faster alternative to the two-level FETI method, Internat. J. Numer. Methods Engrg. 50 (7) (2001) 1523–1544.

[7] Y. Fragakis, M. Papadrakakis, The mosaic of high performance domain decomposition methods for structural mechanics: Formulation, interrelation and numerical efficiency of primal and dual methods, Comput. Methods Appl. Mech. Engrg. 192 (2003) 3799–3830.

[8] I.C.F. Ipsen, C.D. Meyer, The angle between complementary subspaces, Amer. Math. Monthly 102 (10) (1995) 904–911.

[9] A. Klawonn, O.B. Widlund, A domain decomposition method with Lagrange multipliers and inexact solvers for linear elasticity, SIAM J. Sci. Comput. 22 (4) (2000) 1199–1219.

[10] A. Klawonn, O.B. Widlund, FETI and Neumann–Neumann iterative substructuring methods: Connections and new results, Comm. Pure Appl. Math. 54 (1) (2001) 57–90.

[11] A. Klawonn, O.B. Widlund, Selecting constraints in dual-primal FETI methods for elasticity in three dimensions, in: R. Kornhuber, R. Hoppe, J. Périaux, O. Pironneau, O. Widlund, J. Xu (Eds.), Domain Decomposition Methods in Science and Engineering, Lecture Notes in Computational Science and Engineering, vol. 40, Springer, Berlin, 2004, pp. 67–81.

[12] A. Klawonn, O.B. Widlund, M. Dryja, Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients, SIAM J. Numer. Anal. 40 (1) (2002) 159–179.

[13] P. LeTallec, J. Mandel, M. Vidrascu, A Neumann–Neumann domain decomposition algorithm for solving plate and shell problems, SIAM J. Numer. Anal. 35 (1998) 836–867.

[14] M. Lesoinne, A FETI-DP corner selection algorithm for three-dimensional problems, in: I. Herrera, D.E. Keyes, O.B. Widlund, R. Yates (Eds.), Fourteenth International Conference on Domain Decomposition Methods, UNAM, Mexico City, Mexico, 2003, pp. 217–223.

[15] J. Mandel, Balancing domain decomposition, Comm. Numer. Methods Engrg. 9 (3) (1993) 233–241.

[16] J. Mandel, M. Brezina, Balancing domain decomposition for problems with large jumps in coefficients, Math. Comp. 65 (216) (1996) 1387–1401.

[17] J. Mandel, C.R. Dohrmann, Convergence of a balancing domain decomposition by constraints and energy minization, Numer. Linear Algebra Appl. 10 (2003) 639–659.

[18] J. Mandel, R. Tezaur, Convergence of a substructuring method with Lagrange multipliers, Numer. Math. 73 (4) (1996) 473–487.

[19] J. Mandel, R. Tezaur, On the convergence of a dual-primal substructuring method, Numer. Math. 88 (2001) 543–558.

[20] J. Mandel, R. Tezaur, C. Farhat, A scalable substructuring sethod by Lagrange multipliers for plate bending problems, SIAM J. Numer. Anal. 36 (5) (1999) 1370–1391.

[21] R. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc. 51 (1955) 406–413.

[22] D. Rixen, C. Farhat, A simple and efficient extension of a class of substructure based preconditioners to heterogeneous structural mechanics problems, Internat. J. Numer. Methods Engrg. 46 (1999) 489–516.

[23] D.J. Rixen, C. Farhat, R. Tezaur, J. Mandel, Theoretical comparison of the FETI and algebraically partitioned FETI methods, and performance comparisons with a direct sparse solver, Internat. J. Numer. Methods Engrg. 46 (1999) 501–534.