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Asynchronous multisplitting two-stage iterations for systems of weakly nonlinear equations[1]

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Abstract

For the large sparse systems of weakly nonlinear equations arising in the discretizations of many classical differential and integral equations, this paper presents a class of asynchronous parallel multisplitting two-stage iteration methods for getting their solutions by the high-speed multiprocessor systems. Under suitable assumptions, we study the global convergence properties of these asynchronous multisplitting two-stage iteration methods. Moreover, for this class of new methods, we establish their local convergence theories, and precisely estimate their asymptotic convergence factors under some reasonable assumptions when the involved nonlinear mapping is only assumed to be directionally differentiable. Numerical computations show that our new methods are feasible and efficient for parallely solving the system of weakly nonlinear equations. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The system of weakly nonlinear equations

$$Ax = G(x), \quad A \in L(R^n), \quad G: R^n \to R^n \tag{1.1}$$

often arises in the finite difference or the finite element discretizations of many classical differential and integral equations, where $A \in L(R^n)$ is a large sparse monotone matrix, and $G: R^n \to R^n$ is a

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P-bounded nonlinear mapping but has no derivative in the ordinary meaning. Since this class of weakly nonlinear systems has bounteous applicable backgrounds and special mathematical structure, Bai [1, 2] has studied in depth the serial and parallel two-stage iterative methods for approximately getting its solution, and described in detail the convergence properties of these methods.

To exploit the efficiency of the high-speed multiprocessor systems, in this paper, we further present a class of asynchronous multisplitting two-stage iteration methods for parallely solving the system of weakly nonlinear equations (1.1). Since the designs of these methods sufficiently utilize the concrete characteristics of the multiprocessor systems and the special structure of the weakly nonlinear system (1.1), no mutual waits among the processors of the multiprocessor system are necessarily required. Therefore, these new methods are rather efficient for implementing in the asynchronous parallel computing environments. When the system matrix $A\in L(R^n)$ is an H-matrix and the nonlinear mapping $G:R^n\rightarrow R^n$ is a P-bounded mapping, we establish the global convergence theories of these new asynchronous multisplitting two-stage iteration methods. Moreover, for these new methods, we establish their local convergence theories, and precisely estimate their asymptotic convergence factors under some suitable assumptions when the involved nonlinear mapping $G:R^n\rightarrow R^n$ is only assumed to be directionally differentiable. At last, some numerical computations show that our new methods are feasible and efficient for parallely solving the system of weakly nonlinear equations (1.1).

2. The asynchronous multisplitting two-stage iterations

For a given positive integer α, let $A=B_i-C_i$ $(i=1,2,\ldots,\alpha)$ be α splittings of the matrix $A\in L(R^n)$, i.e., $\det(B_i)\neq 0$ $(i=1,2,\ldots,\alpha)$; $B_i=M_i-N_i$ $(i=1,2,\ldots,\alpha)$ be splittings of the matrices $B_i\in L(R^n)$ $(i=1,2,\ldots,\alpha)$, respectively; and $E_i\in L(R^n)$ $(i=1,2,\ldots,\alpha)$ be nonnegative diagonal matrices satisfying $\sum_{i=1}^{\alpha}E_i=I$ (the identity matrix). Then the collection of triples (B_i,C_i,E_i) $(i=1,2,\ldots,\alpha)$ is called a multisplitting of the matrix $A\in L(R^n)$; and the collection of quintuples $(B_i:M_i,N_i;C_i;E_i)$ $(i=1,2,\ldots,\alpha)$ is called a two-stage multisplitting of the matrix $A\in L(R^n)$. In the sequel, we assume that the considered multiprocessor system is made up of α processors, referred to proc(1), proc(2), $\cdots$, proc(α). In addition, we introduce the following necessary notations: $N_0=\{0,1,2,\cdots\}$; for $\forall p\in N_0$, $J(p)$ is a nonempty subset of the set $\{1,2,\ldots,\alpha\}$; for $\forall i\in\{1,2,\ldots,\alpha\}$ and $\forall p\in N_0$, $\tau_i(p)$ is an infinite sequence of nonnegative integers, such that
(1) for $\forall i\in\{1,2,\ldots,\alpha\}$, the set $\{p\in N_0\,|\,i\in J(p)\}$ is infinite;
(2) for $\forall i\in\{1,2,\ldots,\alpha\}$ and $\forall p\in N_0$, $\tau_i(p)\leqslant p$ holds; and
(3) for $\forall i\in\{1,2,\ldots,\alpha\}$, $\lim_{p\rightarrow\infty}\tau_i(p)=\infty$ holds.
If we denote $\tau(p)=\min_{1\leqslant i\leqslant\alpha}\tau_i(p)$, then it obviously holds that $\tau(p)\leqslant p$ and $\lim_{p\rightarrow\infty}\tau(p)=\infty$.

Now, we consider the following asynchronous multisplitting two-stage iteration method for parallely solving the system of weakly nonlinear equations (1.1).

Method I (Asynchronous multisplitting two-stage iteration method). *Given an initial vector $x^0\in R^n$. Suppose that we have got approximations $x^0,x^1,\ldots,x^p$ to the solution $x^*\in R^n$ of the system of weakly nonlinear equations* (1.1). *Then the $(p+1)$th approximation x^{p+1} to the solution x^* is calculated by*

$$x^{p+1}=x^p+\sum_{i\in J(p)}E_i(x^{p+1,i}-x^p),$$

where $x^{p+1,i}$ $(i \in J(p))$ *is computed by the following process*:

Begin
$x^{p,i,0} = x^{\tau_i(p)}$;
FOR $k=0$ *TO* $s_i(p)-1$ *DO*
$x^{p,i,k+1} = M_i^{-1}(N_i x^{p,i,k} + C_i x^{\tau_i(p)} + G(x^{\tau_i(p)}))$;
$x^{p+1,i} = x^{p,i,s_i(p)}$
End.

Here $\{s_i(p)\}_{p\in N_0}$ $(i=1,2,\ldots,\alpha)$ *are positive integer sequences.*

Clearly, in this method, each of the processors proc(i) $(i=1,2,\ldots,\alpha)$ is allowed to update the global approximate solution, or retrieve any subset of the elements of the global approximate solution at any time. Hence, new information can be used on time once they are available. Moreover, considerable savings in computational workload are possible, since a component of $x^{p+1,i}$ need not be computed if the corresponding diagonal entry of E_i is zero. The role of the weighting matrices E_i $(i=1,2,\ldots,\alpha)$ may be regarded as determining the distribution of the computational work to the individual processors.

Just as in [1, 2, 8], we will also separate this asynchronous multisplitting two-stage iteration method into the stationary method, in which the numbers of local inner iterations $s_i(p)$ $(i=1,2,\ldots,\alpha)$ stay fixed in each of the global outer steps, and the non-stationary method, in which the numbers of local inner iterations $s_i(p)$ $(i=1,2,\ldots,\alpha)$ change with p, the global outer iteration index. Evidently, for $\forall p\in N_0$ and $\forall i\in J(p)$, when $J(p)=\{1,2,\ldots,\alpha\}$ and $\tau_i(p)=p$, Method I recovers the synchronous parallel multisplitting two-stage iteration method in [2], and in particular, as $\alpha=1$ the serial two-stage iteration method in [1], for the system of weakly nonlinear equations (1.1); and when $G(x)\equiv b$ (a constant vector in R^n) for all $x\in R^n$, and $s_i(p)=1(\forall i\in J(p),\ \forall p\in N_0)$, it naturally leads to the asynchronous parallel matrix multisplitting iteration method in [20] for solving the system of linear equations $Ax=b$.

After direct manipulations, Method I can be briefly expressed in the matrix–vector form:

$$x^{p+1} = \sum_{i\in J(p)} E_i \left[(M_i^{-1}N_i)^{s_i(p)} x^{\tau_i(p)} + \sum_{j=0}^{s_i(p)-1} (M_i^{-1}N_i)^j M_i^{-1} (C_i x^{\tau_i(p)} + G(x^{\tau_i(p)})) \right] + \sum_{i\notin J(p)} E_i x^p, \tag{2.1}$$

or, alternatively,

$$x^{p+1} = \sum_{i\in J(p)} E_i[(M_i^{-1}N_i)^{s_i(p)} x^{\tau_i(p)} + (I-(M_i^{-1}N_i)^{s_i(p)})\, B_i^{-1}(C_i x^{\tau_i(p)} + G(x^{\tau_i(p)}))] + \sum_{i\notin J(p)} E_i x^p. \tag{2.2}$$

As a matter of fact, there are various kinds of two-stage multisplittings. For example, if, in the two-stage multisplitting $(B_i : M_i, N_i; C_i; E_i)$ $(i=1,2,\ldots,\alpha)$ of the matrix $A\in L(R^n)$, for each $i\in\{1,2,\ldots,\alpha\}$ we take $M_i = D_i - L_i$ with $D_i = \mathrm{diag}(B_i)$ being a nonsingular matrix and $L_i \in L(R^n)$ being a strictly lower triangular matrix of the matrix $(-B_i)$, and $N_i = U_i$ with $U_i \in L(R^n)$ being zero-diagonal matrix such that $B_i = D_i - L_i - U_i$, then a special but practical two-stage multisplitting, say $(B_i : D_i - L_i, U_i; C_i; E_i)$ $(i = 1,2,\ldots,\alpha)$, of the matrix $A \in L(R^n)$ can be naturally induced. Based upon this special two-stage multisplitting, we can set up the following asynchronous multisplitting two-stage accelerated overrelaxation (AOR) method for parallely solving the system of weakly nonlinear equations (1.1).

Method II (Asynchronous multisplitting two-stage AOR method). *Given an initial vector $x^0 \in R^n$. Suppose that we have got approximations $x^0, x^1, \ldots, x^p$ to the solution $x^* \in R^n$ of the system of weakly nonlinear equations* (1.1). *Then the $(p+1)$th approximation x^{p+1} to the solution x^* is calculated by*

$$x^{p+1} = x^p + \sum_{i \in J(p)} E_i(x^{p+1,i} - x^p),$$

where $x^{p+1,i}$ $(i \in J(p))$ is computed by the following process:

$$\begin{aligned}
&\textit{Begin}\\
&\quad x^{p,i,0} = x^{\tau_i(p)};\\
&\quad \textit{FOR } k = 0 \textit{ TO } s_i(p) - 1 \textit{ DO}\\
&\qquad x^{p,i,k+1} = (D_i - \gamma L_i)^{-1}\{[(1-\omega)D_i + (\omega - \gamma)L_i + \omega U_i]x^{p,i,k}\\
&\qquad\qquad + \omega(C_i x^{\tau_i(p)} + G(x^{\tau_i(p)}))\};\\
&\quad x^{p+1,i} = x^{p,i,s_i(p)}\\
&\textit{End.}
\end{aligned}$$

Here γ is called a relaxation factor, ω an acceleration factor, and $\{s_i(p)\}_{p \in N_0}$ $(i = 1,2,\ldots,\alpha)$ are positive integer sequences.

Clearly, Method II reduces to the synchronous parallel multisplitting two-stage AOR method discussed in [2] as $J(p) = \{1,2,\ldots,\alpha\}$ and $s_i(p) = p$, for $\forall i \in J(p)$ and $\forall p \in N_0$; in particular, the serial two-stage AOR method investigated in [1] when $\alpha = 1$. In addition, if $G(x) \equiv b$ (a constant vector in R^n) for all $x \in R^n$ and $s_i(p) \equiv 1$ $(i = 1,2,\ldots,\alpha; p \in N_0)$, it naturally recovers the asynchronous parallel matrix multisplitting AOR method for the system of linear equations $Ax = b$, which has been deeply investigated in [20]. Note that Method II includes two arbitrary parameters γ and ω, their suitable adjustments can greatly improve the convergence property of this method. Moreover, following special choices of these two relaxation parameters, Method II can yield a series of practical and efficient asynchronous multisplitting two-stage relaxation methods.

If we define matrices

$$\begin{cases} M_i(\gamma,\omega) = \dfrac{1}{\omega}(D_i - \gamma L_i), \\ N_i(\gamma,\omega) = \dfrac{1}{\omega}[(1-\omega)D_i + (\omega-\gamma)L_i + \omega U_i], \end{cases} \qquad i = 1,2,\ldots,\alpha, \tag{2.3}$$

then Method II can be analogously expressed in the matrix–vector form:

$$x^{p+1} = \sum_{i\in J(p)} E_i\left[M_i(\gamma,\omega)^{-1}N_i(\gamma,\omega)\right]^{s_i(p)} x^{\tau_i(p)} + \sum_{i\notin J(p)} E_i x^p$$
$$+ \sum_{i\in J(p)} E_i \sum_{j=0}^{s_i(p)-1} [M_i(\gamma,\omega)^{-1}N_i(\gamma,\omega)]^j M_i(\gamma,\omega)^{-1}(C_i x^{\tau_i(p)} + G(x^{\tau_i(p)})), \tag{2.4}$$

or, equivalently,

$$x^{p+1} = \sum_{i\in J(p)} E_i\left[M_i(\gamma,\omega)^{-1}N_i(\gamma,\omega)\right]^{s_i(p)} x^{\tau_i(p)} + \sum_{i\notin J(p)} E_i x^p$$
$$+ \sum_{i\in J(p)} E_i(I - (M_i(\gamma,\omega)^{-1}N_i(\gamma,\omega))^{s_i(p)})B_i^{-1}(C_i x^{\tau_i(p)} + G(x^{\tau_i(p)})). \tag{2.5}$$

In fact, Methods I and II are asynchronous extensions of the synchronous multisplitting two-stage iteration methods for the system of weakly nonlinear equations (see [2]).

3. Global convergence theories

In this section, we will mainly discuss the global convergence properties of Methods I and II presented in Section 2. For this purpose, we will closely follow the notations and concepts introduced in [1, 2] in the sequel. In particular, we denote by $|\bullet|$ the absolute value of either a vector or a matrix, and $\langle\bullet\rangle$ the comparison matrix of the corresponding matrix. We call a mapping $G: R^n \to R^n$ P-bounded if there exists a nonnegative matrix $P\in L(R^n)$ such that

$$|G(x) - G(y)| \leqslant P|x-y| \quad \text{for all } x, y\in R^n.$$

Besides, we state the following theorem, which is crucial for the establishments of the global convergence theorem of our new asynchronous multisplitting two-stage iteration methods. Since this theorem can be demonstrated quite similar to Lemmas 3.1 and 3.2 in [5], its proof is omitted here.

Theorem 3.1. *Let $H_{p,i}$ $(i=1,2,\dots,\alpha;\ p\in N_0)$ be nonnegative matrices, E_i $(i=1,2,\dots,\alpha)$ be nonnegative diagonal matrices such that $\sum_{i=1}^{\alpha} E_i = I$, and $\{\varepsilon^p\}_{p\in N_0}$ be a sequence defined by*

$$\varepsilon^{p+1} = \sum_{i\in J(p)} E_i H_{p,i}\varepsilon^{\tau_i(p)} + \sum_{i\notin J(p)} E_i\varepsilon^p, \quad p=0,1,2,\dots$$

with $\{J(p)\}_{p\in N_0}$ and $\{\tau_i(p)\}_{p\in N_0}$ $(i=1,2,\dots,\alpha)$ being given as in Section 2. Then there holds $\lim_{p\to\infty}\varepsilon^p = 0$ for any $\varepsilon^0\in R^n$ provided there exists a nonnegative number $\theta\in[0,1)$ and a positive vector $u\in R^n$ such that $H_{p,i}u\leqslant\theta u$ $(i=1,2,\dots,\alpha;\ p\in N_0)$.

The existence and uniqueness of the solution of the system of weakly nonlinear equations (1.1) can be described by the following theorem, which has been proved in Bai [1] (see also [19]).

Theorem 3.2. *Let $A \in L(R^n)$ be a nonsingular matrix, and $G: R^n \to R^n$ be P-bounded. Then the system of weakly nonlinear equations* (1.1) *has a unique solution $x^* \in R^n$ provided either of the following two conditions is satisfied:*
(a) *$A \in L(R^n)$ is a monotone matrix and $\rho(A^{-1}P) < 1$;*
(b) *$A \in L(R^n)$ is an H-matrix and $\rho(\langle A \rangle^{-1}P) < 1$.*

Now, we demonstrate the global convergence of Methods I and II when the system matrix $A \in L(R^n)$ is an H-matrix and the nonlinear mapping $G: R^n \to R^n$ is a P-bounded mapping.

Theorem 3.3. *Let $A \in L(R^n)$ be an H-matrix with $D = \mathrm{diag}(A)$ and $A = D - B$, and (B_i, C_i, E_i) $(i = 1, 2, \ldots, \alpha)$ be its multisplitting satisfying $\langle A \rangle = \langle B_i \rangle - |C_i|$ $(i = 1, 2, \ldots, \alpha)$. Assume $G: R^n \to R^n$ be a P-bounded mapping and $\rho(\langle A \rangle^{-1}P) < 1$ hold. For any starting vector $x^0 \in R^n$ and any number sequences $\{s_i(p)\}_{p \in N_0}$ $(i = 1, 2, \ldots, \alpha)$ of the local inner iterations satisfying $s_i(p) \geqslant 1$ $(i = 1, 2, \ldots, \alpha;$ $p \in N_0)$,*
(i) *if $(B_i : M_i, N_i; C_i; E_i)$ $(i = 1, 2, \ldots, \alpha)$ is a two-stage multisplitting of the matrix $A \in L(R^n)$ with $\mathrm{diag}(B_i)$, $\mathrm{diag}(M_i)$ $(i = 1, 2, \ldots, \alpha)$ being nonsingular and with $\langle B_i \rangle = \langle M_i \rangle - |N_i|$ $(i = 1, 2, \ldots, \alpha)$, then the sequence $\{x^p\}$ generated by Method I converges to the unique solution $x^* \in R^n$ of the system of weakly nonlinear equations* (1.1);
(ii) *if $(B_i : D_i - L_i, U_i; C_i; E_i)$ $(i = 1, 2, \ldots, \alpha)$ is a two-stage multisplitting of the matrix $A \in L(R^n)$ with $D_i \equiv D$ $(i = 1, 2, \ldots, \alpha)$ and $\langle B_i \rangle = |D| - |L_i| - |U_i|$ $(i = 1, 2, \ldots, \alpha)$, then the sequence $\{x^p\}$ generated by Method II converges to the unique solution $x^* \in R^n$ of the system of weakly nonlinear equations* (1.1) *provided the relaxation parameters γ and ω satisfy*

$$0 \leqslant \gamma \leqslant \omega, \quad 0 < \omega < \frac{2}{1 + \rho(|D|^{-1}(|B| + P))}.$$

Proof. We first demonstrate (i). By Theorem 3.2 we know that there exists a unique vector $x^* \in R^n$ such that $Ax^* = G(x^*)$. Since Method I is consistent with the weakly nonlinear system (1.1), corresponding to (2.1), x^* also satisfies

$$x^* = \sum_{i \in J(p)} E_i \left[(M_i^{-1}N_i)^{s_i(p)} x^* + \sum_{j=0}^{s_i(p)-1} (M_i^{-1}N_i)^j M_i^{-1}(C_i x^* + G(x^*)) \right] + \sum_{i \notin J(p)} E_i x^*. \tag{3.1}$$

Now, if we introduce error vectors

$$\varepsilon^{\tau_i(p)} = x^{\tau_i(p)} - x^*, \quad \varepsilon^p = x^p - x^*, \quad i = 1, 2, \ldots, \alpha; \quad p \in N_0,$$

by subtracting (3.1) from (2.1) we can easily get the recursive formula:

$$\begin{aligned} \varepsilon^{p+1} = {} & \sum_{i \in J(p)} E_i \left[(M_i^{-1}N_i)^{s_i(p)} \varepsilon^{\tau_i(p)} + \sum_{j=0}^{s_i(p)-1} (M_i^{-1}N_i)^j M_i^{-1}(C_i \varepsilon^{\tau_i(p)} + G(x^{\tau_i(p)}) - G(x^*)) \right] \\ & + \sum_{i \notin J(p)} E_i \varepsilon^p. \end{aligned} \tag{3.2}$$

Noticing that $A \in L(R^n)$ is an H-matrix, $\mathrm{diag}(B_i)$ and $\mathrm{diag}(M_i)$ $(i=1,2,\dots,\alpha)$ are nonsingular matrices, and

$$\begin{cases} \langle A\rangle = \langle B_i\rangle - |C_i| \leqslant \langle B_i\rangle \leqslant |\mathrm{diag}(B_i)|, \\ \langle B_i\rangle = \langle M_i\rangle - |N_i| \leqslant \langle M_i\rangle \leqslant |\mathrm{diag}(M_i)|, \end{cases} \quad i=1,2,\dots,\alpha,$$

we know that the matrices B_i and M_i $(i=1,2,\dots,\alpha)$ are all H-matrices. Hence,

$$|B_i^{-1}| \leqslant \langle B_i\rangle^{-1}, \quad |M_i^{-1}| \leqslant \langle M_i\rangle^{-1}, \; i=1,2,\dots,\alpha$$

hold. By applying the P-bounded property of the mapping $G: R^n \to R^n$, based on (3.2) we obtain

$$\begin{aligned} |\varepsilon^{p+1}| &= \Bigg| \sum_{i\in J(p)} E_i \Bigg[(M_i^{-1}N_i)^{s_i(p)} \varepsilon^{\tau_i(p)} \\ &\quad + \sum_{j=0}^{s_i(p)-1} (M_i^{-1}N_i)^j M_i^{-1}(C_i\varepsilon^{\tau_i(p)} + G(x^{\tau_i(p)}) - G(x^*)) \Bigg] + \sum_{i\notin J(p)} E_i\varepsilon^p \Bigg| \\ &\leqslant \sum_{i\in J(p)} E_i \Bigg[|M_i^{-1}N_i|^{s_i(p)} |\varepsilon^{\tau_i(p)}| \\ &\quad + \sum_{j=0}^{s_i(p)-1} |M_i^{-1}N_i|^j |M_i^{-1}| (|C_i||\varepsilon^{\tau_i(p)}| + |G(x^{\tau_i(p)}) - G(x^*)|) \Bigg] + \sum_{i\notin J(p)} E_i|\varepsilon^p| \\ &\leqslant \sum_{i\in J(p)} E_i \Bigg[(\langle M_i\rangle^{-1}|N_i|)^{s_i(p)} + \sum_{j=0}^{s_i(p)-1} (\langle M_i\rangle^{-1}|N_i|)^j \langle M_i\rangle^{-1}(|C_i| + P) \Bigg] |\varepsilon^{\tau_i(p)}| \\ &\quad + \sum_{i\notin J(p)} E_i|\varepsilon^p|. \end{aligned}$$

For $\forall i \in \{1,2,\dots,\alpha\}$ and $\forall p \in N_0$, denote $\hat{\varepsilon}^p = |\varepsilon^p|$, $\hat{\varepsilon}^{\tau_i(p)} = |\varepsilon^{\tau_i(p)}|$ and

$$H_{p,i} = (\langle M_i\rangle^{-1}|N_i|)^{s_i(p)} + \sum_{j=0}^{s_i(p)-1} (\langle M_i\rangle^{-1}|N_i|)^j \langle M_i\rangle^{-1}(|C_i| + P).$$

Then the above estimate turns to

$$\hat{\varepsilon}^{p+1} \leqslant \sum_{i\in J(p)} E_i H_{p,i} \hat{\varepsilon}^{\tau_i(p)} + \sum_{i\notin J(p)} E_i \hat{\varepsilon}^p, \quad p=0,1,2,\dots .$$

Now, defining the infinite sequence $\{\varepsilon^p\}_{p\in N_0}$ in accordance with

$$\varepsilon^0 = \hat{\varepsilon}^0, \quad \varepsilon^{p+1} = \sum_{i\in J(p)} E_i H_{p,i} \varepsilon^{\tau_i(p)} + \sum_{i\notin J(p)} E_i \varepsilon^p, \qquad p \in N_0,$$

we can immediately deduce that $\{\varepsilon^p\}_{p\in N_0}$ is a majorizing sequence of $\{\hat{\varepsilon}^p\}_{p\in N_0}$. That is to say, $\hat{\varepsilon}^p \leqslant \varepsilon^p (\forall p \in N_0)$ holds. Therefore, to fulfill our proof we only need to verify the validity of the relation $\lim_{p\to\infty} \varepsilon^p = 0$. Moreover, Theorem 3.1 guarantees that this limit relation holds if there exists a nonnegative number $\theta \in [0,1)$ and a positive vector $u \in R^n$ such that

$$H_{p,i}u \leqslant \theta u, \quad i=1,2,\ldots,\alpha; \;\; p \in N_0. \tag{3.3}$$

As a matter of fact, because $A \in L(R^n)$ is an H-matrix and $\rho(\langle A\rangle^{-1}P)<1$, it is evident that $(\langle A\rangle - P)$ is an M-matrix. Therefore, there exists a positive vector $u \in R^n$ such that $(\langle A\rangle - P)u = e$, where $e=(1,1,\ldots,1)^{\mathrm{T}} \in R^n$. Since $\langle M_i\rangle^{-1} \geqslant 0$ $(i=1,2,\ldots,\alpha)$, we have $\langle M_i\rangle^{-1}e>0$ $(i=1,2,\ldots,\alpha)$. Hence, there exists a constant $\theta \in [0,1)$ such that $\langle M_i\rangle^{-1}e \geqslant (1-\theta)u$ $(i=1,2,\ldots,\alpha)$. Noticing $H_{p,i}\geqslant 0$ $(i=1,2,\ldots,\alpha;\, p\in N_0)$ and

$$\begin{aligned} H_{p,i} &= (\langle M_i\rangle^{-1}|N_i|)^{s_i(p)} + \sum_{j=0}^{s_i(p)-1} (\langle M_i\rangle^{-1}|N_i|)^j \langle M_i\rangle^{-1}\langle B_i\rangle - \sum_{j=0}^{s_i(p)-1} (\langle M_i\rangle^{-1}|N_i|)^j \langle M_i\rangle^{-1}(\langle A\rangle - P) \\ &= I - \sum_{j=0}^{s_i(p)-1} (\langle M_i\rangle^{-1}|N_i|)^j \langle M_i\rangle^{-1}(\langle A\rangle - P), \end{aligned}$$

we obtain

$$\begin{aligned} H_{p,i}u &= u - \sum_{j=0}^{s_i(p)-1} (\langle M_i\rangle^{-1}|N_i|)^j \langle M_i\rangle^{-1}(\langle A\rangle - P)u \\ &= u - \sum_{j=0}^{s_i(p)-1} (\langle M_i\rangle^{-1}|N_i|)^j \langle M_i\rangle^{-1}e \\ &\leqslant u - \langle M_i\rangle^{-1}e \leqslant u - (1-\theta)u = \theta u. \end{aligned}$$

This obviously shows the validity of (3.3). Therefore, $\lim_{p\to\infty} x^p = x^*$.

We now turn to demonstrate (ii). Analogous to (3.2), for Method II we have

$$\begin{aligned} \varepsilon^{p+1} = \sum_{i\in J(p)} E_i \Bigg[& (M_i(\gamma,\omega)^{-1}N_i(\gamma,\omega))^{s_i(p)}\varepsilon^{\tau_i(p)} \\ & + \sum_{j=0}^{s_i(p)-1} (M_i(\gamma,\omega)^{-1}N_i(\gamma,\omega))^j M_i(\gamma,\omega)^{-1}\left(C_i\varepsilon^{\tau_i(p)} + G(x^{\tau_i(p)}) - G(x^*)\right)\Bigg] \\ & + \sum_{i\notin J(p)} E_i\varepsilon^p, \end{aligned} \tag{3.4}$$

where the matrices $M_i(\gamma,\omega)$ and $N_i(\gamma,\omega)$, are defined by (2.3). Noticing that

$$|M_i(\gamma,\omega)^{-1}| = |\omega(D-\gamma L_i)^{-1}| \leqslant \omega(|D| - \gamma|L_i|)^{-1} \equiv \widehat{M}_i(\gamma,\omega)^{-1}$$

and

$$\begin{aligned} |N_i(\gamma,\omega)| &= \left|\frac{1}{\omega}[(1-\omega)D + (\omega-\gamma)L_i + \omega U_i]\right| \\ &\leqslant \frac{1}{\omega}[|1-\omega||D| + (\omega-\gamma)|L_i| + \omega|U_i|] \equiv \widehat{N}_i(\gamma,\omega), \end{aligned}$$

by applying the P-bounded property of the mapping $G: R^n \to R^n$ in (3.4) again we obtain

$$
\begin{aligned}
|\varepsilon^{p+1}| = \Bigg| \sum_{i\in J(p)} E_i \Bigg[& (M_i(\gamma,\omega)^{-1}N_i(\gamma,\omega))^{s_i(p)}\varepsilon^{\tau_i(p)} \\
& + \sum_{j=0}^{s_i(p)-1} (M_i(\gamma,\omega)^{-1}N_i(\gamma,\omega))^j M_i(\gamma,\omega)^{-1}(C_i\varepsilon^{\tau_i(p)} + G(x^{\tau_i(p)}) - G(x^*))\Bigg] + \sum_{i\notin J(p)} E_i\varepsilon^p \Bigg| \\
\leqslant \sum_{i\in J(p)} E_i \Bigg[& |M_i(\gamma,\omega)^{-1}N_i(\gamma,\omega)|^{s_i(p)}|\varepsilon^{\tau_i(p)}| \\
& + \sum_{j=0}^{s_i(p)-1} |M_i(\gamma,\omega)^{-1}N_i(\gamma,\omega)|^j |M_i(\gamma,\omega)^{-1}|(|C_i||\varepsilon^{\tau_i(p)}| + |G(x^{\tau_i(p)}) - G(x^*)|)\Bigg] \\
& + \sum_{i\notin J(p)} E_i|\varepsilon^p| \\
\leqslant \sum_{i\in J(p)} E_i \Bigg[& (\widehat{M}_i(\gamma,\omega)^{-1}\widehat{N}_i(\gamma,\omega))^{s_i(p)} \\
& + \sum_{j=0}^{s_i(p)-1} (\widehat{M}_i(\gamma,\omega)^{-1}\widehat{N}_i(\gamma,\omega))^j \widehat{M}_i(\gamma,\omega)^{-1}(|C_i| + P)\Bigg] |\varepsilon^{\tau_i(p)}| + \sum_{i\notin J(p)} E_i|\varepsilon^p|.
\end{aligned}
$$

For $\forall i \in \{1,2,\ldots,\alpha\}$ and $\forall p \in N_0$, represent $\widehat{\varepsilon}^p = |\varepsilon^p|$, $\widehat{\varepsilon}^{\tau_i(p)} = |\varepsilon^{\tau_i(p)}|$ and

$$
\begin{aligned}
H_{p,i}(\gamma,\omega) = & (\widehat{M}_i(\gamma,\omega)^{-1}\widehat{N}_i(\gamma,\omega))^{s_i(p)} \\
& + \sum_{j=0}^{s_i(p)-1} (\widehat{M}_i(\gamma,\omega)^{-1}\widehat{N}_i(\gamma,\omega))^j \widehat{M}_i(\gamma,\omega)^{-1}(|C_i| + P).
\end{aligned}
$$

Then the above estimate can be briefly expressed as

$$\widehat{\varepsilon}^{p+1} \leqslant \sum_{i\in J(p)} E_i H_{p,i}(\gamma,\omega)\widehat{\varepsilon}^{\tau_i(p)} + \sum_{i\notin J(p)} E_i\widehat{\varepsilon}^p, \quad p = 0,1,2,\ldots.$$

Now, defining the infinite sequence $\{\varepsilon^p\}_{p\in N_0}$ in accordance with

$$\varepsilon^0 = \widehat{\varepsilon}^0, \quad \varepsilon^{p+1} = \sum_{i\in J(p)} E_i H_{p,i}(\gamma,\omega)\varepsilon^{\tau_i(p)} + \sum_{i\notin J(p)} E_i\varepsilon^p, \; p \in N_0,$$

we can immediately deduce that $\{\varepsilon^p\}_{p\in N_0}$ is a majorizing sequence of $\{\widehat{\varepsilon}^p\}_{p\in N_0}$. That is to say, $\widehat{\varepsilon}^p \leqslant \varepsilon^p (\forall p \in N_0)$ holds. Therefore, to fulfill our proof we only need to verify the validity of the relation $\lim_{p\to\infty} \varepsilon^p = 0$. Moreover, Theorem 3.1 guarantees that this limit relation holds if there exists a nonnegative number $\theta \in [0,1)$ and a positive vector $u \in R^n$ such that

$$H_{p,i}(\gamma,\omega)u \leqslant \theta u, \quad i = 1,2,\ldots,\alpha; \; p \in N_0. \tag{3.5}$$

In fact, if we define matrices

$$\begin{cases} \widehat{A}(\omega) = \dfrac{1-\omega-|1-\omega|}{\omega}|D| + \langle A\rangle - P, & \\ \widehat{B}_i(\omega) = \dfrac{1-\omega-|1-\omega|}{\omega}|D| + \langle B_i\rangle, & i=1,2,\ldots,\alpha, \\ \widehat{C}_i(\omega) = |C_i| + P, & \end{cases}$$

then

$$\widehat{A}(\omega)=\widehat{B}_i(\omega)-\widehat{C}_i(\omega),\quad \widehat{B}_i(\omega)=\widehat{M}_i(\gamma,\omega)-\widehat{N}_i(\gamma,\omega),\ i=1,2,\ldots,\alpha$$

holds. Clearly, $\widehat{C}_i(\omega)\geqslant 0$ $(i=1,2,\ldots,\alpha)$, and from the previous investigations we know that

$$\widehat{B}_i(\omega)=\widehat{M}_i(\gamma,\omega)-\widehat{N}_i(\gamma,\omega),\ i=1,2,\ldots,\alpha$$

are M-splittings. So, in light of the verification of (3.3) we know that (3.5) holds provided $\widehat{A}(\omega)$ and $\widehat{B}_i(\omega)$ $(i=1,2,\ldots,\alpha)$ are monotone matrices under the conditions. Noticing

$$\widehat{A}(\omega)=\widehat{B}_i(\omega)-(|C_i|+P)\leqslant\widehat{B}_i(\omega)\leqslant\langle B_i\rangle\leqslant|D|,\ i=1,2,\ldots,\alpha,$$

we therefore only need to test whether $\widehat{A}(\omega)$ is a monotone matrix.

As a matter of fact, let $|Q|=|B|+P$ and $\widehat{R}=|D|-|Q|$. Since

$$\widehat{R}=|D|-|Q|=|D|-|B|-P=\langle A\rangle-P$$

and $\rho(\langle A\rangle^{-1}P)<1$, we easily see that $\widehat{R}$ is a monotone matrix. Hence, $\rho(|D|^{-1}|Q|)<1$. Considering

$$\widehat{A}(\omega)=\frac{1-|1-\omega|}{\omega}|D|-|Q|,$$

we immediately know that $\widehat{A}(\omega)$ is a monotone matrix when $\omega\in(0,2/(1+\rho(|D|^{-1}|Q|)))$. Thereby, (3.5) holds and the demonstration of (ii) is accomplished. □

More specifically, when the system matrix $A\in L(R^n)$ is a monotone matrix, Theorem 3.3 directly results in the following global convergence theorem for the new asynchronous multisplitting two-stage iteration methods.

Theorem 3.4. *Let $A\in L(R^n)$ be a monotone matrix with $D=\mathrm{diag}(A)$ and $A=D-B$, and (B_i,C_i,E_i) $(i=1,2,\ldots,\alpha)$ be its multisplitting with $A=B_i-C_i$ $(i=1,2,\ldots,\alpha)$ being regular splittings. Assume $G:R^n\to R^n$ be a P-bounded mapping and $\rho(A^{-1}P)<1$ hold. For any starting vector $x^0\in R^n$ and any number sequences $\{s_i(p)\}_{p\in N_0}$ $(i=1,2,\ldots,\alpha)$ of the local inner iterations satisfying $s_i(p)\geqslant 1$ $(i=1,2,\ldots,\alpha;\ p\in N_0)$,*

(i) *if $(B_i:M_i,N_i;C_i;E_i)$ $(i=1,2,\ldots,\alpha)$ is a two-stage multisplitting of the matrix $A\in L(R^n)$ with $B_i=M_i-N_i$ $(i=1,2,\ldots,\alpha)$ being weak regular splittings, then the sequence $\{x^p\}$ generated by Method I converges to the unique solution $x^*\in R^n$ of the system of weakly nonlinear equations* (1.1);

(ii) *if* $(B_i : D_i - L_i, U_i; C_i; E_i)$ $(i=1,2,\dots,\alpha)$ *is a two-stage multisplitting of the matrix* $A \in L(R^n)$ *with* $D_i \geqslant 0$, $L_i \geqslant 0$ *and* $U_i \geqslant 0$, $i=1,2,\dots,\alpha$, *then the sequence* $\{x^p\}$ *generated by Method II converges to the unique solution* $x^* \in R^n$ *of the system of weakly nonlinear equations* (1.1) *provided the relaxation parameters* γ *and* ω *satisfy* $0 \leqslant \gamma \leqslant \omega$ *and* $0 < \omega \leqslant 1$.

4. Asymptotic convergence analyses

In this section, we will emphatically discuss the asymptotic convergence properties of the previously presented asynchronous multisplitting two-stage iteration methods for the system of weakly nonlinear equations (1.1) when the nonlinear mapping $G : R^n \to R^n$ is only assumed to be Lipschitz continuous and directionally differentiable. For this purpose, we require the following known results proved in [5, 20].

Lemma 4.1. *Let* $x \in R^n$ *be a positive vector* $(x > 0)$. *If a sequence* $\{\varepsilon^p\}_{p \in N_0}$ *satisfies*

$$|\varepsilon^{p+1}| \leqslant I_p^{(1)} x + I_p^{(2)} |\varepsilon^p|, \quad p=0,1,2,\dots,$$

then for any nonnegative integer $k \leqslant p-1$,

$$|\varepsilon^{p+1}| \leqslant \left(I - \prod_{j=p-k-1}^{p} I_j^{(2)} \right) x + \prod_{j=p-k-1}^{p} I_j^{(2)} |\varepsilon^{p-k-1}|,$$

where $\{I_p^{(1)}\}_{p \in N_0}$ *and* $\{I_p^{(2)}\}_{p \in N_0}$ *are positive diagonal matrix sequences defined by*

$$I_p^{(1)} = \sum_{i \in J(p)} E_i, \quad I_p^{(2)} = \sum_{i \notin J(p)} E_i, \quad p=0,1,2,\dots$$

with E_i $(i=1,2,\dots,\alpha)$ *being the weighting matrices.*

Lemma 4.2. *Let*

$$I^{(0)} = \prod_{p=0}^{m_0-1} I_p^{(2)}, \quad I^{(l+1)} = \prod_{p=m_l}^{m_{l+1}-1} I_p^{(2)}, \quad l=0,1,2,\dots .$$

Then for any positive vector $x \in R^n$, *there exists a constant* $\tilde{\gamma}_0 \in [0,1)$, *uniformly independent of* l *and* x, *such that* $I^{(l)} x \leqslant \tilde{\gamma}_0 x$ $(l \in N_0)$, *where the infinite number sequence* $\{m_l\}_{l \in N_0}$ *is defined in accordance with the following rule*:

m_0 *is the least positive integer such that*

$$\bigcup_{0 \leqslant \tau(p) \leqslant p < m_0} J(p) = \{1,2,\dots,\alpha\},$$

in general, m_{l+1} is the least positive integer such that

$$\bigcup_{m_l \leqslant \tau(p) \leqslant p < m_{l+1}} J(p) = \{1,2,\dots,\alpha\}, \quad l=0,1,2,\dots .$$

For the meaning of the sequence $\{m_l\}_{l\in N_0}$, one can refer to [5, 20] *for details.*

In addition, given a positive vector $u \in R^n$, we define a vector norm on R^n by the functional

$$\|y\|_u = \inf\{\tilde{\sigma} > 0 \mid -\tilde{\sigma}u \leqslant y \leqslant \tilde{\sigma}u\} = \max_{1\leqslant j\leqslant n} \frac{|y_j|}{u_j}.$$

This norm is monotonic in the sense that $|y| \leqslant |x|$ implies that $\|y\|_u \leqslant \|x\|_u$. It is well known that $\||U|u\|_u = \|U\|_u$, where $\|U\|_u$ denotes the matrix norm of $U \in L(R^n)$ induced by the monotonic norm $\|\bullet\|_u$ (see [6]). It easily follows that if $u \in R^n$ is a positive vector, $\tilde{\sigma} > 0$ is a scalar and $U \in L(R^n)$ for which $|U|u \leqslant \tilde{\sigma}u$, then $\|U\|_u \leqslant \tilde{\sigma}$. Moreover, for any $x \in R^n$ and $U \in L(R^n)$, $|x| \leqslant \|x\|_u u$ and $|U|u \leqslant \|U\|_u u$ hold. Now, by Lemma 4.2, we immediately know that there exist nonnegative constants

$$\underline{m} = \liminf_{l\to\infty} \frac{l}{m_l}, \qquad \gamma_0 = \inf_{x>0} \sup_{l\in N_0} \|I^{(l)}\|_x.$$

In the subsequent discussions, we will adopt the notations introduced here without special illustrations.

Theorem 4.3. *Let $A \in L(R^n)$ be a nonsingular matrix, and $(B_i : M_i, N_i; C_i; E_i)$ $(i=1,2,\dots,\alpha)$ be its two-stage multisplitting. Let $G : R^n \to R^n$ be Lipschitz continuous and directionally differentiable and its directional derivative be Lipschitz continuous at a solution $x^* \in R^n$ of the system of weakly nonlinear equations* (1.1). *Assume that there exists a positive vector $u \in R^n$ such that*

$$\max_{1\leqslant i\leqslant \alpha} \|B_i^{-1}(C_i + G'(x^*))\|_u \equiv \sigma < 1,$$

and $B_i = M_i - N_i$ $(i=1,2,\dots,\alpha)$ are convergent splittings. Also, assume $\lim_{p\to\infty} s_i(p) = \infty$ $(i=1, 2,\dots,\alpha)$. Then there exists a $\delta > 0$ such that if $\|x^0 - x^*\|_u \leqslant \delta$,
(i) *the sequence $\{x^p\}$ generated by Method I is well-defined and converges to x^*;*
(ii) *the R_1-factor of the sequence $\{x^p\}$ is at most $\rho^* = (\sigma + (1-\sigma)\gamma_0)^{\underline{m}}$.*

Proof. Because $x^* \in R^n$ is a solution of the system of weakly nonlinear equations (1.1), and Method I is consistent with this weakly nonlinear system, in accordance with (2.2) we have

$$x^* = \sum_{i\in J(p)} E_i[(M_i^{-1}N_i)^{s_i(p)}x^* + (I - (M_i^{-1}N_i)^{s_i(p)})B_i^{-1}(C_i x^* + G(x^*))] + \sum_{i\notin J(p)} E_i x^*.$$

By subtracting this identity from (2.2) and representing

$$\varepsilon^p = x^p - x^*, \quad \varepsilon^{\tau_i(p)} = x^{\tau_i(p)} - x^*, \quad i=1,2,\dots,\alpha; \ p \in N_0,$$

we can directly obtain the recursive formula

$$\varepsilon^{p+1} = \sum_{i\in J(p)} E_i B_i^{-1}(C_i + G'(x^*))\varepsilon^{\tau_i(p)} + \sum_{i\notin J(p)} E_i \varepsilon^p$$

$$+ \sum_{i\in J(p)} E_i (M_i^{-1}N_i)^{s_i(p)} B_i^{-1}[(A - G'(x^*))\varepsilon^{\tau_i(p)} - y(x^*; x^{\tau_i(p)})]$$

$$+ \sum_{i\in J(p)} E_i B_i^{-1} y(x^*; x^{\tau_i(p)}), \tag{4.1}$$

where we have defined $y(x^*; x^{\tau_i(p)})$ by

$$y(x^*; x) = G(x) - G(x^*) - G'(x^*)(x - x^*) \quad \text{for all } x \in R^n.$$

Since by the hypothesis $\lim_{p\to\infty}(M_i^{-1}N_i)^{s_i(p)}B_i^{-1} = 0$ $(i = 1,2,\ldots,\alpha)$ holds, we know that for any $\varepsilon > 0$ there exists a positive integer p_0 such that

$$\|(M_i^{-1}N_i)^{s_i(p)}B_i^{-1}\|_u \leqslant \varepsilon, \quad i = 1,2,\ldots,\alpha$$

holds for all $p \geqslant p_0$. Besides, the Lipschitz continuity of the directional derivative of the nonlinear mapping $G: R^n \to R^n$ at x^* implies that there exists a constant $\kappa > 0$ such that $\|y(x^*; x)\|_u \leqslant \kappa \|x - x^*\|_u^2$ holds when $\|x - x^*\|_u \leqslant \varepsilon$. Now, writing $\max_{1\leqslant i\leqslant\alpha} \|B_i^{-1}\|_u \leqslant \beta$, $\|A - G'(x^*)\|_u \leqslant \eta$, and taking absolute values on both sides of (4.1), with direct manipulations we have

$$|\varepsilon^{p+1}| \leqslant \sum_{i\in J(p)} E_i |B_i^{-1}(C_i + G'(x^*))| |\varepsilon^{\tau_i(p)}| + \sum_{i\notin J(p)} E_i |\varepsilon^p|$$

$$+ \sum_{i\in J(p)} E_i |(M_i^{-1}N_i)^{s_i(p)} B_i^{-1}| [|A - G'(x^*)| |\varepsilon^{\tau_i(p)}| + |y(x^*; x^{\tau_i(p)})|]$$

$$+ \sum_{i\in J(p)} E_i |B_i^{-1}| |y(x^*; x^{\tau_i(p)})|. \tag{4.2}$$

Based upon this inequality, we can assert that

$$|\varepsilon^p| \leqslant \varepsilon u, \quad p \in N_0 \tag{4.3}$$

holds provided $\delta > 0$ is small enough such that $\delta \leqslant \tilde{c}^{-p_0}\varepsilon$, where

$$\tilde{c} = \max\{1, \sigma + \hat{\varepsilon}(\eta + \kappa\varepsilon) + \beta\varepsilon\}, \quad \hat{\varepsilon} = \max_{\substack{1\leqslant i\leqslant\alpha \\ 0\leqslant p\leqslant p_0}} \|(M_i^{-1}N_i)^{s_i(p)}B_i^{-1}\|_u.$$

In fact, (4.3) is obviously trivial when $p = 0$. Suppose that for some positive integer p such that $p \leqslant p_0 - 1$ we have verified the correctness of the estimates $|\varepsilon^k| \leqslant \tilde{c}^{-(p_0-k)}\varepsilon u$ $(k = 0,1,2,\ldots,p)$. Then $|\varepsilon^{\tau_i(p)}| \leqslant \tilde{c}^{-(p_0-p)}\varepsilon u$ $(i = 1,2,\ldots,\alpha)$ holds because of $\tau_i(p) \leqslant p$ $(i = 1,2,\ldots,\alpha)$. Thereby, from (4.2) we can obtain that

$$\begin{aligned}
|\varepsilon^{p+1}| &\leqslant \sum_{i\in J(p)} E_i|B_i^{-1}(C_i+G'(x^*))|\tilde{c}^{-(p_0-p)}\varepsilon u+\sum_{i\notin J(p)} E_i|\varepsilon^p| \\
&\quad+\sum_{i\in J(p)} E_i|(M_i^{-1}N_i)^{s_i(p)}B_i^{-1}|[|A-G'(x^*)|+\tilde{c}^{-(p_0-p)}\varepsilon]\tilde{c}^{-(p_0-p)}\varepsilon u \\
&\quad+\sum_{i\in J(p)} E_i|B_i^{-1}|\tilde{c}^{-2(p_0-p)}\varepsilon^2 u \\
&\leqslant \sum_{i\in J(p)} E_i[\sigma+\widehat{\varepsilon}(\eta+\tilde{c}^{-(p_0-p)}\varepsilon)+\beta\tilde{c}^{-(p_0-p)}\varepsilon]\tilde{c}^{-(p_0-p)}\varepsilon u+\sum_{i\notin J(p)} E_i|\varepsilon^p| \\
&\leqslant \sum_{i\in J(p)} E_i\tilde{c}^{-(p_0-(p+1))}\varepsilon u+\sum_{i\notin J(p)} E_i|\varepsilon^p| \\
&\leqslant \sum_{i\in J(p)} E_i\tilde{c}^{-(p_0-(p+1))}\varepsilon u+\sum_{i\notin J(p)} E_i\tilde{c}^{-(p_0-p)}\varepsilon u \\
&\leqslant \tilde{c}^{-(p_0-(p+1))}\varepsilon u.
\end{aligned}$$

This demonstration shows that we have inductively proved the validity of (4.3) for $0\leqslant p\leqslant p_0$.

Let $\varepsilon>0$ be further small such that $q(\varepsilon)=\sigma+(\beta+\eta)\varepsilon+\varepsilon^2<1$. Then, for $p\geqslant p_0+1$, from (4.2) we can analogously get the relation

$$\begin{aligned}
|\varepsilon^{p+1}| &\leqslant \sum_{i\in J(p)} E_i[\sigma+\varepsilon(\eta+\varepsilon)+\beta\varepsilon]\varepsilon u+\sum_{i\notin J(p)} E_i|\varepsilon^p| \\
&\leqslant \sum_{i\in J(p)} E_i\varepsilon u+\sum_{i\notin J(p)} E_i\varepsilon u=\varepsilon u
\end{aligned} \tag{4.4}$$

by making use of the induction. Therefore, (4.3) is valid.

Moreover, for the above chosen δ and ε, we can further confirm the following estimates:

$$|\varepsilon^p|\leqslant\Delta_l u,\quad \forall p\geqslant m_l,\quad l=0,1,2,\ldots,\,. \tag{4.5}$$

where $\Delta_l=[q(\varepsilon)+(1-q(\varepsilon))\gamma_0]^{l+1}\varepsilon$, $l=0,1,2,\ldots$

As a matter of fact, for $l=0$, by applying (4.2) and (4.3) again we can deduce, analogously to (4.4), that

$$|\varepsilon^{p+1}|\leqslant I_p^{(1)}q(\varepsilon)\varepsilon u+I_p^{(2)}|\varepsilon^p|,\quad p=0,1,2,\ldots$$

holds. From these inequalities we have

$$|\varepsilon^p|\leqslant\left(I-\prod_{j=0}^{p-1}I_j^{(2)}\right)q(\varepsilon)\varepsilon u+\prod_{j=0}^{p-1}I_j^{(2)}|\varepsilon^0|$$

$$\leqslant \left[\left(I - \prod_{j=0}^{p-1} I_j^{(2)}\right) q(\varepsilon) + \prod_{j=0}^{p-1} I_j^{(2)}\right] \varepsilon u$$

$$= \left[q(\varepsilon) + (1 - q(\varepsilon)) \prod_{j=0}^{p-1} I_j^{(2)}\right] \varepsilon u$$

$$\leqslant [q(\varepsilon) + (1 - q(\varepsilon))I^{(0)}]\varepsilon u$$

$$\leqslant [q(\varepsilon) + (1 - q(\varepsilon))\gamma_0]\varepsilon u = \Delta_0 u,$$

which evidently shows that (4.5) is true for $l = 0$. Here, Lemmas 4.1 and 4.2 have been used in the first and the fifth inequalities, respectively.

Suppose now that (4.5) is true for some positive integer $l \geqslant 1$. Then by starting from (4.2) and making use of Lemmas 4.1 and 4.2 again we can obtain

$$|\varepsilon^p| \leqslant I_{p-1}^{(1)} q(\varepsilon)\Delta_l u + I_{p-1}^{(2)} |\varepsilon^{p-1}|$$

$$\leqslant \left(I - \prod_{j=m_l}^{p-1} I_j^{(2)}\right) q(\varepsilon)\Delta_l u + \prod_{j=m_l}^{p-1} I_j^{(2)} |\varepsilon^{m_l}|$$

$$\leqslant \left[\left(I - \prod_{j=m_l}^{p-1} I_j^{(2)}\right) q(\varepsilon) + \prod_{j=m_l}^{p-1} I_j^{(2)}\right] \Delta_l u$$

$$= \left[q(\varepsilon) + (1 - q(\varepsilon)) \prod_{j=m_l}^{p-1} I_j^{(2)}\right] \Delta_l u$$

$$\leqslant [q(\varepsilon) + (1 - q(\varepsilon))I^{(l+1)}]\Delta_l u$$

$$\leqslant [q(\varepsilon) + (1 - q(\varepsilon))\gamma_0]\Delta_l u = \Delta_{l+1} u,$$

which immediately implies the validity of (4.5) for $l + 1$. By induction, we can conclude that (4.5) is true.

Noticing that $\Delta_l \to 0$ $(l \to \infty)$ holds, (4.5) immediately gives $\lim_{p\to\infty} x^p = x^*$.

On the other hand, from (4.5), again we see that when $m_l \leqslant p \leqslant m_{l+1} - 1 (l \in N_0)$, we have

$$\|\varepsilon^p\|_u \leqslant [q(\varepsilon) + (1 - q(\varepsilon))\gamma_0]^{l+1} \varepsilon$$

$$= [q(\varepsilon) + (1 - q(\varepsilon))\gamma_0]^p [q(\varepsilon) + (1 - q(\varepsilon))\gamma_0]^{l+1-p} \varepsilon$$

$$\leqslant [q(\varepsilon) + (1 - q(\varepsilon))\gamma_0]^p [(q(\varepsilon) + (1 - q(\varepsilon))\gamma_0)^{(l+1)/(m_{l+1}-1)-1}]^p \varepsilon.$$

Therefore,

$$\limsup_{p\to\infty} \|\varepsilon^p\|_u^{1/p} \leqslant [q(\varepsilon)+(1-q(\varepsilon))\gamma_0]^{\underline{m}},$$

and (ii) follows at once since $\lim_{\varepsilon\to 0}[q(\varepsilon)+(1-q(\varepsilon))\gamma_0]=\sigma+(1-\sigma)\gamma_0$. □

This result shows that Method I converges asymptotically at least as fast as the asynchronous multisplitting iterative method induced by the only outer multiple splittings, provided $\lim_{p\to\infty} s_i(p)=\infty$ $(i=1,2,\ldots,\alpha)$. However, if this condition is violated, Method I still converges provided $s_i(p)$ $(i=1,2,\ldots,\alpha)$ are sufficiently large. For this case, the speed of asymptotic convergence of the asynchronous multisplitting two-stage iteration method for the system of weakly nonlinear equations (1.1) may then be slower than that of the outer asynchronous multisplitting method. We state this fact more precisely in the following theorem, the proof of which follows almost verbatim that of Theorem 4.3 and is therefore omitted.

Theorem 4.4. *Let $A\in L(R^n)$ be a nonsingular matrix, and $(B_i:M_i,N_i;C_i;E_i)$ $(i=1,2,\ldots,\alpha)$ be its two-stage multisplitting. Let $G:R^n\to R^n$ be Lipschitz continuous and directionally differentiable and its directional derivative be Lipschitz continuous at a solution $x^*\in R^n$ of the system of weakly nonlinear equations* (1.1). *Assume that there exists a positive vector $u\in R^n$ such that*

$$\max_{1\leqslant i\leqslant\alpha} \|B_i^{-1}(C_i+G'(x^*))\|_u \equiv \sigma<1,$$

and $B_i=M_i-N_i$ $(i=1,2,\ldots,\alpha)$ are convergent splittings. Let $\tilde{s}$ be a positive integer such that

$$\|(M_i^{-1}N_i)^s B_i^{-1}\|_u \leqslant \tilde{\alpha} < \frac{1-\sigma}{\|A-G'(x^*)\|_u} \quad \textit{for all } s\geqslant\tilde{s}.$$

Also, assume $\liminf_{p\to\infty} s_i(p)>\tilde{s}$ $(i=1,2,\ldots,\alpha)$. Then there exists a $\delta>0$ such that if $\|x^0-x^*\|_u\leqslant\delta$,

(i) *the sequence $\{x^p\}$ generated by Method* I *is well-defined and converges to x^*;*

(ii) *the R_1-factor of the sequence $\{x^p\}$ is at most $\rho_0^*=(q+(1-q)\gamma_0)^{\underline{m}}$, where $q=\sigma+\tilde{\alpha}\|A-G'(x^*)\|_u$.*

Applying Theorems 4.3 and 4.4 to Method II, we can immediately get the following conclusions.

Theorem 4.5. *Let $A\in L(R^n)$ be a nonsingular matrix, and $(B_i:D_i-L_i,U_i;C_i;E_i)$ $(i=1,2,\ldots,\alpha)$ be its two-stage multisplitting. Let $G:R^n\to R^n$ be Lipschitz continuous and directionally differentiable and its directional derivative be Lipschitz continuous at a solution $x^*\in R^n$ of the system of weakly nonlinear equations* (1.1). *Assume that there exists a positive vector $u\in R^n$ such that*

$$\max_{1\leqslant i\leqslant\alpha} \|B_i^{-1}(C_i+G'(x^*))\|_u \equiv \sigma<1,$$

and $\rho(\mathscr{L}_{\mathrm{AOR}}^{(i)}(\gamma,\omega))<1$ $(i=1,2,\ldots,\alpha)$ with

$$\mathscr{L}_{\mathrm{AOR}}^{(i)}(\gamma,\omega)=(D_i-\gamma L_i)^{-1}[(1-\omega)D_i+(\omega-\gamma)L_i+\omega U_i], \quad i=1,2,\ldots,\alpha. \tag{4.6}$$

Also, assume $\lim_{p\to\infty} s_i(p)=\infty$ $(i=1,2,\ldots,\alpha)$. *Then there exists a* $\delta>0$ *such that if* $\|x^0-x^*\|_u\leqslant\delta$,
(i) *the sequence* $\{x^p\}$ *generated by Method* II *is well-defined and converges to* x^*;
(ii) *the* R_1*-factor of the sequence* $\{x^p\}$ *is at most* $\rho^*=(\sigma+(1-\sigma)\gamma_0)^{\frac{m}{}}$.

Theorem 4.6. *Let* $A\in L(R^n)$ *be a nonsingular matrix, and* $(B_i:D_i-L_i,U_i;C_i;E_i)$ $(i=1,2,\ldots,\alpha)$ *be its two-stage multisplitting. Let* $G:R^n\to R^n$ *be Lipschitz continuous and directionally differentiable and its directional derivative be Lipschitz continuous at a solution* $x^*\in R^n$ *of the system of weakly nonlinear equations* (1.1). *Assume that there exists a positive vector* $u\in R^n$ *such that*

$$\max_{1\leqslant i\leqslant\alpha}\|B_i^{-1}(C_i+G'(x^*))\|_u\equiv\sigma<1,$$

and $\rho(\mathscr{L}_{\mathrm{AOR}}^{(i)}(\gamma,\omega))<1$ $(i=1,2,\ldots,\alpha)$, *where* $\mathscr{L}_{\mathrm{AOR}}^{(i)}(\gamma,\omega)$ $(i=1,2,\ldots,\alpha)$ *are given by* (4.6). *Let* $\tilde{s}$ *be a positive integer such that*

$$\|[\mathscr{L}_{\mathrm{AOR}}^{(i)}(\gamma,\omega)]^sB_i^{-1}\|_u\leqslant\tilde{\alpha}(\gamma,\omega)<\frac{1-\sigma}{\|A-G'(x^*)\|_u}\quad \textit{for all } s\geqslant\tilde{s}.$$

Also, assume $\liminf_{p\to\infty} s_i(p)>\tilde{s}$ $(i=1,2,\ldots,\alpha)$. *Then there exists a* $\delta>0$ *such that if* $\|x^0-x^*\|_u\leqslant\delta$,
(i) *the sequence* $\{x^p\}$ *generated by Method* II *is well defined and converges to* x^*;
(ii) *the* R_1*-factor of the sequence* $\{x^p\}$ *is at most* $\rho_0^*(\gamma,\omega)=(q(\gamma,\omega)+(1-q(\gamma,\omega))\gamma_0)^{\frac{m}{}}$, *where*

$$q(\gamma,\omega)=\sigma+\tilde{\alpha}(\gamma,\omega)\|A-G'(x^*)\|_u.$$

We end this section with the following remark:

Remark. Note that $A=G'(x^*)$ iff $B_i^{-1}(C_i+G'(x^*))=I$ $(i=1,2,\ldots,\alpha)$, we see that the conditions $\|B_i^{-1}(C_i+G'(x^*))\|_u\leqslant\sigma<1$ $(i=1,2,\ldots,\alpha)$ are sufficient for guaranteeing the validity of the inequality $A\neq G'(x^*)$, or in other words, $\|A-G'(x^*)\|_u\neq 0$. Therefore, the quantity $(1-\sigma)/\|A-G'(x^*)\|_u$ in Theorems 4.4 and 4.6 is well-defined.

5. Numerical examples

For a given positive integer $\tilde{n}$, let $n=\tilde{n}^2$ and consider the system of weakly nonlinear equations (1.1) with

$$A=\mathrm{BlockTridiag}(-I,\tilde{B},-I)\in L(R^n),$$

$$G(x)=(g_1(x),g_2(x),\ldots,g_n(x))^{\mathrm{T}}:R^n\to R^n,$$

$$g_1(x)=h^2(|x_1|-|x_1-10|+(1-c)\mathrm{e}^{c|x_1|}\sin x_1),$$

$$g_j(x)=h^2(|x_j|-|x_j-10|+(1-c)\mathrm{e}^{c|x_j|}\sin x_j\cos x_{j-1}),\quad j=2,3,\ldots,n,$$

where $\tilde{B}=\mathrm{tridiag}(-1,4,-1)\in L(R^{\tilde{n}})$, $h=1/(\tilde{n}+1)$ and c is a parameter. This example comes from the finite difference discretization of a Dirichlet problem on the unit square $[0,1]\times[0,1]$; see [14] for details. We solve this system of weakly nonlinear equations by the stationary asynchronous multisplitting two-stage AOR method (AMTS AOR method), as well as its special case, i.e., the asynchronous multisplitting two-stage SOR method (AMTS SOR method) which is given by taking (γ,ω) to be (ω,ω).

In our computations, with $(2\alpha-1)$ positive integers $n_1,n_2,\ldots,n_{2\alpha-1}$ satisfying $n_k=(k\tilde{n}/(2\alpha-1))$ $(k=1,2,\ldots,2\alpha-1)$ we let processor i solve the variables x_j $(j=\tilde{n}n_{2i-3}+1,\tilde{n}n_{2i-3}+2,\ldots,\tilde{n}n_{2i})$. Here, we stipulate that $n_{-1}=0$ and $n_{2\alpha}=\tilde{n}$. The inner iteration numbers are taken to be $s_i(p)\equiv s$ $(i=1,2,\ldots,\alpha,\ p\in N_0)$, and the splitting and the weighting matrices are taken to be $C_i=B_i-A$ and

$$B_i=\mathrm{diag}(\overbrace{4I,\ldots,4I}^{\tilde{n}n_{2i-3}}\ \overbrace{\tilde{B},\ldots,\tilde{B}}^{\tilde{n}(n_{2i}-n_{2i-3})},4I,\ldots,4I),$$

$$E_i=\mathrm{diag}(\overbrace{0,\ldots,0}^{\tilde{n}n_{2i-3}},\overbrace{\mu_{\tilde{n}n_{2i-3}+1}I,\ldots,\mu_{\tilde{n}n_{2i}}I}^{\tilde{n}(n_{2i}-n_{2i-3})},0,\ldots,0),$$

$$L_i=\text{the strictly lower triangular matrix of }(-B_i),$$

$$U_i=\text{the strictly upper triangular matrix of }(-B_i),$$

respectively, where

$$\mu_j=\begin{cases}0.5 & \text{if } \tilde{n}n_{2i-3}+1\leqslant j\leqslant\tilde{n}n_{2i-2},\\ 1.0 & \text{if } \tilde{n}n_{2i-2}+1\leqslant j\leqslant\tilde{n}n_{2i-1},\\ 0.5 & \text{if } \tilde{n}n_{2i-1}+1\leqslant j\leqslant\tilde{n}n_{2i}.\end{cases}$$

Computations are done corresponding to $n=6400$, and various processor numbers α and relaxation parameter pairs (γ,ω). All our computations are started from an initial vector having all components equal to -100, and terminated once the current iterations x^p obey

$$\frac{\|Ax^p-G(x^p)\|_1}{\|Ax^0-G(x^0)\|_1}\leqslant 10^{-7}$$

or the stopping criterion is not satisfied after 8000 iteration steps. For $\alpha=4$, the corresponding sequential CPU time (CPU) in seconds and parallel speed-up (SP) are listed in Tables 1–6. Here, the SP is defined to be the ratio of the sequential CPU to the corresponding parallel runnings. We remark that the parallel CPU time is not listed in the numerical tables since it can be easily obtained by dividing the sequential CPU by the corresponding parallel SP. From our computations we see that suitable choices of the relaxation parameters γ and ω can greatly accelerate the convergence rates of the relaxation methods, and the asynchronous multisplitting two-stage relaxation methods have better numerical behaviour than the ordinary asynchronous multisplitting relaxation methods. Roughly speaking, $s=3,4$, or sometimes, $s=2$ will be good choices. Moreover, the asynchronous multisplitting two-stage AOR method has larger convergence domain than the asynchronous multisplitting two-stage SOR method, and the convergence rate of the former is, in general, not slower than that of the later. Evidently, the numerical results further confirm the correctness of the theoretical results established in the previous sections, and also show that our new methods are feasible and efficient for parallely solving the system of weakly nonlinear equations (1.1).

Table 1
AMTS SOR method ($c = 1.0$)

ω		0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6
	CPU	∞	∞	260.1	249.8	210.7	187.3	169.8	143.1	∞	∞	∞
$s=1$	SP	—	—	3.1	3.1	3.1	3.1	3.2	3.1	—	—	—
	CPU	301.2	266.7	236.3	203.1	184.3	171.0	161.8	168.3	207.5	∞	∞
$s=2$	SP	3.1	3.1	3.0	3.1	3.1	3.2	3.1	3.1	3.2	—	—
	CPU	270.9	244.8	212.2	208.5	189.7	198.2	197.8	183.9	193.5	∞	∞
$s=3$	SP	3.2	3.1	3.3	3.1	3.1	3.1	3.2	3.1	3.1	—	—
	CPU	255.0	229.8	232.3	222.3	210.4	211.7	216.0	221.0	217.3	206.0	218.2
$s=4$	SP	3.1	3.1	3.1	3.1	3.0	3.1	3.2	3.1	3.2	3.2	3.1
	CPU	278.7	248.2	253.1	248.1	237.7	237.7	233.3	244.3	240.7	243.7	∞
$s=5$	SP	3.1	3.1	3.1	3.1	3.1	3.1	3.2	3.1	3.2	3.2	—

Table 2
AMTS AOR method ($c = 1.0$)

γ		0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.1	1.2	1.2	1.1
ω		0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6
	CPU	∞	∞	270.2	226.2	190.3	168.7	152.5	149.4	∞	∞	∞
$s=1$	SP	—	—	3.0	3.1	3.3	3.1	3.1	3.1	—	—	—
	CPU	278.3	247.0	219.5	189.4	172.9	161.7	154.6	164.9	161.4	154.0	145.2
$s=2$	SP	3.1	3.1	3.1	3.1	3.1	3.1	2.9	3.1	3.2	3.2	3.1
	CPU	253.2	229.9	200.3	198.0	181.3	190.6	191.4	178.6	187.7	202.5	∞
$s=3$	SP	3.1	3.1	3.0	3.1	3.1	3.1	3.1	3.1	3.2	3.2	—
	CPU	240.5	218.0	221.7	213.3	202.9	204.9	209.4	214.4	210.8	199.5	209.2
$s=4$	SP	3.1	3.1	3.1	3.1	3.2	3.1	3.2	3.1	3.2	3.2	3.1
	CPU	264.9	237.2	243.1	239.4	230.0	230.4	226.3	237.0	233.5	236.5	233.2
$s=5$	SP	3.1	3.1	3.1	3.1	3.1	3.1	3.2	3.1	3.2	3.2	3.1

Table 3
AMTS SOR method ($c = 0.0$)

ω		0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6
	CPU	∞	∞	226.2	199.1	174.0	156.4	134.2	128.6	∞	∞	∞
$s=1$	SP	—	—	3.0	3.1	3.1	3.1	3.2	3.1	—	—	—
	CPU	239.457	208.6	180.1	174.6	154.4	144.9	146.2	137.9	167.5	∞	∞
$s=2$	SP	3.1	3.1	3.1	3.1	3.1	3.1	3.2	3.1	3.2	—	—
	CPU	222.9	189.3	186.3	175.6	170.1	178.6	167.4	165.0	173.8	∞	∞
$s=3$	SP	3.1	3.1	3.0	3.1	3.1	3.1	3.2	3.1	3.2	—	—
	CPU	229.4	221.7	197.0	198.2	187.0	190.6	191.1	190.6	197.0	183.7	∞
$s=4$	SP	3.1	3.1	3.1	3.1	3.1	3.1	3.2	3.1	3.2	3.2	—
	CPU	242.7	228.2	230.7	211.5	209.5	216.9	209.7	223.1	206.7	209.3	∞
$s=5$	SP	3.1	3.1	3.0	3.1	3.1	3.1	3.2	3.1	3.2	3.2	—

Table 4
AMTS AOR method ($c=0.0$)

γ		0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.1	1.2	1.2	1.1
ω		0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6
	CPU	248.7	220.7	205.3	180.3	157.2	140.9	120.6	134.3	∞	∞	∞
$s=1$	SP	3.1	3.1	3.1	3.1	3.1	3.1	3.2	3.1	—	—	—
	CPU	221.3	193.1	167.3	162.8	144.8	137.0	139.7	135.0	130.3	147.3	148.4
$s=2$	SP	3.1	3.1	3.1	3.1	3.1	3.1	3.2	3.1	3.2	3.2	3.1
	CPU	208.3	177.8	175.8	166.8	162.6	171.8	162.0	160.2	168.6	173.2	∞
$s=3$	SP	3.1	3.1	3.0	3.1	3.1	3.1	3.2	3.1	3.2	3.2	—
	CPU	216.3	210.3	187.9	190.2	180.3	184.5	185.3	184.9	191.0	177.9	182.7
$s=4$	SP	3.1	3.1	3.1	3.1	3.1	3.1	3.2	3.1	3.2	3.2	3.1
	CPU	230.6	218.1	221.6	204.1	202.7	210.2	203.4	216.4	200.5	203.1	211.8
$s=5$	SP	3.1	3.1	3.0	3.1	3.1	3.1	3.2	3.1	3.2	3.2	3.1

Table 5
AMTS SOR method ($c=-1.0$)

ω		0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6
	CPU	∞	∞	468.8	547.8	481.6	255.3	164.5	149.9	∞	∞	∞
$s=1$	SP	—	—	3.1	3.1	3.1	3.1	3.2	3.1	—	—	—
	CPU	295.9	250.4	218.5	209.2	186.3	185.9	175.6	169.4	203.3	∞	∞
$s=2$	SP	3.1	3.1	3.0	3.1	3.1	3.1	3.2	3.1	3.2	—	—
	CPU	256.6	233.5	221.3	206.3	200.4	195.1	189.7	186.0	178.9	∞	∞
$s=3$	SP	3.1	3.1	3.1	3.1	3.1	3.1	3.2	3.1	3.2	—	—
	CPU	261.6	235.1	235.3	216.2	213.9	209.3	212.1	211.1	221.6	221.6	219.0
$s=4$	SP	3.1	3.1	3.1	3.1	3.1	3.1	3.2	3.1	3.2	3.2	3.1
	CPU	273.5	261.6	250.5	249.1	242.3	241.9	237.3	234.9	238.1	247.1	∞
$s=5$	SP	3.1	3.1	3.1	3.1	3.1	3.1	3.2	3.1	3.2	3.2	—

Table 6
AMTS AOR method ($c=-1.0$)

γ		0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.1	1.2	1.2	1.1
ω		0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6
	CPU	289.2	281.1	425.3	496.0	435.0	230.1	147.8	156.5	∞	∞	∞
$s=1$	SP	3.1	3.1	3.0	3.1	3.1	3.1	3.2	3.1	—	—	—
	CPU	273.4	231.8	203.1	195.1	174.7	175.7	167.8	165.9	158.2	243.4	144.6
$s=2$	SP	3.1	3.1	3.0	3.1	3.1	3.1	3.2	3.1	3.2	3.2	3.1
	CPU	239.9	219.3	208.8	196.0	191.6	187.7	183.5	180.6	173.6	184.8	∞
$s=3$	SP	3.1	3.1	3.1	3.1	3.1	3.1	3.2	3.1	3.2	3.2	—
	CPU	246.7	223.1	224.5	207.1	206.3	202.6	205.6	204.8	214.9	214.6	209.9
$s=4$	SP	3.1	3.1	3.0	3.1	3.1	3.1	3.2	3.1	3.2	3.2	3.1
	CPU	259.9	250.0	240.6	240.4	234.5	234.5	230.2	227.8	230.9	239.8	234.4
$s=5$	SP	3.1	3.1	3.1	3.1	3.1	3.1	3.2	3.1	3.2	3.2	3.1

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