CONVERGENCE OF STATIONARY ITERATIVE METHODS FOR HERMITIAN SEMIDEFINITE LINEAR SYSTEMS AND APPLICATIONS TO SCHWARZ METHODS*

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Abstract. A simple proof is presented of a quite general theorem on the convergence of stationary iterations for solving singular linear systems whose coefficient matrix is Hermitian and positive semidefinite. In this manner, elegant proofs are obtained of some known convergence results, including the necessity of the P-regular splitting result due to Keller, as well as recent results involving generalized inverses. Other generalizations are also presented. These results are then used to analyze the convergence of several versions of algebraic additive and multiplicative Schwarz methods for Hermitian positive semidefinite systems.

Key words. linear systems, Hermitian semidefinite systems, singular systems, stationary iterative methods, seminorm, convergence analysis, algebraic Schwarz methods

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1. Introduction.

We consider the linear system

$$Ax = b, \tag{1.1}$$

where the coefficient matrix $A \in \mathbb{C}^{n\times n}$ is assumed to be singular and Hermitian positive semidefinite. Denoting by $\mathrm{Null}(A)$ the nullspace of A and by $\mathrm{Range}(A)$ its range, we assume that $b \in \mathrm{Range}(A)$. This implies that the solution set of (1.1) is nonempty and it is given as an affine space $x^* + \mathrm{Null}(A)$ for some $x^* \in \mathbb{C}^n$ solution of (1.1).

If A is large and sparse, iterative methods for solving (1.1) are the standard approach. In this paper, we focus on stationary iterative methods, including, for example, certain algebraic multigrid methods and additive and multiplicative Schwarz methods. Sometimes, these iterations are accelerated by using them as preconditioners to Krylov subspace methods like Conjugate Gradients. While we do not consider the latter aspect in any detail in this work, let us just mention that one usually assumes convergence of the preconditioner as a prerequisite in this context, so our work is relevant in this case as well.

We consider the very general situation in which we are given an iteration matrix H for (1.1) of the form

$$H = I - \widetilde{M}A \tag{1.2}$$

where $\widetilde{M} \in \mathbb{C}^{n\times n}$ is a matrix which might be singular but it is injective on $\mathrm{Range}(A)$, i.e.,

$$\mathrm{Null}(\widetilde{M}A) = \mathrm{Null}(A). \tag{1.3}$$

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The matrices H and $\widetilde{M}$ induce the iteration

$$x^{k+1} = Hx^k + \widetilde{M}b. \tag{1.4}$$

Since any solution x^* of (1.1) satisfies $\widetilde{M}Ax^* = \widetilde{M}b$, we see that each such x^* is a fixed point of the iteration (1.4). Conversely, if x^* is a fixed point of (1.4), then $0 = -\widetilde{M}Ax^* + \widetilde{M}b$, and since $\widetilde{M}$ is injective on Range(A) we get $Ax^* = b$. We conclude that under the conditions (1.2) and (1.3), x^* is a solution of (1.1) if and only if x^* is a fixed point of (1.4).

The rest of this paper is devoted to the analysis of situations where we can guarantee that the iteration (1.4) converges to a fixed point. Due to the singularity of A, such a limiting fixed point usually depends on the starting vector x^0. Actually, condition (1.3), implies that that convergence of the iteration (1.4) is equivalent to H being semiconvergent according to the following definition[1]; see, e.g., [3], [7], [18].

Definition 1.1. *A matrix $H \in \mathbb{C}^{n\times n}$ is called semiconvergent, if $\rho(H) = 1$, $\lambda = 1$ is the only eigenvalue of modulus 1 and $\lambda = 1$ is a semisimple eigenvalue of H, i.e., its geometric multiplicity is equal to its algebraic multiplicity.*

It follows then, that one goal is to find simple conditions for which we can show that H of the form (1.2) is such that (1.3) holds and it is semiconvergent.

Our general form of the iteration operator from (1.2) applies in particular to iterations induced by splittings of the form $A = M - N$, M nonsingular, in which $\widetilde{M}$ is taken to be M^{-1}. Then condition (1.3) is automatically satisfied. There are iterations which can be interpreted as being of the form (1.2) with $\widetilde{M} = M^\dagger$, the Moore-Penrose pseudoinverse of some singular matrix M; see [7], [12], [13], where such iterations are studied. This situation occurs in particular in the analysis of Schwarz iterations where the artificial boundary conditions between subdomains are of Neumann type; see, e.g., [17], [19].

The rest of the paper is organized as follows. In Section 2 we derive a fundamental convergence result based on an estimate in the energy seminorm. In Section 3 the fundamental result is used in two directions: We obtain simple and elegant proofs for some known convergence results and we develop new convergence results which improve over some that have been published previously. We then consider algebraic additive and multiplicative Schwarz methods. The paper finishes with a conclusion in Section 4. We mention that applications of the fundamental result to algebraic multigrid methods are presented in the forthcoming paper [8].

2. A fundamental result. In the analysis to follow, we use the bilinear form $\langle\cdot,\cdot\rangle_A$ defined for a Hermitian matrix $A \in \mathbb{C}^{n\times n}$ as

$$\langle\cdot,\cdot\rangle_A : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C},\ \ (x,x) \mapsto \langle x,y\rangle_A = \langle Ax,y\rangle\ (= \langle x,Ay\rangle).$$

Here, $\langle x,y\rangle$ denotes the standard Euclidian inner product. Since in our context A is only positive semidefinite, the bilinear form is only semidefinite as well. We collect some trivial properties of $\langle\cdot,\cdot\rangle_A$ in the following lemma.

Lemma 2.1. *Assume that A is Hermitian and positive semidefinite. Then,*

(i) For all $x \in \mathbb{C}^n$ we have $\langle x,x\rangle_A \geq 0$.
(ii) $\langle x,x\rangle_A = 0$ if and only if $x \in$ Null(A).
(iii) If $x \in$ Null(A) or $y \in$ Null(A) then $\langle x,y\rangle_A = 0$.

[1] We note that in some papers such a matrix is simply called *convergent*.

In the sequel, $\|x\|_A$ denotes the seminorm $\langle x, x\rangle_A^{1/2}$.

We now turn to formulate a fundamental result on the convergence of the iteration (1.4). We include a simple proof of this result before discussing how the result is related to similar results in recent publications.

THEOREM 2.2. *Let $H = I - \widetilde{M}A \in \mathbb{C}^{n\times n}$ be the iteration operator of the iteration (1.4). Assume that the following holds:*

$$x \notin \mathrm{Null}(A) \Longrightarrow \|Hx\|_A < \|x\|_A. \tag{2.1}$$

Then,

(i) $\mathrm{Null}(\widetilde{M}A) = \mathrm{Null}(A)$*, i.e.,* $\widetilde{M}$ *is injective on* $\mathrm{Range}(A)$.

(ii) H *is semiconvergent.*

As a consequence, for $b \in \mathrm{Range}(A)$ the iteration (1.4) converges to a solution of (1.1) for any starting vector x^0.

Proof. First observe that $\mathrm{Null}(\widetilde{M}A) = \mathrm{Null}(I - H)$. For $y \notin \mathrm{Null}(A)$ the hypothesis (2.1) gives $Hy \neq y$, i.e., $y \notin \mathrm{Null}(I - H)$. On the other hand $y \in \mathrm{Null}(A)$ implies $y \in \mathrm{Null}(I - H)$, by the definition of H. This shows $\mathrm{Null}(\widetilde{M}A) = \mathrm{Null}(I - H) = \mathrm{Null}(A)$, i.e., (i) holds.

To prove (ii), let x be an eigenvector for an eigenvalue λ of H. If $x \notin \mathrm{Null}(A)$, we have $\|x\|_A > 0$, and from (2.1) we get $|\lambda| \cdot \|x\|_A < \|x\|_A$ which implies $|\lambda| < 1$. If $x \in \mathrm{Null}(A)$, we know that $Hx = x$, i.e., $\lambda = 1$. So $\rho(H) = 1$, and $\lambda = 1$ is the only eigenvalue of modulus 1. It remains to show that $\lambda = 1$ is semisimple.

Assume that, on the contrary, $\lambda = 1$ is not a semisimple eigenvalue of H. Then, there exists a level-2 generalized eigenvector for the eigenvalue $\lambda = 1$, i.e., a vector $v \neq 0$ satisfying

$$Hv = v + u, \text{ where } Hu = u, u \neq 0.$$

Since v is not an eigenvector of H we have $v \notin \mathrm{Null}(A)$. We also have $u \in \mathrm{Null}(A)$, since u is an eigenvector of H for the eigenvalue $\lambda = 1$ and $\widetilde{M}$ is injective on $\mathrm{Range}(A)$. Thus, using parts (ii) and (iii) of Lemma 2.1, we get

$$\langle Hv, Hv\rangle_A = \langle v, v\rangle_A + \langle v, u\rangle_A + \langle u, v\rangle_A + \langle u, u\rangle_A = \langle v, v\rangle_A$$

which contradicts (2.1). Therefore, there is no level-2 generalized eigenvector for the eigenvalue $\lambda = 1$, i.e., $\lambda = 1$ is semisimple. ∎

REMARK 2.3. Since $Hx = x$ for $x \in \mathrm{Null}(A)$, the operator H canonically induces a linear operator $\mathcal{H}$ on the quotient space $\mathcal{Q} = \mathbb{C}^n/\mathrm{Null}(A)$ on which $\|\cdot\|_A$ canonically induces a true norm $\|x + \mathrm{Null}(A)\|_A := \|x\|_A$. Therefore, the implication (2.1) is actually equivalent to

$$\|Hx\|_A \leq \|\mathcal{H}\|_A \cdot \|x\|_A \text{ with } \|\mathcal{H}\|_A < 1. \tag{2.2}$$

We will write $\|H\|_A$ for $\|\mathcal{H}\|_A$ in the sequel and, for simplicity, we will always formulate our convergence results to come by stating that $\|H\|_A < 1$, having in mind that this means that $\widetilde{M}$ is injective on $\mathrm{Range}(A)$ and that the iteration (1.4) converges to a solution of (1.1) whenever $b \in \mathrm{Range}(A)$.

In [12], it was observed that $\|H\|_A < 1$ is sufficient for $\lim_{k\to\infty}(Ax^k - b) = 0$ for the iterates of (1.4) in the case that $\widetilde{M} = M^\dagger$, the Moore-Penrose inverse of a matrix M satisfying $\mathrm{Range}(A) \subseteq \mathrm{Range}(M)$ and $b \in \mathrm{Range}(A)$. It was then shown in [7] that this kind of "quotient convergence" is actually equivalent to "usual" convergence, i.e., we also have $\lim_{k\to\infty} x^k = x^*$ with $Ax^* = b$. This is precisely the assertion of Theorem 2.2 except that we do not require $\widetilde{M}$ to be a Moore-Penrose pseudoinverse. The references [7], [12] use such pseudoinverses since they view the iteration (1.4) as arising from a splitting $A = M - N$ of A. Since every matrix $\widetilde{M}$ is the Moore-Penrose inverse of its own Moore-Penrose inverse, i.e., $\widetilde{M} = (\widetilde{M}^\dagger)^\dagger$, we see that there is nothing special in requiring $\widetilde{M}$ to be a Moore-Penrose pseudoinverse. The crucial condition is (2.1) (or, equivalently, (2.2)), implying (1.3).

3. Applications of the fundamental result. As first applications of Theorem 2.2 we give simple proofs of the necessity of a well-known result of Keller [11] (see also [7]), and a generalization which contains as a special case a recent result from [13].

THEOREM 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite. Let M be nonsingular and let $H = I - M^{-1}A$. Then $M + M^H - A$ is positive definite on* $\mathrm{Range}(M^{-1}A)$ *if and only if $\|H\|_A < 1$.*

Here, M^H denotes the conjugate transpose of the matrix M.

Proof. Using the identity

$$H^H AH = A - AM^{-H}(M + M^H - A)M^{-1}A,$$

we see that

$$\langle Hx, Hx\rangle_A = x^H H^H AHx = \langle x, x\rangle_A - \langle M^{-1}Ax, (M + M^H - A)M^{-1}Ax\rangle. \tag{3.1}$$

For $x \notin \mathrm{Null}(A)$ the vector $M^{-1}Ax$ is non-zero, so that due to the positive definiteness of $M + M^H - A$ on $\mathrm{Range}(M^{-1}A)$ we obtain

$$x \notin \mathrm{Null}(A) \Longrightarrow \langle Hx, Hx\rangle_A < \langle x, x\rangle_A,$$

and $\|H\|_A < 1$ follows by Remark 2.3.

On the other hand, if $\|H\|_A < 1$, then $\langle Hx, Hx\rangle_A < \langle x, x\rangle_A$ for all $x \notin \mathrm{Null}(A)$, so that (3.1) gives

$$\langle M^{-1}Ax, (M + M^H - A)M^{-1}Ax\rangle = \langle x, x\rangle_A - \langle Hx, Hx\rangle_A > 0.$$

Since every nonzero $y \in \mathrm{Range}(M^{-1}A)$ can be expressed as $y = M^{-1}Ax$ with $x \notin \mathrm{Null}(A)$, this shows that $M + M^H - A$ is positive definite on $\mathrm{Range}(M^{-1}A)$. ∎

Recall that by Theorem 2.2 and Remark 2.3, $\|H\|_A < 1$ implies that the iteration (1.4) converges towards a solution of (1.1) for every starting vector. It is in these terms that the above theorem was originally formulated in [11].

One application of Theorem 3.1 is for the relaxed Gauss-Seidel iteration. With $A = D - L - L^H$ denoting the canonical decomposition of A into its diagonal part D, its lower triangular part $-L$ and its upper triangular part $-L^H$, one then has $M = \frac{1}{\omega}D - L$. This matrix M is nonsingular if no diagonal element of A is zero, and $M + M^H - A = \frac{2-\omega}{\omega}D$ is positive definite on the whole space for $\omega \in (0, 2)$.

We now turn to the announced generalization, where M is allowed to be singular.

THEOREM 3.2. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite. Let $M, \widetilde{M} \in \mathbb{C}^{n\times n}$ satisfy*

$$M\widetilde{M}A = A \tag{3.2}$$

and put $H = I - \widetilde{M}A$. Then $\|H\|_A < 1$ if and only if $M + M^H - A$ is positive definite on $\mathrm{Range}(\widetilde{M}A)$.

Proof. We first observe that since $M\widetilde{M}A = A$ we have $\mathrm{Null}(\widetilde{M}A) = \mathrm{Null}(A)$, i.e., $\widetilde{M}$ is injective on $\mathrm{Range}(A)$. We also have $A(\widetilde{M})^H M^H = A$ and thus

$$\begin{aligned} H^H A H &= A - A\widetilde{M}A - A(\widetilde{M})^H A + A(\widetilde{M})^H A\widetilde{M}A \\ &= A - A(\widetilde{M})^H \cdot \big(M + M^H - A\big) \cdot \widetilde{M}A. \end{aligned} \tag{3.3}$$

So, if $M+M^H-A$ is positive definite on $\mathrm{Range}(\widetilde{M}A)$, we see that for $x \notin \mathrm{Null}(\widetilde{M}A) = \mathrm{Null}(A)$ one has

$$\|Hx\|_A < \|x\|_A$$

so that $\|H\|_A < 1$ follows again from Remark 2.3.

The converse follows in the same manner as in the proof of Theorem 3.1, so we do not reproduce it here. ∎

This result allows to use for $\widetilde{M}$ various generalized inverses of M. In the case that $\widetilde{M}$ is the Moore-Penrose inverse $M^\dagger$ of M, a sufficient condition for $M\widetilde{M}A = A$ is to require $\mathrm{Range}(A) \subseteq \mathrm{Range}(M)$. With this more restrictive condition, Theorem 3.2 was essentially proved in [12, Theorem 4.4], see also [7]. The paper [13], too, uses the same condition $\mathrm{Range}(A) \subseteq \mathrm{Range}(M)$, but allows $\widetilde{M}$ to just be an inner inverse of M, i.e., an operator satisfying $M\widetilde{M}M = M$. The convergence results there, however, come in a quite different flavor. Instead of assuming the positive definiteness of $M + M^H - A$ on $\mathrm{Range}(\widetilde{M}A)$, they require a further, more indirectly defined matrix to be an inner inverse.

We note that Cao [7] presents the following example indicating that condition (3.2) is essential for the necessity part of Theorem 3.2.

EXAMPLE 3.3. *Let*

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad M = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad \widetilde{M} = M^\dagger = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix},$$

for which it holds that $\widetilde{M}A = A$, and thus $M\widetilde{M}A = MA = M \neq A$. On the other hand, we have that

$$H = I - \widetilde{M}A = I - A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

so that $\|H\|_A = 0 < 1$, but it holds that $(\widetilde{M}A)^H(M^H + M - A)(\widetilde{M}A) = 0$, and thus $M^H + M - A$ is not positive definite on $\mathrm{Range}(\widetilde{M}A)$.

Theorem 2.2 can be used to derive further conditions implying the convergence of iteration (1.4). The following result has the same spirit as Theorem 3.1, but note that the hypothesis (3.2) is not needed here. This result is used later in the paper.

THEOREM 3.4. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite. Let $H = I - \widetilde{M}A$. Then*

$$\widetilde{M} + \widetilde{M}^H - \widetilde{M}^H A \widetilde{M} \text{ is positive definite on } \mathrm{Range}(A) \tag{3.4}$$

if and only if $\|H\|_A < 1$.

Proof. We have

$$H^H A H = A - A(\widetilde{M} + \widetilde{M}^H - \widetilde{M}^H A \widetilde{M})A. \tag{3.5}$$

So if $\widetilde{M} + \widetilde{M}^H - \widetilde{M}^H A \widetilde{M}$ is positive definite on Range(A) we immediately get that for $x \notin \text{Null}(A)$ we have $\|Hx\|_A < \|x\|_A$, i.e., $\|H\|_A < 1$. On the other hand, if $\|H\|_A < 1$ and $x \notin \text{Null}(A)$, then $\|Hx\|_A^2 < \|x\|_A^2$. From (3.5) we see that this means $\langle Ax, (\widetilde{M} + \widetilde{M}^H - \widetilde{M}^H A \widetilde{M})Ax \rangle > 0$, i.e., $(\widetilde{M} + \widetilde{M}^H - \widetilde{M}^H A \widetilde{M})$ is positive definite on Range(A). $\square$

We note that for the matrices of Example 3.3 we have that

$$\widetilde{M} + \widetilde{M}^H - \widetilde{M}^H A \widetilde{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is the identity on Range(A), and thus positive definite on Range(A).

3.1. Application to additive Schwarz. We start this section with a general result where the generic operator $\widetilde{M}$ is decomposed into p operators and involves a positive damping factor θ, i.e., we have $\widetilde{M} = \theta \sum_{i=1}^{p} \widetilde{M}_i$ and

$$H = I - \widetilde{M}A = I - \theta \sum_{i=1}^{p} \widetilde{M}_i A. \tag{3.6}$$

As we shall see, this general formulation applies in particular to several variants of additive Schwarz iterations.

One of the hypothesis we use is that there exists a number $\gamma > 0$ such that

$$\Re\langle x, \widetilde{M}_i A x\rangle_A \geq \gamma \cdot \langle \widetilde{M}_i A x, \widetilde{M}_i A x\rangle_A \text{ for all } x \in \mathbb{C}^n \text{ and for } i = 1, \ldots, p. \tag{3.7}$$

Here, $\Re z$ denotes the real part of a complex number z. It is easy to see that (3.7) is equivalent to the hypothesis (cf. (3.4))

$$\widetilde{M}_i + \widetilde{M}_i^H - 2\gamma \widetilde{M}_i^H A \widetilde{M}_i \text{ is positive semidefinite on Range}(A). \tag{3.8}$$

In the following theorems we give convergence results requiring upper bounds for the damping factor θ in (3.6). These upper bounds are given in terms of p (usually representing the number of subdomains). Nevertheless, the bounds can be enlarged in the same way as it is done in the convergence analysis for classical additive Schwarz methods for Hermitian positive definite matrices, where q "colors" are used; see Remark 3.14.

THEOREM 3.5. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite. Let $\widetilde{M}_i \in \mathbb{C}^{n\times n}$, $i = 1, \ldots, p$, be such that*

(i) There exists a number $\gamma > 0$ such that (3.7) holds.

(ii) $\cap_{i=1}^{p} \text{Null}(A\widetilde{M}_i A) = \text{Null}(A)$.

Then, there exists $\bar{\theta} \geq \frac{2\gamma}{p}$ such that if $0 < \theta < \bar{\theta}$, the matrix H from (3.6) satisfies $\|H\|_A < 1$, and $\widetilde{M}$ satisfies (3.4).

Moreover, if the following strengthened Cauchy-Schwarz inequalities

$$|\langle \widetilde{M}_i A x, \widetilde{M}_j A x\rangle_A| \leq c_{ij} \cdot \|\widetilde{M}_i A x\|_A \cdot \|\widetilde{M}_j A x\|_A \text{ for all } x \in \mathbb{C}^n,\ i, j = 1, \ldots, p, \tag{3.9}$$

hold with $0 \leq c_{ij} = c_{ji} \leq 1$ and $c_{ii} = 1$, then $\bar{\theta}$ can be taken such that $\bar{\theta} \geq (2\gamma)/\lambda_{max}(C)$, where $\lambda_{max}(C)$ is the largest eigenvalue of the matrix $C = (c_{ij})$.

Proof. For all $x \in \mathbb{C}^n$ we have

$$\langle Hx, Hx\rangle_A = \langle x, x\rangle_A - 2\theta \sum_{i=1}^{p} \Re\langle x, \widetilde{M}_i A x\rangle_A + \theta^2 \sum_{i,j=1}^{p} \langle \widetilde{M}_i A x, \widetilde{M}_j A x\rangle_A. \quad (3.10)$$

For ease of notation we put

$$m_i = \|\widetilde{M}_i A x\|_A,\ i = 1, \dots, p, \quad (3.11)$$

and observe that using hypothesis (i) it holds that

$$\gamma \cdot m_i^2 = \gamma \cdot \langle \widetilde{M}_i A x, \widetilde{M}_i A x\rangle_A \leq \Re\langle x, \widetilde{M}_i A x\rangle_A. \quad (3.12)$$

Also, using the Cauchy-Schwarz inequality, one has $\langle \widetilde{M}_i A x, \widetilde{M}_j A x\rangle_A \leq m_i m_j$. Let now $m = (m_1, \dots, m_p)^T$ and $E \in \mathbb{C}^{p \times p}$ be the matrix of all ones. Then from (3.10) we obtain

$$\begin{aligned} \langle Hx, Hx\rangle_A &\leq \langle x, x\rangle_A - 2\theta\gamma \sum_{i=1}^{p} m_i^2 + \theta^2 \sum_{i,j=1}^{p} m_i m_j \\ &= \langle x, x\rangle_A - \theta \cdot \langle m, (2\gamma I - \theta E) m\rangle. \quad (3.13) \end{aligned}$$

For $\theta < (2\gamma)/p$ the matrix $2\gamma I - \theta E$ is strictly diagonally dominant and thus Hermitian and positive definite. Therefore, once we have shown that $m \neq 0$ for $x \notin \mathrm{Null}(A)$ we will have proven the first part of the theorem, since then, by (3.13), we have $\langle Hx, Hx\rangle_A < \langle x, x\rangle_A$, i.e., $\|H\|_A < 1$. But if $m_i = 0$ for $i = 1, \dots, p$, we have $\widetilde{M}_i A x \in \mathrm{Null}(A)$ and thus $x \in \mathrm{Null}(A\widetilde{M}_i A)$ for $i = 1, \dots, p$. By (ii) this gives $x \in \mathrm{Null}(A)$.

The fact that $\widetilde{M}$ fulfills (3.4) follows directly from Theorem 3.4.

If the strengthened Cauchy-Schwarz inequalities (3.9) hold, we can replace (3.13) by the stronger

$$\langle Hx, Hx\rangle_A \leq \langle x, x\rangle_A - \theta \cdot \langle m, (2\gamma I - \theta C) m\rangle.$$

Since $2\gamma I - \theta C$ is Hermitian and positive definite for $\theta < (2\gamma)/\lambda_{\max}(C)$, the same arguments as before prove the last part of the theorem. ☐

THEOREM 3.6. *Assume that $A \in \mathbb{C}^{n\times n}$ is Hermitian and positive semidefinite and that (i) and (ii) of Theorem 3.5 hold. With the notation from Theorem 3.5, assume that there exists a natural number $q < p$ such that for each $i \in \{1, \dots, p\}$ the space* $\mathrm{Range}(\widetilde{M}_i)$ *is orthogonal to all spaces* $\mathrm{Range}(\widetilde{M}_j)$*, $j = 1, \dots, p$, $j \neq i$, except for at most $q - 1$ such indices. Then $\bar{\theta}$ can be taken such that $\bar{\theta} \geq (2\gamma)/q$.*

Proof. By the hypothesis, we have strengthened Cauchy-Schwarz inequalities (3.9), where for each i at most q of the c_{ij} are non-zero, and the non-zero ones can be taken to be equal to 1. Therefore, all row sums of C are bounded by q, and thus by Gershgorin's theorem (see, e.g., [20]), we have $\lambda_{\max}(C) \leq q$. ☐

In the results presented so far, the operators $\widetilde{M}_i$ were allowed to be of quite general nature; in particular, they may be non-Hermitian. In many situations, however, the operators $\widetilde{M}_i$ are Hermitian and positive semidefinite on $\mathrm{Range}(A)$. In this case, $x \in \mathrm{Null}(A\widetilde{M}_i A)$ implies $0 = \langle x, (A\widetilde{M}_i A)x\rangle = \langle Ax, \widetilde{M}_i A x\rangle$ and thus $Ax \in \mathrm{Null}(\widetilde{M}_i)$, i.e., $x \in \mathrm{Null}(\widetilde{M}_i A)$. In this situation we consequently have $\cap_{i=1}^{p} \mathrm{Null}(A\widetilde{M}_i A) = \cap_{i=1}^{p} \mathrm{Null}(\widetilde{M}_i A)$, which directly gives the following corollary to Theorem 3.5

THEOREM 3.7. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite. Let $\widetilde{M}_i \in \mathbb{C}^{n\times n}$, $i = 1, \dots, p$, be such that* *(i)* $\widetilde{M}_i$ *is Hermitian,* $i = 1, \ldots, p$,

(ii) There exists a number $\gamma > 0$ *such that (3.7) holds.*

(iii) $\cap_{i=1}^{p} \mathrm{Null}(\widetilde{M}_i A) = \mathrm{Null}(A)$.

Then all conclusions of Theorem 3.5 hold.

To study additive Schwarz methods, we need some further notation. We consider a decomposition of $\mathbb{C}^n$ into p subspaces of dimensions n_i, $i = 1, \ldots, p$, represented by $\mathbb{C}^{n_i}$. By R_i we denote the projections ('restrictions') onto these subspaces, represented as matrices $R_i \in \mathbb{C}^{n_i \times n}$ having full rank n_i. We define the Galerkin operators

$$A_i = R_i A R_i^H \in \mathbb{C}^{n_i \times n_i}, \; i = 1, \ldots, p.$$

The following result on the range of the Galerkin operator will be useful later.

LEMMA 3.8. *If A is Hermitian and positive semidefinite, then*

$$\mathrm{Range}(A_i) = \mathrm{Range}(R_i A).$$

Proof. Since A_i is Hermitian, the assertion is equivalent to $\mathrm{Null}(A_i) = \mathrm{Null}(A R_i^H)$. Clearly, $\mathrm{Null}(A_i) \supseteq \mathrm{Null}(A R_i^H)$. On the other hand, if $x \in \mathrm{Null}(A_i)$, it satisfies $0 = \langle A_i x, x \rangle = \langle A R_i^H x, R_i^H x \rangle$, which implies $R_i^H x \in \mathrm{Null}(A)$, i.e., $x \in \mathrm{Null}(A R_i^H)$, showing that we also have $\mathrm{Null}(A_i) \subseteq \mathrm{Null}(A R_i^H)$. □

For the moment, let us assume that, although A is only Hermitian positive semidefinite, all Galerkin operators are nonsingular (and thus positive definite), i.e., we assume that $\mathrm{Range}(R_i^H) \cap \mathrm{Null}(A) = \{0\}$ for all i; see, e.g., [4], [6], [15], [16], for examples when this situation occurs. The additive (damped) Schwarz iteration for solving $Ax = b$ is then given as (1.4) with

$$\widetilde{M} = \theta \sum_{i=1}^{p} R_i^H A_i^{-1} R_i \text{ and } H = I - \widetilde{M} A. \tag{3.14}$$

We refer the reader, e.g., to [17], [19], and references therein for details on Schwarz methods, and to [2], [9], for algebraic formulations.

THEOREM 3.9. *Let $A \in \mathbb{C}^{n \times n}$ be Hermitian and positive semidefinite. Moreover, assume that the projection operators R_i satisfy*

$$\cap_{i=1}^{p} \mathrm{Null}(R_i) = \{0\}, \tag{3.15}$$

and that $A_i = R_i A R_i^H$ is nonsingular, $i = 1, \ldots, p$. Finally, let $0 < \theta < \frac{2}{p}$. Then H from (3.14) satisfies $\|H\|_A < 1$, and $\widetilde{M}$ satisfies (3.4).

Proof. We show that $\widetilde{M}_i = R_i^H A_i^{-1} R_i$ satisfies hypotheses (i)–(iii) of Theorem 3.7 with $\gamma = 1$. Obviously, $\widetilde{M}_i$ is Hermitian. For (ii) we have

$$\widetilde{M}_i A \widetilde{M}_i A = R_i^H A_i^{-1} R_i A R_i^H A_i^{-1} R_i A = R_i^H A_i^{-1} A_i A_i^{-1} R_i A = R_i^H A_i^{-1} R_i A = \widetilde{M}_i A,$$

which shows that the $\widetilde{M}_i A$ are projections, i.e., (ii) holds with $\gamma = 1$. For (iii) let $x \in \cap_{i=1}^{p} \mathrm{Null}(\widetilde{M}_i A)$, then,

$$0 = \langle R_i^H A_i^{-1} R_i A x, x \rangle_A = \langle A_i^{-1} R_i A x, R_i A x \rangle,$$

which, since A_i is Hermitian positive definite, implies $R_i A x = 0$, $i = 1, \ldots, p$, i.e.

$$Ax \in \cap_{i=1}^{p} \mathrm{Null}(R_i) = \{0\}.$$

Thus, $x \in \mathrm{Null}(A)$. So we have $\cap_{i=1}^p \mathrm{Null}(\widetilde{M}_i A) \subseteq \mathrm{Null}(A)$, and since the opposite inclusion is trivial we have (iii). ◻

REMARK 3.10. The restriction operators R_i in our formulation of Schwarz methods are very general. In the special case when they are Boolean gather operators (i.e., their rows being rows of the identity), using Theorem 3.9 we recover the convergence part of [16, Theorem 4.2].

Let us also note that for the "prolongation" operators $P_i = R_i^H$, we have $\mathrm{Range}(P_i) = \mathrm{Null}(R_i)^\perp$. Thus, condition (3.15) can equivalently be stated as

$$\sum_{i=1}^{p} \mathrm{Range}(P_i) = \mathbb{C}^n,$$

as is done, e.g., in [10].

Theorem 3.9 can be extended to the case where the Galerkin matrices A_i are singular, if we replace their inverses by the Moore-Penrose pseudoinverses $A_i^\dagger$.

THEOREM 3.11. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite. Let $0 < \theta < \frac{2}{p}$ and put*

$$H = I - \widetilde{M}A, \text{ with } \widetilde{M} = \theta \sum_{i=1}^{p} R_i^H A_i^\dagger R_i, \text{ where } A_i = R_i A R_i^H. \tag{3.16}$$

Finally, assume that the projection operators R_i satisfy

$$\cap_{i=1}^p \mathrm{Null}(R_i^H A_i^\dagger R_i A) = \mathrm{Null}(A).$$

Then H from (3.16) satisfies $\|H\|_A < 1$, and $\widetilde{M}$ satisfies (3.4).

Proof. All we need to do is show that with $\widetilde{M}_i = R_i^H A_i^\dagger R_i, i = 1, \ldots, p$, the hypotheses (i) and (ii) of Theorem 3.7 (with $\gamma = 1$) are satisfied, (iii) being assumed. Since A_i is Hermitian positive semidefinite, so is $A_i^\dagger$, and therefore also is $\widetilde{M}_i$. For (ii), we have

$$R_i^H A_i^\dagger R_i A R_i^H A_i^\dagger R_i A = R_i^H A_i^\dagger A_i A_i^\dagger R_i A = R_i^H A_i^\dagger R_i A \tag{3.17}$$

showing that the matrices $R_i^H A_i^\dagger R_i A$ are again projections, i.e., (ii) holds with equality. ◻

We next consider a situation usually referred to as inexact solution of the local problems; see, e.g., [1], [5], [17], [19]. This is the situation, e.g., when the solution of the local problem

$$A_i y_i = z_i \tag{3.18}$$

is not obtained exactly. Thus one replaces $A_i^{-1} z_i$ or $A_i^\dagger z_i$ with a vector other than a solution of (3.18), and this is represented by $\tilde{A}_i z_i$. In this case we have $\widetilde{M}_i = R_i^H \tilde{A}_i R_i$; and using Lemma 3.8 it is easy to see that the hypothesis (3.8), or equivalently (3.7), can be rewritten as

$$\tilde{A}_i + \tilde{A}_i^H - 2\gamma \tilde{A}_i^H A_i \tilde{A}_i \text{ is positive semidefinite on } \mathrm{Range}(A_i). \tag{3.19}$$

Observe that here $\tilde{A}_i$ is not assumed to be symmetric, and thus neither is $\widetilde{M}_i$.

We are ready now to establish the convergence of (damped) additive Schwarz iterations with inexact local solvers, which follows directly from Theorem 3.5.

THEOREM 3.12. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite, and let $A_i = R_i A R_i^H$, $i = 1,\dots,p$. Let $0 < \theta < \frac{2}{p}$ and put*

$$H = I - \widetilde{M}A, \ \text{with}\ \widetilde{M} = \theta \sum_{i=1}^{p} \widetilde{M}_i = \theta \sum_{i=1}^{p} R_i^H \tilde{A}_i R_i,$$

where $\tilde{A}_i$ is such that there exists a number γ for which (3.19) holds. Finally, assume that the projection operators R_i satisfy

$$\cap_{i=1}^{p} \mathrm{Null}(AR_i^H \tilde{A}_i R_i A) = \mathrm{Null}(A).$$

Then H satisfies $\|H\|_A < 1$, and $\widetilde{M}$ satisfies (3.4).

Note that condition (3.19) is fulfilled with $\gamma = \frac{1}{2}$ if $\tilde{A}_i + \tilde{A}_i^H - \tilde{A}_i^H A_i \tilde{A}_i$ is positive definite on $\mathrm{Range}(A_i)$, which is precisely (3.4) from Theorem 3.4. So in this special case, by Theorem 3.4, we have $\|I - \tilde{A}_i A_i\|_{A_i} < 1$, or that an iteration for the solution of the local problem (3.18) with iteration matrix $I - \tilde{A}_i A_i$ is convergent. One particular general example of this situation is when one uses a splitting of $A_i = B_i - C_i$, and the solution of the system (3.18) is approximated by κ classical stationary iterations associated with this splitting. Thus, for this example

$$\tilde{A}_i = \sum_{j=0}^{\kappa-1} (B_i^{-1} C_i)^j B_i^{-1}. \tag{3.20}$$

Of course one can have different values of κ for different local problems. As a particular case, consider the canonical decompositions $A_i = D_i - L_i - L_i^H$ and put $B_i = \frac{1}{\omega} D_i - L_i$, i.e., relaxed Gauss-Seidel. If one sets $\tilde{A}_i = B_i$ the local solutions are approximated by one step of the relaxed Gauss-Seidel, i.e., $\kappa = 1$. Assuming that no diagonal element of A_i is zero and that $\omega \in (0, 2)$, a simple calculation shows that (3.19) is fulfilled with $\gamma = \frac{1}{2}$. Since we then have that the relaxed Gauss-Seidel iteration is convergent, using Theorem 3.4, we see that that $\tilde{A}_i$ of (3.20) also fulfills (3.19) with $\gamma = \frac{1}{2}$ for all integer values of κ.

REMARK 3.13. We note that a special case of Theorem 3.12 when R_i are Boolean gather operators, and $\tilde{A}_i$ are symmetric and nonsingular, is [16, Theorem 6.1], where the hypothesis used there is equivalent to

$$\langle z, \tilde{A}_i z\rangle \le \langle z, A_i^{-1} z\rangle \quad \text{for all} \ \ z \in \mathbb{C}^{n_i}, \ \ \text{and for} \ \ i = 1,\dots,p,$$

which implies (3.19) with $\gamma \le 1$. Indeed, in this case, we have that the difference $\tilde{A}_i^{-1} - A_i$ is positive semidefinite, and we write

$$2\tilde{A}_i - 2\gamma \tilde{A}_i^H A_i \tilde{A}_i = 2\tilde{A}_i \left(\tilde{A}_i^{-1} - \gamma A_i\right) \tilde{A}_i.$$

REMARK 3.14. If in Theorems 3.7, 3.9, 3.11, and 3.12, we add the hypothesis that there exists a natural number $q < p$ such that for each $i \in \{1,\dots,p\}$ the space $\mathrm{Range}(R_i^H)$ is orthogonal to all spaces $\mathrm{Range}(R_j^H)$, $j = 1,\dots,p$, $j \ne i$, except for at most $q-1$ such indices, then, using Theorem 3.6, the results hold for $\theta < 2/q$; cf. [10, Ch. 11.2.4], where this is done for classical additive Schwarz for A Hermitian positive definite. See also [2], [9], and [16] for other such situations.

We note that Hermitian positive semidefinite matrices $\widetilde{M}_i$ different than those considered in Theorems 3.9, 3.11, and 3.12, also do appear in other Schwarz contexts and our general Theorem 3.5 would apply to such cases as well. For example, in [14] matrices of the form $\widetilde{M}_i = R_i^H (A_i + G_i) R_i$ are used, where G_i derives from the Robin boundary conditions.

3.2. Multiplicative Schwarz. Instead of an additive we now consider a multiplicative combination of p operators $\widetilde{M}_i$ resulting in

$$H = (I - \widetilde{M}_p A)(I - \widetilde{M}_{p-1} A) \cdots (I - \widetilde{M}_1 A) = \prod_{i=p}^{1} (I - \widetilde{M}_i A). \tag{3.21}$$

Of course, the iteration operator H can be written in the form $H = I - \widetilde{M}A$, but an explicit formula for $\widetilde{M}$ is not needed in our convergence analysis. As in the additive case, the general formulation (3.21) applies, for particular choices of the matrices $\widetilde{M}_i$, to several variants of multiplicative Schwarz methods, including, for example, those corresponding to Robin boundary conditions [14].

As we did in the additive case, we first state a general theorem which we then apply to the multiplicative Schwarz setting.

THEOREM 3.15. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite. Let $\widetilde{M}_i \in \mathbb{C}^{n\times n}$, $i = 1, \ldots, p$, be such that*

(i) A is injective on $\widetilde{M}_i A$ for $i = 1, \ldots, p$, i.e., $\mathrm{Null}(A\widetilde{M}_i A) = \mathrm{Null}(\widetilde{M}_i A)$ for $i = 1, \ldots, p$.

(ii) There exists a number $\gamma > \frac{1}{2}$ such that (3.7) holds.

(iii) $\cap_{i=1}^{p} \mathrm{Null}(\widetilde{M}_i A) = \mathrm{Null}(A)$.

Let H be as in (3.21). Then H satisfies $\|H\|_A < 1$. Furthermore, if we write $H = I - \widetilde{M}A$, then $\widetilde{M}$ satisfies (3.4).

Proof. We first note that by (ii) we have for $x \in \mathbb{C}^n$ and $i = 1, \ldots, p$,

$$\begin{aligned} \langle (I - \widetilde{M}_i A)x, (I - \widetilde{M}_i A)x \rangle_A &= \langle x, x\rangle_A - 2\Re \langle x, \widetilde{M}_i A x\rangle_A + \langle \widetilde{M}_i A x, \widetilde{M}_i A x\rangle_A \\ &\le \langle x, x\rangle_A - (2\gamma - 1)\langle \widetilde{M}_i A x, \widetilde{M}_i A x\rangle_A. \end{aligned} \tag{3.22}$$

Now, let $x^{(1)} = z$ and $x^{(i+1)} = (I - \widetilde{M}_i A)x^{(i)}$, $i = 1, \ldots, p$, so that $x^{(p+1)} = Hx^{(1)} = Hz$. Using (3.22) repeatedly we obtain

$$\langle Hz, Hz\rangle_A - \langle z, z\rangle_A = -(2\gamma - 1) \sum_{i=1}^{p} \langle \widetilde{M}_i A x^{(i)}, \widetilde{M}_i A x^{(i)} \rangle_A. \tag{3.23}$$

The right hand side of (3.23) is nonpositive. It remains to show that it is zero only when $z \in \mathrm{Null}(A)$. Now, the right hand side of (3.23) is zero if and only if $\langle \widetilde{M}_i A x^{(i)}, \widetilde{M}_i A x^{(i)} \rangle_A = 0$ for $i = 1, \ldots, p$. This is equivalent to $\widetilde{M}_i A x^{(i)} \in \mathrm{Null}(A)$, i.e., $x^{(i)} \in \mathrm{Null}(A\widetilde{M}_i A)$, which, by assumption (i) implies $x^{(i)} \in \mathrm{Null}(\widetilde{M}_i A)$ for $i = 1, \ldots, p$. But then $x^{(i+1)} = (I - \widetilde{M}_i A)x^{(i)} = x^{(i)}$ for $i = 1, \ldots, p$, resulting in $x^{(i)} = z$ for $i = 1, \ldots, p$, and $z \in \mathrm{Null}(\widetilde{M}_i A)$ for $i = 1, \ldots, p$. By assumption (iii) this means $z \in \mathrm{Null}(A)$. So we have shown $\|H\|_A < 1$. The fact that $\widetilde{M}$ fulfills condition (3.4) follows directly from Theorem 3.4. ☐

We now use Theorem 3.15 for the analysis of multiplicative Schwarz methods. We use the notation introduced in Section 3.1. As in the additive case, we first consider the case where the Galerkin operators $A_i = R_i A R_i^H$ are nonsingular, i.e., we have

$$\widetilde{M}_i = R_i^H A_i^{-1} R_i, \; i = 1, \ldots, p. \tag{3.24}$$

THEOREM 3.16. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite. Moreover, assume that the projection operators R_i satisfy*

$$\cap_{i=1}^{p} \mathrm{Null}(R_i) = \{0\},$$

and that $A_i = R_i A R_i^H$ is nonsingular, $i = 1, \dots, p$. Then H from (3.21) with $\widetilde{M}_i$ from (3.24) satisfies $\|H\|_A < 1$. Furthermore, if we write $H = I - \widetilde{M}A$, then $\widetilde{M}$ satisfies (3.4).

Proof. We need to show that the hypotheses (i)-(iii) of Theorem 3.15 are fulfilled with $\gamma = 1$. For (ii) and (iii), this was already done in the proof of Theorem 3.9. To show that (i) holds, we first note that, trivially, $\mathrm{Null}(A\widetilde{M}_i A) \supseteq \mathrm{Null}(\widetilde{M}_i A)$. On the other hand, $x \in \mathrm{Null}(A\widetilde{M}_i A)$ implies $0 = \langle x, A\widetilde{M}_i Ax\rangle = \langle Ax, \widetilde{M}_i Ax\rangle$, which, since $\widetilde{M}_i$ is Hermitian positive semidefinite, yields $Ax \in \mathrm{Null}(\widetilde{M}_i)$, i.e., $x \in \mathrm{Null}(\widetilde{M}_i A)$. ◻

The next theorem considers the case where the Galerkin operators can be singular.

THEOREM 3.17. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite. Let $A_i = R_i A R_i^H$ and $\widetilde{M}_i = R_i^H A_i^\dagger R_i$ for $i = 1, \dots, p$, and let H be as in (3.21). Finally, assume that the projection operators R_i satisfy*

$$\cap_{i=1}^p \mathrm{Null}(R_i^H A_i^\dagger R_i A) = \mathrm{Null}(A).$$

Then H satisfies $\|H\|_A < 1$. Furthermore, if we write $H = I - \widetilde{M}A$, then $\widetilde{M}$ satisfies (3.4).

Proof. The proof follows again by showing that assumptions (i) to (iii) of Theorem 3.15 are fulfilled with $\gamma = 1$. But (iii) is assumed and (i) follows in exactly the same manner as in the proof of the preceding Theorem 3.16, whereas (ii) holds with $\gamma = 1$ since the $\widetilde{M}_i A$ are projections as shown in (3.17). ◻

We end this section considering multiplicative Schwarz iterations with inexact solutions of the local problems (3.18), i.e., when $\widetilde{M}_i = R_i^H \tilde{A}_i R_i$.

THEOREM 3.18. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite, and let $A_i = R_i A R_i^H$, $i = 1, \dots, p$. Let $\widetilde{M}_i = R_i^H \tilde{A}_i R_i$. Assume that each operator $\tilde{A}_i$ fulfills one of the two following conditions*

(a) $\tilde{A}_i$ is Hermitian positive semidefinite

or

(b) $\tilde{A}_i + \tilde{A}_i^H$ is positive definite on $\mathrm{Range}(A_i)$.

and that there exists a number $\gamma > \frac{1}{2}$ for which (3.19) holds. Finally, assume that the projection operators R_i satisfy

$$\cap_{i=1}^p \mathrm{Null}(R_i^H \tilde{A}_i R_i A) = \mathrm{Null}(A).$$

Then H from (3.21) satisfies $\|H\|_A < 1$. Furthermore, if we write $H = I - \widetilde{M}A$, then $\widetilde{M}$ satisfies (3.4).

Proof. We have to prove that assumptions (i) and (ii) of Theorem 3.15 hold, (iii) being part of the assumptions. Recall that (i) from Theorem 3.15 reads $\mathrm{Null}(A\widetilde{M}_i A) = \mathrm{Null}(\widetilde{M}_i A)$, where only the inclusion $\mathrm{Null}(A\widetilde{M}_i A) \subseteq \mathrm{Null}(\widetilde{M}_i A)$ is nontrivial. In the case that $\tilde{A}_i$ is Hermitian positive definite, $\widetilde{M}_i$ is Hermitian positive definite, too, and (i) of Theorem 3.15 follows as in the proof of Theorem 3.16. In the case that $\tilde{A}_i + \tilde{A}_i^H$ is positive definite on $\mathrm{Range}(A_i)$, assume that $A\widetilde{M}_i Ax = 0$. Then

$$\begin{aligned} 0 = \langle x, AR_i^H \tilde{A}_i R_i Ax\rangle \;&=\; \langle R_i Ax, \tilde{A}_i R_i Ax\rangle, \\ 0 = \langle AR_i^H \tilde{A}_i R_i Ax, x\rangle \;&=\; \langle R_i Ax, \tilde{A}_i^H R_i Ax\rangle, \end{aligned}$$

and thus $0 = \langle R_i Ax, (\tilde{A}_i + \tilde{A}_i^H) R_i Ax\rangle$. By Lemma 3.8, we have $R_i Ax \in \mathrm{Range}(A_i)$, and since $\tilde{A}_i + \tilde{A}_i^H$ is positive definite on that space we get $R_i Ax = 0$. This yields $R_i^H \tilde{A}_i R_i Ax = 0$, i.e., $x \in \mathrm{Null}(\widetilde{M}_i A)$, so that we have again shown that (i) of Theorem 3.15 holds.

Finally, since (3.19) is equivalent to (3.7), we also have (ii). ☐

We observe again that assuming $\|I - \tilde{A}_i A_i\|_{A_i} < 1$ is sufficient for (3.19) to hold. Indeed, by Theorem 3.4, this assumption is equivalent to that $\tilde{A}_i + \tilde{A}_i^H - \tilde{A}_i^H A_i \tilde{A}_i$ is positive definite on Range(A_i), which implies that $\tilde{A}_i + \tilde{A}_i^H$ is positive definite on Range(A_i) and that $\tilde{A}_i + \tilde{A}_i^H - 2\gamma \tilde{A}_i^H A_i \tilde{A}_i$ is still positive semidefinite on Range(A_i) for $\gamma > \frac{1}{2}$ sufficiently close to $\frac{1}{2}$. Hence, we have the following corollary.

Corollary 3.19. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite, and let $A_i = R_i A R_i^H$, $i = 1, \ldots, p$. Let $\widetilde{M}_i = R_i^H \tilde{A}_i R_i$. Assume that each operator $\tilde{A}_i$ satisfies*

$$\|I - \tilde{A}_i A_i\|_{A_i} < 1. \tag{3.25}$$

Finally, assume that the projection operators R_i satisfy

$$\cap_{i=1}^{p} \mathrm{Null}(R_i^H \tilde{A}_i R_i A) = \mathrm{Null}(A).$$

Then H from (3.21) satisfies $\|H\|_A < 1$. Furthermore, if we write $H = I - \widetilde{M}A$, then $\widetilde{M}$ satisfies (3.4).

Again, one can use relaxed Gauss-Seidel, i.e., $\tilde{A}_i = B_i = \frac{1}{\omega} D_i - L_i$ where $A_i = D_i - L_i - L_i^H$. We have $\tilde{A}_i + \tilde{A}_i^H = \frac{2-\omega}{\omega} D + A$, which is positive definite for $\omega \in (0, 2)$ if A_i has no zero diagonal elements. Thus, assumption (ii) of Theorem 3.18 and assumption (3.25) of Corollary 3.19 are fulfilled in this case, as well as for $\tilde{A}_i$ as in (3.20).

We note also that for $\tilde{A}_i$ nonsingular, [16, Theorem 6.4] follows from Theorem 3.18, since in [16] it is assumed that $\tilde{A}_i^{-1} + \tilde{A}_i^{-H} - A_i$ is positive definite. This assumption can be written as $\tilde{A}_i^{-H}(\tilde{A}_i^H + \tilde{A}_i - \tilde{A}_i^H A_i \tilde{A}_i)\tilde{A}_i^{-1}$ being positive definite, so that (3.19) holds for some $\gamma > 1/2$.

4. Conclusions. We presented a very general convergence result for stationary iterative methods for linear systems whose coefficient matrix A is Hermitian and positive semidefinite. It is shown that if for $x \notin \mathrm{Null}(A)$, $\langle x, Hx\rangle_A < \langle x, x\rangle_A$, with $H = I - \widetilde{M}A$, then $\widetilde{M}$ is injective on Range(A), and H is semiconvergent. This result allowed us to give simple proofs of well-known results, as well as to generalize them in several directions. We further used these new results to give convergence proofs of several variants of additive and multiplicative Schwarz iterations. These variants include those with local problems with Neumann or Robin boundary conditions, as well as the inexact solution of the local problems.

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