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Weighted max norms, splittings, and overlapping additive Schwarz iterations

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Dedicated to Prof. Olof B. Widlund on the occasion of his 60th birthday

Summary. Weighted max-norm bounds are obtained for Algebraic Additive Schwarz Iterations with overlapping blocks for the solution of $Ax = b$, when the coefficient matrix A is an M-matrix. The case of inexact local solvers is also covered. These bounds are analogous to those that exist using A-norms when the matrix A is symmetric positive definite. A new theorem concerning P-regular splittings is presented which provides a useful tool for the A-norm bounds. Furthermore, a theory of splittings is developed to represent Algebraic Additive Schwarz Iterations. This representation makes a connection with multisplitting methods. With this representation, and using a comparison theorem, it is shown that a coarse grid correction improves the convergence of Additive Schwarz Iterations when measured in weighted max norm.

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1. Introduction

Consider the solution of a linear system of n algebraic equations of the form

$$Ax = b \tag{1.1}$$

in $\mathbb{R}^n$ (A nonsingular) by Additive Schwarz Iterations with overlapping blocks. We consider two distinct – though not mutually exclusive – cases.

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Namely, when the matrix A is symmetric positive definite, denoted $A \succ O$ (this implies that $A^{-1} \succ O$), and when the matrix A is monotone, i.e., it has a nonnegative inverse, denoted $A^{-1} \geq O$, where this last inequality is understood to be componentwise. The concept of Additive Schwarz was introduced by Dryja and Widlund [13]; see also [9, 12, 14, 20, 21, (Ch. 11), 38], and the extensive bibliography therein.

To that end, let $V = \mathbb{R}^n$, and p subspaces $V_i \subset V$, $i = 1, \ldots, p$ such that

$$\sum_{i=1}^{p} V_i := \left\{ x \in V : x = \sum_{i=1}^{p} v_i, v_i \in V_i \right\} = V, \tag{1.2}$$

i.e., the combined bases of the subspaces span the whole space. If the system (1.1) corresponds to a discretization of a differential equation over a domain Ω, and the nodal values of a solution are in the space V, the subpaces V_i can be thought of as the nodal values in a subdomain $\Omega_i \subset \Omega$. In most of this paper, this interpretation is not necessary, and therefore most of our results correspond to what is known as *Algebraic Additive Schwarz*, i.e., when no underlying mesh is necessarily present; see [6,16,18,19,36,37] for different variants of Algebraic Additive Schwarz. As pointed out in [6], this algebraic approach is important for the case of unstructured mesh problems as well. Let $n_i = \dim V_i$. We concentrate our study to the case of overlapping subdomains, or in the case of Algebraic Additive Schwarz, of overlapping blocks, i.e., when $\sum_{i=1}^{p} n_i > n$.

We identify V_i with $\mathbb{R}^{n_i}$. Let $R_i : V \to \mathbb{R}^{n_i}$ be the restriction operator. In our context R_i is an $n_i \times n$ matrix. We assume that the rank of R_i is n_i. Its transpose R_i^{T} is a prolongation operator from $\mathbb{R}^{n_i}$ to V. One can use other prolongations Q_i from $\mathbb{R}^{n_i}$ to V, but it is convenient to use $Q_i = R_i^{\mathrm{T}}$, e.g., for symmetry, when $A \succ O$. Also, as we show in Sect. 5, the choice $Q_i = R_i^{\mathrm{T}}$ is also natural in the monotone case; see also [22, Remark 4.13]. Let $A_i = R_i A R_i^{\mathrm{T}}$ (the restriction of A to V_i). In the cases considered here we will have A_i nonsingular. If the bases of V_i, and thus that of V are chosen appropriately, then the application of the prolongation operator R_i^{T} consists of taking the entries of an n_i-dimensional vector and position them in the locations of an n-dimensional vector, according to the position of the basis of V_i in the basis of V, i.e., each column of R_i^{T} is a different column of the identity matrix, e.g., as in the following 3×8 example

$$R_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Formally, such a matrix R_i can be expressed as

(1.3) $$R_i = [I_i|O]\,\pi_i$$

with I_i the identity on $\mathbb{R}^{n_i}$ and π_i a permutation matrix on $\mathbb{R}^n$. In this case, it follows that A_i is an $n_i \times n_i$ principal minor of A.

Given an initial approximation x^0 to the solution of (1.1), the damped additive Schwarz iteration reads, for $k = 0, \ldots,$

(1.4) $$x^{k+1} = x^k + \theta \sum_{i=1}^{p} R_i^{\mathrm{T}} A_i^{-1} R_i (b - Ax^k),$$

where $\theta > 0$ is the damping factor. In other words, the iteration consists of the following: on each subspace, restrict the current residual, solve the local problem, prolongate the approximation of the error, and add the correction; cf. [45]. It follows that the iteration matrix for the process (1.4) is

(1.5) $$T_\theta = I - \theta \sum_{i=1}^{p} R_i^{\mathrm{T}} A_i^{-1} R_i A,$$

and that the error $e^{k+1} = x^{k+1} - x$ satisfies $e^{k+1} = T_\theta e^k$.

Often in practice, instead of solving the local problems $A_i y_i = R_i(b - Ax^k)$ exactly, such linear systems are approximated; see, e.g. [4, 9, 21, (Ch. 11), 38]. Let $\tilde{A}_i$ denote the approximation to A_i used, i.e., the inexact, or approximate, local solver is $\tilde{A}_i^{-1}$. By replacing A_i with $\tilde{A}_i$ in (1.4) and (1.5) one obtains the damped additive Schwarz iteration with inexact local solvers, and its iteration matrix is then

(1.6) $$\tilde{T}_\theta = I - \theta \sum_{i=1}^{p} R_i^{\mathrm{T}} \tilde{A}_i^{-1} R_i A.$$

Additive Schwarz is often used as a preconditioner for a Krylov subspace iterative method, such as conjugate gradients; see, e.g., [38]. In this case, what is important is to have bounds on the condition number of $M^{-1}A = I - T$, where T is the representation of the additive Schwarz iteration matrix, e.g., (1.5) or (1.6). We will provide some bounds in this direction, though we will concentrate mostly on bounds on the norm of T. Our theory complements the heuristic study of algebraic additive Schwarz of [6].

The Schwarz Alternating Method, of which the method (1.4) is a variant, is used in several cases, especially when the matrix A is nonsymmetric; see, e.g., [8, 29], and the references given therein. In these cases, the convergence is studied using max norms and the analytic tool is some maximum principle; see also [26, 27]. As described below, we provide weighted max norms convergence bounds in a general algebraic setting. Thus, our result apply to discretized nonsymmetric equations rather than the continuous case; cf. [29].

In the special case of nonoverlap, i.e., when $\sum_{i=1}^{p} n_i = n$, and when

$$R_i = [O \cdots I \cdots O], \tag{1.7}$$

the iteration (1.4) is the standard block Jacobi iteration [3,42]. As is well known, there might not be convergence for $\theta = 1$. Hackbusch [21, Ch. 11.2.4] explicitly showed that, in this special case and when $A \succ O$, the iteration (1.4) converges, i.e., the spectral radius $\rho(T_\theta) < 1$ provided that $\theta < 1/p$. In Sect. 2 we show under the same condition on θ that

$$\|T_\theta\|_A \leq \gamma < 1, \tag{1.8}$$

for the general overlapping case.

If A corresponds to a discretization of a partial differential equation on $\Omega = \cup_{i=1}^{p} \Omega_i$, the subdomains Ω_i can usually be colored using $q \ll p$ colors in such a way that if $\Omega_i \cap \Omega_j \neq \emptyset$ then Ω_i and Ω_j have different colors. Similarly, we color the subspaces V_i so that V_i and V_j have a different color whenever $V_i \cap V_j \neq \{0\}$. The bound (1.8) is extended to hold under the less restrictive condition $\theta < 1/q$, as well as for the iteration matrix with inexact solvers (1.6). As we shall see, the results in Sect. 2 are not entirely new. We present them here to set the stage for similar results using weighted max norms shown in Sect. 3, to show clearly the correspondence between the symmetric positive definite and monotone cases, and to present some new tools used in our proofs.

Given a positive vector $w \in \mathbb{R}^n$, denoted $w > 0$, the weighted max-norm is defined for any $y \in \mathbb{R}^n$ as $\|y\|_w = \max_{j=1,\cdots,n} |\frac{1}{w_j} y_j|$; see, e.g., [24, 35]. Weighted max norms have played a fundamental role in the study of asynchronous methods (see [19,39]), and are obvious generalizations of the usual max norm. Most of our estimates hold for all positive vectors w of the form $w = A^{-1}e$, where e is any positive vector, i.e., for any positive vector w such that Aw is positive. In particular this would hold for $w = A^{-1}e$ and $e = (1, \ldots, 1)^{\mathrm{T}}$, i.e., with w being the row sums of A^{-1}. Recall that we are assuming $A^{-1} \geq O$, and that since A^{-1} is nonsingular no row of it can be a zero row. This guarantees that $w = A^{-1}e > O$. The same logic is used to conclude that $M^{-1}e > O$ for any monotone matrix M, and this is also used in our proofs.

In Sect. 3 we also develop an interpretation of additive Schwarz iterations similar to multisplitting methods; see [5,33]. This interpretation is important for our estimates, but it is also interesting in its own right.

In Sect. 4 we present a comparison theorem between two splittings, using the weighted max norm, and use this theorem in Sect. 5 to show that the addition of a coarse grid correction can improve the convergence of the additive Schwarz iteration. This analysis of the coarse grid correction applies also to the inexact version (1.6), and to the asynchronous version presented in [19].

2. A-norm bounds

In this section we assume that $A \succ O$, i.e., that $A^{\mathrm{T}} = A$ and that $x^{\mathrm{T}}Ax > 0$ for $x \neq 0$, although some results presented here, such as Lemma 2.6, do not depend on this assumption. It follows directly that each A_i is symmetric positive definite in $\mathbb{R}^{n_i}$, and thus nonsingular, $i = 1, \ldots, p$.

We first develop a new tool, which we use in our proofs. To that end, we first review a result from [34]. By $A \succeq O$ we denote a symmetric positive semidefinite matrix. The notation $A \succ B$ stands for $A - B \succ O$, while $A \prec B$ stands for $B - A \succ O$, and similarly with the symbols $\succeq$ and $\preceq$ when the difference is semidefinite.

Definition 2.1 [34] *A splitting $A = M - N$ is called* P-regular *if $M^{\mathrm{T}} + N$ is positive definite.*

Note that this is equivalent to requiring that the symmetric matrix $M^{\mathrm{T}} + M - A$ be positive definite.

Lemma 2.2 [34] *Let $A \succ O$, let $A = M - N$ be P-regular and let $H = M^{-1}N$. Then $A \succ H^{\mathrm{T}}AH$.*

Lemma 2.3 *Let $A \succ O$ and $H \in \mathbb{R}^{n\times n}$. Then $A \succ H^{\mathrm{T}}AH$ if and only if $\|H\|_A < 1$.*

Proof. Assume $A \succ H^{\mathrm{T}}AH$ and let $u \in \mathbb{R}^n$, $u \neq 0$. Then

$$
\begin{aligned}
\|Hu\|_A^2 &= u^{\mathrm{T}}H^{\mathrm{T}}AHu \\
&= u^{\mathrm{T}}(H^{\mathrm{T}}AH - A)u + u^{\mathrm{T}}Au \\
&< u^{\mathrm{T}}Au = \|u\|_A^2.
\end{aligned}
\tag{2.1}
$$

This shows $\|H\|_A < 1$. For the other part, $\|H\|_A < 1$ implies that $\|u\|_A > \|Hu\|_A$ for all $u \neq 0$, i.e., $u^{\mathrm{T}}Au > u^{\mathrm{T}}H^{\mathrm{T}}AHu$, which means that $A \succ H^{\mathrm{T}}AH$. $\square$

Lemma 2.3 can be regarded as a version of Stein's Theorem [34] relying on A-norms.

Theorem 2.4 *Let A be symmetric positive definite. Then $A = M - N$ is a P-regular splitting if and only if $\|M^{-1}N\|_A < 1$.*

Proof. Let $H = M^{-1}N$ and assume that the splitting is P-regular. Then, by Lemma 2.2, $A \succ H^{\mathrm{T}}AH$, and by Lemma 2.3, $\|H\|_A < 1$.
For the proof in the other direction, by Lemma 2.3, $A \succ H^{\mathrm{T}}AH$. Then $M^{\mathrm{T}} + N$ is positive definite from the following identity

$$\begin{aligned} A - H^{\mathrm{T}}AH &= A - (M^{-1}N)^{\mathrm{T}}A(M^{-1}N) \\ &= A - (I - M^{-1}A)^{\mathrm{T}}A(I - M^{-1}A) \\ &= (M^{-1}A)^{\mathrm{T}}(M + M^{\mathrm{T}} - A)M^{-1}A \\ &= (M^{-1}A)^{\mathrm{T}}(M^{\mathrm{T}} + N)M^{-1}A. \qquad \square \end{aligned}$$

We remark that results in the spirit of this theorem can be found in [7, 31]. We will use Theorem 2.4, together with several results found in [21] to show the bound (1.8).

Lemma 2.5 [21] *Let $A \succ O$, then the matrix $B = \sum_{i=1}^{p} R_i^{\mathrm{T}} A_i^{-1} R_i$ is nonsingular.*

Lemma 2.6 [21] *Given T_θ from (1.5), there exist a pair of matrices M_θ and N_θ such that $A = M_\theta - N_\theta$, M_θ is nonsingular, and $T_\theta = M_\theta^{-1}N_\theta$. Furthermore*

$$M_\theta^{-1} = \theta B = \theta \sum_{i=1}^{p} R_i^{\mathrm{T}} A_i^{-1} R_i. \tag{2.2}$$

We point out that this splitting is unique; see, e.g., [2]. If $\theta = 1$, i.e., in the absence of damping, we simply denote the matrix M_θ as M, i.e.,

$$M^{-1} = \sum_{i=1}^{p} R_i^{\mathrm{T}} A_i^{-1} R_i. \tag{2.3}$$

It can be seen then that the matrices B, M, and M_θ are all symmetric positive definite.

Lemma 2.7 [21] *Let $A \succ O$, then the inequality $A \preceq pM$ holds.*

We mention that the proof of this lemma uses the fact that $P_i = R_i^{\mathrm{T}} A_i^{-1} R_i A$ is an orthogonal projection (with respect to the A-inner product) from V to V_i. The inequality from Lemma 2.7 can be improved to

$$A \preceq qM; \tag{2.4}$$

i.e., from the number of subspaces (or subdomains) p, to the number of colors q [21]. We note that (2.4) is equivalent to $M^{-1/2}AM^{-1/2} \preceq qI$, and this implies that $\|M^{-1/2}AM^{-1/2}\|_2 = \max\{\lambda,\ \lambda \in \sigma(M^{-1}A)\} \leq q$, where $\sigma(M^{-1}A)$ is the spectrum of $M^{-1}A$, which is of course useful to bound the condition number $\kappa(M^{-1}A)$.

Lemma 2.8 *Let $A \succ O$, and let $A = M_\theta - N_\theta$ be the splitting defined in Lemma 2.6. If $\theta < 1/q$, then this splitting is a P-regular splitting, and therefore $\|T_\theta\|_A < 1$.*

Proof. As pointed out earlier, $M_\theta \succ O$. Using (2.4) we also have that $N_\theta = M_\theta - A = \frac{1}{\theta}M - A$ is symmetric positive definite if $\theta < 1/q$. Thus, $M_\theta + N_\theta \succ O$, and the splitting is P-regular. Thus, by Theorem 2.4 the proof is complete. □

This last bound also follows from the fact that since P_i is an A-orthogonal projection $\|P_i\|_A = 1$ (see, e.g., [9]), and can also be obtained as a corollary to results in Frommer and Renaut [17]. This last bound implies the following bound on the condition number (see, e.g., [45])

$$\kappa\left(M_\theta^{-1}A\right) \leq \frac{1 + \|T_\theta\|_A}{1 - \|T_\theta\|_A} \ .$$

We turn now our attention to the case of inexact local solvers. With an appropriate assumption on the local solver on each subspace, we can follow the same arguments as in the exact case. If each of the inexact solvers is symmetric positive definite, then one can prove in a way similar to Lemma 2.5 that $\tilde{B} = \sum_{i=1}^{p} R_i^{\mathrm{T}} \tilde{A}_i^{-1} R_i$ is nonsingular.

Lemma 2.9 [21] *Let $A \succ O$. If for some $\mu > 0$, $A_i \preceq \mu\tilde{A}_i$, for $i = 1, \ldots, p$, then $A \preceq \mu q\tilde{M}$, where $\tilde{M}^{-1} = \sum_{i=1}^{p} R_i^{\mathrm{T}} \tilde{A}_i^{-1} R_i$.*

From this lemma it follows that the unique splitting induced by $\tilde{T}_\theta$ of (1.6), $A = \tilde{M}_\theta - \tilde{N}_\theta$ (with $\tilde{M}_\theta^{-1} = \theta\tilde{B}$) is P-regular as long as $\theta < 1/(\mu q)$, and thus under this condition $\|\tilde{T}_\theta\|_A < 1$.

3. Weighted max norm bounds

In this section, and in the rest of the paper, we assume that A is monotone, i.e., $A^{-1} \geq O$. Thus, we do not assume that A is necessarily symmetric or positive definite. A special case of monotone matrices are nonsingular M-matrices, i.e., nonsingular matrices with nonpositive off-diagonals and nonnegative inverses; see, e.g., [3,42].

In the theory of nonnegative matrices, $A = M - N$ is called a weak regular splitting if $M^{-1} \geq O$ and $M^{-1}N \geq O$ [3]. As before, these inequalities are componentwise. In multisplitting theory, several splittings $A = M_i - N_i$, $i = 1, \ldots, p$ are used, and the resulting iterates are added up using diagonal nonnegative weighting matrices E_i such that

$$\sum_{i=1}^{p} E_i = I; \tag{3.1}$$

see, e.g., [5,33], and the references therein. In this paper, unlike the Schwarz descriptions in [18] and [19], the condition (3.1) is not imposed, instead we will only assume that $\sum_{i=1}^{p} E_i$ is non singular.

The following lemma is used in Theorem 3.2 below. The notation $|A|$, and $v \leq w$, for matrices and vectors, respectively, is again componentwise. Then, we represent additive Schwarz using weak regular splittings and apply Theorem 3.2 to this representation.

Lemma 3.1 *Let $A \in \mathbb{R}^{n\times n}$, $w \in \mathbb{R}^n$, $w > 0$, and $\gamma > 0$ such that*

$$|A| w \leq \gamma w \,. \tag{3.2}$$

Then, $||A||_w \leq \gamma$. In particular, $||Ax||_w \leq \gamma ||x||_w$ for all $x \in \mathbb{R}^n$. If the inequality in (3.2) *is strict, then the bound on the norm is also strict.*

Proof. The lemma with the hypothesis (3.2) can be found in [19]. The extension to the strict inequality follows in a similar manner. □

Theorem 3.2 *Let $A^{-1} \geq O$. Let $A = M_i - N_i$ be weak regular splittings, $i = 1, \ldots, p$. Let $E_i \geq O$ be diagonal matrices ($i = 1, \ldots, p$) such that $\sum_{i=1}^{p} E_i$ is nonsingular. Then,* (a) *$P = \sum_{i=1}^{p} E_i - \sum_{i=1}^{p} E_i M_i^{-1} A \geq O$, and there exists a positive vector $w \in \mathbb{R}^n$, such that*

$$\left\| \left(\sum_{i=1}^{p} E_i \right)^{-1} P \right\|_w < 1 \,,$$

which implies that

$$\rho \left(\left(\sum_{i=1}^{p} E_i \right)^{-1} P \right) < 1 \,,$$

and

(b) *$\sum_{i=1}^{p} E_i M_i^{-1}$ is nonsingular.*

Proof. (a) $P = \sum_{i=1}^{p} E_i \left(I - M_i^{-1}A\right) = \sum_{i=1}^{p} E_i M_i^{-1} N_i \geq O$. Let $e = (1,\ldots,1)^{\mathrm{T}} \in \mathbb{R}^n$ (or any other positive vector), and $w = A^{-1}e > 0$. Then, since $M_i^{-1}e > 0$, $i = 1,\ldots,p$, we have

$$Pw = \sum_{i=1}^{p} E_i w - \sum_{i=1}^{p} E_i M_i^{-1} e < \sum_{i=1}^{p} E_i w.$$

This implies, using the nonnegativity of the diagonal weighting matrices, that

$$\left(\sum_{i=1}^{p} E_i\right)^{-1} Pw < w,$$

and by Lemma 3.1 part (a) follows.
(b) By part (a),

$$\rho\left(I - \left(\sum_{i=1}^{p} E_i\right)^{-1} \sum_{i=1}^{p} E_i M_i^{-1} A\right) < 1.$$

This implies that

$$\left(\sum_{i=1}^{p} E_i\right)^{-1} \left(\sum_{i=1}^{p} E_i M_i^{-1} A\right)$$

is nonsingular. This in turn implies that the second factor, which we can rewrite as $\left(\sum_{i=1}^{p} E_i M_i^{-1}\right) A$, is nonsingular, and since A is nonsingular, the proof is complete. □

For our representation of additive Schwarz using weak regular splittings, we assume that the matrix A is a nonsingular M-matrix, and thus all its principal minors are also nonsingular M-matrices [3]. We further assume that R_i is of the form (1.3). Then

$$A_i = [I_i|O] \cdot \pi_i \cdot A \cdot \pi_i^{\mathrm{T}} \cdot [I_i|O]^{\mathrm{T}}$$

and

$$R_i^{\mathrm{T}} A_i^{-1} R_i = \pi_i^{\mathrm{T}} \begin{bmatrix} A_i^{-1} & O \\ O & O \end{bmatrix} \pi_i.$$

For each $i = 1,\ldots,p$, we construct matrices $M_i, E_i \in \mathbb{R}^{n\times n}$ associated with R_i from (1.3) as follows

$$E_i = R_i^{\mathrm{T}} R_i, \quad M_i = \pi_i^{\mathrm{T}} \begin{bmatrix} A_i & O \\ O & A_{\neg i} \end{bmatrix} \pi_i, \tag{3.3}$$

where $A_{\neg i}$ is the principal minor of A 'complementary' to A_i, i.e.

$$A_{\neg i} = [O|I_{\neg i}] \cdot \pi_i \cdot A \cdot \pi_i^{\mathrm{T}} \cdot [O|I_{\neg i}]^{\mathrm{T}}$$

with $I_{\neg i}$ the identity on $\mathbb{R}^{n-n_i}$.

Note that

$$M_i^{-1} = \pi_i^{\mathrm{T}} \begin{bmatrix} A_i^{-1} & O \\ O & A_{\neg i}^{-1} \end{bmatrix} \pi_i$$

is nonnegative since the principal minors A_i and $A_{\neg i}$ of A are M-matrices [3]. Moreover $N_i = M_i - A$ is nonnegative, since it is a symmetric permutation of a matrix with a 2 by 2 block structure, the off-diagonal blocks being nonnegative and the diagonal blocks being zero. So the splittings $A = M_i - N_i$ are (weak) regular; see also [18] for a general example of splittings satisfying our conditions.

The diagonal matrices $E_i = R_i^{\mathrm{T}} R_i$ from (3.3) have ones on the diagonal in every row where R_i^{T} has nonzeros. Note that in the case of overlapping blocks, we have here that each diagonal entry of $\sum_{i=1}^{p} E_i$ is greater than or equal to one, which implies nonsingularity; cf. [1]. Only in the rows corresponding to overlap this matrix has an entry different from one. In that case, the maximum that these entries can attain is q, the number of colors. We thus have that

$$\sum_{i=1}^{p} E_i \leq qI. \tag{3.4}$$

Moreover, for E_i and M_i from (3.3) we have that

$$E_i M_i^{-1} = R_i^{\mathrm{T}} A_i^{-1} R_i, \; i = 1, \ldots, p. \tag{3.5}$$

We note that the representation of additive Schwarz presented here is different than that found in [25, 40], where the matrices are augmented to account for the overlap.

Lemma 3.3 *Let A be a nonsingular M-matrix. Let the matrices R_i be of the form* (1.3)*. Then, $B = \sum_{i=1}^{p} R_i^{\mathrm{T}} A_i^{-1} R_i$ is nonsingular. Moreover, $A_i^{-1} \leq R_i A^{-1} R_i^{\mathrm{T}}$ for $i = 1, \ldots, p$.*

Proof. The matrix $\sum_{i=1}^{p} E_i = \sum_{i=1}^{p} R_i^{\mathrm{T}} R_i$ is of full rank because of (1.2). The first part of the lemma then follows from (3.5), and Theorem 3.2 (b). To prove the inequality for A_i^{-1}, let us write

$$\pi_i^{\mathrm{T}} A \pi_i = \begin{bmatrix} A_i & F \\ G & A_{\neg i} \end{bmatrix} \leq \begin{bmatrix} A_i & O \\ O & A_{\neg i} \end{bmatrix} =: \hat{A}. \tag{3.6}$$

Since $\pi_i^{\mathrm{T}} A \pi_i$ is an M-matrix and $\hat{A}$ has the sign structure of an M-matrix, $\hat{A}$ itself is an M-matrix as well [3], and upon multiplication with the nonnegative matrices $(\pi_i^{\mathrm{T}} A \pi_i)^{-1}$ and $\hat{A}^{-1}$ in the above inequality we arrive at $\hat{A}^{-1} \leq (\pi_i^{\mathrm{T}} A \pi_i)^{-1} = \pi_i^{\mathrm{T}} A^{-1} \pi_i$. Multiplying with $[I_i|O]$ and $[I_i|O]^{\mathrm{T}}$ we finally get $A_i^{-1} \leq R_i A^{-1} R_i^{\mathrm{T}}$. □

The same argument used in the proof of the second part of Lemma 3.3 can be used to show that $A \leq M_i$; compare (3.6) and (3.3). Since both A, and M_i are monotone, we also obtain $M_i^{-1} \leq A^{-1}$ and $M_i^{-1} A \leq I$, which in turn implies $M^{-1} A = \displaystyle\sum_{i=1}^{p} M_i^{-1} A \leq pI$, or $M^{-1} \leq pA^{-1}$; cf. Lemma 2.7.

Theorem 3.4 *Let A be a nonsingular M-matrix. Let the matrices R_i be of the form* (1.3)*. Then, if $\theta \leq 1/q$, the damped additive Schwarz iteration* (1.4) *converges to the solution of* (1.1), *and there is a positive vector w and $0 < \gamma < 1$ for which $\|T_\theta\|_w \leq \gamma$.*

Proof. Using (1.5), (3.4) , and Theorem 3.2 (a), we have

(3.7)
$$T_\theta = I - \theta BA \geq \frac{1}{q} \sum_{i=1}^{p} E_i - \theta BA \geq \theta P \geq O.$$

As in the proof of Theorem 3.2, let $e = (1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$ (or any other positive vector), and $w = A^{-1} e > 0$. Since by Lemma 3.3, B is nonsingular, and it is nonnegative, then $BAw = Be > 0$. Thus,

(3.8)
$$T_\theta w = w - \theta BAw < w$$

and the theorem follows from Lemma 3.1. □

We note that convergence, i.e., $\rho(T_\theta) < 1$, also follows from the fact that by (3.5), $M_\theta^{-1} = \theta \displaystyle\sum_{i=1}^{p} E_i M_i^{-1} \geq O$ and together with (3.7) this implies that the induced splitting [2] is weak regular, and therefore convergent [3].

The bound in Theorem 3.4 can be improved and quantified. To that end, note that if $e = (1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$, w is the vector of row sums of the nonnegative nonsingular matrix A^{-1}. Due to the second part of Lemma 3.3 we have $R_i^{\mathrm{T}} A_i^{-1} R_i e \leq R_i^{\mathrm{T}} (R_i A^{-1} R_i^{\mathrm{T}}) R_i e$. Here, the vector to the right represents the vector of row sums of a principal minor of the nonnegative matrix A^{-1}, so its entries cannot exceed the corresponding row sums of A^{-1}. When these vectors are added, there are at most q (number of colors) contributions to any row sum. This yields $Be \leq qw$, which instead of (3.8) leads to the inequality $T_\theta w \geq (1-\theta q)w$. Since $\|T_\theta\|_w = \|T_\theta w\|_w$ [35], this implies $\|T_\theta\|_w \geq (1-\theta q)$ for $\theta q < 1$.

At the same time, the sum of the terms $R_i^{\mathrm{T}} A_i^{-1} R_i e$ is a positive vector, so that it is never smaller than a fixed portion of the row sums of A^{-1}. This implies that there exists $\delta > 0$ such that $Be > \delta w$. Using this inequality in (3.8) we obtain the improved bound $\|T_\theta\|_w < 1 - \theta\delta$.

Theorem 3.4 can be extended to the case of inexact local solvers, i.e., to the iteration matrix (1.6). All that is necessary to that end is to replace A_i with $\tilde{A}_i$ in (3.3), i.e., replace in each matrix M_i the entries of A_i with those of $\tilde{A}_i$. Call these new matrices $\tilde{M}_i$.

Theorem 3.5 *Let A be a nonsingular M-matrix. Let the matrices R_i be of the form (1.3). Assume that $\tilde{A}_i$ are monotone matrices, $i = 1, \ldots, p$, and that*

$$\tilde{A}_i^{-1}\left(\tilde{A}_i - A_i\right) \geq O, \; i = 1, \ldots, p. \tag{3.9}$$

Then, if $\theta \leq 1/q$, the damped additive Schwarz iteration with inexact local solvers converges to the solution of (1.1), and there is a positive vector w and $0 < \gamma < 1$ for which $\|T_\theta\|_w \leq \gamma$.

Proof. The definition of $\tilde{M}_i$ implies that $E_i \tilde{M}_i^{-1} = R_i^{\mathrm{T}} \tilde{A}_i^{-1} R_i$, $i = 1, \ldots, p$, cf. (3.5). Let $\tilde{N}_i = \tilde{M}_i - A$, $i = 1, \ldots, p$. From the hypothesis on $\tilde{A}_i$ it follows that $A = \tilde{M}_i - \tilde{N}_i$ are weak regular splittings, and thus the hypotheses of Theorem 3.2 hold. The proofs follows now using the same arguments as in Lemma 3.3 and Theorem 3.4. □

We point out that the inequalities (3.9) hold, for example, if

$$\tilde{A}_i \geq A_i; \tag{3.10}$$

cf. Lemma 2.9 and the comment following it. This condition is easily satisfied. This is the case, for example, if $\tilde{A}_i$ has a subset of the nonzeros of A_i (including the diagonal). This last case includes many standard splittings such as the diagonal, tridiagonal, or triangular part, as well as block versions of them. The other notable example is incomplete factorizations $\tilde{A}_i = L_i U_i$ where the nonzeros of the factors are in the locations of the nonzeros of A_i, and in particular ILU(0) [30]. In these cases, the inequality (3.10) holds, or equivalently, we have (weak) regular splittings [30,41].

4. Comparison theorem

This section can be read independently of the rest of the paper.

Theorem 4.1 *Let $A^{-1} \geq O$. Let $A = \bar{M} - \bar{N} = M - N$ be two weak regular splittings such that*

$$\bar{M}^{-1} \geq M^{-1}. \tag{4.1}$$

Let $w > 0$ be such that $w = A^{-1}e$ for some $e > 0$. Then,

$$\|\bar{M}^{-1}\bar{N}\|_w \leq \|M^{-1}N\|_w. \tag{4.2}$$

If the inequality in (4.1) *is strict, then, the inequality in* (4.2) *is also strict.*

Proof. For any nonnegative matrix T, $\|T\|_w = \|Tw\|_w$ [35]. The theorem is then obtained from the following inequality.

$$\begin{aligned} \bar{M}^{-1}\bar{N}w &= w - \bar{M}^{-1}Aw = w - \bar{M}^{-1}e \leq w - M^{-1}e \\ &= M^{-1}Nw. \end{aligned} \tag{4.3}$$

The case of strict inequality is obtained using strict inequality in (4.3). □

It follows directly from this theorem that $\rho(\bar{M}^{-1}\bar{N}) \leq \|M^{-1}N\|_w$, with the inequality being strict if the hypothesis of strict inequality holds in (4.1). Recall that the Perron eigenvector of a nonnegative matrix is the nonnegative vector corresponding to its spectral radius.

Corollary 4.2 *Let the hypotheses of Theorem 4.1 hold. Assume further that w is the Perron eigenvector of $M^{-1}N$. Then $\rho(\bar{M}^{-1}\bar{N}) \leq \rho(M^{-1}N)$, with the inequality being strict if the hypothesis of strict inequality holds in* (4.1).

Proof. It follows by equating the right hand side in (4.3) with $\rho(M^{-1}N)w$ and from the fact that $\|w\|_w = 1$. □

Theorem 4.1 and Corollary 4.2 are very similar to [32, Lemma 2.2], and are analogous to comparison theorems for spectral radii due to Woźnicki [43,44]; see also [10,11,28]. We end the section with another result in the spirit of [28, Theorem 3.11], for which we need the following result from [28].

Lemma 4.3 *Let $T \geq O$, and let $w \geq 0$ be such that $\alpha w \leq Tw$. Then $\alpha \leq \rho(T)$, and if $\alpha w < Tw$, then $\alpha < \rho(T)$.*

Theorem 4.4 *Let $A^{-1} \geq O$. Let $A = \bar{M} - \bar{N} = M - N$ be two weak regular splittings such that* (4.1) *holds. Let w be the Perron eigenvector of $\bar{M}^{-1}\bar{N}$, assume that $w > 0$ and that $\bar{N}w > 0$. Then $\rho(\bar{M}^{-1}\bar{N}) \leq \rho(M^{-1}N)$, with the inequality being strict if the hypothesis of strict inequality holds in* (4.1).

Proof. We have that $\bar{N}w = \rho(\bar{M}^{-1}\bar{N})\bar{M}w > 0$. Since $A = \bar{M} - \bar{N}$ is a weak regular splitting and $A^{-1} \geq 0$, $\rho(\bar{M}^{-1}\bar{N}) < 1$ [3], and thus $Aw = (\bar{M} - \bar{N})w = (1 - \rho(\bar{M}^{-1}\bar{N})\bar{M}w > 0$. Therefore w satisfies the hypothesis of Theorem 4.1. We rewrite (4.3) as $\rho(\bar{M}^{-1}\bar{N})w \leq M^{-1}Nw$, and use Lemma 4.3 to complete the proof. □

We remark that Theorems 4.1 and 4.4, and Corollary 4.2 are also valid for splittings called nonnegative of the second kind by some authors, namely $A = M - N$, $M^{-1} \geq O$, $NM^{-1} \geq O$ [10,44]. The proofs are analogous.

5. Coarse grid correction

In order to accelerate the convergence of Additive Schwarz iterations, a 'global' solver is usually added to communicate information from each subspace to all others. This global solver often corresponds to a solver on a coarse grid when the mesh of the discretized underlying differential equation allows this; see, e.g., [15,38]. In the pure algebraic case, one can choose an additional subspace V_0, taking selected variables from each of the other subspaces, and obtaining $A_0 = R_0AR_0^{\mathrm{T}}$, which is a principal minor of A; see, e.g., the formulation in [6, Sect. 4]. In this case, we can have a representation of the form (3.5). Other choices are possible, and it is not always clear how to choose the global solver in the most effective way [6]. When actual grids are present, R_0 is the restriction from the fine grid to the coarse grid, and the solver on the coarse grid A_0 is usually not a principal minor of A; see an example further below. In either case the coarse grid correction is added to the other corrections, and the iterative method becomes, for $k = 0, \ldots,$

$$\begin{aligned} x^{k+1} &= x^k + \theta \sum_{i=0}^{p} R_i^{\mathrm{T}} A_i^{-1} R_i \left(b - Ax^k\right) \\ &= x^k + \theta \left(R_0^{\mathrm{T}} A_0^{-1} R_0 + \sum_{i=1}^{p} R_i^{\mathrm{T}} A_i^{-1} R_i\right) \left(b - Ax^k\right). \qquad (5.1) \end{aligned}$$

It is evident from (5.1) that the choice of the global solver should be such that the matrix in parenthesis in this iteration be nonsingular. We show in this section that if A_0 is a principal minor of A, the iteration with the global solver converges at least as fast as the standard damped additive Schwarz, when measured in some weighted max norm.

Let

$$\bar{T}_\theta = I - \theta \left(R_0^{\mathrm{T}} A_0^{-1} R_0 + \sum_{i=1}^{p} R_i^{\mathrm{T}} A_i^{-1} R_i\right) A \qquad (5.2)$$

be the iteration matrix of the iteration (5.1), with the corresponding induced splitting $A = \bar{M}_\theta - \bar{N}_\theta$. Since $R_0^{\mathrm{T}} A_0^{-1} R_0 \geq O$, it follows from (2.2) that

$$\bar{M}_\theta^{-1} = \theta R_0^{\mathrm{T}} A_0^{-1} R_0 + M_\theta^{-1} \geq M_\theta^{-1}. \tag{5.3}$$

Theorem 5.1 *Let A be a nonsingular M-matrix. Let the matrices R_i be of the form* (1.3), *for $i = 1, \ldots, p$. Let the global solver be such that $A_0^{-1} \geq O$, and the corresponding restriction operator be such that $R_0 \geq O$. Let $\bar{M}_\theta$ as defined in* (5.3) *be nonsingular. Assume further that there is a weak regular splitting $A = M_0 - N_0$, and a diagonal matrix E_0 such that*

$$0 \leq E_0 \leq I, \tag{5.4}$$

and such that

$$R_0^{\mathrm{T}} A_0^{-1} R_0 = E_0 M_0^{-1}. \tag{5.5}$$

Then, if $\theta \leq 1/(q+1)$, the damped additive Schwarz iteration (5.1) *converges to the solution of* (1.1), *and there is a positive vector w for which $\|\bar{T}_\theta\|_w < 1$. Furthermore, $\|\bar{T}_\theta\|_w \leq \|T_\theta\|_w$.*

Proof. By the hypotheses we have that (5.3) holds. To complete the proof all we need to do is satisfy the remaining hypotheses of Theorem 4.1, namely, that $A = \bar{M}_\theta - \bar{N}_\theta$ is a weak regular splitting. From (5.3) it follows that $\bar{M}_\theta^{-1} \geq O$. From (5.4) and (3.4) it follows that $\sum_{i=0}^{p} E_i \leq (q+1)I$ and this matrix is nonsingular. Using Theorem 3.2 (a) (now with $p+1$ splittings), (5.5) and (3.5), we have that

$$\begin{aligned}
\bar{M}_\theta^{-1}\bar{N}_\theta = \bar{T}_\theta &= -\theta R_0^{\mathrm{T}} A_0^{-1} R_0 A + \theta B A \\
&= I - \theta E_0 M_0^{-1} A - \theta \sum_{i=1}^{p} E_i M_i^{-1} A \\
&\geq \frac{1}{q+1} \sum_{i=0}^{p} E_i - \theta \sum_{i=0}^{p} E_i M_i^{-1} A \\
&\geq \theta \left(\sum_{i=0}^{p} E_i - \sum_{i=0}^{p} E_i M_i^{-1} A \right) \geq 0
\end{aligned}$$

and the proof is complete. □

We comment now on the hypotheses of Theorem 5.1. If R_0 is of the form (1.3), i.e., if the A_0 is a principal minor of the monotone matrix A, then one can build M_0 as in (3.3), and the hypotheses of the theorem hold with $E_0 = R_0^{\mathrm{T}} R_0$. As we show in detail at the end of the section, this case is essentially the only 'matrix independent' one for which the hypotheses of the theorem holds. In the usual coarse grid correction, where A_0 is the discretization in a coarse grid of the differential equation, then A_0 is *not* a principal minor of A (the discretization on a fine grid of the differential equation). As an example, consider the discretization (with mesh h) of a one-dimensional problem in [38, Sect. 2.7]. Simple algebraic manipulations show that A_0 (with a grid of size $H > h$) is not a principal minor of A. In this case, $R_0^{\mathrm{T}} R_0$ is not a diagonal matrix, and we cannot produce a representation for which (5.5) holds. To be specific, let us consider the case $n = 5$, i.e.

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad R_0 = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix},$$

so that

$$A_0 = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}, \quad R_0^{\mathrm{T}} R_0 = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 0 \\ 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix},$$

$$R_0^{\mathrm{T}} A_0^{-1} R_0 = \frac{1}{6} \begin{bmatrix} 2 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 6 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 2 \end{bmatrix}, \quad R_0^{\mathrm{T}} A_0^{-1} R A = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Note that in this example $R_0^{\mathrm{T}} A_0^{-1} R_0$ is still nonnegative, but

$$I - R_0^{\mathrm{T}} A_0^{-1} R_0 A \tag{5.6}$$

is not. Nevertheless, the conclusion of Theorem 5.1 still holds if the new iteration matrix (5.2) is nonnegative, i.e., if the negative (off-diagonal) entries in (5.6) are smaller in absolute value than the corresponding entries in $M^{-1}A$. In this case, the induced splitting $A = \bar{M}_\theta - \bar{N}_\theta$ is a weak regular splitting, and this is all we need.

We note that when the hypotheses of Theorem 5.1 hold, i.e., when R_0 is of the form (1.3), then, the associated splitting $A = M_0 - N_0$ is weak regular, and the asynchronous Schwarz method of [19] converges with the addition of the coarse grid correction as well.

We also note that arguing as in Sect. 3, and again using the comparison theorem 4.1, Additive Schwarz iteration with inexact solvers also converges when $\tilde{A}_0$ satisfies (3.10). Furthermore, if all inexact solvers satisfy this relation, then we can conclude that this iteration is at least as fast as the one with no coarse grid correction using the same inexact solvers.

We end the paper showing that it is quite natural that our discussion is confined to restrictions R_i of the form (1.3). As we shall see, weak regular splittings can only be expected to arise for restrictions of this form.

To make our discussion precise, let us first place ourselves in the most general situation, where we have an arbitrary linear surjective restriction $R_i : V \to \mathbb{R}^{n_i}$ and an injective linear prolongation $Q_i : \mathbb{R}^{n_i} \to V$. The operator $D_i = R_i Q_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ is thus invertible. We first take a closer look at $Q_i A_i^{-1} R_i$ with $A_i = R_i A Q_i$. The following lemma shows that this operator only depends on the projection $\Pi_i = Q_i D_i^{-1} R_i$, but not on Q_i, R_i individually.

Lemma 5.2 *Let A be nonsingular. Then A_i is nonsingular and $Q_i A_i^{-1} R_i = q(\Pi_i A)\Pi_i$, where $q(z)$ is a polynomial.*

Proof. A_i is nonsingular because Q_i and R_i have full rank. The inverse of $D_i^{-1} A_i$ can be expressed as a polynomial $q(D_i^{-1} A_i)$ [23, Corollary 2.4.4]. But then

$$\begin{aligned} Q_i A_i^{-1} R_i &= Q_i \left(D_i^{-1} A_i\right)^{-1} D_i^{-1} R_i \\ &= Q_i q\left(D_i^{-1} A_i\right) D_i^{-1} R_i = q\left(\Pi_i A\right) \Pi_i. \end{aligned} \qquad \square$$

The next theorem states the central result of this final discussion.

Theorem 5.3 (i) *$Q_i A_i^{-1} R_i \geq O$ for every nonsingular M-matrix A if and only if $\Pi_i \geq O$.*
(ii) *Assume that $\Pi_i \geq O$. Then $Q_i A_i^{-1} R_i A \leq I$ for every nonsingular M-matrix A if and only if $\Pi_i \leq I$.*

Proof. (i): If $Q_i A_i^{-1} R_i \geq O$ for every nonsingular M-matrix A, then particularly so for $A = I$. In that case $A_i^{-1} = D_i^{-1}$ so that $Q_i A_i^{-1} R_i \geq O$ is equivalent to $\Pi_i \geq O$. To show the other direction, we represent A as $A = \beta I - C$ with $C \geq 0$ and $\rho(C) < \beta$. Such a representation exists for any nonsingular M-matrix [3]. Then $A_i = D_i(\beta I_i - C_i)$ with $C_i = D_i^{-1} R_i C Q_i$. But $\rho(D_i^{-1} R_i C Q_i) \leq \rho(C)$ which can be seen as follows: For all $\tau > \rho(C)$ we know that $\lim_{k\to\infty}(\frac{1}{\tau}C)^k = O$, and since $(\frac{1}{\tau}C_i)^k = D_i^{-1} R_i (\frac{1}{\tau}C)^k Q_i$ this implies $\lim_{k\to\infty}(\frac{1}{\tau}C_i)^k = O$ which proves $\rho(C_i) < \tau$.
Therefore, using the Neumann series we can express $A_i^{-1} = (\beta I - C_i)^{-1} D_i^{-1}$ as

$$A_i^{-1} = \frac{1}{\beta} \sum_{\nu=0}^{\infty} \left(\frac{1}{\beta} D_i^{-1} R_i C Q_i\right)^{\nu} D_i^{-1}$$

which yields

$$Q_iA_i^{-1}R_i = \frac{1}{\beta}\sum_{\nu=0}^{\infty}\left(\frac{1}{\beta}\Pi_iC\right)^{\nu}\Pi_i. \tag{5.7}$$

Clearly, this sum is nonnegative if $\Pi_i \geq O$.
(ii): Let us write $A = \beta I - C$ as in (i). From (5.7) we get

$$\begin{aligned} Q_iA_i^{-1}R_iA &= \frac{1}{\beta}\sum_{\nu=0}^{\infty}\left(\frac{1}{\beta}\Pi_iC\right)^{\nu}\Pi_i(\beta I - C) \\ &= \Pi_i - \sum_{\nu=1}^{\infty}\left(\frac{1}{\beta}\Pi_iC\right)^{\nu}(I - \Pi_i) \end{aligned}$$

So evidently, $Q_iA_i^{-1}R_iA \leq I$ if $O \leq \Pi_i \leq I$. The other direction follows in a trivial manner by taking $A = I$, i.e. $\beta = 1$ and $C = O$. □

Note that the proof for part (ii) of Theorem 5.3 shows that in the case $O \leq \Pi_i \leq I$ we actually have $Q_iA_i^{-1}R_iA \leq \Pi_i \leq I$ for any nonsingular M-matrix A.

As a consequence of Theorem 5.3 we see that $Q_iA_i^{-1}R_i \geq O$ together with $I - Q_iA_i^{-1}R_i \geq O$ holds for all nonsingular M-matrices A only if the projections Π_i satisfy $O \leq \Pi_i \leq I$ (cf. (5.4)), i.e., these projections can be represented as diagonal matrices with 0's and 1's, their eigenvalues, as diagonal entries. Such projections, however, are exactly those which can be represented as $R_i^{\mathrm{T}}R_i$ with R_i of the form (1.3).

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