David Colton • Rainer Kress

Inverse Acoustic and Electromagnetic Scattering Theory

Third Edition

Springer

David Colton
Department Mathematical Science
University of Delaware
Newark, DE
USA

Rainer Kress
Institut für Numerische und Angewandte Mathematik
Georg-August-Universität Göttingen
Göttingen, Germany

ISBN 978-1-4614-4941-6 ISBN 978-1-4614-4942-3 (eBook)
DOI 10.1007/978-1-4614-4942-3
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2012943988

Mathematics Subject Classification (2010): 35J05, 35P25, 35R25, 35R30, 45A05, 65M30, 65R20, 65R30, 76Q05, 78A45

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

From Rainer for Marcus

As far as my eyes can see
There are shadows approaching me
And to those I left behind
I wanted you to know
You've always shared my deepest thoughts
I'll miss you when I go
And oh when I'm old and wise
Bitter words mean little to me
Like autumn winds will blow right through me
And someday in the mist of time
When they asked me if I knew you
I'd smile and say you were a friend of mine
And the sadness would be lifted from my eyes

Alan Parsons

Preface to the Third Edition

Since the second edition of our book appeared fourteen years ago, the field of inverse scattering theory has continued to be an active and growing area of applied mathematics. In this third edition we have tried to bring our book up to date by including many of the new developments in the field that have taken place during this period. We again have made no effort to cover all of the many new directions in inverse scattering theory but rather have restricted ourselves to a selection of those developments that we have either participated in or are a natural development of material discussed in previous editions. We have also continued to emphasize simplicity over generality, e.g. smooth domains instead of domains with corners, isotropic media rather than anisotropic media, standard boundary conditions rather than more generalized ones, etc. By so doing, we hope that our book will continue to serve as a basic introduction to the field of inverse scattering theory.

In order to bring our book up to date, considerable changes have been made to the second edition. In particular, new sections have been added on the linear sampling and factorization methods for solving the inverse scattering problem as well as expanded treatments of iteration methods and uniqueness theorems for the inverse obstacle problem.These additions have also required us to expand our presentation on both transmission eigenvalues and boundary integral equations in Sobolev spaces. These changes in turn suggest a more integrated view of inverse scattering theory. In particular, what was previously referred to as the Colton–Monk and Kirsch–Kress methods, respectively, are now viewed as two examples of what are called decomposition methods. From this point of view the techniques of iteration, decomposition and sampling form a natural trilogy of methods for solving inverse scattering problems. Although a few results from the second edition have been removed due to the fact that we now consider them to be obsolete, for historical reasons we have tended to do so sparingly.

We hope that this new edition of our book will continue to serve readers who are already in the field of inverse scattering theory as well as to attract newcomers to this beautiful area of applied mathematics.

Newark, Delaware — David Colton
Göttingen, Germany — Rainer Kress

Preface to the Second Edition

In the five years since the first edition of this book appeared, the field of inverse scattering theory has continued to grow and flourish. Hence, when the opportunity for a second edition presented itself, we were pleased to have the possibility of updating our monograph to take into account recent developments in the area. As in the first edition, we have been motivated by our own view of inverse scattering and have not attempted to include all of the many new directions in the field. However, we feel that this new edition represents a state of the art overview of the basic elements of the mathematical theory of acoustic and electromagnetic inverse scattering.

In addition to making minor corrections and additional comments in the text and updating the references, we have added new sections on Newton's method for solving the inverse obstacle problem (Section 5.3), the spectral theory of the far field operator (Section 8.4), a proof of the uniqueness of the solution to the inverse medium problem for acoustic waves (Section 10.2) and a method for determining the support of an inhomogeneous medium from far field data by solving a linear integral equation of the first kind (Section 10.7).

We hope that this second edition will attract new readers to the beautiful and intriguing field of inverse scattering.

Newark, Delaware David Colton
Göttingen, Germany Rainer Kress

Preface to the First Edition

It has now been almost ten years since our first book on scattering theory appeared [64]. At that time we claimed that “in recent years the development of integral equation methods for the direct scattering problem seems to be nearing completion, whereas the use of such an approach to study the inverse scattering problem has progressed to an extent that a ‘state of the art’ survey appears highly desirable”. Since we wrote these words, the inverse scattering problem for acoustic and electromagnetic waves has grown from being a few theoretical considerations with limited numerical implementations to a well developed mathematical theory with tested numerical algorithms. This maturing of the field of inverse scattering theory has been based on the realization that such problems are in general not only nonlinear but also improperly posed in the sense that the solution does not depend continuously on the measured data. This was emphasized in [64] and treated with the ideas and tools available at that time. Now, almost ten years later, these initial ideas have developed to the extent that a monograph summarizing the mathematical basis of the field seems appropriate. This book is our attempt to write such a monograph.

The inverse scattering problem for acoustic and electromagnetic waves can broadly be divided into two classes, the inverse obstacle problem and the inverse medium problem. In the inverse obstacle problem, the scattering object is a homogeneous obstacle with given boundary data and the inverse problem is to determine the obstacle from a knowledge of the scattered field at infinity, i.e., the far field pattern. The inverse medium problem, in its simplest form, is the situation when the scattering object is an inhomogeneous medium such that the constitutive parameters vary in a continuous manner and the inverse problem is to determine one or more of these parameters from the far field pattern. Only the inverse obstacle problem was considered in [64]. In this book we shall consider both the inverse obstacle and the inverse medium problem using two different methods. In the first method one looks for an obstacle or parameters whose far field pattern best fits the measured data whereas in the second method one looks for an obstacle or parameters whose far field pattern has the same weighted averages as the measured data. The theoretical and numerical development of these two methods for solving the inverse scattering problem for acoustic and electromagnetic waves is the basic subject matter of this book.

We make no claim to cover all the many topics in inverse scattering theory for acoustic and electromagnetic waves. Indeed, with the rapid growth of the field, such a task would be almost impossible in a single volume. In particular, we have emphasized the nonlinear and improperly posed nature of the inverse scattering problem and have paid only passing attention to the various linear methods which are applicable in certain cases. This view of inverse scattering theory has been arrived at through our work in collaboration with a number of mathematicians over the past ten years, in particular Thomas Angell, Peter Hähner, Andreas Kirsch, Ralph Kleinman, Peter Monk, Lassi Päivärinta, Lutz Wienert and Axel Zinn.

As with any book on mathematics, a basic question to answer is where to begin, i.e., what degree of mathematical sophistication is expected of the reader? Since the inverse scattering problem begins with the asymptotic behavior of the solution to the direct scattering problem, it seems reasonable to start with a discussion of the existence and uniqueness of a solution to the direct problem. We have done this for both the Helmholtz and the Maxwell equations. Included in our discussion is a treatment of the numerical solution of the direct problem. In addition to a detailed presentation of the direct scattering problem, we have also included as background material the rudiments of the theory of spherical harmonics, spherical Bessel functions, operator valued analytic functions and ill-posed problems (This last topic has been considerably expanded from the brief discussion given in [64]). As far as more general mathematical background is concerned, we assume that the reader has a basic knowledge of classical and functional analysis.

We have been helped by many people in the course of preparing this book. In particular, we would like to thank Wilhelm Grever, Rainer Hartke and Volker Walther for reading parts of the manuscript and Peter Hähner for his many valuable suggestions for improvements. Thanks also go to Ginger Moore for doing part of the typing. We would like to acknowledge the financial support of the Air Force Office of Scientific Research and the Deutsche Forschungsgemeinschaft, both for the long-term support of our research as well as for the funds made available to us for regular visits between Newark and Göttingen to nurture our collaboration. Finally, we want to give special thanks to our friends and colleagues Andreas Kirsch and Peter Monk. Many of the results of this book represent joint work with these two mathematicians and their insights, criticism and support have been an indispensable component of our research efforts.

Newark, Delaware David Colton
Göttingen, Germany Rainer Kress

Contents

Chapter 1
Introduction

The purpose of this chapter is to provide a survey of our book by placing what we have to say in a historical context. We obviously cannot give a complete account of inverse scattering theory in a book of only a few hundred pages, particularly since before discussing the inverse problem we have to give the rudiments of the theory of the direct problem. Hence, instead of attempting the impossible, we have chosen to present inverse scattering theory from the perspective of our own interests and research program. This inevitably means that certain areas of scattering theory are either ignored or given only cursory attention. In view of this fact, and in fairness to the reader, we have therefore decided to provide a few words at the beginning of our book to tell the reader what we are going to do, as well as what we are not going to do, in the forthcoming chapters.

Scattering theory has played a central role in twentieth century mathematical physics. Indeed, from Rayleigh's explanation of why the sky is blue, to Rutherford's discovery of the atomic nucleus, through the modern medical applications of computerized tomography, scattering phenomena have attracted, perplexed and challenged scientists and mathematicians for well over a hundred years. Broadly speaking, scattering theory is concerned with the effect an inhomogeneous medium has on an incident particle or wave. In particular, if the total field is viewed as the sum of an incident field u^i and a scattered field u^s then the *direct scattering problem* is to determine u^s from a knowledge of u^i and the differential equation governing the wave motion. Of possibly even more interest is the *inverse scattering problem* of determining the nature of the inhomogeneity from a knowledge of the asymptotic behavior of u^s, i.e., to reconstruct the differential equation and/or its domain of definition from the behavior of (many of) its solutions. The above oversimplified description obviously covers a huge range of physical concepts and mathematical ideas and for a sample of the many different approaches that have been taken in this area the reader can consult the monographs of Chadan and Sabatier [46], Colton and Kress [64], Jones [168], Lax and Phillips [220], Leis [224], Martin [236], Müller [256], Newton [261], Reed and Simon [292] and Wilcox [335].

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences 93, DOI 10.1007/978-1-4614-4942-3_1,

1.1 The Direct Scattering Problem

The two basic problems in classical scattering theory (as opposed to quantum scattering theory) are the scattering of time-harmonic acoustic or electromagnetic waves by a penetrable inhomogeneous medium of compact support and by a bounded impenetrable obstacle. Considering first the case of acoustic waves, assume the incident field is given by the time-harmonic acoustic plane wave

$$u^i(x,t) = e^{i(k\,x\cdot d - \omega t)}$$

where $k = \omega/c_0$ is the wave number, ω the frequency, c_0 the speed of sound and d the direction of propagation. Then the simplest scattering problem for the case of an inhomogeneous medium is to find the total field u such that

$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

$$u(x) = e^{ik\,x\cdot d} + u^s(x), \tag{1.2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{1.3}$$

where $r = |x|$, $n = c_0^2/c^2$ is the refractive index given by the ratio of the square of the sound speeds, $c = c_0$ in the homogeneous host medium and $c = c(x)$ in the inhomogeneous medium and (1.3) is the *Sommerfeld radiation condition* which guarantees that the scattered wave is outgoing. It is assumed that $1 - n$ has compact support. If the medium is absorbing, n is complex valued and no longer is simply the ratio of the sound speeds. Turning now to the case of scattering by an impenetrable obstacle D, the simplest problem is to find the total field u such that

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}, \tag{1.4}$$

$$u(x) = e^{ik\,x\cdot d} + u^s(x), \tag{1.5}$$

$$u = 0 \quad \text{on } \partial D, \tag{1.6}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{1.7}$$

where the differential equation (1.4) is the *Helmholtz equation* and the Dirichlet boundary condition (1.6) corresponds to a *sound-soft* obstacle. Boundary conditions other than (1.6) can also be considered, for example the Neumann or *sound-hard* boundary condition or the impedance boundary condition

$$\frac{\partial u}{\partial \nu} + ik\lambda u = 0 \quad \text{on } \partial D$$

where ν is the unit outward normal to ∂D and λ is a positive constant. Although problems (1.1)–(1.3) and (1.4)–(1.7) are perhaps the simplest examples of physically realistic problems in acoustic scattering theory, they still cannot be considered completely solved, particularly from a numerical point of view, and remain the subject matter of much ongoing research.

Considering now the case of electromagnetic waves, assume the incident field is given by the (normalized) time-harmonic electromagnetic plane wave

$$E^i(x,t) = \frac{i}{k}\,\operatorname{curl}\operatorname{curl} p\, e^{i(k\,x\cdot d-\omega t)} = ik\,(d\times p)\times d\, e^{i(k\,x\cdot d-\omega t)},$$

$$H^i(x,t) = \operatorname{curl} p\, e^{i(k\,x\cdot d-\omega t)} = ik\, d\times p\, e^{i(k\,x\cdot d-\omega t)},$$

where $k=\omega\sqrt{\varepsilon_0\mu_0}$ is the wave number, ω the frequency, ε_0 the electric permittivity, μ_0 the magnetic permeability, d the direction of propagation and p the polarization. Then the electromagnetic scattering problem corresponding to (1.1)–(1.3) (assuming variable permittivity but constant permeability) is to find the electric field E and magnetic field H such that

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikn(x)E = 0 \quad \text{in } \mathbb{R}^3, \tag{1.8}$$

$$E(x) = \frac{i}{k}\,\operatorname{curl}\operatorname{curl} p\, e^{ik\,x\cdot d} + E^s(x), \quad H(x) = \operatorname{curl} p\, e^{ik\,x\cdot d} + H^s(x), \tag{1.9}$$

$$\lim_{r\to\infty}(H^s\times x - rE^s) = 0, \tag{1.10}$$

where $n=\varepsilon/\varepsilon_0$ is the refractive index given by the ratio of the permittivity $\varepsilon = \varepsilon(x)$ in the inhomogeneous medium and ε_0 in the homogeneous host medium and where (1.10) is the *Silver–Müller radiation condition*. It is again assumed that $1-n$ has compact support and if the medium is conducting then n is complex valued. Similarly, the electromagnetic analogue of (1.4)–(1.7) is scattering by a perfectly conducting obstacle D which can be mathematically formulated as the problem of finding an electromagnetic field E, H such that

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0 \quad \text{in } \mathbb{R}^3\setminus\bar{D}, \tag{1.11}$$

$$E(x) = \frac{i}{k}\,\operatorname{curl}\operatorname{curl} p\, e^{ik\,x\cdot d} + E^s(x), \quad H(x) = \operatorname{curl} p\, e^{ik\,x\cdot d} + H^s(x), \tag{1.12}$$

$$\nu\times E = 0 \quad \text{on } \partial D, \tag{1.13}$$

$$\lim_{r\to\infty}(H^s\times x - rE^s) = 0, \tag{1.14}$$

where (1.11) are the time-harmonic *Maxwell equations* and ν is again the unit outward normal to ∂D. As in the case of (1.4)–(1.7), more general boundary conditions than (1.13) can also be considered, for example the impedance boundary condition

$$\nu\times\operatorname{curl} E - i\lambda\,(\nu\times E)\times\nu = 0 \quad \text{on } \partial D$$

where λ is again a positive constant.

This book is primarily concerned with the inverse scattering problems associated with the direct scattering problems formulated above. However, before we can consider the inverse problems, we must say more about the direct problems. The mathematical methods used to investigate the direct scattering problems for acoustic and electromagnetic waves depend heavily on the frequency of the wave motion. In particular, if the wavelength $\lambda = 2\pi/k$ is very small compared with the smallest distance which can be observed with the available apparatus, the scattering obstacle produces a shadow with an apparently sharp edge. Closer examination reveals that the edge of the shadow is not sharply defined but breaks up into fringes. This phenomenon is known as diffraction. At the other end of the scale, obstacles which are small compared with the wavelength disrupt the incident wave without producing an identifiable shadow. Hence, we can distinguish two different frequency regions corresponding to the magnitude of ka where a is a typical dimension of the scattering object. More specifically, the set of values of k such that $ka >> 1$ is called the *high frequency region* whereas the set of values of k such that ka is less than or comparable to unity is called the *resonance region*. As suggested by the observed physical differences, the mathematical methods used to study scattering phenomena in the resonance region differ sharply from those used in the high frequency region. Because of this reason, as well as our own mathematical preferences, we have decided that in this book we will be primarily concerned with scattering problems in the resonance region.

The first question to ask about the direct scattering problem is that of the uniqueness of a solution. The basic tools used to establish uniqueness are Green's theorems and the unique continuation property of solutions to elliptic equations. Since equations (1.4) and (1.11) have constant coefficients, the uniqueness question for problems (1.4)–(1.7) and (1.11)–(1.14) are the easiest to handle, with the first results being given by Sommerfeld in 1912 for the case of acoustic waves [308]. Sommerfeld's work was subsequently generalized by Rellich [293] and Vekua [322], all under the assumption that $\operatorname{Im} k \geq 0$. The corresponding uniqueness result for problem (1.11)–(1.14) was first established by Müller [252]. The uniqueness of a solution to the scattering problems (1.1)–(1.3) and (1.8)–(1.10) is more difficult since use must now be made of the unique continuation principle for elliptic equations with variable, but non-analytic, coefficients. The first results in this direction were given by Müller [252], again for the case $\operatorname{Im} k \geq 0$. When $\operatorname{Im} k < 0$ then for each of the above problems there can exist values of k for which uniqueness no longer holds. Such values of k are called *resonance states* and are intimately involved with the asymptotic behavior of the time dependent wave equation. Although we shall not treat resonance states in this book, the subject is of considerable interest and we refer the reader to Dolph [93], Lax and Phillips [220] and Melrose [241] for further information.

Having established uniqueness, the next question to turn to is the existence and numerical approximation of the solution. The most popular approach to existence has been through the method of integral equations. In particular, for problem (1.1)–(1.3), it is easily verified that for all positive values of k the total field u is the unique solution of the *Lippmann–Schwinger equation*

$$u(x) = e^{ik\,x\cdot d} - k^2 \int_{\mathbb{R}^3} \Phi(x,y)m(y)u(y)\,dy, \quad x \in \mathbb{R}^3, \tag{1.15}$$

where $m := 1 - n$ and

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y,$$

is the fundamental solution to the Helmholtz equation. The corresponding integral equation for (1.8)–(1.10) is also easily obtained and is given by

$$\begin{aligned} E(x) = \frac{i}{k} \operatorname{curl} \operatorname{curl} p\, e^{ik\, x\cdot d} - k^2 \int_{\mathbb{R}^3} \Phi(x, y) m(y) E(y)\, dy \\ + \operatorname{grad} \int_{\mathbb{R}^3} \frac{1}{n(y)} \operatorname{grad} n(y) \cdot E(y)\, \Phi(x, y)\, dy, \quad x \in \mathbb{R}^3, \end{aligned} \tag{1.16}$$

where again $m := 1 - n$ and, if E is the solution of (1.16), we define

$$H(x) := \frac{1}{ik} \operatorname{curl} E(x).$$

The application of integral equation methods to problems (1.4)–(1.7) and (1.11)–(1.14) is more subtle. To see why this is so, suppose, by analogy to Laplace's equation, we look for a solution of problem (1.4)–(1.7) in the form of a double-layer potential

$$u^s(x) = \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)} \varphi(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{1.17}$$

where φ is a continuous density to be determined. Then, letting x tend to the boundary ∂D, it can be shown that φ must be a solution of a boundary integral equation of the second kind in order to obtain a solution of (1.4)–(1.7). Unfortunately, this integral equation is not uniquely solvable if k^2 is a Neumann eigenvalue of the negative Laplacian in D. Similar difficulties occur if we look for a solution of problem (1.11)–(1.14) in the form

$$E^s(x) = \operatorname{curl} \int_{\partial D} \Phi(x, y) a(y)\, ds(y), \quad H^s(x) = \frac{1}{ik} \operatorname{curl} E^s(x), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{1.18}$$

where a is a tangential density to be determined. These difficulties in acoustic and electromagnetic scattering theory were first resolved by Vekua [322], Weyl [331] and Müller [254]. A more satisfying approach to this problem was initiated by Werner [327] who suggested modifying the representations (1.17) and (1.18) to include further source terms. This idea was further developed by Brakhage and Werner [26], Leis [223], Panich [268], Knauff and Kress [195] and Kress [198] among others. Finally, we note that the numerical solution of boundary integral equations in scattering theory is fraught with difficulties and we refer the reader to Sections 3.5 and 3.6 of this book and the monograph by Rjasanow and Steinbach [295] for more information on this topic.

Of particular interest in scattering theory is the *far field pattern*, or scattering amplitude, of the scattered acoustic or electromagnetic wave. More specifically, if u^s is the scattered field of (1.1)–(1.3) or (1.4)–(1.7) then u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{r}\, u_\infty(\hat{x}, d) + O\left(\frac{1}{r^2}\right), \quad r = |x| \to \infty,$$

where $\hat{x} = x/|x|$ and u_∞ is the far field pattern of u^s. Similarly, if E^s, H^s is the scattered field of (1.8)–(1.10) or (1.11)–(1.14), then E^s has the asymptotic behavior

$$E^s(x) = \frac{e^{ikr}}{r}\, E_\infty(\hat{x}, d)p + O\left(\frac{1}{r^2}\right), \quad r = |x| \to \infty,$$

where the 3 by 3 matrix E_∞ is the electric far field pattern of E^s. The interest in far field patterns lies in the fact that the basic inverse problem in scattering theory, and, in fact, the one we shall consider in this book, is to determine either n or D from a knowledge of u_∞ or E_∞ for $\hat{x}$ and d on the unit sphere. Until the 1980's, very little was known concerning the mathematical properties of far field patterns, other than the fact that they are analytic functions of their independent variables. However, in the past two decades results have been obtained concerning the completeness properties of far field patterns considered as functions of $\hat{x}$ in $L^2(\mathbb{S}^2)$ where $L^2(\mathbb{S}^2)$ is the space of square integrable functions on the unit sphere $\mathbb{S}^2$. Since these results are of particular relevance to methods developed for solving the inverse scattering problem, we shall now briefly consider some of the ideas involved in this investigation.

The problem of completeness of far field patterns was first considered by Colton and Kirsch [57] for the case of problem (1.4)–(1.7). In particular, they showed that if $\{d_n : n = 1, 2, \ldots\}$ is a dense set of vectors on the unit sphere $\mathbb{S}^2$ then the set $\{u_\infty(\cdot, d_n) : n = 1, 2, \ldots\}$ is complete in $L^2(\mathbb{S}^2)$ if and only if k^2 is not an eigenvalue of the interior Dirichlet problem for the negative Laplacian in D or, if k^2 is an eigenvalue, none of the eigenfunctions is a *Herglotz wave function*, i.e., a solution v of the Helmholtz equation in $\mathbb{R}^3$ such that

$$\sup_{R>0} \frac{1}{R} \int_{|x| \le R} |v(x)|^2 dx < \infty.$$

This result was extended to the case of problem (1.11)–(1.14) by Colton and Kress [65] who also introduced the concept of *electromagnetic Herglotz pairs* which are the analogue for the Maxwell equations of Herglotz wave functions for the Helmholtz equation. The completeness of far field patterns for problem (1.1)–(1.3) is more complicated and was first studied by Kirsch [176], Colton and Monk [75] and Colton, Kirsch and Päivärinta [62]. The result is that the far field patterns are complete provided there does not exist a nontrivial solution of the *interior transmission problem*

$$\Delta w + k^2 n(x) w = 0, \quad \Delta v + k^2 v = 0 \quad \text{in } D, \tag{1.19}$$

$$w = v, \quad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D, \tag{1.20}$$

such that v is a Herglotz wave function where $D := \{x \in \mathbb{R}^3 : m(x) \neq 0\}$ and again $m := 1 - n$. Values of $k > 0$ for which a nontrivial solution to (1.19), (1.20) exist are called *transmission eigenvalues*. The generalization of this result to problem (1.8)–(1.10) was given by Colton and Päivärinta [83, 84], whereas the inverse spectral problem associated with (1.19), (1.20) has been considered by McLaughlin and Polyakov [238], McLaughlin, Polyakov and Sacks [239], Cakoni, Colton and Gintides [33] and Aktosun, Gintides and Papanicolaou [4]. At the time that the second edition of this book was written, research on the transmission eigenvalue problem mainly focused on showing that transmission eigenvalues form at most a discrete set. From a practical point of view the question of discreteness was important to answer since the sampling methods discussed in the next section for reconstructing the support of an inhomogeneous medium fail if the interrogating frequency corresponds to a transmission eigenvalue. On the other hand, due to the non-selfadjointness of (1.19), (1.20), the existence of transmission eigenvalues for non-spherically stratified media remained open for more than twenty years until Paivarinta and Sylvester [267] showed the existence of at least one transmission eigenvalue provided that the contrast m in the medium is large enough. The story of the existence of transmission eigenvalues was completed by Cakoni, Gintides and Haddar [39] where the existence of an infinite set of transmission eigenvalues was proven only under the assumption that the contrast m in the medium does not change sign and is bounded away from zero. It was then shown by Cakoni, Colton and Haddar [34] that transmission eigenvalues could be determined from the scattering data and since they provide information about the material properties of the scattering object can play an important role in a variety of problems in target identification.

Values of the wave number k for which the far field patterns are not complete can be viewed as a value of k for which the *far field operator* $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ defined (in the case of acoustic waves) by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2, \tag{1.21}$$

has zero as an eigenvalue, that is, F is not injective. The question of when the far field operator has non–zero eigenvalues and where they lie in the complex plane has been investigated by Colton and Kress [68, 69] for the scattering of acoustic and electromagnetic waves by both an obstacle and an inhomogeneous medium.

1.2 The Inverse Scattering Problem

As indicated above, the direct scattering problem has been thoroughly investigated and a considerable amount of information is available concerning its solution. In contrast, the inverse scattering problem has only since the 1980's progressed from a collection of ad hoc techniques with little rigorous mathematical basis to an area of intense activity with a solid mathematical foundation. The reason for this is that the inverse scattering problem is inherently nonlinear and, more seriously from the point of view of numerical computations, improperly posed. In particular, small perturbations of the far field pattern in any reasonable norm lead to a function which lies outside the class of far field patterns and, unless regularization methods are used, small variations in the measured data can lead to large errors in the reconstruction of the scatterer. Nevertheless, the inverse scattering problem is basic in areas such as radar, sonar, geophysical exploration, medical imaging and nondestructive testing. Indeed, it is safe to say that the inverse problem is at least of equal interest as the direct problem and, armed with a knowledge of the direct scattering problem, is currently in the foreground of mathematical research in scattering theory [48].

As with the direct scattering problem, the first question to ask about the inverse scattering problem is uniqueness. The first result in this direction was due to Schiffer (see Lax and Phillips [220]) who showed that for problem (1.4)–(1.7) the far field pattern $u_\infty(\hat{x}, d)$ for all $\hat{x}, d \in \mathbb{S}^2$ and k fixed uniquely determines the scattering obstacle D. The corresponding result for problem (1.1)–(1.3) was obtained by Nachman [257], Novikov [263] and Ramm [289]. Uniqueness theorems for the electromagnetic problems (1.8)–(1.10) and (1.11)–(1.14) were first presented in the first edition of this book (c.f. Section 7.1) and by Colton and Päivärinta [85], Ola, Päivärinta and Somersalo [264] and Ola and Somersalo [265]. Closely related to the uniqueness theorem for the inverse scattering problem is Karp's theorem [172], which states for problem (1.4)–(1.7) that if $u_\infty(\hat{x}, d) = u_\infty(Q\hat{x}, Qd)$ for all rotations Q and all $\hat{x}, d \in \mathbb{S}^2$ then D is a ball centered at the origin. The analogues of this result for problems (1.1)–(1.3) and (1.11)–(1.14) have been given by Colton and Kirsch [58] and Colton and Kress [66] (see also Ramm [290]).

Turning to the question of the existence of a solution to the inverse scattering problem, we first note that this is the wrong question to ask. This is due to the observations made above that the inverse scattering problem is improperly posed, i.e., in any realistic situation the measured data is not exact and hence a solution to the inverse scattering problem does not exist. The proper question to ask is how can the inverse problem be stabilized and approximate solutions found to the stabilized problem. (However, for a different point of view, see Newton [262].) Initial efforts in this direction attempted to linearize the problem by reducing it to the problem of solving a linear integral equation of the first kind. The main techniques used to accomplish this were the Born or Rytov approximation for problems (1.1)–(1.3) and (1.8)–(1.10) and the Kirchhoff, or physical optics, approximation to the solution of problems (1.4)–(1.7) and (1.11)–(1.14). While such linearized models are attractive because of their mathematical simplicity, they have the defect of ignoring the basic nonlinear nature of the inverse scattering problem, e.g., multiple reflections are essentially ignored. For detailed presentations of the linearized approach to inverse scattering theory, including a glimpse at the practical applications of such an approach, we refer the reader to Bleistein [22], Chew [50], Devaney [92] and Langenberg [218].

The earliest attempts to treat the inverse scattering problem without linearizing it were due to Imbriale and Mittra [151] for problem (1.4)–(1.7) and Weston and Boerner [330] for problem (1.11)–(1.14). Their methods were based on analytic continuation with little attention being given to issues of stabilization. Then, beginning in the 1980's, a number of methods were given for solving the inverse scattering problem which explicitly acknowledged the nonlinear and ill-posed nature of the problem. In particular, the *acoustic inverse obstacle problem* of determining ∂D in (1.1)–(1.3) from a knowledge of the far field data u_∞ was considered. Either integral equations or Green's formulas were used to reformulate the inverse obstacle problem as a nonlinear optimization problem that required the solution of the direct scattering problem for different domains at each step of the iteration procedure used to arrive at a solution. In this framework, Roger [296] was the first to employ Newton type iterations for the approximate solution of inverse obstacle problems and was followed by Angell, Colton and Kirsch [7], Hanke, Hettlich and Scherzer [130], Hettlich [138], Hohage [142], Ivanyshyn and Kress [160], Johansson and Sleeman [166], Kirsch [181], Kress [204], Kress and Rundell [208], Mönch [243], Potthast [273] and many other researchers.

Approaches which avoid the problem of solving a direct scattering problem at each iteration step and, furthermore, attempt to separate the ill-posedness and nonlinearity of the inverse obstacle problem were introduced and theoretically and numerically analyzed by Kirsch and Kress in [187, 188, 189] and Colton and Monk in [72, 73, 74]. A method that is closely related to the approach suggested by Kirsch and Kress was also introduced by Angell, Kleinman and Roach [11]. These methods are collectively called *decomposition methods* and their main idea is to break up the inverse obstacle scattering problem into two parts: the first part deals with the ill-posedness by constructing the scattered wave from its far field pattern and the second part deals with the nonlinearity by determining the unknown boundary of the scatterer as the location where the boundary condition for the total field is satisfied in a least-squares sense. We shall now briefly outline the original approach of Kirsch and Kress.

We assume a priori that enough information is known about the unknown scattering obstacle D so we can place a surface Γ inside D such that k^2 is not a Dirichlet eigenvalue of the negative Laplacian for the interior of Γ. For fixed wave number k and fixed incident direction d, we then try to represent the scattered field u^s as a single-layer potential

$$u^s(x) = \int_\Gamma \Phi(x, y)\varphi(y)\, ds(y) \tag{1.22}$$

with unknown density $\varphi \in L^2(\Gamma)$ and note that in this case the far field pattern u_∞ has the representation

$$u_\infty(\hat{x}, d) = \frac{1}{4\pi} \int_\Gamma e^{-ik\,\hat{x}\cdot y}\varphi(y)\, ds(y), \quad \hat{x} \in \mathbb{S}^2. \tag{1.23}$$

Given the (measured) far field pattern u_∞ the density φ is now found by solving the ill-posed integral equation of the first kind (1.23). The unknown boundary ∂D is then determined by requiring (1.22) to assume the boundary data (1.5), (1.6) on the surface ∂D. For example, if we assume that ∂D is starlike, i.e., $x(a) = r(a)\,a$ for $x \in \partial D$ and $a \in \mathbb{S}^2$, then this last requirement means solving the nonlinear equation

$$e^{ikr(a)\,a\cdot d} + \int_\Gamma \Phi(r(a)\,a, y)\varphi(y)\,ds(y) = 0, \quad a \in \mathbb{S}^2, \tag{1.24}$$

for the unknown function r. Numerical examples of the use of this method in three dimensions have been provided by Kress and Zinn [216]. The extension of the method of Kirsch and Kress to the case of the *electromagnetic inverse obstacle problem* was carried out by Blöhbaum [23] and used by Haas, Rieger, Rucker and Lehner [116] for fully three dimensional numerical reconstructions.

A more recently developed decomposition method is the *point source method* of Potthast [281]. The hybrid method suggested by Kress and Serranho [206, 305] combines ideas of the method of Kirsch and Kress with Newton type iterations as mentioned above.

The *inverse medium problem* is to determine n in (1.1)–(1.3) or (1.8)–(1.10) from a knowledge of the far field pattern u_∞ or E_∞. As with the inverse obstacle problem, the scalar problem (1.1)–(1.3) has received the most attention. Essentially two methods have been proposed for the determination of n (sampling methods, which we shall discuss shortly, only determine the support of $m := 1 - n$). The first method is based on noting that from (1.15) we have

$$u_\infty(\hat{x}, d) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\,\hat{x}\cdot y} m(y)u(y)\,dy, \quad \hat{x} \in \mathbb{S}^2, \tag{1.25}$$

and then seeking a solution m and u of (1.15) such that the constraint (1.25) is satisfied. This is done by reformulating (1.15) and (1.25) as a nonlinear optimization problem subject to a priori constraints on m such that the optimization problem has a solution that depends continuously on the data. Variations of this approach have been used by van den Berg and Kleinman [321, 193, 194] in acoustics and Abubakar and van den Berg [1] in electromagnetics, among others. There are, of course, important differences in the practical implementation in each of these efforts, for example the far field constraint (1.25) is sometimes replaced by an analogous near field condition and the optimization scheme is numerically solved by different methods, e.g., successive over-relaxation, sinc basis moment methods, steepest descent, etc. Newton type iterations for the inverse medium problem based on the Lippmann–Schwinger equation have been considered by Hohage and Langer [144, 145, 146]. It is also possible to work directly with the scattering problem (1.1)–(1.3) instead of rewriting it as the Lippmann–Schwinger equation and this has been pursued by Gutman and Klibanov [112, 113, 114] and Natterer and Wübbeling [259] and Vögeler [323].

A second method for solving the acoustic inverse medium problem was introduced by Colton and Monk [76, 77] and can be viewed as a decomposition method for approaching the inverse medium problem. The first version of this method [76] begins by determining a function $g \in L^2(\mathbb{S}^2)$ such that, for k fixed and p an integer,

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d)g(d)\,ds(d) = \frac{1}{ki^{p+1}}\,Y_p(\hat{x}), \quad \hat{x} \in \mathbb{S}^2, \tag{1.26}$$

where u_∞ is the far field pattern corresponding to problem (1.1)–(1.3) and Y_p is a spherical harmonic of order p and then constructing the Herglotz wave function v defined by

$$v(x) = \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d)\,ds(d), \quad x \in \mathbb{R}^3. \tag{1.27}$$

The solution of the inverse medium problem is now found by looking for a solution m and w of the Lippmann–Schwinger equation

$$w(x) = v(x) - k^2 \int_{\mathbb{R}^3} \Phi(x,y)m(y)w(y)\,dy, \quad x \in \mathbb{R}^3, \tag{1.28}$$

such that m and w satisfy the constraint

$$-k^2 \int_{\mathbb{R}^3} \Phi(x,y)m(y)w(y)\,dy = h_p^{(1)}(k|x|)\,Y_p(\hat{x}) \tag{1.29}$$

for $|x| = a$ where a is the radius of a ball B centered at the origin and containing the support of m and $h_p^{(1)}$ is the spherical Hankel function of the first kind of order p and again $\hat{x} = x/|x|$. We note that if m is real valued, difficulties can occur in the numerical implementation of this method due to the presence of *transmission eigenvalues*, i.e., values of $k > 0$ such that there exists a nontrivial solution of (1.19), (1.20). The second version of the method of Colton and Monk [77, 78] is designed to overcome this problem by replacing the integral equation (1.26) by

$$\int_{\mathbb{S}^2} [u_\infty(\hat{x},d) - h_\infty(\hat{x},d)]\,g(\hat{x})\,ds(\hat{x}) = \frac{i^{p-1}}{k}\,Y_p(d), \quad d \in \mathbb{S}^2, \tag{1.30}$$

where h_∞ is the (known) far field pattern corresponding to an exterior impedance boundary value problem for the Helmholtz equation in the exterior of the ball B and replacing the constraint (1.29) by

$$\left(\frac{\partial}{\partial \nu} + ik\lambda\right)(w(x) - h_p^{(1)}(k|x|)\,Y_p(\hat{x})) = 0, \quad x \in \partial B, \tag{1.31}$$

where λ is the impedance. A numerical comparison of the two versions of the method due to Colton and Monk can be found in [79]. Alternate methods of modifying the method of Colton and Monk than that described above can be found in [56, 80, 81]. The theoretical basis of this decomposition method for electromagnetic waves has been developed by Colton and Päivärinta [83], Colton and Kress [67] and Colton and Hähner [56].

A different approach to solving inverse scattering problems than the use of iterative methods is the use of sampling methods, c.f. [32, 36, 186]. These methods have the advantage of requiring less a priori information than iterative methods (e.g. it is not necessary to know the topology of the scatterer or the boundary conditions satisfied by the total field) and in addition reduces the nonlinear problem to a non-iterative series of linear problems. On the other hand, the implementation of such methods often requires more data than iterative methods do and in the case of a penetrable inhomogeneous medium only recover the support of the scatterer together with some estimates on its material properties.

The two best known sampling methods are the *linear sampling method* and the *factorization method*. In the case of acoustic waves each of these methods is based on constructing a linear integral equation using the far field operator F as given by (1.21). In particular, the linear sampling method looks for a regularized solution of the far field equation

$$Fg = \Phi_\infty(\cdot\,, z) \tag{1.32}$$

where $z \in \mathbb{R}^3$ and

$$\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi}\, e^{-ik\,\hat{x}\cdot z}, \quad \hat{x} \in \mathbb{S}^2,$$

is the far field pattern of the radiating fundamental solution $\Phi(x, z)$. Then if g_z^α is the solution of (1.32) obtained by Tikhonov regularization and the scatterer is non-absorbing it can be shown that if $v_{g_z^\alpha}$ is the Herglotz wave function with kernel g_z^α then, as $\alpha \to 0$, the sequence $v_{g_z^\alpha}(z)$ is bounded if and only if $z \in D$. On the other hand, the factorization method looks for a solution of the equation of the first kind

$$(F^*F)^{1/4}g = \Phi_\infty(\cdot\,, z) \tag{1.33}$$

where F^* is the adjoint of F in $L^2(\mathbb{S}^2)$. It can then be shown that (1.33) has a solution if and only if $z \in D$ and consequently the solution g_z^α of (1.33) obtained by Tikhonov regularization converges as $\alpha \to 0$ if and only if $z \in D$. Both these sampling methods have extensions to certain classes of electromagnetic wave scattering problems, see [36, 186].

Chapter 2
The Helmholtz Equation

Studying an inverse problem always requires a solid knowledge of the theory for the corresponding direct problem. Therefore, the following two chapters of our book are devoted to presenting the foundations of obstacle scattering problems for time-harmonic acoustic waves, i.e., to exterior boundary value problems for the scalar Helmholtz equation. Our aim is to develop the analysis for the direct problems to an extent which is needed in the subsequent chapters on inverse problems.

In this chapter we begin with a brief discussion of the physical background to scattering problems. We will then derive the basic Green representation theorems for solutions to the Helmholtz equation. Discussing the concept of the Sommerfeld radiation condition will already enable us to introduce the idea of the far field pattern which is of central importance in our book. For a deeper understanding of these ideas, we require sufficient information on spherical wave functions. Therefore, we present in two sections those basic properties of spherical harmonics and spherical Bessel functions that are relevant in scattering theory. We will then be able to derive uniqueness results and expansion theorems for solutions to the Helmholtz equation with respect to spherical wave functions. We also will gain a first insight into the ill-posedness of the inverse problem by examining the smoothness properties of the far field pattern. The study of the boundary value problems will be the subject of the next chapter.

2.1 Acoustic Waves

Consider the propagation of sound waves of small amplitude in a homogeneous isotropic medium in $\mathbb{R}^3$ viewed as an inviscid fluid. Let $v = v(x,t)$ be the velocity field and let $p = p(x,t)$, $\rho = \rho(x,t)$ and $S = S(x,t)$ denote the pressure, density and specific entropy, respectively, of the fluid. The motion is then governed by *Euler's equation*

$$\frac{\partial v}{\partial t} + (v \cdot \operatorname{grad})\, v + \frac{1}{\rho}\, \operatorname{grad} p = 0,$$

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences 93, DOI 10.1007/978-1-4614-4942-3_2,

the *equation of continuity*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0,$$

the *state equation*

$$p = f(\rho, S),$$

and the *adiabatic hypothesis*

$$\frac{\partial S}{\partial t} + v \cdot \operatorname{grad} S = 0,$$

where f is a function depending on the nature of the fluid. We assume that v, p, ρ and S are small perturbations of the static state $v_0 = 0$, $p_0 = \text{constant}$, $\rho_0 = \text{constant}$ and $S_0 = \text{constant}$ and linearize to obtain the linearized Euler equation

$$\frac{\partial v}{\partial t} + \frac{1}{\rho_0} \operatorname{grad} p = 0,$$

the linearized equation of continuity

$$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} v = 0,$$

and the linearized state equation

$$\frac{\partial p}{\partial t} = \frac{\partial f}{\partial \rho}(\rho_0, S_0) \frac{\partial \rho}{\partial t}.$$

From this we obtain the *wave equation*

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \Delta p$$

where the *speed of sound* c is defined by

$$c^2 = \frac{\partial f}{\partial \rho}(\rho_0, S_0).$$

From the linearized Euler equation, we observe that there exists a velocity potential $U = U(x, t)$ such that

$$v = \frac{1}{\rho_0} \operatorname{grad} U$$

and

$$p = -\frac{\partial U}{\partial t}.$$

Clearly, the velocity potential also satisfies the wave equation

$$\frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = \Delta U.$$

For time-harmonic acoustic waves of the form

$$U(x,t) = \mathrm{Re}\left\{u(x)\, e^{-i\omega t}\right\}$$

with frequency $\omega > 0$, we deduce that the complex valued space dependent part u satisfies the *reduced wave equation* or *Helmholtz equation*

$$\Delta u + k^2 u = 0$$

where the *wave number* k is given by the positive constant $k = \omega/c$. This equation carries the name of the physicist Hermann Ludwig Ferdinand von Helmholtz (1821 – 1894) for his contributions to mathematical acoustics and electromagnetics.

In the first part of this book we will be concerned with the scattering of time-harmonic waves by obstacles surrounded by a homogeneous medium, i.e., with exterior boundary value problems for the Helmholtz equation. However, studying the Helmholtz equation in some detail is also required for the second part of our book where we consider wave scattering from an inhomogeneous medium since we always will assume that the medium is homogeneous outside some sufficiently large sphere.

In obstacle scattering we must distinguish between the two cases of impenetrable and penetrable objects. For a *sound-soft* obstacle the pressure of the total wave vanishes on the boundary. Consider the scattering of a given incoming wave u^i by a sound-soft obstacle D. Then the total wave $u = u^i + u^s$, where u^s denotes the scattered wave, must satisfy the wave equation in the exterior $\mathbb{R}^3 \setminus \bar{D}$ of D and a Dirichlet boundary condition $u = 0$ on ∂D. Similarly, the scattering from *sound-hard* obstacles leads to a Neumann boundary condition $\partial u/\partial \nu = 0$ on ∂D where ν is the unit outward normal to ∂D since here the normal velocity of the acoustic wave vanishes on the boundary. More generally, allowing obstacles for which the normal velocity on the boundary is proportional to the excess pressure on the boundary leads to an *impedance boundary condition* of the form

$$\frac{\partial u}{\partial \nu} + ik\lambda u = 0 \quad \text{on } \partial D$$

with a positive constant λ.

The scattering by a penetrable obstacle D with constant density ρ_D and speed of sound c_D differing from the density ρ and speed of sound c in the surrounding medium $\mathbb{R}^3 \setminus \bar{D}$ leads to a transmission problem. Here, in addition to the superposition $u = u^i + u^s$ of the incoming wave u^i and the scattered wave u^s in $\mathbb{R}^3 \setminus \bar{D}$ satisfying the Helmholtz equation with wave number $k = \omega/c$, we also have a transmitted wave v in D satisfying the Helmholtz equation with wave number $k_D = \omega/c_D \neq k$. The continuity of the pressure and of the normal velocity across the interface leads to the *transmission conditions*

$$u = v, \quad \frac{1}{\rho}\frac{\partial u}{\partial \nu} = \frac{1}{\rho_D}\frac{\partial v}{\partial \nu} \quad \text{on } \partial D.$$

In addition to the transmission conditions, more general *resistive boundary conditions* have been introduced and applied. For their description and treatment we refer to [10].

In order to avoid repeating ourselves by considering all possible types of boundary conditions, we have decided to confine ourselves to working out the basic ideas only for the case of a sound-soft obstacle. On occasion, we will mention modifications and extensions to the other cases.

For the scattered wave u^s, the radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |x|,$$

introduced by Sommerfeld [308] in 1912 will ensure uniqueness for the solutions to the scattering problems. From the two possible spherically symmetric solutions

$$\frac{e^{ik|x|}}{|x|} \quad \text{and} \quad \frac{e^{-ik|x|}}{|x|}$$

to the Helmholtz equation, only the first one satisfies the radiation condition. Since via

$$\mathrm{Re}\left\{\frac{e^{ik|x|-i\omega t}}{|x|}\right\} = \frac{\cos(k|x| - \omega t)}{|x|}$$

this corresponds to an outgoing spherical wave, we observe that physically speaking the *Sommerfeld radiation condition* characterizes outgoing waves. Throughout the book by $|x|$ we denote the Euclidean norm of a point x in $\mathbb{R}^3$.

For more details on the physical background of linear acoustic waves, we refer to the article by Morse and Ingard [251] in the Encyclopedia of Physics and to Jones [168] and Werner [326].

2.2 Green's Theorem and Formula

We begin by giving a brief outline of some basic properties of solutions to the Helmholtz equation $\Delta u + k^2 u = 0$ with positive wave number k. Most of these can be deduced from the *fundamental solution*

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x - y|}, \quad x \neq y. \tag{2.1}$$

Straightforward differentiation shows that for fixed $y \in \mathbb{R}^3$ the fundamental solution satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus \{y\}$.

A domain $D \subset \mathbb{R}^3$, i.e., an open and connected set, is said to be of *class* C^k, $k \in \mathbb{N}$, if for each point z of the boundary ∂D there exists a neighborhood V_z of z with the following properties: the intersection $V_z \cap \bar{D}$ can be mapped bijectively onto the half ball $\{x \in \mathbb{R}^3 : |x| < 1,\ x_3 \geq 0\}$, this mapping and its inverse are k–times continuously differentiable and the intersection $V_z \cap \partial D$ is mapped onto the disk $\{x \in \mathbb{R}^3 : |x| < 1,\ x_3 = 0\}$. On occasion, we will express the property of a domain D to be of class C^k also by saying that its boundary ∂D is of class C^k. By $C^k(D)$ we denote the linear space of real or complex valued functions defined on the domain D which are k–times continuously differentiable. By $C^k(\bar{D})$ we denote the subspace of all functions in $C^k(D)$ which together with all their derivatives up to order k can be extended continuously from D into the closure $\bar{D}$.

One of the basic tools in studying the Helmholtz equation is provided by Green's integral theorems. Let D be a bounded domain of class C^1 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D. Then, for $u \in C^1(\bar{D})$ and $v \in C^2(\bar{D})$ we have *Green's first theorem*

$$\int_D (u\Delta v + \operatorname{grad} u \cdot \operatorname{grad} v)\, dx = \int_{\partial D} u \frac{\partial v}{\partial \nu}\, ds, \tag{2.2}$$

and for $u, v \in C^2(\bar{D})$ we have *Green's second theorem*

$$\int_D (u\Delta v - v\Delta u)\, dx = \int_{\partial D} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) ds. \tag{2.3}$$

For two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ or $\mathbb{C}^3$ we will denote by $a \cdot b := a_1 b_1 + a_2 b_2 + a_3 b_3$ the bilinear scalar product and by $|a| := \sqrt{a \cdot \bar{a}}$ the Euclidean norm. For complex numbers or vectors the bar indicates the complex conjugate. Note that our regularity assumptions on D are sufficient conditions for the validity of Green's theorems and can be weakened (see Kellogg [174]).

Theorem 2.1. *Let D be a bounded domain of class C^2 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D. Assume that $u \in C^2(D) \cap C(\bar{D})$ is a function which possesses a normal derivative on the boundary in the sense that the limit*

$$\frac{\partial u}{\partial \nu}(x) = \lim_{h \to +0} \nu(x) \cdot \operatorname{grad} u(x - h\nu(x)), \quad x \in \partial D,$$

exists uniformly on ∂D. Then we have Green's formula

$$\begin{aligned} u(x) = &\int_{\partial D} \left\{ \frac{\partial u}{\partial \nu}(y)\, \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y) \\ &- \int_D \left\{ \Delta u(y) + k^2 u(y) \right\} \Phi(x, y)\, dy, \quad x \in D, \end{aligned} \tag{2.4}$$

where the volume integral exists as improper integral. In particular, if u is a solution to the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } D,$$

then

$$u(x) = \int_{\partial D} \left\{ \frac{\partial u}{\partial \nu}(y)\, \Phi(x,y) - u(y)\, \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right\} ds(y), \qquad x \in D. \tag{2.5}$$

Proof. First, we assume that $u \in C^2(\bar{D})$. We circumscribe the arbitrary fixed point $x \in D$ with a sphere $S(x;\rho) := \{y \in \mathbb{R}^3 : |x-y| = \rho\}$ contained in D and direct the unit normal ν to $S(x;\rho)$ into the interior of $S(x;\rho)$. We now apply Green's theorem (2.3) to the functions u and $\Phi(x,\cdot)$ in the domain $D_\rho := \{y \in D : |x-y| > \rho\}$ to obtain

$$\begin{aligned} &\int_{\partial D \cup S(x;\rho)} \left\{ \frac{\partial u}{\partial \nu}(y)\, \Phi(x,y) - u(y)\, \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right\} ds(y) \\ &\qquad = \int_{D_\rho} \left\{ \Delta u(y) + k^2 u(y) \right\} \Phi(x,y)\, dy. \end{aligned} \tag{2.6}$$

Since on $S(x;\rho)$ we have

$$\Phi(x,y) = \frac{e^{ik\rho}}{4\pi\rho}$$

and

$$\operatorname{grad}_y \Phi(x,y) = \left(\frac{1}{\rho} - ik \right) \frac{e^{ik\rho}}{4\pi\rho}\, \nu(y),$$

a straightforward calculation, using the mean value theorem, shows that

$$\lim_{\rho \to 0} \int_{S(x;\rho)} \left\{ u(y)\, \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y)\, \Phi(x,y) \right\} ds(y) = u(x),$$

whence (2.4) follows by passing to the limit $\rho \to 0$ in (2.6). The existence of the volume integral as an improper integral is a consequence of the fact that its integrand is weakly singular.

The case where u belongs only to $C^2(D) \cap C(\bar{D})$ and has a normal derivative in the sense of uniform convergence is treated by first integrating over parallel surfaces to the boundary of D and then passing to the limit ∂D. For the concept of parallel surfaces, we refer to [64], [205] and [235]. We note that the parallel surfaces for $\partial D \in C^2$ belong to C^1. □

In the literature, Green's formula (2.5) is also known as the *Helmholtz representation*. Obviously, Theorem 2.1 remains valid for complex values of k.

Theorem 2.2. *If u is a two times continuously differentiable solution to the Helmholtz equation in a domain D, then u is analytic.*

Proof. Let $x \in D$ and choose a closed ball contained in D with center x. Then Theorem 2.1 can be applied in this ball and the statement follows from the analyticity of the fundamental solution for $x \neq y$. □

As a consequence of Theorem 2.2, a solution to the Helmholtz equation that vanishes in an open subset of its domain of definition must vanish everywhere.

In the sequel, by saying u is a solution to the Helmholtz equation we tacitly imply that u is twice continuously differentiable, and hence analytic, in the interior of its domain of definition.

The following theorem is a special case of a more general result for partial differential equations known as *Holmgren's theorem*.

Theorem 2.3. *Let D be as in Theorem 2.1 and let $u \in C^2(D) \cap C^1(\bar{D})$ be a solution to the Helmholtz equation in D such that*

$$u = \frac{\partial u}{\partial \nu} = 0 \quad on\ \varGamma \tag{2.7}$$

for some open subset $\varGamma \subset \partial D$. Then u vanishes identically in D.

Proof. In view of (2.7), we use Green's representation formula (2.5) to extend the definition of u by setting

$$u(x) := \int_{\partial D \setminus \varGamma} \left\{ \frac{\partial u}{\partial \nu}(y)\, \varPhi(x,y) - u(y)\, \frac{\partial \varPhi(x,y)}{\partial \nu(y)} \right\} ds(y)$$

for $x \in (\mathbb{R}^3 \setminus \bar{D}) \cup \varGamma$. Then, by Green's second integral theorem (2.3), applied to u and $\varPhi(x,\cdot)$, we have $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. By G we denote a component of $\mathbb{R}^3 \setminus \bar{D}$ with $\varGamma \cap \partial G \neq \emptyset$. Clearly u solves the Helmholtz equation in $(\mathbb{R}^3 \setminus \partial D) \cup \varGamma$ and therefore $u = 0$ in D, since D and G are connected through the gap $\varGamma$ in ∂D. □

Definition 2.4 *A solution u to the Helmholtz equation whose domain of definition contains the exterior of some sphere is called* radiating *if it satisfies the* Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{2.8}$$

where $r = |x|$ and the limit is assumed to hold uniformly in all directions $x/|x|$.

Theorem 2.5. *Assume the bounded set D is the open complement of an unbounded domain of class C^2 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D. Let $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ be a radiating solution to the Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad in\ \mathbb{R}^3 \setminus \bar{D}$$

which possesses a normal derivative on the boundary in the sense that the limit

$$\frac{\partial u}{\partial \nu}(x) = \lim_{h\to+0} \nu(x) \cdot \operatorname{grad} u(x + h\nu(x)), \quad x \in \partial D,$$

exists uniformly on ∂D. Then we have Green's formula

$$u(x) = \int_{\partial D} \left\{ u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y)\, \Phi(x,y) \right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}. \tag{2.9}$$

Proof. We first show that

$$\int_{S_r} |u|^2 ds = O(1), \quad r \to \infty, \tag{2.10}$$

where S_r denotes the sphere of radius r and center at the origin. To accomplish this, we observe that from the radiation condition (2.8) it follows that

$$\int_{S_r} \left\{ \left| \frac{\partial u}{\partial \nu} \right|^2 + k^2 |u|^2 + 2k \operatorname{Im}\left(u \frac{\partial \bar{u}}{\partial \nu} \right) \right\} ds = \int_{S_r} \left| \frac{\partial u}{\partial \nu} - iku \right|^2 ds \to 0, \quad r \to \infty,$$

where ν is the unit outward normal to S_r. We take r large enough such that D is contained in S_r and apply Green's theorem (2.2) in $D_r := \{y \in \mathbb{R}^3 \setminus \bar{D} : |y| < r\}$ to obtain

$$\int_{S_r} u \frac{\partial \bar{u}}{\partial \nu} ds = \int_{\partial D} u \frac{\partial \bar{u}}{\partial \nu} ds - k^2 \int_{D_r} |u|^2 dy + \int_{D_r} |\operatorname{grad} u|^2 dy.$$

We now insert the imaginary part of the last equation into the previous equation and find that

$$\lim_{r \to \infty} \int_{S_r} \left\{ \left| \frac{\partial u}{\partial \nu} \right|^2 + k^2 |u|^2 \right\} ds = -2k \operatorname{Im} \int_{\partial D} u \frac{\partial \bar{u}}{\partial \nu} ds. \tag{2.11}$$

Both terms on the left hand side of (2.11) are nonnegative. Hence, they must be individually bounded as $r \to \infty$ since their sum tends to a finite limit. Therefore, (2.10) is proven.

Now from (2.10) and the radiation condition

$$\frac{\partial \Phi(x,y)}{\partial \nu(y)} - ik\Phi(x,y) = O\left(\frac{1}{r^2} \right), \quad r \to \infty,$$

which is valid uniformly for $y \in S_r$, by the Cauchy–Schwarz inequality we see that

$$I_1 := \int_{S_r} u(y) \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - ik\Phi(x,y) \right\} ds(y) \to 0, \quad r \to \infty,$$

and the radiation condition (2.8) for u and $\Phi(x,y) = O(1/r)$ for $y \in S_r$ yield

$$I_2 := \int_{S_r} \Phi(x,y) \left\{ \frac{\partial u}{\partial \nu}(y) - iku(y) \right\} ds(y) \to 0, \quad r \to \infty.$$

Hence,

$$\int_{S_r} \left\{ u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y)\, \Phi(x,y) \right\} ds(y) = I_1 - I_2 \to 0, \quad r \to \infty.$$

The proof is now completed by applying Theorem 2.1 in the bounded domain D_r and passing to the limit $r \to \infty$. □

From Theorem 2.5 we deduce that radiating solutions u to the Helmholtz equation automatically satisfy Sommerfeld's finiteness condition

$$u(x) = O\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \tag{2.12}$$

uniformly for all directions and that the validity of the Sommerfeld radiation condition (2.8) is invariant under translations of the origin. Wilcox [333] first established that the representation formula (2.9) can be derived without the additional condition (2.12) of finiteness. Our proof of Theorem 2.5 has followed Wilcox's proof. It also shows that (2.8) can be replaced by the weaker formulation

$$\int_{S_r} \left| \frac{\partial u}{\partial r} - iku \right|^2 ds \to 0, \quad r \to \infty,$$

with (2.9) still being valid. Of course, (2.9) then implies that (2.8) also holds.

Solutions to the Helmholtz equation which are defined in all of $\mathbb{R}^3$ are called *entire solutions*. An entire solution to the Helmholtz equation satisfying the radiation condition must vanish identically. This follows immediately from combining Green's formula (2.9) and Green's theorem (2.3).

We are now in a position to introduce the definition of the *far field pattern* or the *scattering amplitude* which plays a central role in this book.

Theorem 2.6. *Every radiating solution u to the Helmholtz equation has the asymptotic behavior of an outgoing spherical wave*

$$u(x) = \frac{e^{ik|x|}}{|x|} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty, \tag{2.13}$$

uniformly in all directions $\hat{x} = x/|x|$ where the function u_∞ defined on the unit sphere $\mathbb{S}^2$ is known as the far field pattern *of u. Under the assumptions of Theorem 2.5 we have*

$$u_\infty(\hat{x}) = \frac{1}{4\pi} \int_{\partial D} \left\{ u(y) \frac{\partial e^{-ik\,\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y)\, e^{-ik\,\hat{x}\cdot y} \right\} ds(y), \quad \hat{x} \in \mathbb{S}^2. \tag{2.14}$$

Proof. From

$$|x-y| = \sqrt{|x|^2 - 2\,x\cdot y + |y|^2} = |x| - \hat{x}\cdot y + O\left(\frac{1}{|x|}\right),$$

we derive

$$\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|}\left\{e^{-ik\,\hat{x}\cdot y} + O\left(\frac{1}{|x|}\right)\right\}, \tag{2.15}$$

and

$$\frac{\partial}{\partial\nu(y)}\,\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|}\left\{\frac{\partial e^{-ik\,\hat{x}\cdot y}}{\partial\nu(y)} + O\left(\frac{1}{|x|}\right)\right\} \tag{2.16}$$

uniformly for all $y \in \partial D$. Inserting this into Green's formula (2.9), the theorem follows. □

One of the main themes of our book will be to recover radiating solutions of the Helmholtz equation from a knowledge of their far field patterns. In terms of the mapping $A : u \mapsto u_\infty$ transferring the radiating solution u into its far field pattern u_∞, we want to solve the equation $Au = u_\infty$ for a given u_∞. In order to establish uniqueness for determining u from its far field pattern u_∞ and to understand the strong ill-posedness of the equation $Au = u_\infty$, we need to develop some facts on spherical wave functions. This will be the subject of the next two sections. We already can point out that the mapping A is extremely smoothing since from (2.14) we see that the far field pattern is an analytic function on the unit sphere.

2.3 Spherical Harmonics

For convenience and to introduce notations, we summarize some of the basic properties of spherical harmonics which are relevant in scattering theory and briefly indicate their proofs. For a more detailed study we refer to Lebedev [221].

Recall that solutions u to the Laplace equation $\Delta u = 0$ are called harmonic functions. The restriction of a homogeneous harmonic polynomial of degree n to the unit sphere $\mathbb{S}^2$ is called a *spherical harmonic* of order n.

Theorem 2.7. *There exist exactly* $2n + 1$ *linearly independent spherical harmonics of order n.*

Proof. By the maximum-minimum principle for harmonic functions it suffices to show that there exist exactly $2n + 1$ linearly independent homogeneous harmonic polynomials H_n of degree n. We can write

$$H_n(x_1, x_2, x_3) = \sum_{k=0}^{n} a_{n-k}(x_1, x_2)\, x_3^k$$

where the a_k are homogeneous polynomials of degree k in the two variables x_1 and x_2. Then, straightforward calculations show that H_n is harmonic if and only if the coefficients satisfy

$$a_{n-k} = -\frac{\Delta a_{n-k+2}}{k(k-1)}\,, \quad k = 2, \dots, n.$$

Therefore, choosing the two coefficients a_n and a_{n-1} uniquely determines H_n, and by setting

$$a_n(x_1, x_2) = x_1^{n-j} x_2^j, \quad a_{n-1}(x_1, x_2) = 0, \quad j = 0, \dots, n,$$
$$a_n(x_1, x_2) = 0, \quad a_{n-1}(x_1, x_2) = x_1^{n-1-j} x_2^j, \quad j = 0, \dots, n-1,$$

clearly we obtain $2n + 1$ linearly independent homogeneous harmonic polynomials of degree n. □

In principle, the proof of the preceding theorem allows a construction of all spherical harmonics. However, it is more convenient and appropriate to use polar coordinates for the representation of spherical harmonics. In polar coordinates (r, θ, φ), homogeneous polynomials clearly are of the form

$$H_n = r^n Y_n(\theta, \varphi),$$

and $\Delta H_n = 0$ is readily seen to be satisfied if

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial Y_n}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2 Y_n}{\partial\varphi^2} + n(n+1)Y_n = 0. \tag{2.17}$$

From Green's theorem (2.3), applied to two homogeneous harmonic polynomials H_n and $H_{n'}$, we have

$$0 = \int_{\mathbb{S}^2} \left\{ \overline{H}_{n'} \frac{\partial H_n}{\partial r} - H_n \frac{\partial \overline{H}_{n'}}{\partial r} \right\} ds = (n - n') \int_{\mathbb{S}^2} Y_n \overline{Y}_{n'} \, ds.$$

Therefore spherical harmonics satisfy the orthogonality relation

$$\int_{\mathbb{S}^2} Y_n \overline{Y}_{n'} \, ds = 0, \quad n \neq n'. \tag{2.18}$$

We first construct spherical harmonics which only depend on the polar angle θ. Choose points x and y with $r = |x| < |y| = 1$, denote the angle between x and y by θ and set $t = \cos\theta$. Consider the function

$$\frac{1}{|x - y|} = \frac{1}{\sqrt{1 - 2tr + r^2}} \tag{2.19}$$

which for fixed y is a solution to Laplace's equation with respect to x. Since for fixed t with $-1 \leq t \leq 1$ the right hand side is an analytic function in r, we have the Taylor series

$$\frac{1}{\sqrt{1 - 2tr + r^2}} = \sum_{n=0}^{\infty} P_n(t) r^n. \tag{2.20}$$

The coefficients P_n in this expansion are called *Legendre polynomials* and the function on the left hand side consequently is known as the *generating function* for the Legendre polynomials. For each $0 < r_0 < 1$ the Taylor series

$$\frac{1}{\sqrt{1 - r\exp(\pm i\theta)}} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}\, r^n\, e^{\pm in\theta} \tag{2.21}$$

and all its term by term derivatives with respect to r and θ are absolutely and uniformly convergent for all $0 \le r \le r_0$ and all $0 \le \theta \le \pi$. Hence, by multiplying the equations (2.21) for the plus and the minus sign, we note that the series (2.20) and all its term by term derivatives with respect to r and θ are absolutely and uniformly convergent for all $0 \le r \le r_0$ and all $-1 \le t = \cos\theta \le 1$. Setting $\theta = 0$ in (2.21) obviously provides a majorant for the series for all θ. Therefore, the geometric series is a majorant for the series in (2.20) and we obtain the inequality

$$|P_n(t)| \le 1, \quad -1 \le t \le 1, \quad n = 0, 1, 2, \dots. \tag{2.22}$$

Differentiating (2.20) with respect to r, multiplying by $1 - 2tr + r^2$, inserting (2.20) on the left hand side and then equating powers of r shows that the P_n satisfy the recursion formula

$$(n+1)P_{n+1}(t) - (2n+1)tP_n(t) + nP_{n-1}(t) = 0, \quad n = 1, 2, \dots. \tag{2.23}$$

Since, as easily seen from (2.20), we have $P_0(t) = 1$ and $P_1(t) = t$, the recursion formula shows that P_n indeed is a polynomial of degree n and that P_n is an even function if n is even and an odd function if n is odd.

Since for fixed y the function (2.19) is harmonic, differentiating (2.20) term by term, we obtain that

$$\sum_{n=0}^{\infty} \left\{ \frac{1}{\sin\theta} \frac{d}{d\theta} \sin\theta \frac{dP_n(\cos\theta)}{d\theta} + n(n+1)P_n(\cos\theta) \right\} r^{n-2} = 0.$$

Equating powers of r shows that the Legendre polynomials satisfy the *Legendre differential equation*

$$(1-t^2)P_n''(t) - 2tP_n'(t) + n(n+1)P_n(t) = 0, \quad n = 0, 1, 2, \dots, \tag{2.24}$$

and that the homogeneous polynomial $r^n P_n(\cos\theta)$ of degree n is harmonic. Therefore, $P_n(\cos\theta)$ represents a spherical harmonic of order n. The orthogonality (2.18) implies that

$$\int_{-1}^{1} P_n(t)P_{n'}(t)\,dt = 0, \quad n \ne n'.$$

Since we have uniform convergence, we may integrate the square of the generating function (2.20) term by term and use the preceding orthogonality to arrive at

$$\int_{-1}^{1} \frac{dt}{1 - 2tr + r^2} = \sum_{n=0}^{\infty} \int_{-1}^{1} [P_n(t)]^2\,dt\; r^{2n}.$$

On the other hand, we have

$$\int_{-1}^{1} \frac{dt}{1-2tr+r^2} = \frac{1}{r} \ln \frac{1+r}{1-r} = \sum_{n=0}^{\infty} \frac{2}{2n+1} r^{2n}.$$

Thus, we have proven the orthonormality relation

$$\int_{-1}^{1} P_n(t)P_m(t)\,dt = \frac{2}{2n+1}\,\delta_{nm}, \qquad n,m = 0,1,2,\dots, \tag{2.25}$$

with the usual meaning for the Kronecker symbol δ_{nm}. Since $\operatorname{span}\{P_0,\dots,P_n\} = \operatorname{span}\{1,\dots,t^n\}$ the Legendre polynomials P_n, $n = 0,1,\dots$, form a complete orthogonal system in $L^2[-1,1]$.

We now look for spherical harmonics of the form

$$Y_n^m(\theta,\varphi) = f(\cos\theta)\,e^{im\varphi}.$$

Then (2.17) is satisfied provided f is a solution of the *associated Legendre differential equation*

$$(1-t^2)f''(t) - 2tf'(t) + \left\{ n(n+1) - \frac{m^2}{1-t^2} \right\} f(t) = 0. \tag{2.26}$$

Differentiating the Legendre differential equation (2.24) m–times shows that $g = P_n^{(m)}$ satisfies

$$(1-t^2)g''(t) - 2(m+1)tg'(t) + (n-m)(n+m+1)g(t) = 0.$$

From this it can be deduced that the *associated Legendre functions*

$$P_n^m(t) := (1-t^2)^{m/2}\,\frac{d^m P_n(t)}{dt^m}, \qquad m = 0,1,\dots,n, \tag{2.27}$$

solve the associated Legendre equation for $n = 0,1,2,\dots$. In order to make sure that the functions $Y_n^m(\theta,\varphi) = P_n^m(\cos\theta)\,e^{im\varphi}$ are spherical harmonics, we have to prove that the harmonic functions $r^n Y_n^m(\theta,\varphi) = r^n P_n^m(\cos\theta)\,e^{im\varphi}$ are homogeneous polynomials of degree n. From the recursion formula (2.23) for the P_n and the definition (2.27) for the P_n^m, we first observe that

$$P_n^m(\cos\theta) = \sin^m\theta\, u_n^m(\cos\theta)$$

where u_n^m is a polynomial of degree $n-m$ which is even if $n-m$ is even and odd if $n-m$ is odd. Since in polar coordinates we have

$$r^m \sin^m\theta\, e^{im\varphi} = (x_1 + ix_2)^m,$$

it follows that

$$r^n\, Y_n^m(\theta,\varphi) = (x_1+ix_2)^m\, r^{n-m}\, u_n^m(\cos\theta).$$

For $n-m$ even we can write

$$r^{n-m}\,u_n^m(\cos\theta) = r^{n-m}\sum_{k=0}^{\frac{1}{2}(n-m)} a_k \cos^{2k}\theta = \sum_{k=0}^{\frac{1}{2}(n-m)} a_k\, x_3^{2k}\,(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}(n-m)-k}$$

which is a homogeneous polynomial of degree $n-m$ and this is also true for $n-m$ odd. Putting everything together, we see that the $r^n\,Y_n^m(\theta,\varphi)$ are homogeneous polynomials of degree n.

Theorem 2.8. *The spherical harmonics*

$$Y_n^m(\theta,\varphi) := \sqrt{\frac{2n+1}{4\pi}\,\frac{(n-|m|)!}{(n+|m|)!}}\;P_n^{|m|}(\cos\theta)\,e^{im\varphi} \tag{2.28}$$

for $m=-n,\dots,n,\ n=0,1,2,\dots,$ *form a complete orthonormal system in* $L^2(\mathbb{S}^2)$.

Proof. Because of (2.18) and the orthogonality of the $e^{im\varphi}$, the Y_n^m given by (2.28) are orthogonal. For $m>0$ we evaluate

$$A_n^m := \int_0^\pi [P_n^m(\cos\theta)]^2 \sin\theta\,d\theta$$

by m partial integrations to get

$$A_n^m = \int_{-1}^1 (1-t^2)^m\left[\frac{d^m P_n(t)}{dt^m}\right]^2 dt = \int_{-1}^1 P_n(t)\,\frac{d^m}{dt^m}\,g_n^m(t)\,dt$$

where

$$g_n^m(t) = (t^2-1)^m\,\frac{d^m P_n(t)}{dt^m}\,.$$

Hence

$$\frac{d^m}{dt^m}\,g_n^m(t) = \frac{(n+m)!}{(n-m)!}\,a_n t^n + \cdots$$

is a polynomial of degree n with a_n the leading coefficient in $P_n(t) = a_n t^n + \cdots$. Therefore, by the orthogonality (2.25) of the Legendre polynomials we derive

$$\frac{(n-m)!}{(n+m)!}\,A_n^m = \int_{-1}^1 a_n t^n\,P_n(t)\,dt = \int_{-1}^1 [P_n(t)]^2\,dt = \frac{2}{2n+1}\,,$$

and the proof of the orthonormality of the Y_n^m is finished.

For fixed m the associated Legendre functions P_n^m for $n=m,m+1,\dots$ are orthogonal and they are complete in $L^2[-1,1]$ since we have

$$\operatorname{span}\{P_m^m,\dots,P_{m+n}^m\} = (1-t^2)^{m/2}\;\operatorname{span}\{1,\dots,t^n\}\,.$$

Writing $Y := \operatorname{span}\{Y_n^m : m = -n, \dots, n,\ n = 0, 1, 2, \dots\}$, it remains to show that Y is dense in $L^2(\mathbb{S}^2)$. Let $g \in C(\mathbb{S}^2)$. For fixed θ we then have Parseval's equality

$$2\pi \sum_{m=-\infty}^{\infty} |g_m(\theta)|^2 = \int_0^{2\pi} |g(\theta, \varphi)|^2 \, d\varphi \tag{2.29}$$

for the Fourier coefficients

$$g_m(\theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\theta, \varphi) e^{-im\varphi} \, d\varphi$$

with respect to φ. Since the g_m and the right hand side of (2.29) are continuous in θ, by Dini's theorem the convergence in (2.29) is uniform with respect to θ. Therefore, given $\varepsilon > 0$ there exists $M = M(\varepsilon) \in \mathbb{N}$ such that

$$\int_0^{2\pi} \left| g(\theta, \varphi) - \sum_{m=-M}^{M} g_m(\theta) e^{im\varphi} \right|^2 d\varphi = \int_0^{2\pi} |g(\theta, \varphi)|^2 \, d\varphi - 2\pi \sum_{m=-M}^{M} |g_m(\theta)|^2 < \frac{\varepsilon}{4\pi}$$

for all $0 \le \theta \le \pi$. The finite number of functions g_m, $m = -M, \dots, M$, can now be simultaneously approximated by the associated Legendre functions, i.e., there exist $N = N(\varepsilon)$ and coefficients a_n^m such that

$$\int_0^{\pi} \left| g_m(\theta) - \sum_{n=|m|}^{N} a_n^m \, P_n^{|m|}(\cos\theta) \right|^2 \sin\theta \, d\theta < \frac{\varepsilon}{8\pi(2M+1)^2}$$

for all $m = -M, \dots, M$. Then, combining the last two inequalities with the help of the Cauchy–Schwarz inequality, we find

$$\int_0^{\pi} \int_0^{2\pi} \left| g(\theta, \varphi) - \sum_{m=-M}^{M} \sum_{n=|m|}^{N} a_n^m \, P_n^{|m|}(\cos\theta) \, e^{im\varphi} \right|^2 \sin\theta \, d\varphi d\theta < \varepsilon.$$

Therefore, Y is dense in $C(\mathbb{S}^2)$ with respect to the L^2 norm and this completes the proof since $C(\mathbb{S}^2)$ is dense in $L^2(\mathbb{S}^2)$. □

We conclude our brief survey of spherical harmonics by proving the important *addition theorem.*

Theorem 2.9. *Let Y_n^m, $m = -n, \dots, n$, be any system of $2n+1$ orthonormal spherical harmonics of order n. Then for all $\hat{x}, \hat{y} \in \mathbb{S}^2$ we have*

$$\sum_{m=-n}^{n} Y_n^m(\hat{x}) \, \overline{Y_n^m(\hat{y})} = \frac{2n+1}{4\pi} \, P_n(\cos\theta) \tag{2.30}$$

where θ denotes the angle between $\hat{x}$ and $\hat{y}$.

Proof. We abbreviate the left hand side of (2.30) by $Y(\hat{x}, \hat{y})$ and first show that Y only depends on the angle θ. Each orthogonal matrix Q in $\mathbb{R}^3$ transforms homogeneous harmonic polynomials of degree n again into homogeneous harmonic polynomials of degree n. Hence, we can write

$$Y_n^m(Q\hat{x}) = \sum_{k=-n}^{n} a_{mk} Y_n^k(\hat{x}), \quad m = -n, \dots, n.$$

Since Q is orthogonal and the Y_n^m are orthonormal, we have

$$\int_{\mathbb{S}^2} Y_n^m(Q\hat{x})\, \overline{Y_n^{m'}(Q\hat{x})}\, ds = \int_{\mathbb{S}^2} Y_n^m(\hat{x})\, \overline{Y_n^{m'}(\hat{x})}\, ds = \delta_{mm'}.$$

From this it can be seen that the matrix $A = (a_{mk})$ also is orthogonal and we obtain

$$Y(Q\hat{x}, Q\hat{y}) = \sum_{m=-n}^{n} \sum_{k=-n}^{n} a_{mk} Y_n^k(\hat{x}) \sum_{l=-n}^{n} \overline{a_{ml} Y_n^l(\hat{y})} = \sum_{k=-n}^{n} Y_n^k(\hat{x})\, \overline{Y_n^k(\hat{y})} = Y(\hat{x}, \hat{y})$$

whence $Y(\hat{x}, \hat{y}) = f(\cos\theta)$ follows. Since for fixed $\hat{y}$ the function Y is a spherical harmonic, by introducing polar coordinates with the polar axis given by $\hat{y}$ we see that $f = a_n P_n$ with some constant a_n. Hence, we have

$$\sum_{m=-n}^{n} Y_n^m(\hat{x})\, \overline{Y_n^m(\hat{y})} = a_n P_n(\cos\theta).$$

Setting $\hat{y} = \hat{x}$ and using $P_n(1) = 1$ (this follows from the generating function (2.20)) we obtain

$$a_n = \sum_{m=-n}^{n} |Y_n^m(\hat{x})|^2.$$

Since the Y_n^m are normalized, integrating the last equation over $\mathbb{S}^2$ we finally arrive at $4\pi a_n = 2n + 1$ and the proof is complete. □

2.4 Spherical Bessel Functions

We continue our study of spherical wave functions by introducing the basic properties of spherical Bessel functions. For a more detailed analysis we again refer to Lebedev [221].

We look for solutions to the Helmholtz equation of the form

$$u(x) = f(k|x|)\, Y_n\left(\frac{x}{|x|}\right)$$

where Y_n is a spherical harmonic of order n. From the differential equation (2.17) for the spherical harmonics, it follows that u solves the Helmholtz equation provided f is a solution of the *spherical Bessel differential equation*

$$t^2 f''(t) + 2tf'(t) + [t^2 - n(n+1)]f(t) = 0. \tag{2.31}$$

We note that for any solution f to the spherical Bessel differential equation (2.31) the function $g(t) := \sqrt{t}\, f(t)$ solves the Bessel differential equation with half integer order $n + 1/2$ and vice versa. By direct calculations, we see that for $n = 0, 1, \ldots$ the functions

$$j_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p t^{n+2p}}{2^p p!\, 1 \cdot 3 \cdots (2n + 2p + 1)} \tag{2.32}$$

and

$$y_n(t) := -\frac{(2n)!}{2^n n!} \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p-n-1}}{2^p p!(-2n + 1)(-2n + 3) \cdots (-2n + 2p - 1)} \tag{2.33}$$

represent solutions to the spherical Bessel differential equation (the first coefficient in the series (2.33) has to be set equal to one). By the ratio test, the function j_n is seen to be analytic for all $t \in \mathbb{R}$ whereas y_n is analytic for all $t \in (0, \infty)$. The functions j_n and y_n are called *spherical Bessel functions* and *spherical Neumann functions* of order n, respectively, and the linear combinations

$$h_n^{(1,2)} := j_n \pm i y_n$$

are known as *spherical Hankel functions* of the first and second kind of order n.

From the series representation (2.32) and (2.33), by equating powers of t, it is readily verified that both $f_n = j_n$ and $f_n = y_n$ satisfy the recurrence relation

$$f_{n+1}(t) + f_{n-1}(t) = \frac{2n + 1}{t} f_n(t), \quad n = 1, 2, \ldots. \tag{2.34}$$

Straightforward differentiation of the series (2.32) and (2.33) shows that both $f_n = j_n$ and $f_n = y_n$ satisfy the differentiation formulas

$$f_{n+1}(t) = -t^n \frac{d}{dt} \left\{t^{-n} f_n(t)\right\}, \quad n = 0, 1, 2, \ldots, \tag{2.35}$$

and

$$t^{n+1} f_{n-1}(t) = \frac{d}{dt} \left\{t^{n+1} f_n(t)\right\}, \quad n = 1, 2, \ldots. \tag{2.36}$$

Finally, from (2.31), the Wronskian

$$W(j_n(t), y_n(t)) := j_n(t) y_n'(t) - y_n(t) j_n'(t)$$

is readily seen to satisfy

$$W' + \frac{2}{t}\, W = 0,$$

whence $W(j_n(t), y_n(t)) = C/t^2$ for some constant C. This constant can be evaluated by passing to the limit $t \to 0$ with the result

$$j_n(t)y_n'(t) - j_n'(t)y_n(t) = \frac{1}{t^2}\,. \tag{2.37}$$

From the series representation of the spherical Bessel and Neumann functions, it is obvious that

$$j_n(t) = \frac{t^n}{1 \cdot 3 \cdots (2n+1)} \left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, \tag{2.38}$$

uniformly on compact subsets of $\mathbb{R}$ and

$$h_n^{(1)}(t) = \frac{1 \cdot 3 \cdots (2n-1)}{it^{n+1}} \left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, \tag{2.39}$$

uniformly on compact subsets of $(0, \infty)$. With the aid of Stirling's formula $n! = \sqrt{2\pi n}\,(n/e)^n\,(1 + o(1))$, $n \to \infty$, which implies that

$$\frac{(2n)!}{n!} = 2^{2n+\frac{1}{2}} \left(\frac{n}{e}\right)^n (1 + o(1)\,), \quad n \to \infty,$$

from (2.39) we obtain

$$h_n^{(1)}(t) = O\left(\frac{2n}{et}\right)^n, \quad n \to \infty, \tag{2.40}$$

uniformly on compact subsets of $(0, \infty)$.

The spherical Bessel and Neumann functions can be expressed in terms of trigonometric functions. Setting $n = 0$ in the series (2.32) and (2.33) we have that

$$j_0(t) = \frac{\sin t}{t}\,, \quad y_0(t) = -\frac{\cos t}{t}$$

and consequently

$$h_0^{(1,2)}(t) = \frac{e^{\pm it}}{\pm it}\,. \tag{2.41}$$

Hence, by induction, from (2.41) and (2.35) it follows that the spherical Hankel functions are of the form

$$h_n^{(1)}(t) = (-i)^n\, \frac{e^{it}}{it} \left\{1 + \sum_{p=1}^{n} \frac{a_{pn}}{t^p}\right\}$$

and

$$h_n^{(2)}(t) = i^n \, \frac{e^{-it}}{-it} \left\{ 1 + \sum_{p=1}^{n} \frac{\bar{a}_{pn}}{t^p} \right\}$$

with complex coefficients $a_{1n}, \dots, a_{nn}$. From this we readily obtain the following asymptotic behavior of the spherical Hankel functions for large argument

$$\begin{aligned} h_n^{(1,2)}(t) &= \frac{1}{t} \, e^{\pm i\left(t - \frac{n\pi}{2} - \frac{\pi}{2}\right)} \left\{ 1 + O\left(\frac{1}{t}\right) \right\}, \quad t \to \infty, \\ h_n^{(1,2)\prime}(t) &= \frac{1}{t} \, e^{\pm i\left(t - \frac{n\pi}{2}\right)} \left\{ 1 + O\left(\frac{1}{t}\right) \right\}, \quad t \to \infty. \end{aligned} \tag{2.42}$$

Taking the real and the imaginary part of (2.42) we also have asymptotic formulas for the spherical Bessel and Neumann functions.

For solutions to the Helmholtz equation in polar coordinates, we can now state the following theorem on *spherical wave functions*.

Theorem 2.10. *Let Y_n be a spherical harmonic of order n. Then*

$$u_n(x) = j_n(k|x|) \, Y_n\left(\frac{x}{|x|}\right)$$

is an entire solution to the Helmholtz equation and

$$v_n(x) = h_n^{(1)}(k|x|) \, Y_n\left(\frac{x}{|x|}\right)$$

is a radiating solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \{0\}$.

Proof. Since we can write $j_n(kr) = k^n r^n w_n(r^2)$ with an analytic function $w_n : \mathbb{R} \to \mathbb{R}$ and since $r^n Y_n(\hat{x})$ is a homogeneous polynomial in x_1, x_2, x_3, the product $j_n(kr) \, Y_n(\hat{x})$ for $\hat{x} = x/|x|$ is regular at $x = 0$, i.e., u_n also satisfies the Helmholtz equation at the origin. That the radiation condition is satisfied for v_n follows from the asymptotic behavior (2.42) of the spherical Hankel functions of the first kind. □

We conclude our brief discussion of spherical wave functions by the following *addition theorem* for the fundamental solution.

Theorem 2.11. *Let Y_n^m, $m = -n, \dots, n$, $n = 0, 1, \dots,$ be a set of orthonormal spherical harmonics. Then for $|x| > |y|$ we have*

$$\frac{e^{ik|x-y|}}{4\pi|x-y|} = ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n^{(1)}(k|x|) \, Y_n^m\left(\frac{x}{|x|}\right) \, j_n(k|y|) \, \overline{Y_n^m\left(\frac{y}{|y|}\right)}. \tag{2.43}$$

The series and its term by term first derivatives with respect to $|x|$ and $|y|$ are absolutely and uniformly convergent on compact subsets of $|x| > |y|$.

Proof. We abbreviate $\hat{x} = x/|x|$ and $\hat{y} = y/|y|$. From Green's theorem (2.3) applied to $u_n^m(z) = j_n(k|z|)\,Y_n^m(\hat{z})$ with $\hat{z} = z/|z|$ and $\Phi(x,z)$, we have

$$\int_{|z|=r} \left\{ u_n^m(z)\,\frac{\partial \Phi(x,z)}{\partial \nu(z)} - \frac{\partial u_n^m}{\partial \nu}(z)\,\Phi(x,z) \right\} ds(z) = 0, \quad |x| > r,$$

and from Green's formula (2.9), applied to $v_n^m(z) = h_n^{(1)}(k|z|)\,Y_n^m(\hat{z})$, we have

$$\int_{|z|=r} \left\{ v_n^m(z)\,\frac{\partial \Phi(x,z)}{\partial \nu(z)} - \frac{\partial v_n^m}{\partial \nu}(z)\,\Phi(x,z) \right\} ds(z) = v_n^m(x), \quad |x| > r.$$

From the last two equations, noting that on $|z| = r$ we have

$$u_n^m(z) = j_n(kr)\,Y_n^m(\hat{z}), \qquad \frac{\partial u_n^m}{\partial \nu}(z) = k j_n'(kr)\,Y_n^m(\hat{z})$$

and

$$v_n^m(z) = h_n^{(1)}(kr)\,Y_n^m(\hat{z}), \qquad \frac{\partial v_n^m}{\partial \nu}(z) = k h_n^{(1)\prime}(kr)\,Y_n^m(\hat{z})$$

and using the Wronskian (2.37), we see that

$$\frac{1}{ikr^2}\int_{|z|=r} Y_n^m(\hat{z})\,\Phi(x,z)\,ds(z) = j_n(kr)\,h_n^{(1)}(k|x|)\,Y_n^m(\hat{x}), \quad |x| > r,$$

and by transforming the integral into one over the unit sphere we get

$$\int_{\mathbb{S}^2} Y_n^m(\hat{z})\,\Phi(x, r\hat{z})\,ds(\hat{z}) = ik\, j_n(kr)\,h_n^{(1)}(k|x|)\,Y_n^m(\hat{x}), \quad |x| > r. \tag{2.44}$$

We can now apply Theorem 2.8 to obtain from the orthogonal expansion

$$\Phi(x,y) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \int_{\mathbb{S}^2} Y_n^m(\hat{z})\,\Phi(x, r\hat{z})\,ds(\hat{z})\,\overline{Y_n^m(\hat{y})}$$

and (2.44) that the series (2.43) is valid for fixed x with $|x| > r$ and with respect to y in the L^2 sense on the sphere $|y| = r$ for arbitrary r. With the aid of the Cauchy–Schwarz inequality, the Addition Theorem 2.9 for the spherical harmonics and the inequalities (2.22), (2.38) and (2.39) we can estimate

$$\sum_{m=-n}^{n} \left| h_n^{(1)}(k|x|)\,Y_n^m(\hat{x})\,j_n(k|y|)\,\overline{Y_n^m(\hat{y})} \right|$$

$$\leq \frac{2n+1}{4\pi}\,|\,h_n^{(1)}(k|x|)\,j_n(k|y|)\,| \;=\; O\left(\frac{|y|^n}{|x|^n}\right), \quad n \to \infty,$$

uniformly on compact subsets of $|x| > |y|$. Hence, we have a majorant implying absolute and uniform convergence of the series (2.43). The absolute and uniform convergence of the derivatives with respect to $|x|$ and $|y|$ can be established analogously with the help of estimates for the derivatives j_n' and $h_n^{(1)\prime}$ corresponding to (2.38) and (2.39) which follow readily from (2.35). □

Passing to the limit $|x| \to \infty$ in (2.44) with the aid of (2.15) and (2.42), we arrive at the *Funk–Hecke formula*

$$\int_{\mathbb{S}^2} e^{-ikr\,\hat{x}\cdot\hat{z}}\, Y_n(\hat{z})\, ds(\hat{z}) = \frac{4\pi}{i^n}\, j_n(kr)\, Y_n(\hat{x}), \quad \hat{x} \in \mathbb{S}^2, \quad r > 0,$$

for spherical harmonics Y_n of order n. Obviously, this may be rewritten in the form

$$\int_{\mathbb{S}^2} e^{-ik\,x\cdot\hat{z}}\, Y_n(\hat{z})\, ds(\hat{z}) = \frac{4\pi}{i^n}\, j_n(k|x|)\, Y_n\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^3. \tag{2.45}$$

Proceeding as in the proof of the previous theorem, from (2.45) and Theorem 2.9 we can derive the *Jacobi–Anger expansion*

$$e^{ik\,x\cdot d} = \sum_{n=0}^{\infty} i^n (2n+1)\, j_n(k|x|)\, P_n(\cos\theta), \quad x \in \mathbb{R}^3, \tag{2.46}$$

where d is a unit vector, θ denotes the angle between x and d and the convergence is uniform on compact subsets of $\mathbb{R}^3$.

2.5 The Far Field Pattern

In this section we first establish the one-to-one correspondence between radiating solutions to the Helmholtz equation and their far field patterns.

Lemma 2.12 (Rellich) *Assume the bounded set D is the open complement of an unbounded domain and let $u \in C^2(\mathbb{R}^3 \setminus \bar{D})$ be a solution to the Helmholtz equation satisfying*

$$\lim_{r\to\infty} \int_{|x|=r} |u(x)|^2 ds = 0. \tag{2.47}$$

Then $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$.

Proof. For sufficiently large $|x|$, by Theorem 2.8 we have a Fourier expansion

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m(|x|)\, Y_n^m(\hat{x})$$

with respect to spherical harmonics where $\hat{x} = x/|x|$. The coefficients are given by

$$a_n^m(r) = \int_{\mathbb{S}^2} u(r\hat{x})\overline{Y_n^m(\hat{x})}\, ds(\hat{x})$$

and satisfy Parseval's equality

$$\int_{|x|=r} |u(x)|^2 ds = r^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|a_n^m(r)\right|^2 .$$

Our assumption (2.47) implies that

$$\lim_{r\to\infty} r^2 \left|a_n^m(r)\right|^2 = 0 \tag{2.48}$$

for all n and m.

Since $u \in C^2(\mathbb{R}^3 \setminus \bar{D})$, we can differentiate under the integral and integrate by parts using $\Delta u + k^2 u = 0$ and the differential equation (2.17) to conclude that the a_n^m are solutions to the spherical Bessel equation

$$\frac{d^2 a_n^m}{dr^2} + \frac{2}{r}\frac{da_n^m}{dr} + \left(k^2 - \frac{n(n+1)}{r^2}\right) a_n^m = 0,$$

that is,

$$a_n^m(r) = \alpha_n^m h_n^{(1)}(kr) + \beta_n^m h_n^{(2)}(kr)$$

where α_n^m and β_n^m are constants. Substituting this into (2.48) and using the asymptotic behavior (2.42) of the spherical Hankel functions yields $\alpha_n^m = \beta_n^m = 0$ for all n and m. Therefore, $u = 0$ outside a sufficiently large sphere and hence $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ by analyticity (Theorem 2.2). □

Rellich's lemma ensures uniqueness for solutions to exterior boundary value problems through the following theorem.

Theorem 2.13. *Let D be as in Lemma 2.12, let ∂D be of class C^2 with unit normal ν directed into the exterior of D and assume $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ is a radiating solution to the Helmholtz equation with wave number $k > 0$ which has a normal derivative in the sense of uniform convergence and for which*

$$\operatorname{Im} \int_{\partial D} u \frac{\partial \bar{u}}{\partial \nu}\, ds \geq 0.$$

Then $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$.

Proof. From the identity (2.11) and the assumption of the theorem, we conclude that (2.47) is satisfied. Hence, the theorem follows from Rellich's Lemma 2.12. □

Rellich's lemma also establishes the one-to-one correspondence between radiating waves and their far field patterns.

Theorem 2.14. *Let D be as in Lemma 2.12 and let $u \in C^2(\mathbb{R}^3 \setminus \bar{D})$ be a radiating solution to the Helmholtz equation for which the far field pattern vanishes identically. Then $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$.*

Proof. Since from (2.13) we deduce

$$\int_{|x|=r} |u(x)|^2 ds = \int_{\mathbb{S}^2} |u_\infty(\hat{x})|^2 ds + O\left(\frac{1}{r}\right), \quad r \to \infty,$$

the assumption $u_\infty = 0$ on $\mathbb{S}^2$ implies that (2.47) is satisfied. Hence, the theorem follows from Rellich's Lemma 2.12. □

Theorem 2.15. *Let u be a radiating solution to the Helmholtz equation in the exterior $|x| > R > 0$ of a sphere. Then u has an expansion with respect to spherical wave functions of the form*

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m \, h_n^{(1)}(k|x|) \, Y_n^m\left(\frac{x}{|x|}\right) \tag{2.49}$$

that converges absolutely and uniformly on compact subsets of $|x| > R$. Conversely, if the series (2.49) converges in the mean square sense on the sphere $|x| = R$ then it also converges absolutely and uniformly on compact subsets of $|x| > R$ and u represents a radiating solution to the Helmholtz equation for $|x| > R$.

Proof. For a radiating solution u to the Helmholtz equation, we insert the addition theorem (2.43) into Green's formula (2.9), applied to the boundary surface $|y| = \tilde{R}$ with $R < \tilde{R} < |x|$, and integrate term by term to obtain the expansion (2.49).

Conversely, L^2 convergence of the series (2.49) on the sphere $|x| = R$, implies by Parseval's equality that

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|h_n^{(1)}(kR)\right|^2 \left|a_n^m\right|^2 < \infty.$$

Using the Cauchy–Schwarz inequality, the asymptotic behavior (2.39) and the addition theorem (2.30) for $R < R_1 \le |x| \le R_2$ and for $N \in \mathbb{N}$ we can estimate

$$\left[\sum_{n=0}^{N} \sum_{m=-n}^{n} \left|h_n^{(1)}(k|x|) \, a_n^m Y_n^m\left(\frac{x}{|x|}\right)\right|\right]^2$$

$$\le \sum_{n=0}^{N} \left|\frac{h_n^{(1)}(k|x|)}{h_n^{(1)}(kR)}\right|^2 \sum_{m=-n}^{n} \left|Y_n^m\left(\frac{x}{|x|}\right)\right|^2 \sum_{n=0}^{N} \sum_{m=-n}^{n} \left|h_n^{(1)}(kR)\right|^2 \left|a_n^m\right|^2$$

$$\le C \sum_{n=0}^{N} (2n+1) \left(\frac{R}{|x|}\right)^{2n}$$

for some constant C depending on R, R_1 and R_2. From this we conclude absolute and uniform convergence of the series (2.49) on compact subsets of $|x| > R$. Similarly, it can be seen that the term by term first derivatives with respect to $|x|$ are absolutely and uniformly convergent on compact subsets of $|x| > R$. To establish that

u solves the Helmholtz equation and satisfies the Sommerfeld radiation condition, we show that Green's formula is valid for u. Using the addition Theorem 2.11, the orthonormality of the Y_n^m and the Wronskian (2.37), we indeed find that

$$\int_{|y|=\tilde{R}} \left\{ u(y)\, \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}\,(y)\, \Phi(x,y) \right\} ds(y)$$

$$= ik\tilde{R}^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m\, h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right) k\, W(h_n^{(1)}(k\tilde{R}), j_n(k\tilde{R}))$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m\, h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right) \;=\; u(x)$$

for $|x| > \tilde{R} > R$. From this it is now obvious that u represents a radiating solution to the Helmholtz equation. □

Let R be the radius of the smallest closed ball with center at the origin containing the bounded domain D. Then, by the preceding theorem, each radiating solution $u \in C^2(\mathbb{R}^3 \setminus \bar{D})$ to the Helmholtz equation has an expansion with respect to spherical wave functions of the form (2.49) that converges absolutely and uniformly on compact subsets of $|x| > R$. Conversely, the expansion (2.49) is valid in all of $\mathbb{R}^3 \setminus \bar{D}$ if the origin is contained in D and if u can be extended as a solution to the Helmholtz equation in the exterior of the largest closed ball with center at the origin contained in $\bar{D}$.

Theorem 2.16. *The far field pattern of the radiating solution to the Helmholtz equation with the expansion (2.49) is given by the uniformly convergent series*

$$u_\infty = \frac{1}{k} \sum_{n=0}^{\infty} \frac{1}{i^{n+1}} \sum_{m=-n}^{n} a_n^m\, Y_n^m. \tag{2.50}$$

The coefficients in this expansion satisfy the growth condition

$$\sum_{n=0}^{\infty} \left(\frac{2n}{ker}\right)^{2n} \sum_{m=-n}^{n} \left|a_n^m\right|^2 < \infty \tag{2.51}$$

for all $r > R$.

Proof. We cannot pass to the limit $|x| \to \infty$ in (2.49) by using the asymptotic behavior (2.42) because the latter does not hold uniformly in n. Since by Theorem 2.6 the far field pattern u_∞ is analytic, we have an expansion

$$u_\infty = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_n^m\, Y_n^m$$

with coefficients

$$b_n^m = \int_{\mathbb{S}^2} u_\infty(\hat{x})\, \overline{Y_n^m(\hat{x})}\, ds(\hat{x}).$$

On the other hand, the coefficients a_n^m in the expansion (2.49) clearly are given by

$$a_n^m\, h_n^{(1)}(kr) = \int_{\mathbb{S}^2} u(r\hat{x})\, \overline{Y_n^m(\hat{x})}\, ds(\hat{x}).$$

Therefore, with the aid of (2.42) we find that

$$\begin{aligned} b_n^m &= \int_{\mathbb{S}^2} \lim_{r\to\infty} r\, e^{-ikr} u(r\hat{x})\, \overline{Y_n^m(\hat{x})}\, ds(\hat{x}) \\ &= \lim_{r\to\infty} r\, e^{-ikr} \int_{\mathbb{S}^2} u(r\hat{x})\, \overline{Y_n^m(\hat{x})}\, ds(\hat{x}) \;=\; \frac{a_n^m}{k\, i^{n+1}}\,, \end{aligned}$$

and the expansion (2.50) is valid in the L^2 sense.

Parseval's equation for the expansion (2.49) reads

$$r^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|a_n^m\right|^2 \left|h_n^{(1)}(kr)\right|^2 = \int_{|x|=r} |u(x)|^2 ds(x).$$

From this, using the asymptotic behavior (2.40) of the Hankel functions for large order n, the condition (2.51) follows. In particular, by the Cauchy–Schwarz inequality, we can now conclude that (2.50) is uniformly valid on $\mathbb{S}^2$. □

Theorem 2.17. *Let the Fourier coefficients b_n^m of $u_\infty \in L^2(\mathbb{S}^2)$ with respect to the spherical harmonics satisfy the growth condition*

$$\sum_{n=0}^{\infty} \left(\frac{2n}{keR}\right)^{2n} \sum_{m=-n}^{n} \left|b_n^m\right|^2 < \infty \tag{2.52}$$

with some $R > 0$. Then

$$u(x) = k \sum_{n=0}^{\infty} i^{n+1} \sum_{m=-n}^{n} b_n^m h_n^{(1)}(k|x|)\, Y_n^m\!\left(\frac{x}{|x|}\right), \quad |x| > R, \tag{2.53}$$

is a radiating solution of the Helmholtz equation with far field pattern u_∞.

Proof. By the asymptotic behavior (2.40), the assumption (2.52) implies that the series (2.53) converges in the mean square sense on the sphere $|x| = R$. Hence, by Theorem 2.15, u is a radiating solution to the Helmholtz equation. The fact that the far field pattern coincides with the given function u_∞ follows from Theorem 2.16. □

The last two theorems indicate that the equation

$$Au = u_\infty \tag{2.54}$$

with the linear operator A mapping a radiating solution u to the Helmholtz equation onto its far field u_∞ is ill-posed. Following Hadamard [118], a problem is called *properly posed* or *well-posed* if a solution exists, if the solution is unique and if the solution continuously depends on the data. Otherwise, the problem is called *improperly posed* or *ill-posed*. Here, for equation (2.54), by Theorem 2.14 we have uniqueness of the solution. However, since by Theorem 2.16 the existence of a solution requires the growth condition (2.51) to be satisfied, for a given function u_∞ in $L^2(\mathbb{S}^2)$ a solution of equation (2.54) will, in general, not exist. Furthermore, if a solution u does exist it will not continuously depend on u_∞ in any reasonable norm. This is illustrated by the fact that for the radiating solutions

$$u_n(x) = \frac{1}{n}\, h_n^{(1)}(k|x|)\, Y_n\left(\frac{x}{|x|}\right)$$

where Y_n is a normalized spherical harmonic of degree n the far field patterns are given by

$$u_{n,\infty} = \frac{1}{k i^{n+1} n}\, Y_n.$$

Hence, we have convergence $u_{n,\infty} \to 0$, $n \to \infty$, in the L^2 norm on $\mathbb{S}^2$ whereas, as a consequence of the asymptotic behavior (2.40) of the Hankel functions for large order n, the u_n will not converge in any suitable norm. Later in this book we will study the ill-posedness of the reconstruction of a radiating solution of the Helmholtz equation from its far field pattern more closely. In particular, we will describe stable methods for approximately solving improperly posed problems such as this one.

Chapter 3
Direct Acoustic Obstacle Scattering

This chapter is devoted to the solution of the direct obstacle scattering problem for acoustic waves. As in [64], we choose the method of integral equations for solving the boundary value problems. However, we decided to leave out some of the details in the analysis. In particular, we assume that the reader is familiar with the Riesz–Fredholm theory for operator equations of the second kind in dual systems as described in [64] and [205]. We also do not repeat the technical proofs for the jump relations and regularity properties for single- and double-layer potentials. Leaving aside these two restrictions, however, we will present a rather complete analysis of the forward scattering problem. For the reader interested in a more comprehensive treatment of the direct problem, we suggest consulting our previous book [64] on this subject.

We begin by listing the jump and regularity properties of surface potentials in the classical setting of continuous and Hölder continuous functions and later present their extensions to the case of Sobolev spaces. We then proceed to establish the existence of the solution to the exterior Dirichlet problem via boundary integral equations and also describe some results on the regularity of the solution. In particular, we will establish the well-posedness of the Dirichlet to Neumann map in the Hölder and Sobolev space settings. Coming back to the far field pattern, we prove reciprocity relations that will be of importance in the study of the inverse scattering problem. We then use one of the reciprocity relations to derive some completeness results on the set of far field patterns corresponding to the scattering of incident plane waves propagating in different directions. For this we need to introduce and examine Herglotz wave functions and the far field operator which will both be of central importance later on for the inverse scattering problem.

Our presentation is in $\mathbb{R}^3$. For the sake of completeness, we include a section where we list the necessary modifications for the two-dimensional theory. We also add a section advertising a Nyström method for the numerical solution of the boundary integral equations in two dimensions by a spectral method based on approximations via trigonometric polynomials. Finally, we present the main ideas of a spectral

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences 93, DOI 10.1007/978-1-4614-4942-3_3,

method based on approximations via spherical harmonics for the numerical solution of the boundary integral equations in three dimensions that was developed and investigated by Wienert [332] and by Ganesh, Graham and Sloan [99, 109].

3.1 Single- and Double-Layer Potentials

In this chapter, if not stated otherwise, we always will assume that the bounded set D is the open complement of an unbounded domain of class C^2, that is, we include scattering from more than one obstacle in our analysis noting that the C^2 smoothness implies that D has only a finite number of components.

We first briefly review the basic jump relations and regularity properties of acoustic single- and double-layer potentials. Given an integrable function φ, the integrals

$$u(x) := \int_{\partial D} \varphi(y)\Phi(x,y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D,$$

and

$$v(x) := \int_{\partial D} \varphi(y)\,\frac{\partial \Phi(x,y)}{\partial \nu(y)}\,ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D,$$

are called, respectively, *acoustic single-layer* and *acoustic double-layer potentials* with density φ. They are solutions to the Helmholtz equation in D and in $\mathbb{R}^3 \setminus \bar{D}$ and satisfy the Sommerfeld radiation condition. Green's formulas (2.5) and (2.9) show that any solution to the Helmholtz equation can be represented as a combination of single- and double-layer potentials. For continuous densities, the behavior of the surface potentials at the boundary is described by the following *jump relations*. By $\|\cdot\|_\infty = \|\cdot\|_{\infty,G}$ we denote the usual supremum norm of real or complex valued functions defined on a set $G \subset \mathbb{R}^3$.

Theorem 3.1. *Let ∂D be of class C^2 and let φ be continuous. Then the single-layer potential u with density φ is continuous throughout $\mathbb{R}^3$ and*

$$\|u\|_{\infty,\mathbb{R}^3} \le C\|\varphi\|_{\infty,\partial D}$$

for some constant C depending on ∂D. On the boundary we have

$$u(x) = \int_{\partial D} \varphi(y)\Phi(x,y)\,ds(y), \quad x \in \partial D, \tag{3.1}$$

$$\frac{\partial u_\pm}{\partial \nu}(x) = \int_{\partial D} \varphi(y)\,\frac{\partial \Phi(x,y)}{\partial \nu(x)}\,ds(y) \mp \frac{1}{2}\,\varphi(x), \quad x \in \partial D, \tag{3.2}$$

where

$$\frac{\partial u_\pm}{\partial \nu}(x) := \lim_{h\to+0} \nu(x)\cdot \operatorname{grad} u(x \pm h\nu(x))$$

is to be understood in the sense of uniform convergence on ∂D and where the integrals exist as improper integrals. The double-layer potential v with density φ can be *continuously extended from D to $\bar{D}$ and from $\mathbb{R}^3 \setminus \bar{D}$ to $\mathbb{R}^3 \setminus D$ with limiting values*

$$v_\pm(x) = \int_{\partial D} \varphi(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, ds(y) \pm \frac{1}{2}\varphi(x), \quad x \in \partial D, \tag{3.3}$$

where

$$v_\pm(x) := \lim_{h \to +0} v(x \pm h\nu(x))$$

and where the integral exists as an improper integral. Furthermore,

$$\|v\|_{\infty,\overline{D}} \le C\|\varphi\|_{\infty,\partial D}, \quad \|v\|_{\infty,\mathbb{R}^3\setminus D} \le C\|\varphi\|_{\infty,\partial D}$$

for some constant C depending on ∂D and

$$\lim_{h\to+0}\left\{\frac{\partial v}{\partial \nu}(x + h\nu(x)) - \frac{\partial v}{\partial \nu}(x - h\nu(x))\right\} = 0, \quad x \in \partial D, \tag{3.4}$$

uniformly on ∂D.

Proof. For a proof, we refer to Theorems 2.12, 2.13, 2.19 and 2.21 in [64]. Note that the estimates on the double-layer potential follow from Theorem 2.13 in [64] by using the maximum-minimum principle for harmonic functions in the limiting case $k = 0$ and Theorems 2.7 and 2.15 in [64]. □

An appropriate framework for formulating additional regularity properties of these surface potentials is provided by the concept of Hölder spaces. A real or complex valued function φ defined on a set $G \subset \mathbb{R}^3$ is called *uniformly Hölder continuous* with *Hölder exponent* $0 < \alpha \le 1$ if there is a constant C such that

$$|\varphi(x) - \varphi(y)| \le C|x - y|^\alpha \tag{3.5}$$

for all $x, y \in G$. We define the *Hölder space* $C^{0,\alpha}(G)$ to be the linear space of all functions defined on G which are bounded and uniformly Hölder continuous with exponent α. It is a Banach space with the norm

$$\|\varphi\|_\alpha := \|\varphi\|_{\alpha,G} := \sup_{x\in G}|\varphi(x)| + \sup_{\substack{x,y\in G\\ x\ne y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha}. \tag{3.6}$$

Clearly, for $\alpha < \beta$ each function $\varphi \in C^{0,\beta}(G)$ is also contained in $C^{0,\alpha}(G)$. For this imbedding, from the Arzelà–Ascoli theorem, we have the following compactness property (for a proof we refer to [64], p. 38 or [205], p. 84).

Theorem 3.2. *Let $0 < \alpha < \beta \le 1$ and let G be compact. Then the* imbedding operators

$$I^\beta : C^{0,\beta}(G) \to C(G), \quad I^{\alpha,\beta} : C^{0,\beta}(G) \to C^{0,\alpha}(G)$$

are compact.

For a vector field, Hölder continuity and the Hölder norm are defined analogously by replacing absolute values in (3.5) and (3.6) by Euclidean norms. We can then introduce the Hölder space $C^{1,\alpha}(G)$ of uniformly Hölder continuously differentiable functions as the space of differentiable functions φ for which $\operatorname{grad}\varphi$ (or the surface gradient $\operatorname{Grad}\varphi$ in the case $G = \partial D$) belongs to $C^{0,\alpha}(G)$. With the norm

$$\|\varphi\|_{1,\alpha} := \|\varphi\|_{1,\alpha,G} := \|\varphi\|_\infty + \|\operatorname{grad}\varphi\|_{0,\alpha}$$

the Hölder space $C^{1,\alpha}(G)$ is again a Banach space and we also have an imbedding theorem corresponding to Theorem 3.2.

Extending Theorem 3.1, we can now formulate the following regularity properties of single- and double-layer potentials in terms of Hölder continuity.

Theorem 3.3. *Let ∂D be of class C^2 and let $0 < \alpha < 1$. Then the single-layer potential u with density $\varphi \in C(\partial D)$ is uniformly Hölder continuous throughout $\mathbb{R}^3$ and*

$$\|u\|_{\alpha,\mathbb{R}^3} \le C_\alpha \|\varphi\|_{\infty,\partial D}.$$

The first derivatives of the single-layer potential u with density $\varphi \in C^{0,\alpha}(\partial D)$ can be uniformly Hölder continuously extended from D to $\bar{D}$ and from $\mathbb{R}^3 \setminus \bar{D}$ to $\mathbb{R}^3 \setminus D$ with boundary values

$$\operatorname{grad} u_\pm(x) = \int_{\partial D} \varphi(y) \operatorname{grad}_x \Phi(x,y)\, ds(y) \mp \frac{1}{2}\varphi(x)\nu(x), \quad x \in \partial D, \tag{3.7}$$

where

$$\operatorname{grad} u_\pm(x) := \lim_{h\to+0} \operatorname{grad} u(x \pm h\nu(x))$$

and we have

$$\|\operatorname{grad} u\|_{\alpha,\overline{D}} \le C_\alpha \|\varphi\|_{\alpha,\partial D}, \quad \|\operatorname{grad} u\|_{\alpha,\mathbb{R}^3\setminus D} \le C_\alpha \|\varphi\|_{\alpha,\partial D}.$$

The double-layer potential v with density $\varphi \in C^{0,\alpha}(\partial D)$ can be uniformly Hölder continuously extended from D to $\bar{D}$ and from $\mathbb{R}^3 \setminus \bar{D}$ to $\mathbb{R}^3 \setminus D$ such that

$$\|v\|_{\alpha,\overline{D}} \le C_\alpha \|\varphi\|_{\alpha,\partial D}, \quad \|v\|_{\alpha,\mathbb{R}^3\setminus D} \le C_\alpha \|\varphi\|_{\alpha,\partial D}.$$

The first derivatives of the double-layer potential v with density $\varphi \in C^{1,\alpha}(\partial D)$ can be uniformly Hölder continuously extended from D to $\bar{D}$ and from $\mathbb{R}^3 \setminus \bar{D}$ to $\mathbb{R}^3 \setminus D$ such that

$$\|\operatorname{grad} v\|_{\alpha,\overline{D}} \le C_\alpha \|\varphi\|_{1,\alpha,\partial D}, \quad \|\operatorname{grad} v\|_{\alpha,\mathbb{R}^3\setminus D} \le C_\alpha \|\varphi\|_{1,\alpha,\partial D}.$$

In all inequalities, C_α denotes some constant depending on ∂D and α.

Proof. For a proof, we refer to the Theorems 2.12, 2.16, 2.17 and 2.23 in [64]. □

For the direct values of the single- and double-layer potentials on the boundary ∂D, we have more regularity. This can be conveniently expressed in terms of the mapping properties of the single- and double-layer operators S and K, given by

$$(S\varphi)(x) := 2\int_{\partial D} \Phi(x,y)\varphi(y)\,ds(y), \quad x \in \partial D, \tag{3.8}$$

$$(K\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\,\varphi(y)\,ds(y), \quad x \in \partial D, \tag{3.9}$$

and the normal derivative operators K' and T, given by

$$(K'\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\,\varphi(y)\,ds(y), \quad x \in \partial D, \tag{3.10}$$

$$(T\varphi)(x) := 2\,\frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\,\varphi(y)\,ds(y), \quad x \in \partial D. \tag{3.11}$$

Theorem 3.4. *Let ∂D be of class C^2. Then the operators S, K and K' are bounded operators from $C(\partial D)$ into $C^{0,\alpha}(\partial D)$, the operators S and K are also bounded from $C^{0,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$, and the operator T is bounded from $C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$.*

Proof. The statements on S and T are contained in the preceding theorem and proofs for the operators K and K' can be found in Theorems 2.15, 2.22, and 2.30 of [64]. □

We wish to point out that all these jump and regularity properties essentially are deduced from the corresponding results for the classical single- and double-layer potentials for the Laplace equation by smoothness arguments on the difference between the fundamental solutions for the Helmholtz and the Laplace equation.

Clearly, by interchanging the order of integration, we see that S is self-adjoint and K and K' are adjoint with respect to the bilinear form

$$\langle \varphi, \psi \rangle := \int_{\partial D} \varphi\psi\,ds,$$

that is,

$$\langle S\varphi, \psi \rangle = \langle \varphi, S\psi \rangle \quad \text{and} \quad \langle K\varphi, \psi \rangle = \langle \varphi, K'\psi \rangle$$

for all $\varphi, \psi \in C(\partial D)$. To derive further properties of the boundary integral operators, let u and v denote the double-layer potentials with densities φ and ψ in $C^{1,\alpha}(\partial D)$, respectively. Then by the jump relations of Theorem 3.1, Green's theorem (2.3) and the radiation condition we find that

$$\int_{\partial D} T\varphi\,\psi\,ds = 2\int_{\partial D} \frac{\partial u}{\partial \nu}\,(v_+ - v_-)\,ds = 2\int_{\partial D} (u_+ - u_-)\,\frac{\partial v}{\partial \nu}\,ds = \int_{\partial D} \varphi T\psi\,ds,$$

that is, T also is self-adjoint. Now, in addition, let w denote the single-layer potential with density $\varphi \in C(\partial D)$. Then

$$\int_{\partial D} S\varphi\,T\psi\,ds = 4\int_{\partial D} w\,\frac{\partial v}{\partial \nu}\,ds = 4\int_{\partial D} v_-\,\frac{\partial w_-}{\partial \nu}\,ds = \int_{\partial D} (K - I)\psi(K' + I)\varphi\,ds,$$

whence

$$\int_{\partial D} \varphi\, ST\psi\, ds = \int_{\partial D} \varphi(K^2 - I)\psi\, ds$$

follows for all $\varphi \in C(\partial D)$ and $\psi \in C^{1,\alpha}(\partial D)$. Thus, we have proven the relation

$$ST = K^2 - I \tag{3.12}$$

and similarly it can be shown that the adjoint relation

$$TS = K'^2 - I \tag{3.13}$$

is also valid. Throughout the book I stands for the identity operator.

Looking at the regularity and mapping properties of surface potentials, we think it is natural to start with the classical Hölder space case. As worked out in detail by Kirsch [178], the corresponding results in the Sobolev space setting can be deduced from these classical results through the use of a functional analytic tool provided by Lax [219], that is, the classical results are stronger. Since we shall be referring to Lax's theorem several times in the sequel, we prove it here.

Theorem 3.5. *Let X and Y be normed spaces both of which are equipped with a scalar product $(\cdot\,,\cdot)$ and assume that there exists a positive constant c such that*

$$|(\varphi, \psi)| \le c\|\varphi\|\,\|\psi\| \tag{3.14}$$

for all $\varphi, \psi \in X$. Let $U \subset X$ be a subspace and let $A : U \to Y$ and $B : Y \to X$ be bounded linear operators satisfying

$$(A\varphi, \psi) = (\varphi, B\psi) \tag{3.15}$$

for all $\varphi \in U$ and $\psi \in Y$. Then $A : U \to Y$ is bounded with respect to the norms induced by the scalar products.

Proof. We denote the norms induced by the scalar products by $\|\cdot\|_s$. Consider the bounded operator $M : U \to X$ given by $M := BA$ with $\|M\| \le \|B\|\,\|A\|$. Then, as a consequence of (3.15), M is self-adjoint, that is, $(M\varphi, \psi) = (\varphi, M\psi)$ for all $\varphi, \psi \in U$. Therefore, using the Cauchy–Schwarz inequality, we obtain

$$\|M^n\varphi\|_s^2 = (M^n\varphi, M^n\varphi) = (\varphi, M^{2n}\varphi) \le \|M^{2n}\varphi\|_s$$

for all $\varphi \in U$ with $\|\varphi\|_s \le 1$ and all $n \in \mathbb{N}$. From this, by induction, it follows that

$$\|M\varphi\|_s \le \|M^{2^n}\varphi\|_s^{2^{-n}}.$$

By (3.14) we have $\|\varphi\|_s \le \sqrt{c}\,\|\varphi\|$ for all $\varphi \in X$. Hence,

$$\|M\varphi\|_s \le \left\{\sqrt{c}\,\|M^{2^n}\varphi\|\right\}^{2^{-n}} \le \left\{\sqrt{c}\,\|\varphi\|\,\|M\|^{2^n}\right\}^{2^{-n}} = \left\{\sqrt{c}\,\|\varphi\|\right\}^{2^{-n}} \|M\|.$$

Passing to the limit $n \to \infty$ now yields

$$\|M\varphi\|_s \le \|M\|$$

for all $\varphi \in U$ with $\|\varphi\|_s \le 1$. Finally, for all $\varphi \in U$ with $\|\varphi\|_s \le 1$, we again have from the Cauchy–Schwarz inequality that

$$\|A\varphi\|_s^2 = (A\varphi, A\varphi) = (\varphi, M\varphi) \le \|M\varphi\|_s \le \|M\|.$$

From this the statement follows. □

We now use Lax's Theorem 3.5 to prove the mapping properties of surface potentials in Sobolev spaces. For an introduction into the classical Sobolev spaces $H^1(D)$ and $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ for domains and the Sobolev spaces $H^p(\partial D)$, $p \in \mathbb{R}$, on the boundary ∂D we refer to Adams [2] and McLean [240]. We note that $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ is the space of all functions $u : \mathbb{R}^3 \setminus \bar{D} \to \mathbb{C}$ such that $u \in H^1((\mathbb{R}^3 \setminus \bar{D}) \cup B)$ for all open balls containing $\bar{D}$. For an introduction of the spaces $H^p(\partial D)$ in two dimensions using a Fourier series approach we also refer to [205].

Theorem 3.6. *Let ∂D be of class C^2 and let $H^1(\partial D)$ denote the usual Sobolev space. Then the operator S is bounded from $L^2(\partial D)$ into $H^1(\partial D)$. Assume further that ∂D belongs to $C^{2,\alpha}$. Then the operators K and K' are bounded from $L^2(\partial D)$ into $H^1(\partial D)$ and the operator T is bounded from $H^1(\partial D)$ into $L^2(\partial D)$.*

Proof. We prove the boundedness of $S : L^2(\partial D) \to H^1(\partial D)$. Let $X = C^{0,\alpha}(\partial D)$ and $Y = C^{1,\alpha}(\partial D)$ be equipped with the usual Hölder norms and introduce scalar products on X by the L^2 scalar product and on Y by the H^1 scalar product

$$(u, v)_{H^1(\partial D)} := \int_{\partial D} \{\varphi\bar{\psi} + \operatorname{Grad}\varphi \cdot \operatorname{Grad}\bar{\psi}\}\, ds.$$

By interchanging the order of integration, we have

$$\int_{\partial D} S\varphi\,\psi\, ds = \int_{\partial D} \varphi\, S\psi\, ds \tag{3.16}$$

for all $\varphi, \psi \in C(\partial D)$. For $\varphi \in C^{0,\alpha}(\partial D)$ and $\psi \in C^2(\partial D)$, by Gauss' surface divergence theorem and (3.16) we have

$$\int_{\partial D} \operatorname{Grad} S\varphi \cdot \operatorname{Grad}\psi\, ds = -\int_{\partial D} \varphi\, S\,(\operatorname{Div}\operatorname{Grad}\psi)\, ds. \tag{3.17}$$

(For the reader who is not familiar with vector analysis on surfaces, we refer to Section 6.3.) Using again Gauss' surface divergence theorem and the relation $\operatorname{grad}_x \Phi(x, y) = -\operatorname{grad}_y \Phi(x, y)$, we find that

$$\int_{\partial D} \Phi(x, y)\operatorname{Div}\operatorname{Grad}\psi(y)\, ds(y) = \operatorname{div}\int_{\partial D} \Phi(x, y)\operatorname{Grad}\psi(y)\, ds(y), \quad x \notin \partial D.$$

Hence, with the aid of the jump relations of Theorem 3.1 and 3.3 (see also Theorem 6.13), for $\psi \in C^2(\partial D)$ we obtain

$$S\,(\mathrm{Div}\,\mathrm{Grad}\,\psi) = \tilde{S}\,(\mathrm{Grad}\,\psi)$$

where the bounded operator $\tilde{S} : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ is given by

$$(\tilde{S}\,a)(x) := 2\,\mathrm{div} \int_{\partial D} \Phi(x,y)a(y)\,ds(y), \quad x \in \partial D,$$

for Hölder continuous tangential fields a on ∂D. Therefore, from (3.17) we have

$$\int_{\partial D} \mathrm{Grad}\,S\varphi \cdot \mathrm{Grad}\,\psi\,ds = -\int_{\partial D} \varphi\,\tilde{S}\,(\mathrm{Grad}\,\psi)\,ds \tag{3.18}$$

for all $\varphi \in C^{0,\alpha}(\partial D)$ and $\psi \in C^2(\partial D)$. Since, for fixed φ, both sides of (3.18) represent bounded linear functionals on $C^{1,\alpha}(\partial D)$, (3.18) is also true for all $\varphi \in C^{0,\alpha}(\partial D)$ and $\psi \in C^{1,\alpha}(\partial D)$. Hence, from (3.16) and (3.18) we have that the operators $S : C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ and $S^* : C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ given by

$$S^*\psi := \overline{S\bar{\psi}} - \overline{\tilde{S}\;\mathrm{Grad}\,\bar{\psi}}$$

are adjoint, i.e.,

$$(S\varphi,\psi)_{H^1(\partial D)} = (\varphi, S^*\psi)_{L^2(\partial D)}$$

for all $\varphi \in C^{0,\alpha}(\partial D)$ and $\psi \in C^{1,\alpha}(\partial D)$. By Theorem 3.3, both S and S^* are bounded with respect to the Hölder norms. Hence, from Lax's Theorem 3.5 we see that there exists a positive constant C such that

$$\|S\varphi\|_{H^1(\partial D)} \le C\|\varphi\|_{L^2(\partial D)}$$

for all $\varphi \in C^{0,\alpha}(\partial D)$. The proof of the boundedness of $S : L^2(\partial D) \to H^1(\partial D)$ is now finished by observing that $C^{0,\alpha}(\partial D)$ is dense in $L^2(\partial D)$.

The proofs of the assertions on K, K' and T are similar in structure and for details we refer the reader to [178]. □

Corollary 3.7 *If ∂D is of class C^2 then the operator S is bounded from $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$. Assume further that ∂D belongs to $C^{2,\alpha}$. Then the operators K and K' are bounded from $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$ and the operator T is bounded from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$.*

Proof. We prove the statement on S. The L^2 adjoint S^* of S has kernel $2\overline{\Phi(x,y)}$ and therefore also is bounded from $L^2(\partial D)$ into $H^1(\partial D)$. By duality, this implies that S is bounded from $H^{-1}(\partial D)$ into $L^2(\partial D)$. Now the boundedness of $S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ follows by the interpolation property of the Sobolev spaces $H^{1/2}(\partial D)$ (see Theorem 8.13 in [205]). The proofs of the assertions on K, K' and T are analogous. □

In view of the compactness of the *imbedding operators* $I^{p,q}$ from $H^p(\partial D)$ into $H^q(\partial D)$ for $p > q$, from Corollary 3.7 we observe that the operators S, K and K' are compact from $H^{-1/2}(\partial D)$ into $H^{-1/2}(\partial D)$ and from $H^{1/2}(\partial D)$ into $H^{1/2}(\partial D)$. For the following corollary we make use of the *trace theorem*, i.e., the boundedness of the boundary *trace operator* in the sense of

$$\|\psi\|_{H^{1/2}(\partial D)} \le C\,\|\psi\|_{H^1(D)} \tag{3.19}$$

for all $\psi \in H^1(D)$ and some positive constant C (see [240]).

Corollary 3.8 *Let ∂D be of class $C^{2,\alpha}$. The single-layer potential defines bounded linear operators from $H^{-1/2}(\partial D)$ into $H^1(D)$ and into $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$. The double-layer potential defines bounded linear operators from $H^{1/2}(\partial D)$ into $H^1(D)$ and into $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$.*

Proof. Let u be the single-layer potential with density $\varphi \in C^{0,\alpha}(\partial D)$. Then, by Green's theorem and the jump relations of Theorem 3.3, we have

$$\int_D \left\{|\operatorname{grad} u|^2 - k^2|u|^2\right\} dx = \int_{\partial D} \bar{u}\,\frac{\partial u}{\partial \nu}\,ds = \frac{1}{4}\int_{\partial D} \overline{S\varphi}\,(\varphi + K'\varphi)\,ds.$$

Therefore, by the preceding Corollary 3.7, we can estimate

$$\|\operatorname{grad} u\|^2_{L^2(D)} - k^2\|u\|^2_{L^2(D)} \le \frac{1}{4}\,\|S\varphi\|_{H^{1/2}(\partial D)}\,\|\varphi + K'\varphi\|_{H^{-1/2}(\partial D)} \le c_1\,\|\varphi\|^2_{H^{-1/2}(\partial D)}$$

for some positive constant c_1. In terms of the volume potential operator V as introduced in Theorem 8.2, interchanging orders of integration we have

$$(u, u)_{L^2(D)} = (\varphi, \overline{V\bar{u}})_{L^2(\partial D)}$$

and estimating with the aid of the trace theorem and the mapping property of Theorem 8.2 for the volume potential operator V yields

$$\|u\|^2_{L^2(D)} \le C\|\varphi\|_{H^{-1/2}(\partial D)}\|\overline{V\bar{u}}\|_{H^1(D)} \le c_2\|\varphi\|_{H^{-1/2}(\partial D)}\|u\|_{L^2(D)}$$

for some positive constant c_2. Now the statement on the single-layer potential for the interior domain D follows by combining the last two inequalities and using the denseness of $C^{0,\alpha}(\partial D)$ in $H^{-1/2}(\partial D)$. The proof carries over to the exterior domain $\mathbb{R}^3 \setminus \bar{D}$ by considering the product χu for some smooth cut-off function χ with compact support.

The case of the double-layer potential v with density φ is treated analogously through using

$$\int_D \left\{|\operatorname{grad} v|^2 - k^2|v|^2\right\} dx = \frac{1}{4}\int_{\partial D} T\bar{\varphi}\,(K\varphi - \varphi)\,ds,$$

which follows from Green's theorem and the jump relations, and

$$(v, v)_{L^2(D)} = \left(\varphi, \overline{\frac{\partial}{\partial \nu} V \bar{v}}\right)_{L^2(\partial D)}$$

which is obtained by interchanging orders of integration. □

Finally we note that the above analysis also implies that the jump relations for the boundary trace and the normal derivative trace of the single- and double-layer potential remain valid in the Sobolev space setting. For a different approach to proving Theorem 3.6 and its two corollaries we refer to McLean [240] and to Nédélec [260].

The jump relations of Theorem 3.1 can also be extended through the use of Lax's theorem from the case of continuous densities to L^2 densities. This was done by Kersten [175]. In the L^2 setting, the jump relations (3.1)–(3.4) have to be replaced by

$$\lim_{h\to+0} \int_{\partial D} |2u(x \pm h\nu(x)) - (S\varphi)(x)|^2 ds(x) = 0, \tag{3.20}$$

$$\lim_{h\to+0} \int_{\partial D} \left|2\,\frac{\partial u}{\partial \nu}\,(x \pm h\nu(x)) - (K'\varphi)(x) \pm \varphi(x)\right|^2 ds(x) = 0 \tag{3.21}$$

for the single-layer potential u with density $\varphi \in L^2(\partial D)$ and

$$\lim_{h\to+0} \int_{\partial D} |2v(x \pm h\nu(x)) - (K\varphi)(x) \mp \varphi(x)|^2 ds(x) = 0, \tag{3.22}$$

$$\lim_{h\to+0} \int_{\partial D} \left|\frac{\partial v}{\partial \nu}\,(x + h\nu(x)) - \frac{\partial v}{\partial \nu}\,(x - h\nu(x))\right|^2 ds(x) = 0 \tag{3.23}$$

for the double-layer potential v with density $\varphi \in L^2(\partial D)$. Using Lax's theorem, Hähner [121] has also established that

$$\lim_{h\to+0} \int_{\partial D} \left|\operatorname{grad} u(\cdot \pm h\nu) - \int_{\partial D} \operatorname{grad}_x \Phi(\cdot\,, y)\varphi(y)\,ds(y) \pm \frac{1}{2}\,\varphi\nu\right|^2 ds = 0 \tag{3.24}$$

for single-layer potentials u with $L^2(\partial D)$ density φ, extending the jump relation (3.7).

3.2 Scattering from a Sound-Soft Obstacle

The scattering of time-harmonic acoustic waves by sound-soft obstacles leads to the following problem.

Direct Acoustic Obstacle Scattering Problem. *Given an entire solution u^i to the Helmholtz equation representing an incident field, find a solution*

$$u = u^i + u^s$$

to the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ such that the scattered field u^s satisfies the Sommerfeld radiation condition and the total field u satisfies the boundary condition

$$u = 0 \quad \text{on } \partial D.$$

Clearly, after renaming the unknown functions, this direct scattering problem is a special case of the following Dirichlet problem.

Exterior Dirichlet Problem. *Given a continuous function f on ∂D, find a radiating solution $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ to the Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}$$

which satisfies the boundary condition

$$u = f \quad \text{on } \partial D.$$

We briefly sketch uniqueness, existence and well-posedness for this boundary value problem.

Theorem 3.9. *The exterior Dirichlet problem has at most one solution.*

Proof. We have to show that solutions to the homogeneous boundary value problem $u = 0$ on ∂D vanish identically. If u had a normal derivative in the sense of uniform convergence, we could immediately apply Theorem 2.13 to obtain $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. However, in our formulation of the exterior Dirichlet problem we require u only to be continuous up to the boundary which is the natural assumption for posing the Dirichlet boundary condition. There are two possibilities to overcome this difficulty: either we can use the fact that the solution to the Dirichlet problem belongs to $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$ provided the given boundary data is in $C^{1,\alpha}(\partial D)$ (c.f. [64] or [227]), or we can justify the application of Green's theorem by a more direct argument using convergence theorems for Lebesgue integration. Despite the fact that later we will also need the result on the smoothness of solutions to the exterior Dirichlet problem up to the boundary, we briefly sketch a variant of the second alternative based on an approximation idea due to Heinz (see [104] and also [325] and [169], p. 144). It is more satisfactory since it does not rely on techniques used in the existence results. Thus, we state and prove the following lemma which then justifies the application of Theorem 2.13. Note that this uniqueness result for the Dirichlet problem requires no regularity assumptions on the boundary ∂D. □

Lemma 3.10 *Let $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ be a solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ which satisfies the homogeneous boundary condition $u = 0$ on ∂D. Define $D_R := \{y \in \mathbb{R}^3 \setminus \bar{D} : |y| < R\}$ and $S_R := \{y \in \mathbb{R}^3 : |y| = R\}$ for sufficiently large R. Then* $\operatorname{grad} u \in L^2(D_R)$ *and*

$$\int_{D_R} |\operatorname{grad} u|^2 \, dx - k^2 \int_{D_R} |u|^2 dx = \int_{S_R} u \, \frac{\partial \bar{u}}{\partial \nu} \, ds. \tag{3.25}$$

Proof. We first assume that u is real valued. We choose an odd function $\psi \in C^1(\mathbb{R})$ such that $\psi(t) = 0$ for $0 \le t \le 1$, $\psi(t) = t$ for $t \ge 2$ and $\psi'(t) \ge 0$ for all t, and set $u_n := \psi(nu)/n$. We then have uniform convergence $\|u - u_n\|_\infty \to 0$, $n \to \infty$. Since $u = 0$ on the boundary ∂D, the functions u_n vanish in a neighborhood of ∂D and we can apply Green's theorem (2.2) to obtain

$$\int_{D_R} \operatorname{grad} u_n \cdot \operatorname{grad} u \, dx = k^2 \int_{D_R} u_n u \, dx + \int_{S_R} u_n \, \frac{\partial u}{\partial \nu} \, ds.$$

It can be easily seen that

$$0 \le \operatorname{grad} u_n(x) \cdot \operatorname{grad} u(x) = \psi'(nu(x)) \, |\operatorname{grad} u(x)|^2 \to |\operatorname{grad} u(x)|^2, \quad n \to \infty,$$

for all x not contained in $\{x \in D_R : u(x) = 0, \ \operatorname{grad} u(x) \neq 0\}$. Since as a consequence of the implicit function theorem the latter set has Lebesgue measure zero, Fatou's lemma tells us that $\operatorname{grad} u \in L^2(D_R)$.

Now assume $u = v + i\,w$ with real functions v and w. Then, since v and w also satisfy the assumptions of our lemma, we have $\operatorname{grad} v$, $\operatorname{grad} w \in L^2(D_R)$. From

$$\operatorname{grad} v_n + i \ \operatorname{grad} w_n = \psi'(nv) \ \operatorname{grad} v + i\,\psi'(nw) \ \operatorname{grad} w$$

we can estimate

$$|(\operatorname{grad} v_n + i \ \operatorname{grad} w_n) \cdot \operatorname{grad} \bar{u}| \le 2\|\psi'\|_\infty \left\{ |\operatorname{grad} v|^2 + |\operatorname{grad} w|^2 \right\}.$$

Hence, by the Lebesgue dominated convergence theorem, we can pass to the limit $n \to \infty$ in Green's theorem

$$\int_{D_R} \{(\operatorname{grad} v_n + i \ \operatorname{grad} w_n) \cdot \operatorname{grad} \bar{u} + (v_n + i w_n) \Delta \bar{u}\} \, dx = \int_{S_R} (v_n + i\,w_n) \, \frac{\partial \bar{u}}{\partial \nu} \, ds$$

to obtain (3.25). □

The existence of a solution to the exterior Dirichlet problem can be based on boundary integral equations. In the so-called *layer approach*, we seek the solution in the form of acoustic surface potentials. Here, we choose an approach in the form of a combined acoustic double- and single-layer potential

$$u(x) = \int_{\partial D} \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\eta \Phi(x,y) \right\} \varphi(y) \, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D, \tag{3.26}$$

with a density $\varphi \in C(\partial D)$ and a real coupling parameter $\eta \neq 0$. Then from the jump relations of Theorem 3.1 we see that the potential u given by (3.26) in $\mathbb{R}^3 \setminus \bar{D}$ solves the exterior Dirichlet problem provided the density is a solution of the integral equation

$$\varphi + K\varphi - i\eta S\varphi = 2f. \tag{3.27}$$

Combining Theorems 3.2 and 3.4, the operators $S, K : C(\partial D) \to C(\partial D)$ are seen to be compact. Therefore, the existence of a solution to (3.27) can be established by the Riesz–Fredholm theory for equations of the second kind with a compact operator.

Let φ be a continuous solution to the homogeneous form of (3.27). Then the potential u given by (3.26) satisfies the homogeneous boundary condition $u_+ = 0$ on ∂D whence by the uniqueness for the exterior Dirichlet problem $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows. The jump relations (3.1)–(3.4) now yield

$$-u_- = \varphi, \quad -\frac{\partial u_-}{\partial \nu} = i\eta\varphi \quad \text{on } \partial D.$$

Hence, using Green's theorem (2.2), we obtain

$$i\eta \int_{\partial D} |\varphi|^2 ds = \int_{\partial D} \bar{u}_- \frac{\partial u_-}{\partial \nu}\, ds = \int_D \left\{ |\operatorname{grad} u|^2 - k^2|u|^2 \right\} dx. \tag{3.28}$$

Taking the imaginary part of the last equation shows that $\varphi = 0$. Thus, we have established uniqueness for the integral equation (3.27), i.e., injectivity of the operator $I + K - i\eta S : C(\partial D) \to C(\partial D)$. Therefore, by the Riesz–Fredholm theory, $I + K - i\eta S$ is bijective and the inverse $(I + K - i\eta S)^{-1} : C(\partial D) \to C(\partial D)$ is bounded. Hence, the inhomogeneous equation (3.27) possesses a solution and this solution depends continuously on f in the maximum norm. From the representation (3.26) of the solution as a combined double- and single-layer potential, with the aid of the regularity estimates in Theorem 3.1, the continuous dependence of the density φ on the boundary data f shows that the exterior Dirichlet problem is well-posed, i.e., small deviations in f in the maximum norm ensure small deviations in u in the maximum norm on $\mathbb{R}^3 \setminus D$ and small deviations of all its derivatives in the maximum norm on closed subsets of $\mathbb{R}^3 \setminus \bar{D}$.

We summarize these results in the following theorem.

Theorem 3.11. *The exterior Dirichlet problem has a unique solution and the solution depends continuously on the boundary data with respect to uniform convergence of the solution on* $\mathbb{R}^3 \setminus D$ *and all its derivatives on closed subsets of* $\mathbb{R}^3 \setminus \bar{D}$.

Note that for $\eta = 0$ the integral equation (3.27) becomes non-unique if k is a so-called irregular wave number or internal resonance, i.e., if there exist nontrivial solutions u to the Helmholtz equation in the interior domain D satisfying homogeneous Neumann boundary conditions $\partial u/\partial \nu = 0$ on ∂D. The approach (3.26) was introduced independently by Brakhage and Werner [26], Leis [223], and Panich [268] in order to remedy this non-uniqueness deficiency of the classical double-layer approach due to Vekua [322] and Weyl [331]. For an investigation on the proper choice of the coupling parameter η with respect to the condition of the integral equation (3.27), we refer to Kress [197] and Chandler-Wilde, Graham, Langdon, and Lindner [47].

In the literature, a variety of other devices have been designed for overcoming the non-uniqueness difficulties of the double-layer integral equation. The combined single- and double-layer approach seems to be the most attractive method from a theoretical point of view since its analysis is straightforward as well as from a numerical point of view since it never fails and can be implemented without additional computational cost as compared with the double-layer approach.

In order to be able to use Green's representation formula for the solution of the exterior Dirichlet problem, we need its normal derivative. However, assuming the given boundary values to be merely continuous means that in general the normal derivative will not exist. Hence, we need to impose some additional smoothness condition on the boundary data.

From Theorems 3.2 and 3.4 we also have compactness of the operators $S, K : C^{1,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$. Hence, by the Riesz–Fredholm theory, the injective operator $I + K - i\eta S : C^{1,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ again has a bounded inverse $(I + K - i\eta S)^{-1} : C^{1,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$. Therefore, given a right hand side f in $C^{1,\alpha}(\partial D)$, the solution φ of the integral equation (3.27) belongs to $C^{1,\alpha}(\partial D)$ and depends continuously on f in the $\|\cdot\|_{1,\alpha}$ norm. Using the regularity results of Theorem 3.3 for the derivatives of single- and double-layer potentials, from (3.26) we now find that u belongs to $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$ and depends continuously on f. In particular, the normal derivative $\partial u/\partial\nu$ of the solution u exists and belongs to $C^{0,\alpha}(\partial D)$ if $f \in C^{1,\alpha}(\partial D)$ and is given by

$$\frac{\partial u}{\partial \nu} = \mathcal{A}f$$

where

$$\mathcal{A} := (i\eta I - i\eta K' + T)(I + K - i\eta S)^{-1} : C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$$

is bounded. The operator $\mathcal{A}$ transfers the boundary values, i.e., the Dirichlet data, into the normal derivative, i.e., the Neumann data, and therefore it is called the *Dirichlet to Neumann map*.

For the sake of completeness, we wish to show that $\mathcal{A}$ is bijective and has a bounded inverse. This is equivalent to showing that

$$i\eta I - i\eta K' + T : C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$$

is bijective and has a bounded inverse. Since T is not compact, the Riesz–Fredholm theory cannot be employed in a straightforward manner. In order to regularize the operator, we first examine the exterior Neumann problem.

Exterior Neumann Problem. *Given a continuous function g on ∂D, find a radiating solution $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ to the Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}$$

which satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} = g \quad \text{on } \partial D$$

in the sense of uniform convergence on ∂D*.*

The exterior Neumann problem describes acoustic scattering from sound-hard obstacles. Uniqueness for the Neumann problem follows from Theorem 2.13. To prove existence we again use a combined single- and double-layer approach. We overcome the problem that the normal derivative of the double-layer potential in general does not exist if the density is merely continuous by incorporating a smoothing operator, that is, we seek the solution in the form

$$u(x) = \int_{\partial D} \left\{ \Phi(x,y)\,\varphi(y) + i\eta\,\frac{\partial \Phi(x,y)}{\partial \nu(y)}\,(S_0^2\varphi)(y) \right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{3.29}$$

with continuous density φ and a real coupling parameter $\eta \neq 0$. By S_0 we denote the single-layer operator (3.8) in the potential theoretic limit case $k = 0$. Note that by Theorem 3.4 the density $S_0^2\varphi$ of the double-layer potential belongs to $C^{1,\alpha}(\partial D)$. The idea of using a smoothing operator as in (3.29) was first suggested by Panich [268]. From Theorem 3.1 we see that (3.29) solves the exterior Neumann problem provided the density is a solution of the integral equation

$$\varphi - K'\varphi - i\eta T S_0^2 \varphi = -2g. \tag{3.30}$$

By Theorems 3.2 and 3.4 both $K' + i\eta T S_0^2 : C(\partial D) \to C(\partial D)$ and $K' + i\eta T S_0^2 : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ are compact. Hence, the Riesz–Fredholm theory is available in both spaces.

Let φ be a continuous solution to the homogeneous form of (3.30). Then the potential u given by (3.29) satisfies the homogeneous Neumann boundary condition $\partial u_+/\partial\nu = 0$ on ∂D whence by the uniqueness for the exterior Neumann problem $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows. The jump relations (3.1)–(3.4) now yield

$$-u_- = i\eta S_0^2\varphi, \quad -\frac{\partial u_-}{\partial \nu} = -\varphi \quad \text{on } \partial D$$

and, by interchanging the order of integration and using Green's integral theorem as above in the proof for the Dirichlet problem, we obtain

$$i\eta \int_{\partial D} |S_0\varphi|^2 ds = i\eta \int_{\partial D} \varphi S_0^2 \bar{\varphi}\, ds = \int_{\partial D} \bar{u}_- \frac{\partial u_-}{\partial \nu}\, ds = \int_D \left\{ |\operatorname{grad} u|^2 - k^2 |u|^2 \right\} dx$$

whence $S_0\varphi = 0$ on ∂D follows. The single-layer potential w with density φ and wave number $k = 0$ is continuous throughout $\mathbb{R}^3$, harmonic in D and in $\mathbb{R}^3 \setminus \bar{D}$ and vanishes on ∂D and at infinity. Therefore, by the maximum-minimum principle for harmonic functions, we have $w = 0$ in $\mathbb{R}^3$ and the jump relation (3.2) yields $\varphi = 0$. Thus, we have established injectivity of the operator $I - K' - i\eta T S_0^2$ and, by the Riesz–Fredholm theory, $(I - K' - i\eta T S_0^2)^{-1}$ exists and is bounded in both $C(\partial D)$ and $C^{0,\alpha}(\partial D)$. From this we conclude the existence of the solution to the Neumann problem for continuous boundary data g and the continuous dependence of the solution on the boundary data.

Theorem 3.12. *The exterior Neumann problem has a unique solution and the solution depends continuously on the boundary data with respect to uniform convergence of the solution on $\mathbb{R}^3 \setminus D$ and all its derivatives on closed subsets of $\mathbb{R}^3 \setminus \bar{D}$.*

In the case when $g \in C^{0,\alpha}(\partial D)$, the solution φ to the integral equation (3.30) belongs to $C^{0,\alpha}(\partial D)$ and depends continuously on g in the norm of $C^{0,\alpha}(\partial D)$. Using the regularity results of Theorem 3.3 for the single- and double-layer potentials, from (3.29) we now find that u belongs to $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$. In particular, the boundary values u on ∂D are given by

$$u = \mathcal{B}g$$

where

$$\mathcal{B} = (i\eta S_0^2 + i\eta K S_0^2 + S)(K' - I + i\eta T S_0^2)^{-1} : C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$$

is bounded. Clearly, the operator $\mathcal{B}$ is the inverse of $\mathcal{A}$. Thus, we can summarize our regularity analysis as follows.

Theorem 3.13. *The Dirichlet to Neumann map $\mathcal{A}$ which transfers the boundary values of a radiating solution to the Helmholtz equation into its normal derivative is a bijective bounded operator from $C^{1,\alpha}(\partial D)$ onto $C^{0,\alpha}(\partial D)$ with bounded inverse. The solution to the exterior Dirichlet problem belongs to $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$ if the boundary values are in $C^{1,\alpha}(\partial D)$ and the mapping of the boundary data into the solution is continuous from $C^{1,\alpha}(\partial D)$ into $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$.*

Instead of looking for classical solutions in the spaces of continuous or Hölder continuous functions one can also pose and solve the boundary value problems for the Helmholtz equation in a weak formulation for the boundary condition either in an L^2 sense or in a Sobolev space setting. This then leads to existence results under weaker regularity assumptions on the given boundary data and to continuous dependence in different norms. The latter, in particular, can be useful in the error analysis for approximate solution methods.

In the Sobolev space setting, the solution to the exterior Dirichlet problem is required to belong to the energy space $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ and the boundary condition $u = f$ on ∂D for a given $f \in H^{1/2}(\partial D)$ has to be understood in the sense of the trace operator. This simplifies the uniqueness issue since the identity (3.25) is obvious for functions in $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$. The existence analysis via the combined double- and single-layer potential (3.26) with a density $\varphi \in H^{1/2}(\partial D)$ and the integral equation (3.27) can be carried over in a straightforward manner. For the exterior Neumann problem the boundary condition $\partial u/\partial \nu = g$ on ∂D for $g \in H^{-1/2}(\partial D)$ has to be understood in the sense of the second trace operator. Again the existence analysis via the combined single- and double-layer potential (3.29) with a density $\varphi \in H^{-1/2}(\partial D)$

and the integral equation (3.30) carries over. Corollary 3.7 implies well-posedness in the sense that the mapping from the boundary values $f \in H^{1/2}(\partial D)$ onto the solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ is continuous. Further, we note that analogous to Theorem 3.13 the Dirichlet to Neumann map $\mathcal{A}$ is a bijective bounded operator from $H^{1/2}(\partial D)$ onto $H^{-1/2}(\partial D)$ with a bounded inverse.

A major drawback of the integral equation approach to constructively proving existence of solutions for scattering problems is the relatively strong regularity assumption on the boundary to be of class C^2. It is possible to slightly weaken the regularity and allow *Lyapunov boundaries* instead of C^2 boundaries and still remain within the framework of compact operators in the spaces of Hölder continuous functions. The boundary is said to satisfy a Lyapunov condition if at each point $x \in \partial D$ the normal vector ν to the surface exists and if there are positive constants L and α such that for the angle $\theta(x, y)$ between the normal vectors at x and y there holds $\theta(x, y) \le L|x - y|^\alpha$ for all $x, y \in \partial D$. For the treatment of the Dirichlet problem for Lyapunov boundaries, which does not differ essentially from that for C^2 boundaries, we refer to Mikhlin [242].

However, the situation changes considerably if the boundary is allowed to have edges and corners since this affects the compactness of the double-layer integral operator in the space of continuous functions. Here, under suitable assumptions on the nature of the edges and corners, the double-layer integral operator can be decomposed into the sum of a compact operator and a bounded operator with norm less than one reflecting the behavior at the edges and corners, and then the Riesz–Fredholm theory still can be employed. For details, we refer to Section 3.5 for the two-dimensional case. Resorting to single-layer potentials in the Sobolev space setting as introduced above is another efficient option to handle edges and corners (see Hsiao and Wendland [148] and McLean [240]).

Explicit solutions for the direct scattering problem are only available for special geometries and special incoming fields. In general, to construct a solution one must resort to numerical methods, for example, the numerical solution of the boundary integral equations. An introduction into numerical approximation for integral equations of the second kind by the Nyström method, collocation method and Galerkin method is contained in [205]. We will describe in some detail Nyström methods for the two- and three-dimensional case at the end of this chapter.

For future reference, we present the solution for the scattering of a plane wave

$$u^i(x) = e^{ik\,x\cdot d}$$

by a sound-soft ball of radius R with center at the origin. The unit vector d describes the direction of propagation of the incoming wave. In view of the Jacobi–Anger expansion (2.46) and the boundary condition $u^i + u^s = 0$, we expect the scattered wave to be given by

$$u^s(x) = -\sum_{n=0}^{\infty} i^n(2n + 1)\,\frac{j_n(kR)}{h_n^{(1)}(kR)}\,h_n^{(1)}(k|x|)\,P_n(\cos\theta) \tag{3.31}$$

where θ denotes the angle between x and d. By the asymptotic behavior (2.38) and (2.39) of the spherical Bessel and Hankel functions for large n, we have

$$\frac{j_n(kR)}{h_n^{(1)}(kR)}\, h_n^{(1)}(k|x|) = O\left(\frac{n!\,(2kR)^n}{(2n+1)!}\,\frac{R^n}{|x|^n}\right), \quad n \to \infty,$$

uniformly on compact subsets of $\mathbb{R}^3 \setminus \{0\}$. Therefore, the series (3.31) is uniformly convergent on compact subsets of $\mathbb{R}^3 \setminus \{0\}$. Hence, by Theorem 2.15 the series represents a radiating field in $\mathbb{R}^3 \setminus \{0\}$, and therefore indeed solves the scattering problem for the sound-soft ball.

For the far field pattern, we see by Theorem 2.16 that

$$u_\infty(\hat{x}) = \frac{i}{k} \sum_{n=0}^{\infty} (2n+1)\, \frac{j_n(kR)}{h_n^{(1)}(kR)}\, P_n(\cos\theta). \tag{3.32}$$

Clearly, as we expect from symmetry reasons, it depends only on the angle θ between the observation direction $\hat{x}$ and the incident direction d.

In general, for the scattering problem the boundary values are as smooth as the boundary since they are given by the restriction of the analytic function u^i to ∂D. In particular, for domains D of class C^2 our regularity analysis shows that the scattered field u^s is in $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$. Therefore, we may apply Green's formula (2.9) with the result

$$u^s(x) = \int_{\partial D} \left\{ u^s(y)\, \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u^s}{\partial \nu}(y)\, \Phi(x,y) \right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}.$$

Green's theorem (2.3), applied to the entire solution u^i and $\Phi(x,\cdot)$, gives

$$0 = \int_{\partial D} \left\{ u^i(y)\, \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u^i}{\partial \nu}(y)\, \Phi(x,y) \right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}.$$

Adding these two equations and using the boundary condition $u^i + u^s = 0$ on ∂D gives the following theorem. The representation for the far field pattern is obtained with the aid of (2.15).

Theorem 3.14. *For the scattering of an entire field u^i from a sound-soft obstacle D we have*

$$u(x) = u^i(x) - \int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, \Phi(x,y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{3.33}$$

and the far field pattern of the scattered field u^s is given by

$$u_\infty(\hat{x}) = -\frac{1}{4\pi} \int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, e^{-ik\,\hat{x}\cdot y}\, ds(y), \quad \hat{x} \in \mathbb{S}^2. \tag{3.34}$$

In physics, the representation (3.33) for the scattered field through the so-called *secondary sources* on the boundary is known as *Huygens' principle*.

We conclude this section by briefly giving the motivation for the *Kirchhoff* or *physical optics approximation* which is frequently used in applications as a physically intuitive procedure to simplify the direct scattering problem. The solution for the scattering of a plane wave with incident direction d at a plane $\Gamma := \{x \in \mathbb{R}^3 : x \cdot \nu = 0\}$ through the origin with normal vector ν is described by

$$u(x) = u^i(x) + u^s(x) = e^{ik\,x\cdot d} - e^{ik\,x\cdot \tilde{d}}$$

where $\tilde{d} = d - 2\,\nu\cdot d\,\nu$ denotes the reflection of d at the plane Γ. Clearly, $u^i + u^s = 0$ is satisfied on Γ and we evaluate

$$\frac{\partial u}{\partial \nu} = ik\{\nu\cdot d\,u^i + \nu\cdot\tilde{d}\,u^s\} = 2ik\,\nu\cdot d\,u^i = 2\,\frac{\partial u^i}{\partial \nu}\,.$$

For large wave numbers k, i.e., for small wavelengths, in a first approximation a convex object D locally may be considered at each point x of ∂D as a plane with normal $\nu(x)$. This leads to setting

$$\frac{\partial u}{\partial \nu} = 2\,\frac{\partial u^i}{\partial \nu}$$

on the region $\partial D_- := \{x \in \partial D : \nu(x)\cdot d < 0\}$ which is illuminated by the plane wave with incident direction d, and

$$\frac{\partial u}{\partial \nu} = 0$$

in the shadow region $\partial D_+ := \{x \in \partial D : \nu(x)\cdot d \geq 0\}$. Thus, the Kirchhoff approximation for the scattering of a plane wave with incident direction d at a convex sound-soft obstacle is approximated by

$$u(x) \approx e^{ik\,x\cdot d} - 2\int_{\partial D_-} \frac{\partial e^{ik\,y\cdot d}}{\partial \nu(y)}\,\Phi(x,y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{3.35}$$

and the far field pattern of the scattered field is approximated by

$$u_\infty(\hat{x}) \approx -\frac{1}{2\pi}\int_{\partial D_-} \frac{\partial e^{ik\,y\cdot d}}{\partial \nu(y)}\,e^{-ik\,\hat{x}\cdot y}\,ds(y), \quad \hat{x} \in \mathbb{S}^2. \tag{3.36}$$

In this book, the Kirchhoff approximation does not play an important role since we are mainly interested in scattering at low and intermediate values of the wave number.

3.3 Herglotz Wave Functions and the Far Field Operator

In the sequel, for an incident plane wave $u^i(x) = u^i(x, d) = e^{ik\,x\cdot d}$ we will indicate the dependence of the scattered field, of the total field and of the far field pattern on the incident direction d by writing, respectively, $u^s(x, d)$, $u(x, d)$ and $u_\infty(\hat{x}, d)$.

Theorem 3.15. *The far field pattern for sound-soft obstacle scattering satisfies the* reciprocity relation

$$u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x}), \quad \hat{x}, d \in \mathbb{S}^2. \tag{3.37}$$

Proof. By Green's theorem (2.3), the Helmholtz equation for the incident and the scattered wave and the radiation condition for the scattered wave we find

$$\int_{\partial D} \left\{ u^i(\cdot\,, d)\, \frac{\partial}{\partial \nu}\, u^i(\cdot\,, -\hat{x}) - u^i(\cdot\,, -\hat{x})\, \frac{\partial}{\partial \nu}\, u^i(\cdot\,, d) \right\} ds = 0$$

and

$$\int_{\partial D} \left\{ u^s(\cdot\,, d)\, \frac{\partial}{\partial \nu}\, u^s(\cdot\,, -\hat{x}) - u^s(\cdot\,, -\hat{x})\, \frac{\partial}{\partial \nu}\, u^s(\cdot\,, d) \right\} ds = 0.$$

From (2.14) we deduce that

$$4\pi u_\infty(\hat{x}, d) = \int_{\partial D} \left\{ u^s(\cdot\,, d)\, \frac{\partial}{\partial \nu}\, u^i(\cdot\,, -\hat{x}) - u^i(\cdot\,, -\hat{x})\, \frac{\partial}{\partial \nu}\, u^s(\cdot\,, d) \right\} ds$$

and, interchanging the roles of $\hat{x}$ and d,

$$4\pi u_\infty(-d, -\hat{x}) = \int_{\partial D} \left\{ u^s(\cdot\,, -\hat{x})\, \frac{\partial}{\partial \nu}\, u^i(\cdot\,, d) - u^i(\cdot\,, d)\, \frac{\partial}{\partial \nu}\, u^s(\cdot\,, -\hat{x}) \right\} ds.$$

We now subtract the last equation from the sum of the three preceding equations to obtain

$$\begin{aligned} &4\pi\{u_\infty(\hat{x}, d) - u_\infty(-d, -\hat{x})\} \\ &= \int_{\partial D} \left\{ u(\cdot\,, d)\, \frac{\partial}{\partial \nu}\, u(\cdot\,, -\hat{x}) - u(\cdot\,, -\hat{x})\, \frac{\partial}{\partial \nu}\, u(\cdot\,, d) \right\} ds \end{aligned} \tag{3.38}$$

whence (3.37) follows by using the boundary condition $u(\cdot\,, d) = u(\cdot\,, -\hat{x}) = 0$ on ∂D. □

In the derivation of (3.38), we only used the Helmholtz equation for the incident field in $\mathbb{R}^3$ and for the scattered field in $\mathbb{R}^3 \setminus \bar{D}$ and the radiation condition. Therefore, we can conclude that the reciprocity relation (3.37) is also valid for the sound-hard, impedance and transmission boundary conditions. It states that the far field pattern is unchanged if the direction of the incident field and the observation directions are interchanged.

For the scattering of a point source $w^i(x,z) = \Phi(x,z)$ located at $z \in \mathbb{R}^3 \setminus \bar{D}$ we denote the scattered field by $w^s(x,z)$, the total field by $w(x,z)$, and the far field pattern of the scattered wave by $w^s_\infty(\hat{x},z)$.

Theorem 3.16. *For obstacle scattering of point sources and plane waves we have the* mixed reciprocity relation

$$4\pi w^s_\infty(-d,z) = u^s(z,d), \quad z \in \mathbb{R}^3 \setminus \bar{D},\ d \in \mathbb{S}^2. \tag{3.39}$$

Proof. The statement follows by combining Green's theorems

$$\int_{\partial D} \left\{ w^i(\cdot\,,z)\,\frac{\partial}{\partial \nu}\,u^i(\cdot\,,d) - u^i(\cdot\,,d)\,\frac{\partial}{\partial \nu}\,w^i(\cdot\,,z) \right\} ds = 0$$

and

$$\int_{\partial D} \left\{ w^s(\cdot\,,z)\,\frac{\partial}{\partial \nu}\,u^s(\cdot\,,d) - u^s(\cdot\,,d)\,\frac{\partial}{\partial \nu}\,u^s(\cdot\,,z) \right\} ds = 0$$

and the representations

$$\int_{\partial D} \left\{ w^s(\cdot\,,z)\,\frac{\partial}{\partial \nu}\,u^i(\cdot\,,d) - u^i(\cdot\,,d)\,\frac{\partial}{\partial \nu}\,w^s(\cdot\,,z) \right\} ds = 4\pi w^s_\infty(-d,z)$$

and

$$\int_{\partial D} \left\{ u^s(\cdot\,,d)\,\frac{\partial}{\partial \nu}\,w^i(\cdot\,,z) - w^i(\cdot\,,z)\,\frac{\partial}{\partial \nu}\,u^s(\cdot\,,d) \right\} ds = u^s(z,d)$$

as in the proof of Theorem 3.15. □

Again the statement of Theorem 3.16 is valid for all boundary conditions. Since the far field pattern Φ_∞ of the incident field Φ is given by

$$\Phi_\infty(d,z) = \frac{1}{4\pi}\,e^{-ik\,d\cdot z}, \tag{3.40}$$

from (3.39) we conclude that

$$w_\infty(d,z) = \frac{1}{4\pi}\,u(z,-d) \tag{3.41}$$

for the far field pattern w_∞ of the total field w.

The proof of the following theorem is analogous to that of the two preceding theorems.

Theorem 3.17. *For obstacle scattering of point sources we have the* symmetry relation

$$w^s(x,y) = w^s(y,x), \quad x,y \in \mathbb{R}^3 \setminus \bar{D}. \tag{3.42}$$

We now ask the question if the far field patterns for a fixed sound-soft obstacle D and all incident plane waves are complete in $L^2(\mathbb{S}^2)$. We call a subset U of a Hilbert space X *complete* if the linear combinations of elements from U are dense in X, that is, if $X = \overline{\operatorname{span} U}$. Recall that U is complete in the Hilbert space X if and only if $(u, \varphi) = 0$ for all $u \in U$ implies that $\varphi = 0$ (see [90]).

Definition 3.18 *A* Herglotz wave function *is a function of the form*

$$v(x) = \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d)\,ds(d), \quad x \in \mathbb{R}^3, \tag{3.43}$$

where $g \in L^2(\mathbb{S}^2)$. *The function* g *is called the Herglotz kernel of* v.

Herglotz wave functions are clearly entire solutions to the Helmholtz equation. We note that for a given $g \in L^2(\mathbb{S}^2)$ the function

$$v(x) = \int_{\mathbb{S}^2} e^{-ik\,x\cdot d} g(d)\,ds(d), \quad x \in \mathbb{R}^3,$$

also defines a Herglotz wave function. The following theorem establishes a one-to-one correspondence between Herglotz wave functions and their kernels.

Theorem 3.19. *Assume that the Herglotz wave function* v *with kernel* g *vanishes in all of* $\mathbb{R}^3$. *Then* $g = 0$.

Proof. From $v(x) = 0$ for all $x \in \mathbb{R}^3$ and the Funk–Hecke formula (2.45), we see that

$$\int_{\mathbb{S}^2} g\, Y_n\, ds = 0$$

for all spherical harmonics Y_n of order $n = 0, 1, \dots$. Now $g = 0$ follows from the completeness of the spherical harmonics (Theorem 2.8). □

Lemma 3.20 *For a given function* $g \in L^2(\mathbb{S}^2)$ *the solution to the scattering problem for the incident wave*

$$v^i(x) = \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d)\,ds(d), \quad x \in \mathbb{R}^3,$$

is given by

$$v^s(x) = \int_{\mathbb{S}^2} u^s(x, d) g(d)\,ds(d), \quad x \in \mathbb{R}^3 \setminus \bar{D},$$

and has the far field pattern

$$v_\infty(\hat{x}) = \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\,ds(d), \quad \hat{x} \in \mathbb{S}^2.$$

Proof. Multiply (3.26) and (3.27) by g, integrate with respect to d over $\mathbb{S}^2$ and interchange orders of integration. □

Now, the rather surprising answer to our completeness question, due to Colton and Kirsch [57], will be that the far field patterns are complete in $L^2(\mathbb{S}^2)$ if and only if there does not exist a nontrivial Herglotz wave function v that vanishes on ∂D. A nontrivial Herglotz wave function that vanishes on ∂D, of course, is a Dirichlet eigenfunction, i.e., a solution to the Dirichlet problem in D with zero boundary condition, and this is peculiar since from physical considerations the eigenfunctions corresponding to the *Dirichlet eigenvalues* of the negative Laplacian in D should have nothing to do with the exterior scattering problem at all.

Theorem 3.21. *Let (d_n) be a sequence of unit vectors that is dense on $\mathbb{S}^2$ and define the set $\mathcal{F}$ of far field patterns by*

$$\mathcal{F} := \{u_\infty(\cdot\,, d_n) : n = 1, 2, \ldots\}.$$

Then $\mathcal{F}$ is complete in $L^2(\mathbb{S}^2)$ if and only if there does not exist a Dirichlet eigenfunction for D which is a Herglotz wave function.

Proof. Deviating from the original proof by Colton and Kirsch [57], we make use of the reciprocity relation. By the continuity of u_∞ as a function of d and Theorem 3.15, the completeness condition

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d_n) h(\hat{x})\, ds(\hat{x}) = 0, \quad n = 1, 2, \ldots,$$

for a function $h \in L^2(\mathbb{S}^2)$ is equivalent to the condition

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\, ds(d) = 0, \quad \hat{x} \in \mathbb{S}^2, \tag{3.44}$$

for $g \in L^2(\mathbb{S}^2)$ with $g(d) = h(-d)$.

By Theorem 3.19 and Lemma 3.20, the existence of a nontrivial function g satisfying (3.44) is equivalent to the existence of a nontrivial Herglotz wave function v^i (with kernel g) for which the far field pattern of the corresponding scattered wave v^s is $v_\infty = 0$. By Theorem 2.14, the vanishing far field $v_\infty = 0$ on $\mathbb{S}^2$ is equivalent to $v^s = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. By the boundary condition $v^i + v^s = 0$ on ∂D and the uniqueness for the exterior Dirichlet problem, this is equivalent to $v^i = 0$ on ∂D and the proof is finished. □

Clearly, by the Funk–Hecke formula (2.45), the spherical wave functions

$$u_n(x) = j_n(k|x|)\, Y_n\left(\frac{x}{|x|}\right)$$

provide examples of Herglotz wave functions. The spherical wave functions also describe Dirichlet eigenfunctions for a ball of radius R centered at the origin with the eigenvalues k^2 given in terms of the zeros $j_n(kR) = 0$ of the spherical Bessel functions. By arguments similar to those used in the proof of Rellich's Lemma 2.12, an expansion with respect to spherical harmonics shows that all the eigenfunctions for a ball are indeed spherical wave functions. Therefore, the eigenfunctions for balls are always Herglotz wave functions and by Theorem 3.21 the far field patterns for plane waves are not complete for a ball D when k^2 is a Dirichlet eigenvalue.

The corresponding completeness results for the transmission problem were given by Kirsch [176] and for the resistive boundary condition by Hettlich [135]. For extensions to Sobolev and Hölder norms we refer to Kirsch [177].

We can also express the result of Theorem 3.21 in terms of a far field operator.

Theorem 3.22. *The far field operator $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ defined by*

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2, \tag{3.45}$$

is injective and has dense range if and only if there does not exist a Dirichlet eigenfunction for D which is a Herglotz wave function.

Proof. For the L^2 adjoint $F^* : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ the reciprocity relation (3.37) implies that

$$F^* g = \overline{RFR\bar{g}}, \tag{3.46}$$

where $R : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is defined by

$$(Rg)(d) := g(-d). \tag{3.47}$$

Hence, the operator F is injective if and only if its adjoint F^* is injective. Observing that in a Hilbert space we have $N(F^*)^\perp = \overline{F(L^2(\mathbb{S}^2))}$ for bounded operators F (see Theorem 4.6), the statement of the corollary is indeed seen to be a reformulation of the preceding theorem. □

The far field operator F will play an essential role in our investigations of the inverse scattering problem in Chapter 5. For the preparation of this analysis we proceed by presenting some of its main properties.

Lemma 3.23 *The far field operator satisfies*

$$2\pi\,\{(Fg, h) - (g, Fh)\} = ik(Fg, Fh), \quad g, h \in L^2(\mathbb{S}^2), \tag{3.48}$$

where $(\cdot\,,\cdot)$ denotes the inner product in $L^2(\mathbb{S}^2)$.

Proof. If v^s and w^s are radiating solutions of the Helmholtz equation with far field patterns v_∞ and w_∞, then from the far field asymptotics and Green's second integral theorem we deduce that

$$\int_{\partial D} \left(v^s\, \frac{\partial \overline{w^s}}{\partial \nu} - \overline{w_s}\, \frac{\partial v^s}{\partial \nu} \right) ds = -2ik \int_{\mathbb{S}^2} v_\infty \overline{w_\infty}\, ds. \tag{3.49}$$

From the far field representation of Theorem 2.6 we see that if w_h^i is a Herglotz wave function with kernel h then

$$\begin{aligned}
&\int_{\partial D}\left(v^s(x)\,\frac{\partial\overline{w_h^i}}{\partial\nu}(x)-\overline{w_h^i}(x)\,\frac{\partial v^s}{\partial\nu}(x)\right)ds(x)\\
&=\int_{\mathbb{S}^2}\overline{h(d)}\int_{\partial D}\left(v^s(x)\,\frac{\partial e^{-ik\,x\cdot d}}{\partial\nu(x)}-e^{-ik\,x\cdot d}\,\frac{\partial v^s}{\partial\nu}(x)\right)ds(x)\,ds(d)\\
&=4\pi\int_{\mathbb{S}^2}\overline{h(d)}\,v_\infty(d)\,ds(d).
\end{aligned}$$

Now let v_g^i and v_h^i be the Herglotz wave functions with kernels $g,h\in L^2(\mathbb{S}^2)$, respectively, and let v_g and v_h be the solutions to the obstacle scattering problem with incident fields v_g^i and v_h^i, respectively. We denote by $v_{g,\infty}$ and $v_{h,\infty}$ the far field patterns corresponding to v_g and v_h, respectively. Then we can combine the two previous equations to obtain

$$\begin{aligned}
&-2ik\,(Fg,Fh)+4\pi(Fg,h)-4\pi(g,Fh)\\
&=-2ik\int_{\mathbb{S}^2}v_{g,\infty}\,\overline{v_{h,\infty}}\,ds+4\pi\int_{\mathbb{S}^2}v_{g,\infty}\,\bar{h}\,ds-4\pi\int_{\mathbb{S}^2}g\,\overline{v_{h,\infty}}\,ds\\
&=\int_{\partial D}\left(v_g\,\frac{\partial\overline{v_h}}{\partial\nu}-\overline{v_h}\,\frac{\partial v_g}{\partial\nu}\right)ds.
\end{aligned}\tag{3.50}$$

From this the statement follows in view of the boundary condition. □

Theorem 3.24. *The far field operator F is compact and normal, i.e., $FF^*=F^*F$, and hence has a countable number of eigenvalues.*

Proof. Since F is an integral operator with continuous kernel, it is compact. From (3.48) we obtain that

$$(g,ikF^*Fh)=2\pi\,\{(g,Fh)-(g,F^*h)\}$$

for all $g,h\in L^2(\mathbb{S}^2)$ and therefore

$$ikF^*F=2\pi(F-F^*).\tag{3.51}$$

Using (3.46) we can deduce that

$$(F^*g,F^*h)=(FR\bar{h},FR\bar{g})$$

and hence, from (3.48), it follows that

$$ik(F^*g,F^*h)=2\pi\,\{(g,F^*h)-(F^*g,h)\}$$

for all $g, h \in L^2(\mathbb{S}^2)$. If we now proceed as in the derivation of (3.51), we find that

$$ikFF^* = 2\pi(F - F^*) \tag{3.52}$$

and the proof is finished. □

Corollary 3.25 *The scattering operator* $S : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ *defined by*

$$S := I + \frac{ik}{2\pi}F \tag{3.53}$$

is unitary.

Proof. From (3.51) and (3.52) we see that $SS^* = S^*S = I$. □

In view of (3.53), the unitarity of S implies that the eigenvalues of F lie on the circle with center at $(0, 2\pi/k)$ on the positive imaginary axis and radius $2\pi/k$.

The question of when we can find a superposition of incident plane waves such that the resulting far field pattern coincides with a prescribed far field is answered in terms of a solvability condition for an integral equation of the first kind in the following theorem.

Theorem 3.26. *Let* v^s *be a radiating solution to the Helmholtz equation with far field pattern* v_∞. *Then the integral equation of the first kind*

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d)g(d)\, ds(d) = v_\infty(\hat{x}), \quad \hat{x} \in \mathbb{S}^2, \tag{3.54}$$

possesses a solution $g \in L^2(\mathbb{S}^2)$ *if and only if* v^s *is defined in* $\mathbb{R}^3 \setminus \bar{D}$, *is continuous in* $\mathbb{R}^3 \setminus D$ *and the interior Dirichlet problem for the Helmholtz equation*

$$\Delta v^i + k^2 v^i = 0 \quad \text{in } D \tag{3.55}$$

and

$$v^i + v^s = 0 \quad \text{on } \partial D \tag{3.56}$$

is solvable with a solution v^i *being a Herglotz wave function.*

Proof. By Theorem 3.19 and Lemma 3.20, the solvability of the integral equation (3.54) for g is equivalent to the existence of a Herglotz wave function v^i (with kernel g) for which the far field pattern for the scattering by the obstacle D coincides with the given v_∞, i.e., the scattered wave coincides with the given v^s. This completes the proof. □

Special cases of Theorem 3.26 include the radiating spherical wave function

$$v^s(x) = h_n^{(1)}(k|x|)Y_n\left(\frac{x}{|x|}\right)$$

of order n with far field pattern

$$v_\infty = \frac{1}{ki^{n+1}}\, Y_n.$$

Here, for solvability of (3.54) it is necessary that the origin is contained in D.

The integral equation (3.54) will play a role in our analysis of the inverse scattering problem in Section 5.6. By reciprocity, the solvability of (3.54) is equivalent to the solvability of

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d)h(\hat{x})\, ds(\hat{x}) = v_\infty(-d), \quad d \in \mathbb{S}^2, \tag{3.57}$$

where $h(\hat{x}) = g(-\hat{x})$. Since the Dirichlet problem (3.55), (3.56) is solvable provided k^2 is not a Dirichlet eigenvalue, the crucial condition in Theorem 3.26 is the property of the solution to be a Herglotz wave function, that is, a strong regularity condition. In the special case $v_\infty = 1$, the connection between the solution to the integral equation (3.57) and the interior Dirichlet problem (3.55), (3.56) as described in Theorem 3.26 was first established by Colton and Monk [73] without, however, making use of the reciprocity Theorem 3.15.

The original proof for Theorem 3.21 by Colton and Kirsch [57] is based on the following completeness result which we include for its own interest.

Theorem 3.27. *Let (d_n) be a sequence of unit vectors that is dense on $\mathbb{S}^2$. Then the normal derivatives of the total fields*

$$\left\{ \frac{\partial}{\partial \nu}\, u(\cdot\,, d_n) : n = 1, 2, \dots \right\}$$

corresponding to incident plane waves with directions (d_n) are complete in $L^2(\partial D)$.

Proof. The weakly singular operators $K - iS$ and $K' - iS$ are both compact from $C(\partial D)$ into $C(\partial D)$ and from $L^2(\partial D)$ into $L^2(\partial D)$ and they are adjoint with respect to the L^2 bilinear form, i.e.,

$$\int_{\partial D} (K - iS)\varphi\, \psi\, ds = \int_{\partial D} \varphi\, (K' - iS)\psi\, ds$$

for all $\varphi, \psi \in L^2(\partial D)$. From the proof of Theorem 3.11, we know that the operator $I + K - iS$ has a trivial nullspace in $C(\partial D)$. Therefore, by the Fredholm alternative applied in the dual system $\langle C(\partial D), L^2(\partial D)\rangle$ with the L^2 bilinear form, the adjoint operator $I + K' - iS$ has a trivial nullspace in $L^2(\partial D)$. Again by the Fredholm alternative, but now applied in the dual system $\langle L^2(\partial D), L^2(\partial D)\rangle$ with the L^2 bilinear form, the operator $I + K - iS$ also has a trivial nullspace in $L^2(\partial D)$. Hence, by the Riesz–Fredholm theory for compact operators, both the operators $I + K - iS : L^2(\partial D) \to L^2(\partial D)$ and $I + K' - iS : L^2(\partial D) \to L^2(\partial D)$ are bijective and have a bounded inverse. This idea to employ the Fredholm alternative in two different dual systems for showing that the dimensions of the nullspaces for weakly singular integral operators of the second kind in the space of continuous functions and in the L^2 space coincide is due to Hähner [122].

From the representation (3.33), the boundary condition $u = 0$ on ∂D and the jump relations of Theorem 3.1 we deduce that

$$\frac{\partial u}{\partial \nu} + K' \frac{\partial u}{\partial \nu} - iS \frac{\partial u}{\partial \nu} = 2 \frac{\partial u^i}{\partial \nu} - 2iu^i.$$

Now let $g \in L^2(\partial D)$ satisfy

$$\int_{\partial D} g \, \frac{\partial u(\cdot\,, d_n)}{\partial \nu} \, ds = 0, \quad n = 1, 2, \ldots.$$

This, by the continuity of the Dirichlet to Neumann map (Theorem 3.13), implies

$$\int_{\partial D} g \, \frac{\partial u(\cdot\,, d)}{\partial \nu} \, ds = 0$$

for all $d \in \mathbb{S}^2$. Then from

$$\frac{\partial u}{\partial \nu} = 2(I + K' - iS)^{-1} \left\{ \frac{\partial u^i}{\partial \nu} - iu^i \right\}$$

we obtain

$$\int_{\partial D} g \, (I + K' - iS)^{-1} \left\{ \frac{\partial}{\partial \nu} \, u^i(\cdot\,, d) - iu^i(\cdot\,, d) \right\} ds = 0$$

for all $d \in \mathbb{S}^2$, and consequently

$$\int_{\partial D} \varphi(y) \left\{ \frac{\partial}{\partial \nu(y)} \, e^{ik\, y\cdot d} - ie^{ik\, y\cdot d} \right\} ds(y) = 0$$

for all $d \in \mathbb{S}^2$ where we have set

$$\varphi := (I + K - iS)^{-1} g.$$

Therefore, since $I + K - iS$ is bijective, our proof will be finished by showing that $\varphi = 0$. To this end, by (2.15) and (2.16), we deduce from the last equation that the combined single- and double-layer potential

$$v(x) := \int_{\partial D} \varphi(y) \left\{ \frac{\partial \Phi(x, y)}{\partial \nu(y)} - i\Phi(x, y) \right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D},$$

has far field pattern

$$v_\infty(\hat{x}) = \frac{1}{4\pi} \int_{\partial D} \varphi(y) \left\{ \frac{\partial}{\partial \nu(y)} \, e^{-ik\, y\cdot \hat{x}} - ie^{-ik\, y\cdot \hat{x}} \right\} ds(y) = 0, \quad \hat{x} \in \mathbb{S}^2.$$

By Theorem 2.14, this implies $v = 0$ in $\mathbb{R}^3 \setminus \bar{D}$, and letting x tend to the boundary ∂D with the help of the L^2 jump relations (3.20) and (3.22) yields $\varphi + K\varphi - iS\varphi = 0$, whence $\varphi = 0$ follows. □

With the tools involved in the proof of Theorem 3.27, we can establish the following result which we shall also need in our analysis of the inverse problem in Chapter 5.

Theorem 3.28. *The operator $A : C(\partial D) \to L^2(\mathbb{S}^2)$ which maps the boundary values of radiating solutions $w \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ to the Helmholtz equation onto the far field pattern w_∞ can be extended to an injective bounded linear operator $A : L^2(\partial D) \to L^2(\mathbb{S}^2)$ with dense range.*

Proof. From the solution (3.26) to the exterior Dirichlet problem, for $\hat{x} \in \mathbb{S}^2$ we derive

$$w_\infty(\hat{x}) = \frac{1}{2\pi} \int_{\partial D} \left\{ \frac{\partial}{\partial \nu(y)} \, e^{-ik\, y\cdot\hat{x}} - ie^{-ik\, y\cdot\hat{x}} \right\} \left((I + K - iS)^{-1} f \right)(y)\, ds(y)$$

with the boundary values $w = f$ on ∂D. From this, given the boundedness of the operator $(I + K - iS)^{-1} : L^2(\partial D) \to L^2(\partial D)$ from the proof of Theorem 3.27, it is obvious that A is bounded from $L^2(\partial D) \to L^2(\mathbb{S}^2)$. The injectivity of A is also immediate from the proof of Theorem 3.27.

In order to show that A has dense range we rewrite it as an integral operator. To this end we note that in terms of the plane waves $u^i(x, d) = e^{ik\, x\cdot d}$ the far field representation (2.14) for a radiating solution w of the Helmholtz equation can be written in the form

$$w_\infty(\hat{x}) = \frac{1}{4\pi} \int_{\partial D} \left\{ \frac{\partial u^i(y, -\hat{x})}{\partial \nu(y)} \, w(y) - u^i(y, -\hat{x}) \, \frac{\partial w}{\partial \nu}(y) \right\} ds(y), \quad \hat{x} \in \mathbb{S}^2.$$

(See also the proof of Theorem 3.16.) From this, with the aid of Green's integral theorem and the radiation condition, using the sound-soft boundary condition for the total wave $u = u^i + u^s$ on ∂D we conclude that

$$w_\infty(\hat{x}) = \frac{1}{4\pi} \int_{\partial D} \frac{\partial u(y, -\hat{x})}{\partial \nu(y)} \, w(y)\, ds(y), \quad \hat{x} \in \mathbb{S}^2,$$

that is,

$$(Af)(d) = \frac{1}{4\pi} \int_{\partial D} \frac{\partial u(y, -d)}{\partial \nu(y)} \, f(y)\, ds(y), \quad d \in \mathbb{S}^2. \tag{3.58}$$

Consequently the adjoint operator $A^* : L^2(\mathbb{S}^2) \to L^2(\partial D)$ can be expressed as the integral operator

$$(A^* g)(x) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \frac{\partial \overline{u(x, -d)}}{\partial \nu(x)} \, g(d)\, ds(d), \quad x \in \partial D. \tag{3.59}$$

If for $g \in L^2(\mathbb{S}^2)$ we define the Herglotz wave function v_g^i in the form

$$v_g^i(x) = \int_{\mathbb{S}^2} e^{-ik\,x\cdot d}\,\overline{g(d)}\,ds(d) = \int_{\mathbb{S}^2} u^i(x,-d)\overline{g(d)}\,ds(d), \quad x \in \mathbb{R}^3,$$

then from Lemma 3.20 we have that

$$v_g(x) = \int_{\mathbb{S}^2} u(x,-d)\overline{g(d)}\,ds(d), \quad x \in \mathbb{R}^3,$$

is the total wave for scattering of v_g^i from D. Hence,

$$\overline{A^*g} = \frac{1}{4\pi}\,\frac{\partial v_g}{\partial \nu} = \frac{1}{4\pi}\left\{\frac{\partial v_g^i}{\partial \nu} - \mathcal{A} v_g^i|_{\partial D}\right\} \tag{3.60}$$

with the Dirichlet to Neumann operator $\mathcal{A}$. Now let g satisfy $A^*g = 0$. Then (3.59) implies that $\partial v_g/\partial\nu = 0$ on ∂D. By definition we also have $v_g = 0$ on ∂D and therefore, by Holmgren's Theorem 2.3, it follows that $v_g = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. Thus the entire solution v_g^i satisfies the radiation condition and therefore must vanish identically. Thus $g = 0$, i.e., A^* is injective. Hence A has dense range by Theorem 4.6. □

Theorem 3.29. *For the far field operator F we have the factorization*

$$F = -2\pi A S^* A^*. \tag{3.61}$$

Proof. For convenience we introduce the Herglotz operator $H : L^2(\mathbb{S}^2) \to L^2(\partial D)$ by

$$(Hg)(x) := \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d)\,ds(d), \quad x \in \partial D. \tag{3.62}$$

Since Fg represents the far field pattern of the scattered wave corresponding to Hg as incident field, we clearly have

$$F = -AH. \tag{3.63}$$

The L^2 adjoint $H^* : L^2(\partial D) \to L^2(\mathbb{S}^2)$ is given by

$$(H^*\varphi)(\hat{x}) = \int_{\partial D} e^{-ik\,\hat{x}\cdot y}\varphi(y)\,ds(y), \quad \hat{x} \in \mathbb{S}^2,$$

and represents the far field pattern of the single-layer potential with density $4\pi\varphi$. Therefore

$$H^* = 2\pi A S \tag{3.64}$$

and consequently

$$H = 2\pi S^* A^*. \tag{3.65}$$

Now the statement follows by combining (3.63) and (3.65). □

We now wish to study Herglotz wave functions more closely. The concept of the growth condition in the following theorem for solutions to the Helmholtz equation was introduced by Herglotz in a lecture in 1945 in Göttingen and was studied further by Magnus [233] and Müller [253]. The equivalence stated in the theorem was shown by Hartman and Wilcox [133].

Theorem 3.30. *An entire solution v to the Helmholtz equation possesses the growth property*

$$\sup_{R>0} \frac{1}{R} \int_{|x|\le R} |v(x)|^2 dx < \infty \tag{3.66}$$

if and only if it is a Herglotz wave function, i.e., if and only if there exists a function $g \in L^2(\mathbb{S}^2)$ such that v can be represented in the form (3.43).

Proof. Before we can prove this result, we need to note two properties for integrals containing spherical Bessel functions. From the asymptotic behavior (2.42), that is, from

$$j_n(t) = \frac{1}{t}\,\cos\left(t - \frac{n\pi}{2} - \frac{\pi}{2}\right)\left\{1 + O\left(\frac{1}{t}\right)\right\}, \quad t \to \infty,$$

we readily find that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T t^2 [j_n(t)]^2 dt = \frac{1}{2}\,, \quad n = 0, 1, 2, \dots. \tag{3.67}$$

We now want to establish that the integrals in (3.67) are uniformly bounded with respect to T and n. This does not follow immediately since the asymptotic behavior for the spherical Bessel functions is not uniformly valid with respect to the order n. If we multiply the differential formula (2.35) rewritten in the form

$$j_{n+1}(t) = -\frac{1}{\sqrt{t}}\,\frac{d}{dt}\,\sqrt{t}\,j_n(t) + \left(n + \frac{1}{2}\right)\frac{1}{t}\,j_n(t)$$

by two and subtract it from the recurrence relation (2.34), that is, from

$$j_{n-1}(t) + j_{n+1}(t) = \frac{2n+1}{t}\,j_n(t),$$

we obtain

$$j_{n-1}(t) - j_{n+1}(t) = \frac{2}{\sqrt{t}}\,\frac{d}{dt}\,\sqrt{t}\,j_n(t).$$

Hence, from the last two equations we get

$$\int_0^T t^2 \left\{[j_{n-1}(t)]^2 - [j_{n+1}(t)]^2\right\} dt = (2n+1)T\,[j_n(T)]^2$$

for $n = 1, 2, \ldots$ and all $T > 0$. From this monotonicity, together with (3.67) for $n = 0$ and $n = 1$, it is now obvious that

$$\sup_{\substack{T>0 \\ n=0,1,2,\ldots}} \frac{1}{T} \int_0^T t^2 [j_n(t)]^2 dt < \infty. \tag{3.68}$$

For the proof of the theorem, we first observe that any entire solution v of the Helmholtz equation can be expanded in a series

$$v(x) = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n \, a_n^m \, j_n(k|x|) \, Y_n^m\left(\frac{x}{|x|}\right) \tag{3.69}$$

and the series converges uniformly on compact subsets of $\mathbb{R}^3$. This follows from Green's representation formula (2.5) for v in a ball with radius R and center at the origin and inserting the addition theorem (2.43) with the roles of x and y interchanged, that is,

$$\Phi(x, y) = ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} j_n(k|x|) \, Y_n^m\left(\frac{x}{|x|}\right) \, h_n^{(1)}(k|y|) \, \overline{Y_n^m\left(\frac{y}{|y|}\right)}, \quad |x| < |y|.$$

Since the expansion derived for two different radii represent the same function in the ball with the smaller radius, the coefficients a_n^m do not depend on the radius R. Because of the uniform convergence, we can integrate term by term and use the orthonormality of the Y_n^m to find that

$$\frac{1}{R} \int_{|x| \le R} |v(x)|^2 dx = \frac{16\pi^2}{R} \sum_{n=0}^{\infty} \int_0^R r^2 [j_n(kr)]^2 dr \sum_{m=-n}^{n} \left|a_n^m\right|^2. \tag{3.70}$$

Now assume that v satisfies

$$\frac{1}{R} \int_{|x| \le R} |v(x)|^2 dx \le C$$

for all $R > 0$ and some constant $C > 0$. This, by (3.70), implies that

$$\frac{16\pi^2}{R} \sum_{n=0}^{N} \int_0^R r^2 [j_n(kr)]^2 dr \sum_{m=-n}^{n} \left|a_n^m\right|^2 \le C$$

for all $R > 0$ and all $N \in \mathbb{N}$. Hence, by first passing to the limit $R \to \infty$ with the aid of (3.67) and then letting $N \to \infty$ we obtain

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|a_n^m\right|^2 \le \frac{k^2 C}{8\pi^2}.$$

Therefore,

$$g := \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m \, Y_n^m$$

defines a function $g \in L^2(\mathbb{S}^2)$. From the Jacobi–Anger expansion (2.46) and the addition theorem (2.30), that is, from

$$e^{ik\,x\cdot d} = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n \, j_n(k|x|) \, Y_n^m\left(\frac{x}{|x|}\right) \overline{Y_n^m(d)}$$

we now derive

$$\int_{\mathbb{S}^2} g(d) e^{ik\,x\cdot d} ds(d) = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n \, a_n^m \, j_n(k|x|) \, Y_n^m\left(\frac{x}{|x|}\right) = v(x)$$

for all $x \in \mathbb{R}^3$, that is, we have shown that v can be represented in the form (3.43).

Conversely, for a given $g \in L^2(\mathbb{S}^2)$ we have an expansion

$$g = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m \, Y_n^m,$$

where, by Parseval's equality, the coefficients satisfy

$$\|g\|^2_{L^2(\mathbb{S}^2)} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|a_n^m\right|^2 < \infty. \tag{3.71}$$

Then for the entire solution v to the Helmholtz equation defined by

$$v(x) := \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d) \, ds(d), \quad x \in \mathbb{R}^3,$$

we again see by the Jacobi–Anger expansion that

$$v(x) = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n \, a_n^m \, j_n(k|x|) \, Y_n^m\left(\frac{x}{|x|}\right) \tag{3.72}$$

and from (3.68), (3.70) and (3.72) we conclude that the growth condition (3.66) is fulfilled for v. The proof is now complete. □

With the help of (3.68), we observe that the series (3.70) has a convergent majorant independent of R. Hence, it is uniformly convergent for all $R > 0$ and we may interchange the limit $R \to \infty$ with the series and use (3.67) and (3.71) to obtain that for the Herglotz wave function v with kernel g we have

$$\lim_{R\to\infty} \frac{1}{R} \int_{|x|\le R} |v(x)|^2 dx = \frac{8\pi^2}{k^2} \, \|g\|^2_{L^2(\mathbb{S}^2)}.$$

3.4 The Two-Dimensional Case

The scattering from infinitely long cylindrical obstacles leads to exterior boundary value problems for the Helmholtz equation in $\mathbb{R}^2$. The two-dimensional case can be used as an approximation for the scattering from finitely long cylinders, and more important, it can serve as a model case for testing numerical approximation schemes in direct and inverse scattering. Without giving much of the details, we would like to show how all the results of this chapter remain valid in two dimensions after appropriate modifications of the fundamental solution, the radiation condition and the spherical wave functions.

We note that in two dimensions there exist two linearly independent spherical harmonics of order n which can be represented by $e^{\pm in\varphi}$. Correspondingly, looking for solutions to the Helmholtz equation of the form

$$u(x) = f(kr)\,e^{\pm in\varphi}$$

in polar coordinates (r,φ) leads to the *Bessel differential equation*

$$t^2 f''(t) + tf'(t) + [t^2 - n^2]f(t) = 0 \tag{3.73}$$

with integer order $n = 0, 1, \dots$. The analysis of the Bessel equation which is required for the study of the two-dimensional Helmholtz equation, in particular the asymptotics of the solutions for large argument, is more involved than the corresponding analysis for the spherical Bessel equation (2.31). Therefore, here we will list only the relevant results without proofs. For a concise treatment of the Bessel equation for the purpose of scattering theory, we refer to Colton [52] or Lebedev [221].

By direct calculations and the ratio test, we can easily verify that for $n = 0, 1, 2, \dots$ the functions

$$J_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,(n+p)!} \left(\frac{t}{2}\right)^{n+2p} \tag{3.74}$$

represent solutions to Bessel's equation which are analytic for all $t \in \mathbb{R}$ and these are known as *Bessel functions* of order n. As opposed to the spherical Bessel equation, here it is more complicated to construct a second linearly independent solution. Patient, but still straightforward, calculations together with the ratio test show that

$$\begin{aligned} Y_n(t) := {}& \frac{2}{\pi}\left\{\ln\frac{t}{2} + C\right\} J_n(t) - \frac{1}{\pi}\sum_{p=0}^{n-1} \frac{(n-1-p)!}{p!}\left(\frac{2}{t}\right)^{n-2p} \\ & - \frac{1}{\pi}\sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,(n+p)!}\left(\frac{t}{2}\right)^{n+2p}\{\psi(p+n)+\psi(p)\} \end{aligned} \tag{3.75}$$

for $n = 0, 1, 2, \ldots$ provide solutions to Bessel's equation which are analytic for all $t \in (0, \infty)$. Here, we define $\psi(0) := 0$,

$$\psi(p) := \sum_{m=1}^{p} \frac{1}{m}, \quad p = 1, 2, \ldots,$$

let

$$C := \lim_{p \to \infty} \left\{ \sum_{m=1}^{p} \frac{1}{m} - \ln p \right\}$$

denote Euler's constant, and if $n = 0$ the finite sum in (3.75) is set equal to zero. The functions Y_n are called *Neumann functions* of order n and the linear combinations

$$H_n^{(1,2)} := J_n \pm iY_n$$

are called *Hankel functions* of the first and second kind of order n respectively.

From the series representation (3.74) and (3.75), by equating powers of t, it is readily verified that both $f_n = J_n$ and $f_n = Y_n$ satisfy the recurrence relation

$$f_{n+1}(t) + f_{n-1}(t) = \frac{2n}{t} f_n(t), \quad n = 1, 2, \ldots. \tag{3.76}$$

Straightforward differentiation of the series (3.74) and (3.75) shows that both $f_n = J_n$ and $f_n = Y_n$ satisfy the differentiation formulas

$$f_{n+1}(t) = -t^n \frac{d}{dt} \left\{ t^{-n} f_n(t) \right\}, \quad n = 0, 1, 2, \ldots, \tag{3.77}$$

and

$$t^n f_{n-1}(t) = \frac{d}{dt} \left\{ t^n f_n(t) \right\}, \quad n = 1, 2, \ldots. \tag{3.78}$$

The Wronskian

$$W(J_n(t), Y_n(t)) := J_n(t)Y_n'(t) - Y_n(t)J_n'(t)$$

satisfies

$$W' + \frac{1}{t} W = 0.$$

Therefore, $W(J_n(t), Y_n(t)) = C/t$ for some constant C and by passing to the limit $t \to 0$ it follows that

$$J_n(t)Y_n'(t) - J_n'(t)Y_n(t) = \frac{2}{\pi t}. \tag{3.79}$$

From the series representation of the Bessel and Neumann functions, it is obvious that

$$J_n(t) = \frac{t^n}{2^n n!} \left(1 + O\left(\frac{1}{n} \right) \right), \quad n \to \infty, \tag{3.80}$$

uniformly on compact subsets of $\mathbb{R}$ and

$$H_n^{(1)}(t) = \frac{2^n(n-1)!}{\pi i t^n}\left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, \tag{3.81}$$

uniformly on compact subsets of $(0, \infty)$.

For large arguments, we have the following asymptotic behavior of the Hankel functions

$$\begin{aligned} H_n^{(1,2)}(t) &= \sqrt{\frac{2}{\pi t}}\, e^{\pm i\left(t - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \left\{1 + O\left(\frac{1}{t}\right)\right\}, \quad t \to \infty, \\ H_n^{(1,2)\prime}(t) &= \sqrt{\frac{2}{\pi t}}\, e^{\pm i\left(t - \frac{n\pi}{2} + \frac{\pi}{4}\right)} \left\{1 + O\left(\frac{1}{t}\right)\right\}, \quad t \to \infty. \end{aligned} \tag{3.82}$$

For a proof, we refer to Lebedev [221]. Taking the real and the imaginary part of (3.82) we also have asymptotic formulas for the Bessel and Neumann functions.

Now we have listed all the necessary tools for carrying over the analysis of Chapters 2 and 3 for the Helmholtz equation from three to two dimensions. The fundamental solution to the Helmholtz equation in two dimensions is given by

$$\Phi(x, y) := \frac{i}{4}\, H_0^{(1)}(k|x - y|), \quad x \neq y. \tag{3.83}$$

For fixed $y \in \mathbb{R}^2$, it satisfies the Helmholtz equation in $\mathbb{R}^2 \setminus \{y\}$. From the expansions (3.74) and (3.75), we deduce that

$$\Phi(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|} + \frac{i}{4} - \frac{1}{2\pi} \ln \frac{k}{2} - \frac{C}{2\pi} + O\left(|x - y|^2 \ln \frac{1}{|x - y|}\right) \tag{3.84}$$

for $|x - y| \to 0$. Therefore, the fundamental solution to the Helmholtz equation in two dimensions has the same singular behavior as the fundamental solution of Laplace's equation. As a consequence, Green's formula (2.5) and the jump relations and regularity results on single- and double-layer potentials of Theorems 3.1 and 3.3 can be carried over to two dimensions. From (3.84) we note that, in contrast to three dimensions, the fundamental solution does not converge for $k \to 0$ to the fundamental solution for the Laplace equation. This leads to some difficulties in the investigation of the convergence of the solution to the exterior Dirichlet problem as $k \to 0$ (see Werner [329] and Kress [199]).

In $\mathbb{R}^2$ the Sommerfeld radiation condition has to be replaced by

$$\lim_{r \to \infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \quad r = |x|, \tag{3.85}$$

uniformly for all directions $x/|x|$. From (3.82) it is obvious that the fundamental solution satisfies the radiation condition uniformly with respect to y on compact sets. Therefore, Green's representation formula (2.9) can be shown to be valid for two-dimensional radiating solutions. According to the form (3.85) of the radiation condition, the definition of the far field pattern (2.13) has to be replaced by

$$u(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty, \tag{3.86}$$

and, due to (3.82), the representation (2.14) has to be replaced by

$$u_\infty(\hat{x}) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_{\partial D} \left\{ u(y) \frac{\partial e^{-ik\,\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y)\, e^{-ik\,\hat{x}\cdot y} \right\} ds(y) \tag{3.87}$$

for $|\hat{x}| = x/|x|$. We explicitly write out the addition theorem

$$H_0^{(1)}(k|x-y|) = H_0^{(1)}(k|x|)\, J_0(k|y|) + 2\sum_{n=1}^{\infty} H_n^{(1)}(k|x|)\, J_n(k|y|) \cos n\theta \tag{3.88}$$

which is valid for $|x| > |y|$ in the sense of Theorem 2.11 and where θ denotes the angle between x and y. The proof is analogous to that of Theorem 2.11. We note that the entire spherical wave functions in $\mathbb{R}^2$ are given by $J_n(kr)e^{\pm in\varphi}$ and the radiating spherical wave functions by $H_n^{(1)}(kr)e^{\pm in\varphi}$. Similarly, the Jacobi–Anger expansion (2.46) assumes the form

$$e^{ik\,x\cdot d} = J_0(k|x|) + 2\sum_{n=1}^{\infty} i^n\, J_n(k|x|) \cos n\theta, \quad x \in \mathbb{R}^2. \tag{3.89}$$

With all these prerequisites, it is left as an exercise to establish that, with minor adjustments in the proofs, all the results of Sections 2.5, 3.2 and 3.3 remain valid in two dimensions.

3.5 On the Numerical Solution in $\mathbb{R}^2$

We would like to include in our presentation an advertisement for what we think is the most efficient method for the numerical solution of the boundary integral equations for two-dimensional problems. Since it seems to be safe to state that the boundary curves in most practical applications are either analytic or piecewise analytic with corners, we restrict our attention to approximation schemes which are the most appropriate under these regularity assumptions. We begin with the analytic case where we recommend the Nyström method based on weighted trigonometric interpolation quadratures on an equidistant mesh. To support our preference for using trigonometric polynomial approximations we quote from Atkinson [16]: *...the most efficient numerical methods for solving boundary integral equations on smooth planar boundaries are those based on trigonometric polynomial approximations, and such methods are sometimes called spectral methods. When calculations using* *piecewise polynomial approximations are compared with those using trigonometric polynomial approximations, the latter are almost always the more efficient.*

We first describe the necessary parametrization of the integral equation (3.27) in the two-dimensional case. We assume that the boundary curve ∂D possesses a regular analytic and 2π–periodic parametric representation of the form

$$x(t) = (x_1(t), x_2(t)), \quad 0 \le t \le 2\pi, \tag{3.90}$$

in counterclockwise orientation satisfying $|x'(t)|^2 > 0$ for all t. Then, by straightforward calculations using $H_1^{(1)} = -H_0^{(1)\prime}$, we transform (3.27) into the parametric form

$$\psi(t) - \int_0^{2\pi} \{L(t,\tau) + i\eta M(t,\tau)\}\,\psi(\tau)\,d\tau = g(t), \quad 0 \le t \le 2\pi,$$

where we have set $\psi(t) := \varphi(x(t))$, $g(t) := 2f(x(t))$ and the kernels are given by

$$L(t,\tau) := \frac{ik}{2}\,\{x_2'(\tau)[x_1(\tau) - x_1(t)] - x_1'(\tau)[x_2(\tau) - x_2(t)]\}\,\frac{H_1^{(1)}(k|x(t) - x(\tau)|)}{|x(t) - x(\tau)|}\ ,$$

$$M(t,\tau) := \frac{i}{2}\,H_0^{(1)}(k|x(t) - x(\tau)|)\,|x'(\tau)|$$

for $t \neq \tau$. From the expansion (3.75) for the Neumann functions, we see that the kernels L and M have logarithmic singularities at $t = \tau$. Hence, for their proper numerical treatment, following Martensen [234] and Kussmaul [217], we split the kernels into

$$L(t,\tau) = L_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + L_2(t,\tau),$$

$$M(t,\tau) = M_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + M_2(t,\tau),$$

where

$$L_1(t,\tau) := \frac{k}{2\pi}\,\{x_2'(\tau)[x_1(t) - x_1(\tau)] - x_1'(\tau)[x_2(t) - x_2(\tau)]\}\,\frac{J_1(k|x(t) - x(\tau)|)}{|x(t) - x(\tau)|}\ ,$$

$$L_2(t,\tau) := L(t,\tau) - L_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right),$$

$$M_1(t,\tau) := -\frac{1}{2\pi}\,J_0(k|x(t) - x(\tau)|)\,|x'(\tau)|,$$

$$M_2(t,\tau) := M(t,\tau) - M_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right).$$

The kernels L_1, L_2, M_1, and M_2 turn out to be analytic. In particular, using the expansions (3.74) and (3.75) we can deduce the diagonal terms

$$L_2(t,t) = L(t,t) = \frac{1}{2\pi} \frac{x_1'(t)x_2''(t) - x_2'(t)x_1''(t)}{|x'(t)|^2}$$

and

$$M_2(t,t) = \left\{ \frac{i}{2} - \frac{C}{\pi} - \frac{1}{\pi} \ln\left(\frac{k}{2} |x'(t)|\} \right) \right\} |x'(t)|$$

for $0 \le t \le 2\pi$. We note that despite the continuity of the kernel L, for numerical accuracy it is advantageous to separate the logarithmic part of L since the derivatives of L fail to be continuous at $t = \tau$.

Hence, we have to numerically solve an integral equation of the form

$$\psi(t) - \int_0^{2\pi} K(t,\tau)\psi(\tau)\,d\tau = g(t), \quad 0 \le t \le 2\pi, \tag{3.91}$$

where the kernel can be written in the form

$$K(t,\tau) = K_1(t,\tau) \ln\left(4 \sin^2 \frac{t-\tau}{2}\right) + K_2(t,\tau) \tag{3.92}$$

with analytic functions K_1 and K_2 and with an analytic right hand side g. Here we wish to point out that it is essential to split off the logarithmic singularity in a fashion which preserves the 2π–periodicity for the kernels K_1 and K_2.

For the numerical solution of integral equations of the second kind, in principle, there are three basic methods available, the *Nyström method*, the *collocation method* and the *Galerkin method*. In the case of one-dimensional integral equations, the Nyström method is more practical than the collocation and Galerkin methods since it requires the least computational effort. In each of the three methods, the approximation requires the solution of a finite dimensional linear system. In the Nyström method, for the evaluation of each of the matrix elements of this linear system only an evaluation of the kernel function is needed, whereas in the collocation and Galerkin methods the matrix elements are single or double integrals demanding numerical quadratures. In addition, the Nyström method is generically stable in the sense that it preserves the condition of the integral equation whereas in the collocation and Galerkin methods the condition can be disturbed by a poor choice of the basis (see [205]).

In the case of integral equations for periodic analytic functions, using global approximations via trigonometric polynomials is superior to using local approximations via low order polynomial splines since the trigonometric approximations yield much better convergence. By choosing the appropriate basis, the computational effort for the global approximation is comparable to that for local approximations.

The Nyström method consists in the straightforward approximation of the integrals by quadrature formulas. In our case, for the 2π–periodic integrands, we choose an equidistant set of knots $t_j := \pi j/n,\ j = 0, \dots, 2n-1$, and use the quadrature rule

$$\int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right) f(\tau)d\tau \approx \sum_{j=0}^{2n-1} R_j^{(n)}(t) f(t_j), \quad 0 \le t \le 2\pi, \tag{3.93}$$

with the quadrature weights given by

$$R_j^{(n)}(t) := -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t-t_j) - \frac{\pi}{n^2}\cos n(t-t_j), \quad j = 0, \dots, 2n-1,$$

and the trapezoidal rule

$$\int_0^{2\pi} f(\tau)d\tau \approx \frac{\pi}{n}\sum_{j=0}^{2n-1} f(t_j). \tag{3.94}$$

Both these numerical integration formulas are obtained by replacing the integrand f by its trigonometric interpolation polynomial and then integrating exactly. The quadrature formula (3.93) was first used by Martensen [234] and Kussmaul [217]. Provided f is analytic, according to derivative-free error estimates for the remainder term in trigonometric interpolation for periodic analytic functions (see [196, 205]), the errors for the quadrature rules (3.93) and (3.94) decrease at least exponentially when the number $2n$ of knots is increased. More precisely, the error is of order $O(\exp(-n\sigma))$ where σ denotes half of the width of a parallel strip in the complex plane into which the real analytic function f can be holomorphically extended.

Of course, it is also possible to use quadrature rules different from (3.93) and (3.94) obtained from other approximations for the integrand f. However, due to their simplicity and high approximation order we strongly recommend the application of (3.93) and (3.94).

In the Nyström method, the integral equation (3.91) is replaced by the approximating equation

$$\psi^{(n)}(t) - \sum_{j=0}^{2n-1}\left\{R_j^{(n)}(t)K_1(t,t_j) + \frac{\pi}{n}K_2(t,t_j)\right\}\psi^{(n)}(t_j) = g(t) \tag{3.95}$$

for $0 \le t \le 2\pi$. Equation (3.95) is obtained from (3.91) by applying the quadrature rule (3.93) to $f = K_1(t,.)\psi$ and (3.94) to $f = K_2(t,.)\psi$. The solution of (3.95) reduces to solving a finite dimensional linear system. In particular, for any solution of (3.95) the values $\psi_i^{(n)} = \psi^{(n)}(t_i)$, $i = 0, \dots, 2n-1$, at the quadrature points trivially satisfy the linear system

$$\psi_i^{(n)} - \sum_{j=0}^{2n-1}\left\{R_{|i-j|}^{(n)}K_1(t_i,t_j) + \frac{\pi}{n}K_2(t_i,t_j)\right\}\psi_j^{(n)} = g(t_i), \quad i = 0, \dots, 2n-1, \tag{3.96}$$

where

$$R_j^{(n)} := R_j^{(n)}(0) = -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos\frac{mj\pi}{n} - \frac{(-1)^j\pi}{n^2}, \quad j = 0,\dots,2n-1.$$

Conversely, given a solution $\psi_i^{(n)}$, $i = 0,\dots,2n-1$, of the system (3.96), the function $\psi^{(n)}$ defined by

$$\psi^{(n)}(t) := \sum_{j=0}^{2n-1}\left\{R_j^{(n)}(t)K_1(t,t_j) + \frac{\pi}{n}K_2(t,t_j)\right\}\psi_j^{(n)} + g(t), \quad 0 \le t \le 2\pi, \tag{3.97}$$

is readily seen to satisfy the approximating equation (3.95). The formula (3.97) may be viewed as a natural interpolation of the values $\psi_i^{(n)}$, $i = 0,\dots,2n-1$, at the quadrature points to obtain the approximating function $\psi^{(n)}$ and goes back to Nyström.

For the solution of the large linear system (3.96), we recommend the use of the fast iterative two-grid or multi-grid methods as described in [205] or, in more detail, in [117].

Provided the integral equation (3.91) itself is uniquely solvable and the kernels K_1 and K_2 and the right hand side g are continuous, a rather involved error analysis (for the details we refer to [201, 205]) shows that

1. the approximating linear system (3.96), i.e., the approximating equation (3.95), is uniquely solvable for all sufficiently large n;
2. as $n \to \infty$ the approximate solutions $\psi^{(n)}$ converge uniformly to the solution ψ of the integral equation;
3. the convergence order of the quadrature errors for (3.93) and (3.94) carries over to the error $\psi^{(n)} - \psi$.

The latter, in particular, means that in the case of analytic kernels K_1 and K_2 and analytic right hand sides g the approximation error decreases exponentially, i.e., there exist positive constants C and σ such that

$$|\psi^{(n)}(t) - \psi(t)| \le C\, e^{-n\sigma}, \quad 0 \le t \le 2\pi, \tag{3.98}$$

for all n. In principle, the constants in (3.98) are computable but usually they are difficult to evaluate. In most practical cases, it is sufficient to judge the accuracy of the computed solution by doubling the number $2n$ of knots and then comparing the results for the coarse and the fine grid with the aid of the exponential convergence order, i.e., by the fact that doubling the number $2n$ of knots will double the number of correct digits in the approximate solution.

For a numerical example, we consider the scattering of a plane wave by a cylinder with a non-convex kite-shaped cross section with boundary ∂D illustrated in Fig. 3.1 and described by the parametric representation

$$x(t) = (\cos t + 0.65\cos 2t - 0.65,\ 1.5\sin t), \quad 0 \le t \le 2\pi.$$

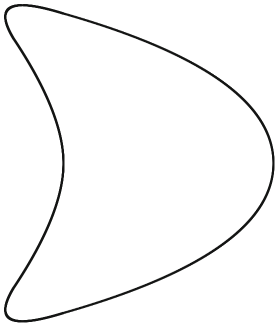

Fig. 3.1 Kite-shaped domain for numerical example

From the asymptotics (3.82) for the Hankel functions, analogous to (3.87) it can be deduced that the far field pattern of the combined potential (3.26) in two dimensions is given by

$$u_\infty(\hat{x}) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_{\partial D} \{k\,\nu(y)\cdot\hat{x} + \eta\} e^{-ik\,\hat{x}\cdot y} \varphi(y)\,ds(y), \quad |\hat{x}| = 1,$$

which can be evaluated again by the trapezoidal rule after solving the integral equation for φ. Table 3.1 gives some approximate values for the far field pattern $u_\infty(d)$ and $u_\infty(-d)$ in the forward direction d and the backward direction $-d$. The direction d of the incident wave is $d = (1, 0)$ and, as recommended in [197], the coupling parameter is $\eta = k$. Note that the exponential convergence is clearly exhibited.

Table 3.1 Numerical results for Nyström's method

	n	Re $u_\infty(d)$	Im $u_\infty(d)$	Re $u_\infty(-d)$	Im $u_\infty(-d)$
$k = 1$	8	-1.62642413	0.60292714	1.39015283	0.09425130
	16	-1.62745909	0.60222343	1.39696610	0.09499454
	32	-1.62745750	0.60222591	1.39694488	0.09499635
	64	-1.62745750	0.60222591	1.39694488	0.09499635
$k = 5$	8	-2.30969119	1.52696566	-0.30941096	0.11503232
	16	-2.46524869	1.67777368	-0.19932343	0.06213859
	32	-2.47554379	1.68747937	-0.19945788	0.06015893
	64	-2.47554380	1.68747937	-0.19945787	0.06015893

The corresponding quadrature method including its error and convergence analysis for the Neumann boundary condition has been described by Kress [202].

For domains D with corners, a uniform mesh yields only poor convergence and therefore has to be replaced by a graded mesh. We suggest to base this grading upon the idea of substituting an appropriate new variable and then using the Nyström method as described above for the transformed integral equation. With a suitable choice for the substitution, this will lead to high order convergence.

Without loss of generality, we confine our presentation to a boundary curve ∂D with one corner at the point x_0 and assume $\partial D \setminus \{x_0\}$ to be C^2 and piecewise analytic. We do not allow cusps in our analysis, i.e., the angle γ at the corner is assumed to satisfy $0 < \gamma < 2\pi$.

Using the fundamental solution

$$\Phi_0(x, y) := \frac{1}{2\pi} \ln \frac{1}{|x - y|}, \quad x \neq y,$$

to the Laplace equation in $\mathbb{R}^2$ to subtract a vanishing term, we rewrite the combined double- and single-layer potential (3.26) in the form

$$u(x) = \int_{\partial D} \left[\left\{ \frac{\partial \Phi(x, y)}{\partial \nu(y)} - i\eta \Phi(x, y) \right\} \varphi(y) - \frac{\partial \Phi_0(x, y)}{\partial \nu(y)} \varphi(x_0) \right] ds(y)$$

for $x \in \mathbb{R}^2 \setminus \bar{D}$. This modification is notationally advantageous for the corner case and it makes the error analysis for the Nyström method work. The integral equation (3.27) now becomes

$$\begin{aligned} \varphi(x) - \varphi(x_0) + 2 \int_{\partial D} \left\{ \frac{\partial \Phi(x, y)}{\partial \nu(y)} - i\eta \Phi(x, y) \right\} \varphi(y)\, ds(y) \\ - 2 \int_{\partial D} \frac{\partial \Phi_0(x, y)}{\partial \nu(y)} \varphi(x_0)\, ds(y) = 2f(x), \quad x \in \partial D. \end{aligned} \tag{3.99}$$

Despite the corner at x_0, there is no change in the residual term in the jump relations since the density $\varphi - \varphi(x_0)$ of the leading term in the singularity vanishes at the corner. However, the kernel of the integral equation (3.99) at the corner no longer remains weakly singular. For a C^2 boundary, the weak singularity of the kernel of the double-layer operator rests on the inequality

$$|\nu(y) \cdot (x - y)| \leq L|x - y|^2, \quad x, y \in \partial D, \tag{3.100}$$

for some positive constant L. This inequality expresses the fact that the vector $x - y$ for x close to y is almost orthogonal to the normal vector $\nu(y)$. For a proof, we refer to [64]. However, in the vicinity of a corner (3.100) does not remain valid.

After splitting off the operator $K_0 : C(\partial D) \to C(\partial D)$ defined by

$$(K_0\varphi)(x) := 2 \int_{\partial D} \frac{\partial \Phi_0(x, y)}{\partial \nu(y)} [\varphi(y) - \varphi(x_0)]\, ds(y), \quad x \in \partial D,$$

from (3.84) we see that the remaining integral operator in (3.99) has a weakly singular kernel and therefore is compact. For the further investigation of the non-compact part K_0, we choose a sufficiently small positive number r and denote the two arcs of the boundary ∂D contained in the disk of radius r and center at the corner x_0 by

A and B (see Fig. 3.2). These arcs intersect at x_0 with an angle γ and without loss of generality we restrict our presentation to the case where $\gamma < \pi$. By elementary geometry and continuity, we can assume that r is chosen such that both A and B have length less than $2r$ and for the angle $\alpha(x, B)$ between the two straight lines connecting the points $x \in A \setminus \{x_0\}$ with the two endpoints of the arc B we have

$$0 < \alpha(x, B) \leq \pi - \frac{1}{2}\gamma, \quad x \in A \setminus x_0,$$

and analogously with the roles of A and B interchanged. For the sake of brevity, we confine ourselves to the case where the boundary ∂D in a neighborhoud of the corner x_0 consists of two straight lines intersecting at x_0. Then we can assume that r is chosen such that the function $(x, y) \mapsto \nu(y) \cdot (y - x)$ does not change its sign for all $(x, y) \in A \times B$ and all $(x, y) \in B \times A$. Finally, for the two C^2 arcs A and B, there exists a constant L independent of r such that the estimate (3.100) holds for all $(x, y) \in A \times A$ and all $(x, y) \in B \times B$.

We now choose a continuous cut-off function $\psi : \mathbb{R}^2 \to [0, 1]$ such that $\psi(x) = 1$ for $0 \leq |x-x_0| \leq r/2$, $\psi(x) = 0$ for $r \leq |x-x_0| < \infty$ and define $K_{0,r} : C(\partial D) \to C(\partial D)$ by

$$K_{0,r}\varphi := \psi\, K_0(\psi\varphi).$$

Then, the kernel of $K_0 - K_{0,r}$ vanishes in a neighborhood of (x_0, x_0) and therefore is weakly singular.

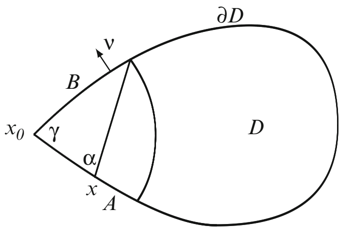

Fig. 3.2 Domain with a corner

We introduce the norm

$$\|\varphi\|_{\infty,0} := \max_{x\in\partial D} |\varphi(x) - \varphi(x_0)| + |\varphi(x_0)|$$

which obviously is equivalent to the maximum norm. We now show that r can be chosen such that $\|K_{0,r}\|_{\infty,0} < 1$. Then, by the Neumann series, the operator $I + K_{0,r}$

has a bounded inverse and the results of the Riesz–Fredholm theory are available for the corner integral equation (3.99).

By our assumptions on the choice of r, we can estimate

$$|(K_{0,r}\varphi)(x_0)| \le \frac{4Lr}{\pi}\,\|\varphi\|_{\infty,0} \tag{3.101}$$

since (3.100) holds for $x = x_0$ and all $y \in A \cup B$. For $x \in A \setminus \{x_0\}$ we split the integral into the parts over A and over B and evaluate the second one by using Green's integral theorem and our assumptions on the geometry to obtain

$$2\int_B \left|\frac{\partial \Phi_0(x,y)}{\partial \nu(y)}\right| ds(y) = 2\left|\int_B \frac{\partial \Phi_0(x,y)}{\partial \nu(y)}\,ds(y)\right| = \frac{\alpha(x,B)}{\pi}\,, \qquad x \in A \setminus \{x_0\},$$

and consequently

$$|(K_{0,r}\varphi)(x)| \le \left\{\frac{2Lr}{\pi} + 1 - \frac{\gamma}{2\pi}\right\}\|\varphi\|_{\infty,0}, \tag{3.102}$$

which by symmetry is valid for all $x \in A \cup B \setminus \{x_0\}$. Summarizing, from the inequalities (3.101) and (3.102) we deduce that we can choose r small enough such that $\|K_{0,r}\|_{\infty,0} < 1$. For an analysis for more general domains with corners we refer to Ruland [297] and the literature therein.

The above analysis establishes the existence of a continuous solution to the integral equation (3.99). However, due to the singularities of elliptic boundary value problems in domains with corners (see [110]), this solution will have singularities in the derivatives at the corner. To take proper care of this corner singularity, we replace our equidistant mesh by a graded mesh through substituting a new variable in such a way that the derivatives of the new integrand vanish up to a certain order at the endpoints and then use the quadrature rules (3.93) and (3.94) for the transformed integrals.

We describe this numerical quadrature rule for the integral $\int_0^{2\pi} f(t)\,dt$ where the integrand f is analytic in $(0, 2\pi)$ but has singularities at the endpoints $t = 0$ and $t = 2\pi$. Let the function $w : [0, 2\pi] \to [0, 2\pi]$ be one-to-one, strictly monotonically increasing and infinitely differentiable. We assume that the derivatives of w at the endpoints $t = 0$ and $t = 2\pi$ vanish up to an order $p \in \mathbb{N}$. We then substitute $t = w(s)$ to obtain

$$\int_0^{2\pi} f(t)\,dt = \int_0^{2\pi} w'(s)\, f(w(s))\,ds.$$

Applying the trapezoidal rule to the transformed integral now yields the quadrature formula

$$\int_0^{2\pi} f(t)\,dt \approx \frac{\pi}{n}\sum_{j=1}^{2n-1} a_j f(s_j) \tag{3.103}$$

with the weights and mesh points given by

$$a_j = w'\left(\frac{j\pi}{n}\right), \quad s_j = w\left(\frac{j\pi}{n}\right), \quad j = 1, \dots, 2n-1.$$

A typical example for such a substitution is given by

$$w(s) = 2\pi \, \frac{[v(s)]^p}{[v(s)]^p + [v(2\pi - s)]^p}\,, \quad 0 \le s \le 2\pi, \tag{3.104}$$

where

$$v(s) = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{\pi - s}{\pi}\right)^3 + \frac{1}{p}\,\frac{s - \pi}{\pi} + \frac{1}{2}$$

and $p \ge 2$. Note that the cubic polynomial v is chosen such that $v(0) = 0$, $v(2\pi) = 1$ and $w'(\pi) = 2$. The latter property ensures, roughly speaking, that one half of the grid points is equally distributed over the total interval, whereas the other half is accumulated towards the two end points.

For an error analysis for the quadrature rule (3.103) with substitutions of the form described above and using the Euler–MacLaurin expansion, we refer to Kress [200]. Assume f is $2q + 1$–times continuously differentiable on $(0, 2\pi)$ such that for some $0 < \alpha < 1$ with $\alpha p \ge 2q + 1$ the integrals

$$\int_0^{2\pi} \left[\sin \frac{t}{2}\right]^{m-\alpha} |f^{(m)}(t)|\, dt$$

exist for $m = 0, 1, \dots, 2q + 1$. The error $E^{(n)}(f)$ in the quadrature (3.103) can then be estimated by

$$|E^{(n)}(f)| \le \frac{C}{n^{2q+1}} \tag{3.105}$$

with some constant C. Thus, by choosing p large enough, we can obtain almost exponential convergence behavior.

For the numerical solution of the corner integral equation (3.99), we choose a parametric representation of the form (3.90) such that the corner x_0 corresponds to the parameter $t = 0$ and rewrite (3.99) in the parameterized form

$$\begin{aligned} \psi(t) - \psi(0) - \int_0^{2\pi} K(t,\tau)\psi(\tau)\, d\tau \\ - \int_0^{2\pi} H(t,\tau)\psi(0)\, d\tau = g(t), \quad 0 \le t \le 2\pi, \end{aligned} \tag{3.106}$$

where K is given as above in the analytic case and where

$$H(t,\tau) = \begin{cases} \dfrac{1}{\pi}\, \dfrac{x_2'(\tau)[x_1(t) - x_1(\tau)] - x_1'(\tau)[x_2(t) - x_2(\tau)]}{|x(t) - x(\tau)|^2}\,, & t \ne \tau, \\[2ex] \dfrac{1}{\pi}\, \dfrac{x_2'(t)x_1''(t) - x_1'(t)x_2''(t)}{|x'(t)|^2}\,, & t = \tau,\ t \ne 0, 2\pi, \end{cases}$$

corresponds to the additional term in (3.99). For the numerical solution of the integral equation (3.106) by Nyström's method on the graded mesh, we also have to take into account the logarithmic singularity. We set $t = w(s)$ and $\tau = w(\sigma)$ to obtain

$$\int_0^{2\pi} K(t, \tau)\, \psi(\tau)\, d\tau = \int_0^{2\pi} K(w(s), w(\sigma))\, w'(\sigma)\psi(w(\sigma))\, d\sigma$$

and then write

$$K(w(s), w(\sigma)) = \tilde{K}_1(s, \sigma) \ln\left(4 \sin^2 \frac{s - \sigma}{2}\right) + \tilde{K}_2(s, \sigma).$$

This decomposition is related to (3.92) by

$$\tilde{K}_1(s, \sigma) = K_1(w(s), w(\sigma))$$

and

$$\tilde{K}_2(s, \sigma) = K(w(s), w(\sigma)) - \tilde{K}_1(s, \sigma) \ln\left(4 \sin^2 \frac{s - \sigma}{2}\right), \quad s \neq \sigma.$$

From

$$K_2(s, s) = \lim_{\sigma \to s}\left[K(s, \sigma) - K_1(s, \sigma) \ln\left(4 \sin^2 \frac{s - \sigma}{2}\right)\right]$$

we deduce the diagonal term

$$\tilde{K}_2(s, s) = K_2(w(s), w(s)) + 2 \ln w'(s)\, K_1(w(s), w(s)).$$

Now, proceeding as in the derivation of (3.96), for the approximate values $\psi_i^{(n)} = \psi^{(n)}(s_i)$ at the quadrature points s_i for $i = 1, \dots, 2n - 1$ and $\psi_0^{(n)} = \psi^{(n)}(0)$ at the corner $s_0 = 0$ we arrive at the linear system

$$\begin{aligned} \psi_i^{(n)} - \psi_0^{(n)} - \sum_{j=1}^{2n-1} \left\{R_{|i-j|}^{(n)} \tilde{K}_1(s_i, s_j) + \frac{\pi}{n}\, \tilde{K}_2(s_i, s_j)\right\} a_j \psi_j^{(n)} & \\ - \sum_{j=1}^{2n-1} \frac{\pi}{n}\, H(s_i, s_j)\, a_j \psi_0^{(n)} = g(s_i), \quad i = 0, \dots, 2n - 1. & \end{aligned} \tag{3.107}$$

A rigorous error analysis carrying over the error behavior (3.105) to the approximate solution of the integral equation obtained from (3.107) for the potential theoretic case $k = 0$ has been worked out by Kress [200]. Related substitution methods have been considered by Jeon [165] and by Elliott and Prössdorf [94, 95].

For a numerical example, we used the substitution (3.104) with order $p = 8$. We consider a drop-shaped domain with the boundary curve ∂D illustrated by Fig. 3.3 and given by the parametric representation

$$x(t) = (2 \sin \frac{t}{2}\,,\, - \sin t), \quad 0 \leq t \leq 2\pi.$$

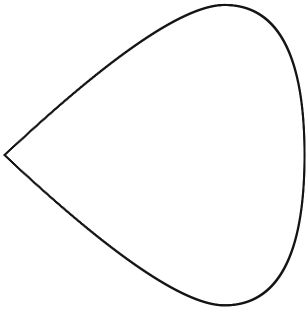

Fig. 3.3 Drop-shaped domain for numerical example

It has a corner at $t = 0$ with interior angle $\gamma = \pi/2$. The direction d of the incoming plane wave and the coupling parameter η are chosen as in our previous example. Table 3.2 clearly exhibits the fast convergence of the method.

Table 3.2 Nyström's method for a domain with corner

	n	$\operatorname{Re} u_\infty(d)$	$\operatorname{Im} u_\infty(d)$	$\operatorname{Re} u_\infty(-d)$	$\operatorname{Im} u_\infty(-d)$
$k = 1$	16	-1.28558226	0.30687170	-0.53002440	-0.41033666
	32	-1.28549613	0.30686638	-0.53020518	-0.41094518
	64	-1.28549358	0.30686628	-0.53021014	-0.41096324
	128	-1.28549353	0.30686627	-0.53021025	-0.41096364
$k = 5$	16	-1.73779647	1.07776749	-0.18112826	-0.20507986
	32	-1.74656264	1.07565703	-0.19429063	-0.19451172
	64	-1.74656303	1.07565736	-0.19429654	-0.19453324
	128	-1.74656304	1.07565737	-0.19429667	-0.19453372

3.6 On the Numerical Solution in $\mathbb{R}^3$

In three dimensions, for the numerical solution of the boundary integral equation (3.27) the Nyström, collocation and Galerkin methods are still available. However, for surface integral equations we have to modify our statements on comparing the efficiency of the three methods. Firstly, there is no straightforward simple quadrature rule analogous to (3.93) available that deals appropriately with the singularity of the three-dimensional fundamental solution. Hence, the Nyström method looses some of its attraction. Secondly, for the surface integral equations there is no immediate choice for global approximations like the trigonometric polynomials in the one-dimensional periodic case. Therefore, local approximations by low order polynomial splines have been more widely used and the collocation method is the most important numerical approximation method. To implement the collocation method, the boundary surface is first subdivided into a finite number of segments, like curved triangles and squares. The approximation space is then chosen to consist of low order polynomial splines with respect to these surface elements. The simplest choices are piecewise constants or piecewise linear functions. Within each segment, depending on the degree of freedom in the chosen splines, a number of collocation points is selected. Then, the integrals for the matrix elements in the collocation system are evaluated using numerical integration. Due to the weak singularity of the kernels, the calculation of the improper integrals for the diagonal elements of the matrix, where the collocation points and the surface elements coincide, needs special attention. For a detailed description of this so-called *boundary element method* we refer to Rjasanow and Steinbach [295] and to Sauter and Schwab [300].

Besides these local approximations via boundary elements there are also global approaches available in the sense of spectral methods. For surfaces which can be mapped onto spheres, Atkinson [15] has developed a Galerkin method for the Laplace equation using spherical harmonics as the counterpart of the trigonometric polynomials. This method has been extended to the Helmholtz equation by Lin [228]. Based on spherical harmonics and transforming the boundary surface to a sphere as in Atkinson's method, Wienert [332] has developed a Nyström type method for the boundary integral equations for three-dimensional Helmholtz problems which exhibits exponential convergence for analytic boundary surfaces. Wienert's method has been further developed into a fully discrete Galerkin type method through the work of Ganesh, Graham and Sloan [99, 109]. We conclude this chapter by introducing the main ideas of this method.

We begin by describing a numerical quadrature scheme for the integration of analytic functions over closed analytic surfaces Γ in $\mathbb{R}^3$ which are homeomorphic to the unit sphere $\mathbb{S}^2$ and then we proceed to corresponding quadratures for acoustic single- and double-layer potentials. To this end, we first introduce a suitable projection operator Q_N onto the linear space H_{N-1} of all spherical harmonics of order less than N. We denote by $-1 < t_1 < t_2 < \cdots < t_N < 1$ the zeros of the Legendre polynomial P_N (the existence of N distinct zeros of P_N in the interval $(-1, 1)$ is a consequence of the orthogonality relation (2.25), see [90], p. 236) and by

$$\alpha_j := \frac{2(1-t_j^2)}{[NP_{N-1}(t_j)]^2}\,, \quad j = 1, \dots, N,$$

the weights of the Gauss–Legendre quadrature rule which are uniquely determined by the property

$$\int_{-1}^{1} p(t)\,dt = \sum_{j=1}^{N} \alpha_j p(t_j) \tag{3.108}$$

for all polynomials p of degree less than or equal to $2N - 1$ (see [91], p. 89). We then choose a set of points x_{jk} on the unit sphere $\mathbb{S}^2$ given in polar coordinates by

$$x_{jk} := (\sin\theta_j \cos\varphi_k, \sin\theta_j \sin\varphi_k, \cos\theta_j)$$

for $j = 1, \dots, N$ and $k = 0, \dots, 2N - 1$ where $\theta_j := \arccos t_j$ and $\varphi_k = \pi k/N$ and define $Q_N : C(\mathbb{S}^2) \to H_{N-1}$ by

$$Q_N f := \frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j f(x_{jk}) \sum_{n=0}^{N-1} \sum_{m=-n}^{n} Y_n^{-m}(x_{jk})\, Y_n^m \tag{3.109}$$

where the spherical harmonics Y_n^m are given by (2.28). By orthogonality we clearly have

$$\int_{\mathbb{S}^2} Q_N f\, Y_n^{-m}\, ds = \frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j f(x_{jk}) Y_n^{-m}(x_{jk}) \tag{3.110}$$

for $|m| \le n < N$. Since the trapezoidal rule with $2N$ knots integrates trigonometric polynomials of degree less than N exactly, we have

$$\frac{\pi}{N} \sum_{k=0}^{2N-1} Y_n^m(x_{jk}) Y_{n'}^{-m'}(x_{jk}) = \int_0^{2\pi} Y_n^m(\theta_j, \varphi)\, Y_{n'}^{-m'}(\theta_j, \varphi)\, d\varphi$$

for $|m|, |m'| \le n < N$ and these integrals, in view of (2.28), vanish if $m \ne m'$. For $m = m'$, by (2.27) and (2.28), $Y_n^m\, Y_{n'}^{-m}$ is a polynomial of degree less than $2N$ in $\cos\theta$. Hence, by the property (3.108) of the Gauss–Legendre quadrature rule, summing the previous equation we find

$$\frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j Y_n^m(x_{jk})\, Y_{n'}^{-m'}(x_{jk}) = \int_{\mathbb{S}^2} Y_n^m\, Y_{n'}^{-m'}\, ds,$$

that is, $Q_N Y_n^m = Y_n^m$ for $|m| \le n < N$ and therefore Q_N is indeed a projection operator onto H_{N-1}. We note that Q_N is not an interpolation operator since by Theorem 2.7 we have $\dim H_{N-1} = N^2$ whereas we have $2N^2$ points x_{jk}. Therefore, it is also called a *hyperinterpolation operator*. With the aid of (2.22), the addition theorem (2.30) and (3.108) we can estimate

$$\|Q_N f\|_\infty \le \frac{1}{4N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j \sum_{n=0}^{N-1} (2n+1)\, \|f\|_\infty = N^2 \|f\|_\infty$$

whence

$$\|Q_N\|_\infty \le N^2 \tag{3.111}$$

follows. However, this straightforward estimate is suboptimal and can be improved into

$$c_1\, N^{1/2} \le \|Q_N\|_\infty \le c_2\, N^{1/2} \tag{3.112}$$

with positive constants $c_1 < c_2$ (see [109, 307]). For analytic functions $f : \mathbb{S}^2 \to \mathbb{C}$, Wienert [332] has shown that the approximation error $f - Q_N f$ decreases exponentially, that is, there exist positive constants C and σ depending on f such that

$$\|f - Q_N f\|_{\infty,\mathbb{S}^2} \leq C e^{-N\sigma} \tag{3.113}$$

for all $N \in \mathbb{N}$.

Integrating the approximation $Q_N f$ instead of f we obtain the so-called Gauss trapezoidal product rule

$$\int_{\mathbb{S}^2} f\, ds \approx \frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j f(x_{jk}) \tag{3.114}$$

for the numerical integration over the unit sphere. For analytic surfaces Γ which can be mapped bijectively through an analytic function $q : \mathbb{S}^2 \to \Gamma$ onto the unit sphere, (3.114) can also be used after the substitution

$$\int_{\Gamma} g(\xi)\, ds(\xi) = \int_{\mathbb{S}^2} g(q(x)) J_q(x)\, ds(x)$$

where J_q stands for the Jacobian of the mapping q. For analytic functions, the exponential convergence (3.113) carries over to the quadrature (3.114).

By passing to the limit $k \to 0$ in (2.44), with the help of (2.32) and (2.33), we find

$$\int_{\mathbb{S}^2} \frac{Y_n(y)}{|x-y|}\, ds(y) = \frac{4\pi}{2n+1}\, Y_n(x), \quad x \in \mathbb{S}^2,$$

for spherical harmonics Y_n of order n. This can be used together with the addition formula (2.30) to obtain the approximation

$$\int_{\mathbb{S}^2} \frac{f(y)}{|x-y|}\, ds(y) \approx \frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j f(x_{jk}) \sum_{n=0}^{N-1} P_n(x_{jk} \cdot x), \quad x \in \mathbb{S}^2,$$

which again is based on replacing f by $Q_N f$. In particular, for the north pole $x_0 = (0, 0, 1)$ this reads

$$\int_{\mathbb{S}^2} \frac{f(y)}{|x_0 - y|}\, ds(y) \approx \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \beta_j f(x_{jk}) \tag{3.115}$$

where

$$\beta_j := \frac{\pi \alpha_j}{N} \sum_{n=0}^{N-1} P_n(t_j), \quad j = 1, \ldots, N.$$

The exponential convergence for analytic densities $f : \mathbb{S}^2 \to \mathbb{C}$ again carries over from (3.113) to the numerical quadrature (3.115) of the harmonic single-layer potential.

For the extension of this quadrature scheme to more general surfaces Γ, we need to allow more general densities and we can do this without losing the rapid convergence order. Denote by $\tilde{\mathbb{S}}^2$ the cylinder

$$\tilde{\mathbb{S}}^2 := \{(\cos\varphi, \sin\varphi, \theta) : 0 \le \theta \le \pi,\ 0 \le \varphi \le 2\pi\}.$$

Then we can identify functions defined on $\tilde{\mathbb{S}}^2$ with functions on $\mathbb{S}^2$ through the mapping

$$(\cos\varphi, \sin\varphi, \theta) \mapsto (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

and, loosely speaking, in the sequel we refer to functions on $\tilde{\mathbb{S}}^2$ as functions on $\mathbb{S}^2$ depending on the azimuth φ at the poles. As Wienert [332] has shown, the exponential convergence is still true for the application of (3.115) to analytic functions $f : \tilde{\mathbb{S}}^2 \to \mathbb{C}$.

For the general surface Γ as above, we write

$$\int_\Gamma \frac{g(\eta)}{|q(x) - \eta|}\, ds(\eta) = \int_{\mathbb{S}^2} \frac{F(x, y) f(y)}{|x - y|}\, ds(y)$$

where we have set $f(y) := g(q(y)) J_q(y)$ and

$$F(x, y) := \frac{|x - y|}{|q(x) - q(y)|}\,, \qquad x \neq y. \tag{3.116}$$

Unfortunately, as can be seen from simple examples, the function F in general cannot be extended as a continuous function on $\mathbb{S}^2 \times \mathbb{S}^2$. However, since on the unit sphere we have

$$|x - y|^2 = 2(1 - x \cdot y)$$

from the estimate (see the proof of Theorem 2.2 in [64])

$$c_1 |x - y|^2 \le |q(x) - q(y)|^2 \le c_2 |x - y|^2$$

which is valid for all $x, y \in \mathbb{S}^2$ and some constants $0 < c_1 < c_2$ it can be seen that $F(x_0, \cdot)$ is analytic on $\tilde{\mathbb{S}}^2$.

For $\psi \in \mathbb{R}$, we define the orthogonal transformations

$$D_P(\psi) := \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$D_T(\psi) := \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}.$$

Then for $x = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in \mathbb{S}^2$ the orthogonal transformation

$$T_x := D_P(\varphi) D_T(\theta) D_P(-\varphi)$$

has the property

$$T_x x = (0, 0, 1), \quad x \in \mathbb{S}^2.$$

Therefore

$$\int_{\mathbb{S}^2} \frac{F(x, y) f(y)}{|x - y|} \, ds(y) \approx \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \beta_j F(x, T_x^{-1} x_{jk}) f(T_x^{-1} x_{jk}) \tag{3.117}$$

is exponentially convergent for analytic densities f in the sense of (3.113) since x is the north pole for the set of quadrature points $T_x^{-1} x_{jk}$. It can be shown that the exponential convergence is uniform with respect to $x \in \mathbb{S}^2$.

By decomposing

$$\frac{e^{ik|x-y|}}{|x - y|} = \frac{\cos k|x - y|}{|x - y|} + i \, \frac{\sin k|x - y|}{|x - y|} \, ,$$

we see that the integral equation (3.27) for the exterior Dirichlet problem is of the form

$$g(\xi) - \int_{\Gamma} \left\{ \frac{h_1(\xi, \eta)}{|\xi - \eta|} + \frac{\nu(\eta) \cdot (\xi - \eta)}{|\xi - \eta|^2} \, h_2(\xi, \eta) + h_3(\xi, \eta) \right\} g(\eta) \, ds(\eta) = w(\xi)$$

for $\xi \in \Gamma$ with analytic kernels h_1, h_2 and h_3. For our purpose of exposition, it suffices to consider only the singular part, that is, the case when $h_2 = h_3 = 0$. Using the substitution $\xi = q(x)$ and $\eta = q(y)$, the integral equation over Γ can be transformed into an integral equation over $\mathbb{S}^2$ of the form

$$f(x) - \int_{\mathbb{S}^2} \frac{k(x, y) F(x, y)}{|x - y|} \, f(y) \, ds(y) = v(x), \quad x \in \mathbb{S}^2, \tag{3.118}$$

with the functions f, k and v appropriately defined through g, h_1 and w and with F given as in (3.116). We write $A : C(\mathbb{S}^2) \to C(\mathbb{S}^2)$ for the weakly singular integral operator

$$(Af)(x) := \int_{\mathbb{S}^2} \frac{k(x, y) F(x, y)}{|x - y|} \, f(y) \, ds(y), \quad x \in \mathbb{S}^2,$$

occuring in (3.118). By using the quadrature rule (3.117), we arrive at an approximating quadrature operator $A_N : C(\mathbb{S}^2) \to C(\tilde{\mathbb{S}}^2)$ given by

$$(A_N f)(x) := \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \beta_j k(x, T_x^{-1} x_{jk}) F(x, T_x^{-1} x_{jk}) f(T_x^{-1} x_{jk}), \quad x \in \mathbb{S}^2. \tag{3.119}$$

We observe that the quadrature points $T_x^{-1} x_{jk}$ depend on x. Therefore, we cannot reduce the solution of the approximating equation

$$\tilde{f}_N - A_N \tilde{f}_N = v \tag{3.120}$$

to a linear system in the usual fashion of Nyström interpolation. A possible remedy for this difficulty is to apply the projection operator Q_N a second time. For this, two variants have been proposed. Wienert [332] suggested

$$f_N^w - A_N Q_N f_N^w = v \tag{3.121}$$

as the final approximating equation for the solution of (3.120). Analogously to the presentation in the first edition of this book, Graham and Sloan [109] considered solving (3.120) through the projection method with the final approximating equation of the form

$$f_N - Q_N A_N f_N = Q_N v. \tag{3.122}$$

As observed in [109] there is an immediate one-to-one correspondence between the solutions of (3.121) and (3.122) via

$$f_N = Q_N f_N^w$$

and

$$f_N^w = v + A_N f_N.$$

Therefore, we restrict our outline on the numerical implementation to the second variant (3.122). Representing

$$f_N := \sum_{n=0}^{N-1} \sum_{m=-n}^{n} a_n^m \, Y_n^m$$

and using (3.110) and (3.119) we find that solving (3.122) is equivalent to solving the linear system

$$a_n^m - \sum_{n'=0}^{N-1} \sum_{m'=-n'}^{n'} R_{nn'}^{mm'} \, a_{n'}^{m'} = \frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j v(x_{jk}) Y_n^{-m}(x_{jk}) \tag{3.123}$$

for $n = 0, \dots, N-1,\ m = -n, \dots, n$, where

$$R_{nn'}^{mm'} := \frac{\pi}{N} \sum_{j_1=1}^{N} \sum_{k_1=0}^{2N-1} \sum_{j_2=1}^{N} \sum_{k_2=0}^{2N-1} \alpha_{j_1} \beta_{j_2} K(x_{j_1k_1}, x_{j_2k_2}) \, Y_n^{-m}(x_{j_1k_1}) \, Y_{n'}^{m'}(T_{x_{j_1k_1}}^{-1} x_{j_2k_2})$$

and

$$K(x, y) := k(x, T_x^{-1} y) F(x, T_x^{-1} y).$$

Since orthogonal transformations map spherical harmonics of order n into spherical harmonics of order n, we have

$$Y_{n'}^{m'}(D_T(-\theta) y) = \sum_{\mu=-n'}^{n'} Z_0(n', m', \mu, \theta) Y_{n'}^{\mu}(y)$$

with

$$Z_0(n', m', \mu, \theta) = \int_{\mathbb{S}^2} Y_{n'}^{m'}(D_T(-\theta)y)\, Y_{n'}^{-\mu}(y)\, ds(y)$$

and from (2.28) we clearly have

$$Y_{n'}^{m'}(D_P(-\varphi)y) = e^{-im'\varphi} Y_{n'}^{m'}(y).$$

From this we find that the coefficients in (3.123) can be evaluated recursively through the scheme

$$Z_1(j_1, k_1, j_2, \mu) := \sum_{k_2=0}^{2N-1} \beta_{j_2} e^{i\mu(\varphi_{k_2} - \varphi_{k_1})} K(x_{j_1 k_1}, x_{j_2 k_2}),$$

$$Z_2(j_1, k_1, n', \mu) := \sum_{j_2=1}^{N} Y_{n'}^{\mu}(x_{j_2,0}) Z_1(j_1, k_1, j_2, \mu),$$

$$Z_3(j_1, k_1, n', m') := \sum_{\mu=-n'}^{n'} Z_0(n', m', \mu, \theta_{j_1}) Z_2(j_1, k_1, n', \mu) e^{im'\varphi_{k_1}},$$

$$Z_4(j_1, m, n', m') := \sum_{k_1=0}^{2N-1} e^{-im\varphi_{k_1}} Z_3(j_1, k_1, n', m'),$$

$$R_{nn'}^{mm'} := \frac{\pi}{N} \sum_{j_1=1}^{N} \alpha_{j_1} Y_n^{-m}(x_{j_1,0}) Z_4(j_1, m, n', m'),$$

by $O(N^5)$ multiplications provided the numbers $Z_0(n', m', \mu, \theta_{j_1})$ (which do not depend on the surface) are precalculated. The latter calculations can be based on

$$\begin{aligned} Z_0(n', m', \mu, \theta) &= \int_{\mathbb{S}^2} (Q_N(Y_{n'}^{m'} \circ D_T(-\theta)))(y)\, Y_{n'}^{-\mu}(y)\, ds(y) \\ &= \frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j Y_{n'}^{m'}(D_T(-\theta)x_{jk})\, Y_{n'}^{-\mu}(x_{jk}). \end{aligned}$$

For further details we refer to [99, 332]. To obtain a convergence result, a further modification of (3.121) and (3.122) was required by using different orders N and N' for the projection operator Q_N and the approximation operator $A_{N'}$ such that

$$N' = \kappa N \tag{3.124}$$

for some $\kappa > 1$. Under this assumption Graham and Sloan [109] established superalgebraic convergence. We note that for the proof it is crucial that the exponent in the estimate (3.112) is less than one.

Table 3.3 gives approximate values for the far field pattern in the forward and backward direction for scattering of a plane wave with incident direction $d = (1, 0, 0)$ from a pinched ball with representation

$$r(\theta, \varphi) = \sqrt{1.44 + 0.5 \cos 2\varphi(\cos 2\theta - 1)}, \quad \theta \in [0, \pi], \; \varphi \in [0, 2\pi],$$

in polar coordinates. (For the shape of the pinched ball see Fig. 5.2.) The results were provided to us by Olha Ivanyshyn and obtained by using the combined double- and single-layer potential integral equation (3.27) with coupling parameter $\eta = k$ and applying the Graham and Sloan variant (3.122) of Wienert's method with $N' = 2N$. The rapid convergence behavior is clearly exhibited. For further numerical examples we refer to [99].

Table 3.3 Numerical results for Wienert's method

	N	Re $u_\infty(d)$	Im $u_\infty(d)$	Re $u_\infty(-d)$	Im $u_\infty(-d)$
k=1	8	-1.43201720	1.40315084	0.30954060	0.93110842
	16	-1.43218545	1.40328665	0.30945849	0.93112270
	32	-1.43218759	1.40328836	0.30945756	0.93112274
	64	-1.43218759	1.40328836	0.30945756	0.93112274
k=5	8	-1.73274564	5.80039242	1.86060183	0.92743363
	16	-2.10055735	5.86052809	1.56336545	1.07513529
	32	-2.10058191	5.86053941	1.56328188	1.07513840
	64	-2.10058191	5.86053942	1.56328188	1.07513841

Chapter 4
Ill-Posed Problems

As previously mentioned, for problems in mathematical physics Hadamard [118] postulated three requirements: a solution should exist, the solution should be unique, and the solution should depend continuously on the data. The third postulate is motivated by the fact that in all applications the data will be measured quantities. Therefore, one wants to make sure that small errors in the data will cause only small errors in the solution. A problem satisfying all three requirements is called *well-posed*. Otherwise, it is called *ill-posed*.

For a long time, research on ill-posed problems was neglected since they were not considered relevant to the proper treatment of applied problems. However, it eventually became apparent that a growing number of important problems fail to be well-posed, for example Cauchy's problem for the Laplace equation and the initial boundary value problem for the backward heat equation. In particular, a large number of inverse problems for partial differential equations turn out to be ill-posed. Most classical problems where one assumes the partial differential equation, its domain and its initial and/or boundary data completely prescribed are well-posed in a canonical setting. Usually, such problems are referred to as *direct problems*. However, if the problem consists in determining part of the differential equation or its domain or its initial and/or boundary data then this *inverse problem* quite often will be ill-posed in any reasonable setting. In this sense, there is a close linkage and interaction between research on inverse problems and ill-posed problems.

This chapter is intended as an introduction into the basic ideas on ill-posed problems and regularization methods for their stable approximate solution. We mainly confine ourselves to linear equations of the first kind with compact operators in Hilbert spaces and base our presentation on the singular value decomposition. From the variety of regularization concepts, we will discuss only the spectral cut-off, Tikhonov regularization, the discrepancy principle and quasi-solutions. At the end of the chapter, we will include some material on nonlinear problems.

For a more comprehensive study of ill-posed problems, we refer to Baumeister [18], Engl, Hanke and Neubauer [96], Groetsch [111], Kabanikhin [170], Kaltenbacher, Neubauer and Scherzer [171], Kirsch [182], Kress [205], Louis [232], Morozov [249], Tikhonov and Arsenin [316] and Wang, Yagola and Yang [324].

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*,
Applied Mathematical Sciences 93, DOI 10.1007/978-1-4614-4942-3_4,

4.1 The Concept of Ill-Posedness

We will first make Hadamard's concept of well-posedness more precise.

Definition 4.1 *Let $A : U \subset X \to V \subset Y$ be an operator from a subset U of a normed space X into a subset V of a normed space Y. The equation*

$$A(\varphi) = f \tag{4.1}$$

is called well-posed *or* properly posed *if $A : U \to V$ is bijective and the inverse operator $A^{-1} : V \to U$ is continuous. Otherwise the equation is called* ill-posed *or* improperly posed.

According to this definition we may distinguish three types of ill-posedness. If A is not surjective, then equation (4.1) is not solvable for all $f \in V$ (*nonexistence*). If A is not injective, then equation (4.1) may have more than one solution (*nonuniqueness*). Finally, if $A^{-1} : V \to U$ exists but is not continuous then the solution φ of equation (4.1) does not depend continuously on the data f (*instability*). The latter case of instability is the one of primary interest in the study of ill-posed problems. We note that the three properties, in general, are not independent. For example, if $A : X \to Y$ is a bounded linear operator mapping a Banach space X bijectively onto a Banach space Y, then by the inverse mapping theorem the inverse operator $A^{-1} : Y \to X$ is bounded and therefore continuous. Note that the well-posedness of a problem is a property of the operator A together with the solution space X and the data space Y including the norms on X and Y. Therefore, if an equation is ill-posed one could try and restore stability by changing the spaces X and Y and their norms. But, in general, this approach is inadequate since the spaces X and Y including their norms are determined by practical needs. In particular, the space Y and its norm must be suitable to describe the measured data and the data error.

The typical example of an ill-posed problem is a completely continuous operator equation of the first kind. Recall that an operator $A : U \subset X \to Y$ is called compact if it maps bounded sets from U into relatively compact sets in Y and that A is called *completely continuous* if it is continuous and compact. Since linear compact operators are always continuous, for linear operators there is no need to distinguish between compactness and complete continuity.

Theorem 4.2. *Let $A : U \subset X \to Y$ be a completely continuous operator from a subset U of a normed space X into a subset V of a normed space Y. Then the equation of the first kind $A\varphi = f$ is improperly posed if U is not of finite dimension.*

Proof. Assume that $A^{-1} : V \to U$ exists and is continuous. Then from $I = A^{-1}A$ we see that the identity operator on U is compact since the product of a continuous and a compact operator is compact. Hence U must be finite dimensional. □

The ill-posed nature of an equation, of course, has consequences for its numerical treatment. We may view a numerical approximation of a given equation as the solution to perturbed data. Therefore, straightforward application of the classical methods for the approximate solution of operator equations to ill-posed problems usually will generate numerical nonsense. In terms of condition numbers, the fact that a bounded linear operator A does not have a bounded inverse means that the condition numbers of its finite dimensional approximations grow with the quality of the approximation. Hence, a careless discretization of ill-posed problems leads to a numerical behavior which at a first glance seems to be paradoxical. Namely, increasing the degree of discretization, i.e., increasing the accuracy of the approximation for the operator A will cause the approximate solution to the equation $A\varphi = f$ to become less and less reliable.

4.2 Regularization Methods

Methods for constructing a stable approximate solution of an ill-posed problem are called *regularization methods*. We shall now introduce the classical regularization concepts for linear equations of the first kind. In the sequel, we mostly will assume that the linear operator $A : X \to Y$ is injective. This is not a significant loss of generality since uniqueness for a linear equation always can be achieved by a suitable modification of the solution space X. We wish to approximate the solution φ to the equation $A\varphi = f$ from a knowledge of a perturbed right hand side f^δ with a known error level

$$\|f^\delta - f\| \le \delta. \tag{4.2}$$

When f belongs to the range $A(X) := \{A\varphi : \varphi \in X\}$ then there exists a unique solution φ of $A\varphi = f$. For a perturbed right hand side, in general, we cannot expect $f^\delta \in A(X)$. Using the erroneous data f^δ, we want to construct a reasonable approximation φ^δ to the exact solution φ of the unperturbed equation $A\varphi = f$. Of course, we want this approximation to be stable, i.e., we want φ^δ to depend continuously on the actual data f^δ. Therefore, our task requires finding an approximation of the unbounded inverse operator $A^{-1} : A(X) \to X$ by a bounded linear operator $R : Y \to X$.

Definition 4.3 *Let X and Y be normed spaces and let $A : X \to Y$ be an injective bounded linear operator. Then a family of bounded linear operators $R_\alpha : Y \to X$, $\alpha > 0$, with the property of pointwise convergence*

$$\lim_{\alpha \to 0} R_\alpha A\varphi = \varphi \tag{4.3}$$

for all $\varphi \in X$ is called a regularization scheme *for the operator A. The parameter α is called the* regularization parameter.

Of course, (4.3) is equivalent to $R_\alpha f \to A^{-1}f$, $\alpha \to 0$, for all $f \in A(X)$. The following theorem shows that for regularization schemes for compact operators this convergence cannot be uniform.

Theorem 4.4. *Let X and Y be normed spaces, let $A : X \to Y$ be a compact linear operator, and let $\dim X = \infty$. Then for a regularization scheme the operators R_α cannot be uniformly bounded with respect to α and the operators $R_\alpha A$ cannot be norm convergent as $\alpha \to 0$.*

Proof. For the first statement, assume $\|R_\alpha\| \le C$ for all $\alpha > 0$ and some constant C. Then from $R_\alpha f \to A^{-1}f$, $\alpha \to 0$, for all $f \in A(X)$ we deduce $\|A^{-1}f\| \le C\|f\|$, i.e., $A^{-1} : A(X) \to X$ is bounded. By Theorem 4.2 this is a contradiction to $\dim X = \infty$.

For the second statement, assume that we have norm convergence. Then there exists $\alpha > 0$ such that $\|R_\alpha A - I\| < 1/2$. Now for all $f \in A(X)$ we can estimate

$$\|A^{-1}f\| \le \|A^{-1}f - R_\alpha AA^{-1}f\| + \|R_\alpha f\| \le \frac{1}{2}\,\|A^{-1}f\| + \|R_\alpha\|\,\|f\|,$$

whence $\|A^{-1}f\| \le 2\|R_\alpha\|\,\|f\|$ follows. Therefore, $A^{-1} : A(X) \to X$ is bounded and we have the same contradiction as above. □

The regularization scheme approximates the solution φ of $A\varphi = f$ by the regularized solution

$$\varphi_\alpha^\delta := R_\alpha f^\delta. \tag{4.4}$$

Then, for the approximation error, writing

$$\varphi_\alpha^\delta - \varphi = R_\alpha f^\delta - R_\alpha f + R_\alpha A\varphi - \varphi,$$

by the triangle inequality we have the estimate

$$\|\varphi_\alpha^\delta - \varphi\| \le \delta\|R_\alpha\| + \|R_\alpha A\varphi - \varphi\|. \tag{4.5}$$

This decomposition shows that the error consists of two parts: the first term reflects the influence of the incorrect data and the second term is due to the approximation error between R_α and A^{-1}. Under the assumptions of Theorem 4.4, the first term cannot be estimated uniformly with respect to α and the second term cannot be estimated uniformly with respect to φ. Typically, the first term will be increasing as $\alpha \to 0$ due to the ill-posed nature of the problem whereas the second term will be decreasing as $\alpha \to 0$ according to (4.3). Every regularization scheme requires a strategy for choosing the parameter α in dependence on the error level δ and on the given data f^δ in order to achieve an acceptable total error for the regularized solution. On one hand, the accuracy of the approximation asks for a small error $\|R_\alpha A\varphi - \varphi\|$, i.e., for a small parameter α. On the other hand, the stability requires a small $\|R_\alpha\|$, i.e., a large parameter α. An optimal choice would try and make the right hand side of (4.5) minimal. The corresponding parameter effects a compromise between accuracy and stability. For a reasonable regularization strategy we expect the regularized solution to converge to the exact solution when the error level tends to zero. We express this requirement through the following definition.

Definition 4.5 *A* strategy *for a regularization scheme R_α, $\alpha > 0$, that is, the choice of the regularization parameter $\alpha = \alpha(\delta, f^\delta)$ depending on the error level δ and on f^δ is called* regular *if for all $f \in A(X)$ and all $f^\delta \in Y$ with $\|f^\delta - f\| \leq \delta$ we have*

$$R_{\alpha(\delta,f^\delta)} f^\delta \to A^{-1} f, \quad \delta \to 0.$$

In the discussion of regularization schemes, one usually has to distinguish between an *a priori* or an *a posteriori* choice of the regularization parameter α. An a priori choice would be based on some information on smoothness properties of the exact solution which, in practical problems, in general will not be available. Therefore, a posteriori strategies based on some considerations of the data error level δ are more practical.

A natural a posteriori strategy is given by the *discrepancy* or *residue principle* introduced by Morozov [247, 248]. Its motivation is based on the consideration that, in general, for erroneous data the residual $\|A\varphi - f\|$ should not be smaller than the accuracy of the measurements of f, i.e., the regularization parameter α should be chosen such that

$$\|AR_\alpha f^\delta - f^\delta\| = \gamma\delta$$

with some fixed parameter $\gamma \geq 1$ multiplying the error level δ. In the case of a regularization scheme R_m with a regularization parameter $m = 1, 2, 3, \ldots$ taking only discrete values, m should be chosen as the smallest integer satisfying

$$\|AR_m f^\delta - f^\delta\| \leq \gamma\delta.$$

Finally, we also need to note that quite often the only choice for selecting the regularization parameter will be *trial and error*, that is, one uses a few different parameters α and then picks the most reasonable result based on appropriate information on the expected solution.

4.3 Singular Value Decomposition

We shall now describe some regularization schemes in a Hilbert space setting. Our approach will be based on the singular value decomposition for compact operators which is a generalization of the spectral decomposition for compact self-adjoint operators.

Let X be a Hilbert space and let $A : X \to X$ be a self-adjoint compact operator, that is, $(A\varphi, \psi) = (\varphi, A\psi)$ for all $\varphi, \psi \in X$. Then all eigenvalues of A are real. $A \neq 0$ has at least one eigenvalue different from zero and at most a countable set of eigenvalues accumulating only at zero. All nonzero eigenvalues have finite multiplicity, that is, the corresponding eigenspaces are finite dimensional, and eigenelements corresponding to different eigenvalues are orthogonal. Assume the sequence (λ_n) of the nonzero eigenvalues is ordered such that

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$$

where each eigenvalue is repeated according to its multiplicity and let (φ_n) be a sequence of corresponding orthonormal eigenelements. Then for each $\varphi \in X$ we can expand

$$\varphi = \sum_{n=1}^{\infty} (\varphi, \varphi_n)\varphi_n + Q\varphi \tag{4.6}$$

where $Q : X \to N(A)$ denotes the orthogonal projection operator of X onto the nullspace $N(A) := \{\varphi \in X : A\varphi = 0\}$ and

$$A\varphi = \sum_{n=1}^{\infty} \lambda_n (\varphi, \varphi_n)\varphi_n. \tag{4.7}$$

For a proof of this *spectral decomposition* for self-adjoint compact operators see for example [205], Theorem 15.12.

We will now describe modified forms of the expansions (4.6) and (4.7) for arbitrary compact operators in a Hilbert space. Recall that for each bounded linear operator $A : X \to Y$ between two Hilbert spaces X and Y there exists a uniquely determined bounded linear operator $A^* : Y \to X$ called the *adjoint operator* of A such that $(A\varphi, \psi) = (\varphi, A^*\psi)$ for all $\varphi \in X$ and $\psi \in Y$.

Occasionally, we will make use of the following basic connection between the nullspaces and the ranges of A and A^*. Therefore, we include the simple proof.

Theorem 4.6. *For a bounded linear operator we have*

$$A(X)^\perp = N(A^*) \quad \textit{and} \quad N(A^*)^\perp = \overline{A(X)}.$$

Proof. $g \in A(X)^\perp$ means $(A\varphi, g) = 0$ for all $\varphi \in X$. This is equivalent to $(\varphi, A^*g) = 0$ for all $\varphi \in X$, which in turn is equivalent to $A^*g = 0$, that is, $g \in N(A^*)$. Hence, $A(X)^\perp = N(A^*)$. We abbreviate $U = A(X)$ and, trivially, have $\bar{U} \subset (U^\perp)^\perp$. Denote by $P : Y \to \bar{U}$ the orthogonal projection operator. Then for arbitrary $\varphi \in (U^\perp)^\perp$ we have orthogonality $P\varphi - \varphi \perp U$. But we also have $P\varphi - \varphi \perp U^\perp$ since we already know that $\bar{U} \subset (U^\perp)^\perp$. Therefore, it follows that $\varphi = P\varphi \in \bar{U}$, whence $\bar{U} = (U^\perp)^\perp$, i.e., $\overline{A(X)} = N(A^*)^\perp$. □

Now let $A : X \to Y$ be a compact linear operator. Then its adjoint operator $A^* : Y \to X$ is also compact. The nonnegative square roots of the eigenvalues of the nonnegative self-adjoint compact operator $A^*A : X \to X$ are called *singular values* of A.

Theorem 4.7. *Let* (μ_n) *denote the sequence of the nonzero singular values of the compact linear operator A (with* $A \neq 0$*) ordered such that*

$$\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots$$

and repeated according to their multiplicity, that is, according to the dimension of the nullspaces $N(\mu_n^2 I - A^*A)$. Then there exist orthonormal sequences (φ_n) in X and (g_n) in Y such that

$$A\varphi_n = \mu_n g_n, \quad A^* g_n = \mu_n \varphi_n \tag{4.8}$$

for all $n \in \mathbb{N}$. For each $\varphi \in X$ we have the singular value decomposition

$$\varphi = \sum_{n=1}^{\infty} (\varphi, \varphi_n)\varphi_n + Q\varphi \tag{4.9}$$

with the orthogonal projection operator $Q : X \to N(A)$ and

$$A\varphi = \sum_{n=1}^{\infty} \mu_n (\varphi, \varphi_n) g_n. \tag{4.10}$$

Each system (μ_n, φ_n, g_n), $n \in \mathbb{N}$, with these properties is called a singular system *of A. When there are only finitely many singular values the series (4.9) and (4.10) degenerate into finite sums.*

Proof. Let (φ_n) denote an orthonormal sequence of the eigenelements of A^*A, that is,

$$A^*A\varphi_n = \mu_n^2 \varphi_n$$

and define a second orthonormal sequence by

$$g_n := \frac{1}{\mu_n} A\varphi_n.$$

Straightforward computations show that the system (μ_n, φ_n, g_n), $n \in \mathbb{N}$, satisfies (4.8). Application of the expansion (4.6) to the self-adjoint compact operator A^*A yields

$$\varphi = \sum_{n=1}^{\infty} (\varphi, \varphi_n)\varphi_n + Q\varphi$$

for all $\varphi \in X$ where Q denotes the orthogonal projection operator from X onto $N(A^*A)$. Let $\psi \in N(A^*A)$. Then $(A\psi, A\psi) = (\psi, A^*A\psi) = 0$ and this implies that $N(A^*A) = N(A)$. Therefore, (4.9) is proven and (4.10) follows by applying A to (4.9). □

Note that the singular value decomposition implies that for all $\varphi \in X$ we have

$$\|\varphi\|^2 = \sum_{n=1}^{\infty} |(\varphi, \varphi_n)|^2 + \|Q\varphi\|^2, \tag{4.11}$$

$$\|A\varphi\|^2 = \sum_{n=1}^{\infty} \mu_n^2 |(\varphi, \varphi_n)|^2. \tag{4.12}$$

In the following theorem, we express the solution to an equation of the first kind with a compact operator in terms of a singular system.

Theorem 4.8. (Picard) *Let $A : X \to Y$ be a compact linear operator with singular system (μ_n, φ_n, g_n). The equation of the first kind*

$$A\varphi = f \tag{4.13}$$

is solvable if and only if f belongs to the orthogonal complement $N(A^*)^\perp$ and satisfies

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} \, |(f, g_n)|^2 < \infty. \tag{4.14}$$

In this case a solution is given by

$$\varphi = \sum_{n=1}^{\infty} \frac{1}{\mu_n} \, (f, g_n)\varphi_n. \tag{4.15}$$

Proof. The necessity of $f \in N(A^*)^\perp$ follows from Theorem 4.6. If φ is a solution of (4.13) then

$$\mu_n(\varphi, \varphi_n) = (\varphi, A^* g_n) = (A\varphi, g_n) = (f, g_n)$$

and (4.11) implies

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} \, |(f, g_n)|^2 = \sum_{n=1}^{\infty} |(\varphi, \varphi_n)|^2 \le \|\varphi\|^2,$$

whence the necessity of (4.14) follows.

Conversely, assume that $f \in N(A^*)^\perp$ and (4.14) is fulfilled. Then, by considering the partial sums of (4.14), we see that the series (4.15) converges in the Hilbert space X. We apply A to (4.15), use (4.9) with the singular system (μ_n, g_n, φ_n) of the operator A^* and observe $f \in N(A^*)^\perp$ to obtain

$$A\varphi = \sum_{n=1}^{\infty} (f, g_n) g_n = f.$$

This ends the proof. □

Picard's theorem demonstrates the ill-posed nature of the equation $A\varphi = f$. If we perturb the right hand side by setting $f^\delta = f + \delta g_n$ we obtain a perturbed solution $\varphi^\delta = \varphi + \delta\varphi_n/\mu_n$. Hence, the ratio $\|\varphi^\delta - \varphi\|/\|f^\delta - f\| = 1/\mu_n$ can be made arbitrarily large due to the fact that the singular values tend to zero. The influence of errors in the data f is obviously controlled by the rate of this convergence. In this sense, we may say that the equation is *mildly ill-posed* if the singular values decay slowly to zero and that it is *severely ill-posed* if they decay very rapidly.

Coming back to the far field mapping introduced in Section 2.5, we may consider it as a compact operator $A : L^2(S_R) \to L^2(\mathbb{S}^2)$ transferring the restriction of a radiating solution u to the Helmholtz equation to the sphere S_R with radius R and center at the origin onto its far field pattern u_∞. From Theorems 2.15 and 2.16 we see that $\varphi \in L^2(S_R)$ with the expansion

$$\varphi(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m Y_n^m\left(\frac{x}{|x|}\right), \quad |x| = R,$$

is mapped onto

$$(A\varphi)(\hat{x}) = \frac{1}{k} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{a_n^m}{i^{n+1} h_n^{(1)}(kR)}\, Y_n^m(\hat{x}), \quad \hat{x} \in \mathbb{S}^2.$$

Therefore, the singular values of A are given by

$$\mu_n = \frac{1}{kR^2 |h_n^{(1)}(kR)|}, \quad n = 0, 1, 2, \ldots,$$

and from (2.39) we have the asymptotic behavior

$$\mu_n = O\left(\frac{ekR}{2n}\right)^n, \quad n \to \infty,$$

indicating severe ill-posedness.

As already pointed out, Picard's Theorem 4.8 illustrates the fact that the ill-posedness of an equation of the first kind with a compact operator stems from the behavior of the singular values $\mu_n \to 0$, $n \to \infty$. This suggests to try to regularize the equation by damping or filtering out the influence of the factor $1/\mu_n$ in the solution formula (4.15).

Theorem 4.9. *Let $A : X \to Y$ be an injective compact linear operator with singular system (μ_n, φ_n, g_n), $n \in \mathbb{N}$, and let $q : (0, \infty) \times (0, \|A\|] \to \mathbb{R}$ be a bounded function such that for each $\alpha > 0$ there exists a positive constant $c(\alpha)$ with*

$$|q(\alpha, \mu)| \le c(\alpha)\mu, \quad 0 < \mu \le \|A\|, \tag{4.16}$$

and

$$\lim_{\alpha \to 0} q(\alpha, \mu) = 1, \quad 0 < \mu \le \|A\|. \tag{4.17}$$

Then the bounded linear operators $R_\alpha : Y \to X$, $\alpha > 0$, defined by

$$R_\alpha f := \sum_{n=1}^{\infty} \frac{1}{\mu_n}\, q(\alpha, \mu_n)\, (f, g_n)\varphi_n, \quad f \in Y, \tag{4.18}$$

describe a regularization scheme with

$$\|R_\alpha\| \le c(\alpha). \tag{4.19}$$

Proof. From (4.11) and (4.16) we have

$$\|R_\alpha f\|^2 = \sum_{n=1}^{\infty} \frac{1}{\mu_n^2}\,[q(\alpha,\mu_n)]^2\,|(f,g_n)|^2 \le [c(\alpha)]^2 \sum_{n=1}^{\infty} |(f,g_n)|^2 \le [c(\alpha)]^2 \|f\|^2$$

for all $f \in Y$, whence the bound (4.19) follows. With the aid of

$$(R_\alpha A\varphi, \varphi_n) = \frac{1}{\mu_n}\, q(\alpha,\mu_n)\,(A\varphi, g_n) = q(\alpha,\mu_n)\,(\varphi,\varphi_n)$$

and the singular value decomposition for $R_\alpha A\varphi - \varphi$ we obtain

$$\|R_\alpha A\varphi - \varphi\|^2 = \sum_{n=1}^{\infty} |(R_\alpha A\varphi - \varphi, \varphi_n)|^2 = \sum_{n=1}^{\infty} [q(\alpha,\mu_n) - 1]^2\,|(\varphi,\varphi_n)|^2.$$

Here we have used the fact that A is injective. Let $\varphi \in X$ with $\varphi \neq 0$ and $\varepsilon > 0$ be given and let M denote a bound for q. Then there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} |(\varphi,\varphi_n)|^2 < \frac{\varepsilon}{2(M+1)^2}\ .$$

By the convergence condition (4.17), there exists $\alpha_0(\varepsilon) > 0$ such that

$$[q(\alpha,\mu_n) - 1]^2 < \frac{\varepsilon}{2\|\varphi\|^2}$$

for all $n = 1,\dots,N$ and all $0 < \alpha \le \alpha_0$. Splitting the series in two parts and using (4.11), it now follows that

$$\|R_\alpha A\varphi - \varphi\|^2 < \frac{\varepsilon}{2\|\varphi\|^2} \sum_{n=1}^{N} |(\varphi,\varphi_n)|^2 + \frac{\varepsilon}{2} \le \varepsilon$$

for all $0 < \alpha \le \alpha_0$. Thus we have established that $R_\alpha A\varphi \to \varphi$, $\alpha \to 0$, for all $\varphi \in X$ and the proof is complete. □

We now describe two basic regularization schemes, namely the spectral cut-off and the Tikhonov regularization, obtained by choosing the damping or filter function q appropriately.

Theorem 4.10. *Let $A : X \to Y$ be an injective compact linear operator with singular system (μ_n, φ_n, g_n), $n \in \mathbb{N}$. Then the* spectral cut-off

$$R_m f := \sum_{\mu_n \ge \mu_m} \frac{1}{\mu_n}\,(f, g_n)\varphi_n \tag{4.20}$$

describes a regularization scheme with regularization parameter $m \to \infty$ and $\|R_m\| = 1/\mu_m$.

Proof. The function q with $q(m,\mu) = 1$ for $\mu \geq \mu_m$ and $q(m,\mu) = 0$ for $\mu < \mu_m$ satisfies the conditions (4.16) and (4.17). For the norm, by Bessel's inequality, we can estimate

$$\|R_m f\|^2 = \sum_{\mu_n \geq \mu_m} \frac{1}{\mu_n^2} \, |(f, g_n)|^2 \leq \frac{1}{\mu_m^2} \sum_{\mu_n \geq \mu_m} |(f, g_n)|^2 \leq \frac{1}{\mu_m^2} \, \|f\|^2,$$

whence $\|R_m\| \leq 1/\mu_m$. Equality follows from $R_m g_m = \varphi_m/\mu_m$. □

The regularization parameter m determines the number of terms in the sum (4.20). Accuracy of the approximation requires this number to be large and stability requires it to be small. In particular, the following discrepancy principle turns out to be a regular a posteriori strategy for determining the stopping point for the spectral cut-off.

Theorem 4.11. *Let $A : X \to Y$ be an injective compact linear operator with dense range in Y, let $f \in Y$ and let $\delta > 0$. Then there exists a smallest integer m such that*

$$\|AR_m f - f\| \leq \delta.$$

Proof. By Theorem 4.6, the dense range $\overline{A(X)} = Y$ implies that A^* is injective. Hence, the singular value decomposition (4.9) with the singular system (μ_n, g_n, φ_n) for the adjoint operator A^*, applied to an element $f \in Y$, yields

$$f = \sum_{n=1}^{\infty} (f, g_n) g_n \tag{4.21}$$

and consequently

$$\|(AR_m - I)f\|^2 = \sum_{\mu_n < \mu_m} |(f, g_n)|^2 \to 0, \quad m \to \infty. \tag{4.22}$$

From this we conclude that there exists a smallest integer $m = m(\delta)$ such that $\|AR_m f - f\| \leq \delta$. □

From (4.21) and (4.22), we see that

$$\|AR_m f - f\|^2 = \|f\|^2 - \sum_{\mu_n \geq \mu_m} |(f, g_n)|^2,$$

which allows a stable determination of the stopping parameter $m(\delta)$ by terminating the sum when the right hand side becomes smaller than or equal to δ^2 for the first time.

The regularity of the discrepancy principle for the spectral cut-off described through Theorem 4.11 is established in the following theorem.

Theorem 4.12. *Let $A : X \to Y$ be an injective compact linear operator with dense range in Y. Let $f \in A(X)$, $f^\delta \in Y$ satisfy $\|f^\delta - f\| \le \delta$ with $\delta > 0$ and let $\gamma > 1$. Then there exists a smallest integer $m = m(\delta)$ such that*

$$\|AR_{m(\delta)} f^\delta - f^\delta\| \le \gamma\delta \tag{4.23}$$

is satisfied and

$$R_{m(\delta)} f^\delta \to A^{-1} f, \quad \delta \to 0. \tag{4.24}$$

Proof. In view of Theorem 4.11, we only need to establish the convergence (4.24). We first note that (4.22) implies $\|I - AR_m\| = 1$ for all $m \in \mathbb{N}$. Therefore, writing

$$(AR_m f^\delta - f^\delta) - (AR_m f - f) = (AR_m - I)(f^\delta - f)$$

we have the triangle inequalities

$$\|AR_m f - f\| \le \delta + \|AR_m f^\delta - f^\delta\|, \tag{4.25}$$

$$\|AR_m f^\delta - f^\delta\| \le \delta + \|AR_m f - f\|. \tag{4.26}$$

From (4.23) and (4.25) we obtain

$$\|AR_{m(\delta)} f - f\| \le \delta + \|AR_{m(\delta)} f^\delta - f^\delta\| \le (1 + \gamma)\delta \to 0, \quad \delta \to 0.$$

Therefore, from the expansion (4.22), we conclude that either the number of terms in the sum (4.20) tends to infinity, $m(\delta) \to \infty$, $\delta \to 0$, or the expansion for f degenerates into a finite sum

$$f = \sum_{\mu_n \ge \mu_{m_0}} (f, g_n) g_n$$

and $m(\delta) \ge m_0$. In the first case, from $\|AR_{m(\delta)-1} f^\delta - f^\delta\| > \gamma\delta$ and the triangle inequality (4.26), we conclude

$$\gamma\delta < \delta + \|A(R_{m(\delta)-1} f - A^{-1} f)\|$$

whence

$$\delta < \frac{1}{\gamma - 1}\, \|A(R_{m(\delta)-1} f - A^{-1} f)\|$$

follows. In order to establish the convergence (4.24), in this case, from (4.5) it suffices to show that

$$\|R_m\|\, \|A(R_{m-1} A\varphi - \varphi)\| \to 0, \quad m \to \infty,$$

for all $\varphi \in X$. But the latter property is obvious from

$$\|R_m\|^2\, \|A(R_{m-1} A\varphi - \varphi)\|^2 = \frac{1}{\mu_m^2} \sum_{\mu_n < \mu_{m-1}} \mu_n^2 |(\varphi, \varphi_n)|^2 \le \sum_{\mu_n \le \mu_m} |(\varphi, \varphi_n)|^2.$$

In the case where f has a finite expansion then clearly

$$A^{-1}f = \sum_{\mu_n \geq \mu_{m_0}} \frac{1}{\mu_n}\,(f, g_n)\varphi_n = R_m f$$

for all $m \geq m_0$. Hence

$$\|AR_m f^\delta - f^\delta\| = \|(AR_m - I)(f^\delta - f)\| \leq \|f^\delta - f\| \leq \delta < \gamma\delta,$$

and therefore $m(\delta) \leq m_0$. This implies equality $m(\delta) = m_0$ since $m(\delta) \geq m_0$ as already noted above. Now observing

$$\|R_{m(\delta)} f^\delta - A^{-1}f\| = \|R_{m_0}(f^\delta - f)\| \leq \frac{\delta}{\mu_{m_0}} \to 0, \quad \delta \to 0,$$

the proof is finished. □

4.4 Tikhonov Regularization

We continue our study of regularization methods by introducing Tikhonov's [314, 315] regularization scheme first as a special case of Theorem 4.9 and then also as a penalized residual minimization.

Theorem 4.13. *Let $A : X \to Y$ be a compact linear operator. Then for each $\alpha > 0$ the operator $\alpha I + A^*A : X \to X$ is bijective and has a bounded inverse. Furthermore, if A is injective then*

$$R_\alpha := (\alpha I + A^*A)^{-1}A^* \tag{4.27}$$

describes a regularization scheme with $\|R_\alpha\| \leq 1/2\sqrt{\alpha}$.

Proof. From

$$\alpha\|\varphi\|^2 \leq (\alpha\varphi + A^*A\varphi, \varphi) \tag{4.28}$$

for all $\varphi \in X$ we conclude that for $\alpha > 0$ the operator $\alpha I + A^*A$ is injective. Let (μ_n, φ_n, g_n), $n \in \mathbb{N}$, be a singular system for A and let $Q : X \to N(A)$ denote the orthogonal projection operator. Then the operator $T : X \to X$ defined by

$$T\varphi := \sum_{n=1}^{\infty} \frac{1}{\alpha + \mu_n^2}\,(\varphi, \varphi_n)\varphi_n + \frac{1}{\alpha}\,Q(\varphi)$$

can be easily seen to be bounded and to satisfy $(\alpha I + A^*A)T = T(\alpha I + A^*A) = I$, i.e., $T = (\alpha I + A^*A)^{-1}$.

If A is injective then for the unique solution φ_α of

$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f \tag{4.29}$$

we deduce from the above expression for $(\alpha I + A^*A)^{-1}$ and $(A^*f, \varphi_n) = \mu_n(f, g_n)$ that

$$\varphi_\alpha = \sum_{n=1}^{\infty} \frac{\mu_n}{\alpha + \mu_n^2} (f, g_n)\varphi_n. \tag{4.30}$$

Hence, R_α can be brought into the form (4.18) with

$$q(\alpha, \mu) = \frac{\mu^2}{\alpha + \mu^2} .$$

This function q is bounded by $0 < q(\alpha, \mu) < 1$ and satisfies the conditions (4.16) and (4.17) with

$$c(\alpha) = \frac{1}{2\sqrt{\alpha}}$$

because of the arithmetic-geometric mean inequality

$$\sqrt{\alpha}\mu \leq \frac{\alpha + \mu^2}{2} .$$

The proof of the theorem now follows from Theorem 4.9. □

The following theorem presents another aspect of the Tikhonov regularization complementing its introduction via Theorems 4.9 and 4.13.

Theorem 4.14. *Let $A : X \to Y$ be a compact linear operator and let $\alpha > 0$. Then for each $f \in Y$ there exists a unique $\varphi_\alpha \in X$ such that*

$$\|A\varphi_\alpha - f\|^2 + \alpha\|\varphi_\alpha\|^2 = \inf_{\varphi \in X} \left\{ \|A\varphi - f\|^2 + \alpha\|\varphi\|^2 \right\}. \tag{4.31}$$

The minimizer φ_α is given by the unique solution of (4.29) and depends continuously on f.

Proof. From the equation

$$\|A\varphi - f\|^2 + \alpha\|\varphi\|^2 = \|A\varphi_\alpha - f\|^2 + \alpha\|\varphi_\alpha\|^2$$

$$+2\,\mathrm{Re}(\varphi - \varphi_\alpha, \alpha\varphi_\alpha + A^*(A\varphi_\alpha - f)) + \|A(\varphi - \varphi_\alpha)\|^2 + \alpha\|\varphi - \varphi_\alpha\|^2,$$

which is valid for all $\varphi \in X$, we observe that the condition (4.29) is necessary and sufficient for φ_α to minimize the *Tikhonov functional* defined by (4.31). The theorem now follows from the Riesz–Fredholm theory or from the first part of the statement of the previous Theorem 4.13. □

We note that Theorem 4.14 remains valid for bounded operators. This follows from the Lax–Milgram theorem since by (4.28) the operator $\alpha I + A^*A$ is strictly coercive (see [205]).

By the interpretation of the Tikhonov regularization as minimizer of the Tikhonov functional, its solution keeps the residual $\|A\varphi_\alpha - f\|^2$ small and is stabilized through the penalty term $\alpha\|\varphi_\alpha\|^2$. Although Tikhonov regularization itself is not a penalty method, such a view nevertheless suggests the following constrained optimization problems:

a) For given $\delta > 0$, minimize the norm $\|\varphi\|$ subject to the constraint that the defect is bounded by $\|A\varphi - f\| \le \delta$.

b) For given $\rho > 0$, minimize the defect $\|A\varphi - f\|$ subject to the constraint that the norm is bounded by $\|\varphi\| \le \rho$.

The first interpretation leads to the *discrepancy principle* and the second to the concept of *quasi-solutions*. We begin by discussing the discrepancy principle.

Theorem 4.15. *Let $A : X \to Y$ be an injective compact linear operator with dense range in Y and let $f \in Y$ with $0 < \delta < \|f\|$. Then there exists a unique parameter α such that*

$$\|AR_\alpha f - f\| = \delta. \tag{4.32}$$

Proof. We have to show that the function $F : (0, \infty) \to \mathbb{R}$ defined by

$$F(\alpha) := \|AR_\alpha f - f\|^2 - \delta^2$$

has a unique zero. From (4.21) and (4.30) we find

$$F(\alpha) = \sum_{n=1}^{\infty} \frac{\alpha^2}{(\alpha + \mu_n^2)^2}\, |(f, g_n)|^2 - \delta^2.$$

Therefore, F is continuous and strictly monotonically increasing with the limits $F(\alpha) \to -\delta^2 < 0$, $\alpha \to 0$, and $F(\alpha) \to \|f\|^2 - \delta^2 > 0$, $\alpha \to \infty$. Hence, F has exactly one zero $\alpha = \alpha(\delta)$. □

In general, we will have data satisfying $\|f\| > \delta$, i.e., data exceeding the error level. Then the regularization parameter satisfying (4.32) can be obtained numerically by Newton's method for solving $F(\alpha) = 0$. With the unique solution φ_α of (4.29), we can write $F(\alpha) = \|A\varphi_\alpha - f\|^2 - \delta^2$ and get

$$F'(\alpha) = 2\,\mathrm{Re}\left(A\,\frac{d\varphi_\alpha}{d\alpha}\,, A\varphi_\alpha - f\right)$$

from which, again using (4.29), we deduce that

$$F'(\alpha) = -2\alpha\,\mathrm{Re}\left(\frac{d\varphi_\alpha}{d\alpha}\,, \varphi_\alpha\right).$$

Differentiating (4.29) with respect to the parameter α yields

$$\alpha\,\frac{d\varphi_\alpha}{d\alpha} + A^*A\,\frac{d\varphi_\alpha}{d\alpha} = -\varphi_\alpha$$

as an equation for $d\varphi_\alpha/d\alpha$ which has to be solved for the evaluation of $F'(\alpha)$.

The regularity of the discrepancy principle for the Tikhonov regularization is established through the following theorem.

Theorem 4.16. *Let $A : X \to Y$ be an injective compact linear operator with dense range in Y. Let $f \in A(X)$ and $f^\delta \in Y$ satisfy*

$$\|f^\delta - f\| \le \delta < \|f^\delta\|$$

with $\delta > 0$. Then there exists a unique parameter $\alpha = \alpha(\delta)$ such that

$$\|AR_{\alpha(\delta)}f^\delta - f^\delta\| = \delta \tag{4.33}$$

is satisfied and

$$R_{\alpha(\delta)}f^\delta \to A^{-1}f, \quad \delta \to 0. \tag{4.34}$$

Proof. In view of Theorem 4.15, we only need to establish the convergence (4.34). Since $\varphi^\delta = R_{\alpha(\delta)}f^\delta$ minimizes the Tikhonov functional for the right hand side f^δ, we have

$$\begin{aligned}\delta^2 + \alpha\|\varphi^\delta\|^2 &= \|A\varphi^\delta - f^\delta\|^2 + \alpha\|\varphi^\delta\|^2\\ &\le \|AA^{-1}f - f^\delta\|^2 + \alpha\|A^{-1}f\|^2\\ &\le \delta^2 + \alpha\|A^{-1}f\|^2,\end{aligned}$$

whence

$$\|\varphi^\delta\| \le \|A^{-1}f\| \tag{4.35}$$

follows. Now let $g \in Y$ be arbitrary. Then we can estimate

$$|(A\varphi^\delta - f, g)| \le \left\{\|A\varphi^\delta - f^\delta\| + \|f^\delta - f\|\right\}\|g\| \le 2\delta\|g\| \to 0, \quad \delta \to 0.$$

This implies weak convergence $\varphi^\delta \rightharpoonup A^{-1}f$, $\delta \to 0$, since for the injective operator A the range $A^*(Y)$ is dense in X by Theorem 4.6 and φ^δ is bounded by (4.35). Then, again using (4.35), we obtain

$$\begin{aligned}\|\varphi^\delta - A^{-1}f\|^2 &= \|\varphi^\delta\|^2 - 2\,\mathrm{Re}(\varphi^\delta, A^{-1}f) + \|A^{-1}f\|^2\\ &\le 2\{\|A^{-1}f\|^2 - \mathrm{Re}(\varphi^\delta, A^{-1}f)\} \to 0, \quad \delta \to 0,\end{aligned}$$

which finishes the proof. □

The principal idea underlying the concept of quasi-solutions as introduced by Ivanov [154, 155] is to stabilize an ill-posed problem by restricting the solution set to some subset $U \subset X$ exploiting suitable a priori information on the solution of $A\varphi = f$. For perturbed right hand sides, in general we cannot expect a solution in U. Therefore, instead of trying to solve the equation exactly, we minimize the residual. For simplicity, we restrict our presentation to the case where U is a ball of radius ρ for some $\rho > 0$.

Theorem 4.17. *Let $A : X \to Y$ be a compact injective linear operator and let $\rho > 0$. Then for each $f \in Y$ there exists a unique element $\varphi_0 \in X$ with $\|\varphi_0\| \le \rho$ satisfying*

$$\|A\varphi_0 - f\| \le \|A\varphi - f\|$$

for all $\varphi \in X$ with $\|\varphi\| \le \rho$. The element φ_0 is called the quasi-solution *of $A\varphi = f$ with constraint ρ.*

Proof. Note that φ_0 is a quasi-solution with constraint ρ if and only if $A\varphi_0$ is a best approximation to f with respect to the set $V := \{A\varphi : \|\varphi\| \le \rho\}$. Since A is linear, the set V is clearly convex. In the Hilbert space Y there exists at most one best approximation to f with respect to the convex set V. Since A is injective this implies uniqueness of the quasi-solution.

To prove existence of the quasi-solution, let (φ_n) be a minimizing sequence, that is, $\|\varphi_n\| \le \rho$ and

$$\lim_{n\to\infty} \|A\varphi_n - f\| = \inf_{\|\varphi\|\le\rho} \|A\varphi - f\|.$$

Without loss of generality, for the bounded sequence (φ_n) we may assume weak convergence $\varphi_n \rightharpoonup \varphi_0$, $n \to \infty$, for some $\varphi_0 \in X$. Since A is compact this implies convergence $\|A\varphi_n - A\varphi_0\| \to 0$, $n \to \infty$, and therefore

$$\|A\varphi_0 - f\| = \inf_{\|\varphi\|\le\rho} \|A\varphi - f\|.$$

From

$$\|\varphi_0\|^2 = \lim_{n\to\infty} (\varphi_n, \varphi_0) \le \rho\|\varphi_0\|$$

we obtain $\|\varphi_0\| \le \rho$ and the proof is complete. □

The connection of the quasi-solution to Tikhonov regularization is described through the following theorem.

Theorem 4.18. *Let $A : X \to Y$ be a compact injective linear operator with dense range in Y and assume $f \notin V := \{A\varphi : \|\varphi\| \le \rho\}$. Then the quasi-solution φ_0 assumes the constraint*

$$\|\varphi_0\| = \rho \tag{4.36}$$

and there exists a unique parameter $\alpha > 0$ such that

$$\alpha\varphi_0 + A^*A\varphi_0 = A^*f. \tag{4.37}$$

Proof. If φ_0 satisfies (4.36) and (4.37) then

$$\|A\varphi - f\|^2 = \|A\varphi_0 - f\|^2 + 2\alpha\,\mathrm{Re}(\varphi_0 - \varphi, \varphi_0) + \|A(\varphi - \varphi_0)\|^2 \ge \|A\varphi_0 - f\|^2$$

for all $\|\varphi\| \le \rho$ and therefore φ_0 is a quasi-solution of $A\varphi = f$ with constraint ρ. Therefore, in view of the preceding Theorem 4.17, the proof is established by showing existence of a solution to equations (4.36) and (4.37) with $\alpha > 0$.

For this we define the function $G : (0, \infty) \to \mathbb{R}$ by

$$G(\alpha) := \|\varphi_\alpha\|^2 - \rho^2,$$

where φ_α denotes the unique solution of (4.29), and show that $G(\alpha)$ has a unique zero. From (4.30) we obtain

$$G(\alpha) = \sum_{n=1}^{\infty} \frac{\mu_n^2}{(\alpha + \mu_n^2)^2} \, |(f, g_n)|^2 - \rho^2.$$

Therefore, G is continuous and strictly monotonically decreasing with

$$G(\alpha) \to -\rho^2 < 0, \quad \alpha \to \infty,$$

and

$$G(\alpha) \to \sum_{n=1}^{\infty} \frac{1}{\mu_n^2} \, |(f, g_n)|^2 - \rho^2, \quad \alpha \to 0,$$

where the series may diverge. The proof is now completed by showing that the latter limit is positive or infinite. Assume, to the contrary, that

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} \, |(f, g_n)|^2 \leq \rho^2.$$

Then by Picard's Theorem 4.8 (note that $N(A^*) = \{0\}$ by Theorem 4.6) the equation $A\varphi = f$ has the solution

$$\varphi = \sum_{n=1}^{\infty} \frac{1}{\mu_n} \, (f, g_n)\varphi_n$$

with

$$\|\varphi\|^2 = \sum_{n=1}^{\infty} \frac{1}{\mu_n^2} \, |(f, g_n)|^2 \leq \rho^2,$$

which is a contradiction to $f \notin V$. □

In the sense of Theorem 4.18, we may view the quasi-solution as a possibility of an a posteriori choice of the regularization parameter in the Tikhonov regularization. We also note that Theorems 4.15 to 4.18 remain valid for injective linear operators which are merely bounded (see [205]).

We conclude this section with considering a generalized discrepancy principle due to Morozov that also allows erroneous operators in the Tikhonov regularization. We begin with an analogue of Theorem 4.15.

Theorem 4.19. *Let $A : X \to Y$ be an injective compact linear operator with dense range and let $f \in Y$ with $f \neq 0$. Then there exists a unique parameter α such that*

$$\|AR_\alpha f - f\| = \delta \|R_\alpha f\|. \tag{4.38}$$

Proof. We show that the function $H : (0, \infty) \to \mathbb{R}$ defined by

$$H(\alpha) := \|AR_\alpha f - f\|^2 - \delta^2 \|R_\alpha f\|^2$$

has a unique zero. From the continuity and monotonicity of the functions F and G used in the proofs of the Theorems 4.15 and 4.18 we observe that H is continuous and strictly monotonically increasing with limits

$$H(\alpha) \to -\delta^2 \sup_{\alpha>0} \|R_\alpha f\|^2 < 0, \quad \alpha \to 0,$$

and

$$H(\alpha) \to \|f\|^2 > 0, \quad \alpha \to \infty.$$

Hence, H has exactly one zero $\alpha = \alpha(\delta)$. □

Theorem 4.20. *Let $A_\delta : X \to Y$, $\delta \geq 0$, be a family of injective compact linear operators with dense range and $\|A_\delta - A_0\| \leq \delta$ for all $\delta > 0$. Furthermore, let $f \in Y$ with $f \neq 0$ and let $(\alpha_\delta, \varphi^\delta) \in \mathbb{R}^+ \times X$ be the Tikhonov regularized solutions of $A_\delta = f$ subject to the generalized discrepancy principle, i.e.,*

$$\alpha_\delta \varphi^\delta + A_\delta^* A_\delta \varphi^\delta = A_\delta^* f \tag{4.39}$$

and

$$\|A_\delta \varphi^\delta - f\| = \delta \|\varphi^\delta\|. \tag{4.40}$$

Then

a) if the equation $A_0\varphi = f$ has a solution $\varphi \in X$ then $\varphi^\delta \to \varphi$ as $\delta \to 0$,
b) if the equation $A_0\varphi = f$ has no solution then $\|\varphi^\delta\| \to \infty$ as $\delta \to 0$.

Proof. We begin by observing that for any null sequence (δ_n) with corresponding $\varphi_n := \varphi^{\delta_n}$ boundedness of the sequence (φ_n) implies that

$$A_0 \varphi_n \to f, \quad n \to \infty, \tag{4.41}$$

as a consequence of $\|A_\delta - A_0\| \leq \delta$ and (4.40).

a) Since φ^δ minimizes the Tikhonov functional for A_δ, in view of $A_0\varphi = f$ we have

$$\begin{aligned}(\delta^2 + \alpha_\delta)\|\varphi^\delta\|^2 &= \|A_\delta \varphi^\delta - f\|^2 + \alpha_\delta \|\varphi^\delta\|^2 \\ &\leq \|A_\delta \varphi - A_0 \varphi\|^2 + \alpha_\delta \|\varphi\|^2 \\ &\leq (\delta^2 + \alpha_\delta)\|\varphi\|^2,\end{aligned}$$

whence

$$\|\varphi^\delta\| \leq \varphi\| \tag{4.42}$$

follows for all $\delta > 0$. Now assume that the assertion of a) is not valid. Then there exists a null sequence (δ_n) such that $\varphi_n := \varphi^{\delta_n}$ does not converge to φ as $n \to \infty$. Because of (4.42) we may assume weak convergence $\varphi_n \rightharpoonup \varphi_0 \in X$ as $n \to \infty$.

The compactness of A_0 implies norm convergence $A_0\varphi_n \to A_0\varphi_0$ as $n \to \infty$. Hence, in view of (4.41) we have that $A_0\varphi_0 = f$ whence $\varphi_0 = \varphi$ follows by the injectivity of A_0. With the aid of (4.42) we can estimate

$$0 \leq \|\varphi_n - \varphi\|^2 \leq 2\,\mathrm{Re}(\varphi - \varphi_n, \varphi)$$

and passing to the limit yields a contradiction to our assumption that φ_n does not converge to φ as $n \to \infty$.

b) Assume that for $f \notin A_0(X)$ the assertion b) is not valid. Then there exists a null sequence (δ_n) such that $\varphi_n := \varphi^{\delta_n}$ is bounded. Again we may assume $\varphi_n \rightharpoonup \varphi_0 \in X$ as $n \to \infty$ and compactness of A_0 implies $A_0\varphi_n \to A_0\varphi_0$ as $n \to \infty$. From this (4.41) now implies $A_0\varphi_0 = f$ which contradicts $f \notin A_0(X)$. □

4.5 Nonlinear Operators

We conclude our introduction to ill-posed problems by making a few comments on nonlinear problems. We first show that the ill-posedness of a nonlinear problem is inherited by its linearization. This implies that whenever we try to approximately solve an ill-posed nonlinear equation by linearization, for example by a Newton method, we obtain ill-posed linear equations for which a regularization must be enforced, for example by one of the methods from the previous sections.

Theorem 4.21. *Let $A : U \subset X \to Y$ be a completely continuous operator from an open subset U of a normed space X into a Banach space Y and assume A to be Fréchet differentiable at $\psi \in U$. Then the derivative A'_ψ is compact.*

Proof. We shall use the fact that a subset V of a Banach space is relatively compact if and only if it is totally bounded, i.e., for every $\varepsilon > 0$ there exists a finite system of elements $\varphi_1, \dots, \varphi_n$ in V such that each element $\varphi \in V$ has a distance smaller than ε from at least one of the $\varphi_1, \dots, \varphi_n$. We have to show that

$$V := \left\{A'_\psi(\varphi) : \varphi \in X,\ \|\varphi\| \leq 1\right\}$$

is relatively compact. Given $\varepsilon > 0$, by the definition of the Fréchet derivative there exists $\delta > 0$ such that for all $\|\varphi\| \leq \delta$ we have $\psi + \varphi \in U$ and

$$\|A(\psi + \varphi) - A(\psi) - A'_\psi(\varphi)\| \leq \frac{\varepsilon}{3}\,\|\varphi\|. \tag{4.43}$$

Since A is compact, the set

$$\{A(\psi + \delta\varphi) : \varphi \in X,\ \|\varphi\| \leq 1\}$$

is relatively compact and hence totally bounded, i.e., there are finitely many elements $\varphi_1, \dots, \varphi_n \in X$ with norm less than or equal to one such that for each $\varphi \in X$

with $\|\varphi\| \le 1$ there exists $j = j(\varphi)$ such that

$$\|A(\psi + \delta\varphi) - A(\psi + \delta\varphi_j)\| < \frac{\varepsilon\delta}{3}\,. \tag{4.44}$$

By the triangle inequality, using (4.43) and (4.44), we now have

$$\delta\|A'_\psi(\varphi) - A'_\psi(\varphi_j)\| \le \|A(\psi + \delta\varphi) - A(\psi + \delta\varphi_j)\|$$

$$+ \|A(\psi + \delta\varphi) - A(\psi) - A'_\psi(\delta\varphi)\| + \|A(\psi + \delta\varphi_j) - A(\psi) - A'_\psi(\delta\varphi_j)\| < \delta\varepsilon$$

and therefore V is totally bounded. This ends the proof. □

In the following, we will illustrate how the classical concepts of Tikhonov regularization and quasi-solutions can be directly applied to ill-posed nonlinear problems. We begin with the nonlinear counterpart of Theorem 4.14.

Theorem 4.22. *Let $A : U \subset X \to Y$ be a weakly sequentially closed operator from a subset U of a Hilbert space X into a Hilbert space Y, i.e., for any sequence (φ_n) from U weak convergence $\varphi_n \rightharpoonup \varphi \in X$ and $A(\varphi_n) \rightharpoonup g \in Y$ implies $\varphi \in U$ and $A(\varphi) = g$. Let $\alpha > 0$. Then for each $f \in Y$ there exists $\varphi_\alpha \in U$ such that*

$$\|A(\varphi_\alpha) - f\|^2 + \alpha\|\varphi_\alpha\|^2 = \inf_{\varphi \in U}\left\{\|A(\varphi) - f\|^2 + \alpha\|\varphi\|^2\right\}. \tag{4.45}$$

Proof. We abbreviate the Tikhonov functional by

$$\mu(\varphi, \alpha) := \|A(\varphi) - f\|^2 + \alpha\|\varphi\|^2$$

and set

$$m(\alpha) := \inf_{\varphi \in U} \mu(\varphi, \alpha).$$

Let (φ_n) be a minimizing sequence in U, i.e.,

$$\lim_{n\to\infty} \mu(\varphi_n, \alpha) = m(\alpha).$$

Since $\alpha > 0$, the sequences (φ_n) and $(A(\varphi_n))$ are bounded. Hence, by selecting subsequences and relabeling, we can assume weak convergence $\varphi_n \rightharpoonup \varphi_\alpha$ as $n \to \infty$ for some $\varphi_\alpha \in X$ and $A(\varphi_n) \rightharpoonup g$ as $n \to \infty$ for some $g \in Y$. Since A is assumed to be weakly sequentially closed φ_α belongs to U and we have $g = A(\varphi_\alpha)$. This now implies

$$\|A(\varphi_n) - A(\varphi_\alpha)\|^2 + \alpha\|\varphi_n - \varphi_\alpha\|^2 \to m(\alpha) - \mu(\varphi_\alpha, \alpha), \quad n \to \infty,$$

whence $m(\alpha) \ge \mu(\varphi_\alpha, \alpha)$ follows. Now observing that trivially $m(\alpha) \le \mu(\varphi_\alpha, \alpha)$, we have shown that $m(\alpha) = \mu(\varphi_\alpha, \alpha)$ and the proof is complete. □

We continue with the following result on quasi-solutions in the nonlinear case.

Theorem 4.23. *Let $A : U \subset X \to Y$ be a continuous operator from a compact subset U of a Hilbert space X into a Hilbert space Y. Then for each $f \in Y$ there exists a $\varphi_0 \in U$ such that*

$$\|A(\varphi_0) - f\|^2 = \inf_{\varphi \in U} \left\{ \|A(\varphi) - f\|^2 \right\}. \tag{4.46}$$

The element φ_0 is called the quasi-solution *of $A\varphi = f$ with respect to U.*

Proof. This is an immediate consequence of the compactness of U and the continuity of A and $\| \cdot \|$. □

We conclude with a short discussion of some concepts on the iterative solution of the ill-posed nonlinear equation

$$A(\varphi) = f \tag{4.47}$$

from a knowledge of an erroneous right hand side f^δ with an error level $\|f^\delta - f\| \le \delta$ where we assume that A is completely continuous and Fréchet differentiable. The classical Newton method applied to the perturbed equation $A\varphi^\delta = f^\delta$ consists in solving the linearized equation

$$B_n h_n = f^\delta - A(\varphi_n^\delta) \tag{4.48}$$

for h_n to update the approximation φ_n^δ into $\varphi_{n+1}^\delta = \varphi_n^\delta + h_n$ where we have set

$$B_n := A'_{\varphi_n^\delta}.$$

Since by Theorem 4.21 the linearized equation (4.48) inherits the ill-posedness from (4.47) regularization is required. For this, in principle, all the methods discussed in the three previous sections can be applied. For brevity here we only consider Tikhonov regularization assuming that $A : X \to Y$ is a nonlinear operator between Hilbert spaces X and Y. In this case, the regularized solution of (4.48) is given by

$$h_n = (\alpha_n I + B_n^* B_n)^{-1} B_n^* \left(f^\delta - A(\varphi_n^\delta) \right). \tag{4.49}$$

By Theorem 4.14 the update h_n is the unique solution of the minimization problem

$$\|B_n h_n + A(\varphi_\alpha^\delta) - f^\delta\|^2 + \alpha_n \|h_n\|^2 = \inf_{h \in X} \left\{ \|B_n h + A(\varphi_\alpha^\delta) - f^\delta\|^2 + \alpha_n \|h\|^2 \right\}. \tag{4.50}$$

This method is known as the *Levenberg–Marquardt algorithm*. Its convergence was first analyzed by Hanke [128] who proposed to choose the regularization parameter α_n such that

$$\|B_n h_n + A(\varphi_\alpha^\delta) - f^\delta\|^2 = \tau \|A(\varphi_n^\delta) - f^\delta\| \tag{4.51}$$

for some $\tau < 1$, i.e., the Newton equation is only satisfied up to a residual of size $\tau \|A(\varphi_n^\delta) - f^\delta\|$. Therefore the method is also referred to as an inexact Newton method. In addition to choosing the regularization parameter for each iteration step a stopping rule is also required since for erroneous data the approximations will start deteriorating after a certain number of iterations. Here again a natural strategy is provided by the discrepancy principle, i.e., the iterations are stopped at the first index $N = N(\delta, f^\delta)$ such that

$$\|A(\varphi_N^\delta) - f^\delta\| \le \gamma\delta \tag{4.52}$$

with some fixed parameter $\gamma \ge 1$. Under the assumption that (4.47) has a unique solution φ and that A satisfies the condition

$$\|A(\chi) - A(\psi) - A'_\chi(\chi - \psi) \le c\,\|\chi - \psi\|\,\|A(\chi) - A(\psi)\| \tag{4.53}$$

for some $c > 0$ and all χ and ψ in some neighborhood of φ, the Levenberg–Marquardt algorithm with the regularization parameter determined by (4.51) and the stopping rule (4.52) with $\gamma\tau > 1$ terminates after finitely many iterations $N^* = N^*(\delta, f^\delta)$ and $\varphi_{N^*}^\delta \to \varphi$ as $\delta \to 0$, i.e., this strategy is regular in the sense of Definition 4.5 (extended to nonlinear problems).

Changing the penalty $\|h\|^2$ in the penalized minimization formulation (4.50) for equation (4.48) leads to the *iteratively regularized Gauss–Newton iteration* in the form

$$h_n = (\alpha_n I + B_n^* B_n)^{-1}\left(B_n^*\left(f^\delta - A(\varphi_n^\delta)\right) + \alpha_n(\varphi_0 - \varphi_n^\delta)\right) \tag{4.54}$$

with a priori guess φ_0. Substituting $g = h + \varphi_n^\delta - \varphi_0$, by straighforward calculations it follows from Theorem 4.14 that $\varphi_{n+1}^\delta = \varphi_n^\delta + h_n$ is the unique solution of the minimization problem

$$\begin{aligned} &\|B_n h_n + A(\varphi_\alpha^\delta) - f^\delta\|^2 + \alpha_n\|h_n + \varphi_n^\delta - \varphi_0\|^2 \\ &= \inf_{h\in X}\left\{\|B_n h + A(\varphi_\alpha^\delta) - f^\delta\|^2 + \alpha_n\|h + \varphi_n^\delta - \varphi_0\|^2\right\}. \end{aligned} \tag{4.55}$$

The modified penalty term in (4.55) as compared with (4.50) has an additional regularization effect by preventing the iterations to move too far away from the initial guess. This also allows the incorporation of a priori information on the true solution into the iteration scheme. Furthermore the convergence analysis for (4.54) turns out to be slightly easier than for (4.49).

This method was first proposed and analyzed by Bakushinksii [17] who suggested an a priori choice of the regularization parameters α_n such that

$$\alpha_{n+1} \le \alpha_n \le \tau\alpha_{n+1} \quad \text{and} \quad \lim_{n\to\infty}\alpha_n = 0$$

for some $\tau > 1$. In particular, these condition are satisfied for the choice

$$\alpha_n = \tau^{-n}\alpha_0 \tag{4.56}$$

and for this case it can again be shown that under an appropriate nonlinearity assumption on A the regularized Gauss–Newton iterations with the discrepancy principle (4.52) terminate after finitely many iterations $N^* = N^*(\delta, f^\delta)$ and that $\varphi^\delta_{N^*} \to \varphi$ as $\delta \to 0$ (see Blaschke (Kaltenbacher), Neubauer and Scherzer [21]).

Another possibility to obtain iterative methods for the nonlinear ill-posed operator equation (4.47) is to apply gradient methods to the minimization of the defect

$$\frac{1}{2}\,\|A(\varphi) - f\|^2.$$

The negative gradient of this functional is given by $[A'_\varphi]^*(f - A(\varphi))$ and therefore gradient or steepest descent methods are of the form

$$\varphi_{n+1} = \varphi_n + \mu_n [A'_{\varphi_n}]^*(f - A(\varphi_n))$$

with a step size parameter $\mu_n > 1$. Keeping $\mu_n = \mu$ constant during the iteration we obtain the nonlinear Landweber iteration in the form

$$\varphi^\delta_{n+1} = \varphi^\delta_n + \mu[A'_{\varphi^\delta_n}]^*\left(f^\delta - A(\varphi^\delta_n)\right) \tag{4.57}$$

for inexact data f^δ. The parameter μ now must be understood as a scaling factor to be chosen such that $\mu\,\|A'_\varphi\| < 1$ in a neighborhood of the solution of (4.47). The Landweber iteration has been studied and applied extensively for the solution of linear ill-posed equations [205]. For results on the regularity of the discrepancy principle for the nonlinear Landweber iteration we refer to Hanke, Neubauer and Scherzer [131].

For an extensive study on nonlinear Tikhonov regularization, regularized Newton iterations, nonlinear Landweber iterations and related topics we refer to the monographs [96, 171] and the survey [28].

Chapter 5
Inverse Acoustic Obstacle Scattering

With the analysis of the preceding chapters, we now are well prepared for studying inverse acoustic obstacle scattering problems. We recall that the direct scattering problem is, given information on the boundary of the scatterer and the nature of the boundary condition, to find the scattered wave and in particular its behavior at large distances from the scatterer, i.e., its far field. The inverse problem starts from this answer to the direct problem, i.e., a knowledge of the far field pattern, and asks for the nature of the scatterer. Of course, there is a large variety of possible inverse problems, for example, if the boundary condition is known, find the shape of the scatterer, or, if the shape is known, find the boundary condition, or, if the shape and the type of the boundary condition are known for a penetrable scatterer, find the space dependent coefficients in the transmission or resistive boundary condition, etc. Here, following the main guideline of our book, we will concentrate on one model problem for which we will develop ideas which in general can also be used to study a wider class of related problems. The inverse problem we consider is, given the far field pattern for one or several incident plane waves and knowing that the scatterer is sound-soft, to determine the shape of the scatterer. We want to discuss this inverse problem for frequencies in the *resonance region*, that is, for scatterers D and wave numbers k such that the wavelengths $2\pi/k$ is less than or of a comparable size to the diameter of the scatterer. This inverse problem turns out to be nonlinear and improperly posed. Although both of these properties make the inverse problem hard to solve, it is the latter which presents the more challenging difficulties. The inverse obstacle problem is improperly posed since, as we already know, the determination of the scattered wave u^s from a given far field pattern u_∞ is improperly posed. It is nonlinear since, given the incident wave u^i and the scattered wave u^s, the problem of finding the boundary of the scatterer as the location of the zeros of the total wave $u^i + u^s$ is nonlinear.

We begin this chapter with results on uniqueness for the inverse obstacle problem, that is, we investigate the question whether knowing the far field pattern provides enough information to completely determine the boundary of the scatterer. In the section on uniqueness, we also include an explicitly solvable problem in inverse obstacle scattering known as *Karp's theorem*.

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences 93, DOI 10.1007/978-1-4614-4942-3_5,

We then proceed to briefly describe a linearization method based on the physical optics approximation and indicate its limitations. After that, we shall provide a detailed analysis of the dependence of the far field mapping on variations of the boundary. Here, by the far field mapping we mean the mapping which for a given incident wave maps the boundary of the scatterer onto the far field of the scattered wave. In particular, we will establish continuity and differentiability of this mapping by using both weak solution and boundary integral equation techniques. This provides the necessary prerequisites for iterative methods for solving the inverse problem that will be the next theme of this chapter. Iterative methods, in principle, interpret the inverse obstacle scattering problem as a nonlinear ill-posed operator equation in terms of the above boundary to far field map and apply iterative schemes such as regularized Newton methods for its solution.

A common feature of these methods is that their iterative numerical implementation requires the numerical solution of the direct scattering problem for different domains at each iteration step. In contrast to this, the second group of reconstruction methods that we are going to discuss, i.e., decomposition methods, circumvent this problem. These methods, in principle, separate the inverse scattering problem into an ill-posed linear problem to reconstruct the scattered wave from its far field and the subsequent determination of the boundary of the scatterer from the boundary condition.

In numerical tests, all of the above methods have been shown to yield satisfactory reconstructions. However, in general, their successful numerical implementation requires sufficient a priori information on the scatterer. This drawback is avoided by the more recently developed sampling methods that we are going to present in the last section of this chapter. These methods, in principle, develop and evaluate criteria in terms of indicator functions obtained from solutions to certain ill-posed linear operator equations that decide on whether a point lies inside or outside the scatterer.

Although a wealth of reconstruction methods in inverse obstacle scattering is now available, there is still work to be done on the improvement of their numerical performance, particularly for three-dimensional problems. In this sense, we provide only a state of the art survey on inverse obstacle scattering in the resonance region rather than an exposition on a subject which has already been brought to completion.

5.1 Uniqueness

In this section, we investigate under what conditions an obstacle is uniquely determined by a knowledge of its far field patterns for incident plane waves. We note that by analyticity the far field pattern is completely determined on the whole unit sphere by only knowing it on some surface patch. We first give a uniqueness result for sound-soft obstacles based on the ideas of Schiffer (see [220]).

Theorem 5.1. *Assume that D_1 and D_2 are two sound-soft scatterers such that the far field patterns coincide for an infinite number of incident plane waves with distinct directions and one fixed wave number. Then $D_1 = D_2$.*

Proof. Assume that $D_1 \neq D_2$. Since by Rellich's lemma, i.e., Theorem 2.14 the far field pattern uniquely determines the scattered field, for each incident wave $u^i(x) = e^{ik\,x\cdot d}$ the scattered wave u^s for both obstacles coincides in the unbounded component G of the complement of $D_1 \cup D_2$ and the total wave vanishes on ∂G. Without loss of generality, we can assume that $D^* := (\mathbb{R}^3 \setminus G) \setminus \bar{D}_2$ is nonempty. Then u^s is defined in D^* since it describes the scattered wave for D_2, that is, $u = u^i + u^s$ satisfies the Helmholtz equation in D^* and the homogeneous boundary condition $u = 0$ on ∂D^*. Hence, u is a Dirichlet eigenfunction for the negative Laplacian in the domain D^* with eigenvalue k^2. From Lemma 3.10 and the approximation technique used in its proof, we know that u belongs to the Sobolev space $H_0^1(D^*)$ without needing any regularity requirements on D^* (besides the assumption that the solution to the scattering problem exists, that is, the scattered wave is continuous up to the boundary). The proof of our theorem is now completed by showing that the total fields for distinct incoming plane waves are linearly independent and that for a fixed wave number k there exist only finitely many linearly independent Dirichlet eigenfunctions in $H_0^1(D^*)$.

Recall that we indicate the dependence of the scattered wave and the total wave on the incident direction by writing $u^s(x, d)$ and $u(x, d)$. Assume that

$$\sum_{n=1}^{N} c_n u(\cdot\,, d_n) = 0 \tag{5.1}$$

in D^* for some constants c_n and N distinct incident directions d_n, $n = 1, \dots, N$. Then, by analyticity (Theorem 2.2), equation (5.1) is also satisfied in the exterior of some sphere containing D_1 and D_2. Writing

$$u(x, d_n) = e^{ik\,x\cdot d_n} + u^s(x, d_n)$$

and using the asymptotic behavior $u^s(x, d_n) = O(1/|x|)$, from (5.1) we obtain

$$\frac{1}{R^2} \sum_{n=1}^{N} c_n \int_{|x|=R} e^{ik\,x\cdot(d_n - d_m)} ds(x) = O\left(\frac{1}{R}\right), \quad R \to \infty, \tag{5.2}$$

for $m = 1, \dots, N$. Now we apply the Funk–Hecke theorem (2.45), that is,

$$\int_{|x|=R} e^{ik\,x\cdot(d_n - d_m)} ds(x) = \frac{4\pi R \sin(kR|d_n - d_m|)}{k|d_n - d_m|}\,, \quad n \neq m,$$

to see that

$$\frac{1}{R^2} \sum_{n=1}^{N} c_n \int_{|x|=R} e^{ik\,x\cdot(d_n - d_m)} ds(x) = 4\pi c_m + O\left(\frac{1}{R}\right), \quad R \to \infty.$$

Hence, passing to the limit $R \to \infty$ in (5.2) yields $c_m = 0$ for $m = 1, \dots, N$, i.e., the functions $u(\cdot\,, d_n)$, $n = 1, \dots, N$, are linearly independent.

We now show that there are only finitely many linearly independent eigenfunctions u_n possible. Assume to the contrary that we have infinitely many. Then, by the Gram–Schmidt orthogonalization procedure, we may assume that

$$\int_{D^*} u_n \bar{u}_m \, dx = \delta_{nm}$$

where δ_{nm} denotes the Kronecker delta symbol. From Green's theorem (2.2) we observe that

$$\int_{D^*} |\operatorname{grad} u_n|^2 dx = k^2 \int_{D^*} |u_n|^2 dx = k^2,$$

that is, the sequence (u_n) is bounded with respect to the norm in $H_0^1(D^*)$. By the Rellich selection theorem, that is, by the compactness of the imbedding of the Sobolev space $H_0^1(D^*)$ into $L^2(D^*)$, we can choose a convergent subsequence of (u_n) with respect to the norm in $L^2(D^*)$. But this is a contradiction to $\|u_n - u_m\|^2_{L^2(D^*)} = 2$ for all $n \neq m$ for the orthonormal sequence (u_n). □

According to the following theorem (Colton and Sleeman [88]), the scatterer is uniquely determined by the far field pattern of a finite number of incident plane waves provided a priori information on the size of the obstacle is available. For $n = 0, 1, \dots,$ we denote the positive zeros of the spherical Bessel functions j_n by t_{nl}, $l = 0, 1, \dots,$ i.e., $j_n(t_{nl}) = 0$.

Theorem 5.2. *Let D_1 and D_2 be two sound-soft scatterers which are contained in a ball of radius R, let*

$$N := \sum_{t_{nl} < kR} (2n + 1),$$

and assume that the far field patterns coincide for $N + 1$ incident plane waves with distinct directions and one fixed wave number. Then $D_1 = D_2$.

Proof. We proceed as in the previous proof, recalling the definition of D^*. As a consequence of the Courant maximum-minimum principle for compact symmetric operators, the eigenvalues of the negative Laplacian under Dirichlet boundary conditions have the following strong monotonicity property (see [224], Theorem 4.7): the n–th eigenvalue for a ball B containing the domains D_1 and D_2 is always smaller than the n–th eigenvalue for the subdomain $D^* \subset B$ where the eigenvalues are arranged according to increasing magnitude and taken with their respective multiplicity. Hence, if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m = k^2$ are the eigenvalues of D^* that are less than or equal to k^2 and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$ are the first m eigenvalues of the ball of radius R, then $\mu_m < \lambda_m = k^2$. In particular, the multiplicity M of λ_m is less than or equal to the sum of the multiplicities of the eigenvalues for the ball which are less than k^2. However, from the discussion of the example after Theorem 3.21, we know that the eigenfunctions for a ball of radius R are the spherical wave functions $j_n(k|x|)\, Y_n(\hat{x})$ with the eigenvalues given in terms of the zeros of the spherical Bessel functions by $\mu_{nl} = t_{nl}^2/R^2$. By Theorem 2.7, the multiplicity of the eigenvalues μ_{nl} is $2n + 1$ whence $M \leq N$ follows with N as defined in the theorem. Since D^* is nonempty, this leads to a contradiction because the $N + 1$ different incident waves yield $N + 1$ linearly independent eigenfunctions with eigenvalue k^2 for D^*. Hence, we can now conclude that $D_1 = D_2$. □

Essentially the same arguments as in the proof of Theorem 5.1 show that a sound-soft scatterer is uniquely determined by the far field patterns for an infinite number of incident plane waves with distinct wave numbers that do not accumulate at infinity and one fixed incident direction. There is also an analogue of Theorem 5.2 for a finite number of wave numbers with one fixed incident direction.

A challenging open problem is to determine if the far field pattern for scattering of one incident plane wave at one single wave number completely determines the scatterer. However, under additional geometric assumptions uniqueness for one incident plane wave can be established. As a corollary of Theorem 5.2 we immediately have a uniqueness result for one incident plane wave under an a priori assumption on the size of the scatterer.

Corollary 5.3 *Let D_1 and D_2 be two sound-soft scatterers which are contained in a ball of radius R such that $kR < \pi$ and assume that the far field patterns coincide for one incident plane wave with wave number k. Then $D_1 = D_2$.*

Proof. From (2.36) and Rolle's theorem, we see that between two zeros of j_n there lies a zero of j_{n-1}. Since $j_n(0) = 0$ for $n = 1, 2, \dots$, this implies that the sequence (t_{n0}) is strictly monotonically increasing and therefore the smallest positive zero of the spherical Bessel functions is given by the smallest zero of j_0, that is, $t_{00} = \pi$ since $j_0(t) = \sin t/t$. □

Exploiting the fact that the wave functions are complex-valued and consequently the corresponding eigenvalue is of multiplicity larger than one, Gintides [106] was able to improve the bound in Corollary 5.3 to $kR < t_{10} \dots$, that is, to $kR < 4.49 \dots$. A sound soft scatterer D is also uniquely determined if instead of assuming that D is contained in a ball of sufficiently small radius it is assumed that D is close to a given obstacle as shown by Stefanov and Uhlmann [309].

Uniqueness can also be established under a priori assumptions on the shape of the scatterer. If it is known a priori that the scatterer is a ball, uniqueness for one incident wave was first established by Liu [229] and can be proven as follows with the aid of a translation property of the far field pattern for a general domain D. For the shifted domain $D_h := \{x + h : x \in D\}$ with $h \in \mathbb{R}^3$ and boundary $\partial D_h := \{x + h : x \in \partial D\}$ the scattered field u_h^s for plane wave incidence $u^i(x) = e^{ik\, x\cdot d}$ is given by

$$u_h^s(x) = e^{ikh\cdot d} u^s(x - h), \quad x \in \mathbb{R}^3 \setminus D_h,$$

in terms of the scattered field u^s for D as can be seen by checking the boundary condition $u_h^s + u^i = 0$ on ∂D_h. In view of (2.15), the corresponding far field pattern is

$$u_{\infty,h}(\hat{x}) = e^{ik\, h\cdot(d-\hat{x})} u_\infty(\hat{x}), \quad \hat{x} \in \mathbb{S}^2, \tag{5.3}$$

in terms of the far field pattern u_∞ for D.

Theorem 5.4. *A sound-soft ball is uniquely determined by the far field pattern for one incident plane wave.*

Proof. Assume that two balls D_1 and D_2 with centers z_1 and z_2 have the same far field pattern $u_{\infty,1} = u_{\infty,2}$ for scattering of one incident plane wave. Then by Rellich's lemma, i.e., Theorem 2.14, the scattered waves coincide in $\mathbb{R}^3 \setminus (D_1 \cup D_2)$ and we can identify $u^s = u_1^s = u_2^s$ in $\mathbb{R}^2 \setminus (D_1 \cup D_2)$. Now assume that $z_1 \neq z_2$. Then from the explicit solution (3.31) we observe that u_1^s has an extension into $\mathbb{R}^3 \setminus \{z_1\}$ and u_2^s an extension into $\mathbb{R}^3 \setminus \{z_2\}$. Therefore, u^s can be extended from $\mathbb{R}^3 \setminus (D_1 \cup D_2)$ into all of $\mathbb{R}^3$, that is, u^s is an entire solution to the Helmholtz equation. Consequently, since u^s also satisfies the radiation condition it must vanish identically in all of $\mathbb{R}^3$. Then the total field coincides with the incident field and this leads to a contradiction since the plane wave cannot satisfy the boundary condition.

Therefore the two balls D_1 and D_2 must have the same center. From (3.32) we observe that the far field pattern for scattering from a ball centered at the origin only depends on the angle between the incident and the observation direction. Therefore, in view of (5.3), coincidence of the far field patterns for scattering from the two balls D_1 and D_2 with the same center for one incident plane wave implies coincidence of the far field patterns for all incident directions. Now the statement of the theorem follows from Theorem 5.1. □

Finally, if it is assumed that D is a polyhedron, then a single incident plane wave is sufficient to uniquely determine D as established by Cheng and Yamamoto [49], by Alessandrini and Rondi [5] and by Liu and Zou [231]. For convenience we only present a proof for the special case of a convex polyhedron, since the general case is technically more involved.

Theorem 5.5. *A sound-soft convex polyhedron is uniquely determined by the far field pattern for one incident plane wave.*

Proof. Assume that two convex polyhedrons D_1 and D_2 have the same far field pattern $u_{\infty,1} = u_{\infty,2}$ for scattering of one incident plane wave. As in the proof of the previous theorem, by Rellich's lemma the total waves coincide in $\mathbb{R}^3 \setminus (D_1 \cup D_2)$ and we can identify $u = u_1 = u_2$ in $\mathbb{R}^2 \setminus (D_1 \cup D_2)$. Now assume that $D_1 \neq D_2$. Then, because of convexity, there exists a corner z of one of the polyhedrons, let's say D_1, and an open neighborhood V of z such that V does not intersect with the closure of the other polyhedron D_2. Consider an open face Γ of D_1 that has z as one of its corners and denote the plane containing Γ by Λ. Then $\underline{u_2}$ is analytic on $V \cap \Lambda$ as the total field for the obstacle D_2 and u_1 is analytic on $\Lambda \setminus \overline{\Gamma}$ as the total field for the obstacle D_1. Since u vanishes on $V \cap \Gamma$ as the total field for the obstacle D_1 the analyticity of u_2 implies $u = 0$ in $V \cap \Lambda$. From this and the analyticity of u_1 we finally obtain $u = 0$ on all of the plane Λ. However this is a contradiction to $|u^i(x)| = 1$ for all $x \in \mathbb{R}^3$ and $u^s(x) \to 0$ as $x \to \infty$. □

We note that for other than the Dirichlet boundary condition there is no analogue to Theorem 5.2 and Corollary 5.3 since there is no monotonicity property for the eigenvalues of the negative Laplacian with respect to the domain for other boundary conditions. However, Theorems 5.4 and 5.5 can be extended to other boundary conditions, for the latter see [231].

Difficulties arise in attempting to generalize Schiffer's approach to other boundary conditions. This is due to the fact that the finiteness of the dimension of the eigenspaces for eigenvalues of $-\Delta$ for the Neumann or impedance boundary condition requires the boundary of the intersection D^* from the proof of Theorem 5.1 to be sufficiently smooth. Therefore, a different approach is required for establishing uniqueness for the inverse obstacle scattering problem for other boundary conditions. Assuming two different scatterers producing the same far field patterns for all incident directions, Isakov [152, 153] obtained a contradiction by considering a sequence of solutions with a singularity moving towards a boundary point of one scatterer that is not contained in the other scatterer. He used weak solutions and the proofs are technically involved. Later on, Kirsch and Kress [190] realized that these proofs can be simplified by using classical solutions rather than weak solutions and by obtaining the contradiction by considering pointwise limits of the singular solutions rather than limits of L^2 integrals. Only after this new uniqueness proof was published, it was also observed by the authors that for scattering from impenetrable objects it is not required to know the boundary condition of the scattered wave on the boundary of the scatterer. Furthermore, as stated in the following theorem, one can conclude that in addition to the shape ∂D of the scatterer the boundary condition is also uniquely determined by the far field pattern for infinitely many incident plane waves (see also Alves and Ha-Duong [6], Cakoni and Colton [31] and Kress and Rundell [211]).

We consider boundary conditions of the form $Bu = 0$ on ∂D, where $Bu = u$ for a sound-soft scatterer and $Bu = \partial u/\partial \nu + ik\lambda u$ for the impedance boundary condition. In the latter case, the real-valued function λ is assumed to be continuous and nonnegative to ensure well-posedness of the direct scattering problem as can be proven analogous to Theorem 3.12.

In the proof of the uniqueness theorem, in addition to scattering of plane waves, we also need to consider scattering of point sources $\Phi(\cdot\,, z)$ with source location z in $\mathbb{R}^3 \setminus \bar{D}$. As in the mixed reciprocity relation we denote the corresponding scattered wave by $w^s(\cdot\,, z)$ and its far field pattern by $w_\infty(\cdot\,, z)$.

Theorem 5.6. *Assume that D_1 and D_2 are two scatterers with boundary conditions B_1 and B_2 such that the far field patterns coincide for an infinite number of incident plane waves with distinct directions and one fixed wave number. Then $D_1 = D_2$ and $B_1 = B_2$.*

Proof. Following Potthast [281] we simplify the approach of Kirsch and Kress through the use of the mixed reciprocity relation (3.39). Let $u_{\infty,1}$ and $u_{\infty,2}$ be the far field patterns for plane wave incidence and let w_1^s and w_2^s be the scattered waves for point source incidence corresponding to D_1 and D_2, respectively. With (3.39) and two applications of Theorem 2.14, first for scattering of plane waves and then for scattering of point sources, from the assumption $u_{\infty,1}(\hat{x}, d) = u_{\infty,2}(\hat{x}, d)$ for all $\hat{x}, d \in \mathbb{S}^2$ it can be concluded that $w_1^s(x, z) = w_2^s(x, z)$ for all $x, z \in G$. Here, as in the proof of Theorem 5.1, G denotes the unbounded component of the complement of $D_1 \cup D_2$.

Now assume that $D_1 \neq D_2$. Then, without loss of generality, there exists $x^* \in \partial G$ such that $x^* \in \partial D_1$ and $x^* \notin \bar{D}_2$. In particular we have

$$z_n := x^* + \frac{1}{n}\,\nu(x^*) \in G, \quad n = 1, 2, \ldots,$$

for sufficiently large n. Then, on one hand we obtain that

$$\lim_{n\to\infty} B_1 w_2^s(x^*, z_n) = B_1 w_2^s(x^*, x^*),$$

since $w_2^s(x^*, \cdot)$ is continuously differentiable in a neighborhood of $x^* \notin \bar{D}_2$ due to the reciprocity Theorem 3.17 and the well-posedness of the direct scattering problem with boundary condition B_2 on ∂D_2. On the other hand we find that

$$\lim_{n\to\infty} B_1 w_1^s(x^*, z_n) = \infty,$$

because of the boundary condition $B_1 w_1^s(x^*, z_n) = -B_1 \Phi(x^*, z_n)$ on ∂D_1. This contradicts $w_1^s(x^*, z_n) = w_2^s(x^*, z_n)$ for all sufficiently large n, and therefore $D_1 = D_2$.

Finally, to establish that $\lambda_1 = \lambda_2$ for the case of two impedance boundary conditions B_1 and B_2 we set $D = D_1 = D_2$ and assume that $\lambda_1 \neq \lambda_2$. Then from Rellich's lemma, i.e., Theorem 2.14, and the boundary conditions (considering one incident field) we have that

$$\frac{\partial u}{\partial \nu} + ik\lambda_1 u = \frac{\partial u}{\partial \nu} + ik\lambda_2 u = 0 \quad \text{on } \partial D$$

for the total wave $u = u_1 = u_2$. Hence, $(\lambda_1 - \lambda_2)u$ on ∂D. From this, in view of the fact that $\lambda_1 \neq \lambda_2$, by Holmgren's Theorem 2.3 and the boundary condition we obtain that $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. This leads to the contradiction that the incident field must satisfy the radiation condition. Hence, $\lambda_1 = \lambda_2$. The case when one of the boundary conditions is the sound-soft boundary condition is dealt with analogously. □

The above method has been employed by Kirsch and Kress [190] to prove an analogue of Theorem 5.6 for the transmission problem and by Hettlich [136] and by Gerlach and Kress [103] for the conductive boundary condition. Later on in Section 5.6 we will see that Potthast's [279, 281, 283] *singular source method* for solving the inverse obstacle scattering problem can be viewed as a straightforward numerical implementation of the uniqueness proof for Theorem 5.6 whereas the *probe or needle method* as suggested by Ikehata [149, 150] follows the uniqueness proof of Isakov.

Closely related to the uniqueness question is the following example which gives an explicit solution to the inverse obstacle problem. If the sound-soft scatterer D is a ball centered at the origin, then in the proof of Theorem 5.4 we have already used the fact that in view of (3.32) the far field pattern only depends on the angle between the incident and the observation direction. Hence, we have

$$u_\infty(\hat{x}, d) = u_\infty(Q\hat{x}, Qd) \tag{5.4}$$

for all $\hat{x}, d \in \mathbb{S}^2$ and all rotations, i.e., all real orthogonal matrices Q with $\det Q = 1$. *Karp's theorem* [172] says that the converse of this statement is also true. We shall now show how the approach of Colton and Kirsch [58] to proving this result can be considerably simplified. For this, in view of the Funk–Hecke formula (2.45), we consider the superposition of incident plane waves given by

$$v^i(x) = \int_{\mathbb{S}^2} e^{ik\,x\cdot d} ds(d) = \frac{4\pi \sin k|x|}{k|x|}\,. \tag{5.5}$$

Then, by Lemma 3.20, the corresponding scattered wave v^s has the far field pattern

$$v_\infty(\hat{x}) = \int_{\mathbb{S}^2} u_\infty(\hat{x}, d)\, ds(d),$$

and the condition (5.4) implies that $v_\infty(\hat{x}) = c$ for some constant c. Hence,

$$v^s(x) = c\,\frac{e^{ik|x|}}{|x|} \tag{5.6}$$

follows. From (5.5) and (5.6) together with the boundary condition $v^i + v^s = 0$ on ∂D we now have

$$\sin k|x| + \frac{kc}{4\pi}\, e^{ik|x|} = 0$$

for all $x \in \partial D$. From this we conclude that $|x| = \text{constant}$ for all $x \in \partial D$, i.e., D is a ball with center at the origin.

5.2 Physical Optics Approximation

From the boundary integral equation approach to the solution of the direct scattering problem, it is obvious that the far field pattern depends nonlinearly on the boundary of the scatterer. Therefore, the inverse problem to determine the boundary of the scatterer from a knowledge of the far field pattern is a nonlinear problem. We begin our discussion on the solution of the inverse problem by presenting a linearized method based on the Kirchhoff or physical optics approximation.

In the physical optics approximation for a convex sound-soft scatterer D for large wave numbers k, by (3.36) the far field pattern is approximately given by

$$u_\infty(\hat{x}, d) = -\frac{ik}{2\pi} \int_{\partial D_-} \nu(y)\cdot d\; e^{ik(d-\hat{x})\cdot y}\, ds(y), \quad \hat{x} \in \mathbb{S}^2,$$

where $\partial D_- := \{y \in \partial D : \nu(y) \cdot d < 0\}$ denotes the part of the boundary which is illuminated by the plane wave with incident direction d. In particular, for $\hat{x} = -d$, i.e., for the far field in the *back scattering* direction we have

$$u_\infty(-d, d) = -\frac{1}{4\pi} \int_{\nu(y)\cdot d<0} \frac{\partial}{\partial \nu(y)} \, e^{2ik\, d\cdot y} \, ds(y).$$

Analogously, replacing d by $-d$, we have

$$u_\infty(d, -d) = -\frac{1}{4\pi} \int_{\nu(y)\cdot d>0} \frac{\partial}{\partial \nu(y)} \, e^{-2ik\, d\cdot y} \, ds(y).$$

Combining the last two equations we find

$$u_\infty(-d, d) + \overline{u_\infty(d, -d)} = -\frac{1}{4\pi} \int_{\partial D} \frac{\partial}{\partial \nu(y)} \, e^{2ik\, d\cdot y} \, ds(y)$$

whence by Green's theorem

$$u_\infty(-d, d) + \overline{u_\infty(d, -d)} = -\frac{1}{4\pi} \int_D \Delta e^{2ik\, d\cdot y} \, dy = \frac{k^2}{\pi} \int_D e^{2ik\, d\cdot y} \, dy$$

follows. Denoting by χ the characteristic function of the domain D, we rewrite this equation in the form

$$\int_{\mathbb{R}^3} \chi(y) e^{2ik\, d\cdot y} \, dy = \frac{\pi}{k^2} \left\{ u_\infty(-d, d) + \overline{u_\infty(d, -d)} \right\} \tag{5.7}$$

which is known as the *Bojarski identity* [24, 25]. Hence, in the physical optics approximation the Fourier transform of the characteristic function of the scatterer, in principle, can be completely obtained from measurements of the back scattering far field for all incident directions $d \in \mathbb{S}^2$ and all wave numbers $k > 0$. Then, by inverting the Fourier transform (which is a bounded operator from $L^2(\mathbb{R}^3)$ onto $L^2(\mathbb{R}^3)$ with bounded inverse) one can determine χ and therefore D. Thus, the physical optics approximation leads to a linearization of the inverse problem. For details on the implementation, we refer to Bleistein [22], Devaney [92], Langenberg [218] and Ramm [287].

However, there are several drawbacks to this procedure. Firstly, we need the far field data for all wave numbers. But the physical optics approximation is valid only for large wave numbers. Therefore, in practice, the Fourier transform of χ is available only for wave numbers $k \geq k_0$ for some $k_0 > 1$. This means that we have to invert a Fourier transform with incomplete data. This may cause uniqueness ambiguities and it leads to severe ill-posedness of the inversion as is known from corresponding situations in the inversion of the Radon transform in computerized tomography (see Natterer [258]). Thus, the ill-posedness which seemed to have disappeared through the inversion of the Fourier transform is back on stage. Secondly, and particularly important in the context of the scope of this book, the physical optics approximation will not work at all in situations where far field data are available only for frequencies in the resonance region. Therefore, after this brief mentioning of the physical optics approximation we turn our attention to a solution of the full nonlinear inverse scattering problem.

5.3 Continuity and Differentiability of the Far Field Mapping

The solution to the direct scattering problem with a fixed incident plane wave u^i defines an operator

$$\mathcal{F} : \partial D \mapsto u_\infty$$

which maps the boundary ∂D of the sound-soft scatterer D onto the far field pattern u_∞ of the scattered wave. In terms of this operator, given a far field pattern u_∞, the inverse problem just consists in solving the nonlinear and ill-posed equation

$$\mathcal{F}(\partial D) = u_\infty \tag{5.8}$$

for the unknown surface ∂D. Hence, it is quite natural to try some of the regularization methods mentioned at the end of the previous chapter. For this we need to establish some properties of the operator $\mathcal{F}$.

So far, we have not defined the solution and data space for the operator $\mathcal{F}$. Having in mind that for ill-posed problems the norm in the data space has to be suitable for describing the measurement errors, our natural choice is the Hilbert space $L^2(\mathbb{S}^2)$ of square integrable functions on the unit sphere. For the solution space we have to choose a class of admissible surfaces described by some suitable parametrization and equipped with an appropriate norm. For the sake of simplicity, we restrict ourselves to the case of domains D which are starlike with respect to the origin, that is, we assume that ∂D is represented in the parametric form

$$\partial D = \left\{ r(\hat{x})\,\hat{x} : \hat{x} \in \mathbb{S}^2 \right\} \tag{5.9}$$

with a positive function $r \in C^1(\mathbb{S}^2)$. Then we may view the operator $\mathcal{F}$ as a mapping from $C^1(\mathbb{S}^2)$ into $L^2(\mathbb{S}^2)$. We also will write $\mathcal{F}(r)$ synonymously for $\mathcal{F}(\partial D)$.

The main question we wish to address in this section is that of continuity, differentiability and compactness of the operator $\mathcal{F}$. For this, we have to investigate the dependence of the solution to the direct problem on the boundary surface. In principle, this analysis can be based either on the boundary integral equation method described in Section 3.2 or on weak solution methods. In both approaches, in order to compare the solution operators for different domains, we have to transform the boundary value problem for the variable domain into one for a fixed reference domain. Since each method has its merits, we will present them both. We postpone the presentation of the continuous dependence and differentiability results via integral equation methods until the end of this section and first follow Pironneau [272] in using a weak solution approach. In doing this, as in Sections 3.1 and 3.2, we expect the reader to be familiar with the basic theory of Sobolev spaces. For an introduction to Sobolev spaces, we refer to Adams [2], Gilbarg and Trudinger [105], McLean [240] and Treves [317].

Theorem 5.7. *For a fixed incident wave u^i, the operator $\mathcal{F} : \partial D \mapsto u_\infty$ which maps the boundary ∂D onto the far field pattern u_∞ of the scattered wave u^s is continuous from $C^1(\mathbb{S}^2)$ into $L^2(\mathbb{S}^2)$.*

Proof. Due to the unbounded domain and the radiation condition, we have to couple the weak solution technique either with a boundary integral equation method or a spectral method. Our proof consists of two parts. First, for a fixed domain, we will establish that the linear operator which maps the incident field u^i onto the normal derivative $\partial u^s/\partial \nu$ of the scattered field is bounded from the Sobolev space $H^{1/2}(S_R)$ into its dual space $H^{-1/2}(S_R)$. Here, S_R is a sphere of radius R centered at the origin where R is chosen large enough such that D is contained in the interior of $S_{R/2}$. In the second step, we will show that this mapping depends continuously on the boundary ∂D whence the statement of the theorem follows by using Theorem 2.6.

We denote $D_R := \{x \in \mathbb{R}^3 \setminus \bar{D} : |x| < R\}$ and introduce the Sobolev space $\tilde{H}_0^1(D_R) := \{v \in H^1(D_R) : v = 0 \text{ on } \partial D\}$ where the boundary condition $v = 0$ on ∂D has to be understood in the sense of the trace operator. As in the proof of Theorem 5.1, from Lemma 3.10 we see that the solution u to the direct scattering problem belongs to $\tilde{H}_0^1(D_R)$. Then, by Green's theorem (2.2), we see that u satisfies

$$\int_{D_R} \left\{ \operatorname{grad} u \cdot \operatorname{grad} \bar{v} - k^2 u \bar{v} \right\} dx = \int_{S_R} \frac{\partial u}{\partial \nu}\, \bar{v}\, ds \tag{5.10}$$

for all $v \in \tilde{H}_0^1(D_R)$ where ν denotes the exterior unit normal to S_R.

We recall the Dirichlet to Neumann map $\mathcal{A}$ for radiating solutions w to the Helmholtz equation in the exterior of S_R as introduced in Theorem 3.13. It transforms the boundary values into the normal derivative on the boundary

$$\mathcal{A} : w \mapsto \frac{\partial w}{\partial \nu} \quad \text{on } S_R.$$

As pointed out after the proof of Theorem 3.13, the operator $\mathcal{A}$ is bounded from $H^{1/2}(S_R)$ into $H^{-1/2}(S_R)$ and has a bounded inverse. However, for the simple shape of the sphere this result and further properties of $\mathcal{A}$ can also be established by expansion of w with respect to spherical wave functions as we will briefly indicate.

From the expansion (2.49) of radiating solutions to the Helmholtz equation with respect to spherical wave functions, we see that $\mathcal{A}$ maps

$$w = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m \, Y_n^m$$

with coefficients a_n^m onto

$$\mathcal{A}w = \sum_{n=0}^{\infty} \gamma_n \sum_{m=-n}^{n} a_n^m \, Y_n^m$$

where

$$\gamma_n := \frac{k h_n^{(1)\prime}(kR)}{h_n^{(1)}(kR)}, \quad n = 0, 1, \dots . \tag{5.11}$$

The spherical Hankel functions and their derivatives do not have real zeros since otherwise the Wronskian (2.37) would vanish. From this we observe that $\mathcal{A}$ is bijective. In view of the differentiation formula (2.35) and the asymptotic formula (2.39) for the spherical Hankel functions, we see that

$$c_1(n+1) \le |\gamma_n| \le c_2(n+1)$$

for all n and some constants $0 < c_1 < c_2$. From this the boundedness of $\mathcal{A} : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ is obvious since for $p \in \mathbb{R}$ the norm on $H^p(S_R)$ can be described in terms of the Fourier coefficients by

$$\|w\|_p^2 = \sum_{n=0}^{\infty} (n+1)^{2p} \sum_{m=-n}^{n} \left|a_n^m\right|^2 .$$

For the limiting operator $\mathcal{A}_0 : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ given by

$$\mathcal{A}_0 w = -\frac{1}{R} \sum_{n=0}^{\infty} (n+1) \sum_{m=-n}^{n} a_n^m \, Y_n^m,$$

we clearly have

$$-\int_{S_R} \mathcal{A}_0 w \, \bar{w} \, ds = \frac{1}{R} \sum_{n=0}^{\infty} (n+1) \sum_{m=-n}^{n} |a_n^m|^2$$

with the integral to be understood as the duality pairing between $H^{1/2}(S_R)$ and $H^{-1/2}(S_R)$. Hence,

$$-\int_{S_R} \mathcal{A}_0 w \, \bar{w} \, ds \ge c \|w\|^2_{H^{1/2}(S_R)}$$

for some constant $c > 0$, that is, the operator $-\mathcal{A}_0$ is strictly coercive. Finally, from the power series expansions (2.32) and (2.33) for the spherical Hankel functions, for fixed k we derive

$$\gamma_n = -\frac{n+1}{R} \left\{ 1 + O\left(\frac{1}{n}\right) \right\}, \quad n \to \infty.$$

This implies that $\mathcal{A} - \mathcal{A}_0$ is compact from $H^{1/2}(S_R)$ into $H^{-1/2}(S_R)$ since it is bounded from $H^{1/2}(S_R)$ into $H^{1/2}(S_R)$ and the imbedding from $H^{1/2}(S_R)$ into $H^{-1/2}(S_R)$ is compact.

For the following transformation of the scattering problem into a sesquilinear equation we follow Kirsch [180] who simplified our analysis from the first edition of this book. From (5.10) it can be deduced that if u is the solution to the scattering problem, then $u \in \tilde{H}_0^1(D_R)$ satisfies the sesquilinear equation

$$\int_{D_R} \{\operatorname{grad} u \cdot \operatorname{grad} \bar{v} - k^2 u\bar{v}\} dx - \int_{S_R} \mathcal{A}u\,\bar{v}\,ds = \int_{S_R} \left\{\frac{\partial u^i}{\partial \nu} - \mathcal{A}u^i\right\} \bar{v}\,ds \tag{5.12}$$

for all $v \in \tilde{H}_0^1(D_R)$. The sesquilinear form T on $\tilde{H}_0^1(D_R)$ defined by the left hand side of (5.12) can be written as $T = V_0 + V_1$ where

$$V_0(u, v) := \int_{D_R} \{\operatorname{grad} u \cdot \operatorname{grad} \bar{v} + u\bar{v}\}\; dx - \int_{S_R} \mathcal{A}_0 v\,\bar{v}\,ds$$

and

$$V_1(u, v) := -(k^2 + 1) \int_{D_R} u\bar{v}\,dx - \int_{S_R} (\mathcal{A} - \mathcal{A}_0)u\,\bar{v}\,ds.$$

Note that V_0 and V_1 are well defined since by the trace theorem the restriction of u on S_R belongs to $H^{1/2}(S_R)$ if $u \in \tilde{H}_0^1(D_R)$. Clearly, as a consequence of the boundedness of $\mathcal{A} : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ and the trace theorem, V_0 is bounded, and since

$$V_0(u, u) = \|u\|^2_{\tilde{H}_0^1(D_R)} - \int_{S_R} \mathcal{A}_0 u\,\bar{u}\,ds \geq \|u\|^2_{\tilde{H}_0^1(D_R)} + c\|u\|^2_{H^{1/2}(S_R)} \geq \|u\|^2_{\tilde{H}_0^1(D_R)}$$

for all $u \in \tilde{H}_0^1(D_R)$ we have that V_0 is strictly coercive. By the compactness of $\mathcal{A} - \mathcal{A}_0 : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ and the Rellich selection theorem, that is, the compact imbedding of $\tilde{H}_0^1(D_R)$ into $L^2(D_R)$, the term V_1 is compact. Through the Riesz representation theorem we can write

$$T(u, v) = (Lu, v) \tag{5.13}$$

with a bounded linear operator L mapping $\tilde{H}_0^1(D_R)$ into itself. Corresponding to $T = V_0 + V_1$, we have $L = K_0 + K_1$ where K_0 is strictly coercive and K_1 is compact. Hence, by the Lax–Milgram theorem (see [205]) and the Riesz–Fredholm theory for compact operators, in order to establish unique solvability of the sesquilinear equation (5.12) and continuous dependence of the solution on the right hand side, that is, the existence of a bounded inverse L^{-1} to L, it suffices to prove uniqueness for the homogeneous form of (5.12).

This can be shown as a consequence of the uniqueness for the direct scattering problem. Assume that $u \in \tilde{H}_0^1(D_R)$ satisfies

$$T(u, v) = 0$$

for all $v \in \tilde{H}_0^1(D_R)$, that is,

$$\int_{D_R} \left\{\operatorname{grad} u \cdot \operatorname{grad} \bar{v} - k^2 u\bar{v}\right\} dx - \int_{S_R} \mathcal{A}u\,\bar{v}\,ds = 0 \tag{5.14}$$

for all $v \in \tilde{H}_0^1(D_R)$. Substituting $v = u$ into (5.14) and taking the imaginary part we obtain

$$\operatorname{Im} \int_{S_R} \mathcal{A} u \, \bar{u} \, ds = 0,$$

that is,

$$\sum_{n=0}^{\infty} \operatorname{Im} \gamma_n \sum_{m=-n}^{n} |a_n^m|^2 = 0$$

for

$$u = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m \, Y_n^m \quad \text{on } S_R.$$

From this we deduce $u = 0$ on S_R since from (5.11) and the Wronskian (2.37) we have that

$$\operatorname{Im} \gamma_n = \frac{1}{kR^2 \left| h_n^{(1)}(kR) \right|^2} > 0$$

for all n. Now (5.14) reads

$$\int_{D_R} \left\{ \operatorname{grad} u \cdot \operatorname{grad} \bar{v} - k^2 u \bar{v} \right\} dx = 0,$$

i.e., u is a weak solution to the Helmholtz equation in D_R satisfying weakly a homogeneous Dirichlet condition on ∂D and a homogeneous Neumann condition on S_R. In addition we have $u = 0$ on S_R. By the classical regularity properties of weak solutions to elliptic boundary value problems (see [105]), it follows that u also is a classical solution. Then by Holmgren's Theorem 2.3 the vanishing Cauchy data on S_R implies that $u = 0$ in D_R. This completes the uniqueness proof for the sesquilinear equation (5.12).

We now wish to study the dependence of the sesquilinear form T on the shape of the domain D. For this we map D_R onto a fixed reference domain B_R. We represent the starlike boundary surface ∂D in the parametric form (5.9) with a positive radial function $r \in C^1(\mathbb{S}^2)$. Without loss of generality, we may assume that $r(\hat{x}) > 1$ for all $\hat{x} \in \mathbb{S}^2$. After setting $R = 2\|r\|_\infty$, the mapping ψ_r defined by

$$\Psi_r(y) := y + h_r(y), \quad |y| \geq 1, \tag{5.15}$$

where

$$h_r(y) := \begin{cases} \left(r\left(\dfrac{y}{|y|} \right) - 1 \right) \left(\dfrac{R - |y|}{R - 1} \right)^2 \dfrac{y}{|y|}, & 0 \leq |y| \leq R, \\[2ex] y, & R \leq |y| < \infty, \end{cases}$$

is a diffeomorphism from the closed exterior of the unit sphere onto $\mathbb{R}^3 \setminus D$ such that $B_R := \{y \in \mathbb{R}^3 : 1 < |y| < R\}$ is mapped onto D_R. We denote its Jacobian by Ψ_r'

and substitute $x = \Psi_r(y)$ to transform

$$\int_{D_R} \left\{ \operatorname{grad} u \cdot \operatorname{grad} \bar{v} - k^2 u\bar{v} \right\} dx = T_r(\tilde{u}, \tilde{v})$$

where $\tilde{u} = u \circ \Psi_r$, $\tilde{v} = v \circ \Psi_r$, and the sesquilinear form T_r on $\tilde{H}_0^1(B_R)$ is defined by

$$T_r(u, v) := \int_{B_R} \left\{ [\Psi_r']^{-1} \operatorname{grad} u \cdot [\Psi_r']^{-1} \operatorname{grad} \bar{v} - k^2\, u\bar{v} \right\} \det \Psi_r'\, dy \tag{5.16}$$

for $u, v \in \tilde{H}_0^1(B_R)$. After this transformation we can consider the sesquilinear equation on $\tilde{H}_0^1(B_R)$ with the fixed domain B_R independent of the boundary. Obviously, in any matrix norm on $\mathbb{R}^3$, we can estimate

$$\left\| [\Psi_{r+q}'(y)]^{-1} - [\Psi_r'(y)]^{-1} \right\| \leq c_1 \|q\|_{C^1(\mathbb{S}^2)}$$

and

$$\left| \det \Psi_{r+q}'(y) - \det \Psi_r'(y) \right| \leq c_1 \|q\|_{C^1(\mathbb{S}^2)}$$

for all $y \in B_R$, all q sufficiently small and some constant $c_1 > 0$ depending on r. From this we can conclude that

$$\sup_{\|u\|, \|v\| \leq 1} \left\| T_{r+q}(u, v) - T_r(u, v) \right\| \leq c_2 \|h\|_{C^1(\mathbb{S}^2)}$$

for q sufficiently small and some constant $c_2 > 0$ depending on r. Using (5.13), in terms of the operator L this can be written as

$$\left\| L_{r+q} - L_r \right\| \leq c_3 \|q\|_{C^1(\mathbb{S}^2)} \tag{5.17}$$

with some constant c_3. Therefore, a perturbation argument based on the Neumann series shows that the inverse operator also satisfies an estimate of the form

$$\left\| L_{r+q}^{-1} - L_r^{-1} \right\| \leq c_4 \|q\|_{C^1(\mathbb{S}^2)} \tag{5.18}$$

for q sufficiently small and some constant $c_4 > 0$ depending on r. Through the sesquilinear equation (5.12) this inequality carries over to the scattered waves on S_R, that is, we have

$$\left\| u_{r+q}^s - u_r^s \right\|_{H^{1/2}(S_R)} \leq c_5 \|q\|_{C^1(\mathbb{S}^2)}$$

for sufficiently small q and some constant c_5 depending on r. The theorem now follows from the boundedness of the Dirichlet to Neumann operator $\mathcal{A} : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ and the integral representation of the far field pattern in Theorem 2.6 applied in the exterior of S_R. □

We actually can pursue the ideas of the preceding proof one step further and show that $\mathcal{F} : \partial D \mapsto u_\infty$ has a Fréchet derivative. To this end, denoting by h_r' the Jacobian of h_r and exploiting the affine linearity of h_r with respect to r, from (5.15) we observe that

$$\left\|[\Psi'_{r+q}(y)]^{-1} - [\Psi'_r(y)]^{-1} + [\Psi'_r(y)]^{-1}h'_{q+1}(y)[\Psi'_r(y)]^{-1}\right\| \le c_5\|q\|^2_{C^1(\mathbb{S}^2)}$$

and

$$\left|\det \Psi'_{r+q}(y) - \det \Psi'_r(y) - \operatorname{div} h_{q+1}(y) - b\left(h'_r(y), h'_{q+1}(y)\right)\right| \le c_5\|q\|^2_{C^1(\mathbb{S}^2)}$$

for all $y \in B_R$, all q sufficiently small and some constant $c_5 > 0$ depending on r. Here, b denotes a bilinear form which we refrain from writing down explicitly. From this after setting

$$(T'_r q)(u, v) := \int_{B_R} \left\{\operatorname{grad}\tilde{u}\cdot\operatorname{grad}\bar{v} - k^2\, u\bar{v}\right\}\left\{\operatorname{div} h_{q+1} + b\left(h'_r, h'_{q+1}\right)\right\}\, dy$$
$$- \int_{B_R} [\Psi'^{\perp}_r]^{-1}\left\{[\Psi'_r]^{-1}h'_{q+1} + h'^{\perp}_{q+1}[\Psi'^{\perp}_r]^{-1}\right\}[\Psi'_r]^{-1}\operatorname{grad} u\cdot\operatorname{grad} v\, dy$$

we obtain that

$$\sup_{\|u\|,\|v\|\le 1}\left\|T_{r+q}(u, v) - T_r(u, v) - (T'_r q)(u, v)\right\| \le c_6\|q\|^2_{C^1(\mathbb{S}^2)}$$

for all $y \in B_R$, all q sufficiently small and some constant $c_6 > 0$ depending on r, that is, the mapping $r \mapsto T_r$ is Fréchet differentiable. By the Riesz representation theorem, this implies that there exists a bounded linear operator L'_r from $C^1(\mathbb{S}^2)$ into the space of bounded linear operators $\mathcal{L}\left(\tilde{H}^1_0(D_R)\right)$ from $\tilde{H}^1_0(D_R)$ into itself such that

$$\left\|L_{r+q} - L_r - L'_r q\right\| \le c_7\|q\|^2_{C^1(\mathbb{S}^2)}, \tag{5.19}$$

that is, the mapping $r \mapsto L_r$ from $C^1(\mathbb{S}^2)$ into $\mathcal{L}\left(\tilde{H}^1_0(D_R)\right)$ is also Fréchet differentiable. Then from

$$L_r\left\{L^{-1}_{r+q} - L^{-1}_r + L^{-1}_r L'_r q L^{-1}_r\right\}L_r = (L_{r+q} - L_r)L^{-1}_{r+q}(L_{r+q} - L_r) - (L_{r+q} - L_r - L'_r q)$$

and (5.17)–(5.19) we see that

$$\left\|L^{-1}_{r+q} - L^{-1}_r + L^{-1}_r L'_r q L^{-1}_r\right\| \le c_8\|q\|^2_{C^1(\mathbb{S}^2)}$$

for some constant $c_8 > 0$ depending on r, that is, the mapping $r \mapsto L^{-1}_r$ is also Fréchet differentiable with the derivative given by

$$q \mapsto -L^{-1}_r L'_r q L^{-1}_r. \tag{5.20}$$

As in the previous proof, this now implies that the mapping $\partial D \mapsto u^s$ is Fréchet differentiable from C^1 into $H^{1/2}(S_R)$. Since $u^s \mapsto u_\infty$ is linear and bounded from $H^{1/2}(S_R)$ into $L^2(\mathbb{S}^2)$, we have established the following result.

Theorem 5.8. *The mapping* $\mathcal{F} : \partial D \mapsto u_\infty$ *is Fréchet differentiable from* $C^1(\mathbb{S}^2)$ *into* $L^2(\mathbb{S}^2)$.

Kirsch [180] has used the above weak solution approach to characterize the Fréchet derivative through a Dirichlet boundary condition. Here, we will derive this characterization by the boundary integral equation approach (see Theorem 5.14). For the Neumann boundary condition Fréchet differentiability of the boundary to far field operator via the above Hilbert space method has been established by Hettlich [137].

The following theorem indicates that solving the equation $\mathcal{F}(r) = u_\infty$ is improperly posed.

Theorem 5.9. *The mapping* $\mathcal{F} : \partial D \mapsto u_\infty$ *is locally compact from* $C^1(\mathbb{S}^2)$ *into* $L^2(\mathbb{S}^2)$*, that is, for each* $r \in C^1(\mathbb{S}^2)$ *(describing* ∂D*) there exists a neighborhood* U *of* r *such that* $\mathcal{F} : U \to L^2(\mathbb{S}^2)$ *is compact.*

Proof. From the proof of Theorem 5.7 we know that the mapping $r \mapsto u_r^s$ is continuous from $C^1(\mathbb{S}^2)$ into $H^{1/2}(S_R)$. This implies that for each $r \in C^1(\mathbb{S}^2)$ there exists a neighborhood U and a constant C such that

$$\|u_q^s\|_{H^{1/2}(S_R)} \le C$$

for all $q \in U$. The statement of the theorem now follows from the boundedness of the Dirichlet to Neumann operator $\mathcal{A} : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ and the analyticity of the kernel in the integral representation of the far field pattern in Theorem 2.6. □

Differentiability of the far field mapping $\mathcal{F}$ via boundary integral equations was first established by Potthast [273, 274]. In this approach, as a first and major step, the differentiability of the boundary integral operators has to be established. As in the weak solution approach, the operators need to be transformed onto a fixed reference surface. Again we assume that the boundary surface is starlike with respect to the origin and represented in the form (5.9) with a positive function in $r \in C^2(\mathbb{S}^2)$. For notational convenience we assosiate with each scalar function q on $\mathbb{S}^2$ a vector function p_q on $\mathbb{S}^2$ by setting

$$p_q(\hat{x}) := q(\hat{x})\,\hat{x}, \quad \hat{x} \in \mathbb{S}^2. \tag{5.21}$$

We note that the function p_r maps $\mathbb{S}^2$ bijectively onto ∂D. Substituting $x = p_r(\hat{x})$ and $y = p_r(\hat{y})$ into the expressions (3.8)–(3.10) for the single- and double-layer operators S and K and the normal derivative operator K', we obtain the transformed operators $S_r, K_r, K_r^* : C(\mathbb{S}^2) \to C(\mathbb{S}^2)$ given by

$$(S_r\psi)(\hat{x}) = 2\int_{\mathbb{S}^2} \Psi(p_r(\hat{x}) - p_r(\hat{y}))J_r(\hat{y})\psi(\hat{y})\,ds(\hat{y}),$$

$$(K_r\psi)(\hat{x}) = 2\int_{\mathbb{S}^2} \nu_r(\hat{y}) \cdot \operatorname{grad} \Psi(p_r(\hat{y}) - p_r(\hat{x}))J_r(\hat{y})\psi(\hat{y})\,ds(\hat{y}),$$

$$(K_r^*\psi)(\hat{x}) = 2\int_{\mathbb{S}^2} \nu_r(\hat{x}) \cdot \operatorname{grad} \Psi(p_r(\hat{x}) - p_r(\hat{y}))J_r(\hat{y})\psi(\hat{y})\,ds(\hat{y})$$

for $\hat{x} \in \mathbb{S}^2$. Here, we have set

$$\Psi(z) := \frac{1}{4\pi}\,\frac{e^{ik|z|}}{|z|}, \quad z \in \mathbb{R}^3 \setminus \{0\},$$

and straightforward calculations show that the determinant J_r of the Jacobian of the transformation and the normal vector ν_r are given by

$$J_r = r\sqrt{r^2 + |\operatorname{Grad} r|^2} \quad \text{and} \quad \nu_r = \frac{p_r - \operatorname{Grad} r}{\sqrt{r^2 + |\operatorname{Grad} r|^2}}\,. \tag{5.22}$$

For the further analysis we need the following two technical results.

Lemma 5.10 *The inequality*

$$|\{p_r(\hat{x}) - \operatorname{Grad} r(\hat{x})\} \cdot \{p_r(\hat{x}) - p_r(\hat{y})\}| \le C\|r\|_{1,\alpha}^2 |\hat{x} - \hat{y}|^{1+\alpha} \tag{5.23}$$

is valid for all $r \in C^{1,\alpha}(\mathbb{S}^2)$, all $\hat{x}, \hat{y} \in \mathbb{S}^2$ and some constant C.

Proof. Using $\operatorname{Grad} r(\hat{x}) \cdot \hat{x} = 0$, we can write

$$\{r(\hat{x})\,\hat{x} - \operatorname{Grad} r(\hat{x})\} \cdot \{r(\hat{x})\,\hat{x} - r(\hat{y})\,\hat{y}\} = a_1 + a_2 + a_3$$

where

$$a_1 := \operatorname{Grad} r(\hat{x}) \cdot \{\hat{y} - \hat{x}\}\,\{r(\hat{y}) - r(\hat{x})\},$$

$$a_2 := r(\hat{x})\{r(\hat{x}) - r(\hat{y}) + \operatorname{Grad} r(\hat{x}) \cdot (\hat{y} - \hat{x})\},$$

$$a_3 := r(\hat{x})r(\hat{y})\,\hat{x} \cdot \{\hat{x} - \hat{y}\}.$$

By writing

$$r(\hat{y}) - r(\hat{x}) = \int_\Gamma \frac{\partial r}{\partial s}(\hat{z})\,ds(\hat{z})$$

where Γ denotes the shorter great circle arc on $\mathbb{S}^2$ connecting $\hat{x}$ and $\hat{y}$, we can estimate

$$|r(\hat{y}) - r(\hat{x})| = \left|\int_\Gamma \frac{\partial r}{\partial s}(\hat{z})\,ds(\hat{z})\right| \le \|\operatorname{Grad} r\|_\infty \theta$$

where θ is the angle between $\hat{x}$ and $\hat{y}$. With the aid of the elementary inequality $2\theta \le \pi|\hat{y} - \hat{x}|$ we now have

$$|r(\hat{y}) - r(\hat{x})| \le \frac{\pi}{2} \, \|\operatorname{Grad} r\|_\infty |\hat{y} - \hat{x}| \tag{5.24}$$

and consequently

$$|a_1| \le \frac{\pi}{2} \, \|\operatorname{Grad} r\|_\infty^2 |\hat{y} - \hat{x}|^2 \le \frac{\pi}{2} \, \|r\|_{1,\alpha}^2 |\hat{y} - \hat{x}|^2.$$

We can also write

$$r(\hat{y}) - r(\hat{x}) - \operatorname{Grad} r(\hat{x}) \cdot (\hat{y} - \hat{x}) = \int_\Gamma \left\{ \frac{\partial r}{\partial s}(\hat{z}) - \frac{\partial r}{\partial s}(\hat{x}) \right\} ds(\hat{z}) + (\theta - \sin\theta) \, \frac{\partial r}{\partial s}(\hat{x}) \, .$$

From this, using the definition of the Hölder norm and $|\theta - \sin\theta| \le \theta^2/2$ which follows from Taylor's formula, we can estimate

$$|r(\hat{y}) - r(\hat{x}) - \operatorname{Grad} r(\hat{x}) \cdot (\hat{y} - \hat{x})| \le c\|r\|_{1,\alpha} |\hat{y} - \hat{x}|^{1+\alpha}$$

for some constant c, whence

$$|a_2| \le c\|r\|_{1,\alpha}^2 |\hat{y} - \hat{x}|^{1+\alpha}$$

follows. Finally, since on the unit sphere $2\,\hat{x} \cdot (\hat{x} - \hat{y}) = |\hat{x} - \hat{y}|^2$, we have

$$|a_3| \le \frac{1}{2} \, \|r\|_\infty^2 |\hat{y} - \hat{x}|^2 \le \frac{1}{2} \, \|r\|_{1,\alpha}^2 |\hat{y} - \hat{x}|^2.$$

We can now sum our three inequalities to obtain the estimate (5.23). □

Note that for our analysis of the Fréchet differentiability of the boundary integral operators S_r, K_r, and K_r^* the limiting case $\alpha = 1$ of Lemma 5.10 would be sufficient. However, we shall need the more general form for a closely related result which is required for the analysis of two reconstruction methods in Section 5.5.

Lemma 5.11 *The inequalities*

$$\frac{1}{2} \, \min_{\hat{z} \in \mathbb{S}^2} |r(\hat{z})| \, |\hat{x} - \hat{y}| \le |p_r(\hat{x}) - p_r(\hat{y})| \le \pi \, \|r\|_{C^1(\mathbb{S}^2)} |\hat{x} - \hat{y}| \tag{5.25}$$

are valid for all $r \in C^1(\mathbb{S}^2)$ and all $\hat{x}, \hat{y} \in \mathbb{S}^2$.

Proof. The first inequality in (5.25) follows from

$$|r(\hat{x})(\hat{x} - \hat{y})| \le |p_r(\hat{x}) - p_r(\hat{y})| + |\{r(\hat{y}) - r(\hat{x})\}\hat{y}|$$

and the triangle inequality $|r(\hat{y}) - r(\hat{x})| \leq |p_r(\hat{x}) - p_r(\hat{y})|$. The second inequality follows with (5.24) from

$$|p_r(\hat{x}) - p_r(\hat{y})| \leq |\{r(\hat{x}) - r(\hat{y})\}\hat{x}| + |r(\hat{y})\{\hat{x} - \hat{y}\}|$$

and the proof is finished. □

In particular, the estimates (5.23) and (5.25) ensure that the kernels of the integral operators S_r, K_r and K_r^* are weakly singular if $r \in C^2(\mathbb{S}^2)$, i.e., S_r, K_r and K_r^* are compact operators from $C(\mathbb{S}^2)$ into $C(\mathbb{S}^2)$.

We are now ready to establish the differentiability of the boundary integral operators with respect to r. Formally, we obtain the derivatives of the operators by differentiating their kernels with respect to r. Clearly, the mapping $r \to J_r$ is Fréchet differentiable from $C^1(\mathbb{S}^2)$ into $C(\mathbb{S}^2)$ with the derivative given by

$$J_r'q = q\sqrt{r^2 + |\operatorname{Grad} r|^2} + r\,\frac{rq + \operatorname{Grad} r \cdot \operatorname{Grad} q}{\sqrt{r^2 + |\operatorname{Grad} r|^2}} \tag{5.26}$$

since r is positive.

Theorem 5.12. *The mapping $r \mapsto S_r$ is Fréchet differentiable from $C^2(\mathbb{S}^2)$ into the space of bounded linear operators $\mathcal{L}\big(C(\mathbb{S}^2), C(\mathbb{S}^2)\big)$ of $C(\mathbb{S}^2)$ into itself.*

Proof. Let $r \in C^2(\mathbb{S}^2)$ be fixed but arbitrary such that $r > 0$. We denote the kernel of the integral operator S_r by

$$s_r(\hat{x}, \hat{y}) = 2\Psi\big(p_r(\hat{x}) - p_r(\hat{y})\big)J_r(\hat{y}), \quad \hat{x}, \hat{y} \in \mathbb{S}^2,\ \hat{x} \neq \hat{y}.$$

Its derivative with respect to r is given by

$$[s_r'q](\hat{x}, \hat{y}) = 2\operatorname{grad}\Psi\big(p_r(\hat{x}) - p_r(\hat{y})\big) \cdot (p_q(\hat{x}) - p_q(\hat{y}))\,J_r(\hat{y})$$

$$+2\Psi\big(p_r(\hat{x}) - p_r(\hat{y})\big)(J_r'q)(\hat{y}).$$

From (5.25) it follows that the kernel $s_r'q$ is weakly singular, i.e.,

$$|[s_r'q](\hat{x}, \hat{y})| \leq \gamma\,\frac{1}{|\hat{x} - \hat{y}|}\,\|q\|_{C^1(\mathbb{S}^2)}$$

for all $\hat{x}, \hat{y} \in \mathbb{S}^2$ with $\hat{x} \neq \hat{y}$ and some constant γ depending on r. Therefore

$$([S_r'q]\psi)(\hat{x}) := 2\int_{\mathbb{S}^2} [s_r'q](\hat{x}, \hat{y})\psi(\hat{y})\,ds(\hat{y}), \quad \hat{x} \in \mathbb{S}^2, \tag{5.27}$$

defines a bounded operator from $C(\mathbb{S}^2)$ into itself with norm

$$\|S_r'q\|_\infty \leq C\|q\|_{C^1(\mathbb{S}^2)} \tag{5.28}$$

where C is some constant depending on r. In particular, the estimate (5.28) implies that the mapping $q \mapsto S'_r q$ is a bounded linear operator from $C^2(\mathbb{S}^2)$ into $\mathcal{L}\big(C(\mathbb{S}^2), C(\mathbb{S}^2)\big)$.

Provided $0 \notin \{z + \lambda\zeta : \lambda \in [0,1]\}$, by Taylor's formula we have that

$$\Psi(z+\zeta) - \Psi(z) - \operatorname{grad}\Psi(z)\cdot\zeta = \int_0^1 (1-\lambda)\Psi''(z+\lambda\zeta)\,\zeta\cdot\zeta\,d\lambda \tag{5.29}$$

where Ψ'' denotes the Hessian. Elementary estimates show that there exists a constant c such that

$$\left|\Psi''(z)\,\zeta\cdot\zeta\right| \le c\,\frac{|\zeta|^2}{|z|^3} \tag{5.30}$$

for all $\zeta \in \mathbb{R}^3$ and $z \in \mathbb{R}^3 \setminus \{0\}$ with $|z| \le 4\|r\|_\infty$. By setting $z = p_r(\hat{x}) - p_r(\hat{y})$ and $\zeta = p_q(\hat{x}) - p_q(\hat{y})$ in (5.29), from (5.25) and (5.30) it follows that

$$|\psi\big(p_{r+q}(\hat{x}) - p_{r+q}(\hat{y})\big) - \psi\big(p_r(\hat{x}) - p_r(\hat{y})\big)$$

$$-\operatorname{grad}\Psi\big(p_r(\hat{x}) - p_r(\hat{y})\big)\cdot\big(p_q(\hat{x}) - p_q(\hat{y})\big)| \le C_1\,\frac{\|q\|^2_{C^1(\mathbb{S}^2)}}{|\hat{x}-\hat{y}|}$$

for all sufficiently small $q \in C^{1,\alpha}(\mathbb{S}^2)$, all $\hat{x}, \hat{y} \in \mathbb{S}^2$ with $\hat{x} \neq \hat{y}$ and some positive constant C_1 depending on r. From this, proceeding as in the derivation of the product rule in differentiation and using the Fréchet differentiability of J_r and (5.26), it can be deduced that

$$|s_{r+q}(\hat{x},\hat{y}) - s_r(\hat{x},\hat{y}) - [s'_r q](\hat{x},\hat{y})| \le C_2\,\frac{\|q\|^2_{C^1(\mathbb{S}^2)}}{|\hat{x}-\hat{y}|}$$

for all sufficiently small $q \in C^2(\mathbb{S}^2)$, all $\hat{x}, \hat{y} \in \mathbb{S}^2$ with $\hat{x} \neq \hat{y}$ and some positive constant C_2 depending on r. Integrating this inequality with respect to $\hat{y}$, we obtain that

$$\|S_{r+q} - S_r - S'_r q\|_\infty \le M\|q\|^2_{C^1(\mathbb{S}^2)} \le M\|q\|^2_{C^2(\mathbb{S}^2)} \tag{5.31}$$

for all sufficiently small $q \in C^2(\mathbb{S}^2)$ and some constant M depending on r. This completes the proof. □

Theorem 5.13. *The mappings $r \mapsto K_r$ and $r \mapsto K^*_r$ are Fréchet differentiable from $C^2(\mathbb{S}^2)$ into the space of bounded linear operators $\mathcal{L}\big(C(\mathbb{S}^2), C(\mathbb{S}^2)\big)$.*

Proof. We proceed analogously to the previous proof. After introducing the functions

$$\tilde{\Psi}(z) := -\frac{1}{4\pi}\,\frac{e^{ik|z|}}{|z|^3}\,(1-k|z|), \quad z \in \mathbb{R}^3 \setminus \{0\},$$

and

$$M_r(\hat{x},\hat{y}) := \{p_r(\hat{x}) - \operatorname{Grad} r(\hat{x})\}\cdot\{p_r(\hat{x}) - p_r(\hat{y})\}, \quad \hat{x}, \hat{y} \in \mathbb{S}^2,$$

we can write the kernel of the operator K_r in the form

$$k_r(\hat{x},\hat{y}) = 2M_r(\hat{x},\hat{y})\tilde{\Psi}(p_r(\hat{x}) - p_r(\hat{y}))\, r(\hat{y}), \quad \hat{x},\hat{y} \in \mathbb{S}^2,\ \hat{x} \neq \hat{y}.$$

The derivative of k_r is given by

$$k'_r(\hat{x},\hat{y};r,q) = 2M_r(\hat{x},\hat{y})\ \operatorname{grad}\tilde{\Psi}(p_r(\hat{x}) - p_r(\hat{y}))\cdot(p_q(\hat{x}) - p_q(\hat{y}))\, r(\hat{y})$$

$$+2(M'_r q)(\hat{x},\hat{y})\tilde{\Psi}(p_r(\hat{x}) - p_r(\hat{y}))\, r(\hat{y}) + 2M_r(\hat{x},\hat{y})\tilde{\Psi}(p_r(\hat{x}) - p_r(\hat{y}))\, q(\hat{y}).$$

The derivative of the quadratic mapping $r \mapsto M_r$ is given by

$$(M'_r q)(\hat{x},\hat{y}) = \{p_q(\hat{y}) - \operatorname{Grad} q(\hat{y})\}\cdot\{p_r(\hat{y}) - p_r(\hat{x})\}$$

$$+\{p_r(\hat{y}) - \operatorname{Grad} r(\hat{y})\}\cdot\{p_q(\hat{y}) - p_q(\hat{x})\}$$

and satisfies the relation

$$M_{r+q} - M_r - M'_r q = M_q. \tag{5.32}$$

In view of (5.23), the equality (5.32) implies that

$$|(M'_r q)(\hat{x},\hat{y})|| \leq \gamma\left\{\|r\|^2_{1,\alpha} + \|q\|^2_{1,\alpha}\right\}|\hat{x} - \hat{y}|^{1+\alpha} \tag{5.33}$$

for all $\hat{x},\hat{y} \in \mathbb{S}^2$ and some constant γ. With the aid of (5.23), (5.25) and (5.33) it can be seen that

$$([K'_r q]\psi)(\hat{x}) := 2\int_{\mathbb{S}^2}[k'_r q](\hat{x},\hat{y})\psi(\hat{y})\, ds(\hat{y}), \quad \hat{x} \in \mathbb{S}^2,$$

defines a bounded operator from $C(\mathbb{S}^2)$ into itself with norm

$$\|K'_r q\|_\infty \leq C\|q\|_{1,\alpha} \tag{5.34}$$

where C is some constant depending on r. In particular, the estimate (5.34) implies that the mapping $q \mapsto K'_r q$ is a bounded linear operator from $C^2(\mathbb{S}^2)$ into $\mathcal{L}\big(C(\mathbb{S}^2), C(\mathbb{S}^2)\big)$.

Corresponding to (5.30) we have an estimate

$$\left|\Psi''(z)\,\zeta\cdot\zeta\right| \leq c\,\frac{|\zeta|^2}{|z|^4} \tag{5.35}$$

for all $\zeta \in \mathbb{R}^3$ and $z \in \mathbb{R}^3 \setminus \{0\}$ with $|z| \leq 4\|r\|_\infty$. Proceeding as in the previous proof for the function Ψ, using (5.35) it can be seen that there exists a positive constant

C_1 depending on r such that

$$|\tilde{\Psi}(p_{r+q}(\hat{x}) - p_{r+q}(\hat{y})) - \tilde{\Psi}(p_r(\hat{x}) - p_r(\hat{y}))$$

$$- \operatorname{grad} \tilde{\Psi}(p_r(\hat{x}) - p_r(\hat{y})) \cdot (p_q(\hat{x}) - p_q(\hat{y}))| \leq C_1 \frac{\|q\|_{1,\alpha}^2}{|\hat{x} - \hat{y}|^2}$$

for all sufficiently small $q \in C^{1,\alpha}(\mathbb{S}^2)$ and all $\hat{x}, \hat{y} \in \mathbb{S}^2$ with $\hat{x} \neq \hat{y}$. From this with the help of (5.23) and (5.32) it can be deduced that

$$|k_{r+q}(\hat{x}, \hat{y}) - k_r(\hat{x}, \hat{y}) - [k_r' q](\hat{x}, \hat{y})| \leq C \, \frac{\|q\|_{1,\alpha}^2}{|\hat{x} - \hat{y}|}$$

for all sufficiently small $q \in C^{1,\alpha}(\mathbb{S}^2)$, all $\hat{x}, \hat{y} \in \mathbb{S}^2$ with $\hat{x} \neq \hat{y}$ and some positive constant C. As in the proof of Theorem 5.12, this now implies that

$$\|K_{r+q} - K_r - K_r' q\|_\infty \leq N \|q\|_{1,\alpha}^2 \leq N \|q\|_{C^2(\mathbb{S}^2)}^2 \tag{5.36}$$

for all sufficiently small $q \in C^2(\mathbb{S}^2)$ and some constant N depending on r. This finishes the proof for the operator K. The differentiability of the operator K^* can be proven analogously. □

We also need to transform the single-layer potential into the operator $P_r : C(\mathbb{S}^2) \to C^2(\mathbb{R}^3 \setminus \bar{D})$ defined by

$$(P_r \psi)(x) := \int_{\mathbb{S}^2} \Psi(x - p_r(\hat{y})) J_r(\hat{y}) \psi(\hat{y}) \, ds(\hat{y}), \quad x \in \mathbb{R}^3 \setminus \bar{D}.$$

For each compact set $U \subset \mathbb{R}^3 \setminus \bar{D}$ the mapping $r \mapsto P_r$ is Fréchet differentiable from $C(\mathbb{S}^2)$ into $\mathcal{L}\big(C(\mathbb{S}^2), C(U)\big)$ with the derivative given by

$$([P_r' q]\psi)(x) = \int_{\mathbb{S}^2} \Big\{ \operatorname{grad} \Psi(p_r(\hat{y}) - x) \cdot p_q(\hat{y}) \, J_r(\hat{y}) + \Psi(x - p_r(\hat{y}))[J_r' q](\hat{y}) \Big\} \, \psi(\hat{y}) \, ds(\hat{y})$$

for $x \in \mathbb{R}^3 \setminus \bar{D}$. This follows by a straightforward application of Taylor's formula applied to the kernel of P_r which is smooth on $\mathbb{S}^2 \times U$. Obviously $[P_r' q]\psi$ represents a radiating solution to the Helmholtz equation.

After defining the restriction operator $R_r : C(\mathbb{R}^3 \setminus D) \to \mathbb{S}^2$ by

$$(R_r w)(\hat{x}) := w(r(\hat{x})\hat{x}), \quad \hat{x} \in \mathbb{S}^2,$$

we have the relation

$$S_r = 2 R_r P_r \tag{5.37}$$

for the boundary values of the single-layer potential. To derive an expression for the boundary values of $P_r' q$, we view $[P_r' q]\psi$ as a linear combination of derivatives of single-layer potentials. With the aid of the jump relations of Theorem 3.3 and the expression (5.27) for the derivative S'_r, it follows that

$$[S'_r q]\psi = 2R_r[P'_r q]\psi + 2\,p_q \cdot R_r \operatorname{grad} P_r\psi \tag{5.38}$$

for all $\psi \in C^{0,\alpha}(\mathbb{S}^2)$.

Theorem 5.14. *The far field mapping $\mathcal{F} : r \to u_\infty$ is Fréchet differentiable from $C^2(\mathbb{S}^2)$ into $L^2(\mathbb{S}^2)$. The derivative is given by*

$$\mathcal{F}'_r q = v_\infty$$

where v_∞ denotes the far field pattern of the solution v to the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ satisfying the Sommerfeld radiation condition and the boundary condition

$$v = -\nu \cdot \left(p_q \circ p_r^{-1}\right) \frac{\partial u}{\partial \nu} \quad \text{on } \partial D. \tag{5.39}$$

Proof. From the representation (3.33), by using the boundary condition $u = 0$ on ∂D and the jump relations of Theorem 3.1, we deduce that the normal derivative

$$\psi_r(\hat{x}) := \frac{\partial u}{\partial \nu}(r(\hat{x})\hat{x}), \quad \hat{x} \in \mathbb{S}^2,$$

satisfies the integral equation

$$S_r\psi_r = 2R_r u^i \tag{5.40}$$

of the first kind and the integral equation

$$\psi_r + K_r^*\psi_r - iS_r\psi_r = 2\nu_r \cdot R_r \operatorname{grad} u^i - 2iR_r u^i \tag{5.41}$$

of the second kind. Since the inverse operator $(I + K_r^* - iS_r)^{-1} : C(\mathbb{S}^2) \to C(\mathbb{S}^2)$ exists (see the proof of Theorem 3.27), proceeding as above for the operator L_r in the derivation of (5.20), it can be seen that the inverse $(I + K_r^* - iS_r)^{-1}$ is Fréchet differentiable with respect to r. Hence, from (5.41) we can conclude that the normal derivative ψ_r is Fréchet differentiable with respect to r since the right hand side of (5.41) is.

In view of (3.33), for the scattered wave $u^s = u_r^s$ we now write

$$u_r^s = -P_r\psi_r \tag{5.42}$$

and observe from the above that the mapping $r \mapsto u_r^s$ is Fréchet differentiable from $C^2(\mathbb{S}^2) \to C(U)$ for each compact $U \subset \mathbb{R}^3 \setminus \bar{D}$. By the chain rule, the derivative $v = u_r^{s\prime} q$ is given by

$$v = -P_r\psi'_r q - [P'_r q]\psi_r \tag{5.43}$$

and describes a radiating solution to the Helmholtz equation. Using (5.37) and (5.38), from (5.42) and (5.43) we obtain the boundary values

$$-2R_r v = S_r\psi'_r q + [S'_r q]\psi_r - 2\,p_q \cdot R_r \operatorname{grad} P_r\psi_r = (S_r\psi_r)' q + 2\,p_q \cdot R_r \operatorname{grad} u^s.$$

Differentiating the integral equation (5.40) with respect to r yields

$$(S_r\psi_r)'q = 2\, p_q \cdot R_r \operatorname{grad} u^i.$$

Now the boundary condition (5.39) for v follows by combining the last two equations and noting that $u = 0$ on ∂D.

The Fréchet differentiability of $r \mapsto u^s$ as expressed above means that

$$\lim_{q\to 0} \frac{1}{\|q\|_{C^2(\mathbb{S}^2)}} \, \|u^s_{r+q} - u^s_q - v\|_{C(U)} = 0.$$

Choosing $U = \{x \in \mathbb{R}^3 : |x| = a\}$ with sufficiently large $a > 0$, by the well-posedness of the exterior Dirichlet problem (Theorem 3.11) and the far field representation (2.14) for the Helmholtz equation the above limit implies that for the far field patterns we have that

$$\lim_{q\to 0} \frac{1}{\|q\|_{C^2(\mathbb{S}^2)}} \, \|\mathcal{F}(r+q) - \mathcal{F}(r) - v_\infty\|_{L^2(\mathbb{S}^2)} = 0.$$

This finishes the proof. □

In his original proof Potthast in [273, 274] used the integral equation (3.26) from the combined double- and single-layer potential approach. Here we applied the boundary integral equations from the representation theorem in order to simplify the verification of the boundary condition for v. Note, that (5.39) of course can be obtained formally by differentiating the boundary condition $u_r \circ p_r = 0$ with respect to r by the chain rule. Proceeding this way, (5.39) was initially obtained by Roger [296] who first employed Newton type iterations for the approximate solution of inverse obstacle scattering problems.

The above results can be carried over to arbitrary C^2 boundary surfaces (see [273, 274]). For our presentation we restricted ourselves to starlike boundaries in order to be consistent with the analysis of the two inverse methods of Section 5.5. The investigations on Fréchet differentiability have also been extended to the Neumann boundary condition and to electromagnetic scattering from a perfect conductor by Potthast [274, 275, 276]. For the Neumann problem in two dimensions we also refer to [243]. Fréchet differentiability with respect to scattering from two-dimensional cracks has been investigated by Kress [203] and by Mönch [244]. Here the analysis has to be based on boundary integral equations of the first kind via a single-layer approach. Alternative approaches to differentiation of the far field pattern with respect to the boundary were contributed by Kress and Päivärinta [207] based on Green's theorems and a factorization of the difference of the far field pattern for the two domains with radial functions r and $r + q$, and by Hohage [143] and Schormann [303] via the implicit function theorem.

Theorem 5.15. *The linear operator $\mathcal{F}_r'$ is injective and has dense range.*

Proof. Assume that $\mathcal{F}_r' q = 0$. Then the solution v to the Dirichlet problem with boundary condition (5.39) has a vanishing far field pattern. Hence, by Rellich's lemma we have $v = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ and consequently $v = 0$ on ∂D. By the form of the boundary condition (5.39) this in turn implies $\nu \circ p_r \cdot p_q = 0$ since the normal derivative $\partial u / \partial \nu$ cannot vanish on open subsets of ∂D as a consequence of Holmgren's Theorem 2.3 and the boundary condition $u = 0$ on ∂D. By (5.22) the condition $(\nu \circ p_r) \cdot p_q = 0$ implies that $q = 0$.

The above arguments also imply that the set

$$\left\{ (\nu \circ p_r) \cdot p_q \, \frac{\partial u}{\partial \nu} \circ p_r : q \in L^2(\mathbb{S}^2) \right\}$$

is dense in $L^2(\mathbb{S}^2)$. Therefore, writing

$$\mathcal{F}_r' q = -A \left(\nu \cdot \left(p_q \circ p_r^{-1} \right) \frac{\partial u}{\partial \nu} \right)$$

in terms of the boundary data to far field operator $A : L^2(\partial D) \to L^2(\mathbb{S}^2)$ from Theorem 3.28, the dense range of the latter implies dense range of $\mathcal{F}_r'$. □

Theorem 5.14 may be viewed as an extension of a continuous dependence result due to Angell, Colton and Kirsch [7]. An L^2 version of this result will be needed in the convergence analysis of the inverse methods described in Section 5.5. For its formulation we consider the set of all surfaces Λ which are starlike with respect to the origin and represented in the form (5.9) with a positive function r from the Hölder space $C^{1,\alpha}(\mathbb{S}^2)$ with $0 < \alpha < 1$. For a sequence of such surfaces, by convergence $\Lambda_n \to \Lambda$, $n \to \infty$, we mean the convergence $\|r_n - r\|_{1,\alpha} \to 0$, $n \to \infty$, of the representing functions in the $C^{1,\alpha}$ Hölder norm on $\mathbb{S}^2$. We say that a sequence of functions f_n from $L^2(\Lambda_n)$ is L^2 convergent to a function f in $L^2(\Lambda)$ if

$$\lim_{n\to\infty} \int_{\mathbb{S}^2} |f_n(r_n(\hat{x})\,\hat{x}) - f(r(\hat{x})\,\hat{x})|^2 ds(\hat{x}) = 0.$$

We can now state the following convergence theorem. (In the theorem we can replace C^2 surfaces by $C^{1,\alpha}$ surfaces. However, for consistency with the rest of our book, we have used the more restrictive assumption of C^2 surfaces.)

Theorem 5.16. *Let (Λ_n) be a sequence of starlike C^2 surfaces which converges with respect to the $C^{1,\alpha}$ norm to a C^2 surface Λ as $n \to \infty$ and let u_n and u be radiating solutions to the Helmholtz equation in the exterior of Λ_n and Λ, respectively. Assume that the continuous boundary values of u_n on Λ_n are L^2 convergent to the boundary values of u on Λ. Then the sequence (u_n), together with all its derivatives, converges to u uniformly on compact subsets of the open exterior of Λ.*

Proof. For the solution to the exterior Dirichlet problem, we use the combined double- and single-layer potential approach (3.26), that is, we represent u in the form

$$u(x) = \int_\Lambda \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\Phi(x,y) \right\} \varphi(y)\, ds(y)$$

with a density $\varphi \in C(\Lambda)$ and, analogously, we write u_n as a combined potential with density $\varphi_n \in C(\Lambda_n)$. Transformation of the integral equation (3.27) onto $\mathbb{S}^2$ leads to

$$\psi_r + K_r\psi_r - iS_r\psi_r = f_r$$

where we have set $\psi_r(\hat{x}) := \varphi(r(\hat{x})\,\hat{x})$ and $f_r(\hat{x}) := 2u(r(\hat{x})\,\hat{x})$. The corresponding integral equations for the densities φ_n representing u_n are obtained by replacing r by r_n. For abbreviation we write $A_r := K_r - iS_r$. From (5.28), (5.31), (5.34) and (5.36) we have that

$$\|A_{r_n} - A_r\|_\infty \le C\|r_n - r\|_{1,\alpha}$$

for some constant C depending on r. The same inequality can also be established for the L^2 adjoint of A_r. Therefore, by Lax's Theorem 3.5 it follows that

$$\|A_{r_n} - A_r\|_{L^2(\mathbb{S}^2)} \le C\|r_n - r\|_{1,\alpha}. \tag{5.44}$$

A Neumann series argument now shows that $\|(I + A_{r_n})^{-1}\|_{L^2(\mathbb{S}^2)}$ is uniformly bounded. From

$$(I + A_{r_n})(\psi_{r_n} - \psi_r) = f_{r_n} - f_r - (A_{r_n} - A_r)\psi_r$$

we then derive

$$\|\psi_{r_n} - \psi_r\|_{L^2(\mathbb{S}^2)} \le \tilde{C}\|f_{r_n} - f_r\|_{L^2(\mathbb{S}^2)} + \|A_{r_n} - A_r\|_{L^2(\mathbb{S}^2)}\|\psi_r\|_{L^2(\mathbb{S}^2)}$$

for some constant $\tilde{C}$ whence L^2 convergence of the densities $\varphi_n \to \varphi$, $n \to \infty$, follows. The convergence of (u_n) on compact subsets of the exterior of Λ is now obtained by substituting the densities into the combined double- and single-layer potentials for u_n and u and then using the Cauchy–Schwarz inequality. □

5.4 Iterative Solution Methods

The analysis of the previous Section 5.3 on the continuity, differentiability and compactness of the far field mapping may be considered as the theoretical foundation for the application of Newton's method and related schemes for the approximate solution of (5.8). In this method, given a far field pattern u_∞, the nonlinear equation

$$\mathcal{F}(r) = u_\infty$$

is replaced by the linearized equation

$$\mathcal{F}(r) + \mathcal{F}'_r q = u_\infty \tag{5.45}$$

which has to be solved for q in order to improve an approximate boundary given by the radial function r into the new approximation given by $\tilde{r} = r + q$. In the usual fashion, Newton's method consists in iterating this procedure (see Section 4.5). The question of uniqueness for the linear equation (5.45) is settled through Theorem 5.15. In view of Theorem 4.21, a regularization has to be incorporated in the solution of (5.45) since by Theorems 5.8 and 5.9 the operator $\mathcal{F}$ is completely continuous. Of course, the compactness of the derivative $\mathcal{F}_r'$ can also be deduced from Theorem 5.14. Dense range of $\mathcal{F}_r'$ as a prerequisite for regularization schemes is guaranteed by Theorem 5.15.

For practical computations q is taken from a finite dimensional subspace W_N of $C^2(\mathbb{S}^2)$ with dimension N and equation (5.45) is approximately solved by projecting it onto a finite dimensional subspace of $L^2(\mathbb{S}^2)$. The most convenient projection is given through collocation at M points $\hat{x}_1, \dots, \hat{x}_M \in \mathbb{S}^2$. Then writing

$$q = \sum_{j=1}^{N} a_j q_j$$

where $q_1, \dots, q_N$ denotes a basis of W_N, one has to solve the linear system

$$\sum_{j=1}^{N} a_j \, (\mathcal{F}_r' \, q_j)(\hat{x}_i) = u_\infty(\hat{x}_i) - (\mathcal{F}(r))(\hat{x}_i), \quad i = 1, \dots, M, \tag{5.46}$$

for the real coefficients $a_1, \dots, a_N$. In general, i.e., when $2M > N$, the system (5.46) is overdetermined and has to be solved approximately by a least squares method. In addition, since we have to stabilize the ill-posed linearized equation (5.45), we replace (5.46) by the least squares problem of minimizing the penalized defect

$$\sum_{i=1}^{M} \left| \sum_{j=1}^{N} a_j (\mathcal{F}_r' q_j)(\hat{x}_i) - u_\infty(\hat{x}_i) + (\mathcal{F}(r))(\hat{x}_i) \right|^2 + \alpha \sum_{j=1}^{N} a_j^2 \tag{5.47}$$

with some regularization parameter $\alpha > 0$, that is, we employ a Tikhonov regularization in the spirit of the Levenberg–Marquardt algorithm that we briefly discussed in Section 4.5. Assuming that the basis functions $q_1, \dots, q_N$ are orthonormal in $L^2(\mathbb{S}^2)$, for example spherical harmonics, the penalty term in (5.47) corresponds to L^2 penalization. However, numerical evidence strongly suggests to replace the L^2 penalization by a Sobolev penalization, that is, by considering $\mathcal{F}'$ as an operator from $H^m(\mathbb{S}^2)$ into $L^2(\mathbb{S}^2)$ for $m = 1$ or $m = 2$.

In order to compute the right hand sides of the linear system (5.46), in each iteration step the direct problem for the boundary ∂D given by the radial function r has to be solved for the evaluation of $(\mathcal{F}(r))(\hat{x}_i)$. For this, we suggest numerically solving the integral equation (5.41) for the normal derivative and evaluating the corresponding far field expression (3.34). Using (5.41) has the advantage of immediately yielding approximate values for the normal derivative of the total field, which enters the boundary condition (5.39) for the Fréchet derivatives. To compute the matrix entries $(\mathcal{F}_r' q_j)(\hat{x}_i)$ we need to solve N additional direct problems for the same boundary ∂D and different boundary values given by (5.39) for the basis functions $q = q_j,\ j = 1, \dots, N$. For this we suggest using the combined double- and single-layer potential approach (3.26), i.e., the integral equation (3.27), and evaluating the corresponding far field expression. If here we used the integral equation derived from the Green's representation formula we would be faced with the problem that the corresponding right hand sides require the numerical computation of the normal derivative of a combined double- and single-layer potential (see [64], p. 103). Note that the same problem would come up in the computation of the boundary values in (5.39) if we used (3.27) instead of (5.41) for the evaluation of $(\mathcal{F}(r))(\hat{x}_i)$. To avoid the need to set up the matrix for the numerical evaluation of the normal derivative of the combined double- and single-layer potential, we think it is legitimate to pay the cost of having to solve two different linear systems by using two adjoint integral equations (which simply lead to transposed matrices in the approximating linear systems). The second linear system has to be solved simultaneously for N different right hand sides corresponding to setting $q = q_j,\ j = 1, \dots, N$. Of course, any boundary element method for the numerical solution of the two adjoint boundary integral equations (3.27) and (5.41) can be employed in this procedure. However, we recommend using the spectral methods described in Sections 3.5 and 3.6.

For the details on possible implementations of this regularized Newton method in two dimensions, with various choices for the approximating subspace W_N and also more sophisticated regularizations than indicated above, we refer to [142, 180, 204, 206] for sound-soft obstacles and to [243] for sound-hard obstacles and the references therein. Inverse scattering problems for two-dimensional cracks have been solved using Newton's method in [203, 244]. For numerical examples in three dimensions we refer to Farhat et al [98] and to Harbrecht and Hohage [132]. We also refer to Kress and Rundell [208] who investigated a frozen Newton method where an explicit expression for the Fréchet derivative for the unit circle is used and kept fixed throughout the Newton iterations. Kress and Rundell also have used Newton's method for successful reconstructions of obstacles from the amplitude of the far field pattern only [209], from backscattering data [210] and for the simultaneous reconstruction of the shape and impedance of a scatterer [211]. For second degree Newton iterations in inverse obstacle scattering we refer to Hettlich and Rundell [139].

In closing our analysis on Newton iterations for the boundary to far field operator $\mathcal{F}$ we note as main advantages that this approach is conceptually simple and, as numerical examples indicate, leads to highly accurate reconstructions with reasonable stability against errors in the far field pattern. On the other hand, it should be noted that for the numerical implementation an efficient forward solver is needed for the solution of the direct scattering problem for each iteration step. Furthermore, good a priori information is required in order to be able to choose an initial guess that ensures numerical convergence. In addition, on the theoretical side, although some progress has been made through the work of Hohage [143] and Potthast [280] the convergence of regularized Newton iterations for the operator $\mathcal{F}$ has not been completely settled. At the time this is being written it remains an open problem whether the convergence results on the Levenberg–Marquardt algorithm and the iteratively regularized Gauss–Newton iterations as mentioned in Section 4.5 are applicable to inverse obstacle scattering or, more generally, to inverse boundary value problems.

Numerical implementations of the nonlinear Landweber iteration as explained in Section 4.5 for the two-dimensional inverse scattering problem have been given by Hanke, Hettlich and Scherzer [130] for sound-soft obstacles and by Hettlich [138] for sound-hard obstacles.

For modified Newton type iterations with reduced computational costs we recall Huygens' principle from Theorem 3.14. In view of the sound-soft boundary condition, from (3.33) we conclude that

$$u^i(x) = \int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, \Phi(x,y)\, ds(y), \quad x \in \partial D. \tag{5.48}$$

Now we can interpret (5.48) and (3.34), that is,

$$u_\infty(\hat{x}) = -\frac{1}{4\pi} \int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, e^{-ik\,\hat{x}\cdot y}\, ds(y), \quad \hat{x} \in \mathbb{S}^2, \tag{5.49}$$

as a system of two integral equations for the unknown boundary ∂D of the scatterer and the unknown normal derivative

$$\varphi := -\frac{\partial u}{\partial \nu} \quad \text{on } \partial D$$

of the total field. For the sequel it is convenient to call (5.49) the *data equation* since it contains the given far field for the inverse problem and (5.48) the *field equation* since it represents the boundary condition. Both equations are linear with respect to φ and nonlinear with respect to ∂D. Equation (5.49) is severely ill-posed whereas (5.48) is only mildly ill-posed.

Obviously there are three options for an iterative solution of (5.48) and (5.49). In a first method, given an approximation for the boundary ∂D one can solve the mildly ill-posed integral equation of the first kind (5.48) for φ. Then, keeping φ fixed, equation (5.49) is linearized with respect to ∂D to update the boundary approximation. In a second approach, one also can solve the system (5.48) and (5.49) simultaneously for ∂D and φ by Newton iterations, i.e., by linearizing both equations with respect to both unknowns. Whereas in the first method the burden of the ill-posedness and nonlinearity is put on one equation, in a third method a more even distribution of the difficulties is obtained by reversing the roles of (5.49) and (5.48), i.e., by solving the severely ill-posed equation (5.49) for φ and then linearizing (5.48) to obtain the boundary update. We will consider a slight modification of the latter alternative in the following section on decomposition methods.

For a more detailed description of these ideas, using the parameterization (5.9) for starlike ∂D and recalling the mapping p_r from (5.21), we introduce the parameterized single-layer operator and far field operators $A, A_\infty : C^2(\mathbb{S}^2) \times L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ by

$$A(r,\psi)(\hat{x}) := \int_{\mathbb{S}^2} \Phi(p_r(\hat{x}), p_r(\hat{y}))\psi(\hat{y})\, ds(\hat{y}), \quad \hat{x} \in \mathbb{S}^2,$$

and

$$A_\infty(r,\psi)(\hat{x}) := \frac{1}{4\pi}\int_{\mathbb{S}^2} e^{-ikr(\hat{y})\,\hat{x}\cdot\hat{y}}\psi(\hat{y})\,ds(\hat{y}), \quad \hat{x}\in\mathbb{S}^2.$$

Then (5.48) and (5.49) can be written in the operator form

$$A(r,\psi) = -u^i \circ p_r \tag{5.50}$$

and

$$A_\infty(r,\psi) = u_\infty \tag{5.51}$$

where we have incorporated the surface element into the density function via

$$\psi := J_r\,\varphi\circ p_r \tag{5.52}$$

with the determinant J_r of the Jacobian of the mapping p_r given in (5.22). The linearization of these equations requires the Fréchet derivatives of the operators A and A_∞ with respect to r. Analogously to Theorem 5.12 these can be obtained by formally differentiating their kernels with respect to r, i.e.,

$$(A'(r,\psi)q)\,(\hat{x}) = \int_{\mathbb{S}^2} \operatorname{grad}_x \Phi(p_r(\hat{x}),p_r(\hat{y}))\cdot[p_q(\hat{x})-p_q(\hat{y})]\,\psi(\hat{y})\,ds(\hat{y}), \quad \hat{x}\in\mathbb{S}^2,$$

and

$$(A'_\infty(r,\psi)q)\,(\hat{x}) = -\frac{ik}{4\pi}\int_{\mathbb{S}^2} e^{-ikr(\hat{y})\,\hat{x}\cdot\hat{y}}\,\hat{x}\cdot\hat{y}\,q(\hat{y})\psi(\hat{y})\,ds(\hat{y}), \quad \hat{x}\in\mathbb{S}^2.$$

Note that as opposed to the Fréchet derivative of the boundary to far-field operator $\mathcal{F}$ as given in Theorem 5.14 the derivatives of the integral operators are available in an explicit form which offers computational advantages.

For convenience and also for later use in Section 5.6, we include the outline of a proof for the invertibility of the single-layer potential operator in a Sobolev space setting.

Theorem 5.17. *Assume that k^2 is not a Dirichlet eigenvalue of the negative Laplacian in D. Then the single-layer potential operator $S : H^{-1/2}(\partial D)\to H^{1/2}(\partial D)$ is bijective with a bounded inverse.*

Proof. Let $\varphi\in H^{-1/2}(\partial D)$ satisfy $S\varphi = 0$. Then the single-layer potential u with density φ belongs to $H^1(D)$ and $H^1_{\mathrm{loc}}(\mathbb{R}^3\setminus\bar{D})$ and has vanishing trace on ∂D. By Rellich's lemma, i.e., uniqueness for the exterior Dirichlet problem, u vanishes in $\mathbb{R}^3\setminus\bar{D}$ and by the assumption on k^2 it also vanishes in D. Now the jump relations for the single-layer potential for the normal derivative, which remain valid in the trace sense for $H^{-1/2}$ densities, imply $\varphi = 0$. Hence S is injective.

To prove surjectivity, we choose a second wave number $k_0 > 0$ such that k_0^2 is not a Neumann eigenvalue for the negative Laplacian in D and distinguish boundary integral operators for the two different wave numbers by indices k and k_0. By Rellich's lemma and the jump relations it can be seen that $T_{k_0} : H^{1/2}(\partial D)\to H^{-1/2}(\partial D)$ is injective. Consequently, given $f\in H^{1/2}(\partial D)$ the equations $S_k\varphi = f$

and $T_{k_0} S_k \varphi = T_{k_0} f$ for $\varphi \in H^{-1/2}(\partial D)$ are equivalent. In view of (3.13) we have $T_{k_0} S_k = C - I$ where

$$C := K_{k_0}'^2 - T_{k_0}(S_k - S_{k_0}).$$

From the increased smoothness of the kernel of $S_k - S_{k_0}$ as compared with that of S_k it follows that $S_k - S_{k_0}$ is compact from $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$ and consequently $C : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is compact (see Corollary 3.7 and note that due to [240] the assumption that ∂D belongs to $C^{2,\alpha}(\partial D)$ is dispensable). Therefore, the Riesz–Fredholm theory can be applied and injectivity of $T_{k_0} S_k$ implies solvability of $T_{k_0} S_k \varphi = T_{k_0} f$ and consequently also of $S_k \varphi = f$. Hence, we have bijectivity of S_k and the Banach open mapping theorem implies the boundedness of the inverse $S_k^{-1} : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$. □

Transforming this theorem to the operator A, provided k^2 is not a Dirichlet eigenvalue of the negative Laplacian in the corresponding D, for fixed r the operator $A(r, \cdot) : H^{-1/2}(\mathbb{S}^2) \to H^{1/2}(\mathbb{S}^2)$ is bijective. In this case, given an approximation to the boundary parameterization r, the field equation (5.50) can be solved for the density ψ. Then, keeping ψ fixed, linearizing the data equation (5.51) with respect to r leads to the linear equation

$$A'_\infty(r, \psi) q = u_\infty - A_\infty(r, \psi) \tag{5.53}$$

for q to update the radial function r via $r + q$. This procedure can be iterated. For fixed r and ψ the operator $A'_\infty(r, \psi)$ has a smooth kernel and therefore is severely ill-posed. This requires stabilization, for example via Tikhonov regularization. For corresponding results on injectivity and dense range as prerequisites for Tikhonov regularization we refer to [164].

This approach for solving the inverse obstacle scattering problem has been proposed by Johansson and Sleeman [166]. It can be related to the Newton iterations for the boundary to far field operator $\mathcal{F}$. In the case when k^2 is not a Dirichlet eigenvalue of the negative Laplacian in D we can write

$$\mathcal{F}(r) = -A_\infty \left(r, [A(r, \cdot)]^{-1} (u^i \circ p_r) \right).$$

By the product and chain rule this implies

$$\begin{aligned} \mathcal{F}_r' q = & -A'_\infty \left(r, [A(r, \cdot)]^{-1} (u^i \circ p_r) \right) q \\ & + A_\infty \left(r, [A(r, \cdot)]^{-1} A' \left(r, [A(r, \cdot)]^{-1} (u^i \circ p_r) \right) q \right) \\ & - A_\infty \left(r, [A(r, \cdot)]^{-1} (\operatorname{grad} u^i \circ p_r) \cdot p_q \right). \end{aligned} \tag{5.54}$$

Hence, the iteration scheme proposed by Johansson and Sleeman can be interpreted as Newton iterations for (5.8) with the derivative of $\mathcal{F}$ approximated by the first term in the representation (5.54). As to be expected from this close relation, the quality of the reconstructions via (5.53) can compete with those of Newton iterations for (5.8) with the benefit of reduced computational costs.

Following ideas first developed for the Laplace equation by Kress and Rundell [212], our second approach for iteratively solving the system (5.50) and (5.51) consists in simultaneously linearizing both equations with respect to both unknowns. In this case, given approximations r and ψ both for the boundary parameterization and the density, the system of linear equations

$$A'(r,\psi)q + (\operatorname{grad} u^i \circ p_r) \cdot p_q + A(r,\chi) = -A(r,\psi) - u^i \circ p_r \tag{5.55}$$

and

$$A'_\infty(r,\psi)q + A_\infty(q,\chi) = -A_\infty(r,\psi) + u_\infty \tag{5.56}$$

has to be solved for q and χ in order to obtain updates $r+q$ for the boundary parameterization and $\psi+\chi$ for the density. This procedure again is iterated and coincides with traditional Newton iterations for the system (5.50) and (5.51). It has been analyzed and tested by Ivanyshyn and Kress, see for example [157, 160, 162]. Due to the smoothness of the kernels in the second equation the system (5.55) and (5.56) is severely ill-posed and requires regularization with respect to both unknowns. For corresponding results on injectivity and dense range again we refer to [164]. In particular, as in the Newton iterations for $\mathcal{F}$, for the parameterization update it is appropriate to incorporate Sobolev penalties.

The simultaneous iterations (5.55) and (5.56) again exhibit connections to the Newton iteration for (5.8) as expressed through the following theorem which is proven in [164].

Theorem 5.18. *Assume that k^2 is not a Dirichlet eigenvalue of the negative Laplacian in D and set $\psi := -[A(r,\cdot)]^{-1}(u^i \circ p_r)$. If q satisfies the linearized boundary to far field equation (5.45) then q and*

$$\chi := -[A(r,\cdot)]^{-1}\left(A'(r,\psi)q + (\operatorname{grad} u^i \circ p_r)\cdot p_q\right)$$

satisfy the linearized data and field equations (5.55) and (5.56). Conversely, if q and χ satisfy (5.55) and (5.56) then q satisfies (5.45).

Theorem 5.18 illustrates the difference between the iteration method based on (5.55) and (5.56) and the Newton iterations for (5.8). In general when performing (5.55) and (5.56) in the sequence of updates the relation $A(r,\psi) = -(u^i \circ p_r)$ between the approximations r and ψ for the parameterization and the density will not be satisfied. This observation also indicates a possibility to use (5.55) and (5.56) for implementing a Newton scheme for (5.8). It is only necessary to replace the update $\psi+\chi$ for the density by $-[A(r+q,\cdot)]^{-1}(u^i \circ (p_r + p_q))$, i.e., at the expense of throwing away χ and solving the field equation for the updated boundary with representation $r+q$ for a new density.

We present two examples for reconstructions via (5.55) and (5.56) that were provided to us by Olha Ivanyshyn. In both cases, the synthetic data were obtained by applying the spectral method of Section 3.6 for the combined double- and single-layer method from (3.26) and consisted of 128 values of the far field. Correspondingly, for the reconstruction the number of collocation points for the far field on $\mathbb{S}^2$ also was chosen as 128. For the discretization of the weakly singular integral equation (5.55) the version of the spectral method described by (3.122) with the modification (3.124) was used whereas for the smooth kernels of (5.55) the Gauss trapezoidal rule (3.114) was applied. For both discretizations the number of quadrature points was 242 corresponding to $N' = 10$ in (3.124) and spherical harmonics up to order $N = 7$ as approximation spaces for the density. For the approximation space for the radial function representing the boundary of the scatterer spherical harmonics up to order six were chosen.

The wave number was $k = 1$ and the incident direction $d = (0, 1, 0)$ is indicated in the figures by an arrow. The iterations were started with a ball of radius $3.5Y_0^0 = 0.9873$ centered at the origin. Both for regularizing the density update and the surface update H^1 penalization was applied with the regularization parameters selected by trial and error as $\alpha_n = \alpha\gamma^n$ and $\beta_n = \beta\gamma^n$ depending on the iteration number n with with $\alpha = 10^{-6}$, $\beta = 0.5$ and $\gamma = 2/3$. Both to the real and imaginary part of the far field data 2% of normally distributed noise was added, i.e.,

$$\frac{\|u_\infty - u_\infty^\delta\|_{L^2(\mathbb{S}^2)}}{\|u_\infty\|_{L^2(\mathbb{S}^2)}} \leq 0.02.$$

In terms of the relative data error

$$\epsilon_r := \frac{\|u_\infty - u_{r,\infty}\|_{L^2(\mathbb{S}^2)}}{\|u_\infty\|_{L^2(\mathbb{S}^2)}}$$

with the given far field data u_∞ and the far field $u_{r,\infty}$ corresponding to the radial function r, a stopping criterion was incorporated such that the iteration was carried on as long as $\epsilon_r > \epsilon_{r+q}$. The figures show the exact shape on the left and the reconstruction on the right.

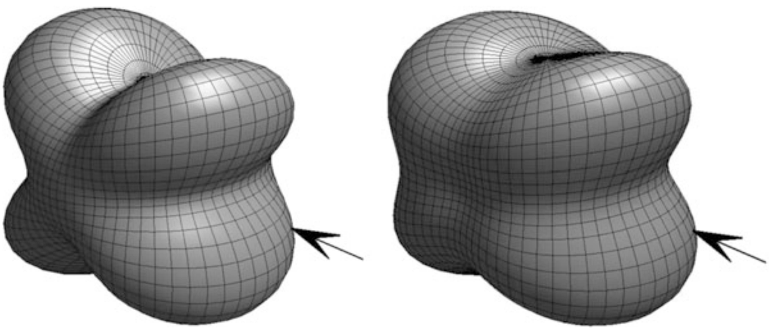

Fig. 5.1 Reconstruction of a cushion from noisy data

The first example is a cushion shaped scatterer that is starlike with radial function

$$r(\theta, \varphi) = \sqrt{0.8 + 0.5(\cos 2\varphi - 1)(\cos 4\theta - 1)}, \quad \theta \in [0, \pi],\ \varphi \in [0, 2\pi].$$

Fig. 5.1 shows the reconstruction after 12 iteration steps with the final data error $\epsilon_r = 0.020$. The second example is a pinched ball with radial function

$$r(\theta, \varphi) = \sqrt{1.44 + 0.5 \cos 2\varphi(\cos 2\theta - 1)}, \quad \theta \in [0, \pi],\ \varphi \in [0, 2\pi].$$

Fig. 5.2 shows the reconstruction after 13 iteration steps with data error $\epsilon_r = 0.019$.

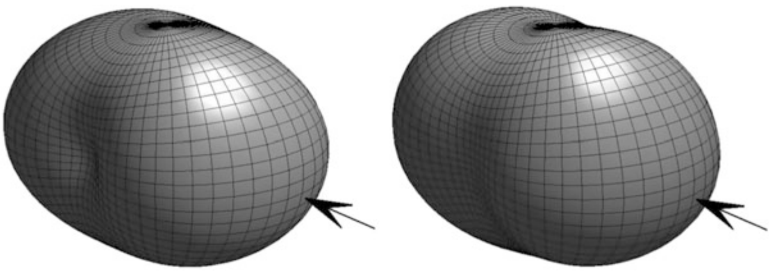

Fig. 5.2 Reconstruction of a pinched ball from noisy data

The method of simultaneos linearization (5.50) and (5.51) has been extended to the case of sound-soft [161] and sound-hard [222] cracks in two dimensions. In addition, it has been also applied to reconstructions of sound-soft or sound-hard scatterers from the modulus of the far field pattern [156, 162]. In order to avoid the exceptional values for k^2, modifications using combined single- and double-layer potentials in the spirit of the existence analyis of Theorem 3.11 were suggested in [159]. The numerical performance of the simultaneous linearizations has been compared with the method of Johansson and Sleeman in [158].

Since there are no explicit solutions of the direct scattering problem available, as in the above examples numerical tests of approximate methods for the inverse problem usually rely on synthetic far field data obtained through the numerical solution of the forward scattering problem. Here we take the opportunity to put up a warning sign against *inverse crimes*. In order to avoid trivial inversion of finite dimensional problems, for reliably testing the performance of an approximation method for the inverse problem it is crucial that the synthetic data be obtained by a forward solver which has no connection to the inverse solver under consideration. Unfortunately, not all of the numerical reconstructions which have appeared in the literature meet with this obvious requirement. To be more precise about our objections, consider a

m–parameter family G_m of boundary surfaces and use a numerical method $\mathcal{M}$ for the solution of the direct problem to obtain a number n of evaluations of the far field pattern u_∞, for example, point evaluations or Fourier coefficients. This obviously may be considered as defining some function $g : \mathbb{R}^m \to \mathbb{C}^n$. Now use the method $\mathcal{M}$ to create the synthetic data for a boundary surface $\partial D \in G_m$, that is, evaluate g for a certain parameter $a_0 \in \mathbb{R}^m$. Then, for example in Newton's method, incorporating the same method $\mathcal{M}$ in the inverse solver now just means nothing else than solving the finite dimensional nonlinear problem $g(a) = g(a_0)$. Hence, it is no surprise, in particular if m and n are not too large, that the surface ∂D is recovered pretty well.

5.5 Decomposition Methods

The main idea of the decomposition methods that we are considering in this chapter is to break up the inverse obstacle scattering problem into two parts: the first part deals with the ill-posedness by constructing the scattered wave u^s from a knowledge of its far field pattern u_∞ and the second part deals with the nonlinearity by determining the unknown boundary ∂D of the scatterer as the location where the boundary condition for the total field $u^i + u^s$ is satisfied. We begin with a detailed analysis of the method introduced in a series of papers by Kirsch and Kress [187, 188, 189] that we outlined already in the introduction. We confine our analysis to inverse scattering from a three-dimensional sound-soft scatterer. However, we note that the method can be carried over to the two-dimensional case and to other boundary conditions.

For the first part, we seek the scattered wave in the form of a surface potential. We choose an auxiliary closed C^2 surface Γ contained in the unknown scatterer D. The knowledge of such an internal surface Γ requires weak a priori information about D. Since the choice of Γ is at our disposal, without loss of generality we may assume that it is chosen such that the Helmholtz equation $\Delta u + k^2 u = 0$ in the interior of Γ with homogeneous Dirichlet boundary condition $u = 0$ on Γ admits only the trivial solution $u = 0$, i.e., k^2 is not a Dirichlet eigenvalue for the negative Laplacian in the interior of Γ. For example, we may choose Γ to be a sphere of radius R such that kR does not coincide with a zero of one of the spherical Bessel functions j_n, $n = 0, 1, 2, \dots$.

Given the internal surface Γ, we try to represent the scattered field as an acoustic single-layer potential

$$u^s(x) = \int_\Gamma \Phi(x, y)\varphi(y)\, ds(y) \tag{5.57}$$

with an unknown density $\varphi \in L^2(\Gamma)$. From (2.15) we see that the asymptotic behavior of this single-layer potential is given by

$$u^s(x) = \frac{1}{4\pi}\, \frac{e^{ik|x|}}{|x|} \int_\Gamma e^{-ik\,\hat{x}\cdot y}\varphi(y)\, ds(y) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,$$

uniformly for all directions $\hat{x} = x/|x|$. Hence, given the far-field pattern u_∞, we have to solve the integral equation of the first kind

$$S_\infty \varphi = u_\infty \tag{5.58}$$

for the density φ where the integral operator $S_\infty : L^2(\Gamma) \to L^2(\mathbb{S}^2)$ is defined by

$$(S_\infty \varphi)(\hat{x}) := \frac{1}{4\pi} \int_\Gamma e^{-ik\,\hat{x}\cdot y} \varphi(y)\, ds(y), \quad \hat{x} \in \mathbb{S}^2. \tag{5.59}$$

The integral operator S_∞ has an analytic kernel and therefore equation (5.58) is severely ill-posed. We first establish some properties of S_∞.

Theorem 5.19. *The far field integral operator S_∞, defined by (5.59), is injective and has dense range provided k^2 is not a Dirichlet eigenvalue for the negative Laplacian in the interior of Γ.*

Proof. Let $S_\infty \varphi = 0$ and define the acoustic single-layer potential by (5.57). Then u^s has far field pattern $u_\infty = 0$, whence $u^s = 0$ in the exterior of Γ follows by Theorem 2.14. Analogous to (3.10), we introduce the normal derivative of the single-layer operator $K' : L^2(\Gamma) \to L^2(\Gamma)$. By the L^2 jump relation (3.21), we find that

$$\varphi - K'\varphi = 0.$$

Employing the argument used in the proof of Theorem 3.27, by the Fredholm alternative we see that the nullspaces of $I - K'$ in $L^2(\Gamma)$ and in $C(\Gamma)$ coincide. Therefore, φ is continuous and by the jump relations for continuous densities u^s represents a solution to the homogeneous Dirichlet problem in the interior of Γ. Hence, by our assumption on the choice of Γ, we have $u^s = 0$ everywhere in $\mathbb{R}^3$. The jump relations of Theorem 3.1 now yield $\varphi = 0$ on Γ, whence S_∞ is injective.

The adjoint operator $S_\infty^* : L^2(\mathbb{S}^2) \to L^2(\Gamma)$ of S_∞ is given by

$$(S_\infty^* g)(y) = \frac{1}{4\pi} \int_{\mathbb{S}^2} e^{ik\,\hat{x}\cdot y} g(\hat{x})\, ds(\hat{x}), \quad y \in \Gamma. \tag{5.60}$$

Let $S_\infty^* g = 0$. Then

$$v(y) := \int_{\mathbb{S}^2} e^{ik\,\hat{x}\cdot y} g(\hat{x})\, ds(\hat{x}), \quad y \in \mathbb{R}^3,$$

defines a Herglotz wave function which vanishes on Γ, i.e., it solves the homogeneous Dirichlet problem in the interior of Γ. Hence, it vanishes there by our choice of Γ and since v is analytic in $\mathbb{R}^3$ it follows that $v = 0$ everywhere. Theorem 3.19 now yields $g = 0$ on $\mathbb{S}^2$, whence S_∞^* is injective and by Theorem 4.6 the range of S_∞ is dense in $L^2(\mathbb{S}^2)$. □

For later use we state a corresponding theorem for the acoustic single-layer operator $S : L^2(\Gamma) \to L^2(\Lambda)$ defined by

$$(S\varphi)(x) := \int_\Gamma \Phi(x, y)\varphi(y)\, ds(y), \quad x \in \Lambda, \tag{5.61}$$

where Λ denotes a closed C^2 surface containing Γ in its interior. The proof is similar to that of Theorem 5.19 and therefore is left as an exercise for the reader.

Theorem 5.20. *The single-layer operator S, defined by (5.61), is injective and has dense range provided k^2 is not a Dirichlet eigenvalue for the negative Laplacian in the interior of Γ.*

Also for later use, recalling (3.62) we introduce the Herglotz operator H : $L^2(\mathbb{S}^2) \to L^2(\Lambda)$ as the restriction of the Herglotz function with kernel g to a closed surface Λ by

$$(Hg)(x) := \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d)\, ds(d), \quad x \in \Lambda. \tag{5.62}$$

Theorem 5.21. *The Herglotz operator H, defined by (5.62), is injective and has dense range provided k^2 is not a Dirichlet eigenvalue for the negative Laplacian in the interior of Λ.*

Proof. Up to a factor of 4π, the operator H is the adjoint of the far field integral operator given by (5.59). Therefore, the statement of the theorem is equivalent to Theorem 5.19. □

We note that if H is viewed as an operator from $L^2(\mathbb{S}^2)$ into the space of solutions to the Helmholtz equation in $H^1(G)$ where G is the interior of Λ then $H(L^2(\mathbb{S}^2))$ is dense in this space without any restriction on the wave number [70, 89], see also Corollary 5.31.

We now know that by our choice of Γ the integral equation of the first kind (5.58) has at most one solution. Its solvability is related to the question of whether or not the scattered wave can be analytically extended as a solution to the Helmholtz equation across the boundary ∂D. Clearly, we can expect (5.58) to have a solution $\varphi \in L^2(\Gamma)$ only if u_∞ is the far field of a radiating solution to the Helmholtz equation in the exterior of Γ and by Theorem 3.6 the boundary data must belong to the Sobolev space $H^1(\Gamma)$. Conversely, it can be shown that if u_∞ satisfies these conditions then (5.58) is indeed solvable. Hence, the solvability of (5.58) is related to regularity properties of the scattered field which, in general, cannot be known in advance for an unknown obstacle D.

We wish to illustrate the degree of ill-posedness of the equation (5.58) by looking at the singular values of S_∞ in the special case where Γ is the unit sphere. Here, from the Funk–Hecke formula (2.45), we deduce that the singular values of S_∞ are given by

$$\mu_n = |j_n(k)|, \quad n = 0, 1, \ldots.$$

Therefore, from the asymptotic formula (2.38) and Stirling's formula, we have the extremely rapid decay

$$\mu_n = O\left(\frac{ek}{2n}\right)^n, \quad n \to \infty, \tag{5.63}$$

indicating severe ill-posedness.

We may apply the Tikhonov regularization from Section 4.4, that is, we may solve

$$\alpha\varphi_\alpha + S_\infty^* S_\infty \varphi_\alpha = S_\infty^* u_\infty \tag{5.64}$$

with regularization parameter $\alpha > 0$ instead of (5.58). Through the solution φ_α of (5.64) we obtain the approximation

$$u_\alpha^s(x) = \int_\Gamma \Phi(x, \cdot)(\alpha I + S_\infty^* S_\infty)^{-1} S_\infty^* u_\infty \, ds \tag{5.65}$$

for the scattered field. However, by passing to the limit $\alpha \to 0$ in (5.64), we observe that we can expect convergence of the unique solution φ_α to the regularized equation only if the original equation (5.58) is solvable. Therefore, even if u_∞ is the exact far field pattern of a scatterer D, in general u_α^s will not converge to the exact scattered field u^s since, as mentioned above, the original equation (5.58) may not be solvable.

Given the approximation u_α^s, we can now seek the boundary of the scatterer D as the location of the zeros of $u^i + u_\alpha^s$ in a minimum norm sense, i.e., we can approximate ∂D by minimizing the defect

$$\|u^i + u_\alpha^s\|_{L^2(\Lambda)} \tag{5.66}$$

over some suitable class U of admissible surfaces Λ. Instead of solving this minimization problem one can also visualize ∂D by color coding the values of the modulus $|u|$ of the total field $u \approx u^i + u_\alpha^s$ on a sufficiently fine grid over some domain containing the scatterer. For the minimization problem, we will choose U to be a compact subset (with respect to the $C^{1,\beta}$ norm, $0 < \beta < 1$,) of the set of all starlike closed C^2 surfaces, described by

$$\Lambda = \left\{ r(\hat{x})\,\hat{x} : \hat{x} \in \mathbb{S}^2 \right\}, \quad r \in C^2(\mathbb{S}^2), \tag{5.67}$$

satisfying the a priori assumption

$$0 < r_i(\hat{x}) \le r(\hat{x}) \le r_e(\hat{x}), \quad \hat{x} \in \mathbb{S}^2, \tag{5.68}$$

with given functions r_i and r_e representing surfaces Λ_i and Λ_e such that the internal auxiliary surface Γ is contained in the interior of Λ_i and the boundary ∂D of the unknown scatterer D is contained in the annulus between Λ_i and Λ_e. We recall from Section 5.3 that for a sequence of surfaces we understand convergence $\Lambda_n \to \Lambda$, $n \to \infty$, in the sense that $\|r_n - r\|_{C^{1,\beta}(\mathbb{S}^2)} \to 0$, $n \to \infty$, for the functions r_n and r representing Λ_n and Λ via (5.67).

For a satisfactory reformulation of the inverse scattering problem as an optimization problem, we want some convergence properties when the regularization parameter α tends to zero. Therefore, recalling the definition (5.61) of the single-layer operator S, we combine the minimization of the Tikhonov functional for (5.58) and the defect minimization (5.66) into one cost functional

$$\mu(\varphi, \Lambda; \alpha) := \|S_\infty\varphi - u_\infty\|^2_{L^2(\mathbb{S}^2)} + \alpha\|\varphi\|^2_{L^2(\Gamma)} + \gamma\|u^i + S\varphi\|^2_{L^2(\Lambda)}. \tag{5.69}$$

Here, $\alpha > 0$ denotes the regularization parameter for the Tikhonov regularization of (5.58) represented by the first two terms in (5.69) and $\gamma > 0$ denotes a coupling parameter which has to be chosen appropriately for the numerical implementation in order to make the first and third term in (5.69) of the same magnitude. In the sequel, for theoretical purposes we always may assume $\gamma = 1$.

Definition 5.22 *Given the incident field u^i, a (measured) far field $u_\infty \in L^2(\mathbb{S}^2)$ and a regularization parameter $\alpha > 0$, a surface Λ_0 from the compact set U is called* optimal *if there exists $\varphi_0 \in L^2(\Gamma)$ such that φ_0 and Λ_0 minimize the cost functional (5.69) simultaneously over all $\varphi \in L^2(\Gamma)$ and $\Lambda \in U$, that is, we have*

$$\mu(\varphi_0, \Lambda_0; \alpha) = m(\alpha)$$

where

$$m(\alpha) := \inf_{\varphi \in L^2(\Gamma),\, \Lambda \in U} \mu(\varphi, \Lambda; \alpha).$$

For this reformulation of the inverse scattering problem into a nonlinear optimization problem, we can now state the following results. Note that in the existence Theorem 5.23 we need not assume that u_∞ is an exact far field pattern.

Theorem 5.23. *For each $\alpha > 0$ there exists an optimal surface $\Lambda \in U$.*

Proof. Let (φ_n, Λ_n) be a minimizing sequence in $L^2(\Gamma) \times U$, i.e.,

$$\lim_{n\to\infty} \mu(\varphi_n, \Lambda_n; \alpha) = m(\alpha).$$

Since U is compact, we can assume that $\Lambda_n \to \Lambda \in U$, $n \to \infty$. From

$$\alpha\|\varphi_n\|^2_{L^2(\Gamma)} \le \mu(\varphi_n, \Lambda_n; \alpha) \to m(\alpha), \quad n \to \infty,$$

and $\alpha > 0$ we conclude that the sequence (φ_n) is bounded, i.e., $\|\varphi_n\|_{L^2(\Gamma)} \le c$ for all n and some constant c. Hence, we can assume that it converges weakly $\varphi_n \rightharpoonup \varphi$ in $L^2(\Gamma)$ as $n \to \infty$. Since $S_\infty : L^2(\Gamma) \to L^2(\mathbb{S}^2)$ and $S : L^2(\Gamma) \to L^2(\Lambda)$ represent compact operators, it follows that $S_\infty\varphi_n \to S_\infty\varphi$ and $S\varphi_n \to S\varphi$ as $n \to \infty$. We indicate the dependence of $S : L^2(\Gamma) \to L^2(\Lambda_n)$ on n by writing S_n. With functions r_n and r representing Λ_n and Λ via (5.67), by Taylor's formula we can estimate

$$|\Phi(r_n(\hat{x})\,\hat{x}, y) - \Phi(r(\hat{x})\,\hat{x}, y)| \le L\,|r_n(\hat{x}) - r(\hat{x})|$$

for all $\hat{x} \in \mathbb{S}^2$ and all $y \in \Gamma$. Here, L denotes a bound on $\operatorname{grad}_x \Phi$ on $W \times \Gamma$ where W is the closed annular domain between the two surfaces Λ_i and Λ_e. Then, using the Cauchy–Schwarz inequality, we find that

$$\left| \int_\Gamma \{\Phi(r_n(\hat{x})\,\hat{x}, y) - \Phi(r(\hat{x})\,\hat{x}, y)\}\, \varphi_n(y)\, ds(y) \right| \leq cL\,|\Gamma|\,|r_n(\hat{x}) - r(\hat{x})|$$

for all $\hat{x} \in \mathbb{S}^2$. Therefore, from $\|S\varphi_n - S\varphi\|^2_{L^2(\Lambda)} \to 0$, $n \to \infty$, we can deduce that

$$\|u^i + S_n\varphi_n\|^2_{L^2(\Lambda_n)} \to \|u^i + S\varphi\|^2_{L^2(\Lambda)}, \quad n \to \infty.$$

This now implies

$$\alpha\|\varphi_n\|^2_{L^2(\Gamma)} \to m(\alpha) - \|S_\infty\varphi - u_\infty\|^2_{L^2(\mathbb{S}^2)} - \|u^i + S\varphi\|^2_{L^2(\Lambda)} \leq \alpha\|\varphi\|^2_{L^2(\Gamma)}$$

for $n \to \infty$. Since we already know weak convergence $\varphi_n \rightharpoonup \varphi$, $n \to \infty$, it follows that

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|^2_{L^2(\Gamma)} = \lim_{n\to\infty} \|\varphi_n\|^2_{L^2(\Gamma)} - \|\varphi\|^2_{L^2(\Gamma)} \leq 0,$$

i.e., we also have norm convergence $\varphi_n \to \varphi$, $n \to \infty$. Finally, by continuity

$$\mu(\varphi, \Lambda; \alpha) = \lim_{n\to\infty} \mu(\varphi_n, \Lambda_n; \alpha) = m(\alpha),$$

and this completes the proof. □

Theorem 5.24. *Let u_∞ be the exact far field pattern of a domain D such that ∂D belongs to U. Then we have convergence of the cost functional*

$$\lim_{\alpha\to 0} m(\alpha) = 0. \tag{5.70}$$

Proof. By Theorem 5.20, given $\varepsilon > 0$ there exists $\varphi \in L^2(\Gamma)$ such that

$$\|S\varphi + u^i\|_{L^2(\partial D)} < \varepsilon.$$

Since by Theorem 5.16 and the far field representation (2.14) the far field pattern of a radiating solution of the Helmholtz equation depends continuously on the boundary data, we can estimate

$$\|S_\infty\varphi - u_\infty\|_{L^2(\mathbb{S}^2)} \leq c\|S\varphi - u^s\|_{L^2(\partial D)}$$

for some constant c. From $u^i + u^s = 0$ on ∂D we then deduce that

$$\mu(\varphi, \partial D; \alpha) \leq (1 + c^2)\varepsilon^2 + \alpha\|\varphi\|^2_{L^2(\Gamma)} \to (1 + c^2)\varepsilon^2, \quad \alpha \to 0.$$

Since ε is arbitrary, (5.70) follows. □

Based on Theorem 5.24, we can state the following convergence result.

Theorem 5.25. *Let (α_n) be a null sequence and let (Λ_n) be a corresponding sequence of optimal surfaces for the regularization parameter α_n. Then there exists a convergent subsequence of (Λ_n). Assume that u_∞ is the exact far field pattern of a domain D such that ∂D is contained in U. Then every limit point Λ^* of (Λ_n) represents a surface on which the total field vanishes.*

Proof. The existence of a convergent subsequence of (Λ_n) follows from the compactness of U. Let Λ^* be a limit point. Without loss of generality, we can assume that $\Lambda_n \to \Lambda^*$, $n \to \infty$. Let u^* denote the solution to the direct scattering problem with incident wave u^i for the obstacle with boundary Λ^*, i.e., the boundary condition reads

$$u^* + u^i = 0 \quad \text{on } \Lambda^*. \tag{5.71}$$

Since Λ_n is optimal for the parameter α_n, there exists $\varphi_n \in L^2(\Gamma)$ such that

$$\mu(\varphi_n, \Lambda_n; \alpha_n) = m(\alpha_n)$$

for $n = 1, 2, \ldots$. We denote by u_n the single-layer potential with density φ_n and interpret u_n as the solution to the exterior Dirichlet problem with boundary values $S_n\varphi_n$ on the boundary Λ_n. By Theorem 5.24, these boundary data satisfy

$$\|u_n + u^i\|_{L^2(\Lambda_n)} \to 0, \quad n \to \infty. \tag{5.72}$$

By Theorem 5.16, from (5.71) and (5.72) we now deduce that the far field patterns $S_\infty\varphi_n$ of u_n converge in $L^2(\mathbb{S}^2)$ to the far field pattern u^*_∞ of u^*. By Theorem 5.24, we also have $\|S_\infty\varphi_n - u_\infty\|_{L^2(\mathbb{S}^2)} \to 0$, $n \to \infty$. Therefore, we conclude $u_\infty = u^*_\infty$, whence $u^s = u^*$ follows. Because of (5.71), the total field $u^s + u^i$ must vanish on Λ^*. □

Since we do not have uniqueness either for the inverse scattering problem or for the optimization problem, in the above convergence analysis we cannot expect more than convergent subsequences. In addition, due to the lack of a uniqueness result for one wave number and one incident plane wave, we cannot assume that we always have convergence to the boundary of the unknown scatterer. However, if we have the a priori information that the diameter of the unknown obstacle is less than $2\pi/k$, then by Corollary 5.3 we can sharpen the result of Theorem 5.25 and, by a standard argument, we have convergence of the total sequence to the boundary of the unknown scatterer.

Before we turn to related decomposition methods, we wish to mention some modifications and extensions of the Kirsch–Kress method. We can try to achieve more accurate reconstructions by using more incident plane waves $u^i_1, \ldots, u^i_n$ with different directions $d_1, \ldots, d_n$ and corresponding far field patterns $u_{\infty,1}, \ldots, u_{\infty,n}$. Then we have to minimize the sum

$$\sum_{j=1}^{n} \left\{ \|S_\infty\varphi_j - u_{\infty,j}\|^2_{L^2(\mathbb{S}^2)} + \alpha\|\varphi_j\|^2_{L^2(\Gamma)} + \gamma\|u^i_j + S\varphi_j\|^2_{L^2(\Lambda)} \right\} \tag{5.73}$$

over all $\varphi_1, \dots, \varphi_n \in L^2(\Gamma)$ and all $\Lambda \in U$. Obviously, the results of the three preceding theorems carry over to the minimization of (5.73). Of course, the use of more than one wave will lead to a tremendous increase in the computational costs. These costs can be reduced by using suitable linear combinations of incident plane waves as suggested by Zinn [337].

In addition to the reconstruction of the scatterer D from far field data, we also can consider the reconstruction from *near field* data, i.e., from measurements of the scattered wave u^s on some closed surface Γ_{meas} containing D in its interior. By the uniqueness for the exterior Dirichlet problem, knowing u^s on the closed surface Γ_{meas} implies knowing the far field pattern u_∞ of u^s. Therefore, the uniqueness results for the reconstruction from far field data immediately carry over to the case of near field data. In particular, since the measurement surface Γ_{meas} trivially provides a priori information on the size of D, a finite number of incident plane waves will always uniquely determine D. In the case of near field measurements, the integral equation (5.58) has to be replaced by

$$S\varphi = u^s_{\text{meas}} \tag{5.74}$$

where in a slight abuse of notations analogous to (5.61) the integral operator $S : L^2(\Gamma) \to L^2(\Gamma_{\text{meas}})$ is given by

$$(S\varphi)(x) := \int_\Gamma \Phi(x, y)\varphi(y)\, ds(y), \quad x \in \Gamma_{\text{meas}}. \tag{5.75}$$

Correspondingly, for given u^i and u^s_{meas}, the optimization problem has to be modified into minimizing the sum

$$\|S\varphi - u^s_{\text{meas}}\|^2_{L^2(\Gamma_{\text{meas}})} + \alpha\|\varphi\|^2_{L^2(\Gamma)} + \gamma\|u^i + S\varphi\|^2_{L^2(\Lambda)} \tag{5.76}$$

simultaneously over all $\varphi \in L^2(\Gamma)$ and $\Lambda \in U$. The results of Theorems 5.23–5.25 again carry over to the near field case.

The integral operator S given by (5.75) has an analytic kernel and therefore equation (5.74) is ill-posed. In the special case where Γ is the unit sphere and where Γ_{meas} is a concentric sphere with radius R, from the addition theorem (2.43) we deduce that the singular values of S are given by

$$\mu_n = kR|j_n(k)|\left|h_n^{(1)}(kR)\right|, \quad n = 0, 1, \dots.$$

Therefore, from the asymptotic formulas (2.38) and (2.39) we have

$$\mu_n = O\left(\frac{R^{-n}}{2n+1}\right), \quad n \to \infty,$$

indicating an ill-posedness which is slightly less severe than the ill-posedness of the corresponding far field case as indicated by the asymptotics (5.63). However, numerical experiments (see [216]) have shown that unfortunately this does not lead to a highly noticable increase in the accuracy of the reconstructions.

So far we have assumed the far field to be known for all observation directions $\hat{x}$. Due to analyticity, for uniqueness it suffices to know the far field pattern on a subset $\Omega \subset \mathbb{S}^2$ with a nonempty interior. Zinn [337] has shown that after modifying the far field integral operator S_∞ given by (5.59) into an operator from $L^2(\Gamma)$ into $L^2(\Omega)$) and replacing $\mathbb{S}^2$ by Ω in the Tikhonov part of the cost functional μ given by (5.69) the results of Theorems 5.23–5.25 remain valid. However, as one would expect, the quality of the reconstructions decreases drastically for this so called *limited-aperture problem*. For two-dimensional problems the numerical experiments in [337] indicate that satisfactory reconstructions need an aperture not smaller than 180 degrees and more than one incident wave.

We also want to mention that, in principle, we may replace the approximation of the scattered field u^s through a single-layer potential by any other convenient approximation. For example, Angell, Kleinman and Roach [11] have suggested using an expansion with respect to radiating spherical wave functions. Using a single-layer potential approximation on an auxiliary internal surface Γ has the advantage of allowing the incorporation of a priori information on the unknown scatterer by a suitable choice of Γ. Furthermore, by the addition theorem (2.43), from a theoretical point of view we may consider the spherical wave function approach as a special case of the single-layer potential with Γ a sphere. We also want to mention that the above methods may be considered as having some of their roots in the work of Imbriale and Mittra [151] who described the first reconstruction algorithm in inverse obstacle scattering for frequencies in the resonance region.

For the numerical solution, we must of course discretize the optimization problem. This is achieved through replacing $L^2(\Gamma)$ and U by finite dimensional subspaces. Denote by (X_n) a sequence of finite dimensional subspaces $X_{n-1} \subset X_n \subset L^2(\Gamma)$ such that $\bigcup_{n=1}^\infty X_n$ is dense in $L^2(\Gamma)$. Similarly, let (U_n) be a sequence of finite dimensional subsets $U_{n-1} \subset U_n \subset U$ such that $\bigcup_{n=1}^\infty U_n$ is dense in U. We then replace the optimization problem of Definition 5.22 by the finite dimensional problem where we minimize over the finite dimensional set $X_n \times U_n$ instead of $L^2(\Gamma) \times U$.

The finite dimensional optimization problem is now a nonlinear least squares problem with $\dim X_n + \dim U_n$ unknowns. For its numerical solution, we suggest using a Levenberg–Marquardt algorithm [246] as one of the most efficient nonlinear least squares routines. It does not allow the imposition of constraints but we found in practice that the constraints are unnecessary due to the increase in the cost functional as Λ approaches Γ or tends to infinity.

The numerical evaluation of the cost functional (5.69), including the integral operators S_∞ and S, in general requires the numerical evaluation of integrals with analytic integrands over analytic surfaces $\mathbb{S}^2$, Γ and Λ. The integrals over the unit sphere can be numerically approximated by the Gauss trapezoidal product rule (3.114) described in Section 3.6. The integrals over Γ and Λ can be transformed into integrals over $\mathbb{S}^2$ through appropriate substitutions and then again approximated via (3.114). Canonical subspaces for the finite dimensional optimization problems are given in terms of spherical harmonics as follows. Denote by Z_n the linear space of all spherical harmonics of order less than or equal to n. Let $p : \Gamma \to \mathbb{S}^2$ be bijective and choose $X_n \subset L^2(\Gamma)$ by

$$X_n := \{\varphi = Y \circ p : Y \in Z_n\}.$$

Choose U_n to be the set of all starlike surfaces described through (5.67) and (5.68) with $r \in Z_n$. Then, by Theorem 2.7, the degree of freedom in the optimization problem is $2(n+1)^2$.

The above convergence analysis requires to combine the minimization of the Tikhonov functional for (5.58) and the defect minimization (5.66) into one cost functional (5.69). However, numerical tests have shown that satisfactory results can also be obtained when the two steps are carried out separately in order to reduce the computational costs.

For further details on the numerical implementation and examples for reconstructions we refer to [187, 188, 189, 191] for two dimensions and to [216] for three dimensions including shapes which are not rotationally symmetric. Instead of a single-layer potential on an auxiliary internal surface, of course, also a double-layer potential can be used. Corresponding numerical reconstructions were obtained by Haas and Lehner [115].

Note that the potential approach of the Kirsch–Kress method can also be employed for the inverse problem to recover the impedance given the shape of the scatterer. In this case the far field equation (5.58) is solved with Γ replaced by the known boundary ∂D. After the density φ is obtained via (5.64) the impedance function λ can be determined in a least-squares sense from the impedance boundary condition after evaluating the trace and the normal derivative of the single-layer potential (5.65) on ∂D (see [3, 163]).

A hybrid method combining ideas of the above decomposition method and Newton iterations of the previous section has been suggested and investigated in a series of papers by Kress and Serranho [206, 213, 214, 304, 305]. In principle, this approach may be considered as a modification of the Kirsch–Kress method in the sense that the auxiliary surface Γ is viewed as an approximation for the unknown boundary of the scatterer. Then, keeping the potential u^s_α resulting via (5.65) from a regularized solution of (5.58) fixed, Γ is updated via linearizing the boundary condition $u^i + u^s_\alpha = 0$ around Γ.

If we assume again that Γ is starlike with radial function r and look for an update $\tilde{\Gamma}$ that is starlike with radial function $r + q$ the update is found by linearizing the boundary condition $\left(u^i + u^s_\alpha\right)\big|_{\tilde{\Gamma}} = 0$, that is, by solving

$$\left(u^i + u^s_\alpha\right)\big|_\Gamma + \operatorname{grad}(u^i + u^s_\alpha)\big|_\Gamma \cdot \left(p_q \circ p_r^{-1}\right) = 0 \tag{5.77}$$

for q. Recall the notation introduced in (5.21). In an obvious way, the two steps of alternatingly solving (5.58) by Tikhonov regularization and solving (5.77) in the least squares sense are iterated. For the numerical implementation, the terms $u^s_\alpha\big|_\Gamma$ and $\operatorname{grad} u^s_\alpha\big|_\Gamma$ in (5.77) are evaluated with the aid of the jump relations and the update q, for example, as above in the Kirsch–Kress method is taken from the linear space Z_n of all spherical harmonics of order less than or equal to some appropriately chosen n. From numerical examples in two [206, 213, 304] and three dimensions [305] it can be concluded that the quality of the reconstructions is similar to that of Newton iterations.

On the theoretical side, in [305] Serranho obtained results analogous to Theorems 5.23–5.25 and in [304] he achieved a convergence result in the spirit of Potthast's analysis [280]. Furthermore, this iterative variant of the Kirsch–Kress method is closely related to the third alternative pointed out in the previous section for an iterative solution of the system (5.48) and (5.49) by first solving the ill-posed data equation (5.49) for the density and then solving the linearized field equation (5.48) to update the boundary. Alternating the solutions of (5.58) and (5.77) can be seen to coincide with alternating the solutions of (5.49) and the linearization of (5.48) where the derivative of A with respect to r is replaced by a derivative where one linearizes only with respect to the evaluation surface for the single-layer potential but not with respect to the integration surface [164]. A second degree method where (5.77) is replaced by a Taylor formula up to second order has been investigated in [215].

We now proceed with a brief description of the *point source method* due to Potthast [277, 278, 281] as our second example of a decomposition method. For this we begin with a few comments on the *interior Dirichlet problem*. The classical approach for solving the interior Dirichlet problem is to seek the solution in the form of a double-layer potential

$$u(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in D,$$

with a continuous density φ. Then, given a continuous function f on ∂D, by the jump relations of Theorem 3.1 the double-layer potential u satisfies the boundary condition $u = f$ on ∂D if the density solves the integral equation

$$\varphi - K\varphi = -2f$$

with the double-layer integral operator K defined by (3.9). If we assume that k^2 is not a Dirichlet eigenvalue for D, i.e., the homogeneous Dirichlet problem in D has only the trivial solution, then with the aid of the jump relations it can be seen that $I-K$ has a trivial null space in $C(\partial D)$ (for details see [64]). Hence, by the Riesz–Fredholm theory $I - K$ has a bounded inverse $(I - K)^{-1}$ from $C(\partial D)$ into $C(\partial D)$. This implies solvability and well-posedness of the interior Dirichlet problem. In particular, we have the following theorem. For our discussion below on the point source method it would be sufficient to require norm convergence of the boundary data. The stronger result will be needed later on in the presentation of the Colton–Monk method.

Theorem 5.26. *Assume that k^2 is not a Dirichlet eigenvalue for the bounded domain D. Let (u_n) be a sequence of $C^2(D) \cap C(\bar{D})$ solutions to the Helmholtz equation in D such that the boundary data $f_n = u_n$ on ∂D are weakly convergent in $L^2(\partial D)$. Then the sequence (u_n) converges uniformly (together with all its derivatives) on compact subsets of D to a solution u of the Helmholtz equation.*

Proof. Following the argument used in Theorem 3.27, the Fredholm alternative can be employed to show that $(I - K)^{-1}$ is bounded from $L^2(\partial D)$ into $L^2(\partial D)$. Hence, the sequence $\varphi_n := 2(K-I)^{-1}f_n$ converges weakly to $\varphi := 2(K-I)^{-1}f$ as $n \to \infty$ provided (f_n) converges weakly towards f. Substituting this into the double-layer potential, we see that the sequence (u_n) converges pointwise in D to the double-layer potential u with density φ. Applying the Cauchy–Schwarz inequality to the double-layer potential, we see that

$$|u_n(x_1) - u_n(x_2)| \le |\partial D|^{1/2} \sup_{y \in \partial D} \left| \frac{\partial \Phi(x_1, y)}{\partial \nu(y)} - \frac{\partial \Phi(x_2, y)}{\partial \nu(y)} \right| \|2(K-I)^{-1} f_n\|_{L^2(\partial D)}$$

for all $x_1, x_2 \in D$. From this we deduce that the u_n are equicontinuous on compact subsets of D since weakly convergent sequences are bounded. This, together with the pointwise convergence, implies uniform convergence of the sequence (u_n) on compact subsets of D. Convergence of the derivatives follows in an analogous manner. □

Our motivation of the point source method is based on Huygens' principle from Theorem 3.14, i.e., the scattered field representation

$$u^s(x) = -\int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, \Phi(x, y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{5.78}$$

and the far field representation

$$u_\infty(\hat{x}) = -\frac{1}{4\pi} \int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, e^{-ik\,\hat{x}\cdot y}\, ds(y), \quad \hat{x} \in \mathbb{S}^2. \tag{5.79}$$

As in the Kirsch–Kress method we choose an auxiliary closed C^2 surface Λ. However, we now require that the unknown scatterer D is contained in the interior of Λ. We try to approximate the point source $\Phi(x, \cdot)$ for x in the exterior of Λ by a Herglotz wave function such that

$$\Phi(x, y) \approx \frac{1}{4\pi} \int_{\mathbb{S}^2} e^{ik\,y\cdot d} g_x(d)\, ds(d) \tag{5.80}$$

for all y in the interior of Λ and some $g_x \in L^2(\mathbb{S}^2)$. Under the assumption that k^2 is not a Dirichlet eigenvalue for the negative Laplacian in the interior of Λ, by Theorem 5.21 the Herglotz wave functions are dense in $L^2(\Lambda)$. Consequently, by Theorem 5.26 the approximation (5.80) can be achieved uniformly with respect to y up to derivatives of second order on compact subsets of the interior of Λ. We can now insert (5.80) into (5.78) and use (5.79) to obtain

$$u^s(x) \approx \int_{\mathbb{S}^2} g_x(d) u_\infty(-d)\, ds(d) \tag{5.81}$$

as an approximation for the scattered wave u^s. The expression on the right hand side of (5.81) may be viewed as a backprojection of the far field pattern by a weight function g. Knowing an approximation for the scattered wave, in principle, the boundary

∂D can be found as above in the Kirsch–Kress method from the boundary condition $u^i + u^s = 0$ on ∂D.

The approximation (5.80) can be obtained in practice by solving the ill-posed linear integral equation

$$\int_{\mathbb{S}^2} e^{ik\,y\cdot d} g_x(d)\,ds(d) = 4\pi\Phi(x,y), \quad y \in \Lambda, \tag{5.82}$$

via Tikhonov regularization and Morozov's discrepancy principle Theorem 4.16. Note that although the integral equation (5.82) is in general not solvable, the approximation property (5.81) is ensured through the denseness result of Theorem 5.21 on Herglotz wave functions.

Since the concept of the point source method requires the unknown scatterer D to be contained in the interior of Λ and the source point x is located in the exterior of Λ, in order to obtain approximations for the scattered field at locations close to the boundary ∂D in the numerical implementation it is necessary to solve (5.82) for a number of surfaces Λ_x associated with a grid of source points x. The computational effort for doing this can be substantially reduced by fixing a reference surface Λ not containing the origin in its interior, for example a sphere, and then choosing

$$\Lambda_x = M\Lambda + x$$

that is, first apply an orthogonal matrix M to the reference surface Λ and then translate it. Straightforward calculations show that if the Herglotz wave function with kernel g approximates the point source $\Phi(0,\cdot)$ located at the origin with error less than ε with respect to $L^2(\Lambda)$ then the Herglotz wave function with kernel

$$g_x(d) = e^{-ik\,x\cdot d} g(M^* d)$$

approximates the point source $\Phi(x,\cdot)$ located at x with error less than ε with respect to $L^2(\Lambda_x)$. Hence, it suffices to solve (5.82) via Tikhonov regularization only once for $x = 0$ and Λ.

In the practical implementation for a grid of points x_ℓ, $\ell = 1, \dots, L$, the above procedure is carried out for a finite number of matrices M_j, $j = 1, \dots, J$, representing various directions that are used to move the approximating domain around. As an indicator to decide whether the crucial condition that D is contained in the interior of Λ_x is satisfied, one can use two different error levels in the solution of the integral equation (5.82), that is, two different regularization parameters in the Tikhonov regularization and keep the approximation obtained via (5.81) only if the two results are close. For more details on the mathematics and numerics of the point source method we refer to [281, 282, 283].

Here we conclude our short discussion of the point source method by pointing out a duality to the Kirsch–Kress method as observed by Potthast and Schulz [284]. In view of (5.60) we rewrite (5.82) in the operator form $S_\infty^* g_x = \Phi(x,\cdot)$ and obtain via Tikhonov regularization and (5.81) the approximation

$$u_\alpha^s(x) = \int_{\mathbb{S}^2} Ru_\infty(\alpha I + S_\infty S_\infty^*)^{-1} S_\infty \Phi(x, \cdot)\, ds \tag{5.83}$$

for the scattered field with the reflection operator R from (3.47). Using the relations $S_\infty^*(\alpha I + S_\infty^* S_\infty)^{-1} = (\alpha I + S_\infty^* S_\infty)^{-1} S_\infty^*$ and $\overline{S_\infty^* S_\infty \varphi} = S_\infty^* S_\infty \bar{\varphi}$ and $S_\infty^* R\bar{u}_\infty = \overline{S_\infty^* u_\infty}$ we can transform

$$\begin{aligned}
&\int_{\mathbb{S}^2} Ru_\infty(\alpha I + S_\infty S_\infty^*)^{-1} S_\infty \Phi(x, \cdot)\, ds \\
&= ((\alpha I + S_\infty S_\infty^*)^{-1} S_\infty \Phi(x, \cdot), R\bar{u}_\infty)_{L^2(\mathbb{S}^2)} \\
&= (\Phi(x, \cdot), (\alpha I + S_\infty^* S_\infty)^{-1} S_\infty^* R\bar{u}_\infty)_{L^2(\Lambda)} \\
&= \int_\Lambda \Phi(x, \cdot)(\alpha I + S_\infty^* S_\infty)^{-1} S_\infty^* u_\infty\, ds,
\end{aligned}$$

i.e., the approximations (5.65) and (5.83) for the scattered field coincide.

We conclude this section on decomposition methods with an approximation method that was developed by Colton and Monk [72, 73, 74]. We will present this method in a manner which stresses its close connection to the method of Kirsch and Kress. Its analysis is related to the completeness properties for far field patterns of Section 3.3. In the first step of this method we look for superpositions of incident fields with different directions which lead to simple far field patterns, for example to far fields belonging to radiating spherical wave functions. To be more precise, we consider as incident wave v^i a superposition of plane waves of the form

$$v^i(x) = \int_{\mathbb{S}^2} e^{ik\, x\cdot d} g(d)\, ds(d), \quad x \in \mathbb{R}^3, \tag{5.84}$$

with weight function $g \in L^2(\mathbb{S}^2)$, i.e., the incident wave is a Herglotz wave function. By Lemma 3.20, the corresponding far field pattern

$$v_\infty(\hat{x}) = \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2,$$

is obtained by superposing the far field patterns $u_\infty(\cdot\,, d)$ for the incoming directions d. We note that by the Reciprocity Theorem 3.15 we may also consider (5.84) as a superposition with respect to the observation directions instead of the incident directions. Therefore, if we fix d and superpose with respect to the observation directions, we can view this method as one of determining a linear functional having prescribed values on the set of far field patterns. Viewed in this way the method of Colton and Monk is sometimes referred to as a *dual space method*.

If we now want the scattered wave to become a prescribed radiating solution v^s to the Helmholtz equation with far field pattern v_∞, given the far field patterns $u_\infty(\cdot\,, d)$

for all incident directions d we need to solve the integral equation of the first kind

$$Fg = v_\infty \tag{5.85}$$

where the far field operator $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is defined by (3.45). Since F has an analytic kernel, the equation (5.85) is ill-posed. Once we have constructed the incident field v^i by (5.84) and the solution of (5.85), in the second step we determine the boundary as the location of the zeros of the total field $v^i + v^s$.

We have already investigated the operator F in Section 3.3. In particular, by Corollary 3.22, we know that F is injective and has dense range if and only if there does not exist a Dirichlet eigenfunction for D which is a Herglotz wave function. Therefore, for the sequel we will make the restricting assumption that k^2 is not a Dirichlet eigenvalue for the negative Laplacian in the unknown scatterer D.

Now we assume that $\mathbb{R}^3 \setminus D$ is contained in the domain of definition for v^s. In particular, if we choose radiating spherical waves for v^s this means that the origin is contained in D. We associate the following uniquely solvable interior Dirichlet problem

$$\Delta v^i + k^2 v^i = 0 \quad \text{in } D, \tag{5.86}$$

with boundary condition

$$v^i + v^s = 0 \quad \text{on } \partial D \tag{5.87}$$

to the inverse scattering problem. From Theorem 3.26 we know that the solvability of the integral equation (5.85) is connected to this interior boundary value problem, i.e., (5.85) is solvable for $g \in L^2(\mathbb{S}^2)$ if and only if the solution v^i to (5.86), (5.87) is a Herglotz wave function with kernel g. Therefore, the solvability of (5.85) depends on the question of whether or not the solution to the interior Dirichlet problem (5.86), (5.87) can be analytically extended as a Herglotz wave function across the boundary ∂D and this question again cannot be answered in advance for an unknown obstacle D.

We again illustrate the degree of ill-posedness of the equation (5.85) by looking at the singular values in the special case where the scatterer D is the unit ball. Here, from the explicit form (3.32) of the far field pattern and the addition theorem (2.30), we find that the singular values of F are given by

$$\mu_n = \frac{4\pi}{k} \frac{|j_n(k)|}{|h_n^{(1)}(k)|}, \quad n = 0, 1, \ldots,$$

with the asymptotic behavior

$$\mu_n = O\left(\frac{ek}{2n}\right)^{2n}, \quad n \to \infty.$$

This estimate is the square of the corresponding estimate (5.63) for the singular values of S_∞ in the Kirsch–Kress method.

We can again use Tikhonov regularization with regularization parameter α to obtain an approximate solution g_α of (5.85). This then leads to an approximation

v^i_α for the incident wave v^i and we can try to find the boundary of the scatterer D as the set of points where the boundary condition (5.87) is satisfied. We do this by requiring that

$$v^i_\alpha + v^s = 0 \tag{5.88}$$

is satisfied in the minimum norm sense. However, as in the Kirsch–Kress method, for a satisfactory reformulation of the inverse scattering problem as an optimization problem we need to combine a regularization for the integral equation (5.85) and the defect minimization for (5.88) into one cost functional. If we use the standard Tikhonov regularization as in Definition 5.22, i.e., if we use as penalty term $\|g\|^2_{L^2(\mathbb{S}^2)}$, then it is easy to prove results analogous to those of Theorems 5.23 and 5.24. However, we would not be able to obtain a convergence result corresponding to Theorem 5.25. Therefore, we follow Blöhbaum [23] and choose a penalty term for (5.85) as follows. We recall the description of the set U of admissible surfaces from p. 158 and pick a closed C^2 surface Γ_e such that Λ_e is contained in the interior of Γ_e. In addition, without loss of generality, we assume that k^2 is not a Dirichlet eigenvalue for the negative Laplacian in the interior of Γ_e. Then we define the combined cost functional by

$$\mu(g, \Lambda; \alpha) := \|Fg - v_\infty\|^2_{L^2(\mathbb{S}^2)} + \alpha\|Hg\|^2_{L^2(\Gamma_e)} + \gamma\|Hg + v^s\|^2_{L^2(\Lambda)}. \tag{5.89}$$

The coupling parameter γ is again necessary for numerical purposes and for the theory we set $\gamma = 1$. Since the operator H in the penalty term does not have a bounded inverse, we have to slightly modify the notion of an optimal surface.

Definition 5.27 *Given the (measured) far field $u_\infty \in L^2(\mathbb{S}^2 \times \mathbb{S}^2)$ for all incident and observation directions and a regularization parameter $\alpha > 0$, a surface Λ_0 from the compact set U is called* optimal *if*

$$\inf_{g \in L^2(\mathbb{S}^2)} \mu(g, \Lambda_0; \alpha) = m(\alpha)$$

where

$$m(\alpha) := \inf_{g \in L^2(\mathbb{S}^2),\ \Lambda \in U} \mu(g, \Lambda; \alpha).$$

Note that the measured far field u_∞ enters in the operator F through (3.45). For this reformulation of the inverse scattering problem into a nonlinear optimization problem, we have results similar to those for the Kirsch–Kress method. We note that in the original version of their method, Colton and Monk [72, 73] chose the cost functional (5.89) without the penalty term and minimized over all $g \in L^2(\mathbb{S}^2)$ with $\|g\|_{L^2(\mathbb{S}^2)} \le \rho$, i.e., the Tikhonov regularization with regularization parameter $\alpha \to 0$ was replaced by the quasi-solution with regularization parameter $\rho \to \infty$.

Theorem 5.28. *For each $\alpha > 0$ there exists an optimal surface $\Lambda \in U$.*

Proof. Let (g_n, Λ_n) be a minimizing sequence from $L^2(\mathbb{S}^2) \times U$, i.e.,

$$\lim_{n\to\infty} \mu(g_n, \Lambda_n; \alpha) = m(\alpha).$$

Since U is compact, we can assume that $\Lambda_n \to \Lambda \in U$, $n \to \infty$. Because of the boundedness

$$\alpha \|Hg_n\|^2_{L^2(\Gamma_e)} \leq \mu(g_n, \Lambda_n; \alpha) \to m(\alpha), \quad n \to \infty,$$

we can assume that the sequence (Hg_n) is weakly convergent in $L^2(\Gamma_e)$. By Theorem 5.26, applied to the interior of Γ_e, the weak convergence of the boundary data Hg_n on Γ_e then implies that the Herglotz wave functions v_n with kernel g_n converge to a solution v of the Helmholtz equation uniformly on compact subsets of the interior of Γ_e. This, together with $\Lambda_n \to \Lambda \in U$, $n \to \infty$, implies that (indicating the dependence of $H : L^2(\mathbb{S}^2) \to L^2(\Lambda_n)$ on n by writing H_n)

$$\lim_{n\to\infty} \|H_n g_n + v^s\|_{L^2(\Lambda_n)} = \lim_{n\to\infty} \|Hg_n + v^s\|_{L^2(\Lambda)},$$

whence

$$\lim_{n\to\infty} \mu(g_n, \Lambda_n; \alpha) = \lim_{n\to\infty} \mu(g_n, \Lambda; \alpha)$$

follows. This concludes the proof. □

For the following, we assume that u_∞ is the exact far field pattern and first give a reformulation of the integral equation (5.85) for the Herglotz kernel g in terms of the Herglotz function Hg. For this purpose, we recall the operator $A : L^2(\partial D) \to L^2(\mathbb{S}^2)$ from Theorem 3.28 which maps the boundary values of radiating solutions onto their far field pattern and recall that A is bounded and injective. Clearly, by the definition of A, we have $Av^s = v_\infty$ and, by (3.63), we conclude that $AHg = -Fg$ for exact far field data u_∞. Hence

$$v_\infty - Fg = A(Hg + v^s). \tag{5.90}$$

Theorem 5.29. *For all incident directions d, let $u_\infty(\cdot\,, d)$ be the exact far field pattern of a domain D such that ∂D belongs to U. Then we have convergence of the cost functional*

$$\lim_{\alpha\to 0} m(\alpha) = 0.$$

Proof. By Theorem 5.21, given $\varepsilon > 0$ there exists $g \in L^2(\mathbb{S}^2)$ such that

$$\|Hg + v^s\|_{L^2(\partial D)} < \varepsilon.$$

From (5.90) we have

$$\|Fg - v_\infty\|_{L^2(\mathbb{S}^2)} \leq \|A\| \, \|Hg + v^s\|_{L^2(\partial D)}.$$

Therefore, we have

$$\mu(g, \partial D; \alpha) \leq (1 + \|A\|^2)\varepsilon^2 + \alpha\|Hg\|^2_{L^2(\Gamma_e)} \to (1 + \|A\|^2)\varepsilon^2, \quad \alpha \to 0,$$

and the proof is completed as in Theorem 5.24. □

Theorem 5.30. *Let (α_n) be a null sequence and let (Λ_n) be a corresponding sequence of optimal surfaces for the regularization parameter α_n. Then there exists a* *convergent subsequence of* (Λ_n). *Assume that for all incident directions* $u_\infty(\cdot\,, d)$ *is the exact far field pattern of a domain* D *such that* ∂D *belongs to* U. *Assume further that the solution* v^i *to the associated interior Dirichlet problem (5.86), (5.87) can be extended as a solution to the Helmholtz equation across the boundary* ∂D *into the interior of* Γ_e *with continuous boundary values on* Γ_e. *Then every limit point* Λ^* *of* (Λ_n) *represents a surface on which the boundary condition (5.87) is satisfied, i.e.,* $v^i + v^s = 0$ *on* Λ^*.

Proof. The existence of a convergent subsequence of (Λ_n) follows from the compactness of U. Let Λ^* be a limit point. Without loss of generality we can assume that $\Lambda_n \to \Lambda^*$, $n \to \infty$.

By Theorem 5.21, there exists a sequence (g_j) in $L^2(\mathbb{S}^2)$ such that

$$\|Hg_j - v^i\|_{L^2(\Gamma_e)} \to 0, \quad j \to \infty.$$

By Theorem 5.26, this implies the uniform convergence of the Herglotz wave functions with kernel g_j to v^i on compact subsets of the interior of Γ_e, whence in view of the boundary condition for v^i on ∂D we obtain

$$\|Hg_j + v^s\|_{L^2(\partial D)} \to 0, \quad j \to \infty.$$

Therefore, by passing to the limit $j \to \infty$ in

$$m(\alpha) \le \|Fg_j - v_\infty\|^2_{L^2(\mathbb{S}^2)} + \alpha\|Hg_j\|^2_{L^2(\Gamma_e)} + \|Hg_j + v^s\|^2_{L^2(\partial D)},$$

with the aid of (5.90) we find that

$$m(\alpha) \le \alpha\|v^i\|^2_{L^2(\Gamma_e)} \tag{5.91}$$

for all $\alpha > 0$.

By Theorem 5.28, for each n there exists $g_n \in L^2(\mathbb{S}^2)$ such that

$$\|Fg_n - v_\infty\|^2_{L^2(\mathbb{S}^2)} + \alpha_n\|Hg_n\|^2_{L^2(\Gamma_e)} + \|Hg_n + v^s\|^2_{L^2(\Lambda_n)} \le m(\alpha_n) + \alpha_n^2.$$

From this inequality and (5.91), we conclude that

$$\|Hg_n\|^2_{L^2(\Gamma_e)} \le \|v^i\|^2_{L^2(\Gamma_e)} + \alpha_n$$

for all n and therefore we may assume that the sequence (Hg_n) converges weakly in $L^2(\Gamma_e)$. Then, by Theorem 5.26, the Herglotz wave functions v_n with kernels g_n converge uniformly on compact subsets of the interior of Γ_e to a solution v^* of the Helmholtz equation. By Theorem 5.29, we have convergence of the cost functional $m(\alpha_n) \to 0$, $n \to \infty$. In particular, using (5.90), this yields

$$\|A(v_n + v^s)\|^2_{L^2(\mathbb{S}^2)} = \|Fg_n - v_\infty\|^2_{L^2(\mathbb{S}^2)} \le m(\alpha_n) + \alpha_n^2 \to 0, \quad n \to \infty,$$

whence $A(v^* + v^s) = 0$ follows. Since, by Theorem 3.28, the operator A is injective we conclude that $v^* + v^s = 0$ on ∂D, that is, v^i and v^* satisfy the same boundary condition on ∂D. Since k^2 is assumed not to be a Dirichlet eigenvalue for D, v^i and v^* must coincide. Finally, from

$$\|v_n + v^s\|^2_{L^2(\Lambda_n)} \leq m(\alpha_n) + \alpha_n^2 \to 0, \quad n \to \infty,$$

we see that $v^i + v^s = 0$ on Λ^* and the proof is finished. □

For details on the numerical implementation and examples we refer to [72, 73, 74]. A comparison of the numerical performance for the methods of Kirsch and Kress and of Colton and Monk in two dimensions is contained in [191].

5.6 Sampling Methods

The iterative and decomposition methods discussed in the two previous sections, in general, rely on some a priori information for obtaining initial approximations to start the corresponding iterative procedures or to place auxiliary surfaces in the interior of the obstacle. In this final section of this chapter we will outline the main ideas of the so-called sampling methods that do not need any a priori information on the geometry of the obstacle and its physical nature, i.e., on the boundary condition. However, these methods require the knowledge of the far field pattern for a large number of incident waves, whereas the methods of the previous sections, in general, work with one incident field. In this section we will confine ourselves to the case of a sound-soft scatterer, i.e., the Dirichlet boundary condition. But we emphasize that the analysis carries over to other boundary conditions in a way that the numerical implementation of the sampling methods do not require the boundary condition to be known in advance.

Roughly speaking, sampling methods are based on choosing an appropriate indicator function f on $\mathbb{R}^3$ such that its value $f(z)$ decides on whether z lies inside or outside the scatterer D. For Potthast's [279, 281] *singular source method* this indicator function is given by $f(z) := w^s(z, z)$ through the value of the scattered wave $w^s(\cdot\,, z)$ for the singular source $\Phi(\cdot\,, z)$ as incident field evaluated at the source point z. The values $w^s(z, z)$ will be small for points $z \in \mathbb{R}^3 \setminus \bar{D}$ that are away from the boundary and will blow up when z approaches the boundary due to the singularity of the incident field. Clearly, the singular source method can be viewed as a numerical implementation of the uniqueness proof for Theorem 5.6.

Assuming the far field pattern for plane wave incidence to be known for all incident and observation directions, the indicator function $w^s(z, z)$ can be obtained by two applications of the backprojection (5.81) and the mixed reciprocity principle (3.39). Combining (3.39) and (5.81) we obtain the approximation

$$w_\infty(-d, z) = \frac{1}{4\pi}\, u^s(z, d) \approx \frac{1}{4\pi} \int_{\mathbb{S}^2} g_z(\hat{x}) u_\infty(-\hat{x}, d)\, ds(\hat{x}).$$

Inserting this into the backprojection (5.81) as applied to w^s yields the approximation

$$w_s(z,z) \approx \frac{1}{4\pi} \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} g_z(d) g_z(\hat{x}) u_\infty(-\hat{x}, d)\, ds(\hat{x})\, ds(d). \tag{5.92}$$

We explicitly mention that, as opposed to the point source method described in the previous section, for the singular source method the boundary condition does not need to be known. We also note that if we use the reflection operator R from (3.47) the approximation (5.92) can be expressed in terms of the far field operator F by the $L^2(\mathbb{S}^2)$ inner product

$$w_s(z,z) \approx \frac{1}{4\pi}\,(Fg_z, R\overline{g}_z).$$

The *probe method* as suggested by Ikehata [149, 150] uses as indicator function an energy integral for $w^s(\cdot\,, z)$ instead of the point evaluation $w_s(z,z)$. In this sense, it follows the uniqueness proof of Isakov whereas the singular source method mimics the uniqueness proof of Kirsch and Kress.

The *linear sampling method* was first proposed by Colton and Kirsch [61]. Its basic idea is to find a Herglotz wave function v^i with kernel g, i.e., a superposition of plane waves, such that the corresponding scattered wave v^s coincides with a point source $\Phi(\cdot\,, z)$ located at a point z in the interior of the scatterer D. Hence, the decomposition method of Colton and Monk from the previous section may be considered as predecessor to the linear sampling method (see [71]).

In terms of the far field operator F we have to find the kernel g_z as a solution to the integral equation of the first kind

$$Fg_z = \Phi_\infty(\cdot\,, z) \tag{5.93}$$

with the far field

$$\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi}\, e^{-ik\,\hat{x}\cdot z} \tag{5.94}$$

of the fundamental solution $\Phi(\cdot\,, z)$. From Theorem 3.26 we conclude that for any solution g of (5.93) the Herglotz wave function

$$v(x) = \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g_z(d)\, ds(d), \quad x \in \mathbb{R}^3,$$

solves the interior Dirichlet problem

$$\Delta v + k^2 v = 0 \quad \text{in } D \tag{5.95}$$

with boundary condition

$$v + \Phi(\cdot\,, z) = 0 \quad \text{on } \partial D. \tag{5.96}$$

Conversely, if the Herglotz wave function v solves (5.95)–(5.96) then its kernel g_z is a solution of (5.93). Hence, if a solution to the integral equation (5.93) of the first kind exists for all $z \in D$, then from the boundary condition (5.96) for the Herglotz wave function we conclude that $\|g_z\|_{L^2(\mathbb{S}^2)} \to \infty$ as the source point z approaches the boundary ∂D. Therefore, in principle, the boundary ∂D may be found by solving the integral equation (5.93) for z taken from a sufficiently fine grid in $\mathbb{R}^3$ and determining ∂D as the location of those points z where $\|g_z\|_{L^2(\mathbb{S}^2)}$ becomes large.

However, in general, the solution to the interior Dirichlet problem (5.95)–(5.96) will have an extension as a Herglotz wave function across the boundary ∂D only in very special cases (for example if D is a ball with center at z). Hence, the integral equation of the first kind (5.93), in general, will have no solution. Nevertheless, by making use of denseness properties of the Herglotz wave functions, the following mathematical foundation of the linear sampling method can be provided.

To this end, we first present modified versions of the denseness results of Theorems 5.21 and 3.28.

Corollary 5.31 *The Herglotz operator $H : L^2(\mathbb{S}^2) \to H^{1/2}(\partial D)$ defined by*

$$(Hg)(x) := \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d)\,ds(d), \quad x \in \partial D, \tag{5.97}$$

is injective and has dense range provided k^2 is not a Dirichlet eigenvalue for the negative Laplacian for D.

Proof. In view of Theorem 5.21 we only need to be concerned with the denseness of $H(L^2(\mathbb{S}^2))$ in $H^{1/2}(\partial D)$. From (5.97) in view of the duality pairing for $H^{1/2}(\partial D)$ and its dual space $H^{-1/2}(\partial D)$ interchanging the order of integration we observe that analogous to (3.64) for $\varphi \in L^2(\partial D)$ the dual operator $H^\top : H^{-1/2}(\partial D) \to L^2(\mathbb{S}^2)$ of H is given by

$$H^\top \varphi = 2\pi \overline{AS\overline{\varphi}}, \quad \varphi \in H^{-1/2}(\partial D), \tag{5.98}$$

in terms of the boundary data to far field operator $A : H^{1/2}(\partial D) \to L^2(\mathbb{S}^2)$ and the single-layer operator $S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$. Since $L^2(\partial D)$ is dense in $H^{-1/2}(\partial D)$ and A and S are bounded, (5.98) represents the dual operator on $H^{-1/2}(\partial D)$. Both A and S are injective, the latter because of our assumption on k. Hence $H^\top$ is injective and the dense range of H follows by the Hahn–Banach theorem. □

Corollary 5.32 *The operator $A : H^{1/2}(\partial D) \to L^2(\mathbb{S}^2)$ which maps the boundary values of radiating solutions $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ to the Helmholtz equation onto the far field pattern u_∞ is bounded, injective and has dense range.*

Proof. In view of Theorem 3.28 again we only need to be concerned with the denseness of $A(H^{1/2}(\partial D))$ in $L^2(\mathbb{S}^2)$. To this end, from the representation (3.58) of A as an integral operator, we observe that analogous to (3.60) the dual operator $A^\top : L^2(\mathbb{S}^2) \to H^{-1/2}(\partial D)$ of A is given by

$$A^\top g = \overline{A^* \overline{g}}, \quad g \in L^2(\mathbb{S}^2),$$

in terms of the L^2 adjoint A^*. From the proof of Theorem 3.28 we know that A^* is injective. Consequently $A^\top$ is injective and therefore the dense range of A follows. □

We recall the far field pattern (5.94) of the fundamental solution $\Phi(\cdot, z)$ with source point z. Then we we have the following lemma.

Lemma 5.33 $\Phi_\infty(\cdot, z) \in A(H^{1/2}(\partial D))$ *if and only if* $z \in D$.

Proof. If $z \in D$ then clearly $\Phi_\infty(\cdot, z) = A\,(\Phi(\cdot, z)|_{\partial D})$ and $\Phi(\cdot, z)|_{\partial D} \in H^{1/2}(\partial D)$. Conversely, let $z \notin D$ and assume there exists $f \in H^{1/2}(\partial D)$ such that $Af = \Phi_\infty(\cdot, z)$. Then by Rellich's lemma and analyticity the solution u to the exterior Dirichlet problem with boundary trace $u|_{\partial D} = f$ must coincide with $\Phi(\cdot, z)$ in $(\mathbb{R}^3 \setminus \bar{D}) \setminus \{z\}$. If $z \in \mathbb{R}^3 \setminus \bar{D}$ this contradicts the analyticity of u. If $z \in \partial D$ from the boundary condition it follows that $\Phi(\cdot, z)|_{\partial D} \in H^{1/2}(\partial D)$ which is a contradiction to $\Phi(\cdot, z) \notin H_1(D)$ for $z \in \partial D$. □

Theorem 5.34. *Assume that k^2 is not a Dirichlet eigenvalue of the negative Laplacian in D and let F be the far field operator (3.45) for scattering from a sound-soft obstacle. Then the following hold:*

1. *For $z \in D$ and a given $\varepsilon > 0$ there exists a function $g_z^\varepsilon \in L^2(\mathbb{S}^2)$ such that*

$$\|Fg_z^\varepsilon - \Phi_\infty(\cdot\,, z)\|_{L^2(\mathbb{S}^2)} < \varepsilon \tag{5.99}$$

and the Herglotz wave function $v_{g_z^\varepsilon}$ with kernel g_z^ε converges to the solution $w \in H^1(D)$ of the Helmholtz equation with $w + \Phi(\cdot\,, z) = 0$ on ∂D as $\varepsilon \to 0$.

2. *For $z \notin D$ every $g_z^\varepsilon \in L^2(\mathbb{S}^2)$ that satisfies (5.99) for a given $\varepsilon > 0$ is such that*

$$\lim_{\varepsilon \to 0} \|v_{g_z^\varepsilon}\|_{H^1(D)} = \infty.$$

Proof. We note that under the assumption on k well-posedness of the interior Dirichlet problem in the $H^1(D)$ setting can be concluded from Theorem 5.17. Given $\varepsilon > 0$, by Corollary 5.31 we can choose $g_z \in L^2(\mathbb{S}^2)$ such that

$$\|Hg_z^\varepsilon + \Phi(\cdot, z)\|_{H^{1/2}(\partial D)} < \frac{\varepsilon}{\|A\|}$$

where A denotes the boundary data to far field operator from Corollary 5.32. Then (5.99) follows from $F = -AH$, see (3.63). Now if $z \in D$ then by the well-posedness of the Dirichlet problem the convergence $Hg_z^\varepsilon + \Phi(\cdot, z) \to 0$ as $\varepsilon \to 0$ in $H^{1/2}(\partial D)$ implies convergence $v_{g_z^\varepsilon} \to w$ as $\varepsilon \to 0$ in $H^1(D)$ and the first statement is proven.

In order to prove the second statement, for $z \notin D$ assume to the contrary that there exists a null sequence (ε_n) and corresponding Herglotz wave functions v_n with kernels $g_n = g_z^{\varepsilon_n}$ such that $\|v_n\|_{H^1(D)}$ remains bounded. Then without loss of generality we may assume weak convergence $v_n \rightharpoonup v \in H^1(D)$ as $n \to \infty$. Denote by $v^s \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ the solution to the exterior Dirichlet problem for the Helmholtz equation with $v^s = v$ on ∂D and by v_∞ its far field pattern. Since Fg_n is the far field pattern of the scattered wave for the incident field $-v_n$ from (5.99) we conclude that $v_\infty = -\Phi_\infty(\cdot\,, z)$ and therefore $\Phi_\infty(\cdot\,, z)$ in $A(H^{1/2}(\partial D))$. But this contradicts Lemma 5.33. □

From Theorem 5.34 it can be expected that solving the integral equation (5.93) and scanning the values for $\|g_z\|_{L^2(\mathbb{S}^2)}$ will yield an approximation for ∂D through those points where the norm of g is large. A possible procedure with noisy data

$$\|u_{\infty,\delta} - u_\infty\|_{L^2(\mathbb{S}^2\times\mathbb{S}^2)} \le \delta$$

with error level δ is as follows. Denote by F_δ the far field operator F with the kernel u_∞ replaced by the data $u_{\infty,\delta}$. Then for each z from a grid in $\mathbb{R}^3$ determine $g^\delta = g^\delta(\cdot\,, z)$ by minimizing the Tikhonov functional

$$\|F_\delta g^\delta(\cdot\,, z) - \Phi_\infty(\cdot\,, z)\|^2_{L^2(\mathbb{S}^2)} + \alpha\|g^\delta(\cdot\,, z)\|^2_{L^2(\mathbb{S}^2)},$$

where the regularization parameter α is chosen according to Morozov's generalized discrepancy principle for noisy operators as opposed to noisy right hand sides (c.f. Theorem 4.20), i.e., $\alpha = \alpha(z)$ is chosen such that

$$\|F_\delta g^\delta(\cdot\,, z) - \Phi_\infty(\cdot\,, z)\|_{L^2(\mathbb{S}^2)} \approx \delta\|g^\delta(\cdot\,, z)\|_{L^2(\mathbb{S}^2)}.$$

Then the unknown boundary is determined by those points where $\|g^\delta(\cdot\,, z)\|_{L^2(\mathbb{S}^2)}$ sharply increases.

We note that the arguments used to establish Theorem 5.34 do not depend in an essential way on the fact that the obstacle is sound-soft. In particular the conclusion of the theorem remains valid for both the Neumann and impedance boundary conditions as well as for mixed boundary conditions as long as the corresponding interior problem is well posed, i.e., this method for solving the inverse scattering problem does not depend on knowing the boundary conditions a priori. In addition the number of components of the scatterer does not have to be known in advance. For details and numerical examples of this approach to solving the inverse scattering problem we refer the reader to [32].

A problem with the linear sampling method as described above is that, in general, there does not exist a solution of (5.93) for noise free data and hence it is not clear what solution is obtained by using Tikhonov regularization. In particular, it is not clear whether Tikhonov regularization indeed leads to the approximations predicted by the above Theorem 5.34. This question has been addressed and clarified by Arens and Lechleiter [13, 14] using ideas of Kirsch's factorization method. We will resume this issue at the end of this section. Kirsch [183] proposed to replace (5.93) by

$$(F^*F)^{1/4} g_z = \Phi_\infty(\cdot\,, z) \tag{5.100}$$

and was able to completely characterize the range of $(F^*F)^{1/4}$. This method is called the factorization method since it relies on the factorization of the far field operator from Theorem 3.29. As compared to the original paper [183], the theory has been largely modified and extended. For an extensive study we refer to the monograph by Kirsch and Grinberg [186]. We begin our short outline of the basic analysis of the factorization method with one of its main theoretical foundations given by the following optimization theorem from [186].

Theorem 5.35. *Let X and H be Hilbert spaces with inner products $(\cdot,\cdot)$, let X^* be the dual space of X and assume that $F : H \to H$, $B : X \to H$ and $T : X^* \to X$ are bounded linear operators that satisfy*

$$F = BTB^* \tag{5.101}$$

where $B^* : H \to X^*$ is the antilinear adjoint of B defined by

$$\langle \varphi, B^* g\rangle = (B\varphi, g), \quad g \in H,\ \varphi \in X, \tag{5.102}$$

in terms of the bilinear duality pairing of X and X^*. Assume further that

$$|\langle Tf, f\rangle| \geq c\|f\|^2_{X^*} \tag{5.103}$$

for all $f \in B^*(H)$ and some $c > 0$. Then for any $g \in H$ with $g \neq 0$ we have that $g \in B(X)$ if and only if

$$\inf\{|(F\psi,\psi)| : \psi \in H,\ (g,\psi) = 1\} > 0. \tag{5.104}$$

Proof. From (5.101)–(5.103) we obtain that

$$|(F\psi,\psi)| = |\langle TB^*\psi, B^*\psi\rangle| \geq c\|B^*\psi\|^2_{X^*} \tag{5.105}$$

for all $\psi \in H$. Now assume that $g = B\varphi$ for some $\varphi \in X$ and $g \neq 0$. Then for each $\psi \in H$ with $(g,\psi) = 1$ we can estimate

$$c = c|(B\varphi,\psi)|^2 = c|\langle\varphi, B^*\psi\rangle|^2 \leq c\,\|\varphi\|^2_X\,\|B^*\psi\|^2_{X^*} \leq \|\varphi\|^2_X|(F\psi,\psi)|$$

and consequently (5.104) is satisfied.

Conversely let (5.104) be satisfied and assume that $g \notin B(X)$. We define $V := [\mathrm{span}\{g\}]^\perp$ and show that $B^*(V)$ is dense in $\overline{B^*(H)}$. Via the antilinear isomorphism J from the Riesz representation theorem given by

$$\langle\varphi, f\rangle = (\varphi, Jf), \quad \varphi \in X,\ f \in X^*,$$

we can identify $X = J(X^*)$. In particular, then $JB^* : H \to X$ is the Hilbert space adjoint of $B : X \to H$ and it suffices to show that $JB^*(V)$ is dense in $\overline{JB^*(H)}$. Let $\varphi = \lim_{n\to\infty} JB^*\psi_n$ with $\psi_n \in H$ be orthogonal $\varphi \perp JB^*(V)$. Then

$$(B\varphi,\psi) = (\varphi, JB^*\psi) = 0$$

for all $\psi \in V$ and hence $B\varphi \in V^\perp = \mathrm{span}\{g\}$. Since $g \notin B(X)$, this implies $B\varphi = 0$. But then

$$\|\varphi\|^2 = \lim_{n\to\infty}(\varphi, JB^*\psi_n) = \lim_{n\to\infty}(B\varphi,\psi_n) = 0$$

and hence $JB^*(V)$ is dense in $\overline{JB^*(H)}$.

Now we can choose a sequence $(\widetilde{\psi}_n)$ in V such that

$$B^*\widetilde{\psi}_n \to -\frac{1}{\|g\|^2}\, B^* g, \quad n \to \infty.$$

Setting

$$\psi_n := \widetilde{\psi}_n + \frac{1}{\|g\|^2}\, g$$

we have $(g, \psi_n) = 1$ for all n and $B^*\psi_n \to 0$ for $n \to \infty$. Then from the first equation in (5.105) we observe that

$$|(F\psi_n, \psi_n)| \le \|T\|\, \|B^*\psi_n\|_{X^*}^2 \to 0, \quad n \to \infty,$$

which is a contradiction to the assumption that (5.104) is satisfied. Hence g must belong to $B(X)$ and this concludes the proof. □

We note that an equivalent formulation of Theorem 5.35 can be stated without referring to the dual space of the Hilbert space X via the Riesz representation theorem as in the above proof. The corresponding formulation is for a factorization $F = B\widetilde{T}\widetilde{B}^*$ where $\widetilde{T} : X \to X$ and $\widetilde{B}^* : H \to X$ is the Hilbert space adjoint of $B : X \to H$. Both formulations are connected via $\widetilde{B}^* = JB^*$ and $\widetilde{T} := TJ^{-1}$. The condition (5.103) becomes

$$|(\widetilde{T}\varphi, \varphi)| \ge c\|\varphi\|_X^2 \tag{5.106}$$

for all $\varphi \in \widetilde{B}^*(H)$ and some $c > 0$. For the special case where $X = H$ in the sequel, for example in the proof of Theorem 5.39 below, we always will refer to this second variant of Theorem 5.35. We also note that for spaces of complex valued functions Theorem 5.35 remains valid if the bilinear duality pairing of X and X^* is replaced by a sesquilinear pairing.

The following lemma provides a tool for checking the assumption (5.103) in Theorem 5.35.

Lemma 5.36 *In the setting of Theorem 5.35 let $T : X^* \to X$ satisfy*

$$\operatorname{Im}\langle Tf, f\rangle \ne 0 \tag{5.107}$$

for all $f \in \overline{B^*(H)}$ with $f \ne 0$ and be of the form $T = T_0 + C$ where C is compact such that

$$\langle T_0 f, f\rangle \in \mathbb{R} \tag{5.108}$$

and

$$\langle T_0 f, f\rangle \ge c_0\|f\|_{X^*}^2 \tag{5.109}$$

for all $f \in \overline{B^*(H)}$ and some $c_0 > 0$. Then T satisfies (5.103).

Proof. Assume to the contrary that (5.103) is not satisfied. Then there exists a sequence (f_n) in $B^*(H)$ with $\|f_n\| = 1$ for all n and

$$\langle Tf_n, f_n\rangle \to 0, \quad n \to \infty.$$

We can assume that (f_n) converges weakly to some $f \in \overline{B^*(H)}$. From the compactness of C, writing

$$\langle Cf_n, f_n\rangle = \langle Cf_n - Cf, f_n\rangle + \langle Cf, f_n\rangle$$

we observe that

$$\langle Cf_n, f_n\rangle \to \langle Cf, f\rangle, \quad n \to \infty,$$

and consequently

$$\langle T_0 f_n, f_n\rangle \to -\langle Cf, f\rangle, \quad n \to \infty.$$

Taking the imaginary part implies $\operatorname{Im}\langle Cf, f\rangle = 0$ because of assumption (5.108). Therefore $\operatorname{Im}\langle Tf, f\rangle = 0$, whence $f = 0$ by assumption (5.107). This yields $\langle T_0 f_n, f_n\rangle \to 0$ for $n \to \infty$ which contradicts $\|f_n\| = 1$ for all n and the assumption (5.109). □

We will apply Theorem 5.35 to the factorization $F = -2\pi A S^* A^*$ of Theorem 3.29. We choose the spaces $H = L^2(\mathbb{S}^2)$ and $X = H^{1/2}(\partial D)$ with the dual space $H^{-1/2}(\partial D)$. Then we have to establish the assumptions of Theorem 5.35 for the operators $B = A : H^{1/2}(\partial D) \to L^2(\mathbb{S}^2)$, $B^* = A^* : L^2(\mathbb{S}^2) \to H^{-1/2}(\partial D)$ and $T = -2\pi S^* : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ with the adjoints to be understood with respect to the sesquilinear duality pairings in the sense of the inner products on $L^2(\mathbb{S}^2)$ and $L^2(\partial D)$. For this we need the following lemma.

Lemma 5.37 *Assume that k^2 is not a Dirichlet eigenvalue of the negative Laplacian in D. Then*

1. *$S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ is bijective with a bounded inverse.*
2. *$\operatorname{Im}(S\varphi, \varphi) \neq 0$ for all $\varphi \in H^{-1/2}(\partial D)$ with $\varphi \neq 0$.*
3. *Denote by S_i the single-layer operator corresponding to the wave number $k = i$. Then S_i is self-adjoint with respect to $L^2(\partial D)$ and coercive .*
4. *The difference $S - S_i : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ is compact.*

Proof. 1. See Theorem 5.17.

2. Define the single-layer potential u with density $\varphi \in C(\partial D)$ and let B_R be a sufficiently large ball with radius R centered at the origin with exterior unit normal ν. Then from the jump relations and by Green's integral theorem we have that

$$\operatorname{Im}(S\varphi, \varphi) = 2\operatorname{Im}\int_{\partial D} u\left\{\frac{\partial \bar{u}_-}{\partial \nu} - \frac{\partial \bar{u}_+}{\partial \nu}\right\} ds = -2\operatorname{Im}\int_{\partial D_R} u\,\frac{\partial \bar{u}}{\partial \nu}\, ds \tag{5.110}$$

and by a denseness argument this is also true for $\varphi \in H^{-1/2}(\partial D)$. Now assume that $\operatorname{Im}(S\varphi, \varphi) = 0$. Then from (5.110), Theorem 2.13 and analyticity we conclude $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. Therefore, by the trace theorem, $S\varphi = 0$ and consequently $\varphi = 0$.

3. S_i is self-adjoint since its kernel is real and symmetric. For the single-layer potential u, using the Green's theorem as above and the exponential decay at infinity of the fundamental solution for $k = i$ we find that

$$(S_i\varphi, \varphi) = 2\int_{\mathbb{R}^3}\left\{|\operatorname{grad} u|^2 + |u|^2\right\} dx = 2\|u\|^2_{H^1(\mathbb{R}^3)}.$$

The trace theorem and the boundedness of the inverse S_i^{-1} yields

$$(S_i\varphi, \varphi) \geq c\|S_i\varphi\|^2_{H^{1/2}(\partial D)} \geq c_0\|\varphi\|^2_{H^{-1/2}(\partial D)}$$

for all $\varphi \in H^{-1/2}(\partial D)$ and some positive constants c and c_0.

4. As in the proof of Theorem 5.17, this follows from the increased smoothness of the kernel of $S - S_i$ as compared with that of S. □

Now combining Theorems 3.29 and 5.35 and the Lemmas 5.33, 5.36 and 5.37 and noting that $(S^*\varphi, \varphi) = \overline{(S\varphi, \varphi)}$ we arrive at the following characterization of the domain D.

Corollary 5.38 *Let F be the far field operator and assume that k^2 is not a Dirichlet eigenvalue of the negative Laplacian in D. Then $z \in D$ if and only if*

$$\inf\left\{|(F\psi, \psi)| : \psi \in L^2(\mathbb{S}^2),\ (\psi, \Phi_\infty(\cdot, z)) = 1\right\} > 0.$$

This corollary provides a variational method for determining D from a knowledge of the far field pattern u_∞ for all incident and observation directions. However such an approach is very time consuming since it involves solving a minimization problem for every sampling point z. A more efficient approach, and one more closely related to the linear sampling method, is described in the following range identity theorem.

Theorem 5.39. *Let X and H be Hilbert spaces and let the operators F, T and B satisfy the assumptions of Theorem 5.35 with the condition (5.103) replaced by the assumptions of Lemma 5.36. In addition let the operator $F : H \to H$ be compact, injective and assume that $I + i\gamma F$ is unitary for some $\gamma > 0$. Then the ranges $B(X)$ and $(F^*F)^{1/4}(H)$ coincide.*

Proof. First we note that by Lemma 5.36 the operator T satisfies the assumption (5.103) of Theorem 5.35. Since $I + i\gamma F$ is unitary F is normal. Therefore, by the spectral theorem for compact normal operators, there exists a complete set of orthonormal eigenelements $\psi_n \in H$ with corresponding eigenvalues λ_n, $n = 1, 2, \dots$. In particular, the spectral theorem also provides the expansion

$$F\psi = \sum_{n=1}^{\infty} \lambda_n(\psi, \psi_n)\psi_n, \quad \psi \in H. \tag{5.111}$$

From this we observe that F has a second factorization in the form

$$F = (F^*F)^{1/4}\widetilde{F}(F^*F)^{1/4} \tag{5.112}$$

where the operator $(F^*F)^{1/4} : H \to H$ is given by

$$(F^*F)^{1/4}\psi = \sum_{n=1}^{\infty} \sqrt{|\lambda_n|}(\psi, \psi_n)\psi_n, \quad \psi \in H, \tag{5.113}$$

and $\widetilde{F} : H \to H$ is given by

$$\widetilde{F}\psi = \sum_{n=1}^{\infty} \frac{\lambda_n}{|\lambda_n|} (\psi, \psi_n)\psi_n, \quad \psi \in H. \tag{5.114}$$

We will show that $\widetilde{F}$ also satisfies the assumption (5.103) of Theorem 5.35. Then the statement of the theorem follows by applying Theorem 5.35 to both factorizations of F.

Since the operator $I + i\gamma F$ is unitary the eigenvalues λ_n lie on the circle of radius $r := 1/\gamma$ and center ri. We set

$$s_n := \frac{\lambda_n}{|\lambda_n|}, \quad n \in \mathbb{N}, \tag{5.115}$$

and from $|\lambda_n - ri| = r$ and the only accumulation point $\lambda_n \to 0$, $n \to \infty$, we conclude that 1 and -1 are the only possible accumulation points of the sequence (s_n). We will show that 1 is the only accumulation point. To this end we define $\varphi_n \in X^*$ by

$$\varphi_n := \frac{1}{\sqrt{\lambda_n}} B^*\psi_n, \quad n \in \mathbb{N},$$

where the branch of the square root is chosen such that $\operatorname{Im}\sqrt{\lambda_n} > 0$. Then from $BTB^*\psi_n = F\psi_n = \lambda_n\psi_n$ we readily observe that

$$(T\varphi_n, \varphi_n) = s_n, \quad n \in \mathbb{N}. \tag{5.116}$$

Consequently, since T satisfies the assumption (5.103) of Theorem 5.35, we can estimate

$$c\|\varphi_n\|^2 \le |(T\varphi_n, \varphi_n)| = |s_n| = 1$$

for all $n \in \mathbb{N}$ and some positive constant c, that is, the sequence (φ_n) is bounded.

Now we assume that -1 is an accumulation point of the sequence (s_n). Then, by the boundedness of the sequence (φ_n), without loss of generalization, we may assume that $s_n \to -1$ and $\varphi_n \rightharpoonup \varphi \in X^*$ for $n \to \infty$. From (5.116) we then have that

$$(T_0\varphi_n, \varphi_n) + (C\varphi_n, \varphi_n) = (T\varphi_n, \varphi_n) \to -1, \quad n \to \infty, \tag{5.117}$$

and the compactness of C implies

$$C\varphi_n \to C\varphi, \quad n \to \infty.$$

Consequently

$$|(C\varphi_n - C\varphi, \varphi_n)| \le \|C\varphi_n - C\varphi\| \, \|\varphi_n\| \to 0, \quad n \to \infty,$$

whence

$$(C\varphi_n, \varphi_n) \to (C\varphi, \varphi), \quad n \to \infty,$$

follows. By taking the imaginary part of (5.117), this now leads to $\mathrm{Im}(T\varphi, \varphi) = \mathrm{Im}(C\varphi, \varphi) = 0$ and therefore $\varphi = 0$ by the assumptions of the theorem. Then (5.117) implies

$$(T_0\varphi_n, \varphi_n) \to -1, \quad n \to \infty,$$

and this contradicts the coercivity of T_0.

Now we can write $s_n = e^{it_n}$ where $0 \le t_n \le \pi - 2\delta$ for all $n \in \mathbb{N}$ and some $0 < \delta \le \pi/2$. Then

$$\mathrm{Im}\left\{e^{i\delta} s_n\right\} \ge \sin\delta, \quad n \in \mathbb{N},$$

and using $|(\widetilde{F}\psi, \psi)| = |e^{i\delta}(\widetilde{F}\psi, \psi)|$ we can estimate

$$|(\widetilde{F}\psi, \psi)| \ge \mathrm{Im} \sum_{n=1}^{\infty} e^{i\delta} s_n \, |(\psi, \psi_n)|^2 \ge \sin\delta \sum_{n=1}^{\infty} |(\psi, \psi_n)|^2 = \sin\delta \, \|\psi\|^2$$

for all $\psi \in H$ and the proof is finished. □

Putting Corollary 3.25, Lemma 5.33 and Theorem 5.39 together we arrive at the following final characterization of the scatterer D.

Corollary 5.40 *Let F be the far field operator and assume that k^2 is not a Dirichlet eigenvalue of the negative Laplacian in D. Then $z \in D$ if and only if*

$$(F^*F)^{1/4} g_z = \Phi_\infty(\cdot\,, z) \tag{5.118}$$

is solvable in $L^2(\mathbb{S}^2)$.

This explicit characterization of the scatterer in terms of the range of $(F^*F)^{1/4}$ can be used for a reconstruction with the aid of a singular system $(|\lambda_n|, \psi_n, \psi_n)$ of the operator F. Then, by Picard's Theorem 4.8, we have that $z \in D$ if and only if

$$\sum_{n=1}^{\infty} \frac{|(\psi_n, \Phi_\infty(\cdot\,, z))|^2}{|\lambda_n|} < \infty. \tag{5.119}$$

At first glance Corollary 5.40 seems to imply that the nonlinear inverse problem has been completely solved through a linear problem. However, determining a singular system of F is nonlinear and there is still a nonlinear problem involved for finding those points z where (5.119) is satisfied. Of course an obvious way to approximately solve this nonlinear problem is by truncating the series (5.119) through a finite sum for z on a grid in $\mathbb{R}^3$ and determining ∂D as the location of those points z where this sum becomes large.

We also note that the norm $\|g_z\|_{L^2(\mathbb{S}^2)}$ of the solution to (5.118) tends to infinity as z approaches ∂D. Assume to the contrary that $\|g_{z_n}\|_{L^2(\mathbb{S}^2)}$ remains bounded for a sequence (z_n) in D with $z_n \to z \in \partial D$ for $n \to \infty$. Then without loss of generality we may assume weak convergence $g_{z_n} \rightharpoonup g_z \in L^2(\mathbb{S}^2)$ as $n \to \infty$. The compactness of $(F^*F)^{1/4}$ implies

$$(F^*F)^{1/4}g_z = \lim_{n\to\infty}(F^*F)^{1/4}g_{z_n} = \lim_{n\to\infty}\Phi_\infty(\cdot\,, z_n) = \Phi_\infty(\cdot\,, z)$$

i.e., we have a contradiction.

We emphasize that in the derivation of Corollary 5.40 the scatterer is not required to be connected, i.e., it may consist of a finite number of components that does not need to be known in advance. Furthermore, for the application of the factorization method it is not necessary to know whether the scatterer is sound-soft or sound-hard. Using the above tools it can be proven that Corollary 5.40 is also valid for sound-hard scatterers with the obvious modification in the assumption that k^2 is not a Neumann eigenvalue of the negative Laplacian in D (see [186]). For numerical examples of the implementation of the factorization method in these cases also see [186].

However, for the case of the impedance boundary condition the far field operator F no longer is normal nor is the scattering operator S unitary, i.e., Theorem 5.39 cannot be applied in this case. For modifications of the factorization method and Corollary 5.40 that consider the case where F is not normal, for example the case of an impedance scatterer, we refer again to Kirsch and Grinberg [186].

We conclude this section by using Corollary 5.40 for a rigorous justification of the linear sampling method provided by Arens and Lechleiter [13, 14].

Theorem 5.41. *Under the assumptions of Theorem 5.39 on the operator $F : H \to H$, for $\alpha > 0$ let g_α denote the Tikhonov regularized solution of the equation $Fg = \varphi$ for $\varphi \in H$, i.e., the solution of*

$$\alpha g_\alpha + F^*Fg_\alpha = F^*\varphi.$$

If $\varphi \in (F^*F)^{1/4}(H)$, that is, $\varphi = (F^*F)^{1/4}g$ for some $g \in H$ then $\lim_{\alpha\to 0}(g_\alpha, \varphi)$ exists and

$$c\,\|g\|^2 \le \lim_{\alpha\to 0}|(g_\alpha, \varphi)| \le \|g\|^2 \tag{5.120}$$

for some $c > 0$ depending only on F. If $\varphi \notin (F^*F)^{1/4}(H)$ then $\lim_{\alpha\to 0}|(g_\alpha, \varphi)| = \infty$.

Proof. The expansion (5.111) implies

$$F^*\psi = \sum_{n=1}^{\infty}\bar{\lambda}_n(\psi, \psi_n)\psi_n, \quad \psi \in H,$$

and consequently we have that

$$g_\alpha = \sum_{n=1}^{\infty}\frac{\bar{\lambda}_n}{\alpha + |\lambda_n|^2}\,(\varphi, \psi_n)\psi_n$$

and

$$(g_\alpha, \varphi) = \sum_{n=1}^{\infty}\frac{\bar{\lambda}_n}{\alpha + |\lambda_n|^2}\,|(\varphi, \psi_n)|^2. \tag{5.121}$$

If $\varphi = (F^*F)^{1/4}g$ for some $g \in H$ then

$$(\varphi, \psi_n) = ((F^*F)^{1/4}g, \psi_n) = (g, (F^*F)^{1/4}\psi_n) = \sqrt{|\lambda_n|}\,(g, \psi_n)$$

whence

$$(g_\alpha, \varphi) = \sum_{n=1}^{\infty} \frac{\bar{\lambda}_n\,|\lambda_n|}{\alpha + |\lambda_n|^2}\,|(g, \psi_n)|^2 \tag{5.122}$$

follows. Proceeding as in the proof of Theorem 4.9, from (5.122) convergence

$$\lim_{\alpha\to 0}(g_\alpha, \varphi) = \sum_{n=1}^{\infty} \bar{s}_n\,|(g, \psi_n)|^2 \tag{5.123}$$

with the complex numbers s_n defined in (5.115) can be established. By Parseval's equality, (5.122) implies $|(g_\alpha, \varphi)| \le \|g\|^2$ and the second inequality in (5.120) is obvious. For $g \neq 0$, by Parseval's equality from (5.123) we observe that

$$\frac{1}{\|g\|^2}\,\lim_{\alpha\to 0}(g_\alpha, \varphi)$$

belongs to the closure M of the convex hull of $\{\bar{s}_n : n \in \mathbb{N}\} \subset \mathbb{C}$. From the proof of Theorem 5.39 we know that the s_n lie on the upper half circle $\{e^{it} : 0 \le t \le \pi - 2\delta\}$ for some $0 < \delta \le \pi/2$. This implies that the set M has a positive lower bound c depending on the operator F and this proves the first inequality in (5.120).

Conversely, assume that $\lim_{\alpha\to 0}(g_\alpha, \varphi)$ exists. Then from (5.121) we have that

$$\left|\sum_{n=1}^{\infty} \frac{\lambda_n}{\alpha + |\lambda_n|^2}\,|(\varphi, \psi_n)|^2\right| \le C \tag{5.124}$$

for all $\alpha > 0$ and some $C > 0$. Since 1 is the only accumulation point of the sequence (s_n) there exists $n_0 \in \mathbb{N}$ such that $\operatorname{Re}\lambda_n \ge 0$ for all $n \ge n_0$. From (5.124) and the triangle inequality it follows that

$$\left|\sum_{n=n_0}^{\infty} \frac{\lambda_n}{\alpha + |\lambda_n|^2}\,|(\varphi, \psi_n)|^2\right| \le C_1$$

for all $\alpha > 0$ and some $C_1 > 0$, because the remaining finite sum is bounded. From this we can estimate

$$\sum_{n=n_0}^{\infty} \frac{|\lambda_n|}{\alpha + |\lambda_n|^2}\,|(\varphi, \psi_n)|^2 \le \sum_{n=n_0}^{\infty} \frac{\operatorname{Re}\lambda_n + \operatorname{Im}\lambda_n}{\alpha + |\lambda_n|^2}\,|(\varphi, \psi_n)|^2$$

$$\le \sqrt{2}\left|\sum_{n=n_0}^{\infty} \frac{\lambda_n}{\alpha + |\lambda_n|^2}\,|(\varphi, \psi_n)|^2\right| \le \sqrt{2}\,C_1.$$

Proceeding as in the proof of Theorem 4.9 we can pass to the limit $\alpha \to 0$ and conclude that the series

$$\sum_{n=n_0}^{\infty} \frac{1}{|\lambda_n|} \, |(\varphi, \psi_n)|^2$$

converges. Therefore by Picard's Theorem 4.8 the equation $(F^*F)^{1/4} g = \varphi$ has a solution $g \in H$ and this concludes the proof of the second statement. □

Corollary 5.42 *Let F be the far field operator and assume that k^2 is not a Dirichlet eigenvalue of the negative Laplacian in D. For $z \in D$ denote by g_z the solution of $(F^*F)^{1/4} g_z = \Phi_\infty(\cdot, z)$ and for $\alpha > 0$ and $z \in \mathbb{R}^3$ let g_z^α denote the solution of the far field equation (5.93) obtained by Tikhonov regularization, i.e., the solution of*

$$\alpha g_z^\alpha + F^* F g_z^\alpha = F^* \Phi_\infty(\cdot, z)$$

and let $v_{g_z^\alpha}$ denote the Herglotz wave function with kernel g_z^α. If $z \in D$ then $\lim_{\alpha \to 0} v_{g_z^\alpha}(z)$ exists and

$$c \, \|g_z\|^2 \leq \lim_{\alpha \to 0} |v_{g_z^\alpha}(z)| \leq \|g_z\|^2 \tag{5.125}$$

for some positive c depending only on D. If $z \notin D$ then $\lim_{\alpha \to 0} v_{g_z^\alpha}(z) = \infty$.

Proof. Observing that $v_{g_z^\alpha}(z) = (g_z^\alpha, \Phi_\infty(\cdot, z))_{L^2(\mathbb{S}^2)}$ the statement follows from Theorem 5.41 and Corollary 5.40. □

As pointed out above, the norm $\|g_z\|_{L^2(\mathbb{S}^2)}$ of the solution to (5.119) tends to infinity as $z \to \partial D$. Therefore, in view of (5.125) also the limit $\lim_{\alpha \to 0} |v_{g_z^\alpha}(z)|$ tends to infinity when z approaches the boundary, i.e., the main feature of the linear sampling method is verified.

In addition to the above mathematical justification of the linear sampling method provided by Arens and Lechleiter there have been a number of other attempts to justify the linear sampling method on either physical or mathematical grounds (c.f. [12, 44, 129]).

Chapter 6
The Maxwell Equations

Up until now, we have considered only the direct and inverse obstacle scattering problem for time-harmonic acoustic waves. In the following two chapters, we want to extend these results to obstacle scattering for time-harmonic electromagnetic waves. As in our analysis on acoustic scattering, we begin with an outline of the solution of the direct problem.

After a brief discussion of the physical background of electromagnetic wave propagation, we derive the Stratton–Chu representation theorems for solutions to the Maxwell equations in a homogeneous medium. We then introduce the Silver–Müller radiation condition, show its connection with the Sommerfeld radiation condition and introduce the electric and magnetic far field patterns. The next section then extends the jump relations and regularity properties of surface potentials from the acoustic to the electromagnetic case both for Hölder spaces and Sobolev spaces. For their appropriate presentation, we find it useful to introduce a weak formulation of the notion of a surface divergence and a surface curl of tangential vector fields.

We then proceed to solve the electromagnetic scattering problem for the perfect conductor boundary condition. Our approach differs from the treatment of the Dirichlet problem in acoustic scattering since we start with a formulation requiring Hölder continuous boundary regularity for both the electric and the magnetic field. We then obtain a solution under the weaker regularity assumption of continuity of the electric field up to the boundary and also in Sobolev spaces.

For orthonormal expansions of radiating electromagnetic fields and their far field patterns, we need to introduce vector spherical harmonics and vector spherical wave functions as the analogues of the spherical harmonics and spherical wave functions. Here again, we deviate from the route taken for acoustic waves. In particular, in order to avoid lengthy manipulations with special functions, we use the results on the well-posedness of the direct obstacle scattering problem to justify the convergence of the expansions with respect to vector spherical wave functions.

The last section of this chapter presents reciprocity relations for electromagnetic waves and completeness results for the far field patterns corresponding to the scattering of electromagnetic plane waves with different incident directions and

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences 93, DOI 10.1007/978-1-4614-4942-3_6,

polarizations. For this, and for later use in the analysis of the inverse problem, we need to examine Herglotz wave functions for electromagnetic waves and also the electromagnetic far field operator.

For the Maxwell equations, we only need to be concerned with the study of three-dimensional problems since the two-dimensional case can be reduced to the two-dimensional Helmholtz equation. In order to numerically solve the boundary value problem for a three-dimensional perfect conductor we suggest using an obvious extension of Wienert's Nyström method described for the three-dimensional Helmholtz equation in Section 3.6 applied to the cartesian components of the unknown tangential field (see [100, 269]). For algorithms that use vector spherical harmonics as discussed in Section 6.5 and reduce the number of unknowns by one third we refer to Ganesh and Hawkins [101, 102], see also Pieper [271].

6.1 Electromagnetic Waves

Consider electromagnetic wave propagation in an isotropic medium in $\mathbb{R}^3$ with space independent electric permittivity ε, magnetic permeability μ and electric conductivity σ. The electromagnetic wave is described by the electric field $\mathcal{E}$ and the magnetic field $\mathcal{H}$ satisfying the *Maxwell equations*

$$\operatorname{curl}\mathcal{E} + \mu\,\frac{\partial\mathcal{H}}{\partial t} = 0,$$

$$\operatorname{curl}\mathcal{H} - \varepsilon\,\frac{\partial\mathcal{E}}{\partial t} = \sigma\mathcal{E}.$$

For time-harmonic electromagnetic waves of the form

$$\mathcal{E}(x,t) = \operatorname{Re}\left\{\left(\varepsilon + \frac{i\sigma}{\omega}\right)^{-1/2} E(x)\,e^{-i\omega t}\right\},$$

$$\mathcal{H}(x,t) = \operatorname{Re}\left\{\mu^{-1/2} H(x)\,e^{-i\omega t}\right\}$$

with frequency $\omega > 0$, we deduce that the complex valued space dependent parts E and H satisfy the *reduced Maxwell equations*

$$\begin{aligned} \operatorname{curl} E - ikH &= 0, \\ \operatorname{curl} H + ikE &= 0 \end{aligned} \tag{6.1}$$

where the wave number k is a constant given by

$$k^2 = \left(\varepsilon + \frac{i\sigma}{\omega}\right)\mu\omega^2$$

with the sign of k chosen such that $\operatorname{Im} k \geq 0$. The equations carry the name of the physicist James Clerk Maxwell (1831 – 1879) for his fundamental contributions to electromagnetic theory.

We will consider the scattering of time-harmonic waves by obstacles surrounded by a homogeneous medium with vanishing conductivity $\sigma = 0$, that is, with exterior boundary value problems for the Maxwell equations with a positive wave number. As in the case of acoustic waves, studying the Maxwell equations with constant coefficients is also a prerequisite for studying electromagnetic wave scattering by an inhomogeneous medium.

As for the Helmholtz equation, in electromagnetic obstacle scattering we also must distinguish between the two cases of impenetrable and penetrable objects. For a perfectly conducting obstacle, the tangential component of the electric field of the total wave vanishes on the boundary. Consider the scattering of a given incoming wave E^i, H^i by a perfect conductor D. Then the total wave $E = E^i + E^s$, $H = H^i + H^s$ where E^s, H^s denotes the scattered wave must satisfy the Maxwell equations in the exterior $\mathbb{R}^3 \setminus \bar{D}$ of D and the *perfect conductor* boundary condition $\nu \times E = 0$ on ∂D where ν is the unit outward normal to the boundary ∂D. The scattering by an obstacle that is not perfectly conducting but that does not allow the electromagnetic wave to penetrate deeply into the obstacle is modeled by an *impedance boundary condition* of the form

$$\nu \times \operatorname{curl} E - i\lambda\,(\nu \times E) \times \nu = 0 \quad \text{on } \partial D$$

with a positive constant λ. Throughout this book, for two vectors a and b in $\mathbb{R}^3$ or $\mathbb{C}^3$ we will denote the vector product by $a \times b$.

The scattering by a penetrable obstacle D with constant electric permittivity ε_D, magnetic permeability μ_D and electric conductivity σ_D differing from the electric permittivity ε, magnetic permeability μ and electric conductivity $\sigma = 0$ of the surrounding medium $\mathbb{R}^3 \setminus \bar{D}$ leads to a transmission problem. Here, in addition to the superposition of the incoming wave and the scattered wave in $\mathbb{R}^3 \setminus \bar{D}$ satisfying the Maxwell equations with wave number $k^2 = \varepsilon\mu\omega^2$, we also have a transmitted wave in D satisfying the Maxwell equations with wave number $k_D^2 = (\varepsilon_D + i\sigma_D/\omega)\mu_D\omega^2$. The continuity of the tangential components of the electric field $\mathcal{E}$ and the magnetic field $\mathcal{H}$ across the interface leads to transmission conditions on ∂D. In addition to the transmission conditions, more general *resistive boundary conditions* and *conductive boundary conditions* have also been introduced. For their description and treatment we refer to [9].

As in the treatment of acoustic waves, we will consider in detail only one boundary condition namely that of a perfect conductor. For more details on the physical background of electromagnetic waves, we refer to Jones [168], Müller [256] and van Bladel [319].

6.2 Green's Theorem and Formula

We start with a brief outline of some basic properties of solutions to the time-harmonic Maxwell equations (6.1) with positive wave number k. We first note the vector form of Green's integral theorems. Let D be a bounded domain of class C^1 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D. Then, for $E \in C^1(\bar{D})$ and $F \in C^2(\bar{D})$, we have *Green's first vector theorem*

$$\begin{aligned} &\int_D \{E \cdot \Delta F + \operatorname{curl} E \cdot \operatorname{curl} F + \operatorname{div} E \operatorname{div} F\} \, dx \\ &\qquad = \int_{\partial D} \{\nu \times E \cdot \operatorname{curl} F + \nu \cdot E \operatorname{div} F\} \, ds \end{aligned} \tag{6.2}$$

and for $E, F \in C^2(\bar{D})$ we have *Green's second vector theorem*

$$\begin{aligned} &\int_D \{E \cdot \Delta F - F \cdot \Delta E\} \, dx \\ &= \int_{\partial D} \{\nu \times E \cdot \operatorname{curl} F + \nu \cdot E \operatorname{div} F - \nu \times F \cdot \operatorname{curl} E - \nu \cdot F \operatorname{div} E\} \, ds. \end{aligned} \tag{6.3}$$

Both of these integral theorems follow easily from the Gauss divergence integral theorem applied to $E \times \operatorname{curl} F + E \operatorname{div} F$ with the aid of the vector identities

$$\operatorname{div} uE = \operatorname{grad} u \cdot E + u \operatorname{div} E$$

and

$$\operatorname{div} E \times F = \operatorname{curl} E \cdot F - E \cdot \operatorname{curl} F$$

for continuously differentiable scalars u and vector fields E and F and

$$\operatorname{curl} \operatorname{curl} E = -\Delta E + \operatorname{grad} \operatorname{div} E \tag{6.4}$$

for twice continuously differentiable vector fields E. We also note the formula $\operatorname{curl} uE = \operatorname{grad} u \times E + u \operatorname{curl} E$ for later use.

Recalling the fundamental solution to the Helmholtz equation

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y,$$

we now prove a basic representation theorem for vector fields due to Stratton and Chu [310].

Theorem 6.1. *Let D be a bounded domain of class C^2 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D. For vector fields $E, H \in C^1(D) \cap C(\bar{D})$ we have the* Stratton–Chu formula

$$
\begin{aligned}
E(x) = &- \operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y) \\
&+ \operatorname{grad} \int_{\partial D} \nu(y) \cdot E(y)\, \Phi(x,y)\, ds(y) \\
&-ik \int_{\partial D} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y) \\
&+ \operatorname{curl} \int_D \{\operatorname{curl} E(y) - ikH(y)\}\, \Phi(x,y)\, dy \\
&- \operatorname{grad} \int_D \operatorname{div} E(y)\, \Phi(x,y)\, dy \\
&+ik \int_D \{\operatorname{curl} H(y) + ikE(y)\}\, \Phi(x,y)\, dy, \quad x \in D,
\end{aligned}
\tag{6.5}
$$

where the volume integrals exist as improper integrals. A similar formula holds with the roles of E and H interchanged.

Proof. We first assume that $E, H \in C^1(\bar{D})$. We circumscribe the arbitrary fixed point $x \in D$ with a sphere $S(x;\rho) := \{y \in \mathbb{R}^3 : |x-y| = \rho\}$ contained in D and direct the unit normal ν to $S(x;\rho)$ into the interior of $S(x;\rho)$. From the relation $\operatorname{grad}_x \Phi(x,y) = -\operatorname{grad}_y \Phi(x,y)$ for vector fields $E, H \in C^1(\bar{D})$, we have

$$
\begin{aligned}
\operatorname{curl}_y \{\Phi(x,y)\, E(y)\} &= \Phi(x,y) \operatorname{curl} E(y) - \operatorname{curl}_x \{\Phi(x,y)\, E(y)\}, \\
\operatorname{div}_y \{\Phi(x,y)\, E(y)\} &= \Phi(x,y) \operatorname{div} E(y) - \operatorname{div}_x \{\Phi(x,y)\, E(y)\}, \\
\operatorname{curl}_y \{\Phi(x,y)\, H(y)\} &= \Phi(x,y) \operatorname{curl} H(y) - \operatorname{curl}_x \{\Phi(x,y)\, H(y)\}
\end{aligned}
$$

for $x \neq y$. Taking curl_x of the first equation, $-\operatorname{grad}_x$ of the second equation, multiplying the third equation by ik and adding the resulting three equations with the aid of (6.4) now gives

$$
\begin{aligned}
&\operatorname{curl}_x \operatorname{curl}_y \{\Phi(x,y)\, E(y)\} - \operatorname{grad}_x \operatorname{div}_y \{\Phi(x,y)\, E(y)\} + ik \operatorname{curl}_y \{\Phi(x,y)\, H(y)\} \\
&= \operatorname{curl}_x \{\Phi(x,y)\, [\operatorname{curl} E(y) - ikH(y)]\} - \operatorname{grad}_x \{\Phi(x,y)\ \operatorname{div} E(y)\} \\
&\qquad + ik\Phi(x,y)\, \{\operatorname{curl} H(y) + ikE(y)\}.
\end{aligned}
$$

Integrating this identity over the domain $D_\rho := \{y \in D : |x - y| > \rho\}$ and interchanging differentiation and integration, the Gauss integral theorem yields

$$
\begin{aligned}
&\operatorname{curl} \int_{\partial D \cup S(x;\rho)} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y) \\
&- \operatorname{grad} \int_{\partial D \cup S(x;\rho)} \nu(y) \cdot E(y)\, \Phi(x,y)\, ds(y) \\
&+ ik \int_{\partial D \cup S(x;\rho)} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y) \\
= &\operatorname{curl} \int_{D_\rho} \{\operatorname{curl} E(y) - ikH(y)\}\, \Phi(x,y)\, dy \\
&- \operatorname{grad} \int_{D_\rho} \operatorname{div} E(y)\, \Phi(x,y)\, dy \\
&+ ik \int_{D_\rho} \{\operatorname{curl} H(y) + ikE(y)\}\, \Phi(x,y)\, dy.
\end{aligned}
\tag{6.6}
$$

Since on $S(x;\rho)$ we have

$$
\Phi(x,y) = \frac{e^{ik\rho}}{4\pi\rho}, \quad \operatorname{grad}_x \Phi(x,y) = -\left(\frac{1}{\rho} - ik\right) \frac{e^{ik\rho}}{4\pi\rho}\, \nu(y),
$$

straightforward calculations show that

$$
\begin{aligned}
&\operatorname{curl}_x \{\nu(y) \times E(y)\, \Phi(x,y)\} - \operatorname{grad}_x \{\nu(y) \cdot E(y)\, \Phi(x,y)\} \\
&\qquad = \frac{E(y)}{4\pi\rho^2} + O\left(\frac{1}{\rho}\right), \quad \rho \to 0.
\end{aligned}
\tag{6.7}
$$

Passing to the limit, with the help of the mean value theorem, it now follows from (6.7) that

$$
\begin{aligned}
&\operatorname{curl} \int_{S(x;\rho)} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y) - \operatorname{grad} \int_{S(x;\rho)} \nu(y) \cdot E(y)\, \Phi(x,y)\, ds(y) \\
&\qquad + ik \int_{S(x;\rho)} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y) \to E(x), \quad \rho \to 0,
\end{aligned}
$$

and (6.5) is obtained from (6.6).

The case where E and H only belong to $C^1(D) \cap C(\bar{D})$ is treated by first integrating over parallel surfaces to the boundary of D and then passing to the limit ∂D. □

In the case when E, H solve the Maxwell equations, it is convenient to transform the remaining boundary terms as described in the following theorem.

Theorem 6.2. *Let D be as in Theorem 6.1 and let $E, H \in C^1(D) \cap C(\bar{D})$ be a solution to the Maxwell equations*

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0 \quad \text{in } D.$$

Then we have the Stratton–Chu formulas

$$\begin{aligned} E(x) = &- \operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y) \\ &+ \frac{1}{ik} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y), \quad x \in D, \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} H(x) = &- \operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y) \\ &- \frac{1}{ik} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y), \quad x \in D. \end{aligned} \tag{6.9}$$

Proof. From

$$\operatorname{div}_x \{\nu(y) \times H(y)\, \Phi(x,y)\} = \nu(y) \cdot \operatorname{curl}_y \{H(y)\, \Phi(x,y)\} - \Phi(x,y)\, \nu(y) \cdot \operatorname{curl} H(y),$$

with the help of the Stokes theorem and the second Maxwell equation we see that

$$\operatorname{div} \int_{\partial D} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y) = ik \int_{\partial D} \nu(y) \cdot E(y)\, \Phi(x,y)\, ds(y).$$

Hence, with the aid of (6.4), we have

$$\begin{aligned} &\frac{1}{ik} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y) \\ &= -ik \int_{\partial D} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y) + \operatorname{grad} \int_{\partial D} \nu(y) \cdot E(y)\, \Phi(x,y)\, ds(y). \end{aligned} \tag{6.10}$$

Equation (6.8) now follows by inserting the Maxwell equations into (6.5) and using (6.10). Finally, the representation (6.9) follows from (6.8) by using $H = \operatorname{curl} E / ik$. □

Theorem 6.2 obviously remains valid for complex values of the wave number k. From the proofs of Theorems 6.1 and 6.2, it can also be seen that the identities

$$\begin{aligned}&\operatorname{curl}\int_{\partial D}\nu(y)\times E(y)\,\Phi(x,y)\,ds(y)\\&-\frac{1}{ik}\operatorname{curl}\operatorname{curl}\int_{\partial D}\nu(y)\times H(y)\,\Phi(x,y)\,ds(y)=0,\quad x\in\mathbb{R}^3\setminus\bar{D},\end{aligned}\tag{6.11}$$

and

$$\begin{aligned}&\operatorname{curl}\int_{\partial D}\nu(y)\times H(y)\,\Phi(x,y)\,ds(y)\\&+\frac{1}{ik}\operatorname{curl}\operatorname{curl}\int_{\partial D}\nu(y)\times E(y)\,\Phi(x,y)\,ds(y)=0,\quad x\in\mathbb{R}^3\setminus\bar{D},\end{aligned}\tag{6.12}$$

are valid since for $x \in \mathbb{R}^3 \setminus \bar{D}$ the integrands are twice continuously differentiable in D.

Analogous to Theorem 2.2, we now can state the following theorem.

Theorem 6.3. *Any continuously differentiable solution to the Maxwell equations has analytic cartesian components.*

In particular, the cartesian components of solutions to the Maxwell equations are automatically two times continuously differentiable. Therefore, we can employ the vector identity (6.4) to prove the following result.

Theorem 6.4. *Let E, H be a solution to the Maxwell equations. Then E and H are divergence free and satisfy the* vector Helmholtz equation

$$\Delta E + k^2 E = 0 \quad \textit{and} \quad \Delta H + k^2 H = 0.$$

Conversely, let E (or H) be a solution to the vector Helmholtz equation satisfying $\operatorname{div} E = 0$ *(or* $\operatorname{div} H = 0$*). Then E and $H := \operatorname{curl} E/ik$ (or H and $E := -\operatorname{curl} H/ik$) satisfy the Maxwell equations.*

The following theorem extends Holmgren's Theorem 2.3 to the Maxwell equations.

Theorem 6.5. *Let D be as in Theorem 6.1 and let $E, H \in C^1(D)\cap C(\bar{D})$ be a solution to the Maxwell equations in D such that*

$$\nu \times E = \nu \times H = 0 \quad \textit{on } \Gamma \tag{6.13}$$

for some open subset $\Gamma \subset \partial D$. Then E and H vanish identically in D.

Proof. In view of (6.13), we use the Stratton–Chu formulas (6.8) and (6.9) to extend the definition of E and H by setting

$$E(x) := -\operatorname{curl} \int_{\partial D \setminus \Gamma} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y)$$
$$+ \frac{1}{ik} \operatorname{curl}\operatorname{curl} \int_{\partial D \setminus \Gamma} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y),$$

$$H(x) := \frac{1}{ik} \operatorname{curl} E(x)$$

for $x \in (\mathbb{R}^3 \setminus \bar{D}) \cup \Gamma$. Then, by (6.11) we have $E = H = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. By G we denote a component of $\mathbb{R}^3 \setminus \bar{D}$ with $\Gamma \cap \partial G \neq \emptyset$. Clearly E, H solves the Maxwell equations in $(\mathbb{R}^3 \setminus \partial D) \cup \Gamma$ and therefore $E = H = 0$ in D, since D and G are connected through the gap Γ. □

We now formulate the Silver–Müller radiation conditions (see Müller [252] and Silver [306]) as the counterpart of the Sommerfeld radiation condition for electromagnetic waves.

Definition 6.6 *A solution E, H to the Maxwell equations whose domain of definition contains the exterior of some sphere is called* radiating *if it satisfies one of the* Silver–Müller radiation conditions

$$\lim_{r\to\infty} (H \times x - rE) = 0 \tag{6.14}$$

or

$$\lim_{r\to\infty} (E \times x + rH) = 0 \tag{6.15}$$

where $r = |x|$ and where the limit is assumed to hold uniformly in all directions $x/|x|$.

Theorem 6.7. *Assume the bounded set D is the open complement of an unbounded domain of class C^2 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D. Let $E, H \in C^1(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ be a radiating solution to the Maxwell equations*

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}.$$

Then we have the Stratton–Chu formulas

$$E(x) = \operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y)$$
$$- \frac{1}{ik} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{6.16}$$

and

$$H(x) = \operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y)$$
$$+\frac{1}{ik} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}. \tag{6.17}$$

Proof. We first assume that condition (6.14) is satisfied and show that

$$\int_{S_r} |E|^2 ds = O(1), \quad r \to \infty, \tag{6.18}$$

where S_r denotes the sphere of radius r and center at the origin. To accomplish this, we observe that from (6.14) it follows that

$$\int_{S_r} \left\{ |H \times \nu|^2 + |E|^2 - 2\operatorname{Re}(\nu \times E \cdot \bar{H}) \right\} ds = \int_{S_r} |H \times \nu - E|^2 ds \to 0, \quad r \to \infty,$$

where ν is the unit outward normal to S_r. We take r large enough so that D is contained in the interior of S_r and apply Gauss' divergence theorem in the domain $D_r := \{y \in \mathbb{R}^3 \setminus \bar{D} : |y| < r\}$ to obtain

$$\int_{S_r} \nu \times E \cdot \bar{H}\, ds = \int_{\partial D} \nu \times E \cdot \bar{H}\, ds + ik \int_{D_r} \left\{ |H|^2 - |E|^2 \right\} dy.$$

We now insert the real part of the last equation into the previous equation and find that

$$\lim_{r\to\infty} \int_{S_r} \left\{ |H \times \nu|^2 + |E|^2 \right\} ds = 2\operatorname{Re} \int_{\partial D} \nu \times E \cdot \bar{H}\, ds. \tag{6.19}$$

Both terms on the left hand side of (6.19) are nonnegative. Hence, they must be individually bounded as $r \to \infty$ since their sum tends to a finite limit. Therefore, (6.18) is proven.

From (6.18) and the radiation conditions

$$\operatorname{grad}_y \Phi(x,y) \times \nu(y) = O\left(\frac{1}{r^2}\right), \quad r \to \infty,$$

and

$$\frac{\partial \Phi(x,y)}{\partial \nu(y)} - ik\Phi(x,y) = O\left(\frac{1}{r^2}\right), \quad r \to \infty,$$

which for fixed $x \in \mathbb{R}^3$ are valid uniformly for $y \in S_r$, by the Cauchy–Schwarz inequality we see that

$$I_1 := \int_{S_r} E(y) \times \left\{ \operatorname{grad}_y \Phi(x,y) \times \nu(y) \right\} ds(y) \to 0, \quad r \to \infty,$$

and

$$I_2 := \int_{S_r} E(y) \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - ik\Phi(x,y) \right\} ds(y) \to 0, \quad r \to \infty.$$

The radiation condition (6.14) and $\Phi(x,y) = O(1/r)$ for $y \in S_r$ yield

$$I_3 := ik \int_{S_r} \Phi(x,y) \left\{ \nu(y) \times H(y) + E(y) \right\} ds(y) \to 0, \quad r \to \infty.$$

Analogously to (6.10), we derive

$$\operatorname{curl} \int_{S_r} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y) - \frac{1}{ik} \operatorname{curl} \operatorname{curl} \int_{S_r} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y)$$

$$= \operatorname{curl} \int_{S_r} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y) - \operatorname{grad} \int_{S_r} \nu(y) \cdot E(y)\, \Phi(x,y)\, ds(y)$$

$$+ik \int_{S_r} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y) = I_1 + I_2 + I_3 \to 0, \quad r \to \infty.$$

The proof is now completed by applying Theorem 6.2 in the bounded domain D_r and passing to the limit $r \to \infty$.

Finally, let (6.15) be satisfied. Then $\tilde{E} := -H$ and $\tilde{H} := E$ solve the Maxwell equations and satisfy

$$\lim_{r \to \infty} (\tilde{H} \times x - r\tilde{E}) = 0.$$

Hence, establishing the representation (6.16) and (6.17) under the assumption of the radiation condition (6.15) is reduced to the case of assuming the radiation condition (6.14). □

From Theorem 6.7 we deduce that radiating solutions E, H to the Maxwell equations automatically satisfy the finiteness condition

$$E(x) = O\left(\frac{1}{|x|}\right), \quad H(x) = O\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \tag{6.20}$$

uniformly for all directions and that the validity of the Silver–Müller radiation conditions (6.14) and (6.15) is invariant under translations of the origin. Our proof has followed Wilcox [334] who first established the Stratton–Chu formulas (6.16) and (6.17) without assuming the conditions (6.20) of finiteness. From the proof of Theorem 6.7 it is obvious that (6.14) and (6.15) can be replaced by the weaker formulation

$$\int_{S_r} |H \times \nu - E|^2 ds \to 0, \quad \int_{S_r} |E \times \nu + H|^2 ds \to 0, \quad r \to \infty.$$

Let a be a constant vector. Then

$$E_m(x) := \operatorname{curl}_x a\Phi(x,y), \quad H_m(x) := \frac{1}{ik} \operatorname{curl} E_m(x) \tag{6.21}$$

represent the electromagnetic field generated by a magnetic dipole located at the point y and solve the Maxwell equations for $x \neq y$. Similarly,

$$H_e(x) := \operatorname{curl}_x a\Phi(x,y), \quad E_e(x) := -\frac{1}{ik} \operatorname{curl} H_e(x) \tag{6.22}$$

represent the electromagnetic field generated by an electric dipole. Theorems 6.2 and 6.7 obviously give representations of solutions to the Maxwell equations in terms of electric and magnetic dipoles distributed over the boundary. In this sense, the fields (6.21) and (6.22) may be considered as fundamental solutions to the Maxwell equations. By straightforward calculations, it can be seen that both pairs E_m, H_m and E_e, H_e satisfy

$$H(x) \times x - rE(x) = O\left(\frac{|a|}{|x|}\right), \quad E(x) \times x + rH(x) = O\left(\frac{|a|}{|x|}\right), \quad r = |x| \to \infty,$$

uniformly for all directions $x/|x|$ and all $y \in \partial D$. Hence, from the representations (6.16) and (6.17) we can deduce that the radiation condition (6.14) implies the radiation condition (6.15) and vice versa.

Straightforward calculations show that the cartesian components of the fundamental solutions (6.21) and (6.22) satisfy the Sommerfeld radiation condition (2.8) uniformly for all $y \in \partial D$. Therefore, again from (6.16) and (6.17), we see that the cartesian components of solutions to the Maxwell equations satisfying the Silver–Müller radiation condition also satisfy the Sommerfeld radiation condition. Similarly, elementary asymptotics show that

$$\operatorname{curl} a\Phi(x,y) \times x + x \operatorname{div} a\Phi(x,y) - ik|x|a\Phi(x,y) = O\left(\frac{|a|}{|x|}\right), \quad |x| \to \infty,$$

uniformly for all directions $x/|x|$ and all $y \in \partial D$. The same inequality also holds with $\Phi(x,y)$ replaced by $\partial\Phi(x,y)/\partial\nu(y)$. Hence, from Theorems 2.5 and 6.4, we conclude that solutions of the Maxwell equations for which the cartesian components satisfy the Sommerfeld radiation condition also satisfy the Silver–Müller radiation condition. Therefore we have proven the following result.

Theorem 6.8. *For solutions to the Maxwell equations, the Silver–Müller radiation condition is equivalent to the Sommerfeld radiation condition for the cartesian components.*

Solutions to the Maxwell equations which are defined in all of $\mathbb{R}^3$ are called *entire solutions*. An entire solution to the Maxwell equations satisfying the Silver–Müller radiation condition must vanish identically. This is a consequence of Theorems 6.4 and 6.8 and the fact that entire solutions to the Helmholtz equation satisfying the Sommerfeld radiation condition must vanish identically.

The following theorem deals with the *far field pattern* or *scattering amplitude* of radiating electromagnetic waves.

Theorem 6.9. *Every radiating solution E, H to the Maxwell equations has the asymptotic form*

$$\begin{aligned} E(x) &= \frac{e^{ik|x|}}{|x|}\left\{E_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \\ H(x) &= \frac{e^{ik|x|}}{|x|}\left\{H_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \end{aligned} \tag{6.23}$$

uniformly in all directions $\hat{x} = x/|x|$ where the vector fields E_∞ and H_∞ defined on the unit sphere $\mathbb{S}^2$ are known as the electric far field pattern *and* magnetic far field pattern*, respectively. They satisfy*

$$H_\infty = \nu \times E_\infty \quad and \quad \nu \cdot E_\infty = \nu \cdot H_\infty = 0 \tag{6.24}$$

with the unit outward normal ν on $\mathbb{S}^2$. Under the assumptions of Theorem 6.7, we have

$$\begin{aligned} E_\infty(\hat{x}) &= \frac{ik}{4\pi}\,\hat{x} \times \int_{\partial D} \{\nu(y) \times E(y) + [\nu(y) \times H(y)] \times \hat{x}\}\, e^{-ik\,\hat{x}\cdot y}\, ds(y), \\ H_\infty(\hat{x}) &= \frac{ik}{4\pi}\,\hat{x} \times \int_{\partial D} \{\nu(y) \times H(y) - [\nu(y) \times E(y)] \times \hat{x}\}\, e^{-ik\,\hat{x}\cdot y}\, ds(y). \end{aligned} \tag{6.25}$$

Proof. As in the proof of Theorem 2.6, for a constant vector a we derive

$$\operatorname{curl}_x a\, \frac{e^{ik|x-y|}}{|x-y|} = ik\, \frac{e^{ik|x|}}{|x|}\left\{e^{-ik\,\hat{x}\cdot y}\, \hat{x} \times a + O\left(\frac{|a|}{|x|}\right)\right\}, \tag{6.26}$$

and

$$\operatorname{curl}_x \operatorname{curl}_x a\, \frac{e^{ik|x-y|}}{|x-y|} = k^2\, \frac{e^{ik|x|}}{|x|}\left\{e^{-ik\,\hat{x}\cdot y}\, \hat{x} \times (a \times \hat{x}) + O\left(\frac{|a|}{|x|}\right)\right\} \tag{6.27}$$

as $|x| \to \infty$ uniformly for all $y \in \partial D$. Inserting this into (6.16) and (6.17) we obtain (6.25). Now (6.23) and (6.24) are obvious from (6.25). □

Rellich's lemma establishes a one-to-one correspondence between radiating electromagnetic waves and their far field patterns.

Theorem 6.10. *Assume the bounded domain D is the open complement of an unbounded domain and let $E, H \in C^1(\mathbb{R}^3 \setminus \bar{D})$ be a radiating solution to the Maxwell equations for which the electric or magnetic far field pattern vanishes identically. Then $E = H = 0$ in $\mathbb{R}^3 \setminus \bar{D}$.*

Proof. This is a consequence of the corresponding Theorem 2.14 for the Helmholtz equation and Theorems 6.4 and 6.8. □

Rellich's lemma also ensures uniqueness for solutions to exterior boundary value problems through the following theorem.

Theorem 6.11. *Assume the bounded set D is the open complement of an unbounded domain of class C^2 with unit normal ν to the boundary ∂D directed into the exterior of D and let $E, H \in C^1(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ be a radiating solution to the Maxwell equations with wave number $k > 0$ satisfying*

$$\mathrm{Re} \int_{\partial D} \nu \times E \cdot \bar{H}\, ds \leq 0.$$

Then $E = H = 0$ in $\mathbb{R}^3 \setminus \bar{D}$.

Proof. From the identity (6.19) and the assumption of the theorem, we conclude that (2.47) is satisfied for the cartesian components of E. Hence, $E = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ by Rellich's Lemma 2.12. □

6.3 Vector Potentials

For the remainder of this chapter, if not stated otherwise, we always will assume that D is the open complement of an unbounded domain of class C^2. In this section, we extend our review of the basic jump relations and regularity properties of surface potentials from the scalar to the vector case. Given an integrable vector field a on the boundary ∂D, the integral

$$A(x) := \int_{\partial D} \Phi(x, y) a(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D, \tag{6.28}$$

is called the *vector potential* with density a. Analogous to Theorem 3.1, we have the following *jump relations* for the behavior at the boundary.

Theorem 6.12. *Let ∂D be of class C^2 and let a be a continuous tangential field. Then the vector potential A with density a is continuous throughout $\mathbb{R}^3$. On the boundary, we have*

$$A(x) = \int_{\partial D} \Phi(x, y) a(y)\, ds(y), \tag{6.29}$$

$$\nu(x) \times \operatorname{curl} A_{\pm}(x) = \int_{\partial D} \nu(x) \times \operatorname{curl}_x \{\Phi(x, y) a(y)\}\, ds(y) \pm \frac{1}{2}\, a(x) \tag{6.30}$$

for $x \in \partial D$ where

$$\nu(x) \times \operatorname{curl} A_\pm(x) := \lim_{h\to+0} \nu(x) \times \operatorname{curl} A(x \pm h\nu(x))$$

is to be understood in the sense of uniform convergence on ∂D and where the integrals exist as improper integrals. Furthermore,

$$\lim_{h\to+0} \nu(x) \times [\operatorname{curl}\operatorname{curl} A(x + h\nu(x)) - \operatorname{curl}\operatorname{curl} A(x - h\nu(x))] = 0 \tag{6.31}$$

uniformly for all $x \in \partial D$.

Proof. The continuity of the vector potential is an immediate consequence of Theorem 3.1. The proof of the jump relation for the curl of the vector potential follows in the same manner as for the double-layer potential after observing that the kernel

$$\nu(x) \times \operatorname{curl}_x \{\Phi(x, y)a(y)\} = \operatorname{grad}_x \Phi(x, y)[\nu(x) - \nu(y)] \cdot a(y) - a(y)\, \frac{\partial\Phi(x, y)}{\partial\nu(x)} \tag{6.32}$$

has the same type of singularity for $x = y$ as the kernel of the double-layer potential. It is essential that a is a tangential vector, that is, $\nu \cdot a = 0$ on ∂D. For the details, we refer to [64]. The proof for the continuity (6.31) of the double curl can be found in [65]. □

In the Hölder space setting, we can deduce from Theorem 3.3 the following result.

Theorem 6.13. *Let ∂D be of class C^2 and let $0 < \alpha < 1$. Then the vector potential A with a (not necessarily tangential) density $a \in C(\partial D)$ is uniformly Hölder continuous throughout $\mathbb{R}^3$ and*

$$\|A\|_{\alpha,\mathbb{R}^3} \le C_\alpha \, \|a\|_{\infty,\partial D}$$

for some constant C_α depending on ∂D and α. For densities $a \in C^{0,\alpha}(\partial D)$, the first derivatives of the vector potential can be uniformly Hölder continuously extended from D to $\bar{D}$ and from $\mathbb{R}^3 \setminus \bar{D}$ to $\mathbb{R}^3 \setminus D$ with boundary values

$$\operatorname{div} A_\pm(x) = \int_{\partial D} \operatorname{grad}_x \Phi(x, y) \cdot a(y)\, ds(y) \mp \frac{1}{2}\, \nu(x) \cdot a(x), \quad x \in \partial D,$$

$$\operatorname{curl} A_\pm(x) = \int_{\partial D} \operatorname{grad}_x \Phi(x, y) \times a(y)\, ds(y) \mp \frac{1}{2}\, \nu(x) \times a(x), \quad x \in \partial D,$$

where

$$\operatorname{div} A_\pm(x) := \lim_{h\to+0} \operatorname{div} A(x \pm h\nu(x)), \quad \operatorname{curl} A_\pm(x) := \lim_{h\to+0} \operatorname{curl} A(x \pm h\nu(x)).$$

Furthermore, we have

$$\| \operatorname{div} A\|_{\alpha,\overline{D}} \le C_\alpha \, \|a\|_{\alpha,\partial D}, \quad \| \operatorname{div} A\|_{\alpha,\mathbb{R}^3\setminus D} \le C_\alpha \, \|a\|_{\alpha,\partial D}$$

and

$$\| \operatorname{curl} A\|_{\alpha,\overline{D}} \le C_\alpha \, \|a\|_{\alpha,\partial D}, \quad \| \operatorname{curl} A\|_{\alpha,\mathbb{R}^3\setminus D} \le C_\alpha \, \|a\|_{\alpha,\partial D}.$$

for some constant C_α depending on ∂D and α.

For the tangential component of the curl on the boundary ∂D, we have more regularity which can be expressed in terms of mapping properties for the magnetic dipole operator M given by

$$(Ma)(x) := 2 \int_{\partial D} \nu(x) \times \operatorname{curl}_x \{\Phi(x,y)a(y)\} \, ds(y), \quad x \in \partial D. \tag{6.33}$$

The operator M describes the tangential component of the electric field of a magnetic dipole distribution. For convenience we denote by $C_t(\partial D)$ and $C_t^{0,\alpha}(\partial D)$, $0 < \alpha \le 1$, the spaces of all continuous and uniformly Hölder continuous tangential fields a equipped with the supremum norm and the Hölder norm, respectively.

Theorem 6.14. *The operator M is bounded from $C_t(\partial D)$ into $C_t^{0,\alpha}(\partial D)$.*

Proof. For a proof, we refer to Theorem 2.32 of [64]. □

In order to develop further regularity properties of the vector potential, we need to introduce the concept of the surface divergence of tangential vector fields. For a continuously differentiable function φ on ∂D, the *surface gradient* $\operatorname{Grad}\varphi$ is defined as the vector pointing into the direction of the maximal increase of φ with the modulus given by the value of this increase. In terms of a parametric representation

$$x(u) = (x_1(u_1,u_2), x_2(u_1,u_2), x_3(u_1,u_2))$$

of a surface patch of ∂D, the surface gradient can be expressed by

$$\operatorname{Grad}\varphi = \sum_{i,j=1}^{2} g^{ij} \, \frac{\partial \varphi}{\partial u_i} \, \frac{\partial x}{\partial u_j} \tag{6.34}$$

where g^{ij} is the inverse of the first fundamental matrix

$$g_{ij} := \frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j}\,, \quad i,j = 1,2,$$

of differential geometry. We note that for a continuously differentiable function φ defined in a neighborhood of ∂D we have the relation

$$\operatorname{grad}\varphi = \operatorname{Grad}\varphi + \frac{\partial \varphi}{\partial \nu} \, \nu \tag{6.35}$$

between the spatial gradient grad and the surface gradient Grad.

Let S be a connected surface contained in ∂D with C^2 boundary ∂S and let ν_0 denote the unit normal vector to ∂S that is perpendicular to the surface normal ν to ∂D and directed into the exterior of S. Then for any continuously differentiable tangential field a with the representation

$$a = a_1 \frac{\partial x}{\partial u_1} + a_2 \frac{\partial x}{\partial u_2} ,$$

by Gauss' integral theorem applied in the parameter domain (c.f. [235], p. 74) it can be readily shown that

$$\int_S \operatorname{Div} a \, ds = \int_{\partial S} \nu_0 \cdot a \, ds \tag{6.36}$$

with the *surface divergence* $\operatorname{Div} a$ given by

$$\operatorname{Div} a = \frac{1}{\sqrt{g}} \left\{ \frac{\partial}{\partial u_1} (\sqrt{g}\, a_1) + \frac{\partial}{\partial u_2} (\sqrt{g}\, a_2) \right\}. \tag{6.37}$$

Here, g denotes the determinant of the matrix g_{ij}. In particular, from (6.36) we have the special case

$$\int_{\partial D} \operatorname{Div} a \, ds = 0. \tag{6.38}$$

We call (6.36) the *Gauss surface divergence theorem* and may view it as a coordinate independent definition of the surface divergence.

For our purposes, this definition is not yet adequate and has to be generalized. This generalization, in principle, can be done in two ways. One possibility (see [64, 256]) is to use (6.36) as motivation for a definition in the limit integral sense by letting the surface S shrink to a point. Here, as a second possibility, we use the concept of weak derivatives. From (6.34) and (6.37) we see that for continuously differentiable functions φ and tangential fields a we have the product rule $\operatorname{Div} \varphi a = \operatorname{Grad} \varphi \cdot a + \varphi \operatorname{Div} a$ and consequently by (6.38) we have

$$\int_{\partial D} \varphi \operatorname{Div} a \, ds = - \int_{\partial D} \operatorname{Grad} \varphi \cdot a \, ds. \tag{6.39}$$

This now leads to the following definition.

Definition 6.15 *We say that an integrable tangential field a has a* weak surface divergence *if there exists an integrable scalar denoted by* $\operatorname{Div} a$ *such that (6.39) is satisfied for all $\varphi \in C^1(\partial D)$.*

It is left as an exercise to show that the weak surface divergence, if it exists, is unique. In the sequel, we will in general suppress the adjective weak and just speak of the surface divergence.

For a continuously differentiable tangential field a from (6.36) we observe that

$$-\int_S \operatorname{Div}(\nu \times a)\, ds = \int_{\partial S} \tau_0 \cdot a\, ds \tag{6.40}$$

where τ_0 denotes the unit tangential vector to ∂S with counter clockwise orientation with respect to the unit normal vector ν to ∂D. Hence, in view of Stokes' theorem, we say that an integrable tangential field has a weak *surface curl* if $\nu \times a$ has a weak surface divergence and define

$$\operatorname{Curl} a := -\operatorname{Div}(\nu \times a). \tag{6.41}$$

Again, in the sequel we will suppress the adjective weak. Note that (6.40) may be also interpreted as Stokes' theorem for $\operatorname{Curl} a$.

Let $E \in C^1(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ and assume that

$$\nu(x) \cdot \operatorname{curl} E(x) := \lim_{h \to +0} \nu(x) \cdot \operatorname{curl} E(x + h\nu(x)), \quad x \in \partial D,$$

exists in the sense of uniform convergence on ∂D. Then by applying Stokes' integral theorem on parallel surfaces and passing to the limit it can be seen that

$$\int_{\partial D} \nu \cdot \operatorname{curl} E\, ds = 0. \tag{6.42}$$

By setting $\psi(x + h\nu(x)) := \varphi(x)$, $x \in \partial D, -h_0 \le h \le h_0$, with h_0 sufficiently small, any continuously differentiable function φ on ∂D can be considered as the restriction of a function ψ which is continuously differentiable in a neighborhood of ∂D. Then from the product rule $\operatorname{curl} \psi E = \operatorname{grad} \psi \times E + \psi \operatorname{curl} E$, (6.35) and Stokes' theorem (6.42) applied to ψE, we find that

$$\int_{\partial D} \varphi\, \nu \cdot \operatorname{curl} E\, ds = \int_{\partial D} \operatorname{Grad} \varphi \cdot \nu \times E\, ds$$

for all $\varphi \in C^1(\partial D)$ and from this we obtain the important identity

$$\operatorname{Div}(\nu \times E) = -\nu \cdot \operatorname{curl} E. \tag{6.43}$$

We introduce normed spaces of tangential fields possessing a surface divergence by

$$C(\operatorname{Div}, \partial D) := \{a \in C_t(\partial D) : \operatorname{Div} a \in C(\partial D)\}$$

and

$$C^{0,\alpha}(\operatorname{Div}, \partial D) := \left\{a \in C_t^{0,\alpha}(\partial D) : \operatorname{Div} a \in C^{0,\alpha}(\partial D)\right\}$$

equipped with the norms

$$\|a\|_{C(\operatorname{Div},\partial D)} := \|a\|_{\infty,\partial D} + \|\operatorname{Div} a\|_{\infty.\partial D}, \quad \|a\|_{C^{0,\alpha}(\operatorname{Div},\partial D)} := \|a\|_{\alpha,\partial D} + \|\operatorname{Div} a\|_{\alpha,\partial D}.$$

Theorem 6.16. *Let* $0 < \alpha < \beta \le 1$. *Then the imbedding operators*

$$I^\beta : C^{0,\beta}(\mathrm{Div}, \partial D) \to C(\mathrm{Div}, \partial D), \quad I^{\beta,\alpha} : C^{0,\beta}(\mathrm{Div}, \partial D) \to C^{0,\alpha}(\mathrm{Div}, \partial D)$$

are compact.

Proof. Let (a_n) be a bounded sequence in $C^{0,\beta}(\mathrm{Div}, \partial D)$. Then by Theorem 3.2 there exists a subsequence $(a_{n(j)})$, a tangential field $a \in C_t(\partial D)$ and a scalar $\psi \in C(\partial D)$ such that $\|a_{n(j)} - a\|_\infty \to 0$ and $\|\,\mathrm{Div}\, a_{n(j)} - \psi\|_\infty \to 0$ as $j \to \infty$. Passing to the limit in

$$\int_{\partial D} \varphi \,\mathrm{Div}\, a_{n(j)}\, ds = -\int_{\partial D} \mathrm{Grad}\,\varphi \cdot a_{n(j)}\, ds$$

for $\varphi \in C^1(\partial D)$ shows that $a \in C(\mathrm{Div}, \partial D)$ with $\mathrm{Div}\, a = \psi$. This finishes the proof for the compactness of I^β since we now have $\|a_{n(j)} - a\|_{C(\mathrm{Div},\partial D)} \to 0$ as $j \to \infty$. The proof for $I^{\beta,\alpha}$ is analogous. □

We now extend Theorem 6.14 by proving the following result.

Theorem 6.17. *The operator* M *is bounded from* $C(\mathrm{Div}, \partial D)$ *into* $C^{0,\alpha}(\mathrm{Div}, \partial D)$.

Proof. This follows from the boundedness of S and K' from $C(\partial D)$ into $C^{0,\alpha}(\partial D)$ and of M from $C_t(\partial D)$ into $C_t^{0,\alpha}(\partial D)$ with the aid of

$$\mathrm{Div}\, Ma = -k^2\, \nu \cdot S a - K' \,\mathrm{Div}\, a \tag{6.44}$$

for all $a \in C(\mathrm{Div}, \partial D)$. To establish (6.44), we first note that using the symmetry relation $\mathrm{grad}_x\, \Phi(x, y) = -\,\mathrm{grad}_y\, \Phi(x, y)$, (6.35) and (6.39) for the vector potential A with density a in $C(\mathrm{Div}, \partial D)$ we can derive

$$\mathrm{div}\, A(x) = \int_{\partial D} \Phi(x, y) \,\mathrm{Div}\, a(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D.$$

Then, using the identity (6.4), we find that

$$\begin{aligned} \mathrm{curl}\,\mathrm{curl}\, A(x) = {} & k^2 \int_{\partial D} \Phi(x, y) a(y)\, ds(y) \\ & + \mathrm{grad} \int_{\partial D} \Phi(x, y) \,\mathrm{Div}\, a(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D. \end{aligned} \tag{6.45}$$

Applying the jump relations of Theorems 3.1 and 6.12, we find that

$$2\, \nu \times \mathrm{curl}\, A_\pm = Ma \pm a \quad \text{on } \partial D \tag{6.46}$$

and

$$2\, \nu \cdot \mathrm{curl}\,\mathrm{curl}\, A_\pm = k^2 \nu \cdot S a + K' \,\mathrm{Div}\, a \mp \mathrm{Div}\, a \quad \text{on } \partial D. \tag{6.47}$$

Hence, by using the identity (6.43) we now obtain (6.44) by (6.46) and (6.47). □

Corollary 6.18 *The operator M is a compact operator from $C_t(\partial D)$ into $C_t(\partial D)$ and from $C_t^{0,\alpha}(\partial D)$ into $C_t^{0,\alpha}(\partial D)$. Furthermore, M is also compact from $C(\mathrm{Div}, \partial D)$ into $C(\mathrm{Div}, \partial D)$ and from $C^{0,\alpha}(\mathrm{Div}, \partial D)$ into $C^{0,\alpha}(\mathrm{Div}, \partial D)$.*

Proof. This is a consequence of Theorems 6.14 and 6.17 and the embedding Theorems 3.2 and 6.16. □

In our analysis of boundary value problems, we will also need the electric dipole operator N given by

$$(Nb)(x) := 2\,\nu(x) \times \operatorname{curl}\operatorname{curl} \int_{\partial D} \Phi(x, y)\nu(y) \times b(y)\, ds(y), \qquad x \in \partial D. \tag{6.48}$$

The operator N describes the tangential component of the electric field of an electric dipole distribution. After introducing the normed space

$$C^{0,\alpha}(\mathrm{Curl}, \partial D) := \left\{ b \in C_t^{0,\alpha}(\partial D) : \mathrm{Curl}\, b \in C^{0,\alpha}(\partial D) \right\},$$

that is, $C^{0,\alpha}(\mathrm{Curl}, \partial D) = \left\{ b \in C_t^{0,\alpha}(\partial D) : \nu \times b \in C^{0,\alpha}(\mathrm{Div}, \partial D) \right\}$ with the norm

$$\|b\|_{C^{0,\alpha}(\mathrm{Curl}, \partial D)} := \|\nu \times b\|_{C^{0,\alpha}(\mathrm{Div}, \partial D)}$$

we can state the following mapping property.

Theorem 6.19. *The operator N is bounded from $C^{0,\alpha}(\mathrm{Curl}, \partial D)$ into $C^{0,\alpha}(\mathrm{Div}, \partial D)$.*

Proof. From the decomposition (6.45) and Theorems 3.3 and 6.13, we observe that N is bounded from $C^{0,\alpha}(\mathrm{Curl}, \partial D)$ into $C_t^{0,\alpha}(\partial D)$. Furthermore, from (6.43) and (6.45), we also deduce that

$$\mathrm{Div}\, Nb = k^2\, \mathrm{Div}(\nu \times S(\nu \times b)). \tag{6.49}$$

Hence, in view of Theorem 6.13 and (6.43), there exists a constant C such that

$$\|\,\mathrm{Div}\, Nb\|_{0,\alpha} \le C\|b\|_{0,\alpha}$$

and this implies that N is also bounded from $C^{0,\alpha}(\mathrm{Curl}, \partial D)$ into $C^{0,\alpha}(\mathrm{Div}, \partial D)$. □

By interchanging the order of integration, we see that the adjoint operator M' of the weakly singular operator M with respect to the bilinear form

$$\langle a, b\rangle := \int_{\partial D} a \cdot b\, ds$$

is given by

$$M'a := \nu \times M(\nu \times a). \tag{6.50}$$

After defining the operator R by

$$Ra := a \times \nu,$$

we may rewrite (6.50) as $M' = RMR$.

To show that N is self-adjoint, let $a, b \in C^{0,\alpha}(\mathrm{Curl}, \partial D)$ and denote by A and B the vector potentials with densities $\nu \times a$ and $\nu \times b$, respectively. Then by the jump relations of Theorem 6.12, Green's vector theorem (6.3) applied to $E = \operatorname{curl} A$ and $F = \operatorname{curl} B$, and the radiation condition we find that

$$\int_{\partial D} Na \cdot b\, ds = 2 \int_{\partial D} \nu \times \operatorname{curl}\operatorname{curl} A \cdot (\operatorname{curl} B_+ - \operatorname{curl} B_-)\, ds$$

$$= 2 \int_{\partial D} \nu \times \operatorname{curl}\operatorname{curl} B \cdot (\operatorname{curl} A_+ - \operatorname{curl} A_-)\, ds = \int_{\partial D} Nb \cdot a\, ds,$$

that is, N indeed is self-adjoint. Furthermore, applying Green's vector theorem (6.3) to $E = \operatorname{curl}\operatorname{curl} A$ and $F = \operatorname{curl} B$, we derive

$$\int_{\partial D} Na \cdot Nb \times \nu\, ds = 4 \int_{\partial D} \nu \times \operatorname{curl}\operatorname{curl} A \cdot \operatorname{curl}\operatorname{curl} B\, ds$$

$$= 4k^2 \int_{\partial D} \nu \times \operatorname{curl} B_- \cdot \operatorname{curl} A_-\, ds = k^2 \int_{\partial D} (I - M)(\nu \times b) \cdot (I + M')a\, ds,$$

whence

$$\int_{\partial D} a \cdot N(Nb \times \nu)\, ds = k^2 \int_{\partial D} a \cdot (I - M^2)(\nu \times b)\, ds$$

follows for all $a, b \in C^{0,\alpha}(\mathrm{Curl}, \partial D)$. Thus, setting $c = \nu \times b$, we have proven the relation

$$N(N(c \times \nu) \times \nu) = k^2(I - M^2)c$$

for all $c \in C^{0,\alpha}(\mathrm{Div}, \partial D)$, that is,

$$NRNR = k^2(I - M^2). \tag{6.51}$$

As in the scalar case, corresponding mapping properties for the two vector operators M and N in a Sobolev space setting can again be deduced from the classical results in the Hölder space setting by using Lax's Theorem 3.5. For convenience we introduce the space

$$L^2_t(\partial D) := \left\{a : \partial D \to \mathbb{C}^3 : a \in L^2(\partial D),\ a \cdot \nu = 0\right\}$$

of tangential L^2 fields on ∂D. The appropriate energy space for the Maxwell equations is given by the Sobolev space $H(\operatorname{curl}, D)$ defined by

$$H(\operatorname{curl}, D) := \left\{E \in L^2(D) : \operatorname{curl} E \in L^2(D)\right\}$$

with the inner product given by

$$(E, F)_{H(\operatorname{curl},D)} := (E, F)_{L^2(D)} + (\operatorname{curl} E, \operatorname{curl} F)_{L^2(D)}\,.$$

For $s \in \mathbb{R}$ by $H_t^s(\partial D) := \{a \in H^s(\partial D) : a \cdot \nu = 0\}$ we denote the Sobolev spaces of tangential vector fields on ∂D and by

$$H^s(\mathrm{Div}, \partial D) := \{a \in H_t^s(\partial D) : \mathrm{Div}\, a \in H^s(\partial D)\}$$

the corresponding Sobolev space of tangential fields whose surface divergence is in $H^s(\partial D)$ with the norm

$$\|a\|_{H^s(\mathrm{Div},\partial D)} := \left\{\|a\|^2_{H^s(\partial D)} + \|\,\mathrm{Div}\, a\|^2_{H^s(\partial D)}\right\}^{1/2}.$$

The tangential trace $\nu \times E|_{\partial D}$ of a vector field $E \in H(\mathrm{curl}, D)$ is in $H^{-1/2}(\mathrm{Div}, \partial D)$ with the corresponding trace theorem ensuring boundedness of the tangential trace operator in the sense of

$$\|\nu \times E\|_{H^{-1/2}(\mathrm{Div}\, a,\partial D)} \le C\, \|E\|_{H(\mathrm{curl},D)} \tag{6.52}$$

for some constant $C > 0$ and all $E \in H(\mathrm{curl}, D)$. The dual space of $H^{-1/2}(\mathrm{Div}\, a, \partial D)$ is

$$H^{-1/2}(\mathrm{Curl}, \partial D) := \left\{a \in H_t^{-1/2}(\partial D) : \mathrm{Curl}\, a \in H^{-1/2}(\partial D)\right\}$$

with the duality pairing given by the L^2 bilinear form on $L_t^2(\partial D)$. For proofs of the above statements and further details on these Sobolev spaces we refer to [245, 260].

Proceeding as in Theorem 3.6 with the aid of Lax's Theorem 3.5 it can be shown that $M : L_t^2(\partial D) \to H_t^1(\partial D)$ is bounded (see [121, 178]) provided the boundary ∂D is of class $C^{2,\alpha}$. As in Corollary 3.7 with the aid of duality and interpolation it follows that $M : H_t^{-1/2}(\partial D) \to H_t^{1/2}(\partial D)$ is bounded. From (6.44) and Corollary 3.7 we then obtain that $M : H^{-1/2}(\mathrm{Div}, \partial D) \to H^{1/2}(\mathrm{Div}, \partial D)$ is bounded. Employing the ideas of the proof of Thereorm 6.16 from the compact imbedding of $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$ we finally observe that

$$M : H^{-1/2}(\mathrm{Div}, \partial D) \to H^{-1/2}(\mathrm{Div}, \partial D)$$

is compact. In view of (6.45) we have that

$$Nb = k^2\, \nu \times S\,(\nu \times b) + \nu \times \mathrm{Grad}\, S\ \mathrm{Div}(\nu \times b)$$

and from Corollary 3.7 it follows that $N : H^{-1/2}(\mathrm{Curl}, \partial D) \to H_t^{-1/2}(\partial D)$ is bounded. Then (6.49) and Corollary 3.7 imply boundedness of

$$N : H^{-1/2}(\mathrm{Curl}, \partial D) \to H^{-1/2}(\mathrm{Div}, \partial D).$$

Furthermore, based on the vector Green's integral theorem (6.2) applied to $\mathrm{curl}\, A$, that is,

$$\int_D \left\{|\,\mathrm{curl}\,\mathrm{curl}\, A|^2 - k^2 |\,\mathrm{curl}\, A|^2\right\} dx = \frac{1}{4} \int_{\partial D} (Ma - a) \cdot [\nu \times N(\nu \times a)]\, ds$$

and proceeding analogously to the proof of Corollary 3.8 it can be shown that the curl of the vector potential A given by (6.28) defines bounded linear operators from $H^{-1/2}(\mathrm{Div}, \partial D)$ into $H^1(\mathrm{curl}, D)$ and into $H^1_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$. Here, as usual, $H^1_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$ denotes the space of all fields $A : \mathbb{R}^3 \setminus \bar{D} \to \mathbb{C}^3$ such that $A \in H^1(\mathrm{curl}, (\mathbb{R}^3 \setminus \bar{D}) \cup B)$ for all open balls B containing the closure of D. Correspondingly, the double curl of the vector potential A defines bounded linear operators from $H^{-1/2}(\mathrm{Curl}, \partial D)$ into $H^1(\mathrm{curl}, D)$ and into $H^1_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$.

Using Lax's theorem, the jump relations of Theorem 6.12 can also be extended from continuous densities to L^2 densities. As a simple consequence of the L^2 jump relation (3.24), Hähner [121] has shown that for the vector potential with tangential L^2 density a the jump relation (6.30) has to be replaced by

$$\lim_{h\to+0} \int_{\partial D} |2\,\nu \times \mathrm{curl}\, A(\cdot \pm h\nu) - Ma \mp a|^2 ds = 0 \tag{6.53}$$

and he has further verified that (6.31) can be replaced by

$$\lim_{h\to+0} \int_{\partial D} |\nu \times [\mathrm{curl}\,\mathrm{curl}\, A(\cdot + h\nu) - \mathrm{curl}\,\mathrm{curl}\, A(\cdot - h\nu)]|^2 ds = 0, \tag{6.54}$$

since the singularity of the double curl is similar to the singularity of the normal derivative of the double-layer potential in (3.4) and (3.23).

6.4 Scattering from a Perfect Conductor

The scattering of time-harmonic electromagnetic waves by a perfectly conducting body leads to the following problem.

Direct Electromagnetic Obstacle Scattering Problem. *Given an entire solution E^i, H^i to the Maxwell equations representing an incident electromagnetic field, find a solution*

$$E = E^i + E^s, \quad H = H^i + H^s$$

to the Maxwell equations in $\mathbb{R}^3 \setminus \bar{D}$ such that the scattered field E^s, H^s satisfies the Silver–Müller radiation condition and the total electric field E satisfies the boundary condition

$$\nu \times E = 0 \quad \text{on } \partial D$$

where ν is the unit outward normal to ∂D.

Clearly, after renaming the unknown fields, this direct scattering problem is a special case of the following problem.

Exterior Maxwell Problem. *Given a tangential field $c \in C^{0,\alpha}(\mathrm{Div}, \partial D)$, find a radiating solution $E, H \in C^1(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ to the Maxwell equations*

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}$$

which satisfies the boundary condition

$$\nu \times E = c \quad \text{on } \partial D.$$

From the vector formula (6.43), we observe that the continuity of the magnetic field H up to the boundary requires the differentiability of the tangential component $\nu \times E$ of the electric field, i.e., the given tangential field c must have a continuous surface divergence. The Hölder continuity of the boundary data is necessary for our integral equation approach to solving the exterior Maxwell problem.

Theorem 6.20. *The exterior Maxwell problem has at most one solution.*

Proof. This follows from Theorem 6.11. □

Theorem 6.21. *The exterior Maxwell problem has a unique solution. The solution depends continuously on the boundary data in the sense that the operator mapping the given boundary data onto the solution is continuous from $C^{0,\alpha}(\mathrm{Div}, \partial D)$ into $C^{0,\alpha}(\mathbb{R}^3 \setminus D) \times C^{0,\alpha}(\mathbb{R}^3 \setminus D)$.*

Proof. We seek the solution in the form of the electromagnetic field of a combined magnetic and electric dipole distribution

$$\begin{aligned} E(x) = {}& \operatorname{curl} \int_{\partial D} \Phi(x, y) a(y)\, ds(y) \\ & + i\eta \operatorname{curl}\operatorname{curl} \int_{\partial D} \Phi(x, y)\, \nu(y) \times (S_0^2 a)(y)\, ds(y), \end{aligned} \tag{6.55}$$

$$H(x) = \frac{1}{ik} \operatorname{curl} E(x), \quad x \in \mathbb{R}^3 \setminus \partial D,$$

with a density $a \in C^{0,\alpha}(\mathrm{Div}, \partial D)$ and a real coupling parameter $\eta \neq 0$. By S_0 we mean the single-layer operator (3.8) in the potential theoretic limit case $k = 0$. From Theorems 6.4 and 6.8 and the jump relations, we see that E, H defined by (6.55) in $\mathbb{R}^3 \setminus \bar{D}$ solves the exterior Maxwell problem provided the density solves the integral equation

$$a + Ma + i\eta NPS_0^2 a = 2c. \tag{6.56}$$

Here, the operator P stands for the projection of a vector field defined on ∂D onto the tangent plane, that is,

$$Pb := (\nu \times b) \times \nu.$$

By Theorems 3.2 and 3.4, the operator S_0 is compact from $C^{0,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$ and, with the aid of Theorem 3.3 and the identity (6.43), the operator $PS_0 : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\mathrm{Curl}, \partial D)$ can be seen to be bounded. Therefore, combining Theorems 6.16–6.19, the operator $M + i\eta NPS_0^2 : C^{0,\alpha}(\mathrm{Div}, \partial D) \to C^{0,\alpha}(\mathrm{Div}, \partial D)$ turns out to be compact. Hence, the existence of a solution to (6.56) can be established by the Riesz–Fredholm theory.

Let $a \in C^{0,\alpha}(\mathrm{Div}, \partial D)$ be a solution to the homogeneous form of (6.56). Then the electromagnetic field E, H given by (6.55) satisfies the homogeneous boundary condition $\nu \times E_+ = 0$ on ∂D whence $E = H = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows by Theorem 6.20. The jump relations together with the decomposition (6.45) now yield

$$-\nu \times E_- = a, \quad -\nu \times \operatorname{curl} E_- = i\eta k^2 \nu \times S_0^2 a \quad \text{on } \partial D.$$

Hence, from Gauss' divergence theorem we have

$$\begin{aligned} i\eta k^2 \int_{\partial D} |S_0 a|^2 ds = i\eta k^2 \int_{\partial D} \bar{a} \cdot S_0^2 a \, ds \\ = \int_{\partial D} \nu \times \bar{E}_- \cdot \operatorname{curl} E_- \, ds \\ = \int_D \left\{ |\operatorname{curl} E|^2 - k^2 |E|^2 \right\} dx, \end{aligned}$$

whence $S_0 a = 0$ follows. This implies $a = 0$ as in the proof of Theorem 3.12. Thus, we have established injectivity of the operator $I + M + i\eta NPS_0^2$ and, by the Riesz–Fredholm theory, the inverse operator $(I + M + i\eta NPS_0^2)^{-1}$ exists and is bounded from $C^{0,\alpha}(\mathrm{Div}, \partial D)$ into $C^{0,\alpha}(\mathrm{Div}, \partial D)$. This, together with the regularity results of Theorems 3.3 and 6.13 and the decomposition (6.45), shows that E and H both belong to $C^{0,\alpha}(\mathbb{R}^3 \setminus D)$ and depend continuously on c in the norm of $C^{0,\alpha}(\mathrm{Div}, \partial D)$. □

From

$$\begin{aligned} ikH(x) = \operatorname{curl} \operatorname{curl} \int_{\partial D} \Phi(x, y) a(y) \, ds(y) \\ + i\eta k^2 \operatorname{curl} \int_{\partial D} \Phi(x, y) \, \nu(y) \times (S_0^2 a)(y) \, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \end{aligned}$$

and the jump relations we find that

$$2ik \, \nu \times H = -NRa + i\eta k^2 (RS_0^2 a + MRS_0^2 a)$$

with the bounded operator

$$NR - i\eta k^2 (I + M) RS_0^2 : C^{0,\alpha}(\mathrm{Div}, \partial D) \to C^{0,\alpha}(\mathrm{Div}, \partial D).$$

Therefore, we can write

$$\nu \times H = \mathcal{A}(\nu \times E)$$

where

$$\mathcal{A} := \frac{i}{k}\,\{NR - i\eta k^2 (I+M)RS_0^2\}(I+M+i\eta NPS_0^2)^{-1} : C^{0,\alpha}(\mathrm{Div}, \partial D) \to C^{0,\alpha}(\mathrm{Div}, \partial D)$$

is bounded. The operator $\mathcal{A}$ transfers the tangential component of the electric field on the boundary onto the tangential component of the magnetic field and therefore we call it the *electric to magnetic boundary component map*. It is bijective and has a bounded inverse since it satisfies

$$\mathcal{A}^2 = -I.$$

This equation is a consequence of the fact that for any radiating solution E, H of the Maxwell equations the fields $\tilde{E} := -H$ and $\tilde{H} := E$ solve the Maxwell equations and satisfy the Silver–Müller radiation condition. Hence, by the uniqueness Theorem 6.20, the solution to the exterior Maxwell problem with given electric boundary data $\mathcal{A}c$ has magnetic boundary data $-c$ so that $\mathcal{A}^2 c = -c$. Thus, we can state the following result.

Theorem 6.22. *The electric to magnetic boundary component map $\mathcal{A}$ is a bijective bounded operator from $C^{0,\alpha}(\mathrm{Div}, \partial D)$ onto $C^{0,\alpha}(\mathrm{Div}, \partial D)$ with bounded inverse.*

Note that, analogous to the acoustic case, the integral equation (6.56) is not uniquely solvable if $\eta = 0$ and k is a *Maxwell eigenvalue*, i.e., a value of k such that there exists a nontrivial solution E, H to the Maxwell equations in D satisfying the homogeneous boundary condition $\nu \times E = 0$ on ∂D. An existence proof for the exterior Maxwell problem based on seeking the solution as the electromagnetic field of a combined magnetic and electric dipole distribution was first accomplished by Knauff and Kress [195] in order to overcome the non-uniqueness difficulties of the classical approach by Müller [254] and Weyl [331]. The combined field approach was also independently suggested by Jones [167] and Mautz and Harrington [237]. The idea to incorporate a smoothing operator in (6.55) analogous to (3.29) was first used by Kress [198].

Since the electric and the magnetic fields occur in the Maxwell equations in a symmetric fashion, it seems appropriate to assume the same regularity properties for both fields up to the boundary as we have done in our definition of the exterior Maxwell problem. On the other hand, only the electric field is involved in the formulation of the boundary condition. Therefore, it is quite natural to ask whether the boundary condition can be weakened by seeking a solution $E, H \in C^1(\mathbb{R}^3 \setminus \bar{D})$ to the Maxwell equation such that

$$\lim_{h \to +0} \nu(x) \times E(x + h\nu(x)) = c(x) \tag{6.57}$$

uniformly for all $x \in \partial D$. But then as far as proving uniqueness is concerned we are in a similar position as for the Dirichlet problem: there is a gap between the regularity properties of the solution and the requirements for the application of the Gauss divergence theorem which is involved in the proof of Theorem 6.11. Unfortunately, for the Maxwell problem there is no elegant circumvention of this difficulty available as in Lemma 3.10. Nevertheless, due to an idea going back to Calderón [42], which has been rediscovered and extended more recently by Hähner [122], it is still possible to establish uniqueness for the exterior Maxwell problem with the boundary condition (6.57). The main idea is to represent the solution with homogeneous boundary condition by surface potentials of the form (6.55) with magnetic and electric dipole distributions on surfaces parallel to ∂D as suggested by the formulation (6.57) and then pass to the limit $h \to +0$. Of course, this procedure requires the existence analysis in the Hölder space setting which we developed in Theorem 6.21. Since the details of this analysis are beyond the aims of this book, the reader is referred to [42] and [122].

Once uniqueness under the weaker conditions is established, existence can be obtained by solving the integral equation (6.56) in the space $C_t(\partial D)$ of continuous tangential fields. By Theorems 3.2 and 3.4, the operator S_0^2 is bounded from $C(\partial D)$ into $C^{1,\alpha}(\partial D)$. Therefore, again as a consequence of Theorems 6.16–6.19, the operator $M + i\eta NPS_0^2 : C_t(\partial D) \to C_t(\partial D)$ is seen to be compact. Hence, the existence of a continuous solution to (6.56) for any given continuous right hand side c can be established by the Fredholm alternative. We now proceed to do this.

The adjoint operator of $M + i\eta NPS_0^2$ with respect to the L^2 bilinear form is given by $M' + i\eta PS_0^2 N$ and, by Theorems 3.2, 3.4 and 6.16–6.19, the adjoint is seen to be compact from $C^{0,\alpha}(\mathrm{Curl}, \partial D) \to C^{0,\alpha}(\mathrm{Curl}, \partial D)$. Working with the two dual systems $\langle C^{0,\alpha}(\mathrm{Div}, \partial D), C^{0,\alpha}(\mathrm{Curl}, \partial D)\rangle$ and $\langle C_t(\partial D), C^{0,\alpha}(\mathrm{Curl}, \partial D)\rangle$ as in the proof of Theorem 3.27, the Fredholm alternative tells us that the operator $I + M + i\eta NPS_0^2$ has a trivial nullspace in $C_t(\partial D)$. Hence, by the Riesz–Fredholm theory it has a bounded inverse $(I + M + i\eta NPS_0^2)^{-1} : C_t(\partial D) \to C_t(\partial D)$. Then, given $c \in C_t(\partial D)$ and the unique solution $a \in C_t(\partial D)$ of (6.56), the electromagnetic field defined by (6.55) is seen to solve the exterior Maxwell problem in the sense of (6.57). For this we note that the jump relations of Theorem 6.12 are valid for continuous densities and that $PS_0^2 a$ belongs to $C^{0,\alpha}(\mathrm{Curl}, \partial D)$. From the representation (6.55) of the solution, the continuous dependence of the density a on the boundary data c shows that the exterior Maxwell problem is also well-posed in this setting, i.e., small deviations in c in the maximum norm ensure small deviations in E and H and all their derivatives in the maximum norm on closed subsets of $\mathbb{R}^3 \setminus \bar{D}$. Thus, leaving aside the uniqueness part of the proof, we have established the following theorem.

Theorem 6.23. *The exterior Maxwell problem with continuous boundary data has a unique solution and the solution depends continuously on the boundary data with respect to uniform convergence of the solution and all its derivatives on closed subsets of* $\mathbb{R}^3 \setminus \bar{D}$.

In the Sobolev space setting, the solution E to the exterior Maxwell problem is required to belong to the energy space $H^1_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$ and the boundary condition $\nu \times E = c$ on ∂D for a given $c \in H^{-1/2}(\mathrm{Div}, \partial D)$ has to be understood in the sense of the tangential trace operator in $H^1_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$. The existence analysis via the electromagnetic field of a combined magnetic and electric dipole distribution (6.55) with a density $a \in H^{-1/2}(\mathrm{Div}, \partial D)$ can be carried over. In particular, we have well-posedness in the sense that the mapping from the boundary values $c \in H^{-1/2}(\mathrm{Div}, \partial D)$ onto the solution $E \in H^1_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$ is continuous. Further, we note that analogous to Theorem 6.22 the electric to magnetic boundary component map $\mathcal{A}$ is is a bijective bounded operator from $H^{-1/2}(\mathrm{Div}, \partial D)$ onto $H^{-1/2}(\mathrm{Div}, \partial D)$ with a bounded inverse.

For the scattering problem, the boundary values are the restriction of an analytic field to the boundary and therefore they are as smooth as the boundary. Hence, for domains D of class C^2 there exists a solution in the sense of Theorem 6.21. Therefore, we can apply the Stratton–Chu formulas (6.16) and (6.17) for the scattered field E^s, H^s and the Stratton–Chu formulas (6.11) and (6.12) for the incident field E^i, H^i. Then, adding both formulas and using the boundary condition $\nu \times (E^i + E^s) = 0$ on ∂D, we have the following theorem known as *Huygens' principle*. The representation for the far field pattern is obtained with the aid of (6.26) and (6.27).

Theorem 6.24. *For the scattering of an entire electromagnetic field* E^i, H^i *by a perfect conductor* D *we have*

$$\begin{aligned} E(x) &= E^i(x) - \frac{1}{ik}\,\mathrm{curl}\,\mathrm{curl} \int_{\partial D} \nu(y) \times H(y)\, \Phi(x, y)\, ds(y), \\ H(x) &= H^i(x) + \mathrm{curl} \int_{\partial D} \nu(y) \times H(y)\, \Phi(x, y)\, ds(y) \end{aligned} \tag{6.58}$$

for $x \in \mathbb{R}^3 \setminus \bar{D}$ *where* E, H *is the total field. The far field pattern is given by*

$$\begin{aligned} E_\infty(\hat{x}) &= \frac{ik}{4\pi}\, \hat{x} \times \int_{\partial D} [\nu(y) \times H(y)] \times \hat{x}\, e^{-ik\,\hat{x}\cdot y}\, ds(y), \\ H_\infty(\hat{x}) &= \frac{ik}{4\pi}\, \hat{x} \times \int_{\partial D} \nu(y) \times H(y)\, e^{-ik\,\hat{x}\cdot y}\, ds(y) \end{aligned} \tag{6.59}$$

for $\hat{x} \in \mathbb{S}^2$.

6.5 Vector Wave Functions

For any orthonormal system Y_n^m, $m = -n, \dots, n$, of spherical harmonics of order $n > 0$, the tangential fields on the unit sphere

$$U_n^m := \frac{1}{\sqrt{n(n+1)}}\,\operatorname{Grad} Y_n^m, \quad V_n^m := \nu \times U_n^m \tag{6.60}$$

are called *vector spherical harmonics* of order n. Since in spherical coordinates (θ, φ) we have

$$\operatorname{Div}\operatorname{Grad} f = \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\,\sin\theta\,\frac{\partial f}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2 f}{\partial\varphi^2}\,,$$

we can rewrite (2.17) in the form

$$\operatorname{Div}\operatorname{Grad} Y_n + n(n+1)\,Y_n = 0 \tag{6.61}$$

for spherical harmonics Y_n of order n. Then, from (6.39) we deduce that

$$\int_{\mathbb{S}^2} \operatorname{Grad} Y_n^m \cdot \operatorname{Grad} \overline{Y_{n'}^{m'}}\,ds = n(n+1)\int_{\mathbb{S}^2} Y_n^m \overline{Y_{n'}^{m'}}\,ds.$$

Hence, in view of Stokes' theorem

$$\int_{\mathbb{S}^2} \nu \times \operatorname{Grad} U \cdot \operatorname{Grad} V\,ds = 0$$

for functions $U, V \in C^1(\mathbb{S}^2)$, the vector spherical harmonics are seen to form an orthonormal system in the space

$$L_t^2(\mathbb{S}^2) := \left\{a : \mathbb{S}^2 \to \mathbb{C}^3 : a \in L^2(\mathbb{S}^2),\ a\cdot\nu = 0\right\}$$

of tangential L^2 fields on the unit sphere. Analogous to Theorem 2.8, we wish to establish that this system is complete.

We first show that for a function $f \in C^2(\mathbb{S}^2)$ the Fourier expansion

$$f = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n^m Y_n^m$$

with Fourier coefficients

$$a_n^m = \int_{\mathbb{S}^2} f\,\overline{Y_n^m}\,ds$$

converges uniformly. Setting $\hat{y} = \hat{x}$ in the addition theorem (2.30) yields

$$\sum_{m=-n}^{n} \left|Y_n^m(\hat{x})\right|^2 = \frac{2n+1}{4\pi} \tag{6.62}$$

whence, by applying Div Grad and using (6.61),

$$\sum_{m=-n}^{n} \left|\operatorname{Grad} Y_n^m(\hat{x})\right|^2 = \frac{1}{4\pi}\, n(n+1)(2n+1) \tag{6.63}$$

readily follows. From (6.39) and (6.61) we see that

$$n(n+1)\int_{\mathbb{S}^2} f\,\overline{Y_n^m}\,ds = -\int_{\mathbb{S}^2} \operatorname{Div}\operatorname{Grad} f\,\overline{Y_n^m}\,ds$$

and therefore Parseval's equality, applied to Div Grad f, shows that

$$\sum_{n=0}^{\infty} n^2(n+1)^2 \sum_{m=-n}^{n} \left|a_n^m\right|^2 = \int_{\mathbb{S}^2} |\operatorname{Div}\operatorname{Grad} f|^2 ds.$$

The Cauchy–Schwarz inequality

$$\left[\sum_{n=1}^{N}\sum_{m=-n}^{n} \left|a_n^m Y_n^m(\hat{x})\right|\right]^2 \le \sum_{n=1}^{N} n^2(n+1)^2 \sum_{m=-n}^{n} \left|a_n^m\right|^2 \sum_{n=1}^{N} \frac{1}{n^2(n+1)^2} \sum_{m=-n}^{n} \left|Y_n^m(\hat{x})\right|^2$$

with the aid of (6.62) now yields a uniformly convergent majorant for the Fourier series of f.

Theorem 6.25. *The vector spherical harmonics U_n^m and V_n^m for $m = -n, \dots, n$, $n = 1, 2, \dots,$ form a complete orthonormal system in $L_t^2(\mathbb{S}^2)$.*

Proof. Assume that the tangential field a belongs to $C^3(\mathbb{S}^2)$ and denote by

$$\alpha_n^m := \int_{\mathbb{S}^2} \operatorname{Div} a\,\overline{Y_n^m}\,ds$$

the Fourier coefficients of Div a. Since Div $a \in C^2(\mathbb{S}^2)$, we have uniform convergence of the series representation

$$\operatorname{Div} a = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \alpha_n^m Y_n^m. \tag{6.64}$$

Note that $\alpha_0^0 = 0$ since $\int_{\mathbb{S}^2} \operatorname{Div} a\,ds = 0$. Now define

$$u := -\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=-n}^{n} \alpha_n^m Y_n^m. \tag{6.65}$$

Then proceeding as above in the series for f with the aid of Parseval's equality for Div Grad Div a and (6.63) we can show that the term by term derivatives of the series for u are uniformly convergent. Hence, (6.65) defines a function $u \in C^1(\mathbb{S}^2)$. From the uniform convergence of the series (6.64) for $\operatorname{Div} a$ and (6.61), we observe that

$$\operatorname{Div} \operatorname{Grad} u = \operatorname{Div} a \tag{6.66}$$

in the sense of Definition 6.15.

Analogously, with the Fourier coefficients

$$\beta_n^m := \int_{\mathbb{S}^2} \operatorname{Curl} a \, \overline{Y_n^m} \, ds$$

of the uniformly convergent expansion

$$\operatorname{Curl} a = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \beta_n^m Y_n^m \tag{6.67}$$

we define a second function $v \in C^1(\mathbb{S}^2)$ by

$$v := -\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=-n}^{n} \beta_n^m Y_n^m . \tag{6.68}$$

It satisfies

$$\operatorname{Div} \operatorname{Grad} v = \operatorname{Curl} a. \tag{6.69}$$

Then the tangential field

$$b := \operatorname{Grad} u + \nu \times \operatorname{Grad} v$$

is continuous and in view of (6.43), (6.66) and (6.69) it satisfies

$$\operatorname{Div} b = \operatorname{Div} a, \quad \operatorname{Curl} b = \operatorname{Curl} a. \tag{6.70}$$

In view of Stokes' theorem (6.40), the second equation of (6.70) implies that $a - b = \operatorname{Grad} f$ for some $f \in C^1(\mathbb{S}^2)$. Then the first equation of (6.70) requires f to satisfy $\operatorname{Div} \operatorname{Grad} f = 0$ and from (6.39) we see that

$$\int_{\mathbb{S}^2} |\operatorname{Grad} f|^2 ds = 0$$

which implies $a = b$.

Thus, we have established that three times continuously differentiable tangential fields can be expanded into a uniformly convergent series with respect to the vector spherical harmonics. The proof is now completed by a denseness argument. □

We now formulate the analogue of Theorem 2.10 for *spherical vector wave functions*.

Theorem 6.26. *Let Y_n be a spherical harmonic of order $n \geq 1$. Then the pair*

$$M_n(x) = \operatorname{curl}\left\{x j_n(k|x|)\, Y_n\left(\frac{x}{|x|}\right)\right\}, \qquad \frac{1}{ik}\operatorname{curl} M_n(x)$$

is an entire solution to the Maxwell equations and

$$N_n(x) = \operatorname{curl}\left\{x h_n^{(1)}(k|x|)\, Y_n\left(\frac{x}{|x|}\right)\right\}, \qquad \frac{1}{ik}\operatorname{curl} N_n(x)$$

is a radiating solution to the Maxwell equations in $\mathbb{R}^3 \setminus \{0\}$.

Proof. We use (6.4) and Theorems 2.10 and 6.4 to verify that the Maxwell equations are satisfied in $\mathbb{R}^3$ and $\mathbb{R}^3 \setminus \{0\}$, respectively. Setting $\hat{x} := x/|x|$ we also compute

$$\begin{aligned} M_n(x) &= j_n(k|x|)\operatorname{Grad} Y_n(\hat{x}) \times \hat{x}, \\ N_n(x) &= h_n^{(1)}(k|x|)\operatorname{Grad} Y_n(\hat{x}) \times \hat{x}, \end{aligned} \tag{6.71}$$

and

$$\begin{aligned} \hat{x} \times \operatorname{curl} M_n(x) &= \frac{1}{|x|}\,\{j_n(k|x|) + k|x|j_n'(k|x|)\}\,\hat{x} \times \operatorname{Grad} Y_n(\hat{x}), \\ \hat{x} \times \operatorname{curl} N_n(x) &= \frac{1}{|x|}\,\left\{h_n^{(1)}(k|x|) + k|x|h_n^{(1)\prime}(k|x|)\right\}\hat{x} \times \operatorname{Grad} Y_n(\hat{x}). \end{aligned} \tag{6.72}$$

Hence, the Silver–Müller radiation condition for N_n, $\operatorname{curl} N_n/ik$ follows with the aid of the asymptotic behavior (2.42) of the spherical Hankel functions. □

Theorem 6.27. *Let E, H be a radiating solution to the Maxwell equations for $|x| > R > 0$. Then E has an expansion with respect to spherical vector wave functions of the form*

$$\begin{aligned} E(x) = &\sum_{n=1}^{\infty}\sum_{m=-n}^{n} a_n^m \operatorname{curl}\left\{x h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right)\right\} \\ &+ \sum_{n=1}^{\infty}\sum_{m=-n}^{n} b_n^m \operatorname{curl}\operatorname{curl}\left\{x h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right)\right\} \end{aligned} \tag{6.73}$$

that (together with its derivatives) converges uniformly on compact subsets of $|x| > R$. Conversely, if the tangential component of the series (6.73) converges in the mean square sense on the sphere $|x| = R$ then the series itself converges (together with its derivatives) uniformly on compact subsets of $|x| > R$ and E, $H = \operatorname{curl} E/ik$ represent a radiating solution to the Maxwell equations.

Proof. By Theorem 6.3, the tangential component of the electric field E on a sphere $|x| = \tilde{R}$ with $\tilde{R} > R$ is analytic. Hence, as shown before the proof of Theorem 6.25, it can be expanded in a uniformly convergent Fourier series with respect to spherical vector harmonics. The spherical Hankel functions $h_n^{(1)}(t)$ and $h_n^{(1)}(t) + t h_n^{(1)\prime}(t)$ do not have real zeros since the Wronskian (2.37) does not vanish. Therefore, in view of (6.71) and (6.72), setting $\hat{x} := x/|x|$ we may write the Fourier expansion in the form

$$\hat{x} \times E(x) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} a_n^m \, \hat{x} \times \operatorname{curl} \left\{ x h_n^{(1)}(k|x|) \, Y_n^m(\hat{x}) \right\}$$

$$+ \sum_{n=1}^{\infty} \sum_{m=-n}^{n} b_n^m \, \hat{x} \times \operatorname{curl} \operatorname{curl} \left\{ x h_n^{(1)}(k|x|) \, Y_n^m(\hat{x}) \right\}$$

for $|x| = \tilde{R}$. But now, by the continuous dependence of the solution to the exterior Maxwell problem with continuous tangential component in the maximum norm (see Theorem 6.23), the uniform convergence of the latter series implies the uniform convergence of the series (6.73) (together with its derivatives) on compact subsets of $|x| > \tilde{R}$ and the first statement of the theorem is proven.

Conversely, proceeding as in the proof of Theorem 2.15, with the help of (6.63) it can be shown that L^2 convergence of the tangential component of the series (6.73) on the sphere $|x| = R$ implies uniform convergence of the tangential component on any sphere $|x| = \tilde{R} > R$. Therefore, the second part of the theorem follows from the first part applied to the solution to the exterior Maxwell problem with continuous tangential components given on a sphere $|x| = \tilde{R}$. □

Theorem 6.28. *The electric far field pattern of the radiating solution to the Maxwell equations with the expansion (6.73) is given by*

$$E_\infty = \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{i^{n+1}} \sum_{m=-n}^{n} \{ ikb_n^m \operatorname{Grad} Y_n^m - a_n^m \, \nu \times \operatorname{Grad} Y_n^m \} . \tag{6.74}$$

The coefficients in this expansion satisfy the growth condition

$$\sum_{n=1}^{\infty} \left(\frac{2n}{ker} \right)^{2n} \sum_{m=-n}^{n} \left| a_n^m \right|^2 + \left| b_n^m \right|^2 < \infty \tag{6.75}$$

for all $r > R$.

Proof. Since by Theorem 6.9 the far field pattern E_∞ is analytic, we have an expansion

$$E_\infty = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \{ c_n^m \operatorname{Grad} Y_n^m + d_n^m \, \nu \times \operatorname{Grad} Y_n^m \}$$

with coefficients

$$n(n+1) \, c_n^m = \int_{\mathbb{S}^2} E_\infty(\hat{x}) \cdot \operatorname{Grad} \overline{Y_n^m(\hat{x})} \, ds(\hat{x}),$$

$$n(n+1) \, d_n^m = \int_{\mathbb{S}^2} E_\infty(\hat{x}) \cdot \hat{x} \times \operatorname{Grad} \overline{Y_n^m(\hat{x})} \, ds(\hat{x}).$$

On the other hand, in view of (6.71) and (6.72), the coefficients a_n^m and b_n^m in the expansion (6.73) satisfy

$$n(n+1)\, a_n^m\, h_n^{(1)}(k|x|) = \int_{\mathbb{S}^2} E(r\hat{x}) \cdot \operatorname{Grad} \overline{Y_n^m(\hat{x})} \times \hat{x}\, ds(\hat{x}),$$

$$n(n+1)\, b_n^m \left\{ k h_n^{(1)\prime}(k|x|) + \frac{1}{|x|}\, h_n^{(1)}(k|x|) \right\} = \int_{\mathbb{S}^2} E(r\hat{x}) \cdot \operatorname{Grad} \overline{Y_n^m(\hat{x})}\, ds(\hat{x}).$$

Therefore, with the aid of (2.42) we find that

$$n(n+1)\, d_n^m = \int_{\mathbb{S}^2} \lim_{r\to\infty} r\, e^{-ikr} E(r\hat{x}) \cdot \hat{x} \times \operatorname{Grad} \overline{Y_n^m(\hat{x})}\, ds(\hat{x})$$

$$= \lim_{r\to\infty} r\, e^{-ikr} \int_{\mathbb{S}^2} E(r\hat{x}) \cdot \hat{x} \times \operatorname{Grad} \overline{Y_n^m(\hat{x})}\, ds(\hat{x}) = -\frac{n(n+1)\, a_n^m}{k\, i^{n+1}}$$

and

$$n(n+1)\, c_n^m = \int_{\mathbb{S}^2} \lim_{r\to\infty} r\, e^{-ikr} E(r\hat{x}) \operatorname{Grad} \overline{Y_n^m(\hat{x})}\, ds(\hat{x})$$

$$= \lim_{r\to\infty} r\, e^{-ikr} \int_{\mathbb{S}^2} E(r\hat{x}) \operatorname{Grad} \overline{Y_n^m(\hat{x})}\, ds(\hat{x}) = \frac{n(n+1)\, b_n^m}{i^n}\ .$$

In particular, this implies that the expansion (6.74) is valid in the L^2 sense. Parseval's equality for the expansion (6.73) reads

$$r^2 \sum_{n=1}^{\infty} \sum_{m=-n}^{n} n(n+1) \left\{ \left|a_n^m\right|^2 \left|h_n^{(1)}(kr)\right|^2 + \left|b_n^m\right|^2 \left| k h_n^{(1)\prime}(kr) + \frac{1}{r}\, h_n^{(1)}(kr)\right|^2 \right\}$$

$$= \int_{|x|=r} |\nu \times E|^2 ds(x).$$

From this, using the asymptotic behavior (2.40) of the Hankel functions for large order n, the condition (6.75) follows. □

Analogously to Theorem 2.17, it can be shown that the growth condition (6.75) for the Fourier coefficients of a tangential field $E_\infty \in L_t^2(\mathbb{S}^2)$ is sufficient for E_∞ to be the far field pattern of a radiating solution to the Maxwell equations.

Concluding this section, we wish to apply Theorem 6.27 to derive the vector analogue of the addition theorem (2.43). From (6.71) and (6.72), we see that the coefficients in the expansion (6.73) can be expressed by

$$n(n+1) R^2 h_n^{(1)}(kR) a_n^m = \int_{|x|=R} E(x) \cdot \operatorname{Grad} \overline{Y_n^m(\hat{x})} \times \hat{x}\, ds(x), \tag{6.76}$$

$$n(n+1) R \{h_n^{(1)}(kR) + kR h_n^{(1)\prime}(kR)\} b_n^m = \int_{|x|=R} E(x) \cdot \operatorname{Grad} \overline{Y_n^m(\hat{x})}\, ds(x). \tag{6.77}$$

Given a set Y_n^m, $m = -n, \dots, n$, $n = 0, 1, \dots$, of orthonormal spherical harmonics, we recall the functions

$$u_n^m(x) := j_n(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right), \quad v_n^m(x) := h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right)$$

and set

$$M_n^m(x) := \operatorname{curl}\{x u_n^m(x)\}, \quad N_n^m(x) := \operatorname{curl}\{x v_n^m(x)\}.$$

For convenience, we also write

$$\tilde{M}_n^m(x) := \operatorname{curl}\left\{x\, j_n(k|x|)\, \overline{Y_n^m\left(\frac{x}{|x|}\right)}\right\}, \quad \tilde{N}_n^m(x) := \operatorname{curl}\left\{x h_n^{(1)}(k|x|)\, \overline{Y_n^m\left(\frac{x}{|x|}\right)}\right\}.$$

From the Stratton–Chu formula (6.8) applied to $\tilde{M}_n^m$ and $\operatorname{curl} \tilde{M}_n^m / ik$, we have

$$\frac{1}{k^2} \operatorname{curl}\operatorname{curl} \int_{|x|=R} \nu(x) \times \operatorname{curl} \tilde{M}_n^m(x)\, \Phi(y, x)\, ds(x)$$

$$+ \operatorname{curl} \int_{|x|=R} \nu(x) \times \tilde{M}_n^m(x)\, \Phi(y, x)\, ds(x) = -\tilde{M}_n^m(y), \quad |y| < R,$$

and from the Stratton–Chu formula (6.16) applied to $\tilde{N}_n^m$ and $\operatorname{curl} \tilde{N}_n^m / ik$, we have

$$\frac{1}{k^2} \operatorname{curl}\operatorname{curl} \int_{|x|=R} \nu(x) \times \operatorname{curl} \tilde{N}_n^m(x)\, \Phi(y, x)\, ds(x)$$

$$+ \operatorname{curl} \int_{|x|=R} \nu(x) \times \tilde{N}_n^m(x)\, \Phi(y, x)\, ds(x) = 0, \quad |y| < R.$$

Using (6.71), (6.72) and the Wronskian (2.37), from the last two equations we see that

$$\frac{i}{k^3 R^2} \operatorname{curl}\operatorname{curl} \int_{|x|=R} \nu(x) \times \operatorname{Grad} \overline{Y_n^m\left(\frac{x}{|x|}\right)}\, \Phi(y, x)\, ds(x) = h_n^{(1)}(kR) \tilde{M}_n^m(y), \quad |y| < R,$$

whence

$$\begin{aligned} &\int_{|x|=R} \operatorname{Grad} \overline{Y_n^m\left(\frac{x}{|x|}\right)} \times \nu(x) \cdot \operatorname{curl}_x \operatorname{curl}_x \{p \Phi(y, x)\}\, ds(x) \\ &\qquad = i k^3 R^2 h_n^{(1)}(kR)\, p \cdot \overline{M_n^m(y)}, \quad |y| < R, \end{aligned} \tag{6.78}$$

follows for all $p \in \mathbb{R}^3$ with the aid of the vector identity

$$p \cdot \operatorname{curl}_y \operatorname{curl}_y \{c(x) \Phi(y, x)\} = c(x) \cdot \operatorname{curl}_x \operatorname{curl}_x \{p \Phi(y, x)\}.$$

Analogously, from the Stratton–Chu formulas (6.9) and (6.17) for the magnetic field, we can derive that

$$\begin{aligned}&\int_{|x|=R} \operatorname{Grad} \overline{Y_n^m\left(\frac{x}{|x|}\right)} \cdot \operatorname{curl}_x \operatorname{curl}_x\{p\Phi(y,x)\}\, ds(x)\\&= ikR\{h_n^{(1)}(kR) + kRh_n^{(1)\prime}(kR)\}\, p \cdot \operatorname{curl} \overline{M_n^m(y)}, \quad |y| < R.\end{aligned} \tag{6.79}$$

Therefore, from (6.76)–(6.79) we can derive the expansion

$$\begin{aligned}&\frac{1}{k^2} \operatorname{curl}_x \operatorname{curl}_x\{p \cdot \Phi(y,x)\}\\&= ik \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=-n}^{n} N_n^m(x)\, \overline{M_n^m(y)} \cdot p\\&+ \frac{i}{k} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=-n}^{n} \operatorname{curl} N_n^m(x)\, \operatorname{curl} \overline{M_n^m(y)} \cdot p\end{aligned} \tag{6.80}$$

which, for fixed y, converges uniformly (together with its derivatives) with respect to x on compact subsets of $|x| > |y|$. Interchanging the roles of x and y and using a corresponding expansion of solutions to the Maxwell equations in the interior of a sphere, it can be seen that for fixed x the series (6.80) also converges uniformly (together with its derivatives) with respect to y on compact subsets of $|x| > |y|$.

Using the vector identity

$$\operatorname{div}_x p \cdot \Phi(y,x) = -p \cdot \operatorname{grad}_y \Phi(y,x),$$

from the addition theorem (2.43) we find that

$$\frac{1}{k^2} \operatorname{grad}_x \operatorname{div}_x\{p \cdot \Phi(y,x)\} = -\frac{i}{k} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \operatorname{grad} v_n^m(x)\, \operatorname{grad} \overline{u_n^m(y)} \cdot p. \tag{6.81}$$

In view of the continuous dependence results of Theorem 3.11, it can be seen the series (6.81) has the same convergence properties as (6.80). Finally, using the vector identity (6.4), we can use (6.80) and (6.81) to establish the following *vector addition theorem* for the fundamental solution.

Theorem 6.29. *We have*

$$\begin{aligned}\Phi(x,y)I &= ik \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=-n}^{n} N_n^m(x)\, \overline{M_n^m(y)}^{\top}\\&+ \frac{i}{k} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=-n}^{n} \operatorname{curl} N_n^m(x)\, \operatorname{curl} \overline{M_n^m(y)}^{\top}\\&+ \frac{i}{k} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \operatorname{grad} v_n^m(x)\, \operatorname{grad} \overline{u_n^m(y)}^{\top}\end{aligned} \tag{6.82}$$

where the series and its term by term derivatives are uniformly convergent for fixed y with respect to x and, conversely, for fixed x with respect to y on compact subsets of $|x| > |y|$.

By taking the transpose of (6.82) and interchanging the roles of x and y we obtain the alternate form

$$\begin{aligned}\Phi(x,y)I = ik\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{m=-n}^{n}\overline{M_n^m(x)}\,N_n^m(y)^\top\\ +\frac{i}{k}\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{m=-n}^{n}\operatorname{curl}\overline{M_n^m(x)}\,\operatorname{curl}N_n^m(y)^\top\\ +\frac{i}{k}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\operatorname{grad}\overline{u_n^m(x)}\,\operatorname{grad}v_n^m(y)^\top\end{aligned}\tag{6.83}$$

of the vector addition theorem for $|x| < |y|$.

We conclude this section with the vector analogue of the Jacobi–Anger expansion (2.46). For two vectors $d, p \in \mathbb{R}^3$ with $|d| = 1$ and $p \cdot d = 0$ we have the expansion

$$p\, e^{ik\, x\cdot d} = \sum_{n=1}^{\infty}\frac{(2n+1)\, i^n}{n(n+1)}\,\{D(x,d)\, x j_n(x)P_n(\cos\theta)\}p \tag{6.84}$$

with the differential operator

$$D(x,d) := -\operatorname{curl}_x\{d\times \operatorname{Grad}_d\}^\top - \frac{i}{k}\operatorname{curl}_x\operatorname{curl}_x\operatorname{Grad}_d^\top .$$

Here P_n denotes the Legendre polynomial of order n and θ the angle between x and d. The series (6.84) can be obtained by passing to the limit $|y| \to \infty$ in the vector addition theorem (6.83) with the help of Theorem 6.28 and the addition theorem for the spherical harmonics (2.30). It can be shown to converge uniformly on compact subsets of $\mathbb{R}^3 \times \mathbb{S}^2$ (together with all its derivatives). Clearly, (6.84) represents an expansion of electromagnetic plane waves with respect to the entire solutions of the Maxwell equations of Theorem 6.26.

The expansion (6.84) can be used to obtain an explicit solution for scattering of a plane wave

$$E^i(x) = p\, e^{ik\, x\cdot d}, \quad H^i(x) = d\times p\, e^{ik\, x\cdot d}$$

with incident direction d and polarization $p \perp d$ by a perfectly conducting ball of radius R centered at the origin in terms of the so-called *Mie series*. In view of (6.71) and (6.72), from (6.84) and the boundary condition $\nu\times(E^i + E^s) = 0$ on $|x| = R$, we expect the scattered wave to be given by

$$E^s(x) = -\sum_{n=1}^{\infty}\frac{(2n+1)i^n}{n(n+1)}\left\{D_n(x,d)\, x h_n^{(1)}(x)P_n(\cos\theta)\right\}p \tag{6.85}$$

with the differential operators

$$D_n(x,d) := -\frac{j_n(kR)}{h_n^{(1)}(kR)}\ \operatorname{curl}_x \{d \times \operatorname{Grad}_d\}^\top$$
$$-\frac{i}{k}\ \frac{j_n(kR) + kRj_n'(kR)}{h_n^{(1)}(kR) + kRh_n^{(1)\prime}(kR)}\ \operatorname{curl}\operatorname{curl}_x \operatorname{Grad}_d^\top, \quad n \in \mathbb{N}.$$

By the asymptotic behavior (2.38) and (2.39) of the spherical Bessel and Hankel functions for large n the above series can be shown to converge uniformly on compact subsets of $\mathbb{R}^3 \setminus \{0\}$. Consequently, by Theorem 6.27, together with the corresponding series for the magnetic field it represents a radiating solution to the Maxwell equations in $\mathbb{R}^3 \setminus \{0\}$, and hence indeed solves the scattering problem for a perfectly conducting ball.

6.6 Herglotz Pairs and the Far Field Operator

We consider the scattering of electromagnetic plane waves with incident direction $d \in \mathbb{S}^2$ and polarization vector p as described by the matrices $E^i(x,d)$ and $H^i(x,d)$ defined by

$$\begin{aligned} E^i(x,d)p &:= \frac{i}{k}\operatorname{curl}\operatorname{curl} p\, e^{ik\,x\cdot d} = ik\,(d\times p)\times d\, e^{ik\,x\cdot d}, \\ H^i(x,d)p &= \operatorname{curl} p\, e^{ik\,x\cdot d} = ik\,d\times p\, e^{ik\,x\cdot d}. \end{aligned} \tag{6.86}$$

Because of the linearity of the direct scattering problem with respect to the incident field, we can express the scattered waves by matrices $E^s(x,d)$ and $H^s(x,d)$, the total waves by matrices $E(x,d)$ and $H(x,d)$, and the far field patterns by $E_\infty(\hat{x},d)$ and $H_\infty(\hat{x},d)$, respectively. The latter map the polarization vector p onto the far field patterns $E_\infty(\hat{x},d)p$ and $H_\infty(\hat{x},d)p$, respectively.

Analogously to Theorem 3.15, we can establish the following reciprocity result for the electromagnetic case.

Theorem 6.30. *The electric far field pattern for the scattering of plane electromagnetic waves by a perfect conductor satisfies the* reciprocity relation

$$E_\infty(\hat{x},d) = [E_\infty(-d,-\hat{x})]^\top, \quad \hat{x}, d \in \mathbb{S}^2. \tag{6.87}$$

Proof. From Gauss' divergence theorem, the Maxwell equations for the incident and the scattered fields and the radiation condition for the scattered field we have

$$\int_{\partial D} \{\nu \times E^i(\cdot\,,d)p \cdot H^i(\cdot\,,-\hat{x})q + \nu \times H^i(\cdot\,,d)p \cdot E^i(\cdot\,,-\hat{x})q\}ds = 0$$

and

$$\int_{\partial D} \{\nu \times E^s(\cdot\,,d)p \cdot H^s(\cdot\,,-\hat{x})q + \nu \times H^s(\cdot\,,d)p \cdot E^s(\cdot\,,-\hat{x})q\}ds = 0$$

for all $p, q \in \mathbb{R}^3$. With the aid of

$$\operatorname{curl}_y q\, e^{-ik\,\hat{x}\cdot y} = ik\, q \times \hat{x}\, e^{-ik\,\hat{x}\cdot y},$$

and

$$\operatorname{curl}_y \operatorname{curl}_y q\, e^{-ik\,\hat{x}\cdot y} = k^2\, \hat{x} \times \{q \times \hat{x}\}\, e^{-ik\,\hat{x}\cdot y},$$

from the far field representation (6.25) we derive

$$\begin{aligned} &4\pi\, q \cdot E_\infty(\hat{x}, d)p \\ &= \int_{\partial D} \{\nu \times E^s(\cdot\,,d)p \cdot H^i(\cdot\,,-\hat{x})q + \nu \times H^s(\cdot\,,d)p \cdot E^i(\cdot\,,-\hat{x})q\}ds \end{aligned}$$

and from this by interchanging the roles of d and $\hat{x}$ and of p and q, respectively,

$$\begin{aligned} &4\pi\, p \cdot E_\infty(-d, -\hat{x})q \\ &= \int_{\partial D} \{\nu \times E^s(\cdot\,,-\hat{x})q \cdot H^i(\cdot\,,d)p + \nu \times H^s(\cdot\,,-\hat{x})q \cdot E^i(\cdot\,,d)p\}ds. \end{aligned}$$

We now subtract the last integral from the sum of the three preceding integrals to obtain

$$\begin{aligned} &4\pi\{q \cdot E_\infty(\hat{x}, d)p - p \cdot E_\infty(-d, -\hat{x})q\} \\ &= \int_{\partial D} \{\nu \times E(\cdot\,,d)p \cdot H(\cdot\,,-\hat{x})q + \nu \times H(\cdot\,,d)p \cdot E(\cdot\,,-\hat{x})q\}ds, \end{aligned} \tag{6.88}$$

whence the reciprocity relation (6.87) follows in view of the boundary condition $\nu \times E(\cdot\,,d)p = \nu \times E(\cdot\,,-\hat{x})q = 0$ on ∂D. □

Since in the derivation of (6.88) we only made use of the Maxwell equations for the incident wave in $\mathbb{R}^3$ and for the scattered wave in $\mathbb{R}^3 \setminus \bar{D}$ and the radiation condition, it is obvious that the reciprocity relation is also valid for the impedance boundary condition and the transmission boundary condition.

For the scattering of an electric dipole of the form (6.22), i.e.,

$$\begin{aligned} E_e^i(x, z)p &:= \frac{i}{k} \operatorname{curl}_x \operatorname{curl}_x p\Phi(x, z), \\ H_e^i(x, z)p &:= \operatorname{curl}_x p\Phi(x, z) \end{aligned} \tag{6.89}$$

with a polarization vector p we denote the scattered fields by $E^s_e(x,z)$ and $H^s_e(x,z)$, the total fields by $E_e(x,z)$ and $H_e(x,z)$ and the far field patterns of the scattered wave by $E^s_{e,\infty}(\hat{x},z)$ and $H^s_{e,\infty}(\hat{x},z)$. Note that as above for plane wave incidence all these quantities are matrices.

Theorem 6.31. *For scattering of electric dipoles and plane waves we have the* mixed reciprocity relation

$$4\pi E^s_{e,\infty}(-d,z) = [E^s(z,d)]^\top, \quad z \in \mathbb{R}^3 \setminus \bar{D},\ d \in \mathbb{S}^2. \tag{6.90}$$

Proof. As in the proof of Theorem 6.30 from Gauss' divergence theorem we have

$$\int_{\partial D} \{\nu \times E^i_e(\cdot\,,z)p \cdot H^i(\cdot\,,d)q + \nu \times H^i_e(\cdot\,,z)p \cdot E^i(\cdot\,,d)q\}ds = 0$$

and

$$\int_{\partial D} \{\nu \times E^s_e(\cdot\,,z)p \cdot H^s(\cdot\,,d)q + \nu \times H^s_e(\cdot\,,z)p \cdot E^s(\cdot\,,d)q\}ds = 0$$

for all $p,q \in \mathbb{R}^3$. From the far field representation (6.25) we obtain

$$4\pi\, q \cdot E_\infty(-d,z)p = \int_{\partial D} \{\nu \times E^s_e(\cdot\,,z)p \cdot H^i(\cdot\,,d)q + \nu \times H^s_e(\cdot\,,z)p \cdot E^i(\cdot\,,d)q\}ds$$

and the Stratton-Chu formula (6.16) yields

$$p \cdot E^s(z,d)q = \int_{\partial D} \{\nu \times E^s(\cdot\,,d)q \cdot H^i_e(\cdot\,,z)p + \nu \times H^s(\cdot\,,d)q \cdot E^i_e(\cdot\,,z)p\}ds.$$

Now the proof can be completed as in the previous theorem. □

Again the statement of Theorem 6.31 is valid for all boundary conditions. Since the far field pattern $E^i_{e,\infty}$ of the incident field E^i_e is given by

$$E^i_{e,\infty}(d,z) = \frac{1}{4\pi}\, E^i(z,-d) = \frac{1}{4\pi}\, [E^i(z,-d)]^\top, \tag{6.91}$$

from (6.90) we conclude that

$$E_{e,\infty}(d,z) = \frac{1}{4\pi}\, [E(z,-d)]^\top \tag{6.92}$$

for the far field pattern $E_{e,\infty}$ of the total field E_e.

The proof of the following theorem is analogous to that of the two preceding theorems.

Theorem 6.32. *For scattering of electric dipoles we have the* symmetry relation

$$E^s_e(x,y) = [E^s_e(y,x)]^\top, \quad x,y \in \mathbb{R}^3 \setminus \bar{D}. \tag{6.93}$$

Before we can state results on the completeness of electric far field patterns corresponding to Theorem 3.21, we have to introduce the concept of electromagnetic Herglotz pairs. Consider the vector Herglotz wave function

$$E(x) = \int_{\mathbb{S}^2} e^{ik\,x\cdot d} a(d)\,ds(d), \quad x \in \mathbb{R}^3,$$

with a vector Herglotz kernel $a \in L^2(\mathbb{S}^2)$, that is, the cartesian components of E are Herglotz wave functions. From

$$\operatorname{div} E(x) = ik \int_{\mathbb{S}^2} e^{ik\,x\cdot d}\, d\cdot a(d)\,ds(d), \quad x \in \mathbb{R}^3,$$

and Theorem 3.19 we see that the property of the kernel a to be tangential is equivalent to $\operatorname{div} E = 0$ in $\mathbb{R}^3$.

Definition 6.33 *An* electromagnetic Herglotz pair *is a pair of vector fields of the form*

$$E(x) = \int_{\mathbb{S}^2} e^{ik\,x\cdot d} a(d)\,ds(d), \quad H(x) = \frac{1}{ik}\operatorname{curl} E(x), \quad x \in \mathbb{R}^3, \tag{6.94}$$

where the square integrable tangential field a on the unit sphere $\mathbb{S}^2$ is called the Herglotz kernel of E, H.

Herglotz pairs obviously represent entire solutions to the Maxwell equations. For any electromagnetic Herglotz pair E, H with kernel a, the pair $H, -E$ is also an electromagnetic Herglotz pair with kernel $d \times a$. Using Theorem 3.30, we can characterize Herglotz pairs through a growth condition in the following theorem.

Theorem 6.34. *An entire solution E, H to the Maxwell equations possesses the growth property*

$$\sup_{R>0} \frac{1}{R} \int_{|x|\le R} \left\{ |E(x)|^2 + |H(x)|^2 \right\} dx < \infty \tag{6.95}$$

if and only if it is an electromagnetic Herglotz pair.

Proof. For a pair of entire solutions E, H to the vector Helmholtz equation, by Theorem 3.30, the growth condition (6.95) is equivalent to the property that E can be represented in the form (6.94) with a square integrable field a and the property $\operatorname{div} E = 0$ in $\mathbb{R}^3$ is equivalent to a being tangential. □

In the following analysis, we will utilize the analogue of Lemma 3.20 for the superposition of solutions to the electromagnetic scattering problem.

Lemma 6.35 *For a given L^2 field g on $\mathbb{S}^2$ the solution to the perfect conductor scattering problem for the incident wave*

$$\tilde{E}^i(x) = \int_{\mathbb{S}^2} E^i(x,d) g(d)\,ds(d), \quad \tilde{H}^i(x) = \int_{\mathbb{S}^2} H^i(x,d) g(d)\,ds(d)$$

is given by

$$\tilde{E}^s(x) = \int_{\mathbb{S}^2} E^s(x,d)g(d)\,ds(d), \quad \tilde{H}^s(x) = \int_{\mathbb{S}^2} H^s(x,d)g(d)\,ds(d)$$

for $x \in \mathbb{R}^3 \setminus \bar{D}$ *and has the far field pattern*

$$\tilde{E}_\infty(\hat{x}) = \int_{\mathbb{S}^2} E_\infty(\hat{x},d)g(d)\,ds(d), \quad \tilde{H}_\infty(\hat{x}) = \int_{\mathbb{S}^2} H_\infty(\hat{x},d)g(d)\,ds(d)$$

for $\hat{x} \in \mathbb{S}^2$.

Proof. Multiply (6.55) and (6.56) by g, integrate with respect to d over $\mathbb{S}^2$ and interchange orders of integration. □

We note that for a tangential field $g \in L^2_t(\mathbb{S}^2)$ we can write

$$\tilde{E}^i(x) = ik\int_{\mathbb{S}^2} g(d)\,e^{ik\,x\cdot d}\,ds(d), \quad \tilde{H}^i(x) = \operatorname{curl}\int_{\mathbb{S}^2} g(d)\,e^{ik\,x\cdot d}\,ds(d), \quad x \in \mathbb{R}^3,$$

that is, $\tilde{E}^i, \tilde{H}^i$ represents an electromagnetic Herglotz pair with kernel ikg.

Variants of the following completeness results were first obtained by Colton and Kress [65] and by Blöhbaum [23].

Theorem 6.36. *Let* (d_n) *be a sequence of unit vectors that is dense on* $\mathbb{S}^2$ *and define the set* $\mathcal{F}$ *of electric far field patterns by*

$$\mathcal{F} := \{E_\infty(\cdot\,, d_n)e_j : n = 1,2,\dots,\ j = 1,2,3\}$$

with the cartesian unit vectors e_j*. Then* $\mathcal{F}$ *is complete in* $L^2_t(\mathbb{S}^2)$ *if and only if there does not exist a nontrivial electromagnetic Herglotz pair* E, H *satisfying* $\nu \times E = 0$ *on* ∂D*.*

Proof. By the continuity of E_∞ as a function of d and the Reciprocity Theorem 6.30, the completeness condition

$$\int_{\mathbb{S}^2} h(\hat{x}) \cdot E_\infty(\hat{x},d_n)e_j\,ds(\hat{x}) = 0, \quad n = 1,2,\dots,\ j = 1,2,3,$$

for a tangential field $h \in L^2_t(\mathbb{S}^2)$ is equivalent to

$$\int_{\mathbb{S}^2} E_\infty(\hat{x},d)g(d)\,ds(d) = 0, \quad \hat{x} \in \mathbb{S}^2, \tag{6.96}$$

for $g \in L^2_t(\mathbb{S}^2)$ with $g(d) = h(-d)$.

By Theorem 3.19 and Lemma 6.35, the existence of a nontrivial tangential field g satisfying (6.96) is equivalent to the existence of a nontrivial Herglotz pair $\tilde{E}^i, \tilde{H}^i$ (with kernel ikg) for which the electric far field pattern of the corresponding scattered wave $\tilde{E}^s$ fulfills $\tilde{E}_\infty = 0$. By Theorem 6.10, the vanishing electric far field

$\tilde{E}_\infty = 0$ on $\mathbb{S}^2$ is equivalent to $\tilde{E}^s = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. This in turn, by the boundary condition $\nu \times \tilde{E}^i + \nu \times \tilde{E}^s = 0$ on ∂D and the uniqueness of the solution to the exterior Maxwell problem, is equivalent to $\nu\times\tilde{E}^i = 0$ on ∂D and the proof is finished. □

We note that the set $\mathcal{F}$ of electric far field patterns is linearly dependent. Since for $p = d$ the incident fields (6.86) vanish, the corresponding electric far field pattern also vanishes. This implies

$$\sum_{j=1}^{3} d_j E_\infty(\cdot\,, d) e_j = 0 \tag{6.97}$$

for $d = (d_1, d_2, d_3)$. Of course, the completeness result of Theorem 6.36 also holds for the magnetic far field patterns.

A nontrivial electromagnetic Herglotz pair for which the tangential component of the electric field vanishes on the boundary ∂D is a Maxwell eigensolution, i.e., a nontrivial solution E, H to the Maxwell equations in D with homogeneous boundary condition $\nu\times E = 0$ on ∂D. Therefore, as in the acoustic case, we have the surprising result that the eigensolutions are connected to the exterior scattering problem.

From Theorem 6.34, with the aid of the differentiation formula (2.35), the integral (3.68) and the representations (6.71) and (6.72) for M_n and $\operatorname{curl} M_n$, we conclude that M_n, $\operatorname{curl} M_n/ik$ provide examples of electromagnetic Herglotz pairs. From (6.71) we observe that the spherical vector wave functions M_n describe Maxwell eigensolutions for a ball of radius R centered at the origin with the eigenvalues k given by the zeros of the spherical Bessel functions. The pairs $\operatorname{curl} M_n$, $-ikM_n$ are also electromagnetic Herglotz pairs. From (6.72) we see that the spherical vector wave functions $\operatorname{curl} M_n$ also yield Maxwell eigensolutions for the ball with the eigenvalues k given through $j_n(kR) + kRj_n'(kR) = 0$. By expansion of an arbitrary eigensolution with respect to vector spherical harmonics, and arguing as in the proof of Rellich's Lemma 2.12, it can be seen that all Maxwell eigensolutions in a ball must be spherical vector wave functions. Therefore, the eigensolutions for balls are always electromagnetic Herglotz pairs and by Theorem 6.36 the electric far field patterns for plane waves are not complete for a ball D when k is a Maxwell eigenvalue.

We can express the result of Theorem 6.36 also in terms of a far field operator.

Theorem 6.37. *The far field operator $F : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ defined by*

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} E_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2, \tag{6.98}$$

is injective and has dense range if and only if there does not exist a Maxwell eigensolution for D which is an electromagnetic Herglotz pair.

Proof. From the reciprocity relation (6.87), we easily derive that the L^2 adjoint $F^* : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ of F is given by

$$F^* g = \overline{RFR\bar{g}}, \tag{6.99}$$

where $R : L_t^2(\mathbb{S}^2) \to L_t^2(\mathbb{S}^2)$ is defined by

$$(Rg)(d) := g(-d).$$

The proof is now completed as in Theorem 3.22. □

In view of the linear dependence expressed by (6.97), in order to construct the operator F for each $d \in \mathbb{S}^2$, we obviously need to have the electric far field pattern for two linearly independent polarization vectors orthogonal to d. As in acoustic scattering the far field operator will play an important role in our analysis of the inverse electromagnetic obstacle scattering problem. Therefore we proceed with presenting its main properties.

Lemma 6.38 *The far field operator satisfies*

$$2\pi \{(Fg, h) + (g, Fh)\} = -(Fg, Fh), \tag{6.100}$$

where $(\cdot\,, \cdot)$ *denotes the inner product in* $L_t^2(\mathbb{S}^2)$.

Proof. If E_1^s, H_1^s and E_2^s, H_2^s are radiating solutions of the Maxwell equations with electric far field patterns $E_{1,\infty}$ and $E_{2,\infty}$, then from the far field asymptotics and Gauss' divergence theorem we deduce that

$$\int_{\partial D} \left\{\overline{H_2^s} \cdot (\nu \times E_1^s) - \overline{E_2^s} \cdot (\nu \times H_1^s)\right\} ds = 2 \int_{\mathbb{S}^2} E_{1,\infty} \cdot \overline{E_{2,\infty}}\, ds. \tag{6.101}$$

If E^s, H^s is a radiating solution to the Maxwell equations with electric far field pattern E_∞ and E_h^i, H_h^i is a Herglotz pair with kernel h then

$$\int_{\partial D} \overline{H_h^i} \cdot (\nu \times E^s)\, ds = \int_{\mathbb{S}^2} \overline{h(d)} \cdot \int_{\partial D} [H^i(\cdot\,, -d)]^\top [\nu \times E^s)]\, ds\, ds(d)$$

and

$$-\int_{\partial D} \overline{E_h^i} \cdot (\nu \times H^s)\, ds = \int_{\mathbb{S}^2} \overline{h(d)} \cdot \int_{\partial D} [E^i(\cdot\,, -d)]^\top [\nu \times H^s]\, ds\, ds(d).$$

Adding these two equations, with the aid of the far field representation of Theorem 6.9, we obtain that

$$\int_{\partial D} \left\{\overline{H_h^i} \cdot (\nu \times E^s) - \overline{E_h^i} \cdot (\nu \times H^s)\right\} ds = 4\pi \int_{\mathbb{S}^2} E_\infty \cdot \bar{h}\, ds. \tag{6.102}$$

Now let E_g^i, H_g^i and E_h^i, H_h^i be the Herglotz pairs with kernels $g, h \in L_t^2(\mathbb{S}^2)$, respectively, and let E_g, H_g and E_h, H_h be the total fields to the scattering problems with incident fields E_g^i, H_g^i and E_h^i, H_h^i , respectively. We denote by $E_{g,\infty}$ and $E_{h,\infty}$ the electric far field patterns corresponding to E_g, H_g and E_h, H_h, respectively. Then we can combine (6.101) and (6.102) to obtain

$$\begin{aligned} &2(Fg, Fh) + 4\pi(Fg, h) + 4\pi(g, Fh) \\ &= 2\int_{\mathbb{S}^2} E_{g,\infty}\, \overline{E_{h,\infty}}\, ds + 4\pi \int_{\mathbb{S}^2} E_{g,\infty}\, \bar{h}\, ds + 4\pi \int_{\mathbb{S}^2} \overline{E_{h,\infty}}\, ds \\ &= \int_{\partial D} \left\{ \overline{H_h} \cdot (\nu \times E_g) - \overline{E_h} \cdot (\nu \times H_g) \right\} ds. \end{aligned} \tag{6.103}$$

From this the lemma follows by using the boundary condition. □

Theorem 6.39. *The far field operator F is compact and normal, i.e., $FF^* = F^*F$, and hence has a countable number of eigenvalues.*

Proof. Since F is an integral operator with continuous kernel, it is compact. From (6.100) we obtain that

$$(g, F^*Fh) = -2\pi\{(g, Fh) + (g, F^*h)\}$$

for all $g, h \in L_t^2(\mathbb{S}^2)$ and therefore

$$F^*F = -2\pi(F + F^*). \tag{6.104}$$

Using (6.99) we can deduce that $(F^*g, F^*h) = (FR\bar{h}, FR\bar{g})$ and hence, from (6.100), it follows that

$$(F^*g, F^*h) = -2\pi\{(g, F^*h) + (F^*g, h)\}$$

for all $g, h \in L_t^2(\mathbb{S}^2)$. If we now proceed as in the derivation of (6.104), we find that

$$FF^* = -2\pi(F + F^*) \tag{6.105}$$

and the proof is finished. □

Corollary 6.40 *The scattering operator $S : L_t^2(\mathbb{S}^2) \to L_t^2(\mathbb{S}^2)$ defined by*

$$S := I + \frac{1}{2\pi} F \tag{6.106}$$

is unitary.

Proof. From (6.104) and (6.105) we see that $SS^* = S^*S = I$. □

In view of (6.106), the unitarity of S implies that the eigenvalues of F lie on the circle with center at $(-2\pi, 0)$ on the negative real axis and radius 2π.

The question of when we can find a superposition of incident electromagnetic plane waves of the form (6.86) such that the resulting far field pattern becomes a prescribed far field $\tilde{E}_\infty$ is examined in the following theorem.

Theorem 6.41. *Let $\tilde{E}^s, \tilde{H}^s$ be a radiating solution to the Maxwell equations with electric far field pattern $\tilde{E}_\infty$. Then the linear integral equation of the first kind*

$$\int_{\mathbb{S}^2} E_\infty(\hat{x}, d) g(d)\, ds(d) = \tilde{E}_\infty(\hat{x}), \quad \hat{x} \in \mathbb{S}^2, \tag{6.107}$$

possesses a solution $g \in L^2_t(\mathbb{S}^2)$ if and only if $\tilde{E}^s, \tilde{H}^s$ is defined in $\mathbb{R}^3 \setminus \bar{D}$ and continuous in $\mathbb{R}^3 \setminus D$ and the interior boundary value problem for the Maxwell equations

$$\operatorname{curl} \tilde{E}^i - ik\tilde{H}^i = 0, \quad \operatorname{curl} \tilde{H}^i + ik\tilde{E}^i = 0 \quad \text{in } D, \tag{6.108}$$

and

$$\nu \times (\tilde{E}^i + \tilde{E}^s) = 0 \quad \text{on } \partial D \tag{6.109}$$

is solvable and a solution $\tilde{E}^i, \tilde{H}^i$ is an electromagnetic Herglotz pair.

Proof. By Theorem 3.19 and Lemma 6.35, the solvability of the integral equation (6.107) for g is equivalent to the existence of a Herglotz pair $\tilde{E}^i, \tilde{H}^i$ (with kernel ikg) for which the electric far field pattern for the scattering by the perfect conductor D coincides with the given $\tilde{E}_\infty$, that is, the scattered electromagnetic field coincides with the given $\tilde{E}^s, \tilde{H}^s$. □

By reciprocity, the solvability of (6.107) is equivalent to the solvability of

$$\int_{\mathbb{S}^2} [E_\infty(\hat{x}, d)]^\top h(\hat{x})\, ds(\hat{x}) = \tilde{E}_\infty(-d), \quad d \in \mathbb{S}^2, \; j = 1, 2, 3, \tag{6.110}$$

where $h(\hat{x}) = g(-\hat{x})$. In the special cases where the prescribed scattered wave is given by the electromagnetic field

$$\tilde{E}^s(x) = \operatorname{curl} a\Phi(x, 0), \quad \tilde{H}^s(x) = \frac{1}{ik} \operatorname{curl} \tilde{E}^s(x)$$

of a magnetic dipole at the origin with electric far field pattern

$$\tilde{E}_\infty(\hat{x}) = \frac{ik}{4\pi}\, \hat{x} \times a,$$

or by the electromagnetic field

$$\tilde{E}^s(x) = \operatorname{curl} \operatorname{curl} a\Phi(x, 0), \quad \tilde{H}^s(x) = \frac{1}{ik} \operatorname{curl} \tilde{E}^s(x)$$

of an electric dipole with electric far field pattern

$$\tilde{E}_\infty(\hat{x}) = \frac{k^2}{4\pi}\, \hat{x} \times (a \times \hat{x}),$$

the connection between the solution to the integral equation (6.110) and the interior Maxwell problem (6.108), (6.109) was first obtained by Blöhbaum [23]. Corresponding completeness results for the impedance boundary condition were derived by Angell, Colton and Kress [8].

We now follow Colton and Kirsch [60] and indicate how a suitable linear combination of the electric and the magnetic far field patterns with orthogonal polarization vectors are complete without exceptional interior eigenvalues.

Theorem 6.42. *Let (d_n) be a sequence of unit vectors that is dense on $\mathbb{S}^2$ and let λ and μ be fixed nonzero real numbers. Then the set*

$$\left\{\lambda E_\infty(\cdot\,, d_n)e_j + \mu H_\infty(\cdot\,, d_n)(e_j \times d_n) : n = 1, 2, \dots,\ j = 1, 2, 3\right\}$$

is complete in $L^2_t(\mathbb{S}^2)$.

Proof. We use the continuity of E_∞ and H_∞ as a function of d, the orthogonality (6.24) and the reciprocity relation (6.87) to find that for a tangential field $h \in L^2_t(\mathbb{S}^2)$ the completeness condition

$$\int_{\mathbb{S}^2} h(\hat{x}) \cdot \{\lambda E_\infty(\hat{x}, d_n)e_j + \mu H_\infty(\hat{x}, d_n)(e_j \times d_n)\}\, ds(\hat{x}) = 0$$

for $n = 1, 2, \dots$ and $j = 1, 2, 3$ is equivalent to

$$\int_{\mathbb{S}^2} \{\lambda\, p \cdot E_\infty(-d, -\hat{x})h(\hat{x}) + \mu\, p \times d \cdot E_\infty(-d, -\hat{x})(h(\hat{x}) \times \hat{x})\}\, ds(\hat{x}) = 0$$

for all $p \in \mathbb{R}^3$, that is,

$$\int_{\mathbb{S}^2} \{\lambda E_\infty(-d, -\hat{x})h(\hat{x}) - \mu H_\infty(-d, -\hat{x})(h(\hat{x}) \times \hat{x})\}\, ds(\hat{x}) = 0, \quad d \in \mathbb{S}^2.$$

After relabeling and setting $g(d) := h(-d)$ this reads

$$\int_{\mathbb{S}^2} \{\lambda E_\infty(\hat{x}, d)g(d) + \mu H_\infty(\hat{x}, d)g(d) \times d)\}\, ds(d) = 0, \quad \hat{x} \in \mathbb{S}^2. \tag{6.111}$$

We define incident waves E^i_1, H^i_1 as Herglotz pair with kernel g and E^i_2, H^i_2 as Herglotz pair with kernel $g \times d$ and note that from

$$\frac{1}{ik}\,\operatorname{curl} \int_{\mathbb{S}^2} g(d) \times d\, e^{ik\, x\cdot d}\, ds(d) = \int_{\mathbb{S}^2} g(d)\, e^{ik\, x\cdot d}\, ds(d)$$

we have

$$H^i_2 = E^i_1 \quad \text{and} \quad E^i_2 = -H^i_1. \tag{6.112}$$

By Lemma 6.35, the condition (6.111) implies that $\lambda E^s_1 + \mu H^s_2 = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ whence by applying the curl we have

$$\lambda H^s_1 - \mu E^s_2 = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}. \tag{6.113}$$

Using Gauss' divergence theorem, the boundary conditions $\nu \times (E_1^i + E_1^s) = 0$ and $\nu \times (E_2^i + E_2^s) = 0$ on ∂D and the equations (6.112) and (6.113) we now deduce that

$$\lambda \int_{\partial D} \nu \times E_1^s \cdot \overline{H_1^s}\, ds = \mu \int_D \operatorname{div}\left\{E_1^i \times \overline{E_2^i}\right\} dx = i\mu k \int_D \left\{|E_1^i|^2 - |H_1^i|^2\right\} dx,$$

and this implies $E_1^s = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ by Theorem 6.11. Then (6.113) also yields $E_2^s = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. As a consequence of the boundary conditions, we obtain $\nu \times E_1^i = 0$ on ∂D and $\nu \times H_1^i = \nu \times E_2^i = 0$ on ∂D and from this we have $E_1^i = 0$ in D by the representation Theorem 6.2. It now follows that $E_1^i = 0$ in $\mathbb{R}^3$ by analyticity (Theorem 6.3). From this we finally arrive at $g = 0$, that is, $h = 0$ by Theorem 3.19. □

We continue with the vector analogue of the completeness Theorem 3.27.

Theorem 6.43. *Let (d_n) be a sequence of unit vectors that is dense on $\mathbb{S}^2$. Then the tangential components of the total magnetic fields*

$$\left\{\nu \times H(\cdot\,, d_n)e_j : n = 1, 2, \dots,\ j = 1, 2, 3\right\}$$

for the incident plane waves of the form (6.86) with directions d_n are complete in $L_t^2(\partial D)$.

Proof. From the decomposition (6.45), it can be deduced that the operator N maps tangential fields from the Sobolev space $H^1(\partial D)$ boundedly into $L_t^2(\partial D)$. Therefore, the composition $NPS_0^2 : L_t^2(\partial D) \to L_t^2(\partial D)$ is compact since by Theorem 3.6 the operator S_0 is bounded from $L^2(\partial D)$ into $H^1(\partial D)$. Since M has a weakly singular kernel, we thus have compactness of $M + iNPS_0^2$ from $L_t^2(\partial D)$ into $L_t^2(\partial D)$ and proceeding as in the proof of Theorem 6.23 we can show that $I + M + iNPS_0^2$ has a trivial nullspace in $L_t^2(\partial D)$. Hence, by the Riesz–Fredholm theory, $I + M + iNPS_0^2 : L_t^2(\partial D) \to L_t^2(\partial D)$ is bijective and has a bounded inverse.

From the representation formulas (6.58), the boundary condition $\nu \times E = 0$ on ∂D and the jump relations of Theorem 6.12, in view of the definitions (6.48) and (6.50) we deduce that the tangential component $b := PH$ of H solves the integral equation

$$b + M'b + iPS_0^2Nb = 2\{\nu \times H^i\} \times \nu - 2kPS_0^2\{\nu \times E^i\}.$$

Now let $g \in L_t^2(\partial D)$ satisfy

$$\int_{\partial D} g \cdot H(\cdot\,, d_n)e_j\, ds = 0, \quad n = 1, 2, \dots,\ j = 1, 2, 3.$$

This, by the continuity of the electric to magnetic boundary component map (Theorem 6.22), implies that

$$\int_{\partial D} g \cdot H(\cdot\,, d)p\, ds = 0, \quad d \in \mathbb{S}^2,\ p \in \mathbb{R}^3.$$

We now set $a := (I + M + iNPS_0^2)^{-1}g$ and obtain

$$\int_{\partial D} a \cdot (I + M' + iPS_0^2 N)PH(\cdot\,, d)p\, ds = 0$$

and consequently

$$\int_{\partial D} \left\{ a \cdot H^i(\cdot\,, d)p + k\{\nu \times S_0^2 a\} \cdot E^i(\cdot\,, d)p \right\} ds = 0$$

for all $d \in \mathbb{S}^2$ and $p \in \mathbb{R}^3$. This in turn, by elementary vector algebra, implies that

$$\int_{\partial D} \left\{ a(y) \times d + k\, d \times [\{\nu(y) \times (S_0^2 a)(y)\} \times d] \right\} e^{ik\, y\cdot d}\, ds(y) = 0 \tag{6.114}$$

for all $d \in \mathbb{S}^2$. Now consider the electric field

$$E(x) = \operatorname{curl} \int_{\partial D} \Phi(x, y) a(y)\, ds(y)$$

$$+ i \operatorname{curl}\operatorname{curl} \int_{\partial D} \Phi(x, y)\, \nu(y) \times (S_0^2 a)(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D.$$

By (6.26) and (6.27), its far field pattern is given by

$$E_\infty(\hat{x}) = \frac{ik}{4\pi} \int_{\partial D} \left\{ \hat{x} \times a(y) + k\, \hat{x} \times [\{\nu(y) \times (S_0^2 a)(y)\} \times \hat{x}] \right\} e^{-ik\, \hat{x}\cdot y}\, ds(y), \quad \hat{x} \in \mathbb{S}^2.$$

Hence, the condition (6.114) implies that $E_\infty = 0$ on $\mathbb{S}^2$ and Theorem 6.10 yields $E = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. From this, with the help of the decomposition (6.45) and the L^2 jump relations (3.24) and (6.53), we derive that $a + Ma + iNPS_0^2 a = 0$ whence $g = 0$ follows and the proof is finished. □

With the tools involved in the proof of the preceding theorem we can also establish the following result which we shall need in our analysis of the inverse problem.

Theorem 6.44. *The operator $A : C^{0,\alpha}(\mathrm{Div}, \partial D) \to L_t^2(\mathbb{S}^2)$ which maps the electric tangential components of radiating solutions $E, H \in C^1(\mathbb{R}^3 \setminus D)$ to the Maxwell equations onto the electric far field pattern E_∞ can be extended to an injective bounded linear operator $A : L_t^2(\partial D) \to L_t^2(\mathbb{S}^2)$ with dense range.*

Proof. From the form (6.55) of the solution to the exterior Maxwell problem and (6.26) and (6.27), we derive

$$E_\infty(\hat{x}) = \frac{ik}{2\pi} \int_{\partial D} \left\{ \hat{x} \times a(y) + k\, \hat{x} \times [\{\nu(y) \times (S_0^2 a)(y)\} \times \hat{x}] \right\} e^{-ik\, \hat{x}\cdot y}\, ds(y), \quad \hat{x} \in \mathbb{S}^2,$$

where $a = (I + M + iNPS_0^2)^{-1}(\nu \times E)$. Boundedness and injectivity of A now follow by using the analysis of the previous proof.

In order to show that A has dense range we rewrite it as an integral operator. To this end we note that in terms of the electromagnetic plane waves E^i, H^i introduced in (6.86) the far field representation (6.25) for a radiating solution $\mathcal{E}, \mathcal{H}$ of the Maxwell equations can be written in the form

$$\mathcal{E}_\infty(\hat{x}) = \frac{1}{4\pi}\int_{\partial D}\left\{[H^i(y,-\hat{x})]^\top[\nu(y)\times\mathcal{E}(y)] + [E^i(y,-\hat{x})]^\top[\nu(y)\times\mathcal{H}(y)]\right\}ds(y)$$

for $\hat{x} \in \mathbb{S}^2$. From this, with the aid of the vector Green's theorem (6.3) (applied component wise) and the radiation condition, using the perfect conductor boundary condition for the total electric field $E = E^i + E^s$ on ∂D we conclude that

$$\mathcal{E}_\infty(\hat{x}) = \frac{1}{4\pi}\int_{\partial D}[H(y,-\hat{x})]^\top[\nu(y)\times\mathcal{E}(y)]\,ds(y), \quad \hat{x}\in\mathbb{S}^2,$$

with the total magnetic field $H = H^i + H^s$, that is,

$$(Ac)(d) = \frac{1}{4\pi}\int_{\partial D}[H(y,-d)]^\top c(y)\,ds(y), \quad d\in\mathbb{S}^2. \tag{6.115}$$

By interchanging the order of integration, from (6.115) we observe that the adjoint operator $A^* : L^2_t(\mathbb{S}^2) \to L^2_t(\partial D)$ can be represented as an integral operator by

$$(A^*g)(y) = \frac{1}{4\pi}\,\nu(y)\times\left\{\int_{\mathbb{S}^2}\overline{[H(y,-d)]}g(d)]\,ds(d)\times\nu(y)\right\}, \quad y\in\partial D. \tag{6.116}$$

If for $g \in L^2_t(\mathbb{S}^2)$ we define the vector Herglotz wave function E^i_g in the form

$$E^i_g(x) = ik\int_{\mathbb{S}^2}e^{-ik\,x\cdot d}\,\overline{g(d)}\,ds(d) = \int_{\mathbb{S}^2}E^i(x,-d)\overline{g(d)}\,ds(d), \quad x\in\mathbb{R}^3,$$

then from Lemma 6.35 we have that

$$(H_g)(x) = \int_{\mathbb{S}^2}H(x,-d)\overline{g(d)}\,ds(d), \quad x\in\mathbb{R}^3,$$

is the total magnetic field for scattering of E^i_g from D. Hence, we have that

$$\overline{A^*g} = \frac{1}{4\pi}\left\{\nu\times E_g - \mathcal{A}(\nu\times E_g|_{\partial D})\right\}\times\nu \tag{6.117}$$

with the electric to magnetic boundary component operator $\mathcal{A}$. Now let g satisfy $A^*g = 0$. Then (6.117) implies that $\nu\times H_g = 0$ on ∂D. By definition we also have $\nu\times E_g = 0$ on ∂D and therefore, by Holmgren's Theorem 6.5, it follows that $E_g = 0$ in $\mathbb{R}^3\setminus\bar{D}$. Hence, the entire solution E^i_g satisfies the radiation condition and therefore must vanish identically. Thus $g = 0$, i.e., A^* is injective. From this we can conclude denseness of the range of A by Theorem 4.6. □

Chapter 7
Inverse Electromagnetic Obstacle Scattering

This last chapter on obstacle scattering is concerned with the extension of the results from Chapter 5 on inverse acoustic scattering to inverse electromagnetic scattering. In order to avoid repeating ourselves, we keep this chapter short by referring back to the corresponding parts of Chapter 5 when appropriate. In particular, for notations and for the motivation of our analysis we urge the reader to get reacquainted with the corresponding analysis in Chapter 5 on acoustics. We again follow the general guideline of our book and consider only one of the many possible inverse electromagnetic obstacle problems: given the electric far field pattern for one or several incident plane electromagnetic waves and knowing that the scattering obstacle is perfectly conducting, find the shape of the scatterer.

We begin the chapter with a uniqueness result. Due to the lack of an appropriate selection theorem, we do not follow Schiffer's proof as in acoustics. Instead of this, we prove a uniqueness result following Isakov's approach and, in addition, we use a method based on differentiation with respect to the wave number. We also include the electromagnetic version of Karp's theorem.

We then proceed to establish a continuous dependence result on the boundary based on the integral equation approach. As an alternative for establishing Fréchet differentiability with respect to the boundary we present the electromagnetic version of an approach proposed by Kress and Päivärinta [207]. The following three sections then will present extensions of some of the iterative methods, decomposition methods and sampling methods considered in Chapter 5 from acoustics to electromagnetics. In particular we will present the electromagnetic versions of the iterative method due to Johansson and Sleeman, the decomposition methods of Kirsch and Kress and of Colton and Monk and conclude with a discussion of the linear sampling method in electromagnetic obstacle scattering.

7.1 Uniqueness

For the investigation of uniqueness in inverse electromagnetic obstacle scattering, as in the case of the Neumann and the impedance boundary condition in acous-

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences 93, DOI 10.1007/978-1-4614-4942-3_7,

tics, Schiffer's method of Theorem 5.1 cannot be applied since the appropriate selection theorem in electromagnetics requires the boundary to be sufficiently smooth (see [224]). However, the methods used in Theorem 5.6 for inverse acoustic scattering can be extended to the case of inverse electromagnetic scattering from perfect and impedance conductors. We consider boundary conditions of the form $BE = 0$ on ∂D, where $BE = \nu \times E$ for a perfect conductor and $BE = \nu \times \operatorname{curl} E - i\lambda\,(\nu \times E) \times \nu$ for the impedance boundary condition. In the latter case, the real-valued function λ is assumed to be continuous and positive to ensure well-posedness of the direct scattering problem as proven in Theorem 9.12.

Theorem 7.1. *Assume that D_1 and D_2 are two scatterers with boundary conditions B_1 and B_2 such that for a fixed wave number the electric far field patterns for both scatterers coincide for all incident directions and all polarizations. Then $D_1 = D_2$ and $B_1 = B_2$.*

Proof. The proof is completely analogous to that of Theorem 5.6 for the acoustic case which was based on the reciprocity relations from Theorems 3.16 and 3.17 and Holmgren's Theorem 2.3. In the electromagnetic case we have to use the reciprocity relations from Theorems 6.31 and 6.32 and Holmgren's Theorem 6.5 and instead of point sources $\Phi(\cdot\,, z)$ as incident fields we use electric dipoles $\operatorname{curl}\operatorname{curl} p\Phi(\cdot\,, z)$. □

A corresponding uniqueness result for the inverse electromagnetic transmission problem has been proven by Hähner [124].

For diversity, we now prove a uniqueness theorem for fixed direction and polarization.

Theorem 7.2. *Assume that D_1 and D_2 are two perfect conductors such that for one fixed incident direction and polarization the electric far field patterns of both scatterers coincide for all wave numbers contained in some interval $0 < k_1 < k < k_2 < \infty$. Then $D_1 = D_2$.*

Proof. We will use the fact that the scattered wave depends analytically on the wave number k. Deviating from our usual notation, we indicate the dependence on the wave number by writing $E^i(x; k)$, $E^s(x; k)$, and $E(x; k)$. Since the fundamental solution to the Helmholtz equation depends analytically on k, the integral operator $I + M + iNPS_0^2$ in the integral equation (6.56) is also analytic in k. (For the reader who is not familiar with analytic operators, we refer to Section 8.5.) From the fact that for each $k > 0$ the inverse operator of $I+M+iNPS_0^2$ exists, by using a Neumann series argument it can be deduced that the inverse $(I + M + iNPS_0^2)^{-1}$ is also analytic in k. Therefore, the analytic dependence of the right hand side $c = 2\,E^i(\cdot\,; k) \times \nu$ of (6.56) for the scattering problem implies that the solution a also depends analytically on k and consequently from the representation (6.55) it can be seen that the scattered field $E^s(\cdot\,; k)$ also depends analytically on k. In addition, from (6.55) it also follows that the derivatives of E^s with respect to the space variables and with respect to the wave number can be interchanged. Therefore, from the vector Helmholtz equation $\Delta E + k^2 E = 0$ for the total field $E = E^i + E^s$ we derive the inhomogeneous vector Helmholtz equation

$$\Delta F + k^2 F = -2kE$$

for the derivative

$$F := \frac{\partial E}{\partial k}\,.$$

Let k_0 be an accumulation point of the wave numbers for the incident waves and assume that $D_1 \neq D_2$. By Theorem 6.10, the electric far field pattern uniquely determines the scattered field. Hence, for any incident wave E^i the scattered wave E^s for both obstacles coincide in the unbounded component G of the complement of $D_1 \cup D_2$. Without loss of generality, we assume that $(\mathbb{R}^3 \setminus G) \setminus \bar{D}_2$ is a nonempty open set and denote by D^* a connected component of $(\mathbb{R}^3 \setminus G) \setminus \bar{D}_2$. Then E is defined in D^* since it describes the total wave for D_2, that is, E satisfies the vector Helmholtz equation in D^* and fulfills homogeneous boundary conditions $\nu \times E = 0$ and $\operatorname{div} E = 0$ on ∂D^* for each k with $k_1 < k < k_2$. By differentiation with respect to k, it follows that $F(\cdot\,;k_0)$ satisfies the same homogeneous boundary conditions. Therefore, from Green's vector theorem (6.3) applied to $E(\cdot\,;k_0)$ and $F(\cdot\,;k_0)$ we find that

$$2k_0 \int_{D^*} |E|^2 dx = \int_{D^*} \left\{ \bar{F}\Delta E - E\Delta\bar{F} \right\} dx = 0,$$

whence $E = 0$ first in D^* and then by analyticity everywhere outside $D_1 \cup D_2$. This implies that E^i satisfies the radiation condition whence $E^i = 0$ in $\mathbb{R}^3$ follows (c.f. p. 198). This is a contradiction. □

Concerning uniqueness for one incident wave under a priori assumptions on the shape of the scatterer we note that analogous to Theorem 5.4 using the explicit solution (6.85) it can be shown that a perfectly conducting ball is uniquely determined by the far field pattern for plane wave incidence with one direction d and polarization p. In the context of Theorem 5.5 it has been shown by Liu, Yamamoto and Zou [230] that a perfectly conducting polyhedron is uniquely determined by the far field pattern for plane wave incidence with one direction d and two polarizations p_1 and p_2. We note that our proof of Theorem 5.5 for a convex polyhedron can be carried over to the perfect conductor case.

We include in this section on uniqueness the electromagnetic counterpart of Karp's theorem for acoustics. If the perfect conductor D is a ball centered at the origin, it is obvious from symmetry considerations that the electric far field pattern for incoming plane waves of the from (6.86) satisfies

$$E_\infty(Q\hat{x}, Qd)Qp = QE_\infty(\hat{x}, d)p \tag{7.1}$$

for all $\hat{x}, d \in \mathbb{S}^2$, all $p \in \mathbb{R}^3$ and all rotations Q, i.e., for all real orthogonal matrices Q with $\det Q = 1$. As shown by Colton and Kress [66], the converse of this statement is also true. We include a simplified version of the original proof.

The vectors $\hat{x}$, $p \times \hat{x}$ and $\hat{x} \times (p \times \hat{x})$ form a basis in $\mathbb{R}^3$ provided $p \times \hat{x} \neq 0$. Hence, since the electric far field pattern is orthogonal to $\hat{x}$, we can write

$$E_\infty(\hat{x}, d)p = [e_1(\hat{x}, d)p]\; p \times \hat{x} + [e_2(\hat{x}, d)p]\; \hat{x} \times (p \times \hat{x})$$

where

$$[e_1(\hat{x},d)p] = [p \times \hat{x}] \cdot E_\infty(\hat{x},d)p$$

and

$$[e_2(\hat{x},d)p] = [\hat{x} \times (p \times \hat{x})] \cdot E_\infty(\hat{x},d)p$$

and the condition (7.1) is equivalent to

$$e_j(Q\hat{x},Qd)Qp = e_j(\hat{x},d)p, \quad j = 1,2.$$

This implies that

$$\int_{\mathbb{S}^2} e_j(\hat{x},d)p\,ds(d) = \int_{\mathbb{S}^2} e_j(Q\hat{x},d)Qp\,ds(d)$$

and therefore

$$\int_{\mathbb{S}^2} E_\infty(\hat{x},d)p\,ds(d) = c_1(\theta)\; p \times \hat{x} + c_2(\theta)\; \hat{x} \times (p \times \hat{x}) \tag{7.2}$$

for all $\hat{x} \in \mathbb{S}^2$ and all $p \in \mathbb{R}^3$ with $p \times \hat{x} \neq 0$ where c_1 and c_2 are functions depending only on the angle θ between $\hat{x}$ and p. Given $p \in \mathbb{R}^3$ such that $0 < \theta < \pi/2$, we also consider the vector

$$q := 2\,\hat{x} \cdot p\;\hat{x} - p$$

which clearly makes the same angle with $\hat{x}$ as p. From the linearity of the electric far field pattern with respect to polarization, we have

$$E_\infty(\hat{x},d)(\lambda p + \mu q) = \lambda E_\infty(\hat{x},d)p + \mu E_\infty(\hat{x},d)q \tag{7.3}$$

for all $\lambda, \mu \in \mathbb{R}$. Since $q \times \hat{x} = -p \times \hat{x}$, from (7.2) and (7.3) we can conclude that

$$c_j(\theta_{\lambda\mu}) = c_j(\theta), \quad j = 1,2,$$

for all $\lambda, \mu \in \mathbb{R}$ with $\lambda \neq \mu$ where $\theta_{\lambda\mu}$ is the angle between $\hat{x}$ and $\lambda p + \mu q$. This now implies that both functions c_1 and c_2 are constants since, by choosing λ and μ appropriately, we can make $\theta_{\lambda\mu}$ to be any angle between 0 and π. With these constants, by continuity, (7.2) is valid for all $\hat{x} \in \mathbb{S}^2$ and all $p \in \mathbb{R}^3$.

Choosing a fixed but arbitrary vector $p \in \mathbb{R}^3$ and using the Funk–Hecke formula (2.45), we consider the superposition of incident plane waves given by

$$\widetilde{E}^i(x) = \frac{i}{k}\,\operatorname{curl}\operatorname{curl} p \int_{\mathbb{S}^2} e^{ik\,x\cdot d}\,ds(d) = \frac{4\pi i}{k^2}\,\operatorname{curl}\operatorname{curl} p\;\frac{\sin k|x|}{|x|}\,. \tag{7.4}$$

Then, by Lemma 6.35 and (7.2), the corresponding scattered wave $\widetilde{E}^s$ has the electric far field pattern

$$\widetilde{E}_\infty(\hat{x}) = c_1\, p \times \hat{x} + c_2\, \hat{x} \times (p \times \hat{x}).$$

From this, with the aid of (6.26) and (6.27), we conclude that

$$\widetilde{E}^s(x) = \frac{ic_1}{k}\,\operatorname{curl} p\,\frac{e^{ik|x|}}{|x|} + \frac{c_2}{k^2}\,\operatorname{curl}\operatorname{curl} p\,\frac{e^{ik|x|}}{|x|}\,. \tag{7.5}$$

Using (7.4) and (7.5) and setting $r = |x|$, the boundary condition $\nu \times (\widetilde{E}^i + \widetilde{E}^s) = 0$ on ∂D can be brought into the form

$$\nu(x) \times \{g_1(r)\,p + g_2(r)\,p \times x + g_3(r)\,(p \cdot x)\,x\} = 0, \quad x \in \partial D, \tag{7.6}$$

for some functions g_1, g_2, g_3. In particular,

$$g_1(r) = \frac{4\pi i}{k^2 r}\left\{\frac{d}{dr}\,\frac{\sin kr}{r} + k^2 \sin kr + \frac{c_2}{4\pi i}\,\frac{d}{dr}\,\frac{e^{ikr}}{r} + \frac{c_2 k^2}{4\pi i}\,e^{ikr}\right\}.$$

For a fixed, but arbitrary $x \in \partial D$ with $x \neq 0$ we choose p to be orthogonal to x and take the scalar product of (7.6) with $p \times x$ to obtain

$$g_1(r)\,x \cdot \nu(x) = 0, \quad x \in \partial D.$$

Assume that $g_1(r) \neq 0$. Then $x \cdot \nu(x) = 0$ and inserting $p = x \times \nu(x)$ into (7.6) we arrive at the contradiction $g_1(r)\,x = 0$. Hence, since $x \in \partial D$ can be chosen arbitrarily, we have that $g_1(r) = 0$ for all $x \in \partial D$ with $x \neq 0$. Since g_1 does not vanish identically and is analytic, it can have only discrete zeros. Therefore, $r = |x|$ must be constant for all $x \in \partial D$, i.e., D is a ball with center at the origin.

7.2 Continuity and Differentiability of the Far Field Mapping

In this section, as in the case of acoustic obstacle scattering, we wish to study some of the properties of the far field mapping

$$\mathcal{F} : \partial D \mapsto E_\infty$$

which for a fixed incident plane wave E^i maps the boundary ∂D of the perfect conductor D onto the electric far field pattern E_∞ of the scattered wave.

We first briefly wish to indicate why the weak solution methods used in the proof of Theorems 5.7 and 5.8 have no immediate counterpart for the electromagnetic case. Recall the electric to magnetic boundary component map $\mathcal{A}$ from Theorem 6.22 that for radiating solutions to the Maxwell equations transforms the tangential trace of the electric field onto the tangential trace of the magnetic field. In the remark after Theorem 6.23 we noted that $\mathcal{A}$ is is a bijective bounded operator from $H^{-1/2}(\mathrm{Div}, \partial D)$ onto $H^{-1/2}(\mathrm{Div}, \partial D)$ with a bounded inverse.

Now let S_R denote the sphere of radius R centered at the origin and recall the definition (6.60) of the vector spherical harmonics U_n^m and V_n^m. Then for the tangential field

$$a = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \{a_n^m \, U_n^m + b_n^m \, V_n^m\}$$

with Fourier coefficients a_n^m and b_n^m the norm on $H^{-1/2}(S_R)$ can be written as

$$\|a\|^2_{H^{-1/2}(S_R)} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=-n}^{n} \left\{ \left|a_n^m\right|^2 + \left|b_n^m\right|^2 \right\}.$$

Since $\operatorname{Div} U_n^m = -\sqrt{n(n+1)}\, Y_n^m$ and $\operatorname{Div} V_n^m = 0$, the norm on the Sobolev space $H^{-1/2}(\operatorname{Div}, S_R)$ is equivalent to

$$\|a\|^2_{H^{-1/2}(\operatorname{Div}, S_R)} = \sum_{n=1}^{\infty} \left\{ n \sum_{m=-n}^{n} \left|a_n^m\right|^2 + \frac{1}{n} \sum_{m=-n}^{n} \left|b_n^m\right|^2 \right\}.$$

From the expansion (6.73) for radiating solutions to the Maxwell equations, we see that $\mathcal{A}$ maps the tangential field a with Fourier coefficients a_n^m and b_n^m onto

$$\mathcal{A}a = \frac{1}{ik} \sum_{n=1}^{\infty} \left\{ \delta_n \sum_{m=-n}^{n} a_n^m \, V_n^m + \frac{k^2}{\delta_n} \sum_{m=-n}^{n} b_n^m \, U_n^m \right\} \tag{7.7}$$

where

$$\delta_n := \frac{k h_n^{(1)\prime}(kR)}{h_n^{(1)}(kR)} + \frac{1}{R}, \quad n = 1, 2, \ldots.$$

Comparing this with (5.11), we note that

$$\delta_n = \gamma_n + \frac{1}{R},$$

that is, we can use the results from the proof of Theorem 5.7 on the coefficients γ_n. There does not exist a positive t such that $h_n^{(1)}(t) = 0$ or $h_n^{(1)}(t) + t h_n^{(1)\prime}(t) = 0$ since the Wronskian (2.37) does not vanish. Therefore, we have confirmed that the operator $\mathcal{A}$ in the spacial case of a ball indeed is bijective. Furthermore, from

$$c_1 n \le |\delta_n| \le c_2 n$$

which is valid for all n and some constants $0 < c_1 < c_2$, it is confirmed that $\mathcal{A}$ maps $H^{-1/2}(\operatorname{Div}, S_R)$ boundedly onto itself.

However, different from the acoustic case, due to the factor k^2 in the second term of the expansion (7.7) the operator $ik\mathcal{A}$ in the limiting case $k = 0$ no longer remains bijective. This reflects the fact that for $k = 0$ the Maxwell equations decouple. Therefore, there is no obvious way of splitting $\mathcal{A}$ into a strictly coercive and a compact operator as was done for the Dirichlet to Neumann map in the proof of Theorem 5.7.

Hence, for the continuous dependence on the boundary in electromagnetic obstacle scattering, we rely on the integral equation approach. For this, we describe a modification of the boundary integral equations used for proving existence of a solution to the exterior Maxwell problem in Theorem 6.21 which was introduced by Werner [328] and simplified by Hähner [121, 123]. In addition to surface potentials, it also contains volume potentials which makes it less satisfactory from a numerical point of view. However, it will make the investigation of the continuous dependence on the boundary easier since it avoids dealing with the more complicated second term in the approach (6.55) containing the double curl. We recall the notations introduced in Sections 6.3 and 6.4. After choosing an open ball B such that $\bar{B} \subset D$, we try to find the solution to the exterior Maxwell problem in the form

$$\begin{aligned} E(x) &= \operatorname{curl} \int_{\partial D} \Phi(x,y)a(y)\,ds(y) \\ &\quad - \int_{\partial D} \Phi(x,y)\varphi(y)\nu(y)\,ds(y) - \int_B \Phi(x,y)b(y)\,dy, \qquad (7.8) \\ H(x) &= \frac{1}{ik}\operatorname{curl} E(x), \quad x \in \mathbb{R}^3 \setminus \partial D. \end{aligned}$$

We assume that the densities $a \in C^{0,\alpha}(\mathrm{Div}, \partial D)$, $\varphi \in C^{0,\alpha}(\partial D)$ and $b \in C^{0,\alpha}(B)$ satisfy the three integral equations

$$\begin{aligned} a + M_{11}a + M_{12}\varphi + M_{13}b &= 2c \\ \varphi + M_{22}\varphi + M_{23}b &= 0 \qquad (7.9) \\ b + M_{31}a + M_{32}\varphi + M_{33}b &= 0 \end{aligned}$$

where the operators are given by $M_{11} := M$, $M_{22} := K$, $M_{12}\varphi := -\nu \times S(\nu\varphi)$, and

$$(M_{13}b)(x) := -2\,\nu(x) \times \int_B \Phi(x,y)b(y)\,dy, \quad x \in \partial D,$$

$$(M_{23}b)(x) := -2\int_B b(y)\cdot \operatorname{grad}_x \Phi(x,y)\,dy, \quad x \in \partial D,$$

$$(M_{31}a)(x) := i\eta(x)\operatorname{curl}\int_{\partial D} \Phi(x,y)a(y)\,ds(y), \quad x \in B,$$

$$(M_{32}\varphi)(x) := -i\eta(x)\int_{\partial D} \Phi(x,y)\varphi(y)\nu(y)\,ds(y), \quad x \in B,$$

$$(M_{33}b)(x) := -i\eta(x)\int_B \Phi(x,y)b(y)\,dy, \quad x \in B,$$

and where $\eta \in C^{0,\alpha}(\mathbb{R}^3)$ is a function with $\eta > 0$ in B and $\operatorname{supp}\eta = \bar{B}$.

First assume that we have a solution to these integral equations. Then clearly $\operatorname{div} E$ is a radiating solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ and, by the jump relations, the second integral equation implies $\operatorname{div} E = 0$ on ∂D. Hence, $\operatorname{div} E = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ because of the uniqueness for the exterior Dirichlet problem. Now, with the aid of Theorems 6.4 and 6.8, we conclude that E, H is a radiating solution to the Maxwell equations in $\mathbb{R}^3 \setminus \bar{D}$. By the jump relations, the first integral equation ensures the boundary condition $\nu \times E = c$ on ∂D is satisfied.

We now establish that the system (7.9) of integral equations is uniquely solvable. For this, we first observe that all the integral operators M_{ij} are compact. The compactness of $M_{11} = M$ and $M_{22} = K$ is stated in Theorems 6.17 and 3.4 and the compactness of M_{33} follows from the fact that the volume potential operator maps $C(\bar{B})$ boundedly into $C^{1,\alpha}(\bar{B})$ (see Theorem 8.1) and the imbedding Theorem 3.2. The compactness of $M_{12} : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\operatorname{Div}, \partial D)$ follows from Theorem 3.4 and the representation

$$(\operatorname{Div} M_{12}\varphi)(x) = 2\,\nu(x) \cdot \int_{\partial D} \varphi(y)\{\nu(x) - \nu(y)\} \times \operatorname{grad}_x \Phi(x, y)\, ds(y), \quad x \in \partial D,$$

which can be derived with the help of (6.43). The term $\nu(x) - \nu(y)$ makes the kernel weakly singular in a way such that Corollary 2.9 from [64] can be applied. For the other terms, compactness is obvious since the kernels are sufficiently smooth. Hence, by the Riesz–Fredholm theory it suffices to show that the homogeneous system only allows the trivial solution.

Assume that a, φ, b solve the homogeneous form of (7.9) and define E, H by (7.8). Then, by the above analysis, we already know that E, H solve the homogeneous exterior Maxwell problem whence $E = H = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows. The jump relations then imply that

$$\nu \times \operatorname{curl} E_- = 0, \quad \nu \cdot E_- = 0 \quad \text{on } \partial D. \tag{7.10}$$

From the third integral equation and the conditions on η, we observe that we may view b as a field in $C^{0,\alpha}(\mathbb{R}^3)$ with support in $\bar{B}$. Therefore, by the jump relations for volume potentials (see Theorem 8.1), we have $E \in C^2(D)$ and, in view of the third integral equation,

$$\Delta E + k^2 E = b = -i\eta E \quad \text{in } D. \tag{7.11}$$

From (7.10) and (7.11) with the aid of Green's vector theorem (6.2), we now derive

$$\int_D \left\{ |\operatorname{curl} E|^2 + |\operatorname{div} E|^2 - (k^2 + i\eta)|E|^2 \right\} dx = 0,$$

whence, taking the imaginary part,

$$\int_B \eta |E|^2 dx = 0$$

follows. This implies $E = 0$ in B and from (7.11) we obtain $b = \varDelta E + k^2 E = 0$ in D. Since solutions to the Helmholtz equation are analytic, from $E = 0$ in B we obtain $E = 0$ in D. The jump relations now finally yield $a = \nu \times E_+ - \nu \times E_- = 0$ and $\varphi = \operatorname{div} E_+ - \operatorname{div} E_- = 0$. Thus, we have established unique solvability for the system (7.9).

We are now ready to outline the proof of the electromagnetic analogue to the continuous dependence result of Theorem 5.16. We again consider surfaces $\varLambda$ which are starlike with respect to the origin and represented in the form (5.9) with a positive function $r \in C^{1,\alpha}(\mathbb{S}^2)$ with $0 < \alpha < 1$. We recall from Section 5.3 what we mean by convergence of surfaces $\varLambda_n \to \varLambda$, $n \to \infty$, and L^2 convergence of functions f_n from $L^2(\varLambda_n)$ to a function f in $L^2(\varLambda)$. (For consistency with the rest of our book, we again choose C^2 surfaces $\varLambda_n$ and $\varLambda$ instead of $C^{1,\alpha}$ surfaces.)

Theorem 7.3. *Let $(\varLambda_n)$ be a sequence of starlike C^2 surfaces which converges with respect to the $C^{1,\alpha}$ norm to a C^2 surface $\varLambda$ as $n \to \infty$ and let E_n, H_n and E, H be radiating solutions to the Maxwell equations in the exterior of $\varLambda_n$ and $\varLambda$, respectively. Assume that the continuous tangential components of E_n on $\varLambda_n$ are L^2 convergent to the tangential components of E on $\varLambda$. Then the sequence (E_n), together with all its derivatives, converges to E uniformly on compact subsets of the open exterior of $\varLambda$.*

Proof. As in the proof of Theorem 5.16, we transform the boundary integral equations in (7.9) onto a fixed reference surface by substituting $x = r(\hat{x})\,\hat{x}$ to obtain integral equations over the unit sphere for the surface densities

$$\tilde{a}(\hat{x}) := \hat{x} \times a(r(\hat{x})\,\hat{x}), \quad \tilde{\varphi}(\hat{x}) := \varphi(r(\hat{x})\,\hat{x}).$$

Since the weak singularities of the operators M_{11}, M_{22} and M_{12} are similar in structure to those of K and S which enter into the combined double- and single-layer approach to the exterior Dirichlet problem, proceeding as in Theorem 5.16 it is possible to establish an estimate of the form (5.44) for the boundary integral terms in the transformed equations corresponding to (7.9). For the mixed terms like M_{31} and M_{13}, estimates of the type (5.44) follow trivially from Taylor's formula and the smoothness of the kernels. Finally, the volume integral term corresponding to M_{33} does not depend on the boundary at all. Based on these estimates, the proof is now completely analogous to that of Theorem 5.16. □

Without entering into details we wish to mention that the above approach can also be used to show Fréchet differentiability with respect to the boundary analogously to Theorem 5.14 (see [276]).

In an alternate approach for establishing Fréchet differentiability, we extend a technique due to Kress and Päivärinta [207] from acoustic to electromagnetic scattering. For this, in a slightly more general setting, we consider a family of scatterers D_h with boundaries represented in the form

$$\partial D_h = \{x + h(x) : x \in \partial D\} \tag{7.12}$$

where $h : \partial D \to \mathbb{R}^3$ is of class C^2 and sufficiently small in the C^2 norm on ∂D. Then we may consider the operator $\mathcal{F}$ as a mapping from a ball

$$V := \{h \in C^2(\partial D) : \|h\|_{C^2} < \delta\} \subset C^2(\partial D)$$

with sufficiently small radius $\delta > 0$ into $L^2_t(\mathbb{S}^2)$.

From Theorem 6.44 we recall the bounded linear operator $A : L^2_t(\partial D) \to L^2_t(\mathbb{S}^2)$ which maps the electric tangential components on ∂D of radiating solutions to the Maxwell equations in $\mathbb{R}^3 \setminus \bar{D}$ onto the electric far field pattern. Further we denote by A_h the operator A with ∂D replaced by ∂D_h and define the integral operator $G_h : L^2_t(\partial D_h) \to L^2_t(\partial D_h)$ by

$$(G_h a)(x) := \nu(x) \times \int_{\partial D_h} E_e(x,y)[\nu(y) \times a(y)]\, ds(y), \quad x \in \partial D,$$

in terms of the total electric field E_e for scattering of the electric dipole field E^i_e, H^i_e given by (6.89) from D.

Lemma 7.4 *Assume that $\bar{D} \subset D_h$. Then for the far fields E_∞ and $E_{h,\infty}$ for scattering of an incident field E^i, H^i from D and D_h, respectively, we have the factorization*

$$E_{h,\infty} - E_\infty = A_h G_h \left(\nu \times H_h|_{\partial D_h}\right) \tag{7.13}$$

where H_h denotes the total magnetic field for scattering from D_h.

Proof. As indicated in the formulation of the lemma, we distinguish the solution to the scattering problem for the domain D_h by the subscript h, that is, $E_h = E^i + E^s_h$ and $H_h = H^i + H^s_h$. By Huygens' principle, i.e., Theorem 6.24, the scattered field can be written as

$$E^s(x) = \int_{\partial D} [E^i_e(\cdot,x)]^\top [\nu \times H]\, ds, \quad x \in \mathbb{R}^3 \setminus \bar{D}_h. \tag{7.14}$$

From this we obtain that

$$\begin{aligned}
-E^s(x) &= \int_{\partial D} \left\{[E^s_e(\cdot,x)]^\top [\nu \times H] + [H^s_e(\cdot,x)]^\top [\nu \times E]\right\} ds \\
&= \int_{\partial D_h} \left\{[E^s_e(\cdot,x)]^\top [\nu \times H] + [H^s_e(\cdot,x)]^\top [\nu \times E]\right\} ds \\
&= \int_{\partial D_h} \left\{[E^s_e(\cdot,x)]^\top [\nu \times H^i] + [H^s_e(\cdot,x)]^\top [\nu \times E^i]\right\} ds \\
&= \int_{\partial D_h} \left\{[E^s_e(\cdot,x)]^\top [\nu \times H^i] - [H^s_e(\cdot,x)]^\top [\nu \times E^s_h]\right\} ds \\
&= \int_{\partial D_h} [E^s_e(\cdot,x)]^\top [\nu \times H_h]\, ds, \quad x \in \mathbb{R}^3 \setminus \bar{D}_h,
\end{aligned}$$

where we have used the perfect conductor boundary condition, the vector Green's theorem (applied component-wise) and the radiation condition.

On the other hand, the representation (7.14) applied to D_h yields

$$E_h^s(x) = \int_{\partial D_h} [E_e^i(\cdot, x)]^\top [\nu \times H_h\, ds, \quad x \in \mathbb{R}^3 \setminus \bar{D}_h,$$

and (7.13) follows by adding the last two equations and passing to the far field. □

Theorem 7.5. *The boundary to far field mapping $\mathcal{F} : \partial D_h \mapsto E_\infty$ is Fréchet differentiable. The derivative is given by*

$$\mathcal{F}'(\partial D) : h \mapsto \mathcal{E}_{h,\infty},$$

where $\mathcal{E}_{h,\infty}$ is the electric far field pattern of the uniquely determined radiating solution $\mathcal{E}_h, \mathcal{H}_h$ to the Maxwell equations in $\mathbb{R}^3 \setminus \bar{D}$ satisfying the boundary condition

$$\nu \times \mathcal{E}_h = -ik\, \nu \times (H \times \nu)\, \nu \cdot h - \nu \times \operatorname{Grad}\{(\nu \cdot h)\,(\nu \cdot E)\} \quad \textit{on } \partial D \tag{7.15}$$

in terms of the total field $E = E^i + E^s, H = H^i + H^s$.

Proof. We use the notations introduced in connection with Lemma 7.4. For simplicity we assume that ∂D is analytic. Then, by the regularity results on elliptic boundary value problems, the fields E, H and E_e, H_e can be extended as solutions to the Maxwell equations across the boundary ∂D. (This follows from Sections 6.1 and 6.6 in [250] by considering the boundary value problem for the Maxwell equations equivalently as a boundary value problem for the vector Helmholtz equation with boundary condition for the tangential components and the divergence.) Hence (7.13) remains valid also if $\bar{D} \not\subset D_h$ provided that h is sufficiently small.

For simplicity we confine ourselves to the case where k is not a Maxwell eigenvalue. As in the proof of Theorem 6.43, from Huygen's principle (6.58) we obtain the integral equation

$$b + M'b = 2\{\nu \times H^i\} \times \nu \tag{7.16}$$

for the tangential component $b = \{\nu \times H\} \times \nu$ of the magnetic field. Because of our assumption on k the operator $I + M' : L_t^2(\partial D) \to L_t^2(\partial D)$ has a trivial nullspace and consequently a bounded inverse. By M_h' we denote the operator M' with ∂D replaced by ∂D_h and interpret it as an operator $M_h' : C_t(\partial D) \to C_t(\partial D)$ by substituting $x = \xi + h(\xi)$ and $y = \eta + h(\eta)$. With the aid of the decomposition (6.32) of the kernel of M, proceeding as in the proof of Theorem 5.13 it can be shown that

$$\|M_h' - M'\|_\infty \le c\|h\|_{C^2(\partial D)}$$

for some constant c depending on ∂D. Hence, by a Neumann series argument, from (7.16) it can be deduced that we have continuity

$$|\nu_h(y + h(y)) \times H_h(y + h(y)) - \nu(y) \times H(y)| \to 0, \quad \|h\|_{C^2(\partial D)} \to 0,$$

uniformly for all $y \in \partial D$. From this, in view of the continuity of H, it follows that

$$\int_{\partial D_h} E_e(x,\cdot)[\nu \times H_h]\,ds = \int_{\partial D_h} E_e(x,\cdot)[\nu \times H]\,ds + o\left(\|h\|_{C^2(\partial D)}\right)$$

uniformly for all sufficiently large $|x|$.

Using the symmetry relation (6.93) and the boundary condition $\nu \times E_e(x,\cdot) = 0$ on ∂D, from Gauss' divergence theorem we obtain

$$\int_{\partial D_h} E_e(x,\cdot)[\nu \times H]\,ds - ik \int_{D_h^*} \left\{[E_e(\cdot\,,x)]^\top E + [H_e(\cdot\,,x)]^\top H\right\}\chi\,dy,$$

where

$$D_h^* := \{y \in D_h : y \notin D\} \cup \{y \in D : y \notin D_h\}$$

and $\chi(y) = 1$ if $y \in D_h$ and $y \notin D$ and $\chi(y) = -1$ if $y \in D$ and $y \notin D_h$. With the aid of the boundary condition $\nu \times E = 0$ on ∂D it can be shown that the volume integral over D_h^* can be approximated by a surface integral over ∂D through

$$\int_{D_h^*} \left\{[E_e(\cdot\,,x)]^\top E + [H_e(\cdot\,,x)]^\top H\right\}\chi\,dy$$

$$= \int_{\partial D} \left\{[E_e(\cdot\,,x)]^\top [\nu\,(E\cdot\nu)] + [H_e(\cdot\,,x)]^\top [\nu \times (H \times \nu)]\right\} \nu\cdot h\,ds + o\left(\|h\|_{C^1(\partial D)}\right)$$

uniformly for all sufficiently large $|x|$. Note that $\nu \times E = 0$ on ∂D implies that $\nu\cdot H = 0$ on ∂D as consequence of the Maxwell equations and the identity (6.43). Also as a consequence of the latter identity, with the help of the surface divergence theorem we can deduce that

$$k\int_{\partial D} \nu\cdot h\,[E_e(\cdot\,,x)]^\top[\nu\,(E\cdot\nu)]\,ds = -i\int_{\partial D}[H_e(\cdot\,,x)]^\top[\nu \times \operatorname{Grad}\{(\nu\cdot h)\,(\nu\cdot E)\}]\,ds.$$

Hence, putting the preceding four equations together and using the boundary condition (7.15) we find that

$$\int_{\partial D_h} E_e(x,\cdot)[\nu \times H_h]\,ds = \int_{\partial D}[H_e(\cdot\,,x)]^\top[\nu \times \mathcal{E}]\,ds + o\left(\|h\|_{C^2(\partial D)}\right). \tag{7.17}$$

On the other hand, from the Stratton–Chu formula (6.16) applied to $\mathcal{E}_h, \mathcal{H}_h$, the radiation condition and the boundary condition $\nu \times E_e = 0$ on ∂D we conclude that

$$\mathcal{E}_h(x) = \int_{\partial D}[H_e(\cdot\,,x)]^\top[\nu \times \mathcal{E}]\,ds, \quad x \in \mathbb{R}^3 \setminus \bar{D}.$$

Therefore, we can rewrite (7.17) as

$$\int_{\partial D_h} E_e(x,\cdot)[\nu \times H_h]\,ds = \mathcal{E}_h(x) + o\left(\|h\|_{C^2(\partial D)}\right)$$

and passing to the far field, with the aid of the identity (7.13), it follows that

$$E_{h,\infty} - E_\infty = \mathcal{E}_{h,\infty} + o\left(\|h\|_{C^2(\partial D)}\right).$$

This completes the proof. □

This approach to proving Fréchet differentiability has been extended to the impedance boundary condition by Haddar and Kress [119].

7.3 Iterative Solution Methods

All the iterative methods for solving the inverse obstacle problem in acoustics described in Section 5.4, in principle, have extensions to electromagnetic inverse obstacle scattering. For an implementation of regularized Newton iterations for the boundary to far field mapping $\mathcal{F}$ based on the presentation of the Fréchet derivative in Theorem 7.5 and using spectral methods in the spirit of [101, 102] we refer to LeLouër [225].

Here, in order to avoid repetitions, we only present an electromagnetic version of the method due to Johannson and Sleeman as suggested by Pieper [270]. We recall Huygens' principle from Theorem 6.24 and to circumvent the use of the hypersingular operator N we start from the representation

$$H(x) = H^i(x) + \operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{7.18}$$

for the total magnetic field H in terms of the incident field H^i and the representation

$$H_\infty(\hat{x}) = \frac{ik}{4\pi}\, \hat{x} \times \int_{\partial D} \nu(y) \times H(y)\, e^{-ik\,\hat{x}\cdot y}\, ds(y), \quad \hat{x} \in \mathbb{S}^2, \tag{7.19}$$

for the magnetic far field pattern H_∞. From (7.18), as in the proof of Theorem 6.43, from the jump relations we find that the tangential component

$$a := \nu \times H \quad \text{on } \partial D$$

satisfies

$$a(x) - 2\int_{\partial D} \nu(x) \times \{\operatorname{curl}_x \Phi(x,y) a(y)\}\, ds(y) = 2\,\nu(x) \times H^i(x), \quad x \in \partial D, \tag{7.20}$$

and (7.19) can be written as

$$\frac{ik}{4\pi}\, \hat{x} \times \int_{\partial D} a(y) e^{-ik\,\hat{x}\cdot y}\, ds(y) = H_\infty(\hat{x}), \quad \hat{x} \in \mathbb{S}^2. \tag{7.21}$$

We call (7.20) the field equation and (7.21) the data equation and interpret them as two integral equations for the unknown boundary and the unknown tangential component a of the total magnetic field on the boundary. Both equations are linear with respect to a and nonlinear with respect to ∂D. Equation (7.21) is severely ill-posed whereas (7.20) is well-posed provided k is not a Maxwell eigenvalue for D.

As in Section 5.4 there are three possible options for an iterative solution of the system (7.20)–(7.21). Here, from these we only briefly discuss the case where, given an approximation for the boundary ∂D, we solve the well-posed equation of the second kind (7.20) for a. Since the perfect conductor boundary condition $\nu \times E = 0$ on ∂D by the identity (6.43) implies that $\nu \cdot H = 0$ on ∂D, the full three-dimensional field H on ∂D is available via $H = a \times \nu$. Then, keeping H fixed, equation (7.21) is linearized with respect to ∂D to update the boundary approximation.

To describe this linearization in more detail, using the parameterization (5.9) for starlike ∂D and recalling the notation (5.21), we introduce the parameterized far field operator

$$A_\infty : C^2(\mathbb{S}^2) \times L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$$

by

$$A_\infty(r, b)(\hat{x}) := \frac{ik}{4\pi}\, \hat{x} \times \int_{\mathbb{S}^2} \nu_r(\hat{y}) \times b(\hat{y}) e^{-ikr(\hat{y})\, \hat{x}\cdot\hat{y}} ds(\hat{y}), \quad \hat{x} \in \mathbb{S}^2. \tag{7.22}$$

Here ν_r denotes the transformed normal vector as given by (5.22) in terms of the transformation $p_r : \mathbb{S}^2 \to \partial D$. Now the data equation (7.21) can be written in the operator form

$$A_\infty(r, b) = H_\infty \tag{7.23}$$

where we have set

$$b := J_r\,(a \circ p_r) \times \nu_r \tag{7.24}$$

with the Jacobian J_r of p_r. To update the boundary, the linearization

$$A'_\infty(r, b)q = H_\infty - A_\infty(r, b)$$

of (7.23) needs to be solved for q. The derivative A'_∞ is given by

$$(A'_\infty(r, \psi)q)\,(\hat{x}) = \frac{k^2}{4\pi}\, \hat{x} \times \int_{\mathbb{S}^2} \nu_r(\hat{y}) \times b(\hat{y}) e^{-ikr(\hat{y})\, \hat{x}\cdot\hat{y}}\, \hat{x}\cdot\hat{y}\, q(\hat{y}) ds(\hat{y})$$

$$+\frac{ik}{4\pi}\, \hat{x} \times \int_{\mathbb{S}^2} (\nu'_r q)(\hat{y}) \times b(\hat{y}) e^{-ikr(\hat{y})\, \hat{x}\cdot\hat{y}} ds(\hat{y}), \quad \hat{x} \in \mathbb{S}^2,$$

where

$$(\nu_r)'q = \frac{p_q - \operatorname{Grad} q}{\sqrt{r^2 + |\operatorname{Grad} r|^2}} - \frac{rq + \operatorname{Grad} r \cdot \operatorname{Grad} q}{r^2 + |\operatorname{Grad} r|^2}\, \nu_r$$

denotes the derivative of ν_r, see (5.22) and (5.26).

We present two examples for reconstructions that were provided to us by Olha Ivanyshyn. The synthetic data were obtained by applying the spectral method of Section 3.6 to the integral equation (6.56) for $\eta = k$. For this, the unknown tangential vector field was represented in terms of its three cartesian components and (6.56) was interpreted as a system of three scalar integral equations and the variant (3.122) of Wienert's method was applied. In both examples the synthetic data consisted of 242 values of the far field.

Correspondingly, for the reconstruction the number of collocation points on $\mathbb{S}^2$ for the data equation (7.21) also was chosen as 242. For the field equation (7.20) again Wienert's spectral method (3.122) was applied with 242 collocation points and 338 quadrature points corresponding to $N = 10$ and $N' = 12$ in (3.124). For the approximation space for the radial function representing the boundary of the scatterer, spherical harmonics up to order six were chosen.

The wave number was $k = 1$ and the incident direction $d = (0, 0, -1)$ and the polarization $p = (1, 0, 0)$ are indicated in the figures by a solid and a dashed arrow, respectively. The iterations were started with a ball of radius $3.5Y_0^0 = 0.9873$ centered at the origin. For the surface update H^1 penalization was applied with the regularization parameter selected by trial and error as $\alpha_n = \alpha\gamma^n$ depending on the iteration number n with $\alpha = 0.5$ and $\gamma = 2/3$.

Both to the real and imaginary part of the far field data 1% of normally distributed noise was added, i.e.,

$$\frac{\|H_\infty - H_\infty^\delta\|_{L^2(\mathbb{S}^2)}}{\|H_\infty\|_{L^2(\mathbb{S}^2}} \leq 0.01.$$

In terms of the relative data error

$$\varepsilon_r := \frac{\|H_\infty - H_{r,\infty}\|_{L^2(\mathbb{S}^2)}}{\|H_\infty\|_{L^2(\mathbb{S}^2)}}$$

with the given far field data H_∞ and the far field $H_{r,\infty}$ corresponding to the radial function r, a stopping criterion was incorporated such that the iteration was carried on as long as $\varepsilon_r > 0.05$ or $\varepsilon_r > \varepsilon_{r+q}$. The figures show the exact shape on the left and the reconstruction on the right.

The first example is a cushion shaped scatterer with radial function

$$r(\theta, \varphi) = \sqrt{0.8 + 0.5(\cos 2\varphi - 1)(\cos 4\theta - 1)}, \quad \theta \in [0, \pi], \ \varphi \in [0, 2\pi].$$

Fig. 7.1 shows the reconstruction after 19 iteration steps with the final data error $\varepsilon_r = 0.026$.

The second example is a pinched ball with radial function

$$r(\theta, \varphi) = \sqrt{1.44 + 0.5\cos 2\varphi(\cos 2\theta - 1)}, \quad \theta \in [0, \pi], \ \varphi \in [0, 2\pi].$$

Fig. 7.2 shows the reconstruction after 9 iteration steps with data error $\varepsilon_r = 0.012$.

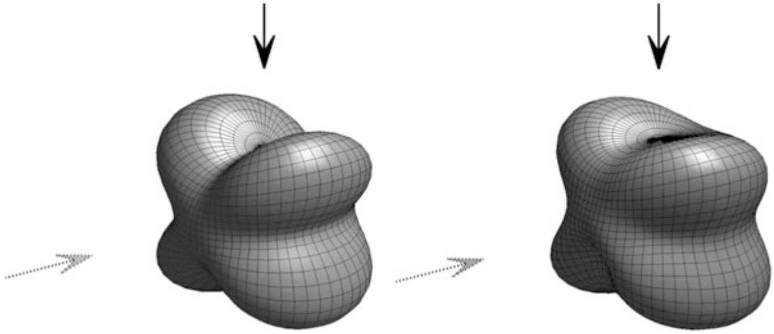

Fig. 7.1 Reconstruction of a cushion from noisy data

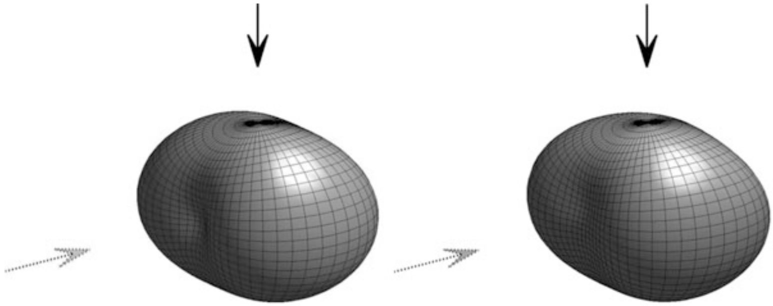

Fig. 7.2 Reconstruction of a pinched ball from noisy data

In passing we note that, in principle, instead of (7.24) one also could substitute $b := (a \circ p_r) \times \nu_r$, i.e., linearize also with respect to the surface element. However, numerical examples indicate that this variant is less stable.

7.4 Decomposition Methods

We begin this section by describing the electromagnetic version of the decomposition method proposed by Kirsch and Kress for inverse acoustic obstacle scattering. We confine our analysis to inverse scattering from a perfectly conducting obstacle. Extensions to other boundary conditions are also possible.

We again first construct the scattered wave E^s from a knowledge of its electric far field pattern E_∞. To this end, we choose an auxiliary closed C^2 surface Γ with unit outward normal ν contained in the unknown scatterer D such that k is not a Maxwell eigenvalue for the interior of Γ. For example, we can choose Γ to be a sphere of radius R such that $j_n(kR) \neq 0$ and $j_n(kR) + kRj_n'(kR) \neq 0$ for $n = 0, 1, \dots$. Given the internal surface Γ, we try to represent the scattered field as the electromagnetic field

$$E^s(x) = \operatorname{curl} \int_\Gamma \Phi(x, y) a(y)\, ds(y), \quad H^s(x) = \frac{1}{ik} \operatorname{curl} E^s(x) \tag{7.25}$$

of a magnetic dipole distribution a from the space $L^2_t(\Gamma)$ of tangential L^2 fields on Γ. From (6.26) we see that the electric far field pattern of E^s is given by

$$E_\infty(\hat{x}) = \frac{ik}{4\pi}\,\hat{x} \times \int_\Gamma e^{-ik\,\hat{x}\cdot y} a(y)\,ds(y), \quad \hat{x} \in \mathbb{S}^2.$$

Hence, given the (measured) electric far field pattern E_∞, we have to solve the ill-posed integral equation of the first kind

$$M_\infty a = E_\infty \tag{7.26}$$

for the density a where the integral operator $M_\infty : L^2_t(\Gamma) \to L^2_t(\mathbb{S}^2)$ is defined by

$$(M_\infty a)(\hat{x}) := \frac{ik}{4\pi}\,\hat{x} \times \int_\Gamma e^{-ik\,\hat{x}\cdot y} a(y)\,ds(y), \quad \hat{x} \in \mathbb{S}^2. \tag{7.27}$$

As for the corresponding operator (5.59) in acoustics, the operator (7.27) has an analytic kernel and therefore the integral equation (7.26) is severely ill-posed. We now establish some properties of M_∞.

Theorem 7.6. *The far field operator M_∞ defined by (7.27) is injective and has dense range provided k is not a Maxwell eigenvalue for the interior of Γ.*

Proof. Let $M_\infty a = 0$ and define an electromagnetic field by (7.25). Then E^s has vanishing electric far field pattern $E_\infty = 0$, whence $E^s = 0$ in the exterior of Γ follows by Theorem 6.10. After introducing, analogous to (6.33), the magnetic dipole operator $M : L^2(\Gamma) \to L^2(\Gamma)$, by the L^2 jump relation (6.53) we find that

$$a + Ma = 0.$$

Employing the argument used in the proof of Theorem 6.23, by the Fredholm alternative we see that the nullspaces of $I + M$ in $L^2(\Gamma)$ and in $C(\Gamma)$ coincide. Therefore, a is continuous and, by the jump relations of Theorem 6.12 for continuous densities, $H^s, -E^s$ represents a solution to the Maxwell equations in the interior of Γ satisfying the homogeneous boundary condition $\nu \times H^s = 0$ on Γ. Hence, by our assumption on the choice of Γ we have $H^s = E^s = 0$ everywhere in $\mathbb{R}^3$. The jump relations now yield $a = 0$ on Γ, whence M_∞ is injective.

The adjoint operator $M^*_\infty : L^2_t(\mathbb{S}^2) \to L^2_t(\Gamma)$ of M_∞ is given by

$$(M^*_\infty g)(y) = \left(\nu(y) \times \frac{ik}{4\pi} \int_{\mathbb{S}^2} e^{ik\,\hat{x}\cdot y}\,\hat{x} \times g(\hat{x})\,ds(\hat{x})\right) \times \nu(y), \quad y \in \Gamma.$$

Let $M^*_\infty g = 0$. Then

$$E(y) := \int_{\mathbb{S}^2} e^{ik\,\hat{x}\cdot y}\,\hat{x} \times g(\hat{x})\,ds(\hat{x}), \quad H(y) := \frac{1}{ik}\,\operatorname{curl} E(y), \quad y \in \mathbb{R}^3,$$

defines an electromagnetic Herglotz pair satisfying $\nu \times E = 0$ on Γ. Hence, $E = H = 0$ in the interior of Γ by our assumption on the choice of Γ. Since E

and H are analytic in $\mathbb{R}^3$, it follows that $E = H = 0$ everywhere. Theorem 3.19 now yields $g = 0$ on Γ, whence M_∞^* is also injective and by Theorem 4.6 the range of M_∞ is dense in $L_t^2(\mathbb{S}^2)$. □

We now define a magnetic dipole operator $\widetilde{M} : L_t^2(\Gamma) \to L_t^2(\Lambda)$ by

$$(\widetilde{M}a)(x) := \nu(x) \times \operatorname{curl} \int_\Gamma \Phi(x, y)a(y)\, ds(y), \quad x \in \Lambda, \tag{7.28}$$

where Λ denotes a closed C^2 surface with unit outward normal ν containing Γ in its interior. The proof of the following theorem is similar to that of Theorem 7.6.

Theorem 7.7. *The operator $\widetilde{M}$ defined by (7.28) is injective and has dense range provided k is not a Maxwell eigenvalue for the interior of Γ.*

Now we know that by our choice of Γ the integral equation of the first kind (7.26) has at most one solution. Analogous to the acoustic case, its solvability is related to the question of whether or not the scattered wave can be analytically extended as a solution to the Maxwell equations across the boundary ∂D.

For the same reasons as in the acoustic case, we combine a Tikhonov regularization for the integral equation (7.26) and a defect minimization for the boundary search into one cost functional. We proceed analogously to Definition 5.22 and choose a compact (with respect to the $C^{1,\beta}$ norm, $0 < \beta < 1$,) subset U of the set of all starlike closed C^2 surfaces described by

$$\Lambda = \left\{ r(\hat{x})\, \hat{x} : \hat{x} \in \mathbb{S}^2 \right\}, \quad r \in C^2(\mathbb{S}^2),$$

satisfying the a priori assumption

$$0 < r_i(\hat{x}) \le r(\hat{x}) \le r_e(\hat{x}), \quad \hat{x} \in \mathbb{S}^2,$$

with given functions r_i and r_e representing surfaces Λ_i and Λ_e such that the internal auxiliary surface Γ is contained in the interior of Λ_i and that the boundary ∂D of the unknown scatterer D is contained in the annulus between Λ_i and Λ_e. We now introduce the cost functional

$$\mu(a, \Lambda; \alpha) := \|M_\infty a - E_\infty\|^2_{L_t^2(\mathbb{S}^2)} + \alpha\|a\|^2_{L_t^2(\Gamma)} + \gamma\|\nu \times E^i + \widetilde{M}a\|^2_{L_t^2(\Lambda)}. \tag{7.29}$$

Again, $\alpha > 0$ denotes the regularization parameter for the Tikhonov regularization of (7.26) and $\gamma > 0$ denotes a suitable coupling parameter which for theoretical purposes we always assume equal to one.

Definition 7.8 *Given the incident field E^i, a (measured) electric far field pattern $E_\infty \in L_t^2(\mathbb{S}^2)$, and a regularization parameter $\alpha > 0$, a surface Λ_0 from the compact set U is called* optimal *if there exists $a_0 \in L_t^2(\Gamma)$ such that a_0 and Λ_0 minimize the cost functional (7.29) simultaneously over all $a \in L_t^2(\Gamma)$ and $\Lambda \in U$, that is, we have*

$$\mu(a_0, \Lambda_0; \alpha) = m(\alpha)$$

where

$$m(\alpha) := \inf_{a \in L^2_t(\Gamma),\, \Lambda \in U} \mu(a, \Lambda; \alpha).$$

For this reformulation of the electromagnetic inverse obstacle problem as a nonlinear optimization problem, we can state the following counterparts of Theorems 5.23–5.25. We omit the proofs since, except for minor adjustments, they literally coincide with those for the acoustic case. The use of Theorems 5.16 and 5.20, of course, has to be replaced by the corresponding electromagnetic versions given in Theorems 7.3 and 7.7.

Theorem 7.9. *For each* $\alpha > 0$ *there exists an optimal surface* $\Lambda \in U$.

Theorem 7.10. *Let* E_∞ *be the exact electric far field pattern of a domain* D *such that* ∂D *belongs to* U. *Then we have convergence of the cost functional* $\lim_{\alpha \to 0} m(\alpha) = 0$.

Theorem 7.11. *Let* (α_n) *be a null sequence and let* (Λ_n) *be a corresponding sequence of optimal surfaces for the regularization parameter* α_n. *Then there exists a convergent subsequence of* (Λ_n). *Assume that* E_∞ *is the exact electric far field pattern of a domain* D *such that* ∂D *is contained in* U. *Then every limit point* Λ^* *of* (Λ_n) *represents a surface on which the total field vanishes.*

Variants of these results were first established by Blöhbaum [23]. We will not repeat all the possible modifications mentioned in Section 5.5 for acoustic waves such as using more than one incoming wave, the limited aperture problem or using near field data. It is also straightforward to replace the magnetic dipole distribution on the internal surface for the approximation of the scattered field by an electric dipole distribution.

The above method for the electromagnetic inverse obstacle problem has been numerically implemented and tested by Haas, Rieger, Rucker and Lehner [116].

We now proceed with briefly describing the extension of Potthast's point source method to electromagnetic obstacle scattering where again we start from Huygens' principle. By Theorem 6.24 the scattered field is given by

$$E^s(x) = \frac{i}{k}\,\operatorname{curl}\operatorname{curl}\int_{\partial D} \nu(y) \times H(y)\, \Phi(x, y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{7.30}$$

and the far field pattern by

$$E_\infty(\hat{x}) = \frac{ik}{4\pi}\, \hat{x} \times \int_{\partial D} [\nu(y) \times H(y)] \times \hat{x}\, e^{-ik\,\hat{x}\cdot y}\, ds(y), \quad \hat{x} \in \mathbb{S}^2, \tag{7.31}$$

in terms of the total magnetic field H. We choose an auxiliary closed C^2 surface Λ such that the scatterer D is contained in the interior of Λ and approximate the point source $\Phi(x, \cdot)$ for x in the exterior of Λ by a Herglotz wave function such that

$$\Phi(x, y) \approx \frac{1}{4\pi} \int_{\mathbb{S}^2} e^{ik\,y\cdot d} g_x(d)\, ds(d) \tag{7.32}$$

for all y in the interior of Λ and some scalar kernel function $g_x \in L^2(\mathbb{S}^2)$. In Section 5.5 we have described how such an approximation can be achieved uniformly with respect to y up to derivatives of second order on compact subsets of the interior of Λ by solving the ill-posed linear integral equation (5.82). With the aid of (6.4) and $\operatorname{grad}_x \Phi(x, y) = -\operatorname{grad}_y \Phi(x, y)$, we first transform (7.30) into

$$\begin{aligned} E^s(x) = \frac{i}{k} \int_{\partial D} \Big([\nu(y)) \times H(y)] \cdot \operatorname{grad}_y \Big) \operatorname{grad}_y \Phi(x, y)\, ds(y) \\ + ik \int_{\partial D} \nu(y) \times H(y)\, \Phi(x, y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}. \end{aligned} \tag{7.33}$$

With the aid of

$$\Big(a(y) \cdot \operatorname{grad}_y \Big) \operatorname{grad}_y e^{ik\, y \cdot d} + k^2 a(y)\, e^{ik\, y \cdot d} = k^2 d \times [a(y) \times d]\, e^{ik\, y \cdot d}$$

for $a = \nu \times H$ we now insert (7.32) into (7.33) and use (7.31) to obtain

$$E^s(x) \approx \int_{\mathbb{S}^2} g_x(d) E_\infty(-d)\, ds(d) \tag{7.34}$$

as an approximation for the scattered wave E^s. Knowing an approximation for the scattered wave, in principle the boundary ∂D can be found from the boundary condition $\nu \times (E^i + E^s) = 0$ on ∂D. For further details we refer to [19].

We conclude this section on decomposition methods with a short presentation of the electromagnetic version of the method of Colton and Monk. Again we confine ourselves to scattering from a perfect conductor and note that there are straightforward extensions to other boundary conditions.

As in the acoustic case, we try to find a superposition of incident plane electromagnetic fields with different directions and polarizations which lead to simple scattered fields and far field patterns. Starting from incident plane waves of the form (6.86), we consider as incident wave a superposition of the form

$$\widetilde{E}^i(x) = \int_{\mathbb{S}^2} ik\, e^{ik\, x \cdot d} g(d)\, ds(d), \quad \widetilde{H}^i(x) = \frac{1}{ik} \operatorname{curl} \widetilde{E}^i(x), \quad x \in \mathbb{R}^3, \tag{7.35}$$

with a tangential field $g \in L^2_t(\mathbb{S}^2)$, i.e., the incident wave is an electromagnetic Herglotz pair. By Lemma 6.35, the corresponding electric far field pattern

$$\widetilde{E}_\infty(\hat{x}) = \int_{\mathbb{S}^2} E_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2,$$

is obtained by superposing the far field patterns $E_\infty(\cdot, d)g(d)$ for the incoming directions d with polarization $g(d)$. We note that by the Reciprocity Theorem 6.30 we may consider (7.35) also as a superposition with respect to the observation directions instead of the incident directions and in this case the method we are considering is sometimes referred to as dual space method.

If we want the scattered wave to become a prescribed radiating solution $\widetilde{E}^s, \widetilde{H}^s$ with explicitly known electric far field pattern $\widetilde{E}_\infty$, given the (measured) far field patterns for all incident directions and polarizations, we need to solve the linear integral equation of the first kind

$$Fg = \widetilde{E}_\infty \tag{7.36}$$

with the far field operator $F : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ defined by (6.98).

We need to assume the prescribed field $\widetilde{E}^s, \widetilde{H}^s$ is defined in the exterior of the unknown scatterer. For example, if we have the a priori information that the origin is contained in D, then for actual computations obvious choices for the prescribed scattered field would be the electromagnetic field

$$\widetilde{E}^s(x) = \operatorname{curl} a\Phi(x, 0), \quad \widetilde{H}^s(x) = \frac{1}{ik} \operatorname{curl} \widetilde{E}^s(x)$$

of a magnetic dipole at the origin with electric far field pattern

$$\widetilde{E}_\infty(\hat{x}) = \frac{ik}{4\pi}\, \hat{x} \times a,$$

or the electromagnetic field

$$\widetilde{E}^s(x) = \operatorname{curl}\operatorname{curl} a\Phi(x, 0), \quad \widetilde{H}^s(x) = \frac{1}{ik} \operatorname{curl} \widetilde{E}^s(x)$$

of an electric dipole with far field pattern

$$\widetilde{E}_\infty(\hat{x}) = \frac{k^2}{4\pi}\, \hat{x} \times (a \times \hat{x}).$$

Another more general possibility is to choose the radiating vector wave functions of Section 6.5 with the far field patterns given in terms of vector spherical harmonics (see Theorem 6.28).

We have already investigated the far field operator F. From Corollary 6.37, we know that F is injective and has dense range if and only if there does not exist an electromagnetic Herglotz pair which satisfies the homogeneous perfect conductor boundary condition on ∂D. Therefore, for the sequel we will make the assumption that k is not a Maxwell eigenvalue for D. This then implies that the inhomogeneous interior Maxwell problem for D is uniquely solvable. The classical approach to solve this boundary value problem is to seek the solution in the form of the electromagnetic field of a magnetic dipole distribution

$$E(x) = \operatorname{curl} \int_{\partial D} a(y)\Phi(x, y)\, ds(y), \quad H(x) = \frac{1}{ik} \operatorname{curl} E(x), \quad x \in D,$$

with a tangential field $a \in C^{0,\alpha}(\mathrm{Div}, \partial D)$. Then, given $c \in C^{0,\alpha}(\mathrm{Div}, \partial D)$, by the jump relations of Theorem 6.12 the electric field E satisfies the boundary condition

$\nu \times E = c$ on ∂D if the density a solves the integral equation

$$a - Ma = -2c \tag{7.37}$$

with the magnetic dipole operator M defined by (6.33). The assumption that there exists no nontrivial solution to the homogeneous interior Maxwell problem in D now can be utilized to show with the aid of the jump relations that $I - M$ has a trivial nullspace in $C^{0,\alpha}(\text{Div}, \partial D)$ (for the details see [64]). Hence, by the Riesz–Fredholm theory $I - M$ has a bounded inverse $(I - M)^{-1}$ from $C^{0,\alpha}(\text{Div}, \partial D)$ into $C^{0,\alpha}(\text{Div}, \partial D)$. This implies solvability and well-posedness of the interior Maxwell problem. The proof of the following theorem is now completely analogous to that of Theorem 5.26.

Theorem 7.12. *Assume that k is not a Maxwell eigenvalue for D. Let (E_n, H_n) be a sequence of $C^1(D) \cap C(\bar{D})$ solutions to the Maxwell equations in D such that the boundary data $c_n = \nu \times E_n$ on ∂D are weakly convergent in $L^2_t(\partial D)$. Then the sequence (E_n, H_n) converges uniformly on compact subsets of D to a solution E, H to the Maxwell equations.*

From now on, we assume that $\mathbb{R}^3 \setminus D$ is contained in the domain of definition of $\widetilde{E}^s, \widetilde{H}^s$, that is, for the case of the above examples for $\widetilde{E}^s, \widetilde{H}^s$ with singularities at $x = 0$ we assume the origin to be contained in D. We associate the following uniquely solvable interior Maxwell problem

$$\operatorname{curl} \widetilde{E}^i - ik\widetilde{H}^i = 0, \quad \operatorname{curl} \widetilde{H}^i + ik\widetilde{E}^i = 0 \quad \text{in } D, \tag{7.38}$$

$$\nu \times (\widetilde{E}^i + \widetilde{E}^s) = 0 \quad \text{on } \partial D \tag{7.39}$$

to the inverse scattering problem. From Theorem 6.41 we know that the solvability of the integral equation (7.36) is connected to this interior boundary value problem, i.e., (7.36) is solvable for $g \in L^2_t(\mathbb{S}^2)$ if and only if the solution $\widetilde{E}^i, \widetilde{H}^i$ to (7.38), (7.39) is an electromagnetic Herglotz pair with kernel ikg.

The Herglotz integral operator $H : L^2_t(\mathbb{S}^2) \to L^2_t(\Lambda)$ defined by

$$(Hg)(x) := ik\,\nu(x) \times \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d)\, ds(d), \quad x \in \Lambda, \tag{7.40}$$

where ν denotes the unit outward normal to the surface Λ represents the tangential component of the electric field on Λ for the Herglotz pair with kernel ikg.

Theorem 7.13. *The Herglotz operator H defined by (7.40) is injective and has dense range provided k is not a Maxwell eigenvalue for the interior of Λ.*

Proof. The operator H is related to the adjoint of the far field integral operator given by (7.27). Therefore, the statement of the theorem is equivalent to Theorem 7.6. □

We are now ready to reformulate the inverse scattering problem as a nonlinear optimization problem analogous to Definition 5.27 in the acoustic case. We recall the description of the set U of admissible surfaces from p. 254 and pick a closed C^2 surface Γ_e such that Λ_e is contained in the interior of Γ_e where we assume that k is not a Maxwell eigenvalue for the interior of Γ_e. We now introduce a cost functional by

$$\mu(g, \Lambda; \alpha) := \|Fg - \widetilde{E}_\infty\|^2_{L^2_t(\mathbb{S}^2)} + \alpha\|Hg\|^2_{L^2_t(\Gamma_e)} + \gamma\|Hg + \nu \times \widetilde{E}^s\|^2_{L^2_t(\Lambda)}. \tag{7.41}$$

Definition 7.14 *Given the (measured) electric far field $E_\infty \in L^2_t(\mathbb{S}^2 \times \mathbb{S}^2)$ for all incident and observation directions and all polarizations and a regularization parameter $\alpha > 0$, a surface Λ_0 from the compact set U is called* optimal *if*

$$\inf_{g \in L^2_t(\mathbb{S}^2)} \mu(g, \Lambda_0; \alpha) = m(\alpha)$$

where

$$m(\alpha) := \inf_{g \in L^2_t(\mathbb{S}^2),\ \Lambda \in U} \mu(\varphi, \Lambda; \alpha).$$

For this electromagnetic optimization problem, we can state the following counterparts to Theorems 5.28–5.30. Variants of these results were first established by Blöhbaum [23].

Theorem 7.15. *For each $\alpha > 0$, there exists an optimal surface $\Lambda \in U$.*

Proof. The proof is analogous to that of Theorem 5.28 with the use of Theorem 7.12 instead of Theorem 5.26. □

Theorem 7.16. *For all incident and observation directions and all polarizations let E_∞ be the exact electric far field pattern of a domain D such that ∂D belongs to U. Then we have convergence of the cost functional* $\lim_{\alpha \to 0} m(\alpha) = 0$.

Proof. The proof is analogous to that of Theorem 5.29. Instead of Theorem 5.21 we use Theorem 7.13 and instead of (5.89) we use the corresponding relation

$$\widetilde{E}_\infty - Fg = A(Hg + \nu \times \widetilde{E}^s) \tag{7.42}$$

where $A : L^2_t(\partial D) \to L^2_t(\mathbb{S}^2)$ is the bounded injective operator introduced in Theorem 6.44 that maps the electric tangential component of radiating solutions to the Maxwell equations in $\mathbb{R}^3 \setminus D$ onto the electric far field pattern. □

Theorem 7.17. *Let (α_n) be a null sequence and let (Λ_n) be a corresponding sequence of optimal surfaces for the regularization parameter α_n. Then there exists a convergent subsequence of (Λ_n). Assume that for all incident and observation directions and all polarizations E_∞ is the exact electric far field pattern of a domain D such that ∂D belongs to U. Assume further that the solution $\widetilde{E}^i, \widetilde{H}^i$ to the associated interior Maxwell problem (7.38), (7.39) can be extended as a solution to the Maxwell equations across the boundary ∂D into the interior of Γ_e with continuous boundary values on Γ_e. Then every limit point Λ^* of (Λ_n) represents a surface on which the boundary condition (7.39) is satisfied, i.e., $\nu \times (\widetilde{E}^i + \widetilde{E}^s) = 0$ on Λ^*.*

Proof. The proof is analogous to that of Theorem 5.30 with the use of Theorems 7.12 and 7.13 instead of Theorems 5.26 and 5.21 and of (7.42) instead of (5.89). □

Using the completeness result of Theorem 6.42, it is possible to design a variant of the above method for which one does not have to assume that k is not a Maxwell eigenvalue for D.

As in acoustics, the decomposition method of Colton and Monk is closely related to the linear sampling method that we are going to discuss in the next section. For numerical examples using the latter method to solve three dimensional electromagnetic inverse scattering problems we refer the reader to [36].

7.5 Sampling Methods

Analogous to Section 5.6, based on the far field operator F which in the case of electromagnetic waves is defined in Theorem 6.37, i.e.,

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} E_\infty(\hat{x}, d)g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2, \tag{7.43}$$

a factorization method can be considered in terms of the ill-posed linear operator equation

$$(F^*F)^{1/4} g_z = E_{e,\infty}(\cdot, z)p. \tag{7.44}$$

Here the right hand side $E_{e,\infty}(\cdot, z)p$ is the far field pattern of an electric dipole with source z and polarization p. However, at the time this is being written, this method has not yet been justified, for example, by proving an analogue of Corollary 5.40 although the far field operator is also compact and normal in the electromagnetic case (see Theorem 6.39). In addition a factorization of the far field operator also is available in the form

$$F = \frac{2\pi i}{k}\, AN^*A^*$$

in terms of the tangential component to far field operator A of Theorem 6.44 and the hypersingular boundary integral operator N defined in (6.48). However, for establishing an obvious analogue of Lemma 5.37 coercivity of N_i (the operator N with k replaced by i) remains open. Nevertheless, for the case of a ball the above factorization method has been justified by Collino, Fares and Haddar [51].

In Section 5.6 we described the linear sampling method for solving the inverse scattering problem for a sound-soft obstacle. Our analysis was based on first presenting the factorization method for solving this inverse scattering problem and then deriving Corollary 5.42 as the final result on the linear sampling method as a consequence of the factorization method. As just pointed out, the factorization method has not been established for the case of a perfect conductor and hence we can develop the linear sampling method for electromagnetic obstacle scattering only up to the analogue of Theorem 5.34 (c.f. [32, 36, 55]). Although we shall not do so here, the inverse scattering problem with limited aperture data can also be handled [30, 36].

Analogous to the scalar case, our analysis is based on an examination of the equation

$$Fg = E_{e,\infty}(\cdot, z)p, \tag{7.45}$$

where now the far field operator F is given by (7.43) and

$$E_{e,\infty}(\hat{x}, z)p = \frac{ik}{4\pi}\,(\hat{x} \times p) \times \hat{x}\, e^{-ik\,\hat{x}\cdot z}$$

is the far field pattern of an electric dipole with source z and polarization p (we could also have considered the right hand side of (7.45) to be the far field pattern of a magnetic dipole). Equation (7.45) is known as the *far field equation*. If $z \in D$, it is seen that if g_z is a solution of the far field equation, then by Theorem 6.41 the scattered field E_g^s due to the vector Herglotz wave function ikE_g as incident field and the electric dipole $E_e(\cdot, z)p$ coincide in $\mathbb{R}^3 \setminus \bar{D}$. Hence, by the trace theorem, the tangential traces $\nu \times E_g^s = -ik\,\nu \times E_g$ and $\nu \times E_e(\cdot, z)p$ coincide on ∂D. As $z \in D$ tends to ∂D we have that

$$\|\nu \times E_e(\cdot, z)p\|_{H^{-1/2}(\mathrm{Div},\partial D)} \to \infty,$$

and hence $\|\nu \times E_g\|_{H^{-1/2}(\mathrm{Div},\partial D)} \to \infty$ also. Thus $\|g\|_{L_t^2(\mathbb{S}^2)} \to \infty$ and this behavior determines ∂D. Unfortunately, the above argument is only heuristic since it is based on the assumption that g satisfies the far field equation for $z \in D$, and in general the far field equation has no solution for $z \in D$. This follows from the fact that by Theorem 6.41 if g satisfies the far field equation, then the Herglotz pair $E = ikE_g$, $H = \operatorname{curl} E_g$ is the solution of the interior Maxwell problem

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0 \quad \text{in } D, \tag{7.46}$$

and

$$\nu \times [E + E_e(\cdot, z)p] = 0 \quad \text{on } \partial D, \tag{7.47}$$

which in general is not possible. However, using denseness properties of Herglotz pairs the following foundation of the linear sampling method can be established.

To achieve this, we first present modified versions of the denseness results of Theorems 6.44 and 7.13.

Corollary 7.18 *The operator* $A : H^{-1/2}(\mathrm{Div}, \partial D) \to L_t^2(\mathbb{S}^2)$ *which maps the electric tangential component of radiating solutions* $E, H \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$ *to the Maxwell equations onto the electric far field pattern* E_∞ *is bounded, injective and has dense range.*

Proof. Injectivity of A is a consequence of Rellich's lemma and the trace estimate (6.52) . Boundedness of A follows from the representation (6.115) via duality pairing in view of the continuous dependence of the solution to the scattering problem on the incident direction d of the plane waves. From (6.115) we also observe that the dual operator $A^\top : L_t^2(\mathbb{S}^2) \to H^{-1/2}(\mathrm{Curl}, \partial D)$ of A is given by

$$A^\top g = \overline{A^* \overline{g}}, \quad g \in L_t^2(\mathbb{S}^2),$$

in terms of the L^2 adjoint A^*. From the proof of Theorem 6.44 we know that A^* is injective. Consequently $A^\top$ is injective and therefore A has dense range by the Hahn–Banach theorem. □

Corollary 7.19 *The Herglotz operator $H : L_t^2(\mathbb{S}^2) \to H^{-1/2}(\mathrm{Div}, \partial D)$ defined by*

$$(Hg)(x) := ik\, \nu(x) \times \int_{\mathbb{S}^2} e^{ik\, x\cdot d} g(d)\, ds(d), \quad x \in \partial D, \tag{7.48}$$

is injective and has dense range provided k is not a Maxwell eigenvalue for D.

Proof. In view of Theorem 7.13 we only need to be concerned with the denseness of $H(L_t^2(\mathbb{S}^2))$ in $H^{-1/2}(\mathrm{Div}, \partial D)$. From (7.48), in view of the duality pairing for $H^{-1/2}(\mathrm{Div}, \partial D)$ and its dual space $H^{-1/2}(\mathrm{Curl}, \partial D)$, interchanging the order of integration we observe that the dual operator $H^\top : H^{-1/2}(\mathrm{Curl}, \partial D) \to L_t^2(\mathbb{S}^2)$ of H is given by

$$H^\top a = \frac{2\pi}{ik}\, \overline{AN\overline{a}}, \quad a \in H^{-1/2}(\mathrm{Curl}, \partial D), \tag{7.49}$$

in terms of the boundary data to far field operator $A : H^{-1/2}(\mathrm{Div}, \partial D) \to L_t^2(\mathbb{S}^2)$ and the electric dipole operator $N : H^{-1/2}(\mathrm{Curl}, \partial D) \to H^{-1/2}(\mathrm{Div}, \partial D)$. Since A and N are bounded, (7.49) represents the dual operator on $H^{-1/2}(\mathrm{Curl}, \partial D)$. Both A and N are injective, the latter because of our assumption on k. Hence $H^\top$ is injective and the dense range of H follows by the Hahn–Banach theorem. □

When k is not a Maxwell eigenvalue, well-posedness of the interior Maxwell problem (7.46)–(7.47) in $H(\mathrm{curl}, D)$ with the tangential trace of the electric dipole replaced by an arbitrary $c \in H^{-1/2}(\mathrm{Div}, \partial D)$ can be established by solving the integral equation (7.37) in $H^{-1/2}(\mathrm{Div}, \partial D)$. Now Corollary 7.19 can be interpreted as denseness of Herglotz pairs in the space of solutions to the Maxwell equations in D with respect to $H(\mathrm{curl}, D)$. For an alternate proof we refer to [70].

Lemma 7.20 *$E_{e,\infty}(\cdot\,, z)p \in A\left(H^{-1/2}(\mathrm{Div}, \partial D)\right)$ if and only if $z \in D$.*

Proof. If $z \in D$, then clearly $A(-\nu \times E_e(\cdot, z)p) = E_{e,\infty}(\cdot\,, z)p$. Now let $z \in \mathbb{R}^3 \setminus \bar{D}$ and assume that there is a tangential vector field $c \in H^{-1/2}(\mathrm{Div}, \partial D)$ such that $Ac = E_{e,\infty}(\cdot, z)p$. Then by Theorem 6.11 the radiating field E^s corresponding to the boundary data c and the electric dipole $E_e(\cdot, z)p$ coincide in $(\mathbb{R}^3 \setminus \bar{D}) \setminus \{z\}$. But this is a contradiction since $E^s \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$ but $E_e(\cdot, z)p$ is not. □

Now we are ready to establish our main result on the linear sampling method in inverse electromagnetic obstacle scattering.

Theorem 7.21. *Assume that k is not a Maxwell eigenvalue for D and let F be the far field operator (7.43) for scattering from a perfect conductor. Then the following hold:*

1. For $z \in D$ and a given $\varepsilon > 0$ there exists a $g_z^\varepsilon \in L_t^2(\mathbb{S}^2)$ such that

$$\|Fg_z^\varepsilon - E_{e,\infty}(\cdot, z)p\|_{L_t^2(\mathbb{S}^2)} < \varepsilon \tag{7.50}$$

and the Herglotz wave field $E_{g_z^\varepsilon}$ with kernel g_z^ε converges to the solution of (7.46)–(7.47) in $H(\mathrm{curl}, D)$ as $\varepsilon \to 0$

2. For $z \notin D$ every $g_z^\varepsilon \in L_t^2(\mathbb{S}^2)$ that satisfies (7.50) for a given $\varepsilon > 0$ is such that

$$\lim_{\varepsilon \to 0} \|H_{g_z^\varepsilon}\|_{H(\mathrm{curl},D)} = \infty.$$

Proof. As pointed out above, under the assumption on k we have well-posedness of the interior Meaxwell problem in the $H(\mathrm{curl}, D)$ setting. Given $\varepsilon > 0$, by Corollary 7.19 we can choose $g_z \in L_t^2(\mathbb{S}^2)$ such that

$$\|Hg_z^\varepsilon + \nu \times E_e(\cdot, z)p\|_{H^{-1/2}(\mathrm{Div},\partial D)} < \frac{\varepsilon}{\|A\|}$$

where A denotes the boundary component to far field operator from Corollary 7.18. Then (7.50) follows from observing that

$$F = -AH.$$

Now if $z \in D$ then by the well-posedness of the interior Maxwell problem the convergence $Hg_z^\varepsilon + \nu \times E_e(\cdot, z)p \to 0$ as $\varepsilon \to 0$ in $H^{-1/2}(\mathrm{Div}, \partial D)$ implies convergence $E_{g_z^\varepsilon} \to E$ as $\varepsilon \to 0$ in $H(\mathrm{curl}, D)$ where E is the solution to (7.46)–(7.47). Hence, the first statement is proven.

In order to prove the second statement, for $z \notin D$ assume to the contrary that there exists a null sequence (ε_n) and corresponding Herglotz wave functions E_n with kernels $g_n = g_z^{\varepsilon_n}$ such that $\|E_n\|_{H(\mathrm{curl},D)}$ remains bounded. Then without loss of generality we may assume weak convergence $E_n \rightharpoonup E \in H(\mathrm{curl}, D)$ as $n \to \infty$. Denote by $E^s, H^s \in H_{\mathrm{loc}}(\mathrm{curl}\, \mathbb{R}^3 \setminus \bar{D})$ the solution to the exterior Maxwell problem with $\nu \times E^s = \nu \times E$ on ∂D and denote its electric far field pattern by E_∞. Since Fg_n is the far field pattern of the scattered wave for the incident field $-E_n$ from (7.50) we conclude that $E_\infty = -E_{e,\infty}(\cdot, z)p$ and therefore $E_{e,\infty}(\cdot, z)p$ in $A(H^{-1/2}(\mathrm{Div}, \partial D))$. But this contradicts Lemma 7.20. □

In particular we expect from the above theorem that $\|g_z^\varepsilon\|_{L_t^2(\mathbb{S}^2)}$ will be larger for $z \in \mathbb{R}^3 \setminus \bar{D}$ than it is for $z \in D$. We note that the assumption that k is not a Maxwell eigenvalue can be removed if the far field operator F is replaced by the combined far field operator (c.f. Theorem 6.42)

$$(Fg)(\hat{x}) = \lambda \int_{L_t^2(\mathbb{S}^2)} E_\infty(\hat{x}, d)g(d)\, ds(d) + \mu \int_{L_t^2(\mathbb{S}^2)} H_\infty(\hat{x}, d)\, [g(d) \times d]\, ds(d), \quad \hat{x} \in \mathbb{S}^2,$$

where H_∞ is the magnetic far field pattern [30]. We also observe that in contrast to the scalar case of the linear sampling method an open question in the present case is how to obtain numerically the ε-approximate solution g_z^ε of the far field equation given by Theorem 7.21. In all numerical experiments implemented to date, Tikhonov regularization combined with the Morozov discrepancy principle is used to solve the far field equation and this procedure leads to a solution that exhibits the same behavior as g_z^ε (c.f. [36]).

Chapter 8
Acoustic Waves in an Inhomogeneous Medium

Until now, we have only considered the scattering of acoustic and electromagnetic time-harmonic waves in a homogeneous medium in the exterior of an impenetrable obstacle. For the remaining chapters of this book, we shall be considering the scattering of acoustic and electromagnetic waves by an inhomogeneous medium of compact support, and in this chapter we shall consider the direct scattering problem for acoustic waves. We shall content ourselves with the simplest case when the velocity potential has no discontinuities across the boundary of the inhomogeneous medium and shall again use the method of integral equations to investigate the direct scattering problem. However, since boundary conditions are absent, we shall make use of volume potentials instead of surface potentials as in the previous chapters.

We begin the chapter by deriving the linearized equations governing the propagation of small amplitude sound waves in an inhomogeneous medium. We then reformulate the direct scattering problem for such a medium as an integral equation known as the *Lippmann–Schwinger equation*. In order to apply the Riesz–Fredholm theory to this equation, we need to prove a unique continuation principle for second order elliptic partial differential equations. Having used this result to show the existence of a unique solution to the Lippmann–Schwinger equation, we then proceed to investigate the set $\mathcal{F}$ of far field patterns of the scattered fields corresponding to incident time-harmonic plane waves moving in arbitrary directions. By proving a reciprocity relation for far field patterns, we show that the completeness of the set $\mathcal{F}$ is equivalent to the non-existence of eigenvalues to a new type of boundary value problem for the reduced wave equation called the *interior transmission problem*. We then show that if absorption is present there are no eigenvalues whereas if the inhomogeneous medium is non-absorbing and spherically symmetric then there do exist eigenvalues. A transmission eigenvalue can also be viewed as a value of the wave number such that the far field operator (see Theorem 3.22) has zero as an eigenvalue. This fact motivates us to examine the spectral properties of the far field operator for an inhomogeneous medium. Continuing in this direction, we present the elements of the theory of operator valued analytic functions and apply this theory

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences 93, DOI 10.1007/978-1-4614-4942-3_8,

to investigate the interior transmission problem for inhomogeneous media that are neither absorbing nor spherically symmetric. Transmission eigenvalues will again make their appearance in Section 10.5.

We conclude the chapter by presenting some results on the numerical solution of the direct scattering problem by combining the methods of finite elements and integral equations.

8.1 Physical Background

We begin by again considering the propagation of sound waves of small amplitude in $\mathbb{R}^3$ viewed as a problem in fluid dynamics. Let $v(x,t)$, $x \in \mathbb{R}^3$, be the velocity vector of a fluid particle in an inviscid fluid and let $p(x,t)$, $\rho(x,t)$ and $S(x,t)$ denote the pressure, density and specific entropy, respectively, of the fluid. If no external forces are acting on the fluid, then from Section 2.1 we have the equations

$$\begin{aligned}
\frac{\partial v}{\partial t} + (v \cdot \operatorname{grad})\, v + \frac{1}{\rho}\ \operatorname{grad} p &= 0 && \text{(Euler's equation)} \\
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) &= 0 && \text{(equation of continuity)} \\
p &= f(\rho, S) && \text{(equation of state)} \\
\frac{\partial S}{\partial t} + v \cdot \operatorname{grad} S &= 0 && \text{(adiabatic hypothesis)}
\end{aligned} \tag{8.1}$$

where f is a function depending on the fluid. Assuming $v(x,t)$, $p(x,t)$, $\rho(x,t)$ and $S(x,t)$ are small, we perturb these quantities around the static state $v = 0$, $p = p_0 =$ constant, $\rho = \rho_0(x)$ and $S = S_0(x)$ with $p_0 = f(\rho_0, S_0)$ and write

$$\begin{aligned}
v(x,t) &= \epsilon v_1(x,t) + \cdots \\
p(x,t) &= p_0 + \epsilon p_1(x,t) + \cdots \\
\rho(x,t) &= \rho_0(x) + \epsilon \rho_1(x,t) + \cdots \\
S(x,t) &= S_0(x) + \epsilon S_1(x,t) + \cdots
\end{aligned} \tag{8.2}$$

where $0 < \epsilon \ll 1$ and the dots refer to higher order terms in ϵ. We now substitute (8.2) into (8.1), retaining only the terms of order ϵ. Doing this gives us the linearized equations

$$\frac{\partial v_1}{\partial t} + \frac{1}{\rho_0}\ \operatorname{grad} p_1 = 0$$

$$\frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_0 v_1) = 0$$

$$\frac{\partial p_1}{\partial t} = c^2(x)\left(\frac{\partial \rho_1}{\partial t} + v_1 \cdot \operatorname{grad} \rho_0\right)$$

where the *sound speed* c is defined by

$$c^2(x) = \frac{\partial}{\partial \rho}\ f(\rho_0(x), S_0(x)).$$

From this we deduce that p_1 satisfies

$$\frac{\partial^2 p_1}{\partial t^2} = c^2(x)\rho_0(x)\ \operatorname{div}\left(\frac{1}{\rho_0(x)}\ \operatorname{grad} p_1\right).$$

If we now assume that terms involving $\operatorname{grad} \rho_0$ are negligible and that p_1 is time harmonic,

$$p_1(x,t) = \operatorname{Re}\{u(x)\, e^{-i\omega t}\},$$

we see that u satisfies

$$\Delta u + \frac{\omega^2}{c^2(x)}\ u = 0. \tag{8.3}$$

Equation (8.3) governs the propagation of time harmonic acoustic waves of small amplitude in a slowly varying inhomogeneous medium. We still must prescribe how the wave motion is initiated and what is the boundary of the region containing the fluid. We shall only consider the simplest case when the inhomogeneity is of compact support, the region under consideration is all of $\mathbb{R}^3$ and the wave motion is caused by an incident field u^i satisfying the unperturbed linearized equations being scattered by the inhomogeneous medium. Assuming the inhomogeneous region is contained inside a ball B, i.e., $c(x) = c_0 =$ constant for $x \in \mathbb{R}^3 \setminus B$, we see that the scattering problem under consideration is now modeled by

$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^3, \tag{8.4}$$

$$u = u^i + u^s, \tag{8.5}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{8.6}$$

where $k = \omega/c_0 > 0$ is the *wave number*,

$$n(x) := \frac{c_0^2}{c^2(x)}$$

is the *refractive index*, u^i is an entire solution of the Helmholtz equation $\Delta u + k^2 u = 0$ and u^s is the scattered field which, as discussed in Section 2.2, satisfies the Sommerfeld radiation condition (8.6) uniformly in all directions. The refractive index is always positive and in our case $n(x) = 1$ for $x \in \mathbb{R}^3 \setminus B$. Occasionally, we would also like to include the possibility that the medium is absorbing, i.e., the refractive index has an imaginary component. This is often modeled in the literature by adding a term that is proportional to v in Euler's equation which implies that n is now of the form

$$n(x) = n_1(x) + i\,\frac{n_2(x)}{k}\,. \tag{8.7}$$

8.2 The Lippmann–Schwinger Equation

The aim of this section is to derive an integral equation that is equivalent to the scattering problem (8.4)–(8.6) where we assume the refractive index n of the general form (8.7) to be piecewise continuous in $\mathbb{R}^3$ such that

$$m := 1 - n$$

has compact support and

$$n_1(x) > 0 \quad \text{and} \quad n_2(x) \geq 0$$

for all $x \in \mathbb{R}^3$. Throughout this chapter, we shall always assume that these assumptions are valid and let $D := \{x \in \mathbb{R}^3 : m(x) \neq 0\}$.

To derive an integral equation equivalent to (8.4)–(8.6), we shall need to consider the *volume potential*

$$u(x) := \int_{\mathbb{R}^3} \Phi(x, y)\varphi(y)\,dy, \quad x \in \mathbb{R}^3, \tag{8.8}$$

where

$$\Phi(x, y) := \frac{1}{4\pi}\,\frac{e^{ik|x-y|}}{|x-y|}\,, \quad x \neq y,$$

is the fundamental solution to the Helmholtz equation and φ is a continuous function in $\mathbb{R}^3$ with compact support, i.e., $\varphi \in C_0(\mathbb{R}^3)$. Extending the definitions given in Section 3.1, for a domain $G \subset \mathbb{R}^3$ the Hölder spaces $C^{p,\alpha}(G)$ are defined as the subspaces of $C^p(G)$ consisting of bounded functions whose p–th order derivates are uniformly Hölder continuous with exponent α. They are Banach spaces with the norms recursively defined by

$$\|\varphi\|_{p,\alpha} := \|\varphi\|_\infty + \|\operatorname{grad}\varphi\|_{p-1,\alpha}.$$

We can now state the following theorem (c.f. [105]; the first mapping property follows from a slight modification of the proof of Theorem 2.7 of [64]).

Theorem 8.1. *The volume potential u given by (8.8) exists as an improper integral for all $x \in \mathbb{R}^3$ and has the following properties. If $\varphi \in C_0(\mathbb{R}^3)$ then $u \in C^{1,\alpha}(\mathbb{R}^3)$ and the orders of differentiation and integration can be interchanged. If $\varphi \in C_0(\mathbb{R}^3) \cap C^{0,\alpha}(\mathbb{R}^3)$ then $u \in C^{2,\alpha}(\mathbb{R}^3)$ and*

$$\Delta u + k^2 u = -\varphi \quad \text{in } \mathbb{R}^3. \tag{8.9}$$

In addition, we have

$$\|u\|_{2,\alpha,\mathbb{R}^3} \leq C\|\varphi\|_{\alpha,\mathbb{R}^3}$$

for some positive constant C depending only on the support of φ. Furthermore, if $\varphi \in C_0(\mathbb{R}^3) \cap C^{1,\alpha}(\mathbb{R}^3)$ then $u \in C^{3,\alpha}(\mathbb{R}^3)$.

Since for piecewise continuous n we cannot expect C^2 solutions of (8.5) we require the solutions to belong to the Sobolev space $H^2_{\mathrm{loc}}(\mathbb{R}^3)$ of functions with locally square integrable weak derivatives, i.e, derivatives in the distributional sense up to second order. As a tool for establishing existence in this setting, we shall now use Lax's Theorem 3.5 to deduce a mapping property for the volume potential in Sobolev spaces from the classical property in Hölder spaces given above.

Theorem 8.2. *Given two bounded domains D and G, the volume potential*

$$(V\varphi)(x) := \int_D \Phi(x,y)\varphi(y)\,dy, \quad x \in \mathbb{R}^3,$$

defines a bounded operator $V : L^2(D) \to H^2(G)$.

Proof. We choose an open ball B such that $\bar{G} \subset B$ and a nonnegative function $\gamma \in C^2_0(B)$ such that $\gamma(x) = 1$ for all $x \in G$. Consider the spaces $X = C^{0,\alpha}(D)$ and $Y = C^{2,\alpha}(B)$ equipped with the usual Hölder norms. Introduce scalar products on X by the usual L^2 scalar product and on Y by the weighted Sobolev scalar product

$$(u,v)_Y := \int_B \gamma \left\{ u\bar{v} + \sum_{i=1}^{3} \frac{\partial u}{\partial x_i}\,\frac{\partial \bar{v}}{\partial x_i} + \sum_{i,j=1}^{3} \frac{\partial^2 u}{\partial x_i \partial x_j}\,\frac{\partial^2 \bar{v}}{\partial x_i \partial x_j} \right\} dx.$$

We first note that by using $\operatorname{grad}_x \Phi(x,y) = -\operatorname{grad}_y \Phi(x,y)$ and interchanging the order of integration we have

$$\int_B \gamma V\varphi\,\psi\,dx = \int_D \varphi\,V^*(\gamma\psi)\,dx \tag{8.10}$$

and

$$\int_B \gamma\,\frac{\partial}{\partial x_i}\,V\varphi\,\frac{\partial\psi}{\partial x_i}\,dx = -\int_D \varphi\,\frac{\partial}{\partial x_i}\,V^*\left(\gamma\,\frac{\partial\psi}{\partial x_i}\right)dx \tag{8.11}$$

for $\varphi \in X$ and $\psi \in Y$ where

$$(V^*\psi)(x) := \int_B \psi(y)\Phi(x,y)\,dy, \quad x \in \mathbb{R}^3.$$

Using Gauss' divergence theorem, for $\varphi \in C_0^1(D)$ we have

$$\frac{\partial}{\partial x_i}\int_D \Phi(x,y)\varphi(y)\,dy = \int_D \Phi(x,y)\,\frac{\partial\varphi}{\partial y_i}(y)\,dy, \quad x \in \mathbb{R}^3,$$

that is,

$$\frac{\partial}{\partial x_i}\,V\varphi = V\,\frac{\partial\varphi}{\partial x_i}$$

and consequently, by (8.11) and Gauss' theorem,

$$\int_B \gamma\,\frac{\partial^2 V\varphi}{\partial x_i \partial x_j}\,\frac{\partial^2\psi}{\partial x_i\partial x_j}\,dx = \int_D \varphi\,\frac{\partial^2}{\partial x_i\partial x_j}\,V^*\left(\gamma\,\frac{\partial^2\psi}{\partial x_i\partial x_j}\right)dx \tag{8.12}$$

for $\varphi \in C_0^1(D)$ and $\psi \in Y$. Hence, after setting $U = C_0^1(D) \subset X$, from (8.10)–(8.12) we have that the operators $V : U \to Y$ and $W : Y \to X$ given by

$$W\psi := \overline{V^*\bar\psi} - \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\,\overline{V^*\left(\gamma\,\frac{\partial\bar\psi}{\partial x_i}\right)} + \sum_{i,j=1}^{3}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\,\overline{V^*\left(\gamma\,\frac{\partial^2\bar\psi}{\partial x_i\partial x_j}\right)}$$

are adjoint, i.e.,

$$(V\varphi,\psi)_X = (\varphi, W\psi)_Y$$

for all $\varphi \in U$ and $\psi \in Y$. By Theorem 8.1, both V and W are bounded with respect to the Hölder norms. Hence, from Lax's Theorem 3.5 and using the fact that the norm on Y dominates the H^2 norm over G, we see that there exists a positive constant c such that

$$\|V\varphi\|_{H^2(G)} \le c\|\varphi\|_{L^2(D)}$$

for all $\varphi \in C_0^1(D)$. The proof is now finished by observing that $C_0^1(D)$ is dense in $L^2(D)$. □

Approximating an L^2 density φ with compact support by a sequence of $C^{0,\alpha}$ functions with compact support, from Theorem 8.2 we can deduce that (8.9) remains valid in the H^2 sense. In $\mathbb{R}^3$ for a bounded domain D with C^2 boundary by the Sobolev imbedding theorem $H^2(D)$ functions are continuous (see [105]). Furthermore, for functions $u \in H^2(D)$ the traces $u|_{\partial D} \in H^{3/2}(\partial D)$ and $\partial u/\partial\nu|_{\partial D} \in H^{1/2}(\partial D)$ exist (see [240]) and consequently Green's integral theorem remains valid. Therefore, the proof of Green's formula (2.4) can be carried over to H^2 functions. In particular, (2.5) remains valid for H^2 solutions to the Helmholtz equation. This implies that H^2 solutions to the Helmholtz equation automatically are C^2 solutions. Therefore the Sommerfeld radiation condition is well defined for H^2 solutions.

We now show that the scattering problem (8.4)–(8.6) is equivalent to the problem of solving the integral equation

$$u(x) = u^i(x) - k^2\int_{\mathbb{R}^3}\Phi(x,y)m(y)u(y)\,dy, \quad x \in \mathbb{R}^3, \tag{8.13}$$

for u which is known as the *Lippmann–Schwinger equation*.

Theorem 8.3. *If $u \in H^2_{loc}(\mathbb{R}^3)$ is a solution of (8.4)–(8.6), then u is a solution of (8.13). Conversely, if $u \in C(\mathbb{R}^3)$ is a solution of (8.13) then $u \in H^2_{loc}(\mathbb{R}^3)$ and u is a solution of (8.4)–(8.6).*

Proof. Let $u \in H^2_{loc}(\mathbb{R}^3)$ be a solution of (8.4)–(8.6). Let $x \in \mathbb{R}^3$ be an arbitrary point and choose an open ball B with exterior unit normal ν containing the support of m such that $x \in B$. From Green's formula (2.4) applied to u, we have

$$u(x) = \int_{\partial B} \left\{ \frac{\partial u}{\partial \nu}\, \Phi(x,\cdot) - u\, \frac{\partial \Phi(x,\cdot)}{\partial \nu} \right\} ds - k^2 \int_B \Phi(x,\cdot) m u \, dy \tag{8.14}$$

since $\Delta u + k^2 u = mk^2 u$. Note that in the volume integral over B we can integrate over all of $\mathbb{R}^3$ since m has support in B. Green's formula (2.5), applied to u^i, gives

$$u^i(x) = \int_{\partial B} \left\{ \frac{\partial u^i}{\partial \nu}\, \Phi(x,\cdot) - u^i\, \frac{\partial \Phi(x,\cdot)}{\partial \nu} \right\} ds. \tag{8.15}$$

Finally, from Green's theorem (2.3) and the radiation condition (8.6) we see that

$$\int_{\partial B} \left\{ \frac{\partial u^s}{\partial \nu}\, \Phi(x,\cdot) - u^s\, \frac{\partial \Phi(x,\cdot)}{\partial \nu} \right\} ds = 0. \tag{8.16}$$

With the aid of $u = u^i + u^s$ we can now combine (8.14)–(8.16) to conclude that (8.13) is satisfied.

Conversely, let $u \in C(\mathbb{R}^3)$ be a solution of (8.13) and define u^s by

$$u^s(x) := -k^2 \int_{\mathbb{R}^3} \Phi(x,y) m(y) u(y)\, dy, \quad x \in \mathbb{R}^3.$$

Since Φ satisfies the Sommerfeld radiation condition (8.6) uniformly with respect to y on compact sets and m has compact support, it is easily verified that u^s satisfies (8.6). Since m is piecewise continuous and has compact support we can conclude from Theorem 8.2 and (8.9) that $u^s \in H^2_{loc}(\mathbb{R}^3)$ with $\Delta u^s + k^2 u^s = k^2 mu$. Finally, since $\Delta u^i + k^2 u^i = 0$, we have that

$$\Delta u + k^2 u = (\Delta u^i + k^2 u^i) + (\Delta u^s + k^2 u^s) = k^2 mu,$$

that is, $\Delta u + k^2 nu = 0$ in $\mathbb{R}^3$ and the proof is completed. □

We note that in (8.13) we can replace the region of integration by any domain G such that the support of m is contained in $\bar{G}$ and look for solutions in $C(\bar{G})$. Then for $x \in \mathbb{R}^3 \setminus \bar{G}$ we define $u(x)$ by the right hand side of (8.13) and obviously obtain a continuous solution u to the Lippmann–Schwinger equation in all of $\mathbb{R}^3$.

We shall show shortly that (8.13) is uniquely solvable for all values of $k > 0$. This result is nontrivial since it will be based on a unique continuation principle for solutions of (8.4). However, for k sufficiently small we can show the existence of a unique solution to (8.13) by the simple method of successive approximations.

Theorem 8.4. *Suppose that $m(x) = 0$ for $|x| \geq a$ with some $a > 0$ and $k^2 < 2/Ma^2$ where $M := \sup_{|x|\leq a} |m(x)|$. Then there exists a unique solution to the integral equation (8.13).*

Proof. As already pointed out, it suffices to solve (8.13) for $u \in C(\bar{B})$ with the ball $B := \{x \in \mathbb{R}^3 : |x| < a\}$. On the Banach space $C(\bar{B})$, define the operator $T_m : C(\bar{B}) \to C(\bar{B})$ by

$$(T_m u)(x) := \int_B \Phi(x, y)m(y)u(y)\, dy, \quad x \in \bar{B}. \tag{8.17}$$

By the method of successive approximations, our theorem will be proved if we can show that $\|T_m\|_\infty \leq Ma^2/2$. To this end, we have

$$|(T_m u)(x)| \leq \frac{M\|u\|_\infty}{4\pi} \int_B \frac{dy}{|x - y|}\,, \quad x \in \bar{B}. \tag{8.18}$$

To estimate the integral in (8.18), we note that (see Theorem 8.1)

$$h(x) := \int_B \frac{dy}{|x - y|}\,, \quad x \in \bar{B},$$

is a solution of the Poisson equation $\Delta h = -4\pi$ and is a function only of $r = |x|$. Hence, h solves the differential equation

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{dh}{dr}\right) = -4\pi$$

which has the general solution

$$h(r) = -\frac{2}{3}\,\pi r^2 + \frac{C_1}{r} + C_2$$

where C_1 and C_2 are arbitrary constants. Since h is continuous in a neighborhood of the origin, we must have $C_1 = 0$ and, letting $r \to 0$, we see that

$$C_2 = h(0) = \int_B \frac{dy}{|y|} = 4\pi \int_0^a \rho\, d\rho = 2\pi a^2.$$

Hence, $h(r) = 2\pi(a^2 - r^2/3)$ and thus $\|h\|_\infty = 2\pi a^2$. From (8.18) we can now conclude that

$$|(T_m u)(x)| \leq \frac{Ma^2}{2}\, \|u\|_\infty, \quad x \in \bar{B},$$

i.e., $\|T_m\|_\infty \leq Ma^2/2$ and the proof is completed. □

8.3 The Unique Continuation Principle

In order to establish the existence of a unique solution to the scattering problem (8.4)–(8.6) for all positive values of the wave number k, we see from the previous section that it is necessary to establish the existence of a unique solution to the Lippmann–Schwinger equation (8.13). To this end, we would like to apply the Riesz–Fredholm theory since the integral operator (8.17) has a weakly singular kernel and hence is a compact operator $T_m : C(\bar{B}) \to C(\bar{B})$ where B is a ball such that $\bar{B}$ contains the support of m. In order to achieve this aim, we must show that the homogeneous equation has only the trivial solution, or, equivalently, that the only solution of

$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^3, \tag{8.19}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{8.20}$$

is u identically zero. To prove this, the following unique continuation principle is fundamental. The unique continuation principle for elliptic equations has a long history and we refer the reader to [20] for a historical discussion. The proof of this principle for (8.19) dates back to Müller [252, 255]. Our proof is based on the ideas of Protter [285] and Leis [224].

Lemma 8.5 *Let G be a domain in $\mathbb{R}^3$ and let $u_1, \dots, u_P \in H^2(G)$ be real valued functions satisfying*

$$|\Delta u_p| \le c \sum_{q=1}^{P} \{|u_q| + |\operatorname{grad} u_q|\} \quad \text{in } G \tag{8.21}$$

for $p = 1, \dots, P$ and some constant c. Assume that u_p vanishes in a neighborhood of some $x_0 \in G$ for $p = 1, \dots, P$. Then u_p is identically zero in G for $p = 1, \dots, P$.

Proof. For $0 < R \le 1$, let $B[x_0; R]$ be the closed ball of radius R centered at x_0. Choose R such that $B[x_0, R] \subset G$. We shall show that $u_p(x) = 0$ for $x \in B[x_0; R/2]$ and $p = 1, \dots, P$. The theorem follows from this since any other point $x_1 \in G$ can be connected to x_0 by a finite number of overlapping balls. Without loss of generality, we shall assume that $x_0 = 0$ and for convenience we temporarily write $u = u_p$.

For $r = |x|$ and n an arbitrary positive integer, we define $v \in H^2(G)$ by

$$v(x) := \begin{cases} e^{r^{-n}} u(x), & x \ne 0, \\ 0, & x = 0. \end{cases}$$

Then

$$\Delta u = e^{-r^{-n}} \left\{ \Delta v + \frac{2n}{r^{n+1}} \frac{\partial v}{\partial r} + \frac{n}{r^{n+2}} \left(\frac{n}{r^n} - n + 1 \right) v \right\}.$$

Using the inequality $(a+b)^2 \geq 2ab$ and calling the middle term in the above expression in brackets b, we see that

$$(\Delta u)^2 \geq \frac{4n\, e^{-2r^{-n}}}{r^{n+1}} \,\frac{\partial v}{\partial r}\left\{\Delta v + \frac{n}{r^{n+2}}\left(\frac{n}{r^n} - n + 1\right)v\right\}.$$

We now let $\varphi \in C^2(\mathbb{R}^3)$ be such that $\varphi(x) = 1$ for $|x| \leq R/2$ and $\varphi(x) = 0$ for $|x| \geq R$. Then if we define $\hat{u}$ and $\hat{v}$ by $\hat{u} := \varphi u$ and $\hat{v} := \varphi v$ respectively, we see that the above inequality is also valid for u and v replaced by $\hat{u}$ and $\hat{v}$ respectively. In particular, we have the inequality

$$\int_G r^{n+2}\, e^{2r^{-n}} (\Delta \hat{u})^2\, dx \geq 4n \int_G r\, \frac{\partial \hat{v}}{\partial r}\left\{\Delta \hat{v} + \frac{n}{r^{n+2}}\left(\frac{n}{r^n} - n + 1\right)\hat{v}\right\} dx. \tag{8.22}$$

We now proceed to integrate by parts in (8.22), noting that by our choice of φ the boundary terms all vanish. Using the vector identity

$$2\,\mathrm{grad}\{x \cdot \mathrm{grad}\,\hat{v}\} \cdot \mathrm{grad}\,\hat{v} = \mathrm{div}\{x |\,\mathrm{grad}\,\hat{v}|^2\} - |\,\mathrm{grad}\,\hat{v}|^2,$$

from Green's theorem and Gauss' divergence theorem we find that

$$\int_G r\, \frac{\partial \hat{v}}{\partial r}\, \Delta \hat{v}\, dx = -\int_G \mathrm{grad}\{x \cdot \mathrm{grad}\,\hat{v}\} \cdot \mathrm{grad}\,\hat{v}\, dx = \frac{1}{2} \int_G |\,\mathrm{grad}\,\hat{v}|^2\, dx,$$

that is,

$$\int_G r\, \frac{\partial \hat{v}}{\partial r}\, \Delta \hat{v}\, dx = \frac{1}{2} \int_G |\,\mathrm{grad}\,\hat{v}|^2\, dx. \tag{8.23}$$

Furthermore, for m an integer, by partial integration with respect to r we have

$$\begin{aligned}\int_G \frac{1}{r^m}\, \hat{v}\, \frac{\partial \hat{v}}{\partial r}\, dx &= -\int_G \hat{v}\, \frac{\partial}{\partial r}\left(\frac{1}{r^{m-2}}\, \hat{v}\right) \frac{dx}{r^2}\\ &= -\int_G \frac{1}{r^m}\, \hat{v}\, \frac{\partial \hat{v}}{\partial r}\, dx + (m-2)\int_G \frac{\hat{v}^2}{r^{m+1}}\, dx,\end{aligned}$$

that is,

$$\int_G \frac{1}{r^m}\, \hat{v}\, \frac{\partial \hat{v}}{\partial r}\, dx = \frac{1}{2}\,(m-2)\int_G \frac{\hat{v}^2}{r^{m+1}}\, dx. \tag{8.24}$$

We can now insert (8.23) and (8.24) (for $m = 2n+1$ and $m = n+1$) into the inequality (8.22) to arrive at

$$\int_G r^{n+2}\, e^{2r^{-n}} (\Delta \hat{u})^2 dx \geq 2n \int_G |\,\mathrm{grad}\,\hat{v}|^2 dx + 2n^2(n^2 + n - 1)\int_G \frac{\hat{v}^2}{r^{2n+2}}\, dx. \tag{8.25}$$

Here we have also used the inequality

$$\int_G \frac{\hat{v}^2}{r^{2n+2}}\,dx \geq \int_G \frac{\hat{v}^2}{r^{n+2}}\,dx$$

which follows from $\hat{v}(x) = 0$ for $r = |x| \geq R$ and $0 < R \leq 1$. From

$$\operatorname{grad}\hat{u} = e^{-r^{-n}}\left\{\operatorname{grad}\hat{v} + \frac{n}{r^{n+1}}\,\frac{x}{r}\,\hat{v}\right\}$$

we can estimate

$$e^{2r^{-n}}|\operatorname{grad}\hat{u}|^2 \leq 2|\operatorname{grad}\hat{v}|^2 + \frac{2n^2}{r^{2n+2}}\,|\hat{v}|^2$$

and with this and (8.25) we find that

$$\int_G r^{n+2}\,e^{2r^{-n}}|\Delta\hat{u}|^2 dx \geq n\int_G e^{2r^{-n}}|\operatorname{grad}\hat{u}|^2 dx + n^4\int_G \frac{e^{2r^{-n}}}{r^{2n+2}}\,\hat{u}^2 dx. \tag{8.26}$$

Up to now, we have not used the inequality (8.21). Now we do, relabeling u by u_p. From (8.21) and the Cauchy–Schwarz inequality, we clearly have that

$$|\Delta u_p(x)|^2 \leq 2Pc^2\sum_{q=1}^{P}\left\{\frac{|\operatorname{grad}u_q(x)|^2}{r^{n+2}} + \frac{|u_q(x)|^2}{r^{3n+4}}\right\}, \quad |x| \leq \frac{R}{2}\,,$$

since $R \leq 1$. We further have

$$|\Delta\hat{u}_p(x)|^2 \leq \frac{|\Delta\hat{u}_p(x)|^2}{r^{3n+4}}\,, \quad \frac{R}{2} \leq |x| \leq R,$$

since $R \leq 1$. Observing that $u_p(x) = \hat{u}_p(x)$ for $|x| \leq R/2$, from (8.26) we now have

$$\begin{aligned} n\int_{|x|\leq R/2} e^{2r^{-n}}|\operatorname{grad}u_p|^2 dx + n^4\int_{|x|\leq R/2}\frac{e^{2r^{-n}}}{r^{2n+2}}\,u_p^2\,dx &\leq \int_G r^{n+2}\,e^{2r^{-n}}|\Delta\hat{u}_p|^2 dx \\ \leq 2Pc^2\sum_{q=1}^{P}\left\{\int_{|x|\leq R/2} e^{2r^{-n}}|\operatorname{grad}u_q|^2 dx + \int_{|x|\leq R/2}\frac{e^{2r^{-n}}}{r^{2n+2}}\,u_q^2\,dx\right\} & \\ + \int_{R/2\leq|x|\leq R}\frac{e^{2r^{-n}}|\Delta\hat{u}_p(x)|^2}{r^{2n+2}}\,dx, & \end{aligned}$$

i.e., for sufficiently large n we have

$$n^4\int_{|x|\leq R/2}\frac{e^{2r^{-n}}}{r^{2n+2}}\,u_p^2\,dx \leq C\int_{R/2\leq|x|\leq R}\frac{e^{2r^{-n}}|\Delta\hat{u}_p(x)|^2}{r^{2n+2}}\,dx, \quad p = 1, \ldots, P,$$

for some constant C. From this, since the function

$$r \mapsto \frac{e^{2r^{-n}}}{r^{2n+2}}, \quad r > 0,$$

is monotonically decreasing, for sufficiently large n we have

$$n^4 \int_{|x|\le R/2} u_p^2\, dx \le C \int_{R/2\le|x|\le R} |\Delta \hat{u}_p(x)|^2 dx, \quad p = 1, \dots, P.$$

Letting n tend to infinity now shows that $u_p(x) = 0$ for $|x| \le R/2$ and $p = 1, \dots, P$ and the theorem is proved. □

Theorem 8.6. *Let G be a domain in $\mathbb{R}^3$ and suppose $u \in H^2(G)$ is a solution of*

$$\Delta u + k^2 n(x) u = 0$$

in G such that n is piecewise continuous in G and u vanishes in a neighborhood of some $x_0 \in G$. Then u is identically zero in G.

Proof. Apply Lemma 8.5 to $u_1 := \operatorname{Re} u$ and $u_2 := \operatorname{Im} u$. □

We are now in a position to show that for all $k > 0$ there exists a unique solution to the scattering problem (8.4)–(8.6).

Theorem 8.7. *For each $k > 0$ there exists a unique solution $u \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ to (8.4)–(8.6) and u depends continuously with respect to the maximum norm on the incident field u^i.*

Proof. As previously discussed, to show existence and uniqueness it suffices to show that the only solution of (8.19), (8.20) is u identically zero. If this is done, by the Riesz–Fredholm theory the integral equation (8.13) can be inverted in $C(\bar{B})$ and the inverse operator is bounded. From this, it follows that u depends continuously on the incident field u^i with respect to the maximum norm. Hence we only must show that the only solution of (8.19), (8.20) is $u = 0$.

Recall that B is chosen to be a ball of radius a centered at the origin such that m vanishes outside of B. As usual ν denotes the exterior unit normal to ∂B. We begin by noting from Green's theorem (2.2) and (8.19) that

$$\int_{|x|=a} u \,\frac{\partial \bar{u}}{\partial \nu}\, ds = \int_{|x|\le a} \left\{|\operatorname{grad} u|^2 - k^2\, \overline{n}\, |u|^2\right\} dx.$$

From this, since $\operatorname{Im} n \ge 0$, it follows that

$$\operatorname{Im} \int_{|x|=a} u \,\frac{\partial \bar{u}}{\partial \nu}\, ds = k^2 \int_{|x|\le a} \operatorname{Im} n\, |u|^2 dx \ge 0. \tag{8.27}$$

Theorem 2.13 now shows that $u(x) = 0$ for $|x| \ge a$ and it follows by Theorem 8.6 that $u(x) = 0$ for all $x \in \mathbb{R}^3$. □

8.4 The Far Field Pattern

From (8.13) we see that

$$u^s(x) = -k^2 \int_{\mathbb{R}^3} \Phi(x,y) m(y) u(y)\, dy, \quad x \in \mathbb{R}^3.$$

Hence, letting $|x|$ tend to infinity, with the help of (2.15) we see that

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\, u_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,$$

where the *far field pattern* u_∞ is given by

$$u_\infty(\hat{x}) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\,\hat{x}\cdot y} m(y) u(y)\, dy \tag{8.28}$$

for $\hat{x} = x/|x|$ on the unit sphere $\mathbb{S}^2$. We note that by Theorem 8.4, for k sufficiently small, u can be obtained by the method of successive approximations. If in (8.28) we replace u by the first term in this iterative process, we obtain the *Born approximation*

$$u_\infty(\hat{x}) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\,\hat{x}\cdot y} m(y) u^i(y)\, dy. \tag{8.29}$$

We shall briefly return to this approximation in Chapter 10 where it will provide the basis of a linear approach to the inverse scattering problem.

We now consider the case when the incident field u^i is a plane wave, i.e., $u^i(x) = e^{ik\,x\cdot d}$ where d is a unit vector giving the direction of propagation. We denote the dependence of the far field pattern u_∞ on d by writing $u_\infty(\hat{x}) = u_\infty(\hat{x}, d)$ and, similarly, we write $u^s(x) = u^s(x,d)$ and $u(x) = u(x,d)$. Then, analogous to Theorem 3.15, we have the following *reciprocity relation*.

Theorem 8.8. *The far field pattern satisfies the reciprocity relation*

$$u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x})$$

for all $\hat{x}, d$ on the unit sphere $\mathbb{S}^2$.

Proof. By the relation (3.38) from the proof of Theorem 3.15, we have

$$4\pi\{u_\infty(\hat{x}, d) - u_\infty(-d, -\hat{x})\}$$

$$= \int_{|y|=a} \left\{ u(y,d)\, \frac{\partial}{\partial\nu(y)}\, u(y,-\hat{x}) - u(y,-\hat{x})\, \frac{\partial}{\partial\nu(y)}\, u(y,d) \right\} ds(y)$$

whence the statement follows with the aid of Green's theorem (2.3). □

As in Chapter 3, we are again concerned with the question if the far field patterns corresponding to all incident plane waves are complete in $L^2(\mathbb{S}^2)$. The reader will recall from Section 3.3 that for the case of obstacle scattering the far field patterns are complete provided k^2 is not a Dirichlet eigenvalue having an eigenfunction that is a Herglotz wave function. In the present case of scattering by an inhomogeneous medium we have a similar result except that the Dirichlet problem is replaced by a new type of boundary value problem introduced by Kirsch in [176] (see also [76]) called the interior transmission problem. This name is motivated by the fact that, as in the classical transmission problem, we have two partial differential equations linked together by their Cauchy data on the boundary but, in this case, the partial differential equations are both defined in the same interior domain instead of in an interior and exterior domain as for the classical transmission problem (c.f. [64]). In particular, let $\{d_n : n = 1, 2, \dots\}$ be a countable dense set of vectors on the unit sphere $\mathbb{S}^2$ and define the class $\mathcal{F}$ of far field patterns by

$$\mathcal{F} := \{u_\infty(\cdot\,, d_n) : n = 1, 2, \dots\}.$$

Then we have the following theorem. For the rest of this chapter, we shall assume that $D := \{x \in \mathbb{R}^3 : m(x) \neq 0\}$ is connected with a connected C^2 boundary ∂D and D contains the origin.

Theorem 8.9. *The orthogonal complement of $\mathcal{F}$ in $L^2(\mathbb{S}^2)$ consists of the conjugate of those functions $g \in L^2(\mathbb{S}^2)$ for which there exists $w \in H^2(D)$ and a Herglotz wave function*

$$v(x) = \int_{\mathbb{S}^2} e^{-ik\,x\cdot d} g(d)\, ds(d), \qquad x \in \mathbb{R}^3,$$

such that the pair v, w is a solution to

$$\Delta w + k^2 n(x) w = 0, \quad \Delta v + k^2 v = 0 \quad \text{in } D \tag{8.30}$$

satisfying

$$w = v, \quad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D. \tag{8.31}$$

Proof. Let $\mathcal{F}^\perp$ denote the orthogonal complement to $\mathcal{F}$. We will show that $\bar{g} \in \mathcal{F}^\perp$ if and only if g satisfies the assumptions stated in the theorem. From the continuity of u_∞ as a function of d and Theorem 8.8, we have that the property $\bar{g} \in \mathcal{F}^\perp$, i.e.,

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d_n) g(\hat{x})\, ds(\hat{x}) = 0$$

for $n = 1, 2, \dots$ is equivalent to

$$\int_{\mathbb{S}^2} u_\infty(-d, -\hat{x}) g(\hat{x})\, ds(\hat{x}) = 0$$

for all $d \in \mathbb{S}^2$, i.e.,

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(-d)\, ds(d) = 0 \tag{8.32}$$

for all $\hat{x} \in \mathbb{S}^2$. From the Lippmann–Schwinger equation (8.13) it can be seen that the left hand side of (8.32) is the far field pattern of the scattered field w^s corresponding to the incident field

$$w^i(x) := \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(-d)\,ds(d) = \int_{\mathbb{S}^2} e^{-ik\,x\cdot d} g(d)\,ds(d).$$

But now (8.32) is equivalent to a vanishing far field pattern of w^s and hence by Theorem 2.14 equivalent to $w^s = 0$ in all of $\mathbb{R}^3 \setminus D$, i.e., if $v = w^i$ and $w = w^i + w^s$ then $w = v$ on ∂D and $\partial w/\partial\nu = \partial v/\partial\nu$ on ∂D. Conversely, if v and w satisfy the conditions of the theorem, by setting $w = v$ in $\mathbb{R}^3 \setminus \bar{D}$ and using Green's formula we see that w can be extended into all of $\mathbb{R}^3$ as a H^2 solution of $\Delta w + k^2 n(x) w = 0$. The theorem now follows. □

From the above proof, we see that the boundary conditions (8.31) are equivalent to the condition that $w = v$ in $\mathbb{R}^3 \setminus D$ (In particular, we can impose the boundary conditions (8.31) on the boundary of any domain with C^2 boundary that contains D). We have chosen to impose the boundary conditions on ∂D since it is necessary for the more general scattering problem where the density in D is different from that in $\mathbb{R}^3 \setminus D$ [177]. However, when we later consider weak solutions of (8.30), (8.31), we shall replace (8.31) by the more managable condition $w = v$ in $\mathbb{R}^3 \setminus D$.

Analogous to Theorem 3.26, we also have the following theorem, the proof of which is the same as that of Theorem 8.9 except that w^s is now equal to the spherical wave function $v_p(x) = h_p^{(1)}(k|x|)\,Y_p(\hat{x})$. Note that, in contrast to Theorem 3.26, we are now integrating with respect to $\hat{x}$ instead of d. However, by the reciprocity relation, these two procedures are equivalent.

Theorem 8.10. *Let $v_p(x) = h_p^{(1)}(k|x|)\,Y_p(\hat{x})$ be a spherical wave function of order p. The integral equation of the first kind*

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}; d) g_p(\hat{x})\,ds(\hat{x}) = \frac{i^{p-1}}{k}\,Y_p(d), \quad d \in \mathbb{S}^2,$$

has a solution $g_p \in L^2(\mathbb{S}^2)$ if and only if there exists $w \in H^2(D)$ and a Herglotz wave function

$$v(x) = \int_{\mathbb{S}^2} e^{-ik\,x\cdot d} g_p(d)\,ds(d), \quad x \in \mathbb{R}^3,$$

such that v, w is a solution to

$$\Delta w + k^2 n(x) w = 0, \quad \Delta v + k^2 v = 0 \quad \text{in } D \tag{8.33}$$

satisfying

$$w - v = v_p, \quad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \frac{\partial v_p}{\partial \nu} \quad \text{on } \partial D. \tag{8.34}$$

Motivated by Theorems 8.9 and 8.10, we now define the interior transmission problem. (In Chapter 10 we will consider the interior transmission problem under weaker conditions on v and w.)

Interior Transmission Problem. *Find two functions $v, w \in H^2(D)$ such that (8.33), (8.34) are satisfied. The boundary value problem (8.30), (8.31) is called the homogeneous interior transmission problem.*

In this chapter, we shall only be concerned with the homogeneous interior transmission problem. (We return to this problem in Section 10.5.) For information on the inhomogeneous interior transmission problem, we refer the reader to Colton and Kirsch [59], Colton, Kirsch and Päivärinta [62], Kedzierawski [173] and Rynne and Sleeman [298] and Section 10.5. Of primary concern to us in this chapter will be the existence of positive values of the wave number k such that nontrivial solutions exist to the homogeneous interior transmission problem since it is only in this case that there is a possibility that $\mathcal{F}$ is not complete in $L^2(\mathbb{S}^2)$. This motivates the following definition.

Definition 8.11 *If $k > 0$ is such that the homogeneous interior transmission problem has a nontrivial solution then k is called a* transmission eigenvalue*.*

We note that complex transmission eigenvalues can also exist [33, 226]. However in our definition we consider only real, positive transmission eigenvalues since this corresponds to the fact that the wave number k is positive.

Theorem 8.12. *Suppose* $\operatorname{Im} n \neq 0$. *Then $k > 0$ is not a transmission eigenvalue, i.e., the set $\mathcal{F}$ of far field patterns is complete in $L^2(\mathbb{S}^2)$ for each $k > 0$.*

Proof. Let v, w be a solution to (8.30), (8.31). Then by Green's theorem (2.3) we have

$$0 = \int_{\partial D} \left(v \, \frac{\partial \bar{v}}{\partial \nu} - \bar{v} \, \frac{\partial v}{\partial \nu} \right) ds = \int_{\partial D} \left(w \, \frac{\partial \bar{w}}{\partial \nu} - \bar{w} \, \frac{\partial w}{\partial \nu} \right) ds$$

$$= \int_D (w \Delta \bar{w} - \bar{w} \Delta w) \, dx = 2ik^2 \int_D \operatorname{Im} n \, |w|^2 \, dx.$$

Hence, w vanishes identically in the open set $D_0 := \{x \in D : \operatorname{Im} n(x) > 0\}$ and by the unique continuation principle (Theorem 8.6) we see that $w = 0$ in D. Hence, v has vanishing Cauchy data $v = \partial v / \partial \nu = 0$ on ∂D and by Theorem 2.1 this implies $v = 0$ in D. Hence $k > 0$ cannot be a transmission eigenvalue. □

In the case when $\operatorname{Im} n = 0$, there may exist values of k for which $\mathcal{F}$ is not complete and we shall present partial results in the direction later on in this chapter. In the special case of a spherically stratified medium, i.e., (with a slight abuse of notation) $n(x) = n(r)$, $r = |x|$, Colton and Monk [76] have given a rather complete answer to the question of when the set $\mathcal{F}$ is complete. To motivate the hypothesis of the following theorem, note that the case when $n = 1$ is singular since in this case if $h \in C^2(D) \cap C^1(\bar{D})$ is any solution of the Helmholtz equation in D then $v = w = h$ defines a solution of (8.30), (8.31). The case when $n = 1$ corresponds to the case when the sound speed in the inhomogeneous medium is equal to the sound speed in the host medium. Hence, the hypothesis of the following theorem is equivalent to saying that the sound speed in the inhomogeneous medium is always greater than the sound speed in the host medium. An analogous result is easily seen to hold for the case when $n(x) > 1$ for $x \in D$. For the case when n longer belongs to C^2 we refer the reader to [86].

Theorem 8.13. *Suppose that $n(x) = n(r)$,* $\operatorname{Im} n = 0$, $0 < n(r) < 1$ *for* $0 \le r < a$ *and $n(r) = 1$ for $r \ge a$ for some $a > 0$ and, as a function of r, $n \in C^2$. Then transmission eigenvalues exist, and if $k > 0$ is a transmission eigenvalue there exists a solution v of (8.30), (8.31) that is a Herglotz wave function, i.e., the set $\mathcal{F}$ is not complete in $L^2(\mathbb{S}^2)$.*

Proof. We postpone the existence question and assume that there exists a nontrivial solution v, w of the homogeneous interior transmission problem, i.e., $k > 0$ is a transmission eigenvalue. We want to show that v is a Herglotz wave function. To this end, we expand v and w in a series of spherical harmonics

$$v(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m \, j_l(kr) \, Y_l^m(\hat{x})$$

and

$$w(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_l^m(r) \, Y_l^m(\hat{x})$$

where j_l is the spherical Bessel function of order l (see the proof of Rellich's Lemma 2.12). Then, by the orthogonality of the spherical harmonics, the functions

$$v_l^m(x) := a_l^m \, j_l(kr) \, Y_l^m(\hat{x})$$

and

$$w_l^m(x) := b_l^m(r) \, Y_l^m(\hat{x})$$

also satisfy (8.30), (8.31). By the Funk–Hecke formula (2.45) each of the v_l^m is clearly a Herglotz wave function. At least one of them must be different from zero because otherwise v would vanish identically.

To prove that transmission eigenvalues exist, we confine our attention to a solution of (8.30), (8.31) depending only on $r = |x|$. Then clearly v must be of the form

$$v(x) = a_0 \, j_0(kr)$$

with a constant a_0. Writing

$$w(x) = b_0 \, \frac{y(r)}{r}$$

with a constant b_0, straightforward calculations show that if y is a solution of

$$y'' + k^2 n(r) \, y = 0$$

satisfying the initial conditions

$$y(0) = 0, \quad y'(0) = 1,$$

then w satifies (8.30). We note that in order for w to satisfy (8.30) at the origin it suffices to construct a solution $y \in C^1[0,a] \cap C^2(0,a]$ to the initial value problem. This can be seen by applying Green's formula (2.4) for w in a domain where we exclude the origin by a small sphere centered at the origin and letting the radius of this sphere tend to zero. Following Erdélyi [97], p. 79, we use the Liouville transformation

$$\xi := \int_0^r [n(\rho)]^{1/2} d\rho, \quad z(\xi) := [n(r)]^{1/4} y(r)$$

to arrive at the initial-value problem for

$$z'' + [k^2 - p(\xi)]z = 0 \tag{8.35}$$

with initial conditions

$$z(0) = 0, \quad z'(0) = [n(0)]^{-1/4} \tag{8.36}$$

where

$$p(\xi) := \frac{n''(r)}{4\,[n(r)]^2} - \frac{5}{16}\,\frac{[n'(r)]^2}{[n(r)]^3}\,.$$

Rewriting (8.35), (8.36) as a Volterra integral equation

$$z(\xi) = \frac{\sin k\xi}{k\,[n(0)]^{1/4}} + \frac{1}{k}\int_0^\xi \sin k(\eta - \xi) z(\eta) p(\eta)\, d\eta$$

and using the method of successive approximations, we see that the solution of (8.35), (8.36) satisfies

$$z(\xi) = \frac{\sin k\xi}{k\,[n(0)]^{1/4}} + O\left(\frac{1}{k^2}\right) \quad \text{and} \quad z'(\xi) = \frac{\cos k\xi}{[n(0)]^{1/4}} + O\left(\frac{1}{k}\right),$$

that is,

$$y(r) = \frac{1}{k\,[n(0)\,n(r)]^{1/4}}\ \sin\left(k\int_0^r [n(\rho)]^{1/2}\, d\rho\right) + O\left(\frac{1}{k^2}\right)$$

and

$$y'(r) = \left[\frac{n(r)}{n(0)}\right]^{1/4} \cos\left(k\int_0^r [n(\rho)]^{1/2}\, d\rho\right) + O\left(\frac{1}{k}\right)$$

uniformly on $[0,a]$.

The boundary condition (8.31) now requires

$$\begin{aligned} b_0\,\frac{y(a)}{a} \qquad &- \ a_0 j_0(ka) \ = \ 0 \\ b_0\,\frac{d}{dr}\left(\frac{y(r)}{r}\right)_{r=a} &- a_0 k j_0'(ka) = 0. \end{aligned}$$

A nontrivial solution of this system exists if and only if

$$d := \det \begin{pmatrix} \dfrac{y(a)}{a} & -j_0(ka) \\ \dfrac{d}{dr}\left(\dfrac{y(r)}{r}\right)_{r=a} & -kj_0'(ka) \end{pmatrix} = 0. \tag{8.37}$$

Since $j_0(kr) = \sin kr/kr$, from the above asymptotics for $y(r)$, we find that

$$d = \frac{1}{a^2 k\,[n(0)]^{1/4}} \left\{ \sin k \left(a - \int_0^a [n(r)]^{1/2} dr \right) + O\left(\frac{1}{k}\right) \right\}. \tag{8.38}$$

Since $0 < n(r) < 1$ for $0 \le r < a$ by hypothesis, we see that

$$a - \int_0^a [n(r)]^{1/2} dr \ne 0.$$

Hence, from (8.38) we see that for k sufficiently large there exists an infinite set of values of k such that (8.37) is true. Each such k is a transmission eigenvalue and this completes the proof of the theorem. □

In the discussion so far, we have related the completeness of the set $\mathcal{F}$ of far field patterns to the existence of a nontrivial solution to the homogeneous interior transmission problem. An alternate way of viewing this question of completeness is suggested by the proof of Theorem 8.9, i.e., $\mathcal{F}$ is complete if and only if zero is not an eigenvalue of the *far field operator* $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ defined by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2. \tag{8.39}$$

This connection motivates us to conclude this section by briefly examining the spectral properties of the operator F. This analysis is based on the fact that F is a compact operator on $L^2(\mathbb{S}^2)$ and, even more, is a *trace class operator* as defined below.

Definition 8.14 *An operator T is a* trace class operator *on a Hilbert space if there exists a sequence of operators T_n having finite rank not greater than n such that*

$$\sum_{n=1}^{\infty} \|T - T_n\| < \infty.$$

The fact that F is a trace class operator on $L^2(\mathbb{S}^2)$ follows easily from the definition if we define the operators F_n by (8.39) with u_∞ replaced by its truncated spherical harmonic expansion and then use the estimates of Theorem 2.16 (see [69]). The importance of trace class operators for our investigation is the following theorem due to Lidski [294].

Theorem 8.15. *Let T be a trace class operator on a Hilbert space X such that T has finite dimensional nullspace and* $\operatorname{Im}(Tg, g) \geq 0$ *for every $g \in X$. Then T has an infinite number of eigenvalues.*

The first step in using Lidski's theorem to examine the far field operator F is to show that F has a finite dimensional nullspace. If $\operatorname{Im} n \neq 0$, this is true by Theorem 8.12. Hence we will restrict ourselves to the case when $\operatorname{Im} n = 0$ and, as in Theorem 8.13, only consider the case when $m(x) := 1 - n(x) > 0$ for $x \in D$. As in Theorem 8.13, an analogous result is easily seen to hold for the case when $m(x) < 0$ for $x \in D$.

Theorem 8.16. *Suppose* $\operatorname{Im} n = 0$ *and $m(x) > 0$ for $x \in D$. Then the dimension of the nullspace of the far field operator F is finite.*

Proof. From Theorem 8.9 we see that if $Fg = 0$ then there exist a function w in $C^2(D) \cap C^1(\bar{D})$ and a Herglotz wave function v with kernel g such that the pair v, w is a solution of the homogeneous interior transmission problem (8.30), (8.31). Using Theorem 2.1 we see that

$$v = w + k^2 T_m w \tag{8.40}$$

where T_m is defined as in (8.17) by

$$(T_m f)(x) := \int_D \Phi(x, y) m(y) f(y)\, dy, \quad x \in \mathbb{R}^3.$$

Clearly, $T_m w$ is a radiating solution of the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$. From (8.30), (8.31) and (8.40) it follows that $T_m w$ has zero Cauchy data on ∂D and therefore we can conclude that $T_m w = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. Hence, $T_m w$ has vanishing far field pattern, i.e.,

$$\int_D e^{-ik\,\hat{x}\cdot y} m(y) w(y)\, dy = 0, \quad \hat{x} \in \mathbb{S}^2, \tag{8.41}$$

(see (8.28)). For each Herglotz wave function v_h with kernel h, multiplying (8.41) by $\bar{h}$, integrating over $\mathbb{S}^2$ and interchanging the order of integration, we obtain

$$\int_D m w \overline{v_h}\, dy = 0. \tag{8.42}$$

Now let H be the linear space of all Herglotz wave functions and consider H as a subspace of the weighted L^2 space $L^2_m(D)$. Then (8.42) implies that $w \in H^\perp$. Hence, if $P : L^2_m(D) \to H^\perp$ is the orthogonal projection operator onto the closed subspace $H^\perp$ then from (8.40) we have that $0 = w + k^2 P T_m w$. Furthermore, it is easily verified that $T_m : L^2_m(D) \to L^2_m(D)$ is compact. Since the orthogonal projection P is bounded, it now follows from the Fredholm alternative that $I + k^2 PT$ has finite dimensional nullspace. The conclusion of the theorem now follows from (8.40) and Theorem 3.19. $\square$

The next tool we will need in our investigation of the spectral properties of the far field operator F is the identity stated in the following theorem [68].

Theorem 8.17. *Let v_g^i and v_h^i be Herglotz wave functions with kernels $g, h \in L^2(\mathbb{S}^2)$, respectively, and let v_g, v_h be the solutions of (8.4)–(8.6) with u^i equal to v_g^i and v_h^i, respectively. Then*

$$ik^2 \int_D \operatorname{Im} n v_g \bar{v}_h \, dx = 2\pi(Fg, h) - 2\pi(g, Fh) - ik(Fg, Fh)$$

where $(\cdot\,,\cdot)$ denotes the inner product on $L^2(\mathbb{S}^2)$.

Proof. From Green's theorem we have that

$$2ik^2 \int_D \operatorname{Im} n \, v_g \overline{v_h} \, dx = \int_{\partial D} \left(v_g \, \frac{\partial \overline{v_h}}{\partial \nu} - \overline{v_h} \, \frac{\partial v_g}{\partial \nu} \right) ds$$

and combining this with (3.50) the statement follows. □

We can now use Theorem 8.17 to deduce a series of results on the spectral theory of the far field operator F. We begin with the existence of eigenvalues of F.

Corollary 8.18 *Assume that either $\operatorname{Im} n \neq 0$ or $\operatorname{Im} n = 0$ and $m(x) > 0$ for $x \in D$. Then the far field operator has an infinite number of eigenvalues.*

Proof. By Theorems 8.12, 8.15 and 8.16, it suffices to show that $\operatorname{Im}(Fg, g) \geq 0$ for every $g \in L^2(\mathbb{S}^2)$. But from Theorem 8.17 we have that (recalling that $\operatorname{Im} n(x) \geq 0$ for $x \in D$)

$$\operatorname{Im}(Fg, g) = \frac{1}{2i} \left\{ (Fg, g) - (g, Fg) \right\} = \frac{k^2}{4\pi} \int_D \operatorname{Im} n \, |v_g|^2 dx + \frac{k}{4\pi} \, \|Fg\|^2 \geq 0$$

and the corollary is proved. □

Corollary 8.19 *If $\operatorname{Im} n \neq 0$ the eigenvalues of the far field operator F lie in the disk*

$$|\lambda|^2 - \frac{4\pi}{k} \operatorname{Im} \lambda < 0$$

whereas if $\operatorname{Im} n = 0$ and $m(x) > 0$ for $x \in D$ they lie on the circle

$$|\lambda|^2 - \frac{4\pi}{k} \operatorname{Im} \lambda = 0$$

in the complex plane.

Proof. This follows from Theorem 8.17 by setting $g = h$ and $Fg = \lambda g$. □

Corollary 8.20 *If* $\operatorname{Im} n = 0$ *then the far field operator* F *is normal.*

Proof. From Theorem 8.17 we have that

$$ik(Fg, Fh) = 2\pi\{(Fg, h) - (g, Fh)\}$$

and from this the statement follows as in the proof of Theoreml 3.24, using the reciprocity Theorem 8.8. □

We note that if we define the *scattering operator* S by

$$S = I + \frac{ik}{2\pi} F \tag{8.43}$$

then as in Corollary 3.25 we have that S is unitary if $\operatorname{Im} n = 0$. Finally, the above analysis of the spectral properties of the far field operator for acoustic waves in an inhomogeneous medium can be extended to scattering of acoustic waves by an obstacle with impedance boundary condition [69], electromagnetic waves in an isotropic inhomogeneous medium [68] and electromagnetic waves in an anisotropic inhomogeneous medium [54].

8.5 The Analytic Fredholm Theory

Our aim, in the next section of this chapter, is to show that under suitable conditions on the refractive index the transmission eigenvalues form at most a discrete set. Our proof will be based on the theory of operator valued analytic functions. Hence, in this section we shall present the rudiments of the theory.

Definition 8.21 *Let* D *be a domain in the complex plane* $\mathbb{C}$ *and* $f : D \to X$ *a function from* D *into the (complex) Banach space* X. f *is said to be* strongly holomorphic *in* D *if for every* $z \in D$ *the limit*

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists in X. f *is said to be* weakly holomorphic D *if for every bounded linear functional* ℓ *in the dual space* $X^\star$ *we have that* $z \mapsto \ell(f(z))$ *is a holomorphic function of* z *for* $z \in D$.

Strongly holomorphic functions are obviously continuous. As we shall see in the next section of this chapter, it is often easier to verify that a function is weakly holomorphic than that it is strongly holomorphic. What is surprising is that these two definitions of holomorphic functions are in fact equivalent. Note that strongly holomorphic functions are clearly weakly holomorphic.

Theorem 8.22. *Every weakly holomorphic function is strongly holomorphic.*

Proof. Let $f : D \to X$ be weakly holomorphic in D. Let $z_0 \in D$ and let Γ be a circle of radius $r > 0$ centered at z_0 with counterclockwise orientation whose closed interior is contained in D. Then if $\ell \in X^*$, the function $z \mapsto \ell(f(z))$ is holomorphic in D. Since $\ell(f)$ is continuous on Γ, we have that

$$|\ell(f(\zeta))| \le C(\ell) \tag{8.44}$$

for all $\zeta \in \Gamma$ and some positive number $C(\ell)$ depending on ℓ. Now, for each $\zeta \in \Gamma$ let $\Lambda(\zeta)$ be the linear functional on $X^\star$ which assigns to each $\ell \in X^\star$ the number $\ell(f(\zeta))$. From (8.44), for each $\ell \in X^\star$ we have that $|\Lambda(\zeta)(\ell)| \le C(\ell)$ for all $\zeta \in \Gamma$ and hence by the uniform boundedness principle we have that $\|\Lambda(\zeta)\| \le C$ for all $\zeta \in \Gamma$ for some positive constant C. From this, using the Hahn–Banach theorem, we conclude that

$$\|f(\zeta)\| = \sup_{\|\ell\|=1} |\ell(f(\zeta))| = \sup_{\|\ell\|=1} |\Lambda(\zeta)(\ell)| = \|\Lambda(\zeta)\| \le C \tag{8.45}$$

for all $\zeta \in \Gamma$.

For $|h| \le r/2$, by Cauchy's integral formula, we have

$$\ell\left(\frac{f(z_0+h) - f(z_0)}{h}\right) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{h}\left(\frac{1}{\zeta - (z_0+h)} - \frac{1}{\zeta - z_0}\right)\ell(f(\zeta))\, d\zeta.$$

Since for $\zeta \in \Gamma$ and $|h_1|, |h_2| \le r/2$ we have

$$\left|\frac{1}{h_1}\left(\frac{1}{\zeta - (z_0+h_1)} - \frac{1}{\zeta - z_0}\right) - \frac{1}{h_2}\left(\frac{1}{\zeta - (z_0+h_2)} - \frac{1}{\zeta - z_0}\right)\right|$$

$$= \left|\frac{h_1 - h_2}{(\zeta - z_0)(\zeta - z_0 - h_1)(\zeta - z_0 - h_2)}\right| \le \frac{4\,|h_1 - h_2|}{r^3},$$

using (8.45) and Cauchy's integral formula we can estimate

$$\left|\ell\left(\frac{f(z_0+h_1) - f(z_0)}{h_1}\right) - \ell\left(\frac{f(z_0+h_2) - f(z_0)}{h_2}\right)\right| \le \frac{4C}{r^2}\;|h_1 - h_2|$$

for all $\ell \in X^*$ with $\|\ell\| \le 1$. Again by the Hahn–Banach theorem, this implies that

$$\left\|\frac{f(z_0+h_1) - f(z_0)}{h_1} - \frac{f(z_0+h_2) - f(z_0)}{h_2}\right\| \le \frac{4C}{r^2}\;|h_1 - h_2| \tag{8.46}$$

for all h_1, h_2 with $|h_1|, |h_2| \le r/2$. Therefore,

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

exists since the Banach space X is complete. Since z_0 was an arbitrary point of D, the theorem follows. □

Corollary 8.23 *Let X and Y be two Banach spaces and denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators mapping X into Y. Let D be a domain in $\mathbb{C}$ and let $A : D \to \mathcal{L}(X, Y)$ be an operator valued function such that for each $\varphi \in X$ the function $A\varphi : D \to Y$ is weakly holomorphic. Then A is strongly holomorphic.*

Proof. For each $\varphi \in X$, we apply the analysis of the previous proof to the weakly holomorphic function $z \mapsto f(z) := A(z)\varphi$. By (8.45), we have

$$\|A(\zeta)\varphi\| = \|f(\zeta)\| \le C_\varphi$$

for all $\zeta \in \Gamma$ and some positive constant C_φ depending on φ. This, again by the uniform boundedness principle, implies that

$$\|A(\zeta)\| \le C$$

for all $\zeta \in \Gamma$ and some constant $C > 0$. Hence, we can estimate in Cauchy's formula for $z \mapsto \ell(A(z)\varphi)$ with the aid of

$$\|\ell(A(\zeta)\varphi)\| \le C$$

for all $\zeta \in \Gamma$, all $\ell \in Y^*$ with $\|\ell\| \le 1$ and all $\varphi \in X$ with $\|\varphi\| \le 1$ and obtain the inequality (8.46) for A in the operator norm. This concludes the proof. □

Definition 8.24 *Let D be a domain in the complex plane $\mathbb{C}$ and $f : D \to X$ a function from D into the (complex) Banach space X. f is said to be* analytic *in D if for every $z_0 \in D$ there exists a power series expansion*

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m$$

that converges in the norm on X uniformly for all z in a neighborhood of z_0 and where the cofficients a_m are elements from X.

As in classical complex function theory, the concepts of holomorphic and analytic functions coincide as stated in the following theorem. Therefore, we can synonymously talk about (weakly and strongly) holomorphic and analytic functions.

Theorem 8.25. *Every analytic function is holomorphic and vice versa.*

Proof. Let $f : D \to X$ be analytic. Then, for each ℓ in the dual space X^*, the function $z \mapsto \ell(f(z))$, by the continuity of ℓ, is a complex valued analytic function. Therefore, by classical function theory it is a holomorphic complex valued function. Hence, f is weakly holomorphic and thus by Theorem 8.22 it is strongly holomorhic.

Conversely, let $f : D \to X$ be holomorphic. Then, by Definition 8.21, for each $z \in D$ the derivative

$$f'(z) := \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists. By continuity, for every $\ell \in X^*$ the function $z \mapsto \ell(f'(z))$ clearly represents the derivative of the complex valued function $z \mapsto \ell(f(z))$. Classical function theory again implies that $z \mapsto f'(z)$ is weakly holomorphic and hence by Theorem 8.22

strongly holomorphic. Therefore, by induction the derivatives $f^{(m)}$ of order m exist and for each $\ell \in X^*$ the m–th derivative of $\ell(f)$ is given by $\ell(f^{(m)})$. Then, using the notation of the proof of Theorem 8.22, by Cauchy's integral formula we have

$$\ell\left(f(z) - \sum_{m=0}^{n} \frac{1}{m!}\, f^{(m)}(z_0)(z-z_0)^m\right)$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{\ell(f(\zeta))}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\sum_{m=0}^{n}(z-z_0)^m \int_\Gamma \frac{\ell(f(\zeta))}{(\zeta-z_0)^{m+1}}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{\ell(f(\zeta))}{\zeta - z}\left(\frac{z-z_0}{\zeta-z_0}\right)^{n+1} d\zeta.$$

From this we have the estimate

$$\left|\ell\left(f(z) - \sum_{m=0}^{n} \frac{1}{m!}\, f^{(m)}(z_0)(z-z_0)^m\right)\right| \le \sup_{\zeta\in\Gamma} |\ell(f(\zeta))|\,\frac{1}{2^n}$$

for all $\ell \in X^*$ and all $z \in D$ with $|z - z_0| \le r/2$. Using the uniform boundedness principle and the Hahn–Banach theorem as in the proof of Theorem 8.22, we can now conclude uniform convergence of the series

$$f(z) = \sum_{m=0}^{\infty} \frac{1}{m!}\, f^{(m)}(z_0)(z-z_0)^m$$

for all $z \in D$ with $|z - z_0| \le r/2$ and the proof is finished. □

We now want to establish the analytic Riesz–Fredholm theory for compact operators in a Banach space. For this we recall that for a single compact linear operator $A : X \to X$ mapping a Banach space X into itself either the inverse operator $(I - A)^{-1} : X \to X$ exists and is bounded or the operator $I - A$ has a nontrivial nullspace of finite dimension (see [205]). In the latter case, it can be proved (see Theorem 1.21 in [64] or Theorem 3.9 in [205]) that there exists a bounded operator P on X with finite dimensional range such that the inverse of $I - A - P : X \to X$ exists and is bounded. Actually, P can be chosen to be the projection of X onto the generalized nullspace of $I - A$.

We can now prove the following theorem, where $\mathcal{L}(X)$ denotes the Banach space of bounded linear operators mapping the Banach space X into itself.

Theorem 8.26. *Let D be a domain in $\mathbb{C}$ and let $A : D \to \mathcal{L}(X)$ be an operator valued analytic function such that $A(z)$ is compact for each $z \in D$. Then either*

a) $(I - A(z))^{-1}$ does not exist for any $z \in D$ or

b) $(I - A(z))^{-1}$ exists for all $z \in D \setminus S$ where S is a discrete subset of D.

Proof. Given an arbitrary $z_0 \in D$, we shall show that for z in a neighborhood of z_0 either a) or b) holds. The theorem will then follow by a straightforward connectedness argument. As mentioned above, for fixed z_0 either the inverse of $I - A(z_0)$ exists and is bounded or the operator $I - A(z_0)$ has a nontrivial nullspace of finite dimension.

In the case where $I - A(z_0)$ has a bounded inverse, since A is continuous we can choose $r > 0$ such that

$$\|A(z) - A(z_0)\| < \frac{1}{\|(I - A(z_0))^{-1}\|}$$

for all $z \in B_r := \{z \in \mathbb{C} : |z - z_0| < r\}$. Then the Neumann series for

$$[I - (I - A(z_0))^{-1}(A(z) - A(z_0))]^{-1}$$

converges and we can conclude that the inverse operator $(I - A(z))^{-1}$ exists for all $z \in B_r$, is bounded and depends continuously on z. Hence, in B_r property b) holds. In particular, from

$$\frac{1}{h}\left\{(I - A(z+h))^{-1} - (I - A(z))^{-1}\right\}$$

$$= \frac{1}{h}\ (I - A(z+h))^{-1}\ (A(z+h) - A(z))\ (I - A(z))^{-1}$$

we observe that $z \mapsto (I - A(z))^{-1}$ is holomorphic and hence analytic in B_r.

In the case where $I - A(z_0)$ has a nontrivial nullspace, by the above remark there exists a bounded linear operator of the form

$$P\varphi = \sum_{j=1}^{n} \ell_j(\varphi)\psi_j$$

with linearly independent elements $\psi_1, \dots, \psi_n \in X$ and bounded linear functionals $\ell_1, \dots, \ell_n \in X^*$ such that $I - A(z_0) - P : X \to X$ has a bounded inverse. We now choose $r > 0$ such that

$$\|A(z) - A(z_0)\| < \frac{1}{\|(I - A(z_0) - P)^{-1}\|}$$

for all $z \in B_r := \{z \in \mathbb{C} : |z - z_0| < r\}$. Then as above we have that the inverse $T(z) := (I - A(z) - P)^{-1}$ exists for all $z \in B_r$, is bounded and depends analytically on z. Now define

$$B(z) := P\,(I - A(z) - P)^{-1}\,.$$

Then

$$B(z)\varphi = \sum_{j=1}^{n} \ell_j(T(z)\varphi)\psi_j \tag{8.47}$$

and since

$$I - A(z) = (I + B(z))\,(I - A(z) - P)$$

we see that for $z \in B_r$ the operator $I - A(z)$ is invertible if and only if $I + B(z)$ is invertible.

Since $B(z)$ is an operator with finite dimensional range, the invertibility of $I+B(z)$ depends on whether or not the homogeneous equation $\varphi + B(z)\varphi = 0$ has a nontrivial solution. Given $\psi \in X$, let φ be a solution of

$$\varphi + B(z)\varphi = \psi. \tag{8.48}$$

Then from (8.47) we see that φ must be of the form

$$\varphi = \psi - \sum_{j=1}^{n} \beta_j \psi_j \tag{8.49}$$

where the coefficients $\beta_j := \ell_j(T(z)\varphi)$ satisfy

$$\beta_j + \sum_{i=1}^{n} \ell_j(T(z)\psi_i)\beta_i = \ell_j(T(z)\psi), \quad j = 1, \dots n. \tag{8.50}$$

Conversely, if (8.50) has a solution $\beta_1, \beta_2, \dots, \beta_n$ then φ defined by (8.49) is easily seen to be a solution of (8.48). Hence, $I + B(z)$ is invertible if and only if the linear system (8.50) is uniquely solvable for each right hand side, i.e., if and only if

$$d(z) := \det\left\{\delta_{ij} + \ell_j(T(z)\psi_i)\right\} \neq 0.$$

The analyticity of $z \mapsto T(z)$ implies analyticity of the functions $z \mapsto \ell_j(T(z)\psi_i)$ in B_r. Therefore, d also is analytic, i.e., either $S_r := \{z \in B_r : d(z) = 0\}$ is a discrete set in B_r or $S_r = B_r$. Hence, in the case where $I - A(z_0)$ has a nontrivial nullspace we have also established existence of a neighborhood where either a) or b) holds. This completes the proof of the theorem. □

8.6 Transmission Eigenvalues

From Theorem 8.12 we see that if $\operatorname{Im} n > 0$ then transmission eigenvalues do not exist whereas from Theorem 8.13 they do exist if $\operatorname{Im} n = 0$ and $n(x) = n(r)$ is spherically stratified and twice continuously differentiable. In this section we shall remove the condition of spherical stratification and give sufficient conditions on n such that there exist at most a countable number of transmission eigenvalues (see also Theorem 10.29). By Theorem 8.9 this implies that the set $\mathcal{F}$ of far field patterns is complete in $L^2(\mathbb{S}^2)$ except for possibly a discrete set of values of the wave number. Throughout this section, we shall always assume that $\operatorname{Im} n(x) = 0$ and $m(x) > 0$ for $x \in D$ where $m := 1 - n$ is piecewise continuous in $\bar{D}$. Analogous results are easily seen to hold for the case when $m(x) < 0$ for $x \in D$.

We begin our analysis by introducing the linear space W by

$$W := \left\{ u \in H^2(\mathbb{R}^3) : u = 0 \text{ in } \mathbb{R}^3 \setminus D, \int_D \frac{1}{m} \left(|u|^2 + |\Delta u|^2\right) dx < \infty \right\}$$

where $H^2(\mathbb{R}^3)$ is the usual Sobolev space and on W we define the scalar product

$$(u, v) := \int_D \frac{1}{m} \left(u\bar{v} + \Delta u \Delta \bar{v}\right) dx, \quad u, v \in W.$$

Lemma 8.27 *The space W is a Hilbert space.*

Proof. We first note that there exists a positive constant c such that

$$\int_D \left(|u|^2 + |\Delta u|^2\right) dx \le c \int_D \frac{1}{m} \left(|u|^2 + |\Delta u|^2\right) dx \tag{8.51}$$

for all $u \in W$. Furthermore, from the representation

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\Delta u(y)}{|x - y|} \, dy, \quad x \in \mathbb{R}^3,$$

for $u \in C_0^2(\mathbb{R}^3)$ and Theorem 8.2 we see that

$$\|u\|_{H^2(\mathbb{R}^3)} \le C \|\Delta u\|_{L^2(D)} \tag{8.52}$$

for functions $u \in W$ and some positive constant C (Here we have used the fact that $u = 0$ in $\mathbb{R}^3 \setminus D$). From (8.51) and (8.52) we have that for functions $u \in W$ the norm in W dominates the norm in $H^2(\mathbb{R}^3)$. Now let (u_n) be a Cauchy sequence in W. Then (u_n) is a Cauchy sequence in $H^2(\mathbb{R}^3)$ and hence converges to a function $u \in H^2(\mathbb{R}^3)$ with respect to the norm in $H^2(\mathbb{R}^3)$. Since, by the Sobolev imbedding theorem, the H^2 norm dominates the maximum norm, and each $u_n = 0$ in $\mathbb{R}^3 \setminus D$, we can conclude that $u = 0$ in $\mathbb{R}^3 \setminus D$. Since $(u_n/\sqrt{m})$ is a Cauchy sequence in $L^2(D)$ we have that (u_n) converges in $L^2(D)$ to a function of the form $\sqrt{m}\, v$ for $v \in L^2(D)$. Similarly, (Δu_n) converges in $L^2(D)$ to a function of the form $\sqrt{m}\, w$ for $w \in L^2(D)$. Since (u_n) converges to u in $H^2(\mathbb{R}^3)$, we have that $\sqrt{m}\, v = u$ and $\sqrt{m}\, w = \Delta u$. Hence $u \in W$ and (u_n) converges to u with respect to the norm in W. □

Now let G be Green's function for the Laplacian in D and make the assumption that

$$\int_D \int_D [G(x, y)]^2 \, \frac{m(y)}{m(x)} \, dx dy < \infty. \tag{8.53}$$

The condition (8.53) is clearly true if $m(x) \ge c > 0$ for $x \in D$. It can also be shown that (8.53) is true if $m(x)$ approaches zero sufficiently slowly as x tends to the boundary ∂D.

Lemma 8.28 *Assume that $m(x) > 0$ for $x \in D$ and (8.53) is valid. For $x \in D$, let $d(x, \partial D)$ denote the distance between x and ∂D and for $\delta > 0$ sufficiently small define the set $U_\delta := \{x \in D : d(x, \partial D) < \delta\}$. Then for all $u \in W$ we have*

$$\int_{U_\delta} \frac{1}{m} |u|^2 \, dx \leq C(\delta) \, \|u\|^2$$

where $\lim_{\delta \to 0} C(\delta) = 0$.

Proof. Applying Green's theorem over D to $G(x, \cdot)$ and functions $u \in C_0^2(\mathbb{R}^3)$ and then using a limiting argument shows that

$$u(x) = -\int_D G(x, y) \Delta u(y) \, dy, \quad x \in D, \tag{8.54}$$

for functions $u \in W$. By the Cauchy–Schwarz inequality, we have

$$|u(x)|^2 \leq \int_D m(y) \, [G(x, y)]^2 dy \int_D \frac{1}{m(y)} \, |\Delta u(y)|^2 dy, \quad x \in D. \tag{8.55}$$

It now follows that

$$\int_{U_\delta} \frac{1}{m(x)} \, |u(x)|^2 dx \leq \int_{U_\delta} \frac{1}{m(x)} \int_D m(y) \, [G(x, y)]^2 dy dx \, \|u\|^2.$$

If we define $C(\delta)$ by

$$C(\delta) := \int_D \int_{U_\delta} [G(x, y)]^2 \, \frac{m(y)}{m(x)} \, dx dy,$$

the lemma follows. □

Using Lemma 8.28, we can now prove a version of Rellich's selection theorem for the weighted function spaces W and

$$L^2_{1/m}(D) := \left\{ u : D \to \mathbb{C} : u \text{ measurable}, \int_D \frac{1}{m} \, |u|^2 dx < \infty \right\}.$$

Theorem 8.29. *Assume that $m(x) > 0$ for $x \in D$ and (8.53) is valid. Then the imbedding from W into $L^2_{1/m}(D)$ is compact.*

Proof. For $u \in W$, by Green's theorem and the Cauchy–Schwarz inequality we have

$$\| \operatorname{grad} u\|^2_{L^2(D)} = -\int_D \bar{u} \Delta u \, dx \leq \|u\|^2_{L^2(D)} \|\Delta u\|^2_{L^2(D)}. \tag{8.56}$$

Suppose now that (u_n) is a bounded sequence from W, that is, $\|u_n\| \leq M$ for $n = 1, 2, \dots$ and some positive constant M. Then, from the fact that the norm in $L^2_{1/m}(D)$ dominates the norm in $L^2(D)$ and (8.56), we see that each u_n is in the Sobolev space $H^1(D)$ and there exists a positive constant, which we again designate by M, such that $\|u_n\|_{H^1(D)} \leq M$ for $n = 1, 2, \ldots$. Hence, by Rellich's selection theorem, there exists a subsequence, again denoted by (u_n), such that (u_n) is convergent to u in $L^2(D)$. We now must show that in fact $u_n \to u$, $n \to \infty$, in $L^2_{1/m}(D)$. To this end, let U_δ be as in Lemma 8.28. Let $\varepsilon > 0$ and choose δ such that $C(\delta) \leq \varepsilon/8M^2$ and n_0 such that

$$\int_{D\setminus U_\delta} \frac{1}{m}\, |u_n - u_\ell|^2 dx < \frac{\varepsilon}{2}$$

for $n, l \geq n_0$. Then for $n, l \geq n_0$ we have

$$\begin{aligned}\int_D \frac{1}{m}\, |u_n - u_\ell|^2 dx &= \int_{U_\delta} \frac{1}{m}\, |u_n - u_\ell|^2 dx + \int_{D\setminus U_\delta} \frac{1}{m}\, |u_n - u_\ell|^2 dx\\ &\leq C(\delta)\, \|u_n - u_\ell\|^2 + \frac{\varepsilon}{2} \;\leq\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon.\end{aligned}$$

Hence, (u_n) is a Cauchy sequence in $L^2_{1/m}(D)$ and thus $u_n \to u \in L^2_{1/m}(D)$ for $n \to \infty$. The theorem is now proved. □

To prove that there exist at most a countable number of transmission eigenvalues, we could now use Theorem 8.29 and proceed along the lines of Rynne and Sleeman [298]. However, we choose an alternate route based on analytic projection operators since these operators are of interest in their own right. We begin with two lemmas.

Lemma 8.30 *Assume that $m(x) > 0$ for $x \in D$ and (8.53) is valid. Then for all k there exists a positive constant $\gamma = \gamma(k)$ such that for $u \in W$*

$$\|u\|^2 \leq \gamma(k) \int_D \frac{1}{m}\, |\varDelta u + k^2 u|^2 dx.$$

Proof. We first choose $k = 0$. Integrating (8.55) we obtain

$$\int_D \frac{1}{m(x)}\, |u(x)|^2 dx \leq \int_D \int_D \frac{m(y)}{m(x)}\, [G(x, y)]^2 dy\, dx \int_D \frac{1}{m(y)}\, |\varDelta u(y)|^2 dy,$$

and hence the lemma is true for $k = 0$.

Now assume that $k \neq 0$. From the above analysis for $k = 0$, we conclude that $\varDelta : W \to L^2_{1/m}(D)$ is injective and has closed range. In particular, $\varDelta$ is a semi-Fredholm operator (c.f. [302], p. 125). Since compact perturbations of semi-Fredholm operators are semi-Fredholm ([302], p. 128), we have from Theorem 8.29 that $\varDelta + k^2$ is also semi-Fredholm. Applying Green's formula (2.4) to functions $u \in C_0^2(\mathbb{R}^3)$ and then using a limiting argument shows that

$$u(x) = -\int_D \varPhi(x, y)\{\varDelta u(y) + k^2 u(y)\}\, dy, \quad x \in \mathbb{R}^3,$$

for functions $u \in W$. Hence, by the Cauchy–Schwarz inequality, there exists a positive constant C such that

$$\|u\|_{L^2(D)} \leq C\|\Delta u + k^2 u\|_{L^2_{1/m}(D)}$$

for all $u \in W$. From this we can now conclude that $\Delta + k^2 : W \to L^2_{1/m}(D)$ is injective and, since the range of a semi-Fredholm operator is closed, by the bounded inverse theorem the lemma is now seen to be true for $k \neq 0$. □

Now for $k \geq 0$ and for $u, v \in W$ define the scalar product

$$(u, v)_k = \int_D \frac{1}{m} (\Delta u + k^2 u)(\Delta \bar{v} + k^2 \bar{v})\, dx$$

with norm $\|u\|_k = \sqrt{(u, u)_k}$. By Lemma 8.30 and Minkowski's inequality, the norm $\|\cdot\|_k$ is equivalent to $\|\cdot\|$ for any $k \geq 0$. For arbitrary complex k, we define the sesquilinear form B on W by

$$B(u, v; k) = \int_D \frac{1}{m} (\Delta u + k^2 u)(\Delta \bar{v} + k^2 \bar{v})\, dx.$$

We then have the following result.

Lemma 8.31 *Assume that $m(x) > 0$ for $x \in D$ and (8.53) is valid. Then for every $k_0 \geq 0$ there exists $\varepsilon > 0$ such that if $|k - k_0| \leq \varepsilon$ then*

$$|B(u, v; k) - B(u, v; k_0)| \leq C\|u\|_{k_0}\|v\|_{k_0}$$

for all $u, v \in W$ where C is a constant satisfying $0 < C < 1$.

Proof. We have

$$B(u, v; k) - B(u, v; k_0) = (k^2 - k_0^2) \int_D \frac{1}{m} (u\Delta \bar{v} + \bar{v}\Delta u)\, dx + (k^4 - k_0^4) \int_D \frac{1}{m}\, u\bar{v}\, dx$$

and hence by the Cauchy–Schwarz inequality

$$\begin{aligned}|B(u, v; k) - B(u, v; k_0)| \leq\ & |k^2 - k_0^2|\left(\int_D \frac{1}{m}\, |u|^2 dx\right)^{\frac{1}{2}} \left(\int_D \frac{1}{m}\, |\Delta v|^2 dx\right)^{\frac{1}{2}} \\ & + |k^2 - k_0^2|\left(\int_D \frac{1}{m}\, |v|^2 dx\right)^{\frac{1}{2}} \left(\int_D \frac{1}{m}\, |\Delta u|^2 dx\right)^{\frac{1}{2}} \\ & + |k^4 - k_0^4|\left(\int_D \frac{1}{m}\, |u|^2 dx\right)^{\frac{1}{2}} \left(\int_D \frac{1}{m}\, |v|^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

From Lemma 8.30, we now have that

$$|B(u, v; k) - B(u, v; k_0)| \leq (2|k^2 - k_0^2| + |k^4 - k_0^4|)\gamma(k_0)\|u\|_{k_0}\|v\|_{k_0}.$$

Hence, if $|k - k_0|$ is sufficiently small, then $(2|k^2 - k_0^2| + |k^4 - k_0^4|)\gamma(k_0)$ is less than one and the lemma follows. □

From Lemma 8.31, we see that for $|k - k_0| \leq \varepsilon$ we have

$$|B(u, v; k)| \leq (1 + C)\|u\|_{k_0}\|v\|_{k_0}$$

for all $u, v \in W$, i.e., the sesquilinear form B is bounded, and

$$\operatorname{Re} B(u, u; k) \geq \operatorname{Re} B(u, u; k_0) - |B(u, u; k) - B(u, u; k_0)| \geq (1 - C)\|u\|_{k_0}^2$$

for all $u \in W$, i.e., B is strictly coercive. Hence, by the Lax–Milgram theorem, for each $k \in \mathbb{C}$ with $|k - k_0| \leq \varepsilon$ where ε is defined as in Lemma 8.31 there exists a bounded linear operator $S(k) : W \to W$ with a bounded inverse $S^{-1}(k)$ such that

$$B(u, v; k) = (S(k)u, v)_{k_0} \tag{8.57}$$

holds for all $u, v \in W$. From (8.57), we have that for each $u \in W$ the function $k \mapsto S(k)u$ is weakly analytic and hence from Corollary 8.23 we conclude that $k \mapsto S(k)$ is strongly analytic. Then, in particular, the inverse $S^{-1}(k)$ is also strongly analytic in k. We are now in a position to prove the main result of this section.

Theorem 8.32. *Assume that $m(x) > 0$ for $x \in D$ and (8.53) is valid. Then the set of transmission eigenvalues is either empty or forms a discrete set.*

Proof. Our first aim is to define a projection operator P_k in $L^2_m(D)$ which depends on k in a neighborhood of the positive real axis. To this end, let $f \in L^2_m(D)$ and for $k \in \mathbb{C}$ define the antilinear functional ℓ_f on W by

$$\ell_f(\varphi) = \int_D f(\Delta\bar{\varphi} + k^2\bar{\varphi})\,dx, \quad \varphi \in W. \tag{8.58}$$

Then by Minkowski's inequality, the functional ℓ_f is bounded on W. Therefore, by the Riesz representation theorem, there exists $p_f \in W$ such that for all $\varphi \in W$ and fixed $k_0 \geq 0$ we have

$$\ell_f(\varphi) = (p_f, \varphi)_{k_0}$$

where p_f depends on k and the linear mapping $p(k) : L^2_m(D) \to W$ with $p(k)f := p_f$ is bounded from $L^2_m(D)$ into W. From (8.58) we see that for each $f \in L^2_m(D)$ the mapping $k \mapsto p(k)f = p_f$ is weakly analytic. Hence, by Corollary 8.23, the mapping $k \mapsto p(k)$ is strongly analytic.

For all $k \in \mathbb{C}$ with $|k - k_0| \leq \varepsilon$ where ε is defined as in Lemma 8.31, we now introduce the analytic operator $P_k : L^2_m(D) \to L^2_m(D)$ by

$$P_k f := \frac{1}{m}(\Delta + k^2)S^{-1}(k)p_f, \quad f \in L^2_m(D).$$

Note that $P_k f \in L^2_m(D)$ since, with the aid of Minkowski's inequality and the boundedness of $S^{-1}(k)$ and $p(k)$, we have

$$\int_D m|P_k f|^2 dx = \int_D \frac{1}{m} \, |(\varDelta + k^2)S^{-1}(k)p_f|^2 dx \le C_1 \|S^{-1}(k)p_f\|^2_{k_0} \le C_2 \|f\|^2_{L^2_m(D)}$$

for some positive constants C_1 and C_2.

We now want to show that P_k is an orthogonal projection operator. To this end, let H be the linear space

$$H := \left\{ u \in C^2(\mathbb{R}^3) : \varDelta u + k^2 u = 0 \text{ in } \mathbb{R}^3 \right\},$$

$\bar{H}$ the closure of H in $L^2_m(D)$ and $H^\perp$ the orthogonal complement of H in $L^2_m(D)$. Then for $f \in H$ and $\varphi \in W$, by Green's theorem (2.3) (using our by now familiar limiting argument), we have

$$\ell_f(\varphi) = \int_D f(\varDelta\bar{\varphi} + k^2\bar{\varphi}) \, dx = \int_D \bar{\varphi}(\varDelta f + k^2 f) \, dx = 0.$$

Hence, $p_f = 0$ for $f \in H$ and since $f \mapsto p_f$ is bounded from $L^2_m(D)$ into W we also have $p_f = 0$ for $f \in \bar{H}$ and consequently $P_k f = 0$ for $f \in \bar{H}$.

Recall now the definition (8.17) of the operator

$$(T_m f)(x) := \int_D \varPhi(x, y)m(y) f(y) \, dy, \quad x \in \mathbb{R}^3.$$

Note that we can write

$$T_m f = \tilde{T}_m(\sqrt{m}\, f)$$

where $\tilde{T}_m$ has a weakly singular kernel, i.e., $\tilde{T}_m : L^2(D) \to L^2(D)$ is a compact operator. Therefore, since the L^2 norm dominates the L^2_m norm, we have that $T_m : L^2_m(D) \to L^2_m(D)$ is also compact. Furthermore, from Theorems 8.1 and 8.2 and a limiting argument, we see that

$$mf = -(\varDelta + k^2)T_m f. \tag{8.59}$$

For $f \in H^\perp$, we clearly have $(T_m f)(x) = 0$ for $x \in \mathbb{R}^3 \setminus \bar{D}$ since $\varPhi(x, \cdot) \in H$ for $x \in \mathbb{R}^3 \setminus \bar{D}$. Therefore, from Lemma 8.30 and (8.59) we see that $T_m f$ is in W for $f \in H^\perp$. From (8.57)–(8.59) we have

$$-\ell_f(\varphi) = \int_D \frac{1}{m} \, (\varDelta + k^2)\bar{\varphi} \, (\varDelta + k^2)T_m f \, dx = B(T_m f, \varphi; k) = (S(k)T_m f, \varphi)_{k_0}$$

for all $\varphi \in W$ and $f \in H^\perp$. Since $T_m f \in W$, from the definition of p_f we conclude that

$$p_f = -S(k)T_m f$$

for $f \in H^\perp$ and consequently

$$P_k f = -\frac{1}{m}\,(\varDelta + k^2)T_m f = f$$

for $f \in H^\perp$, i.e., P_k is indeed an orthogonal projection operator (depending analytically on the parameter k).

Having defined the projection operator P_k, we now turn to the homogeneous interior transmission problem (8.30), (8.31) and note that k is real. From the above analysis, we see that if v, w is a solution of (8.30), (8.31) then $P_k v = 0$ since clearly $v \in H$. From (8.30), that is, $(\varDelta + k^2)(w - v) = k^2 m w$, and the homogeneous Cauchy data (8.31), by Green's theorem (2.3), we obtain

$$k^2 \int_D m w \bar{u}\, dx = \int_D \bar{u}(\varDelta + k^2)(w - v)\, dx = \int_D \{\bar{u}\varDelta(w - v) - (w - v)\varDelta\bar{u}\}\, dx = 0$$

for all $u \in H$, that is, $w \in H^\perp$ and consequently

$$P_k w = w.$$

If we now use Green's formula (2.4) to rewrite (8.30), (8.31) in the form

$$w - v = -k^2 T_m w$$

and apply the operator P_k to both sides of this equation, we arrive at the operator equation

$$w + k^2\, P_k T_m w = 0. \tag{8.60}$$

Now consider (8.60) defined for $w \in L^2_m(D)$. Since T_m is compact and P_k is bounded, $P_k T_m$ is compact. Since $P_k T_m$ is an operator valued analytic function of k, we can apply Theorem 8.26 to conclude that $(I + k^2 P_k T_m)^{-1}$ exist for all k in a neighborhood of the positive real axis with the possible exception of a discrete set (We note that for k sufficiently small, $(I + k^2 P_k T_m)^{-1}$ exists by the contraction mapping principle). Hence, we can conclude from (8.60) that $w = 0$ except for possibly a discrete set of values of $k > 0$. From (8.30), (8.31) this now implies that $v = 0$ by Green's formula (2.5) and the theorem follows. □

Corollary 8.33 *Assume that $m(x) > 0$ for $x \in D$ and (8.53) is valid. Then, except possibly for a discrete set of values of $k > 0$, the set $\mathcal{F}$ of far field patterns is complete in $L^2(\mathbb{S}^2)$.*

Proof. This follows from Theorem 8.32 and Theorem 8.9. □

We remind the reader that Theorem 8.32 and Corollary 8.33 remain valid if the condition $m(x) > 0$ is replaced by $m(x) < 0$.

8.7 Numerical Methods

In this final section we shall make some brief remarks on the numerical solution of the scattering problem (8.4)–(8.6) for the inhomogeneous medium with particular emphasis on an approach proposed by Kirsch and Monk [192]. The principle problem associated with the numerical solution of (8.4)–(8.6) is that the domain is unbounded. A variety of methods have been proposed for the numerical solution. Broadly speaking, these methods can be grouped into three categories: 1) volume integral equations, 2) expanding grid methods and 3) coupled finite element and boundary element methods.

The volume integral equation method seeks to numerically solve the Lippmann–Schwinger equation (8.13). The advantage of this method is that the problem of an unbounded domain is handled in a simple and natural way. A disadvantage is that care must be used to approximate the three-dimensional singular integral appearing in (8.13). Furthermore, the discrete problem derived from (8.13) has a non-sparse matrix and hence a suitable iteration scheme must be used to obtain a solution, preferably a multi-grid method. Vainikko [299, 318] has suggested a fast solution method for the Lippmann–Schwinger equation based on periodization, fast Fourier transform techniques and multi-grid methods. For modifications of this approach including also the case of electromagnetic waves we refer to Hohage [144, 145].

The expanding grid method seeks a solution of (8.4)–(8.6) in a ball B_R of radius R centered at the origin where on the boundary ∂B_R the scattered field u^s is required to satisfy

$$\frac{\partial u^s}{\partial r} - iku^s = 0 \quad \text{on } \partial B_R. \tag{8.61}$$

This boundary condition is clearly motivated by the Sommerfeld radiation condition (8.6) where it is understood that $R >> a$ with $n(x) = 1$ for $|x| \geq a$. Having posed the boundary condition (8.61), the resulting interior problem is solved by finite element methods. A complete analysis of this method has been provided by Goldstein [107] who has shown how to choose R as well as how the mesh size must be graded in order to obtain optimal convergence. The expanding grid method has the advantage that any standard code for solving the Helmholtz equation in an interior domain can be used to approximate the infinite domain problem (8.4)–(8.6). A disadvantage of this approach for solving (8.4)–(8.6) is that R must be taken to be large and hence computations must be made over a very large (but bounded) domain.

In order to avoid computing on a large domain, various numerical analysts have suggested combining a finite element method inside the inhomogeneity with an appropriate boundary integral equation outside the inhomogeneity. This leads to a coupled finite element and boundary element method for solving (8.4)–(8.6). In this method, a domain D is chosen which contains the support of m and then u is approximated inside D by finite element methods and outside D by a boundary integral representation of u such that u and its normal derivative are continuous across ∂D. For a survey of such coupled methods, we refer the reader to Hsiao [147]. We shall now present a version of this method due to Kirsch and Monk [192] which uses Nyström's method to approximately solve the boundary integral equation. The advantage of this approach is that Nyström's method is exponentially convergent for analytic boundaries and easy to implement (compare Section 3.5). A difficulty is that Nyström's method is defined pointwise whereas the finite element solution for the interior domain is defined variationally. However, as we shall see, this difficulty can be overcome.

For the sake of simplicity, we shall only present the method of Kirsch and Monk for the nonabsorbing two-dimensional case, i.e., we want to construct a solution to the scattering problem

$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^2, \tag{8.62}$$

$$u(x) = u^i(x) + u^s(x), \tag{8.63}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{8.64}$$

where we assume that $k > 0$ and $n \in C^1(\mathbb{R}^3)$ is real valued such that $m := 1 - n$ has compact support. We choose a simply connected bounded domain D with analytic boundary ∂D that contains the support of m. We shall use the standard notation $L^2(D)$ and $L^2(\partial D)$ for the spaces of square integrable functions defined on D and ∂D, respectively, and the corresponding Sobolev spaces will be denoted by $H^s(D)$ and $H^s(\partial D)$. The inner product on $L^2(D)$ will be denoted by $(\cdot\,,\cdot)$ and on $L^2(\partial D)$ (or the dual pairing between $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$) by $\langle\cdot\,,\cdot\rangle$. Finally, we recall the Sobolev space $H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \bar{D})$ of all functions u for which the restriction onto $D_R := \{x \in \mathbb{R}^2 \setminus \bar{D} : |x| < R\}$ belongs to $H^1(D_R)$ for all sufficiently large R.

To describe the method due to Kirsch and Monk for numerically solving (8.62)–(8.64), we begin by defining two operators $G_i : H^{-1/2}(\partial D) \to H^1(D)$ and $G_e : H^{-1/2}(\partial D) \to H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \bar{D})$ in terms of the following boundary value problems. Given $\psi \in H^{-1/2}(\partial D)$, define $G_i\psi := w \in H^1(D)$ as the weak solution of

$$\Delta w + k^2 n(x) w = 0 \quad \text{in } D, \tag{8.65}$$

$$\frac{\partial w}{\partial \nu} + ikw = \psi \quad \text{on } \partial D, \tag{8.66}$$

where ν is the unit outward normal to ∂D. Similarly, $G_e\psi := w \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \bar{D})$ is the weak solution of

$$\Delta w + k^2 w = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}, \tag{8.67}$$

$$\frac{\partial w}{\partial \nu} + ikw = \psi \quad \text{on } \partial D, \tag{8.68}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial w}{\partial r} - ikw\right) = 0, \tag{8.69}$$

where (8.69) holds uniformly in all directions. Now define $u(\psi)$ by

$$u(\psi) := \begin{cases} G_e\psi + u^i & \text{in } \mathbb{R}^2 \setminus \bar{D}, \\ G_i\psi + G_i\left(\dfrac{\partial u^i}{\partial \nu} + iku^i\right) & \text{in } D. \end{cases}$$

Note that $\partial u/\partial \nu + iku$ has the same limiting values on both sides of ∂D. Furthermore, we see that if $\psi \in H^{-1/2}(\partial D)$ can be chosen such that

$$(G_i - G_e)\,\psi = u^i - G_i\left(\frac{\partial u^i}{\partial \nu} + iku^i\right) \quad \text{on } \partial D, \tag{8.70}$$

then u has the same limiting values on both sides of ∂D. From these facts it can be deduced that u solves the scattering problem (8.62)–(8.64) (c.f. the proof of Theorem 5.7). Hence, we need to construct an approximate solution to the operator equation (8.70).

To solve (8.70), we first choose a finite element space $S_h \subset H^1(D)$ and define $G_i^h\psi$ to be the usual finite element approximation of (8.65), (8.66). In particular, if $\psi \in H^{-1/2}(\partial D)$ then $G_i^h\psi \in S_h$ satisfies (we do not distinguish between $G_i^h\psi$ defined on D and its trace defined on ∂D)

$$(\operatorname{grad} G_i^h\psi, \operatorname{grad}\varphi_h) - k^2(nG_i^h\psi, \varphi_h) + ik\,\langle G_i^h\psi, \varphi_h\rangle - \langle\psi, \varphi_h\rangle = 0 \tag{8.71}$$

for all $\varphi_h \in S_h$. For h sufficiently small, the existence and uniqueness of a solution to (8.71) is well known (Schatz [301]). Having defined $G_i^h\psi$, we next define $G_e^M\psi$ to be the approximate solution of (8.67)–(8.69) obtained by numerically solving an appropriate boundary integral equation using Nyström's method with M knots (c.f. Section 3.5).

We now need to discretize $H^{-1/2}(\partial D)$. To this end, we parameterize ∂D by

$$x = (x_1(t), x_2(t)), \quad 0 \le t \le 2\pi,$$

and define S_N by

$$S_N := \left\{ g : \partial D \to \mathbb{C} : g(x) = \sum_{j=-N+1}^{N} a_j e^{ijt},\ a_j \in \mathbb{C},\ x = (x_1(t), x_2(t)) \right\}.$$

Note that the indices on the sum in this definition are chosen such that S_N has an even number of degrees of freedom which is convenient for using fast Fourier transforms. We now want to define a projection $P_N : L^2(\partial D) \to S_N$. For $g \in L^2(\partial D)$, this is done by defining $P_N\, g \in S_N$ to be the unique solution of

$$\langle g - P_N g, \varphi\rangle = 0$$

for every $\varphi \in S_N$. We next define a projection P_N^M from discrete functions defined on the Nyström's points into functions in S_N. To do this, let x_i for $i = 1, \dots, M$ be the Nyström points on ∂D and let g be a discrete function on ∂D so that $g(x_i) = g_i,\ i = 1, \dots, M$. Then, provided $M \geq 2N$, we define $P_N^M g \in S_N$ by requiring that

$$\langle P_N^M g, \varphi \rangle_M = \langle g, \varphi \rangle_M$$

for every $\varphi \in S_N$ where

$$\langle u, v \rangle_M := \frac{1}{M} \sum_{i=1}^{M} u(x_i)\, \overline{v(x_i)}.$$

Note that $P_N^M g$ is the uniquely determined element in S_N that is the closest to g with respect to the norm $\|\cdot\|_M$ assoziated with $\langle \cdot, \cdot \rangle_M$.

Following Kirsch and Monk, we can now easily define a discrete method for solving the scattering problem (8.62)–(8.64). We first seek $\varphi_N \in S_N$ such that

$$(P_N G_i^h - P_N^M G_e^M)\varphi_N = P_N \left(u^i - G_i^h \left(\frac{\partial u^i}{\partial \nu} + iku^i \right) \right).$$

Then, having found φ_N, we approximate the solution u of (8.62)–(8.64) by

$$u_N^{h,M} := \begin{cases} G_e^M \varphi_N + u^i & \text{in } \mathbb{R}^2 \setminus \bar{D}, \\[2ex] G_i^h \varphi_N + G_i^h \left(\dfrac{\partial u^i}{\partial \nu} + iku^i \right) & \text{in } D. \end{cases}$$

Error estimates and numerical examples of the implementation of this scheme for solving the scattering problem (8.62)–(8.64) can be found in Kirsch and Monk [192].

Chapter 9
Electromagnetic Waves in an Inhomogeneous Medium

In the previous chapter, we considered the direct scattering problem for acoustic waves in an inhomogeneous medium. We now consider the case of electromagnetic waves. However, our aim is not to simply prove the electromagnetic analogue of each theorem in Chapter 8 but rather to select the basic ideas of the previous chapter, extend them when possible to the electromagnetic case, and then consider some themes that were not considered in Chapter 8, but ones that are particularly relevant to the case of electromagnetic waves. In particular, we shall consider two simple problems, one in which the electromagnetic field has no discontinuities across the boundary of the medium and the second where the medium is an imperfect conductor such that the electromagnetic field does not penetrate deeply into the body. This last problem is an approximation to the more complicated transmission problem for a piecewise constant medium and leads to what is called the exterior impedance problem for electromagnetic waves.

After a brief discussion of the physical background to electromagnetic wave propagation in an inhomogeneous medium, we show existence and uniqueness of a solution to the direct scattering problem for electromagnetic waves in an inhomogeneous medium. By means of a reciprocity relation for electromagnetic waves in an inhomogeneous medium, we then show that, for a conducting medium, the set of electric far field patterns corresponding to incident time-harmonic plane waves moving in arbitrary directions is complete in the space of square integrable tangential vector fields on the unit sphere. However, we show that this set of far field patterns is in general not complete for a dielectric medium. Finally, we establish the existence and uniqueness of a solution to the exterior impedance problem and show that the set of electric far field patterns is again complete in the space of square integrable tangential vector fields on the unit sphere. These results for the exterior impedance problem will be used in the next chapter when we discuss the inverse scattering problem for electromagnetic waves in an inhomogeneous medium. We note, as in the case of acoustic waves, that our ideas and methods can be extended to more complicated scattering problems involving discontinuous fields, piecewise continuous refractive indexes, etc. but, for the sake of clarity and brevity, we do not consider these more general problems in this book.

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences 93, DOI 10.1007/978-1-4614-4942-3_9,

9.1 Physical Background

We consider electromagnetic wave propagation in an inhomogeneous isotropic medium in $\mathbb{R}^3$ with electric permittivity $\varepsilon = \varepsilon(x) > 0$, magnetic permeability $\mu = \mu_0$ and electric conductivity $\sigma = \sigma(x)$ where μ_0 is a positive constant. We assume that $\varepsilon(x) = \varepsilon_0$ and $\sigma(x) = 0$ for all x outside some sufficiently large ball where ε_0 is a constant. Then if J is the current density, the electric field $\mathcal{E}$ and magnetic field $\mathcal{H}$ satisfy the *Maxwell equations*, namely

$$\operatorname{curl} \mathcal{E} + \mu_0 \frac{\partial \mathcal{H}}{\partial t} = 0, \quad \operatorname{curl} \mathcal{H} - \varepsilon(x) \frac{\partial \mathcal{E}}{\partial t} = J. \tag{9.1}$$

Furthermore, in an isotropic conductor, the current density is related to the electric field by *Ohm's law*

$$J = \sigma \mathcal{E}. \tag{9.2}$$

For most metals, σ is very large and hence it is often reasonable in many theoretical investigations to approximate a metal by a fictitious *perfect conductor* in which σ is taken to be infinite. However, in this chapter, we shall assume that the inhomogeneous medium is not a perfect conductor, i.e., σ is finite. If σ is nonzero, the medium is called a *conductor*, whereas if $\sigma = 0$ the medium is referred to as a *dielectric*.

We now assume that the electromagnetic field is time-harmonic, i.e., of the form

$$\mathcal{E}(x,t) = \frac{1}{\sqrt{\varepsilon_0}} E(x) \, e^{-i\omega t}, \quad \mathcal{H}(x,t) = \frac{1}{\sqrt{\mu_0}} H(x) \, e^{-i\omega t}$$

where ω is the frequency. Then from (9.1) and (9.2) we see that E and H satisfy the time-harmonic Maxwell equations

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikn(x)E = 0 \tag{9.3}$$

in $\mathbb{R}^3$ where the (positive) wave number k is defined by $k^2 = \varepsilon_0 \mu_0 \omega^2$ and the *refractive index* $n = n(x)$ is given by

$$n(x) := \frac{1}{\varepsilon_0} \left(\varepsilon(x) + i \, \frac{\sigma(x)}{\omega} \right).$$

In order to be able to formulate an integral equation of Lippmann–Schwinger type for the direct scattering problem we assume that $n \in C^{1,\alpha}(\mathbb{R}^3)$ for some $0 < \alpha < 1$ and, as usual, that $m := 1 - n$ has compact support. As in the previous chapter, we define $D := \{x \in \mathbb{R}^3 : m(x) \neq 0\}$. For an integral equation formulation of the direct scattering problem in the case when n is discontinuous across ∂D we refer the reader to [185].

We consider the following scattering problem for (9.3). Let $E^i, H^i \in C^1(\mathbb{R}^3)$ be a solution of the Maxwell equations for a homogeneous medium

$$\operatorname{curl} E^i - ikH^i = 0, \quad \operatorname{curl} H^i + ikE^i = 0 \tag{9.4}$$

in all of $\mathbb{R}^3$. We then want to find a solution $E, H \in C^1(\mathbb{R}^3)$ of (9.3) in $\mathbb{R}^3$ such that if

$$E = E^i + E^s, \quad H = H^i + H^s \tag{9.5}$$

the scattered field E^s, H^s satisfies the *Silver–Müller radiation condition*

$$\lim_{r\to\infty}(H^s \times x - rE^s) = 0 \tag{9.6}$$

uniformly for all directions $x/|x|$ where $r = |x|$.

For the next three sections of this chapter, we shall be concerned with the scattering problem (9.3)–(9.6). The existence and uniqueness of a solution to this problem were first given by Müller [256] for the more general case when $\mu = \mu(x)$. The proof simplifies considerably for the case we are considering, i.e., $\mu = \mu_0$, and we shall present this proof in the next section.

9.2 Existence and Uniqueness

Under the assumptions given in the previous section for the refractive index n, we shall show in this section that there exists a unique solution to the scattering problem (9.3)–(9.6). Our analysis follows that of Colton and Kress [64] and is based on reformulating (9.3)–(9.6) as an integral equation. We first prove the following theorem, where

$$\Phi(x, y) := \frac{1}{4\pi}\,\frac{e^{ik|x-y|}}{|x-y|}\,, \quad x \neq y,$$

as usual, denotes the fundamental solution to the Helmholtz equation and

$$m := 1 - n.$$

Theorem 9.1. *Let $E, H \in C^1(\mathbb{R}^3)$ be a solution of the scattering problem (9.3)–(9.6). Then E satisfies the integral equation*

$$\begin{aligned} E(x) = E^i(x) - k^2 &\int_{\mathbb{R}^3} \Phi(x, y) m(y) E(y)\, dy \\ &+ \operatorname{grad} \int_{\mathbb{R}^3} \frac{1}{n(y)}\, \operatorname{grad} n(y) \cdot E(y)\, \Phi(x, y)\, dy, \quad x \in \mathbb{R}^3. \end{aligned} \tag{9.7}$$

Proof. Let $x \in \mathbb{R}^3$ be an arbitrary point and choose an open ball B with unit outward normal ν such that B contains the support of m and $x \in B$. From the Stratton–Chu formula (6.5) applied to E, H, we have

$$\begin{aligned} E(x) = &- \operatorname{curl} \int_{\partial B} \nu(y) \times E(y)\, \Phi(x, y)\, ds(y) \\ &+ \operatorname{grad} \int_{\partial B} \nu(y) \cdot E(y)\, \Phi(x, y)\, ds(y) \\ &- ik \int_{\partial B} \nu(y) \times H(y)\, \Phi(x, y)\, ds(y) \\ &+ \operatorname{grad} \int_{B} \frac{1}{n(y)}\, \operatorname{grad} n(y) \cdot E(y)\, \Phi(x, y)\, dy \\ &- k^2 \int_{B} m(y) E(y)\, \Phi(x, y)\, dy \end{aligned} \tag{9.8}$$

since $\operatorname{curl} H + ikE = ikmE$ and $n \operatorname{div} E = -\operatorname{grad} n \cdot E$. Note that in the volume integrals over B we can integrate over all of $\mathbb{R}^3$ since m has support in B. The Stratton–Chu formula applied to E^i, H^i gives

$$\begin{aligned} E^i(x) = &- \operatorname{curl} \int_{\partial B} \nu(y) \times E^i(y)\, \Phi(x, y)\, ds(y) \\ &+ \operatorname{grad} \int_{\partial B} \nu(y) \cdot E^i(y)\, \Phi(x, y)\, ds(y) \\ &- ik \int_{\partial B} \nu(y) \times H^i(y)\, \Phi(x, y)\, ds(y). \end{aligned} \tag{9.9}$$

Finally, from the version of the Stratton–Chu formula corresponding to Theorem 6.7, we see that

$$\begin{aligned} &- \operatorname{curl} \int_{\partial B} \nu(y) \times E^s(y)\, \Phi(x, y)\, ds(y) \\ &+ \operatorname{grad} \int_{\partial B} \nu(y) \cdot E^s(y)\, \Phi(x, y)\, ds(y) \\ &- ik \int_{\partial B} \nu(y) \times H^s(y)\, \Phi(x, y)\, ds(y) = 0. \end{aligned} \tag{9.10}$$

With the aid of $E = E^i + E^s$, $H = H^i + H^s$ we can now combine (9.8)–(9.10) to conclude that (9.7) is satisfied. □

We now want to show that every solution of the integral equation (9.7) is also a solution to (9.3)–(9.6).

Theorem 9.2. *Let $E \in C(\mathbb{R}^3)$ be a solution of the integral equation (9.7). Then E and $H := \operatorname{curl} E/ik$ are a solution of (9.3)–(9.6).*

Proof. Since m has compact support, from Theorem 8.1 we can conclude that if $E \in C(\mathbb{R}^3)$ is a solution of (9.7) then $E \in C^{1,\alpha}(\mathbb{R}^3)$. Hence, by the relation $\operatorname{grad}_x \Phi(x,y) = -\operatorname{grad}_y \Phi(x,y)$, Gauss' divergence theorem and Theorem 8.1, we have

$$\operatorname{div} \int_{\mathbb{R}^3} \Phi(x,y) m(y) E(y)\, dy = \int_{\mathbb{R}^3} \operatorname{div}\{m(y)E(y)\} \Phi(x,y)\, dy \tag{9.11}$$

and

$$(\varDelta + k^2) \int_{\mathbb{R}^3} \frac{1}{n(y)} \operatorname{grad} n(y) \cdot E(y)\, \Phi(x,y)\, dy = -\frac{1}{n(x)} \operatorname{grad} n(x) \cdot E(x) \tag{9.12}$$

for $x \in \mathbb{R}^3$. Taking the divergence of (9.7) and using (9.11) and (9.12), we see that

$$u := \frac{1}{n} \operatorname{div}(nE)$$

satisfies the integral equation

$$u(x) + k^2 \int_{\mathbb{R}^3} \Phi(x,y) m(y) u(y)\, dy = 0, \quad x \in \mathbb{R}^3.$$

Hence, from Theorems 8.3 and 8.7 we can conclude that $u(x) = 0$ for $x \in \mathbb{R}^3$, that is,

$$\operatorname{div}(nE) = 0 \quad \text{in } \mathbb{R}^3. \tag{9.13}$$

Therefore, the integral equation (9.7) can be written in the form

$$\begin{aligned} E(x) = E^i(x) - k^2 & \int_{\mathbb{R}^3} \Phi(x,y) m(y) E(y)\, dy \\ & - \operatorname{grad} \int_{\mathbb{R}^3} \Phi(x,y) \operatorname{div} E(y)\, dy, \quad x \in \mathbb{R}^3, \end{aligned} \tag{9.14}$$

and thus for $H := \operatorname{curl} E/ik$ we have

$$H(x) = H^i(x) + ik \operatorname{curl} \int_{\mathbb{R}^3} \Phi(x,y) m(y) E(y)\, dy, \quad x \in \mathbb{R}^3. \tag{9.15}$$

In particular, by Theorem 8.1 this implies $H \in C^{1,\alpha}(\mathbb{R}^3)$ since $E \in C^{1,\alpha}(\mathbb{R}^3)$. We now use the vector identity (6.4), the Maxwell equations (9.4), and (9.11), (9.13)–(9.15) to deduce that

$$\begin{aligned}
\operatorname{curl} H(x) + ikE(x) &= ik(\operatorname{curl}\operatorname{curl} - k^2) \int_{\mathbb{R}^3} \Phi(x,y)m(y)E(y)\,dy \\
&\quad - ik \operatorname{grad} \int_{\mathbb{R}^3} \Phi(x,y) \operatorname{div} E(y)\,dy \\
&= -ik(\Delta + k^2) \int_{\mathbb{R}^3} \Phi(x,y)m(y)E(y)\,dy \\
&\quad - ik \operatorname{grad} \int_{\mathbb{R}^3} \operatorname{div}\{n(y)E(y)\}\Phi(x,y)\,dy \\
&= ikm(x)E(x)
\end{aligned}$$

for $x \in \mathbb{R}^3$. Therefore E, H satisfy (9.3). Finally, the decomposition (9.5) and the radiation condition (9.6) follow readily from (9.7) and (9.15) with the aid of (2.15) and (6.26). $\square$

We note that in (9.7) we can replace the region of integration by any domain G such that the support of m is contained in $\bar{G}$ and look for solutions in $C(\bar{G})$. Then for $x \in \mathbb{R}^3 \setminus \bar{G}$ we define $E(x)$ by the right hand side of (9.7) and obviously obtain a continuous solution to (9.7) in all of $\mathbb{R}^3$.

In order to show that (9.7) is uniquely solvable we need to establish the following unique continuation principle for the Maxwell equations.

Theorem 9.3. *Let G be a domain in $\mathbb{R}^3$ and let $E, H \in C^1(G)$ be a solution of*

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikn(x)E = 0 \tag{9.16}$$

in G such that $n \in C^{1,\alpha}(G)$. Suppose E, H vanishes in a neighborhood of some $x_0 \in G$. Then E, H is identically zero in G.

Proof. From the representation formula (9.8) and Theorem 8.1, since by assumption $n \in C^{1,\alpha}(G)$, we first can conclude that $E \in C^{1,\alpha}(B)$ for any ball B with $\bar{B} \subset G$. Then, using $\operatorname{curl} E = ikH$ from (9.8) we have $H \in C^{2,\alpha}(B)$ whence, in particular, $H \in C^2(G)$ follows.

Using the vector identity (6.4), we deduce from (9.16) that

$$\Delta H + \frac{1}{n(x)} \operatorname{grad} n(x) \times \operatorname{curl} H + k^2 n(x) H = 0 \quad \text{in } G$$

and the proof is completed by applying Lemma 8.5 to the real and imaginary parts of the cartesian components of H. $\square$

Theorem 9.4. *The scattering problem (9.3)–(9.6) has at most one solution E, H in $C^1(\mathbb{R}^3)$.*

Proof. Let E, H denote the difference between two solutions. Then E, H clearly satisfy the radiation condition (9.6) and the Maxwell equations for a homogeneous medium outside some ball B containing the support of m. From Gauss' divergence theorem and the Maxwell equations (9.3), denoting as usual by ν the exterior unit normal to B, we have that

$$\int_{\partial B} \nu \times E \cdot \bar{H}\, ds = \int_B (\operatorname{curl} E \cdot \bar{H} - E \cdot \operatorname{curl} \bar{H})\, dx = ik \int_B (|H|^2 - \bar{n}\, |E|^2)\, dx \tag{9.17}$$

and hence

$$\operatorname{Re} \int_{\partial B} \nu \times E \cdot \bar{H}\, ds = -k \int_B \operatorname{Im} n\, |E|^2 dx \leq 0.$$

Hence, by Theorem 6.11, we can conclude that $E(x) = H(x) = 0$ for $x \in \mathbb{R}^3 \setminus \bar{B}$. By Theorem 9.3 the proof is complete. □

We are now in a position to show that there exists a unique solution to the electromagnetic scattering problem.

Theorem 9.5. *The scattering problem (9.3)–(9.6) for an inhomogeneous medium has a unique solution and the solution E, H depends continuously on the incident field E^i, H^i with respect to the maximum norm.*

Proof. By Theorems 9.2 and 9.4, it suffices to prove the existence of a solution $E \in C(\mathbb{R}^3)$ to (9.7). As in the proof of Theorem 8.7, it suffices to look for solutions of (9.7) in an open ball B containing the support of m. We define an electromagnetic operator $T_e : C(\bar{B}) \to C(\bar{B})$ on the Banach space of continuous vector fields in $\bar{B}$ by

$$\begin{aligned}(T_e E)(x) := -k^2 &\int_B \Phi(x, y) m(y) E(y)\, dy \\ &+ \operatorname{grad} \int_B \frac{1}{n(y)} \operatorname{grad} n(y) \cdot E(y)\, \Phi(x, y)\, dy, \quad x \in \bar{B}.\end{aligned} \tag{9.18}$$

Since T_e has a weakly singular kernel it is a compact operator. Hence, we can apply the Riesz–Fredholm theory and must show that the homogeneous equation corresponding to (9.7) has only the trivial solution. If this is done, equation (9.7) can be solved and the inverse operator $(I - T_e)^{-1}$ is bounded. From this it follows that E, H depend continuously on the incident field with respect to the maximum norm.

By Theorem 9.2, a continuous solution E of $E - T_e E = 0$ solves the homogeneous scattering problem (9.3)–(9.6) with $E^i = 0$ and hence, by Theorem 9.4, it follows that $E = 0$. The theorem is now proved. □

9.3 The Far Field Patterns

We now want to examine the far field patterns of the scattering problem (9.3)–(9.6) where the refractive index $n = n(x)$ again satisfies the assumptions of Section 9.1. As in Section 6.6 the incident electromagnetic field is given by the plane wave described by the matrices $E^i(x,d)$ and $H^i(x,d)$ defined by

$$\begin{aligned} E^i(x,d)p &= \frac{i}{k}\,\operatorname{curl}\operatorname{curl} p\, e^{ik\,x\cdot d} = ik\,(d\times p)\times d\, e^{ik\,x\cdot d},\\ H^i(x,d)p &= \operatorname{curl} p\, e^{ik\,x\cdot d} = ik\, d\times p\, e^{ik\,x\cdot d}, \end{aligned} \tag{9.19}$$

where d is a unit vector giving the direction of propagation and $p \in \mathbb{R}^3$ is a constant vector giving the polarization. Because of the linearity of the direct scattering problem with respect to the incident field, we can also express the scattered waves by matrices. From Theorem 6.9, we see that

$$\begin{aligned} E^s(x,d)p &= \frac{e^{ik|x|}}{|x|}\, E_\infty(\hat{x},d)p + O\left(\frac{1}{|x|^2}\right), \quad |x|\to\infty,\\ H^s(x,d)p &= \frac{e^{ik|x|}}{|x|}\, \hat{x}\times E_\infty(\hat{x},d)p + O\left(\frac{1}{|x|^2}\right), \quad |x|\to\infty, \end{aligned} \tag{9.20}$$

where E_∞ is the electric far field pattern. Furthermore, from (6.88) and Green's vector theorem (6.3), we can immediately deduce the following reciprocity relation.

Theorem 9.6. *Let E_∞ be the electric far field pattern of the scattering problem (9.3)–(9.6) and (9.19). Then for all vectors $\hat{x}, d \in \mathbb{S}^2$ we have*

$$E_\infty(\hat{x},d) = [E_\infty(-d,-\hat{x})]^\top.$$

Motivated by our study of acoustic waves in Chapter 8, we now want to use this reciprocity relation to show the equivalence of the completeness of the set of electric far field patterns and the uniqueness of the solution to an electromagnetic interior transmission problem. In this chapter, we shall only be concerned with the homogeneous problem, defined as follows.

Homogeneous Electromagnetic Interior Transmission Problem. *Find a solution $E_0, E_1, H_0, H_1 \in C^1(D)\cap C(\bar{D})$ of*

$$\begin{aligned} \operatorname{curl} E_1 - ikH_1 = 0, \quad \operatorname{curl} H_1 + ikn(x)E_1 = 0 \quad \text{in } D,\\ \operatorname{curl} E_0 - ikH_0 = 0, \quad \operatorname{curl} H_0 + ikE_0 = 0 \quad \text{in } D, \end{aligned} \tag{9.21}$$

satisfying the boundary condition

$$\nu\times(E_1 - E_0) = 0, \quad \nu\times(H_1 - H_0) = 0 \quad \text{on } \partial D, \tag{9.22}$$

where again $D := \{x \in \mathbb{R}^3 : m(x) \neq 0\}$ and where we assume that D is connected with a connected C^2 boundary.

In order to establish the connection between electric far field patterns and the electromagnetic interior transmission problem, we now recall the definition of the Hilbert space

$$L_t^2(\mathbb{S}^2) := \left\{ g : \mathbb{S}^2 \to \mathbb{C}^3 : g \in L^2(\mathbb{S}^2),\ \nu \cdot g = 0 \text{ on } \mathbb{S}^2 \right\}$$

of square integrable tangential fields on the unit sphere. Let $\{d_n : n = 1, 2, \dots\}$ be a countable dense set of unit vectors on $\mathbb{S}^2$ and consider the set $\mathcal{F}$ of electric far field patterns defined by

$$\mathcal{F} := \left\{ E_\infty(\cdot\,, d_n) e_j : n = 1, 2, \dots,\ j = 1, 2, 3 \right\}$$

where e_1, e_2, e_3 are the cartesian unit coordinate vectors in $\mathbb{R}^3$. Recalling the definition of an electromagnetic Herglotz pair and Herglotz kernel given in Section 6.6, we can now prove the following theorem due to Colton and Päivärinta [84].

Theorem 9.7. *A tangential vector field g is in the orthogonal complement $\mathcal{F}^\perp$ of $\mathcal{F}$ if and only if there exists a solution of the homogeneous electromagnetic interior transmission problem such that $\overline{E_0, H_0}$ is an electromagnetic Herglotz pair with Herglotz kernel ikh where $h(d) = \overline{g(-d)}$.*

Proof. Suppose that $g \in L_t^2(\mathbb{S}^2)$ satisfies

$$\int_{\mathbb{S}^2} E_\infty(\hat{x}, d_n) e_j \cdot \overline{g(\hat{x})}\, ds(\hat{x}) = 0$$

for $n = 1, 2, \dots$ and $j = 1, 2, 3$. By the reciprocity relation, this is equivalent to

$$\int_{\mathbb{S}^2} E_\infty(-d, -\hat{x}) \overline{g(\hat{x})}\, ds(\hat{x}) = 0$$

for all $d \in \mathbb{S}^2$, i.e.,

$$\int_{\mathbb{S}^2} E_\infty(\hat{x}, d) h(d)\, ds(d) = 0 \tag{9.23}$$

for all $\hat{x} \in \mathbb{S}^2$ where $h(d) = \overline{g(-d)}$. Analogous to Lemma 6.35, from the integral equation (9.7) it can be seen that the left hand side of (9.23) represents the electric far field pattern of the scattered wave E_0^s, H_0^s corresponding to the incident wave E_0^i, H_0^i given by the the electromagnetic Herglotz pair

$$E_0^i(x) = \int_{\mathbb{S}^2} E^i(x, d) h(d)\, ds(d) = ik \int_{\mathbb{S}^2} h(d)\, e^{ik\,x\cdot d}\, ds(d),$$

$$H_0^i(x) = \int_{\mathbb{S}^2} H^i(x, d) h(d)\, ds(d) = \operatorname{curl} \int_{\mathbb{S}^2} h(d)\, e^{ik\,x\cdot d}\, ds(d).$$

Hence, (9.23) is equivalent to a vanishing far field pattern of E_0^s, H_0^s and thus, by Theorem 6.10, equivalent to $E_0^s = H_0^s = 0$ in $\mathbb{R}^3 \setminus B$, i.e., with $E_0 := E_0^i$, $H_0 := H_0^i$ and $E_1 := E_0^i + E_0^s$, $H_1 := H_0^i + H_0^s$ we have solutions to (9.21) satisfying the boundary condition (9.22). □

In the case of a conducting medium, i.e., $\operatorname{Im} n \neq 0$, we can use Theorem 9.7 to deduce the following result [84].

Theorem 9.8. *In a conducting medium, the set $\mathcal{F}$ of electric far field patterns is complete in $L_t^2(\mathbb{S}^2)$.*

Proof. Recalling that an electromagnetic Herglotz pair vanishes if and only if its Herglotz kernel vanishes (Theorem 3.19 and Definition 6.33), we see from Theorem 9.7 that it suffices to show that the only solution of the homogeneous electromagnetic interior transmission problem (9.21), (9.22) is $E_0 = E_1 = H_0 = H_1 = 0$. However, analogous to (9.17), from Gauss' divergence theorem and the Maxwell equations (9.21) we have

$$\int_{\partial D} \nu \cdot E_1 \times \bar{H}_1 \, ds = ik \int_D (|H_1|^2 - \bar{n}\,|E_1|^2)\, dx,$$

$$\int_{\partial D} \nu \cdot E_0 \times \bar{H}_0 \, ds = ik \int_D (|H_0|^2 - |E_0|^2)\, dx.$$

From these two equations, using the transmission conditions (9.22) we obtain

$$\int_D (|H_1|^2 - \bar{n}\,|E_1|^2)\, dx = \int_D (|H_0|^2 - |E_0|^2)\, dx$$

and taking the imaginary part of both sides gives

$$\int_D \operatorname{Im} n\, |E_1|^2 dx = 0.$$

From this, we conclude by unique continuation that $E_1 = H_1 = 0$ in D. From (9.22) we now have vanishing tangential components of E_0 and H_0 on the boundary ∂D whence $E_0 = H_0 = 0$ in D follows from the Stratton–Chu formulas (6.8) and (6.9). □

In contrast to Theorem 9.8, the set $\mathcal{F}$ of electric far field patterns is not in general complete for a dielectric medium. We shall show this for a spherically stratified medium in the next section.

9.4 The Spherically Stratified Dielectric Medium

In this section, we shall consider the class $\mathcal{F}$ of electric far field patterns for a spherically stratified dielectric medium. Our aim is to show that in this case there exist wave numbers k such that $\mathcal{F}$ is not complete in $L_t^2(\mathbb{S}^2)$. It suffices to show that when $n(x) = n(r)$, $r = |x|$, $\operatorname{Im} n = 0$ and, as a function of r, $n \in C^2$, there exist values of k such that there exists a nontrivial solution to the homogeneous electromagnetic interior transmission problem

$$\begin{aligned} \operatorname{curl} E_1 - ikH_1 = 0, \quad \operatorname{curl} H_1 + ikn(r)E_1 = 0 \quad \text{in } B, \\ \operatorname{curl} E_0 - ikH_0 = 0, \quad \operatorname{curl} H_0 + ikE_0 = 0 \quad \text{in } B, \end{aligned} \tag{9.24}$$

with the boundary condition

$$\nu \times (E_1 - E_0) = 0, \quad \nu \times (H_1 - H_0) = 0 \quad \text{on } \partial B, \tag{9.25}$$

where E_0, H_0 is an electromagnetic Herglotz pair, where now B is an open ball of radius a with exterior unit normal ν and where $\operatorname{Im} n = 0$. Analogous to the construction of the spherical vector wave functions in Theorem 6.26 from the scalar spherical wave functions, we will develop special solutions to the electromagnetic transmission problem (9.24), (9.25) from solutions to the acoustic interior transmission problem

$$\Delta w + k^2 n(r) w = 0, \quad \Delta v + k^2 v = 0 \quad \text{in } B, \tag{9.26}$$

$$w - v = 0, \quad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial B. \tag{9.27}$$

Assume that w, v are solutions of (9.26), (9.27), are three times continuously differentiable and define

$$\begin{aligned} E_1(x) := \operatorname{curl}\{xw(x)\}, \quad H_1(x) := \frac{1}{ik} \operatorname{curl} E_1(x), \\ E_0(x) := \operatorname{curl}\{xv(x)\}, \quad H_0(x) := \frac{1}{ik} \operatorname{curl} E_0(x). \end{aligned} \tag{9.28}$$

Then, from the identity (6.4) together with

$$\Delta\{xw(x)\} = x\Delta w(x) + 2 \operatorname{grad} w(x)$$

and (9.26) we have that

$$\begin{aligned} ik \operatorname{curl} H_1(x) &= \operatorname{curl}\operatorname{curl}\operatorname{curl}\{xw(x)\} = -\operatorname{curl} \Delta\{xw(x)\} \\ &= k^2 \operatorname{curl}\{xn(r)w(x)\} = k^2 n(r) \operatorname{curl}\{xw(x)\} = k^2 n(r) E_1(x), \end{aligned}$$

that is,

$$\operatorname{curl} H_1 + ikn(r)E_1 = 0,$$

and similarly

$$\operatorname{curl} H_0 + ikE_0 = 0.$$

Hence, E_1, H_1 and E_0, H_0 satisfy (9.24). From $w - v = 0$ on ∂B we have that

$$x \times \{E_1(x) - E_0(x)\} = x \times \{\operatorname{grad}[w(x) - v(x)] \times x\} = 0, \quad x \in \partial B,$$

that is,

$$\nu \times (E_1 - E_0) = 0 \quad \text{on } \partial B.$$

Finally, setting $u = w - v$ in the relation

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\{xu(x)\} &= -\Delta\{xu(x)\} + \operatorname{grad}\operatorname{div}\{xu(x)\} \\ &= -x\Delta u(x) + \operatorname{grad}\left\{u(x) + r\,\frac{\partial u}{\partial r}(x)\right\} \end{aligned}$$

and using the boundary condition (9.27), we deduce that

$$\nu \times (H_1 - H_0) = 0 \quad \text{on } \partial B$$

is also valid. Hence, from a three times continuously differentiable solution w, v to the scalar transmission problem (9.26), (9.27), via (9.28) we obtain a solution E_1, H_1 and E_0, H_0 to the electromagnetic transmission problem (9.24), (9.25). Note, however, that in order to obtain a nontrivial solution through (9.28) we have to insist that w and v are not spherically symmetric.

We proceed as in Section 8.4 and, after introducing spherical coordinates (r, θ, φ), look for solutions to (9.26), (9.27) of the form

$$\begin{aligned} v(r,\theta) &= a_l\, j_l(kr)\, P_l(\cos\theta), \\ w(r,\theta) &= b_l\, \frac{y_l(r)}{r}\, P_l(\cos\theta), \end{aligned} \tag{9.29}$$

where P_l is Legendre's polynomial, j_l is a spherical Bessel function, a_l and b_l are constants to be determined and the function y_l is a solution of

$$y_l'' + \left(k^2 n(r) - \frac{l(l+1)}{r^2}\right) y_l = 0 \tag{9.30}$$

for $r > 0$ such that y_l is continuous for $r \geq 0$. However, in contrast to the analysis of Section 8.4, we are only interested in solutions which are dependent on θ, i.e., in solutions for $l \geq 1$. In particular, the ordinary differential equation (9.30) now has singular coefficients. We shall show that if $n(r) > 1$ for $0 \leq r < a$ or $0 < n(r) < 1$ for

$0 \le r < a$, then for each $l \ge 1$ there exist an infinite set of values of k and constants $a_l = a_l(k)$, $b_l = b_l(k)$, such that (9.29) is a nontrivial solution of (9.26), (9.27). From Section 6.6 we know that E_0, H_0, given by (9.28), is an electromagnetic Herglotz pair. Hence, by Theorem 9.7, for such values of k the set of electric far field patterns is not complete.

To show the existence of values of k such that (9.29) yields a nontrivial solution of (9.26), (9.27), we need to examine the asymptotic behavior of solutions to (9.30). To this end, we use the Liouville transformation

$$\xi := \int_0^r [n(\rho)]^{1/2} d\rho, \quad z(\xi) := [n(r)]^{1/4} y_l(r) \tag{9.31}$$

to transform (9.30) to

$$z'' + [k^2 - p(\xi)]z = 0 \tag{9.32}$$

where

$$p(\xi) := \frac{n''(r)}{4\,[n(r)]^2} - \frac{5}{16}\,\frac{[n'(r)]^2}{[n(r)]^3} + \frac{l(l+1)}{r^2 n(r)}\,.$$

Note that since $n(r) > 0$ for $r \ge 0$ and n is in C^2, the transformation (9.31) is invertible and p is well defined and continuous for $r > 0$. In order to deduce the required asymptotic estimates, we rewrite (9.32) in the form

$$z'' + \left(k^2 - \frac{l(l+1)}{\xi^2} - g(\xi)\right) z = 0 \tag{9.33}$$

where

$$g(\xi) := \frac{l(l+1)}{r^2 n(r)} - \frac{l(l+1)}{\xi^2} + \frac{n''(r)}{4\,[n(r)]^2} - \frac{5}{16}\,\frac{[n'(r)]^2}{[n(r)]^3}\,, \qquad r = r(\xi), \tag{9.34}$$

and note that since $n(r) = 1$ for $r \ge a$ we have

$$\int_1^\infty |g(\xi)|\, d\xi < \infty \quad \text{and} \quad \int_0^1 \xi |g(\xi)|\, d\xi < \infty.$$

For $\lambda > 0$ we now define the functions E_λ and M_λ by

$$E_\lambda(\xi) := \begin{cases} \left[-\dfrac{Y_\lambda(\xi)}{J_\lambda(\xi)} \right]^{1/2}, & 0 < \xi < \xi_\lambda, \\[2ex] 1, & \xi_\lambda \le \xi < \infty, \end{cases}$$

and

$$M_\lambda(\xi) := \begin{cases} [2\,|Y_\lambda(\xi)|\, J_\lambda(\xi)]^{1/2}, & 0 < \xi < \xi_\lambda, \\[2ex] \left[J_\lambda^2(\xi) + Y_\lambda^2(\xi) \right]^{1/2}, & \xi_\lambda \le \xi < \infty, \end{cases}$$

where J_λ is the Bessel function, Y_λ the Neumann function and ξ_λ is the smallest positive root of the equation

$$J_\lambda(\xi) + Y_\lambda(\xi) = 0.$$

Note that ξ_λ is less than the first positive zero of J_λ. For the necessary information on Bessel and Neumann functions of non-integral order we refer the reader to [52] and [221]. We further define G_λ by

$$G_\lambda(k,\xi) := \frac{\pi}{2}\int_0^\xi \rho M_\lambda^2(k\rho)\,|g(\rho)|\,d\rho$$

where g is given by (9.34). Noting that for $k > 0$ and $\lambda \geq 0$ we have that G_λ is finite when r is finite, we can now state the following result from Olver ([266], p. 450).

Theorem 9.9. *Let $k > 0$ and $l \geq -1/2$. Then (9.33) has a solution z which, as a function of ξ, is continuous in $[0,\infty)$, twice continuously differentiable in $(0,\infty)$, and is given by*

$$z(\xi) = \sqrt{\frac{\pi\xi}{2k}}\;\{J_\lambda(k\xi) + \varepsilon_l(k,\xi)\} \tag{9.35}$$

where

$$\lambda = l + \frac{1}{2}$$

and

$$|\varepsilon_l(k,\xi)| \leq \frac{M_\lambda(k\xi)}{E_\lambda(k\xi)}\;\left\{e^{G_\lambda(k,\xi)} - 1\right\}.$$

In order to apply Theorem 9.9 to obtain an asymptotic estimate for a continuous solution y_l of (9.30), we fix $\xi > 0$ and let k be large. Then for $\lambda > 0$ we have that there exist constants C_1 and C_2, both independent of k, such that

$$\begin{aligned}|G_\lambda(k,\xi)| &\leq C\left\{\int_0^1 M_\lambda^2(k\rho)\,d\rho + \frac{1}{k}\int_1^\infty |g(\rho)|\,d\rho\right\}\\ &\leq C_1\left\{\frac{1}{k}\int_{1/k}^1 \frac{d\rho}{\rho} + \frac{1}{k}\right\} \;=\; C_1\left\{\frac{\ln k}{k} + \frac{1}{k}\right\}.\end{aligned} \tag{9.36}$$

Hence, for z defined by (9.35) we have from Theorem 9.9, (9.36) and the asymptotics for the Bessel function J_λ that

$$\begin{aligned}z(\xi) &= \sqrt{\frac{\pi\xi}{2k}}\;\left\{J_\lambda(k\xi) + O\left(\frac{\ln k}{k^{3/2}}\right)\right\}\\ &= \frac{1}{k}\;\cos\left(k\xi - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{\ln k}{k^2}\right)\end{aligned} \tag{9.37}$$

for fixed $\xi > 0$ and λ as defined in Theorem 9.9. Furthermore, it can be shown that the asymptotic expansion (9.37) can be differentiated with respect to ξ, the error estimate being $O(\ln k/k)$. Hence, from (9.31) and (9.37) we can finally conclude that if y_l is defined by (9.31) then

$$y_l(r) = \frac{1}{k[n(r)]^{1/4}[n(0)]^{l/2+1/4}} \cos\left(k\int_0^r [n(\rho)]^{1/2} d\rho - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{\ln k}{k^2}\right) \tag{9.38}$$

where the asymptotic expansion for $[n(r)]^{1/4} y_l(r)$ can be differentiated with respect to r, the error estimate being $O(\ln k/k)$.

We now note that, from the above estimates, w, as defined by (9.29), is a C^2 solution of $\Delta w + k^2 n(r) w = 0$ in $B \setminus \{0\}$ and is continuous in B. Hence, by the removable singularity theorem for elliptic differential equations (c.f. [286],p. 104) we have that $w \in C^2(B)$. Since $n \in C^{1,\alpha}(\mathbb{R}^3)$, we can conclude from Green's formula (8.14) and Theorem 8.1 that $w \in C^3(B)$ and hence E_1 and H_1 are continuously differentiable in B.

We now return to the scalar interior transmission problem (9.26), (9.27) and note that (9.29) will be a nontrivial solution provided there exists a nontrivial solution a_l, b_l of the homogeneous linear system

$$\begin{aligned} b_l \frac{y_l(a)}{a} \quad & - \ a_l j_l(ka) = 0 \\ b_l \frac{d}{dr}\left(\frac{y_l(r)}{r}\right)_{r=a} & - a_l k j_l'(ka) = 0. \end{aligned} \tag{9.39}$$

The system (9.39) will have a nontrivial solution provided the determinant of the coefficients vanishes, that is,

$$d := \det \begin{pmatrix} \dfrac{y_l(a)}{a} & -j_l(ka) \\ \dfrac{d}{dr}\left(\dfrac{y_l(r)}{r}\right)_{r=a} & -k j_l'(ka) \end{pmatrix} = 0. \tag{9.40}$$

Recalling the asymptotic expansions (2.42) for the spherical Bessel functions, i.e.,

$$\begin{aligned} j_l(kr) &= \frac{1}{kr}\cos\left(kr - \frac{l\pi}{2} - \frac{\pi}{2}\right) + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \\ j_l'(kr) &= \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2} + \frac{\pi}{2}\right) + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \end{aligned} \tag{9.41}$$

we see from (9.38), (9.41) and the addition formula for the sine function that

$$d = \frac{1}{a^2 k\,[n(0)]^{l/2+1/4}} \left\{ \sin\left(k\int_0^a [n(r)]^{1/2} dr - ka\right) + O\left(\frac{\ln k}{k}\right)\right\}.$$

Therefore, a sufficient condition for (9.40) to be valid for a discrete set of values of k is that either $n(r) > 1$ for $0 \leq r < a$ or $n(r) < 1$ for $0 \leq r < a$. Hence we have the following theorem [84].

Theorem 9.10. *Assume that* $\operatorname{Im} n = 0$ *and that* $n(x) = n(r)$ *is spherically stratified,* $n(r) = 1$ *for* $r \geq a$, $n(r) > 1$ *or* $0 < n(r) < 1$ *for* $0 \leq r < a$ *and, as a function of* r, $n \in C^2$. *Then there exists an infinite set of wave numbers* k *such that the set* $\mathcal{F}$ *of electric far field patterns is not complete in* $L_t^2(\mathbb{S}^2)$.

9.5 The Exterior Impedance Boundary Value Problem

The mathematical treatment of the scattering of time harmonic electromagnetic waves by a body which is not perfectly conducting but which does not allow the electric and magnetic field to penetrate deeply into the body leads to what is called an exterior impedance boundary value problem for electromagnetic waves (c.f. [167], p. 511, [319], p. 304). In particular, such a model is sometimes used for coated media instead of the more complicated transmission problem. In addition to being an appropriate theme for this chapter, we shall also need to make use of the mathematical theory of the exterior impedance boundary value problem in our later treatment of the inverse scattering problem for electromagnetic waves. The first rigorous proof of the existence of a unique solution to the exterior impedance boundary value problem for electromagnetic waves was given by Colton and Kress in [63]. Here we shall provide a simpler proof of this result by basing our ideas on those developed for a perfect conductor in Chapter 6. We first define the problem under consideration where for the rest of this section D is a bounded domain in $\mathbb{R}^3$ with connected C^2 boundary ∂D with unit outward normal ν.

Exterior Impedance Problem. *Given a Hölder continuous tangential field* c *on* ∂D *and a positive constant* λ, *find a solution* $E, H \in C^1(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ *of the Maxwell equations*

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \tag{9.42}$$

satisfying the impedance boundary condition

$$\nu \times \operatorname{curl} E - i\lambda\,(\nu \times E) \times \nu = c \quad \text{on } \partial D \tag{9.43}$$

and the Silver–Müller radiation condition

$$\lim_{r\to\infty} (H \times x - rE) = 0 \tag{9.44}$$

uniformly for all directions $\hat{x} = x/|x|$.

The uniqueness of a solution to (9.42)–(9.44) is easy to prove.

Theorem 9.11. *The exterior impedance problem has at most one solution provided* $\lambda > 0$.

Proof. If $c = 0$, then from (9.43) and the fact that $\lambda > 0$ we have that

$$\operatorname{Re} k \int_{\partial D} \nu \times E \cdot \bar{H}\, ds = -\lambda \int_{\partial D} |\nu \times E|^2 ds \le 0.$$

We can now conclude from Theorem 6.11 that $E = H = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. □

We now turn to the existence of a solution to the exterior impedance problem, always assuming that $\lambda > 0$. To this end, we recall the definition of the space $C^{0,\alpha}(\partial D)$ of Hölder continuous functions defined on ∂D from Section 3.1 and the space $C_t^{0,\alpha}(\partial D)$ of Hölder continuous tangential fields defined on ∂D from Section 6.3. We also recall from Theorems 3.2, 3.4 and 6.14 that the single-layer operator $S : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ defined by

$$(S\varphi)(x) := 2 \int_{\partial D} \Phi(x, y)\varphi(y)\, ds(y), \quad x \in \partial D,$$

the double-layer operator $K : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ defined by

$$(K\varphi)(x) := 2 \int \frac{\partial \Phi(x, y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in \partial D,$$

and the magnetic dipole operator $M : C_t^{0,\alpha}(\partial D) \to C_t^{0,\alpha}(\partial D)$ defined by

$$(Ma)(x) := 2 \int_{\partial D} \nu(x) \times \operatorname{curl}_x\{a(y)\Phi(x, y)\}\, ds(y), \quad x \in \partial D,$$

are all compact. Furthermore, with the spaces

$$C^{0,\alpha}(\mathrm{Div}, \partial D) = \left\{a \in C_t^{0,\alpha}(\partial D) : \mathrm{Div}\, a \in C^{0,\alpha}(\partial D)\right\}$$

and

$$C^{0,\alpha}(\mathrm{Curl}, \partial D) = \left\{b \in C_t^{0,\alpha}(\partial D) : \mathrm{Curl}\, b \in C^{0,\alpha}(\partial D)\right\}$$

which were also introduced in Section 6.3, the electric dipole operator $N : C^{0,\alpha}(\mathrm{Curl}, \partial D) \to C^{0,\alpha}(\mathrm{Div}, \partial D)$ defined by

$$(Na)(x) := 2\, \nu(x) \times \operatorname{curl}\operatorname{curl} \int_{\partial D} \Phi(x, y)\, \nu(y) \times a(y)\, ds(y), \quad x \in \partial D,$$

is bounded by Theorem 6.19.

With these definitions and facts recalled, following Hähner [120], we now look for a solution of the exterior impedance problem in the form

$$E(x) = \int_{\partial D} \Phi(x,y)b(y)\,ds(y) + i\lambda \operatorname{curl} \int_{\partial D} \Phi(x,y)\,\nu(y) \times (S_0^2 b)(y)\,ds(y)$$

$$+ \operatorname{grad} \int_{\partial D} \Phi(x,y)\varphi(y)\,ds(y) + i\lambda \int_{\partial D} \Phi(x,y)\nu(y)\varphi(y)\,ds(y), \tag{9.45}$$

$$H(x) = \frac{1}{ik} \operatorname{curl} E(x), \quad x \in \mathbb{R}^3 \setminus \bar{D},$$

where S_0 is the single-layer operator in the potential theoretic limit $k = 0$ and the densities $b \in C_t^{0,\alpha}(\partial D)$ and $\varphi \in C^{0,\alpha}(\partial D)$ are to be determined. The vector field E clearly satisfies the vector Helmholtz equation and its cartesian components satisfy the (scalar) Sommerfeld radiation condition. Hence, if we insist that $\operatorname{div} E = 0$ in $\mathbb{R}^3 \setminus \bar{D}$, then by Theorems 6.4 and 6.8 we have that E, H satisfy the Maxwell equations and the Silver–Müller radiation condition. Since $\operatorname{div} E$ satisfies the scalar Helmholtz equation and the Sommerfeld radiation condition, by the uniqueness for the exterior Dirichlet problem it suffices to impose $\operatorname{div} E = 0$ only on the boundary ∂D. From the jump and regularity conditions of Theorems 3.1, 3.3, 6.12 and 6.13, we can now conclude that (9.45) for $b \in C_t^{0,\alpha}(\partial D)$ and $\varphi \in C^{0,\alpha}(\partial D)$ ensures the regularity $E, H \in C^{0,\alpha}(\mathbb{R}^3 \setminus D)$ up to the boundary and that it solves the exterior impedance problem provided b and φ satisfy the integral equations

$$\begin{aligned} b + M_{11}b + M_{12}\varphi &= 2c \\ -i\lambda\varphi + M_{21}b + M_{22}\varphi &= 0 \end{aligned} \tag{9.46}$$

where

$$M_{11}b := Mb + i\lambda NPS_0^2 b - i\lambda PS\,b + \lambda^2\{M(\nu \times S_0^2 b)\} \times \nu + \lambda^2 PS_0^2 b,$$

$$(M_{12}\varphi)(x) := 2i\lambda\,\nu(x) \times \int_{\partial D} \operatorname{grad}_x \Phi(x,y) \times \{\nu(y) - \nu(x)\}\varphi(y)\,ds(y)$$

$$+\lambda^2 (PS\,\nu\varphi)(x), \quad x \in \partial D,$$

$$(M_{21}b)(x) := -2\int_{\partial D} \operatorname{grad}_x \Phi(x,y) \cdot b(y)\,ds(y), \quad x \in \partial D,$$

$$M_{22}\varphi := k^2 S\,\varphi + i\lambda K\varphi,$$

and where P stands for the orthogonal projection of a vector field defined on ∂D onto the tangent plane, that is, $Pa := (\nu \times a) \times \nu$. Noting the smoothing property $S_0 : C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ from Theorem 3.4, as in the proof of Theorem 6.21 it is not difficult to verify that $M_{11} : C_t^{0,\alpha}(\partial D) \to C_t^{0,\alpha}(\partial D)$ is compact. Compactness of the operator $M_{12} : C^{0,\alpha}(\partial D) \to C_t^{0,\alpha}(\partial D)$ follows by applying Corollary 2.9 from [64] to the first term in the definition of M_{12}. Loosely speaking, compactness of M_{12} rests on the fact that the factor $\nu(x) - \nu(y)$ makes the kernel weakly singular. Finally, $M_{22} : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ is compact, whereas $M_{21} : C_t^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ is merely bounded. Writing the system (9.46) in the form

$$\begin{pmatrix} I & 0 \\ M_{21} & -i\lambda I \end{pmatrix} \begin{pmatrix} b \\ \varphi \end{pmatrix} + \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} b \\ \varphi \end{pmatrix} = \begin{pmatrix} 2c \\ 0 \end{pmatrix},$$

we now see that the first of the two matrix operators has a bounded inverse because of its triangular form and the second is compact. Hence, we can apply the Riesz–Fredholm theory to (9.46).

For this purpose, suppose b and φ are a solution to the homogeneous equation corresponding to (9.46) (i.e., $c = 0$). Then the field E, H defined by (9.45) satisfies the homogeneous exterior impedance problem in $\mathbb{R}^3 \setminus \bar{D}$. Since $\lambda > 0$, we can conclude from Theorem 9.11 that $E = H = 0$ in $\mathbb{R}^3 \setminus D$. Viewing (9.45) as defining a solution of the vector Helmholtz equation in D, from the jump relations of Theorems 3.1, 3.3, 6.12 and 6.13 we see that

$$-\nu \times E_- = i\lambda \nu \times S_0^2 b, \quad -\nu \times \operatorname{curl} E_- = b \quad \text{on } \partial D, \tag{9.47}$$

$$-\operatorname{div} E_- = -i\lambda\varphi, \quad -\nu \cdot E_- = -\varphi \quad \text{on } \partial D. \tag{9.48}$$

Hence, with the aid of Green's vector theorem (6.2), we derive from (9.47) and (9.48) that

$$\int_D \left\{ |\operatorname{curl} E|^2 + |\operatorname{div} E|^2 - k^2 |E|^2 \right\} dx = i\lambda \int_{\partial D} \left\{ |S_0 b|^2 + |\varphi|^2 \right\} ds.$$

Taking the imaginary part of the last equation and recalling that $\lambda > 0$ now shows that $S_0 b = 0$ and $\varphi = 0$ on ∂D. Since S_0 is injective (see the proof of Theorem 3.12), we have that $b = 0$ on ∂D. The Riesz–Fredholm theory now implies the following theorem. The statement on the boundedness of the operator $\mathcal{A}$ follows from the fact that by the Riesz–Fredholm theory the inverse operator for (9.46) is bounded from $C_t^{0,\alpha}(\partial D) \times C^{0,\alpha}(\partial D)$ into itself and by applying the mapping properties of Theorems 3.3 and 6.13 to the solution (9.45).

Theorem 9.12. *Suppose $\lambda > 0$. Then for each $c \in C_t^{0,\alpha}(\partial D)$ there exists a unique solution to the exterior impedance problem. The operator $\mathcal{A}$ mapping the boundary data c onto the tangential component $\nu \times E$ of the solution is a bounded operator $\mathcal{A} : C_t^{0,\alpha}(\partial D) \to C^{0,\alpha}(\mathrm{Div}, \partial D)$.*

For technical reasons, we shall need in Chapter 10 sufficient conditions for the invertibility of the operator

$$NR - i\lambda R(I + M) : C^{0,\alpha}(\mathrm{Div}, \partial D) \to C_t^{0,\alpha}(\partial D)$$

where the operator $R : C_t^{0,\alpha}(\partial D) \to C_t^{0,\alpha}(\partial D)$ is given by

$$Ra := a \times \nu.$$

To this end, we first try to express the solution of the exterior impedance problem in the form

$$E(x) = \operatorname{curl} \int_{\partial D} a(y)\Phi(x, y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D},$$

where $a \in C^{0,\alpha}(\text{Div}, \partial D)$. From the jump conditions of Theorems 6.12 and 6.13, this leads to the integral equation

$$NRa - i\lambda RMa - i\lambda Ra = 2c \tag{9.49}$$

for the unknown density a. However, we can interpret the solution of the exterior impedance problem as the solution of the exterior Maxwell problem with boundary condition

$$\nu \times E = \mathcal{A}c \quad \text{on } \partial D,$$

and hence a also is required to satisfy the integral equation

$$a + Ma = 2\mathcal{A}c.$$

The last equation turns out to be a special case of equation (6.56) with $\eta = 0$ (and a different right hand side). From the proof of Theorem 6.21, it can be seen that if k is not a Maxwell eigenvalue for D then $I + M$ has a trivial nullspace. Hence, since by Theorem 6.17 the operator $M : C^{0,\alpha}(\text{Div}, \partial D) \to C^{0,\alpha}(\text{Div}, \partial D)$ is compact, by the Riesz–Fredholm theory $(I + M)^{-1} : C^{0,\alpha}(\text{Div}, \partial D) \to C^{0,\alpha}(\text{Div}, \partial D)$ exists and is bounded. Hence, $(I + M)^{-1}\mathcal{A} : C_t^{0,\alpha}(\partial D) \to C^{0,\alpha}(\text{Div}, \partial D)$ is the bounded inverse of $NR - i\lambda R(I + M)$ and we have proven the following theorem.

Theorem 9.13. *Assume that $\lambda > 0$ and that k is not a Maxwell eigenvalue for D. Then the operator $NR - i\lambda R(I + M) : C^{0,\alpha}(\text{Div}, \partial D) \to C_t^{0,\alpha}(\partial D)$ has a bounded inverse.*

We shall now conclude this chapter by briefly considering the electric far field patterns corresponding to the exterior impedance problem (9.42)–(9.44) with c given by

$$c := -\nu \times \operatorname{curl} E^i + i\lambda\, (\nu \times E^i) \times \nu \quad \text{on } \partial D$$

where E^i and H^i are given by (9.19). This corresponds to the scattering of the incident field (9.19) by the imperfectly conducting obstacle D where the total electric field $E = E^i + E^s$ satisfies the impedance boundary condition

$$\nu \times \operatorname{curl} E - i\lambda\, (\nu \times E) \times \nu = 0 \quad \text{on } \partial D \tag{9.50}$$

and E^s is the scattered electric field. From Theorem 6.9 we see that E^s has the asymptotic behavior

$$E^s(x,d)p = \frac{e^{ik|x|}}{|x|}\, E^\lambda_\infty(\hat{x},d)p + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,$$

where E^λ_∞ is the electric far field pattern. From (6.88) and (9.50) we can easily deduce the following reciprocity relation [8].

Theorem 9.14. *For all vectors $\hat{x}, d \in \mathbb{S}^2$ we have*

$$E^\lambda_\infty(\hat{x},d) = [E^\lambda_\infty(-d,-\hat{x})]^\top.$$

We are now in a position to prove the analogue of Theorem 9.8 for the exterior impedance problem. In particular, recall the Hilbert space $L^2_t(\mathbb{S}^2)$ of tangential L^2 vector fields on the unit sphere, let $\{d_n : n = 1,2,\dots\}$ be a countable dense set of unit vectors on $\mathbb{S}^2$ and denote by e_1, e_2, e_3 the cartesian unit coordinate vectors in $\mathbb{R}^3$. For the electric far field patterns we now have the following theorem due to Angell, Colton and Kress [8].

Theorem 9.15. *Assume $\lambda > 0$. Then the set*

$$\mathcal{F}_\lambda = \left\{E^\lambda_\infty(\cdot\,,d_n)e_j : n = 1,2,\dots,\ j = 1,2,3\right\}$$

of electric far field patterns for the exterior impedance problem is complete in $L^2_t(\mathbb{S}^2)$.

Proof. Suppose that $g \in L^2_t(\mathbb{S}^2)$ satisfies

$$\int_{\mathbb{S}^2} E^\lambda_\infty(\hat{x},d_n)e_j \cdot g(\hat{x})\, ds(\hat{x}) = 0$$

for $n = 1,2,\dots$ and $j = 1,2,3$. We must show that $g = 0$. As in the proof of Theorem 9.7, by the reciprocity Theorem 9.14, we have

$$\int_{\mathbb{S}^2} E^\lambda_\infty(-d,-\hat{x})g(\hat{x})\, ds(\hat{x}) = 0$$

for all $d \in \mathbb{S}^2$, i.e.,

$$\int_{\mathbb{S}^2} E^\lambda_\infty(\hat{x},d)h(d)\, ds(d) = 0 \tag{9.51}$$

for all $\hat{x} \in \mathbb{S}^2$ where $h(d) = g(-d)$.

Now define the electromagnetic Herglotz pair E_0^i, H_0^i by

$$E_0^i(x) = \int_{\mathbb{S}^2} E^i(x,d)h(d)\,ds(d) = ik\int_{\mathbb{S}^2} h(d)\,e^{ik\,x\cdot d}\,ds(d),$$

$$H_0^i(x) = \int_{\mathbb{S}^2} H^i(x,d)h(d)\,ds(d) = \operatorname{curl}\int_{\mathbb{S}^2} h(d)\,e^{ik\,x\cdot d}\,ds(d).$$

Analogous to Lemma 6.35 it can be seen that the left hand side of (9.51) represents the electric far field pattern of the scattered field E_0^s, H_0^s corresponding to the incident field E_0^i, H_0^i. Then from (9.51) we see that the electric far field pattern of E_0^s vanishes and hence, from Theorem 6.10, both E_0^s and H_0^s are identically zero in $\mathbb{R}^3 \setminus D$. We can now conclude that E_0^i, H_0^i satisfies the impedance boundary condition

$$\nu \times \operatorname{curl} E_0^i - i\lambda\,(\nu \times E_0^i) \times \nu = 0 \quad \text{on } \partial D. \tag{9.52}$$

Gauss' theorem and the Maxwell equations (compare (9.17)) now imply that

$$\int_{\partial D} \nu \times E_0^i \cdot \bar{H}_0^i\,ds = ik\int_D \left\{|H_0^i|^2 - |E_0^i|^2\right\} dx$$

and hence from (9.52) we have that

$$\lambda \int_{\partial D} |\nu \times E_0^i|^2 ds = ik^2 \int_D \left\{|E_0^i|^2 - |H_0^i|^2\right\} dx$$

whence $\nu \times E_0^i = 0$ on ∂D follows since $\lambda > 0$. From (9.52) we now see that $\nu \times H_0^i = 0$ on ∂D and hence from the Stratton–Chu formulas (6.8) and (6.9) we have that $E_0^i = H_0^i = 0$ in D and by analyticity (Theorem 6.3) $E_0^i = H_0^i = 0$ in $\mathbb{R}^3$. But now from Theorem 3.19 we conclude that $h = 0$ and consequently $g = 0$. $\square$

Chapter 10
The Inverse Medium Problem

We now turn our attention to the problem of reconstructing the refractive index from a knowledge of the far field pattern of the scattered acoustic or electromagnetic wave. We shall call this problem the *inverse medium problem*. We first consider the case of acoustic waves and the use of the Lippmann–Schwinger equation to reformulate the acoustic inverse medium problem as a problem in constrained optimization. Included here is a brief discussion of the use of the Born approximation to linearize the problem. We then proceed to the proof of a uniqueness theorem for the acoustic inverse medium problem. Our uniqueness result is then followed by a discussion of decomposition methods for solving the inverse medium problem for acoustic waves and the use of sampling methods and transmission eigenvalues to obtain qualitative estimates on the refractive index. We conclude by examining the use of decomposition methods to solve the inverse medium problem for electromagnetic waves followed by some numerical examples illustrating the use of decomposition methods to solve the inverse medium problem for acoustic waves.

10.1 The Inverse Medium Problem for Acoustic Waves

We consider the inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium which we shall from now on refer to as the acoustic *inverse medium problem*. Recall from Chapter 8 that the direct scattering problem we are now concerned with is, given the refractive index

$$n(x) = n_1(x) + i\,\frac{n_2(x)}{k}$$

where $k > 0$ and n is piecewise continuous in $\mathbb{R}^3$ such that

$$m := 1 - n$$

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences 93, DOI 10.1007/978-1-4614-4942-3_10,

has compact support, to determine u such that

$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^3, \tag{10.1}$$

$$u(x, d) = e^{ik\, x\cdot d} + u^s(x, d), \tag{10.2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{10.3}$$

uniformly for all directions. As in Chapter 8, we shall in addition always assume that $n_1(x) > 0$ and $n_2(x) \geq 0$ for $x \in \mathbb{R}^3$. The existence of a unique solution to (10.1)–(10.3) was established in Chapter 8 via the Lippmann–Schwinger equation

$$u(x, d) = e^{ik\, x\cdot d} - k^2 \int_{\mathbb{R}^3} \Phi(x, y) m(y) u(y, d)\, dy, \quad x \in \mathbb{R}^3. \tag{10.4}$$

Further it was also shown that u^s has the asymptotic behavior

$$u^s(x, d) = \frac{e^{ik|x|}}{|x|}\, u_\infty(\hat{x}, d) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,$$

with the far field pattern u_∞ given by

$$u_\infty(\hat{x}, d) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\, \hat{x}\cdot y} m(y) u(y, d)\, dy, \quad \hat{x} \in \mathbb{S}^2. \tag{10.5}$$

The inverse medium problem for acoustic waves is to determine n from $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \mathbb{S}^2$. We shall also often consider data for different values of k and in this case we shall write $u_\infty(\hat{x}, d) = u_\infty(\hat{x}, d, k)$.

The solution of the inverse medium problem becomes particularly simple when use is made of the Born approximation (8.29). In this case, assuming the far field pattern $u_\infty(\hat{x}, d, k)$ is known for $\hat{x} \in \mathbb{S}^2$, $d \in \{d_1, \ldots, d_P\}$ and $k \in \{k_1, \ldots, k_Q\}$, instead of the nonlinear system (10.4), (10.5) we have the linear integral equations

$$u_\infty(\hat{x}, d_p, k_q) = -\frac{k_q^2}{4\pi} \int_{\mathbb{R}^3} e^{ik_q\, (d_p - \hat{x})\cdot y} m(y)\, dy, \quad \hat{x} \in \mathbb{S}^2, \tag{10.6}$$

for $p = 1, \ldots, P$ and $q = 1, \ldots, Q$ to solve for the unknown function m and any one of the linear methods described in Chapter 4 can be used to do this. (Note that since the kernel of each equation is analytic, this problem is severely ill-posed and regularization methods must be used.) The obvious advantage to the Born approximation approach is that the nonlinear inverse medium problem is reduced to considering a set of linear integral equations (albeit of the first kind). The equally obvious disadvantage is that the approach is only valid if $k_q^2\, \|m\|_\infty \ll 1$, a condition that is often not satisfied in applications. For a further discussion of the Born approximation approach to the inverse medium problem, the reader is referred to Bleistein [22], Chew [50], Devaney [92] and Langenberg [218] where additional references may be found.

10.2 Uniqueness

In this section we shall prove a uniqueness theorem for the inverse acoustic medium problem, which is due to Nachman [257], Novikov[263], and Ramm[289, 291] (see also Gosh Roy and Couchman [108]). To motivate our analysis we begin by proving the following theorem due to Calderón [43] (see also Ramm [288]).

Theorem 10.1. *The set of products $h_1 h_2$ of entire harmonic functions h_1 and h_2 is complete in $L^2(D)$ for any bounded domain $D \subset \mathbb{R}^3$.*

Proof. Given $y \in \mathbb{R}^3$ choose a vector $b \in \mathbb{R}^3$ with $b \cdot y = 0$ and $|b| = |y|$. Then for $z := y + ib \in \mathbb{C}^3$ we have $z \cdot z = 0$ and therefore the function $h_z(x) := e^{iz\cdot x}$, $x \in \mathbb{R}^3$, is harmonic. Now assume that $\varphi \in L^2(D)$ is such that

$$\int_D \varphi h_1 h_2 \, dx = 0$$

for all pairs of entire harmonic functions h_1 and h_2. For $h_1 = h_z$ and $h_2 = h_{\bar{z}}$ this becomes

$$\int_D \varphi(x)\, e^{2i\,y\cdot x}\, dx = 0$$

for $y \in \mathbb{R}^3$ and we can now conclude by the Fourier integral theorem that $\varphi = 0$ almost everywhere in D. □

For our uniqueness proof for the inverse medium problem we need a property corresponding to Theorem 10.1 for products $v_1 v_2$ of solutions to $\Delta v_1 + k^2 n_1 v_1 = 0$ and $\Delta v_2 + k^2 n_2 v_2 = 0$ for two different refractive indices n_1 and n_2. Such a result was first established by Sylvester and Uhlmann [313] by using solutions which asymptotically behave like the functions $e^{i\,z\cdot x}$ occurring in the proof of the previous theorem. To construct these solutions Sylvester and Uhlmann employed Fourier integral techniques. Here, however, we will follow Hähner [126] who simplified the analysis considerably by using Fourier series techniques.

Define the set

$$\widetilde{\mathbb{Z}}^3 := \left\{ \alpha = \beta - \left(0, \frac{1}{2}, 0\right) : \beta \in \mathbb{Z}^3 \right\}.$$

Then, in the cube $Q := [-\pi, \pi]^3$ the functions

$$e_\alpha(x) := \frac{1}{\sqrt{2\pi}^{\,3}}\, e^{i\,\alpha\cdot x}, \quad \alpha \in \widetilde{\mathbb{Z}}^3,$$

provide a complete orthonormal system for $L^2(Q)$. We denote the Fourier coefficients of $f \in L^2(Q)$ with respect to this orthonormal system by $\hat{f}_\alpha$.

Lemma 10.2 *Let $t > 0$ and $\zeta = t\,(1, i, 0) \in \mathbb{C}^3$. Then*

$$G_\zeta f := -\sum_{\alpha\in\widetilde{\mathbb{Z}}^3} \frac{\hat{f}_\alpha}{\alpha\cdot\alpha + 2\,\zeta\cdot\alpha}\, e_\alpha$$

defines an operator $G_\zeta : L^2(Q) \to H^2(Q)$ with the properties

$$\|G_\zeta f\|_{L^2(Q)} \le \frac{1}{t}\, \|f\|_{L^2(Q)}$$

and

$$\Delta G_\zeta f + 2i\,\zeta\cdot\operatorname{grad} G_\zeta f = f$$

in the weak sense for all $f \in L^2(Q)$.

Proof. Obviously, we have $|\alpha_2| \ge 1/2$ for all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \widetilde{\mathbb{Z}}^3$, whence

$$|\alpha\cdot\alpha + 2\,\zeta\cdot\alpha| \ge |\operatorname{Im}\{\alpha\cdot\alpha + 2\,\zeta\cdot\alpha\}| = 2t\,|\alpha_2| \ge t$$

follows for all $\alpha \in \widetilde{\mathbb{Z}}^3$. This implies that $G_\zeta : L^2(Q) \to L^2(Q)$ is well defined and by Parseval's equality we have

$$\|G_\zeta f\|^2_{L^2(Q)} = \sum_{\alpha\in\widetilde{\mathbb{Z}}^3} \left|\frac{\hat{f}_\alpha}{\alpha\cdot\alpha + 2\,\zeta\cdot\alpha}\right|^2 \le \frac{1}{t^2}\, \|f\|^2_{L^2(Q)}$$

for all $f \in L^2(Q)$. Clearly there exists a constant c, depending on t, such that

$$\left|\frac{1 + \alpha\cdot\alpha}{\alpha\cdot\alpha + 2\,\zeta\cdot\alpha}\right| \le c$$

for all $\alpha \in \widetilde{\mathbb{Z}}^3$. Hence, for the Fourier coefficients of $G_\zeta f$ we have

$$\sum_{\alpha\in\widetilde{\mathbb{Z}}^3} (1 + \alpha\cdot\alpha)^2 \left|\widehat{G_\zeta f}_\alpha\right|^2 < \infty,$$

i.e., $G_\zeta f \in H^2(Q)$. Finally, we compute

$$\Delta G_\zeta f + 2i\,\zeta\cdot\operatorname{grad} G_\zeta f = -\sum_{\alpha\in\widetilde{\mathbb{Z}}^3} (\alpha\cdot\alpha + 2\,\zeta\cdot\alpha)\widehat{G_\zeta f}_\alpha e_\alpha = \sum_{\alpha\in\widetilde{\mathbb{Z}}^3} \hat{f}_\alpha e_\alpha = f$$

in the weak sense. □

Lemma 10.3 *Let D be an open ball centered at the origin and containing the support of $1 - n$. Then there exists a constant $C > 0$ such that for each $z \in \mathbb{C}^3$ with $z\cdot z = 0$ and $|\operatorname{Re} z| \ge 2k^2\|n\|_\infty$ there exists a solution $v \in H^2(D)$ to the equation*

$$\Delta v + k^2 n v = 0 \quad \textit{in } D$$

of the form

$$v(x) = e^{i\,z\cdot x}[1 + w(x)]$$

where w satisfies

$$\|w\|_{L^2(D)} \le \frac{C}{|\operatorname{Re} z|}\ . \tag{10.7}$$

Proof. Without loss of generality we may assume that $\bar{D}$ is contained in the interior of Q. Since the property $z \cdot z = 0$ implies that $|\operatorname{Re} z| = |\operatorname{Im} z|$ and $\operatorname{Re} z \cdot \operatorname{Im} z = 0$, we can choose a unitary transformation U of $\mathbb{R}^3$ such that

$$U \operatorname{Re} z = (|\operatorname{Re} z|, 0, 0) \quad \text{and} \quad U \operatorname{Im} z = (0, |\operatorname{Im} z|, 0).$$

We set $\zeta = |\operatorname{Re} z|(1, i, 0)$ and $\tilde{n}(x) = n(U^{-1}x)$, and consider the equation

$$u + k^2 G_\zeta(\tilde{n}u) = -k^2 G_\zeta \tilde{n}. \tag{10.8}$$

Provided $|\operatorname{Re} z| \ge 2k^2\|n\|_\infty$, the mapping $u \mapsto k^2 G_\zeta(\tilde{n}u)$ is a contraction in $L^2(Q)$ since in this case, by Lemma 10.2, we have the estimate

$$\|k^2 G_\zeta(\tilde{n}u)\|_{L^2(Q)} \le \frac{k^2}{|\operatorname{Re} z|}\ \|n\|_\infty \|u\|_{L^2(Q)} \le \frac{1}{2}\ \|u\|_{L^2(Q)}.$$

Hence, by the Neumann series, for $|\operatorname{Re} z| \ge 2k^2\|n\|_\infty$ the equation (10.8) has a unique solution $u \in L^2(Q)$ and this solution satisfies

$$\|u\|_{L^2(Q)} \le 2k^2 \|G_\zeta \tilde{n}\|_{L^2(Q)}.$$

From this, by Lemma 10.2, it follows that

$$\|u\|_{L^2(Q)} \le 2k^2 \|\tilde{n}\|_{L^2(Q)}\ \frac{1}{|\operatorname{Re} z|}\ . \tag{10.9}$$

From the fixed point equation (10.8) and Lemma 10.2 we also can conclude that $u \in H^2(Q)$ and

$$\Delta u + 2i\,\zeta \cdot \operatorname{grad} u = -k^2 \tilde{n}(1 + u). \tag{10.10}$$

We now set $w(x) := u(Ux)$ and

$$v(x) := e^{i\,z\cdot x}[1 + w(x)].$$

Then, using $z \cdot z = 0$, $Uz = \zeta$ and (10.10), we compute

$$\begin{aligned}\Delta v(x) &= e^{i\,z\cdot x}(\Delta + 2i\,z \cdot \operatorname{grad})w(x) = e^{i\,z\cdot x}[\Delta u(Ux) + 2i\,Uz \cdot \operatorname{grad} u(Ux)] \\ &= -k^2 e^{i\,z\cdot x}\tilde{n}(Ux)[1 + u(Ux)] = -k^2 e^{i\,z\cdot x} n(x)[1 + w(x)] = -k^2 n(x) v(x)\end{aligned}$$

in the weak sense for $x \in \bar{D}$. Finally, (10.9) implies that (10.7) is satisfied. □

Now we return to the scattering problem.

Lemma 10.4 *Let B and D be two open balls centered at the origin and containing the support of $m = 1 - n$ such that $\bar{B} \subset D$. Then the set of total fields $\{u(\cdot\,, d) : d \in \mathbb{S}^2\}$ satisfying (10.1)–(10.3) is complete in the closure of*

$$H := \left\{v \in H^2(D) : \Delta v + k^2 n v = 0 \ in\ D\right\}$$

with respect to the $L^2(B)$ norm.

Proof. Consider the mapping $A : H \to H^2_{\mathrm{loc}}(\mathbb{R}^3)$ defined by the volume potential

$$(Av)(x) := \int_B \Phi(x, y)\overline{[(I + k^2 T_m^*)^{-1} v](y)}\, dy, \quad x \in \mathbb{R}^3,$$

where $T_m^* : L^2(B) \to L^2(B)$ denotes the adjoint of the Lippmann–Schwinger operator $T_m : L^2(B) \to L^2(B)$ given by

$$(T_m u)(x) := \int_B \Phi(x, y) m(y) u(y)\, dy, \quad x \in B. \tag{10.11}$$

Note that for $v \in H$ we have that $Av \in H^2(B)$. For the density

$$V := (I + k^2 T_m^*)^{-1} v$$

of this potential we have that

$$v(x) = V(x) + k^2 \overline{m(x)} \int_B \overline{\Phi(x, y)} V(y)\, dy, \quad x \in B, \tag{10.12}$$

which, in particular, implies that $V \in H^2(B)$. From Theorem 8.1 we obtain that $\Delta Av + k^2 Av = -\overline{V}$ in B, and from this, using (10.12), it follows that

$$\bar{v} = -\Delta Av - k^2 Av + k^2 m Av = -\Delta Av - k^2 n Av \quad \text{in } B.$$

Multiplying this equation by $w \in H$ and then integrating, by Green's theorem and using the fact that both Av and w solve the Helmholtz equation in $D \setminus \bar{B}$, we deduce that

$$\int_B w\bar{v}\, dx = -\int_B w(\Delta Av + k^2 n Av)\, dx = \int_{\partial D} \left\{ Av\, \frac{\partial w}{\partial \nu} - w\, \frac{\partial Av}{\partial \nu} \right\} ds \tag{10.13}$$

for all $v, w \in H$. Here, ν denotes the outward unit normal to ∂D.

Now let $v \in \bar{H}$, i.e., $v \in L^2(B)$ is the L^2 limit of a sequence (v_j) from H. Assume that

$$(u(\cdot\,, d), v) = \int_B \overline{v(x)} u(x, d)\, dx = 0, \quad d \in \mathbb{S}^2.$$

Then from the Lippmann–Schwinger equation (10.4) we obtain

$$0 = ((I + k^2 T_m)^{-1} u^i(\cdot\,, d), v) = (u^i(\cdot\,, d), (I + k^2 T_m^*)^{-1} v), \quad d \in \mathbb{S}^2. \tag{10.14}$$

As a consequence of (10.14), the potential Av with density $V := (I + k^2 T_m^*)^{-1} v$ has the far field pattern

$$(Av)_\infty(d) = \frac{1}{4\pi} \int_B \overline{V(y)} e^{-ik\,y\cdot d} dy = \frac{1}{4\pi} \, (u^i(\cdot\,, -d), V) = 0, \quad d \in \mathbb{S}^2.$$

By Rellich's Lemma 2.12, this implies $Av = 0$ in $\mathbb{R}^3 \setminus B$. Since $I + k^2 T_m^*$ has a bounded inverse, by the Cauchy–Schwarz inequality the mapping A is bounded from $L^2(B)$ into $H^2(K)$ for each compact set $K \subset \mathbb{R}^3 \setminus \bar{B}$. Hence, inserting v_j into (10.13) and passing to the limit $j \to \infty$ yields

$$\int_B w \bar{v} \, dx = 0$$

for all $w \in H$. Now inserting $w = v_j$ in this equation and again passing to the limit $j \to \infty$ we obtain

$$\int_B |v|^2 dx = 0,$$

whence $v = 0$ follows and the proof is complete. □

Now we are ready to prove the uniqueness result for the inverse medium problem.

Theorem 10.5. *The refractive index n is uniquely determined by a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \mathbb{S}^2$ and a fixed wave number k.*

Proof. Assume that n_1 and n_2 are two refractive indices such that

$$u_{1,\infty}(\cdot\,, d) = u_{2,\infty}(\cdot\,, d), \quad d \in \mathbb{S}^2,$$

and let B and D be two open balls centered at the origin and containing the supports of $1 - n_1$ and $1 - n_2$ such that $\bar{B} \subset D$. Then, by Rellich's Lemma 2.12, it follows that

$$u_1(\cdot\,, d) = u_2(\cdot\,, d) \quad \text{in } \mathbb{R}^3 \setminus \bar{B}$$

for all $d \in \mathbb{S}^2$. Hence $u := u_1 - u_2$ satisfies the boundary conditions

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B \tag{10.15}$$

and the differential equation

$$\Delta u + k^2 n_1 u = k^2 (n_2 - n_1) u_2 \quad \text{in } B.$$

From this and the differential equation for $\tilde{u}_1 := u_1(\cdot\,, \tilde{d})$ we obtain

$$k^2 \tilde{u}_1 u_2 (n_2 - n_1) = \tilde{u}_1 (\Delta u + k^2 n_1 u) = \tilde{u}_1 \Delta u - u \Delta \tilde{u}_1.$$

Now Green's theorem and the boundary values (10.15) imply that

$$\int_B u_1(\cdot\,,\tilde{d})u_2(\cdot\,,d)(n_1 - n_2)\,dx = 0$$

for all $d, \tilde{d} \in \mathbb{S}^2$. In view of Lemma 10.4 from this it follows that

$$\int_B v_1 v_2 (n_1 - n_2)\,dx = 0 \tag{10.16}$$

for all solutions $v_1, v_2 \in H^2(D)$ of $\Delta v_1 + k^2 n_1 v_1 = 0$, $\Delta v_2 + k^2 n_2 v_2 = 0$ in D.

Given $y \in \mathbb{R}^3 \setminus \{0\}$ and $\rho > 0$, we choose vectors $a, b \in \mathbb{R}^3$ such that $\{y, a, b\}$ is an orthogonal basis in $\mathbb{R}^3$ with the properties $|a| = 1$ and $|b|^2 = |y|^2 + \rho^2$. Then for

$$z_1 := y + \rho a + ib, \quad z_2 := y - \rho a - ib$$

we have that

$$z_j \cdot z_j = |\operatorname{Re} z_j|^2 - |\operatorname{Im} z_j|^2 + 2i \operatorname{Re} z_j \cdot \operatorname{Im} z_j = |y|^2 + \rho^2 - |b|^2 = 0$$

and

$$|\operatorname{Re} z_j|^2 = |y|^2 + \rho^2 \geq \rho^2.$$

In (10.16) we now insert the solutions v_1 and v_2 from Lemma 10.3 for the refractive indices n_1 and n_2 and the vectors z_1 and z_2, respectively. In view of $z_1 + z_2 = 2y$ this yields

$$\int_B e^{2i\,y\cdot x}[1 + w_1(x)][1 + w_2(x)][n_1(x) - n_2(x)]\,dx = 0.$$

Passing to the limit $\rho \to \infty$ in this equation, with the help of the inequality (10.7) and $|\operatorname{Re} z_j| \geq \rho$ yields

$$\int_B e^{2i\,y\cdot x}[n_1(x) - n_2(x)]\,dx = 0.$$

Since the latter equation is true for all $y \in \mathbb{R}^3$, by the Fourier integral theorem, we now can conclude that $n_1 = n_2$ in B and the proof is finished. □

The above proof does not work in two dimensions because in this case there is no corresponding decomposition of y into two complex vectors z_1 and z_2 such that $z_1 \cdot z_2 = 0$ and z_1 and z_2 tend to infinity. However, using different methods, it can be shown that Theorem 10.5 is also valid in two dimensions [27].

Although of obvious theoretical interest, a uniqueness theorem such as Theorem 10.5 is often of limited practical interest, other than suggesting the amount of information that is necessary in order to reconstruct the refractive index. The reason for this is that in order to numerically solve the inverse medium problem one usually reduces the problem to a constrained nonlinear optimization problem involving inexact far field data. Hence, the uniqueness question of primary interest is whether or not there exist local minima to this optimization problem and, more specifically, whether or not there exists a unique global minimum. In general, these questions are still unanswered for these optimization problems.

10.3 Iterative Solution Methods

Analogous to the inverse obstacle scattering problem, we can reformulate the inverse medium problem as a nonlinear operator equation. To this end we define the operator $\mathcal{F} : m \mapsto u_\infty$ that maps $m := 1-n$ to the far field pattern u_∞ for plane wave incidence $u^i(x,d) = e^{ik\,x\cdot d}$. Since by Theorem 10.5 we know that m is uniquely determined by a knowledge of $u_\infty(\hat{x},d)$ for all incident and observation directions $d, \hat{x} \in \mathbb{S}^2$, we interpret $\mathcal{F}$ as an operator from $L^2(B)$ into $L^2(\mathbb{S}^2 \times \mathbb{S}^2)$ for a ball B that contains the unknown support of m.

In view of the Lippmann–Schwinger equation (10.4) and the far field representation (10.5) we can write

$$(\mathcal{F}(m))(\hat{x},d) = -\frac{k^2}{4\pi}\int_B e^{-ik\,\hat{x}\cdot y} m(y)u(y,d)\,dy, \quad \hat{x}, d \in \mathbb{S}^2, \tag{10.17}$$

where $u(\cdot,d)$ is the unique solution of

$$u(x,d) + k^2\int_B \Phi(x,y)m(y)u(y,d)\,dy = u^i(x,d), \quad x \in B. \tag{10.18}$$

The representation (10.17) indicates that $\mathcal{F}$ is nonlinear and completely continuous.

From (10.18) it can be seen that the Fréchet derivative $v := u'_m q$ of u with respect to m (in direction q) satisfies the Lippmann–Schwinger equation

$$v(x,d) + k^2\int_B \Phi(x,y)[m(y)v(y,d) + q(y)u(y,d)]\,dy = 0, \quad x \in B. \tag{10.19}$$

From this and (10.17) it follows that the Fréchet derivative of $\mathcal{F}$ is given by

$$(\mathcal{F}'_m q)(\hat{x},d) = -\frac{k^2}{4\pi}\int_B e^{-ik\,\hat{x}\cdot y}[m(y)v(y,d) + q(y)u(y,d)]\,dy, \quad \hat{x}, d \in \mathbb{S}^2,$$

which coincides with the far field pattern of the solution $v(\cdot,d) \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ of (10.19). Hence, we have proven the following theorem.

Theorem 10.6. *The operator $\mathcal{F} : m \mapsto u_\infty$ is Fréchet differentiable. The derivative is given by*

$$\mathcal{F}'_m q = v_\infty$$

where v_∞ is the far field pattern of the radiating solution $v \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ to

$$\Delta v + k^2 nv = -k^2 uq \quad \text{in } \mathbb{R}^3. \tag{10.20}$$

Confirming Theorem 4.21, from the above characterization of the Fréchet derivative it is obvious that $\mathcal{F}_m' : L^2(B) \to L^2(\mathbb{S}^2 \times \mathbb{S}^2)$ is compact. Theorem 10.6 can also be used to establish injectivity of $\mathcal{F}_m'$ in the following theorem (see [127, 144]).

Theorem 10.7. *For piecewise continuous m the operator $\mathcal{F}_m' : L^2(B) \to L^2(\mathbb{S}^2 \times \mathbb{S}^2)$ is injective.*

Proof. Assume that $q \in L^2(B)$ satisfies $\mathcal{F}_m' q = 0$. Then for each $d \in \mathbb{S}^2$ the far field pattern of the solution v of (10.19) vanishes and Rellich's lemma yields $v(\cdot\,, d) = \partial v(\cdot\,, d)/\partial \nu = 0$ on ∂B. Therefore Green's second theorem implies that

$$k^2 \int_B q u(\cdot\,, d) w \, dx = 0$$

for all $d \in \mathbb{S}^2$ and all solutions $w \in H^2(B)$ of $\Delta w + k^2 n w = 0$ in D. In view of Lemma 10.4 this implies

$$\int_B q w \tilde{w} \, dx = 0$$

for all $w, \tilde{w} \in H^2(B)$ satisfying $\Delta w + k^2 n w = 0$ and $\Delta \tilde{w} + k^2 n \tilde{w} = 0$ in D in B. Now the proof can be completed analogous to that of Theorem 10.5. □

Theorems 10.6 and 10.7 provide the theoretical foundation for the application of Newton type iterations such as the Levenberg–Marquardt algorithm and the iteratively regularized Gauss–Newton iteration from Section 4.5 for solving the inverse medium problem. Gutman and Klibanow [112, 113, 114] proposed and analyzed a quasi-Newton scheme where the Fréchet derivative F_m' is kept fixed throughout the iterations and is replaced by F_0' which sort of mimics the Born approximation (see also [127, 182]). For an application of the Levenberg–Marquardt algorithm to the inverse medium problem we refer to Hohage [144]. For a corresponding approach to the electromagnetic inverse medium problem based on the Lippmann–Schwinger equation (9.7) we refer to Hohage and Langer [145, 146].

In view of the Lippmann–Schwinger equation (10.4) and the far field representation (10.5) the inverse medium problem is equivalent to solving the system consisting of the *field equation*

$$u(x, d) + k^2 \int_B \Phi(x, y) m(y) u(y, d) \, dy = u^i(x, d), \quad x \in B, d \in \mathbb{S}^2, \tag{10.21}$$

and the *data equation*

$$-\frac{k^2}{4\pi} \int_B e^{-ik\,\hat{x}\cdot y} m(y) u(y, d) \, dy = u_\infty(\hat{x}, d), \quad \hat{x}, d \in \mathbb{S}^2. \tag{10.22}$$

For a more concise formulation we define the integral operators $T : L^2(B \times \mathbb{S}^2) \to L^2(B \times \mathbb{S}^2)$ and $T_\infty : L^2(B \times \mathbb{S}^2) \to L^2(\mathbb{S}^2 \times \mathbb{S}^2)$ by

$$(Tv)(x, d) := -k^2 \int_B \Phi(x, y) v(y, d) \, dy, \quad x \in B, \; d \in \mathbb{S}^2,$$

and

$$(T_\infty v)(\hat{x}, d) := -\frac{k^2}{4\pi} \int_B e^{-ik\,\hat{x}\cdot y} v(y, d)\, dy, \quad \hat{x}, d \in \mathbb{S}^2,$$

and rewrite the field equation (10.21) as

$$u^i + Tmu = u \tag{10.23}$$

and the data equation (10.22) as

$$T_\infty mu = u_\infty. \tag{10.24}$$

In principle one could try to first solve the ill-posed linear equation (10.24) to determine the source mu from the far field pattern and then solve the nonlinear equation (10.23) to construct the contrast m. Unfortunately this approach is unsatisfactory due to the fact that T_∞ is not injective and has a null space with infinite dimension. In particular, all functions v of the form $v = \Delta w + k^2 w$ for a C^2 function w with compact support contained in B belong to the null space of T_∞ as can be seen from the Green's formula (2.4) applied to w. Hence it is not possible to break up the solution of (10.23) and (10.24) as in a decomposition method and we need to keep them combined. For this we define the cost function

$$\mu(m, u) := \frac{\|u^i + Tmu - u\|^2_{L^2(B\times\mathbb{S}^2)}}{\|u^i\|^2_{L^2(B\times\mathbb{S}^2)}} + \frac{\|u_\infty - T_\infty mu\|^2_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}}{\|u_\infty\|^2_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}} \tag{10.25}$$

and reformulate the inverse medium problem as the optimization problem to minimize μ over the contrast $m \in V$ and the fields $u \in W$ where V and W are appropriately chosen admissible sets. The weights in the cost function are chosen such that the two terms are of the same magnitude.

Since by Theorem 10.5 all incident directions are required, the discrete versions of the optimization problem suffer from a large number of unknowns. Assuming the far field pattern $u_\infty(\hat{x}, d)$ is known for $\hat{x} \in \mathbb{S}^2$, $d \in \{d_1, \dots, d_P\}$, a discrete version of the cost function is given by

$$\mu_P(m, u_1, \dots, u_P) := \sum_{p=1}^{P} \left\{ \frac{\|u^i(\cdot, d_p) + Tmu_p - u_p\|^2_{L^2(B)}}{\|u^i(\cdot, d_p)\|^2_{L^2(B)}} + \frac{\|u_\infty(\cdot, d_p) - T_\infty mu_p\|^2_{L^2(\mathbb{S}^2)}}{\|u_\infty(\cdot, d_p)\|^2_{L^2(\mathbb{S}^2)}} \right\}$$

which has to be minimized over the contrast $m \in V$ and the fields $u_1, \dots, u_P \in W$. One way to reduce the computational complexity is to use a modified conjugate gradient method for this optimization problem as proposed by Kleinman and van den Berg [193, 194]. We also note that multiple frequency data can be incorporated in the cost functionals by integrating or summing in the above definitions over a corresponding range of wave numbers.

In a modified version of this approach van den Berg and Kleinman [320] transformed the Lippmann–Schwinger equation (10.23) into the equation

$$mu^i + mTw = w \tag{10.26}$$

for the contrast sources $w := mu$ and instead of simultaneously updating the contrast m and the fields u the contrast is updated together with the contrast source w. The cost function (10.25) is now changed to

$$\mu(m, w) := \frac{\|mu^i + mTw - w\|^2_{L^2(B\times\mathbb{S}^2)}}{\|u^i\|^2_{L^2(B\times\mathbb{S}^2)}} + \frac{\|u_\infty - T_\infty mu\|^2_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}}{\|u_\infty\|^2_{L^2(\mathbb{S}^2\times\mathbb{S}^2)}}.$$

Via the Lippmann–Schwinger equation (9.7) the above approach for the acoustic inverse medium problem can be adapted to the case of electromagnetic waves.

A different iterative algorithm for the inverse medium problem was developed by Natterer, Vögeler and Wübbeling [259, 323]. Their method is based on solving the direct scattering problem by a marching scheme in space and the inverse problem by Kaczmarz's algorithm adopting ideas from computerized tomography.

10.4 Decomposition Methods

In this section we shall discuss a decomposition method for solving the inverse medium problem for acoustic waves due to Colton and Monk [76] (see also Colton and Kirsch [59]) that is based on an application of Theorem 8.10. This method (as well as the modified method to be discussed below) has the advantage of being able to increase the number of incident fields without increasing the cost of solving the inverse problem. We shall call this approach the *dual space method* since it requires the determination of a function $g_{pq} \in L^2(\mathbb{S}^2)$ such that

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g_{pq}(\hat{x})\, ds(\hat{x}) = \frac{i^{p-1}}{k}\, Y_p^q(d), \quad d \in \mathbb{S}^2, \tag{10.27}$$

i.e., the determination of a linear functional in the dual space of $L^2(\mathbb{S}^2)$ having prescribed values on the class $\mathcal{F} := \{u_\infty(\cdot, d_n) : n = 1, 2, 3, \dots\}$ of far field patterns where $\{d_n : n = 1, 2, 3, \dots\}$ is a countable dense set of vectors on the unit sphere $\mathbb{S}^2$. We shall assume throughout this section that $\operatorname{Im} n(x) > 0$ for all $x \in D := \{x \in \mathbb{R}^3 : m(x) \neq 0\}$ where again $m := 1 - n$. From Theorem 8.12 we see that this implies that, if a solution exists to the integral equation (10.27), then this solution is unique. As in Chapter 8, we shall, for the sake of simplicity, always assume that D is connected with a connected C^2 boundary ∂D and D contains the origin.

We shall begin our analysis by giving a different proof of the "if" part of Theorem 8.10. In particular, assume that there exist functions $v, w \in H^2(D)$ which satisfy the interior transmission problem

$$\Delta w + k^2 n(x) w = 0, \quad \Delta v + k^2 v = 0 \quad \text{in } D \tag{10.28}$$

with the transmission condition

$$w - v = u_p^q, \quad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \frac{\partial u_p^q}{\partial \nu} \quad \text{on } \partial D \tag{10.29}$$

where

$$u_p^q(x) := h_p^{(1)}(k|x|)\, Y_p^q(\hat{x})$$

denotes a radiating spherical wave function and where ν is the unit outward normal to ∂D. If we further assume that v is a Herglotz wave function

$$v(x) = \int_{\mathbb{S}^2} e^{-ik\, x\cdot d}\, g_{pq}(d)\, ds(d), \quad x \in \mathbb{R}^3, \tag{10.30}$$

where $g_{pq} \in L^2(\mathbb{S}^2)$, then from the representation (2.14) for the far field pattern, Green's theorem and the radiation condition we have for fixed k and every $d \in \mathbb{S}^2$ that

$$\begin{aligned}
\int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g_{pq}(\hat{x})\, ds(\hat{x}) &= \frac{1}{4\pi} \int_{\partial D} \left(u \,\frac{\partial v}{\partial \nu} - v\, \frac{\partial u}{\partial \nu} \right) ds \\
= \frac{1}{4\pi} \int_{\partial D} \left(u \,\frac{\partial w}{\partial \nu} - w\, \frac{\partial u}{\partial \nu} \right) ds &- \frac{1}{4\pi} \int_{\partial D} \left(u \,\frac{\partial u_p^q}{\partial \nu} - u_p^q\, \frac{\partial u}{\partial \nu} \right) ds \\
= -\frac{1}{4\pi} \int_{\partial D} \left(e^{ik\, x\cdot d}\, \frac{\partial u_p^q}{\partial \nu}(x) - u_p^q(x)\, \frac{\partial e^{ik\, x\cdot d}}{\partial \nu(x)} \right) ds(x) &= \frac{i^{p-1}}{k}\, Y_p^q(d).
\end{aligned} \tag{10.31}$$

From (10.31) we see that the identity (10.27) is approximately satisfied if there exists a Herglotz wave function v such that the Cauchy data (10.29) for w is approximately satisfied in $L^2(\partial D)$.

The dual space method for solving the inverse acoustic medium problem is to determine $g_{pq} \in L^2(\mathbb{S}^2)$ such that (10.27) is satisfied and, given v defined by (10.30), to determine w and n from the overdetermined boundary value problem (10.28), (10.29). This is done for a finite set of values of k and integers p and q with $q = -p, \dots, p$. To reformulate this scheme as an optimization problem, we define the operator T_m as in (10.11) and the operator $F_k : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ by

$$(F_k g)(d) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d, k) g(\hat{x})\, ds(\hat{x}), \quad d \in \mathbb{S}^2. \tag{10.32}$$

Then, from Green's formula (2.4) we can rewrite the boundary value problem (10.28), (10.29) in the form

$$\begin{aligned} w_{pq} &= v_{pq} - k^2 T_m w_{pq} \quad \text{in } B, \\ -k^2 T_m w_{pq} &= u_p^q \quad \text{on } \partial B \end{aligned} \tag{10.33}$$

where $v = v_{pq}$ and $w = w_{pq}$. Note, that by the uniqueness for the exterior Dirichlet problem for the Helmholtz equation the boundary condition in (10.33) ensures that $k^2 T_m w_{pq} + u_p^q = 0$ first in $\mathbb{R}^3 \setminus B$ and then, by unique continuation $k^2 T_m w_{pq} + u_p^q = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. This implies that both boundary conditions in (10.29) are satisfied.

The dual space method for solving the inverse medium problem can then be formulated as the optimization problem

$$\min_{\substack{g_{pq} \in W \\ w_{pq} \in U_1 \\ m \in U_2}} \left\{ \sum_{p=1}^{P} \sum_{q=-p}^{p} \sum_{r=1}^{R} \sum_{s=1}^{S} \left| (F_{k_s} g_{pq})(d_r) - \frac{i^{p-1}}{k_s} \, Y_p^q(d_r) \right|^2 \right.$$
$$\left. + \sum_{p=1}^{P} \sum_{q=-p}^{p} \sum_{s=1}^{S} \left[\| w_{pq} + k_s^2 T_m w_{pq} - v_{pq} \|^2_{L^2(B)} + \| k_s^2 T_m w_{pq} + u_p^q \|^2_{L^2(\partial B)} \right] \right\}, \tag{10.34}$$

noting that v_{pq}, w_{pq}, u_p^q and the operator T_m all depend on $k = k_s$, $s = 1, \dots S$. Here W, U_1 and U_2 are appropriate (possibly weakly) compact sets. We shall again not dwell on the details of the optimization scheme (10.34) except to note that if exact far field data is used and W, U_1 and U_2 are large enough then the minimum value of the cost functional in (10.34) will be zero, provided the approximation property stated after (10.31) is valid. We now turn our attention to showing that this approximation property is indeed true. In the analysis which follows, we shall occasionally apply Green's theorem to functions in $C^2(D)$ having L^2 Cauchy data. When doing so, we shall always be implicitly appealing to a limiting argument involving parallel surfaces (c.f. [179]).

For the Herglotz wave function

$$v_g(y) = \int_{\mathbb{S}^2} e^{-ik\, y \cdot \hat{x}} g(\hat{x}) \, ds(\hat{x}), \quad y \in \mathbb{R}^3, \tag{10.35}$$

with kernel g and

$$V(D) := \left\{ w \in H^2(D) : \Delta w + k^2 n(x) w = 0 \text{ in } D \right\},$$

we define the subspace $W \subset L^2(\partial D) \times L^2(\partial D)$ by

$$W := \left\{ \left(v_g - w, \frac{\partial}{\partial \nu}(v_g - w) \right) : g \in L^2(\mathbb{S}^2),\, w \in V(D) \right\}.$$

The desired approximation property is valid provided W is dense in $L^2(\partial D)\times L^2(\partial D)$. To this end, we have the following theorem due to Colton and Kirsch [59]. This result has been extended to the case of the Maxwell equations by Hähner [125].

Theorem 10.8. *Suppose* $\operatorname{Im} n(x) > 0$ *for* $x \in D$*. Then the subspace* W *is dense in* $L^2(\partial D) \times L^2(\partial D)$.

Proof. Let $\varphi, \psi \in L^2(\partial D)$ be such that

$$\int_{\partial D} \left\{ \varphi(v_g - w) + \psi \, \frac{\partial}{\partial \nu} (v_g - w) \right\} ds = 0 \tag{10.36}$$

for all $g \in L^2(\mathbb{S}^2)$, $w \in V(D)$. We first set $w = 0$ in (10.36). Then from (10.35) and (10.36) we have that

$$\int_{\mathbb{S}^2} g(\hat{x}) \int_{\partial D} \left\{ \varphi(y)\, e^{-ik\, y\cdot\hat{x}} + \psi(y)\, \frac{\partial e^{-ik\, y\cdot\hat{x}}}{\partial \nu(y)} \right\} ds(y)\, ds(\hat{x}) = 0$$

for all $g \in L^2(\mathbb{S}^2)$ and hence

$$\int_{\partial D} \left\{ \varphi(y)\, e^{-ik\, y\cdot\hat{x}} + \psi(y)\, \frac{\partial e^{-ik\, y\cdot\hat{x}}}{\partial \nu(y)} \right\} ds(y) = 0$$

for $\hat{x} \in \mathbb{S}^2$. Therefore, the far field pattern of the combined single- and double-layer potential

$$u(x) := \int_{\partial D} \left\{ \Phi(x,y)\varphi(y) + \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, \psi(y) \right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D,$$

vanishes, i.e., from Theorem 2.14 we can conclude that $u(x) = 0$ for $x \in \mathbb{R}^3 \setminus \bar{D}$. Since $\varphi, \psi \in L^2(\partial D)$, we can apply the generalized jump relations (3.20)–(3.23) to conclude that

$$\varphi = \frac{\partial u_-}{\partial \nu}\,, \quad \psi = -u_- \quad \text{on } \partial D \tag{10.37}$$

and hence $u \in H^{3/2}(D)$ (c.f. [179]). If we now set $g = 0$ in (10.36), we see from Green's theorem that

$$k^2 \int_D muw\, dx = \int_{\partial D} \left(u\, \frac{\partial w}{\partial \nu} - w\, \frac{\partial u}{\partial \nu} \right) ds = 0 \tag{10.38}$$

for all $w \in V(D)$.

Now consider the boundary value problem

$$\begin{aligned} \Delta v + k^2 n(x) v &= k^2 m(x) u \quad \text{in } D, \\ v &= 0 \quad \text{on } \partial D. \end{aligned} \tag{10.39}$$

Since $\operatorname{Im} n(x) > 0$ for $x \in D$, this problem has a unique solution $v \in H^2(D)$ (c.f. [336]). Then from Green's theorem and (10.38) we have that

$$\int_{\partial D} w \, \frac{\partial v}{\partial \nu} \, ds = \int_D w(\Delta v + k^2 n v) \, dx = k^2 \int_D m u w \, dx = 0 \tag{10.40}$$

for all $w \in V(D)$. Note that from the trace theorem we have that the boundary integral in (10.40) is well defined. Since $\operatorname{Im} n(x) > 0$ for $x \in D$, the boundary values of functions $w \in V(D)$ are dense in $L^2(\partial D)$ and hence we can conclude from (10.40) that $\partial v/\partial \nu = 0$ on ∂D. From (10.39) and Green's theorems we now see that, since $u \in H^{3/2}(D)$ and $\Delta u + k^2 u = 0$ in D, we have

$$\begin{aligned} 0 &= \int_D u(\Delta \bar{v} + k^2 \bar{v}) \, dx = \frac{1}{k^2} \int_D \frac{1}{m} \, (\Delta v + k^2 n v)(\Delta \bar{v} + k^2 \bar{v}) \, dx \\ &= \frac{1}{k^2} \int_D \left(\frac{1}{m} \, \left| \Delta v + k^2 v \right|^2 - k^2 v \, (\Delta \bar{v} + k^2 \bar{v}) \right) dx \\ &= \frac{1}{k^2} \int_D \left(\frac{1}{m} \, \left| \Delta v + k^2 v \right|^2 - k^4 |v|^2 + k^2 |\operatorname{grad} v|^2 \right) dx \end{aligned}$$

and, taking the imaginary part, we see that

$$\int_D \frac{\operatorname{Im} n}{|m|^2} \, \left| \Delta v + k^2 v \right|^2 dx = 0.$$

Hence, $\Delta v + k^2 v = 0$ in D. From the trace theorem we have that Theorem 2.1 remains valid in the present context and therefore since the Cauchy data for v vanish on ∂D we have that $v = 0$ in D. Hence, $u = 0$ in D and thus, from (10.37), $\varphi = \psi = 0$. This completes the proof of the theorem. □

Due to the fact that the far field operator F_k is injective with dense range (provided k is not a transmission eigenvalue) the dual space method described above can be broken up into two problems, i.e., one first solves the linear ill-posed problem

$$F_k \, g_{pq} = \frac{i^{p-1}}{k} \, Y_p^q,$$

for $1 \le p \le P$, $-p \le q \le p$, constructs the Herglotz wave function v_{pq} with kernel g_{pq} and then uses nonlinear optimization methods to find a refractive index such that (10.33) is optimally satisfied. (Note that this decomposition was not possible for the iterative approach of the previous section.) The integer P is chosen by the criteria that P is the order of the spherical harmonic in the last numerically meaningful term in the spherical harmonic expansion of the (noisy) far field pattern. Following this procedure can dramatically reduce the number of unknowns in the nonlinear optimization step for determining the refractive index. For a mathematical justification for breaking up the dual space method into two separate problems in this way, we refer the reader to [45].

In addition to breaking up the dual space method into two separate problems, in practice one often needs to consider point sources as incident fields and near field scattering data. An example of this is the use of microwaves to detect leukemia in the bone marrow of the upper part of the lower leg. The possibility of using such an approach arises from the fact that the presence of cancer cells can cause the refractive index in the bone marrow to change significantly. However, in order for the incident fields to penetrate the body there must be a good impedance match between the leg and the host medium, e.g. the leg should be immersed in water. Since water is a conductor, point sources and near field measurements are more appropriate than plane wave incident fields and far field measurements. A further consideration that needs to be addressed is that the human body is dispersive with a poorly understood dispersion relation, i.e., the refractive index depends on the wave number and this functional relationship is not precisely known. This means that the dual space method, as modified for point sources and near field data, should be restricted to a fixed value of k. For details and numerical examples of the above modifications of the dual space method to the detection of leukemia the reader is referred to [82].

In order for the dual space method presented in this section to work, we have required that $\operatorname{Im} n(x) > 0$ for $x \in D$. In particular, if this is not the case, or $\operatorname{Im} n$ is small, the presence of transmission eigenvalues can contaminate the method to the extent of destroying its ability to reconstruct the refractive index. Numerical examples using this method for solving the inverse medium problem can be found in [76, 79] and Section 10.7 of this book.

As just mentioned, a disadvantage of the dual space method presented above is that the presence of transmission eigenvalues can lead to numerical instabilities and poor reconstructions of the refractive index. We shall now introduce a modified version of the identity (10.27) that leads to a method for solving the acoustic inverse medium problem that avoids this difficulty and that in the sequel we will refer to as the *modified dual space method*.

We begin by considering the following auxiliary problem. Let $\lambda \geq 0$ and let $h \in C^2(\mathbb{R}^3 \setminus \bar{B}) \cap C^1(\mathbb{R}^3 \setminus B)$ be the solution of the exterior impedance boundary value problem

$$\Delta h + k^2 h = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{B}, \tag{10.41}$$

$$h(x) = e^{ik\, x\cdot d} + h^s(x), \tag{10.42}$$

$$\frac{\partial h}{\partial \nu} + ik\lambda h = 0 \quad \text{on } \partial B, \tag{10.43}$$

$$\lim_{r\to\infty} r\left(\frac{\partial h^s}{\partial r} - ikh^s\right) = 0 \tag{10.44}$$

where again B is an open ball centered at the origin containing the support of m and where ν denotes the exterior normal to ∂B. (Domains other than balls could also be used provided they contain the support of m.) The uniqueness of a solution to (10.41)–(10.44) follows from Theorem 2.13 whereas the existence of a solution follows by seeking a solution in the form (3.29) and imitating the proof of Theorem

3.12 (See also the remarks after Theorem 3.12). We note that by the analytic Riesz–Fredholm Theorem 8.26 the integral equation obtained from the use of (3.29) is uniquely solvable for all $\lambda \in \mathbb{C}$ with the possible exception of a countable set of values of λ, i.e., there exists a solution to (10.41)–(10.44) for a range of λ where the condition $\lambda \geq 0$ is violated. Finally, from the representation (3.29), we see that h^s has the asymptotic behavior

$$h^s(x) = \frac{e^{ik|x|}}{|x|}\, h_\infty(\hat{x}, d) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,$$

and from (3.38) we see that the far field pattern h_∞ satisfies the reciprocity relation

$$h_\infty(\hat{x}, d) = h_\infty(-d, -\hat{x}), \quad \hat{x}, d \in \mathbb{S}^2. \tag{10.45}$$

Now let u_∞ be the far field pattern of the scattering problem (10.1)–(10.3) for acoustic waves in an inhomogeneous medium and consider the problem of when there exists a function $g_{pq} \in L^2(\mathbb{S}^2)$ such that

$$\int_{\mathbb{S}^2} [u_\infty(\hat{x}, d) - h_\infty(\hat{x}, d)]\, g_{pq}(\hat{x})\, ds(\hat{x}) = \frac{i^{p-1}}{k}\, Y_p^q(d) \tag{10.46}$$

for all d in a countable dense set of vectors on the unit sphere, i.e., by continuity, for all $d \in \mathbb{S}^2$. By the Reciprocity Theorem 8.8 and (10.45) we see that (10.46) is equivalent to the identity

$$\int_{\mathbb{S}^2} [u_\infty(\hat{x}, d) - h_\infty(\hat{x}, d)]\, g_{pq}(-d)\, ds(d) = \frac{(-i)^{p+1}}{k}\, Y_p^q(\hat{x}) \tag{10.47}$$

for all $\hat{x} \in \mathbb{S}^2$. If we define the Herglotz wave function w^i by

$$w^i(x) := \int_{\mathbb{S}^2} e^{-ik\, x\cdot d} g_{pq}(d)\, ds(d) = \int_{\mathbb{S}^2} e^{ik\, x\cdot d} g_{pq}(-d)\, ds(d), \quad x \in \mathbb{R}^3, \tag{10.48}$$

then

$$w_\infty(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d)\, g_{pq}(-d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2,$$

is the far field pattern corresponding to the scattering problem

$$\Delta w + k^2 n(x) w = 0 \quad \text{in } \mathbb{R}^3,$$

$$w(x) = w^i(x) + w^s(x),$$

$$\lim_{r\to\infty} r\left(\frac{\partial w^s}{\partial r} - ikw^s\right) = 0,$$

and

$$v_\infty(\hat{x}) := \int_{\mathbb{S}^2} h_\infty(\hat{x}, d)\, g_{pq}(-d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2,$$

is the far field pattern corresponding to the scattering problem

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{B},$$

$$v(x) = w^i(x) + v^s(x),$$

$$\frac{\partial v}{\partial \nu} + ik\lambda v = 0 \quad \text{on } \partial B,$$

$$\lim_{r\to\infty} r\left(\frac{\partial v^s}{\partial r} - ikv^s\right) = 0.$$

Hence, from (2.42), (10.47) and Theorem 2.14 we can conclude that

$$w^s(x) - v^s(x) = h_p^{(1)}(k|x|)\, Y_p^q(\hat{x}), \quad x \in \mathbb{R}^3 \setminus \bar{B},$$

i.e., w satisfies the boundary value problem

$$\Delta w + k^2 n(x) w = 0 \quad \text{in } B, \tag{10.49}$$

$$w(x) = w^i(x) + w^s(x), \tag{10.50}$$

$$\left(\frac{\partial}{\partial \nu} + ik\lambda\right)\left(w - u_p^q\right) = 0 \quad \text{on } \partial B, \tag{10.51}$$

$$\lim_{r\to\infty} r\left(\frac{\partial w^s}{\partial r} - ikw^s\right) = 0 \tag{10.52}$$

uniformly for all directions. Again, as in Section 10.4, we have written

$$u_p^q(x) := h_p^{(1)}(k|x|)\, Y_p^q(\hat{x})$$

for the radiating spherical wave function.

The boundary value problem (10.49)–(10.52) can be understood as the problem of first solving the interior impedance problem (10.49), (10.51) and then decomposing the solution in the form (10.50) where w^i satisfies the Helmholtz equation in B and w^s is defined for all of $\mathbb{R}^3$, satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{B}$ and the radiation condition (10.52). Note that for the identity (10.46) to be valid, w^i must be a Herglotz wave function with Herglotz kernel g_{pq}. In particular, from the above analysis we have the following theorem ([77, 78]).

Theorem 10.9. *Assume there exists a solution $w \in H^2_{\rm loc}(\mathbb{R}^3 \setminus \bar{B})$ to the interior impedance problem (10.49), (10.51) such that w has the decomposition (10.50) with w^i and w^s as described in the paragraph above. Then there exists $g_{pq} \in L^2(\mathbb{S}^2)$ such that (10.46) is valid if and only if w^i is a Herglotz wave function with Herglotz kernel g_{pq}.*

We now turn our attention to when there exists a solution to (10.49), (10.51) having the decomposition (10.50). To establish the existence of a unique solution to (10.49), (10.51) we make use of potential theory. In particular, we recall from Section 3.1 the single-layer operator $S : C(\partial B) \to C^{0,\alpha}(\partial B)$ and the normal derivative operator $K' : C(\partial B) \to C(\partial B)$ and from Section 8.2 the volume potential $T_m : C(\bar{B}) \to C(\bar{B})$. We can now prove the following theorem.

Theorem 10.10. *Suppose that $\lambda = 0$ if $\operatorname{Im} n(x) > 0$ for some $x \in B$ and $\lambda > 0$ if $\operatorname{Im} n(x) = 0$ for all $x \in B$. Then there exists a unique solution $w \in H^2(B)$ to the impedance problem (10.49), (10.51) having the decomposition (10.50) where $w^i \in H^2(B)$ is a solution of the Helmholtz equation in B and $w^s \in H^2(\mathbb{R}^3)$ satisfies the radiation condition (10.52).*

Proof. We first establish uniqueness for the boundary value problem (10.49), (10.51). Assume that w is a solution of (10.49) satisfying homogeneous impedance boundary data on ∂B. Then, by Green's theorem, we have that

$$k\lambda \int_{\partial B} |w|^2 ds = \operatorname{Im} \int_{\partial B} w \, \frac{\partial \bar{w}}{\partial \nu} \, ds = -k^2 \operatorname{Im} \int_B \bar{n} \, |w|^2 dx \tag{10.53}$$

where w and $\partial w/\partial \nu$ are to be interpreted in the sense of the trace theorem. If $\operatorname{Im} n(x) > 0$ for some $x \in B$, then by assumption $\lambda = 0$ and we can conclude from (10.53) that $w(x) = 0$ for those points $x \in B$ where $\operatorname{Im} n(x) > 0$. Hence, by the Unique Continuation Theorem 8.6, we have $w(x) = 0$ for $x \in B$. On the other hand, if $\operatorname{Im} n(x) = 0$ for all $x \in B$ then $\lambda > 0$ and (10.53) implies that $w = 0$ on ∂B. Since w satisfies homogeneous impedance boundary data, we have that $\partial w/\partial \nu = 0$ on ∂B and Green's formula (2.4) now tells us that $w + k^2 T_m w = 0$ in B. Hence, we see from the invertibility of $I + k^2 T_m$ in $C(\bar{B})$ that $w(x) = 0$ for $x \in B$.

In order to establish existence for (10.49), (10.51), we look for a solution in the form

$$w(x) = \int_{\partial B} \Phi(x, y)\varphi(y) \, ds(y) - k^2 \int_B \Phi(x, y)m(y)\psi(y) \, dy, \quad x \in B, \tag{10.54}$$

where the densities $\varphi \in C(\partial B)$ and $\psi \in C(\bar{B})$ are assumed to satisfy the two integral equations

$$\psi - \tilde{S}\varphi + k^2 T_m \psi = 0 \tag{10.55}$$

and

$$\varphi + (K' + ik\lambda S)\varphi - 2k^2 T_{m,\lambda} \psi = 2f \tag{10.56}$$

with

$$f := \frac{\partial u_p^q}{\partial \nu} + ik\lambda u_p^q \quad \text{on } \partial B.$$

Here we define $\tilde{S} : C(\partial B) \to C(\bar{B})$ by

$$(\tilde{S}\varphi)(x) := \int_{\partial B} \Phi(x, y)\varphi(y)\, ds(y), \quad x \in \bar{B},$$

and $T_{m,\lambda} : C(\bar{B}) \to C(\partial B)$ by

$$(T_{m,\lambda}\psi)(x) := \int_B \left\{ \frac{\partial \Phi(x, y)}{\partial \nu(x)} + ik\lambda \Phi(x, y) \right\} m(y)\psi(y)\, dy, \quad x \in \partial B.$$

By the regularity of f, from Theorems 3.4 and 8.2 we have that for a continuous solution of (10.55), (10.56) we automatically have $\varphi \in C^{0,\alpha}(\partial B)$ and $\psi \in H^2(B)$. Hence, defining w by (10.54), we see that $w \in H^2(B)$, i.e., w has the required regularity. By Theorem 8.2, the integral equation (10.55) ensures that w solves the differential equation (10.49) and from the jump relations of Theorem 3.1 we see that the integral equation (10.56) implies that the boundary condition (10.51) is satisfied.

All integral operators in the system (10.55), (10.56) clearly have weakly singular kernels and therefore they are compact. Hence, by the Riesz–Fredholm theory, to show the existence of a unique solution of (10.55), (10.56) we must show that the only solution of the homogeneous problem is the trivial solution $\varphi = 0$, $\psi = 0$. If φ, ψ is a solution to (10.55), (10.56) with $f = 0$, then w defined by (10.54) is a solution to the homogeneous problem (10.49), (10.51) and therefore by uniqueness we have $w = 0$ in B. Applying $\Delta + k^2$ to both sides of (10.54) (with $w = 0$) now shows that $m\psi = 0$ and $\tilde{S}\varphi = 0$ in B. Hence, from (10.55), we have that $\psi = 0$ in B. Now, making use of the uniqueness for the exterior Dirichlet problem (Theorem 3.9) and the jump relations of Theorem 3.1, we can also conclude that $\varphi = 0$.

The decomposition (10.50) holds where $w^i = \tilde{S}\varphi$ is in $H^2(B)$ and $w^s = -k^2 T_m \psi$ is in $H^2_{\text{loc}}(\mathbb{R}^3)$. We have already shown above that the solution of (10.49), (10.51) is unique and all that remains is to show the uniqueness of the decomposition (10.50). But this follows from the fact that entire solutions of the Helmholtz equation satisfying the radiation condition must be identically zero (c.f. p. 21).

Corollary 10.11 *In the decomposition (10.50), w^i can be approximated in $C^1(\bar{B})$ by a Herglotz wave function.*

Proof. From the regularity properties of surface potentials we see that if $\psi \in C(\bar{B})$ and $\varphi \in C(\partial B)$ is a solution of (10.55), (10.56) then $\varphi \in C^{2,\alpha}(\partial B)$. The corollary now follows by approximating the surface density φ by a linear combination of spherical harmonics. □

With Theorem 10.10 and Corollary 10.11 at our disposal, we can now formulate a decomposition method for solving the inverse scattering problem such that the problem of transmission eigenvalues is avoided. Indeed, using the same notation as in (10.34) and defining $F_k^\lambda : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ by

$$(F_k^\lambda g)(d) := \int_{\mathbb{S}^2} h_\infty(\hat{x}, d, k)\, g(\hat{x})\, ds(\hat{x}), \quad d \in \mathbb{S}^2, \tag{10.57}$$

we can formulate the optimization problem

$$\min_{\substack{g_{pq}\in W\\ w_{pq}\in U_1\\ m\in U_2}} \left\{ \sum_{p=1}^{P}\sum_{q=-p}^{p}\sum_{r=1}^{R}\sum_{s=1}^{S} \left| ((F_{k_s} - F^{\lambda}_{k_s})g_{pq})(d_r) + \frac{i^{p+1}}{k_s}\, Y_p^q(d_r) \right|^2 \right.$$
$$+ \sum_{p=1}^{P}\sum_{q=-p}^{p}\sum_{s=1}^{S} \|w_{pq} + k_s^2 T_m w_{pq} - v_{pq}\|^2_{L^2(B)} \tag{10.58}$$
$$\left. + \sum_{p=1}^{P}\sum_{q=-p}^{p}\sum_{s=1}^{S} \left\| \left(\frac{\partial}{\partial r} + i\lambda\right)(v_{pq} - k_s^2 T_m w_{pq} + u_p^q) \right\|^2_{L^2(\partial B)} \right\}.$$

From Theorem 10.10 and Corollary 10.11 we see that if exact far field data is used the minimum value of the cost functional in (10.58) will be zero provided W, U_1 and U_2 are large enough. Numerical examples using this method for solving the inverse medium problem can be found in [77, 78] and Section 10.7 of this book.

There are other decomposition methods for the case when $\operatorname{Im} n = 0$ than the one presented in this section and for these alternate methods the reader is referred to [56, 80, 81].

10.5 Sampling Methods and Transmission Eigenvalues

The sampling methods obtained in Section 5.6 for determining the shape of a sound-soft obstacle from a knowledge of the far field pattern of the scattered wave can be extended to the problem of determining the support D of an inhomogeneous medium from the far field data corresponding to (10.1)–(10.3) (see [32, 184, 186]). It is also possible to use this approach to obtain qualitative information about the index of refraction from a knowledge of the transmission eigenvalues associated with this problem [39, 86]. This section is devoted to the development of this set of ideas where we assume that D is bounded with a connected exterior $\mathbb{R}^3 \setminus \bar{D}$ and has C^2 boundary ∂D and that n is piecewise continuous in $\mathbb{R}^3$ such that $n(x) > 1$ for $x \in \bar{D}$ and $n(x) = 1$ for $x \in \mathbb{R}^3 \setminus \bar{D}$.

We first derive the factorization method for inhomogeneous media. To this end, following [182], we rewrite (10.1) as

$$\Delta u^s + k^2 n u^s = k^2(1-n)u^i = k^2 m u^i \quad \text{in } \mathbb{R}^3 \tag{10.59}$$

where $u = u^i + u^s$ and $m := 1 - n$. Then, more generally, we consider

$$\Delta u^s + k^2 n u^s = m f \quad \text{in } \mathbb{R}^3 \tag{10.60}$$

where $f \in L^2(D)$ and u^s satisfies the Sommerfield radiation condition (10.3). The existence of a unique solution u^s to (10.60) in the space $H^2_{\mathrm{loc}}(\mathbb{R}^3)$ follows from the existence of a unique solution to the Lippmann–Schwinger equation (8.13) in $L^2(D)$. This follows in the same way as in Theorem 8.7 by noting that from Theorem 8.2 and the Rellich selection theorem the integral operator in (8.13) is compact in $L^2(D)$. We can now define the operator $G : L^2(D) \to L^2(\mathbb{S}^2)$ which maps $f \in L^2(D)$ onto the far field pattern of the solution u^s of (10.60). We again let $u_\infty \in L^2(\mathbb{S}^2)$ be the far field pattern associated with the scattering problem (10.1)–(10.3) and recall the definition of the far field operator $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ given by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2.$$

The following factorization theorem is fundamental.

Theorem 10.12. *Let $G : L^2(D) \to L^2(\mathbb{S}^2)$ be defined as above and let $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ be the far field operator associated with the scattering problem (10.1)–(10.3). Then*

$$F = 4\pi k^2 G S^* G^* \tag{10.61}$$

where S^* is the adjoint of $S : L^2(D) \to L^2(D)$ defined by

$$(S\psi)(x) := -\frac{\psi(x)}{m(x)} - k^2 \int_D \Phi(x, y)\psi(y)\, dy, \quad x \in D, \tag{10.62}$$

and $G^* : L^2(\mathbb{S}^2) \to L^2(D)$ is the adjoint of G.

Proof. From (10.59) and the definition of G we have that $u_\infty = k^2 G u^i$. We now define the Herglotz operator $H : L^2(\mathbb{S}^2) \to L^2(D)$ by

$$(Hg)(x) := \int_{\mathbb{S}^2} e^{ikx\cdot d} g(d)\, ds(d), \quad x \in D, \tag{10.63}$$

and note that by the analogue of Lemma 3.20 for medium scattering Fg is the far field pattern corresponding to the incident field Hg, that is,

$$F = k^2 GH.$$

The adjoint H^* of H is given by

$$(H^*\psi)(\hat{x}) = \int_D e^{-ik\hat{x}\cdot y} \psi(y)\, dy, \quad \hat{x} \in \mathbb{S}^2,$$

and hence $H^*\psi = 4\pi w_\infty$ where w_∞ is he far field pattern of

$$w(x) := \int_D \Phi(x, y)\psi(y)\, dy, \quad x \in \mathbb{R}^3.$$

But

$$\Delta w + k^2 n w = -m\left(\frac{\psi}{m} + k^2 w\right)$$

and hence

$$H^*\psi = 4\pi w_\infty = -4\pi G\left(\frac{\psi}{m} + k^2 w\right) = 4\pi G S\psi,$$

that is, $H^* = 4\pi G S$. Thus

$$H = 4\pi S^* G^*$$

and since we have previously shown that $F = k^2 G H$, the theorem follows. □

The next step in the derivation of the factorization method for inhomogeneous media is to characterize D in terms of the range of G.

Lemma 10.13 *For $z \in \mathbb{R}^3$ let*

$$\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi}\, e^{-ik\,\hat{x}\cdot z}$$

be the far field of the fundamental solution $\Phi(\cdot, z)$. Then $z \in D$ if and only if $\Phi_\infty(\cdot, z)$ is in the range $G(L^2(D))$ of G.

Proof. First let $z \in D$ and choose ε such that $B_\varepsilon := \{x \in \mathbb{R}^3 : |x - z| < \varepsilon\} \subset D$. Choose $\rho \in C^\infty(\mathbb{R}^3)$ such that $\rho(x) = 0$ for $|x - z| \le \varepsilon/2$ and $\rho(x) = 1$ for $|x - z| \ge \varepsilon$ and set $v(x) := \rho(x)\Phi(x, z)$ for $x \in \mathbb{R}^3$. Then $v \in C^\infty(\mathbb{R}^3)$ and has $\Phi_\infty(\cdot, z)$ as its far field pattern. Hence $\Phi_\infty(\cdot, z) = Gf$ where

$$f := \frac{1}{m}\left(\Delta v + k^2 n v\right) \in L^2(D).$$

Now assume that $z \notin D$ and that $\Phi_\infty(\cdot, z) = Gf$ for some $f \in L^2(D)$. We will show that this leads to a contradiction. Let u^s be the radiating solution of (10.60). Then since $\Phi_\infty(\cdot, z)$ is the far field pattern of $\Phi(\cdot, z)$ and Gf is the far field pattern of u^s, by Rellich's lemma we have that $\Phi(\cdot, z) = u^s$ in the exterior of $D \cup \{z\}$. If $z \notin \bar{D}$ then this is a contradiction since u^s is a smooth function in a neighborhood of z but $\Phi(\cdot, z)$ is singular at z. If $z \in \partial D$, let $C_0 \subset \mathbb{R}^3$ be an open truncated cone with vertex at z such that $C_0 \cap D = \emptyset$. Then $\Phi(\cdot, z) \notin H^1(C_0)$ but $u^s \in H^1(C_0)$ and we are again led to a contradiction. □

To proceed further we need to collect some properties of the operator S defined by (10.62).

Theorem 10.14. *Let $S : L^2(D) \to L^2(D)$ be defined by (10.62) and let $S_0 : L^2(D) \to L^2(D)$ be given by*

$$S_0\psi := -\frac{\psi}{m}\,.$$

Then the following statements are true:

1. S_0 is bounded, self-adjoint and satisfies

$$(S_0\psi, \psi)_{L^2(D)} \geq \frac{1}{\|m\|_\infty} \|\psi\|^2_{L^2(D)}, \quad \psi \in L^2(D). \tag{10.64}$$

2. $S - S_0 : L^2(D) \to L^2(D)$ is compact.

3. S is an isomorphism from $L^2(D)$ onto $L^2(D)$.

4. $\operatorname{Im}(S\psi, \psi)_{L^2(D} \leq 0$ for all $\psi \in L^2(D)$ with strict inequality holding for all $\psi \in \overline{G^*(L^2(D))}$ with $\psi \neq 0$.

Proof. 1. This follows immediately since m is real valued and bounded away from zero in $\bar{D}$.

2. This follows from Theorem 8.2 and the compact embedding of $H^2(D)$ into $L^2(D)$.

3. From the first two statements, it suffices to show that S is injective. Suppose $S\psi = 0$. Then setting $\varphi = -\psi/m$ we see that φ satisfies the homogeneous Lippmann–Schwinger equation and hence by Theorem 8.7 we have that $\varphi = 0$ and hence $\psi = 0$.

4. Let $\psi \in L^2(D)$ and define $f \in L^2(D)$ by

$$f(x) := \psi(x) + k^2 m(x) \int_D \Phi(x, y)\psi(y)\, dy, \quad x \in D.$$

Then $S\psi = -f/m$ and setting

$$w(x) = \int_D \Phi(x, y)\psi(y)\, dy, \quad x \in \mathbb{R}^3,$$

we see that

$$(S\psi, \psi)_{L^2(D)} = -\int_D \frac{f}{m} \left\{\bar{f} - k^2 m\bar{w}\right\} dx = -\int_D \frac{1}{m} |f|^2\, dx + k^2 \int_D f\bar{w}\, dx. \tag{10.65}$$

Since

$$\Delta w + k^2 n w = -\psi - k^2 m w = -f$$

in D, by Green's first integral theorem (which remains valid for $H^2(D)$ functions as pointed out in the remarks after Theorem 8.2) we have

$$\int_D f\bar{w}\, dx = -\int_D \bar{w}\left\{\Delta w + k^2 n w\right\} dx = \int_D \left\{|\operatorname{grad} w|^2 - k^2 n |w|^2\right\} dx - \int_{\partial D} \bar{w}\, \frac{\partial w}{\partial \nu}\, ds.$$

Taking the imaginary part now gives

$$\operatorname{Im}(S\psi, \psi)_{L^2(D)} = -\operatorname{Im} \int_{\partial D} \bar{w}\, \frac{\partial w}{\partial \nu}\, ds \tag{10.66}$$

whence $\operatorname{Im}(S\psi,\psi)_{L^2(D)} \leq 0$ follows from the identity (2.11) (which again remains valid for H^2 functions) since w satisfies $\Delta w + k^2 w = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ and the Sommerfeld radiation condition.

We now prove the last part of the fourth statement. To this end, let $\psi \in L^2(D)$ be in $\overline{G^*(L^2(D))}$ such that $\operatorname{Im}(S\psi,\psi)_{L^2(D)} = 0$. We want to show that $\psi = 0$. Since $\operatorname{Im}(S\psi,\psi)_{L^2(D)} = 0$ we have from (10.66) and Theorem 2.13 (which is also valid for H^2 functions) that $w = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. We now want to show that $w = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ and the constraint $\psi \in \overline{G^*(L^2(D))}$ on the density ψ of w implies that $\psi = 0$ in D. From the proof of Theorem 10.12 we have that

$$H = 4\pi S^* G^*$$

where $H : L^2(\mathbb{S}^2) \to L^2(D)$ is defined by (10.63). Since S^* is an isomorphism from $L^2(D)$ onto $L^2(D)$, the range of G^* coincides with the range of H. Hence there exists a sequence $\psi_j = Hg_j$ with $\psi_j \to \psi \in \overline{G^*(L^2(D))}$. Define v_j by

$$v_j(x) := \int_{\mathbb{S}^2} e^{ikx\cdot d} g_j(d)\, ds(d), \qquad x \in \mathbb{R}^3.$$

Then $v_j|_D = \psi_j$ and v_j satisfies the Helmholtz equation $\Delta v_j + k^2 v_j = 0$ in $\mathbb{R}^3$. Recalling the definition of w, we have that $\Delta w + k^2 w = -\psi$ in D and by Green's second theorem we obtain

$$\int_D \psi\bar{\psi}_j\, dx = \int_D \psi\bar{v}_j\, dx = -\int_D \left\{\Delta w + k^2 w\right\} \bar{v}_j\, dx$$

$$= -\int_D \left\{\Delta\bar{v}_j + k^2\bar{v}_j\right\} w\, dx - \int_{\partial D} \left\{\bar{v}_j\, \frac{\partial w}{\partial \nu} - w\, \frac{\partial \bar{v}_j}{\partial \nu}\right\} ds = 0$$

since we have already shown that $w = 0$ in $\mathbb{R}^3 \setminus D$. Hence, letting j tend to infinity we have that

$$\int_D |\psi|^2\, dx = 0$$

that is, $\psi = 0$. □

We are now in a position to establish the factorization method for determining the support of D of $m = 1 - n$ from a knowledge of the far field pattern of the scattered field corresponding to (10.1)–(10.3). To this end, we assume that k is not a transmission eigenvalue . Then by Theorem 8.9 the far field operator F is injective and by Corollary 8.20 it is normal. In particular, by (8.43) the operator

$$I + \frac{ik}{2\pi}\, F$$

is unitary. Hence, using Theorems 10.12, 10.14, Lemma 10.14 and applying Theorem 5.39, we obtain as in the case of obstacle scattering the factorization method for determining the support D of $m = 1 - n$.

Theorem 10.15. *Assume that $n(x) > 1$ for $x \in \bar{D}$ and that $k > 0$ is not a transmission eigenvalue. Then $z \in D$ if and only if $\Phi_\infty(\cdot, z) \in (F^*F)^{1/4}(L^2(\mathbb{S}^2))$.*

In practice the support D of $m = 1 - n$ can now be determined by using Tikhonov regularization to find a regularized solution of the modified far field equation

$$(F^*F)^{1/4} g = \Phi_\infty(\cdot, z) \tag{10.67}$$

and noting that the regularized solution g_z^α of (10.67) converges in $L^2(\mathbb{S}^2)$ as $\alpha \to 0$ if and only if $z \in D$ (c.f. Theorem 4.20). Although we have assumed that $n(x) > 1$ for $x \in \bar{D}$, the factorization method remains valid for $0 < n(x) < 1$ for $x \in \bar{D}$, the proof being exactly the same as that given above. There also exist extensions of the factorization method for the case when $n(x)$ is no longer real valued for $x \in D$. In this case the far field operator is no longer normal and different techniques must be used [186]. Finally, for either the case when $n(x) > 1$ for $x \in \bar{D}$ or $0 < n(x) < 1$ for $x \in \bar{D}$ one can derive the linear sampling method as a corollary to Theorem 10.15 in precisely the same way as in the case of obstacle scattering (c.f. Corollary 5.42). Of course, as was done in Sections 5.6 and 7.5, the linear sampling method can also be derived in a manner independent of the factorization method [61, 87]. We will now illustrate this by considering the problem of determining the support of an absorbing inhomogeneous medium from a knowledge of the far field pattern of the scattered wave [87].

We begin with a projection theorem. Let X be a Hilbert space with the scalar product $(\cdot\,,\cdot)$ and norm $\|\cdot\|$ induced by $(\cdot\,,\cdot)$. Let $\langle\cdot\,,\cdot\rangle$ be a bounded sesquilinear form on X such that

$$|\langle\varphi, \varphi\rangle| \geq \gamma\|\varphi\|^2 \tag{10.68}$$

for all $\varphi \in X$ where γ is a positive constant. For a subspace $H \subset X$ we define $H^\perp$ to be the orthogonal complement of H with respect to $(\cdot\,,\cdot)$ and $H^{\perp_s}$ to be the orthogonal complement of H with respect to $\langle\cdot\,,\cdot\rangle$. By the Lax–Milgram theorem there exists a unique bounded linear operator $M : X \to X$ such that

$$\langle\varphi, \psi\rangle = (M\varphi, \psi) \tag{10.69}$$

for all $\varphi, \psi \in X$, M is bijective and the norm of M^{-1} is bounded by γ^{-1}.

Lemma 10.16 *For every closed subspace $H \subset X$ we have the decomposition*

$$X = H^\perp + MH$$

where $H^\perp \cap MH = \{0\}$.

Proof. Define $G := H^\perp + MH$ and let $\psi \in G^\perp$. Then $\psi \in H \cup (MH)^\perp$, i.e.,

$$\langle\varphi, \psi\rangle = (M\varphi, \psi) = 0$$

for all $\varphi \in H$. Setting $\varphi = \psi$ from (10.68) we obtain $\psi = 0$ and therefore $X = H^\perp + MH$. Now assume that for $f \in X$ we have $f = \psi_1 + M\varphi_1 = \psi_2 + M\varphi_2$. Then for $\psi := \psi_1 - \psi_2$ and $\varphi := \varphi_1 - \varphi_2$ we have $0 = \psi + M\varphi$ with $\psi \in H^\perp$ and $\varphi \in H$. Therefore

$$0 = (\psi + M\varphi, \varphi) = (M\varphi, \varphi) = \langle \varphi, \varphi \rangle$$

and hence $\varphi = 0$. This implies $\psi = 0$ and the proof is complete. □

Now let P_0 be the orthogonal projection operator in X onto the space H with respect to the scalar product $(\cdot\,,\cdot)$ and let P_M be the projection operator onto MH as defined by Lemma 10.16. By the closed graph theorem, P_M is a bounded operator.

Lemma 10.17 *For every closed subspace $H \subset X$ we have*

$$M^{-1}H^\perp = (M^*H)^\perp = H^{\perp_s}.$$

Proof. The first equality follows from the fact that $\varphi \in (M^*H)^\perp$ if and only if $(\varphi, M^*\psi) = (M\varphi, \psi) = 0$ for every $\psi \in H$ and hence $M\varphi \in H^\perp$, i.e. $\varphi \in M^{-1}H^\perp$. The second equality follows from the fact that $(\varphi, M^*\psi) = (M\varphi, \psi) = \langle \varphi, \psi \rangle$. □

We are now in a position to show that every $\varphi \in X$ can be uniquely written as a sum $\varphi = v + w$ with $v \in H^{\perp_s}$ and $w \in H$, i.e. $X = H^{\perp_s} \oplus_s H$ where $\oplus_s$ is the orthogonal decomposition with respect to the sesquilinear form $\langle \cdot\,,\cdot \rangle$.

Theorem 10.18. *For every closed subspace $H \subset X$ we have the orthogonal decomposition*

$$X = H^{\perp_s} \oplus_s H.$$

The projection operator $P : X \to H^{\perp_s}$ defined by this decomposition is bounded in X.

Proof. For $\varphi \in X$ we define $\hat{\varphi} := M\varphi$. Then from Lemma 10.16 we have that

$$\hat{\varphi} = (1 - P_M)\hat{\varphi} + P_M\hat{\varphi},$$

that is,

$$M\varphi = (1 - P_M)M\varphi + P_M M\varphi.$$

Hence

$$\varphi = M^{-1}(1 - P_M)M\varphi + M^{-1}P_M M\varphi = v + w$$

where

$$v := M^{-1}(1 - P_M)M\varphi \in M^{-1}H^\perp = H^{\perp_s} \quad \text{and} \quad w := M^{-1}P_M M\varphi \in H.$$

We have thus shown that $X = H^{\perp_s} + H$. To show the uniqueness of this decomposition, suppose $v + w = 0$ with $v \in H^{\perp_s}$ and $w \in H$. Then

$$0 = |\langle v, w\rangle| = |\langle w, w\rangle| \geq \gamma \|w\|^2$$

which implies that $w = v = 0$. Finally, since from the above analysis we have that $P = M^{-1}(1 - P_M)M$ and P_M is bounded, we have that P is bounded. □

We will now turn our attention to the problem of showing the existence of a unique solution v, w of the inhomogeneous interior transmission problem

$$\Delta w + k^2 n(x) w = 0, \quad \Delta v + k^2 v = 0 \quad \text{in } D \tag{10.70}$$

with the transmission condition

$$w - v = \Phi(\cdot\,, z), \quad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu}\, \Phi(\cdot, z) \quad \text{on } \partial D \tag{10.71}$$

where n is piecewise continuously differentiable in $\bar{D}$ and, for the sake of simplicity, D is assumed to be connected with a connected C^2 boundary ∂D and $z \in D$. We will further assume that there exists a positive constant c such that

$$\operatorname{Im} n(x) \geq c \tag{10.72}$$

for $x \in D$. Using Theorem 8.2 we can formulate the problem as follows:

Definition 10.19 *Let H be the linear space of all Herglotz wave functions and $\bar{H}$ the closure of H in $L^2(D)$. For $\varphi \in L^2(D)$ define the volume potential by*

$$(T_m\varphi)(x) := \int_D \Phi(x, y) m(y) \varphi(y)\, dy, \quad x \in \mathbb{R}^3.$$

Then a pair v, w with $v \in \bar{H}$ and $w \in L^2(D)$ is said to be a solution of the inhomogeneous interior transmission problem (10.70), (10.71) with point source $z \in D$ *if v and w satisfy the integral equation*

$$w + k^2 T_m w = v \quad \textit{in } D$$

and the boundary condition

$$-k^2 T_m w = \Phi(\cdot, z) \quad \textit{on } \partial B,$$

where B is an open ball centered at the origin with $\bar{D} \subset B$

Before proceeding to establish the existence of a unique solution to the interior inhomogeneous transmission problem (10.70), (10.71), we make a few preliminary observations. We note that condition (10.72) implies that in $L^2(D)$ the sesquilinear form

$$\langle \varphi, \psi \rangle := \int_D m(y) \varphi(y) \overline{\psi(y)}\, dy \tag{10.73}$$

satisfies the assumption (10.68). In this case the operator M is simply the multiplication operator $(M\varphi)(x) := m(x)\varphi(x)$. Finally by the uniqueness of the solution to the exterior Dirichlet problem for the Helmholtz equation and the unique continuation principle, we see that if v, w is a solution of the inhomogeneous interior transmission problem with point source $z \in D$ then $-(k^2 T_m w)(x) = \Phi(x, z)$ for all $x \in \mathbb{R}^2 \backslash D$.

Theorem 10.20. *For every source point $z \in D$ there exists at most one solution of the inhomogeneous interior transmission problem.*

Proof. Let w and v be the difference between two solutions of the inhomogeneous interior transmission problem. Then from the boundary condition $T_m w = 0$ on ∂B we have

$$\int_D \Phi(x, y) m(y) w(y)\, dy = 0, \quad x \in \mathbb{R}^3 \setminus D. \tag{10.74}$$

Hence $T_m w$ has vanishing far field pattern, i.e.,

$$\int_D e^{-ik\,\hat{x}\cdot y} m(y) w(y)\, dy = 0, \quad \hat{x} \in \mathbb{S}^2,$$

(see (8.28)). Multiplying this identity by $\bar{h}$, integrating over $\mathbb{S}^2$ and interchanging the order of integration, we obtain

$$\langle w, v_h \rangle = \int_D m w \overline{v_h}\, dy = 0 \tag{10.75}$$

for each Herglotz wave function v_h with kernel $h \in L^2(\mathbb{S}^2)$. By continuity (10.75) also holds for $v \in \bar{H}$.

Now let $(v_j) \in H$ be a sequence with $v_j \to v$ as $j \to \infty$ in $L^2(D)$ and note that $(I + k^2 T_m)^{-1}$ exists and is a bounded operator in $L^2(D)$. Hence, for

$$w_j := (I + k^2 T_m)^{-1} v_j$$

we have that $w_j \to w \in L^2(D)$ as $j \to \infty$. For $x \notin D$ we define w_j by

$$w_j(x) := v_j(x) - k^2 (T_m w_j)(x)\,, \quad x \in \mathbb{R}^3 \backslash D.$$

The functions v_j and w_j satisfy

$$\Delta v_j + k^2 v_j = 0, \quad \Delta w_j + k^2 n(x) w_j = 0 \tag{10.76}$$

in both D and $B\backslash\bar{D}$. From (10.76), by Green's first theorem applied to D and $B\backslash\bar{D}$, we have

$$\begin{aligned}
&\operatorname{Im}\int_{\partial B}(w_j - v_j)\,\frac{\partial}{\partial\nu}\,(\overline{w_j - v_j})\,ds \\
&= \operatorname{Im}\int_D (w_j - v_j)\Delta(\overline{w_j - v_j})\,dx \qquad (10.77) \\
&= k^2\operatorname{Im}\int_D \overline{m}(w_j - v_j)\overline{w_j}\,dx.
\end{aligned}$$

We now note that from (10.74) and (10.75) we have by the Cauchy–Schwarz inequality that

$$w_j - v_j = -k^2 T_m w_j \to -k^2 T_m w = 0,$$

$$\frac{\partial}{\partial\nu}\,(w_j - v_j) = -k^2\,\frac{\partial}{\partial\nu}\,T_m w_j \to -k^2\,\frac{\partial}{\partial\nu}\,T_m w = 0$$

uniformly on ∂B and

$$\int_D m(y)w_j(y)\overline{v_j(y)}\,dy \to \int_D m(y)w(y)\overline{v(y)}\,dy = 0$$

as $j \to \infty$. Hence, taking the limit $j \to \infty$ in (10.77) we obtain

$$\int_D \operatorname{Im} m\,|w|^2\,dy = 0.$$

From (10.72) we can now conclude that $w(x) = 0$ for $x \in D$. Since $v = w + k^2 T_m w$ in D we can also conclude that $v(x) = 0$ for $x \in D$ and the proof is complete. □

Theorem 10.21. *For every source point $z \in D$ there exists a solution to the inhomogeneous interior transmission problem.*

Proof. Choosing an appropriate coordinate system, we can assume without loss of generality that $z = 0$. We consider the space

$$H_1^0 := \operatorname{span}\left\{j_p(kr)Y_p^q : p = 1, 2, \ldots,\ -p \le q \le q\right\}$$

where j_p is a spherical Bessel function and Y_p^q is a spherical harmonic and denote by H_1 the closure of H_1^0 in $L^2(D)$. We note that from the Jacobi–Anger expansion (2.46) we have that the space

$$\operatorname{span}\left\{j_p(kr)Y_p^q : p = 0, 1, 2, \ldots,\ -p \le q \le q\right\}$$

is a dense subset of $\bar{H}$. From Theorem 10.18 we conclude that $\bar{H} = H_1^{\perp_s} \oplus_s H_1$, and therefore there exists a nontrivial $\psi \in H_1^{\perp_s} \cap \bar{H}$. (It is easily verified that $\bar{H} \neq H_1$, since for sufficiently small a a function $h \in H_1$ satisfies $\int_{|x| \le a} h\,dx = 0$ but this is not true for $h = j_0 \in \bar{H}$.) Then $\langle j_0, \psi \rangle \ne 0$ because otherwise we would have $\langle h, \psi \rangle = 0$ for all $h \in \bar{H}$ which contradicts the fact that the nontrivial ψ belongs to $\bar{H}$.

Now let P be the projection operator from $L^2(D)$ onto $H^{\perp_s}$ as defined by Theorem 10.18. We first consider the integral equation

$$u + k^2 P T_m u = k^2 P T_m \psi \tag{10.78}$$

in $L^2(D)$. Since T_m is compact and P is bounded, the operator PT_m is compact in $L^2(D)$. In order to apply the Riesz theory, we will prove uniqueness for the homogeneous equation. To this end, assume that $w \in L^2(D)$ satisfies

$$w + k^2 P T_m w = 0.$$

Then $w \in H^{\perp_s}$ and

$$v := k^2 (I - P) T_m w \in \bar{H}$$

satisfy

$$w + k^2 T_m w = v.$$

Since $\langle w, \varphi \rangle = 0$ for all $\varphi \in H$, from the Addition Theorem 2.11 we conclude that

$$T_m w = 0 \quad \text{on } \partial B.$$

Now from Theorem 10.20 we have that $v = w = 0$ and by the Riesz theory we obtain the continuous invertibility of $I + PT_m$ in $L^2(D)$.

Now let u be the solution of (10.78) and note that $u \in H^{\perp_s}$. We define the constant c and function $w \in L^2(D)$ by

$$c := \frac{1}{k^2 \langle j_0, \psi \rangle}\,, \quad w := c(u - \psi).$$

Then we compute

$$w + k^2 P T_m w = -c\psi$$

and hence

$$w + k^2 T_m w = v$$

where

$$v := k^2 (I - P) T_m w - c\psi \in \bar{H}.$$

Since

$$\langle h, w \rangle = c \langle h, u - \psi \rangle = 0$$

for all $h \in H_1$ and

$$\langle j_0, w \rangle = c \langle j_0, u - \psi \rangle = -\frac{1}{k^2}$$

we have from the Addition Theorem 2.11 that

$$-k^2(T_m w)(x) = \frac{ik}{4\pi}\, h_0^{(1)}(k|x|) = \Phi(x, 0), \quad x \in \partial B,$$

where h_0^1 is the spherical Hankel function of the first kind of order zero and the proof is complete. □

We are now in a position to show how the support D of m can be determined from the far field pattern u_∞ corresponding to the scattering problem (10.1)–(10.3) by using the linear sampling method. To this end we define the far field operator $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2. \tag{10.79}$$

We then have the following theorem that corresponds to Theorem 5.34 for obstacle scattering (note that by (10.72) $k > 0$ is not a transmission eigenvalue):

Theorem 10.22. *Assume that (10.72) is valid and let F be the far field operator (10.79) for scattering by an inhomogeneous medium. Then the following hold:*

1. For $z \in D$ and a given $\varepsilon > 0$ there exists a function $g_z^\varepsilon \in L^2(\mathbb{S}^2)$ such that

$$\|F g_z^\varepsilon - \Phi_\infty(\cdot\,, z)\|_{L^2(\mathbb{S}^2)} < \varepsilon \tag{10.80}$$

and the Herglotz wave function $v_{g_z^\varepsilon}$ with kernel g_z^ε converges to the solution $v \in \bar{H}$ of the inhomogeneous interior transmission problem as $\varepsilon \to 0$.

2. For $z \notin D$ every $g_z^\varepsilon \in L^2(\mathbb{S}^2)$ that satisfies (10.80) for a given $\varepsilon > 0$ is such that

$$\lim_{\varepsilon \to 0} \|v_{g_z^\varepsilon}\|_{L^2(D)} = \infty.$$

Proof. Let $v(\cdot, z)$, $w(\cdot, z)$ be the unique solution to the inhomogeneous interior transmission problem with source point $z \in D$. By the definition of H, we can approximate $v(\cdot, z) \in \bar{H}$ by a Herglotz wave function v_g with kernel $g = g(\cdot, z)$, i.e., for every $\tilde{\varepsilon} > 0$ and $z \in D$ there exists $g \in L^2(\mathbb{S}^2)$ such that

$$\|v(\cdot, z) - v_g\|_{L^2(D)} \le \tilde{\varepsilon}. \tag{10.81}$$

Then by the continuity of $(I + k^2 T_m)^{-1}$ we have for $u_g := (I + k^2 T_m)^{-1}\, v_g$ that

$$\|w(\cdot, z) - u_g\|_{L^2(D)} \le c\,\tilde{\varepsilon} \tag{10.82}$$

for some positive constant c and by the continuity of $T_m : L^2(D) \to C(\partial B)$ we have that

$$\|k^2 T_m u_g + \Phi(\cdot, z)\|_{C(\partial B)} \le c'\,\tilde{\varepsilon} \tag{10.83}$$

for some positive constant c'. We now note that the far field pattern $k^2 T_{m,\infty} u_g$ of $-k^2 T_m u_g$ is given by

$$k^2 (T_{m,\infty} u_g)(\hat{x}) = -\int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2.$$

Hence, by the continuous dependence of the solution of the exterior Dirichlet problem with respect to the boundary data, we obtain from (10.83) the estimate

$$\|k^2 T_m^\infty u_g - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S}^2)} \le c'' \,\tilde{\varepsilon}$$

for a positive constant c''. Letting $\varepsilon \to 0$ now establishes part one of the theorem.

In order to prove the second part, let $z \notin D$ and, contrary to the statement of the theorem assume that there exists a null sequence (ε_j) and corresponding Herglotz wave functions v_j with kernels $g_j = g_z^{\varepsilon_j}$ such that $\|v_j\|_{L^2(D)}$ remains bounded. Then without loss of generality we may assume weak convergence $v_j \rightharpoonup v \in L^2(D)$ as $n \to \infty$. Denote by $v^s \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ the scattered field field for (10.1)–(10.3) arising from the incident field v instead of $e^{ik\,x\cdot d}$ (obtained by using the Lippmann–Schwinger equation) and denote its far field pattern by v_∞. Since Fg_j is the far field pattern of the scattered wave for the incident field v_j then if $u_j \in L^2(D)$ is the solution of $u_j + k^2 T_m u_j = v_j$ in D we have that $Fg_j = -k^2 T_{m,\infty} u_j$. Letting $j \to \infty$ and using (10.80) shows that $v_\infty = -k^2 T_{m,\infty} u = \Phi_\infty(\cdot\,, z)$ where $u = (I - k^2 T_m)^{-1} v$. But $-k^2 T_m u \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ and $\Phi_\infty(\cdot\,, z)$ is not and hence by Rellich's lemma we arrive at a contradiction. □

Having determined the support of the inhomogeneous scattering object from a knowledge of the far field pattern by using either the factorization method or the linear sampling method, the next question to ask is how to obtain information about the material properties of the scatterer without resorting to nonlinear optimization methods as was done in previous sections of this chapter. One approach to this question is to make use of transmission eigenvalues which up to now have been something to avoid. To this end from now on we assume that the scatterer D is connected with a connected C^2 boundary and $\mathrm{Im}\, n = 0$. In particular, if F is the far field operator, $0 < n(x) < 1$ or $n(x) > 1$ for $x \in \bar{D}$, and a regularized solution g_z^α of the far field equation $Fg = \Phi_\infty(\cdot, z)$ is obtained by using Tikhonov regularization, then if F has dense range it can be shown that for almost every $z \in D$ the Herglotz wave function $v_{g_z^\alpha}$ with kernel g_z^α converges in $L^2(\mathbb{S}^2)$ if and only if $k > 0$ is not a transmission eigenvalue (Note that by Theorem 8.9 the fact that F has dense range is generally the case) [34]. In practice this means that if the noise level is sufficiently small then, under the above assumptions on n, if $\|g_z^\alpha\|_{L^2(\mathbb{S}^2)}$ is plotted against k for a variety of values of z the transmission eigenvalues will appear as sharp peaks in the graph (c.f. [37]). Each of these measured transmission eigenvalues contains information about n and the problem is to extract this information. We shall now proceed to examine this problem for the case of the first (real) transmission eigenvalue, beginning with the problem of showing that such eigenvalues in fact exist. For further results on the existence of transmission eigenvalues see [35, 38, 39, 40, 41, 140, 141, 185]. The first proof of the existence of transmission eigenvalues was given by Sylvester and Päivärinta for the case when $n(x) > 1$ for $x \in D$ and $\|n\|_\infty$ is sufficiently large [267].

Let X be a Hilbert space with scalar product $(\cdot, \cdot)$ and associated norm $\|\cdot\|$, and let $A : X \to X$ be a bounded, self-adjoint and strictly positive definite operator, i.e.,

$$(Au, u) \geq c\|u\|^2$$

for all $u \in X$ and some $c > 0$. We recall that the operators $A^{\pm 1/2}$ are defined by $A^{\pm 1/2} = \int_0^\infty \lambda^{\pm 1/2}\, dE_\lambda$ where dE_λ is the spectral measure associated with A. In particular, $A^{\pm 1/2}$ are also bounded, self-adjoint and strictly positive definite operators on X satisfying $A^{-1/2}A^{1/2} = I$ and $A^{1/2}A^{1/2} = A$. We shall consider the spectral decomposition of the operator A with respect to self-adjoint non-negative compact operators. The next two theorems [40] indicate the main properties of such a decomposition.

Theorem 10.23. *Let $A : X \to X$ be a bounded, self-adjoint and strictly positive definite operator on a Hilbert space X and let $B : X \to X$ be a non-negative, self-adjoint and compact linear operator with null space $N(B)$. There exists an increasing sequence of positive real numbers (λ_j) and a sequence (u_j) of elements of X satisfying*

$$Au_j = \lambda_j B u_j \tag{10.84}$$

and

$$(Bu_j, u_\ell) = \delta_{j\ell} \tag{10.85}$$

such that each $u \in [A(N(B))]^\perp$ can be expanded in a series

$$u = \sum_{j=1}^\infty \gamma_j u_j. \tag{10.86}$$

If $N(B)^\perp$ has infinite dimension then $\lambda_j \to \infty$ as $j \to \infty$.

Proof. This theorem is a direct consequence of the spectral decomposition (4.6) and (4.7) applied to the non-negative, self-adjoint compact operator $\widetilde{B} = A^{-1/2}BA^{-1/2}$. Let (μ_j) be the decreasing sequence of positive eigenvalues and (v_j) the corresponding orthonormal eigenelements of $\widetilde{B}$ that are complete in $\overline{A^{-1/2}BA^{-1/2}(X)}$, that is,

$$v = \sum_{j=1}^\infty (v, v_j) v_j \tag{10.87}$$

for all $v \in \overline{A^{-1/2}BA^{-1/2}(X)}$. Note that zero is the only possible accumulation point for the sequence (μ_j). Straightforward calculations show that $\lambda_j = 1/\mu_j$ and

$$u_j = \sqrt{\lambda_j}\, A^{-1/2} v_j$$

for $j = 1, 2, \ldots$ satisfy (10.84) and (10.85). Since $A^{-1/2}$ is bounded, from (10.87) we conclude that each $u \in \overline{A^{-1}BA^{-1/2}(X)}$ can be expanded in a series of the form (10.86). The bijectivity of $A^{-1/2}$ and Theorem 4.6 imply that

$$\overline{A^{-1}BA^{-1/2}(X)} = \overline{A^{-1}B(X)} = [N(BA^{-1})]^{\perp} = [A(N(B))]^{\perp}$$

since BA^{-1} is the adjoint of $A^{-1}B$ and $N(BA^{-1}) = A(N(B))$. This ends the proof of the theorem. □

Theorem 10.24. *Let A, B and (λ_j) be as in Theorem 10.23 and define the Rayleigh quotient as*

$$R(u) = \frac{(Au, u)}{(Bu, u)}$$

for $u \notin N(B)$, where $(\cdot, \cdot)$ is the scalar product in X. Then the following min-max principles hold

$$\lambda_j = \min_{W \in \mathcal{U}_j^A} \left(\max_{u \in W \setminus \{0\}} R(u) \right) = \max_{W \in \mathcal{U}_{j-1}^A} \left(\min_{u \in [A(W + N(B))]^{\perp} \setminus \{0\}} R(u) \right)$$

where $\mathcal{U}_j^A$ denotes the set of all j-dimensional subspaces of $[A(N(B))]^{\perp}$.

Proof. The proof follows the classical proof of the Courant min-max principle and is given here for the reader's convenience. It is based on the fact that if $u \in [A(N(B))]^{\perp}$ then from Theorem 10.23 we can write $u = \sum_{k=j}^{\infty} \gamma_j u_j$ for some coefficients γ_j, where the u_j are defined in Theorem 10.23 (note that the u_j are orthogonal with respect to the scalar product induced by the self-adjoint invertible operator A). Then using (10.84) and (10.85) it can be seen that

$$R(u) = \frac{1}{\sum_{j=1}^{\infty} |\gamma_j|^2} \sum_{j=1}^{\infty} \lambda_j |\gamma_j|^2.$$

Therefore, if $W_j \in \mathcal{U}_k^A$ denotes the space spanned by $\{u_1, \ldots, u_j\}$ we have that

$$\lambda_j = \max_{u \in W_j \setminus \{0\}} R(u) = \min_{u \in [A(W_{j-1} + N(B))]^{\perp} \setminus \{0\}} R(u).$$

Next, let W be any element of $\mathcal{U}_j^A$. Since W has dimension j and $W \subset [A(N(B)]^{\perp}$, then $W \cap [AW_{j-1} + A(N(B))]^{\perp} \neq \{0\}$. Therefore

$$\max_{u \in W \setminus \{0\}} R(u) \geq \min_{u \in W \cap [A(W_{j-1} + N(B)]^{\perp} \setminus \{0\}} R(u) \geq \min_{u \in [A(W_{j-1} + N(B))]^{\perp} \setminus \{0\}} R(u) = \lambda_j$$

which proves the first equality of the theorem. Similarly, if W has dimension $j - 1$ and $W \subset [A(N(B))]^{\perp}$, then $W_j \cap (AW)^{\perp} \neq \{0\}$. Therefore

$$\min_{u \in [A(W + A(N(B))]^{\perp} \setminus \{0\}} R(u) \leq \max_{u \in W_j \cap (AW)^{\perp} \setminus \{0\}} R(u) \leq \max_{u \in W_j \setminus \{0\}} R(u) = \lambda_j$$

which proves the second equality of the theorem. □

The following corollary shows that it is possible to remove the dependence on A in the choice of the subspaces in the min-max principle for the eigenvalues λ_j.

Corollary 10.25 *Let A, B, λ_j and R be as in Theorem 10.24. Then*

$$\lambda_j = \min_{W \subset \mathcal{U}_j} \left(\max_{u \in W \setminus \{0\}} R(u) \right) \tag{10.88}$$

where $\mathcal{U}_j$ denotes the set of all j-dimensional subspaces W of X such that $W \cap N(B) = \{0\}$.

Proof. From Theorem 10.24 and the fact that $\mathcal{U}_j^A \subset \mathcal{U}_j$ it suffices to prove that

$$\lambda_j \le \min_{W \subset \mathcal{U}_k} \left(\max_{u \in W \setminus \{0\}} R(u) \right).$$

Let $W \in \mathcal{U}_j$ and let $v_1, v_2, \ldots, v_j$ be a basis for W. Each vector v_ℓ can be decomposed into a sum $v_\ell^0 + \tilde{v}_\ell$ where $\tilde{v}_\ell \in [A(N(B))]^\perp$ and $v_\ell^0 \in N(B)$ (which is the orthogonal decomposition with respect to the scalar product induced by A). Since $W \cap N(B) = \{0\}$, the space $\tilde{W}$ spanned by $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_j$ has dimension j. Moreover, $\tilde{W} \subset [A(N(B))]^\perp$. Now let $\tilde{u} \in \tilde{W}$. Obviously $\tilde{u} = u - u^0$ for some $u \in W$ and $u^0 \in N(B)$. Since $Bu^0 = 0$ and $(Au_0, \tilde{u}) = 0$ we have that

$$R(u) = \frac{(A\tilde{u}, \tilde{u}) + (Au^0, u^0)}{(B\tilde{u}, \tilde{u})} = R(\tilde{u}) + \frac{(Au^0, u^0)}{(B\tilde{u}, \tilde{u})} .$$

Consequently, since A is positive definite and B is non-negative, we obtain

$$R(\tilde{u}) \le R(u) \le \max_{u \in W \setminus \{0\}} R(u).$$

Finally, taking the maximum with respect to $\tilde{u} \in \tilde{W} \subset [A(N(B))]^\perp$ in the above inequality, we obtain from Theorem 10.24 that

$$\lambda_j \le \max_{u \in W \setminus \{0\}} R(u),$$

which completes the proof after taking the minimum over all $W \subset \mathcal{U}_j$. □

The following theorem provides the theoretical basis of our analysis of the existence of transmission eigenvalues. This theorem is a simple consequence of Theorem 10.24 and Corollary 10.25.

Theorem 10.26. *Let $\tau \mapsto A_\tau$ be a continuous mapping from $(0, \infty)$ to the set of bounded, self-adjoint and strictly positive definite operators on the Hilbert space X and let B be a self-adjoint and non-negative compact linear operator on X. We assume that there exist two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that*

a) $A_{\tau_0} - \tau_0 B$ is positive on X,
b) $A_{\tau_1} - \tau_1 B$ is non-positive on an ℓ-dimensional subspace W_ℓ of X.
Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \dots, \ell$ has at least one solution in $[\tau_0, \tau_1]$ where $\lambda_j(\tau)$ is the j^{th} eigenvalue (counting multiplicity) of A_τ with respect to B, that is, $N(A_\tau - \lambda_j(\tau)B) \neq \{0\}$.

Proof. First we can deduce from (10.88) that $\lambda_j(\tau)$ is a continuous function of τ for all $j \geq 1$. Assumption a) shows that $\lambda_j(\tau_0) > \tau_0$ for all $j \geq 1$. Assumption b) implies in particular that $W_k \cap N(B) = \{0\}$. Hence, another application of (10.88) implies that $\lambda_j(\tau_1) \leq \tau_1$ for $1 \leq j \leq \ell$. The desired result is now obtained by applying the intermediate value theorem. □

In a Sobolev space setting the *homogeneous interior transmission problem* corresponding to the scattering problem for (10.1)–(10.3) is

$$\begin{aligned} \Delta w + k^2 n w &= 0 \quad \text{in } D, \\ \Delta v + k^2 v &= 0 \quad \text{in } D, \\ w &= v \quad \text{on } \partial D, \\ \frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} \quad \text{on } \partial D \end{aligned} \tag{10.89}$$

for $w, v \in L^2(D)$ such that $w - v \in H_0^2(D)$ where

$$H_0^2(D) = \left\{ u \in H^2(D) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D \right\}.$$

In this setting Definition 8.11 becomes the following:

Definition 10.27 *Values of $k > 0$ for which the homogeneous interior transmission problem (10.89) has nonzero solutions $w, v \in L^2(D)$ such that $w - v \in H_0^2(D)$ are called transmission eigenvalues. If $k > 0$ is a transmission eigenvalue we call $u = w - v$ the corresponding eigenfunction where w and v is a nonzero solution of (10.89).*

It is possible to write (10.89) as an equivalent eigenvalue problem for $u = w - v$ in $H_0^2(D)$ for the fourth order equation

$$(\Delta + k^2 n)\, \frac{1}{n-1}\, (\Delta + k^2) u = 0 \tag{10.90}$$

which in variational form, after integrating by parts, is formulated as finding a function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{n-1}\, (\Delta u + k^2 u)\, (\Delta \bar{v} + k^2 n \bar{v})\, dx = 0 \quad \text{for all } v \in H_0^2(D). \tag{10.91}$$

In our discussion we must distinguish between the two cases $n > 1$ and $0 < n < 1$. To fix our ideas, we consider in detail only the case where $n(x) > 1$ for $x \in \bar{D}$. (Similar results can be obtained for $0 < n(x) < 1$ for $x \in \bar{D}$, c.f. [38, 39].)

In the sequel we set

$$n_{\min} := \min_{x \in \bar{D}} n(x) \quad \text{and} \quad n_{\max} := \max_{x \in \bar{D}} n(x).$$

The following result was first obtained in [86] and provides a Faber–Krahn type inequality for the first transmission eigenvalue.

Theorem 10.28. *Assume that* $n_{\min} > 1$. *Then*

$$k_0^2 > \frac{\lambda_0(D)}{n_{\max}} \tag{10.92}$$

where k_0 *is the smallest transmission eigenvalue and* $\lambda_0(D)$ *is the first Dirichlet eigenvalue of* $-\Delta$ *on* D.

Proof. Taking $v = u$ in (10.91), using Green's theorem and the zero boundary values for u we obtain

$$\begin{aligned} 0 &= \int_D \frac{1}{n-1} \, (\Delta u + k^2 u)\, (\Delta \bar{u} + k^2 n \bar{u})\, dx \\ &= \int_D \frac{1}{n-1} |(\Delta u + k^2 n u)|^2 \, dx + k^2 \int_D \left\{ |\operatorname{grad} u|^2 - k^2 n\, |u|^2 \right\} dx. \end{aligned}$$

Since $n(x) - 1 \geq n_{\min} - 1 > 0$ for all $x \in D$, if

$$\int_D \left\{ |\operatorname{grad} u|^2 - k^2 n\, |u|^2 \right\} dx \geq 0 \tag{10.93}$$

then $\Delta u + k^2 n u = 0$ in D which together with the fact $u \in H_0^2(D)$ implies that $u = 0$ in D. Consequently we obtain $w = v = 0$, whence k is not a transmission eigenvalue. But

$$\inf_{u \in H_0^2(D)} \frac{(\operatorname{grad} u, \operatorname{grad} u)_{L^2(D)}}{(u, u)_{L^2(D)}} \geq \inf_{u \in H_0^1(D)} \frac{(\operatorname{grad} u, \operatorname{grad} u)_{L^2(D)}}{(u, u)_{L^2(D)}} = \lambda_0(D)$$

and hence we have that

$$\int_D \left\{ |\operatorname{grad} u|^2 - k^2 n\, |u|^2 \right\} dx \geq \|u\|^2_{L^2(D)} (\lambda_0(D) - k^2 n_{\max}).$$

Thus, (10.93) is satisfied whenever $k^2 \leq \lambda_0(D)/n_{\max}$. Hence, we have shown that any transmission eigenvalue k (in particular the smallest transmission eigenvalue k_0) satisfies $k^2 > \lambda_0(D)/n_{\max}$. $\square$

Remark. From Theorem 10.28 it follows that if $n_{\min} > 1$ and k_0 is the smallest transmission eigenvalue, then $n_{\max} > \lambda_0(D)/k_0^2$ which provides a lower bound for $n_{\max}$.

To understand the structure of the transmission eigenvalue problem, we first set $\tau := k^2$ in (10.91) to obtain

$$\int_D \frac{1}{n-1}\,(\Delta u + \tau u)\,(\Delta \bar{v} + \tau n \bar{v})\,dx = 0 \quad \text{for all } v \in H_0^2(D) \tag{10.94}$$

which can be written as

$$u - \tau K_1 u + \tau^2 K_2 u = 0, \tag{10.95}$$

where

$$K_1 = T^{-1/2} T_1 T^{-1/2} \quad \text{and} \quad K_2 = T^{-1/2} T_2 T^{-1/2}.$$

Here $T : H_0^2(D) \to H_0^2(D)$ is the bounded, strictly positive definite self-adjoint operator defined by means of the Riesz representation theorem

$$(Tu, v)_{H^2(D)} = \int_D \frac{1}{n-1}\,\Delta u \Delta \bar{v}\,dx,$$

(noting that the $H^2(D)$ norm for a function with zero Cauchy data on ∂D is equivalent to the $L^2(D)$ norm of its Laplacian), $T_1 : H_0^2(D) \to H_0^2(D)$ is the compact self-adjoint operator defined by means of the Riesz representation theorem

$$(T_1 u, v)_{H^2(D)} = -\int_D \frac{1}{n-1}\,(n\bar{v}\Delta u + u\Delta \bar{v})\,dx = -\int_D \frac{1}{n-1}\,(\bar{v}\Delta u + nu\Delta \bar{v})\,dx$$

and $T_2 : H_0^2(D) \to H_0^2(D)$ is the compact non-negative self-adjoint operator defined by means of the Riesz representation theorem

$$(T_2 u, v)_{H^2(D)} = \int_D \frac{n}{n-1}\,u\bar{v}\,dx$$

(compactness of T_1 and T_2 is a consequence of the compact embeddding of $H_0^2(D)$ and $H_0^1(D)$ in $L^2(D)$). Hence, setting $U := \left(u, \tau K_2^{1/2} u\right)$, the transmission eigenvalue problem becomes the eigenvalue problem

$$\left(K - \frac{1}{\tau} I\right) U = 0$$

for the compact non-selfadjoint operator $K : H_0^2(D) \times H_0^2(D) \to H_0^2(D) \times H_0^2(D)$ given by

$$K := \begin{pmatrix} K_1 & -K_2^{1/2} \\ K_2^{1/2} & 0 \end{pmatrix}.$$

From the spectral theory for compact operators we immediately obtain a proof of the discreteness of transmission eigenvalues for the case when $n(x) > 1$ for $x \in \bar{D}$:

Theorem 10.29. *Assume that $n(x) > 1$ for $x \in \bar{D}$. Then the set of transmission eigenvalues is at most discrete with ∞ as the only (possible) accumulation point. Furthermore, the multiplicity of each transmission eigenvalue is finite.*

The non-selfadjointness nature of the interior transmission eigenvalue problem calls for new techniques to prove the existence of transmission eigenvalues. For this reason the existence of transmission eigenvalues remained an open problem until Päivärinta and Sylvester showed in [267] that for large enough index of refraction n there exists at least one transmission eigenvalue. The existence of transmission eigenvalues was completely resolved in [39] where the existence of an infinite set of transmission eigenvalues was proven under the only assumption that $n > 1$ or $0 < n < 1$. Here we present the proof in [39] which makes use of the analytical framework developed above.

Theorem 10.30. *Assume that $n_{\min} > 1$. Then there exist an infinite number of transmission eigenvalues with ∞ as the only accumulation point.*

Proof. We start by defining the bounded sesquilinear forms

$$\mathcal{A}_\tau(u, v) = \int_D \left[\frac{1}{n-1} (\Delta u + \tau u)\, (\Delta \bar{v} + \tau \bar{v}) + \tau^2 u \bar{v} \right] dx$$

and

$$\mathcal{B}(u, v) = \int_D \operatorname{grad} u \cdot \operatorname{grad} \bar{v}\, dx.$$

on $H_0^2(D) \times H_0^2(D)$. Using the Riesz representation theorem we now define the bounded linear operators $A_\tau : H_0^2(D) \to H_0^2(D)$ and $B : H_0^2(D) \to H_0^2(D)$ by

$$(A_\tau u, v)_{H^2(D)} = \mathcal{A}_\tau(u, v)$$

and

$$(Bu, v)_{H^2(D)} = \mathcal{B}(u, v).$$

The operators A and B are clearly self-adjoint. Furthermore, since the sesquilinear form $\mathcal{A}_\tau$ is a coercive sesquilinear form on $H_0^2(D) \times H_0^2(D)$, the operator A is strictly positive definite and hence invertible. Indeed, since

$$\frac{1}{n(x) - 1} > \frac{1}{n_{\max} - 1} = \gamma > 0$$

for $x \in D$ we have

$$\begin{aligned}
\mathcal{A}_\tau(u,u) &\geq \gamma\|\Delta u + \tau u\|^2_{L^2(D)} + \tau^2\|u\|^2_{L^2(D)} \\
&\geq \gamma\|\Delta u\|^2_{L^2(D)} - 2\gamma\tau\|\Delta u\|_{L^2(D)}\|u\|_{L^2(D)} + (\gamma+1)\tau^2\|u\|^2_{L^2(D)} \\
&= \varepsilon\left(\tau\|u\|_{L^2(D)} - \frac{\gamma}{\varepsilon}\|\Delta u\|_{L^2(D)}\right)^2 + \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)\|\Delta u\|^2_{L^2(D)} + (1+\gamma-\varepsilon)\tau^2\|u\|^2_{L^2(D)} \\
&\geq \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)\|\Delta u\|^2_{L^2(D)} + (1+\gamma-\varepsilon)\tau^2\|u\|^2_{L^2(D)}
\end{aligned}$$

for some $\gamma < \varepsilon < \gamma + 1$. Furthermore, since $\operatorname{grad} u \in H^1_0(D)$, using the Poincaré inequality we have that

$$\|\operatorname{grad} u\|^2_{L^2(D)} \leq \frac{1}{\lambda_0(D)}\|\Delta u\|^2_{L^2(D)}$$

where $\lambda_0(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D. Hence we can conclude that

$$(A_\tau u, u)_{H^2(D)} = \mathcal{A}_\tau(u,u) \geq C_\tau\|u\|^2_{H^2(D)}$$

for some positive constant C_τ. We now consider the operator B. By definition B is a non-negative operator, and furthermore, since $H^1_0(D)$ is compactly embedded in $L^2(D)$ and $\operatorname{grad} u \in H^1_0(D)$ we can conclude that $B : H^2_0(D) \to H^2_0(D)$ is a compact operator. Finally, it is obvious by definition that the mapping $\tau \to A_\tau$ is continuous from $(0,\infty)$ to the set of self-adjoint strictly positive definite operators.

In terms of the above operators we can rewrite (10.94) as

$$(A_\tau u - \tau B u, v)_{H^2(D)} = 0 \tag{10.96}$$

for all $v \in H^2_0(D)$, which means that k is a transmission eigenvalue if and only if $\tau = k^2$ is such that the kernel of the operator $A_\tau u - \tau B$ is not trivial. In order to analyze the kernel of this operator we consider the auxiliary eigenvalue problems

$$A_\tau u - \lambda(\tau) B u = 0 \tag{10.97}$$

for $u \in H^2_0(D)$. Thus a transmission eigenvalue $k > 0$ is such that $\tau = k^2$ satisfies $\lambda(\tau) - \tau = 0$ where $\lambda(\tau)$ is an eigenvalue corresponding to (10.97). In order to prove the existence of an infinite set of transmission eigenvalues, we now use Theorem 10.26 for A_τ and B with $X = H^2_0(D)$. Theorem 10.28 states that as long as $0 < \tau_0 \leq \lambda_0(D)/n_{\max}$ the operator $A_{\tau_0} u - \tau_0 B$ is positive on $H^2_0(D)$, whence the assumption a) of Theorem 10.26 is satisfied for such τ_0.

Next let $k_{1,n_{\min}}$ be the first transmission eigenvalue for a ball of radius one and constant index of refraction $n_{\min}$ (i.e. corresponding to (10.89) for $D = \{x \in \mathbb{R}^3 : |x| < 1\}$ and $n(x) = n_{\min}$). This transmission eigenvalue is the smallest zero of

$$W(k) = \det\begin{pmatrix} j_0(k) & j_0(k\sqrt{n_{\min}}) \\ -j_0'(k) & -\sqrt{n_{\min}}\, j_0'(k\sqrt{n_{\min}}) \end{pmatrix} = 0 \tag{10.98}$$

where j_0 is the spherical Bessel function of order zero (if the smallest zero of the above determinant is not the first transmission eigenvalue, the latter will be a zero of a similar determinant corresponding to higher order spherical Bessel functions). By a scaling argument, it is obvious that $k_{\varepsilon,n_{\min}} := k_{1,n_{\min}}/\varepsilon$ is the first transmission eigenvalue corresponding to the ball of radius $\varepsilon > 0$ with index of refraction $n_{\min}$. Now take $\varepsilon > 0$ small enough such that D contains $m = m(\varepsilon) \geq 1$ disjoint balls $B_\varepsilon^1, B_\varepsilon^2, \ldots, B_\varepsilon^m$ of radius ε, i.e., $\overline{B_\varepsilon^j} \subset D$, $j = 1 \ldots m$, and $\overline{B_\varepsilon^j} \cap \overline{B_\varepsilon^i} = \emptyset$ for $j \neq i$. Then $k_{\varepsilon,n_{\min}} := k_{1,n_{\min}}/\varepsilon$ is the first transmission eigenvalue for each of these balls with index of refraction $n_{\min}$ and let $u_j \in H_0^2(B_\varepsilon^j)$, $j = 1, \ldots, m$, be the corresponding eigenfunctions. We have that

$$\int_{B_\varepsilon^j} \frac{1}{n_{\min} - 1} \left(\Delta u_j + k_{\varepsilon,n_{\min}}^2 u_j\right)\left(\Delta \overline{u}_j + k_{\varepsilon,n_{\min}}^2 n_{\min} \overline{u}_j\right) dx = 0. \tag{10.99}$$

The extension by zero $\tilde{u}_j$ of u_j to the whole D is obviously in $H_0^2(D)$ due to the boundary conditions on ∂B_ε^j. Furthermore, the functions $\{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_m\}$ are linearly independent and orthogonal in $H_0^2(D)$ since they have disjoint supports and from (10.99) we have that

$$\begin{aligned} 0 &= \int_D \frac{1}{n_{\min} - 1} (\Delta \tilde{u}_j + k_{\varepsilon,n_{\min}}^2 \tilde{u}_j)(\Delta \overline{\tilde{u}_j} + k_{\varepsilon,n_{\min}}^2 n_{\min} \overline{\tilde{u}_j})\, dx \\ &= \int_D \left\{ \frac{1}{n_{\min} - 1} |\Delta \tilde{u}_j + k_{\varepsilon,n_{\min}}^2 \tilde{u}_j|^2 + k_{\varepsilon,n_{\min}}^4 |\tilde{u}_j|^2 - k_{\varepsilon,n_{\min}}^2 |\operatorname{grad} \tilde{u}_j|^2 \right\} dx \end{aligned} \tag{10.100}$$

for $j = 1, \ldots, m$. Denote by $\mathcal{U}$ the m-dimensional subspace of $H_0^2(D)$ spanned by $\{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_m\}$. Since each $\tilde{u}_j$, $j = 1, \ldots, m$, satisfies (10.100) and they have disjoint supports, we have that for $\tau_1 := k_{\varepsilon,n_{\min}}^2$ and for every $\tilde{u} \in \mathcal{U}$

$$\begin{aligned} (A_{\tau_1}\tilde{u} - \tau_1 B\tilde{u}, \tilde{u})_{H_0^2(D)} &= \int_D \left\{ \frac{1}{n - 1} |\Delta \tilde{u} + \tau_1 \tilde{u}|^2 + \tau_1^2 |\tilde{u}|^2 - \tau_1 |\operatorname{grad} \tilde{u}|^2 \right\} dx \\ &\leq \int_D \left\{ \frac{1}{n_{\min} - 1} |\Delta \tilde{u} + \tau_1 \tilde{u}|^2 + \tau_1^2 |\tilde{u}|^2 - \tau_1 |\operatorname{grad} \tilde{u}|^2 \right\} dx = 0. \end{aligned}$$

This means that assumption b) of Theorem 10.26 is also satisfied and therefore we can conclude that there are $m(\varepsilon)$ transmission eigenvalues (counting multiplicity) inside $[\tau_0, k_{\varepsilon,n_{\min}}]$. Note that $m(\varepsilon)$ and $k_{\varepsilon,n_{\min}}$ both go to ∞ as $\varepsilon \to 0$. Since the multiplicity of each eigenvalue is finite we have shown, by letting $\varepsilon \to 0$, that there exists an infinite countable set of transmission eigenvalues that accumulate at ∞. □

In a similar way [39] it is possible to prove the following theorem:

Theorem 10.31. *Assume that* $0 < n_{\min} \leq n_{\max} < 1$. *Then there exist an infinite number of transmission eigenvalues with* ∞ *as the only accumulation point.*

The above proof of the existence of transmission eigenvalues provides a framework to obtain lower and upper bounds for the first transmission eigenvalue. To this end denote by $k_{0,n} > 0$ the first real transmission eigenvalue corresponding to n and D (we omit the dependence on D in our notations since D is assumed to be known) and set $\tau_{0,n} := k_{0,n}^2$.

Theorem 10.32. *Assume that the index of refraction n satisfies* $1 < n_{\min}$. *Then*

$$0 < k_{0,n_{\max}} \leq k_{0,n} \leq k_{0,n_{\min}}. \tag{10.101}$$

Proof. From the proof of Theorem 10.30 we have that $\tau_{0,n}$ is the smallest zero of

$$\lambda(\tau, n) - \tau = 0$$

where

$$\lambda(\tau, n) = \inf_{\substack{u \in H_0^2(D) \\ \|\operatorname{grad} u\|_{L^2(D)} = 1}} \int_D \left(\frac{1}{n-1} \, |\Delta u + \tau u|^2 + \tau^2 |u|^2 \right) dx. \tag{10.102}$$

(Note that any zero $\tau > 0$ of $\lambda(\tau, n) - \tau = 0$ leads to a transmission eigenvalue $k = \sqrt{\tau}$). Obviously the mapping $\tau \to \lambda(\tau, n)$ is continuous in $(0, \infty)$. We first note that (10.102) yields

$$\lambda(\tau, n_{\max}) \leq \lambda(\tau, n(x)) \leq \lambda(\tau, n_{\min}) \tag{10.103}$$

for all $\tau > 0$. In particular, for $\tau := \tau_{0,n_{\max}}$ we have that

$$0 = \lambda(\tau_{0,n_{\max}}, n_{\max}) - \tau_{0,n_{\max}} \leq \lambda(\tau_{0,n_{\max}}, n(x)) - \tau_{0,n_{\max}}$$

and for $\tau := \tau_{0,n_{\min}}$ we have that

$$\lambda(\tau_{0,n_{\min}}, n(x)) - \tau_{0,n_{\min}} \leq \lambda(\tau_{0,n_{\min}}, n_{\min}) - \tau_{0,n_{\min}} = 0.$$

By continuity of $\tau \to \lambda(\tau, n) - \tau$ we have that there is a zero $\tilde{\tau}$ of $\lambda(\tau, n) - \tau = 0$ such that $\tau_{0,n_{\max}} \leq \tilde{\tau} \leq \tau_{0,n_{\min}}$. In particular, the smallest zero $\tau_{0,n(x)}$ of $\lambda(\tau, n) - \tau = 0$ is such that $\tau_{0,n(x)} \leq \tilde{\tau} \leq \tau_{0,n_{\min}}$. To end the proof we need to show that $\tau_{0,n_{\max}} \leq \tau_{0,n(x)}$, i.e., all the zeros of $\lambda(\tau, n) - \tau = 0$ are larger than or equal to $\tau_{0,n_{\max}}$. Assume by contradiction that $\tau_{0,n(x)} < \tau_{0,n_{\max}}$. Then from (10.103) on one hand we have

$$\lambda(\tau_{0,n(x)}, n_{\max}) - \tau_{0,n(x)} \leq \lambda(\tau_{0,n(x)}, n(x)) - \tau_{0,n(x)} = 0.$$

On the other hand, from the proof of Theorem 10.28 we have that for a sufficiently small $\tau' > 0$ (in fact for all $0 < \tau' < \lambda_0(D)/n_{\max}$, where $\lambda_0(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in D), we have that $\lambda(\tau', n_{\max}) - \tau' > 0$. Hence there exists a zero of $\lambda(\tau, n_{\max}) - \tau = 0$ between τ' and $\tau_{0,n(x)}$ smaller than $\tau_{0,n_{\max}}$, which contradicts the fact that $\tau_{0,n_{\max}}$ is the smallest zero. Thus we have proven that $\tau_{0,n_{\max}} \leq \tau_{0,n(x)} \leq \tau_{0,n_{\min}}$ which establishes (10.101) and thus ends the proof. □

In a similar way [39] one can prove the following:

Theorem 10.33. *Assume that the index of refraction n satisfies* $0 < n_{\min} \leq n_{\max} < 1$. *Then*

$$0 < k_{0,n_{\min}} \leq k_{0,n} \leq k_{0,n_{\max}}. \tag{10.104}$$

Theorem 10.32 and Theorem 10.33 show in particular that for constant index of refraction the first transmission eigenvalue $k_{0,n}$ is monotonically decreasing if $n > 1$ and is monotonically increasing if $0 < n < 1$. In fact in [33] it is shown that this monotonicity is strict which leads to the following uniqueness result for the constant index of refraction in terms of the first transmission eigenvalue.

Theorem 10.34. *A constant index of refraction n is uniquely determined from a knowledge of the corresponding smallest transmission eigenvalue* $k_{0,n} > 0$ *provided that it is known a priori that either* $n > 1$ *or* $0 < n < 1$.

Proof. We show the proof for the case $n > 1$ (see [33] for the case $0 < n < 1$). Assume two homogeneous media with constant index of refraction n_1 and n_2 such that $1 < n_1 < n_2$ and let $u_1 := w_1 - v_1$ where w_1, v_1 is the nonzero solution to (10.89) with $n(x) := n_1$ corresponding to the first transmission eigenvalue k_{0,n_1}. Now, setting $\tau_1 = k_{0,n_1}$ and normalizing u_1 such that $\|\operatorname{grad} u_1\|_{L^2(D)} = 1$, we have from (10.102) that

$$\frac{1}{n_1 - 1}\,\|\Delta u_1 + \tau_1 u_1\|^2_{L^2(D)} + \tau_1^2\|u_1\|^2_{L^2(D)} = \tau_1 = \lambda(\tau_1, n_1).$$

Furthermore, we have

$$\frac{1}{n_2 - 1}\,\|\Delta u + \tau u\|^2_{L^2(D)} + \tau^2\|u\|^2_{L^2(D)} < \frac{1}{n_1 - 1}\,\|\Delta u + \tau u\|^2_{L^2(D)} + \tau^2\|u\|^2_{L^2(D)}$$

for all $u \in H_0^2(D)$ such that $\|\operatorname{grad} u\|_{L^2(D)} = 1$ and all $\tau > 0$. In particular for $u = u_1$ and $\tau = \tau_1$

$$\frac{1}{n_2 - 1}\,\|\Delta u_1 + \tau_1 u_1\|^2_{L^2(D)} + \tau_1^2\|u_1\|^2_{L^2(D)}$$

$$< \frac{1}{n_1 - 1}\,\|\Delta u_1 + \tau_1 u_1\|^2_{L^2(D)} + \tau_1^2\|u_1\|^2_{L^2(D)} = \lambda(\tau_1, n_1).$$

But

$$\lambda(\tau_1, n_2) \leq \frac{1}{n_2 - 1}\,\|\Delta u_1 + \tau_1 u_1\|^2_{L^2(D)} + \tau_1^2\|u_1\|^2_{L^2(D)} < \lambda(\tau_1, n_1)$$

and hence for this τ_1 we have a strict inequality, i.e.,

$$\lambda(\tau_1, n_2) < \lambda(\tau_1, n_1). \tag{10.105}$$

From (10.105) we see that the first zero τ_2 of $\lambda(\tau, n_2) - \tau = 0$ is such that $\tau_2 < \tau_1$ and therefore we have that $k_{0,n_2} < k_{0,n_1}$ for the first transmission eigenvalues k_{0,n_1} and k_{0,n_2} corresponding to n_1 and n_2, respectively. Hence we have shown that if $n_1 > 1$ and $n_2 > 1$ are such that $n_1 \neq n_2$ then $k_{0,n_1} \neq k_{0,n_2}$, which proves the uniqueness. □

We note in passing that Theorem 10.32 provides a significant improvement on the estimate for $n(x)$ than the lower bound for $n_{\max}$ given by Theorem 10.28. In particular if $k_{0,n(x)}$ is the first transmission eigenvalue let $n_0 > 0$ be the unique constant such that $k_{0,n_0} = k_{0,n(x)}$. Then n_0 provides an approximation to $n(x)$ in the sense that $k_{0,n_{\max}} \leq k_{0,n_0} \leq k_{0,n_{\min}}$. In the case of two dimensions, examples of the computation of n_0 are given in [311]. In particular if D is the unit square $(-1/2, 1/2)\times(-1/2, 1/2)$ in $\mathbb{R}^2$ and

$$n(x_1, x_2) = 8 + x_1 - x_2$$

then Theorem 10.28 gives $n_{\max} \geq 2.35$ whereas using Theorem 10.32 as described above gives $n_0 = 7.87$.

10.6 The Inverse Medium Problem for Electromagnetic Waves

We recall from Chapter 9 that the direct scattering problem for electromagnetic waves can be formulated as that of determining the electric field E and magnetic field H such that

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikn(x)E = 0 \quad \text{in } \mathbb{R}^3, \tag{10.106}$$

$$E(x) = \frac{i}{k} \operatorname{curl}\operatorname{curl} p\, e^{ik\,x\cdot d} + E^s(x), \quad H(x) = \operatorname{curl} p\, e^{ik\,x\cdot d} + H^s(x), \tag{10.107}$$

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0 \tag{10.108}$$

uniformly for all directions where $k > 0$ is the wave number, $p \in \mathbb{R}^3$ is the polarization and $d \in \mathbb{S}^2$ the direction of the incident wave. The refractive index $n \in C^{1,\alpha}(\mathbb{R}^3)$ is of the form

$$n(x) = \frac{1}{\varepsilon_0} \left\{ \varepsilon(x) + i\, \frac{\sigma(x)}{\omega} \right\}$$

where $\varepsilon = \varepsilon(x)$ is the permittivity, $\sigma = \sigma(x)$ is the conductivity and ω is the frequency. We assume that $m := 1 - n$ is of compact support and, as usual, define

$$D := \left\{ x \in \mathbb{R}^3 : m(x) \neq 0 \right\}.$$

It is further assumed that D is connected with a connected C^2 boundary ∂D and D contains the origin. The existence of a unique solution to (10.106)–(10.108) was established in Chapter 9. It was also shown there that E^s has the asymptotic behavior

$$E^s(x,d)p = \frac{e^{ik|x|}}{|x|}\, E_\infty(\hat{x},d)p + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty, \tag{10.109}$$

where E_∞ is the electric far field pattern. The *inverse medium problem* for electromagnetic waves is to determine n from $E_\infty(\hat{x},d)p$ for $\hat{x}, d \in \mathbb{S}^2$, $p \in \mathbb{R}^3$, and (possibly) different values of k. It can be shown that for k fixed, $\hat{x}, d \in \mathbb{S}^2$ and $p \in \mathbb{R}^3$, the electric far field pattern E_∞ uniquely determines n [36, 85, 127]. The proof of this fact is similar to the one for acoustic waves given in Theorem 10.5. The main difference is that we must now construct a solution E, H of (10.106) such that E has the form

$$E(x) = e^{i\zeta\cdot x}[\eta + R_\zeta(x)]$$

where $\zeta, \eta \in \mathbb{C}^3$, $\eta \cdot \zeta = 0$ and $\zeta \cdot \zeta = k^2$ and, in contrast to the case of acoustic waves, it is no longer true that R_ζ decays to zero as $|\zeta|$ tends to infinity. This makes the uniqueness proof for electromagnetic waves more complicated than the corresponding proof for acoustic waves and for details we refer to [85]. For uniqueness results in the case when the magnetic permeability is also a function of x, i.e., $\mu = \mu(x)$, we refer the reader to [264, 265, 312].

As with the case of acoustic waves there are a number of methods that can be used to solve the inverse medium problem for electromagnetic waves. One approach is to use sampling methods together with a knowledge of the first transmission eigenvalue as was done in Section 10.5 for acoustic waves. Such an approach yields qualitative information on the index of refraction n and for details of such an approach for electromagnetic waves we refer the reader to the monograph [36]. A second approach to determining the index of refraction n is to use an optimization method applied to the integral equation (9.7) in order to determine n. Since this can be done in precisely the same manner as in the case for acoustic waves (c.f. Section 10.3), we shall forego such an investigation and proceed directly to the derivation of decomposition methods for solving the electromagnetic inverse medium problem that are analogous to those derived for acoustic waves in Section 10.4. To this end, we recall the Hilbert space $L^2_t(\mathbb{S}^2)$ of L^2 tangential fields on the unit sphere $\mathbb{S}^2$, let $\{d_n : n = 1,2,3,\dots\}$ be a dense set of vectors on $\mathbb{S}^2$ and consider the set of electric far field patterns $\mathcal{F} := \{E_\infty(\cdot\,, d_n)e_j : n = 1,2,3,\dots,\ j = 1,2,3\}$ where e_1, e_2, e_3 are the unit coordinate vectors in $\mathbb{R}^3$. Following the proofs of Theorems 6.41 and 9.7, we can immediately deduce the following result.

Theorem 10.35. *For $q \in \mathbb{R}^3$, define the radiating solution E_q, H_q of the Maxwell equations by*

$$E_q(x) := \operatorname{curl} q\Phi(x,0), \quad H_q(x) := \frac{1}{ik}\operatorname{curl}\operatorname{curl} q\Phi(x,0), \quad x \in \mathbb{R}^3 \setminus \{0\}.$$

Then there exists $g \in L^2_t(\mathbb{S}^2)$ such that

$$\int_{\mathbb{S}^2} E_\infty(\hat{x}, d)p \cdot g(\hat{x})\, ds(\hat{x}) = \frac{ik}{4\pi}\, p \cdot q \times d \tag{10.110}$$

for all $p \in \mathbb{R}^3$ and $d \in \mathbb{S}^2$ if and only if there exists a solution E_0, E_1, H_0, H_1 in $C^1(D) \cap C(\bar{D})$ of the electromagnetic interior transmission problem

$$\operatorname{curl} E_1 - ikH_1 = 0, \quad \operatorname{curl} H_1 + ikn(x)E_1 = 0 \quad \textit{in } D, \tag{10.111}$$

$$\operatorname{curl} E_0 - ikH_0 = 0, \quad \operatorname{curl} H_0 + ikE_0 = 0 \quad \textit{in } D, \tag{10.112}$$

$$\nu \times (E_1 - E_0) = \nu \times E_q, \quad \nu \times (H_1 - H_0) = \nu \times H_q \quad \textit{on } \partial D, \tag{10.113}$$

such that E_0, H_0 is an electromagnetic Herglotz pair.

In order to make use of Theorem 10.35, we need to show that there exists a solution to the interior transmission problem (10.111)–(10.113) and that E_0, H_0 can be approximated by an electromagnetic Herglotz pair. Following the ideas of Colton and Päivärinta [83], we shall now proceed to do this for the special case when $\varepsilon(x) = \varepsilon_0$ for all $x \in \mathbb{R}^3$, i.e.,

$$n(x) = 1 + i\, \frac{\sigma(x)}{\varepsilon_0 \omega} \tag{10.114}$$

where $\sigma(x) > 0$ for $x \in D$.

We begin by introducing the Hilbert space $L^2_\sigma(D)$ defined by

$$L^2_\sigma(D) := \left\{ f : D \to \mathbb{C}^3 : f \text{ measurable}, \int_D \sigma |f|^2 < \infty \right\}$$

with scalar product

$$(f, g) := \int_D \sigma\, f \cdot \bar{g}\, dx.$$

Of special importance to us is the subspace $H \subset L^2_\sigma(D)$ defined by

$$H := \operatorname{span}\{M_n^m, \operatorname{curl} M_n^m : n = 1, 2, \dots, \ m = -n, \dots, n\}$$

where, as in Section 6.5,

$$M_n^m(x) := \operatorname{curl}\left\{ x\, j_n(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right) \right\}.$$

Let $\bar{H}$ denote the closure of H in $L^2_\sigma(D)$. Instead of considering solutions of (10.111)–(10.113) in $C^1(D) \cap C(\bar{D})$, it is convenient for our purposes to consider a weak formulation based on Theorem 9.2.

Definition 10.36 *Let $\varepsilon(x) = \varepsilon_0$ for all $x \in \mathbb{R}^3$. Then the pair $E_0, E_1 \in L^2_\sigma(D)$ is said to be a weak solution of the interior transmission problem for electromagnetic* *waves with* $q \in \mathbb{R}^3$ *if* $E_0 \in \bar{H}$, $E_1 \in L^2_\sigma(D)$ *satisfy the integral equation*

$$E_1 = E_0 + T_\sigma E_1 \quad \text{in } D, \tag{10.115}$$

where

$$\begin{aligned}(T_\sigma E)(x) &:= i\mu_0\omega \int_D \Phi(x,y)\sigma(y)E(y)\,dy \\ &\quad + \operatorname{grad} \int_D \frac{1}{n(y)} \operatorname{grad} n(y) \cdot E(y)\,\Phi(x,y)\,dy, \quad x \in \mathbb{R}^3,\end{aligned} \tag{10.116}$$

for $n(x) = 1 + i\sigma(x)/\varepsilon_0\omega$ *and*

$$T_\sigma E_1 = E_q \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \tag{10.117}$$

where $E_q(x) := \operatorname{curl} q\Phi(x,0)$.

Before proceeding, we make some preliminary observations concerning Definition 10.36. To begin with, as in the acoustic case (see Section 8.6), we can view (10.117) as a generalized form of the boundary conditions (10.113). In order to ensure the existence of the second integral in (10.116), for the sake of simplicity, we assume that there exists a positive constant M such that

$$|\operatorname{grad} \sigma(x)|^2 \le M\sigma(x), \quad x \in D.$$

This also implies that we can write $T_\sigma(E) = \tilde{T}(\sqrt{\sigma}\,E)$ where $\tilde{T}$ has a weakly singular kernel. In particular, from this we can see that $T_\sigma : L^2_\sigma(D) \to L^2_\sigma(D)$ is compact. From the proof of Theorem 9.5 we recall that the inverse operator $(I - T_\sigma)^{-1} : C(\bar{D}) \to C(\bar{D})$ exists and is bounded. For our following analysis we also need to establish the boundedness of $(I - T_\sigma)^{-1} : L^2_\sigma(D) \to L^2_\sigma(D)$. To this end, we first note that since T_σ is an integral operator with weakly singular kernel it has an adjoint T^*_σ with respect to the L^2 bilinear form

$$\langle E, F\rangle := \int_D E \cdot F\,dx$$

which again is an integral operator with a weakly singular kernel and therefore compact from $C(\bar{D})$ into $C(\bar{D})$. Hence, by the Fredholm alternative, applied in the two dual systems $\langle C(\bar{D}), C(\bar{D})\rangle$ and $\langle L^2_\sigma(D), C(\bar{D})\rangle$ (see the proof of Theorem 3.27) the nullspaces of the operator $I - T_\sigma$ in $C(\bar{D})$ and $L^2_\sigma(D)$ coincide. Since from the proof of Theorem 9.5 we already know that the nullspace in $C(\bar{D})$ is trivial, by the Riesz–Fredholm theory, we have established existence and boundedness of $(I - T_\sigma)^{-1} : L^2_\sigma(D) \to L^2_\sigma(D)$.

If $E_0 \in H$, then given a solution $E_1 \in L^2_\sigma(D)$ of (10.115), we can use (10.115) and (10.116) to define E_1 also in $\mathbb{R}^3 \setminus \bar{D}$ such that (10.115) is satisfied in all of $\mathbb{R}^3$. From Theorem 9.2 we can then conclude that $E_0, H_0 = \operatorname{curl} E_0/ik$ and $E_1, H_1 = \operatorname{curl} E_1/ik$ satisfy (10.112) and (10.111) respectively. If $E_0 \in \bar{H}$, we choose a sequence $(E_{0,j})$

from H such that $E_{0,j} \to E_0,\ j \to \infty$, and define $E_{1,j}$ by $E_{1,j} := (I-T_\sigma)^{-1}E_{0,j}$. Then, by the boundedness of $(I - T_\sigma)^{-1} : L^2_\sigma(D) \to L^2_\sigma(D)$, we have $E_{1,j} \to E_1,\ j \to \infty$. From $E_{1,j} - E_{0,j} = T_\sigma E_{1,j}$ we see that $\operatorname{div} T_\sigma E_{1,j} = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. Since from (10.116) we have that

$$(\operatorname{div} T_\sigma E_1)(x) = i\mu_0\omega \operatorname{div} \int_D \Phi(x,y)\sigma(y)E_1(y)\,dy$$

$$-k^2 \int_D \frac{1}{n(y)}\ \operatorname{grad} n(y) \cdot E_1(y)\,\Phi(x,y)\,dy, \quad x \in \mathbb{R}^3 \setminus \bar{D},$$

using the Cauchy–Schwarz inequality we can now deduce that $\operatorname{div} T_\sigma E_1 = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. Hence, by Theorems 6.4 and 6.8 we conclude that $T_\sigma E_1$ is a solution to the Maxwell equations in $\mathbb{R}^3 \setminus \bar{D}$ satisfying the Silver–Müller radiation condition.

We now proceed to proving that there exists a unique weak solution to the interior transmission problem for electromagnetic waves.

Theorem 10.37. *Let $\varepsilon(x) = \varepsilon_0$ for all $x \in \mathbb{R}^3$. Then for any $q \in \mathbb{R}^3$ and wave number $k > 0$ there exists at most one weak solution of the interior transmission problem for electromagnetic waves.*

Proof. It suffices to show that if

$$E_1 = E_0 + T_\sigma E_1 \quad \text{in } D, \tag{10.118}$$

$$T_\sigma E_1 = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}, \tag{10.119}$$

where $E_0 \in \bar{H}$ then $E_1 = 0$. We first show that the only solution of (10.118) with $E_0 \in \bar{H}$ and $E_1 \in H^\perp$ is $E_0 = E_1 = 0$. To this end, let $(E_{0,j})$ be a sequence from H with $E_{0,j} \to E_0,\ j \to \infty$, and, as above, define $E_{1,j}$ by $E_{1,j} := (I - T_\sigma)^{-1}E_{0,j}$. Then $E_{1,j} - E_{0,j} = T_\sigma E_{1,j}$ and, with the aid of

$$\operatorname{curl}\operatorname{curl} E_{0,j} = k^2 E_{0,j} \quad \text{and} \quad \operatorname{curl}\operatorname{curl} E_{1,j} = k^2 n E_{1,j},$$

by Green's vector theorem (6.2) using $k^2 = \varepsilon_0\mu_0\omega^2$ we find that

$$\begin{aligned}\int_{\partial B} \nu \cdot \overline{T_\sigma E_{1,j}} \times \operatorname{curl} T_\sigma E_{1,j}\,ds &= \int_B |\operatorname{curl}(E_{0,j} - E_{1,j})|^2 dx \\ -k^2 \int_B |E_{0,j} - E_{1,j}|^2 dx - i\mu_0\omega &\int_B \sigma |E_{1,j}|^2 dx + i\mu_0\omega \int_B \sigma\, E_{1,j} \cdot \overline{E_{0,j}}\,dx\end{aligned} \tag{10.120}$$

where B is an open ball with $\bar{D} \subset B$. Furthermore, from (10.116) we have

$$(\operatorname{curl} T_\sigma E_1)(x) = i\mu_0\omega \operatorname{curl} \int_D \sigma(y)\Phi(x,y)E_1(y)\,dy, \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{10.121}$$

and from the Vector Addition Theorem 6.29 and the fact that $E_1 \in H^\perp$ we obtain that $\operatorname{curl} T_\sigma E_1 = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. Therefore, using the Cauchy–Schwarz inequality in

(10.116) and (10.121), we can now conclude that the integral on the left hand side of (10.120) tends to zero as $j \to \infty$. Hence, taking the imaginary part in (10.120) and letting $j \to \infty$, we see that

$$\int_D \sigma |E_1|^2 dx = 0$$

and hence $E_1 = E_0 = 0$.

We must now show that if E_0, E_1 is a solution of (10.118) and (10.119) with $E_0 \in \bar{H}$ then $E_1 \in H^\perp$. To this end, we note that from (10.119) we trivially have that $\operatorname{curl} T_\sigma E_1 = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. From this, using (10.121) and the Vector Addition Theorem 6.29, we have by orthogonality that

$$\int_D \sigma\, \overline{M_n^m} \cdot E_1 \, dy = 0 \quad \text{and} \quad \int_D \sigma \operatorname{curl} \overline{M_n^m} \cdot E_1 \, dy = 0$$

for $n = 1, 2, \dots$ and $m = -n, \dots, n$, that is, $E_1 \in H^\perp$. □

Having established the uniqueness of a weak solution to the interior transmission problem, we now want to show existence. Without loss of generality, we shall assume that $q = (0, 0, 1)$ and, in this case,

$$E_q(x) := \operatorname{curl} q\Phi(x, 0) = \frac{ik^2}{\sqrt{12\pi}}\, N_1^0(x)$$

where, as in Section 6.5,

$$N_n^m(x) := \operatorname{curl} \left\{ x h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right) \right\}.$$

Theorem 10.38. *Let $\varepsilon(x) = \varepsilon_0$ for all $x \in \mathbb{R}^3$. Then for any $q \in \mathbb{R}^3$ and wave number $k > 0$ there exists a weak solution of the interior transmission problem for electromagnetic waves.*

Proof. We begin by defining H_0 to be the closure in $L^2_\sigma(D)$ of

$$\operatorname{span}\left\{ \operatorname{curl} M_1^0, M_n^m, \operatorname{curl} M_n^m : n = 1, m = \pm 1,\ n = 2, 3, \dots,\ m = -n, \dots, n \right\}$$

and then define the associated orthogonal projection operator $P_0 : \bar{H} \to H_0$. We can then define the vector field $F \in \bar{H}$ by

$$F := M_1^0 - P_0 M_1^0$$

and note that $F \in H_0^\perp$ and $(F, M_1^0) \neq 0$ since $F \neq 0$ and, by orthogonality, $(F, M_1^0) = (F, F)$. Without loss of generality we can assume F is normalized such that

$$(F, M_1^0) = \frac{1}{i\sqrt{3\pi}} \sqrt{\frac{\varepsilon_0}{\mu_0}}\,.$$

We now want to construct a solution $E_0 \in \bar{H}$, $E_1 \in H_0^\perp$ of

$$E_1 = E_0 + T_\sigma E_1 \quad \text{in } D$$

such that

$$(E_1, M_1^0) = \frac{1}{i\sqrt{3\pi}} \sqrt{\frac{\varepsilon_0}{\mu_0}} \,. \tag{10.122}$$

Let $P : L^2_\sigma(D) \to H^\perp$ be the orthogonal projection operator. From the proof of Theorem 10.37, we see that $I - PT_\sigma$ has a trivial nullspace since $E_1 - PT_\sigma E_1 = 0$ implies that $E_1 - T_\sigma E_1 = E_0$ with $E_0 \in \bar{H}$ and $E_1 \in H^\perp$. Hence, by the Riesz–Fredholm theory, the equation

$$\tilde{E}_1 - PT_\sigma \tilde{E}_1 = PT_\sigma F$$

has a unique solution $\tilde{E}_1 \in H^\perp$. Setting

$$E_1 := \tilde{E}_1 + F$$

we have $E_1 - PT_\sigma E_1 = F$ and, since $T_\sigma E_1 = PT_\sigma E_1 + \tilde{E}_0$ with $\tilde{E}_0 \in \bar{H}$, we finally have $E_1 - T_\sigma E_1 = E_0$ with $E_0 = \tilde{E}_0 + F \in \bar{H}$. Since $\tilde{E}_1 \in H^\perp$, the condition (10.122) and $E_1 \in H_0^\perp$ are satisfied.

We will now show that $T_\sigma E_1 = E_q$ in $\mathbb{R}^3 \setminus \bar{D}$ which implies that (10.117) is satisfied, thus completing the proof of the theorem. To show this, we first note that from $E_1 \in H_0^\perp$ and (10.122) we have

$$\begin{aligned} \int_D \sigma\, \overline{M_n^m} \cdot E_1 \, dy &= \delta_{n1}\delta_{m0} \frac{1}{i\sqrt{3\pi}} \sqrt{\frac{\varepsilon_0}{\mu_0}} \,, \\ \int_D \sigma \operatorname{curl} \overline{M_n^m} \cdot E_1 \, dy &= 0 \end{aligned} \tag{10.123}$$

for $n = 1, 2, \dots$ and $m = -n, \dots, n$ where δ_{nm} denotes the Kronecker delta symbol. From (10.121), (10.123) and the Vector Addition Theorem 6.29 we now see that for $|x|$ sufficiently large we have

$$(\operatorname{curl} T_\sigma E_1)(x) = \frac{ik^2}{\sqrt{12\pi}} \operatorname{curl} N_1^0(x) = \operatorname{curl} E_q(x).$$

By unique continuation, this holds for $x \in \mathbb{R}^3 \setminus \bar{D}$. Since from the analysis preceding Theorem 10.37 we know that $T_\sigma E_1$ solves the Maxwell equations in $\mathbb{R}^3 \setminus \bar{D}$ this implies that $T_\sigma E_1 = E_q$ in $\mathbb{R}^3 \setminus \bar{D}$ and we are done. □

Corollary 10.39 *Let $\varepsilon(x) = \varepsilon_0$ for all $x \in \mathbb{R}^3$ and let E_0, E_1 be a weak solution of the interior transmission problem for electromagnetic waves. Then E_0 can be approximated in $L^2_\sigma(D)$ by the electric field of an electromagnetic Herglotz pair.*

Proof. This follows from the facts that $E_0 \in \bar{H}$ and the elements of H are the electric fields of an electromagnetic Herglotz pair. □

Using Theorem 10.35 and Definition 10.36, we can now set up an optimization scheme for solving the inverse scattering problem for electromagnetic waves in the special case when $\varepsilon(x) = \varepsilon_0$ for all $x \in \mathbb{R}^3$ that is analogous to the optimization scheme (10.34) for acoustic waves. We shall spare the reader the details of examining the optimization scheme more closely and instead now proceed to deriving the electromagnetic analogue of the modified scheme used to solve the acoustic inverse medium problem in Section 10.4. An alternate method for modifying the above decomposition method for electromagnetic waves can be found in [56].

The modified dual space method of Section 10.4 was based on two ingredients: the existence of a solution to the interior impedance problem (10.49), (10.51) having the decomposition (10.50) and a characterization of the Herglotz kernel satisfying the identity (10.46) in terms of this impedance problem. Following Colton and Kress [67], we shall now proceed to derive the electromagnetic analogue of these two ingredients from which the electromagnetic analogue of the modified dual space method for solving the acoustic inverse medium problem will follow immediately. Note that in the sequel we no longer assume that $\varepsilon(x) = \varepsilon_0$ for all $x \in \mathbb{R}^3$, i.e., we only assume that $n \in C^{2,\alpha}(\mathbb{R}^3)$ and $\operatorname{Im} n \geq 0$.

We first consider the following interior impedance problem for electromagnetic waves.

Interior Impedance Problem. *Let G be a bounded domain in $\mathbb{R}^3$ containing $D := \{x \in \mathbb{R}^3 : m(x) \neq 0\}$ with connected C^2 boundary ∂G, let c be a given Hölder continuous tangential field on ∂G and λ a complex constant. Find vector fields $E^i, H^i \in C^1(G) \cap C(\bar{G})$ and $E^s, H^s \in C^1(\mathbb{R}^3)$ satisfying*

$$\operatorname{curl} E^i - ikH^i = 0, \quad \operatorname{curl} H^i + ikE^i = 0 \quad \text{in } G, \tag{10.124}$$

$$\operatorname{curl} E^s - ikH^s = 0, \quad \operatorname{curl} H^s + ikE^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{G}, \tag{10.125}$$

and

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0, \tag{10.126}$$

such that $E = E^i + E^s$, $H = H^i + H^s$ satisfies

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikn(x)E = 0 \quad \text{in } G, \tag{10.127}$$

$$\nu \times \operatorname{curl} E - i\lambda\,(\nu \times E) \times \nu = c \quad \text{on } \partial G, \tag{10.128}$$

where, as usual, ν is the unit outward normal to ∂G and the radiation condition (10.126) is assumed to hold uniformly for all directions.

We note that the interior impedance problem can be viewed as the problem of first solving the impedance boundary value problem (10.127), (10.128) and then decomposing the solution such that (10.124)–(10.126) hold.

Theorem 10.40. *Assume $\lambda < 0$. Then the interior impedance problem has at most one solution.*

Proof. Let E, H denote the difference between two solutions. Then from (9.17) and the homogeneous form of the boundary condition (10.128) we see that

$$i\lambda \int_{\partial G} |\nu \times E|^2 ds = -k^2 \int_G (\bar{n}\,|E|^2 - |H|^2)\,dx$$

and hence, taking the imaginary part,

$$\lambda \int_{\partial G} |\nu \times E|^2 ds = k^2 \int_G \operatorname{Im} n\,|E|^2 dx \geq 0.$$

Since $\lambda < 0$ it follows that $\nu \times E = 0$ on ∂G and hence, from the boundary condition, $\nu \times H = 0$ on ∂G. Applying Theorem 6.2 to E and H in the domain $G \setminus \operatorname{supp} m$, we can conclude that E, H can be extended to all of $\mathbb{R}^3$ as a solution to (10.127) satisfying the radiation condition. Hence, by Theorem 9.4, we can conclude that $E = H = 0$ in G.

To show uniqueness for the decomposition (10.124)–(10.126), we assume that $E^i + E^s = 0$ and $H^i + H^s = 0$ are such that (10.124)–(10.126) is valid. Then E^i, H^i can be extended to all of $\mathbb{R}^3$ as an entire solution of the Maxwell equations (10.124) satisfying the radiation condition whence $E^i = H^i = 0$ in $\mathbb{R}^3$ follows (c.f. p. 198). This in turn implies $E^s = H^s = 0$. □

Motivated by the methods of Chapter 9, we now seek a solution of the interior impedance problem by solving the integral equation

$$\begin{aligned} E(x) = \operatorname{curl} \int_{\partial G} \Phi(x,y) a(y)\,ds(y) - k^2 \int_G \Phi(x,y) m(y) E(y)\,dy \\ + \operatorname{grad} \int_G \frac{1}{n(y)} \operatorname{grad} n(y) \cdot E(y)\,\Phi(x,y)\,dy, \quad x \in \bar{G}, \end{aligned} \tag{10.129}$$

where the surface density $a \in C^{0,\alpha}(\operatorname{Div}, \partial G)$ is determined from the boundary condition and, having found a and E, we define H by

$$H(x) := \frac{1}{ik} \operatorname{curl} E(x), \quad x \in G.$$

After recalling the operators M, N and R from our investigation of the exterior impedance problem in Section 9.5, the electromagnetic operator T_e from (9.18) in the proof of Theorem 9.5 (with D replaced by G) and introducing the two additional operators $W : C^{0,\alpha}(\operatorname{Div}, \partial G) \to C(\bar{G})$ and $T_{e,\lambda} : C(\bar{G}) \to C_t^{0,\alpha}(\partial G)$ by

$$(Wa)(x) := \operatorname{curl} \int_{\partial G} \Phi(x,y) a(y)\,ds(y), \quad x \in \bar{G},$$

$$(T_{e,\lambda} E)(x) := \nu(x) \times (\operatorname{curl} T_e E)(x) - i\lambda\,(\nu(x) \times (T_e E)(x)) \times \nu(x), \quad x \in \partial G,$$

we consider the system of integral equations

$$
\begin{aligned}
NRa - i\lambda RMa + i\lambda Ra + T_{e,\lambda}E &= 2c, \\
E - Wa - T_e E &= 0.
\end{aligned}
\tag{10.130}
$$

Analogous to the proof of Theorem 9.13 (c.f. (9.49)), the first integral equation in (10.130) ensures that the impedance boundary condition (10.128) is satisfied. Proceeding as in the proof of Theorem 9.2, it can be seen that the second integral equation guarantees that E and $H = \operatorname{curl} E/ik$ satisfy the differential equations (10.127). The decomposition $E = E^i + E^s$, $H = H^i + H^s$, follows in an obvious way from (10.129). Hence, to show the existence of a solution of the interior impedance problem we must show the existence of a solution to the system of integral equations (10.130).

Theorem 10.41. *Assume $\lambda < 0$. Then there exists a solution to the interior impedance problem.*

Proof. We need to show the existence of a solution to (10.130). To this end, we have by Theorem 3.3 that the operator W is bounded from $C^{0,\alpha}(\mathrm{Div}, \partial G)$ into $C^{0,\alpha}(\bar{G})$ and hence $W : C^{0,\alpha}(\mathrm{Div}, \partial G) \to C(\bar{G})$ is compact by Theorem 3.2. Similarly, using Theorem 8.1, we see that $T_{e,\lambda} : C(\bar{G}) \to C_t^{0,\alpha}(\partial G)$ is bounded.

Now choose a real wave number $\tilde{k}$ which is not a Maxwell eigenvalue for G and denote the operators corresponding to M and N by $\tilde{M}$ and $\tilde{N}$. Then, since $\lambda < 0$, we have from Theorem 9.13 that

$$\tilde{N}R + i\lambda R(I + \tilde{M}) : C^{0,\alpha}(\mathrm{Div}, \partial G) \to C_t^{0,\alpha}(\partial G)$$

has a bounded inverse

$$B := (\tilde{N}R + i\lambda RI + i\lambda R\tilde{M})^{-1} : C_t^{0,\alpha}(\partial G) \to C^{0,\alpha}(\mathrm{Div}, \partial G).$$

Therefore, by setting $b := B^{-1}a$ we can equivalently transform the system (10.130) into the form

$$\begin{pmatrix} I & T_{e,\lambda}B \\ 0 & I \end{pmatrix} \begin{pmatrix} b \\ E \end{pmatrix} - \begin{pmatrix} \tilde{B} & 0 \\ W & T_e \end{pmatrix} \begin{pmatrix} b \\ E \end{pmatrix} = \begin{pmatrix} 2c \\ 0 \end{pmatrix},$$

where

$$\tilde{B} := (\tilde{N} - N)RB + i\lambda R(M + \tilde{M})B.$$

In this system, the first matrix operator has a bounded inverse and the second is compact from $C_t^{0,\alpha}(\partial G) \times C(\bar{G})$ into itself since the operator

$$(N - \tilde{N})R : C^{0,\alpha}(\mathrm{Div}, \partial G) \to C_1^{0,\alpha}(\partial G)$$

is compact by Theorem 2.23 of [64]. Hence, the Riesz–Fredholm theory can be applied.

Suppose a and E is a solution of the homogeneous form of (10.130). Then E and $H := \operatorname{curl} E/ik$ solve the homogeneous interior impedance problem and hence by Theorem 10.40 we have that $E = 0$ in D. This implies from (10.129) that the field $\tilde{E}$ defined by

$$\tilde{E} := \operatorname{curl} \int_{\partial D} \Phi(x, y) a(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D,$$

vanishes in D. By the jump relations of Theorem 6.12 we see that the exterior field $\operatorname{curl} \tilde{E}$ in $\mathbb{R}^3 \setminus \bar{G}$ satisfies $\nu \times \operatorname{curl} \tilde{E} = 0$ on ∂D whence $\operatorname{curl} \tilde{E} = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows by Theorem 6.20. This implies $\tilde{E} = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ since $\operatorname{curl}\operatorname{curl} \tilde{E} - k^2 \tilde{E} = 0$. Hence, by Theorem 6.12 we can conclude that $a = 0$. The proof is now complete. □

We now turn our attention to the second ingredient that is needed in order to extend the modified dual space method of Section 10.4 to the case of electromagnetic waves, i.e., the generalization of the identity (10.46). To this end, let E_∞ be the electric far field pattern corresponding to the scattering problem (10.106)–(10.108) and E_∞^λ the electric far field pattern for the exterior impedance problem (9.42)–(9.44) with

$$c = -\nu \times \operatorname{curl} E^i + i\lambda\, (\nu \times E^i) \times \nu$$

and with H^i and E^i given by (9.19). (We note that by the analytic Riesz–Fredholm Theorem 8.26 applied to the integral equation (9.46) we have that there exists a solution of the exterior impedance problem not only for $\lambda > 0$ but in fact for all $\lambda \in \mathbb{C}$ with the exception of a countable set of values of λ accumulating only at zero and infinity.) Our aim is to find a vector field $g \in L_t^2(\mathbb{S}^2)$ such that, given $q \in \mathbb{R}^3$, we have

$$\int_{\mathbb{S}^2} [E_\infty(\hat{x}, d)p - E_\infty^\lambda(\hat{x}, d)p] \cdot g(\hat{x})\, ds(\hat{x}) = \frac{ik}{4\pi}\, p \cdot q \times d \tag{10.131}$$

for all $d \in \mathbb{S}^2$, $p \in \mathbb{R}^3$. To this end, we have the following theorem.

Theorem 10.42. *Let $q \in \mathbb{R}^3$ and define E_q and H_q by*

$$E_q(x) := \operatorname{curl} q\Phi(x, 0), \quad H_q(x) := \frac{1}{ik} \operatorname{curl}\operatorname{curl} q\Phi(x, 0), \quad x \in \mathbb{R}^3 \setminus \{0\}.$$

Suppose $\lambda < 0$ is such that there exists a solution to the exterior impedance problem. Then there exists $g \in L_t^2(\mathbb{S}^2)$ such that the integral equation (10.131) is satisfied for all $d \in \mathbb{S}^2$, $p \in \mathbb{R}^3$, if and only if the solution of the interior impedance problem for $c = \nu \times \operatorname{curl} E_q - i\lambda(\nu \times E_q) \times \nu$ is such that E^i, H^i is an electromagnetic Herglotz pair.

Proof. First assume that there exists $g \in L_t^2(\mathbb{S}^2)$ such that (10.131) is true. Then, by the reciprocity relations given in Theorems 9.6 and 9.13 we have that

$$p \cdot \int_{\mathbb{S}^2} [E_\infty(-d, -\hat{x}) g(\hat{x}) - E_\infty^\lambda(-d, -\hat{x}) g(\hat{x})]\, ds(\hat{x}) = \frac{ik}{4\pi}\, p \cdot q \times d$$

for every $p \in \mathbb{R}^3$ and hence

$$\int_{\mathbb{S}^2} [E_\infty(-d, -\hat{x})g(\hat{x}) - E_\infty^\lambda(-d, -\hat{x})g(\hat{x})]\, ds(\hat{x}) = \frac{ik}{4\pi}\, q \times d,$$

or

$$\int_{\mathbb{S}^2} [E_\infty(\hat{x}, -d)h(d) - E_\infty^\lambda(\hat{x}; d)h(d)]\, ds(d) = \frac{ik}{4\pi}\, \hat{x} \times q \tag{10.132}$$

where $h(d) = g(-d)$. For this h, we define the electromagnetic Herglotz pair E_0^i, H_0^i by

$$H_0^i(x) = \operatorname{curl} \int_{\mathbb{S}^2} h(d)\, e^{ik\, x\cdot d}\, ds(d), \quad E_0^i(x) = -\frac{1}{ik} \operatorname{curl} H_0^i(x), \tag{10.133}$$

denote by E_0^s, H_0^s the radiating field of (9.3)–(9.6) with incident field given by (10.133) and let $E_{\lambda 0}^s$ be the radiating field of the exterior impedance problem with E^i replaced by E_0^i in the definition of c above. Then the left hand side of (10.132) represents the electric far field pattern of the scattered field $E_0^s - E_{\lambda 0}^s$. Since E_q, H_q is a radiating solution of the Maxwell equations with electric far field given by the right hand side of (10.132), we can conclude from Theorem 6.10 that

$$E_0^s - E_{\lambda 0}^s = E_q \quad \text{in } \mathbb{R}^3 \setminus \bar{D}. \tag{10.134}$$

It can now be verified that $E_0 = E_0^s + E_0^i$, $H_0 = H_0^s + H_0^i$, satisfies the interior impedance problem with c as given in the theorem.

Now suppose there exists a solution to the interior impedance problem with c given as in the theorem such that E^i, H^i is an electromagnetic Herglotz pair. Relabel E^i, H^i by E_0^i, H_0^i and let E_0^s, H_0^s be the relabeled scattered fields. Then, defining $E_{\lambda 0}^s$ as above, we see that (10.134) is valid. Retracing our steps, we see that (10.131) is true, and the proof of the theorem is complete. □

With Theorems 10.40, 10.41 and 10.42 at our disposal, we can now follow Section 10.4 for the case of acoustic waves and formulate an optimization scheme for solving the inverse medium problem for electromagnetic waves. In particular, from the proof of Theorem 10.41, we see that if G is a ball then the solution of the interior impedance problem can be approximated in $C(\bar{G})$ by a continuously differentiable solution such that E^i, H^i is an electromagnetic Herglotz pair (Approximate the surface density by a finite sum of the surface gradients of spherical harmonics and the rotation of these functions by ninety degrees; see Theorem 6.25). This implies that (10.131) can be approximated in $C(\mathbb{S}^2)$. We can now reformulate the inverse medium problem for electromagnetic waves as a problem in constrained optimization in precisely the same manner as in the case of acoustic waves in Section 10.4. By the above remarks, the cost functional of this optimization scheme has infimum equal to zero provided the constraint set is sufficiently large. Since this approach for solving the electromagnetic inverse medium problem is completely analogous to the method for solving the acoustic inverse medium problem given in Section 10.4, we shall omit giving further details.

10.7 Numerical Examples

We shall now proceed to give some simple numerical examples of the methods discussed in this chapter for solving the acoustic inverse medium problem. We shall refer to the dual space method discussed in Section 10.4 as Method A and the modified dual space method of Section 10.4 as Method B. For the sake of simplicity we shall restrict ourself to the case of a spherically stratified medium, i.e., it is known a priori that $m(x) = m(r)$ and $m(r) = 0$ for $r > a$. Numerical examples for the case of non-spherically stratified media in $\mathbb{R}^2$ can be found in [53].

We first consider the direct problem. When $m(x) = m(r)$, the total field u in (10.1)–(10.3) can be written as

$$u(x) = \sum_{p=0}^{\infty} u_p(r)\, P_p(\cos\theta), \tag{10.135}$$

where P_p is Legendre's polynomial, $r = |x|$ and

$$\cos\theta = \frac{x \cdot d}{r}\,.$$

Without loss of generality we can assume that $d = (0, 0, 1)$. Using (10.135) in (10.4), it is seen that u_p satisfies

$$u_p(r) = i^p(2p+1)\, j_p(kr) - ik^3 \int_0^a K_p(r,\rho) m(\rho) u_p(\rho) \rho^2\, d\rho \tag{10.136}$$

where

$$K_p(r,\rho) := \begin{cases} j_p(kr)\, h_p^{(1)}(k\rho), & \rho > r, \\ j_p(k\rho)\, h_p^{(1)}(kr), & \rho \le r, \end{cases}$$

and j_p and $h_p^{(1)}$ are, respectively, the p–th order spherical Bessel and first kind Hankel functions. Then, from Theorem 2.16 and (10.136), we obtain

$$u_\infty(\hat{x}, d) = \sum_{p=0}^{\infty} f_p(k)\, P_p(\cos\,\theta) \tag{10.137}$$

where

$$f_p(k) = (-i)^{p+2} k^2 \int_0^a j_p(k\rho) m(\rho) u_p(\rho) \rho^2\, d\rho. \tag{10.138}$$

The Fourier coefficients $f_p(k)$ of the far field pattern can thus be found by solving (10.136) and using (10.138).

Turning now to the inverse problem, we discuss Method A first and, in the case that m is real, assume that k is not a transmission eigenvalue. From (10.27) and (10.137) we have

$$g_{pq} = i^{p-1} \frac{(2p+1)Y_p^q}{4\pi k f_p(k)} . \tag{10.139}$$

Under the above assumptions, we have that $f_p(k) \neq 0$. As to be expected, g_{pq} depends on q in a way that is independent of u_∞ and hence independent of m. Thus, we only consider the case $q = 0$. Having computed g_{p0}, we can compute v from (10.30) and (10.139) as

$$v_p(x) = -\frac{i(2p+1)j_p(kr)}{kf_p(k)} \sqrt{\frac{2p+1}{4\pi}}\, P_p(\cos\theta). \tag{10.140}$$

From (10.28) and (10.29) we now have that w is given by

$$w(x) = w_p(r) \sqrt{\frac{2p+1}{4\pi}}\, P_p(\cos\theta)$$

where w_p satisfies

$$w_p(r) = -\frac{i(2p+1)j_p(kr)}{kf_p(k)} - ik^3 \int_0^a K_p(r,\rho)m(\rho)w_p(\rho)\rho^2\, d\rho, \tag{10.141}$$

$$w_p(b) + \frac{i(2p+1)j_p(kb)}{kf_p(k)} = h_p^{(1)}(kb) \tag{10.142}$$

and

$$\frac{\partial}{\partial r}\left(w_p(r) + \frac{i(2p+1)j_p(kr)}{kf_p(k)} \right)\Bigg|_{r=b} = \left(\frac{\partial}{\partial r} h_p^{(1)}(kr) \right)\Bigg|_{r=b} \tag{10.143}$$

where $b > a$. Note that (10.142) and (10.143) each imply the other since, if we replace b by r, from (10.141) we see that both sides of (10.142) are radiating solutions to the Helmholtz equation and hence are equal for $r \geq a$. To summarize, Method A consists of finding m and w_p for $p = 0, 1, 2, \ldots$ such that (10.141) and either (10.142) or (10.143) hold for each p and for k in some interval.

A similar derivation to that given above can be carried out for Method B and we only summarize the results here. Let

$$\gamma_p(k) := \frac{i(2p+1)}{k} \left. \frac{\dfrac{\partial}{\partial r} j_p(kr) + i\lambda j_p(kr)}{\dfrac{\partial}{\partial r} h_p^{(1)}(kr) + i\lambda h_p^{(1)}(kr)} \right|_{r=b}, \quad p = 0, 1, 2, \ldots,$$

i.e., the $\gamma_p(k)$ are the Fourier coefficients of h_∞ where h_∞ is the far field pattern associated with (10.41)–(10.44). Then w_p satisfies the integral equation

$$w_p(r) = -\frac{i(2p+1)j_p(kr)}{k[f_p(k)-\gamma_p(k)]} - ik^3 \int_0^a K_p(r,\rho)m(\rho)w_p(\rho)\rho^2\,d\rho \qquad (10.144)$$

and the impedance condition (10.51) becomes

$$\left(\frac{\partial}{\partial r} + ik\lambda\right)(w_p(r) - h_p^{(1)}(kr))\Big|_{r=b} = 0. \qquad (10.145)$$

Thus, for Method B, we must find m and w_p for $p = 0, 1, 2, \ldots$, such that (10.144) and (10.145) are satisfied for an interval of k values.

To construct our synthetic far field data, we use a N_f point trapezoidal Nyström method to approximate u_p as a solution of (10.136). Then we compute the far field pattern for

$$k = k_j = k_{\min} + (k_{\max} - k_{\min})\,\frac{j-1}{N_k - 1}\,, \quad j = 1, 2, \ldots, N_k,$$

by using the trapezoidal rule with N_f points to discretize (10.138). Thus the data for the inverse solver is an approximation to $f_p(k_j)$ for $p = 0, 1, \ldots, P$ and $j = 1, 2, \ldots, N_k$.

We now present some numerical results for Methods A and B applied to the inverse problem using the synthetic far field data obtained above. To discretize m we use a cubic spline basis with N_m equally spaced knots in $[0, a]$ under the constraints that $m(a) = m'(0) = 0$. This implies that the expansion for m has N_m free parameters that must be computed via an appropriate inverse algorithm. To implement Methods A and B, we approximate, respectively, (10.141) and (10.144) using the trapezoidal Nyström method with N_i equally spaced quadrature points on $[0, a]$. The approximate solution to (10.141) or (10.144) can be computed away from the Nyström points by using the N_i point trapezoidal rule to approximate the integral in (10.141) or (10.144). Let $w_p^a(r, \tilde{m}, k)$ represent the Nyström solution of either (10.141) or (10.144) with m replaced by an arbitrary function $\tilde{m}$.

For Method A, we choose to work with (10.141) and (10.143). For this case the inverse algorithm consists of finding $m^\star$ such that the sum of the squares of

$$F^A_{p,j}(\tilde{m}) := \frac{\partial}{\partial r}\left(w_p^a(r, \tilde{m}, k_j) + i\,\frac{(2p+1)j_p(k_j r)}{k_j f_p(k_j)} - h_p^{(1)}(k_j r)\right)\Bigg|_{r=b} \qquad (10.146)$$

is as small as possible when $\tilde{m} = m^\star$. Similarly, from (10.145), Method B consists of finding $m^\star$ such that the sum of the squares of

$$F^B_{p,j}(\tilde{m}) := \left(\frac{\partial}{\partial r} + ik\lambda\right)(w_p^a(r, \tilde{m}, k_j) - h_p^{(1)}(k_j r))\Big|_{r=b} \qquad (10.147)$$

is as small as possible when $\tilde{m} = m^\star$. Since the inverse problem is ill-posed, we use a Tikhonov regularization technique to minimize (10.146) or (10.147). Let

$$J_\alpha(\tilde{m}) := \frac{1}{N_k(P+1)} \sum_{p=0}^{P} \sum_{j=1}^{N_k} |F_{p,j}(\tilde{m})|^2 + \alpha^2 \|\tilde{m}'\|^2_{L^2(0,a)} \tag{10.148}$$

where $F_{p,j}$ is given by either (10.146) or (10.147) and $\alpha > 0$ is a regularization parameter. The approximate solution $m^\star$ for either Method A or B is obtained by minimizing $J_\alpha(\tilde{m})$ over the spline space for $\tilde{m}$ with $F_{p,j}(\tilde{m})$ replaced by (10.146) for Method A or (10.147) for Method B. Since $\tilde{m}$ is given by a cubic spline, the minimization of (10.148) is a finite dimensional optimization problem and we use a Levenberg–Marquardt method to implement the optimization scheme.

Before presenting some numerical examples, some comments are in order regarding the design of numerical tests for our inverse algorithms (c.f. p. 154). Care must be taken that interactions between the inverse and forward solvers do not result in excessively optimistic predictions regarding the stability or accuracy of an inverse algorithm. For example, if (10.5) and (10.4) are discretized and used to generate far field data for a given profile, and the same discretization is used to solve the inverse problem, it is possible that the essential ill-posedness of the inverse problem may not be evident. This is avoided in our example, since different discretizations are used in the forward and the inverse algorithms. A similar problem can occur if the subspace containing the discrete coefficient $\tilde{m}$ contains or conforms with the exact solution. This problem is particularly acute if a piecewise constant approximation is used to approximate a discontinuous coefficient. If the grid lines for the discrete coefficient correspond to actual discontinuities in the exact solution, the inverse solver may again show spurious accuracy. In particular, one can not choose the mesh consistent with the coefficient to be reconstructed. In our examples, we use a cubic spline basis for approximating m, and choose coefficients m that are not contained in the cubic spline space.

We now discuss a few numerical examples. In what follows, we write the refractive index as in (8.7), i.e.,

$$n(x) = n_1(r) + i\,\frac{n_2(r)}{k}\,,$$

and identify n_2 as the absorption coefficient of the inhomogeneous medium.

Example 1. We take

$$\left.\begin{aligned} n_1(r) &= 1 + \frac{1}{2}\cos\frac{9}{2}\pi r, \\ n_2(r) &= \frac{1}{2}(1-r)^2(1+2r), \end{aligned}\right\} \quad 0 \le r \le 1 = a. \tag{10.149}$$

The absorption n_2 is a cubic spline, but n_1 is not. We choose $k_{\min} = 1$, $k_{\max} = 7$, $N_f = 513$, $N_i = 150$, $N_m = 15$ and $\lambda = 0$. The solution is computed for the regularization parameter $\alpha = 0.001$ by first computing $m^\star$ for $\alpha = 0.1$, taking as initial guess for the coefficients of m the value -0.3. The solution for $\alpha = 0.1$ is used as an initial guess for $\alpha = 0.01$ and this solution is used as initial guess for $\alpha = 0.001$. We report only results for $\alpha = 0.001$. In Table 10.1 we report the relative L^2 error in the reconstructions defined by

$$\left[\frac{\|\,\mathrm{Re}(n^\star - n_e)\|^2_{L^2(0,a)} + k^2\|\,\mathrm{Im}(n^\star - n_e)\|^2_{L^2(0,a)}}{\|\,\mathrm{Re}(n_e)\|^2_{L^2(0,a)} + k^2\|\,\mathrm{Im}(n_e)\|^2_{L^2(0,a)}}\right]^{1/2}$$

expressed as a percentage. Here, n_e is the exact solution (in this case given by (10.149) and (10.150)) and $n^\star$ is the approximate reconstruction. Table 10.1 shows that both methods work comparably well on this problem except in the case $P = 1$, when Method A is markedly superior to Method B.

Table 10.1 Numerical reconstructions for (10.149)

	Method A		Method B	
P	$N_k = 9$	$N_k = 15$	$N_k = 9$	$N_k = 15$
0	26.50	26.20	27.2	27.1
1	13.20	13.30	41.0	40.3
2	9.59	9.66	12.0	11.7
3	9.35	8.66	11.7	11.8

Example 2. Our next example is a discontinuous coefficient given by

$$n_1(r) = \begin{cases} 3.5, & 0 \le r < 0.5, \\ 1, & 0.5 \le r \le 1 = a, \end{cases} \tag{10.150}$$

$$n_2(r) = 0.3\,(1 + \cos 3\pi r), \quad 0 \le r \le 1 = a.$$

In this case, neither n_1 nor n_2 are cubic spline functions. Table 10.2 shows results for reconstructing this coefficient (the parameters are the same as in Example 1).

Table 10.2 Numerical reconstructions for (10.150)

	Method A		Method B	
P	$N_k = 9$	$N_k = 15$	$N_k = 9$	$N_k = 15$
0	11.2	10.50	11.10	10.80
1	10.0	9.85	10.50	11.20
2	111.0	9.51	10.10	9.88
3	131.0	9.38	9.67	9.49

When both methods yield satisfactory results, they possess similar errors. However, Method A is somewhat less robust than Method B and fails to work in two cases. This failure may be due to insufficient absorption to stabilize Method A. The effect of absorption on the reconstruction is investigated further in the next example.

Example 3. In this example we allow the maximum value of the absorption to be variable.

$$\begin{aligned} n_1(r) &= 1 + \frac{1}{2}\cos\frac{5}{2}\pi r, \\ & \qquad\qquad\qquad\qquad 0 \le r \le 1 = a. \\ n_2(r) &= \frac{\gamma}{2}(1-r)^2(1+2r), \end{aligned} \tag{10.151}$$

To reconstruct (10.151), we take $N_f = 129$, $N_k = 15$, $N_i = 50$ and $N_m = 15$. Independent of γ, we take $\lambda = k$, since we want to vary only a single parameter in our numerical experiments.

First we investigate changing the amount of absorption in the problem for $k_{\max} = 7$, $k_{\min} = 1$. Table 10.3 shows that results of varying γ between 0 and 0.5. As to be expected, Method B is insensitive to γ (although if γ is made large enough, we would expect that the quality of reconstruction would deteriorate).

Table 10.3 Reconstruction of (10.151) as γ varies

	Method A				Method B			
γ	$P = 0$	$P = 1$	$P = 2$	$P = 3$	$P = 0$	$P = 1$	$P = 2$	$P = 3$
0.0	107.00	87.40	53.50	52.00	5.89	2.44	2.59	2.26
0.01	109.00	88.60	2.72	25.90	–	–	–	–
0.025	105.00	88.90	3.88	2.53	–	–	–	–
0.05	213.00	96.50	2.73	2.56	–	–	–	–
0.1	124.00	165.00	2.76	2.62	5.87	2.44	2.59	2.26
0.15	6.24	320.00	2.78	2.68	–	–	–	–
0.2	6.24	2.60	2.80	2.74	5.81	2.44	2.58	2.26
0.3	6.24	2.65	2.83	2.83	5.73	2.44	2.58	2.24
0.4	6.21	2.70	2.86	2.92	5.62	2.43	2.57	2.25
0.5	6.15	2.76	2.90	3.00	5.50	2.43	2.56	2.25

Method A does not work for $\gamma = 0$. It is believed that this failure is due to transmission eigenvalues in $[k_{\min}, k_{\max}]$. The value of γ below which Method A is unstable depends on P. For example, when $P = 0$ Method A works satisfactorily when $\gamma = 0.15$ but not when $\gamma = 0.1$, whereas when $P = 3$ the method works satisfactorily when $\gamma = 0.025$ but not when $\gamma = 0.01$. These results are the principle reason for preferring Method B over Method A in the case of low absorption.

Finally, we investigate the dependence on $k_{\max}$ and $k_{\min}$ of the reconstruction error. Figure 10.1a shows the relative L^2 error in the reconstruction of (10.151) when $\gamma = 0.5$, $k_{\min} = 1$ and $k_{\max}$ varies. Clearly, in this example, increasing $k_{\max}$ greatly improves the reconstruction. Figure 10.1b shows results in the reconstruction of (10.151) with $\gamma = 0.5$, $k_{\max} = 7$ and $k_{\min}$ variable. For this example, it is clear that lower wave numbers contribute little information to the reconstruction. However, in the presence of large absorption, the lower wave number may well be important.

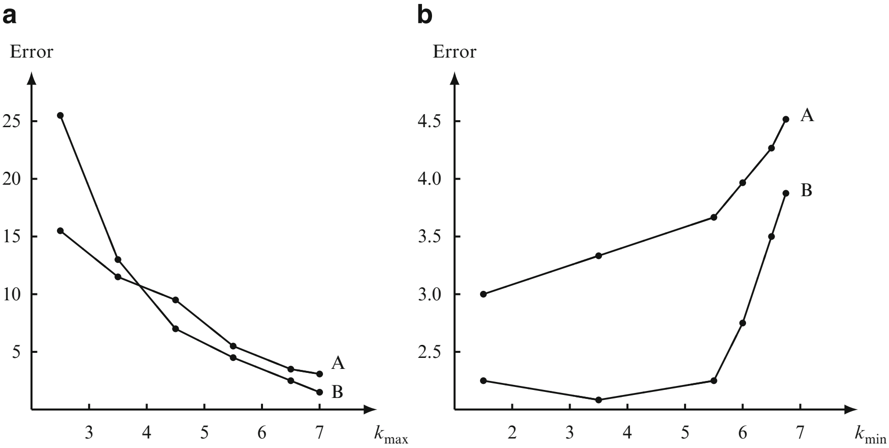

Fig. 10.1 A graph of the percentage relative L^2 error in reconstructing (10.151)

References

1. Abubakar, A., and van den Berg, P.: Iterative forward and inverse algorithms based on domain integral equations for three-dimensional electric and magnetic objects, Jour. Comp. Phys. **195**, 236–262 (2004).
2. Adams, R.A.: *Sobolev Spaces.* Academic Press, New York 1975.
3. Akduman, I., and Kress, R.: Direct and inverse scattering problems for inhomogeneous impedance cylinders of arbitrary shape. Radio Science **38**, 1055–1064 (2003).
4. Aktosun, T., Gintides, G., and Papanicolaou, V.: The uniqueness in the inverse problem for transmission eigenvalues for the spherically symmetric variable-speed wave equation. Inverse Problems **27**, 115004 (2011).
5. Alessandrini, G., and Rondi, L.: Determining a sound–soft polyhedral scatterer by a single far–field measurement. Proc. Amer. Math. Soc. **133**, 1685–1691 (2005).
6. Alves, C.J.S., and Ha-Duong, T.: On inverse scattering by screens. Inverse Problems **13**, 1161–1176 (1997).
7. Angell, T.S., Colton, D., and Kirsch, A.: The three dimensional inverse scattering problem for acoustic waves. J. Diff. Equations **46**, 46–58 (1982).
8. Angell, T.S., Colton, D., and Kress, R.: Far field patterns and inverse scattering problems for imperfectly conducting obstacles. Math. Proc. Camb. Phil. Soc. **106**, 553–569 (1989).
9. Angell, T.S., and Kirsch, A.: The conductive boundary condition for Maxwell's equations. SIAM J. Appl. Math. **52**, 1597–1610 (1992).
10. Angell, T.S., Kleinman, R.E., and Hettlich, F.: The resistive and conductive problems for the exterior Helmholtz equation. SIAM J. Appl. Math. **50**, 1607–1622 (1990).
11. Angell, T.S., Kleinman, R.E., and Roach, G.F.: An inverse transmission problem for the Helmholtz equation. Inverse Problems **3**, 149–180 (1987).
12. Aramini, R., Caviglia, G., Masa, A., and Piana, M.: The linear sampling method and energy conservation. Inverse Problems **26**, 05504 (2010).
13. Arens, T.: Why linear sampling works. Inverse Problems **20**, 163–173 (2004).
14. Arens, T., and Lechleiter, A.: The linear sampling method revisited. Jour. Integral Equations and Applications **21**, 179–202 (2009).
15. Atkinson, K.E.: The numerical solution of Laplace's equation in three dimensions. SIAM J. Numer. Anal. **19**, 263–274 (1982).
16. Atkinson, K.E.: *The Numerical Solution of Integral Equations of the Second Kind.* Cambridge Univ. Press, Cambridge 1997.
17. Bakushinskii, A.B.: The problem of the convergence of the iteratively regularized Gauss–Newton method. Comput. Maths. Maths. Phys. **32**, 1353–1359 (1992).
18. Baumeister, J.: *Stable Solution of Inverse Problems.* Vieweg, Braunschweig 1986.
19. Ben Hassen, F., Erhard, K., and Potthast, R.: The point source method for 3d reconstructions for the Helmholtz and Maxwell equations. Inverse Problems **22**, 331–353 (2006).

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*,
Applied Mathematical Sciences 93, DOI 10.1007/978-1-4614-4942-3,

20. Bers, L., John, F., and Schechter, M.: *Partial Differential Equations.* John Wiley, New York 1964.
21. Blaschke, B., Neubauer, A., and Scherzer, O: On convergence rates for the iteratively regularized Gauss–Newton method. IMA J. Numerical Anal. **17**, 421–436 (1997).
22. Bleistein, N.: *Mathematical Methods for Wave Phenomena.* Academic Press, Orlando 1984.
23. Blöhbaum, J.: Optimisation methods for an inverse problem with time-harmonic electromagnetic waves: an inverse problem in electromagnetic scattering. Inverse Problems **5**, 463–482 (1989).
24. Bojarski, N.N.: Three dimensional electromagnetic short pulse inverse scattering. Spec. Proj. Lab. Rep. Syracuse Univ. Res. Corp., Syracuse 1967.
25. Bojarski, N.N.: A survey of the physical optics inverse scattering identity. IEEE Trans. Ant. Prop. **AP-20**, 980–989 (1982).
26. Brakhage, H., and Werner, P.: Über das Dirichletsche Aussenraumproblem für die Helmholtzsche Schwingungsgleichung. Arch. Math. **16**, 325–329 (1965).
27. Bukhgeim, A.L.: Recovering a potential from Cauchy data in the two dimensional case. Jour. Inverse Ill-Posed Problems **16**, 19–33 (2008).
28. Burger, M., Kaltenbacher, B., and Neubauer, A.: Iterative solution methods. In: *Handbook of Mathematical Methods in Imaging* (Scherzer, ed.) Springer, Berlin 345–384 (2011).
29. Cakoni, F.: Recent developments in the qualitative approach to inverse electromagnetic scattering theory. J. Comp. Appl. Math. **204**, 242–253 (2007).
30. Cakoni, F., and Colton, D.: Combined far field operators in electromagnetic inverse scattering theory. Math. Methods Appl. Sci. **26**, 293–314 (2003).
31. Cakoni, F., and Colton, D.: The determination of the surface impedance of a partially coated obstacle from far field data. SIAM J. Appl. Math. **64**, 709–723 (2004).
32. Cakoni, F., and Colton, D.: *Qualitative Methods in Inverse Scattering Theory*. Springer, Berlin 2006.
33. Cakoni, F., Colton, D., and Gintides, D.: The interior transmission eigenvalue problem,. SIAM J. Math. Anal. **42**, 2912–2921 (2010).
34. Cakoni, F., Colton, D., and Haddar, H.: On the determination of Dirichlet and transmission eigenvalues from far field data. C. R. Math. Acad. Sci. Paris, Ser. 1 **348**, 379–383 (2010).
35. Cakoni, F. , Colton, D. and Haddar, H.: The interior transmission problem for regions with cavities. SIAM J. Math. Anal. **42**, 145–162 (2010).
36. Cakoni, F., Colton, D., and Monk, P.: *The Linear Sampling Method in Inverse Electromagnetic Scattering*. SIAM Publications, Philadelphia, 2011.
37. Cakoni, F., Colton, D., Monk, P., and Sun, J.: The inverse electromagnetic scattering problem for anisotropic media. Inverse Problems **26**, 07404 (2010).
38. Cakoni, F., and Gintides, D.: New results on transmission eigenvalues. Inverse Problems and Imaging **4**, 39–48 (2010).
39. Cakoni, F., Gintides, D., and Haddar, H.: The existence of an infinite discrete set of transmission eigenvalues. SIAM J. Math. Anal. **42**, 237–255 (2010).
40. Cakoni, F., and Haddar, H.: On the existence of transmission eigenvalues in an inhomogeneous medium. Applicable Analysis **88**, 475–493 (2009).
41. Cakoni, F., and Kirsch, A.: On the interior transmission eigenvalue problem. Int. Jour. Comp. Sci. Math. **3**, 142–16 (2010).
42. Calderón, A.P.: The multipole expansions of radiation fields. J. Rat. Mech. Anal. **3**, 523–537 (1954).
43. Calderón, A.P.: On an inverse boundary value problem. In: *Seminar on Numerical Analysis and its Applications to Continuum Mechanics.* Soc. Brasileira de Matemática, Rio de Janerio, 65–73 (1980).
44. Catapano, I., Crocco, L., and Isernia, T.: On simple methods for shape reconstruction of unknown scatterers. IEEE Trans. Antennas Prop. **55**, 1431–1436 (2007).
45. Chadan, K., Colton, D., Päivärinta, L., and Rundell, W.: *An Introduction to Inverse Scattering and Inverse Spectral Problems.* SIAM Publications, Philadelphia 1997.
46. Chadan, K., and Sabatier, P. C.: *Inverse Problems in Quantum Scattering Theory.* Springer, Berlin 1989.

47. Chandler-Wilde, S. N., Graham, I. G., Langdon, S., and Lindner, M.: Condition number estimates for combined potential boundary integral operators in acoustic scattering. Jour. Integral Equations and Appl. **21**, 229–279 (2009).
48. Chavent, G., Papanicolaou, G., Sacks, P., and Symes, W.: *Inverse Problems in Wave Propagation.* Springer, Berlin1997.
49. Cheng, J., and Yamamoto, M.: Uniqueness in an inverse scattering problem within non-trapping polygonal obstacles with at most two incoming waves. Inverse Problems **19**, 1361–1384 (2003).
50. Chew, W: *Waves and Fields in Inhomogeneous Media.* Van Nostrand Reinhold, New York 1990.
51. Collino, C., Fares, M., and Haddar, H.: Numerical and analytical studies of the linear sampling method in electromagnetic inverse scattering problems. Inverse Problems **19**, 1279–1298 (2003).
52. Colton, D.: *Partial Differential Equations.* Dover Publications, New York 2004.
53. Colton, D., Coyle, J., and Monk, P. : Recent developments in inverse acoustic scattering theory. SIAM Review **42**, 369–414 (2000).
54. Colton, D., and Erbe, C.: Spectral theory of the magnetic far field operator in an orthotropic medium. In: *Nonlinear Problems in Applied Mathematics* (Angell et al, eds). SIAM, Philadelphia, 96–103 (1995).
55. Colton, D., Haddar, H., and Monk, P.: The linear sampling method for solving the electromagnetic inverse scattering problem. SIAM J. Sci. Comput. **24**, 719–731 (2002).
56. Colton, D., and Hähner, P.: Modified far field operators in inverse scattering theory. SIAM J. Math. Anal. **24**, 365–389 (1993).
57. Colton, D., and Kirsch, A.: Dense sets and far field patterns in acoustic wave propagation. SIAM J. Math. Anal. **15**, 996–1006 (1984).
58. Colton, D., and Kirsch, A.: Karp's theorem in acoustic scattering theory. Proc. Amer. Math. Soc. **103**, 783–788 (1988).
59. Colton, D., and Kirsch, A.: An approximation problem in inverse scattering theory. Applicable Analysis **41**, 23–32 (1991).
60. Colton, D., and Kirsch, A.: The use of polarization effects in electromagnetic inverse scattering problems. Math. Meth. in the Appl. Sci. **15**, 1–10 (1992).
61. Colton, D., and Kirsch, A.: A simple method for solving inverse scattering problems in the resonance region. Inverse Problems **12**, 383–393 (1996).
62. Colton, D., Kirsch, A., and Päivärinta, L.: Far field patterns for acoustic waves in an inhomogeneous medium. SIAM J. Math. Anal. **20**, 1472–1483 (1989).
63. Colton, D., and Kress, R.: The impedance boundary value problem for the time harmonic Maxwell equations. Math. Meth. in the Appl. Sci. **3**, 475–487 (1981).
64. Colton, D., and Kress, R.: *Integral Equation Methods in Scattering Theory.* Wiley-Interscience Publication, New York 1983.
65. Colton, D., and Kress, R.: Dense sets and far field patterns in electromagnetic wave propagation. SIAM J. Math. Anal. **16**, 1049–1060 (1985).
66. Colton, D., and Kress, R.: Karp's theorem in electromagnetic scattering theory. Proc. Amer. Math. Soc. **104**, 764–769 (1988).
67. Colton, D., and Kress, R.: Time harmonic electromagnetic waves in an inhomogeneous medium. Proc. Royal Soc. Edinburgh **116 A**, 279–293 (1990).
68. Colton, D., and Kress, R.: Eigenvalues of the far field operator and inverse scattering theory. SIAM J. Math. Anal. **26**, 601–615 (1995).
69. Colton, D., and Kress, R.: Eigenvalues of the far field operator for the Helmholtz equation in an absorbing medium. SIAM J. Appl. Math. **55**, 1724–1735 (1995).
70. Colton, D., and Kress, R.: On the denseness of Herglotz wave functions and electromagnetic Herglotz pairs in Sobolev spaces. Math. Methods Applied Science **24**, 1289–1303 (2001).
71. Colton, D., and Kress, R.: Using fundamental solutions in inverse scattering. Inverse Problems **22**, R49–R66 (2006).

72. Colton, D., and Monk, P.: A novel method for solving the inverse scattering problem for time-harmonic acoustic waves in the resonance region. SIAM J. Appl. Math. **45**, 1039–1053 (1985).
73. Colton, D., and Monk, P.: A novel method for solving the inverse scattering problem for time-harmonic acoustic waves in the resonance region II. SIAM J. Appl. Math. **46**, 506–523 (1986).
74. Colton, D., and Monk, P.: The numerical solution of the three dimensional inverse scattering problem for time-harmonic acoustic waves. SIAM J. Sci. Stat. Comp. **8**, 278–291 (1987).
75. Colton, D., and Monk, P: The inverse scattering problem for time harmonic acoustic waves in a penetrable medium. Quart. J. Mech. Appl. Math. **40**, 189–212 (1987).
76. Colton, D., and Monk, P: The inverse scattering problem for acoustic waves in an inhomogeneous medium. Quart. J. Mech. Appl. Math. **41**, 97–125 (1988).
77. Colton, D., and Monk, P: A new method for solving the inverse scattering problem for acoustic waves in an inhomogeneous medium. Inverse Problems **5**, 1013–1026 (1989).
78. Colton, D., and Monk, P: A new method for solving the inverse scattering problem for acoustic waves in an inhomogeneous medium II. Inverse Problems **6**, 935–947 (1990).
79. Colton, D., and Monk, P: A comparison of two methods for solving the inverse scattering problem for acoustic waves in an inhomogeneous medium. J. Comp. Appl. Math. **42**, 5–16 (1992).
80. Colton, D., and Monk, P.: On a class of integral equations of the first kind in inverse scattering theory. SIAM J. Appl. Math. **53**, 847–860 (1993).
81. Colton, D., and Monk, P.: A modified dual space method for solving the electromagnetic inverse scattering problem for an infinite cylinder. Inverse Problems **10**, 87–107 (1994).
82. Colton, D., and Monk, P.: A new approach to detecting leukemia: Using computational electromagnetics. Comp. Science and Engineering **2**, 46–52 (1995).
83. Colton, D., and Päivärinta, L.: Far field patterns and the inverse scattering problem for electromagnetic waves in an inhomogeneous medium. Math. Proc. Camb. Phil. Soc. **103**, 561–575 (1988).
84. Colton, D., and L. Päivärinta, L.: Far-field patterns for electromagnetic waves in an inhomogeneous medium. SIAM J. Math. Anal. **21**, 1537–1549 (1990).
85. Colton, D. and Päivärinta, L.: The uniqueness of a solution to an inverse scattering problem for electromagnetic waves. Arch. Rational Mech. Anal. **119**, 59–70 (1992).
86. Colton, D., Päivärinta, L., and Sylvester, J.: The interior transmission problem. Inverse Problems and Imaging **1**, 13–28 (2007).
87. Colton, D., Piana, M., and Potthast, R.: A simple method using Morozov's discrepancy principle for solving inverse scattering problems. Inverse Problems **13**, 1477–1493 (1997).
88. Colton, D., and Sleeman, B.D.: Uniqueness theorems for the inverse problem of acoustic scattering. IMA J. Appl. Math. **31**, 253–259 (1983).
89. Colton, D., and Sleeman, B.D.: An approximation property of importance in inverse scattering theory. Proc. Edinburgh Math. Soc. **44**, 449–454 (2001).
90. Davis, P.J.: *Interpolation and Approximation.* Blaisdell Publishing Company, Waltham 1963.
91. Davis, P.J., and Rabinowitz, P.: *Methods of Numerical Integration.* Academic Press, New York 1975.
92. Devaney, A.J.: *Mathematical Foundations of Imaging, Tomography and Wavefield Inversion.* Cambridge University Press, Cambridge 2012.
93. Dolph, C. L.: The integral equation method in scattering theory. In: *Problems in Analysis* (Gunning, ed). Princeton University Press, Princeton, 201–227 (1970).
94. Elliott, D.: Sigmoidal transformations and the trapezoidal rule. ANZIAM Jour. **B 40**, E77–E137 (1998).
95. Elliott, D. and Prössdorf, S.: An algorithm for the approximate solution of integral equations of Mellin type. Numer. Math. **70**, 427–452 (1995).
96. Engl, H.W., Hanke, M. and Neubauer, A.: *Regularization of Inverse Problems.* Kluwer Academic Publisher, Dordrecht 1996.
97. Erdélyi, A.: *Asymptotic Expansions.* Dover Publications, New York 1956.

98. Farhat, C., Tezaur, R., and Djellouli, R.: On the solution of three-dimensional inverse obstacle acoustic scattering problems by a regularized Newton method. Inverse Problems **18**, 1229–1246 (2002).
99. Ganesh, M., and Graham, I. G.: A high-order algorithm for obstacle scattering in three dimensions. J. Comput. Phys. **198**, 211–242 (2004).
100. Ganesh, M., and Hawkins, S. C.: A spectrally accurate algorithm for electromagnetic scattering in three dimensions. Numer. Algorithms **43**, 25–60 (2006).
101. Ganesh, M., and Hawkins, S. C.: An efficient surface integral equation method for the time-harmonic Maxwell equations. ANZIAM J. **48**, C17–C33 (2007).
102. Ganesh, M., and Hawkins, S. C.: A high-order tangential basis algorithm for electromagnetic scattering by curved surfaces. J. Comput. Phys. **227**, 4543–4562 (2008).
103. Gerlach, T. and Kress, R.: Uniqueness in inverse obstacle scattering with conductive boundary condition. Inverse Problems **12**, 619–625 (1996).
104. Gieseke, B.: Zum Dirichletschen Prinzip für selbstadjungierte elliptische Differentialoperatoren. Math. Z. **68**, 54–62 (1964).
105. Gilbarg, D., and Trudinger, N.S.: *Elliptic Partial Differential Equations of Second Order.* Springer, Berlin 1977.
106. Gintides, D.: Local uniqueness for the inverse scattering problem in acoustics via the Faber–Krahn inequality. Inverse Problems **21**, 1195–1205 (2005).
107. Goldstein, C.I.: The finite element method with non-uniform mesh sizes applied to the exterior Helmholtz problem. Numer. Math. **38**, 61–82 (1981).
108. Gosh Roy, D.N., and Couchman, L.S. : *Inverse Problems and Inverse Scattering of Plane Waves.* Academic Press, New York 2002.
109. Graham, I. G., and Sloan, I. H.: Fully discrete spectral boundary integral methods for Helmholtz problems on smooth closed surfaces in $\mathbb{R}^3$. Numer. Math. **92**, 289–323 (2002).
110. Grisvard, P.: *Elliptic Problems in Nonsmooth Domains.* Pitman, Boston 1985.
111. Groetsch, C.W.: *The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind.* Pitman, Boston 1984.
112. Gutman, S., and Klibanov, M.: Regularized quasi–Newton method for inverse scattering problems. Math. Comput. Modeling **18**, 5–31 (1993).
113. Gutman, S., and Klibanov, M.: Two versions of quasi–Newton method for multidimensional inverse scattering problem. J. Comput. Acoust. **1**, 197–228 (1993).
114. Gutman, S., and Klibanov, M.: Iterative method for multidimensional inverse scattering problems at fixed frequencies. Inverse Problems **10**, 573–599 (1994).
115. Haas, M., and Lehner, G.: Inverse 2D obstacle scattering by adaptive iteration. IEEE Transactions on Magnetics **33**, 1958–1961 (1997)
116. Haas, M., Rieger, W., Rucker, W., and Lehner, G.: Inverse 3D acoustic and electromagnetic obstacle scattering by iterative adaption. In: *Inverse Problems of Wave Propagation and Diffraction* (Chavent and Sabatier, eds). Springer, Berlin 1997.
117. Hackbusch, W.: *Multi-grid Methods and Applications.* Springer, Berlin 1985.
118. Hadamard, J.: *Lectures on Cauchy's Problem in Linear Partial Differential Equations.* Yale University Press, New Haven 1923.
119. Haddar, H., and Kress. R.: On the Fréchet derivative for obstacle scattering with an impedance boundary condition. SIAM J. Appl. Math. **65**, 194–208 (2004).
120. Hähner, P.: Abbildungseigenschaften der Randwertoperatoren bei Randwertaufgaben für die Maxwellschen Gleichungen und die vektorielle Helmholtzgleichung in Hölder- und L^2–Räumen mit einer Anwendung auf vollständige Flächenfeldsysteme. Diplomarbeit, Göttingen 1987.
121. Hähner, P.: An exterior boundary-value problem for the Maxwell equations with boundary data in a Sobolev space. Proc. Roy. Soc. Edinburgh **109A**, 213–224 (1988).
122. Hähner, P.: Eindeutigkeits- und Regularitätssätze für Randwertprobleme bei der skalaren und vektoriellen Helmholtzgleichung. Dissertation, Göttingen 1990.
123. Hähner, P.: A uniqueness theorem for the Maxwell equations with L^2 Dirichlet boundary conditions. Meth. Verf. Math. Phys. **37**, 85–96 (1991).

124. Hähner, P.: A uniqueness theorem for a transmission problem in inverse electromagnetic scattering. Inverse Problems **9**, 667–678 (1993).
125. Hähner, P.: An approximation theorem in inverse electromagnetic scattering. Math. Meth. in the Appl. Sci. **17**, 293-303 (1994).
126. Hähner, P.: A periodic Faddeev-type solution operator. Jour. of Differential Equations **128**, 300–308 (1996).
127. Hähner, P.: Scattering by media. In: *Scattering* (Pike and Sabatier, eds). Academic Press, New York, 74–94, 2002.
128. Hanke, M.: A regularization Levenberg–Marquardt scheme, with applications to inverse groundwater filtration problems. Inverse Problems **13**, 75–95 (1997).
129. Hanke, M.: Why linear sampling really seems to work. Inverse Problems and Imaging **2**, 373–395 (2008).
130. Hanke, M., Hettlich, F., and Scherzer, O.: The Landweber iteration for an inverse scattering problem. In: *Proceedings of the 1995 Design Engineering Technical Conferences, Vol. 3, Part C* (Wang et al, eds). The American Society of Mechanical Engineers, New York, 909–915 (1995).
131. Hanke, M., Neubauer, A., and Scherzer, O.: A convergence analysis for the Landweber iteration for nonlinear ill-posed problems. Numer. Math. **72**, 21–37 (1995).
132. Harbrecht, H., and Hohage. T.: Fast methods for three-dimensional inverse obstacle scattering problems. Jour. Integral Equations and Appl. **19**, 237–260 (2007).
133. Hartman, P., and Wilcox, C.: On solutions of the Helmholtz equation in exterior domains. Math. Z. **75**, 228–255 (1961).
134. Hellwig, G.: *Partial Differential Equations.* Blaisdell Publishing, New York 1964.
135. Hettlich, F.: Die Integralgleichungsmethode bei Streuung an Körpern mit einer dünnen Schicht. Diplomarbeit, Göttingen 1989.
136. Hettlich, F.: On the uniqueness of the inverse conductive scattering problem for the Helmholtz equation. Inverse Problems **10**, 129–144 (1994).
137. Hettlich, F.: Fréchet derivatives in inverse obstacle scattering. Inverse Problems **11**, 371–382 (1995).
138. Hettlich, F.: An iterative method for the inverse scattering problem from sound-hard obstacles. In: *Proceedings of the ICIAM 95, Vol. II, Applied Analysis* (Mahrenholz and Mennicken, eds). Akademie Verlag, Berlin (1996).
139. Hettlich, F., and Rundell, W.: A second degree method for nonlinear inverse problem. SIAM J. Numer. Anal. **37**, 587–620 (2000).
140. Hitrik, M., Krupchyk, K., Ola, P., and Päivärinta, L.: Transmission eigenvalues for operators with constant coefficients. SIAM J. Math. Anal. **42**, 2965–2986 (2010).
141. Hitrik, M., Krupchyk, K., Ola, P., and Päivärinta, L.: Transmission eigenvalues for elliptic operators. SIAM J. Math. Anal. **43**, 2630–2639 (2011).
142. Hohage, T.: Logarithmic convergence rates of the iteratively regularized Gauss–Newton method for an inverse potential and an inverse scattering problem. Inverse Problems **13**, 1279–1299 (1997).
143. Hohage, T.: *Iterative Methods in Inverse Obstacle Scattering: Regularization Theory of Linear and Nonlinear Exponentially Ill-Posed Problems.* Dissertation, Linz 1999.
144. Hohage, T.: On the numerical solution of a three-dimensional inverse medium scattering problem. Inverse Problems **17**, 1743–1763 (2001).
145. Hohage, T.: Fast numerical solution of the electromagnetic medium scattering problem and applications to the inverse problem. J. Comp. Phys. **214**, 224–238 (2006).
146. Hohage, T., and Langer, S.: Acceleration techniques for regularized Newton methods applied to electromagnetic inverse medium scattering problems. Inverse Problems **26**, 074011 (2010).
147. Hsiao, G.C.: The coupling of boundary element and finite element methods. Z. Angew. Math. Mech. **70**, T493–T503 (1990).
148. Hsiao, G.C., and Wendland, W. L.: *Boundary Integral Equations.* Springer, Berlin 2008.
149. Ikehata, M.: Reconstruction of the shape of an obstacle from the scattering amplitude at a fixed frequency. Inverse Problems **14**, 949–954 (1998).

150. Ikehata, M.: Reconstruction of obstacle from boundary measurements. Wave Motion **30**, 205–223 (1999).
151. Imbriale, W.A., and Mittra, R.: The two-dimensional inverse scattering problem. IEEE Trans. Ant. Prop. **AP-18**, 633–642 (1970).
152. Isakov, V.: On uniqueness in the inverse transmission scattering problem. Comm. Part. Diff. Equa. **15**, 1565–1587 (1990).
153. Isakov, V.: *Inverse Problems for Partial Differential Equations.* 2nd ed, Springer, Berlin 2006.
154. Ivanov, K.V.: Integral equations of the first kind and an approximate solution for the inverse problem of potential. Soviet Math. Doklady **3**, 210–212 (1962) (English translation).
155. Ivanov, K.V.: On linear problems which are not well-posed. Soviet Math. Doklady **3**, 981–983 (1962) (English translation).
156. Ivanyshyn, O.: Shape reconstruction of acoustic obstacles from the modulus of the far field pattern. Inverse Problems and Imaging **1**, 609–622 (2007).
157. Ivanyshyn, O.: *Nonlinear Boundary Integral Equations in Inverse Scattering.* Dissertation, Göttingen, 2007.
158. Ivanyshyn, O., and Johansson, T.: Nonlinear integral equation methods for the reconstruction of an acoustically sound-soft obstacle. J. Integral Equations Appl. **19**, 289–308 (2007).
159. Ivanyshyn, O., and Johansson, T.: A coupled boundary integral equation method for inverse sound-soft scattering. In: *Proceedings of waves 2007. The 8th international conference on mathematical and numerical aspects of waves*, pp. 153–155 University of Reading 2007.
160. Ivanyshyn, O., and Kress, R.: Nonlinear integral equations in inverse obstacle scattering. In: *Mathematical Methods in Scattering Theory and Biomedical Engineering*, Fotiatis, Massalas, editors, pp 39–50 World Scientific, Singapore, 2006.
161. Ivanyshyn, O., and Kress, R.: Inverse scattering for planar cracks via nonlinear integral equations. Math. Meth. Appl. Sciences **31**, 1221–1232 (2007).
162. Ivanyshyn, O., and Kress, R.: Identification of sound-soft 3D obstacles from phaseless data. *Inverse Problems and Imaging* **4**, 131–149 (2010).
163. Ivanyshyn, O., and Kress, R.: Inverse scattering for surface impedance from phase-less far field data. J. Comp. Phys. **230**, 3443–3452 (2011).
164. Ivanyshyn, O., Kress, R., and Serranho, P.: Huygens' principle and iterative methods in inverse obstacle scattering. Adv. Comput. Math. **33**, 413–429 (2010).
165. Jeon, Y.: A Nyström method for boundary integral equations in domains with a piecewise smooth boundary. Jour. Integral Equations Appl. **5**, 221–242 (1993).
166. Johansson, T., and Sleeman, B.: Reconstruction of an acoustically sound-soft obstacle from one incident field and the far field pattern. IMA J. Appl. Math. **72**, 96–112 (2007).
167. Jones, D.S.: *Methods in Electromagnetic Wave Propagation.* Clarendon Press, Oxford 1979.
168. Jones, D.S.: *Acoustic and Electromagnetic Waves.* Clarendon Press, Oxford 1986.
169. Jörgens, K.: *Lineare Integraloperatoren.* Teubner–Verlag, Stuttgart 1970.
170. Kabanikhin, S.I.: *Inverse and Ill-posed Problems: Theory and Applications.* de Gruyter, Berlin-Boston 2011
171. Kaltenbacher, B., Neubauer, A., and Scherzer, O.: *Iterative Regularization Methods for Nonlinear Ill-Posed Problems.* de Gruyter, Berlin, 2008.
172. Karp, S.N.: Far field amplitudes and inverse diffraction theory. In: *Electromagnetic Waves* (Langer, ed). Univ. of Wisconsin Press, Madison, 291-300 (1962).
173. Kedzierawski, A.: The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium with complex refraction index. J. Comp. Appl. Math. **47**, 83-100 (1993).
174. Kellogg, O.D.: *Foundations of Potential Theory.* Springer, Berlin 1929.
175. Kersten, H.: Grenz- und Sprungrelationen für Potentiale mit quadratsummierbarer Dichte. Resultate d. Math. **3**, 17–24 (1980).
176. Kirsch, A.: The denseness of the far field patterns for the transmission problem. IMA J. Appl. Math. **37**, 213–225 (1986).
177. Kirsch, A.: Properties of far field operators in acoustic scattering. Math. Meth. in the Appl. Sci. **11**, 773–787 (1989).

178. Kirsch, A.: Surface gradients and continuity properties for some integral operators in classical scattering theory. Math. Meth. in the Appl. Sci. **11**, 789–804 (1989).
179. Kirsch, A.: Remarks on some notions of weak solutions for the Helmholtz equation. Applicable Analysis **47**, 7–24 (1992).
180. Kirsch, A.: The domain derivative and two applications in inverse scattering. Inverse Problems **9**, 81–96 (1993).
181. Kirsch, A.: Numerical algorithms in inverse scattering theory. In: *Ordinary and Partial Differential Equations, Vol. IV,* (Jarvis and Sleeman, eds). Pitman Research Notes in Mathematics **289**, Longman, London 93–111 (1993).
182. Kirsch, A.: *An Introduction to the Mathematical Theory of Inverse Problems.* 2nd ed, Springer, Berlin 2011.
183. Kirsch, A.: Characterization of the shape of the scattering obstacle by the spectral data of the far field operator. Inverse Problems **14**, 1489–1512 (1998).
184. Kirsch, A.: Factorization of the far field operator for the inhomogeneous medium case and an application to inverse scattering theory. Inverse Problems **15**, 413–429 (1999).
185. Kirsch, A.: On the existence of transmission eigenvalues. Inverse Problems and Imaging **3**, 155–172 (2009).
186. Kirsch, A., and Grinberg, N.: *The Factorization Method for Inverse Problems*. Oxford University Press, Oxford, 2008.
187. Kirsch, A., and Kress, R.: On an integral equation of the first kind in inverse acoustic scattering. In: *Inverse Problems* (Cannon and Hornung, eds). ISNM **77**, 93–102 (1986).
188. Kirsch, A., and Kress, R.: A numerical method for an inverse scattering problem. In: *Inverse Problems* (Engl and Groetsch, eds). Academic Press, Orlando, 279–290 (1987).
189. Kirsch, A., and Kress, R.: An optimization method in inverse acoustic scattering. In: *Boundary elements IX, Vol 3. Fluid Flow and Potential Applications* (Brebbia et al, eds). Springer, Berlin 3–18 (1987).
190. Kirsch, A., and Kress, R.: Uniqueness in inverse obstacle scattering. Inverse Problems **9**, 285–299 (1993).
191. Kirsch, A., Kress, R., Monk, P., and Zinn, A.: Two methods for solving the inverse acoustic scattering problem. Inverse Problems **4**, 749–770 (1988).
192. Kirsch, A., and Monk, P.: An analysis of the coupling of finite element and Nyström methods in acoustic scattering. IMA J. Numerical Anal. **14**, 523–544 (1994).
193. Kleinman, R., and van den Berg, P.: A modified gradient method for two dimensional problems in tomography. J. Comp. Appl. Math. **42**, 17–35 (1992).
194. Kleinman, R., and van den Berg, P.: An extended range modified gradient technique for profile inversion. Radio Science **28**, 877–884 (1993).
195. Knauff, W., and Kress, R.: On the exterior boundary value problem for the time-harmonic Maxwell equations. J. Math. Anal. Appl. **72**, 215–235 (1979).
196. Kress, R.: Ein ableitungsfreies Restglied für die trigonometrische Interpolation periodischer analytischer Funktionen. Numer. Math. **16**, 389–396 (1971).
197. Kress, R.: Minimizing the condition number of boundary integral operators in acoustic and electromagnetic scattering. Q. Jl. Mech. appl. Math. **38**, 323–341 (1985).
198. Kress, R.: On the boundary operator in electromagnetic scattering. Proc. Royal Soc. Edinburgh **103A**, 91–98 (1986).
199. Kress, R.: On the low wave number asymptotics for the two-dimensional exterior Dirichlet problem for the reduced wave equation. Math. Meth. in the Appl. Sci. **9**, 335–341 (1987).
200. Kress, R.: A Nyström method for boundary integral equations in domains with corners. Numer. Math. **58**, 145–161 (1990).
201. Kress, R.: Boundary integral equations in time-harmonic acoustic scattering. Mathl. Comput. Modelling **15**, 229–243 (1991).
202. Kress, R.: On the numerical solution of a hypersingular integral equation in scattering theory. J. Comp. Appl. Math. **61**, 345–360 (1995).
203. Kress, R.: Inverse scattering from an open arc. Math. Meth. in the Appl. Sci. **18**, 267–293 (1995).

204. Kress, R.: Integral equation methods in inverse acoustic and electromagnetic scattering. In: *Boundary Integral Formululations for Inverse Analysis* (Ingham and Wrobel, eds). Computational Mechanics Publications, Southampton, 67–92 (1997).
205. Kress, R.: *Linear Integral Equations.* 2nd ed, Springer, Berlin 1999.
206. Kress, R.: Newton's Method for inverse obstacle scattering meets the method of least squares. Inverse Problems **19**, 91–104 (2003).
207. Kress, R., and Päivärinta, L.: On the far field in obstacle scattering. SIAM J. Appl. Math. **59**, 1413–1426 (1999).
208. Kress, R., and Rundell, W.: A quasi-Newton method in inverse obstacle scattering. Inverse Problems **10**, 1145–1157 (1994).
209. Kress, R., and Rundell, W.: Inverse obstacle scattering with modulus of the far field pattern as data. In: *Inverse Problems in Medical Imaging and Nondestructive Testing* (Engl et al, eds). Springer, Wien 75–92 (1997).
210. Kress, R., and Rundell, W.: Inverse obstacle scattering using reduced data. SIAM J. Appl. Math. **59**, 442–454 (1999).
211. Kress, R., and Rundell, W.: Inverse scattering for shape and impedance. Inverse Problems **17**, 1075–1085 (2001).
212. Kress, R., and Rundell, W.: Nonlinear integral equations and the iterative solution for an inverse boundary value problem. Inverse Problems **21**, 1207–1223 (2005).
213. Kress, R., and Serranho, P.: A hybrid method for two-dimensional crack reconstruction. Inverse Problems **21**, 773–784 (2005)
214. Kress, R., and Serranho, P.: A hybrid method for sound-hard obstacle reconstruction. J. Comput. Appl. Math. **24**, 418–427 (2007).
215. Kress, R., Tezel, N., and Yaman, F.: A second order Newton method for sound soft inverse obstacle scattering. Jour. Inverse and Ill-Posed Problems **17**, 173–185 (2009).
216. Kress, R., and Zinn, A.: On the numerical solution of the three dimensional inverse obstacle scattering problem. J. Comp. Appl. Math. **42**, 49–61 (1992).
217. Kussmaul, R.: Ein numerisches Verfahren zur Lösung des Neumannschen Aussenraumproblems für die Helmholtzsche Schwingungsgleichung. Computing **4**, 246–273 (1969).
218. Langenberg, K.J.: Applied inverse problems for acoustic, electromagnetic and elastic wave scattering. In: *Basic Methods of Tomography and Inverse Problems* (Sabatier, ed). Adam Hilger, Bristol and Philadelphia, 127–467 (1987).
219. Lax, P.D.: Symmetrizable linear transformations. Comm. Pure Appl. Math. **7**, 633–647 (1954).
220. Lax, P.D., and Phillips, R.S.: *Scattering Theory.* Academic Press, New York 1967.
221. Lebedev, N.N.: *Special Functions and Their Applications.* Prentice-Hall, Englewood Cliffs 1965.
222. Lee, K.M.: Inverse scattering via nonlinear integral equations for a Neumann crack. Inverse Problems **22**, 1989–2000 (2006).
223. Leis, R.: Zur Dirichletschen Randwertaufgabe des Aussenraums der Schwingungsgleichung. Math. Z. **90**, 205–211 (1965)
224. Leis, R.: *Initial Boundary Value Problems in Mathematical Physics.* John Wiley, New York 1986.
225. Le Louër, F.: Fast methods for the shape identification problem of a perfectly conducting obstacle. In preparation.
226. Leung, Y.J., and Colton,D.: Complex transmission eigenvalues for spherically stratified media. Inverse Problems **28**, 07505 (2012).
227. Levine, L.M.: A uniqueness theorem for the reduced wave equation. Comm. Pure Appl. Math. **17**, 147–176 (1964).
228. Lin, T.C.: The numerical solution of Helmholtz's equation for the exterior Dirichlet problem in three dimensions. SIAM J. Numer. Anal. **22**, 670–686 (1985).
229. Liu, C.: Inverse obstacle problem: local uniqueness for rougher obstacles and the identification of a ball. Inverse Problems **13**, 1063–1069 (1997).
230. Liu, H., Yamamoto, M. and Zou, J.: Reflection principle for the Maxwell equations and its application to inverse electromagnetic scattering. Inverse Problems **23**, 2357–2366 (2007).

231. Liu, H., and Zou, J.: Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers. Inverse Problems **22**, 515–524 (2006).
232. Louis, A.K.: *Inverse und schlecht gestellte Probleme.* Teubner, Stuttgart 1989.
233. Magnus, W.: Fragen der Eindeutigkeit und des Verhaltens im Unendlichen für Lösungen von $\Delta u + k^2 u = 0$. Abh. Math. Sem. Hamburg **16**, 77–94 (1949).
234. Martensen, E.: Über eine Methode zum räumlichen Neumannschen Problem mit einer Anwendung für torusartige Berandungen. Acta Math. **109**, 75–135 (1963).
235. Martensen, E.: *Potentialtheorie.* Teubner-Verlag, Stuttgart 1968.
236. Martin, P.: *Multiple Scattering: Interaction of Time-harmonic Waves with N Obstacles.* Cambridge University Press, Cambridge 2006.
237. Mautz, J.R., and Harrington, R.F.: A combinded-source solution for radiating and scattering from a perfectly conducting body. IEEE Trans. Ant. and Prop. **AP-27**, 445–454 (1979).
238. McLaughlin, J., and Polyakov, P.: On the uniqueness of a spherically symmetric speed of sound from transmission eigenvalues. J. Diff. Equations **107**, 351–382 (1994).
239. McLaughlin, J., Polyakov, P. and Sacks, P.: Reconstruction of a spherically symmetric speed of sound. SIAM.J. Appl. Math. **54**, 1203–1223 (1994).
240. McLean, W.: *Strongly Elliptic Systems and Boundary Integral Equations.* Cambridge University Press, Cambridge 2000.
241. Melrose, R.B.: *Geometric Scattering Theory.* Cambridge University Press, Cambridge 1995.
242. Mikhlin, S.G.: *Mathematical Physics, an Advanced Course.* North-Holland, Amsterdam 1970.
243. Mönch, L.: A Newton method for solving the inverse scattering problem for a sound-hard obstacle. Inverse Problems **12**, 309–323 (1996).
244. Mönch, L.: On the inverse acoustic scattering problem from an open arc: the sound-hard case. Inverse Problems **13**, 1379–1392 (1997).
245. Monk, P.: *Finite Element Methods for Maxwell's Equations*. Clarendon Press, Oxford, 2003.
246. Moré, J.J.: The Levenberg–Marquardt algorithm, implementatiion and theory. In: *Numerical analysis* (Watson, ed). Springer Lecture Notes in Mathematics **630**, Berlin, 105–116 (1977).
247. Morozov, V.A.: On the solution of functional equations by the method of regularization. Soviet Math. Doklady **7**, 414–417 (1966) (English translation).
248. Morozov, V.A.: Choice of parameter for the solution of functional equations by the regularization method. Soviet Math. Doklady **8**, 1000–1003 (1967) (English translation).
249. Morozov, V.A.: *Methods for Solving Incorrectly Posed Problems.* Springer, Berlin 1984.
250. Morrey, C.M.: *Multiple Integrals in the Calculus of Variations.* Springer, Berlin 1966.
251. Morse, P.M., and Ingard, K.U.: Linear acoustic theory. In: *Encyclopedia of Physics* (Flügge, ed). Springer, Berlin 1–128 (1961).
252. Müller, C.: Zur mathematischen Theorie elektromagnetischer Schwingungen. Abh. deutsch. Akad. Wiss. Berlin **3**, 5–56 (1945/46).
253. Müller, C.: Über die ganzen Lösungen der Wellengleichung. Math. Annalen **124**, 235–264 (1952).
254. Müller, C.: Randwertprobleme der Theorie elektromagnetischer Schwingungen. Math. Z. **56**, 261–270 (1952).
255. Müller, C: On the behavior of solutions of the differential equation $\Delta u = F(x, u)$ in the neighborhood of a point. Comm. Pure Appl. Math. **7**, 505–515 (1954).
256. Müller, C.: *Foundations of the Mathematical Theory of Electromagnetic Waves.* Springer, Berlin 1969.
257. Nachman, A.: Reconstructions from boundary measurements. Annals of Math. **128**, 531–576 (1988).
258. Natterer, F.: *The Mathematics of Computerized Tomography.* Teubner, Stuttgart and Wiley, New York 1986.
259. Natterer, F., and Wübbeling, F.: A propagation-backpropagation method for ultrasound tomography. Inverse Problems **11**, 1225–1232 (1995).
260. Nédélec, J.C.; *Acoustic and Electromagnetic Equations.* Springer, Berlin 2001.
261. Newton, R.G.: *Scattering Theory of Waves and Particles.* Springer, Berlin 1982.

262. Newton, R.G.: *Inverse Schrödinger Scattering in Three Dimensions.* Springer, Berlin 1989.
263. Novikov, R.: Multidimensional inverse spectral problems for the equation $-\Delta\psi + (v(x) - E\,u(x))\,\psi = 0$. Translations in Func. Anal. and its Appl. **22**, 263–272 (1988).
264. Ola, P., Päivärinta, L., and Somersalo, E.: An inverse boundary value problem in electrodynamics. Duke Math. Jour. **70**, 617–653 (1993).
265. Ola, P., and Somersalo, E.: Electromagnetic inverse problems and generalized Sommerfeld potentials. SIAM J. Appl. Math. **56**, 1129–1145 (1996).
266. Olver, F.W.J: *Asymptotics and Special Functions.* Academic Press, New York 1974.
267. Päivärinta, L., and Sylvester, J.: Transmission eigenvalues. SIAM J. Math. Anal. **40**, 738–758 (2008).
268. Panich, O.I.: On the question of the solvability of the exterior boundary-value problems for the wave equation and Maxwell's equations. Usp. Mat. Nauk **20A**, 221–226 (1965) (in Russian).
269. Pieper, M.: Spektralrandintegralmethoden zur Maxwell-Gleichung. Dissertation, Göttingen 2007.
270. Pieper, M.: Nonlinear integral equations for an inverse electromagnetic scattering problem. Journal of Physics: Conference Series **124**, 012040 (2008).
271. Pieper, M.: Vector hyperinterpolation on the sphere. J. Approx. Theory **156**, 173–186 (2009).
272. Pironneau, O.: *Optimal Shape Design for Elliptic Systems.* Springer, Berlin 1984.
273. Potthast, R.: Fréchet differentiability of boundary integral operators in inverse acoustic scattering. Inverse Problems **10**, 431–447 (1994).
274. Potthast, R.: Fréchet Differenzierbarkeit von Randintegraloperatoren und Randwertproblemen zur Helmholtzgleichung und den zeitharmonischen Maxwellgleichungen. Dissertation, Göttingen 1994.
275. Potthast, R.: Fréchet differentiability of the solution to the acoustic Neumann scattering problem with respect to the domain. Jour. on Inverse and Ill-posed Problems **4**, 67–84 (1996) .
276. Potthast, R.: Domain derivatives in electromagnetic scattering. Math. Meth. in the Appl. Sci. **19**, 1157–1175 (1996).
277. Potthast, R.: A fast new method to solve inverse scattering problems. Inverse Problems **12**, 731–742 (1996).
278. Potthast, R.: A point-source method for inverse acoustic and electromagnetic obstacle scattering problems. IMA J. Appl. Math **61**, 119–140 (1998).
279. Potthast, R.: Stability estimates and reconstructions in inverse acoustic scattering using singular sources. J. Comp. Appl. Math. **114**, 247–274 (2000).
280. Potthast, R.: On the convergence of a new Newton-type method in inverse scattering. Inverse Problems **17**, 1419–1434 (2001).
281. Potthast, R.: *Point-Sources and Multipoles in Inverse Scattering Theory.* Chapman & Hall, London 2001.
282. Potthast, R.: Sampling and probe methods – an algorithmical view. Computing **75**, 215–235 (2005).
283. Potthast, R.: A survey on sampling and probe methods for inverse problems. Inverse Problems **22**, R1–R47 (2006).
284. Potthast, R. and Schulz, J.: A multiwave range test for obstacle reconstructions with unknown physical properties. J. Comput. Appl. Math. **205**, 53–71 (2007).
285. Protter, M.H.: Unique continuation for elliptic equations. Trans. Amer. Math. Soc. **95**, 81–90, (1960).
286. Protter, M.H., and Weinberger, H.F.: *Maximum Principles in Differential Equations.* Prentice-Hall, Englewood Cliffs 1967.
287. Ramm, A.G.: *Scattering by Obstacles.* D. Reidel Publishing Company, Dordrecht 1986.
288. Ramm, A.G.: On completeness of the products of harmonic functions. Proc. Amer. Math. Soc. **98**, 253–256 (1986).
289. Ramm, A.G.: Recovery of the potential from fixed energy scattering data. Inverse Problems **4**, 877–886 (1988).

290. Ramm, A.G.: Symmetry properties of scattering amplitudes and applications to inverse problems. J. Math. Anal. Appl. **156**, 333–340 (1991).
291. Ramm, A.G.: *Multidimensional Inverse Scattering Problems.* Longman–Wiley, New York 1992.
292. Reed, M., and Simon, B.: *Scattering Theory.* Academic Press, New York 1979.
293. Rellich, F.: Über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten. Jber. Deutsch. Math. Verein. **53**, 57–65 (1943).
294. Ringrose, J.R.: *Compact Non–Self Adjoint Operators.* Van Nostrand Reinhold, London 1971.
295. Rjasanow, S., and Steinbach, O.: *The Fast Solution of Boundary Integral Equations.* Springer, Berlin 2007.
296. Roger, A.: Newton Kantorovich algorithm applied to an electromagnetic inverse problem. IEEE Trans. Ant. Prop. **AP-29**, 232–238 (1981).
297. Ruland, C.: Ein Verfahren zur Lösung von $(\Delta + k^2)u = 0$ in Aussengebieten mit Ecken. Applicable Analysis **7**, 69–79 (1978).
298. Rynne, B.P., and Sleeman, B.D.: The interior transmission problem and inverse scattering from inhomogeneous media. SIAM J. Math. Anal. **22**, 1755–1762 (1991).
299. Saranen, J., and Vainikko, G.: *Periodic Integral and Pseudodifferential Equations with Numerical Approximation.* Springer, Berlin 2002.
300. Sauter, S., and Schwab, C.: *Boundary Element Methods.* Springer, Berlin 2011.
301. Schatz, A.H.: An observation concerning Ritz–Galerkin methods with indefinite bilinear forms. Math. Comp. **28**, 959–962 (1974).
302. Schechter, M.: *Principles of Functional Analysis.* Academic Press, New York 1971.
303. Schormann, C.: *Analytische und numerische Untersuchungen bei inversen Transmissionsproblemen zur zeitharmonischen Wellengleichung.* Dissertation, Göttingen 2000.
304. Serranho, P.: A hybrid method for inverse scattering for shape and impedance. Inverse Problems **22**, 663–680 (2006).
305. Serranho, P.: A hybrid method for sound-soft obstacles in 3D. Inverse Problems and Imaging **1**, 691–712 (2007).
306. Silver, S.: *Microwave Antenna Theory and Design.* M.I.T. Radiation Laboratory Series Vol. 12, McGraw-Hill, New York 1949.
307. Sloan, I. H., and Womersley, R. S.: Constructive approximations on the sphere. J. Approx. Theory **103**, 91–118 (2000).
308. Sommerfeld, A.: Die Greensche Funktion der Schwingungsgleichung. Jber. Deutsch. Math. Verein. **21**, 309–353 (1912).
309. Stefanov, P., and Uhlmann, G.: Local uniqueness for the fixed energy fixed angle inverse problem in obstacle scattering. Proc. Amer. Math. Soc. **132**, 1351–1354 (2003).
310. Stratton, J.A., and Chu, L.J.: Diffraction theory of electromagnetic waves. Phys. Rev. **56**, 99–107 (1939).
311. Sun, J.: Estimation of transmission eigenvalues and the index of refraction from Cauchy data. Inverse Problems **27**, 015009 (2011).
312. Sun, Z., and Uhlmann, G.: An inverse boundary value problem for Maxwell's equations. Arch. Rational. Mech. Anal. **119**, 71–93 (1992).
313. Sylvester, J. and Uhlmann, G.: A global uniqueness theorem for an inverse boundary value problem. Ann. of Math. **125**, 153–169 (1987).
314. Tikhonov, A.N.: On the solution of incorrectly formulated problems and the regularization method. Soviet Math. Doklady **4**, 1035–1038 (1963) (English translation).
315. Tikhonov, A.N.: Regularization of incorrectly posed problems. Soviet Math. Doklady **4**, 1624–1627 (1963) (English translation).
316. Tikhonov, A.N., and Arsenin, V.Y.: *Solutions of Ill-posed Problems.* Winston and Sons, Washington 1977.
317. Treves, F.: *Basic Linear Partial Differential Equations.* Academic Press, New York 1975.
318. Vainikko, G.: Fast solvers of the Lippmann–Schwinger equation In: *Direct and Inverse Problems of Mathematical Physics* (Gilbert, Kajiwara and Xu, eds). Kluwer, Dordrecht (2000).
319. van Bladel, J.: *Electromagnetic Fields.* Hemisphere Publishing Company, Washington 1985.

320. van den Berg, R. and Kleinman, R.: A contrast source inversion method. Inverse Problems **13**, 1607–1620 (1997).
321. van den Berg, R. and Kleinman, R.: Gradient methods in inverse acoustic and electromagnetic scattering. In: *Large-Scale Optimization with Applications, Part I: Optimization in Inverse Problems and Design* (Biegler et al, eds.) The IMA Volumes in Mathematics and its Applications **92**, Springer, Berlin 173–194 (1977).
322. Vekua, I.N.: Metaharmonic functions. Trudy Tbilisskogo matematichesgo Instituta **12**, 105–174 (1943).
323. Vögeler, M.: Reconstruction of the three-dimensional refractive index in electromagnetic scattering using a propagation-backpropagation method. Inverse Problems **19**, 739-753 (2003).
324. Wang, Y., Yagola, A.G., and Yang, C.: *Optimization and Regularization for Computational Inverse Problems and Applications.* Springer, Berlin 2011.
325. Weck, N.: Klassische Lösungen sind auch schwache Lösungen. Arch. Math. **20**, 628–637 (1969).
326. Werner, P.: Zur mathematischen Theorie akustischer Wellenfelder. Arch. Rational Mech. Anal. **6**, 231–260 (1961).
327. Werner, P.: Randwertprobleme der mathematischen Akustik. Arch. Rational Mech. Anal. **10**, 29–66 (1962).
328. Werner, P. : On the exterior boundary value problem of perfect reflection for stationary electromagnetic wave fields. J. Math. Anal. Appl. **7**, 348–396 (1963).
329. Werner, P.: Low frequency asymptotics for the reduced wave equation in two-dimensional exterior spaces. Math. Meth. in the Appl. Sc. **8**, 134–156 (1986).
330. Weston, V.H., and Boerner, W.M.: An inverse scattering technique for electromagnetic bistatic scattering. Canadian J. Physics **47**, 1177–1184 (1969).
331. Weyl, H.: Kapazität von Strahlungsfeldern. Math. Z. **55**, 187–198 (1952).
332. Wienert, L.: Die numerische Approximation von Randintegraloperatoren für die Helmholtzgleichung im $\mathbb{R}^3$. Dissertation, Göttingen 1990.
333. Wilcox, C.H.: A generalization of theorems of Rellich and Atkinson. Proc. Amer. Math. Soc. **7**, 271–276 (1956).
334. Wilcox, C.H.: An expansion theorem for electromagnetic fields. Comm. Pure Appl. Math. **9**, 115–134 (1956).
335. Wilcox, C.H.: *Scattering Theory for the d'Alembert Equation in Exterior Domains*. Springer Lecture Notes in Mathematics **442**, Berlin 1975.
336. Wloka, J.: *Partial Differential Equations.* University Press, Cambridge 1987.
337. Zinn, A.: On an optimisation method for the full- and limited-aperture problem in inverse acoustic scattering for a sound-soft obstacle. Inverse Problems **5**, 239–253 (1989).