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Algebraic approach to absorbing boundary conditions for the Helmholtz equation

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Recent work has shown that designing absorbing boundary conditions through algebraic approaches may be a nice alternative to the continuous approaches based on a Fourier analysis. In this paper, an original algebraic technique based on the computation of small patches is presented for the Helmholtz equation. This new technique is not directly linked to the continuous equations of the problem, nor to the numerical scheme. These properties make this technique very convenient to implement in a domain decomposition context. The proposed algebraic absorbing boundary conditions are used in a non-overlapping domain decomposition method and are defined on the interface between the subdomains. An additional coarse grid correction is then applied to ensure full scalability of the domain decomposition method upon the number of subdomains. This coarse grid correction involves trigonometric functions defined on the interface between the subdomains. Numerical experiments are presented and illustrate the robustness and parallel efficiency of the proposed method for acoustics applications.

Keywords: Domain decomposition; Algebraic patch substructuring; Coarse grid; Absorbing boundary condition; Acoustics

1. Mathematical formulation

The scalar wave equation $\rho\partial_{tt}^2 u - \mathrm{div}(\mu\nabla u) = f$ is considered where the right-hand side, f, and the unknown, u, are scalar functions of the space variable $x \in \mathbb{R}^2$ and where $t > 0$ is the time variable. The function $f(x,t)$ is assumed to be a time harmonic function, i.e. a function of the form $f(x,t) = f(x)e^{-i\omega t}$, where $\omega > 0$ is called the pulsation. Looking for time harmonic solutions of the wave equation, i.e. solutions of the form $u(x,t) = u(x)e^{-i\omega t}$ and after inserting into the wave equation, it is clear that $u(x)$ must be the solution of the Helmholtz equation: $-\mathrm{div}(\mu\nabla u) - \omega^2\rho u = f$. In the case of homogeneous media, which means that the functions ρ and μ are constants, the celerity $c = \sqrt{\mu/\rho}$ and the wave number $k = \omega/c$ can be introduced. With these notations the Helmholtz equation can be rewritten as $-\triangle u - k^2 u = f$. The variational formulation of this last equation gives: find $u \in H^1(\mathbb{R}^2)$ such that $\forall v \in H^1(\mathbb{R}^2)$,

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$a(k;u,v) = l(v)$ where $a(k;u,v) = \int (\nabla u \nabla v - k^2 uv)\,dx$, and $l(v) = \int f v\,dx$. The finite element discretization of the above boundary value problem leads to the system of complex equations $Ku = b$ where K is the so-called impedance matrix, and b is the right-hand side vector.

Solving the linear system $Ku = b$ by a direct method entails memory and CPU requirements that rapidly overwhelm even the largest resources that are currently available. For this reason, a significant amount of research continues to be invested in the development of Krylov-subspace [1, 2], multigrid [3, 4], and domain decomposition based iterative algorithms [5, 6] for the solution of this problem. Recent work has pointed out the importance of the interface conditions in domain decomposition algorithms. Using the best interface conditions usually requires a lot of computations, and several techniques of approximation have been developed in order to reduce this computational cost. The existing approaches can be classified into two major categories: the continuous approximations (see [5, 7–13]) and the discrete approximations (see [11, 12, 14]). In this paper, an original discrete approximation based on the computation of small patches is presented. This approximation is then used on the interface between the subdomains. An additional coarse grid correction involving trigonometric functions is finally applied to ensure full scalability of the domain decomposition method upon the number of subdomains.

The scope of this paper is as follows. Section 2 introduces the algebraic formulation of the domain decomposition method considered in this paper. The algebraic design of absorbing boundary conditions is derived in section 3. First the optimal choice is recalled, followed by the presentation of the proposed discrete approximation. Section 4 presents the two-level preconditioning technique based on a coarse space correction. Since this paper deals with acoustics, the coarse space basis functions considered here consist of trigonometric functions of different wavelengths. In section 5, new numerical experiments present the convergence behaviour of the domain decomposition method equipped with the proposed algebraic absorbing boundary conditions and with the coarse grid correction. The improvement of the convergence when using a coarse space correction is clearly outlined in these experiments. Section 6 concludes this paper.

2. Algebraic formulation of a domain decomposition method

The global domain Ω is meshed and partitioned into two non-overlapping subdomains $\Omega^{(1)}$ and $\Omega^{(2)}$ with an interface $\Gamma = \Omega^{(1)} \cap \Omega^{(2)}$. Let $K^{(s)}$ and $b^{(s)}$ denote respectively the impedance matrix, and the right-hand side vector associated with the subdomain $\Omega^{(s)}$, $s = 1, 2$, and let $u^{(s)}$ denote the restriction to $\Omega^{(s)}$ of the solution of the global problem. The vector $u^{(s)}$, the local impedance matrix, and the right-hand side vector are partitioned as:

$$u^{(s)} = \begin{pmatrix} u_i^{(s)} \\ u_p^{(s)} \end{pmatrix}, \quad K^{(s)} = \begin{pmatrix} K_{ii}^{(s)} & K_{ip}^{(s)} \\ K_{pi}^{(s)} & K_{pp}^{(s)} \end{pmatrix}, \quad b^{(s)} = \begin{pmatrix} b_i^{(s)} \\ b_p^{(s)} \end{pmatrix}$$

where the subscripts i and p respectively designate internal and interface boundary unknowns. The global problem is a block system obtained by assembling local contribution of each subdomain:

$$\begin{pmatrix} K_{ii}^{(1)} & 0 & K_{ip}^{(1)} \\ 0 & K_{ii}^{(2)} & K_{ip}^{(2)} \\ K_{pi}^{(1)} & K_{pi}^{(2)} & K_{pp} \end{pmatrix} \begin{pmatrix} u_i^{(1)} \\ u_i^{(2)} \\ u_p \end{pmatrix} = \begin{pmatrix} b_i^{(1)} \\ b_i^{(2)} \\ b_p \end{pmatrix}. \tag{1}$$

The matrices $K_{pp}^{(1)}$ and $K_{pp}^{(2)}$ represent the interaction matrices between the nodes on the interface obtained by integration on $\Omega^{(1)}$ and on $\Omega^{(2)}$, and $K_{pp}^{(1)} + K_{pp}^{(2)} = K_{pp}$. In the same way, local integration of the right-hand side over each subdomain and summation on the interface give the term $b_p = b_p^{(1)} + b_p^{(2)}$.

From the previous partition, it is clear that the global problem (1) is equivalent to the local subproblems

$$\begin{pmatrix} K_{ii}^{(1)} & K_{ip}^{(1)} \\ K_{pi}^{(1)} & K_{pp}^{(1)} \end{pmatrix} \begin{pmatrix} u_i^{(1)} \\ u_p^{(1)} \end{pmatrix} = \begin{pmatrix} b_i^{(1)} \\ b_p^{(1)} + \lambda^{(1)} \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} K_{ii}^{(2)} & K_{ip}^{(2)} \\ K_{pi}^{(2)} & K_{pp}^{(2)} \end{pmatrix} \begin{pmatrix} u_i^{(2)} \\ u_p^{(2)} \end{pmatrix} = \begin{pmatrix} b_i^{(2)} \\ b_p^{(2)} + \lambda^{(2)} \end{pmatrix} \tag{3}$$

under the admissibility constraint $u_p^{(1)} = u_p^{(2)}$ and the equilibrium constraint $0 = \lambda^{(1)} + \lambda^{(2)}$, where $\lambda^{(s)}$ is a Lagrange multiplier defined on the interface (see [15]). As already discussed in [11, 12], additional matrices defined on the interface can be introduced. This leads to the equivalence of the global problem with the local subproblems

$$\begin{pmatrix} K_{ii}^{(1)} & K_{ip}^{(1)} \\ K_{pi}^{(1)} & K_{pp}^{(1)} + A_{pp}^{(1)} \end{pmatrix} \begin{pmatrix} u_i^{(1)} \\ u_p^{(1)} \end{pmatrix} = \begin{pmatrix} b_i^{(1)} \\ b_p^{(1)} + \lambda^{(1)} \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} K_{ii}^{(2)} & K_{ip}^{(2)} \\ K_{pi}^{(2)} & K_{pp}^{(2)} + A_{pp}^{(2)} \end{pmatrix} \begin{pmatrix} u_i^{(2)} \\ u_p^{(2)} \end{pmatrix} = \begin{pmatrix} b_i^{(2)} \\ b_p^{(2)} + \lambda^{(2)} \end{pmatrix} \tag{5}$$

under the admissibility constraint $u_p^{(1)} = u_p^{(2)}$ and the modified equilibrium constraint $A_{pp}^{(1)} u_p^{(1)} + A_{pp}^{(2)} u_p^{(2)} = \lambda^{(1)} + \lambda^{(2)}$, where $A_{pp}^{(s)}$ is an additional matrix defined on the interface. A combination of the admissibility constraint and of the modified equilibrium constraint, as performed in [11, 12, 16], leads to the following continuity constraints:

$$\lambda^{(1)} + \lambda^{(2)} - A_{pp}^{(1)} u_p^{(2)} - A_{pp}^{(2)} u_p^{(2)} = 0 \tag{6}$$

$$\lambda^{(1)} + \lambda^{(2)} - A_{pp}^{(1)} u_p^{(1)} - A_{pp}^{(2)} u_p^{(1)} = 0. \tag{7}$$

The variables $u_i^{(1)}$ and $u_i^{(2)}$ can be eliminated from the equations (4)–(5) which leads to

$$[S_{pp}^{(1)} + A_{pp}^{(1)}] u_p^{(1)} = b_p^{(1)} - K_{pi}^{(1)} [K_{ii}^{(1)}]^{-1} b_i^{(1)} + \lambda^{(1)} \tag{8}$$

$$[S_{pp}^{(2)} + A_{pp}^{(2)}] u_p^{(2)} = b_p^{(2)} - K_{pi}^{(2)} [K_{ii}^{(2)}]^{-1} b_i^{(2)} + \lambda^{(2)}. \tag{9}$$

The Schur complement matrix and the condensed right-hand side vector in the subdomain $\Omega^{(s)}$ are denoted by $S_{pp}^{(s)} = K_{pp}^{(s)} - K_{pi}^{(s)} [K_{ii}^{(s)}]^{-1} K_{ip}^{(s)}$ and by $c_p^{(s)} = b_p^{(s)} - K_{pi}^{(s)} [K_{ii}^{(s)}]^{-1} b_i^{(s)}$. Substitution of $u_p^{(1)}$ and $u_p^{(2)}$ obtained from equations (8)–(9) into equations (6)–(7), gives the linear system $F\lambda = d$ where the matrix F, the vector λ and the right-hand side d are equal to:

$$F = \begin{pmatrix} I & F_{12} \\ F_{21} & I \end{pmatrix}, \quad \lambda = \begin{pmatrix} \lambda^{(1)} \\ \lambda^{(2)} \end{pmatrix}, \quad d = \begin{pmatrix} (A_{pp}^{(1)} + A_{pp}^{(2)})[S_{pp}^{(2)} + A_{pp}^{(2)}]^{-1} c_p^{(1)} \\ (A_{pp}^{(1)} + A_{pp}^{(2)})[S_{pp}^{(1)} + A_{pp}^{(1)}]^{-1} c_p^{(2)} \end{pmatrix}$$

where $F_{12} = I - (A_{pp}^{(1)} + A_{pp}^{(2)})[S_{pp}^{(2)} + A_{pp}^{(2)}]^{-1}$, and $F_{21} = I - (A_{pp}^{(1)} + A_{pp}^{(2)})[S_{pp}^{(1)} + A_{pp}^{(1)}]^{-1}$.

3. Algebraic design of absorbing boundary conditions

The augmented matrix $A_{pp}^{(s)}$ and the Lagrange multiplier $\lambda^{(s)}$ correspond to the discrete expression of the absorbing boundary conditions defined on the interface between the subdomains. The choice of these absorbing boundary conditions has a strong influence on the convergence of the iterative algorithm used to solve the problem $F\lambda = d$. In the case of a two-subdomain mesh partitioning, choosing the matrix $A_{pp}^{(s)} = S_{pp}^{(q)}$, $s = 1, 2$, $q = 1, 2$, $s \neq q$, leads to the convergence of the iterative algorithm into one iteration only (see [11, 12, 17]). In the case of a one-way mesh partitioning of the domain Ω, choosing the matrix $A_{pp}^{(s)}$ equal to the complete outer Schur complement leads to the convergence of the iterative algorithm into (number of subdomains – 1) iterations at most (see [11, 12, 17, 18]). Such a choice is called optimal since only local transmission procedures between the subdomains are considered at this stage.

The previous optimal choices could not be considered in practice since the computational cost of the complete outer Schur complement matrix is too expensive. Furthermore this matrix is a dense matrix and using it will considerably increase the bandwidth of the local subproblems matrix. An alternative investigated recently consists in the development of efficient approximations of the complete outer Schur complement matrix. Behind the proposed methods, the continuous approaches based on Fourier analysis have been developed in [7–9] and then optimized in [5, 10–12]. More recently purely algebraic techniques have been investigated in [11, 12, 14]. The new approach introduced in this paper for acoustics is a purely algebraic technique. Opposite to the methods based on a Fourier analysis, this new approach does not require the mesh of the interface between the subdomains.

Anyway, for the shake of clarity, the proposed method will be described with a vocabulary related to the geometry of the subdomains. This vocabulary involves the notion of small patches defined on the interface, and the notion of neighbouring areas. It is fundamental to keep in mind that from a practical point of view, only the local subdomain matrices are required and not the mesh of the interface. In the case of a general domain Ω split into two non-overlapping subdomains $\Omega^{(1)}$ and $\Omega^{(2)}$, this method consists of approximating the neighbour Schur complement matrix $K_{pp}^{(s)} - K_{pi}^{(s)}[K_{ii}^{(s)}]^{-1}K_{ip}^{(s)}$. For this purpose, for each interface node, a patch of radius r is first constructed. Then a neighbouring area of depth d around this patch of radius r is built. Figure 1 presents some configurations of a patch and of the neighbouring area for a radius $r = 1$ and different depths $d = 1, 2$ associated to one interface node.

Let us now describe the method for the subdomain $\Omega^{(2)}$. In order to build the approximation of the neighbour Schur complement matrix $K_{pp}^{(2)} - K_{pi}^{(2)}[K_{ii}^{(2)}]^{-1}K_{ip}^{(2)}$, the following algorithm is applied.

ALGORITHM The complete algorithm to build the proposed algebraic approximation is defined as the following steps:

(i) construction of the patch and of the neighbouring area;
(ii) construction of the sparse structure of the interface matrix $A_{pp}^{(1)}$;
(iii) construction of the sparse structure of the subdomain matrix $K^{(2)}$;
(iv) for all node j on the interface do
  a) extraction of some coefficients $K_{mn}^{(2)}$ for indexes (m, n) belonging to the neighbouring area of the patch, and construction of the sparse matrix $\tilde{K}_j^{(2)}$ with these coefficients;
  b) computation of the dense matrix $\tilde{S}_j^{(2)}$ by condensation of the matrix $\tilde{K}_j^{(2)}$ on the nodes of the patch;
  c) assembly of the matrix $\tilde{S}_j^{(2)}$ inside the matrix $A_{pp}^{(1)}$, using the connectivity of the nodes of the patch;
(v) addition of the matrix $A_{pp}^{(1)}$ to the subdomain matrix $K^{(1)}$.

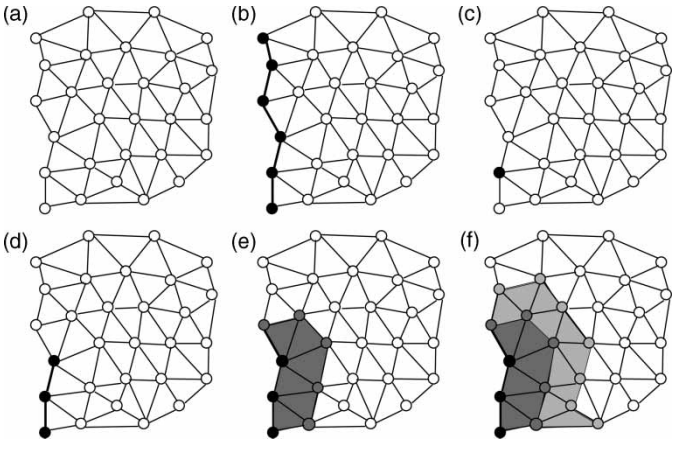

Figure 1. Illustration of the construction of a patch and of its neighbouring area for a given node on the interface. From left to right and from top to bottom: (a) the subdomain $\Omega^{(2)}$; (b) the nodes of the interface (black colour); (c) one node on the interface (black colour); (d) this node and the associated patch of radius one; (e) this patch of radius one and the neighbouring area of depth one; (f) this patch of radius one and the neighbouring area of depth two.

In the previous algorithm, the approximation of the matrix $K_{pp}^{(s)} - K_{pi}^{(s)}[K_{ii}^{(s)}]^{-1}K_{ip}^{(s)}$ has been described with $s = 2$. All the methodology can be applied with $s = 1$ without any difficulties. Similar calculations give the associated matrix $A_{pp}^{(q)}$ with $q = 2$. In summary, each subdomain computes for each interface node a local condensation of some degree of freedom belonging to a given area, on a small patch. These local condensations are then assembled and the result is used to build the matrix involved in the absorbing boundary conditions of the neighbouring subdomain. It is interesting to remark that when the radius is equal to the length of the interface, and when the depth is equal to d, the proposed algorithm becomes similar to the construction of the overlapping area involved in the overlapping Schwarz algorithm.

4. Multilevel preconditioning technique

In the following the condensed interface system $F\lambda = d$ is solved iteratively by the ORTHODIR algorithm [19]. The iterative procedure consists of updating the unknown λ^p with $\lambda^{p+1} = \lambda^p + \varrho^p w^p$ and the gradient g^p with $g^{p+1} = g^p + \varrho^p F w^p$, where w^p represents the descent direction vector. The coefficient ϱ^p is such that it minimizes the $F^T F$-norm of the error,

$$\|\lambda^{p+1} - \lambda\|_{F^T F}^2 = \|F(\lambda^{p+1} - \lambda)\|_2^2 = \|F(\lambda^p + \varrho^p w^p) - d\|_2^2$$

which gives $\varrho^p = -(g^p \mid Fw^p)/(Fw^p \mid Fw^p)$, where $(. \mid .)$ and $\|.\|_2$ denote respectively, the scalar product and the L^2-norm. The descent direction vector, w^p, and its product by the matrix F, Fw^p, are obtained with the relations

$$w^{p+1} = Fw^p + \sum_{i=1}^{p} \gamma_i^p w^i, \quad \text{and} \quad Fw^{p+1} = F(Fw^p) + \sum_{i=1}^{p} \gamma_i^p (Fw^i)$$

where $\gamma_i^p = -(F(Fw^p) \mid Fw^i)/(Fw^i \mid Fw^i)$.

The original ORTHODIR algorithm is now preconditioned. For this purpose, the subspace $\mathcal{W}$, represented by the rectangular matrix W, is introduced. The number of rows of the matrix W is equal to the degrees of freedom on the global interface. At each iteration, the descent direction

vector w^p is replaced by the vector $\tilde{w}^p$ defined as $\tilde{w}^p = w^p + W\sigma^p$. With this constraint, the coefficients ϱ^p and σ^p now minimize the norm

$$\|F(\lambda^p + \varrho^p(w^p + W\sigma^p)) - d\|_2^2$$

and the minimum is obtained when the derivative upon σ^p vanishes, i.e.

$$(FW)^T(F(\lambda^p + \varrho^p(w^p + W\sigma^p)) - d) = (FW)^T g^{p+1} = 0. \tag{10}$$

This equation represents an additional constraint, which is called optional, since the original algorithm converges without it. Since this constraint must be satisfied at each iteration, it becomes:

$$(FW)^T g^p = (FW)^T(F\lambda^p - d) = 0, \quad \forall p.$$

Using the previous expression, equation (10) can be simplified as:

$$(FW)^T FW\sigma^p = -(FW)^T Fw^p. \tag{11}$$

Substitution of σ^p from equation (11) in the expression $\tilde{w}^p = w^p + W\sigma^p$, gives the expression of the new descent direction $\tilde{w}^p$:

$$\tilde{w}^p = Pw^p \text{ where } P = I - W((FW)^T FW)^{-1}(FW)^T F.$$

The quantity P denotes a projection operator which ensures that the gradient g^{p+1} computed from Pw^p satisfies the constraint (10).

The initialization procedure of the projected ORTHODIR algorithm takes place as follows. An initial solution of the form $\tilde{\lambda}^0 = \lambda^0 + W\alpha$, such as:

$$(FW)^T(F(\lambda^0 + W\alpha) - d) = 0 \text{ i.e. } (FW)^T FW\alpha = -(FW)^T(F\lambda^0 - d)$$

is evaluated. The solution of this problem gives $\tilde{\lambda}^0$:

$$\tilde{\lambda}^0 = \lambda^0 - W((FW)^T FW)^{-1}(FW)^T(F\lambda^0 - d)$$

and the initial gradient g^0 and the initial descent direction are obtained as $g^0 = F\tilde{\lambda}^0 - d$ and $w^0 = Pg^0$. The iteration $p+1$ of the modified ORTHODIR algorithm is then similar to the original ORTHODIR algorithm, where the quantities are projected before update.

The computation of the projection of Fw^p consists of solving the additional following problem:

$$(FW)^T FW\sigma^p = -(FW)^T Fw^p. \tag{12}$$

The computational cost of this operation depends on the number of columns of the matrix W. This number is usually chosen small compared to the dimension of the matrix F. If the number of columns of the matrix W is proportional to the number of subdomains, the solution at each iteration of the additional problem (12) induces a global communication of the information between the subdomains. This procedure is called 'coarse grid', since each subdomain is assimilated to an element. The efficiency depends on the choice of the matrix W or similarly of the space $\mathcal{W}$. A possibility is to consider the restriction on each interface of 'particular' functions defined in each subdomain. Such a choice will reduce significantly the computational cost of the projector P since the interfaces $\Gamma^{(sq)}$ and $\Gamma^{(qs)}$ appear twice between the subdomains $\Omega^{(s)}$ and $\Omega^{(q)}$. An interesting choice of these 'particular' functions consists of choosing planar wave as investigated in [16]. In this paper trigonometric functions of different wavelengths are considered as illustrated in figure 2. These trigonometric functions are then multiplied by the matrix $A_{pp}^{(s)}$ in order to keep the homogeneity with the Lagrange multipliers.

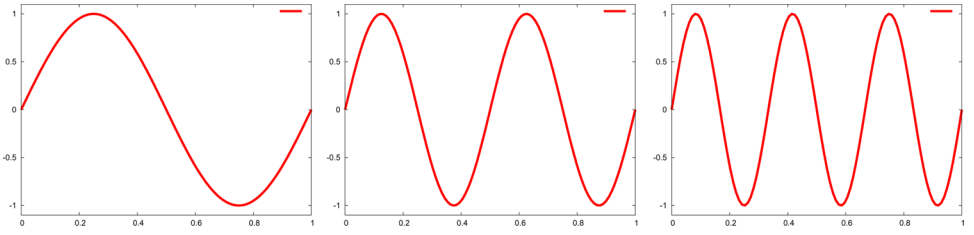

Figure 2. Example of three trigonometric functions composing the coarse space basis.

5. Numerical experiments

The Helmholtz equation is now considered in the domain $\Omega = [0, 1] \times [0, 1]$ with zeroth order approximation of the Sommerfeld radiation condition at $x = 0, x = 1, y = 0$ and $y = 1$.

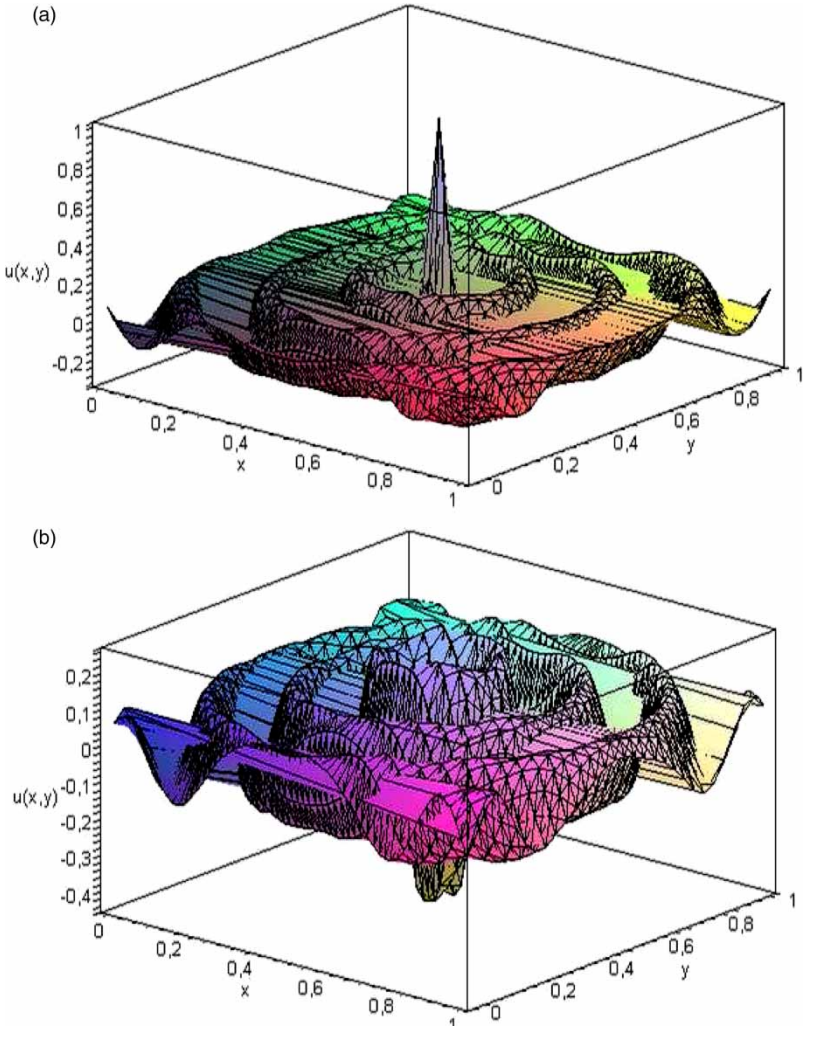

Figure 3. Real part (top) and imaginary part (bottom) of the finite element solution. ($k = 37, h = 1/50$).

A Dirac source is set at the coordinates $(1/2, 1/2)$. The domain Ω is decomposed into four, eight and sixteen non-overlapping subdomains using the METIS [20, 21] software. Figure 3 shows the real part (top) and the imaginary part (bottom) of the finite element solution. It is interesting to notice in the mesh partition, the presence of several cross-points, i.e. points where more than two subdomains intersect.

Both the original and the projected ORTHODIR algorithm are considered for the solution of the condensed interface problem with a stopping criterion set to 10^{-6}. The results reported in table 1 illustrate the respective influence of the radius and of the depth on the number of iteration of the method. The number of iterations decreases when increasing the value of the radius and of the depth. The value of the depth seems to be more important than the value of the radius. Furthermore, since increasing the radius increases the bandwidth of the subdomain matrix, each iteration requires more operations. On the other hand, increasing the depth does not change the bandwidth of the subdomain matrix, but requires more operations during the initial computation of the absorbing boundary conditions. Table 1 shows that $r = 1, d = 2$ seems to be an attractive choice and definitely the best compromise between efficiency, robustness and number of operations involved. This is especially outlined when a two-level preconditioning technique is used, and a very weak dependence upon the number of subdomains can be obtained.

Concerning this point, the projected ORTHODIR algorithm converges faster than the original ORTHODIR algorithm. The reduction of the number of iterations is significant. This property can be noticed for any value of the radius and of the depth. The number of iterations of the projected ORTHODIR algorithm presents a very weak dependence upon the number of subdomains. As already discussed in [16] the number of coarse space functions basis should be proportional to the wavelength of the Helmholtz equation. In table 1, a number of coarse space functions equal to sixteen has been selected and this choice leads to excellent results.

In summary, the proposed projected ORTHODIR ensures the full scalability of the domain decomposition method based on algebraic absorbing transmission conditions.

Table 1. Number of iterations for different value of the radius, of the depth and of the number of subdomains for the model problem. ($k = 37$, $h = 1/50$).

Number of subdomains	Radius	Depth	Number of iterations without a coarse space	Number of iterations with a coarse space
4	4	4	66	8
–	–	6	58	7
–	–	8	50	7
–	6	4	67	8
–	–	6	50	7
–	–	8	51	7
8	4	4	120	9
–	–	6	92	8
–	–	8	86	8
–	6	4	118	9
–	–	6	100	9
–	–	8	87	8
16	4	4	172	10
–	–	6	160	10
–	–	8	126	9
–	6	4	179	10
–	–	6	143	9
–	–	8	127	9

6. Conclusion

In this paper an original domain decomposition method has been presented for the Helmholtz equation. This domain decomposition method uses original absorbing boundary conditions between the subdomains and an additional two-level preconditioning technique. These absorbing boundary conditions consist of algebraic approximations of the theoretical optimal absorbing boundary conditions. The optimal absorbing boundary conditions are associated with an operator equal at the discrete level to the Schur complement of the outer subdomains. The algebraic approximation proposed in this paper is based on the computation of small patches on the interface. An additional coarse grid correction is used to achieve the scalability of the domain decomposition method upon the number of subdomains. This two-level preconditioning technique uses trigonometric functions as coarse space basis functions. Numerical experiments show that the proposed domain decomposition method appears to be a very efficient and robust method for the iterative solution of acoustics problems. This method should now be extended to arbitrarily shaped geometries and to three-dimensional problems.

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