An iterative process for optimizing symmetric successive over-relaxation

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An iterative process for optimizing symmetric successive over-relaxation is described and numerical results are presented to illustrate the practical utility of the process when applied to the solution of elliptic difference equations. This paper is based on one originally presented at the B.C.S. Third National Conference held at Cardiff in September 1962.

Symmetric over-relaxation (SSOR) for solving elliptic difference equations was introduced by Sheldon (1955) who showed that for Laplace's equation, when the σ_2 ordering is used and when combined with linear acceleration techniques, the method converges at a faster rate than successive over-relaxation (SOR) (Young, 1954).

Engeli *et al.* (1959) have also studied the method and report that, for the biharmonic problems they considered, it is superior to SOR. More recently, Habetler and Wachspress (1961) have shown that, for general diffusion equations, Sheldon's results do not hold, but they confirm that for a certain class of problems, SSOR is superior to SOR.

With iterative methods such as SOR and SSOR which involve a parameter, a small deviation from the optimum value of the parameter may greatly decrease the rate of convergence, and without an efficient process for determining a good approximation to the optimum parameter, an iterative scheme is often of little practical value. On the other hand, Engeli *et al.* noted that in their problem where the optimum parameter was determined solely by experiment, the convergence rate was insensitive to small variations from the optimum value.

The problem of optimizing SOR has received extensive attention, ranging from Young's fundamental theoretical work to the recent practical methods of Carré (1961) and Kulsrud (1961), but little has been published on the optimization of SSOR. Useful results were obtained by Habetler and Wachspress, however, who obtained an expression for the optimum parameter by a variational method. At the same time, the process they used to obtain the value numerically, in confirming their theoretical results, is unsuitable as a practical method of optimization as it involves determining the fundamental eigenvector of the SSOR iteration matrix for a series of values of the over-relaxation parameter, and in general this would lead to a prohibitive amount of computing effort.

In this paper we describe an iterative process for optimizing SSOR, which is based on the results obtained by Habetler and Wachspress, but is a computationally feasible method of obtaining a good approximation to the optimum parameter in practical situations. Numerical results are given to illustrate the effectiveness of the method.

Theory of SSOR

Consider the system of equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

where $\boldsymbol{A}$ is an $N \times N$ real, symmetric, positive definite matrix, $\boldsymbol{b}$ a known vector, and $\boldsymbol{x}$ an unknown vector that is to be determined. Assume that the system is normalized so that the diagonal elements of $\boldsymbol{A}$ are unity, and that $\boldsymbol{A}$ is represented in the form

$$\boldsymbol{A} \equiv \boldsymbol{I} - \boldsymbol{L} - \boldsymbol{U}$$

where $\boldsymbol{L}$ and $\boldsymbol{U}$ are respectively lower and upper triangular matrices with null diagonals. Then $\boldsymbol{L} \equiv \boldsymbol{U}^T$.

SSOR is then defined by the iteration

$$\left.\begin{aligned}(\boldsymbol{I} - \omega\boldsymbol{L})\boldsymbol{x}_{k+\frac{1}{2}} &= [\omega\boldsymbol{U} - (\omega - 1)\boldsymbol{I}]\boldsymbol{x}_k + \omega\boldsymbol{b} \\ (\boldsymbol{I} - \omega\boldsymbol{U})\boldsymbol{x}_{k+1} &= [\omega\boldsymbol{L} - (\omega - 1)\boldsymbol{I}]\boldsymbol{x}_{k+\frac{1}{2}} + \omega\boldsymbol{b}\end{aligned}\right\} \quad (1)$$

where ω, the over-relaxation parameter, is a constant for the iteration and is chosen to maximize the convergence rate.

Defining the error vector $\boldsymbol{e}_k$ by

$$\boldsymbol{e}_k = \boldsymbol{x} - \boldsymbol{x}_k$$

we have

$$\boldsymbol{e}_{k+1} = \boldsymbol{M}(\omega)\boldsymbol{e}_k$$

where

$$\boldsymbol{M}(\omega) = (\boldsymbol{I} - \omega\boldsymbol{U})^{-1}\,[\omega\boldsymbol{L} - (\omega - 1)\boldsymbol{I}]\,(\boldsymbol{I} - \omega\boldsymbol{L})^{-1}\,[\omega\boldsymbol{U} - (\omega - 1)\boldsymbol{I}].$$

It may be shown that $\boldsymbol{M}(\omega)$ is positive definite and that for $0 < \omega < 2$, the eigenvalues are all less than one; thus the stationary iteration (1) converges for all ω in this range. Since the eigenvalues are all real the convergence rate may be further increased by the technique of Chebyshev acceleration (see, for example, Sheldon's paper in Ralston and Wilf (1960)). For SSOR to be effective, Chebyshev acceleration must be used, and this requires a knowledge of the largest eigenvalue of $\boldsymbol{M}(\omega)$. In both the stationary and Chebyshev accelerated cases, the rate of convergence is maximized when the largest eigenvalue of $\boldsymbol{M}(\omega)$ is minimized.

Let $\hat{\omega}$ be the value of ω which minimizes the largest eigenvalue of $\boldsymbol{M}(\omega)$, and $\hat{\lambda}$, $\hat{y}$ the largest eigenvalue and

the corresponding normalized eigenvector of $M(\hat{\omega})$. Then

$$\hat{\lambda}\hat{y} = M(\hat{\omega})\hat{y} \tag{2}$$

and Habetler and Wachspress show that $\hat{\omega}$ and $\hat{y}$ satisfy the equation

$$\hat{\omega} = \frac{2}{1 + \sqrt{[P(\hat{y})]}} \tag{3}$$

where

$$P(\hat{y}) \equiv 1 - 2(\hat{y},(I - A)\hat{y}) + 4(\hat{y},LU\hat{y})$$

and (y,z) denotes the scalar product of vectors y and z.

The iteration for the optimum parameter

Equation (3) cannot be solved directly and resort must be made to an iterative process. The method proposed here is an iteration derived from (2) and (3) and is defined below:

$$\bar{y}_{k+1} = M(\omega_k)y_k$$

$$\lambda_{k+1} = (\bar{y}_{k+1}, \bar{y}_{k+1})^{\frac{1}{2}}$$

$$y_{k+1} = \frac{1}{\lambda_{k+1}}\bar{y}_{k+1}$$

$$\omega_{k+1} = \frac{2}{1 + \sqrt{[P(y_{k+1})]}}$$

where ω_0, y_0 are initial approximations to $\hat{\omega}$ and $\hat{y}$. It is seen that if the iteration converges, ω_k, λ_k, and y_k converge to $\hat{\omega}$, $\hat{\lambda}$, and $\hat{y}$, respectively. The values of $\hat{\omega}$ and $\hat{\lambda}$ so obtained may then be used in the Chebyshev accelerated SSOR process.

For the purpose of the computation, $P(y)$ may be written in the form

$$P(y) \equiv [(I - 2U)y, (I - 2U)y]$$

and hence may be accumulated as a single scalar product.

Thus we note that when the method is applied to elliptic difference equations, three sweeps of the difference net are required per iteration, and storage space is required for only a single vector space.

Discussion and numerical results

The method of optimizing SSOR described in the previous Section has been programmed for the Mercury and Atlas computers at Manchester University, and has been tested on the five-point approximation to Laplace's equation, the thirteen-point approximation to the biharmonic equation, and the five-point approximation to a diffusion-type equation.

In all cases that have been tried, involving up to 10,000 interior points, the method has converged, and it has been confirmed that the values of $\hat{\omega}$ and $\hat{\lambda}$ obtained do in fact optimize the SSOR process.

The numerical results given in this Section relate to the usual model problem, i.e. the solution of Laplace's equation over the unit square, but we note that the results obtained for other regions and for the other equations follow the same general pattern.

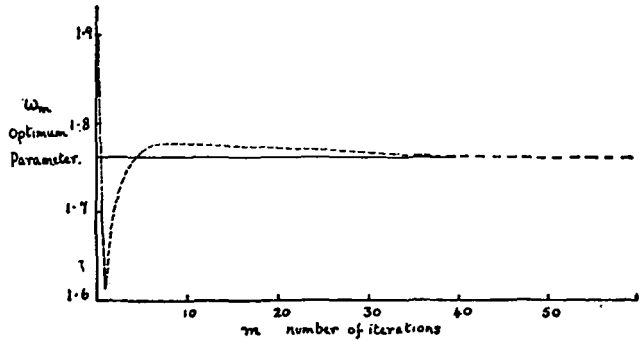

Fig. 1

Table 1

Optimum parameters for model problem

h^{-1}	$\hat{\omega}$	$\hat{\lambda}$	$N1$	$N2$
10	1·575	0·649	17	6
20	1·763	0·810	34	9
40	1·874	0·901	67	13

Fig. 1 illustrates the convergence properties of the process. It gives the values of ω_m obtained after m iterations for a mesh size of $h^{-1} = 20$. An initial value $\omega_0 = 1{\cdot}9$ was taken, but experiment shows that the choice of ω_0 has little bearing on the results. In all cases y_0 was taken as the normalized unit vector. It is seen that the initial convergence of the process is very rapid, but that complete convergence is much slower. Table 1 gives the converged values of $\hat{\omega}$ and $\hat{\lambda}$ for the cases $h^{-1} = 10$, 20 and 40, together with the number of iterations, $N1$, $N2$, respectively, of the stationary and of the Chebyshev accelerated SSOR process necessary to reduce the initial error at each interior point in the model problem by a factor of 10^{-3}.

Where a number of boundary-value problems are to be solved for a particular equation over a particular region, it is worthwhile going to some trouble to obtain $\hat{\omega}$ and $\hat{\lambda}$ accurately, and then, since they depend only on the shape of the region and the mesh size, they may be used for all the cases.

However, for "one-off" problems a quickly obtainable approximation to $\hat{\omega}$ and $\hat{\lambda}$ is often more economical in total computing time. Fig. 2 shows the dependence of $N1$ on ω for the stationary process for $h^{-1} = 20$, and it is seen that small changes from the optimum value cause relatively little change in the rate of convergence.

We may use this fact, together with the rapid initial convergence of the iteration for $\hat{\omega}$, to obtain quickly a reasonable approximation to $\hat{\omega}$.

Table 2 gives the number of iterations m required to obtain a value ω_m which is sufficiently close to $\hat{\omega}$ so as to minimize $N1$ when solving the model problem to the accuracy specified above. Using the values of ω_m, λ_m so obtained, N_m gives the number of iterations of the Chebyshev accelerated SSOR method required to solve the model problem to the same accuracy. It is seen that the values λ_m are lower than the true values given in Table 1, and this will reduce the convergence rate of the Chebyshev accelerated method.

In conclusion, we note that in our experience the method of optimization described in this paper makes it possible to take advantage of the superior convergence rate of SSOR for certain problems, in cases where no prior estimate of the optimum parameter is available.

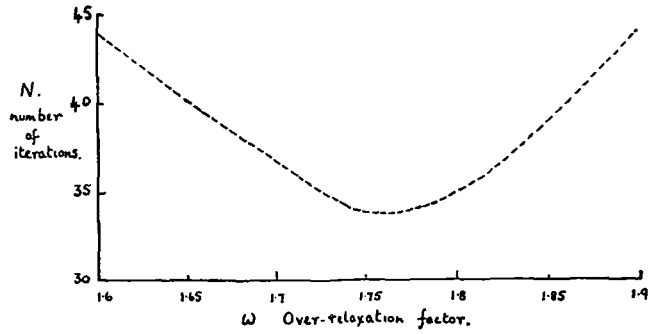

Fig. 2

Table 2

Approximate parameters for model problem

h^{-1}	m	ω_m	λ_m	N_m
10	2	1·566	0·557	9
20	3	1·740	0·781	12
40	8	1·876	0·889	16

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Book Review

Russian Supplement to Elsevier's Dictionary of Automation, Computers, Control and Measuring. By W. E. CLASON, 1962; 90 pages. (Amsterdam and London: *Elsevier Publishing Company*, 60s.)

This supplement is valueless without the main volume, which is in six languages and costs £7 (and was reviewed in this *Journal*, Vol. 4, No. 3). In use, one looks for the Russian term in the first half of the supplement, and obtains the serial number, with which one enters the principal section of the main volume, to find the corresponding term in English, Dutch, French, German, Italian and Spanish, and also a rudimentary definition in English. In reverse, one seeks an English term in the principal section of the main volume (which is in English alphabetical order as well as in order of serial numbers), obtains the serial number, and enters the second half of the supplement to find the corresponding Russian term. This supplement system is a good one, as it results in a monolingual volume—the supplement under notice is set entirely in Cyrillic and decimal digits, apart from the title pages. In fact, it would have been better if the main volume had been produced in six parts on the same system, as this would presumably have made it cheaper for the person who is interested in two languages only.

Like the main volume, the supplement is to some extent oversimplified, since dogmatic equivalent terms are quoted, with very few synonyms and homonyms. On the whole, however, the equivalent terms are very accurate, and the type-setting and production are also of a high standard. The Russian supplement shares with the main volume the distinction of having the largest number of terms (1,400) in automatic data processing of any available dictionary.

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