Thèse de Doctorat

Asynchronous domain decomposition methods for massively parallel computing

Préparée et soutenue par

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Devant le Jury composé de

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Fabienne Jezequel	(Examinatrice)
Pierre Spiteri	(Rapporteur)
Stéphane Vialle	(Examinateur)
Frédéric Magoulès	(Directeur de thèse)

18 décembre 2017

Introduction

Asynchronous iterative methods

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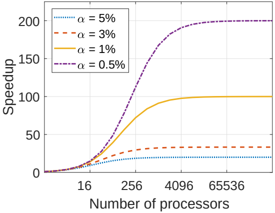

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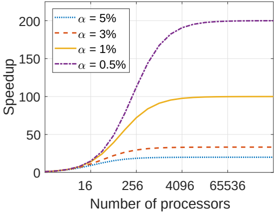

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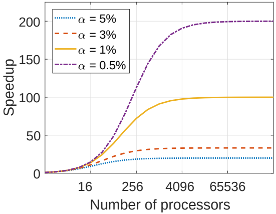

Minimize α: input/output, pre/post-processing, ..., and *inter-process communication*

- But also: load balancing and *fault-tolerance*

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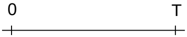
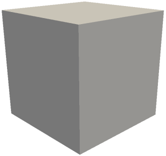


- Numerical simulation

Partial differential equations

$$\delta(u(s,t),s,t)=0, \quad t\in\mathbb{R}^+, \ \ s\in\Omega\subset\mathbb{R}^3$$



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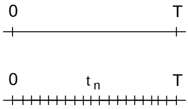

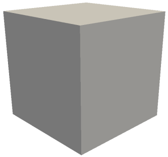

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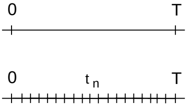

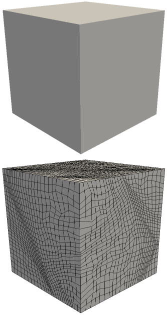

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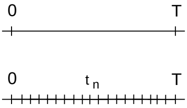

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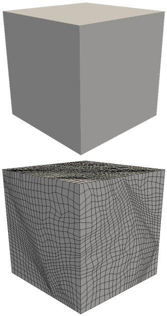

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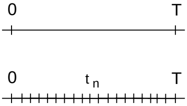

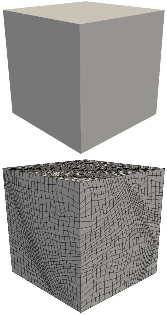

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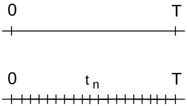

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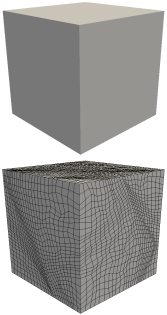
Iterative methods

$$x^{k+1} = f(x^k)$$

$$x_i^{k+1} = f_i(x^k), \; \forall i \in \{1,\ldots,p\}$$

$$x := \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}, \quad p \le m$$



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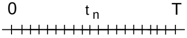

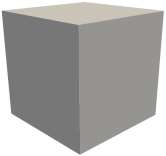

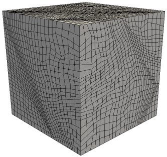

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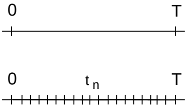
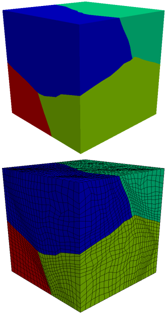
Different subdomains

$$\Omega^{(1)},\dots,\Omega^{(p)}$$





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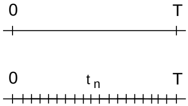

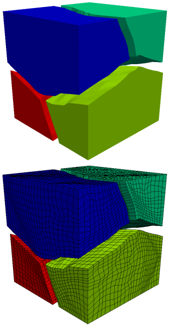

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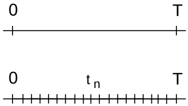

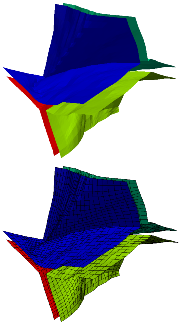

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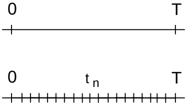
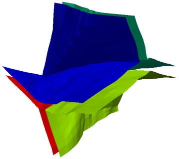
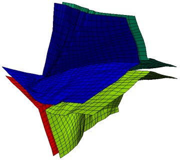
+ parallel input/output
+ parallel pre/post-processing







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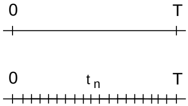
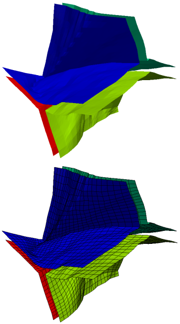
+ parallel input/output
+ parallel pre/post-processing

+ static load balancing





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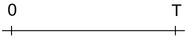
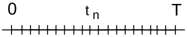
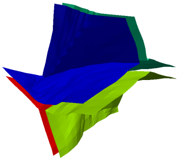


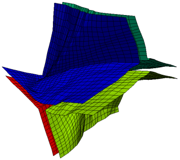


+ parallel input/output
+ parallel pre/post-processing

+ static load balancing

– serial parallel iterations
⇒ blocking inter-process synchronization





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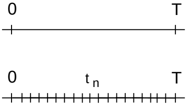
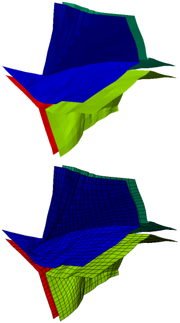
⇒ – non fault-tolerant





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- Asynchronous iterations *[Chazan and Miranker, 1969]*

+ static load balancing
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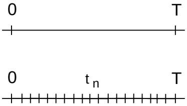

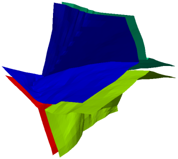

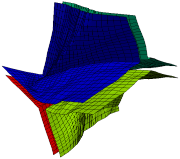

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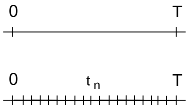

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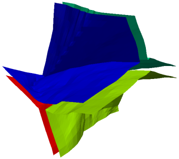
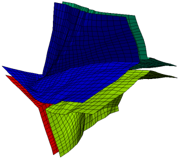
- Asynchronous iterations *[Chazan and Miranker, 1969]*

+ parallel serial iterations

⇒ no speedup limit





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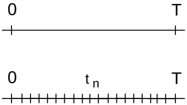

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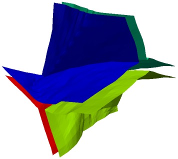
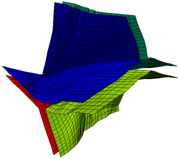
⇒ no speedup limit

+ non-deterministic inter-process communication ⇒ fault-tolerance





Introduction

Parallel computing

- Speedup limit *[Amdahl, 1967]*
- But also: load balancing and fault-tolerance
- Numerical simulation
- Domain decomposition methods

+ static load balancing
– serial parallel iterations
– non fault-tolerant

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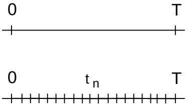
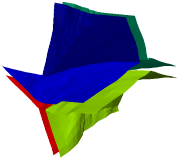
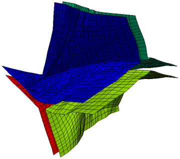
+ dynamic adaptation to unbalanced load







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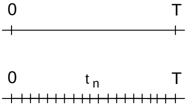

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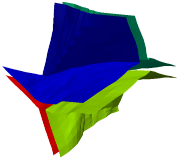

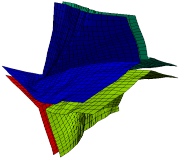

SystemX CentraleSupélec

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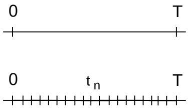
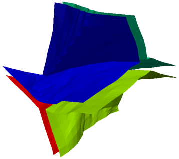
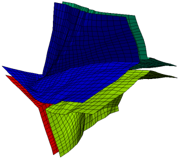
Eventual consistency across interfaces?
(convergence conditions)







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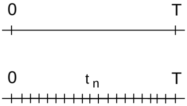

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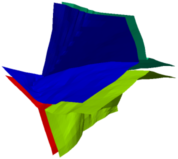

Consistency reached?
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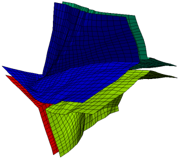

Outline of the presentation

01 Asynchronous iterative methods

02 Space domain decomposition

03 Time domain decomposition

04 Asynchronous convergence detection

05 Conclusion

Outline

01 Asynchronous iterative methods

1 - Asynchronous iterative methods

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

1 - Asynchronous iterative methods

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

Splitting

$$A = M - N$$

Mapping

$$f(x) := M^{-1}Nx + M^{-1}b$$

Fixed-point problem

$$Ax = b \iff x = f(x)$$

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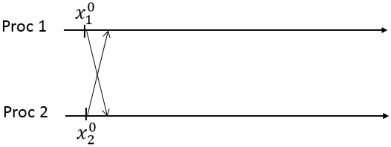

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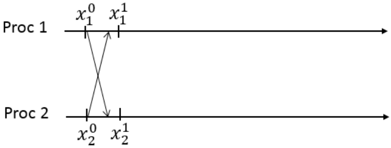

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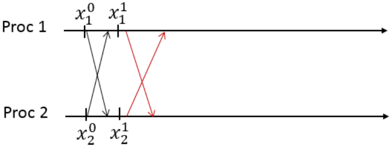

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wait wait

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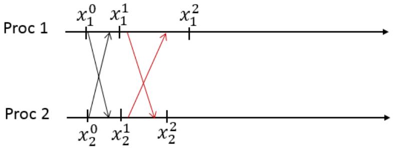

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wait wait

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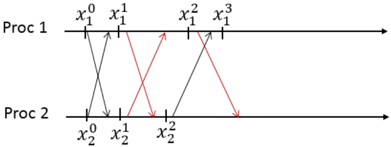

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wait $\quad$ wait

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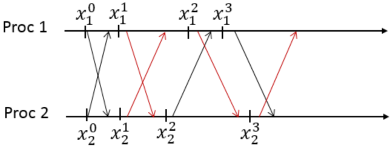

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wait	wait
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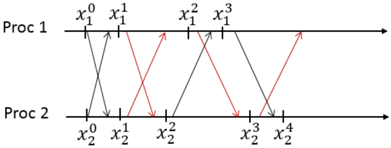
x_1^0 x_1^1 x_1^2 x_1^3
Proc 1
Proc 2
x_2^0 x_2^1 x_2^2 x_2^3 x_2^4

$x_1^1 := f_1(x_1^0, x_2^0)$	$x_2^1 := f_2(x_1^0, x_2^0)$
wait	wait
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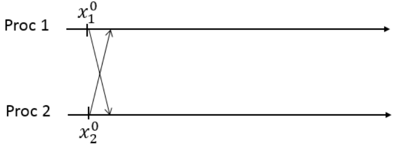
$x_1^3 := f_1(x_1^2, x_2^2)$	wait
wait	$x_2^3 := f_2(x_1^2, x_2^2)$
wait	$x_2^4 := f_2(x_1^3, x_2^3)$

1 - Asynchronous iterative methods

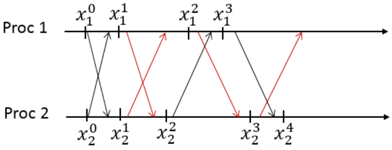
Asynchronous iterations



Synchronous iterations

$$x_i^{k+1} = f_i(x_1^k, \ldots, x_p^k), \quad \forall i \in \{1, \ldots, p\}$$

delay ⇒ speedup limit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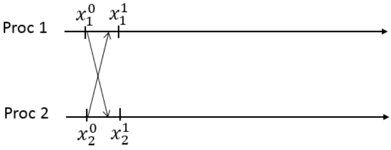
1 - Asynchronous iterative methods

Asynchronous iterations



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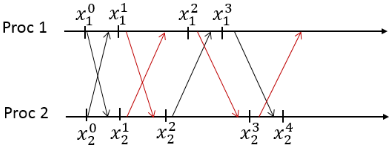

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$x_1^3 := f_1(x_1^2, x_2^2)$	wait
wait	$x_2^3 := f_2(x_1^2, x_2^2)$
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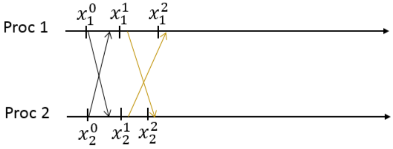
1 - Asynchronous iterative methods

Asynchronous iterations

delay ⇒ low convergence rate

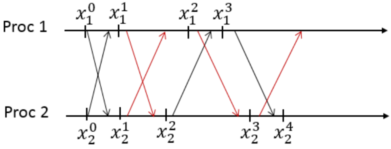


$$\begin{array}{ll} x_1^1 := f_1(x_1^0, x_2^0) & x_2^1 := f_2(x_1^0, x_2^0) \\ x_1^2 := f_1(x_1^1, x_2^0) & x_2^2 := f_2(x_1^0, x_2^1) \end{array}$$

Synchronous iterations

$$x_i^{k+1} = f_i(x_1^k, \ldots, x_p^k), \quad \forall i \in \{1, \ldots, p\}$$

delay ⇒ speedup limit

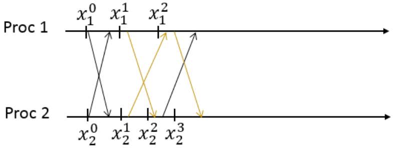


$$\begin{array}{cc} x_1^1 := f_1(x_1^0, x_2^0) & x_2^1 := f_2(x_1^0, x_2^0) \\ \text{wait} & \text{wait} \\ x_1^2 := f_1(x_1^1, x_2^1) & x_2^2 := f_2(x_1^1, x_2^1) \\ x_1^3 := f_1(x_1^2, x_2^2) & \text{wait} \\ \text{wait} & x_2^3 := f_2(x_1^2, x_2^2) \\ \text{wait} & x_2^4 := f_2(x_1^3, x_2^3) \end{array}$$

1 - Asynchronous iterative methods

Asynchronous iterations

delay ⇒ low convergence rate



$$x_1^1 := f_1(x_1^0, x_2^0) \qquad x_2^1 := f_2(x_1^0, x_2^0)$$

$$x_1^2 := f_1(x_1^1, x_2^0) \qquad x_2^2 := f_2(x_1^0, x_2^1)$$

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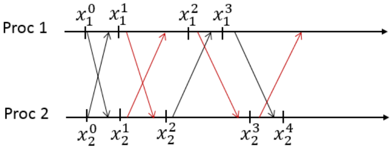
Synchronous iterations

$$x_i^{k+1} = f_i(x_1^k, \ldots, x_p^k), \quad \forall i \in \{1, \ldots, p\}$$

delay ⇒ speedup limit



$$x_1^1 := f_1(x_1^0, x_2^0) \qquad x_2^1 := f_2(x_1^0, x_2^0)$$

wait wait

$$x_1^2 := f_1(x_1^1, x_2^1) \qquad x_2^2 := f_2(x_1^1, x_2^1)$$

$$x_1^3 := f_1(x_1^2, x_2^2)$$

wait

wait

$$x_2^3 := f_2(x_1^2, x_2^2)$$

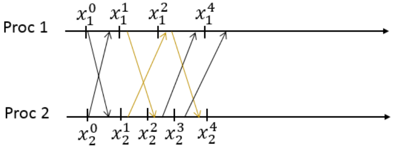
wait

$$x_2^4 := f_2(x_1^3, x_2^3)$$

1 - Asynchronous iterative methods

Asynchronous iterations

delay ⇒ low convergence rate

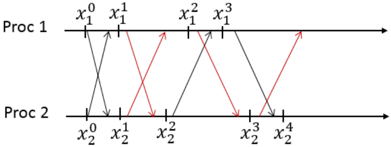

Proc 1
x_1^0 x_1^1 x_1^2 x_1^4
Proc 2
x_2^0 x_2^1 x_2^2 x_2^3 x_2^4

$$\begin{array}{ll}
x_1^1 := f_1(x_1^0, x_2^0) & x_2^1 := f_2(x_1^0, x_2^0) \\
x_1^2 := f_1(x_1^1, x_2^0) & x_2^2 := f_2(x_1^0, x_2^1) \\
x_1^3 := x_1^2 & x_2^3 := f_2(x_1^1, x_2^2) \\
x_1^4 := f_1(x_1^3, x_2^2) & x_2^4 := f_2(x_1^2, x_2^3)
\end{array}$$

Synchronous iterations

$$x_i^{k+1} = f_i(x_1^k, \ldots, x_p^k), \quad \forall i \in \{1, \ldots, p\}$$

delay ⇒ speedup limit

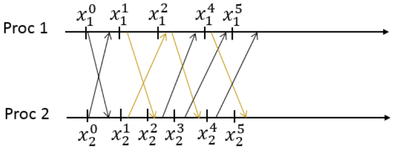

Proc 1
x_1^0 x_1^1 x_1^2 x_1^3
Proc 2
x_2^0 x_2^1 x_2^2 x_2^3 x_2^4

$$\begin{array}{ll}
x_1^1 := f_1(x_1^0, x_2^0) & x_2^1 := f_2(x_1^0, x_2^0) \\
\text{wait} & \text{wait} \\
x_1^2 := f_1(x_1^1, x_2^1) & x_2^2 := f_2(x_1^1, x_2^1) \\
x_1^3 := f_1(x_1^2, x_2^2) & \text{wait} \\
\text{wait} & x_2^3 := f_2(x_1^2, x_2^2) \\
\text{wait} & x_2^4 := f_2(x_1^3, x_2^3)
\end{array}$$

1 - Asynchronous iterative methods

Asynchronous iterations

delay ⇒ low convergence rate

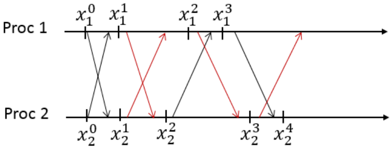


$$\begin{array}{ll}
x_1^1 := f_1(x_1^0, x_2^0) & x_2^1 := f_2(x_1^0, x_2^0) \\
x_1^2 := f_1(x_1^1, x_2^0) & x_2^2 := f_2(x_1^0, x_2^1) \\
x_1^3 := x_1^2 & x_2^3 := f_2(x_1^1, x_2^2) \\
x_1^4 := f_1(x_1^3, x_2^2) & x_2^4 := f_2(x_1^2, x_2^3) \\
x_1^5 := f_1(x_1^4, x_2^3) & x_2^5 := f_2(x_1^2, x_2^4)
\end{array}$$

Synchronous iterations

$$x_i^{k+1} = f_i(x_1^k, \dots, x_p^k), \quad \forall i \in \{1, \dots, p\}$$

delay ⇒ speedup limit



$$\begin{array}{cc}
x_1^1 := f_1(x_1^0, x_2^0) & x_2^1 := f_2(x_1^0, x_2^0) \\
\text{wait} & \text{wait} \\
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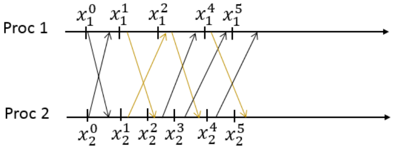
x_1^3 := f_1(x_1^2, x_2^2) & \text{wait} \\
\text{wait} & x_2^3 := f_2(x_1^2, x_2^2) \\
\text{wait} & x_2^4 := f_2(x_1^3, x_2^3)
\end{array}$$

1 - Asynchronous iterative methods

Asynchronous iterations

$$x_i^{k+1} = f_i(x_1^{\tau_1^i(k)}, \dots, x_p^{\tau_p^i(k)}), \quad \forall i \in P^k$$
$$x_i^{k+1} = x_i^k, \quad \forall i \notin P^k$$

delay ⇒ low convergence rate


Proc 1
Proc 2
x_1^0 x_1^1 x_1^2 x_1^4 x_1^5
x_2^0 x_2^1 x_2^2 x_2^3 x_2^4 x_2^5

$$\begin{array}{ll}
x_1^1 := f_1(x_1^0, x_2^0) & x_2^1 := f_2(x_1^0, x_2^0) \\
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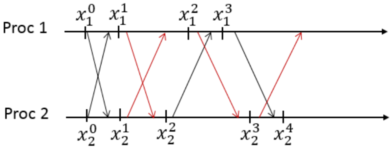
x_1^5 := f_1(x_1^4, x_2^3) & x_2^5 := f_2(x_1^2, x_2^4)
\end{array}$$

$$P^k \subset \{1, \dots, p\}, \qquad \tau_j^i(k) \leq k$$

Synchronous iterations

$$x_i^{k+1} = f_i(x_1^k, \dots, x_p^k), \quad \forall i \in \{1, \dots, p\}$$

delay ⇒ speedup limit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x_1^0 x_1^1 x_1^2 x_1^3
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\end{array}$$

1 - Asynchronous iterative methods

- Linear problems
  $Ax = b \iff M^{-1}Nx + M^{-1}b = x$

Asynchronous iterations

$$\begin{aligned} x_i^{k+1} &= f_i(x_1^{\tau_1^i(k)}, \dots, x_p^{\tau_p^i(k)}), & \forall i \in P^k \\ x_i^{k+1} &= x_i^k, & \forall i \notin P^k \end{aligned}$$

Synchronous iterations

$$x_i^{k+1} = f_i(x_1^k, \dots, x_p^k), \quad \forall i \in \{1, \dots, p\}$$

Convergence condition (necessary and sufficient)

$$\rho(M^{-1}N) < 1$$

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Convergence condition (necessary and sufficient)
[Chazan and Miranker, 1969]

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Convergence condition (necessary and sufficient)
[Chazan and Miranker, 1969]

$$\rho(M^{-1}N) \leq \quad \rho(|M^{-1}N|) < 1$$

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General fixed-point problems

$$f^{(k)}(x, x, \ldots, x) = x, \ \forall k \in \mathbb{N}, \quad f^{(k)} : E^m \mapsto E, \ m \in \mathbb{N}^*$$

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[Miellou, 1975] (sufficient)

$$|f(x) - f(y)| \leq T|x - y|$$

$$T \geq O, \ \rho(T) < 1, \ |x| = (|x_1|, \dots, |x_p|)$$

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$$\|f(x) - f(y)\|_\infty^w \leq \alpha \|x - y\|_\infty^w$$

$$w > 0, \;\; \alpha < 1, \;\; \|x\|_\infty^w = \max_i |x_i|/w_i$$

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$$f(S^{(t)}) \subset S^{(t+1)} \subset S^{(t)}$$

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[Frommer and Szyld, 1994] (sufficient)

$$\|f^{(k)}(x) - f^{(k)}(y)\|_\infty^w \leq \alpha \|x - y\|_\infty^w, \quad \forall k$$

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[Baudet, 1978] (sufficient)

$$|f(X) - f(Y)| \leq T \max\{|x^{(1)} - y^{(1)}|, \ldots, |x^{(m)} - y^{(m)}|\}$$

$$T \geq O, \quad \rho(T) < 1, \quad (\max\{|x|, |y|\})_i = \max\{|x_i|, |y_i|\}$$

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$$\|f(X) - x^*\|_\infty^w \leq \alpha \max\{\|x^{(l)} - x^*\|_\infty^w\}_{1 \leq l \leq m}$$

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- Linear problems
  $Ax = b \iff M^{-1}Nx + M^{-1}b = x$

[Chazan and Miranker, 1969] (necessary and sufficient): $\rho(|M^{-1}N|) < 1$

- General fixed-point problems

$$f^{(k)}(x, x, \ldots, x) = x, \quad \forall k \in \mathbb{N}, \quad f^{(k)} : E^m \mapsto E, \quad m \in \mathbb{N}^*$$

$$m = 1, \quad f^{(k)} \equiv f, \quad \forall k$$

[Miellou, 1975] (sufficient): $|.|$-contraction

[El Tarazi, 1982] (sufficient): $\|.\|_\infty^w$-contraction

[Bertsekas, 1983] (sufficient): $\{S^{(t)}\}$-contraction

$$m = 1$$

[Frommer & Szyld, 1994] (sufficient): $\|.\|_\infty^w$-contraction, $\forall k$

[Frommer & Szyld, 2000] (sufficient): $\{S^{(t)}\}$-contraction, $\forall k$

$$m \geq 1, \quad f^{(k)} \equiv f, \quad \forall k$$

$$X := (x^{(1)}, \ldots, x^{(m)}), \quad Y := (y^{(1)}, \ldots, y^{(m)})$$

[Baudet, 1978] (sufficient): $|.|$-contraction
[El Tarazi, 1982] (sufficient)

$$\|f(X) - x^*\|_\infty^w \leq \alpha \max\{\|x^{(l)} - x^*\|_\infty^w\}_{1 \leq l \leq m}$$

$$w > 0, \quad \alpha < 1, \quad \|x\|_\infty^w = \max_i |x_i|/w_i$$

[Frommer & Szyld, 1998] (sufficient)

$$\|f(x, y) - x^*\|_\infty^w \leq \alpha \max\{\|x - x^*\|_\infty^w, \|y - x^*\|_\infty^w\}$$

$$w > 0, \quad \alpha < 1, \quad \|x\|_\infty^w = \max_i |x_i|/w_i$$

1 - Asynchronous iterative methods

- Linear problems
  $Ax = b \iff M^{-1}Nx + M^{-1}b = x$

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[El Tarazi, 1982] (sufficient)

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$$w > 0, \ \ \alpha < 1, \ \ \|x\|^w_\infty = \max_i |x_i|/w_i$$

[Frommer & Szyld, 1998] (sufficient) $\quad m = 2$

$$\|f(x, y) - x^*\|^w_\infty \leq \alpha \max\{\|x - x^*\|^w_\infty, \|y - x^*\|^w_\infty\}$$

$$w > 0, \ \ \alpha < 1, \ \ \|x\|^w_\infty = \max_i |x_i|/w_i$$

1 - Asynchronous iterative methods

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$$m \geq 1, \quad f^{(k)} \equiv f, \quad \forall k$$

[Baudet, 1978] (sufficient): $|.|$-contraction

$$|f(X) - f(Y)| \leq T \max\{|x^{(1)} - y^{(1)}|, \dots, |x^{(m)} - y^{(m)}|\}$$

$$T \geq O, \quad \rho(T) < 1, \quad (\max\{|x|, |y|\})_i = \max\{|x_i|, |y_i|\}$$

[El Tarazi, 1982] (sufficient): $\|.\|_\infty^w$-contraction

$$\|f(X) - x^*\|_\infty^w \leq \alpha \max\{\|x^{(l)} - x^*\|_\infty^w\}_{1 \leq l \leq m}$$

$$w > 0, \quad \alpha < 1, \quad \|x\|_\infty^w = \max_i |x_i|/w_i$$

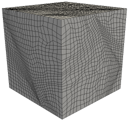
[Frommer & Szyld, 1998] $m = 2$

02 Space domain decomposition

2 - Space domain decomposition

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$



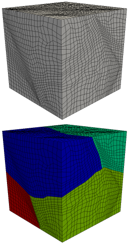
2 - Space domain decomposition

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

Two subdomains

$$\begin{bmatrix} A_{1,1} & O & A_{1,\Gamma} \\ O & A_{2,2} & A_{2,\Gamma} \\ A_{\Gamma,1} & A_{\Gamma,2} & A_{\Gamma,\Gamma}^{(1)} + A_{\Gamma,\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_\Gamma \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_\Gamma^{(1)} + b_\Gamma^{(2)} \end{bmatrix}$$



2 - Space domain decomposition

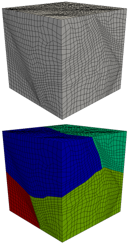
Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

Two subdomains

$$\begin{bmatrix} A_{1,1} & O & A_{1,\Gamma} \\ O & A_{2,2} & A_{2,\Gamma} \\ A_{\Gamma,1} & A_{\Gamma,2} & A_{\Gamma,\Gamma}^{(1)} + A_{\Gamma,\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_\Gamma \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_\Gamma^{(1)} + b_\Gamma^{(2)} \end{bmatrix}$$

$$\begin{aligned} A_{1,1}x_1 + A_{1,\Gamma}x_\Gamma &= b_1 \\ A_{2,2}x_2 + A_{2,\Gamma}x_\Gamma &= b_2 \\ A_{\Gamma,1}x_1 + A_{\Gamma,2}x_2 + \left(A_{\Gamma,\Gamma}^{(1)} + A_{\Gamma,\Gamma}^{(2)}\right) x_\Gamma &= b_\Gamma^{(1)} + b_\Gamma^{(2)} \end{aligned}$$



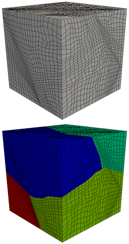
2 - Space domain decomposition

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

Two subdomains

$$\begin{bmatrix} A_{1,1} & O & A_{1,\Gamma} \\ O & A_{2,2} & A_{2,\Gamma} \\ A_{\Gamma,1} & A_{\Gamma,2} & A^{(1)}_{\Gamma,\Gamma} + A^{(2)}_{\Gamma,\Gamma} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_\Gamma \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b^{(1)}_\Gamma + b^{(2)}_\Gamma \end{bmatrix}$$



$$\begin{aligned} A_{1,1}x_1 + A_{1,\Gamma}x_\Gamma &= b_1 \\ A_{2,2}x_2 + A_{2,\Gamma}x_\Gamma &= b_2 \\ A_{\Gamma,1}x_1 + A_{\Gamma,2}x_2 + \left(A^{(1)}_{\Gamma,\Gamma} + A^{(2)}_{\Gamma,\Gamma}\right) x_\Gamma &= b^{(1)}_\Gamma + b^{(2)}_\Gamma \end{aligned}$$

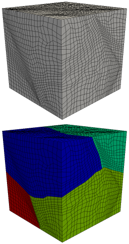
$$\begin{aligned} x_2 &= A^{-1}_{2,2} \left(b_2 - A_{2,\Gamma}x_\Gamma\right) \\ A_{\Gamma,1}x_1 + \left(A^{(1)}_{\Gamma,\Gamma} + A^{(2)}_{\Gamma,\Gamma}\right) x_\Gamma &= b^{(1)}_\Gamma + b^{(2)}_\Gamma - A_{\Gamma,2}x_2 \end{aligned}$$

2 - Space domain decomposition

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

Two subdomains



$$\begin{bmatrix} A_{1,1} & O & A_{1,\Gamma} \\ O & A_{2,2} & A_{2,\Gamma} \\ A_{\Gamma,1} & A_{\Gamma,2} & A_{\Gamma,\Gamma}^{(1)} + A_{\Gamma,\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_\Gamma \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_\Gamma^{(1)} + b_\Gamma^{(2)} \end{bmatrix}$$

$$A_{1,1}x_1 + A_{1,\Gamma}x_\Gamma = b_1$$

$$A_{2,2}x_2 + A_{2,\Gamma}x_\Gamma = b_2$$

$$A_{\Gamma,1}x_1 + A_{\Gamma,2}x_2 + \left(A_{\Gamma,\Gamma}^{(1)} + A_{\Gamma,\Gamma}^{(2)}\right) x_\Gamma = b_\Gamma^{(1)} + b_\Gamma^{(2)}$$

$$x_2 = A_{2,2}^{-1}\left(b_2 - A_{2,\Gamma}x_\Gamma\right)$$

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$$\begin{bmatrix} A_{1,1} & A_{1,\Gamma} \\ A_{\Gamma,1} & A_{\Gamma,\Gamma}^{(1)} + A_{\Gamma,\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_\Gamma \end{bmatrix} = \begin{bmatrix} b_1 \\ b_\Gamma^{(1)} + b_\Gamma^{(2)} - A_{\Gamma,2}x_2 \end{bmatrix}$$

2 - Space domain decomposition

Problem

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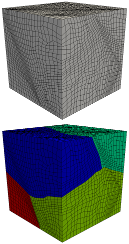
$$\begin{bmatrix} A_{2,2} & A_{2,\Gamma} \\ A_{\Gamma,2} & A_{\Gamma,\Gamma}^{(1)} + A_{\Gamma,\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} x_2 \\ x_\Gamma \end{bmatrix} = \begin{bmatrix} b_2 \\ b_\Gamma^{(1)} + b_\Gamma^{(2)} - A_{\Gamma,1}x_1 \end{bmatrix}$$



2 - Space domain decomposition

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

Two subdomains

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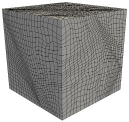
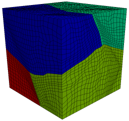




2 - Space domain decomposition

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

Two subdomains

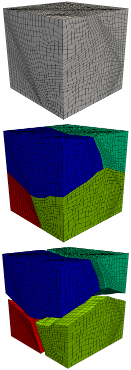
$$\begin{bmatrix} A_{1,1} & A_{1,\Gamma} \\ A_{\Gamma,1} & A^{(1)}_{\Gamma,\Gamma} + A^{(2)}_{\Gamma,\Gamma} \end{bmatrix} \begin{bmatrix} x_1 \\ x_\Gamma \end{bmatrix} = \begin{bmatrix} b_1 \\ b^{(1)}_\Gamma + b^{(2)}_\Gamma - A_{\Gamma,2}x_2 \end{bmatrix}$$

$$\begin{bmatrix} A_{2,2} & A_{2,\Gamma} \\ A_{\Gamma,2} & A^{(1)}_{\Gamma,\Gamma} + A^{(2)}_{\Gamma,\Gamma} \end{bmatrix} \begin{bmatrix} x_2 \\ x_\Gamma \end{bmatrix} = \begin{bmatrix} b_2 \\ b^{(1)}_\Gamma + b^{(2)}_\Gamma - A_{\Gamma,1}x_1 \end{bmatrix}$$

Parallel solutions

$$\begin{bmatrix} A_{1,1} & A_{1,\Gamma} \\ A_{\Gamma,1} & A^{(1)}_{\Gamma,\Gamma} + \Lambda^{(1)}_{\Gamma,\Gamma} \end{bmatrix} \begin{bmatrix} x_1 \\ x^{(1)}_\Gamma \end{bmatrix} = \begin{bmatrix} b_1 \\ b^{(1)}_\Gamma + \lambda^{(1)}_\Gamma \end{bmatrix}$$

$$\begin{bmatrix} A_{2,2} & A_{2,\Gamma} \\ A_{\Gamma,2} & A^{(2)}_{\Gamma,\Gamma} + \Lambda^{(2)}_{\Gamma,\Gamma} \end{bmatrix} \begin{bmatrix} x_2 \\ x^{(2)}_\Gamma \end{bmatrix} = \begin{bmatrix} b_2 \\ b^{(2)}_\Gamma + \lambda^{(2)}_\Gamma \end{bmatrix}$$



2 - Space domain decomposition

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

Two subdomains

$$\begin{bmatrix} A_{1,1} & A_{1,\Gamma} \\ A_{\Gamma,1} & A_{\Gamma,\Gamma}^{(1)} + A_{\Gamma,\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_\Gamma \end{bmatrix} = \begin{bmatrix} b_1 \\ b_\Gamma^{(1)} + b_\Gamma^{(2)} - A_{\Gamma,2}x_2 \end{bmatrix}$$

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Parallel solutions

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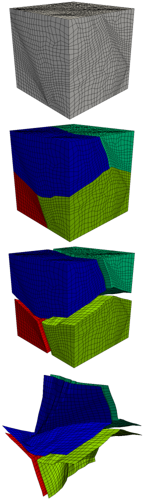
Consistency across interface Γ

$$x_\Gamma^{(1)} = x_\Gamma^{(2)}$$

$$\lambda_\Gamma^{(1)} - \Lambda_{\Gamma,\Gamma}^{(1)} x_\Gamma^{(1)} = -\left(\lambda_\Gamma^{(2)} - \Lambda_{\Gamma,\Gamma}^{(2)} x_\Gamma^{(2)}\right)$$



2 - Space domain decomposition

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

Two subdomains
Parallel solutions

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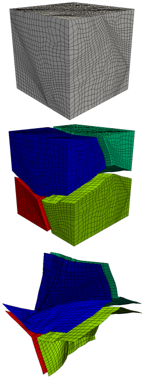


2 - Space domain decomposition

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

Two subdomains
Parallel solutions

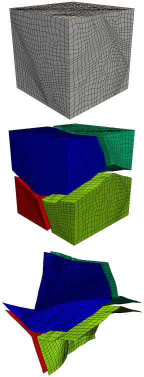
$$\begin{bmatrix} A_{1,1} & A_{1,\Gamma} \\ A_{\Gamma,1} & A^{(1)}_{\Gamma,\Gamma} + \Lambda^{(1)}_{\Gamma,\Gamma} \end{bmatrix} \begin{bmatrix} x_1 \\ x^{(1)}_{\Gamma} \end{bmatrix} = \begin{bmatrix} b_1 \\ b^{(1)}_{\Gamma} + \lambda^{(1)}_{\Gamma} \end{bmatrix}$$

$$\begin{bmatrix} A_{2,2} & A_{2,\Gamma} \\ A_{\Gamma,2} & A^{(2)}_{\Gamma,\Gamma} + \Lambda^{(2)}_{\Gamma,\Gamma} \end{bmatrix} \begin{bmatrix} x_2 \\ x^{(2)}_{\Gamma} \end{bmatrix} = \begin{bmatrix} b_2 \\ b^{(2)}_{\Gamma} + \lambda^{(2)}_{\Gamma} \end{bmatrix}$$

Consistency across interface Γ

$$x^{(1)}_{\Gamma} = x^{(2)}_{\Gamma}$$

$$\lambda^{(1)}_{\Gamma} - \Lambda^{(1)}_{\Gamma,\Gamma} x^{(1)}_{\Gamma} = -\left(\lambda^{(2)}_{\Gamma} - \Lambda^{(2)}_{\Gamma,\Gamma} x^{(2)}_{\Gamma}\right)$$



$$x_1 = A^{-1}_{1,1}\left(b_1 - A_{1,\Gamma} x^{(1)}_{\Gamma}\right)$$

$$A_{\Gamma,1} x_1 + \left(A^{(1)}_{\Gamma,\Gamma} + \Lambda^{(1)}_{\Gamma,\Gamma}\right) x^{(1)}_{\Gamma} = b^{(1)}_{\Gamma} + \lambda^{(1)}_{\Gamma}$$

2 - Space domain decomposition

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

Two subdomains
Parallel solutions

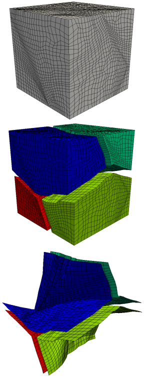
$$\begin{bmatrix} A_{1,1} & A_{1,\Gamma} \\ A_{\Gamma,1} & A^{(1)}_{\Gamma,\Gamma} + \Lambda^{(1)}_{\Gamma,\Gamma} \end{bmatrix} \begin{bmatrix} x_1 \\ x^{(1)}_{\Gamma} \end{bmatrix} = \begin{bmatrix} b_1 \\ b^{(1)}_{\Gamma} + \lambda^{(1)}_{\Gamma} \end{bmatrix}$$

$$\begin{bmatrix} A_{2,2} & A_{2,\Gamma} \\ A_{\Gamma,2} & A^{(2)}_{\Gamma,\Gamma} + \Lambda^{(2)}_{\Gamma,\Gamma} \end{bmatrix} \begin{bmatrix} x_2 \\ x^{(2)}_{\Gamma} \end{bmatrix} = \begin{bmatrix} b_2 \\ b^{(2)}_{\Gamma} + \lambda^{(2)}_{\Gamma} \end{bmatrix}$$

Consistency across interface Γ

$$x^{(1)}_{\Gamma} = x^{(2)}_{\Gamma}$$

$$\lambda^{(1)}_{\Gamma} - \Lambda^{(1)}_{\Gamma,\Gamma} x^{(1)}_{\Gamma} = -\left(\lambda^{(2)}_{\Gamma} - \Lambda^{(2)}_{\Gamma,\Gamma} x^{(2)}_{\Gamma}\right)$$



$$x_1 = A^{-1}_{1,1}\left(b_1 - A_{1,\Gamma} x^{(1)}_{\Gamma}\right)$$

$$A_{\Gamma,1} x_1 + \left(A^{(1)}_{\Gamma,\Gamma} + \Lambda^{(1)}_{\Gamma,\Gamma}\right) x^{(1)}_{\Gamma} = b^{(1)}_{\Gamma} + \lambda^{(1)}_{\Gamma}$$

$$\left(A^{(1)}_{\Gamma,\Gamma} - A_{\Gamma,1} A^{-1}_{1,1} A_{1,\Gamma} + \Lambda^{(1)}_{\Gamma,\Gamma}\right) x^{(1)}_{\Gamma} = b^{(1)}_{\Gamma} + \lambda^{(1)}_{\Gamma} - A_{\Gamma,1} A^{-1}_{1,1} b_1$$

2 - Space domain decomposition

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

Two subdomains
Parallel solutions

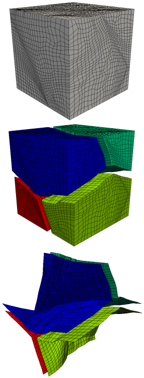
$$\left(A_{\Gamma,\Gamma}^{(1)} - A_{\Gamma,1}A_{1,1}^{-1}A_{1,\Gamma} + \Lambda_{\Gamma,\Gamma}^{(1)}\right) x_{\Gamma}^{(1)} = b_{\Gamma}^{(1)} + \lambda_{\Gamma}^{(1)} - A_{\Gamma,1}A_{1,1}^{-1}b_1$$

$$\left(A_{\Gamma,\Gamma}^{(2)} - A_{\Gamma,2}A_{2,2}^{-1}A_{2,\Gamma} + \Lambda_{\Gamma,\Gamma}^{(2)}\right) x_{\Gamma}^{(2)} = b_{\Gamma}^{(2)} + \lambda_{\Gamma}^{(2)} - A_{\Gamma,2}A_{2,2}^{-1}b_2$$

Consistency across interface Γ

$$x_{\Gamma}^{(1)} = x_{\Gamma}^{(2)}$$

$$\lambda_{\Gamma}^{(1)} - \Lambda_{\Gamma,\Gamma}^{(1)}x_{\Gamma}^{(1)} = -\left(\lambda_{\Gamma}^{(2)} - \Lambda_{\Gamma,\Gamma}^{(2)}x_{\Gamma}^{(2)}\right)$$



2 - Space domain decomposition

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

Two subdomains
Parallel solutions

$$\left(A^{(1)}_{\Gamma,\Gamma} - A_{\Gamma,1}A^{-1}_{1,1}A_{1,\Gamma} + \Lambda^{(1)}_{\Gamma,\Gamma}\right) x^{(1)}_{\Gamma} = b^{(1)}_{\Gamma} + \lambda^{(1)}_{\Gamma} - A_{\Gamma,1}A^{-1}_{1,1}b_1$$

$$\left(A^{(2)}_{\Gamma,\Gamma} - A_{\Gamma,2}A^{-1}_{2,2}A_{2,\Gamma} + \Lambda^{(2)}_{\Gamma,\Gamma}\right) x^{(2)}_{\Gamma} = b^{(2)}_{\Gamma} + \lambda^{(2)}_{\Gamma} - A_{\Gamma,2}A^{-1}_{2,2}b_2$$

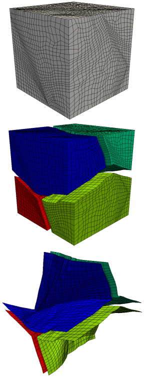
$$\left(S^{(1)}_{\Gamma,\Gamma} + \Lambda^{(1)}_{\Gamma,\Gamma}\right) x^{(1)}_{\Gamma} = d^{(1)}_{\Gamma} + \lambda^{(1)}_{\Gamma}$$

$$\left(S^{(2)}_{\Gamma,\Gamma} + \Lambda^{(2)}_{\Gamma,\Gamma}\right) x^{(2)}_{\Gamma} = d^{(2)}_{\Gamma} + \lambda^{(2)}_{\Gamma}$$

Consistency across interface Γ

$$x^{(1)}_{\Gamma} = x^{(2)}_{\Gamma}$$

$$\lambda^{(1)}_{\Gamma} - \Lambda^{(1)}_{\Gamma,\Gamma}x^{(1)}_{\Gamma} = -\left(\lambda^{(2)}_{\Gamma} - \Lambda^{(2)}_{\Gamma,\Gamma}x^{(2)}_{\Gamma}\right)$$



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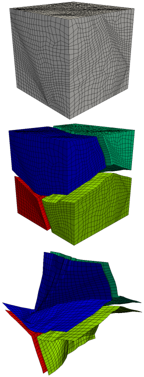


Two subdomains
Parallel solutions

$$\left(A_{\Gamma,\Gamma}^{(1)} - A_{\Gamma,1}A_{1,1}^{-1}A_{1,\Gamma} + \Lambda_{\Gamma,\Gamma}^{(1)}\right) x_\Gamma^{(1)} = b_\Gamma^{(1)} + \lambda_\Gamma^{(1)} - A_{\Gamma,1}A_{1,1}^{-1}b_1$$
$$\left(A_{\Gamma,\Gamma}^{(2)} - A_{\Gamma,2}A_{2,2}^{-1}A_{2,\Gamma} + \Lambda_{\Gamma,\Gamma}^{(2)}\right) x_\Gamma^{(2)} = b_\Gamma^{(2)} + \lambda_\Gamma^{(2)} - A_{\Gamma,2}A_{2,2}^{-1}b_2$$

$$\left(S_{\Gamma,\Gamma}^{(1)} + \Lambda_{\Gamma,\Gamma}^{(1)}\right) x_\Gamma^{(1)} = d_\Gamma^{(1)} + \lambda_\Gamma^{(1)}$$
$$\left(S_{\Gamma,\Gamma}^{(2)} + \Lambda_{\Gamma,\Gamma}^{(2)}\right) x_\Gamma^{(2)} = d_\Gamma^{(2)} + \lambda_\Gamma^{(2)}$$

Consistency across interface Γ

$$x_\Gamma^{(1)} = x_\Gamma^{(2)}$$
$$\lambda_\Gamma^{(1)} - \Lambda_{\Gamma,\Gamma}^{(1)} x_\Gamma^{(1)} = -\left(\lambda_\Gamma^{(2)} - \Lambda_{\Gamma,\Gamma}^{(2)} x_\Gamma^{(2)}\right)$$

Primal approach: $\lambda_\Gamma^{(i)} = \left(S_{\Gamma,\Gamma}^{(i)} + \Lambda_{\Gamma,\Gamma}^{(i)}\right) x_\Gamma^{(i)} - d_\Gamma^{(i)}$

Dual approach: $x_\Gamma^{(i)} = \left(S_{\Gamma,\Gamma}^{(i)} + \Lambda_{\Gamma,\Gamma}^{(i)}\right)^{-1} \left(d_\Gamma^{(i)} + \lambda_\Gamma^{(i)}\right)$

2 - Space domain decomposition

Problem

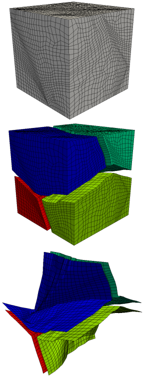
$$Ax = b, \quad x \in \mathbb{C}^n$$

Two subdomains

Parallel solutions

$$\left(A^{(1)}_{\Gamma,\Gamma} - A_{\Gamma,1}A^{-1}_{1,1}A_{1,\Gamma} + \Lambda^{(1)}_{\Gamma,\Gamma}\right) x^{(1)}_{\Gamma} = b^{(1)}_{\Gamma} + \lambda^{(1)}_{\Gamma} - A_{\Gamma,1}A^{-1}_{1,1}b_1$$

$$\left(A^{(2)}_{\Gamma,\Gamma} - A_{\Gamma,2}A^{-1}_{2,2}A_{2,\Gamma} + \Lambda^{(2)}_{\Gamma,\Gamma}\right) x^{(2)}_{\Gamma} = b^{(2)}_{\Gamma} + \lambda^{(2)}_{\Gamma} - A_{\Gamma,2}A^{-1}_{2,2}b_2$$



$$\left(S^{(1)}_{\Gamma,\Gamma} + \Lambda^{(1)}_{\Gamma,\Gamma}\right) x^{(1)}_{\Gamma} = d^{(1)}_{\Gamma} + \lambda^{(1)}_{\Gamma}$$

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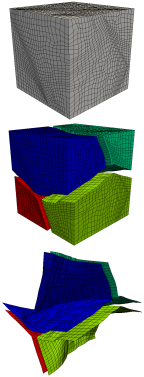

2 - Space domain decomposition

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Primal approach

$$\lambda^{(i)}_{\Gamma} = \left(S^{(i)}_{\Gamma,\Gamma} + \Lambda^{(i)}_{\Gamma,\Gamma}\right) x^{(i)}_{\Gamma} - d^{(i)}_{\Gamma}$$

Interface problem: Schur complement inversion

$$\left(S^{(1)}_{\Gamma,\Gamma} + S^{(2)}_{\Gamma,\Gamma}\right) x^{(1)}_{\Gamma} = d^{(1)}_{\Gamma} + d^{(2)}_{\Gamma}$$

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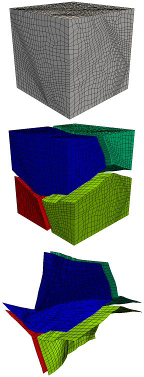

2 - Space domain decomposition

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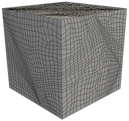
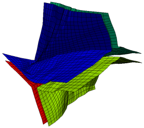
$$Ax = b, \quad x \in \mathbb{C}^n$$

Interface problem: Schur complement inversion

$$\sum_i S^{(i)}_{\Gamma,\Gamma} x_\Gamma = \sum_i d^{(i)}_\Gamma$$





2 - Space domain decomposition

Problem

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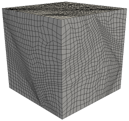
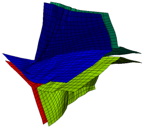
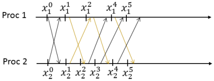
Asynchronous iterative solution

$$\sum_i S^{(i)}_{\Gamma,\Gamma} = M - N$$



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Proc 2
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Asynchronous iterative solution

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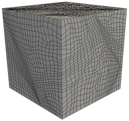
Classical splittings (M, N): (block-)Jacobi, Gauss-Seidel, SOR, ...

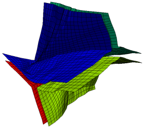
- Minimum requirement: $\mathtt{diag} \sum_i S^{(i)}_{\Gamma,\Gamma}$
- $\Rightarrow$ explicit computation of $\sum_i S^{(i)}_{\Gamma,\Gamma}$
- $\Rightarrow$ vector decomposition

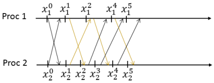
$$x_\Gamma := \begin{bmatrix} (x_\Gamma)_1 & \cdots & (x_\Gamma)_p \end{bmatrix}^\mathsf{T}$$

instead of primal domain decomposition approach

$$x_\Gamma := x^{(1)}_\Gamma = \cdots = x^{(p)}_\Gamma$$







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Problem

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Asynchronous iterative solution

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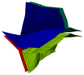
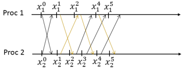
Classical splittings (M, N) $\Rightarrow$ vector decomposition





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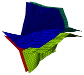

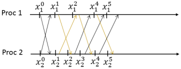
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Proposition 1

$$M := \gamma I$$

Sufficient conditions for vector decomposition

$S_{\Gamma,\Gamma}$ is an $\mathcal{M}$-matrix, $\qquad \gamma \geq \max\ \operatorname{diag} S_{\Gamma,\Gamma}$

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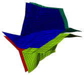

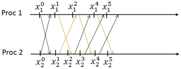
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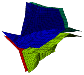

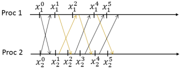
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$\mathcal{M}$-matrix
$(S_{\Gamma,\Gamma})_{i,j\neq i} \leq 0, \quad S_{\Gamma,\Gamma}^{-1} \geq O$
$(S_{\Gamma,\Gamma})_{i,j\neq i} \leq 0, \quad S_{\Gamma,\Gamma}$ symmetric and positive definite

[Crabtree and Haynsworth, 1969]

A is an $\mathcal{M}$-matrix $\implies$ $S_{\Gamma,\Gamma}$ is an $\mathcal{M}$-matrix

2 - Space domain decomposition

Problem

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Interface problem: Schur complement inversion

Asynchronous iterative solution

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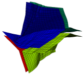
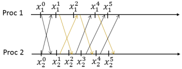
Classical splittings (M, N) ⇒ vector decomposition

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Remark

$$A \text{ is an } \mathcal{M}\text{-matrix} \implies A_{\Gamma,\Gamma} := \sum_i A^{(i)}_{\Gamma,\Gamma} \geq S_{\Gamma,\Gamma}$$

Sufficient conditions for vector decomposition

$$S_{\Gamma,\Gamma} \text{ is an } \mathcal{M}\text{-matrix}, \qquad \gamma \geq \max \ \text{diag}\, S_{\Gamma,\Gamma}$$

⇒ **Sufficient**

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Interface problem: Schur complement inversion
Asynchronous iterative solution

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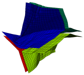
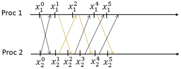
Classical splittings (M, N) ⇒ vector decomposition

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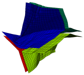

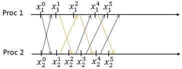
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Sufficient conditions for vector decomposition

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Interface problem: Schur complement inversion
Asynchronous iterative solution

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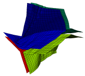
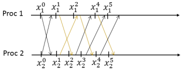
Classical splittings (M, N) $\Rightarrow$ vector decomposition

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Asynchronous iterative solution

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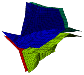
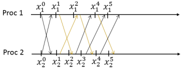
Classical splittings (M, N) ⇒ vector decomposition

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2 - Space domain decomposition

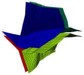
Problem

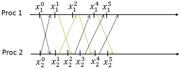
$$Ax = b, \quad x \in \mathbb{C}^n$$

Interface problem: Schur complement inversion
Asynchronous iterative solution

$$S_{\Gamma,\Gamma} := \sum_i S^{(i)}_{\Gamma,\Gamma} = M - N$$

Classical splittings (M, N) $\Rightarrow$ vector decomposition

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Proc 2
x_1^0 x_1^1 x_1^2 x_1^4 x_1^5
x_2^0 x_2^1 x_2^2 x_2^3 x_2^4 x_2^5

Proposition 1

$$M := \gamma I$$

Sufficient conditions for vector decomposition

$S_{\Gamma,\Gamma}$ is an $\mathcal{M}$-matrix, $\quad \gamma \geq \max \text{ diag } S_{\Gamma,\Gamma}$

$S_{\Gamma,\Gamma}$ is an $\mathcal{M}$-matrix, $\quad \gamma \geq \max \text{ diag } A_{\Gamma,\Gamma}$

[Crabtree and Haynsworth, 1969]

A is an $\mathcal{M}$-matrix $\implies$ $S_{\Gamma,\Gamma}$ is an $\mathcal{M}$-matrix

Proposition 2

$$M := M_{\Gamma,\Gamma}, \qquad A_{\Gamma,\Gamma} = M_{\Gamma,\Gamma} - N_{\Gamma,\Gamma}$$

Sufficient conditions for vector decomposition

A is an $\mathcal{M}$-matrix, $\quad (M_{\Gamma,\Gamma}, N_{\Gamma,\Gamma})$ is an $\mathcal{M}$-splitting

A is an $\mathcal{M}$-matrix, $\quad (M_{\Gamma,\Gamma}, N_{\Gamma,\Gamma})$ is a block-Jacobi

- Equivalent sufficient conditions for synchronous convergence of primal Schur decomposition
- What about asynchronous primal Schur decomposition?

2 - Space domain decomposition

Problem

$$Ax = b, \quad x \in \mathbb{C}^n$$

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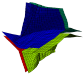
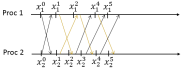
Classical splittings (M, N) $\Rightarrow$ vector decomposition

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Propositions

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Asynchronous vector / Synchronous primal Schur

$S_{\Gamma,\Gamma}$ is an $\mathcal{M}$-matrix, $\quad \gamma \geq \max\ \text{diag}\, S_{\Gamma,\Gamma}$

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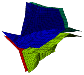

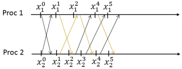
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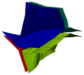

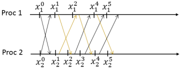
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Theorem

$$\rho(|\widetilde{M}^{-1}\widetilde{N}|) < 1, \quad |\sum_i M^{-1}_{\Gamma,\Gamma} N^{(i)}_{\Gamma,\Gamma}| = \sum_i |M^{-1}_{\Gamma,\Gamma} N^{(i)}_{\Gamma,\Gamma}|$$

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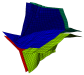
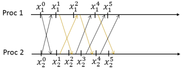
Classical splittings (M, N) ⇒ vector decomposition

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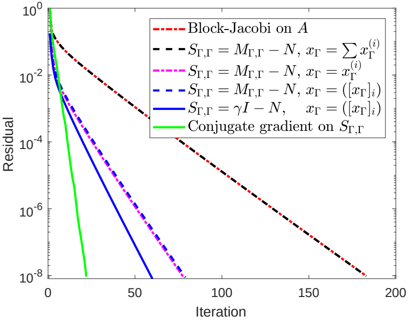
$$N^{(i)}_{\Gamma,\Gamma} = \alpha_i N_{\Gamma,\Gamma}, \qquad \sum_i \alpha_i = 1, \qquad \alpha_i > 0$$

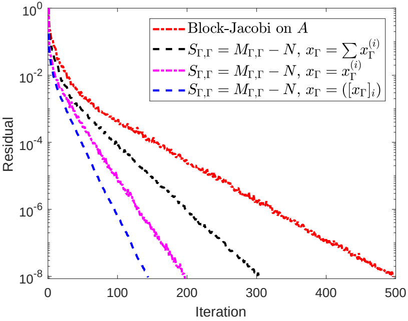
2 - Space domain decomposition

- 2D Laplacian model problem, 16 subdomains
- P^k and τ_j^i pre-generated (same for all methods)
- $S_{\Gamma,\Gamma} = M_{\Gamma,\Gamma} - N$, with $x_\Gamma = \sum_i x_\Gamma^{(i)}$: slight improvement of *[Magoulès & Venet, 2018]*

$$P^k = \{1, \dots, 16\}, \qquad \tau_j^i(k) = k$$



$$P^k \subset \{1, \dots, 16\}, \qquad k - 4 \leq \tau_j^i(k) \leq k$$



Outline

03 Time domain decomposition

3 - Time domain decomposition

Partial differential equation

$$\delta(u(s,t),s,t) = 0, \quad t \in [0,T], \ \ s \in \Omega$$

$u(\Omega, 0)$ given

3 - Time domain decomposition

Partial differential equation

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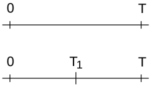

Two time intervals

$$\delta(u_0(s,t),s,t) = 0, \quad t \in [0, T_1] \qquad \delta(u_1(s,t),s,t) = 0, \quad t \in [T_1, T]$$
$$u_0(\Omega,0) = u(\Omega,0) \qquad u_1(\Omega,T_1) = u_0(\Omega,T_1)$$

3 - Time domain decomposition

Partial differential equation

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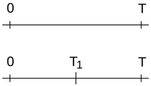

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$$u_0(\Omega,0) = u(\Omega,0) \qquad u_1(\Omega,T_1) = u_0(\Omega,T_1)$$

Parallel solutions

$$\widetilde{u}_0 \equiv u_0 \qquad \delta(\widetilde{u}_1(s,t),s,t) = 0, \quad t \in [T_1,T]$$
$$\lambda_0(\Omega) = u_0(\Omega,0) \qquad \widetilde{u}_1(\Omega,T_1) = \lambda_1(\Omega)$$

3 - Time domain decomposition

Partial differential equation

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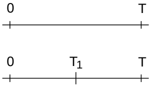

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$$\lambda_0(\Omega) = u_0(\Omega,0) \qquad \widetilde{u}_1(\Omega,T_1) = \lambda_1(\Omega)$$

Consistency across interface T_1

$$\lambda_1(\Omega) = \widetilde{u}_0(\Omega,T_1)$$

3 - Time domain decomposition

Partial differential equation

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Parallel solutions

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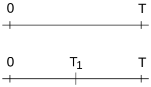

Parareal approach:
Two-level time discretization
[Lions et al, 2001]

3 - Time domain decomposition

Partial differential equation
Two time intervals

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$$u_0(\Omega, 0) = u(\Omega, 0) \qquad u_1(\Omega, T_1) = u_0(\Omega, T_1)$$

Parallel solutions

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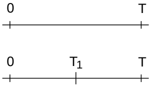

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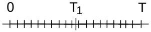


Fine time discretization

$$\alpha(\widetilde{u}_0(s,t_{n+1})) = \beta(\widetilde{u}_0(s,t_n)) \qquad \alpha(\widetilde{u}_1(s,t_{n+1})) = \beta(\widetilde{u}_1(s,t_n))$$

$\Rightarrow$ parallel fine propagator

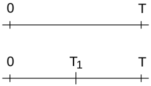
$$F(\lambda_0) := \widetilde{u}_0(\Omega, T_1) \qquad F(\lambda_1) := \widetilde{u}_1(\Omega, T)$$

3 - Time domain decomposition

Partial differential equation

Two time intervals

$$\delta(u_0(s,t),s,t)=0,\quad t\in[0,T_1] \qquad \delta(u_1(s,t),s,t)=0,\quad t\in[T_1,T]$$
$$u_0(\Omega,0)=u(\Omega,0) \qquad u_1(\Omega,T_1)=u_0(\Omega,T_1)$$



Parallel solutions

$$\widetilde{u}_0 \equiv u_0 \qquad \delta(\widetilde{u}_1(s,t),s,t)=0,\quad t\in[T_1,T]$$
$$\lambda_0(\Omega)=u_0(\Omega,0) \qquad \widetilde{u}_1(\Omega,T_1)=\lambda_1(\Omega)$$

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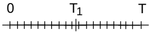
Two-level time discretization
[Lions et al, 2001]

Consistency across interface T_1

$$\lambda_1(\Omega)=\widetilde{u}_0(\Omega,T_1)$$



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$\Rightarrow$ parallel fine propagator

$$F(\lambda_0):=\widetilde{u}_0(\Omega,T_1) \qquad F(\lambda_1):=\widetilde{u}_1(\Omega,T)$$

Coarse time discretization

$$\alpha(u(s,T_1))=\beta(u(s,0))$$
$$\alpha(u(s,T))=\beta(u(s,T_1))$$

$\Rightarrow$ serial coarse propagator

$$G(u(\Omega,0)):=\widetilde{u}(\Omega,T_1),\quad G(\widetilde{u}(\Omega,T_1))=\widetilde{u}(\Omega,T)$$

3 - Time domain decomposition

Partial differential equation
Two time intervals
Parallel solutions

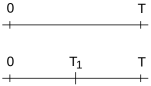
$$\delta(\widetilde{u}_0(s,t),s,t) = 0, \quad t \in [0, T_1] \qquad \delta(\widetilde{u}_1(s,t),s,t) = 0, \quad t \in [T_1, T]$$

$$\lambda_0(\Omega) \text{ given} \qquad \widetilde{u}_1(\Omega, T_1) = \lambda_1(\Omega)$$

Consistency across interface T_1

$$\lambda_1(\Omega) = \widetilde{u}_0(\Omega, T_1)$$

Parareal iterations



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Two-level time discretization
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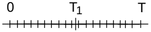

$$F(\lambda_0) := \widetilde{u}_0(\Omega, T_1)$$
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$$G(u(\Omega, 0)) := \widetilde{u}(\Omega, T_1)$$
$$G(\widetilde{u}(\Omega, T_1)) = \widetilde{u}(\Omega, T)$$

3 - Time domain decomposition

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$$\delta(\widetilde{u}_0(s,t),s,t) = 0, \quad t \in [0, T_1] \qquad \delta(\widetilde{u}_1(s,t),s,t) = 0, \quad t \in [T_1, T]$$

$$\lambda_0(\Omega) \text{ given} \qquad \widetilde{u}_1(\Omega, T_1) = \lambda_1(\Omega)$$

Consistency across interface T_1

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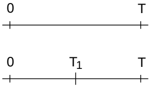
$$\lambda_0^0 := u_0(\Omega, 0)$$

$$\lambda_1^0 := G(\lambda_0^0) \qquad \text{wait} \qquad [0, T_1]$$



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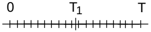

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Prediction

3 - Time domain decomposition

Partial differential equation
Two time intervals
Parallel solutions

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$$\lambda_0(\Omega) \text{ given} \qquad \widetilde{u}_1(\Omega, T_1) = \lambda_1(\Omega)$$

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$$\lambda_1(\Omega) = \widetilde{u}_0(\Omega, T_1)$$

Parareal iterations

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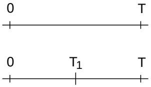
$$\lambda_1^0 := G(\lambda_0^0) \qquad \text{wait} \qquad [0, T_1]$$

$$\lambda_2^0 := G(\lambda_1^0) \qquad [T_1, T]$$



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Two-level time discretization
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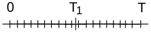

$$F(\lambda_0) := \widetilde{u}_0(\Omega, T_1)$$

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$$G(u(\Omega, 0)) := \widetilde{u}(\Omega, T_1)$$

$$G(\widetilde{u}(\Omega, T_1)) = \widetilde{u}(\Omega, T)$$

Prediction

3 - Time domain decomposition

Partial differential equation

Two time intervals

Parallel solutions

$$\delta(\widetilde{u}_0(s,t),s,t)=0, \quad t\in[0,T_1] \qquad \delta(\widetilde{u}_1(s,t),s,t)=0, \quad t\in[T_1,T]$$

$$\lambda_0(\Omega)\ \text{given} \qquad \widetilde{u}_1(\Omega,T_1)=\lambda_1(\Omega)$$

Consistency across interface T_1

$$\lambda_1(\Omega)=\widetilde{u}_0(\Omega,T_1)$$

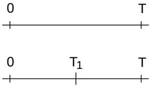
Parareal iterations

$$\lambda_0^0 := u_0(\Omega,0)$$

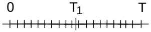
$$\lambda_1^0 := G(\lambda_0^0) \qquad \text{wait} \qquad [0,T_1]$$

$$\lambda_2^0 := G(\lambda_1^0) \qquad [T_1,T]$$

$$F(\lambda_0^0) \qquad F(\lambda_1^0) \qquad [0,T_1]\ [T_1,T]$$



Parareal approach:
Two-level time discretization
[Lions et al, 2001]



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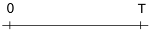
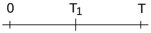
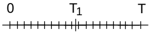
$\lambda_1^1 := G(\lambda_0^1)+F(\lambda_0^0)-G(\lambda_0^0)$	wait	$[0,T_1]$





Parareal approach:
Two-level time discretization
[Lions et al, 2001]



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Prediction

Gap

3 - Time domain decomposition

Partial differential equation

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Parallel solutions

$$\delta(\widetilde{u}_0(s,t),s,t) = 0, \quad t \in [0, T_1] \qquad \delta(\widetilde{u}_1(s,t),s,t) = 0, \quad t \in [T_1, T]$$

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$$F(\lambda_0^0) \qquad F(\lambda_1^0) \qquad [0, T_1]\ [T_1, T]$$

$$\lambda_0^1 := \lambda_0^0$$

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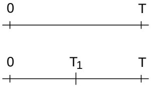

Parareal approach:
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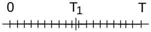

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Prediction

Gap

Corrected prediction

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Partial differential equation
Two time intervals
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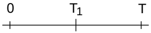

Consistency across interface T_1

$$\lambda_1(\Omega) = \widetilde{u}_0(\Omega, T_1)$$

Parareal approach:
Two-level time discretization
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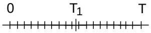
Parareal iterations



$\lambda_0^0 := u_0(\Omega, 0)$			
$\lambda_1^0 := G(\lambda_0^0)$	wait	$[0, T_1]$	
	$\lambda_2^0 := G(\lambda_1^0)$	$[T_1, T]$	$F(\lambda_0) := \widetilde{u}_0(\Omega, T_1)$
$F(\lambda_0^0)$	$F(\lambda_1^0)$	$[0, T_1]$ $[T_1, T]$	$F(\lambda_1) = \widetilde{u}_1(\Omega, T)$
$\lambda_0^1 := \lambda_0^0$			
$\lambda_1^1 := G(\lambda_0^1) + F(\lambda_0^0) - G(\lambda_0^0)$	wait	$[0, T_1]$	$G(u(\Omega, 0)) := \widetilde{u}(\Omega, T_1)$
	$\lambda_2^1 := G(\lambda_1^1) + F(\lambda_1^0) - G(\lambda_1^0)$	$[T_1, T]$	$G(\widetilde{u}(\Omega, T_1)) = \widetilde{u}(\Omega, T)$

Prediction
Gap
Corrected prediction

3 - Time domain decomposition

Partial differential equation
Two time intervals
Parallel solutions

$$\delta(\widetilde{u}_0(s,t),s,t) = 0, \quad t \in [0, T_1] \qquad \delta(\widetilde{u}_1(s,t),s,t) = 0, \quad t \in [T_1, T]$$

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$\lambda_1^0 := G(\lambda_0^0)$	wait	$[0, T_1]$
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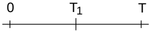
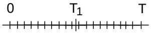
$F(\lambda_0^0)$	$F(\lambda_1^0)$	$[0, T_1]$ $[T_1, T]$
$\lambda_0^1 := \lambda_0^0$		
$\lambda_1^1 := G(\lambda_0^1) + F(\lambda_0^0) - G(\lambda_0^0)$	wait	$[0, T_1]$
	$\lambda_2^1 := G(\lambda_1^1) + F(\lambda_1^0) - G(\lambda_1^0)$	$[T_1, T]$
$\lambda_0^2 := \lambda_0^1$	$F(\lambda_1^1)$	$[T_1, T]$



Parareal approach:
Two-level time discretization
[Lions et al, 2001]



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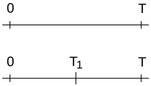
Prediction
Gap
Corrected prediction

3 - Time domain decomposition

Partial differential equation
Two time intervals
Parallel solutions



$$\delta(\widetilde{u}_0(s,t),s,t)=0,\quad t\in[0,T_1]\qquad\qquad \delta(\widetilde{u}_1(s,t),s,t)=0,\quad t\in[T_1,T]$$

$$\lambda_0(\Omega)\text{ given}\qquad\qquad \widetilde{u}_1(\Omega,T_1)=\lambda_1(\Omega)$$

Consistency across interface T_1

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Parareal approach:
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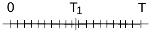

Parareal iterations

$\lambda_0^0 := u_0(\Omega,0)$			
$\lambda_1^0 := G(\lambda_0^0)$	wait	$[0, T_1]$	
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$F(\lambda_0^0)$	$F(\lambda_1^0)$	$[0, T_1]$ $[T_1, T]$	$F(\lambda_1) = \widetilde{u}_1(\Omega,T)$
$\lambda_0^1 := \lambda_0^0$			
$\lambda_1^1 := G(\lambda_0^1)+F(\lambda_0^0)-G(\lambda_0^0)$	wait	$[0, T_1]$	$G(u(\Omega,0)) := \widetilde{u}(\Omega,T_1)$
	$\lambda_2^1 := G(\lambda_1^1)+F(\lambda_1^0)-G(\lambda_1^0)$	$[T_1, T]$	$G(\widetilde{u}(\Omega,T_1)) = \widetilde{u}(\Omega,T)$
$\lambda_0^2 := \lambda_0^1$	$F(\lambda_1^1)$	$[T_1, T]$	
$\lambda_1^2 := \lambda_1^1$	$\lambda_2^2 := G(\lambda_1^2)+F(\lambda_1^1)-G(\lambda_1^1)$	$[T_1, T]$	

Prediction
Gap
Corrected prediction

3 - Time domain decomposition

Partial differential equation
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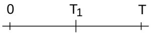
$$\delta(\widetilde{u}_0(s,t),s,t) = 0, \quad t \in [0, T_1] \qquad \delta(\widetilde{u}_1(s,t),s,t) = 0, \quad t \in [T_1, T]$$

$$\lambda_0(\Omega) \text{ given} \qquad \widetilde{u}_1(\Omega, T_1) = \lambda_1(\Omega)$$



Consistency across interface T_1

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Parareal approach:
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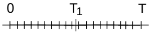

Parareal iterations

$$\lambda_0^0 := u_0(\Omega, 0)$$

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$$\lambda_2^0 := G(\lambda_1^0) \qquad [T_1, T] \qquad F(\lambda_0) := \widetilde{u}_0(\Omega, T_1)$$

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$$\lambda_i^{k+1} = G(\lambda_{i-1}^{k+1}) + F(\lambda_{i-1}^{k}) - G(\lambda_{i-1}^{k}), \qquad [T_{i-1}, T_i]$$

Prediction
Gap
Corrected prediction

3 - Time domain decomposition

Partial differential equation
N time intervals
Parallel solutions and consistency across interfaces T_i
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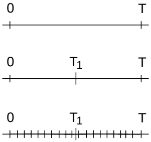

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Parareal approach:
Two-level time discretization
[Lions et al, 2001]

$$F(\lambda_i) = \widetilde{u}_i(\Omega, T_{i+1})$$
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Prediction G
Gap $F - G$
Corrected prediction $G+$

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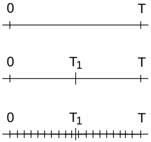

Theorem 1
PDE extension of ODE result
[Gander & Vandewalle, 2007]

Iterative error

$$\|\lambda^k - \lambda^*\|_\infty \leq \alpha^k \|\lambda^0 - \lambda^*\|_\infty$$

$$\alpha = \frac{1-\theta^N}{1-\theta}\|F-G\|, \ \theta \neq 1, \ \theta \geq \|G\|$$

$$\delta(\widetilde{u}_i(s,t), s, t) = 0$$

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Two-level time discretization
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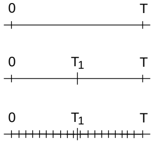

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Sufficient convergence condition for
$\|G\| < 1$ **(stability region)**

$$\|G\| + \|F-G\| < 1 + \|G\|^N \|F-G\|$$

Parareal approach:
Two-level time discretization
[Lions et al, 2001]

$$F(\lambda_i) = \widetilde{u}_i(\Omega, T_{i+1})$$

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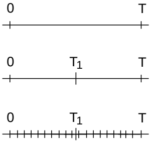

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Sufficient convergence condition for

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Proposition

Parareal approach:

Two-level time discretization

[Lions et al, 2001]

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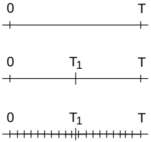

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Sufficient convergence condition for $\|G\| < 1$ (stability region)

$$\|G\| + \|F-G\| < 1 + \|G\|^N \|F-G\|$$

Proposition

Asynchronous convergence (sufficient)

$$\|G\| + \|F-G\| < 1$$

$$\delta(\widetilde{u}_i(s,t), s, t) = 0$$

$$\widetilde{u}_i(\Omega, T_i) = \lambda_i(\Omega)$$

Parareal approach:

Two-level time discretization

[Lions et al, 2001]

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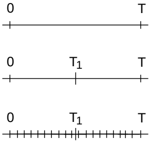

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[Gander & Vandewalle, 2007]

Iterative error

$$\|\lambda^k - \lambda^*\|_\infty \leq \alpha^k \|\lambda^0 - \lambda^*\|_\infty$$

$$\alpha = \frac{1-\theta^N}{1-\theta}\|F-G\|, \; \theta \neq 1, \; \theta \geq \|G\|$$

Sufficient convergence condition for $\|G\| < 1$ (stability region)

$$\|G\| + \|F-G\| < 1 + \|G\|^N \|F-G\|$$

Proposition

Asynchronous convergence (sufficient)

$$\|G\| + \|F-G\| < 1$$

Theorem 2

$$\|\lambda^k - \lambda^*\|_\infty \leq \widetilde{\alpha}^{\tau(k)} \|\lambda^0 - \lambda^*\|_\infty$$

$$\widetilde{\alpha} = \|G\| + \|F-G\|, \quad \lim_{k\to\infty} \tau(k) = \infty$$

$$\delta(\widetilde{u}_i(s,t), s, t) = 0$$

$$\widetilde{u}_i(\Omega, T_i) = \lambda_i(\Omega)$$

Parareal approach:

Two-level time discretization

[Lions et al, 2001]

$$F(\lambda_i) = \widetilde{u}_i(\Omega, T_{i+1})$$

$$G(\widetilde{u}(\Omega, T_i)) = \widetilde{u}(\Omega, T_{i+1})$$

Prediction G

Gap $F - G$

Corrected prediction $G+$

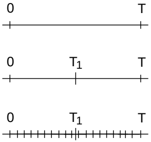
SystemX CentraleSupélec

3 - Time domain decomposition

Partial differential equation
N time intervals
Parallel solutions and consistency across interfaces T_i
Parareal iterations

$$\lambda_i^{k+1} = G(\lambda_{i-1}^{k+1}) + F(\lambda_{i-1}^{k}) - G(\lambda_{i-1}^{k}), \qquad [T_{i-1}, T_i]$$

$$\lambda_i^{k+1} = G\left(\lambda_{i-1}^{\tau_{i-1}^{i,1}(k+1)}\right) + F\left(\lambda_{i-1}^{\tau_{i-1}^{i,2}(k)}\right) - G\left(\lambda_{i-1}^{\tau_{i-1}^{i,2}(k)}\right)$$



Theorem 1
PDE extension of ODE result
[Gander & Vandewalle, 2007]

Iterative error

$$\|\lambda^k - \lambda^*\|_\infty \leq \alpha^k \|\lambda^0 - \lambda^*\|_\infty$$

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Asynchronous convergence (sufficient)

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$$\widetilde{\alpha} = \|G\| + \|F-G\|, \quad \lim_{k\to\infty} \tau(k) = \infty$$

Synchronous convergence for $N \to \infty$

$$\|G\| + \|F-G\| < 1$$

$$\delta(\widetilde{u}_i(s,t), s, t) = 0$$

$$\widetilde{u}_i(\Omega, T_i) = \lambda_i(\Omega)$$

Parareal approach:
Two-level time discretization
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$$F(\lambda_i) = \widetilde{u}_i(\Omega, T_{i+1})$$

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Prediction G
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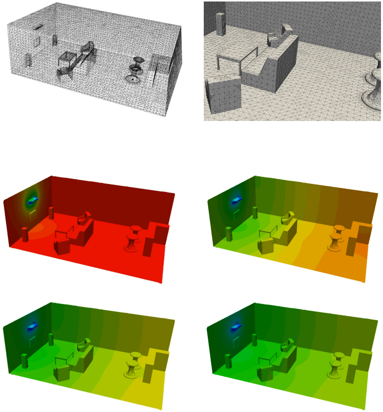
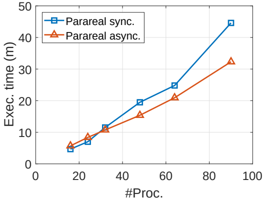
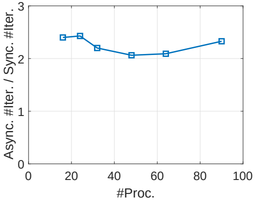
Corrected prediction $G+$

3 - Time domain decomposition







$$t_{n+1} - t_n = 0.002, \qquad T_{i+1} - T_i = 0.2$$

$$N = \#\text{Proc}, \qquad T = N \times (T_{i+1} - T_i)$$

Outline

04 Asynchronous convergence detection

4 - Asynchronous convergence detection

Problem

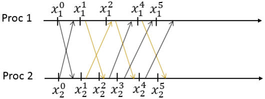
$$f(x) = x, \quad x \in E$$

Asynchronous iterations

$$x_i^{k+1} = f_i(x_1^{\tau_1^i(k)}, \ldots, x_p^{\tau_p^i(k)}), \quad i \in P^k$$

Convergence detection

$$\bar{x} := \left(x_1^{k_1}, \ldots, x_p^{k_p}\right) \simeq x^*$$

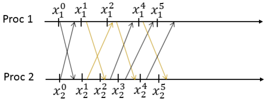


4 - Asynchronous convergence detection

Problem

$$f(x) = x, \quad x \in E$$

Asynchronous iterations


Proc 1
x_1^0 x_1^1 x_1^2 x_1^4 x_1^5
Proc 2
x_2^0 x_2^1 x_2^2 x_2^3 x_2^4 x_2^5

$$x_i^{k+1} = f_i(x_1^{\tau_1^i(k)}, \ldots, x_p^{\tau_p^i(k)}), \quad i \in P^k$$

Convergence detection

$$\bar{x} := \left(x_1^{k_1}, \ldots, x_p^{k_p}\right) \simeq x^*$$

Approach

- Finite time termination *[Bertsekas & Tsitsiklis, 1989], [El Baz, 1996], [Savari & Bertsekas, 1996], ...*
- Snapshot-based supervised termination *[Savari & Bertsekas, 1996]*
- Predicted termination (finite number of iterations) *[Evans & Chikohora, 1998]*
- Local convergence monitoring *[Bahi et al, 2005, 2008]*
- Nested-sets-based supervised termination *[Miellou et al, 2008]*

4 - Asynchronous convergence detection

Problem

$$f(x) = x, \quad x \in E$$

Asynchronous convergence detection
Approach

- Finite time termination *[Bertsekas & Tsitsiklis, 1989], [El Baz, 1996], [Savari & Bertsekas, 1996], ...*
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- Nested-sets-based supervised termination *[Miellou et al, 2008]*

	Intrusiveness	Centralization	Effectiveness	Messages size
Bertsekas & Tsitsiklis, 1989	altered iterations	–	formal	–
El Baz, 1996	altered iterations	–	formal	–
Savari & Bertsekas, 1996	altered iterations	–	formal	–
Savari & Bertsekas, 1996	non-intrusive	two reductions	exact	$\mathcal{O}(n)$
Evans & Chikohora, 1998	non-intrusive	no reduction	heuristic	0
Bahi *et al*, 2005	non-intrusive	one reduction	heuristic	$\mathcal{O}(1)$
Bahi *et al*, 2008	piggybacking	two reductions	heuristic	$\mathcal{O}(1)$
Miellou *et al*, 2008	–	–	formal	–

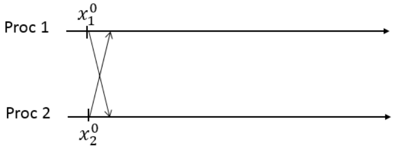
4 - Asynchronous convergence detection

Problem

$$f(x) = x, \quad x \in E$$

Asynchronous convergence detection
Snapshot-based supervised termination

Distributed snapshot *[Chandy & Lamport, 1985]*



- Initiator / First *marker* reception
  1. Record local state
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  3. Start recording neighbors' messages
- On *marker* reception
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4 - Asynchronous convergence detection

Problem

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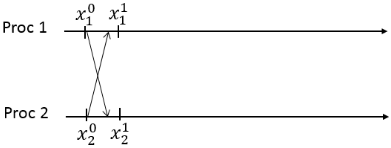
Asynchronous convergence detection
Snapshot-based supervised termination

Distributed snapshot *[Chandy & Lamport, 1985]*



$x_1^1 := f_1(x_1^0, x_2^0)$ $x_2^1 := f_2(x_1^0, x_2^0)$

- Initiator / First *marker* reception
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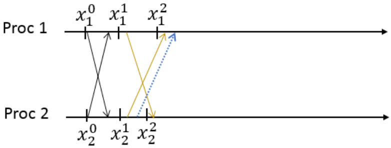

$x_1^1 := f_1(x_1^0, x_2^0)$ x_2^1 $x_2^1 := f_2(x_1^0, x_2^0)$

$x_1^2 := f_1(x_1^1, x_2^0)$ $x_2^2 := f_2(x_1^0, x_2^1)$

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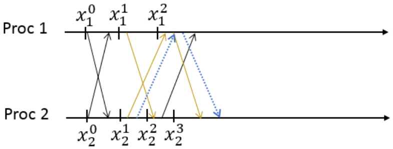

$x_1^1 := f_1(x_1^0, x_2^0)$

$x_1^2 := f_1(x_1^1, x_2^0)$

x_1^2 $x_1^3 := x_1^2$

x_2^1 $x_2^1 := f_2(x_1^0, x_2^0)$

$x_2^2 := f_2(x_1^0, x_2^1)$

$x_2^3 := f_2(x_1^1, x_2^2)$ x_1^1

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Asynchronous convergence detection
Snapshot-based supervised termination

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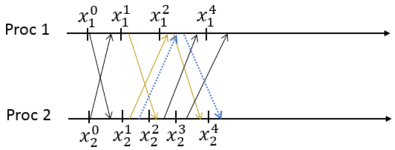

$$\begin{array}{lllll} & x_1^1 := f_1(x_1^0, x_2^0) & x_2^1 & x_2^1 := f_2(x_1^0, x_2^0) & \\ & x_1^2 := f_1(x_1^1, x_2^0) & & x_2^2 := f_2(x_1^0, x_2^1) & \\ x_1^2 & x_1^3 := x_1^2 & & x_2^3 := f_2(x_1^1, x_2^2) & x_1^1 \\ & x_1^4 := f_1(x_1^3, x_2^2) & & x_2^4 := f_2(x_1^2, x_2^3) & x_1^2 \end{array}$$

- Initiator / First *marker* reception
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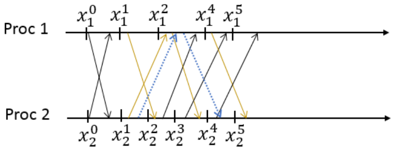
4 - Asynchronous convergence detection

Problem

$$f(x) = x, \quad x \in E$$

Asynchronous convergence detection
Snapshot-based supervised termination

Distributed snapshot *[Chandy & Lamport, 1985]*


x_1^0 x_1^1 x_1^2 x_1^4 x_1^5
Proc 1
Proc 2
x_2^0 x_2^1 x_2^2 x_2^3 x_2^4 x_2^5

$$
\begin{array}{llll}
 & x_1^1 := f_1(x_1^0, x_2^0) & x_2^1 & x_2^1 := f_2(x_1^0, x_2^0) \\
 & x_1^2 := f_1(x_1^1, x_2^0) & & x_2^2 := f_2(x_1^0, x_2^1) \\
x_1^2 & x_1^3 := x_1^2 & & x_2^3 := f_2(x_1^1, x_2^2) \quad x_1^1 \\
 & x_1^4 := f_1(x_1^3, x_2^2) & & x_2^4 := f_2(x_1^2, x_2^3) \quad x_1^2 \\
 & x_1^5 := f_1(x_1^4, x_2^3) & & x_2^5 := f_2(x_1^2, x_2^4)
\end{array}
$$

- Initiator / First *marker* reception
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4 - Asynchronous convergence detection

Problem

$$f(x) = x, \quad x \in E$$

Asynchronous convergence detection
Snapshot-based supervised termination

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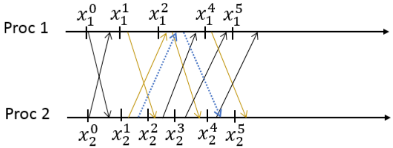
x_1^0 x_1^1 x_1^2 x_1^4 x_1^5
Proc 1
Proc 2
x_2^0 x_2^1 x_2^2 x_2^3 x_2^4 x_2^5

	$x_1^1 := f_1(x_1^0, x_2^0)$	x_2^1	$x_2^1 := f_2(x_1^0, x_2^0)$	
	$x_1^2 := f_1(x_1^1, x_2^0)$		$x_2^2 := f_2(x_1^0, x_2^1)$	
x_1^2	$x_1^3 := x_1^2$		$x_2^3 := f_2(x_1^1, x_2^2)$	x_1^1
	$x_1^4 := f_1(x_1^3, x_2^2)$		$x_2^4 := f_2(x_1^2, x_2^3)$	x_1^2
	$x_1^5 := f_1(x_1^4, x_2^3)$		$x_2^5 := f_2(x_1^2, x_2^4)$	
	$(x_1^2, ?)$		(x_1^2, x_2^1)	

- Initiator / First *marker* reception
  1. Record local state
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4 - Asynchronous convergence detection

Problem

$$f(x) = x, \quad x \in E$$

Asynchronous convergence detection
Snapshot-based supervised termination

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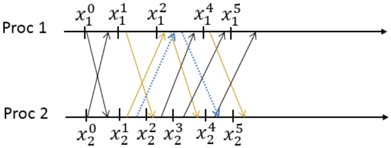
x_1^0 x_1^1 x_1^2 x_1^4 x_1^5
Proc 1
Proc 2
x_2^0 x_2^1 x_2^2 x_2^3 x_2^4 x_2^5

$x_1^1 := f_1(x_1^0, x_2^0)$

$x_1^2 := f_1(x_1^1, x_2^0)$

x_1^2 $x_1^3 := x_1^2$

$x_1^4 := f_1(x_1^3, x_2^2)$

$x_1^5 := f_1(x_1^4, x_2^3)$

$(x_1^2, ?)$

x_2^1 $x_2^1 := f_2(x_1^0, x_2^0)$

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$x_2^5 := f_2(x_1^2, x_2^4)$

(x_1^2, x_2^1)

Proposition 1
Asynchronous iterations snapshot (AIS)

- Initiator / First *marker* reception
  1. Record local state
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  1. Record message

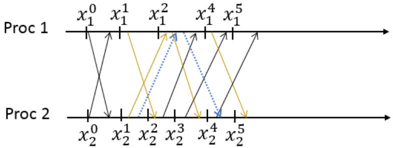
4 - Asynchronous convergence detection

Problem

$$f(x) = x, \quad x \in E$$

Asynchronous convergence detection
Snapshot-based supervised termination

Distributed snapshot *[Chandy & Lamport, 1985]*


x_1^0 x_1^1 x_1^2 x_1^4 x_1^5
Proc 1
Proc 2
x_2^0 x_2^1 x_2^2 x_2^3 x_2^4 x_2^5

$$\begin{array}{lll}
 & x_1^1 & := f_1(x_1^0, x_2^0) \\
 & x_1^2 & := f_1(x_1^1, x_2^0) \\
x_1^2 & x_1^3 & := x_1^2 \\
 & x_1^4 & := f_1(x_1^3, x_2^2) \\
 & x_1^5 & := f_1(x_1^4, x_2^3) \\
 & (x_1^2, ?) &
\end{array}$$

$$\begin{array}{llll}
x_2^1 & x_2^1 & := f_2(x_1^0, x_2^0) & \\
 & x_2^2 & := f_2(x_1^0, x_2^1) & \\
 & x_2^3 & := f_2(x_1^1, x_2^2) & x_1^1 \\
 & x_2^4 & := f_2(x_1^2, x_2^3) & x_1^2 \\
 & x_2^5 & := f_2(x_1^2, x_2^4) & \\
 & (x_1^2, x_2^1) & &
\end{array}$$

Proposition 1
Asynchronous iterations snapshot (AIS)

- Local convergence / First *marker* reception
  1. Record local state
  2. Send *marker* to neighbors
- On *marker* reception
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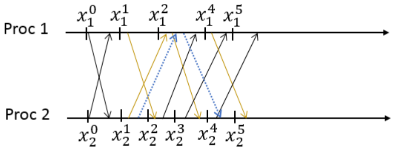
4 - Asynchronous convergence detection

Problem

$$f(x) = x, \quad x \in E$$

Asynchronous convergence detection
Snapshot-based supervised termination

Distributed snapshot *[Chandy & Lamport, 1985]*



$$\begin{array}{clccll}
 & x_1^1 := f_1(x_1^0, x_2^0) & & x_2^1 & x_2^1 := f_2(x_1^0, x_2^0) & \\
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x_1^2 & x_1^3 := x_1^2 & x_2^1 & & x_2^3 := f_2(x_1^1, x_2^2) & \\
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 & (x_1^2, ?) & & & (x_2^1, x_2^1) &
\end{array}$$

Proposition 1
Asynchronous iterations snapshot (AIS)

- Local convergence / First *marker* reception
  1. Record local state
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- On *marker* reception
  1. Record last corresponding neighbor's message
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  1. Record message

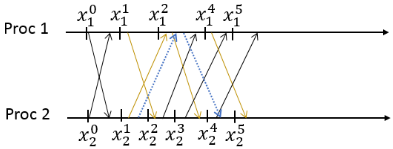
4 - Asynchronous convergence detection

Problem

$$f(x) = x, \quad x \in E$$

Asynchronous convergence detection
Snapshot-based supervised termination

Distributed snapshot *[Chandy & Lamport, 1985]*


x_1^0 x_1^1 x_1^2 x_1^4 x_1^5
Proc 1
Proc 2
x_2^0 x_2^1 x_2^2 x_2^3 x_2^4 x_2^5

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x_1^2 & x_1^3 := x_1^2 & x_2^1 & x_2^3 := f_2(x_1^1, x_2^2) & \\
 & x_1^4 := f_1(x_1^3, x_2^2) & & x_2^4 := f_2(x_1^2, x_2^3) & x_1^2 \\
 & x_1^5 := f_1(x_1^4, x_2^3) & & x_2^5 := f_2(x_1^2, x_2^4) & \\
 & (x_1^2, x_2^1) & & (x_1^2, x_2^1) &
\end{array}$$

Proposition 1
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- Local convergence / First *marker* reception
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Asynchronous convergence detection
Snapshot-based supervised termination

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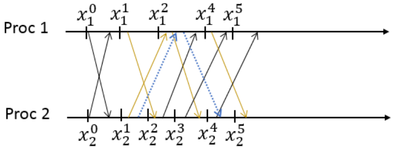
x_1^0 x_1^1 x_1^2 x_1^4 x_1^5
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x_1^2 & x_1^3 := x_1^2 & x_2^1 \\
 & x_1^4 := f_1(x_1^3, x_2^2) & \\
 & x_1^5 := f_1(x_1^4, x_2^3) & \\
 & (x_1^2, ?) &
\end{array}
\qquad
\begin{array}{lll}
x_2^1 & x_2^1 := f_2(x_1^0, x_2^0) & \\
 & x_2^2 := f_2(x_1^0, x_2^1) & \\
 & x_2^3 := f_2(x_1^1, x_2^2) & \\
 & x_2^4 := f_2(x_1^2, x_2^3) & x_1^2 \\
 & x_2^5 := f_2(x_1^2, x_2^4) & \\
 & (?, x_2^1) &
\end{array}$$

Proposition 1
Asynchronous iterations snapshot (AIS)

- Local convergence / First *marker* reception
  1. Record local state
  2. Send *marker* to neighbors
- On *marker* reception
  1. Record last corresponding neighbor's message

Non First-In-First-Out delivering
$\Rightarrow$ marker crossing computation message

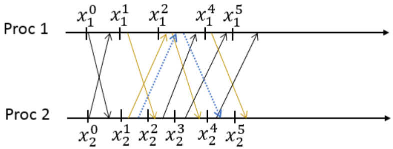
4 - Asynchronous convergence detection

Problem

$$f(x) = x, \quad x \in E$$

Asynchronous convergence detection
Snapshot-based supervised termination

Distributed snapshot *[Chandy & Lamport, 1985]*


Proc 1
x_1^0 x_1^1 x_1^2 x_1^4 x_1^5
Proc 2
x_2^0 x_2^1 x_2^2 x_2^3 x_2^4 x_2^5

$$\begin{array}{llllll}
 & x_1^1 := f_1(x_1^0, x_2^0) & & x_2^1 & x_2^1 := f_2(x_1^0, x_2^0) & \\
 & x_1^2 := f_1(x_1^1, x_2^0) & & & x_2^2 := f_2(x_1^0, x_2^1) & \\
x_1^2 & x_1^3 := x_1^2 & x_2^1 & & x_2^3 := f_2(x_1^1, x_2^2) & \\
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 & x_1^5 := f_1(x_1^4, x_2^3) & & & x_2^5 := f_2(x_1^2, x_2^4) & \\
 & (x_1^2, x_2^1) & & & (x_1^2, x_2^1) &
\end{array}$$

Proposition 1
Asynchronous iterations snapshot (AIS)

Non FIFO case

- *[Savari & Bertsekas, 1996]*
  Embed computation message into marker
  ⇒ marker size: $\mathcal{O}(n)$

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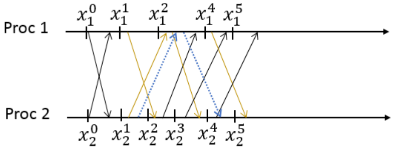
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Non FIFO Asynchronous iterations snapshot (NFAIS)

Non FIFO characterization
A marker can cross at most *m* computation messages

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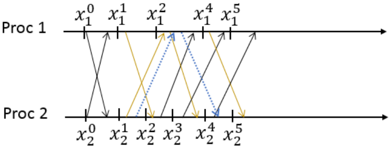
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Proc 1
Proc 2
x_2^0 x_2^1 x_2^2 x_2^3 x_2^4 x_2^5

$$\begin{aligned}
& x_1^1 := f_1(x_1^0, x_2^0) \\
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x_1^2 \quad & x_1^3 := x_1^2 \qquad x_2^1 \\
& x_1^4 := f_1(x_1^3, x_2^2) \\
& x_1^5 := f_1(x_1^4, x_2^3) \\
& (x_1^2, x_2^1)
\end{aligned}$$

$$\begin{aligned}
x_2^1 \quad & x_2^1 := f_2(x_1^0, x_2^0) \\
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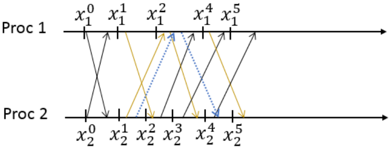

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x_2^0 x_2^1 x_2^2 x_2^3 x_2^4 x_2^5

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$$x_1^4 := f_1(x_1^3, x_2^2) \qquad x_2^4 := f_2(x_1^2, x_2^3) \quad x_1^2$$

$$x_1^5 := f_1(x_1^4, x_2^3) \qquad x_2^5 := f_2(x_1^2, x_2^4)$$

$$(x_1^2, x_2^1) \qquad (x_1^2, x_2^1)$$

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Practical case
Marker transmits faster than computation message
⇒ no need for flag-marker

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Problem

$$f(x) = x, \quad x \in E$$

Asynchronous convergence detection
Snapshot-based supervised termination

Theorem

$$\bar{x} := \begin{bmatrix} \bar{x}_1^{(1)} & \cdots & \bar{x}_p^{(p)} \end{bmatrix}^{\mathsf{T}}$$

Global residual bound

$$r(\bar{x}) < \sigma(r(\bar{x}^{(1)}), \ldots, r(\bar{x}^{(p)})) + Cm\varepsilon$$

constant C depending on r, σ and p
ε used for local convergence

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Corollary
$\| \|_\infty^w$–based residual

$$\sigma(r(\bar{x}^{(1)}), \dots, r(\bar{x}^{(p)})) < \varepsilon \implies r(\bar{x}) < \widetilde{\varepsilon}$$
$$\varepsilon = \widetilde{C}(m, w)\widetilde{\varepsilon}$$

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Asynchronous convergence detection
Approach

- Finite time termination *[Bertsekas & Tsitsiklis, 1989], [El Baz, 1996], [Savari & Bertsekas, 1996], ...*
- Snapshot-based supervised termination *[Savari & Bertsekas, 1996]*, NFAIS
- Predicted termination (finite number of iterations) *[Evans & Chikohora, 1998]*
- Local convergence monitoring *[Bahi et al, 2005, 2008]*
- Nested-sets-based supervised termination *[Miellou et al, 2008]*

	Intrusiveness	Centralization	Effectiveness	Messages size
Bertsekas & Tsitsiklis, 1989	altered iterations	–	formal	–
El Baz, 1996	altered iterations	–	formal	–
Savari & Bertsekas, 1996	altered iterations	–	formal	–
Savari & Bertsekas, 1996	non-intrusive	two reductions	exact	$\mathcal{O}(n)$
Evans & Chikohora, 1998	non-intrusive	no reduction	heuristic	0
Bahi *et al*, 2005	non-intrusive	one reduction	heuristic	$\mathcal{O}(1)$
Bahi *et al*, 2008	piggybacking	two reductions	heuristic	$\mathcal{O}(1)$
Miellou *et al*, 2008	–	–	formal	–
NFAIS	non-intrusive	one reduction	exact	$\mathcal{O}(1)$

4 - Asynchronous convergence detection

NFAIS: $m = 1$

NBS: Non-blocking synchronization (snapshot without local convergence condition)

	Synchronous			NBS		
p	$r \times 10^7$	wt	k	$r \times 10^7$	wt	k
168	8.33	701	281916	5.03	**536**	346226
240	8.31	516	284118	6.18	**378**	366231
360	8.33	382	287557	5.72	**250**	355394
480	8.32	302	289933	5.40	**202**	406611
600	8.32	278	292163	5.23	**168**	432390

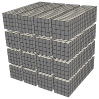
	Savari & Bertsekas, 1996			NFAIS		
p	$r \times 10^7$	wt	k	$r \times 10^7$	wt	k
168	6.55	641	319703	6.54	640	319349
240	6.52	462	342476	6.42	463	343295
360	6.71	310	335008	5.19	314	339204
480	6.43	249	380524	6.63	250	383745
600	6.55	207	404544	6.06	209	410621

Convection-diffusion

$$\frac{\partial u}{\partial t} - \nu \Delta u + a.\nabla u = s$$

Problem size

$$n = 185^3 = 6,331,625$$



Block-Jacobi splitting
+
Gauss-Seidel on blocks

Residual threshold

$$\varepsilon = 10^{-6}$$

p = number of processors
r = global residual after termination

OUTLINE

05 CONCLUSION

5 - Conclusion

- Asynchronous iterations theory
  → Contracting mappings
  → Matrix splittings, maximum norm and spectral radius

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  → Primal approach
  → Two diagonal-free splittings of the interface Schur complement
  → Asynchronous convergence for vector and substructures-based decomposition
  → Journal article in progress

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- ⇒ JACK2: MPI-based communication library
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Future work

- Substructures-based convergence of γI splitting
- Complete interface tearing
- Dual approach
- Coarse space correction

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Next goal
Habilitation à Diriger des Recherches

Thèse de Doctorat

Asynchronous domain decomposition methods for massively parallel computing

Préparée et soutenue par

Guillaume Gbikpi-Benissan

Devant le Jury composé de

Raphaël Couturier	(Rapporteur)
Fabienne Jezequel	(Examinatrice)
Pierre Spiteri	(Rapporteur)
Stéphane Vialle	(Examinateur)
Frédéric Magoulès	(Directeur de thèse)

18 décembre 2017