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New Hermitian and skew-Hermitian splitting methods for non-Hermitian positive-definite linear systems

Hossein Noormohammadi Pour · Hossein Sadeghi Goughery

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Abstract An efficient Hermitian and skew-Hermitian splitting method is presented for solving non-Hermitian and normal positive definite linear systems with strong Hermitian parts. We theoretically prove that this method converges to the unique solution of the system of linear equations. Inexact version of the method which employs conjugate gradient as its inner process is presented. Moreover, we derive an upper bound of the contraction factor of the method. Numerical examples from the finite-difference discretization of a second-order partial differential equation are used to further examine the effectiveness and robustness of both exact and inexact iterations.

Keywords Non-Hermitian positive-definite matrix · Hermitian and skew-Hermitian splitting · Iterative methods

Mathematics Subject Classifications (2010) 65F10 · 65F15

1 Introduction

Many problems in scientific computing need to solve a system of linear equations

$$Ax = b \tag{1.1}$$

with $A \in \mathbb{C}^{n\times n}$ a large sparse non-Hermitian positive-definite matrix and $x, b \in \mathbb{C}^n$. Iterative methods for the system of form (1.1) need efficient splitting of the coefficient matrix A. For example the classic Jacobi and Gauss-Seidel methods

H. Noormohammadi Pour (✉) · H. Sadeghi Goughery
Department of Mathematics, College of Science, Kerman Branch, Islamic Azad University, Kerman, Iran
e-mail: h_noormohammadi@yahoo.com

H. Sadeghi Goughery
e-mail: hsadeghi31@yahoo.com

split A into its diagonal and off-diagonal parts, see [8, 13, 16]. Based on the Hermitian/skew-Hermitian splitting

$$A = H + S$$

of the coefficient matrix A, where

$$H = \frac{1}{2}(A + A^*) \quad and \quad S = \frac{1}{2}(A - A^*)$$

Bai, Golub and Ng in [3] presented the Hermitian/skew-Hermitian splitting (HSS) iteration method as follows:

The HSS iteration method. Given an initial guess $x^{(0)}$ for $k = 0, 1, \ldots$ until $\{x^{(k)}\}$ converges, compute

$$(\alpha I + H)x^{(k+1/2)} = (\alpha I - S)x^{(k)} + b,$$

$$(\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+1/2)} + b,$$

where α is a given positive constant.

They have also proved this method converges unconditionally to the exact solution of the system of linear equation (1.1).

In [7], Benzi and Golub directly applied the HSS iteration technique to the following system of linear equations with a block coefficient matrix

$$A\begin{bmatrix} y \\ z \end{bmatrix} \equiv \begin{bmatrix} B & E \\ -E^* & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv b \tag{1.2}$$

and proved that the resulting iteration method is also convergent for any positive constant α, where α is the involved iteration parameter. This result extends the application region of the original HSS method. Here, the submatrices $B \in \mathbb{C}^{p\times p}$ is Hermitian positive definite and $E \in \mathbb{C}^{p\times q}(p \geq q)$ is of full column rank. Therefore, $A \in \mathbb{C}^{n\times n}$, with $n = p + q$, is a nonsingular, non-Hermitian, and positive semidefinite matrix. In [6], Bai, Golub and Pan by first transforming (or preconditioning) the block system of linear equations (1.2) into an equivalent one, and then applying the HSS method directly to the preconditioned block linear system, they established a class of preconditioned Hermitian/skew-Hermitian splitting (PHSS) iteration methods for the non-Hermitian positive semidefinite system of linear equations (1.2). The PHSS method converges unconditionally to the exact solution of the block system of linear equations (1.2), with a comparable workload and faster convergence speed than the one studied in [7]. In particular, they derived upper bound of its contraction factor and presented optimal choice of the involved iteration parameter. To further improve the efficiency of the HSS method, it is desirable to find a good estimate for the optimal parameter α^*. In [2], Bai, Golub and Li analyzed two-by-two real matrices in detail and obtained the optimal parameter α^* that minimizes the spectral radius of the iteration matrix of the corresponding HSS method. Also a practical formula for computing optimal parameters in the HSS iteration methods was recently given by Huang, see [9].

In [4], Bai, Golub and Ng introduced another approach to accelerate the HSS iteration. They generalized the HS splitting to the normal/skew-Hermitian (NS) splitting

$A = N + S_0$ where $N \in \mathbb{C}^{n\times n}$ is a normal matrix and $S_0 \in \mathbb{C}^{n\times n}$ is a skew-Hermitian matrix, and established a generalized version of the HSS iteration, called as the normal/skew-Hermitian splitting (NSS) iteration method. In [11], Li, Huang and Liu considered asymmetric Hermitian/skew-Hermitian (AHSS) iteration. They theoretically studied the convergence properties of the AHSS method, and derived the contraction factor of the AHSS iteration method. Furthermore Li, Huang and Liu proposed the $LHSS$ iteration (Lopsided Hermitian/skew Hermitian splitting iteration) as follows:

The $LHSS$ iteration method. Given an initial guess $x^{(0)}$, for $k = 0, 1, \ldots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{aligned} Hx^{(k+1/2)} &= -Sx^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} &= (\alpha I - H)x^{(k+1/2)} + b, \end{aligned}$$

where α is a given non-zero constant.

The authors have also proved the convergence properties of the method, for more details see [10, 15]. Recently Li, Yang and Wu based on the preconditioned modified Hermitian and skew-Hermitian splitting (PMHSS) iteration method, proposed a lopsided PMHSS (LPMHSS) iteration method for solving a broad class of complex symmetric linear systems [12]. In this paper we present a different approach to solve equation (1.1) and describe it as follows:

The new iteration method. Given an initial guess $x^{(0)}$, for $k = 0, 1, \ldots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{aligned} Hx^{(k+1/2)} &= -Sx^{(k)} + b, \\ (\alpha I + H)x^{(k+1)} &= (\alpha I - S)x^{(k+1/2)} + b, \end{aligned}$$

where α is a given positive constant.

When the Hermitian part H of the matrix A is dominantly stronger than the skew-Hermitian part S (i.e., $\| H \| \gg \| S \|$ for some matrix norm) since both H and $\alpha I + H$ are Hermitian positive definite, this is a good property for fast convergence. Theoretical analysis shows that if the coefficient matrix A is positive definite and normal the new iteration converges to the unique solution of the linear system (1.1) for any $\alpha > 0$. In this method the upper bound of the contraction factor is dependent on the choice of α, the minimum eigenvalue of the Hermitian part H and the maximum singular value of the skew/Hermitian part S, but is neither dependent on the rest singular values of S nor on the eigenvectors of the matrices H, S and A. Similar to the HSS and $LHSS$ methods the two-half steps at each new iterate require exact solution with the matrices H and $\alpha I + H$. However this is costly to be practical in actual application. To overcome this disadvantage, Bai et al. and Li et al. considered inexact HSS and $LHSS$ iteration methods. They solved the systems of linear equations with coefficient matrices $\alpha I + H$ and H by conjugate gradient and the system of linear equations with coefficient $\alpha I + S$ by krylov subspace method like GMRES [14]. Because in our method, the matrices H and $\alpha I + H$ are both Hermitian positive definite, systems of linear equations with these coefficient matrices can be solved only by conjugate gradient method.

The organization of this paper is as follows. In Section 2, we analyse the convergence properties of the new iteration for non-Hermitian and normal positive-definite linear systems. In Section 3 we establish the inexact new iteration and study its convergence property. Numerical examples are presented in Section 4 to illustrate the effectiveness of our method. Finally, some conclusions are drawn in the last section.

2 Convergence analysis

In this section we study the convergence rate of the new method and derive the upper bound of the contraction factor.This new method can be generalized to the two-step splitting iteration framework, and the following lemma describes a general convergence criterion for a two-step splitting iteration established in [3].

Lemma 2.1 *Let $A \in \mathbb{C}^{n\times n}$, $A = M_i - N_i (i = 1, 2)$ be two splitting of matrix A, and $x^{(0)} \in \mathbb{C}^n$ be a given initial vector. If $\{x^{(k)}\}$ is a two-step iteration sequence defined by*

$$M_1 x^{(k+1/2)} = N_1 x^{(k)} + b$$

$$M_2 x^{(k+1)} = N_2 x^{(k+1/2)} + b$$

then

$$x^{(k+1)} = M_2^{-1} N_2 M_1^{-1} N_1 x^{(k)} + M_2^{-1} \left(I + N_2 M_1^{-1} \right) b, \quad k = 0, 1, \ldots$$

Moreover, if the spectral radius $\rho\left(M_2^{-1} N_2 M_1^{-1} N_1\right) < 1$ then the iterative sequence$\{x^{(k)}\}$ converges to the unique solution $x^* \in \mathbb{C}^n$ of system (1.1) for all initial vector $x^{(0)} \in \mathbb{C}^n$.

Applying this lemma to the new iteration, we obtain the following theorem.

Theorem 2.1 *Let $A \in \mathbb{C}^{n\times n}$ be a positive-definite and normal matrix. $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew Hermitian parts, α be a positive constant. Then the iteration matrix $M(\alpha)$ of the new method is*

$$M(\alpha) = (\alpha I + H)^{-1}(\alpha I - S)H^{-1}(-S) \tag{2.1}$$

and its spectral radius $\rho(M(\alpha))$ is bonded by

$$\delta(\alpha) = \left(\sigma_{max} \sqrt{\alpha^2 + \sigma_{max}^2} \right) / (\lambda_{min}(\alpha + \lambda_{min}))$$

where σ_{max} is the maximum singular value of the matrix S and λ_{min} is the minimum eigenvalue of the matrix H. Moreover if $\sigma_{max} \leq \lambda_{min}$ then the bound $\delta(\alpha) < 1$ i.e. the iteration converges.

Proof By putting

$$M_1 = H, \quad N_1 = -S, \quad M_2 = \alpha I + H, \quad N_2 = \alpha I - S$$

in Lemma 2.1 and since H and $\alpha I + H$ are nonsingular for any positive constant α, we get (2.1). Moreover we have

$$\begin{aligned}
&\rho\left((\alpha I + H)^{-1}(\alpha I - S)H^{-1}(-S)\right)\\
&\leq \| \ (\alpha I + H)^{-1}(\alpha I - S)H^{-1}(-S) \ \|_2\\
&= \| \ (\alpha I + H)^{-1}(\alpha H^{-1} - SH^{-1})(-S) \ \|_2\\
&= \| \ (\alpha I + H)^{-1}(\alpha H^{-1} - H^{-1}S)(-S) \ \|_2\\
&\leq \| \ (\alpha I - S)(-S) \ \|_2 \| \ (\alpha I + H)^{-1}H^{-1} \ \|_2
\end{aligned}$$

But we know that

$$\| \ (\alpha I - S)(-S) \ \|_2 \| \ (\alpha I + H)^{-1}H^{-1} \ \|_2 \ = \max_{\sigma_i \in \sigma(S)} \sigma_i \sqrt{\alpha^2 + \sigma_i^2} \max_{\lambda_i \in \lambda(H)} \frac{1}{\lambda_i(\alpha + \lambda_i)}$$

where $\sigma(S)$ is the singular-value set of matrix S and $\lambda(H)$ is the spectrum of matrix H. obviously

$$\max_{\sigma_i \in \sigma(S)} \sigma_i \sqrt{\alpha^2 + \sigma_i^2} = \sigma_{max} \sqrt{\alpha^2 + \sigma_{max}^2}$$

and

$$\max_{\lambda_i \in \lambda(H)} \frac{1}{\lambda_i(\alpha + \lambda_i)} = \frac{1}{\lambda_{min}(\alpha + \lambda_{min})}$$

so we have the above bound for $\rho(M(\alpha))$, and if $\sigma_{max} \leq \lambda_{min}$ then $\rho(M(\alpha)) \leq \delta(\alpha) < 1$. □

Theorem 2.1 shows that the convergence speed of the new iteration is bounded by $\delta(\alpha)$ which depend on the maximum singular value of the matrix S and minimum eigenvalue of the matrix H. We emphasize that normality is a sufficient criterion and may be omitted in practical problems.

Now we want to find α such that the bound $\delta(\alpha)$ be minimum, that is we want to give the optimal α.

Corollary 2.1 *Let A, H, S, λ_{min} and σ_{max} be defined like Theorem 2.1. Then the optimal parameter α is $\alpha^* = \dfrac{\sigma_{max}^2}{\lambda_{min}}$ and the bound for $\rho(M(\alpha))$ is*

$$\delta(\alpha^*) = \frac{\sigma_{max}^2 \sqrt{\sigma_{max}^2 + \lambda_{min}^2}}{\lambda_{min}\left(\sigma_{max}^2 + \lambda_{min}^2\right)}$$

Proof By differentiating

$$\delta(\alpha) = \left(\sigma_{max} \sqrt{\alpha^2 + \sigma_{max}^2}\right) / (\lambda_{min}(\alpha + \lambda_{min}))$$

and solving the equation $\delta^{'}(\alpha) = 0$ we obtain $\alpha^* = \dfrac{\sigma_{max}^2}{\lambda_{min}}$ and by substituting in $\delta(\alpha)$ we get the result. □

We emphasize that the optimal parameter α^* in Corollary 2.1 minimize only the upper bound $\delta(\alpha)$ of the spectral radius of the iteration matrix but dose not minimize the spectral radius of the iteration matrix.

If we want to compare this method and HSS method the following Lemma from [3] is necessary.

Lemma 2.2 *Let A, H and S be defined as those in Theorem 2.1 and α be a positive constant. Then the spectral radius $\rho(M(\alpha))$ of the iteration matrix of the HSS method is bounded by*

$$\gamma(\alpha) = \max_{\lambda_i \in \lambda(H)} \left| \frac{\alpha - \lambda_i}{\alpha + \lambda_i} \right|$$

where $\lambda(H)$ is the spectrum of the matrix H. Obviously it hold that

$$\rho(M(\alpha)) \leq \gamma(\alpha) < 1 \quad \forall \alpha > 0$$

Moreover if λ_{min} and λ_{max} are the minimum and maximum eigenvalues of the matrix H then

$$\bar{\alpha} = arg \min_{\alpha} \left\{ \max_{\lambda_{min} \leq \lambda \leq \lambda_{max}} \left| \frac{\alpha - \lambda}{\alpha + \lambda} \right| \right\} = \sqrt{\lambda_{min} \lambda_{max}}$$

and

$$\gamma(\bar{\alpha}) = \frac{\sqrt{\lambda_{max}} - \sqrt{\lambda_{min}}}{\sqrt{\lambda_{max}} + \sqrt{\lambda_{min}}}$$

With this Lemma and Corollary 2.1 we will be able to compare these methods.

Theorem 2.2 *Let $\lambda_{min}, \lambda_{max}, \sigma_{max}, \alpha^*$ and $\delta(\alpha^*)$ be defined as those in Corollary 2.1 and $\bar{\alpha}, \gamma(\bar{\alpha})$ those in Lemma 2.2. If*

$$\sigma_{max} \leq \sqrt{2} \lambda_{min} \left(\frac{\sqrt{\lambda_{max}} - \sqrt{\lambda_{min}}}{\sqrt{\lambda_{max}} + \sqrt{\lambda_{min}}} \right)$$

then $\delta(\alpha^*) \leq \gamma(\bar{\alpha})$.

Proof Suppose $\sigma_{max} \leq \sqrt{2} \lambda_{min} \left(\dfrac{\sqrt{\lambda_{max}} - \sqrt{\lambda_{min}}}{\sqrt{\lambda_{max}} + \sqrt{\lambda_{min}}} \right)$.

Then by some simple computations we get

$$\begin{aligned}
\frac{\sigma_{max}^2}{\sqrt{2}\sigma_{max}} &\leq \lambda_{min}\left(\frac{\sqrt{\lambda_{max}}-\sqrt{\lambda_{min}}}{\sqrt{\lambda_{max}}+\sqrt{\lambda_{min}}}\right)\\
\Rightarrow \frac{\sigma_{max}^2}{\sqrt{\sigma_{max}^2+\sigma_{max}^2}} &\leq \lambda_{min}\left(\frac{\sqrt{\lambda_{max}}-\sqrt{\lambda_{min}}}{\sqrt{\lambda_{max}}+\sqrt{\lambda_{min}}}\right)\\
\Rightarrow \frac{\sigma_{max}^2}{\sqrt{\lambda_{min}^2+\sigma_{max}^2}} &\leq \lambda_{min}\left(\frac{\sqrt{\lambda_{max}}-\sqrt{\lambda_{min}}}{\sqrt{\lambda_{max}}+\sqrt{\lambda_{min}}}\right)\\
\Rightarrow \frac{\sigma_{max}^2}{\lambda_{min}\sqrt{\lambda_{min}^2+\sigma_{max}^2}} &\leq \frac{\sqrt{\lambda_{max}}-\sqrt{\lambda_{min}}}{\sqrt{\lambda_{max}}+\sqrt{\lambda_{min}}}
\end{aligned}$$

Therefore we have $\delta(\alpha^*) \leq \gamma(\tilde{\alpha})$. □

3 The inexact iteration

To invert the matrices $\alpha I + H$ and $\alpha I + S$ efficiently at each step of the HSS iteration in actual implementations, Bai et al. [3] further proposed to solve the linear systems with coefficient matrices $\alpha I + H$ and $\alpha I + S$ inexactly by iterative methods, e.g., solving the linear systems with coefficient matrix $\alpha I + H$ by the conjugate gradient method (CG) and those with coefficient matrix $\alpha I + S$ by the Lanczos or the conjugate gradient for normal equations (CGNE) method, to some prescribed accuracies, and obtained two special but quite practical inexact Hermitian/skew-Hermitian splitting (IHSS) iterations, briefly called as IHSS(CG, Lanczos) and IHSS(CG, CGNE). In [5], Bai, Golub and Ng studied the convergence properties of both IHSS(CG, Lanczos) and IHSS(CG, CGNE) in-depth and investigated the optimal numbers of inner iteration steps in detail by considering both global convergence speed and overall computation workload. In particular, they showed that the asymptotic convergence rates of IHSS(CG, Lanczos) and IHSS(CG, CGNE) are essentially the same. In addition, the convergence rates of IHSS(CG, Lanczos) and IHSS(CG, CGNE) tend to the convergence rate of HSS when the tolerances of the inner iterations tend to zero as the outer iterations increase. They also investigated their computational efficiencies of IHSS(CG, Lanczos) and IHSS(CG, CGNE). In the process of this new iteration, we must solve two systems of linear equations with coefficient matrices H and $\alpha I + H$. Since both H and $\alpha I + H$ are Hermitian positive definite, we can solve these systems of linear equations only by employing conjugate gradient (CG) method. The following algorithm describes this situation.

The following lemma which presented by Bai, Golub and Ng [3] show the convergence properties of the two-step iteration. In this lemma we consider a vector norm $|||\ x\ |||_{M_2} = \|\ M_2 x\ \|_2$ $(\forall x \in \mathbb{C}^n)$ which immediately induces the matrix norm $|||\ X\ |||_{M_2} = \|\ M_2 X M_2^{-1}\ \|_2$ $(\forall X \in \mathbb{C}^{n\times n})$

Algorithm (Inexact iteration)

$k = 0$
while (not convergent)

$r^{(k)} = b - Ax^{(k)}$
approximately solve $Hz^{(k)} = r^{(k)}$ by conjugate gradient method such that the residual $p^{(k)} = r^{(k)} - Hz^{(k)}$ of the iteration satisfies $\| p^{(k)} \| \leq \varepsilon_k \| r^{(k)} \|$
$x^{(k+1/2)} = x^{(k)} + z^{(k)}$
$r^{(k+1/2)} = b - Ax^{(k+1/2)}$
approximately solve $(\alpha I + H)z^{(k+1/2)} = r^{(k+1/2)}$ by conjugate gradient method such that residual $q^{(k)} = r^{(k+1/2)} - (\alpha I + H)z^{(k+1/2)}$ of the iteration satisfies $\| q^{(k)} \| \leq \eta_k \| r^{(k+1/2)} \|$
$x^{(k+1)} = x^{(k+1/2)} + z^{(k+1/2)}$
$k = k + 1$

end.

Lemma 3.1 *Let $A \in \mathbb{C}^{n\times n}$ and $A = M_i - N_i (i = 1, 2)$ be two splitting of the matrix A. If $\{x^{(k)}\}$ is an iterative sequence defined as follows:*

$$x^{(k+1/2)} = x^{(k)} + z^{(k)}, with \quad M_1 z^{(k)} = r^{(k)} + p^{(k)}$$

satisfying $\| p^{(k)} \| \leq \epsilon_k \| r^{(k)} \|$, *where* $r^{(k)} = b - Ax^{(k)}$; *and*

$$x^{(k+1)} = x^{(k+1/2)} + z^{(k+1/2)}, with \quad M_2 z^{(k+1/2)} = r^{(k+1/2)} + q^{(k+1/2)}$$

satisfying $\| q^{(k+1/2)} \| \leq \eta_k \| r^{(k+1/2)} \|$, *where* $r^{(k+1/2)} = b - Ax^{(k+1/2)}$ *then* $\{x^{(k)}\}$ *is of the form*

$$x^{(k+1)} = M_2^{-1}N_2M_1^{-1}N_1x^{(k)} + M_2^{-1}(I + N_2M_1^{-1})b + M_2^{-1}\left(N_2M_1^{-1}p^{(k)} + q^{(k+1/2)}\right)$$

Moreover if x^* is the exact solution of the system (1.1) then we have

$$||| x^{(k+1)} - x^* |||_{M_2} \leq (\sigma + \mu\theta\epsilon_k + \theta(\rho + \theta\nu\epsilon_k)\eta_k) ||| x^{(k)} - x^* |||_{M_2}, \quad k = 0, 1, \ldots$$

where

$$\sigma = \| N_2M_1^{-1}N_1M_2^{-1} \|, \rho = \| M_2M_1^{-1}N_1M_2^{-1} \|, \mu = \| N_2M_1^{-1} \|,$$

$$\theta = \| AM_2^{-1} \|, \nu = \| M_2M_1^{-1} \| .$$

In particular if

$$\sigma + \mu\theta\epsilon_{max} + \theta(\rho + \theta\nu\epsilon_{max})\eta_{max} < 1$$

then the iteration sequence $\{x^{(k)}\}$ converges to $x^* \in \mathbb{C}^n$ where

$$\epsilon_{max} = \max_k\{\epsilon_k\} \quad and \quad \eta_{max} = \max_k\{\eta_k\}.$$

According to the above Lemma, we have the following convergence Theorem.

Theorem 3.1 *Let $A \in \mathbb{C}^{n\times n}$ be a positive-definite and normal matrix, H and S be its Hermitian and skew-Hermitian parts, and α be a positive constant. If $\{x^{(k)}\}$ is an iterative sequence generated by the inexact algorithm and if $x^* \in \mathbb{C}^n$ is the exact solution of the system (1.1), then we have*

$$|||\, x^{(k+1)} - x^* \,||| \le \left(\delta(\alpha) + \frac{\alpha\theta\epsilon_k}{\lambda_{min}}(1+\theta\eta_k) + \theta(\rho(\epsilon_k+\eta_k) + \theta\epsilon_k\eta_k\kappa(H))\right) \times |||\, x^{(k)} - x^* \,|||$$

where $|||\, x \,|||=|||\, x \,|||_{(\alpha I+H)}$ and

$$\rho = \| (\alpha I + H)H^{-1}(-S)(\alpha I + H)^{-1} \|_2 \quad \text{and} \quad \theta = \| A(\alpha I + H)^{-1} \|_2$$

Particularly, when

$$\left(\delta(\alpha) + \frac{\alpha\theta\epsilon_{max}}{\lambda_{min}}(1+\theta\eta_{max}) + \theta(\rho(\epsilon_{max}+\eta_{max}) + \theta\epsilon_{max}\eta_{max}\kappa(H))\right) < 1$$

then the sequence $\{x^{(k)}\}$ converges to x^*, where

$$\eta_{max} = \max_k\{\eta_k\}, \quad \epsilon_{max} = \max_k\{\epsilon_k\}$$

and $\kappa(H)$ is the spectral condition number of H.

Proof Replacing $M_i, N_i (i = 1, 2)$ in Lemma 3.1 with

$$M_1 = H,\ N_1 = -S,\ M_2 = \alpha I + H,\ N_2 = \alpha I - S$$

we obtain

$$\sigma = \| (\alpha I - S)H^{-1}(-S)(\alpha I + H)^{-1} \|_2,\ \rho = \| H^{-1}(-S) \|_2,\ \mu = \| (\alpha I - S)H^{-1} \|_2,$$

$$\theta = \| A(\alpha I + H)^{-1} \|_2, \quad \nu = \| (\alpha I + H)H^{-1} \|_2,$$

so we have

$$|||\, x^{(k+1)} - x^* \,||| \le \left(\delta(\alpha) + \left(\frac{\alpha}{\lambda_{min}} + \rho\right)\theta\epsilon_k + \theta\left(\rho + \frac{\theta\epsilon_k(\alpha+\lambda_{max})}{\lambda_{min}}\right)\eta_k\right) \times |||\, x^{(k)} - x^* \,|||$$

$$\Rightarrow |||\, x^{(k+1)} - x^* \,||| \le \left(\delta(\alpha) + \frac{\alpha\theta\epsilon_k}{\lambda_{min}} + \rho\theta\epsilon_k + \rho\theta\eta_k + \frac{\theta^2\epsilon_k\eta_k\alpha}{\lambda_{min}} + \theta^2\epsilon_k\eta_k\kappa(H)\right) \times |||\, x^{(k)} - x^* \,|||$$

$$\Rightarrow |||\, x^{(k+1)} - x^* \,||| \le \left(\delta(\alpha) + \frac{\alpha\theta\epsilon_k}{\lambda_{min}}(1+\theta\eta_k) + \theta(\rho(\epsilon_k+\eta_k) + \theta\epsilon_k\eta_k\kappa(H))\right) \times |||\, x^{(k)} - x^* \,|||$$

Therefore the proof is complete. □

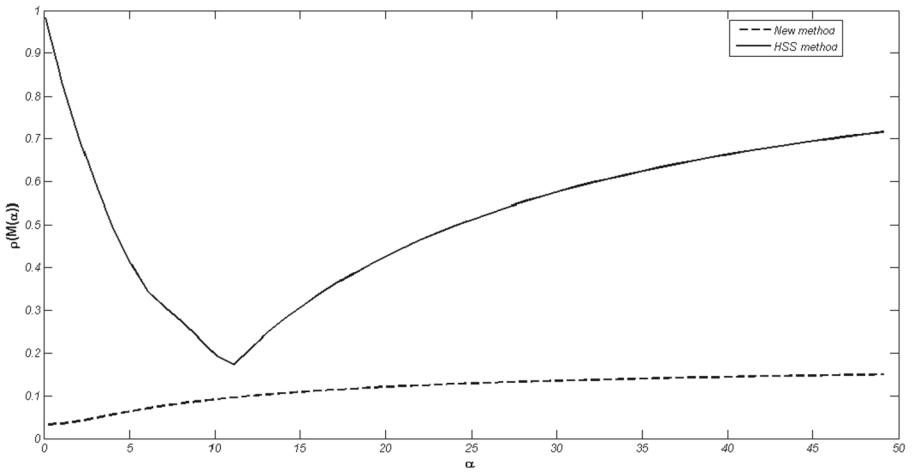

Fig. 1 The spectral radius $\rho(M(\alpha))$ of the iteration matrices of HSS and NEW method for example (a),$N = 64$

4 Numerical examples

To illustrate the effectiveness of both exact and inexact iterations we consider the sparse non-symmetric of linear equation (1.1) with coefficient matrix

$$A = tridiag\left(\eta_k^{(1)} I, T, \mu_k^{(1)} I\right) \in \mathbb{R}^{n\times n} \tag{4.1}$$

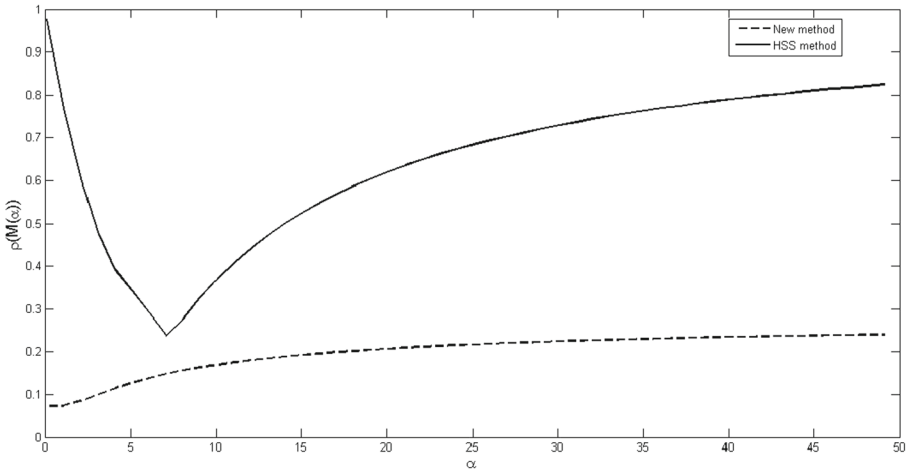

Fig. 2 The spectral radius $\rho(M(\alpha))$ of the iteration matrices of HSS and NEW methods for example (a),$N = 80$

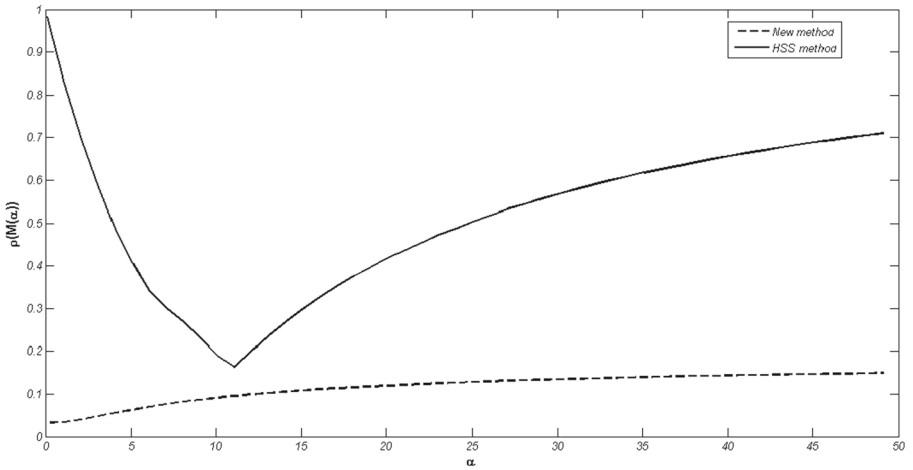

Fig. 3 The spectral radius $\rho(M(\alpha))$ of the iteration matrices of HSS and NEW methods for example (b), $N = 64$

which is an $N \times N$ block tridiagonal matrix. Here

$$T = tridiag\left(\eta_k^{(2)}, \mu^{(0)}, \mu_k^{(2)}\right) \in \mathbb{R}^{N\times N}$$

is an $N \times N$ tridiagonal matrix. The right side is $b = Ab_0$ where $b_0 = (1, 1, ..., 1)^T \in \mathbb{R}^n$. The entries of the matrix $A \in \mathbb{R}^{n\times n}$ are defined by the formulas

$$\begin{aligned} &\mu^{(0)} = 4\left(\alpha + \theta\sigma h^2\right), \quad \mu_i^{(1)} = -\left(\alpha - \tfrac{1}{2}h\beta_i\right), \quad \mu_i^{(2)} = -\left(\alpha - \tfrac{1}{2}h\delta_i\right) \\ &\eta_i^{(1)} = -\left(\alpha + \tfrac{1}{2}h\beta_i\right), \quad \eta_i^{(2)} = -\left(\alpha + \tfrac{1}{2}h\delta_i\right), \quad 1 \leqslant i \leqslant N, \end{aligned} \tag{4.2}$$

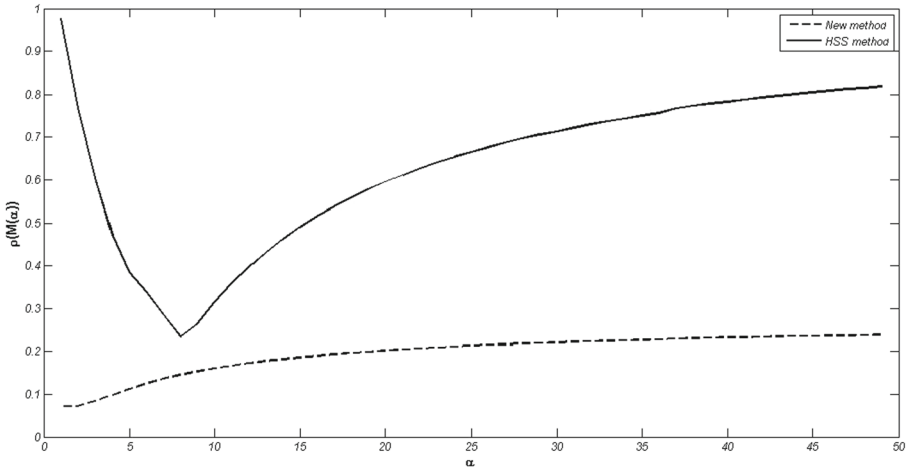

Fig. 4 The spectral radius $\rho(M(\alpha))$ of the iteration matrices of HSS and NEW methods for example (b), $N = 80$

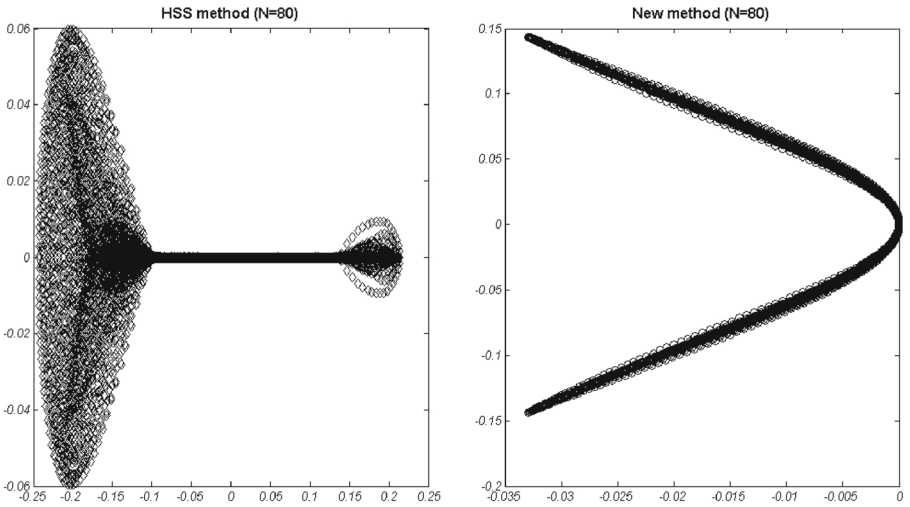

Fig. 5 Eigenvalues distribution of example (**a**), N=80

where $n = N \times N, h = 1/(N+1), \beta_i = \beta(ih), \delta_j = \delta(jh)$are the values of the functions $\beta(t_1), \delta(t_2)$ at the points ih and jh respectively. In fact this system of linear equations arises from central difference discretization of a second order partial differential equation (for more details see [1]). We consider two examples (a) and (b). In example (a) we have $\alpha = 1, \beta = s/10, \delta = s^2, \theta = 10^3, \sigma = 64$, where $s \in (0, 1)$ and in example (b) we have $\alpha = 1, \beta = ln(s/100), \delta = e^s/10, \theta = 10^3, \sigma = 64$, where $s \in (0, 1)$. Through these examples the Hermitian part of coefficient matrix is dominant whereas if θ is reduced to 112 by increasing N, The condition $\| H \| \gg \|$

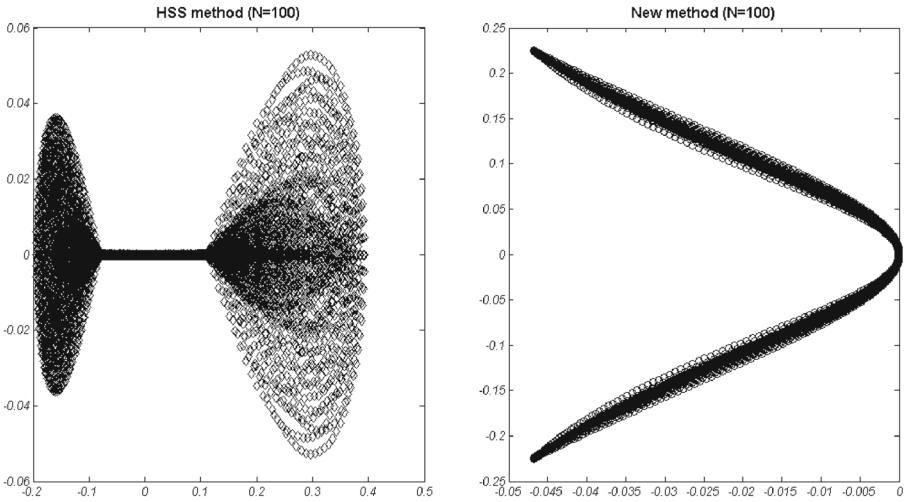

Fig. 6 Eigenvalues distribution of example (**a**), N=100

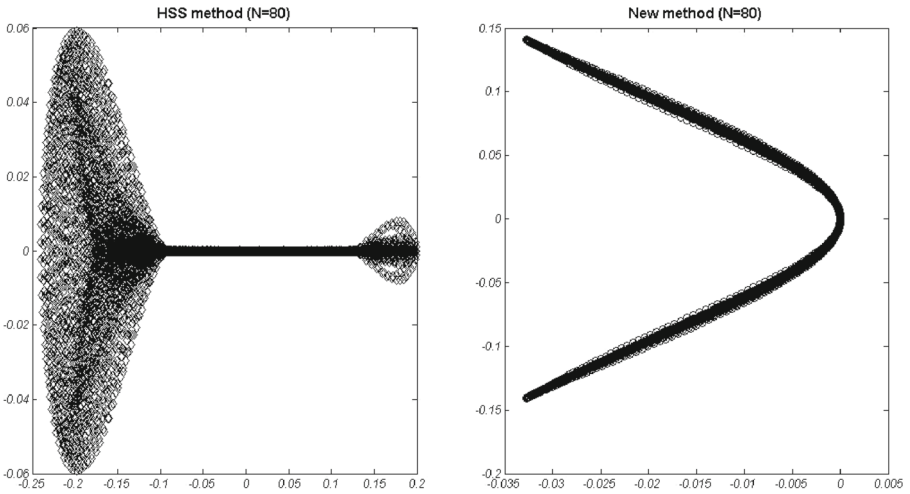

Fig. 7 Eigenvalues distribution of example (**b**), N=80

$S \parallel$ will become weak and the results of both methods will be closed together. For example, Hermitian part is not dominant for $N = 300$, and the results of HSS is better.

4.1 Spectral radius

The spectral radius of the iteration matrix is a good criterion for comparison methods. Figures 1 and 2 show the spectral radius $\rho(M(\alpha))$ of the iteration matrices of both HSS and new methods for example (a) with $N = 64, 80$ also Figs. 3 and 4 show the

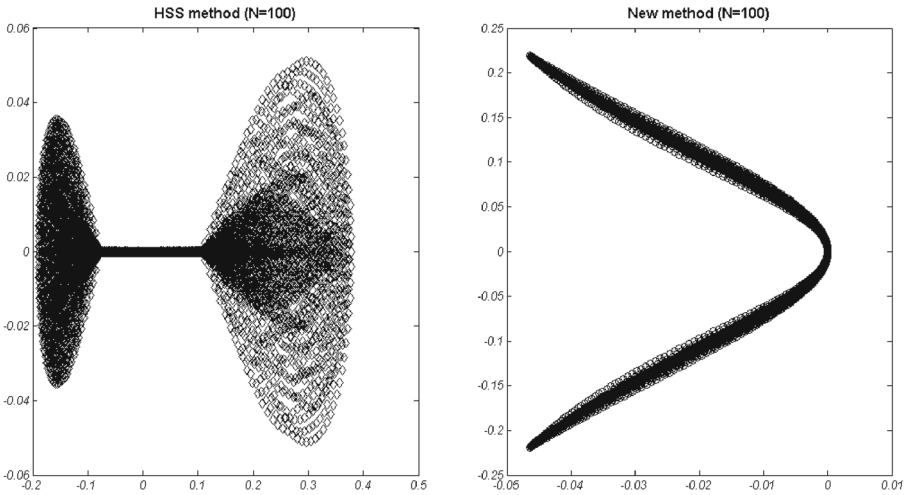

Fig. 8 Eigenvalues distribution of example (**b**), N=100

Table 1 Iteration steps, CPU times and residual of HSS and NEW methods for example (a)

N	Method	IT	CPU	RES	Method	IT	CPU	RES
60	HSS	68	1.749	9.590E–7	NEW	2	0.026	4.687E–7
80	HSS	39	1.727	7.330E–7	IEW	2	0.053	3.723E–7
100	HSS	25	1.915	6.093E–7	NEW	2	0.086	1.850E–7
120	HSS	17	1.864	5.538E–7	NEW	2	0.159	6.799E–7
140	HSS	12	2.024	4.500E–7	NEW	3	0.278	1.238E–7
160	HSS	8	1.723	7.161E–7	NEW	3	0.378	5.124E–7
180	HSS	5	1.456	5.122E–7	NEW	3	0.521	1.765E–7
200	HSS	5	1.786	3.107E–7	NEW	3	0.718	5.259E–7
220	HSS	8	3.584	1.829E–7	NEW	4	1.108	5.891E–7
240	HSS	10	5.887	3.694E–7	NEW	4	1.489	1.879E–7
260	HSS	12	8.736	5.634E–7	NEW	4	1.852	5.372E–7
280	HSS	14	12.665	8.038E–7	NEW	5	2.673	1.890E–7
300	HSS	17	18.538	4.797E–7	NEW	5	3.216	5.579E–7

spectral radius $\rho(M(\alpha))$ of the iteration matrices of both HSS and new methods for example (b) with $N = 64, 80$. From these figures we can see that the spectral radius of the new iteration matrix is smaller than that of HSS method. In Figs. 5, 6, 7 and 8 we depict the distributions of the eigenvalues of the iteration matrices.

Table 2 Iteration steps, CPU times and residual of IHSS and INEW methods for example (a)

N	Method	IT	CPU	RES	Method	IT	CPU	RES
60	IHSS	68	0.998	9.590E–7	INEW	2	0.045	4.687E–7
80	IHSS	39	0.794	7.330E–7	INEW	2	0.014	3.723E–7
100	IHSS	25	0.690	6.093E–7	INEW	2	0.019	1.850E–7
120	IHSS	17	0.603	5.538E–7	INEW	2	0.026	6.799E–7
140	IHSS	12	0.529	4.500E–7	INEW	3	0.052	1.239E–7
160	IHSS	8	0.476	7.161E–7	INEW	3	0.067	5.125E–7
180	IHSS	5	0.350	5.122E–7	INEW	3	0.083	1.765E–7
200	IHSS	5	0.448	3.107E–7	INEW	3	0.097	5.260E–7
220	IHSS	8	0.829	1.829E–7	INEW	4	0.164	5.881E–7
240	IHSS	10	1.293	3.694E–7	INEW	4	0.194	1.879E–7
260	IHSS	12	1.587	5.634E–7	INEW	4	0.232	5.372E–7
280	IHSS	14	2.101	8.038E–7	INEW	5	0.332	1.890E–7
300	IHSS	17	2.938	4.798E–7	INEW	5	0.416	5.579E–7

Table 3 Iteration steps, CPU times and residual of HSS and NEW methods for example (b)

N	Method	IT	CPU	RES	Method	IT	CPU	RES
60	HSS	68	1.687	9.808E–7	NEW	2	0.026	4.635E–7
80	HSS	39	1.831	7.646E–7	NEW	2	0.056	3.687E–7
100	HSS	25	1.872	6.477E–7	NEW	2	0.090	1.827E–7
120	HSS	17	1.718	6.013E–7	NEW	2	0.147	6.700E–7
140	HSS	12	1.881	5.033E–7	NEW	3	0.321	1.211E–7
160	HSS	8	1.639	8.397E–7	NEW	3	0.403	4.996E–7
180	HSS	5	1.448	7.224E–7	NEW	3	0.562	1.715E–7
200	HSS	5	1.761	1.948E–7	NEW	3	0.671	5.094E–7
220	HSS	8	3.581	1.392E–7	NEW	4	1.155	5.678E–7
240	HSS	10	6.012	2.944E–7	NEW	4	1.510	1.804E–7
260	HSS	12	8.566	4.554E–7	NEW	4	1.704	5.137E–7
280	HSS	14	12.438	6.524E–7	NEW	5	2.711	1.808E–7
300	HSS	16	17.276	9.166E–7	NEW	5	3.150	5.313E–7

4.2 Results for HSS and new iteration

In this subsection we solve the system $Ax = b$, where the matrix A is tridiagonal system (4.1). We choose the right-hand side vector b such that the exact solution of the linear system is $x^* = (1, 1, \ldots, 1) \in \mathbb{C}^n$

Table 4 Iteration steps, CPU times and residual of IHSS and INEW methods for example (b)

N	Method	IT	CPU	RES	Method	IT	CPU	RES
60	IHSS	68	1.012	9.808E–7	INEW	2	0.046	4.652E–7
80	IHSS	39	0.812	7.646E–7	INEW	2	0.015	3.686E–7
100	IHSS	25	0.668	6.477E–7	INEW	2	0.018	1.827E–7
120	IHSS	17	0.619	6.013E–7	INEW	2	0.024	6.699E–7
140	IHSS	12	0.563	5.034E–7	INEW	3	0.050	1.242E–7
160	IHSS	8	0.459	8.397E–7	INEW	3	0.062	4.996E–7
180	IHSS	5	0.366	7.224E–7	INEW	3	0.081	1.715E–7
200	IHSS	5	0.437	1.948E–7	INEW	3	0.103	5.094E–7
220	IHSS	8	0.821	1.392E–7	INEW	4	0.150	5.680E–7
240	IHSS	10	1.228	2.944E–7	INEW	4	0.190	1.804E–7
260	IHSS	12	1.587	4.554E–7	INEW	4	0.230	5.137E–7
280	IHSS	14	2.153	6.525E–7	INEW	5	0.354	1.808E–7
300	IHSS	17	2.757	9.166E–7	INEW	5	0.388	5.313E–7

Table 5 Iteration steps, CPU times and residual of HSS and NEW methods for example (a) with quasi-optimal iteration parameter

N	Method	IT	CPU	RES	Method	IT	CPU	RES
60	HSS	4	0.101	5.548E–7	NEW	2	0.024	1.857E–7
80	HSS	5	0.246	2.139E–7	NEW	2	0.059	1.392E–7
100	HSS	6	0.478	1.059E–7	NEW	2	0.094	6.596E–7
120	HSS	6	0.678	6.713E–7	NEW	2	0.137	2.339E–7
140	HSS	7	1.146	3.552E–7	NEW	2	0.196	6.831E–7
160	HSS	8	1.735	2.163E–7	NEW	3	0.346	1.107E–7
180	HSS	8	2.322	8.394E–7	NEW	3	0.481	3.703E–7
200	HSS	9	3.294	5.317E–7	NEW	3	0.589	1.112E–7
220	HSS	10	4.622	3.649E–7	NEW	3	0.747	3.067E–7
240	HSS	11	6.475	2.667E–7	NEW	3	0.904	7.856E–7
260	HSS	11	7.538	7.384E–7	NEW	4	1.435	6.821E–7
280	HSS	12	10.216	5.437E–7	NEW	4	1.756	1.942E–7
300	HSS	13	13.454	4.172E–7	NEW	4	2.060	5.093E–7

Besides, all runs are started from the zero vector, terminated if the current iterates satisfy

$$\mathrm{RES} = \frac{\| b - Ax^{(k)} \|_2}{\| b - Ax^{(0)} \|_2} \leq 10^{-6}$$

and performed in MATLAB with a machine precision 10^{-16}. In Tables 1 and 2 we give number of iteration (denoted by 'IT') , elapsed CPU time in seconds (denoted

Table 6 Iteration steps, CPU times and residual of IHSS and INEW methods for example (a) with quasi-optimal iteration parameter

N	Method	IT	CPU	RES	Method	IT	CPU	RES
60	IHSS	4	0.078	5.548E–7	INEW	2	0.040	2.081E–7
80	IHSS	5	0.084	2.139E–7	INEW	2	0.019	1.139E–7
100	IHSS	6	0.138	1.059E–7	INEW	2	0.027	6.595E–7
120	IHSS	6	0.174	6.713E–7	INEW	2	0.040	2.339E–7
140	IHSS	7	0.287	3.552E–7	INEW	2	0.054	6.831E–7
160	IHSS	8	0.435	2.163E–7	INEW	3	0.110	1.106E–7
180	IHSS	8	0.534	8.394E–7	INEW	3	0.138	3.703E–7
200	IHSS	9	0.829	5.317E–7	INEW	3	0.174	1.112E–7
220	IHSS	10	1.077	3.649E–7	INEW	3	0.228	3.067E–7
240	IHSS	11	1.407	2.667E–7	INEW	3	0.273	7.856E–7
260	IHSS	11	2.088	7.384E–7	INEW	4	0.499	6.823E–7
280	IHSS	12	2.350	5.437E–7	INEW	4	0.551	1.942E–7
300	IHSS	13	3.073	4.172E–7	INEW	4	0.646	5.093E–7

Table 7 Iteration steps, CPU times and residual of HSS and NEW methods for example (b) with quasi-optimal iteration parameter

N	Method	IT	CPU	RES	Method	IT	CPU	RES
60	HSS	4	0.101	4.577E–7	NEW	2	0.028	1.844E–7
80	HSS	5	0.224	1.785E–7	NEW	2	0.054	1.379E–7
100	HSS	6	0.468	8.893E–7	NEW	2	0.084	6.516E–7
120	HSS	6	0.820	5.789E–7	NEW	2	0.138	2.304E–7
140	HSS	7	1.180	3.057E–7	NEW	2	0.164	6.707E–7
160	HSS	8	1.960	1.857E–7	NEW	3	0.374	1.075E–7
180	HSS	8	2.442	7.306E–7	NEW	3	0.505	3.577E–7
200	HSS	9	3.255	4.605E–7	NEW	3	0.663	1.067E–7
220	HSS	10	4.956	3.145E–7	NEW	3	0.821	2.924E–7
240	HSS	10	5.778	9.169E–7	NEW	3	0.930	7.439E–7
260	HSS	11	7.588	6.378E–7	NEW	4	1.397	6.309E–7
280	HSS	12	10.296	4.667E–7	NEW	4	1.700	1.783E–7
300	HSS	13	13.338	3.558E–7	NEW	4	2.055	4.647E–7

by 'CPU') and relative residual error (denoted by 'RES') for HSS and new method. In these tables α is tested and optimal (numerical optimal). From these tables we can see that convergence speed of the new method is faster than HSS method. Table 1 shows the result of exact HSS and new methods for example (a). The result of inexact HSS and new methods for example (a) are in Table 2. Similarly Tables 3 and 4 show the result of exact and inexact methods for example (b). The presented results

Table 8 Iteration steps, CPU times and residual of IHSS and INEW methods for example (b) with quasi-optimal iteration parameter

N	Method	IT	CPU	RES	Method	IT	CPU	RES
60	IHSS	4	0.080	4.577E–7	INEW	2	0.040	2.025E–7
80	IHSS	5	0.079	1.785E–7	INEW	2	0.017	1.382E–7
100	IHSS	6	0.136	8.894E–7	INEW	2	0.025	6.515E–7
120	IHSS	6	0.190	5.789E–7	INEW	2	0.040	2.304E–7
140	IHSS	7	0.287	3.057E–7	INEW	2	0.054	6.707E–7
160	IHSS	8	0.441	1.857E–7	INEW	3	0.105	1.085E–7
180	IHSS	8	0.534	7.306E–7	INEW	3	0.143	3.577E–7
200	IHSS	9	0.794	4.605E–7	INEW	3	0.169	1.067E–7
220	IHSS	10	1.060	3.145E–7	INEW	3	0.218	2.924E–7
240	IHSS	10	1.271	9.169E–7	INEW	3	0.263	7.439E–7
260	IHSS	11	1.724	6.378E–7	INEW	4	0.433	6.311E–7
280	IHSS	12	2.202	4.667E–7	INEW	4	0.529	1.783E–7
300	IHSS	13	2.726	3.559E–7	INEW	4	0.631	4.647E–7

Table 9 Iteration steps, CPU times and residual of HSS and NEW methods for an example with non-dominant Hermitian part($\theta = 112$)

N	Method	IT	CPU	RES	Method	IT	CPU	RES
60	HSS	46	1.247	9.761E–7	NEW	3	0.037	1.502E–7
80	HSS	34	1.547	8.289E–7	NEW	4	0.100	5.606E–7
100	HSS	28	2.413	8.761E–7	NEW	4	0.228	5.617E–7
120	HSS	25	2.867	8.221E–7	NEW	5	0.315	3.196E–7
140	HSS	23	4.283	8.662E–7	NEW	6	0.593	2.753E–7
160	HSS	22	4.839	7.966E–7	NEW	7	1.000	3.275E–7
180	HSS	21	6.247	8.799E–7	NEW	8	1.287	5.022E–7
200	HSS	21	7.526	6.995E–7	NEW	9	1.851	9.430E–7
220	HSS	23	10.802	8.488E–7	NEW	12	3.306	5.877E–7
240	HSS	28	16.324	6.605E–7	NEW	15	5.304	8.266E–7
260	HSS	33	22.188	6.588E–7	NEW	21	8.682	8.774E–7
280	HSS	38	30.667	7.508E–7	NEW	34	15.397	9.961E–7
300	HSS	43	38.592	9.287E–7	NEW	100	51.108	9.797E–7

in Tables 5, 6, 7 and 8 are relevant for when both methods use the quasi-optimal iteration parameters. Also, when Hermitian part is not dominant i.e. $\theta = 112$, the results are given in Tables 9 and 10.

Table 10 Iteration steps, CPU times and residual of IHSS and INEW methods for an example with non-dominant Hermitian part($\theta = 112$)

N	Method	IT	CPU	RES	Method	IT	CPU	RES
60	IHSS	46	1.635	9.759E–7	INEW	3	0.065	1.502E–7
80	IHSS	34	1.665	8.288E–7	INEW	4	0.072	5.608E–7
100	IHSS	28	1.874	8.763E–7	INEW	4	0.099	5.617E–7
120	IHSS	25	2.182	8.221E–7	INEW	5	0.204	3.196E–7
140	IHSS	23	2.610	8.661E–7	INEW	6	0.385	2.753E–7
160	IHSS	22	3.026	7.964E–7	INEW	7	0.511	3.274E–7
180	IHSS	21	3.431	8.800E–7	INEW	8	0.757	5.022E–7
200	IHSS	21	4.323	6.997E–7	INEW	9	1.178	9.429E–7
220	IHSS	23	5.907	8.489E–7	INEW	12	1.897	5.876E–7
240	IHSS	28	8.342	6.607E–7	INEW	15	2.987	8.266E–7
260	IHSS	33	11.023	6.590E–7	INEW	21	5.250	8.744E–7
280	IHSS	38	14.532	7.511E–7	INEW	34	10.172	9.961E–7
300	IHSS	43	18.901	9.290E–7	INEW	100	36.357	9.797E–7

5 Conclusions

In this paper a new method for solving non-Hermitian and normal positive-definite systems of linear equations have introduced. Theoretical analysis shows that if $\sigma_{max} \leq \lambda_{min}$ the method converges to the unique solution of the system (1.1) for any positive α. We also find a bound for the spectral radius of the iteration matrix and derive the α^* which minimizes the bound. For decreasing CPU time in practical problems we present the inexact version of the method. When the Hermitian part of the coefficient matrix is dominant the new method performs very well however as the skew-Hermitian part becomes dominant the HSS method is better. Also this method is very efficient for solving matrix equations which we will study in the future.

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