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Double-step scale splitting iteration method for a class of complex symmetric linear systems☆

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ABSTRACT

In this paper, a double-step scale splitting (DSS) iteration method is proposed to solve a class of complex symmetric of linear equation. Unconditional convergence result is established under appropriate restrictions. Two reciprocal optimal iteration parameters and the corresponding optimal convergence factor are also determined. Numerical experiments on a few model problems are used to verify the effectiveness of the DSS iteration method.

1. Introduction

Consider the iteration solution of the following linear system

$$\mathscr{A}x = b, \qquad \mathscr{A} = W + iT \in \mathbb{C}^{n\times n} \quad \text{and} \quad x, b \in \mathbb{C}^n \tag{1.1}$$

and the $n \times n$ matrices W and T are both symmetric and positive definite which implies that the complex matrix $\mathscr{A}$ is nonsingular. $b \in \mathbb{C}^n$ is a given vector and $x \in \mathbb{C}^n$ is an unknown vector. $i = \sqrt{-1}$ is the imaginary unit.

Such systems appear in a variety of scientific computing and engineering applications, such as diffuse optical tomography, structural dynamics, FFT-based solution of certain time-dependent PDEs and quantum mechanics. For more applications of this class of problems, we refer to [1–6] and references therein.

Many efficient iteration methods have been proposed to solve the complex systems (1.1). Recently, based on the Hermitian and skew-Hermitian splitting (HSS) iteration method [7] and the special structure of the

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complex matrix $\mathscr{A} \in \mathbb{C}^{n\times n}$, Bai et al. designed a modified HSS (MHSS) method and a preconditioned MHSS (PMHSS) method in [6] and [8], respectively, to compute an approximate solution for the complex symmetric linear system (1.1); see also [9,10]. This PMHSS iteration method initially proposed in [8] is algorithmically described as follows:

Method 1.1 (*The PMHSS Iteration Method*). *Let $x_0 \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$, until the sequence of iterates $\{x_k\}_{k=0}^{\infty} \subset \mathbb{C}^n$ converges, compute the next iterate x_{k+1} according to the following procedure:*

$$\begin{cases} (\alpha V + W)x_{k+\frac{1}{2}} = (\alpha V - iT)x_k + b, \\ (\alpha V + T)x_{k+1} = (\alpha V + iW)x_{k+\frac{1}{2}} - ib, \end{cases} \tag{1.2}$$

where α is a given positive constant and $V \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix.

Since matrices $\alpha V + W$ and $\alpha V + T$ are both real symmetric and positive definite, the two sub-systems involved in each step of the PMHSS iteration method (1.2) can be effectively solved either exactly by a sparse Cholesky factorization, or inexactly by a preconditioned conjugate gradient scheme [7]. Further, Bai et al. [8] pointed out that the PMHSS iteration converges to the unique solution of the complex symmetric linear system (1.1) for any positive constant α and initial guess x_0. However, we should mention that the numerical experiments in [8] show that the PMHSS method still has a lot of space for improvement.

In order to solve the complex linear system (1.1) more efficiently, Hezari et al. [1] designed a scale-splitting iteration method by multiplying a complex number $(\alpha - i)$ through both sides of the complex system (1.1); and Zeng et al. [11] studied a parameterized single-step HSS iteration method through the same proportional change strategy. They proved that both of them are convergent to the unique solution of the linear system (1.1) for a loose restriction on the iteration parameter α. Actually, both iteration methods can be regarded as a special single-step HSS method [12] for solving the non-Hermitian positive definite linear systems. Though these two methods can solve the problems well in some specific situations, they are both conditionally convergent. The choice of appropriate relaxation parameter α is also difficult.

In this paper, on the one hand, we use the idea of symmetry of the PMHSS method; on the other hand, we use the technique of scaling to reconstruct complex linear system (1.1), but twice, to design a double-step scale splitting (DSS) iteration method. Moreover, we prove the DSS iteration method is unconditionally convergent, and converges faster than the PMHSS iteration method.

The outline of the paper is as follows. In Section 2 we describe the DSS iteration method, the convergence theory is established and the optimal parameter which minimizes the spectral radius of the DSS iteration method is also discussed. Numerical experiments are given in Section 3 to show the effectiveness of the new iteration method. Finally, our brief concluding remarks are made in Section 4.

2. The DSS iteration method

In this section, firstly, we will build the DSS iteration method and analyze its convergence properties. The optimal relaxation parameter, which is minimizing the spectral radius, will be presented at last.

Firstly, by multiplying a parameter $(\alpha - i)$ for complex linear system (1.1), i.e., $(\alpha - i)\mathscr{A}x = (\alpha - i)b$, we obtain

$$\big((\alpha W + T) + i(\alpha T - W)\big)x = (\alpha - i)b,$$

based on the above matrix splitting, we construct a first half-step iteration formation:

$$(\alpha W + T)x_{k+\frac{1}{2}} = i(W - \alpha T)x_k + (\alpha - i)b. \tag{2.1}$$

Similarly, by multiplying $(1 - i\alpha)$ for (1.1), we have $(1 - i\alpha)\mathscr{A}x = (1 - i\alpha)b$, which is equivalent to

$$\big((\alpha T + W) + i(T - \alpha W)\big)x = (1 - i\alpha)b,$$

then the second half-step iteration formation can be introduced

$$(\alpha T + W)x_{k+1} = i(\alpha W - T)x_{k+\frac{1}{2}} + (1 - i\alpha)b. \tag{2.2}$$

Similar to the idea of the PMHSS iteration method [8], we propose a double scale-splitting (DSS) iteration method by alternating between above iterations (2.1) and (2.2) for solving the complex symmetric linear system (1.1), which is described as follows:

Method 2.1 (*The DSS Iteration Method*). *Let $x_0 \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$, until the sequence of iterates $\{x_k\}_{k=0}^{\infty} \subset \mathbb{C}^n$ converges, compute the next iterate x_{k+1} according to the following procedure:*

$$\begin{cases} (\alpha W + T)x_{k+\frac{1}{2}} = i(W - \alpha T)x_k + (\alpha - i)b, \\ (\alpha T + W)x_{k+1} = i(\alpha W - T)x_{k+\frac{1}{2}} + (1 - i\alpha)b, \end{cases} \tag{2.3}$$

where α is a given positive constant.

As W, T are symmetric and positive definite, and $\alpha \in \mathbb{R}$ is positive, we see that matrices $\alpha W + T$ and $\alpha T + W$ are both symmetric and positive definite. Hence, the two sub-systems involved in each step of the DSS iteration method (2.3) can be solved effectively either by Cholesky factorization or some inner iteration scheme [7].

After straightforward derivations we can reformulate the DSS iteration scheme into the standard form

$$x_{k+1} = \mathscr{L}(\alpha)x_k + 2\alpha(\alpha T + W)^{-1}(W - iT)(\alpha W + T)^{-1}b, \qquad k = 0, 1, 2, \ldots, \tag{2.4}$$

where $\mathscr{L}(\alpha) = (\alpha T + W)^{-1}(\alpha W - T)(\alpha W + T)^{-1}(\alpha T - W)$ is the iteration matrix of the DSS iteration method. The fixed point iteration (2.4) converges to the unique solution of the linear systems (1.1) for an arbitrary initial guess x_0 if and only if $\rho(\mathscr{L}(\alpha)) < 1$, where $\rho(\mathscr{L}(\alpha))$ means the spectral radius of the iteration matrix $\mathscr{L}(\alpha)$. The following theorem describes the convergence result of the DSS iteration method.

Theorem 2.1. *Let $\mathscr{A} = W + iT \in \mathbb{C}^{n\times n}$ be a nonsingular matrix with W and T both symmetric and positive definite. Let α be a positive constant. Then the spectral radius of the DSS iteration method satisfies*

$$\rho\big(\mathscr{L}(\alpha)\big) = \max_{\mu \in sp(W^{-1}T)} \left| \frac{(\alpha + \frac{1}{\alpha}) - (\mu + \frac{1}{\mu})}{(\alpha + \frac{1}{\alpha}) + (\mu + \frac{1}{\mu})} \right|$$

where $sp(W^{-1}T)$ denotes the spectrum of the matrix $W^{-1}T$. Consequently,

$$\rho\big(\mathscr{L}(\alpha)\big) < 1, \qquad \textit{for} \quad \forall \alpha > 0,$$

so the DSS iteration method converges to the unique solution of the linear system (1.1) for any initial guess.

Proof. We first introduce the notation

$$\widehat{T} = W^{-\frac{1}{2}} T W^{-\frac{1}{2}}.$$

Obviously, $\widehat{T}$ is a symmetric and positive definite matrix, then there exist an orthogonal matrix $U \in \mathbb{C}^{n\times n}$ and a positive diagonal matrix $\Sigma = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^{n\times n}$ such that $\widehat{T} = U^T \Sigma U$, where μ_j $(1 \le j \le n)$ being the eigenvalues of the matrix $\widehat{T}$. As $\mu_j > 0$, we have

$$\begin{aligned}\rho\big(\mathscr{L}(\alpha)\big) &= \rho\big((\alpha W - T)(\alpha W + T)^{-1}(\alpha T - W)(\alpha T + W)^{-1}\big)\\ &= \rho\big((\alpha I - \Sigma)(\alpha I + \Sigma)^{-1}(\alpha\Sigma - I)(\alpha\Sigma + I)^{-1}\big)\\ &= \max_{\mu\in sp(\widehat{T})}\Big|\frac{(\alpha-\mu)(\alpha\mu-1)}{(\alpha+\mu)(\alpha\mu+1)}\Big|\\ &= \max_{\mu\in sp(\widehat{T})}\Big|\frac{(\alpha+\frac{1}{\alpha})-(\mu+\frac{1}{\mu})}{(\alpha+\frac{1}{\alpha})+(\mu+\frac{1}{\mu})}\Big| < 1\end{aligned}$$

holds true for any $\alpha > 0$. The DSS iteration method therefore converges unconditionally to the unique solution of the complex symmetric linear system (1.1) for any initial guess. □

Theorem 2.1 gives an unconditional convergence property for the DSS iteration method. As we know, the convergence rate depends on the spectral radius of the iteration matrix $\mathscr{L}(\alpha)$, hence finding the optimal spectral radius is necessary. In order to optimize the spectral radius $\rho(\mathscr{L}(\alpha))$, we introduce a function.

Define

$$f(x) = x + \frac{1}{x}, \quad \text{with} \quad x > 0,$$

then the spectral radius can be rewritten as

$$\rho\big(\mathscr{L}(\alpha)\big) = \max_{\mu\in sp(\widehat{T})}\Big|\frac{f(\alpha)-f(\mu)}{f(\alpha)+f(\mu)}\Big|.$$

In the next, we can demonstrate the following corollary about the optimal iteration parameter and the corresponding optimal convergence factor for Method 2.1.

Corollary 2.2. *Let the conditions of Theorem 2.1 be satisfied, and $f(\mu)_{\min}$ and $f(\mu)_{\max}$ denote the smallest and the largest values of $f(\mu)$, respectively. Then the optimal parameter α that minimizes the spectral radius $\rho(\mathscr{L}(\alpha))$ is*

$$\alpha_1^\star = \frac{2}{\sqrt{f(\mu)_{\min} f(\mu)_{\max}} + \sqrt{(f(\mu)_{\min} f(\mu)_{\max})^2 - 4}}, \quad \text{or} \quad \alpha_2^\star = \frac{1}{\alpha_1^\star},$$

where

$$f(\mu)_{\min} = \begin{cases} \mu_{\max} + \dfrac{1}{\mu_{\max}}, & \mu_{\min} \le \mu_{\max} < 1,\\ 2, & \mu_{\min} \le 1 \le \mu_{\max},\\ \mu_{\min} + \dfrac{1}{\mu_{\min}}, & 1 < \mu_{\min} \le \mu_{\max}, \end{cases}$$

$$f(\mu)_{\max} = \begin{cases} \mu_{\min} + \dfrac{1}{\mu_{\min}}, & \mu_{\min} \le \mu_{\max} < 1,\\ \max\left\{\mu_{\min} + \dfrac{1}{\mu_{\min}},\ \mu_{\max} + \dfrac{1}{\mu_{\max}}\right\}, & \mu_{\min} \le 1 \le \mu_{\max},\\ \mu_{\max} + \dfrac{1}{\mu_{\max}}, & 1 < \mu_{\min} \le \mu_{\max}, \end{cases}$$

with $\mu_{\min}$ and $\mu_{\max}$ be the extreme eigenvalues of the matrix $\widehat{T}$. The corresponding optimal convergence factor is

$$\rho(\mathscr{L}(\alpha_{1,2}^\star)) = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1},$$

where $\kappa = f(\mu)_{\max}/f(\mu)_{\min}$.

Proof. As a result of [7], when $f(\alpha) = \sqrt{f(\mu)_{\min} f(\mu)_{\max}}$, $\rho(\mathscr{L}(\alpha))$ reaches its minimize value, i.e.,

$$\rho\big(\mathscr{L}(\alpha^{\star}_{1,2})\big) = \min_{\alpha} \rho(\mathscr{L}(\alpha)) = \frac{\sqrt{f(\mu)_{\max}} - \sqrt{f(\mu)_{\min}}}{\sqrt{f(\mu)_{\max}} + \sqrt{f(\mu)_{\min}}} = \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}.$$

Under this condition, the iteration parameter α needs to satisfy the following fractional equation:

$$\alpha + \frac{1}{\alpha} = \sqrt{f(\mu)_{\min} f(\mu)_{\max}}. \tag{2.5}$$

By straightforwardly solving (2.5) we immediately obtain the results what we are demonstrating. Furthermore, we note that $\alpha_1^{\star}$ and $\alpha_2^{\star}$ are both positive real numbers as $f(\mu) = \mu + \frac{1}{\mu} \geq 2$. □

3. Numerical experiment

In this section, we use two examples to test the effectiveness of the DSS iteration method in terms of both iteration count (denoted as **IT**) and computing time (in seconds, denoted as **CPU**) for solving the complex symmetric linear system (1.1). Numerical comparisons with the PMHSS iteration method are also presented to show the advantage of the DSS iteration method. In all iteration methods, the two half-steps comprising each iteration are computed exactly by the sparse Cholesky factorization incorporated with the *symamd.m* ordering algorithm.

In our implementations, the initial guess x_0 is chosen to be zero vector and the stopping criteria for all the methods are $\|b - \mathscr{A}x_k\|_2/\|b\|_2 \leq 10^{-6}$, where x_k is the current approximation.

Example 1 (*See [4,6,8]*)**.** Consider the linear system of the form

$$\Big[\big(K + \frac{3-\sqrt{3}}{\tau} I\big) + i\big(K + \frac{3+\sqrt{3}}{\tau} I\big)\Big] x = b,$$

where τ is the time step-size and $K = I \otimes B_m + B_m \otimes I$ with $B_m = \frac{1}{h^2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$ and mesh-size $h = \frac{1}{m+1}$. Hence, K is an $n \times n$ block-tridiagonal matrix with $n = m^2$. The right-hand vector b with its j-th entry $[b]_j$ being given by

$$[b]_j = \frac{(1-i)j}{\tau(j+1)^2}, \qquad j = 1, 2, \ldots, n.$$

In our tests, we take $\tau = h$ and normalize coefficient matrix and right-hand side by multiplying both by h^2.

Example 2 (*See [1,6,8]*)**.** The complex linear system (1.1) is of the form

$$\big[(-\omega^2 M + K) + i(\omega C_V + C_H)\big] x = b,$$

where ω is the driving circular frequency, M and K are the inertia and stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, respectively. We take $C_V = \mu K$ with μ a damping coefficient, $M = I$, $C_V = 10I$. $K = I \otimes B_m + B_m \otimes I$, with $B_m = \frac{1}{h^2} \cdot \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$ and mesh size $h = \frac{1}{m+1}$. Hence, K is an $n \times n$ block-tridiagonal matrix with $n = m^2$. In addition, we set $\omega = \pi$, $\mu = 0.1$, and the right-hand side vector b to be $b = (1+i)\mathscr{A}\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. Furthermore, we normalize the system by multiplying both sides throughout by h^2.

Table 1 shows that the experimental optimal parameters α_{exp} corresponding to the PMHSS and DSS iteration methods with respect to different problem sizes for Examples 1 and 2. These optimal parameters

Table 1
The experimental optimal parameters $\alpha_{\exp}$ for the proposed iteration methods by minimizing iteration steps.

Example	Method	Grid ($m \times m$)			
		64×64	128×128	256×256	512×512
No. 1	PMHSS	[0.87, 1.49]	[0.80, 1.48]	[0.75, 1.47]	[0.97, 1.08]
	DSS	[0.46, 0.56]∪ [1.76, 2.18]	[0.46, 0.56]∪ [1.81, 2.18]	[0.46, 0.56]∪ [1.78, 2.16]	[0.46, 0.59]∪ [1.67, 2.18]
No. 2	PMHSS	[0.42, 0.68]	[0.31, 0.90]	[0.27, 0.88]	[0.23, 0.88]
	DSS	[0.18, 0.19]∪ [5.24, 5.71]	[0.16, 0.18]∪ [5.38, 6.30]	[0.158, 0.167]∪ [5.89, 6.41]	[0.14, 0.17]∪ [5.88, 7.27]

Table 2
The numerical results for the proposed iteration methods.

Example	Method		Grid ($m \times m$)			
			64×64	128×128	256×256	512×512
No. 1	PMHSS	$\alpha_{\exp}$	1.35	1.05	1.44	1.05
		IT	21	21	21	21
		CPU	0.0480	0.3080	1.6917	8.5798
	DSS	$\alpha_{\exp}$	0.50	0.50	0.50	0.50
		IT	7	7	7	7
		CPU	0.0215	0.1352	0.7674	3.8940
No. 2	PMHSS	$\alpha_{\exp}$	0.57	0.78	0.73	0.85
		IT	30	31	31	31
		CPU	0.0631	0.3983	2.3626	12.5069
	DSS	$\alpha_{\exp}$	0.18	0.17	0.16	0.16
		IT	11	11	10	10
		CPU	0.0279	0.1826	0.9829	5.0643

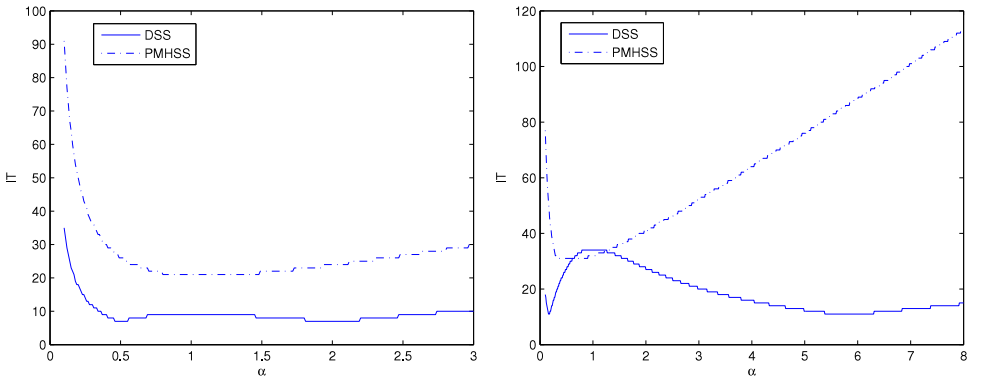

Fig. 1. Pictures of IT versus α for PMHSS and DSS iteration methods with $m = 128$; *left*: Example 1, and *right*: Example 2.

are obtained by minimizing the numbers of iterations with respect to each test example and each spatial mesh-size. From Table 1 we observe that the optimal parameters for both iteration methods form large widths intervals. The optimal parameters of the DSS iteration method are the union of two intervals, which correspond to the two reciprocal optimal parameters in Corollary 2.2. Compared with PMHSS, DSS has a larger optimal parameter interval. This shows that the DSS method is more stable with respect to the iteration parameters α. See Fig. 1.

From Table 2 we see that the ITs for PMHSS and DSS almost remain constant with problem size. The iteration steps of the DSS iteration method are almost one-third of the PMHSS iteration method, and the CPU is no more than half of the PMHSS method. Actually, even though the two methods can deal with these problems well, DSS considerably outperforms PMHSS in solving efficiency, both in terms of iteration counts and in terms of CPU times.

4. Concluding remarks

In this paper, we establish a DSS iteration method for solving the complex symmetric linear system (1.1) with matrices W and T both symmetric and positive definite. The unconditional convergence of the DSS iteration method is proved and two reciprocal optimal iteration parameters are also presented. Numerical examples show that the DSS method is superior to the PMHSS method in terms of the iteration counts and CPU times.

However, we note that the DSS method applies to matrices W and T are both symmetric and positive definite, for a general one, this may not be valid. Hence, the design of an efficient method for general complex symmetric linear systems should be investigated further.

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