Non-commutative extrapolation algorithms

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This paper contains two general results. The first is an extension of the theory of general linear extrapolation methods to a non-commutative field (or even a non-commutative unitary ring). The second one, by exploiting these new results, is to solve an old conjecture about Wynn's vector ε-algorithm. Then, by using designants and Clifford algebras, we show how the vectors $\varepsilon_k^{(n)}$ can be written as a ratio of two designants.
This result allows us to find, as a particular case, some well-known results and some others which are new.

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1. Introduction

Until now, the extrapolation methods used and their applications (solution of systems of linear and nonlinear equations, approximation of the limit or anti-limit of a sequence, acceleration of the convergence of a sequence, ..., see [3]) have $\mathbb{R}$ or $\mathbb{C}$ as the basic fields. The multiplicative law of $\mathbb{R}$ and $\mathbb{C}$ is commutative.

The study of many of these methods is based on the theory of determinants. These determinants are used for building up recursive algorithms corresponding to these extrapolation methods (E-algorithm [4,18], ε-algorithm [28], ...).
In this paper, we shall answer the following question:

Is it possible to extend these extrapolation methods when the field is non-commutative?
There are many difficulties to overcome and, among them, the following: the determinants do not exist when the field (or the unitary ring) is not commutative, i.e.: let $\mathscr{K}$ denote a non-commutative field (or ring), $\mathscr{M}_n(\mathscr{K})$ the set of matrices whose coefficients are in $\mathscr{K}$. By a determinant we mean an application, denoted by *det*

$$det : \mathscr{M}_n(\mathscr{K}) \to \mathscr{K},$$
$$A \to det(A)$$

and satisfying the following conditions

(1) det is multilinear with respect to the rows of A;
(2) $det(A) = 0 \Leftrightarrow A$ singular;
(3) $det(A \times B) = det(A) \cdot det(B)$.

Dyson's theorem [11,21] states that if there exists an application det satisfying (1)–(3) then the multiplicative law on $\mathscr{K}$ is necessarily commutative.

In [10], Dieudonné showed the important role played by the commutativity of the multiplicative law in the theory of determinants.
Let $\mathscr{K}$ be a field, $\mathscr{K}^* = \mathscr{K} - \{0\}$; C : the group of the commutators of $\mathscr{K}^*$. Dieudonné defined an application det

$$det : \mathscr{M}_n(\mathscr{K}) \to (\mathscr{K}^*/C) \cup \{0\}$$

satisfying the conditions (1)–(3) and coinciding with the usual determinant when $\mathscr{K}$ is commutative. In the non-commutative case, the value of det is not in $\mathscr{K}$, but in $(\mathscr{K}^*/C) \cup \{0\}$, i.e. det is an equivalence class. For this reason, Dieudonné's definition is not adapted to our purpose.

In [23], Ore defined some "determinant" of a system of linear equations whose coefficients are in $\mathscr{K}$ (non-commutative). But the disadvantage of this definition is that there is no way of computing recursively these "determinants", which makes this definition not useful in practice.

Many other definitions were proposed by various authors (Dyson [11], Mehta [21], Artin [2]), but they always present some disadvantage for our need. However, there is an old definition which seems particularly adapted to our purpose. It is the notion of designants. They were proposed by Heyting in 1927 [19].

2. Designants

We shall now briefly give the definition and some properties of designants. For more information, see [19,26].

Let $\mathscr{K}$ be a non-commutative field. Consider the system of homogeneous linear equations in the two unknowns $x_1, x_2 \in \mathscr{K}$, with coefficients on the right

$$\begin{cases} x_1 a_{11} + x_2 a_{12} = 0, \\ x_1 a_{21} + x_2 a_{22} = 0, \end{cases} \quad a_{ij} \in \mathscr{K}, \quad i, j = 1, 2. \tag{1}$$

Suppose that $a_{11} \neq 0$; then, by eliminating the unknown x_1 in the second equation of the system, we get:

$$x_2(a_{22} - a_{12}a_{11}^{-1}a_{21}) = 0.$$

Set

$$\Delta_r = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_r = a_{22} - a_{12}a_{11}^{-1}a_{21}.$$

Δ_r is called the right designant of the system (1).
The suffix r indicates that the designant in question is a designant of the right system (1). It indicates also the direction of the computation.

In the same way, consider the system of homogeneous linear equations in the two unknowns x_1, x_2, with coefficients on the left

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = 0, \\ a_{21}x_1 + a_{22}x_2 = 0, \end{cases} \quad a_{ij} \in \mathscr{K}, \quad i, j = 1, 2. \tag{2}$$

Suppose that $a_{11} \neq 0$; then, by eliminating the unknown x_1 in the second equation of the system, we get:

$$(a_{22} - a_{21}a_{11}^{-1}a_{12})x_2 = 0.$$

Set

$$\Delta_l = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_l = (a_{22} - a_{21}a_{11}^{-1}a_{12}).$$

Δ_l is called the left designant of the system (2).
The suffix l indicates that the designant in question is a designant of the left system (2). It indicates also the direction of the computation.
If $\Delta_r \neq 0$ (resp. $\Delta_l \neq 0$) the system (1) (resp. the system (2)) has only the trivial solution.
If $\Delta_r = 0$ (resp. $\Delta_l = 0$) the system (1) (resp. the system (2)) has more than one solution.
In the general case, we proceed in the same way: consider the system of homogeneous linear equations in the n unknowns $x_1, x_2, \ldots, x_n$, with coefficients on the right:

$$\begin{cases} x_1a_{11}+ & x_2a_{12}+ & \cdots & +x_na_{1n} = 0, \\ x_1a_{21}+ & x_2a_{22}+ & \ldots & +x_na_{2n} = 0, \\ \vdots & \vdots & \ddots & \vdots \\ x_1a_{n1}+ & x_2a_{n2}+ & \cdots & +x_na_{nn} = 0, \end{cases} \tag{3}$$

where $a_{ij} \in \mathscr{K}$, $i, j = 1, \cdots, n$.
By eliminating x_1 from the $(n-1)$ last equations, then x_2 from the $(n-2)$ last equations, and so on, we obtain $x_n\Delta_r = 0$. In this way, Δ_r is defined recursively and Δ_r is called the right designant of the system (3); it is denoted by

$$\Delta_r = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}_r .$$

Δ_r has a meaning only if its principal minor

$$\begin{vmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{vmatrix}_r$$

is different from 0. In its turn, this right designant has a meaning only if its principal minor

$$\begin{vmatrix} a_{11} & \cdots & a_{1,n-2} \\ \vdots & \ddots & \vdots \\ a_{n-2,1} & \cdots & a_{n-2,n-2} \end{vmatrix}_r$$

is different from 0, and so on. Finally, Δ_r has a meaning only if

$$a_{11}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_r, \ldots, \begin{vmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{vmatrix}_r$$

are all different from 0. In the sequel, we shall assume that these conditions are satisfied. Similarly, consider the system of homogeneous linear equations in the n unknowns $x_1, x_2, \ldots, x_n$, with coefficients on the left:

$$\begin{cases} a_{11}x_1 + & a_{12}x_2 + & \cdots & +a_{1n}x_n = 0, \\ a_{21}x_1 + & a_{22}x_2 + & \cdots & +a_{2n}x_n = 0, \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}x_1 + & a_{n2}x_2 + & \cdots & +a_{nn}x_n = 0, \end{cases} \tag{4}$$

where $a_{ij} \in \mathscr{K}$, $i, j = 1, \cdots, n$.
By eliminating x_1 from the $(n-1)$ last equations, then x_2 from the $(n-2)$ last equations, and so on, we obtain $\Delta_l x_n = 0$.
Δ_l is called the left designant of the system (4) and it is denoted by

$$\Delta_l = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}_l .$$

Δ_l has a meaning only if

$$a_{11}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_l, \ldots, \begin{vmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{vmatrix}_l$$

are all different from 0.

Now, let us briefly enumerate some fundamental properties of designants. For more details, see [19,26].

Starting with

$$\Delta_r = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}_r ,$$

we define

A_{pq}: the r.designant of order $(n-1)$ obtained from Δ_r by keeping the rows $1, 2, \ldots, n-2, p$ and the columns $1, 2, \ldots, n-2, q$;

A^p : the r.designant of order p obtained from Δ_r by keeping the rows $1, 2, \ldots, p$ and the columns $1, 2, \ldots, p$,

A^p_{qr}: the r.designant of order $p+1$ obtained from Δ_r by keeping the rows $1, 2, \ldots, p, q$ and the columns $1, 2, \ldots, p, r$. Thus $A^p_{p+1,p+1} = A^{p+1}$.

Property 1

$$\Delta_r = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}_r = \begin{vmatrix} A^p_{p+1,p+1} & \cdots & A^p_{p+1,n} \\ \vdots & \ddots & \vdots \\ A^p_{n,p+1} & \cdots & A^p_{n,n} \end{vmatrix}_r .$$

Taking $p = n-2$, we obtain an identity analogous to Sylvester's identity [1] for designants

$$\Delta_r = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}_r = \begin{vmatrix} A^{n-2}_{n-1,n-1} & A^{n-2}_{n-1,n} \\ A^{n-2}_{n,n-1} & A^{n-2}_{n,n} \end{vmatrix}_r .$$

Proof

See [19].

Let D be the determinant of the system (3) where the field $\mathscr{K}$ is commutative and let Δ_r be the r.designant of the same system (here $\Delta_r = \Delta_l$). We have

Property 2

$$D = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \Delta_r \cdot A^{n-1} \ldots A^2 \cdot a_{11}.$$

Conversely, $\Delta_r = \Delta_l = D/D_{nn}$, where D_{nn} denotes the determinant D with its last row and column deleted.

We shall now propose an identity verified by the designants, which is the analogue of Schweins' identity for determinants.

Property 3

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k+1,1} & \cdots & a_{k+1,k} & h_{k+1} \end{vmatrix}_r \begin{vmatrix} a_{11} & \cdots & a_{1k} & a_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k+1,1} & \cdots & a_{k+1,k} & a_{k+1} \end{vmatrix}_r^{-1}$$

$$-\begin{vmatrix} a_{12} & \cdots & a_{1k} & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k,2} & \cdots & a_{k,k} & h_k \end{vmatrix}_r \begin{vmatrix} a_{12} & \cdots & a_{1k} & a_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k,2} & \cdots & a_{k,k} & a_k \end{vmatrix}_r^{-1}$$

$$= \begin{vmatrix} a_{12} & \cdots & a_{1k} & a_1 & h_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{k+1,2} & \cdots & a_{k+1,k} & a_{k+1} & h_{k+1} \end{vmatrix}_r \begin{vmatrix} a_{11} & \cdots & a_{1k} & a_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k+1,1} & \cdots & a_{k+1,k} & a_{k+1} \end{vmatrix}_r^{-1}.$$

Proof
See [26].

To end this section, we give

Property 4

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}_r = 0 \Leftrightarrow \text{the column vectors are linearly dependent on the left,}$$

which means that $\exists \lambda_1, \ldots, \lambda_n \in \mathscr{K}$ not all zero such that:

$$\lambda_1 c^1 + \ldots + \lambda_n c^n = 0,$$

where c^i is the ith column. In the same way, we have:

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}_l = 0 \Leftrightarrow \text{the row vectors are linearly dependent on the right,}$$

which means that $\exists \lambda_1, \ldots, \lambda_n \in \mathscr{K}$ not all zero such that:

$$l^1 \lambda_1 + \ldots + l^n \lambda_n = 0$$

where l^i is the ith row.

Proof
See [26].

Before going to the next section, let us note that since designants do not satisfy condition (1) above, they have some properties completely different from those of determinants, but also some similarities with them (see [19,26]).

3. Extrapolation in a non-commutative field

In this section, where $\mathscr{K}$ is non-commutative, we will show that the designants play an important role for the non-commutative extension of linear extrapolation methods, a role as important as the role of determinants when $\mathscr{K}$ is commutative. These designants are the basis on which the non-commutative extensions of the E-algorithm and the ε-algorithm are built.

3.1. *A general extrapolation method*

In [4], Brezinski proposed a general extrapolation method in $\mathbb{R}$ or $\mathbb{C}$. The aim of this section is to see if it is possible to build an analogous method in a non-commutative field (or even a non-commutative unitary ring).

The main difficulty arises from the fact that, in a non-commutative field, the determinants do not exist. This difficulty will be overcome by the use of designants.

Let (S_n) be a sequence of elements of $\mathscr{K}$. Assume that it satisfies, $\forall n$,

$$S_n = a_1 g_1(n) + \ldots + a_k g_k(n) + S, \tag{5}$$

where the sequences $g_i(n)$ are given sequences.

S is computed by solving the right linear system

$$\left\{\begin{array}{ccccc} S_n & = & a_1 g_1(n)+ & \cdots & +a_k g_k(n) + S, \\ S_{n+1} & = & a_1 g_1(n+1)+ & \cdots & +a_k g_k(n+1) + S, \\ \vdots & & \vdots & \ddots & \vdots \\ S_{n+k} & = & a_1 g_1(n+k)+ & \cdots & +a_k g_k(n+k) + S. \end{array}\right. \tag{6}$$

We shall always assume in this paper that such a system is nonsingular, which is equivalent to the assumption that the r.designant of the system is different from 0.

If the sequence (S_n) does not have the exact form (5), then the value of S obtained by solving the preceding system will depend on the indexed n and k. We shall denote it by ${}^{r}E_k(S_n)$ and from (6), using the r.designants, it is easy to see that

$$
{}^rE_k(S_n) = \begin{vmatrix} g_1(n) & \cdots & g_k(n) & S_n \\ g_1(n+1) & \cdots & g_k(n+1) & S_{n+1} \\ \vdots & \ddots & \vdots & \vdots \\ g_1(n+k) & \cdots & g_k(n+k) & S_{n+k} \end{vmatrix}_r \begin{vmatrix} g_1(n) & \cdots & g_k(n) & 1 \\ g_1(n+1) & \cdots & g_k(n+1) & 1 \\ \vdots & \ddots & \vdots & \vdots \\ g_1(n+k) & \cdots & g_k(n+k) & 1 \end{vmatrix}_r^{-1}. \tag{7}
$$

In general ${}^rE_k(S_n)$ is an approximate value of the limit of the sequence (S_n) if it converges or an approximate value of the antilimit when it diverges.

Similarly, let (S_n) be a sequence of elements of $\mathscr{K}$. Assume that, $\forall n$,

$$
S_n = g_1(n)a_1 + \ldots + g_k(n)a_k + S, \tag{8}
$$

where the sequences $g_i(n)$ are given sequences.

S is computed by solving the left linear system

$$
\begin{cases} S_n & = & g_1(n)a_1 + & \cdots & +g_k(n)a_k + S, \\ S_{n+1} & = & g_1(n+1)a_1 + & \cdots & +g_k(n+1)a_k + S, \\ \vdots & & \vdots & \ddots & \vdots \\ S_{n+k} & = & g_1(n+k)a_1 + & \cdots & +g_k(n+k)a_k + S. \end{cases} \tag{9}
$$

We shall always assume in this paper that such a system is nonsingular, which is equivalent to the assumption that the l.designant of the system is different from 0.

If the sequence (S_n) does not have the exact form (8), then the value of S obtained by solving the preceding system will depend on the indexes n and k. We shall denote it by ${}^lE_k(S_n)$ and from (9), using the l.designants, it is easy to see that

$$
{}^lE_k(S_n) = \begin{vmatrix} g_1(n) & \cdots & g_k(n) & 1 \\ g_1(n+1) & \cdots & g_k(n+1) & 1 \\ \vdots & \ddots & \vdots & \vdots \\ g_1(n+k) & \cdots & g_k(n+k) & 1 \end{vmatrix}_l^{-1} \begin{vmatrix} g_1(n) & \cdots & g_k(n) & S_n \\ g_1(n+1) & \cdots & g_k(n+1) & S_{n+1} \\ \vdots & \ddots & \vdots & \vdots \\ g_1(n+k) & \cdots & g_k(n+k) & S_{n+k} \end{vmatrix}_l. \tag{10}
$$

3.2. *A recursive algorithm*

The main algorithmic problem is now to compute recursively the quantities ${}^rE_k(S_n)$ without computing each of the r.designants appearing in (7).

In this section we shall present a non-commutative extensive of the E-algorithm [4,18]. This extension will allow us to compute recursively the ${}^rE_k(S_n)$'s. This algorithm, which will be called ${}^rE$-algorithm, is the following

$$
{}^rE_0^{(n)} = S_n \qquad n = 0, 1, \ldots,
$$

$$
{}^rg_{0,i}^{(n)} = g_i(n) \qquad i = 1, 2, \ldots \text{ and } n = 1, 2, \ldots .
$$

For $k = 1, 2, \ldots$ and $n = 0, 1, \ldots,$

$$ {}^rE_k^{(n)} = \{ {}^rE_{k-1}^{(n+1)} \times ({}^rg_{k-1,k}^{(n+1)})^{-1} - {}^rE_{k-1}^{(n)} \times ({}^rg_{k-1,k}^{(n)})^{-1} \} \times \{ ({}^rg_{k-1,k}^{(n+1)})^{-1} - ({}^rg_{k-1,k}^{(n)})^{-1} \}^{-1}, $$

$$ {}^rg_{k,i}^{(n)} = \{ {}^rg_{k-1,i}^{(n+1)} \times ({}^rg_{k-1,k}^{(n+1)})^{-1} - {}^rg_{k-1,i}^{(n)} \times ({}^rg_{k-1,k}^{(n)})^{-1} \} \times \{ ({}^rg_{k-1,k}^{(n+1)})^{-1} - ({}^rg_{k-1,k}^{(n)})^{-1} \}^{-1}, $$

$$ i = k+1, k+2, \ldots . $$

Theorem 1

$$ {}^rE_k^{(n)} = {}^rE_k(S_n). $$

Proof

We will simultaneously prove by induction that

$$ {}^rE_k^{(n)} = {}^rE_k(S_n) $$

and

$$ {}^rg_{k,i}^{(n)} = \begin{vmatrix} g_1(n) & \cdots & g_k(n) & g_i(n) \\ \vdots & \vdots & \vdots & \vdots \\ g_1(n+k) & \cdots & g_k(n+k) & g_i(n+k) \end{vmatrix}_r \begin{vmatrix} g_1(n) & \cdots & g_k(n) & 1 \\ \vdots & \vdots & \vdots & \vdots \\ g_1(n+k) & \cdots & g_k(n+k) & 1 \end{vmatrix}_r^{-1}. $$

It is easy to verify that the equality is true for $k = 0$. Assume that it is still true for $k - 1$. Set

$$ A = \begin{vmatrix} g_1(n) & \cdots & g_k(n) & S_n \\ \vdots & \vdots & \vdots & \vdots \\ g_1(n+k) & \cdots & g_k(n+k) & S_{n+k} \end{vmatrix}_r. $$

We have

$$ A = \begin{vmatrix} g_1(n+1) & \cdots & g_k(n+1) & S_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ g_1(n+k-1) & \cdots & g_k(n+k-1) & S_{n+k-1} \\ g_1(n) & \cdots & g_k(n) & S_n \\ g_1(n+k) & \cdots & g_k(n+k) & S_{n+k} \end{vmatrix}_r. $$

By applying Sylvester's identity, we obtain

$$ AJ^{-1} = B - C, $$

with

$$B = \begin{vmatrix} g_1(n+1) & \cdots & g_{k-1}(n+1) & S_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ g_1(n+k) & \cdots & g_{k-1}(n+k) & S_{n+k} \end{vmatrix}_r \begin{vmatrix} g_1(n+1) & \cdots & g_k(n+1) \\ \vdots & \vdots & \vdots \\ g_1(n+k) & \cdots & g_k(n+k) \end{vmatrix}_r^{-1},$$

$$C = \begin{vmatrix} g_1(n) & \cdots & g_{k-1}(n) & S_n \\ \vdots & \vdots & \vdots & \vdots \\ g_1(n+k-1) & \cdots & g_{k-1}(n+k-1) & S_{n+k-1} \end{vmatrix}_r \times \begin{vmatrix} g_1(n) & \cdots & g_k(n) \\ \vdots & \vdots & \vdots \\ g_1(n+k-1) & \cdots & g_k(n+k-1) \end{vmatrix}_r^{-1}$$

and

$$J = \begin{vmatrix} g_1(n+1) & \cdots & g_k(n+1) \\ \vdots & \vdots & \vdots \\ g_1(n+k) & \cdots & g_k(n+k) \end{vmatrix}_r .$$

From the induction assumption, it is easy to see that

$$B = {}^rE_{k-1}^{(n+1)} \cdot ({}^rg_{k-1,k}^{(n+1)})^{-1} \quad \text{and} \quad C = {}^rE_{k-1}^{(n)} \cdot ({}^rg_{k-1,k}^{(n)})^{-1}.$$

Set

$$A' = \begin{vmatrix} g_1(n) & \cdots & g_k(n) & 1 \\ \vdots & \vdots & \vdots & \vdots \\ g_1(n+k) & \cdots & g_k(n+k) & 1 \end{vmatrix}_r .$$

In the same way, by applying Sylvester's identity, we obtain

$$A' = ({}^rg_{k-1,k}^{(n+1)})^{-1} - ({}^rg_{k-1,k}^{(n)})^{-1}.$$

Since ${}^rE_k(S_n) = AA'^{-1} = AJ^{-1}JA'^{-1} = AJ^{-1}(A'J^{-1})^{-1}$, the result follows. The same proof is valid for the ${}^rg_{k,i}^{(n)}$. □

In the same way, the ${}^lE$-algorithm allows us to compute the quantities ${}^lE_k(S_n)$. It is the following

$$ {}^lE_0^{(n)} = S_n \qquad n = 0, 1, \ldots,$$

$$ {}^lg_{0,i}^{(n)} = g_i(n) \qquad i = 1, 2, \ldots \text{ and } n = 1, 2, \ldots .$$

For $k = 1, 2, \ldots$ and $n = 0, 1, \ldots,$

$$
{}^lE_k^{(n)} = \{({}^lg_{k-1,k}^{(n+1)})^{-1} - ({}^lg_{k-1,k}^{(n)})^{-1}\}^{-1} \times \{({}^lg_{k-1,k}^{(n+1)})^{-1} \times {}^lE_{k-1}^{(n+1)} - ({}^lg_{k-1,k}^{(n)})^{-1} \times {}^lE_{k-1}^{(n)}\},
$$

$$
{}^rg_{k,i}^{(n)} = \{({}^lg_{k-1,k}^{(n+1)})^{-1} - ({}^lg_{k-1,k}^{(n)})^{-1}\}^{-1} \times \{({}^lg_{k-1,k}^{(n+1)})^{-1} \times {}^lg_{k-1,i}^{(n+1)} - ({}^lg_{k-1,k}^{(n)})^{-1} \times ({}^lg_{k-1,k}^{(n)})^{-1}\},
$$

$$
i = k+1, k+2, \ldots .
$$

Theorem 2

$$
{}^lE_k^{(n)} = {}^lE_k(S_n).
$$

Proof
Similar to that of theorem 1.

Remarks
(1) The ${}^rE$ and ${}^lE$-algorithms coincide with the E-algorithm when $\mathscr{K}$ is commutative.
(2) It is easy to see that the ${}^rE$-algorithm can be written as follows

$$
{}^rE_0^{(n)} = S_n \qquad \text{for } n = 0, 1, \ldots,
$$

$$
{}^rg_{0,i}^{(n)} = g_i(n) \qquad \text{for } i = 1, 2, \ldots, \; n = 0, 1, \ldots .
$$

For $k = 1, 2, \ldots$ and $n = 0, 1, \ldots,$

$$
{}^rE_k^{(n)} = {}^rE_{k-1}^{(n)} - (\Delta {}^rE_{k-1}^{(n)}) \times (\Delta {}^rg_{k-1,k}^{(n)})^{-1} \times {}^rg_{k-1,k}^{(n)},
$$

$$
{}^rg_{k,i}^{(n)} = {}^rg_{k-1,i}^{(n)} - (\Delta {}^rg_{k-1,i}^{(n)}) \times (\Delta {}^rg_{k-1,k}^{(n)})^{-1} \times {}^rg_{k-1,k}^{(n)} \quad \text{for } i = k+1, \ldots
$$

where the operator Δ acts on the upper indexes n.
(3) Similarly,

$$
{}^lE_0^{(n)} = S_n \qquad \text{for } n = 0, 1, \ldots,
$$

$$
{}^lg_{0,i}^{(n)} = g_i(n) \qquad \text{for } i = 1, 2, \ldots \; n = 0, 1, \ldots .
$$

For $k = 1, 2, \ldots$ and $n = 0, 1, \ldots,$

$$
{}^lE_k^{(n)} = {}^lE_{k-1}^{(n)} - {}^lg_{k-1}^{(n)} \times (\Delta {}^lg_{k-1,k}^{(n)})^{-1} \times (\Delta {}^lE_{k-1}^{(n)}),
$$

$$
{}^lg_{k,i}^{(n)} = {}^lg_{k-1,i}^{(n)} - {}^lg_{k-1,k}^{(n)} \times (\Delta {}^lg_{k-1,k}^{(n)})^{-1} \times (\Delta {}^lg_{k-1,i}^{(n)}) \quad \text{for } i = k+1, \ldots .
$$

In [26], we proved the non-commutative extension of the results given by Brezinski in [4] and Håvie in [18]. The most important ones are

Theorem 3
If

$$S_n = S + a_1 g_1(n) + a_2 g_2(n) + \dots, \qquad \forall n,$$

then

$${}^r E_k^{(n)} = S + a_{k+1} {}^r g_{k,k+1}^{(n)} + a_{k+2} {}^r g_{k,k+2}^{(n)} + \dots, \quad \forall n, k.$$

Theorem 4
If

$$S_n = S + g_1(n) a_1 + g_2(n) a_2 + \dots, \qquad \forall n,$$

then

$${}^l E_k^{(n)} = S + {}^r g_{k,k+1}^{(n)} a_{k+1} + {}^r g_{k,k+2}^{(n)} a_{k,k+2} + \dots, \quad \forall n, k.$$

Theorem 5
There exist coefficients $A_j^{(k,n)}$ such that

$${}^r E_k^{(n)} = \sum_{j=0}^{k} S_{n+j} A_j^{(k,n)} \quad \text{and} \quad {}^r g_{k,i}^{(n)} = \sum_{j=0}^{k} g_i(n+j) A_j^{(k,n)},$$

with $\Sigma_{j=0}^{k} A_j^{(k,n)} = 1$.

4. The vector case of the E-algorithm

Let $\mathscr{K}$ be a non-commutative field. $\mathscr{K}^p$ can be considered as a vector space on the left by defining the external law as follows

$$\mathscr{K} \times \mathscr{K}^p \to \mathscr{K}^p$$

$$\left(\lambda, \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \right) \to \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_p \end{pmatrix}.$$

Let $(\mathscr{K}^p)^*$ be the dual of $\mathscr{K}^p$. It is a vector space on the right (see [13]). Consider now the sequence (S_n) of elements of $\mathscr{K}^p$ such that

$$S_n = S + a_1 g_1(n) + \dots + a_k g_k(n), \tag{11}$$

where $S, g_1(n), \dots, g_k(n) \in \mathscr{K}^p$, $a_1, \dots, a_k \in \mathscr{K}$. To find S in terms of $g_i(n)$,

$i = 1, \dots, k$, we must solve the system

$$\left\{\begin{array}{lllll} a_1 g_1(n) & +\cdots+ & a_k g_k(n) & +S = & S_n, \\ a_1 \Delta g_1(n) & +\cdots+ & a_k \Delta g_k(n) & = & \Delta S_n, \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 \Delta g_1(n+k-1) & +\cdots+ & a_k \Delta g_k(n+k-1) & = & \Delta S_{n+k-1}. \end{array}\right. \tag{12}$$

Let y be an element of $(\mathscr{K}^p)^*$ and x an element of $\mathscr{K}^p$. Denote by (x, y) the value $y(x) \in \mathscr{K}$. Applying y to all the equations of the system (12), except the first one, we obtain the system

$$\left\{\begin{array}{lllll} a_1(\Delta g_1(n), y) & +\cdots+ & a_k(\Delta g_k(n), y) & = & (\Delta S_n, y), \\ \vdots & \vdots & \vdots & & \vdots \\ a_1(\Delta g_1(n+k-1), y) & +\cdots+ & a_k(\Delta g_k(n+k-1), y) & = & (\Delta S_{n+k-1}, y), \\ a_1 g_1(n) & +\cdots+ & a_k g_k(n) & +S = & S_n. \end{array}\right. \tag{13}$$

Definition 1
Let u_{ij}, $i = 1, \dots, k-1$ and $j = 1, \dots, k$, be elements of $\mathscr{K}$ and let v_{kj}, $j = 1, \dots, k$, be elements of $\mathscr{K}^p$.

$$\left|\begin{array}{ccc} u_{11} & \cdots & u_{1k} \\ \vdots & \ddots & \vdots \\ u_{k-1,1} & \cdots & u_{k-1,k} \\ v_{k,1} & \cdots & v_{k,k} \end{array}\right|_r$$

denotes the vector obtained by applying Sylvester's identity and it is called a vector r.designant. In the same way,

$$\left|\begin{array}{ccc} u_{11} & \cdots & u_{1k} \\ \vdots & \ddots & \vdots \\ u_{k-1,1} & \cdots & u_{k-1,k} \\ v_{k,1} & \cdots & v_{k,k} \end{array}\right|_l$$

denotes the vector obtained by applying Sylvester's identity and it is called a vector l.designant.

Remarks
(1) This definition is a generalization of the vector determinants [7] to the non-commutative case.
(2) All the recursive scalar relations remain valid for the vector case.

(3) This definition has no meaning if the vectors v_{ki}, $i = 1, \ldots, k$, are in a row different from the last one.

Now, if the vector sequence (S_n) has not the exact form (11), then the value of S will depend on the indexes n and k, and we shall denote it by ${}^rE_k(S_n)$.

Lemma 1

$$
{}^rE_k(S_n) = \begin{vmatrix} (\Delta S_n, y) & (\Delta g_1(n), y) & \cdots & (\Delta g_k(n), y) \\ \vdots & \vdots & \vdots & \vdots \\ (\Delta S_{n+k-1}, y) & (\Delta g_1(n+k-1), y) & \cdots & (\Delta g_k(n+k-1), y) \\ 1 & 0 & \cdots & 0 \end{vmatrix}_r^{-1}
$$

$$
\times \begin{vmatrix} (\Delta S_n, y) & (\Delta g_1(n), y) & \cdots & (\Delta g_k(n), y) \\ \vdots & \vdots & \vdots & \vdots \\ (\Delta S_{n+k-1}, y) & (\Delta g_1(n+k-1), y) & \cdots & (\Delta g_k(n+k-1), y) \\ S_n & g_1(n) & \cdots & g_k(n) \end{vmatrix}_r .
$$

Proof
See [26].

As in the scalar case, the question is how to build an algorithm which computes recursively the quantities ${}^rE_k^{(n)}(S_n)$.

Consider ${}^rE_k^{(n)}$ and ${}^rg_{k,i}^{(n)}$ computed as follows

$$
{}^rE_0^{(n)} = S_n \qquad n = 0, 1, \ldots,
$$

$$
{}^rg_{0,i}^{(n)} = g_i(n) \qquad i = 1, 2, \ldots, \; n = 0, 1, \ldots .
$$

For $k = 1, 2, \ldots, n = 0, 1, \ldots$

$$
{}^rE_k^{(n)} = {}^rE_{k-1}^{(n)} - (\Delta {}^rE_{k-1}^{(n)}, y)(\Delta {}^rg_{k-1,k}^{(n)}, y)^{-1}({}^rg_{k-1,k}^{(n)}),
$$

$$
{}^rg_{k,i}^{(n)} = {}^rg_{k-1,i}^{(n)} - (\Delta^r g_{k-1,i}^{(n)}, y)(\Delta {}^rg_{k-1,k}^{(n)}, y)^{-1}({}^rg_{k-1,k}^{(n)}).
$$

For these quantities, we have the

Theorem 6

$$
{}^rE_k(S_n) = {}^rE_k^{(n)}.
$$

Proof
See [26, p. 121].

We also have

Theorem 7
If $S_n = S + a_1 g_1(n) + a_2 g_2(n) + \dots$, then

$$^rE_k^{(n)} = S + a_{k+1}\,{}^rg_{k,k+1}^{(n)} + a_{k+2}\,{}^rg_{k,k+2}^{(n)} + \dots, \forall k, n.$$

Proof
See [26].

For the left extension, we obtain similar results:

Lemma 2

$$^lE_k(S_n) = \begin{vmatrix} (\Delta S_n, y) & (\Delta g_1(n), y) & \cdots & (\Delta g_k(n), y) \\ \vdots & \vdots & \vdots & \vdots \\ (\Delta S_{n+k-1}, y) & (\Delta g_1(n+k-1), y) & \cdots & (\Delta g_k(n+k-1), y) \\ S_n & g_1(n) & \cdots & g_k(n) \end{vmatrix}_l \times \begin{vmatrix} (\Delta S_n, y) & (\Delta g_1(n), y) & \cdots & (\Delta g_k(n), y) \\ \vdots & \vdots & \vdots & \vdots \\ (\Delta S_{n+k-1}, y) & (\Delta g_1(n+k-1), y) & \cdots & (\Delta g_k(n+k-1), y) \\ 1 & 0 & \cdots & 0 \end{vmatrix}_l^{-1}.$$

Theorem 8

$$^lE_k(S_n) = {}^lE_k^{(n)}.$$

Theorem 9
If $S_n = S + g_1(n)a_1 + g_2(n)a_2 + \dots$, then

$$^lE_k^{(n)} = S + {}^lg_{k,k+1}^{(n)} a_{k+1} + {}^lg_{k,k+2}^{(n)} a_{k+2} + \dots, \forall k, n.$$

5. Shanks' transformation

We shall now give a non-commutative extension of Shanks' transformation [27].

Let (S_n) be a sequence of elements of a non-commutative field $\mathscr{K}$ satisfying

$$\begin{cases} a_0 S_n + a_1 S_{n+1} + \cdots + a_k S_{n+k} = S, \\ a_0 + a_1 + \cdots + a_k = 1. \end{cases} \tag{14}$$

In order to express S in terms of the elements of the sequence, we solve the system

$$\begin{cases} a_0 + a_1 + \cdots + a_k = 1, \\ a_0 S_n + a_1 S_{n+1} + \cdots + a_k S_{n+k} - S = 0, \\ a_0 \Delta S_n + a_1 \Delta S_{n+1} + \cdots + a_k \Delta S_{n+k} = 0, \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_0 \Delta S_{n+k-1} + a_1 \Delta S_{n+k} + \cdots + a_k \Delta S_{n+2k-1} = 0. \end{cases} \tag{15}$$

If the sequence (S_n) does not have the exact form (14), then the estimated value of S will depend on the indexes n and k. We shall denote it by ${}^r\varepsilon_{2k}(S_n)$.
Assuming that the r.designant of the system is different from zero, we have

$$ {}^r\varepsilon_{2k}(S_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ S_n & S_{n+1} & \cdots & S_{n+k} & 0 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & 0 \end{vmatrix}_r $$
$$ \times \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\ S_n & S_{n+1} & \cdots & S_{n+k} & -1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & 0 \end{vmatrix}_r^{-1} . $$

The transformation

$$ {}^rT : (S_n) \to {}^r\varepsilon_{2k}(S_n) $$

is called the right Shanks transformation.

In the same way, we can build the left Shanks transformation ${}^lT$. Let (S_n) be a sequence of elements of a non-commutative field $\mathscr{K}$ satisfying

$$\begin{cases} S_n a_0 + S_{n+1} a_1 + \cdots + S_{n+k} a_k = S, \\ a_0 + a_1 + \cdots + a_k = 1. \end{cases} \tag{16}$$

In order to express S in terms of the elements of the sequence, we solve the system

$$\begin{cases} a_0 + a_1 + \cdots + a_k = 1, \\ S_n a_0 + S_{n+1} a_1 + \cdots + S_{n+k} a_k - S = 0, \\ \Delta S_n a_0 + \Delta S_{n+1} a_1 + \cdots + \Delta S_{n+k} a_k = 0, \\ \vdots \\ \Delta S_{n+k-1} a_0 + \Delta S_{n+k} a_1 + \cdots + \Delta S_{n+2k-1} a_k = 0. \end{cases} \tag{17}$$

If the sequence (S_n) does not have the exact form (16), then the estimated value of S will depend on the indexes n and k. We shall denote it by ${}^l\varepsilon_{2k}(S_n)$. Assuming that the l.designant of the system is different from zero, we have

$$ {}^l\varepsilon_{2k}(S_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\ S_n & S_{n+1} & \cdots & S_{n+k} & -1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & 0 \end{vmatrix}_l^{-1} \times \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ S_n & S_{n+1} & \cdots & S_{n+k} & 0 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & 0 \end{vmatrix}_l . $$

The transformation

$$ {}^lT : (S_n) \to {}^l\varepsilon_{2k}(S_n) $$

is called the left Shanks transformation.

Now the problem is to build an algorithm which computes recursively the quantities ${}^r\varepsilon_{2k}(S_n)$, ${}^l\varepsilon_{2k}(S_n)$ without computing explicitly the r.designants and the l.designants appearing in the expressions of these quantities.

6. The ε-algorithm

In [28], Wynn gave a recursive algorithm, the ε-algorithm, which allows the recursive implementation of Shanks' transformation (with $\mathscr{K} = \mathbb{R}$ or $\mathbb{C}$).

We will now show that this algorithm is still valid for the computation of ${}^r\varepsilon_{2k}(S_n)$ and ${}^l\varepsilon_{2k}(S_n)$. We define ${}^r\varepsilon_{2k+1}(S_n) = ({}^r\varepsilon_{2k}(\Delta S_n))^{-1}$ and ${}^l\varepsilon_{2k+1}(S_n) = ({}^l\varepsilon_{2k}(\Delta S_n))^{-1}$.

Let $\varepsilon_k^{(n)}$ be the quantities computed by

$$\varepsilon_{-1}^{(n)} = 0 \qquad n = 0, 1, \ldots,$$
$$\varepsilon_0^{(n)} = S_n \qquad n = 0, 1, \ldots .$$

For $k = 0, 1, \ldots$, $n = 0, 1, \ldots$,

$$\varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n+1)} + (\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)})^{-1}.$$

the main result is

Theorem 10

$$\varepsilon_k^{(n)} = {}^r\varepsilon_k(S_n) = {}^l\varepsilon_k(S_n), \quad \forall k, n \in \mathbb{N}.$$

Proof
For $k = 0$, we have

$${}^r\varepsilon_0(S_n) = \begin{vmatrix} 1 & 1 \\ S_n & 0 \end{vmatrix}_r \begin{vmatrix} 1 & 0 \\ S_n & -1 \end{vmatrix}_r^{-1} = (0 - 1 \times 1 \times S_n)(-1 - 0 \times 1 \times S_n)^{-1} = S_n = \varepsilon_0^{(n)}.$$

Similarly, ${}^r\varepsilon_1(S_n) = ({}^r\varepsilon_0(\Delta S_n))^{-1} = (\Delta S_n)^{-1} = \varepsilon_1^{(n)}$.
Now, we will show that $\varepsilon_{2k+2}(S_n) - \varepsilon_{2k}(S_{n+1}) = (\varepsilon_{2k+1}(S_{n+1}) - \varepsilon_{2k+1}(S_n))^{-1}$.
In fact

$$\varepsilon_{2k+2}(S_n) - \varepsilon_{2k}(S_{n+1})$$
$$= \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ S_n & S_{n+1} & \cdots & S_{n+k+1} & 0 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k+1} & 0 \end{vmatrix}_r \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\ S_n & S_{n+1} & \cdots & S_{n+k+1} & -1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k+1} & 0 \end{vmatrix}_r^{-1}$$
$$- \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ S_{n+1} & S_{n+2} & \cdots & S_{n+k+1} & 0 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} & 0 \end{vmatrix}_r \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\ S_{n+1} & S_{n+2} & \cdots & S_{n+k+1} & -1 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k} & \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k} & 0 \end{vmatrix}_r^{-1}$$

Applying property 3 of section 2 (Schweins' identity) and taking the inverse, we have

$$(\varepsilon_{2k+2}(S_n) - \varepsilon_{2k}(S_{n+1}))^{-1}$$

$$= \begin{vmatrix} 1 & \cdots & 1 & 0 \\ S_n & \cdots & S_{n+k+1} & -1 \\ \Delta S_n & \cdots & \Delta S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k+1} & 0 \end{vmatrix}_r \begin{vmatrix} 1 & \cdots & 1 & 0 & 1 \\ S_{n+1} & \cdots & S_{n+k+1} & -1 & 0 \\ \Delta S_{n+1} & \cdots & \Delta S_{n+k+1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k+1} & 0 & 0 \end{vmatrix}_r^{-1} . \tag{18}$$

Since, by definition, $\varepsilon_{2k+1}(S_n) = (\varepsilon_{2k}(\Delta S_n))^{-1}$, we get

$$\varepsilon_{2k+1}(S_{n+1}) - \varepsilon_{2k+1}(S_n)$$

$$= \begin{vmatrix} 1 & \cdots & 1 & 0 \\ \Delta S_{n+1} & \cdots & \Delta S_{n+k+1} & -1 \\ \Delta^2 S_{n+1} & \cdots & \Delta^2 S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^2 S_{n+k} & \cdots & \Delta^2 S_{n+2k} & 0 \end{vmatrix}_r \begin{vmatrix} 1 & \cdots & 1 & 1 \\ \Delta S_{n+1} & \cdots & \Delta S_{n+k+1} & 0 \\ \Delta^2 S_{n+1} & \cdots & \Delta^2 S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^2 S_{n+k} & \cdots & \Delta^2 S_{n+2k} & 0 \end{vmatrix}_r^{-1}$$

$$- \begin{vmatrix} 1 & \cdots & 1 & 0 \\ \Delta S_n & \cdots & \Delta S_{n+k} & -1 \\ \Delta^2 S_n & \cdots & \Delta^2 S_{n+k} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^2 S_{n+k-1} & \cdots & \Delta^2 S_{n+2k-1} & 0 \end{vmatrix}_r \begin{vmatrix} 1 & \cdots & 1 & 1 \\ \Delta S_n & \cdots & \Delta S_{n+k} & 0 \\ \Delta^2 S_n & \cdots & \Delta^2 S_{n+k} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^2 S_{n+k-1} & \cdots & \Delta^2 S_{n+2k-1} & 0 \end{vmatrix}_r^{-1} .$$

By factorizing the second factor of the first term of the second member of the equality and then applying Sylvester's identity, we obtain

$$\varepsilon_{2k+1}(S_{n+1}) - \varepsilon_{2k+1}(S_n)$$

$$= \begin{vmatrix} \Delta S_n & \cdots & \Delta S_{n+k} & -1 \\ \Delta^2 S_n & \cdots & \Delta^2 S_{n+k} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^2 S_{n+k} & \cdots & \Delta^2 S_{n+2k} & 0 \end{vmatrix}_r \begin{vmatrix} 1 & \cdots & 1 & 1 \\ \Delta S_{n+1} & \cdots & \Delta S_{n+k+1} & 0 \\ \Delta^2 S_{n+1} & \cdots & \Delta^2 S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^2 S_{n+k} & \cdots & \Delta^2 S_{n+2k} & 0 \end{vmatrix}_r^{-1} . \tag{19}$$

Carrying out some simple manipulations of designants and comparing (18) to (19) we obtain the result. In the same way, we show that

$$\varepsilon_{2k+3}(S_n) - \varepsilon_{2k+1}(S_{n+1}) = (\varepsilon_{2k+2}(S_{n+1}) - \varepsilon_{2k+2}(S_n))^{-1}.$$

In fact,

$$\varepsilon_{2k+3}(S_n) - \varepsilon_{2k+1}(S_{n+1})$$

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k+1} & -1 \\ \Delta^2 S_n & \Delta^2 S_{n+1} & \cdots & \Delta^2 S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta^2 S_{n+k} & \Delta^2 S_{n+k+1} & \cdots & \Delta^2 S_{n+2k+1} & 0 \end{vmatrix}_r \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k+1} & 0 \\ \Delta^2 S_n & \Delta^2 S_{n+1} & \cdots & \Delta^2 S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta^2 S_{n+k} & \Delta^2 S_{n+k+1} & \cdots & \Delta^2 S_{n+2k+1} & 0 \end{vmatrix}_r^{-1}$$

$$- \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} & -1 \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta^2 S_{n+k} & \Delta^2 S_{n+k+1} & \cdots & \Delta^2 S_{n+2k} & 0 \end{vmatrix}_r \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+k+1} & 0 \\ \Delta^2 S_{n+1} & \Delta^2 S_{n+2} & \cdots & \Delta^2 S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta^2 S_{n+k} & \Delta^2 S_{n+k+1} & \cdots & \Delta^2 S_{n+2k} & 0 \end{vmatrix}_r^{-1}.$$

Applying property 3 of section 2 (Schweins' identity) and taking the inverse, we have

$$(\varepsilon_{2k+3}(S_n) - \varepsilon_{2k+1}(S_{n+1}))^{-1}$$

$$= \begin{vmatrix} 1 & \cdots & 1 & 1 \\ \Delta S_n & \cdots & \Delta S_{n+k+1} & 0 \\ \Delta^2 S_n & \cdots & \Delta^2 S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^2 S_{n+k} & \cdots & \Delta^2 S_{n+2k+1} & 0 \end{vmatrix}_r \begin{vmatrix} 1 & \cdots & 1 & 1 & 0 \\ \Delta S_{n+1} & \cdots & \Delta S_{n+k+1} & 0 & -1 \\ \Delta^2 S_{n+1} & \cdots & \Delta^2 S_{n+k+1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta^2 S_{n+k+1} & \cdots & \Delta^2 S_{n+2k+1} & 0 & 0 \end{vmatrix}_r^{-1}. \tag{20}$$

Then

$$\varepsilon_{2k+2}(S_{n+1}) - \varepsilon_{2k+2}(S_n)$$

$$= \begin{vmatrix} 1 & \cdots & 1 & 1 \\ S_{n+1} & \cdots & S_{n+k+2} & 0 \\ \Delta S_{n+1} & \cdots & \Delta S_{n+k+2} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k+2} & 0 \end{vmatrix}_r \begin{vmatrix} 1 & \cdots & 1 & 0 \\ S_{n+1} & \cdots & S_{n+k+2} & -1 \\ \Delta S_{n+1} & \cdots & \Delta S_{n+k+2} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k+2} & 0 \end{vmatrix}_r^{-1}$$

$$-\begin{vmatrix} 1 & \cdots & 1 & 1 \\ S_n & \cdots & S_{n+k+1} & 0 \\ \Delta S_n & \cdots & \Delta S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k+1} & 0 \end{vmatrix}_r \begin{vmatrix} 1 & \cdots & 1 & 0 \\ S_n & \cdots & S_{n+k+1} & -1 \\ \Delta S_n & \cdots & \Delta S_{n+k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k+1} & 0 \end{vmatrix}_r^{-1}.$$

By factorizing the second factor of the first term of the second member of the equality and then applying Sylvester's identity, we obtain

$$\varepsilon_{2k+2}(S_{n+1}) - \varepsilon_{2k+2}(S_n)$$
$$= \begin{vmatrix} 1 & \cdots & 1 & 0 & 1 \\ S_{n+1} & \cdots & S_{n+k+2} & -1 & 0 \\ \Delta S_{n+1} & \cdots & \Delta S_{n+k+2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k+1} & 0 & 0 \\ \Delta S_n & \cdots & \Delta S_{n+k+1} & 0 & 0 \\ \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k+2} & 0 & 0 \end{vmatrix}_r \begin{vmatrix} 1 & \cdots & 1 & 0 \\ S_{n+1} & \cdots & S_{n+k+2} & -1 \\ \Delta S_{n+1} & \cdots & \Delta S_{n+k+2} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k+1} & \cdots & \Delta S_{n+2k+2} & 0 \end{vmatrix}_r^{-1}. \tag{21}$$

Carrying out some simple manipulations of designants and comparing (20) to (21) we obtain the result.
Since $\varepsilon_{k+1}(S_n) = \varepsilon_{k-1}(S_{n+1}) + (\varepsilon_k(S_{n+1}) - (\varepsilon_k(S_n))^{-1}$, $\forall k, n$, and $\varepsilon_0(S_n) = \varepsilon_0^{(n)}$; $\varepsilon_1(S_n) = \varepsilon_1^{(n)}$, we get $\varepsilon_k^{(n)} = \varepsilon_k(S_n)$, $\forall n, k$. (See [26, pp. 132–141] for a more detailed proof.) □

Remarks

(1) All these constructions remain valid if $\mathscr{K}$ is a ring, but the assumption
H1: element different from zero
must be replaced everywhere by
H2: element invertible.

(2) By simple manipulations of r.designants and l.designants, we can write

$${}^r\varepsilon_{2k}(S_n) = \begin{vmatrix} \Delta S_n & \cdots & \Delta S_{n+k-1} & S_n \\ \Delta S_{n+1} & \cdots & \Delta S_{n+k} & S_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & S_{n+k} \end{vmatrix}_r \begin{vmatrix} \Delta S_n & \cdots & \Delta S_{n+k-1} & 1 \\ \Delta S_{n+1} & \cdots & \Delta S_{n+k} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & 1 \end{vmatrix}_r^{-1}.$$

This expression is similar to the one given in [3] with determinants. In the same way, we get

$$
{}^{l}\varepsilon_{2k}(S_n) = \begin{vmatrix} \Delta S_n & \cdots & \Delta S_{n+k-1} & 1 \\ \Delta S_{n+1} & \cdots & \Delta S_{n+k} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & 1 \end{vmatrix}_l^{-1} \begin{vmatrix} \Delta S_n & \cdots & \Delta S_{n+k-1} & S_n \\ \Delta S_{n+1} & \cdots & \Delta S_{n+k} & S_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & S_{n+k} \end{vmatrix}_l .
$$

(3) From these expressions, we see that the quantities ${}^{r}\varepsilon_{2k}(S_n)$ (resp. ${}^{l}\varepsilon_{2k}(S_n)$) can be computed by the ${}^{r}E$-algorithm (resp. the ${}^{l}E$-algorithm) with the choice $g_i(n) = \Delta S_{n+i-1}$.

In conclusion, designants appear to be a natural generalization of determinants and allow us

(1) to extend extrapolation methods to more general sets (non-commutative field, non-commutative unitary associative ring);
(2) in all the methods exposed above, taking $\mathbb{R}$ or $\mathbb{C}$ or a commutative field instead of $\mathscr{K}$, we obtain the same results as for the methods using determinants.

7. Applications

In this section, we shall show how the use of the general theory of extrapolation, designants and Clifford algebras allows us to answer an old open question on the vector ε-algorithm. Let us first give some properties of Clifford algebras.

7.1. *Real Clifford algebra*

Let V be a real linear space, q its quadratic form. There exists [9] an algebra $(C(V), +, \times, \cdot)$ called a Clifford algebra, which satisfies the following fundamental property:

$$\text{there exists a linear transformation } \varphi : V \to C(V)$$

$$\text{such that } \forall x \in V, (\varphi(x))^2 = q(x).$$

From now on, we take $V = \mathbb{R}^d$ and $q(x) = (x, x)$ where $(\cdot, \cdot)$ denotes the usual scalar product. Let $e_1, \ldots, e_d$ be an orthogonal basis of V and set $E_i = \varphi(e_i)$, $i = 1, \ldots, d$. The Clifford algebra associated to V is generated by d elements E_i, $i = 1, \ldots, d$ [2,25] which satisfy the anti-commutation relations

$$E_i \times E_j + E_j \times E_i = 2\delta_{ij} E_0, \tag{22}$$

E_0 being the identity element, δ_{ij} the Kronecker symbol and $1 \leq i, j \leq d$.
The real linear space spanned by the products

$$E_{j1} \times E_{j2} \times \ldots \times E_{jr}, \quad 0 \leq j_1 < j_2 < \ldots < j_r \leq d,$$

i.e. $E_0, E_1, E_2, \ldots, E_d, E_1 \times E_2, \ldots, E_{d-1} \times E_d, \ldots, E_1 \times E_2 \times \ldots \times E_d$ forms the

associative but non-commutative ($d > 1$) algebra $C(V)$. It is easy to see that $C(V)$ is of dimension 2^d. However, it is not a division algebra ($d \geq 1$) as proved by the relation $(E_0 + E_1)(E_0 - E_1) = 0$.
There are various matrix representations of E_i, e.g. [20,17]. However, all of them involve square matrices whose order rises rapidly with the dimension d. Typically this order is at least proportional to $2^{d/2}$.

Furthermore, we may identify each $x = \Sigma_{i=1}^d x_i e_i \in V$ with $X = \Sigma_{i=1}^d x_i E_i \in C(V)$ and each $\lambda \in \mathbb{R}$ with $\lambda E_0 \in C(V)$ (see [2]). According to this identification, one can consider V and $\mathbb{R}$ as subsets of $C(V)$.
From equation (22), we can easily establish

$$\forall X \in V, \forall Y \in V \quad X \times Y + Y \times X = 2(X \cdot Y), \tag{23}$$

which admits the following two particular rules:

- if $X \perp Y$, that is $(X \cdot Y) = 0$, then

$$X \times Y = -Y \times X; \tag{24}$$

- if we take $X = Y$, then

$$X \times X = (X \cdot X). \tag{25}$$

This last relation allows to conclude that, if $X \neq 0$, then

$$X^{-1} = \frac{X}{(X \cdot X)}. \tag{26}$$

Let us notice that this inverse is the same as the Moore–Penrose pseudo-inverse (see [24]).

7.2. *The vector ε-algorithm*

The vector ε-algorithm is a quite powerful method for accelerating the convergence of vector sequences. It has many other applications: solution of systems of linear equations [5,12], systems of nonlinear equations [8] and the computation of eigenvalues of a matrix [6].
It occurs likewise in approximation theory (vector continued fractions [29,30], vector interpolation and Padé approximants [14–17]). Although it was the subject of intensive research, it still lacks a complete set of theoretical results.

In this section, we will show that the vector ε-algorithm, like other extrapolation algorithms, realizes some extrapolation in $C(V)$, solves some system of linear equations in $C(V)$ and its elements can be written as a "ratio" of two designants.

We saw above that $(C(V), +, \times)$ is an associative unitary ring. It is not commutative when $d > 1$. We also saw that $V \subset C(V)$; $\mathbb{R} \subset C(V)$. For each nonzero element $X = \Sigma_{i=1}^d x_i E_i$ of V, we saw that its inverse is $X^{-1} = X/\|X\|^2$, where

$\|X\| = (\Sigma_{i=1}^{d} x_i^2)^{1/2}$. The vector ε-algorithm of Wynn can be written as follows

$$\begin{array}{ll} \varepsilon_{-1}^{(n)} = 0 & n = 0, 1, \ldots, \\ \varepsilon_{0}^{(n)} = S_n \in V & n = 0, 1, \ldots, \\ \varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n+1)} + (\varepsilon_{k}^{(n+1)} - \varepsilon_{k}^{(n)})^{-1} & k = 0, 1, \ldots; n = 0, 1, \ldots . \end{array}$$

The inverse considered here is the above-mentioned inverse. The $\varepsilon_k^{(n)}$'s belong always to V, this is because the sum and the difference are closed operations in V and the inverse of a nonzero element of V also belongs to V.
Owing to the fact that $(C(V), +, \times, \cdot)$ is an algebra, the elements of $C(V)$ can be regarded as scalar elements of the ring $(C(V), +, \times)$, and as vector elements of the vector space $(C(V), +, \cdot)$.

Let (S_n) be a sequence of vectors of V (scalar elements of the ring $(C(V), +, \times)$) such that

$$\left\{ \begin{array}{ccccccccc} a_0 & + & a_1 & + \cdots + & a_k & = 1, \\ a_0 \times S_n & + & a_1 S_{n+1} & + \cdots + & a_k \times S_{n+k} & = S, \end{array} \right. \tag{27}$$

where $S \in V$; $a_0, a_1, \ldots, a_k \in C(V)$.
By applying theorem 10 for $\mathscr{K} = (C(V), +, \times)$, we have the fundamental result:

Theorem 11
The quantities $\varepsilon_k^{(n)}$ computed by the vector ε-algorithm can be expressed as a "ratio" of two designants as follows

$$\varepsilon_{2k}^{(n)} = \begin{vmatrix} \Delta S_n & \cdots & \Delta S_{n+k-1} & S_n \\ \vdots & \ddots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & S_{n+k} \end{vmatrix}_r \begin{vmatrix} \Delta S_n & \cdots & \Delta S_{n+k-1} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & 1 \end{vmatrix}_r^{-1},$$

$\varepsilon_{2k+1}^{(n)} = (\varepsilon_{2k}(S_n))^{-1}$.

We also have

Theorem 12
$\varepsilon_{2k}^{n} = S \Leftrightarrow \exists A_1, \ldots, A_k \in C(V)$ such that $S_n = S + A_1 \Delta S_n + \ldots + A_k \Delta S_{n+k-1}$.

Proof
It is a direct consequence of theorem 11. In fact,

$$\varepsilon_{2k}^{(n)} = S \Leftrightarrow \begin{vmatrix} \Delta S_n & \cdots & \Delta S_{n+k-1} & S_n \\ \vdots & \ddots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & S_{n+k} \end{vmatrix}_r = S \begin{vmatrix} \Delta S_n & \cdots & \Delta S_{n+k-1} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & 1 \end{vmatrix}_r .$$

Due to properties of designants shown in [26,19], we have

$$\varepsilon_{2k}^{(n)} = S \Leftrightarrow \begin{vmatrix} \Delta S_n & \cdots & \Delta S_{n+k-1} & S_n \\ \vdots & \ddots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & S_{n+k} \end{vmatrix}_r = \begin{vmatrix} \Delta S_n & \cdots & \Delta S_{n+k-1} & S \\ \vdots & \ddots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & S \end{vmatrix}_r,$$

$$\varepsilon_{2k}^{(n)} = S \Leftrightarrow \begin{vmatrix} \Delta S_n & \cdots & \Delta S_{n+k-1} & S_n - S \\ \vdots & \ddots & \vdots & \vdots \\ \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} & S_{n+k} - S \end{vmatrix}_r = 0.$$

By applying the property 4 of section 2, the result follows. □

Remark
The fundamental algebraic result of McLeod [22] is a particular case of theorem 12 by imposing that the A_i's belong to $\mathbb{R}$ ($\subset C(V)$).
From theorem 11, we can also deduce some other interesting results and obtain some other well known results as particular cases.

Theorem 13
Let $\varepsilon_k^{(n)}$ be the vectors obtained by applying the ε-algorithm to the vector sequence (S_n) of elements of V.
Let A, B be two invertible elements of $C(V)$, C an element of V.
Let $\tilde{\varepsilon}_k^{(n)}$ be the vectors obtained by applying the ε-algorithm to the vector sequence $(A \times S_n \times B + C)$ then

$$\tilde{\varepsilon}_{2k}^{(n)} = A \times \varepsilon_{2k}^{(n)} \times B + C,$$

$$\tilde{\varepsilon}_{2k+1}^{(n)} = B^{-1} \times \varepsilon_{2k+1}^{(n)} \times A^{-1}.$$

Proof
See [26, p. 178].

Theorem 14
Let M be a square orthogonal matrix of dimension $d \times d$ ($M^{\mathrm{T}}M = I$). Let $\tilde{\varepsilon}_k^{(n)}$ be the vector obtained by applying the vector ε-algorithm to the vector sequence $MS_n + C$ where C is a vector of V and MS_n the product of the matrix M by the vector S_n, then

$$\tilde{\varepsilon}_{2k}^{(n)} = M{\cdot}\varepsilon_{2k}^{(n)} + C,$$

$$\tilde{\varepsilon}_{2k+1}^{(n)} = M{\cdot}\varepsilon_{2k+1}^{(n)}.$$

Proof
Let u be an endomorphism of V which matrix relatively to the basis $e_1, \ldots, e_d$ is M. Let $x = \Sigma_{i=1}^{d} x_i E_i$; $X = (x_1 \ldots x_d)^{\mathrm{T}}$; $u(x)$ can be written in matrix form as $M \cdot X$. To show that theorem 14 is a particular case of theorem 13, let us use the following well known result (see [2]):
For each orthogonal isomorphism $u : V \to V$, there exists an invertible element $B \in C(V)$ such that $\forall x \in V$, $u(x) = \varepsilon B \times x \times B^{-1}$ where $\varepsilon = 1$ or -1.
So $\varepsilon_{2k}(M \cdot S_n + C)$ can be written as $\varepsilon_{2k}(\varepsilon B \times S_n \times B^{-1} + C)$.
According to theorem 13 we get

$$\varepsilon_{2k}(\varepsilon B \times S_n \times B^{-1} + C) = \varepsilon B \times \varepsilon_{2k}(S_n) \times B^{-1} + C,$$

which can be written in matrix form as $M \cdot \varepsilon_{2k}(S_n) + C$.
In the same way, $\varepsilon_{2k+1}(M \cdot S_n + C)$ can be written as

$$\varepsilon_{2k+1}(\varepsilon B \times S_n \times B^{-1} + C).$$

Using theorem 13, it follows that

$$\varepsilon_{2k+1}(\varepsilon B \times S_n \times B^{-1} + C) = (B^{-1})^{-1} \times \varepsilon_{2k+1}(S_n) \times (\varepsilon B)^{-1}.$$

Now, since $\varepsilon^{-1} = \varepsilon$; $(B^{-1})^{-1} = B$), we obtain

$$\varepsilon_{2k+1}(\varepsilon B \times S_n \times B^{-1} + C) = \varepsilon B \times \varepsilon_{2k+1}(S_n) \times B^{-1}.$$

In matrix form, this last equality can be expressed as $M \cdot \varepsilon_{2k+1}(S_n)$.
We finally have $\tilde{\varepsilon}_{2k}^{(n)} = M \varepsilon_{2k}^{(n)} + C$ and $\tilde{\varepsilon}_{2k+1}^{(n)} = M \varepsilon_{2k+1}^{(n)}$. □

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