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THE BLOCK SYMMETRIC SUCCESSIVE OVERRELAXATION METHOD*

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1. Introduction. The point symmetric successive overrelaxation (PSSOR) method was first introduced by Sheldon [10] and subsequently investigated by Habetler and Wachspress [7]. Several recent papers, noted below, have contributed further. However, other than a brief mention by Arms, Gates and Zondek [1], there appears to be little or no literature on the block symmetric successive overrelaxation (BSSOR) method. This paper is intended to be a step in the direction of investigating BSSOR.

It can be shown that the BSSOR method leads to a matrix which is similar to a symmetric matrix (hence the name) if the matrix of the linear system to be solved is symmetric and positive definite. Thus one can apply semi-iterative techniques [12]. In §2 it is shown that one can gain an improvement in convergence rate of as much as an order of magnitude, by semi-iteration.

In §3 we investigate the BSSOR method. It is seen that most of the results for PSSOR carry over easily to BSSOR. Included also in this section is a new proof of convergence and a monotonicity theorem. Although one can prove the existence of an optimum relaxation factor, there appears to be no explicit expression for this factor, even in the simplest of cases (except for one case noted in §4).

A comparison between BSSOR and the block successive overrelaxation (BSOR) method [1] is given in §4. It is shown that although BSSOR with semi-iteration may not be as fast as BSOR in general, there are special cases where one can gain an order of magnitude improvement in convergence rate.

In §5 the Laplace equation in a square is examined in detail. Although PSSOR is not separable, one can still bound the convergence rate and obtain results consistent with §4. For BSSOR however, if one chooses "row" or "line" iteration, the difference equation is separable. For this form of block iteration, a relaxation factor is given explicit which, although not the optimum, is very near the optimum factor for small mesh size.

Numerical results and conclusion are given, respectively, in the last two sections.

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2. Semi-iterative methods. We propose to solve the linear system

$$Au = b, \tag{2.1}$$

where A is a nonsingular $N \times N$ matrix, b is a known column vector, and u is an unknown column vector. Consider the iterative method

$$u^{(n+1)} = Gu^{(n)} + k, \qquad n \geqq 0, \tag{2.2}$$

where $u^{(0)}$ is arbitrary and $k = (I - G)A^{-1}b$, for some matrix G. We define $S(G)$ to be the *spectral radius* of G and

$$R(G) = -\log S(G) \tag{2.3}$$

to be the *rate of convergence of* G [13] for $S(G) < 1$. (Varga [12] calls this the asymptotic rate of convergence.)

If G is similar to a symmetric matrix and it is known only that the eigenvalues of G, $\lambda_i(G)$, satisfy

$$-1 < a \leqq \lambda_i(G) \leqq b < 1, \tag{2.4}$$

then one can apply a semi-iterative technique (see, e.g., [12, Chap. 5]). Thus

$$v^{(n)} = \sum_{k=0}^{n} \alpha_{n,k} u^{(k)} \tag{2.5}$$

where we choose

$$P_n(x) = \sum_{k=0}^{n} \alpha_{n,k} x^k = \frac{T_n(\gamma)}{T_n(d)}, \tag{2.6}$$

$$\gamma = \frac{2x - (b + a)}{b - a}, \tag{2.7}$$

$$d = \frac{2 - (b + a)}{b - a}, \tag{2.8}$$

$$T_n(\gamma) = \begin{cases} \cos(n \cos^{-1} \gamma), & |\gamma| \leqq 1, \\ \cosh(n \cosh^{-1} \gamma), & |\gamma| \geqq 1, \end{cases} \tag{2.9}$$

i.e., Chebyshev acceleration. If $b = S(G)$, then

$$S(P_n(G)) = \frac{1}{T_n(d)}. \tag{2.10}$$

Let

$$R_n(G) = -\frac{\log S(P_n(G))}{n} \tag{2.11}$$

be the *average rate of convergence* [14] of the semi-iterative method and define

$$R_\infty(G) = \lim_{n\to\infty} R_n(G) \tag{2.12}$$

as the *asymptotic average rate of convergence*, provided the limit exists.

THEOREM 2.1. *If G is similar to a symmetric matrix and if $S(G) < 1$, then for the semi-iterative method of (2.5) and (2.6) we have for $a < 1$,*

$$\lim_{S(G)\to 1^-} \frac{R_\infty(G)}{\sqrt{R(G)}} = \frac{2}{\sqrt{1-a}}, \tag{2.13}$$

where a is defined by (2.4).

Proof. We have

$$\cosh^{-1} d - \frac{\log 2}{n} < \frac{\log T_n(d)}{n} < \cosh^{-1} d$$

and

$$R_\infty(G) = \lim_{n\to\infty} R_n(G) = \lim_{n\to\infty} \frac{\log T_n(d)}{n} = \cosh^{-1} d. \tag{2.14}$$

Hence

$$\begin{aligned}\lim_{S(G)\to 1^-} \frac{R_\infty(G)}{\sqrt{R(G)}} &= \lim_{S(G)\to 1^-} \frac{\cosh^{-1} d}{\sqrt{-\log S(G)}} \\ &= \lim_{S(G)\to 1^-} \frac{\cosh^{-1}\left(\dfrac{2-(S(G)+a)}{S(G)-a}\right)}{\sqrt{-\log S(G)}} \\ &= \frac{2}{\sqrt{1-a}},\end{aligned}$$

by L'Hospital's Rule.

To see the significance of the above theorem, we let $S(G) = 1 - \epsilon$, where ϵ is some small positive number. In a manner similar to that used by Sheldon [11], consider

$$S(G) = e^{\log S(G)}.$$

For $\epsilon/(1-a)$ small, we have approximately

$$n_2 \doteq \frac{\sqrt{\epsilon(1-a)}}{2} n_1,$$

where n_1 is the number of iterations necessary to achieve a given degree of convergence for the original iterative method and n_2 is the number of semi-

iterations to achieve the same degree of convergence. (Sheldon [11] obtains the result for the special case $a = -1$.) Thus, the semi-iterative technique requires a small fraction of the number of iterations of the original method.

We can compare Theorem 2.1 with an existing result. In our notation, Young [13] proves

THEOREM 2.2. *Let r be an integer not less than 2 and let $m = m(d)$ denote an integer such that $m \geqq \log 2^r/\cosh^{-1} d$. Then*

$$\lim_{d\to 1} \frac{R_m(G)}{\sqrt{R_1(G)}} \geqq \left(1 - \frac{1}{r}\right)\sqrt{2}.$$

To relate this to our result we note that

$$\lim_{\substack{S(G)\to 1^- \\ [d\to 1]}} \frac{R_1(G)}{R(G)} = \lim \frac{\log d}{-\log S(G)} = \frac{2}{1-a}, \tag{2.15}$$

which is easily proved by L'Hospital's Rule, and thus obtain

$$\lim_{S(G)\to 1^-} \frac{R_m(G)}{\sqrt{R(G)}} \geqq \frac{2\left(1 - \frac{1}{r}\right)}{\sqrt{1-a}}. \tag{2.16}$$

Equation (2.13) is essentially a limiting case of the above. Also (2.15) introduces the role of the smallest eigenvalue and its effect on convergence.

3. The BSSOR method Let π be a partition of the matrix A of (2.1) with the form

$$A^{(\pi)} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1r} \\ A_{21} & A_{22} & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \\ A_{r1} & & \cdots & & A_{rr} \end{bmatrix},$$

where $1 \leqq r \leqq N$ and where the A_{ii} are all square matrices (or single elements). If all A_{ij} are single elements of A, the corresponding partition will be denoted by π_0 (i.e., a point iterative method). We assume the A_{ii} are nonsingular. Subsequently, we shall use the superscript (π) for emphasis, or to avoid confusion.

Let

$$A = D - C, \tag{3.1}$$

where

$$D = \begin{bmatrix} A_{11} & & & 0 \\ & A_{22} & & \\ & & \ddots & \\ 0 & & & A_{rr} \end{bmatrix}, \tag{3.2}$$

and where

$$C = E + F, \tag{3.3}$$

where E and F are respectively strictly lower and strictly upper triangular matrices. Since D^{-1} exists, we let

$$\begin{aligned} B &= D^{-1}C = L + U, \\ L &= D^{-1}E, \\ U &= D^{-1}F, \end{aligned} \tag{3.4}$$

where L is strictly a lower triangular matrix and U strictly an upper triangular matrix.

The BSSOR method is defined by

$$\begin{aligned} D\bar{u}^{(n+\frac{1}{2})} &= Eu^{(n+\frac{1}{2})} + Fu^{(n)} + b, \\ u^{(n+\frac{1}{2})} &= \omega\bar{u}^{(n+\frac{1}{2})} + (1-\omega)u^{(n)}, \end{aligned} \tag{3.5a}$$

$$\begin{aligned} D\bar{u}^{(n+1)} &= Eu^{(n+\frac{1}{2})} + Fu^{(n+1)} + b, \\ u^{(n+1)} &= \omega\bar{u}^{(n+1)} + (1-\omega)u^{(n+\frac{1}{2})}; \end{aligned} \tag{3.5b}$$

where ω is some real parameter. Eliminating $u^{(n+\frac{1}{2})}$ we obtain

$$u^{(n+1)} = \mathfrak{L}_\omega^{(\pi)} u^{(n)} + \bar{b}, \tag{3.6}$$

where

$$\mathfrak{L}_\omega^{(\pi)} = (I - \omega U)^{-1}(\omega L + (1-\omega)I)(I - \omega L)^{-1} \cdot (\omega U + (1-\omega)I), \tag{3.7}$$

$$\bar{b} = \omega\{(I - \omega U)^{-1}(\omega L + (1-\omega)I)(I - \omega L)^{-1}D^{-1} + (I - \omega U)^{-1}D^{-1}\}b. \tag{3.8}$$

The results of Kahan [9] and Sheldon [10] can be extended to prove the following.

THEOREM 3.1. *Let A be symmetric and D be symmetric and positive definite. Then all eigenvalues η of $\mathfrak{L}_\omega^{(\pi)}$ are in the range $0 \leqq \eta < 1$ if and only if A is positive definite and $0 < \omega < 2$. If $\eta = 0$, then $\omega = 1$.*

If we suppose D of (3.2) to be symmetric and positive definite, then we can define $D^{1/2}$, symmetric and positive definite. Let

$$\begin{aligned} \tilde{B} &= D^{1/2}BD^{-1/2}, \\ \tilde{L} &= D^{1/2}LD^{-1/2}, \\ \tilde{U} &= D^{1/2}UD^{-1/2}. \end{aligned} \tag{3.9}$$

The following which appears to be new will be referred to as the Monotonicity Theorem.

THEOREM 3.2. *Let A be symmetric and positive definite. Let A_* be an $s \times s$ matrix, $1 \leqq s < N$, obtained from A by deleting certain columns and corresponding rows. If D is diagonal (i.e., the partition π_0) and if $\mathfrak{S}_{*\omega}^{(\pi_0)}$ is formed from A_* by (3.7), then*

$$S(\mathfrak{S}_{*\omega}^{(\pi_0)}) \leqq S(\mathfrak{S}_{\omega}^{(\pi_0)}).$$

Proof. By formal manipulation, one can show that

$$\tilde{\mathfrak{S}}_{\omega}^{(\pi_0)} = D^{1/2}(I - \omega U)\,\mathfrak{S}_{\omega}^{(\pi_0)}(I - \omega U)^{-1}D^{-1/2} = I - \omega(2 - \omega)\bar{A},$$

where $\bar{A} = D^{1/2}(D - \omega E)^{-1}A(D - \omega F)^{-1}D^{1/2}$ and where $\tilde{\mathfrak{S}}_{\omega}^{(\pi_0)}$ is similar to $\mathfrak{S}_{\omega}^{(\pi_0)}$. Similarly

$$\tilde{\mathfrak{S}}_{*\omega}^{(\pi_0)} = I_* - \omega(2 - \omega)\bar{A}_*,$$

where $\bar{A}_* = D_*^{1/2}(D_* - \omega E_*)^{-1}A_*(D_* - \omega F_*)^{-1}D_*^{1/2}$.

Let $\underline{\nu}_*$ be the smallest eigenvalue of $\bar{A}_*$ with associated eigenvector x_*. Then $\bar{A}_* x_* = D_*^{1/2}(D_* - \omega E_*)^{-1}A_*(D_* - \omega F_*)^{-1}D_*^{1/2}x_* = \underline{\nu}_* x_*$. Taking inner products with respect to x_* and letting

$$(D_* - \omega F_*)^{-1}D_*^{1/2}x_* = w_* \not\equiv 0, \tag{3.10}$$

we obtain $(A_* w_*, w_*) = \underline{\nu}_*(x_*, x_*)$. From (3.10) we have

$$D_*^{1/2}x_* = (D_* - \omega F_*)w_*. \tag{3.11}$$

We now augment w_* with zeros at the positions which were deleted from A, i.e., if the i_1th row and column were deleted from A to form A_*, then

$$w^T = (w_{*1}, w_{*2}, \cdots, w_{*i_1-1}, 0, w_{*i_1}, w_{*i_1+1}, \cdots, w_{*s})^T$$

and similarly for other rows and columns which were deleted. We now define x such that

$$D^{1/2}x = (D - \omega F)w. \tag{3.12}$$

From the definition of w, $(A_* w_*, w_*) = (Aw, w)$. Further, by definition of w, the right side of (3.12) has the same components as the right side of (3.11) plus additional ones. Since $D^{1/2}$ is diagonal, the components of x are the same as the components of x_* plus additional ones. Hence $(x_*, x_*) \leqq (x, x)$. Therefore

$$\underline{\nu}_* = \frac{(A_* w_*, w_*)}{(x_*, x_*)} \geqq \frac{(Aw, w)}{(x, x)}. \tag{3.13}$$

We now form $(\bar{A}x, x)/(x, x)$. Since $\bar{A}$ is symmetric, we have from the definition of w,

$$\underline{\nu} = \frac{(\bar{A}x, x)}{(x, x)} = \frac{(Aw, w)}{(x, x)} \leqq \frac{(A_* w_*, w_*)}{(x_*, x_*)} = \underline{\nu}_*,$$

where $\underline{\nu}$ is the smallest eigenvalue of $\bar{A}$. Thus since $\bar{A}$ is positive definite,

$$S(\mathcal{S}_\omega^{(\pi_0)}) = 1 - \omega(2 - \omega)\,\underline{\nu} \geqq 1 - \omega(2 - \omega)\underline{\nu}_* = S(\mathcal{S}_{*\omega}^{(\pi_0)}).$$

Unfortunately, this theorem has not been extended to general π although numerical results indicated it might be true, at least for certain other partitions.

LEMMA 3.3. *If A is symmetric and $A = M_1 - M_2$, where M_1 is nonsingular, and if ν is an eigenvalue of $A^{-1}M_2$, then η is an eigenvalue of $M_1^{-1} \cdot M_2$, where $\eta = \nu/(1 + \nu)$, and conversely.*

(See [12, p. 89] from which this easily follows.)

The following is a new proof of convergence for $\mathcal{S}_\omega^{(\pi)}$, for A symmetric and positive definite.

THEOREM 3.4. *If A is symmetric and positive definite and if η is an eigenvalue of $\mathcal{S}_\omega^{(\pi)}$, with $0 < \omega < 2$, then η is real and $0 \leqq \eta < 1$. If $\eta = 0$,* then $\omega = 1$.

Proof. Let

$$A = M_1 - M_2, \tag{3.14}$$

where

$$\begin{aligned} M_1 &= \frac{D(I - \omega L)(I - \omega U)}{\omega(2 - \omega)}, \\ M_2 &= \frac{D(\omega L + (1 - \omega)I)(\omega U + (1 - \omega)I)}{\omega(2 - \omega)} \\ &= \frac{(\omega E + (1 - \omega)D)D^{-1}(\omega F + (1 - \omega)D)}{\omega(2 - \omega)}. \end{aligned} \tag{3.15}$$

Note that M_1 is nonsingular. By direct substitution (3.14) can be verified. Also, one can show that $M_1^{-1}M_2 = \mathcal{S}_\omega^{(\pi)}$. By Lemma (3.3), $\eta = \nu/(1 + \nu)$, where ν is an eigenvalue of $A^{-1}M_2$.

Let $\widetilde{A^{-1}M_2} = D^{1/2}(A^{-1}M_2)D^{-1/2}$. One now considers the eigenvalues of $A^{-1}M_2$, and obtains for some vectors $x \neq 0$,

$$\frac{(\omega\tilde{L} + (1 - \omega)I)(\omega\tilde{U} + (1 - \omega)I)}{\omega(2 - \omega)}\,x = \nu D^{-1/2}AD^{-1/2}x. \tag{3.16}$$

By taking inner products with x, letting $(\omega\tilde{U} + (1 - \omega)I)x = w$ and dividing, we have

$$\nu = \frac{(w, w)}{\omega(2 - \omega)(AD^{-1/2}x,\, D^{-1/2}x)},$$

which is real and nonnegative. Thus $0 \leqq \eta = \nu/(1 + \nu) < 1$. If $w = 0$, we obtain $\omega = 1$.

COROLLARY 3.5.

$$S(\mathcal{S}_\omega^{(\pi)}) = \frac{S(A^{-1}M_2)}{1 + S(A^{-1}M_2)}.$$

This follows from the nondecreasing nature of $x/(1+x)$ for $x > 0$.

COROLLARY 3.6.

$$S(\mathcal{S}_\omega^{(\pi)}) \leqq \frac{S(A^{-1})S(M_2)}{1 + S(A^{-1})S(M_2)}.$$

Since M_2 is symmetric, the proof follows that of Varga [12, p. 90].

Unfortunately the bound on $S(\mathcal{S}_\omega^{(\pi)})$ above is poor for $\omega \neq 1$. Consequently, the result is of more academic than practical interest.

THEOREM 3.7. *Let A be symmetric and positive definite. There exists an optimum ω, say ω_0, in the sense that $S(\mathcal{S}_{\omega_0}^{(\pi)}) < S(\mathcal{S}_\omega^{(\pi)})$, $\omega \neq \omega_0$, for the* BSSOR *method.*

Starting with (3.16) written in the form $Mx = \bar{\nu}\bar{\bar{A}}x$, the proof of this theorem follows the variational approach of Habetler and Wachspress [7]. The final result in our notation is

$$\omega_0 = \frac{2}{1 + \sqrt{P_{\omega_0}}}, \tag{3.17}$$

where

$$P_\omega = 1 + 4\alpha^{(\omega)} - 2\beta^{(\omega)} > 0, \tag{3.18}$$

$$\alpha^{(\omega)} = \frac{(\tilde{L}\tilde{U}x, x)}{(x, x)}, \qquad \beta^{(\omega)} = \frac{(\tilde{B}x, x)}{(x, x)}. \tag{3.19}$$

In general, no explicit expression is known for computing ω_0 even in the simplest of cases (except for the case noted at end of the next section). However, in §5 we present an explicit expression for computing an ω for the case of the Laplace equation in a rectangular region, which appears to be very near ω_0, for the given π.

4. Comparisons with BSOR. Following Habetler and Wachspress [7], we have from (3.16)–(3.19),

$$S(\mathcal{S}_{\omega_0}^{(\pi)}) = \bar{\eta}_{\omega_0} = \frac{1 - (1-\beta)/\sqrt{P_{\omega_0}}}{1 + (1-\beta)/\sqrt{P_{\omega_0}}}, \tag{4.1}$$

where we drop the superscript on β. To bound P_{ω_0} we note that

$$0 \leqq \alpha \leqq S(\tilde{L}\tilde{U}), \qquad -S(\tilde{B}) \leqq \beta \leqq S(\tilde{B}). \tag{4.2}$$

Further since $\tilde{B}$ is symmetric (we assume A symmetric and positive definite), we have

$$S(\tilde{B}) = \| \tilde{B} \| = \| \tilde{L} + \tilde{U} \| \leqq \| \tilde{L} \| + \| \tilde{U} \| = 2\sqrt{S(\tilde{L}\tilde{U})}, \tag{4.3}$$

where $\| M \|$ is the spectral norm of the matrix M (see, e.g., [12]) and we have used $\tilde{L}^T = \tilde{U}$. It is not difficult to show, by considering the trace of the appropriate matrices and using the fact that (see, e.g., [8])

$$\lim_{p \to \infty} (\text{trace } M^p)^{1/p} = S(M),$$

where M is symmetric, positive semi-definite, that

$$S(\tilde{L}\tilde{U}) \leqq S(\tilde{B})^2 < 1. \tag{4.4}$$

(We note that Habetler and Wachspress [7] state that $\alpha \leqq 1$.) Hence

$$P_{\omega_0} \leqq (2\sqrt{S(\tilde{L}\tilde{U})} + 1)^2 \leqq (2S(\tilde{B}) + 1)^2 < 9. \tag{4.5}$$

(This yields the academically interesting result that $\frac{1}{2} < \omega_0 < 2$.)

In order to compare the BSOR and the BSSOR methods, we now suppose A has Property A^π and is consistently ordered (see [15]). Then it can be shown that there exists an ω, say ω_b, such that

$$1 < \omega_b = \frac{2}{1 + \sqrt{1 - [S(B)]^2}} < 2$$

and

$$S(\mathfrak{L}_{\omega_b}^{(\pi)}) = \omega_b - 1 < S(\mathfrak{L}_{\omega}^{(\pi)}), \quad \omega \neq \omega_b, \tag{4.6}$$

where $\mathfrak{L}_{\omega}^{(\pi)}$ is the BSOR iteration matrix.

LEMMA 4.1. *If A is symmetric and positive definite, satisfies Property A^π and is consistently ordered, then*

$$\lim_{\bar{\mu} \to 1^-} \frac{R(\mathfrak{S}_{\omega_0}^{(\pi)})}{R(\mathfrak{L}_{\omega_b}^{(\pi)})^2} \geqq \frac{1}{12},$$

where $\bar{\mu} = S(\tilde{B})$.

Proof. From (4.2) and (4.5),

$$\frac{1 - \beta}{\sqrt{P_{\omega_0}}} \geqq \frac{1 - \beta}{2\bar{\mu} + 1} \geqq \frac{1 - \bar{\mu}}{2\bar{\mu} + 1}.$$

From (4.1),

$$S(\mathfrak{S}_{\omega_0}^{(\pi)}) \leqq \frac{3\bar{\mu}}{\bar{\mu} + 2}.$$

Hence

$$\lim_{\bar{\mu}\to 1^-} \frac{R(\mathcal{S}_{\omega_0}^{(\pi)})}{R(\mathcal{L}_{\omega_b}^{(\pi)})^2} \geqq \lim_{\bar{\mu}\to 1^-} \frac{-\log \dfrac{3\bar{\mu}}{\bar{\mu}+2}}{\left[-2\log\left(\dfrac{\bar{\mu}}{1+\sqrt{1-\bar{\mu}^2}}\right)\right]^2} = \frac{1}{12},$$

by L'Hospital's Rule.

We now apply a semi-iterative method to $\mathcal{S}_{\omega_0}^{(\pi)}$, where we suppose the eigenvalues of $\mathcal{S}_{\omega_0}^{(\pi)}$ lie in the interval $[0, S(\mathcal{S}_{\omega_0}^{(\pi)})]$.

THEOREM 4.2. *Under the hypothesis of Lemma* 4.1,

$$\lim_{\bar{\mu}\to 1^-} \frac{R_\infty(\mathcal{S}_{\omega_0}^{(\pi)})}{R(\mathcal{L}_{\omega_b}^{(\pi)})} \geqq \frac{1}{\sqrt{3}}.$$

Proof. Using Theorem 2.1, we have

$$\lim_{\bar{\mu}\to 1^-} \frac{R_\infty(\mathcal{S}_{\omega_0}^{(\pi)})}{R(\mathcal{L}_{\omega_b}^{(\pi)})} = \lim_{\bar{\mu}\to 1^-} \frac{R_\infty(\mathcal{S}_{\omega_0}^{(\pi)})}{\sqrt{R(\mathcal{S}_{\omega_0}^{(\pi)})}} \frac{\sqrt{R(\mathcal{S}_{\omega_0}^{(\pi)})}}{R(\mathcal{L}_{\omega_b}^{(\pi)})} \geqq 2\sqrt{\frac{1}{12}} = \frac{1}{\sqrt{3}}.$$

This theorem affords a rigorous proof of the results of Habetler and Wachspress [7] for π_0.

THEOREM 4.3. *If A is symmetric and positive definite, satisfies Property A^π and is consistently ordered, and if $\alpha \leqq \beta/4$, where α and β are defined by* (3.19), *then*

$$\lim_{\bar{\mu}\to 1^-} \frac{R_\infty(\mathcal{S}_{\omega_0}^{(\pi)})}{\sqrt{R(\mathcal{L}_{\omega_b}^{(\pi)})}} \geqq 2^{3/4}.$$

We observe that $P_{\omega_0} = 1 + 4\alpha - 2\beta \leqq 1 - \beta$ and $(1-\beta)/\sqrt{P}\omega_0 \geqq \sqrt{1-\beta} \geqq \sqrt{1-\bar{\mu}}$. The rest of the proof follows that of Theorem 4.2.

Thus we see that when $\alpha \leqq \beta/4$, the BSSOR method with semi-iteration affords an order of magnitude improvement in convergence rate over the BSOR method. This condition can be slightly relaxed. It is easy to show that when $S(LU) \leqq \frac{1}{4}$, Theorem 4.3 holds with the $2^{3/4}$ factor on the right replaced by $\sqrt{2}$. In terms of finite difference approximations to elliptic partial differential equations, a sufficient condition for $S(LU) \leqq \frac{1}{4}$ is that the differential equation have constant coefficients and that equal space mesh is used [7]. Furthermore, unlike the BSOR method whose rate of convergence is unaffected by orderings, provided only that they are consistent [13], $S(\mathcal{S}_{\omega_0}^{(\pi)})$ is affected by the ordering, as is $S(LU)$. Numerical evidence indicates that $S(\mathcal{S}_{\omega_0}^{(\pi)})$ is smallest for the "normal" or "page-wise" ordering and is largest for the "red-black" or Property A^{π_0} ordering. D'Sylva and Miles [2]† show that for the "red-black" ordering, $\omega_0 = 1$

† The author is indebted to the referee for this reference. The result also is available in [9] and has been extended to general π in [3].

(this is the only ordering which has been explicitely analyzed) and $S(\mathfrak{L}_{\omega_0}^{(\pi)}) = \bar{\mu}^2$. Thus, for small mesh, for this ordering

$$\lim_{\bar{\mu}\to 1^-} \frac{R_\infty(\mathfrak{L}_1^{(\pi)})}{R(\mathfrak{L}_{\omega_b}^{(\pi)})} = 1,$$

and there is no gain using BSSOR. Hence, conditions for $S(LU) \leqq \frac{1}{4}$ must also include some comment on orderings.

5. A model problem. In this section we shall consider specifically the Laplace difference equation in a square with prescribed boundary values. We assume a uniform (though not necessarily equal) mesh size in each direction. Let $\Delta x = h = H^{-1}$, $\Delta y = k = K^{-1}$, H and K being positive integers $\geqq 2$. The region is shown in Fig. 1.

The Laplace difference equation can be written as

$$\begin{aligned}\alpha_1 u(x+h, y) + \alpha_2 u(x, y+k) + \alpha_3 u(x-h, y) \\ + \alpha_4 u(x, y-k) = \alpha_0 u(x, y),\end{aligned} \tag{5.1}$$

where the coefficients are constant and where

$$\alpha_0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\,, \tag{5.2}$$

with $u(0, y), u(1, y), u(x, 0), u(x, 1)$ given (see, e.g., [15]). We suppose the equations chosen in the ordering shown in Fig. 1.

We shall consider two partitions. The first will be the partition π_0 ; i.e., a point iterative method. If we let $u(x, y) = u(ih, jk) = u_{i,j}$, then the difference equations can be written as:

$$\pi_0 : \begin{cases} u_{i,j}^{(n+\frac{1}{2})} = \omega\{\beta_1 u_{i+1,j}^{(n)} + \beta_2 u_{i,j+1}^{(n)} + \beta_3 u_{i-1,j}^{(n+\frac{1}{2})} + \beta_4 u_{i,j-1}^{(n+\frac{1}{2})}\} \\ \qquad\qquad + (1-\omega) u_{i,j}^{(n)}, \\ u_{i,j}^{(n)} = \omega\{\beta_1 u_{i+1,j}^{(n+1)} + \beta_2 u_{i,j+1}^{(n+1)} + \beta_3 u_{i-1,j}^{(n+\frac{1}{2})} + \beta_4 u_{i,j-1}^{(n+\frac{1}{2})}\} \\ \qquad\qquad + (1-\omega) u_{i,j}^{(n+\frac{1}{2})}, \end{cases} \tag{5.3}$$

where $\beta_i = \alpha_i/\alpha_0$. The second partition will be noted by π_1 . We partition the matrix such that all points with constant y are to be solved simultaneously. This partition is frequently referred to as "line" or "row" iteration (see, e.g., [12]). The equations are:

$$\pi_1 : \begin{cases} \alpha_0 \bar{u}_{ij}^{(n+\frac{1}{2})} - \alpha_1 \bar{u}_{i+1,j}^{(n+\frac{1}{2})} - \alpha_3 \bar{u}_{i-1,j}^{(n+\frac{1}{2})} = \alpha_2 u_{i,j+1}^{(n)} + \alpha_4 u_{i,j-1}^{(n+\frac{1}{2})}, \\ u_{ij}^{(n+\frac{1}{2})} = \omega \bar{u}_{ij}^{(n+\frac{1}{2})} + (1-\omega) u_{ij}^{(n)}, \end{cases} \tag{5.4a}$$

$$\begin{cases} \alpha_0 \bar{u}_{ij}^{(n+1)} - \alpha_1 \bar{u}_{i+1,j}^{(n+1)} - \alpha_3 \bar{u}_{i-1,j}^{(n+1)} = \alpha_2 u_{i,j+1}^{(n+1)} + \alpha_4 u_{i,j-1}^{(n+\frac{1}{2})} \\ u_{ij}^{(n+1)} = \omega \bar{u}_{ij}^{(n+1)} + (1-\omega) u_{ij}^{(n+\frac{1}{2})}. \end{cases} \tag{5.4b}$$

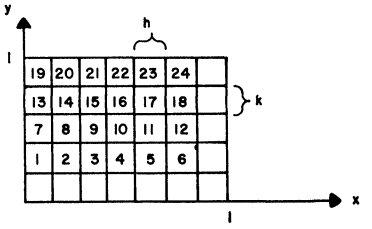

FIG. 1. *Mesh and ordering*

Equations (5.3) cannot be solved by separation of variables even for $h = k$, as was pointed out by Sheldon [10]. Nonetheless, we can bound the rate of convergence by using the results of §3 and §4.

For comparison purposes we consider here equal mesh size in both directions. It can be shown, then, that

$$S(LU) = S(\tilde{L}\tilde{U}) = \frac{1}{4}\cos^2\frac{\pi}{2(H-1)}, \tag{5.5}$$

(see, e.g., [3, Appendix B] and comments at end of §4). This is consistent with a result of Habetler and Wachspress [7] who state that α of (3.19) is equal to $\frac{1}{4} + O(\epsilon)$ where $\epsilon = 1 - S(B)$. Also, we have

$$S(B) = \cos \pi h, \tag{5.6}$$

(see, e.g., [15]). Thus $P_{\omega_0} = 1 + 4\alpha - 2\beta < 2(1-\beta)$, and from (4.1) and (4.2) we obtain

$$S(\mathfrak{L}_{\omega_0}^{(\pi_0)}) < \frac{1 - \sqrt{\frac{1-\beta}{2}}}{1 + \sqrt{\frac{1-\beta}{2}}} \leqq \frac{1 - \sqrt{\frac{1 - S(\tilde{B})}{2}}}{1 + \sqrt{\frac{1 - S(\tilde{B})}{2}}}.$$

For sufficiently small h, we have

$$S(\mathfrak{L}_{\omega_0}^{(\pi_0)}) \leqq 1 - \pi h + \frac{\pi^2 h^2}{2} - \frac{5\pi^3 h^3}{24} + O(h^4),$$

and

$$R)\mathfrak{L}_{\omega_0}^{(\pi_0)}) = -\log S(\mathfrak{L}_{\omega_0}^{(\pi_0)}) \geqq \pi h + \frac{\pi^3 h^3}{24} + O(h^4). \tag{5.7}$$

From (2.14) we obtain

$$R_\infty(\mathfrak{L}_{\omega_0}^{(\pi_0)}) \geqq 2\sqrt{\pi h} + \tfrac{1}{6}(\pi h)^{3/2} + O(h^{5/2}), \tag{5.8}$$

which is consistent with Theorem 2.1. By way of comparison, we note that

$$R(\mathfrak{L}_{\omega_b}^{(\pi_0)}) \sim 2\pi h,$$

(see, e.g., [15]) and

$$\frac{R_\infty(\mathcal{S}_{\omega_0}^{(\pi_0)})}{\sqrt{R(\mathcal{L}_{\omega_b}^{(\pi_0)})}} \geqq \sqrt{2} + \frac{1}{6\sqrt{2}}\pi h + O(h^2),$$

consistent with the remarks under Theorem 4.3.

We might observe at this point that if we consider a subregion of the unit square, the resulting matrix would satisfy the hypothesis of the Monotonicity Theorem 3.2 and we thus have $O(h^{1/2})$ rate of convergence in the subregion.

Unlike (5.3), (5.4) is separable. Since we are interested in $S(\mathcal{S}_{\omega_0}^{(\pi)})$, we shall seek the eigenvalues of $\mathcal{S}_{\omega_0}^{(\pi)}$ in particular. To this end let $e_{ij}^{(n)} = u_{ij}^{(n)} - u_{ij}$, where u_{ij} is the exact solution. Furthermore, we observe that $\alpha_1 = \alpha_3 > 0$ and $\alpha_2 = \alpha_4 > 0$ by the uniformity of the mesh size. By making the appropriate substitution and combining, we obtain

$$\begin{aligned}\alpha_0 e_{ij}^{(n+\frac{1}{2})} - \alpha_1 e_{i+1,j}^{(n+\frac{1}{2})} - \alpha_1 e_{i-1,j}^{(n+\frac{1}{2})} &= \omega\alpha_2(e_{i,j-1}^{(n+\frac{1}{2})} + e_{i,j+1}^{(n)}) \\ &\quad + (1-\omega)[\alpha_0 e_{ij}^{(n)} - \alpha_1(e_{i+1,j}^{(n)} + e_{i-1,j}^{(n)})],\end{aligned} \tag{5.9a}$$

$$\begin{aligned}\alpha_0 e_{ij}^{(n+1)} - \alpha_1 e_{i+1,j}^{(n+1)} - \alpha_1 e_{i-1,j}^{(n+1)} &= \omega\alpha_2(e_{i,j-1}^{(n+\frac{1}{2})} + e_{i,j+1}^{(n+1)}) \\ &\quad + (1-\omega)[\alpha_0 e_{ij}^{(n+\frac{1}{2})} - \alpha_1(e_{i+1,j}^{(n+\frac{1}{2})} + e_{i-1,j}^{(n+\frac{1}{2})})],\end{aligned} \tag{5.9b}$$

$$0 = e_{0,j} = e_{H,j} = e_{i,0} = e_{i,K};$$

$$i = 1, \cdots, H-1; j = 1, \cdots, K-1.$$

We now suppose $e_{ij}^{(n+1)} = \eta e_{ij}^{(n)}$, where η is an eigenvalue. Let $e_{ij}^{(n)} = I(i)\cdot\Phi(j,n)$. We note that $e_{ij}^{(n)} = I(i)\cdot\Phi(j,n+1) = \eta e_{ij}^{(n)} = \eta I(i)\cdot\Phi(j,n)$. Substitution into (5.9) leads to two systems of equations, namely

$$\begin{aligned}I(i+1) - \gamma_p I(i) + I(i-1) &= 0, \\ I(0) = I(H) &= 0, \\ i = 1, 2, \cdots, H-1; p &= 1, 2, \cdots, H-1;\end{aligned} \tag{5.10}$$

where γ_p is the separation constant, and

$$\begin{aligned}&\beta_p[\Phi(j,n+\tfrac{1}{2}) - (1-\omega)\Phi(j,n)] \\ &\qquad = \omega\alpha_2[\Phi(j-1,n+\tfrac{1}{2}) + \Phi(j+1,n)], \\ &\beta_p[\eta\Phi(j,n) - (1-\omega)\Phi(j,n+\tfrac{1}{2})] \\ &\qquad = \omega\alpha_2[\Phi(j-1,n+\tfrac{1}{2}) + \eta\Phi(j+1,n)], \\ &\Phi(0,n) = \Phi(K,n) = 0, \\ &\qquad n \geqq 0, j = 1, 2, \cdots, K-1,\end{aligned} \tag{5.11}$$

where

$$\beta_p = \alpha_0 - \alpha_1\gamma_p\,. \tag{5.12}$$

The solution to (5.10) is

$$\begin{aligned} I_p(i) &= C \sin \pi i p h, \qquad i, p = 1, 2, \cdots, H-1, \\ \gamma_p &= 2\cos \pi p h, \qquad\quad p = 1, 2, \cdots, H-1, \end{aligned} \tag{5.13}$$

where C is an arbitrary constant. Thus (5.12) becomes

$$\beta_p = \alpha_0 - 2\alpha_1 \cos \pi p h, \tag{5.14}$$

and we proceed to solve (5.11) for η. By formal manipulation we eliminate $\Phi(j, n + \frac{1}{2})$ (see, e.g., [3, Appendix D]) and, letting $\Phi(j, n) = \psi_j$, we obtain

$$\begin{aligned} \beta_p{}^2[\eta - (1-\omega)^2]\psi_1 &= \omega\alpha_2\beta_p(\eta + 1 - \omega)\psi_2\,, \\ [\beta_p{}^2(\eta - (1-\omega)^2) + \omega^2\alpha_2{}^2(\eta - 1)]\psi_j & \\ = \omega\alpha_2\beta_p(\eta + 1 - \omega)[\psi_{j-1} + \psi_{j+1}], & \\ j = 2, 3, \cdots, K-1; \qquad p = 1, 2, \cdots, H-1; & \\ \psi_K = 0. & \end{aligned} \tag{5.15}$$

Thus we are lead to the eigenvalues $\eta^{(q,p)}$ and eigenvectors $I_p(i)\psi_j{}^{(q,p)}$, $q = 1, 2, \cdots, K-1$; $p = 1, 2, \cdots, H-1$.

We state the following lemma and two theorems. Although not difficult, the proofs are long and "messy" and can be found in [3].

LEMMA 5.1. *If* $\omega = 1$, *then* $\eta = 0$ *is a simple eigenvalue of* (5.15) (*i.e., multiplicity* 1).

THEOREM 5.2. *If* $\eta = \omega - 1$, $\omega \neq 1$, *then* η *is an eigenvalue of* (5.15) *if and only if*

$$(1-\omega)\beta_p{}^2 + \omega^2\alpha_2{}^2 = 0 \tag{5.16}$$

for fixed p, *where* β_p *is given by* (5.14). *Further, if* η *is an eigenvalue, it is of multiplicity* $(K-2)$ *and the remaining eigenvalue associated with this value of* p *is less than* $\omega - 1$.

THEOREM 5.3. *If*

$$(1-\omega)\beta_p{}^2 + \omega^2\alpha_2{}^2 \neq 0 \tag{5.17}$$

for fixed p, *then the eigenvalues of* (5.15) *are given by*

$$\eta_j = \frac{(1-\omega)\beta_p{}^2 + \omega^2\alpha_2{}^2 + \omega(1-\omega)\alpha_2\beta_p\tau_j}{\beta_p{}^2 + \omega_2{}^2\alpha_2{}^2 - \omega\beta_p\alpha_2\tau_j}\,, \tag{5.18}$$

$$j = 1, \cdots, K-1,$$

where $f_{K-1}(\tau_j) = 0$ *and where* $f_t(\tau)$ *is defined by*

$$
\begin{aligned}
f_1(\tau) &= \beta_p^2(1-\omega)\tau + \omega^2\beta_p\alpha_2, \\
f_2(\tau) &= \beta_p^2(1-\omega)\tau^2 + \omega^2\beta_p\alpha_2\tau - [(1-\omega)\beta_p^2 + \omega^2\alpha_2^2], \\
f_t(\tau) &= \tau f_{t-1}(\tau) - f_{t-2}(\tau), \qquad t \geqq 3.
\end{aligned} \tag{5.19}
$$

The η_j *are all real and distinct.*

Thus in summary we obtain the eigenvalues and eigenvectors of the BSSOR method for the Laplace equation in a rectangle as follows:

For a fixed ω and each $p = 1, 2, \cdots, H-1$,

i) if $(1-\omega)\beta_p^2 + \omega^2\alpha_2^2 = 0$, we use Theorem 5.2 to obtain $(K-1)$ eigenvalues and $(K-1)$ associated linearly independent eigenvectors;

ii) if $(1-\omega)\beta_p^2 + \omega^2\alpha_2^2 \neq 0$, and $\omega \neq 1$, we use Theorem 5.3 to obtain $(K-1)$ eigenvalues and $(K-1)$ associated linearly independent eigenvectors;

iii) if $\omega = 1$, we find one eigenvalue (zero) from Lemma 5.1 and the remaining $(K-2)$ eigenvalues from Theorem 5.3 and also obtain $(K-1)$ associated linearly independent eigenvectors.

It can be shown that we obtain a set of $(H-1)\cdot(K-1)$ linearly independent eigenvectors, and thus all eigenvalues.

We are now in a position to choose an ω explicitly which appears to be near optimum. From the proof of Theorem 5.2 we can prove the following.

THEOREM 5.4. *Let* ω_1 *be the smaller root of* $(1-\omega_1)\beta_1^2 + \omega_1^2\alpha_2^2 = 0$. *Then* $S(\mathcal{S}_{\omega_1}^{(\pi_1)}) = \omega_1 - 1$, *where* $(\omega_1 - 1)$ *is an eigenvalue of multiplicity* $(K-2)$ *and where* $k = 1/K$ *is the mesh size in the y-direction.*

To compare results, we let $h = k$. This implies $\alpha_0 = 4$, $\alpha_1 = \alpha_3 = 1$. Thus $\beta_1 = 2(2 - \cos \pi h)$ and

$$
S(\mathcal{S}_{\omega_1}^{(\pi_1)}) = \frac{\beta_1^2 - \beta_1\sqrt{\beta_1^2 - 4}}{2} - 1.
$$

If h is small we have approximately $\beta_1 \sim 2 + \pi^2h^2$, $S(\mathcal{S}_{\omega_1}^{(\pi_1)}) \sim 1 - 2\pi h$, and $R(\mathcal{S}_{\omega_1}^{(\pi_1)}) = -\log S(\mathcal{S}_{\omega_1}^{(\pi_1)}) \sim 2\pi h$. By applying a semi-iterative technique, we obtain

$$
R_\infty(\mathcal{S}_{\omega_1}^{(\pi_1)}) \sim 2\sqrt{2\pi h}.
$$

For comparison we note that [15]

$$
R(\mathcal{L}_{\omega_b}^{(\pi_1)}) \sim 2\sqrt{2}\pi h.
$$

To see how ω_1 compares to ω_0, we observe that

$$
S(\tilde{L}\tilde{U}) = \frac{1}{16}\left(\frac{2}{2 - \cos \pi h}\right)^2 \leqq \frac{1}{4},
$$

which is easily shown and $S(\tilde{B}) = \cos \pi h/(2 - \cos \pi h)$ (see, e.g., [15]). Thus $P_{\omega_0} = 1 + 4\alpha - 2\beta < 2(1 - \beta)$ and

$$S(\mathfrak{L}_{\omega_0}^{(\pi_1)}) \leqq \frac{1 - \sqrt{\frac{1 - S(\tilde{B})}{2}}}{1 + \sqrt{\frac{1 - S(\tilde{B})}{2}}}.$$

For sufficiently small h, we have approximately

$$S(\mathfrak{L}_{\omega_0}^{(\pi_1)}) \leqq 1 - \sqrt{2}\pi h + \pi^2 h^2 + O(h^3),$$
$$R(\mathfrak{L}_{\omega_0}^{(\pi_1)}) \geqq \sqrt{2}\pi h + O(h^3),$$
$$R_\infty(\mathfrak{L}_{\omega_0}^{(\pi_1)}) \geqq 2^{5/4}\sqrt{\pi h} + O(h^{3/2}).$$

A comparison between the results for ω_0 and ω_1 indicates we could not hope to do much better. Indeed, the numerical results of the next section substantiate this.

6. Numerical results. In this section we present some numerical results obtained from a high-speed digital computer. We consider two equations, each solved over two regions. The equations are

$$u_{xx} + u_{yy} = 0,$$

$$u_{xx} + u_{yy} - \frac{u_y}{y} = 0.$$

These are replaced by the difference equations:

I: $$4u_{ij} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1},$$

II: $$4u_{ij} = u_{i+1,j} + u_{i-1,j} + \left(1 - \frac{1}{2j}\right) u_{i,j+1} + \left(1 + \frac{1}{2j}\right) u_{i,j-1}.$$

The regions considered are shown in Fig. 2.

Tables 1 to 4 summarize the results. All values were obtained on the Control Data 1604 Computer located in the Computation Center of the University of Texas, Austin, Texas. ω_0 was determined experimentally in all cases.

In Table 1 the effectiveness of ω_1 on π_1 is examined. It is seen that there is no distinction in the convergence of the iterative process using either ω_1 or ω_0, even for course mesh. In Table 2, ω_1 is used with π_0. Without semi-iteration, ω_1 is a good estimate of ω_0. With semi-iteration, one still obtains $O(h^{1/2})$ convergence but with more iterations. The reason for this is that the semi-iteration was performed assuming the eigenvalues were in the interval $[0, \omega_1 - 1]$ whereas they actually were in the interval $[0, S(\mathfrak{L}_{\omega_1}^{(\pi_0)})]$. Thus the best semi-iteration available was not used. If it had been, then ω_1

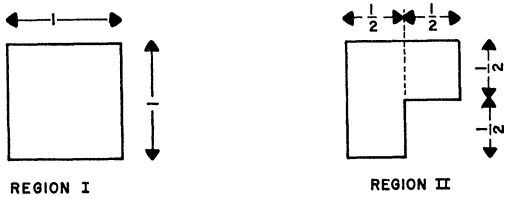

FIG. 2. *Regions*

TABLE 1

Comparison of ω_0 *and* ω_1 *for* π_1 *Region* I, *Equation* I

h^{-1}	ω_0	$S(\mathfrak{L}_{\omega_0}^{(\pi_1)})$	n^*	n_s†	ω_1	$S(\mathfrak{L}_{\omega_1}^{(\pi_1)})$	n	n_s
5	1.20	.2836	12	6	1.296	.296	12	7
10	1.53	.5342	23	9	1.536	.536	23	9
20	1.73	.7305	45	14	1.731	.731	45	14
40	1.85	.8547	89	19	1.855	.855	90	19
80	1.92	.9246	—	28	1.924	.924	—	28

* Number of iterations for convergence without semi-iteration.
† Number of iterations for convergence with semi-iteration.

TABLE 2

Comparison of ω_0 *and* ω_1 *for* π_0 *Region* I, *Equation* I

h^{-1}	ω_0	$S(\mathfrak{L}_{\omega_0}^{(\pi_0)})$	n	n_s	ω_1	$S(\mathfrak{L}_{\omega_1}^{(\pi_0)})$	n	n_s*
5	1.31	.3929	15	7	1.296	.396	16	11
10	1.57	.6492	33	11	1.536	.652	33	19
20	1.76	.8101	67	17	1.731	.813	68	29
40	1.87	.9012	134	24	1.855	.903	138	43
80	1.94	.9497	—	34	1.924	.951	—	61

* With $a = 0$, $b = \omega_1 - 1$.

TABLE 3

Number of iterations with semi-iteration

h^{-1}	Equation I				Equation II			
	Region I		Region II		Region I		Region II	
	π_0	π_1	π_0	π_1	π_0	π_1	π_0	π_1
10	19	9	12	10	16	9	11	9
20	29	14	19	13	24	14	16	14
40	43	19	28	19	34	21	22	19
80	61	28	40	27	48	29	31	29

TABLE 4
BSOR *method*
Equation I, *Region* I

h^{-1}	π_0		π_1	
	ω_b	n	ω_b	n
10	1.54	28	1.44	20
20	1.75	58	1.66	40
40	1.86	117	1.81	83
80	1.93	236	1.90	165

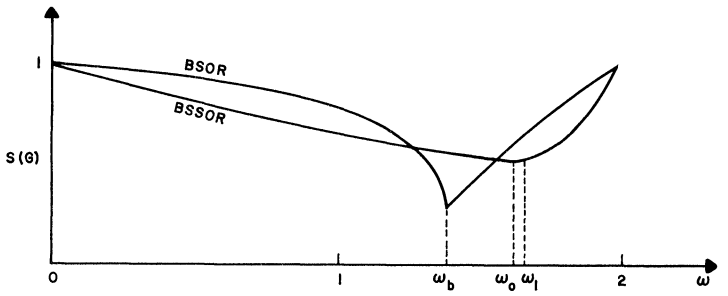

FIG. 3. ω *vs. spectral radius*

would have been competitive with ω_0 even for π_0, at the cost of knowing, computing, or assuming $S(\mathcal{S}_{\omega_1}^{(\pi_0)})$.

Table 3 shows the effect of a subregion on the method. The overall pattern for Region II is similar to that for Region I. Also, the Monotonicity Theorem is substantiated for π_0. However, the results for π_1 are not so sharp. Although the subregion did not take appreciably more iterations than the square, the subregion did not take appreciably less. A monotonicity theorem for general π seems possible, but it may represent a sharper bound than that for π_0.

Table 4 is presented for comparison. It indicates how effective the BSSOR method with semi-iteration can be. One might note, by comparing Table 4 with Tables 1 and 2, that the BSOR method is slightly more effective than the BSSOR method without semi-iteration. This has also been observed by Engeli, Ginsburg, Rutishauser and Stiefel [4]. See also Fig. 3.

7. Summary and conclusions. We note first our result on semi-iteration. Although Theorem 2.1 is an asymptotic result, the convergence is rapid. Thus, using relatively coarse mesh, the theorem can be applied with very good approximation, as the numerical experiments show.

The Monotonicity Theorem 3.2 tells us that the rate of convergence of the PSSOR (π_0) method is not increased by considering subregions of a given region. A proof for general π would be useful. We know from §5 then that for the Laplace difference equation in any subregion of a square using π_0, we have at least $O(h^{1/2})$ convergence. A monotonicity theorem for BSOR for *general* π can be proved in a similar manner [3]. (Varga [12] also has a proof using the Perron-Frobenius theory of nonnegative matrices.)

In general, the BSSOR method may not be an improvement over BSOR. However, under certain conditions, e.g., $S(LU) \leqq \frac{1}{4}$, one may gain an order of magnitude improvement. As noted at the end of §4, this condition depends not only on the differential equation and mesh spacing but also on the ordering of the equations. Indeed, for a region such as a triangle, the ordering could have appreciable effects on $S(LU)$. Further investigation is indicated.

Other than for the simple case of the "red-black" ordering, there is no explicit expression for ω_0. However, unlike the BSOR method where the estimation of ω_b is critical, the BSSOR method can entertain a relatively wide range of ω's about ω_0 without suffering appreciably. (Fig. 3 is representative.) This is partly brought out in Table 2, comparing $S(\mathcal{S}_{\omega_0}^{(\pi_0)})$ and $S(\mathcal{S}_{\omega_1}^{(\pi_0)})$. A more difficult problem is estimating $S(\mathcal{S}_{\omega_1}^{(\pi)})$ to obtain the best semi-iteration available. It might be possible using, say, ω_1, to perform several iterations to obtain an estimate of the spectral radius and then to start semi-iterating. Recently Evans and Forrington [5]† have given an iterative method for estimating ω_0 and $S(\mathcal{S}_{\omega_0}^{(\pi_0)})$ which is also extendable to general π.

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