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On semi-convergence of parameterized SHSS method for a class of singular complex symmetric linear systems☆

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In this paper, we use the parameterized single-step HSS (P-SHSS) iterative method to solve a broad class of singular complex symmetric linear systems. The semi-convergence properties of the P-SHSS method are derived under suitable conditions. Moreover, some properties of the preconditioned matrix and the optimal parameters are analyzed in detail. Numerical experiments are given to support our theoretical results and show the effectiveness of the P-SHSS method either as a solver or as a preconditioner.

1. Introduction

Consider the following complex system of linear equations

$$Ax \equiv (W + iT)x = b, \tag{1.1}$$

where $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices, $b \in \mathbb{C}^n$ is a given vector and $x \in \mathbb{C}^n$ is an unknown vector. Here and in the sequel, we use $i = \sqrt{-1}$ to denote the imaginary unit. Moreover, we assume $W, T \neq 0$, which implies that A is not a skew-Hermitian but non-Hermitian matrix.

Complex linear systems of the form (1.1) arise in a variety of scientific computing and engineering applications, such as diffuse optical tomography [1], FFT-based solution of certain time-dependent PDEs [2], structural dynamics [3], lattice quantum chromodynamics [4], and so on. For more applications of this class of problems, see [5–18] and references therein.

Based on the Hermitian and skew-Hermitian splitting $A = H + S$, where $H = \frac{1}{2}(A + A^*) = W$ and $S = \frac{1}{2}(A - A^*) = iT$ with A^* we denote the conjugate transpose of the matrix A, Bai et al. [19,20] established a class of the Hermitian and skew-Hermitian splitting (HSS) iterative methods for solving non-Hermitian linear systems. When W and T are both symmetric positive semi-definite matrices and at least one of them being positive definite, Bai et al. [21] designed the modified HSS (MHSS) iterative method. Furthermore, Bai et al. [22] discussed the preconditioned MHSS (PMHSS) iterative method, Zeng and Ma [23] established a parameterized variant of the SHSS (P-SHSS) iterative method. More efficient methods to solve nonsingular complex systems can be found in [24–26] and references therein.

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However, when W and T are symmetric positive semi-definite satisfying $null(W) \cap null(T) \neq \{0\}$, then the coefficient matrix A of (1.1) is singular. Therefore, many efficient iterative methods which are designed to solve nonsingular linear systems, could be also efficient for solving singular linear systems. For example, Bai [27] discussed the semi-convergence properties of the HSS method for solving singular, non-Hermitian and positive semi-definite linear systems, Chen and Liu [28], Wu and Li [29] further discussed the semi-convergence properties of the MHSS method for solving singular complex symmetric linear system (1.1). For more methods, see [30–32].

In this paper, we use P-SHSS method to solve singular complex symmetric linear system (1.1) and derive the semi-convergence conditions of the P-SHSS method.

The remainder of this paper is organized as follows. In Section 2, we review the P-SHSS method and its implementations for (1.1). The semi-convergence properties of the P-SHSS method, the spectral properties of the preconditioned matrix and the optimal parameters are discussed in Section 3. Numerical results show that the feasibility and the effectiveness of the P-SHSS method in Section 4. Finally, some concluding remarks are given in Section 5.

2. The P-SHSS method and its implementations

In this section, we review the standard form of the P-SHSS method as well as iterative-based resulting in an efficient P-SHSS preconditioner for (1.1).

According to [23], the P-SHSS method can be described as follows.

Algorithm 2.1 (*The P-SHSS Iterative Method*)**.** Given arbitrary initial guesses $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \dots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty} \in \mathbb{C}^n$ semi-converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$(\alpha I + \omega W + T)x^{(k+1)} = \big(\alpha I - i(\omega T - W)\big)x^{(k)} + (\omega - i)b, \tag{2.1}$$

where α and ω are given positive constants.

Then, the P-SHSS iterative scheme (2.1) can be reformulated into the following standard form

$$x^{(k+1)} = \mathcal{T}_{\alpha,\omega}x^{(k)} + M_{\alpha,\omega}^{-1}b,\ \ k = 0, 1, 2, \dots, \tag{2.2}$$

where

$$\mathcal{T}_{\alpha,\omega} = (\alpha I + \omega W + T)^{-1}\big(\alpha I - i(\omega T - W)\big) \tag{2.3}$$

is the iteration matrix of the P-SHSS method and

$$M_{\alpha,\omega} = \frac{\omega + i}{\omega^2 + 1}(\alpha I + \omega W + T).$$

As a matter of fact, the P-SHSS iterative scheme (2.1) comes also from the following splitting of the coefficient matrix

$$A = M_{\alpha,\omega} - N_{\alpha,\omega},$$

where

$$N_{\alpha,\omega} = \frac{\omega + i}{\omega^2 + 1}\big(\alpha I - i(\omega T - W)\big).$$

Notice that $\mathcal{T}_{\alpha,\omega} = M_{\alpha,\omega}^{-1}N_{\alpha,\omega}$ and the matrix $M_{\alpha,\omega}$ can be used as a preconditioner. Thus, the preconditioned system takes the following form

$$M_{\alpha,\omega}^{-1}Ax = M_{\alpha,\omega}^{-1}b.$$

In every step of the P-SHSS iterative scheme (2.1) or applying the preconditioner $M_{\alpha,\omega}$ to accelerate the convergence rate of Krylov subspace methods [33], it is required to solve a linear system with $M_{\alpha,\omega}$ as the coefficient matrix. In other words, it needs to solve a linear system with $\alpha I + \omega W + T$ as the coefficient matrix. Notice that $\alpha I + \omega W + T$ is a symmetric positive definite matrix, hence, it can be solved exactly by the Cholesky factorization or inexactly by the CG algorithm.

3. Semi-convergence analysis and preconditioning properties

In this section, we discuss the semi-convergence properties of the P-SHSS method and the spectral properties of the preconditioned matrix $M_{\alpha,\omega}^{-1}A$.

Since the matrix A is singular, the iteration matrix $\mathcal{T}_{\alpha,\omega}$ of the P-SHSS method has eigenvalue one, which means its spectral radius cannot be less than one. Based on [34,35], the P-SHSS method is semi-convergent if and only if the following two conditions are satisfied.

1. The elementary divisors of the iteration matrix $\mathcal{T}_{\alpha,\omega}$ associated with its eigenvalue $\lambda = 1$ are linear, i.e., $rank(I-\mathcal{T}_{\alpha,\omega}) = rank((I-\mathcal{T}_{\alpha,\omega})^2)$, or equivalently, $index(I-\mathcal{T}_{\alpha,\omega}) = 1$;
2. The pseudo-spectral radius of the iteration matrix $\mathcal{T}_{\alpha,\omega}$ is less than 1, i.e., $\vartheta(\mathcal{T}_{\alpha,\omega}) \equiv \max\{|\lambda| : \lambda \in \sigma(\mathcal{T}_{\alpha,\omega}), \lambda \neq 1\} < 1$, where $\vartheta(\mathcal{T}_{\alpha,\omega})$ is said to be the semi-convergence factor and $\sigma(\mathcal{T}_{\alpha,\omega})$ denotes the spectrum of $\mathcal{T}_{\alpha,\omega}$.

To obtain the semi-convergence of the P-SHSS method, we need the following lemma.

Lemma 3.1 ([27]). *Assume that $A = W + iT \in \mathbb{C}^{n\times n}$ is a singular matrix, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite matrices. Then*

$$null(A) = null(W) \cap null(T).$$

Based on Lemma 3.1, the sufficient and necessary conditions of the nonsingular matrix A can be described in the following lemma.

Lemma 3.2 ([22]). *Assume that $A = W + iT \in \mathbb{C}^{n\times n}$, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite matrices. Then the matrix A is nonsingular if and only if $null(W) \cap null(T) = \{0\}$.*

By Lemmas 3.1 and 3.2, we have the following conclusions.

Lemma 3.3 ([22]). *Assume that $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices satisfying $null(W) \cap null(T) = \{0\}$, and let ω be a positive constant. Then $\omega W + T$ and $\omega T - W$ be symmetric positive definite and symmetric, respectively.*

Lemma 3.4 ([36,37]). *Assume that $A \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:*

1. *$range(A) = range(A^*)$;*
2. *$null(A) = null(A^*)$;*
3. *There exists a unitary matrix U and nonsingular matrix $\tilde{A} \in \mathbb{C}^{r\times r}$, $r = rank(A)$, which satisfies*

$$A = U \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix} U^*,$$

According to Lemma 3.4, we obtain $index(I - \mathcal{T}_{\alpha,\omega}) = 1$ in the following lemma.

Lemma 3.5. *Assume that $A = W + iT \in \mathbb{C}^{n\times n}$ is a singular matrix, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite matrices. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ and nonsingular matrix $\tilde{A} \in \mathbb{C}^{r\times r}$, $r = rank(A)$, which satisfies*

$$A = U \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix} U^*,$$

where $\tilde{A} = \tilde{W} + i\tilde{T}$, and $\tilde{W}, \tilde{T} \in \mathbb{R}^{r\times r}$. Furthermore, for the iteration matrix $\mathcal{T}_{\alpha,\omega}$ of the P-SHSS method, it holds $index(I - \mathcal{T}_{\alpha,\omega}) = 1$.

Proof. By Lemma 3.4, we only need to prove $null(A) = null(A^*)$ (or $range(A) = range(A^*)$) holds. Suppose that $x \in null(A)$, i.e., $(W + iT)x = 0$, which means that $Wx = Tx = 0$. And we have $(W - iT)x = 0$, thus $x \in null(A^*)$, namely, $null(A) \subseteq null(A^*)$. Similarly, we can obtain $null(A^*) \subseteq null(A)$, namely, $null(A) = null(A^*)$ holds true. Therefore, it follows from Lemma 3.4 that there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A = U \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

It is deduced from this equation that the matrices W and T can be written as

$$W = U \begin{pmatrix} \tilde{W} & 0 \\ 0 & 0 \end{pmatrix} U^*, \quad T = U \begin{pmatrix} \tilde{T} & 0 \\ 0 & 0 \end{pmatrix} U^*,$$

where $\tilde{W}, \tilde{T} \in \mathbb{R}^{r\times r}$ and $\tilde{A} = \tilde{W} + i\tilde{T} \in \mathbb{C}^{r\times r}$ is a nonsingular matrix. According to Lemma 3.2, it holds $null(\tilde{W}) \cap null(\tilde{T}) = \{0\}$ with $\tilde{W} = \frac{1}{2}(\tilde{A} + \tilde{A}^*)$, $i\tilde{T} = \frac{1}{2}(\tilde{A} - \tilde{A}^*)$, yield

$$\mathcal{T}_{\alpha,\omega} = U \begin{pmatrix} \tilde{\mathcal{T}}_{\alpha,\omega} & 0 \\ 0 & I \end{pmatrix} U^*,$$

where $\tilde{\mathcal{T}}_{\alpha,\omega} = (\alpha I + \omega\tilde{W} + \tilde{T})^{-1}\big(\alpha I - (\omega\tilde{T} - \tilde{W})\big)$. Note that $\tilde{\mathcal{T}}_{\alpha,\omega}$ is the iteration matrix of the P-SHSS method for solving the nonsingular complex symmetric linear system $\tilde{A}\tilde{x} = \tilde{b}$, and we have

$$I - \mathcal{T}_{\alpha,\omega} = U\begin{pmatrix} I - \tilde{\mathcal{T}}_{\alpha,\omega} & 0 \\ 0 & 0 \end{pmatrix}U^*,$$

it is straightforward to see that $index(I - \mathcal{T}_{\alpha,\omega}) = 1$. Therefore, we complete the proof. ∎

Next, we will show that $\vartheta(\mathcal{T}_{\alpha,\omega}) < 1$, or equivalently $\rho(\tilde{\mathcal{T}}_{\alpha,\omega}) < 1$ (since $\vartheta(\mathcal{T}_{\alpha,\omega}) = \rho(\tilde{\mathcal{T}}_{\alpha,\omega})$). Let $\tilde{\lambda}$ be an eigenvalue of the matrix $\tilde{\mathcal{T}}_{\alpha,\omega}$ and $\tilde{x}$ be the corresponding eigenvector. Thus, we have $\tilde{\mathcal{T}}_{\alpha,\omega}\tilde{x} = \tilde{\lambda}\tilde{x}$ or equivalently

$$\big(\alpha I - i(\omega\tilde{T} - \tilde{W})\big)\tilde{x} = \tilde{\lambda}(\alpha I + \omega\tilde{W} + \tilde{T})\tilde{x}. \tag{3.1}$$

Then the semi-convergence properties of the P-SHSS method can be derived as follows.

Theorem 3.1. *Assume that $A = W + iT \in \mathbb{C}^{n\times n}$ is a singular matrix, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite matrices, and let α, ω be given positive constants. Let $\tilde{x}$ be an eigenvector of the matrix $\tilde{\mathcal{T}}_{\alpha,\omega}$ corresponding to the eigenvalue $\tilde{\lambda}$, then*

$$\tilde{\lambda} = \frac{\alpha - i\tilde{b}}{\alpha + \tilde{a}},$$

where

$$\tilde{a} = \frac{\tilde{x}^*(\omega\tilde{W} + \tilde{T})\tilde{x}}{\tilde{x}^*\tilde{x}}, \quad \tilde{b} = \frac{\tilde{x}^*(\omega\tilde{T} - \tilde{W})\tilde{x}}{\tilde{x}^*\tilde{x}}. \tag{3.2}$$

Furthermore, the P-SHSS method is semi-convergent if and only if the parameter α satisfies

$$\alpha > \max\left\{0, \frac{\tilde{b}^2 - \tilde{a}^2}{2\tilde{a}}\right\}. \tag{3.3}$$

Proof. Let $(\tilde{\lambda}, \tilde{x})$ be the eigenpair of the matrix $\tilde{\mathcal{T}}_{\alpha,\omega}$, and premultiplying Eq. (3.1) by $\tilde{x}^*$ yields

$$\alpha\tilde{x}^*\tilde{x} - i\tilde{x}(\omega\tilde{T} - \tilde{W})\tilde{x} = \tilde{\lambda}\big(\alpha\tilde{x}^*\tilde{x} + \tilde{x}^*(\omega\tilde{W} + \tilde{T})\tilde{x}\big).$$

Thus, it follows from Eq. (3.2) that

$$\tilde{\lambda} = \frac{\alpha - i\tilde{b}}{\alpha + \tilde{a}}.$$

Notice that $\omega\tilde{W} + \tilde{T}$ is a symmetric positive definite matrix, so $\tilde{a} > 0$. And after simple algebraic manipulations, we obtain the P-SHSS method is semi-convergent if and only if the parameter α satisfies (3.3). Thus, we complete the proof of Theorem 3.1. ∎

According to Lemma 3.3, we can denote

$$\tilde{\eta}_{\max} = \max_{\tilde{\eta}_j\in\sigma(\omega\tilde{W}+\tilde{T})}\{\tilde{\eta}_j\} = \max_{\eta_j\in\sigma(\omega W+T)}\{\eta_j\}, \tag{3.4}$$

$$\tilde{\eta}_{\min} = \min_{\tilde{\eta}_j\in\sigma(\omega\tilde{W}+\tilde{T})}\{\tilde{\eta}_j\} = \min_{\eta_j\in\sigma(\omega W+T)}\{\eta_j \setminus \{0\}\}, \tag{3.5}$$

$$\tilde{\mu}_{\max} = \max_{\tilde{\mu}_j\in\sigma(\omega\tilde{T}-\tilde{W})}\{|\tilde{\mu}_j|\} = \max_{\mu_j\in\sigma(\omega T-W)}\{|\mu_j|\}. \tag{3.6}$$

Then we derive the following practical semi-convergence lemma for the P-SHSS method.

Lemma 3.6. *Under the assumption of Theorem 3.1, the spectral radius $\rho(\tilde{\mathcal{T}}_{\alpha,\omega})$ of the matrix $\tilde{\mathcal{T}}_{\alpha,\omega}$ satisfying $\rho(\tilde{\mathcal{T}}_{\alpha,\omega}) \leqslant \delta_{\alpha,\omega}$, with*

$$\delta_{\alpha,\omega} = \frac{\sqrt{\alpha^2 + \tilde{\mu}_{\max}^2}}{\alpha + \tilde{\eta}_{\min}},$$

where $\tilde{\eta}_{\min}$ and $\tilde{\mu}_{\max}$ are defined as in (3.5)–(3.6). Furthermore, the P-SHSS method is semi-convergent if the parameter α satisfies

$$\alpha > \max\left\{0, \frac{\tilde{\mu}_{\max}^2 - \tilde{\eta}_{\min}^2}{2\tilde{\eta}_{\min}}\right\}. \tag{3.7}$$

Proof. By Theorem 3.1 and using the Courant–Fischer min–max theorem [38], we know that

$$\rho(\tilde{\mathcal{T}}_{\alpha,\omega}) = \max_{\tilde{a},\tilde{b}}\{|\tilde{\lambda}|\} = \max_{\tilde{a},\tilde{b}}\{\frac{\sqrt{\alpha^2+\tilde{b}^2}}{\alpha+\tilde{a}}\} \leqslant \frac{\sqrt{\alpha^2+\tilde{\mu}_{\max}^2}}{\alpha+\tilde{\eta}_{\min}} = \tilde{\delta}_{\alpha,\omega}.$$

Thus, the P-SHSS method is semi-convergent if the parameter α satisfies (3.7). ■

Moreover, in order to estimate the semi-convergence rate of the preconditioned Krylov subspace methods with respect to the P-SHSS preconditioner, we have the following clustering property of the eigenvalues of the preconditioned matrix $M_{\alpha,\omega}^{-1}A$.

Theorem 3.2. *Under the condition of Theorem 3.1, the preconditioned matrix $M_{\alpha,\omega}^{-1}A$ has an eigenvalue 0 with multiplicity $n-r$ and the remaining r eigenvalues are*

$$\tilde{\xi} = \frac{\tilde{a}+i\tilde{b}}{\alpha+\tilde{a}},$$

where $\tilde{a}$ and $\tilde{b}$ are defined as in (3.2). Furthermore, it holds

$$\frac{\tilde{\eta}_{\min}}{\alpha+\tilde{\eta}_{\min}} \leqslant \Re(\tilde{\xi}) \leqslant \frac{\tilde{\eta}_{\max}}{\alpha+\tilde{\eta}_{\max}} \quad \text{and} \quad \left|\Im(\tilde{\xi})\right| \leqslant \frac{\tilde{\mu}_{\max}}{\alpha+\tilde{\eta}_{\min}}, \tag{3.8}$$

where $\tilde{\eta}_{\min}$, $\tilde{\eta}_{\max}$ and $\tilde{\mu}_{\max}$ are defined as in (3.4)–(3.6).

Proof. Let ξ be the eigenvalue of the preconditioned matrix $M_{\alpha,\omega}^{-1}A$, according to Theorem 3.1, we know that the preconditioned matrix $M_{\alpha,\omega}^{-1}A$ has an eigenvalue 0 with multiplicity $n-r$ and the remaining r eigenvalues are

$$\tilde{\xi} = 1-\tilde{\lambda} = \frac{\tilde{a}+i\tilde{b}}{\alpha+\tilde{a}}.$$

It is easy to see that

$$\Re(\tilde{\xi}) = \frac{\tilde{a}}{\alpha+\tilde{a}} \quad \text{and} \quad \Im(\tilde{\xi}) = \frac{\tilde{b}}{\alpha+\tilde{a}}.$$

By making use of the Courant–Fischer min–max theorem [38] again, we obtain the results (3.8). ■

From Theorem 3.2, we know that the preconditioned matrix $M_{\alpha,\omega}^{-1}A$ has an eigenvalue 0 with multiplicity $n-r$, so we only need to discuss the asymptotic behavior of the eigenvalues $\tilde{\xi}$ of the preconditioned matrix $M_{\alpha,\omega}^{-1}A$ and obtain the following practical corollary.

Corollary 3.1. *Under the condition of Theorem 3.1, the asymptotic behavior of the eigenvalue $\tilde{\xi}$ of the preconditioned matrix $M_{\alpha,\omega}^{-1}A$ with respect to the variation of the parameters α and ω based on the following two cases.*

(1) If there exists a parameter ω such that $\tilde{\mu}_{\max}=0$, then $\Im(\tilde{\xi})=0$ and $\tilde{\xi}=\Re(\tilde{\xi})$, and it holds

$$\tilde{\xi} = \Re(\tilde{\xi}) \to 1_-, \quad \text{as} \quad \alpha \to 0_+;$$

(2) If there exists a parameter ω such that $\tilde{\mu}_{\max} \to 0_+$ or $\tilde{\eta}_{\min} \gg \tilde{\mu}_{\max}$, then $\frac{\tilde{\mu}_{\max}}{\tilde{\eta}_{\min}} \to 0_+$, and it holds

$$\Re(\tilde{\xi}) \to 1_- \quad \text{and} \quad \left|\Im(\tilde{\xi})\right| \leqslant \frac{\tilde{\mu}_{\max}}{\tilde{\eta}_{\min}} \to 0_+, \quad \text{as} \ \alpha \to 0_+.$$

Therefore, the spectrum of the preconditioned matrix $M_{\alpha,\omega}^{-1}A$ lies entirely in a circle centered at $(1,0)$ with respect to α is small enough and exists a parameter ω such that $\min_\omega\{\tilde{\mu}_{\max}\} \to 0$ or $\tilde{\eta}_{\min} \gg \tilde{\mu}_{\max}$.

Proof. According to the relationship (3.8) of Theorem 3.2, it is easy to obtain the results. ■

According to the algebraic estimation technique [39], we may expect that $M_{\alpha,\omega}$ is close to A as much as possible or $N_{\alpha,\omega} \approx 0$. If the expectation come true, the P-SHSS method will have fast convergence rates and the preconditioned matrix will have clustered eigenvalue distribution. More precisely, we need to minimize the function $\phi(\alpha,\omega) = \|N_{\alpha,\omega}\|_F^2$ with respect to α, ω. By direct computations, we have

$$\begin{aligned}
\phi(\alpha,\omega) &= \|N_{\alpha,\omega}\|_F^2 = tr(N_{\alpha,\omega}N_{\alpha,\omega}^*) \\
&= \frac{\alpha^2 tr(I_n) + tr((\omega T - W)^2)}{\omega^2+1} \\
&= \frac{\alpha^2 tr(I_n)}{\omega^2+1} + \frac{tr(W^2) - tr(T^2) - 2\omega tr(WT)}{\omega^2+1} + tr(T^2).
\end{aligned}$$

By taking the first-order derivative of $\phi(\alpha,\omega)$ and making use of the necessary condition for extreme value of a function, we have

$$\begin{cases} \dfrac{\partial\phi(\alpha,\omega)}{\partial\alpha} = \dfrac{2\alpha tr(I_n)}{\omega^2+1}, \\ \dfrac{\partial\phi(\alpha,\omega)}{\partial\omega} = \dfrac{2(\omega^2-1)tr(WT) - 2\omega\big(tr(W^2)-tr(T^2)\big) - 2\omega\alpha^2 tr(I_n)}{(\omega^2+1)^2}. \end{cases}$$

Note that the stationary points of $\phi(\alpha,\omega)$ are the roots of $\frac{\partial\phi(\alpha,\omega)}{\partial\alpha}=0$ and $\frac{\partial\phi(\alpha,\omega)}{\partial\omega}=0$. Then we conclude that $\alpha_* \to 0_+$ (since α is a positive constant) and ω^* satisfies the function

$$tr(WT)\omega^2 - \big(tr(W^2)-tr(T^2)\big)\omega - tr(WT) = 0.$$

After a simple algebraic manipulation, we obtain

$$\omega_* = \frac{tr(W^2)-tr(T^2)+\sqrt{\big(tr(W^2)-tr(T^2)\big)^2 + 4\big(tr(WT)\big)^2}}{2tr(WT)}.$$

4. Numerical experiments

In this section, we perform two numerical experiments to illustrate the effectiveness of the P-SHSS method for solving singular complex symmetric linear system (1.1). Moreover, we use left preconditioning with restarted GMRES(10) as the Krylov subspace method. We compare the P-SHSS method with the MHSS [28] and CMHSS [31] methods, and the corresponding preconditioners for the GMRES(10) method from point of view of the number of iteratives (denoted by "IT"), elapsed CPU time in seconds (denoted by "CPU"). In practical implementations, the initial guess is chosen to be zero vector and the stopping criteria for all methods are

$$RES := \frac{\|b-Ax^{(k)}\|_2}{\|b\|_2} < 10^{-6},$$

where $x^{(k)}$ be the current approximate solutions, or the maximum prescribed number of iterative $k_{max} = 600$. All the computations results are run in MATLAB 2017a [version 9.2.0.538062] on a personal computer with 3.20 GHz central processing unit (Intel(R) Core(TM) i5-6500 CPU) and 16.00G memory. In our experiments, the linear sub-systems involved in each step of the compared methods can be solved effectively by the sparse Cholesky factorization [40].

Example 1 (*[27,28]*)**.** Consider the singular linear systems $Ax=b$ with the coefficient matrix $A = W + iT \in \mathbb{C}^{n\times n}$ being given by

$$W = I\otimes V_c + V_c\otimes I \in \mathbb{R}^{n\times n}, \quad T = \frac{\theta}{2m}(I\otimes U_c + U_c\otimes I) \in \mathbb{R}^{n\times n},$$

where

$$\begin{aligned} V_c &= V - (e_1e_m^\top + e_me_1^\top) \in \mathbb{R}^{m\times m}, \\ U_c &= U - (e_1e_{m-1}^\top + e_{m-1}e_1^\top + e_ae_m^\top + e_me_a^\top) \in \mathbb{R}^{m\times m}, \end{aligned}$$

and

$$\begin{aligned} V &= tridiag(-1,2,-1) \in \mathbb{R}^{m\times m}, \\ U &= pentadiag(-1,-1,4,-1,-1) \in \mathbb{R}^{m\times m}, \\ e_1 &= (1,0,\ldots,0) \in \mathbb{R}^m, \\ e_{m-1} &= (0,\ldots,0,1,0) \in \mathbb{R}^m, \\ e_m &= (0,\ldots,1) \in \mathbb{R}^m, \\ e_a &= (1,1,0,\ldots,0) \in \mathbb{R}^m. \end{aligned}$$

The right-hand side vector b is defined as $b = Ax_*$, with $x_* = (1,2,\ldots,n)^\top \in \mathbb{R}^n$.

Example 2 (*[27,28]*)**.** Consider the singular linear system $Ax=b$ with the coefficient matrix $A = W + iT \in \mathbb{C}^{n\times n}$ being given by

$$W = tridiag(c_1,a_1,c_1) \in \mathbb{R}^{n\times n}, \quad T = \gamma\,(I\otimes V_c + V_c\otimes I) \in \mathbb{R}^{n\times n},$$

where

$$V_c = V - (e_1e_m^\top + e_me_1^\top) \in \mathbb{R}^{m\times m}$$

Table 1
The experimental optimal parameters of the proposed methods for Example 1.

	Method		θ			
			10^1	10^2	10^3	10^4
$m=32$	MHSS	α_{exp}	0.38	0.93	1.60	0.46
	CMHSS	α	0.38	0.93	1.60	0.46
		ω_{exp}	$1.9-0.1i$	$1.4-0.6i$	0.9-i	$1.0-0.9i$
	P-SHSS	α	0.01	0.01	0.01	0.01
		ω_*	3.53	0.32	0.032	0.0032
$m=48$	MHSS	α_{exp}	0.23	0.54	1.06	0.69
	CMHSS	α	0.23	0.54	1.06	0.69
		ω_{exp}	$2.1-0.1i$	$1.8-0.6i$	1-1.1i	$0.9-0.9i$
	P-SHSS	α	0.01	0.01	0.01	0.01
		ω_*	5.31	0.49	0.048	0.0048
$m=64$	MHSS	α_{exp}	0.17	0.33	0.83	1.08
	CMHSS	α	0.17	0.33	0.83	1.08
		ω_{exp}	$2.3-0.1i$	$2.1-0.3i$	$1.1-1.1i$	$0.9-0.9i$
	P-SHSS	α	0.01	0.01	0.01	0.01
		ω_*	7.10	0.66	0.064	0.0064

Table 2
Numerical results of different splitting iterative methods for Example 1.

	Method		θ			
			10^1	10^2	10^3	10^4
$m=32$	MHSS	IT	65	44	58	125
		CPU	1.49	1.55	1.33	2.22
	CMHSS	IT	33	24	19	57
		CPU	0.73	0.67	0.83	1.25
	P-SHSS	IT	13	10	4	3
		CPU	0.26	0.23	0.28	0.31
$m=48$	MHSS	IT	94	63	56	112
		CPU	10.88	7.44	6.65	13.42
	CMHSS	IT	44	29	25	41
		CPU	4.53	3.83	5.18	6.01
	P-SHSS	IT	10	11	4	3
		CPU	1.27	1.63	0.95	0.83
$m=64$	MHSS	IT	127	83	66	111
		CPU	47.27	31.21	24.86	36.59
	CMHSS	IT	55	36	26	26
		CPU	15.26	11.03	8.68	8.31
	P-SHSS	IT	8	12	5	3
		CPU	1.60	2.47	1.02	0.64

and

$$\begin{aligned}
V &= tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}, \\
e_1 &= (1, 0, \ldots, 0) \in \mathbb{R}^m, \\
e_m &= (0, \ldots, 1) \in \mathbb{R}^m, \\
a_1 &= (1, 3, 5, 7, \ldots, n-1) \in \mathbb{R}^n, \\
c_1 &= (-1, -2, -3, \ldots, -(n-1)) \in \mathbb{R}^{n-1}.
\end{aligned}$$

Table 3
Numerical results of different preconditioned GMRES methods for Example 1.

	Method		θ			
			10^1	10^2	10^3	10^4
$m = 32$	MHSS-GMRES(10)	IT	2(7)	2(8)	2(6)	3(2)
		CPU	1.31	0.53	0.33	1.50
	CMHSS-GMRES(10)	IT	2(9)	2(4)	1(10)	2(1)
		CPU	0.71	0.53	0.36	0.49
	P-SHSS-GMRES(10)	IT	2(1)	1(9)	1(3)	1(3)
		CPU	0.09	0.11	0.03	0.02
$m = 48$	MHSS-GMRES(10)	IT	3(3)	3(2)	2(10)	3(2)
		CPU	7.53	3.09	3.31	6.32
	CMHSS-GMRES(10)	IT	3(7)	2(7)	2(2)	2(2)
		CPU	5.72	2.83	1.57	1.85
	P-SHSS-GMRES(10)	IT	1(10)	1(10)	1(4)	1(3)
		CPU	0.33	0.46	0.13	0.08
$m = 64$	MHSS-GMRES(10)	IT	4(2)	3(4)	3(3)	2(9)
		CPU	13.86	11.12	9.34	8.19
	CMHSS-GMRES(10)	IT	4(7)	3(3)	2(4)	2(6)
		CPU	9.34	7.98	6.32	6.86
	P-SHSS-GMRES(10)	IT	1(9)	2(2)	1(5)	1(3)
		CPU	1.11	1.56	0.86	0.23

Table 4
Numerical results of different splitting iterative methods for Example 2.

	Method	α_{exp}	ω_*	IT	CPU
$m = 32$	MHSS	4254	–	138	13.64
	CMHSS	4254	$1.0 - 0.9i$	62	3.33
	P-SHSS	0.01	0.0254	4	0.58
$m = 48$	MHSS	2945	–	191	33.35
	CMHSS	2945	$1.0 - 0.9i$	90	15.83
	P-SHSS	0.01	0.0575	5	1.92
$m = 64$	MHSS	2321	–	242	121.08
	CMHSS	2321	$1.0 - 0.9i$	120	63.51
	P-SHSS	0.01	0.1027	7	11.01

The right-hand side vector b is defined as $b = Ax_*$, with $x_* = (1, 2, \ldots, n)^\top \in \mathbb{R}^n$. For the numerical tests we set $\gamma = 10^4$.

In Table 1, we list the optimal parameters (see [31]) of the MHSS and CMHSS methods for Example 1, which are found experimentally. For simplicity, we fix $\alpha = 0.01$ in the P-SHSS method and choose ω_* according to the last part of Section 3. The numerical results about IT and CPU of the tested methods (the iterative methods and the preconditioned-GMRES(10) methods) with respect to different problem sizes are listed in Tables 2 and 3. To better understand the numerical results of Table 3, Fig. 1 shows the eigenvalues distribution of the corresponding preconditioned matrices with $m = 48$ and $\theta = 10^2$.

Using the same strategy in Example 1, we list the numerical results about IT and CPU of the tested methods for Example 2 in Tables 4 and 5. In addition, we plot the eigenvalues distribution of the corresponding preconditioned matrices for Example 2 in Fig. 2.

From the numerical results, it is easy to see that the P-SHSS method as well as the corresponding preconditioner needs less than iterative steps and CPU times to achieve the stopping criterion. Moreover, the eigenvalues distribution of preconditioned matrix $M_{\alpha,\omega}^{-1}A$ is quite clustered according with theoretical analysis. In other words, the numerical results show the correctness of theoretical analyses and the effectiveness of the proposed methods either as a solver or as a preconditioner for solving singular complex symmetric linear system (1.1).

Table 5
Numerical results of different preconditioned GMRES methods for Example 2.

	Method	α_{exp}	ω_*	IT	CPU
$m = 32$	MHSS-GMRES(10)	4254	–	2(10)	2.76
	CMHSS-GMRES(10)	4254	$1.0 - 0.9$i	1(6)	1.02
	P-SHSS-GMRES(10)	0.01	0.0254	1(2)	0.05
$m = 48$	MHSS-GMRES(10)	2945	–	3(3)	12.23
	CMHSS-GMRES(10)	2945	$1.0 - 0.9$i	1(7)	6.63
	P-SHSS-GMRES(10)	0.01	0.0575	1(2)	0.88
$m = 64$	MHSS-GMRES(10)	2321	–	3(8)	52.26
	CMHSS-GMRES(10)	2321	$1.0 - 0.9$i	1(8)	19.32
	P-SHSS-GMRES(10)	0.01	0.1027	1(2)	2.13

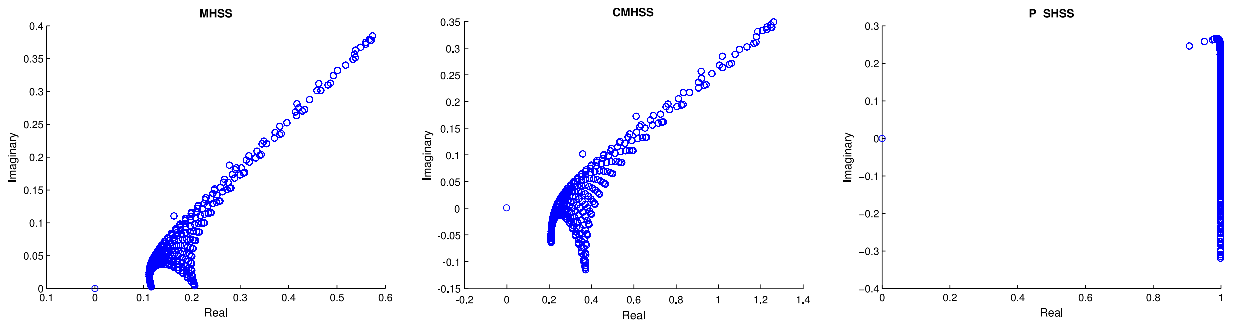

Fig. 1. Eigenvalues distribution of different preconditioned matrices for Example 1 with $m = 48$ and $\theta = 10^2$.

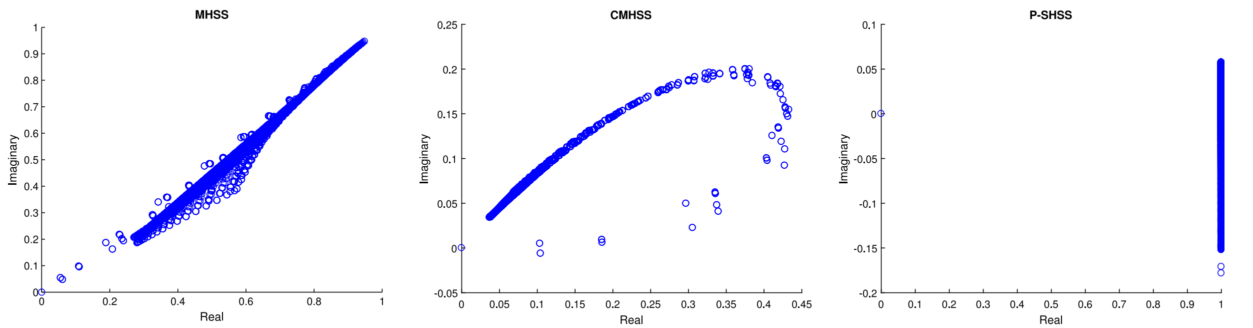

Fig. 2. Eigenvalues distribution of different preconditioned matrices for Example 2 with $m = 48$.

5. Conclusions

In this paper, we discussed the semi-convergence of the P-SHSS method for solving singular complex symmetric linear system (1.1). In addition, some spectral properties of the preconditioned matrix and the optimal parameters are also derived in detail. Numerical results show that the effectiveness of the P-SHSS method either as a solver or as a preconditioner.

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