APPLICATIONS OF M-MATRICES TO NON-NEGATIVE MATRICES

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1. A matrix $B = (b_{ij})$ of order n is called non-negative (written $B \geq 0$) if each element of B is a non-negative number. If each element is positive, then B is called a positive matrix (written $B > 0$). Generalizing some theorems of Perron [14] on positive matrices, Frobenius [7], [8], [9] showed that every non-negative matrix B has a non-negative characteristic root $p(B)$ (the *Perron root* of B) such that each characteristic root β of B satisfies $|\beta| \leq p(B)$.

Since the publication of the results of Perron and Frobenius, the problem of finding estimates for $p(B)$ has been studied extensively. For a history of the problem, see Brauer [1], [2] and Taussky [15]. Some well-known results in this area are as follows (see e.g. [10]). Let $B \geq 0$, and for $i = 1, 2, \cdots, n$ let $R_i(B)$ denote the sum of the elements in row i. Let $R(B)$ be the largest, and $r(B)$ the smallest, of the $R_i(B)$. Then

$$r(B) \leq p(B) \leq R(B). \tag{1}$$

Also

$$p(B) \geq b_{ii} \qquad (i = 1, 2, \cdots, n). \tag{2}$$

If each of B and C is a matrix of order n, we write $B \geq C$ to mean $(B - C) \geq 0$.

$$\text{If} \quad B \geq C \geq 0, \quad \text{then} \quad p(B) \geq p(C). \tag{3}$$

Finally,

$$\text{if } B' \text{ is a principal submatrix of } B, \text{ then } p(B) \geq p(B'). \tag{4}$$

Closely related to non-negative matrices is a class of matrices called M-matrices. A square matrix A is called an M-matrix if it has the form $kI - B$, where B is a non-negative matrix, $k > p(B)$, and I denotes the identity matrix. Ostrowski [12], [13] first studied M-matrices, and they have since been investigated by Fan [4], [5] and Fiedler and Pták [6]. In case A is a real, square matrix with non-positive off-diagonal elements, each of the following is a necessary and sufficient condition for A to be an M-matrix (see e.g. [6]).

$$\text{Each principal minor of } A \text{ is positive.} \tag{5}$$

$$A \text{ is non-singular, and } A^{-1} \geq 0. \tag{6}$$

$$\text{Each real characteristic root of } A \text{ is positive.} \tag{7}$$

Ky Fan [4] proved the following lemma.

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LEMMA A. *Let* $A = (a_{ij})$ *be an M-matrix of order n. Then the matrix* $C = (c_{ij})$ *given by*

$$c_{ij} = a_{ij} - a_{in}a_{nj}(1/a_{nn}) \qquad (i, j = 1, 2, \cdots, n-1)$$

is an M-matrix, and

$$c_{ij} \leq a_{ij} \qquad (i, j = 1, 2, \cdots, n-1).$$

In §2 we generalize Lemma A in order to get new bounds for the Perron root of a non-negative matrix. The bounds obtained improve (2) and (4). Also, in §3, we improve (1) by showing that if B is a non-negative matrix and $k > p(B)$, then

$$k - \frac{1}{r((kI - B)^{-1})} \leq p(B) \leq k - \frac{1}{R((kI - B)^{-1})}. \tag{8}$$

Throughout this paper, the theorems involving various sums along the rows of a matrix have obvious analogues using the columns.

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2. We denote the submatrix of A formed using rows $i_1, i_2, \cdots, i_p$ and columns $j_1, j_2, \cdots, j_p$ by $A(i_1, \cdots, i_p; j_1, \cdots, j_p)$. For principal submatrices we abbreviate this notation to $A(i_1, i_2, \cdots, i_p)$. $A_{i,j}$ denotes the element of A in row i and column j.

THEOREM 1. *Let* $B = (b_{ij})$ *be an* $n \times n$ *non-negative matrix,* $n \geq 2$, *with Perron root* $p(B)$. *If* $k > p(B)$ *and t is an integer such that* $0 \leq t \leq n-2$, *we define (using the Kronecker symbol* δ_{ij}*) a matrix* $D(k, t)$ *by*

$$D(k, t)_{i,j} = k\delta_{ij} - \frac{\det (kI - B)(i, n-t, \cdots, n; j, n-t, \cdots, n)}{\det (kI - B)(n-t, \cdots, n)} \qquad (i, j = 1, 2, \cdots, n-t-1).$$

Then

$$D(k, t) \geq 0, \tag{9}$$

$$D(k, t)_{i,j} \geq b_{ij} \qquad (i, j = 1, 2, \cdots, n-t-1), \tag{10}$$

and

$$p(B) \geq p(D(k, t)). \tag{11}$$

The proof of Theorem 1 depends on two lemmas, the first of which generalizes Lemma A.

LEMMA 1. *Let* $A = (a_{ij})$ *be an M-matrix of order n,* $n \geq 2$, *and let t be an integer such that* $0 \leq t \leq n-2$. *Then the matrix* $C = (c_{ij})$ *given by*

$$c_{ij} = \frac{\det A(i, n-t, \cdots, n; j, n-t, \cdots, n)}{\det A(n-t, \cdots, n)} \qquad (i, j = 1, 2, \cdots, n-t-1),$$

is an M-matrix, and

$$c_{ij} \leq a_{ij} \qquad (i, j = 1, 2, \cdots, n - t - 1). \tag{12}$$

Proof. Let $E = A(n - t, \cdots, n)$ and define a matrix L of order $n - t - 1$ by

$$L_{i,j} = \det A(i, n - t, \cdots, n; j, n - t, \cdots, n) \qquad (i, j = 1, 2, \cdots, n - t - 1).$$

Applying Sylvester's identity (see e.g. [11; 16]) to L, we see that if $1 \leq i_1 < i_2 < \cdots < i_s \leq n - t - 1$, then

$$\det L(i_1, i_2, \cdots, i_s) = (\det E)^{s-1} \det A(i_1, i_2, \cdots, i_s, n - t, \cdots, n).$$

Thus

$$\det C(i_1, i_2, \cdots, i_s) = (\det E)^{-1} \det A(i_1, i_2, \cdots, i_s, n - t, \cdots, n).$$

Hence by (5) we know that

$$\det C(i_1, i_2, \cdots, i_s) > 0,$$

i.e. each principal minor of C is positive.

In order to show that each off-diagonal element of C is non-positive, we prove (12). Expanding the determinant in the numerator of the expression for c_{ij}, we get

$$c_{ij} = a_{ij} + (\det E)^{-1} \sum_{v=n-t}^{n} a_{iv} X(a_{iv}) \tag{13}$$

where $X(a_{iv})$ denotes the cofactor of a_{iv} in the determinant of the matrix $A(i, n - t, \cdots, n; j, n - t, \cdots, n)$. Clearly $X(a_{iv})$ is identical with the cofactor $Y(a_{iv})$ of a_{iv} in $\det A(j, n - t, \cdots, n)$. Since $A(j, n - t, \cdots, n)$ is itself an M-matrix, it has a non-negative inverse. Hence

$$Y(a_{iv})/\det A(j, n - t, \cdots, n) \geq 0 \qquad (v = n - t, \cdots, n),$$

and thus, by (5), $Y(a_{iv}) \geq 0 (v = n - t, \cdots, n)$. So $X(a_{iv}) \geq 0$ for $v = n - t, \cdots, n$. Since also $\det E$ is positive and $a_{iv} \leq 0$ for $v = n - t, \cdots, n$, we see that (12) follows from (13). By (5), the lemma is proved.

LEMMA 2. *Let $B = (b_{ij})$ and t be given as in Theorem* 1. *Let k and h satisfy $k > h > p(B)$. Then $D(h, t) \geq D(k, t)$.*

Proof. Denote the matrix $(kI - B)(n - t, \cdots, n)$ by $E(k)$, and denote the matrix $(hI - B)(n - t, \cdots, n)$ by $E(h)$. Then

$$D(k, t)_{i,j} = k\delta_{ij} - (\det E(k))^{-1} \begin{vmatrix} k\delta_{ij} - b_{i,j} & -b_{i,n-t} \cdots -b_{i,n} \\ -b_{n-t,j} & \\ \vdots & E(k) \\ -b_{n,j} & \end{vmatrix}$$

$$= (\det E(k))^{-1} \begin{vmatrix} b_{i,j} & b_{i,n-t} \cdots b_{i,n} \\ -b_{n-t,j} & \\ \vdots & E(k) \\ -b_{n,j} & \end{vmatrix} \tag{14}$$

Denote the cofactor of b_{iu} in the determinant in the numerator of (14) by $X(b_{iu})$, for $u = n - t, \cdots, n$. Also, let $Y(b_{iu})$ be the cofactor of b_{iu} in the analogous determinant where k is replaced by h. Then we have, for $i, j = 1, 2, \cdots, n - t - 1$,

$$D(k, t)_{i,j} = b_{ij} + \sum_{u=n-t}^{n} b_{iu} X(b_{iu})(\det E(k))^{-1} \tag{15}$$

and

$$D(h, t)_{i,j} = b_{ij} + \sum_{u=n-t}^{n} b_{iu} Y(b_{iu})(\det E(h))^{-1}. \tag{16}$$

Hence the proof will be completed if we show, for $u = n - t, \cdots, n$, that

$$X(b_{iu})(\det E(k))^{-1} \leq Y(b_{iu})(\det E(h))^{-1}. \tag{17}$$

Now for each $u = n - t, \cdots, n$,

$$X(b_{iu}) = (-1)^{u-n+t+3} \begin{vmatrix} -b_{n-t,j} & \\ \vdots & E(k)_u \\ -b_{n,j} & \end{vmatrix}$$

where $E(k)_u$ denotes the $t + 1 \times t$ matrix obtained from $E(k)$ by omitting column number $u - n + t + 1$. Multiply each element in the first column of the above determinant by -1, and interchange adjacent columns in such a way that the column

$$\begin{bmatrix} b_{n-t,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \tag{18}$$

replaces the "missing" column of the matrix $E(k)$. This requires $u - n + t$ such interchanges, so that

$$X(b_{iu}) = (-1)^{u-n+t+3}(-1)(-1)^{u-n+t} \det F(k)_u,$$

that is

$$X(b_{iu}) = \det F(k)_u \qquad (u = n - t, \cdots, n), \tag{19}$$

where $F(k)_u$ is identical with the matrix $E(k)$ except that the column (18) has replaced column $u - n + t + 1$.

Repeating this process with k replaced by h, we have

$$Y(b_{iu}) = \det F(h)_u \qquad (u = n - t, \cdots, n), \tag{20}$$

where $F(h)_u$ agrees with $E(h)$ except that the column (18) has replaced column $u - n + t + 1$. So to verify (17) we show that

$$\det F(k)_u(\det E(k))^{-1} \leq \det F(h)_u(\det E(h))^{-1} \tag{21}$$

for $u = n - t, \cdots, n$.

It is clear from the definition of $F(k)_u$ that for $m = n - t, \cdots, n$ the cofactor of the element b_{mj} in $F(k)_u$ is precisely the cofactor of $E(k)_{m-n+t+1,u-n+t+1}$ in the matrix $E(k)$. Denote this cofactor by $Z(b_{mj}) = Z(E(k)_{m-n+t+1,u-n+t+1})$. Using primes to indicate the replacement of k with h, we have also $Z'(b_{mj}) = Z'(E(h)_{m-n+t+1,u-n+t+1})$. Expanding $\det F(k)_u$ and $\det F(h)_u$ along the column (18), we obtain

$$\frac{\det F(k)_u}{\det E(k)} = \sum_{r=n-t}^{n} b_{rj} \frac{Z(E(k)_{r-n+t+1,\, u-n+t+1})}{\det E(k)} \qquad (u = n - t, \cdots, n). \tag{22}$$

So

$$\frac{\det F(k)_u}{\det E(k)} = \sum_{r=n-t}^{n} b_{rj}(E(k))^{-1}_{u-n+t+1,\, r-n+t+1} \qquad (u = n - t, \cdots, n). \tag{23}$$

Similarly,

$$\frac{\det F(h)_u}{\det E(h)} = \sum_{r=n-t}^{n} b_{rj}(E(h))^{-1}_{u-n+t+1,\, r-n+t+1} \qquad (u = n - t, \cdots, n). \tag{24}$$

Now since $k > p(B)$ and $h > p(B)$, each of $E(k)$ and $E(h)$ is an M-matrix. Also, since $k > h$, it follows that $E(k) \geq E(h)$. So according to a theorem of Ostrowski [12, Theorem III]

$$(E(k))^{-1} \leq (E(h))^{-1}.$$

Since B is non-negative, it follows from (23) and (24) that (21) holds, completing the proof of Lemma 2.

Proof of Theorem 1. Let $A = kI - B$ and let $C = kI - D(k, t)$. Since $k > p(B)$, we know that A is an M-matrix. Thus, by Lemma 1, C is an M-matrix and

$$c_{ij} \leq a_{ij} \qquad (i, j = 1, 2, \cdots, n - t - 1).$$

So

$$\begin{aligned} D(k, t)_{i,j} &= k\delta_{ij} - c_{ij} \\ &\geq k\delta_{ij} - a_{ij} \\ &= b_{ij} \geq 0 \qquad (i, j = 1, 2, \cdots, n - t - 1). \end{aligned}$$

This verifies (9) and (10).

Since C is an M-matrix and $k - p(D(k, t))$ is a real characteristic root of C, it follows from (7) that

$$k > p(D(k, t)).$$

Similarly, if $h > p(B)$, then

$$h > p(D(h, t)). \tag{25}$$

Also, if $k > h > p(B)$, then Lemma 2 and (3) imply

$$p(D(h, t)) \geq p(D(k, t)). \tag{26}$$

Combining (25) and (26), we see that if $k > h > p(B)$, then

$$h > p(D(k, t)).$$

Hence

$$p(B) \geq p(D(k, t)),$$

and the proof is complete.

Any of the various known lower bounds for the Perron root of a non-negative matrix may now be applied to $D(k, t)$ to yield, in accordance with Theorem 1, new lower bounds for $p(B)$. For instance, if we use (2), we obtain the following bound:

THEOREM 2. *For B, k, and t as in Theorem* 1,

$$p(B) \geq k - \frac{\det (kI - B)(i, n - t, \cdots, n)}{\det (kI - B)(n - t, \cdots, n)} \qquad (i = 1, 2, \cdots, n - t - 1). \tag{27}$$

COROLLARY. *If $B = (b_{ij})$ is a non-negative matrix of order n, and if $k > p(B)$, then*

$$p(B) \geq b_{ii} + \frac{b_{in}b_{ni}}{k - b_{nn}} \qquad (i = 1, 2, \cdots, n - 1).$$

Performing the same permutation on the rows and the columns of B, we may obtain a matrix similar to B with the element in the first row and first column maximal among the main diagonal elements. Such a similarity transformation does not change the characteristic roots. Thus we may assume without loss of generality that

$$b_{11} = \max_{i=1,2,\cdots,n} (b_{ii}).$$

Therefore it follows from (10) that (27) improves (2).

Let B' be a principal submatrix of order q of B. By considerations like those just mentioned, we may assume that $B' = B(1, 2, \cdots, q)$. Then

$$D(k, n - q - 1) \geq B'.$$

So by (3) we see that Theorem 1 (11) improves (4). Similarly, by Lemma 2 and (3), we see that (11) improves as the upper bound k for $p(B)$ is improved.

Next we show that for a given matrix $B \geq 0$ and a given number $k > p(B)$, the bound obtained in Theorem 2 improves as the integer t increases. Keeping in mind that we may permute the rows and columns of a matrix without changing its characteristic roots, we see that this improvement is shown in the following theorem.

THEOREM 3. *Let $B = (b_{ij})$ be an $n \times n$ non-negative matrix, $n \geq 3$, and suppose that $0 \leq t \leq n - 3$. Then if $k > p(B)$,*

$$D(k, t)_{i,j} \leq D(k, t + 1)_{i,j} \qquad (i, j = 1, 2, \cdots, n - t - 2). \tag{28}$$

Proof. Let $E(t) = (kI - B)(n - t, \cdots, n)$ and let $E(t + 1) = (kI - B)(n - t - 1, \cdots, n)$. As in the proof of Lemma 2, we have

$$D(k, t)_{i,j} = (\det E(t))^{-1} \begin{vmatrix} b_{ij} & b_{i,n-t} \cdots b_{in} \\ -b_{n-t,j} & \\ \vdots & E(t) \\ -b_{nj} & \end{vmatrix}$$

and

$$D(k, t + 1)_{i,j} = (\det E(t + 1))^{-1} \begin{vmatrix} b_{ij} & b_{i,n-t-1} \cdots b_{in} \\ -b_{n-t-1,j} & \\ \vdots & E(t + 1) \\ -b_{nj} & \end{vmatrix}$$

for $i, j = 1, 2, \cdots, n - t - 2$. Let X denote cofactors of elements in the determinant in the numerator of the expression for $D(k, t)_{i,j}$, and let Y be defined analogously for $D(k, t + 1)_{i,j}$. Then

$$D(k, t)_{i,j} = b_{ij} + \sum_{v=n-t}^{n} b_{iv} X(b_{iv})(\det E(t))^{-1} \tag{29}$$

and

$$D(k, t + 1)_{i,j} = b_{ij} + \sum_{v=n-t-1}^{n} b_{iv} Y(b_{iv})(\det E(t + 1))^{-1} \tag{30}$$

for $i, j = 1, 2, \cdots, n - t - 2$. Now $Y(b_{i,n-t-1})$ is identical with the cofactor of $-b_{j,n-t-1}$ in the M-matrix $(kI - B)(j, n - t - 1, \cdots, n)$. As we have seen earlier, this implies that

$$Y(b_{i,n-t-1}) \geq 0.$$

Also, since $E(t + 1)$ is an M-matrix, its determinant is positive. So since $B \geq 0$, it follows that

$$b_{i,n-t-1} Y(b_{i,n-t-1})(\det E(t + 1))^{-1} \geq 0.$$

Thus in order to verify the inequality (28), it suffices to show that for $v = n - t, \cdots, n$,

$$X(b_{iv})(\det E(t))^{-1} \leq Y(b_{iv})(\det E(t+1))^{-1}. \tag{31}$$

Following the method used to prove Lemma 2, we see that

$$\frac{X(b_{iv})}{\det E(t)} = \sum_{r=n-t}^{n} b_{rj} E(t)^{-1}_{v-n+t+1,\, r-n+t+1} \qquad (v = n - t, \cdots, n) \tag{32}$$

and

$$\frac{Y(b_{iv})}{\det E(t+1)} = \sum_{r=n-t-1}^{n} b_{rj} E(t+1)^{-1}_{v-n+t+2,\, r-n+t+2} \quad (v = n - t, \cdots, n). \tag{33}$$

Since $E(t+1)$ is an M-matrix, its inverse is non-negative. Therefore, we see from (32) and (33) that in order to verify (31), it suffices to show that if $n - t \leq v \leq n$ and $n - t \leq r \leq n$, then

$$E(t)^{-1}_{v-n+t+1,\, r-n+t+1} \leq E(t+1)^{-1}_{v-n+t+2,\, r-n+t+2} \,. \tag{34}$$

Let α be chosen so that $\alpha > k - b_{n-t-1,n-t-1}$ and let

$$F = \begin{bmatrix} \alpha & 0 \cdots 0 \\ 0 & \\ \vdots & E(t) \\ 0 & \end{bmatrix}.$$

Clearly F is an M-matrix, and $F \geq E(t+1)$. So by a theorem of Ostrowski cited earlier [12, Theorem III] we see that $F^{-1} \leq E(t+1)^{-1}$, i.e.

$$\begin{bmatrix} 1/\alpha & 0 \cdots 0 \\ 0 & \\ \vdots & E(t)^{-1} \\ 0 & \end{bmatrix} \leq E(t+1)^{-1}.$$

This implies (34) and completes the proof.

3. In this section, we use the fact that M-matrices have non-negative inverses in order to improve the well-known row sum bounds (1).

THEOREM 4. *Let B be a non-negative matrix, let $k > p(B)$, and let $A = kI - B$. Then*

$$k - \frac{1}{r(A^{-1})} \leq p(B) \leq k - \frac{1}{R(A^{-1})}. \tag{35}$$

Proof. Since A is an M-matrix, $A^{-1} \geq 0$. So by (1)

$$r(A^{-1}) \leq p(A^{-1}) \leq R(A^{-1}), \tag{36}$$

where $r(A^{-1}) > 0$ by (5). Now the real characteristic roots of A are positive, by (7), and the smallest of these is $1/p(A^{-1})$. Since $A = kI - B$, the smallest real characteristic root of A is also $k - p(B)$. Thus from (36) we have

$$\frac{1}{R(A^{-1})} \leq k - p(B) \leq \frac{1}{r(A^{-1})},$$

which proves (35).

The following theorem will be used to evaluate the bounds obtained in Theorem 4.

THEOREM 5. *If $A = (a_{ij})$ is an M-matrix of order n, then*

$$r(A) \leq \frac{1}{R(A^{-1})} \tag{37}$$

and

$$R(A) \geq \frac{1}{r(A^{-1})}. \tag{38}$$

Proof. Suppose that

$$R(A^{-1}) = R_j(A^{-1}) = \sum_{m=1}^{n} \frac{X(a_{mj})}{\det A},$$

where $X(a_{mj})$ denotes the cofactor of a_{mj} in the determinant of A. Then to show (37) we prove that

$$r(A) \sum_{m=1}^{n} X(a_{mj}) \leq \det A.$$

Since $X(a_{mj}) \geq 0$ for all m and j, we see that

$$\begin{aligned} r(A) \sum_{m=1}^{n} X(a_{mj}) &\leq \sum_{m=1}^{n} R_m(A)X(a_{mj}) \\ &= \sum_{m=1}^{n} \sum_{t=1}^{n} a_{mt}X(a_{mj}) \\ &= \sum_{t=1}^{n} \sum_{m=1}^{n} a_{mt}X(a_{mj}) \\ &= \det A + \sum_{\substack{t=1 \\ t\neq j}}^{n} \sum_{m=1}^{n} a_{mt}X(a_{mj}). \end{aligned}$$

If $t \neq j$, then

$$\sum_{m=1}^{n} a_{mt}X(a_{mj})$$

is the determinant of the matrix obtained from A by replacing column j with column t, and is therefore zero. Thus inequality (37) is proved. The same method proves inequality (38) without difficulty.

COROLLARY. *Theorem* 4 *improves* (1); *that is* (*using the notation of Theorem* 4)

$$k - \frac{1}{R(A^{-1})} \leq R(B) \tag{39}$$

and

$$k - \frac{1}{r(A^{-1})} \geq r(B). \tag{40}$$

Proof. Using inequality (37) we have

$$\begin{aligned} k - \frac{1}{R(A^{-1})} &\leq k - r(A) \\ &= k - r(kI - B) \\ &= k - (k - R(B)) \\ &= R(B) \end{aligned}$$

which proves inequality (39). The proof of (40) follows in the same manner from (38).

In the setting of Theorem 4, we may apply bounds other than (1) to estimate $p(A^{-1})$, and hence $p(B)$. It is interesting that such an approach leads to the following

Alternate proof of Theorem 2. For $i = 1, 2, \cdots, n - t - 1$, let B_i denote the principal submatrix $B(i, n - t, \cdots, n)$. Then each B_i is a non-negative matrix, and by (4) we know that $p(B) \geq p(B_i)$. Hence $k > p(B_i)$, so that the matrix $A_i = kI - B_i$ is an M-matrix of order $t + 2$. Then by (6) and (2),

$$p(A_i^{-1}) \geq (A_i^{-1})_{1,1} .$$

Proceeding as in the proof of Theorem 4, we see that

$$k - p(B_i) = \frac{1}{p(A_i^{-1})} \leq \frac{1}{(A_i^{-1})_{1,1}}.$$

So

$$\begin{aligned} p(B_i) &\geq k - \frac{1}{(A_i^{-1})_{1,1}} \\ &= k - \frac{\det A_i}{\det A_i(2, \cdots, t+2)} \\ &= k - \frac{\det (kI - B)(i, n - t, \cdots, n)}{\det (kI - B)(n - t, \cdots, n)}. \end{aligned}$$

Since $p(B) \geq p(B_i)$, this proves Theorem 2.

4. We conclude with two examples illustrating the use of Theorems 1 and 4. Let

$$B = \begin{bmatrix} 10 & 0 & 4 & 2 \\ 1 & 9 & 1 & 1 \\ 5 & 5 & 9 & 0 \\ 1 & 2 & 3 & 7 \end{bmatrix}.$$

B has 10 as its minimum column sum and 17 as its maximum column sum, so the column analogue of (1) implies that $10 \leq p(B) \leq 17$. Moreover, since B is irreducible (see [10; 61]) and the column sums are not equal, it follows (see e.g. [10; 76]) that $p(B) \neq 17$. Letting $k = 17$ and $t = 0$, we obtain from Theorem 1 the matrix

$$D(17, 0) = \begin{bmatrix} 10.2 & 0.4 & 4.6 \\ 1.1 & 9.2 & 1.3 \\ 5 & 5 & 9 \end{bmatrix}.$$

The smallest column sum of $D(17,\ 0)$ is 14.6, so we get the bound $14.6 \leq p(D(17,\ 0)) \leq p(B)$.

If we apply the row sum bounds (1) to the matrix

$$B' = \begin{bmatrix} 5 & 0 & 9 \\ 3 & 0 & 6 \\ 5 & 5 & 4 \end{bmatrix}$$

we obtain the inequalities $9 \leq p(B') \leq 14$. As in the previous example, $p(B') \neq 14$. With $k = 14$ and $A = 14I - B'$, we see that

$$A^{-1} = \frac{1}{225}\begin{bmatrix} 110 & 45 & 126 \\ 60 & 45 & 81 \\ 85 & 45 & 126 \end{bmatrix}.$$

Thus, by Theorem 4, we have the bounds $12.79 < p(B') < 13.20$.

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