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A Generalized Preconditioned MHSS Method for a Class of Complex Symmetric Linear Systems

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Abstract. Based on the MHSS (Modified Hermitian and skew-Hermitian splitting) and preconditioned MHSS methods, we will present a generalized preconditioned MHSS method for solving a class of complex symmetric linear systems. The new method (GPMHSS) is essentially a two-parameter iteration method where the iterative sequence is unconditionally convergent to the unique solution of the linear system. A parameter region of the convergence for our method is provided. An efficient preconditioner is presented for the actual implementation of the new method. Some numerical results are given to show its effectiveness.

Keywords: complex symmetric matrix, Hermitian and skew-Hermitian splitting, preconditioner, inexact generalized preconditioned modified Hermitian and skew-Hermitian splitting (IGPMHSS) method.

AMS Subject Classification: 65F10; 65F50.

1 Introduction

In the paper, we consider the iterative solution of the linear system of the form

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a complex symmetric matrix of the form

$$A = W + iT \tag{1.2}$$

and $W, T \in \mathbb{R}^{n\times n}$ are real symmetric matrices, with W and T being positive definite and positive semidefinite, respectively. We assume $T \neq 0$ which implies A is non-Hermitian. Note that here and in the sequal $i = \sqrt{-1}$ denotes the imaginary unit.

As it is stated in [4], complex symmetric linear systems of this kind appear in several applications in scientific computing and engineering, including quantum mechanics [23], diffuse optimal tomography [2], structural dynamics [18], FFT-based solution of certain time-dependent PDEs [14], molecular scattering [21], and lattice quantum chromodynamics [19]. For more examples and additional references, we refer the interested reader to [4, 11].

The Hermitian and skew-Hermitian parts of the complex symmetric matrix A can be shown by

$$H = \frac{1}{2}(A + A^*) = W \quad \text{and} \quad S = \frac{1}{2}(A - A^*) = iT$$

respectively, hence, A is non-Hermitian positive definite matrix. Based on this Hermitian and skew-Hermitian splitting of the matrix $A \in \mathbb{C}^{n\times n}$, Bai et al. [8] proposed iterative method HSS, to compute an approximate solution for the complex symmetric linear system (1.1)–(1.2). This iterative scheme works as follows.

The HSS iteration method: Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \ldots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - iT)x^{(k)} + b, \\ (\alpha I + iT)x^{(k+1)} = (\alpha I - W)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{1.3}$$

where α is a given positive constant and I is the identity matrix. Since $W \in \mathbb{R}^{n\times n}$ is symmetric positive definite, we know from [8] that the HSS iteration method converges for any positive constant α.

Due to it is promising performance, the HSS scheme has immediately attracted considerable attention. In [7, 10], one can see the preconditioned HSS (PHSS) method and it is systematic analysis. The method was extended to the solution of saddle point problems [12, 13] and the idea of using two parameters to accelerate the HSS iteration method for solving saddle point linear systems was introduced in [6] and deeply discussed in [3].

In applying HSS iteration method for solving system (1.1)–(1.2), a potential difficulty is the need to solve the shifted skew-Hermitian linear subsystem with coefficient matrix $\alpha I + iT$. In [4] a modification of the HSS iteration scheme was presented that has the advantage that the solution of linear system with coefficient matrix $\alpha I + iT$ is avoided and only two linear subsystems with coefficient matrices $\alpha I + W$ and $\alpha I + T$ need to be solved at each step. This iterative scheme works as follows.

The MHSS iteration method: Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \ldots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - iT)x^{(k)} + b, \\ (\alpha I + T)x^{(k+1)} = (\alpha I + iW)x^{(k+\frac{1}{2})} - ib, \end{cases} \tag{1.4}$$

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where α is a given positive constant, and I is the identity matrix. Authors of [4] showed that the MHSS iteration method is unconditionally convergent, i.e., the sequence $\{x^{(k)}\}$ converges to the unique solution $x^* = A^{-1}b$ of the system $Ax = b$, as $k \to \infty$ for all $\alpha > 0$ and for any choice of $x^{(0)}$. Moreover, in the same paper it is shown that choosing $\alpha = \sqrt{\gamma_{min}\gamma_{max}}$, where $\gamma_{min} = \lambda_{min}(W)$ and $\gamma_{max} = \lambda_{max}(W)$ are the extreme eigenvalues of the symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$, minimizes an upper bound on the spectral radius of the iteration matrix associated with MHSS scheme (1.4).

The Generalized MHSS iteration method (denoted GMHSS) is obtained by replacing α with β in the second half-step of the scheme (1.4). The interested reader can see [24] for the main idea behind our approach.

The GMHSS iteration method: Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \ldots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - iT)x^{(k)} + b, \\ (\beta I + T)x^{(k+1)} = (\beta I + iW)x^{(k+\frac{1}{2})} - ib, \end{cases} \tag{1.5}$$

where α and β are given positive constants. when $\alpha = \beta$, the method reduces to the MHSS iteration method.

The rest of the current paper is organized as follows: in Section 2, the GPMHSS method is described and its convergence analysis is presented, and the IGPMHSS iteration is given in Section 3. A few numerical tests are discussed in Section 4. Finally we give a brief concluding remark in Section 5.

2 The GPMHSS Method

Instead of applying the GMHSS iteration methods to the system of linear equations (1.1)–(1.2), we may consider that these methods are applied for solving another preconditioned linear system

$$\hat{A}\hat{x} = \hat{b}, \quad \text{with } \hat{A} = R^{-T}AR^{-1}, \qquad \hat{x} = Rx \quad \text{and} \quad \hat{b} = R^{-T}b,$$

where $R \in \mathbb{R}^{n\times n}$ is a prescribed nonsingular matrix. Let $P = R^TR$, then P is a Hermitian positive definite matrix. The PMHSS method was defined as follows (see [5]).

The PMHSS iteration method: Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \ldots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha P + W)x^{(k+\frac{1}{2})} = (\alpha P - iT)x^{(k)} + b, \\ (\alpha P + T)x^{(k+1)} = (\alpha P + iW)x^{(k+\frac{1}{2})} - ib, \end{cases} \tag{2.1}$$

where α is a given positive constant and P is a prescribed symmetric positive definite matrix.

We introduce two different parameters α and β in the PMHSS scheme, that leads to the following generalized preconditioned MHSS method (GPMHSS method).

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The GPMHSS iteration method: Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \dots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha P + W)x^{(k+\frac{1}{2})} = (\alpha P - iT)x^{(k)} + b, \\ (\beta P + T)x^{(k+1)} = (\beta P + iW)x^{(k+\frac{1}{2})} - ib, \end{cases} \tag{2.2}$$

where α and β are given positive constants, and P is an Hermitian positive definite matrix.

The new method is a two-parameter two-step iterative method. It has MHSS and PMHSS methods as its special cases, respectively, when $(\alpha = \beta \wedge P = I)$ and $\alpha = \beta$. We will show that there exists a reasonable convergence domain of the two parameters for the GPMHSS method.

At each step of the GPMHSS iteration we need to solve the two linear subsystems with their coefficient matrices being symmetric positive definite, so the iterations at each step may be efficiently computed either exactly by the Cholesky factorization or inexactly by the conjugate gradient method [20].

Eliminating $x^{(k+\frac{1}{2})}$ from the second step of (2.2) yields

$$x^{(k+1)} = M_{\alpha,\beta}x^{(k)} + P_{\alpha,\beta}^{-1}b, \tag{2.3}$$

where

$$M_{\alpha,\beta} = I - P_{\alpha,\beta}^{-1}A, \qquad P_{\alpha,\beta} = \frac{1}{\beta - i\alpha}(\alpha P + W)P^{-1}(\beta P + T).$$

Thus $A = P_{\alpha,\beta} - (P_{\alpha,\beta} - A)$ is the splitting induced by the GPMHSS iteration, and $P_{\alpha,\beta}$ can be used as a preconditioning matrix for the complex symmetric matrix A. Note that the multiplicative factor $\frac{1}{\beta - i\alpha}$ has no effect on the preconditioned system and then can be dropped. Hence, the GPMHSS preconditioner is just the real matrix $P_{\alpha,\beta} = (\alpha P + W)P^{-1}(\beta P + T)$. The iteration matrix $M_{\alpha,\beta}$ is given by

$$M_{\alpha,\beta} = (\beta P + T)^{-1}(\beta P + iW)(\alpha P + W)^{-1}(\alpha P - iT).$$

Now the following convergence result can be proved.

Theorem 1. *Let $A = W + iT \in \mathbb{C}^{n\times n}$, with $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semidefinite, respectively, $P = R^T R$ where $R \in \mathbb{R}^{n\times n}$ is a prescribed nonsingular matrix and α be a given positive constant. Let $\hat{W} = R^{-T}WR^{-1}$, $\hat{T} = R^{-T}TR^{-1}$. Suppose $\sigma(\hat{W})$ and $\sigma(\hat{T})$ denote the spectrum of $\hat{W}$ and $\hat{T}$, respectively. Denote*

$$\lambda_{min} = \min_{\lambda_k \in \sigma(\hat{W})} \{\lambda_k\}, \qquad \mu_{min} = \min_{\mu_j \in \sigma(\hat{T})} \{\mu_j\},$$

if $\beta \in [\sqrt{\alpha^2 + \mu_{min}^2} - \mu_{min}, \sqrt{\alpha^2 + 2\alpha\lambda_{min}})$, then the GPMHSS iteration (2.2) *converges unconditionally to the unique solution of the linear system* (1.1)*–*(1.2)*.*

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Proof. The iteration matrix $M(\alpha,\beta)$ is given by

$$M_{\alpha,\beta} = (\beta P + T)^{-1}(\beta P + iW)(\alpha P + W)^{-1}(\alpha P - iT),$$

which is similar to

$$\hat{M}_{\alpha,\beta} = (\beta P + iW)(\alpha P + W)^{-1}(\alpha P - iT)(\beta P + T)^{-1}.$$

Clearly for the spectral radius $\rho(M_{\alpha,\beta})$ of iteration matrix $M_{\alpha,\beta}$, we can write

$$\begin{aligned}
\rho(M_{\alpha,\beta}) &= \rho(\hat{M}_{\alpha,\beta}) = \rho\big((\beta P + iW)(\alpha P + W)^{-1}(\alpha P - iT)(\beta P + T)^{-1}\big)\\
&= \rho\big(R^T\big(\beta I + iR^{-T}WR^{-1}\big)RR^{-1}\big(\alpha I + R^{-T}WR^{-1}\big)^{-1}R^{-T}R^T\\
&\quad \times \big(\alpha I - iR^{-T}TR^{-1}\big)RR^{-1}\big(\beta I + R^{-T}TR^{-1}\big)^{-1}R^{-T}\big)\\
&= \rho\big(\big(\beta I + iR^{-T}WR^{-1}\big)\big(\alpha I + R^{-T}WR^{-1}\big)^{-1}\\
&\quad \times \big(\alpha I - iR^{-T}TR^{-1}\big)\big(\beta I + R^{-T}TR^{-1}\big)^{-1}\big)\\
&\le \big\|(\beta I + i\hat{W})(\alpha I + \hat{W})^{-1}(\alpha I - i\hat{T})(\beta I + \hat{T})^{-1}\big\|_2\\
&\le \big\|(\beta I + i\hat{W})(\alpha I + \hat{W})^{-1}\big\|_2\big\|(\alpha I - i\hat{T})(\beta I + \hat{T})^{-1}\big\|_2.
\end{aligned}$$

It is stated in [4], that because $\hat{W} \in \mathbb{R}^{n\times n}$ and $\hat{T} \in \mathbb{R}^{n\times n}$ are symmetric, there exist orthogonal matrices $U, V \in \mathbb{R}^{n\times n}$ such that

$$U^T\hat{W}U = \Lambda_{\hat{W}}, \quad V^T\hat{T}V = \Lambda_{\hat{T}},$$

where $\Lambda_{\hat{W}} = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\Lambda_{\hat{T}} = diag(\mu_1, \mu_2, \ldots, \mu_n)$, with λ_j $(1 \le j \le n)$ and μ_j $(1 \le j \le n)$ being the eigenvalues of the matrices $\hat{W}$ and $\hat{T}$, respectively. W and T are positive definite and positive semi-definite, respectively, and R is a nonsingular matrix thus $\hat{W}$ and $\hat{T}$ are positive definite and positive semi-definite, respectively, and then we have

$$\lambda_j > 0 \quad \text{and} \quad \mu_j \ge 0, \quad 1 \le j \le n.$$

Now, based on the orthogonal invariance of the Euclidean norm $\|.\|_2$, the following upper bound for the spectral radius of $M(\alpha,\beta)$ can be obtained:

$$\begin{aligned}
\rho(M_{\alpha,\beta}) &\le \big\|(\beta I + i\hat{W})(\alpha I + \hat{W})^{-1}\big\|_2\big\|(\alpha I - i\hat{T})(\beta I + \hat{T})^{-1}\big\|_2\\
&= \big\|(\beta I + i\Lambda_{\hat{W}})(\alpha I + \Lambda_{\hat{W}})^{-1}\big\|_2\big\|(\alpha I - i\Lambda_{\hat{T}})(\beta I + \Lambda_{\hat{T}})^{-1}\big\|_2\\
&= \max_{\lambda_j\in\sigma(\hat{W})}\left|\frac{\beta + i\lambda_j}{\alpha + \lambda_j}\right| \cdot \max_{\mu_j\in\sigma(\hat{T})}\left|\frac{\alpha - i\mu_j}{\beta + \mu_j}\right|\\
&= \max_{\lambda_j\in\sigma(\hat{W})}\frac{\sqrt{\beta^2 + \lambda_j^2}}{\alpha + \lambda_j} \cdot \max_{\mu_j\in\sigma(\hat{T})}\frac{\sqrt{\alpha^2 + \mu_j^2}}{\beta + \mu_j}.
\end{aligned}$$

Denoting

$$\tau(\alpha,\beta) = \max_{\lambda_j\in\sigma(\hat{W})}\frac{\sqrt{\beta^2 + \lambda_j^2}}{\alpha + \lambda_j} \cdot \max_{\mu_j\in\sigma(\hat{T})}\frac{\sqrt{\alpha^2 + \mu_j^2}}{\beta + \mu_j},$$

then the spectral radius of $\rho(M_{\alpha,\beta})$ is bounded by $\tau(\alpha,\beta)$, i.e.,

$$\rho(M_{\alpha,\beta}) \leq \tau(\alpha,\beta).$$

Now with assumption $\beta \in \left[\sqrt{\alpha^2+\mu_{min}^2}-\mu_{min}, \sqrt{\alpha^2+2\alpha\lambda_{min}}\right)$, we show that $\tau(\alpha,\beta) < 1$. The assumption

$$\sqrt{\alpha^2+\mu_{min}^2}-\mu_{min} \leq \beta < \sqrt{\alpha^2+2\alpha\lambda_{min}}$$

yields

$$\beta < \sqrt{\alpha^2+2\alpha\lambda_{min}}, \quad \text{and} \quad \beta \geq \sqrt{\alpha^2+\mu_{min}^2}-\mu_{min},$$

thus we can write

$$\beta^2 < \alpha^2+2\alpha\lambda_{min} \quad \text{and} \quad \alpha^2 \leq \beta^2+2\beta\mu_{min},$$

and hence we obtain

$$\beta^2+\lambda_j^2 < \alpha^2+2\alpha\lambda_j+\lambda_j^2 \quad \text{and} \quad \alpha^2+\mu_j^2 \leq \beta^2+2\beta\mu_j+\mu_j^2$$

for $j=1,\ldots,n$. Therefore

$$\sqrt{\beta^2+\lambda_j^2} < \alpha+\lambda_j \quad \text{and} \quad \sqrt{\alpha^2+\mu_j^2} \leq \beta+\mu_j$$

for $j=1,\ldots,n$. Finally we conclude

$$\frac{\sqrt{\beta^2+\lambda_j^2}}{\alpha+\lambda_j} < 1 \quad \text{and} \quad \frac{\sqrt{\alpha^2+\mu_j^2}}{\beta+\mu_j} \leq 1, \quad j=1,\ldots,n,$$

that yields $\tau(\alpha,\beta) < 1$. Therefore $\rho(M_{\alpha,\beta}) < 1$ for $\alpha > 0$ and

$$\beta \in \left[\sqrt{\alpha^2+\mu_{min}^2}-\mu_{min}, \sqrt{\alpha^2+2\alpha\lambda_{min}}\right),$$

and the GPMHSS iteration converges to the unique solution of the linear system (1.1)–(1.2). $\square$

$\tau(\alpha,\beta)$ defines an upper bound of the contraction factor of the GPMHSS iteration and it is obvious that the convergence rate depends on the choice of the two parameters α and β. So we need to investigate the properties of the function $\tau(\alpha,\beta)$ with respect to the two parameters. We have

$$\tau(\alpha,\beta) = \max_{\lambda_j\in\sigma(\hat{W})} \frac{\sqrt{\beta^2+\lambda_j^2}}{\alpha+\lambda_j} \cdot \max_{\mu_j\in\sigma(\hat{T})} \frac{\sqrt{\alpha^2+\mu_j^2}}{\beta+\mu_j}.$$

Consider the univariate functions $\beta^*(\alpha)$ and $\alpha^*(\beta)$ as follows

$$\beta^*(\alpha) = \frac{\alpha^2(\lambda_{max}+\lambda_{min})+2\alpha\lambda_{max}\lambda_{min}}{2\alpha+\lambda_{max}+\lambda_{min}},$$
$$\alpha^*(\beta) = \frac{\beta^2(\mu_{max}+\mu_{min})+2\beta\mu_{max}\mu_{min}}{2\beta+\mu_{max}+\mu_{min}}.$$

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Since $\alpha > 0$ and $\beta > 0$, it follows that

$$\max_{\lambda_j \in \sigma(\hat{W})} \frac{\sqrt{\beta^2 + \lambda_j^2}}{\alpha + \lambda_j} = \max\left\{ \frac{\sqrt{\beta^2 + \lambda_{max}^2}}{\alpha + \lambda_{max}}, \frac{\sqrt{\beta^2 + \lambda_{min}^2}}{\alpha + \lambda_{min}} \right\}.$$

Now using $\beta^*(\alpha)$ we have

$$\max_{\lambda_j \in \sigma(\hat{W})} \frac{\sqrt{\beta^2 + \lambda_j^2}}{\alpha + \lambda_j} = \begin{cases} \frac{\sqrt{\beta^2+\lambda_{max}^2}}{\alpha+\lambda_{max}}, & \beta < \beta^*(\alpha), \\ \frac{\sqrt{\beta^2+\lambda_{min}^2}}{\alpha+\lambda_{min}}, & \beta \geq \beta^*(\alpha). \end{cases} \tag{2.4}$$

Also we can write

$$\max_{\mu_j \in \sigma(\hat{T})} \frac{\sqrt{\alpha^2 + \mu_j^2}}{\beta + \mu_j} = \begin{cases} \frac{\sqrt{\alpha^2+\mu_{max}^2}}{\beta+\mu_{max}}, & \alpha < \alpha^*(\beta), \\ \frac{\sqrt{\alpha^2+\mu_{min}^2}}{\beta+\mu_{min}}, & \alpha \geq \alpha^*(\beta). \end{cases} \tag{2.5}$$

β^* and α^* are nondecreasing functions that depend on the extreme eigenvalues of $\hat{W}$ and $\hat{T}$. While we don't know the values of these extreme eigenvalues, we will not know the position of two functions with respect to each other.

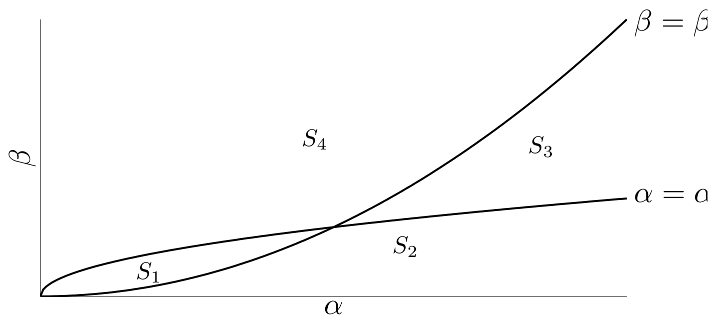

Figure 1. The four subregions of S.

For instance, suppose β^* and α^* be such that, we can divide the first quarter of (α, β)-plane into four subregions $S = \bigcup_{i=1}^{4} S_i$ (see Figure 1). By this assumption and from (2.4) and (2.5), we can write

$$\tau(\alpha, \beta) = \begin{cases} \frac{\sqrt{\beta^2+\lambda_{max}^2}}{\alpha+\lambda_{max}} \cdot \frac{\sqrt{\alpha^2+\mu_{max}^2}}{\beta+\mu_{max}}, & (\alpha, \beta) \in S_1, \\ \frac{\sqrt{\beta^2+\lambda_{max}^2}}{\alpha+\lambda_{max}} \cdot \frac{\sqrt{\alpha^2+\mu_{min}^2}}{\beta+\mu_{min}}, & (\alpha, \beta) \in S_2, \\ \frac{\sqrt{\beta^2+\lambda_{min}^2}}{\alpha+\lambda_{min}} \cdot \frac{\sqrt{\alpha^2+\mu_{min}^2}}{\beta+\mu_{min}}, & (\alpha, \beta) \in S_3, \\ \frac{\sqrt{\beta^2+\lambda_{min}^2}}{\alpha+\lambda_{min}} \cdot \frac{\sqrt{\alpha^2+\mu_{max}^2}}{\beta+\mu_{max}}, & (\alpha, \beta) \in S_4 \end{cases}$$

and then if we know the values of extreme eigenvalues of the matrices $\hat{W}$ and $\hat{T}$, using preliminary calculus, the values of the parameters α and β which minimize the upper bound $\tau(\alpha, \beta)$ can be obtained. Note that without knowing the values of λ_{min}, λ_{max}, μ_{min}, and μ_{max}, we cannot find the optimal values of the iteration parameters in general.

It can be seen that the GPMHSS iteration scheme and its convergence result are applicable also to the case where the matrix W is symmetric positive semi-definite and the matrix T is symmetric positive definite. As it is stated in [4], if there exist real numbers ξ and η such that both matrices $\tilde{W} := \xi W + \eta T$ and $\tilde{T} := \xi T - \eta W$ are symmetric positive semi-definite while at least one of them is positive definite, we can first scale the complex linear system (1.1)–(1.2) by the complex number $\xi - i\eta$, then apply the GPMHSS iteration scheme to compute an approximation solution of the linear system (1.1)–(1.2), for the following equivalent system

$$(\tilde{W} + i\tilde{T})x = \tilde{b}, \quad \text{with } \tilde{b} = (\xi - i\eta)b.$$

3 The IGPMHSS Iteration

The two half-steps at each step of the GPMHSS iteration require finding solution of the two symmetric positive definite systems with coefficient matrices $\alpha P + W$ and $\beta P + T$, which is very costly and impractical in the actual implementation. To overcome this disadvantage and improve the efficiency of the GPMHSS iteration method, similar to the introduced approach in [9] for inexact implementation of HSS iteration method, we can solve the two inner linear systems iteratively. Because of the symmetrically and positive definite properties of the matrices of the two sub-systems, we can use the conjugate gradient (CG) method for solving two sub-problems. This results in the inexact generalized preconditioned modified Hermitian and skew-Hermitian splitting (IGPMHSS) iteration method. Its convergence can be shown in a similar way to that of the IHSS iteration method, using Theorem 3.1 of [8].

For the two half steps at each iteration of GPMHSS method, if good preconditioners for matrices $\alpha P + W$ and $\beta P + T$ are available, we can use the preconditioned conjugate gradient (PCG) method instead of CG for the two inner systems, that yields a better performance of IGPMHSS method. If either $\alpha P + W$ or $\beta P + T$ (or both) be Toeplitz, we can use fast algorithms for solution of the corresponding sub-systems [1, 15, 17].

4 Computational Results

In this section, we present several numerical experiments to show the efficiency of the GPMHSS iteration method, when it is used either as a solver or as a preconditioner for solving the linear system (1.1)–(1.2). We also compare the GPMHSS with the HSS and MHSS schemes as solvers and as preconditioners for the GMRES and GMRES(10), restarted variant of GMRES, [16, 20, 22].

In our experiments, we use $x^{(0)} = 0$ for the initial guess and the stopping criteria for outer iterations (when HSS, MHSS, and GPMHSS methods are used as solvers) is

$$\|b - Ax^{(k)}\|_2 / \|b\|_2 < 10^{-6}.$$

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For the inexact HSS, MHSS, and GPMHSS iteration methods, the stopping criteria for the inner CG iterations is set to be

$$\left|r_k^T r_k\right| < 10^{-11},$$

where r_k is the kth residual.

The numerical examples used to illustrate the effectiveness of the GPMHSS method are given in the following two subsections.

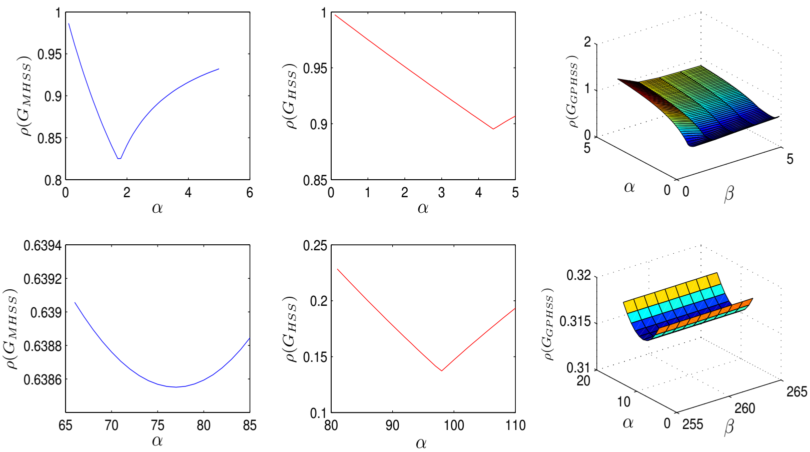

Figure 2. The spectral radius versus parameter α for HSS and MHSS methods and parameters α, β for GPMHSS method; *Top*: Example 4.1 and *Bottom*: Example 4.2.

For the tests reported in this section we used the optimal values of the parameters α and β, denoted by $\alpha^*(HSS)$ for HSS method, $\alpha^*(MHSS)$ for MHSS method, and $\alpha^*(GPMHSS)$, $\beta^*(GPMHSS)$ for GPMHSS method. These parameters are obtained experimentally with the least spectral radius for the iteration matrices of the three methods. See Figure 2 for a graph illustrating of the spectral radius of the three solvers with respect to corresponding parameters for each scheme, (α for HSS and MHSS and α, β for GPMHSS) for the following two examples with grid dimension 20×20.

4.1 Example 1

These examples are taken from [4]. The first example is as follows. The system of linear equations (1.1)–(1.2) is of the form $(W + iT)x = b$ with

$$T = I \otimes V + V \otimes I, \quad \text{and} \quad W = 10(I \otimes V_c + V_c \otimes I) + 9\left(e_1 e_m^T + e_m e_1^T\right) \otimes I,$$

where $V = tridag(-1, 2, -1) \in \mathbb{R}^{m\times m}$, $V_c = V - e_1 e_m^T - e_m e_1^T \in \mathbb{R}^{m\times m}$, and e_1 and e_m are the first and the last unit vectors in $\mathbb{R}^m$, respectively. We take the right hand side vector b to be $b = (1+i)AB$, with B being the vector of all entries equal to 1. Here T and W correspond to the five-point centered difference matrices approximating the negative Laplacian operator with homogeneous

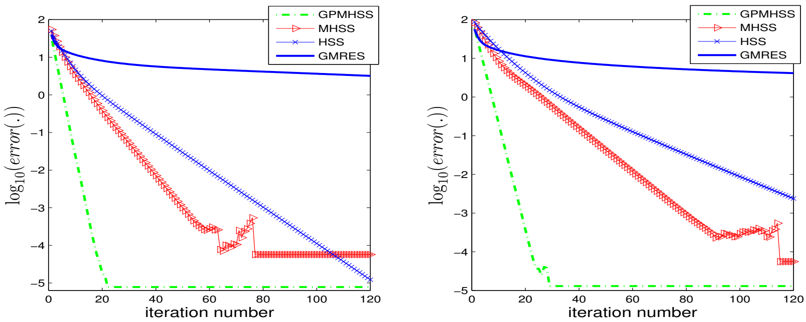

Figure 3. The log 10 of the residual error versus iteration number for Example 1, with $n = 20 \times 20$ (left) and $n = 40 \times 40$ (right).

Dirichlet boundary conditions and periodic boundary conditions, respectively, on a uniform mesh in the unit square $[0,1] \times [0,1]$ with the mesh-size $h = \frac{1}{m+1}$.

For tests reported in this subsection we used $P = W$ for GPMHSS method; see [5]. For each choice of the special mesh-sizes, one can observe the optimal experimental parameters for Example 1 in Table 1.

Table 1. The experimentally optimal parameters for HSS, MHSS, and GPMHSS iteration methods.

Example	Parameter for method	Grid				
		10×10	20×20	30×30	40×40	50×50
No. 1	$\alpha^*(HSS)$	7.9	4.4	3.2	2.5	2.1
	$\alpha^*(MHSS)$	3	1.753	1.29	1	0.8
	$\alpha^*(GPMHSS)$	0.2	0.5	1	0.7	0.7
	$\beta^*(GPMHSS)$	2	1	2	1	1
No. 2	$\alpha^*(HSS)$	98	98	98	98	98
	$\alpha^*(MHSS)$	75	75	75	75	75
	$\alpha^*(GPMHSS)$	11	11	11	11	11
	$\beta^*(GPMHSS)$	260	260	260	260	260

Let $\rho(G_{HSS})$, $\rho(G_{MHSS})$, and $\rho(G_{GPMHSS})$ denote the spectral radius of the iteration matrices for the HSS, MHSS, and GPMHSS methods, respectively. In Table 2, we give a comparison between the spectral radius of the three methods for Example 1, with respect to the experimentally optimal parameters.

In Table 3, we give the iteration numbers for HSS, MHSS, and GPMHSS schemes, and Table 4, shows CPU for these three methods for solving Example 1.

Let $error(k) = \|b - Ax^{(k)}\|_2$, where k denotes the iteration number. Using HSS, MHSS, GPMHSS, and GMRES methods for solving Example 1, the log 10 of the residual error ($\log_{10}(error(.))$) for each method is plotted against the iteration number in Figure 3.

The same results, using HSS, MHSS, and GPMHSS as preconditioner for GMRES(k) method ($k = 1, 10$) are shown in Figures 4 and 5.

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Table 2. The comparison of $\rho(G_{HSS})$, $\rho(G_{MHSS})$, and $\rho(G_{GPMHSS})$.

Example	Spectral radii for method	Grid 10 × 10	20 × 20	30 × 30	40 × 40	50 × 50
No. 1	$\rho(G_{HSS})$	0.8175	0.8952	0.9242	0.9393	0.9488
	$\rho(G_{MHSS})$	0.7464	0.8212	0.8587	0.8847	0.9045
	$\rho(G_{GPMHSS})$	0.3814	0.4948	0.5454	0.5550	0.5768
No. 2	$\rho(G_{HSS})$	0.1363	0.1373	0.1374	0.1375	0.1375
	$\rho(G_{MHSS})$	0.6383	0.6386	0.6386	0.6386	0.6386
	$\rho(G_{GPMHSS})$	0.3144	0.3150	0.3150	0.3151	0.3151

Table 3. The comparison of iteration number.

Example	Method	Grid 10 × 10	20 × 20	30 × 30	40 × 40	50 × 50
No. 1	HSS	61	103	140	167	193
	MHSS	45	64	91	115	134
	GPMHSS	14	18	23	22	23
No. 2	HSS	7	7	7	7	7
	MHSS	31	31	31	31	31
	GPMHSS	9	8	8	8	8

Table 4. The comparison of CPU time (measured in seconds).

Example	Method	Grid 10 × 10	20 × 20	30 × 30	40 × 40	50 × 50
No. 1	HSS	0.1146	7.2167	95.9719	627.5804	2712.3000
	MHSS	0.0983	3.3312	42.5936	195.2957	619.6114
	GPMHSS	0.0468	1.9128	30.4620	129.1083	391.4764
No. 2	HSS	0.0174	0.4870	5.5461	29.8779	105.6924
	MHSS	0.0468	0.5394	4.3229	14.2822	34.4167
	GPMHSS	0.0155	0.1699	1.2361	3.9926	9.7059

4.2 Example 2

The system of linear equations (1.1)–(1.2) is of the form $(W + iT)x = b$, where $W_{n\times n}$ and $T_{n\times n}$ are symmetric positive definite Toeplitz matrices of the form

$$W=\begin{pmatrix} 100 & 5 & -2 & 1.5 & 10 & & 0 \\ 5 & \ddots & \ddots & \ddots & \ddots & \ddots & \\ -2 & \ddots & \ddots & \ddots & \ddots & \ddots & 10 \\ 1.5 & \ddots & \ddots & \ddots & \ddots & \ddots & 1.5 \\ 10 & \ddots & \ddots & \ddots & \ddots & \ddots & -2 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 5 \\ 0 & & 10 & 1.5 & -2 & 5 & 100 \end{pmatrix}, \quad T=\begin{pmatrix} 20 & 2 & -2 & -4 & & 0 \\ 2 & \ddots & \ddots & \ddots & \ddots & \\ -2 & \ddots & \ddots & \ddots & \ddots & -4 \\ -4 & \ddots & \ddots & \ddots & \ddots & -2 \\ & \ddots & \ddots & \ddots & \ddots & 2 \\ 0 & & -4 & -2 & 2 & 20 \end{pmatrix}$$

and $b(j) = 55i + 90$ for $j = 1, 2, \ldots, n$.

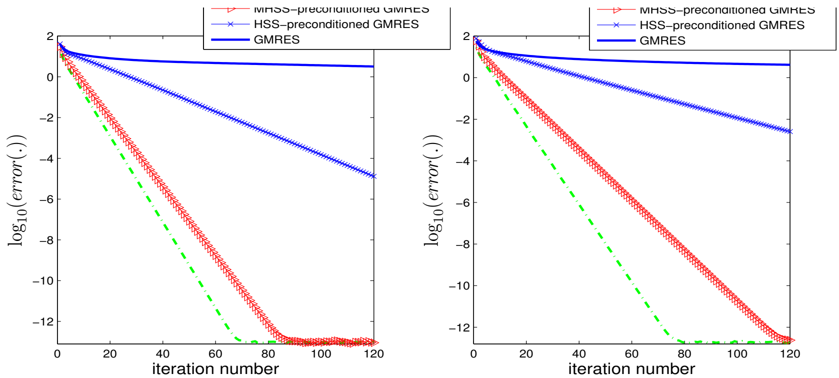

Figure 4. The log 10 of the residual error versus iteration number for Example 1, with $n = 20 \times 20$ (left) and $n = 40 \times 40$ (right).

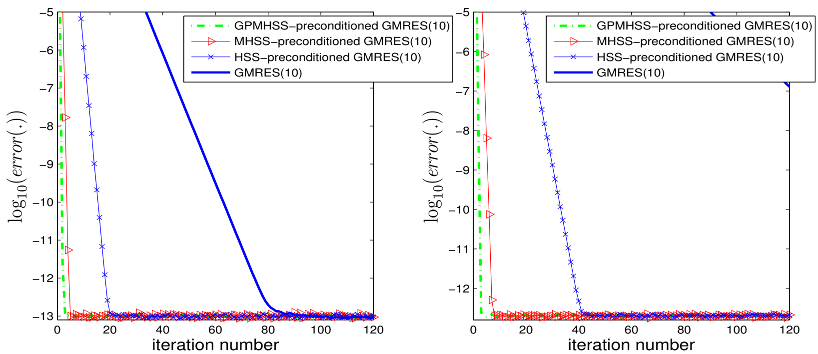

Figure 5. The log 10 of the residual error versus iteration number for Example 1, with $n = 20 \times 20$ (left) and $n = 40 \times 40$ (right).

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For tests reported in this subsection we have used $P = I$ for GPMHSS method. The optimal experimental parameters for Example 2, for different choices of the mesh-sizes are given in Table 1. In Table 2, we give a comparison between the spectral radius of the three methods for Example 2, with respect to the experimentally optimal parameters. The number of required outer iterations for solving Example 2, for each method is given in Table 3. In Table 4, we show CPU for HSS, MHSS, and GPMHSS methods.

Figure 6 shows the log 10 of the residual error ($\log_{10}(error(.))$) against the iteration number using HSS, MHSS, GPMHSS, and GMRES methods for solving Example 2.

Also in Figures 7 and 8, one can see the numerical results for solving Example 2, using HSS, MHSS, and GPMHSS as preconditioner for GMRES(k), ($k = 1, 10$).

From Table 1, we see that for Example 1, $\alpha^*(HSS)$ and $\alpha^*(MHSS)$ decrease with the mesh-size h. As can be seen, the experimentally optimal parameters

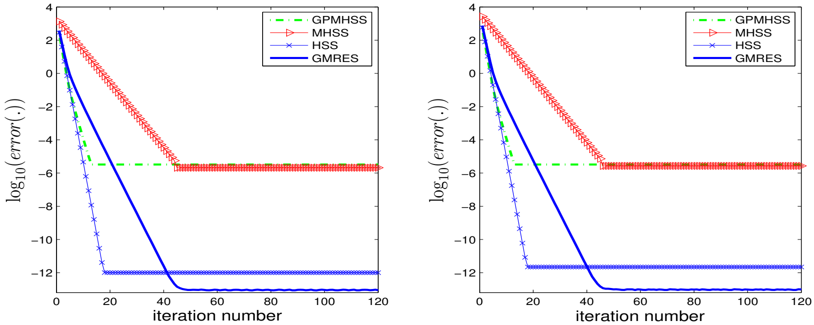

Figure 6. The log 10 of the residual error versus iteration number for Example 2, with $n = 20 \times 20$ (left) and $n = 40 \times 40$ (right).

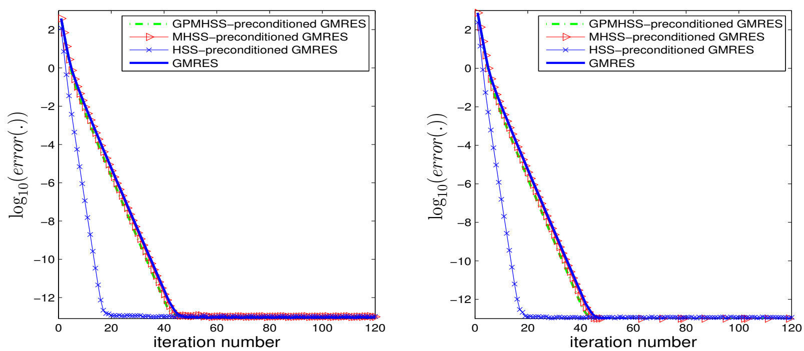

Figure 7. The log 10 of the residual error versus iteration number for Example 2, with $n = 20 \times 20$ (left) and $n = 40 \times 40$ (right).

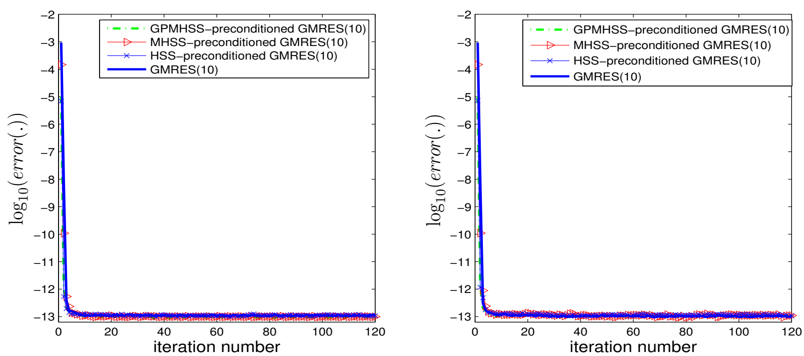

Figure 8. The log 10 of the residual error versus iteration number for Example 2, with $n = 20 \times 20$ (left) and $n = 40 \times 40$ (right).

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for Example 2 are constant. Also note that $\alpha^*(GPMHSS)$ is always less than $\alpha^*(HSS)$ and $\alpha^*(MHSS)$ for the two examples.

In Table 2, we show the spectral radius of iteration matrices for the three methods and for two examples by using the experimentally optimal parameters that are given in Table 1. As you see, for Example 1, in all cases, the spectral radius for the GPMHSS method is less than one for the HSS and MHSS methods. For Example 2, the spectral radius of iteration matrix for the HSS method is less than one for MHSS and GPMHSS methods in all cases and the spectral radius of iteration matrix for the GPMHSS method is almost half of the one for MHSS method. Note that for the two examples and for three methods, the spectral radius increases when the size of the problem increases.

In Table 3, we report the iteration numbers of HSS, MHSS and GPMHSS methods for solving two examples. One can see that for Example 1, the number of iterations for GPMHSS method is less than another two methods and MHSS scheme requires fewer iteration than HSS scheme, also for all methods the number of iterations grows with the problem size. However this growth for the HSS and MHSS is faster than for the GPMHSS method. For Example 2, we don't have any growth in iteration numbers by increasing grid dimension. The iteration number of HSS and GPMHSS methods, approximately are the same for all cases and MHSS has more iteration number than the other two methods. The presented results in Table 4, show that in all cases GPMHHS is superior to the other methods in terms of the CPU time.

From Figure 3, we find that the GPMHSS iteration method for solving Example 1 converges faster than the HSS, MHSS, and GMRES methods, and GMRES method converges slower than other three methods. As can be seen, where the mesh-size is larger ($n = 40 \times 40$), this convergence for GPMHSS method is much faster than the other three methods.

Figure 6 shows that the error of GPMHSS method in the first few repeats decreases faster than those of the HSS, MHSS, and GMRES, but from 20th iteration, the HHS and GMRES give better results. Note that because the spectral radius of the iteration matrix for HSS scheme is very less than MHSS and GPMHSS, so in a low iteration number, it achieves to a good accuracy. However according to Table 4, the CPU time of this method for reaching a prescribed accuracy is much more than MHSS and GPMHSS methods and this is a weakness of it.

From Figure 4, we conclude that the preconditioned GMRES methods converge much faster than the GMRES method for solving Example 1, furthermore, the GPMHSS-preconditioned GMRES method converges faster than one for the HSS- and MHSS-preconditioned GMRES methods. GMRES has more error than the other three methods. In Figure 5, one can see the same results for GMRES(10) and preconditioned GMRES(10).

In Figure 7, we see that the HSS-preconditioned GMRES gives better results than the GMRES and MHSS- and GPMHSS-preconditioned GMRES methods, also GPMHSS-preconditioned GMRES gives better results than MHSS-preconditioned GMRES and GMRES methods. Figure 8 shows that GMRES(10) and preconditioned GMRES(10) techniques are approximately the same.

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5 Concluding Remarks

In this paper, based on the Hermitian and skew-Hermitian splitting of the coefficient matrix, we have established and analyzed a two-parameter generalized preconditioned MHSS method and the corresponding inexact variants. This generalization has the classical MHSS and PMHSS methods as its special cases when we take ($\alpha = \beta \wedge P = I$) and $\alpha = \beta$, respectively. The exact version of the new method has been shown to be unconditionally convergent when P is a symmetric positive definite matrix and the parameters α and β satisfy some moderate conditions. Numerical experiments show the feasibility and effectiveness of the new method.

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References

[1] G.S. Ammar and W.B. Gragg. Superfast solution of real positive definite Toeplitz systems. *SIAM J. Matrix Anal. Appl*, **9**:61–76, 1988. http://dx.doi.org/10.1137/0609005.

[2] S.R. Arridge. Optical tomography in medical imaging. *Inverse Problems*, **15**(2):41–93, 1999. http://dx.doi.org/10.1088/0266-5611/15/2/022.

[3] Z.-Z. Bai. Optimal parameters in the HSS-like methods for saddle-point problems. *Numer. Linear Algebra Appl.*, **16**(6):447–479, 2009. http://dx.doi.org/10.1002/nla.626.

[4] Z.-Z. Bai, M. Benzi and F. Chen. Modified HSS iteration methods for a class of complex symmetric linear systems. *Computing*, **87**(3–4):93–111, 2010. http://dx.doi.org/10.1007/s00607-010-0077-0.

[5] Z.-Z. Bai, M. Benzi and F. Chen. On preconditioned MHSS iteration methods for complex symmetric linear systems. *Numer. Algorithms*, **56**(2):297–317, 2011. http://dx.doi.org/10.1007/s11075-010-9441-6.

[6] Z.-Z. Bai and G.H. Golub. Accelerated Hermitian and skew-Hermitian splitting iteration methods for saddle-point problems. *IMA J. Numer. Anal.*, **27**:1–23, 2007. http://dx.doi.org/10.1093/imanum/drl017.

[7] Z.-Z. Bai, G.H. Golub and C.-K. Li. Convergence properties of preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite matrices. *Math. Comput.*, **76**:287–298, 2007. http://dx.doi.org/10.1090/S0025-5718-06-01892-8.

[8] Z.-Z. Bai, G.H. Golub and M.K. Ng. Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems. *SIAM J. Matrix Anal. Appl.*, **24**:603–626, 2003. http://dx.doi.org/10.1137/S0895479801395458.

[9] Z.-Z. Bai, G.H. Golub and M.K. Ng. On inexact hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems. *Linear Algebra Appl.*, **428**(2-3):413–440, 2008. http://dx.doi.org/10.1016/j.laa.2007.02.018.

[10] Z.-Z. Bai, G.H. Golub and J.-Y. Pan. Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems. *Numer. Math.*, **98**(1):1–32, 2004. http://dx.doi.org/10.1007/s00211-004-0521-1.

[11] M. Benzi and D. Bertaccini. Block preconditioning of real-valued iterative algorithms for complex linear systems. *IMA J. Numer. Anal.*, **28**:598–618, 2008. http://dx.doi.org/10.1093/imanum/drm039.

[12] M. Benzi, M.J. Gander and G. Golub. Optimization of the Hermitian and skew-Hermitian splitting iteration for saddle-point problems. *BIT*, **43**(5):881–900, 2003. http://dx.doi.org/10.1023/B:BITN.0000014548.26616.65.

[13] M. Benzi and G.H. Golub. A preconditioner for generalized saddle point problems. *SIAM J. Matrix Anal. Appl.*, **26**:20–41, 2004. http://dx.doi.org/10.1137/S0895479802417106.

[14] D. Bertaccini. Efficient solvers for sequences of complex symmetric linear systems. *Electr. Trans. Numer. Anal.*, **18**:49–64, 2004.

[15] R. Chan and M.K. Ng. Conjugate gradient methods for Toeplitz systems. *SIAM Rev.*, **38**:427–482, 1996. http://dx.doi.org/10.1137/S0036144594276474.

[16] K. Chen. *Matrix Preconditioning Techniques and Applications.* Cambridge University Press, 2005.

[17] G. Codevico, G. Heinig and M. Van Barel. A superfast solver for real symmetric Toeplitz systems using real trigonometric transformations. *Numer. Linear Algebra Appl.*, **12**(8):699–713, 2005. http://dx.doi.org/10.1002/nla.445.

[18] A. Feriani, F. Perotti and V. Simoncini. Iterative system solvers for the frequency analysis of linear mechanical systems. *Comput. Methods Appl. Mech. Eng.*, **190**:1719–1739, 2000. http://dx.doi.org/10.1016/S0045-7825(00)00187-0.

[19] A. Frommer, T. Lippert, B. Medeke and K. Schilling (eds). *Numerical Challenges in Lattice Quantum Chromodynamics.* Springer, Heidelberg, 2000.

[20] G.H. Golub and C.F. Van Loan. *Matrix Computations, 3rd edn.* The Johns Hopkins University Press, Baltimore, 1996.

[21] B. Poirier. Efficient preconditioning scheme for block partitioned matrices with structured sparsity. *Numer. Linear Algebra Appl.*, **7**:715–726, 2000.

[22] Y. Saad and M.H. Schultz. GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Stat. Comput.*, **7**:856–869, 1986. http://dx.doi.org/10.1137/0907058.

[23] W. van Dijk and F.M. Toyama. Accurate numerical solutions of the time-dependent Schrödinger equation. *Phys. Rev. E*, **75**:036707, 2007. http://dx.doi.org/10.1103/PhysRevE.75.036707.

[24] A.-L. Yang, J. An and Y.-J. Wu. A generalized preconditioned HSS method for non-Hermitian positive definite linear systems. *Appl. Math. Comput.*, **216**:1715–1722, 2010. http://dx.doi.org/10.1016/j.amc.2009.12.032.

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