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Field-of-values analysis of preconditioned iterative methods for nonsymmetric elliptic problems

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Summary. The convergence rate of Krylov subspace methods for the solution of nonsymmetric systems of linear equations, such as GMRES or FOM, is studied. Bounds on the convergence rate are presented which are based on the smallest real part of the field of values of the coefficient matrix and of its inverse. Estimates for these quantities are available during the iteration from the underlying Arnoldi process. It is shown how these bounds can be used to study the convergence properties, in particular, the dependence on the mesh-size and on the size of the skew-symmetric part, for preconditioners for finite element discretizations of nonsymmetric elliptic boundary value problems. This is illustrated for the hierarchical basis and multilevel preconditioners which constitute popular preconditioning strategies for such problems.

Key words: Krylov subspace methods, GMRES, FOM, field of values, hierarchical basis, multilevel preconditioning, nonsymmetric elliptic problems

Mathematics Subject Classification (1991): 65F10, 65N30, 65N55

1. Introduction

The subject of this paper is to provide a framework for the convergence analysis of preconditioning strategies for nonsymmetric elliptic boundary value problems. Bounds on the convergence rate of Krylov subspace methods like GMRES or FOM are established based on the smallest real part of the field of values of the coefficient matrix and of its inverse. These quantities can be bounded in a natural way in the finite element context and can be estimated in the course of the Krylov subspace iteration. Moreover, this gives an improvement of the bounds by Elman [9, Theorem 5.4 and 5.9] (cf. also [8, Theorem 3.3]) which use the smallest real part of the field of values and the norm of the matrix.

A more detailed analysis of iterative methods, in particular, the Chebyshev semi-iterative method, using fields of values was developed by Eiermann in [7]. If one could enclose the fields of values of the preconditioned elliptic operators by appropriate ellipses, the techniques in [7] could be used for a refinement of the convergence analysis given in this paper. We hope to obtain results in this direction in the future.

We apply this framework to hierarchical basis and multilevel preconditioning (see Yserentant [21]). For the hierarchical basis preconditioner this extends the results by Yserentant in [19] who did not explicitly consider the dependence on the size of the convection term (see also [18] for similar bounds for multilevel preconditioning). For the multilevel preconditioner, a similar result to the one presented in Sect. 5 was derived along a different line of proof in [17]. Substructuring preconditioning in the nonsymmetric case can be analyzed along the same lines (cf. [16]).

For convection-dominated problems the standard Galerkin method does not produce physically meaningful approximations, in general. In this case, stabilized discretization techniques like the streamline diffusion method (see, e.g., [12]) have to be used. The hierarchical basis multigrid method in combination with streamline diffusion was studied in [3]. In this paper, we restrict ourselves to moderate convection sizes so that Galerkin discretization is appropriate. Note that our aim is the analysis of the field-of-values framework as a convergence measure and not the construction of efficient preconditioners for convection-dominated problems. It is our hope that the framework developed in this paper can be extended to deal with such problems.

The following section provides the basic facts about Galerkin finite element approximation. In Sect. 3 Krylov subspace methods, in particular GMRES and FOM, are introduced and the convergence analysis based on the field of values is presented. Section 4 shows how the quantities used in the analysis can be approximated in the course of the Krylov subspace iteration. The convergence bounds are applied to hierarchical basis and multilevel preconditioning in Sect. 5. Finally, Sect. 6 shows the results of some computational experiments and the paper ends with some concluding remarks.

2. Background on Galerkin finite element approximation

For simplicity, we restrict ourselves to the solution of second-order elliptic equations in a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$. In weak form, this means that our aim is to find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v)_{0,\Omega} \text{ for all } v \in H_0^1(\Omega) \tag{2.1}$$

holds with a bilinear form $a(\cdot, \cdot)$ on $H_0^1(\Omega)$ which we assume to be elliptic and bounded. More precisely, we split this bilinear form into its symmetric and skew-symmetric parts

$$a^{\mathrm{sy}}(u,v) = \frac{1}{2}(a(u,v)+a(v,u))\,,\; a^{\mathrm{sk}}(u,v) = \frac{1}{2}(a(u,v)-a(v,u))\,,$$

respectively, and assume that

$$\gamma_1 \|u\|_{1,\Omega}^2 \le a(u,u) \le \gamma_2 \|u\|_{1,\Omega}^2 \text{ for all } u \in H_0^1(\Omega) \tag{2.2}$$

and

$$|a^{\mathrm{sk}}(u,v)| \le \tilde{\gamma} \|u\|_{1,\Omega} \|v\|_{0,\Omega} \text{ for all } u,v \in H_0^1(\Omega) \tag{2.3}$$

are fulfilled with positive constants $\gamma_1, \gamma_2, \tilde{\gamma}$. While we assume γ_1 and γ_2 of order 1, we are interested on the dependence of our convergence estimates on $\tilde{\gamma}$ which measures the size of the convection term.

Suppose that our finite element approximation is based on a quasi-uniform family of triangulations $\{\mathscr{T}_l\}_{l=0,1,\ldots}$ of Ω (cf. [10, p. 98] or [6, p. 106]). It is tacitly assumed here that, for all $l \in \mathbb{N}$, $\eta_1 h_{l-1} \le h_l \le \eta_2 h_{l-1}$ for some $\eta_1 < \eta_2 < 1$. For these triangulations we use standard piecewise linear or piecewise bilinear functions on triangles or rectangles, respectively. This results in the nested family of finite element spaces

$$V_0 \subset V_1 \subset \cdots \subset V_{l-1} \subset V_l \subset V_{l+1} \subset \cdots \subset H_0^1(\Omega)\,. \tag{2.4}$$

It is well-known that, under the above assumptions on the finite element spaces, an inverse inequality of the form

$$\|v_l\|_{1,\Omega} \le \frac{c}{h_l} \|v_l\|_{0,\Omega} \text{ for all } v_l \in V_l \tag{2.5}$$

holds (cf. [6, Theorem 4.5.11]). (Throughout this paper, the symbols $c, c_0, c_1, c_2, \ldots$ will be used to denote constants which are independent of h_l and the size of the skew-symmetric part $\tilde{\gamma}$).

The Galerkin approximation $u_l \in V_l$ is defined by

$$a(u_l, v_l) = (f, v_l)_{0,\Omega} \text{ for all } v_l \in V_l\,. \tag{2.6}$$

In the implementation of the finite element method, this approximation is usually represented as linear combination of nodal basis functions. That is, a finite element function $u_l \in V_l$ is stored as a vector $\mathbf{u}_l \in \mathbb{R}^{N_l}$ which contains the values of u_l at the nodes of the triangulation. The matrix $\hat{A}_l \in \mathbb{R}^{N_l \times N_l}$ is defined by

$$\mathbf{w}_l^{\mathrm{T}} \hat{A}_l \mathbf{v}_l = a(v_l, w_l) \text{ for all } v_l, w_l \in V_l\,. \tag{2.7}$$

The representation of $\hat{A}_l$ with respect to the nodal basis leads to the standard stiffness matrix, i.e., the matrix coefficients are given by $a(\Phi_l^{(\nu)}, \Phi_l^{(\mu)})$ where $\{\Phi_l^{(\mu)}\}_{\mu=1}^{N_l}$ are the nodal basis functions. Right-hand sides (and residuals) are represented as moments with respect to the nodal basis, i.e., $\mathbf{f}_l \in \mathbb{R}^{N_l}$ with elements $[(f_l, \Phi_l^{(\mu)})_{0,\Omega}]_{\mu=1}^{N_l}$, and the variational problem (2.6) is equivalent to the system of linear equations $\hat{A}_l \mathbf{u}_l = \mathbf{f}_l$.

3. Krylov subspace methods and fields of values

There are two basic approaches to Krylov subspace methods for nonsymmetric problems. Either a Galerkin condition with respect to the Krylov subspace can be enforced, or the iterates can be computed such that the residual is minimized. Using Arnoldi's method [1] to construct an orthonormal basis for the Krylov subspace, this leads to the FOM (Saad [14]) or the GMRES (Saad and Schultz [15]) algorithms, respectively.

Given a vector $\mathbf{v}^{(1)} \in \mathbb{R}^N$ with $\|\mathbf{v}^{(1)}\| = 1$, Arnoldi's method computes an orthonormal basis for the Krylov subspace

$$\mathscr{K}_n(\mathbf{v}^{(1)}, A) = \mathrm{span}\{\mathbf{v}^{(1)}, A\mathbf{v}^{(1)}, \ldots, A^{n-1}\mathbf{v}^{(1)}\}$$

associated with the matrix A by means of a modified Gram-Schmidt procedure. With $V^{(n)} = [\mathbf{v}^{(1)} \ \ldots \ \mathbf{v}^{(n)}] \in \mathbb{R}^{N\times n}$ and with $H^{(n)} = [h_{i,j}]_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$, the Gram-Schmidt process can be written in matrix notation as

$$AV^{(n)} = V^{(n)}H^{(n)} + \hat{\mathbf{v}}^{(n+1)}\mathbf{e}_n^{\mathrm{T}} = V^{(n)}H^{(n)} + h_{n+1,n}\mathbf{v}^{(n+1)}\mathbf{e}_n^{\mathrm{T}} = V^{(n+1)}\tilde{H}^{(n)} \tag{3.1}$$

(where $\mathbf{e}_n = [0 \ \ldots \ 0 \ 1]^{\mathrm{T}} \in \mathbb{R}^n$ and $\tilde{H}^{(n)} := [h_{i,j}] \in \mathbb{R}^{n+1\times n}$ denotes the extended Hessenberg matrix).

In our applications, A is the preconditioned stiffness matrix $\hat{C}_l^{-1}\hat{A}_l$ and the inner product is defined by $\langle \mathbf{v}_l, \mathbf{w}_l\rangle_l = \mathbf{w}_l^{\mathrm{T}}\hat{C}_l\mathbf{v}_l$ (the corresponding norm is denoted by $|||\cdot|||_l$). Note that this requires the preconditioning matrix $\hat{C}_l$ to be symmetric and positive definite, a condition which is fulfilled for our versions of the hierarchical basis and multilevel preconditioners presented in Sect. 5. In the context of the iterative solution of linear systems, the starting vector is the normalized initial residual, i.e., $\mathbf{v}_l^{(1)} = \hat{C}_l^{-1}\mathbf{r}_l^{(0)}/\beta$ with $\mathbf{r}_l^{(0)} = \mathbf{f}_l - \hat{A}_l\mathbf{u}_l^{(0)}$ and $\beta = |||\hat{C}_l^{-1}\mathbf{r}_l^{(0)}|||_l$. Using Arnoldi's method for the construction of an orthonormal basis (with respect to $\langle\cdot,\cdot\rangle_l$) in a straightforward way would require a matrix-vector product with $\hat{C}_l$ for every computation of an inner product. This can be avoided, at the cost of storing two sequences of vectors, by the following algorithm, starting with $\mathbf{v}_l^{(1)} = \hat{C}_l^{-1}\mathbf{r}_l^{(0)}/\beta$ and $\mathbf{w}_l^{(1)} = \mathbf{r}_l^{(0)}/\beta$ (note that $\beta = ((\mathbf{r}_l^{(0)})^{\mathrm{T}}\hat{C}_l^{-1}\mathbf{r}_l^{(0)})^{1/2}$):

$$
\begin{array}{l}
\texttt{for } j = 1:n,\\
\quad \tilde{\mathbf{w}}_l^{(j+1)} = \hat{A}_l\mathbf{v}_l^{(j)}; \quad \tilde{\mathbf{v}}_l^{(j+1)} = \hat{C}_l^{-1}\tilde{\mathbf{w}}_l^{(j+1)};\\
\quad \texttt{for } i = 1:j,\\
\qquad h_{i,j} = (\mathbf{v}_l^{(i)})^{\mathrm{T}}\tilde{\mathbf{w}}_l^{(j+1)}; \quad \tilde{\mathbf{w}}_l^{(j+1)} = \tilde{\mathbf{w}}_l^{(j+1)} - h_{i,j}\mathbf{w}_l^{(i)};\\
\qquad\qquad \tilde{\mathbf{v}}_l^{(j+1)} = \tilde{\mathbf{v}}_l^{(j+1)} - h_{i,j}\mathbf{v}_l^{(i)};\\
\qquad \texttt{end}\\
\quad h_{j+1,j} = (\tilde{\mathbf{v}}_l^{(j+1)})^{\mathrm{T}}\tilde{\mathbf{w}}_l^{(j+1)}; \quad \mathbf{v}_l^{(j+1)} = \tilde{\mathbf{v}}_l^{(j+1)}/h_{j+1,j};\\
\qquad\qquad\qquad \mathbf{w}_l^{(j+1)} = \tilde{\mathbf{w}}_l^{(j+1)}/h_{j+1,j};\\
\quad \texttt{end}
\end{array}
$$

In matrix notation, this algorithm can be written, similar to (3.1), as

$$\hat{A}_lV_l^{(n)} = W_l^{(n+1)}\tilde{H}^{(n)}, \ V_l^{(n+1)} = \hat{C}_l^{-1}W_l^{(n+1)}. \tag{3.2}$$

Using the orthogonality relation $(V_l^{(n)})^{\mathrm{T}} W_l^{(n+1)} = I_{n,n+1}$ (with $I_{n,n+1} \in \mathbb{R}^{n\times n+1}$ being the n-by-n identity matrix augmented by an additional column of zeros), (3.2) implies

$$(V_l^{(n)})^{\mathrm{T}} \hat{A}_l V_l^{(n)} = H^{(n)} \tag{3.3}$$

(where $H^{(n)} \in \mathbb{R}^{n\times n}$ denotes the square Hessenberg matrix).

The FOM algorithm is based on a Galerkin condition of the form

$$a(u_l - u_l^{(n)}, v_l^{(n)}) = 0 \text{ for all } v_l^{(n)} \in V_l^{(n)} ,$$

where $V_l^{(n)} \subset V_l$ denotes the subspace of finite element functions with corresponding nodal basis representation in the Krylov subspace $\mathscr{K}_n(\mathbf{r}_l^{(0)}, \hat{A}_l)$. Obviously, this leads to

$$0 = (V_l^{(n)})^{\mathrm{T}} \hat{A}_l (\mathbf{u}_l - \mathbf{u}_l^{(n)}) = (V_l^{(n)})^{\mathrm{T}} \mathbf{r}_l^{(n)} = (V_l^{(n)})^{\mathrm{T}} (\mathbf{r}_l^{(0)} - \hat{A}_l V_l^{(n)} \mathbf{y}) \tag{3.4}$$

with $\mathbf{u}_l^{(n)} = \mathbf{u}_l^{(0)} + V_l^{(n)}\mathbf{y}$. Using (3.3) and $\mathbf{r}_l^{(0)} = \beta \mathbf{w}_l^{(1)}$, this is equivalent to $H^{(n)}\mathbf{y} = \beta \mathbf{e}_1$ (with $\mathbf{e}_1 = [1\ 0\ \dots\ 0]^{\mathrm{T}} \in \mathbb{R}^n$). The FOM iterate is then given by

$$\mathbf{u}_l^{(n)} = \mathbf{u}_l^{(0)} + \beta V_l^{(n)} (H^{(n)})^{-1} \mathbf{e}_1 . \tag{3.5}$$

The GMRES algorithm computes its iterates such that the residual is minimized with respect to the norm used in Arnoldi's method. With $\mathbf{u}_l^{(n)} = \mathbf{u}_l^{(0)} + V_l^{(n)}\mathbf{y}$, we have, for the nth residual vector,

$$\mathbf{r}_l^{(n)} = \mathbf{r}_l^{(0)} - \hat{A}_l V_l^{(n)} \mathbf{y} = \beta \mathbf{w}_l^{(1)} - W_l^{(n+1)} \tilde{H}^{(n)} \mathbf{y} = W_l^{(n+1)} (\beta \mathbf{e}_1 - \tilde{H}^{(n)} \mathbf{y}) .$$

Using the orthogonality relation $(W_l^{(n+1)})^{\mathrm{T}} \hat{C}_l^{-1} W_l^{(n+1)} = (W_l^{(n+1)})^{\mathrm{T}} V_l^{(n+1)} = I_{n+1}$, this leads to

$$\begin{aligned} |||\hat{C}_l^{-1}\mathbf{r}_l^{(n)}|||_l^2 &= (\mathbf{r}_l^{(n)})^{\mathrm{T}} \hat{C}_l^{-1} \mathbf{r}_l^{(n)} \\ &= (\beta \mathbf{e}_1 - \tilde{H}^{(n)}\mathbf{y})^{\mathrm{T}} (W_l^{(n+1)})^{\mathrm{T}} \hat{C}_l^{-1} W_l^{(n+1)} (\beta \mathbf{e}_1 - \tilde{H}^{(n)}\mathbf{y}) \\ &= (\beta \mathbf{e}_1 - \tilde{H}^{(n)}\mathbf{y})^{\mathrm{T}} (\beta \mathbf{e}_1 - \tilde{H}^{(n)}\mathbf{y}) . \end{aligned} \tag{3.6}$$

The remaining least-squares problem of minimizing this quantity can be solved using a QR decomposition of the Hessenberg matrix $\tilde{H}^{(n)}$ which is updated from step to step.

The following theorem summarizes the convergence properties of the FOM algorithm (see also [2, Sect. 13.1] for results similar to (3.7).

Theorem 3.1. *Let $V_l^{(n)} \subset V_l$ denote the subspace of finite element functions with nodal basis representation in the Krylov subspace $\mathscr{K}_n(\mathbf{r}_l^{(0)}, \hat{A}_l)$, then the FOM iterate $u_l^{(n)} \in V_l^{(n)}$ satisfies*

$$a(u_l - u_l^{(n)}, u_l - u_l^{(n)}) \le \left(1 + \frac{\tilde{\gamma}}{\gamma_1}\right)^2 \inf_{v_l^{(n)} \in V_l^{(n)}} a(u_l - v_l^{(n)}, u_l - v_l^{(n)}) . \tag{3.7}$$

Moreover, with

$$\text{(3.8)}\quad \tau_l^{\rm FOM} = \inf_{\mathbf{0}\neq\mathbf{w}_l\in\mathbb{R}^{N_l}} \frac{\mathbf{w}_l^{\rm T}\hat{A}_l^{\rm sy}\hat{C}_l^{-1}\hat{A}_l\mathbf{w}_l}{\mathbf{w}_l^{\rm T}\hat{A}_l^{\rm sy}\mathbf{w}_l}\ , \quad \tilde{\tau}_l^{\rm FOM} = \inf_{\mathbf{0}\neq\mathbf{w}_l\in\mathbb{R}^{N_l}} \frac{\mathbf{w}_l^{\rm T}\hat{A}_l^{\rm sy}\hat{A}_l^{-1}\hat{C}_l\mathbf{w}_l}{\mathbf{w}_l^{\rm T}\hat{A}_l^{\rm sy}\mathbf{w}_l}\ ,$$

where $\hat{A}_l^{\rm sy} = (\hat{A}_l + \hat{A}_l^{\rm T})/2$*, we have*

$$\begin{aligned} &a(u_l - u_l^{(n)}, u_l - u_l^{(n)}) \\ \text{(3.9)}\qquad &\le \left(1 + \frac{\tilde{\gamma}}{\gamma_1}\right)^2 \left(1 - \tau_l^{\rm FOM}\cdot\tilde{\tau}_l^{\rm FOM}\right)^n a(u_l - u_l^{(0)}, u_l - u_l^{(0)})\ . \end{aligned}$$

Proof. From the definition of the FOM iterate the associated residual is orthogonal to the Krylov subspace and therefore, for all $v_l^{(n)} \in V_l^{(n)}$,

$$\begin{aligned} a(u_l - u_l^{(n)}, u_l - u_l^{(n)}) &= a(u_l - u_l^{(n)}, u_l - v_l^{(n)}) \\ = a^{\rm sy}(u_l - u_l^{(n)}, u_l - v_l^{(n)}) &+ a^{\rm sk}(u_l - u_l^{(n)}, u_l - v_l^{(n)}) \\ \le a(u_l - u_l^{(n)}, u_l - u_l^{(n)})^{1/2} &a(u_l - v_l^{(n)}, u_l - v_l^{(n)})^{1/2} \\ +\tilde{\gamma}\|u_l - v_l^{(n)}\|_{1,\Omega}&\|u_l - u_l^{(n)}\|_{1,\Omega}\ , \end{aligned}$$

where we used (2.3) for the last inequality. With (2.2) this leads to

$$a(u_l - u_l^{(n)}, u_l - u_l^{(n)})^{1/2} \le (1 + \frac{\tilde{\gamma}}{\gamma_1}) a(u_l - v_l^{(n)}, u_l - v_l^{(n)})^{1/2}$$

which implies (3.7). Translated into matrix notation, (3.7) reads

$$(\mathbf{u}_l - \mathbf{u}_l^{(n)})^{\rm T}\hat{A}_l(\mathbf{u}_l - \mathbf{u}_l^{(n)}) \le (1 + \frac{\tilde{\gamma}}{\gamma_1})^2 (\mathbf{u}_l - \mathbf{v}_l^{(n)})^{\rm T}\hat{A}_l(\mathbf{u}_l - \mathbf{v}_l^{(n)})$$

for all $\mathbf{v}_l^{(n)} \in \mathscr{K}_n(\mathbf{r}_l^{(0)}, \hat{A}_l)$. In particular, this inequality holds for the special sequence of iterates $\tilde{\mathbf{v}}_l^{(j)}$ defined recursively by

$$\begin{aligned} \tilde{\mathbf{v}}_l^{(j+1)} &= \tilde{\mathbf{v}}_l^{(j)} + \alpha_l^{(j)}\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)} \text{ with } \tilde{\mathbf{r}}_l^{(j)} = \mathbf{f}_l - \hat{A}_l\tilde{\mathbf{v}}_l^{(j)}\ , \\ \alpha_l^{(j)} &= \frac{(\mathbf{u}_l - \tilde{\mathbf{v}}_l^{(j)})^{\rm T}\hat{A}_l^{\rm sy}\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)}}{(\tilde{\mathbf{r}}_l^{(j)})^{\rm T}\hat{C}_l^{-1}\hat{A}_l\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)}} \end{aligned}$$

with $\tilde{\mathbf{v}}_l^{(0)} = \mathbf{u}_l^{(0)}$. This leads to

$$\begin{aligned} &(\mathbf{u}_l - \tilde{\mathbf{v}}_l^{(j+1)})^{\rm T}\hat{A}_l(\mathbf{u}_l - \tilde{\mathbf{v}}_l^{(j+1)}) = (\mathbf{u}_l - \tilde{\mathbf{v}}_l^{(j)})^{\rm T}\hat{A}_l(\mathbf{u}_l - \tilde{\mathbf{v}}_l^{(j)}) \\ &\quad - \frac{((\mathbf{u}_l - \tilde{\mathbf{v}}_l^{(j)})^{\rm T}\hat{A}_l^{\rm sy}\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)})^2}{(\tilde{\mathbf{r}}_l^{(j)})^{\rm T}\hat{C}_l^{-1}\hat{A}_l\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)}} = (\mathbf{u}_l - \tilde{\mathbf{v}}_l^{(j)})^{\rm T}\hat{A}_l(\mathbf{u}_l - \tilde{\mathbf{v}}_l^{(j)}) \times \\ &\quad \times \left[1 - \frac{(\mathbf{u}_l - \tilde{\mathbf{v}}_l^{(j)})^{\rm T}\hat{A}_l^{\rm sy}\hat{C}_l^{-1}\hat{A}_l(\mathbf{u}_l - \tilde{\mathbf{v}}_l^{(j)})}{(\mathbf{u}_l - \tilde{\mathbf{v}}_l^{(j)})^{\rm T}\hat{A}_l(\mathbf{u}_l - \tilde{\mathbf{v}}_l^{(j)})} \cdot \frac{(\tilde{\mathbf{r}}_l^{(j)})^{\rm T}\hat{C}_l^{-1}\hat{A}_l^{\rm sy}\hat{A}_l^{-1}\hat{C}_l\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)}}{(\tilde{\mathbf{r}}_l^{(j)})^{\rm T}\hat{C}_l^{-1}\hat{A}_l\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)}}\right] \end{aligned}$$

where the first quotient term in the brackets is bounded from below by $\tau_l^{\rm FOM}$ and the second one is bounded from below by $\tilde{\tau}_l^{\rm FOM}$. □

A similar result for the GMRES algorithm (of minimizing $|||\hat{C}_l^{-1}\mathbf{r}_l^{(n)}|||_l^2$) is given in the following theorem.

Theorem 3.2. *With*

$$
\begin{aligned}
\tau_l^{\mathrm{GMRES}} &= \inf_{\mathbf{0}\neq\mathbf{w}_l\in\mathbb{R}^{N_l}} \frac{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l\mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}}\hat{C}_l\mathbf{w}_l}\,, \\
\tilde{\tau}_l^{\mathrm{GMRES}} &= \inf_{\mathbf{0}\neq\mathbf{w}_l\in\mathbb{R}^{N_l}} \frac{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l^{-1}\mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}}\hat{C}_l^{-1}\mathbf{w}_l}
\end{aligned}
\tag{3.10}
$$

we have

$$
|||\hat{C}_l^{-1}\mathbf{r}_l^{(n)}|||_l \leq \left(1-\tau_l^{\mathrm{GMRES}}\tilde{\tau}_l^{\mathrm{GMRES}}\right)^{n/2}|||\hat{C}_l^{-1}\mathbf{r}_l^{(0)}|||_l\,. \tag{3.11}
$$

Proof. Clearly, the minimal residual property implies that

$$
|||\hat{C}_l^{-1}\mathbf{r}_l^{(n)}|||_l \leq |||\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(n)}|||_l
$$

for any residual $\tilde{\mathbf{r}}_l^{(n)}$ associated with an iterate $\mathbf{u}_l^{(n)} = \mathbf{u}_l^{(0)} + \mathscr{K}_n(\hat{C}_l^{-1}\mathbf{r}_l^{(0)}, \hat{C}_l^{-1}\hat{A}_l)$. In particular, this inequality holds for the special sequence of iterates $\tilde{\mathbf{v}}_l^{(j)}$ defined recursively by

$$
\begin{aligned}
\tilde{\mathbf{v}}_l^{(j+1)} &= \tilde{\mathbf{v}}_l^{(j)} + \alpha_l^{(j)}\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)} \text{ with } \tilde{\mathbf{r}}_l^{(j)} = \mathbf{f}_l - \hat{A}_l\tilde{\mathbf{v}}_l^{(j)}\,, \\
\alpha_l^{(j)} &= \frac{(\tilde{\mathbf{r}}_l^{(j)})^{\mathrm{T}}\hat{C}_l^{-1}\hat{A}_l\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)}}{(\tilde{\mathbf{r}}_l^{(j)})^{\mathrm{T}}\hat{C}_l^{-1}\hat{A}_l^{\mathrm{T}}\hat{C}_l^{-1}\hat{A}_l\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)}}
\end{aligned}
$$

with $\tilde{\mathbf{v}}_l^{(0)} = \mathbf{u}_l^{(0)}$. This leads to

$$
\begin{aligned}
|||\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j+1)}|||_l^2 &= (\tilde{\mathbf{r}}_l^{(j+1)})^{\mathrm{T}}\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j+1)} \\
&= (\tilde{\mathbf{r}}_l^{(j)})^{\mathrm{T}}\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)} - \frac{((\tilde{\mathbf{r}}_l^{(j)})^{\mathrm{T}}\hat{C}_l^{-1}\hat{A}_l\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)})^2}{(\tilde{\mathbf{r}}_l^{(j)})^{\mathrm{T}}\hat{C}_l^{-1}\hat{A}_l^{\mathrm{T}}\hat{C}_l^{-1}\hat{A}_l\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)}} \\
&= |||\tilde{\mathbf{r}}_l^{(j)}|||_l^2\left[1-\frac{(\tilde{\mathbf{s}}_l^{(j)})^{\mathrm{T}}\hat{A}_l\tilde{\mathbf{s}}_l^{(j)}}{(\tilde{\mathbf{s}}_l^{(j)})^{\mathrm{T}}\hat{C}_l\tilde{\mathbf{s}}_l^{(j)}}\cdot\frac{(\tilde{\mathbf{t}}_l^{(j)})^{\mathrm{T}}\hat{A}_l^{-1}\tilde{\mathbf{t}}_l^{(j)}}{(\tilde{\mathbf{t}}_l^{(j)})^{\mathrm{T}}\hat{C}_l^{-1}\tilde{\mathbf{t}}_l^{(j)}}\right]
\end{aligned}
$$

with $\tilde{\mathbf{s}}_l^{(j)} = \hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)}$ and $\tilde{\mathbf{t}}_l^{(j)} = \hat{A}_l\hat{C}_l^{-1}\tilde{\mathbf{r}}_l^{(j)}$. Obviously, this implies (3.11). □

We end this section with a remark on the interpretation of the quantities appearing in Theorems 3.1 and 3.2 in terms of fields of values. As is well-known, the field of values of a matrix A with respect to an inner product defined by a symmetric and positive definite matrix B is given by

$$
W_B(A) = \left\{\frac{\mathbf{x}^{\mathrm{H}}BA\mathbf{x}}{\mathbf{x}^{\mathrm{H}}B\mathbf{x}} : \mathbf{0}\neq\mathbf{x}\in\mathbb{C}^N\right\}
$$

(cf. Horn and Johnson [11, Sect. 1.8]). Theorems 1.5.11 and 1.5.12 in [11] imply that

$$\tau_l^{\mathrm{FOM}} = \inf_{\mathbf{0}\neq\mathbf{w}_l\in\mathbb{R}^{N_l}} \frac{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l^{\mathrm{sy}}\hat{C}_l^{-1}\hat{A}_l\mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l^{\mathrm{sy}}\mathbf{w}_l} = \inf\{\mathrm{Re}z : z \in W_{\hat{A}_l^{\mathrm{sy}}}(\hat{C}_l^{-1}\hat{A}_l)\} ,$$

$$\tilde{\tau}_l^{\mathrm{FOM}} = \inf_{\mathbf{0}\neq\mathbf{w}_l\in\mathbb{R}^{N_l}} \frac{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l^{\mathrm{sy}}\hat{A}_l^{-1}\hat{C}_l\mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l^{\mathrm{sy}}\mathbf{w}_l} = \inf\{\mathrm{Re}z : z \in W_{\hat{A}_l^{\mathrm{sy}}}(\hat{A}_l^{-1}\hat{C}_l)\} ,$$

$$\tau_l^{\mathrm{GMRES}} = \inf_{\mathbf{0}\neq\mathbf{w}_l\in\mathbb{R}^{N_l}} \frac{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l\mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}}\hat{C}_l\mathbf{w}_l} = \inf\{\mathrm{Re}z : z \in W_{\hat{C}_l}(\hat{C}_l^{-1}\hat{A}_l)\}$$

and

$$\tilde{\tau}_l^{\mathrm{GMRES}} = \inf_{\mathbf{0}\neq\mathbf{w}_l\in\mathbb{R}^{N_l}} \frac{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l^{-1}\mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}}\hat{C}_l^{-1}\mathbf{w}_l} = \inf\{\mathrm{Re}z : z \in W_{\hat{C}_l}(\hat{A}_l^{-1}\hat{C}_l)\} .$$

4. Krylov subspace estimates for τ_l and $\tilde{\tau}_l$

The numbers $\tau_l^{\mathrm{FOM}}, \tilde{\tau}_l^{\mathrm{FOM}}$ and $\tau_l^{\mathrm{GMRES}}, \tilde{\tau}_l^{\mathrm{GMRES}}$ can be estimated during the course of the iteration. These estimates will prove to be useful in predicting the dependence of the convergence behavior on parameters like the mesh-size or the size of the convection term $\tilde{\gamma}$. By restricting the infimum to the Krylov subspace so far constructed, we obtain for the FOM algorithm

$$\begin{aligned}
\tau_l^{\mathrm{FOM}} &\approx \inf_{\mathbf{0}\neq\mathbf{w}_l\in\mathscr{K}_n(\hat{C}_l^{-1}\mathbf{r}_l^{(0)},\hat{C}_l^{-1}\hat{A}_l)} \frac{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l^{\mathrm{sy}}\hat{C}_l^{-1}\hat{A}_l\mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l^{\mathrm{sy}}\mathbf{w}_l} \\
&= \inf_{\mathbf{0}\neq\mathbf{z}\in\mathbb{R}^n} \frac{\mathbf{z}^{\mathrm{T}}(V_l^{(n)})^{\mathrm{T}}\hat{A}_l^{\mathrm{sy}}\hat{C}_l^{-1}\hat{A}_l V_l^{(n)}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}(V_l^{(n)})^{\mathrm{T}}\hat{A}_l V_l^{(n)}\mathbf{z}} \\
&= \inf_{\mathbf{0}\neq\mathbf{z}\in\mathbb{R}^n} \frac{\mathbf{z}^{\mathrm{T}}(V_l^{(n)})^{\mathrm{T}}\hat{A}_l^{\mathrm{sy}} V_l^{(n+1)}\tilde{H}^{(n)}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}H^{(n)}\mathbf{z}} \\
&= \inf_{\mathbf{0}\neq\mathbf{z}\in\mathbb{R}^n} \frac{\mathbf{z}^{\mathrm{T}}(I_{n,n+2}\tilde{H}^{(n+1)} + (\tilde{H}^{(n)})^{\mathrm{T}})\tilde{H}^{(n)}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}H^{(n)}\mathbf{z}}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\tau}_l^{\mathrm{FOM}} &\approx \inf_{\mathbf{0}\neq\hat{A}_l^{-1}\hat{C}_l\mathbf{w}_l\in\mathscr{K}_n(\hat{C}_l^{-1}\mathbf{r}_l^{(0)},\hat{C}_l^{-1}\hat{A}_l)} \frac{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l^{\mathrm{sy}}\hat{A}_l^{-1}\hat{C}_l\mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l^{\mathrm{sy}}\mathbf{w}_l} \\
&= \inf_{\mathbf{0}\neq\mathbf{z}\in\mathbb{R}^n} \frac{\mathbf{z}^{\mathrm{T}}(V_l^{(n)})^{\mathrm{T}}\hat{A}_l^{\mathrm{T}}\hat{C}_l^{-1}\hat{A}_l^{\mathrm{sy}}V_l^{(n)}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}(V_l^{(n)})^{\mathrm{T}}\hat{A}_l^{\mathrm{T}}\hat{C}_l^{-1}\hat{A}_l^{\mathrm{sy}}\hat{C}_l^{-1}\hat{A}_l V_l^{(n)}\mathbf{z}} \\
&= \inf_{\mathbf{0}\neq\mathbf{z}\in\mathbb{R}^n} \frac{\mathbf{z}^{\mathrm{T}}(\tilde{H}^{(n)})^{\mathrm{T}}(V_l^{(n+1)})^{\mathrm{T}}\hat{A}_l^{\mathrm{sy}}V_l^{(n)}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}(\tilde{H}^{(n)})^{\mathrm{T}}(V_l^{(n+1)})^{\mathrm{T}}\hat{A}_l^{\mathrm{sy}}V_l^{(n+1)}\tilde{H}^{(n)}\mathbf{z}} \\
&= \inf_{\mathbf{0}\neq\mathbf{z}\in\mathbb{R}^n} \frac{\mathbf{z}^{\mathrm{T}}(I_{n,n+2}\tilde{H}^{(n+1)} + (\tilde{H}^{(n)})^{\mathrm{T}})\tilde{H}^{(n)}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}(\tilde{H}^{(n)})^{\mathrm{T}}H^{(n+1)}\tilde{H}^{(n)}\mathbf{z}} .
\end{aligned}$$

Similarly, for the GMRES algorithm, this leads to

$$\begin{aligned}\tau_l^{\text{GMRES}} &\approx \inf_{\mathbf{0}\neq\mathbf{w}_l\in\mathscr{K}_n(\hat{C}_l^{-1}\mathbf{r}_l^{(0)},\hat{C}_l^{-1}\hat{A}_l)} \frac{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l\mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}}\hat{C}_l\mathbf{w}_l} \\ &= \inf_{\mathbf{0}\neq\mathbf{z}\in\mathbb{R}^n} \frac{\mathbf{z}^{\mathrm{T}}(V_l^{(n)})^{\mathrm{T}}\hat{A}_l V_l^{(n)}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}(V_l^{(n)})^{\mathrm{T}}\hat{C}_l V_l^{(n)}\mathbf{z}} = \inf_{\mathbf{0}\neq\mathbf{z}\in\mathbb{R}^n} \frac{\mathbf{z}^{\mathrm{T}}H^{(n)}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\mathbf{z}}\end{aligned}$$

and

$$\begin{aligned}\tilde{\tau}_l^{\text{GMRES}} &\approx \inf_{\mathbf{0}\neq\hat{A}_l^{-1}\mathbf{w}_l\in\mathscr{K}_n(\hat{C}_l^{-1}\mathbf{r}_l^{(0)},\hat{C}_l^{-1}\hat{A}_l)} \frac{\mathbf{w}_l^{\mathrm{T}}\hat{A}_l^{-1}\mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}}\hat{C}_l^{-1}\mathbf{w}_l} \\ &= \inf_{\mathbf{0}\neq\mathbf{z}\in\mathbb{R}^n} \frac{\mathbf{z}^{\mathrm{T}}(V_l^{(n)})^{\mathrm{T}}\hat{A}_l V_l^{(n)}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}(V_l^{(n)})^{\mathrm{T}}\hat{A}_l^{\mathrm{T}}\hat{C}_l^{-1}\hat{A}_l V_l^{(n)}\mathbf{z}} = \inf_{\mathbf{0}\neq\mathbf{z}\in\mathbb{R}^n} \frac{\mathbf{z}^{\mathrm{T}}H^{(n)}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}(\tilde{H}^{(n)})^{\mathrm{T}}\tilde{H}^{(n)}\mathbf{z}} .\end{aligned}$$

In both cases, these minimal values can be computed by solving eigenvalue problems associated with the symmetric part of the matrices. For example, the Krylov subspace estimate for τ_l^{GMRES},

$$\inf_{\mathbf{0}\neq\mathbf{z}\in\mathbb{R}^n} \frac{\mathbf{z}^{\mathrm{T}}H^{(n)}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\mathbf{z}} ,$$

is given by the smallest eigenvalue of $(H^{(n)} + (H^{(n)})^{\mathrm{T}})/2$. The corresponding estimate for $\tilde{\tau}_l^{\text{GMRES}}$,

$$\inf_{\mathbf{0}\neq\mathbf{z}\in\mathbb{R}^n} \frac{\mathbf{z}^{\mathrm{T}}H^{(n)}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}(\tilde{H}^{(n)})^{\mathrm{T}}\tilde{H}^{(n)}\mathbf{z}} ,$$

is given by the smallest solution λ of the generalized eigenvalue problem

$$\frac{H^{(n)} + (H^{(n)})^{\mathrm{T}}}{2}\mathbf{y} = \lambda(\tilde{H}^{(n)})^{\mathrm{T}}\tilde{H}^{(n)}\mathbf{y} .$$

5. Hierarchical basis and multilevel preconditioning

This section gives an analysis of hierarchical basis and multilevel preconditioning in the nonsymmetric case. Our particular interest is in the dependence of the convergence rate on the size of the skew-symmetric part $\tilde{\gamma}$. For the hierarchical basis preconditioner this extends the results by Yserentant in [19] who did not explicitly consider the dependence on $\tilde{\gamma}$. For the multilevel preconditioner, the main result of this section was derived along a different line of proof in [17].

Following Yserentant [19, 20], the hierarchical basis preconditioner is best described in terms of basis transformations. In Sect. 2 the discretized problem (2.6) was represented as $\hat{A}_l\mathbf{u}_l = \mathbf{f}_l$ with coefficients $\mathbf{u}_l$ and moments $\mathbf{f}_l$ with respect to the nodal basis representation. Let $\hat{T}_l \in \mathbb{R}^{N_l\times N_l}$ denote the transformation from nodal basis to hierarchical basis representation (for details on the definition and implementation of this transformation, see [20]). This implies that $\hat{T}_l^{\mathrm{T}}$ is the corresponding transformation from hierarchical to nodal moment representation. Setting $\mathbf{v}_l = \hat{T}_l \mathbf{u}_l$ and $\mathbf{f}_l = \hat{T}_l^{\mathrm{T}} \mathbf{g}_l$, this leads to

$$\hat{T}_l^{-\mathrm{T}} \hat{A}_l \hat{T}_l^{-1} \mathbf{v}_l = \mathbf{g}_l \tag{5.1}$$

as the linear system of equations arising from (2.6) with respect to the hierarchical basis.

This fits into the preconditioning framework developed in Sect. 3 in the following way. If we set $\hat{C}_l^{-1} = \hat{T}_l^{-1} \hat{T}_l^{-\mathrm{T}}$, then the GMRES algorithm (3.6) minimizes

$$|||\hat{C}_l^{-1} \mathbf{r}_l^{(n)}|||_l = (\mathbf{r}_l^{(n)})^{\mathrm{T}} \hat{C}_l^{-1} \mathbf{r}_l^{(n)} .$$

This is obviously equivalent to minimizing the (Euclidean) norm of residuals

$$\|\mathbf{s}_l^{(n)}\| = \|\mathbf{g}_l - \hat{T}_l^{-\mathrm{T}} \hat{A}_l \mathbf{u}_l^{(n)}\|$$

associated with (5.1). Using the knowledge of the factors in the decomposition $\hat{C}_l^{-1} = \hat{T}_l^{-1} \hat{T}_l^{-\mathrm{T}}$ and applying the GMRES algorithm directly to (5.1) saves the storage of a second Arnoldi basis which was required in the algorithm (3.2).

Theorem 5.1. *For the GMRES algorithm with hierarchical basis preconditioning, we have*

$$\tau_l^{\mathrm{GMRES/HB}} = \inf_{\mathbf{0} \neq \mathbf{w}_l \in \mathbb{R}^{N_l}} \frac{\mathbf{w}_l^{\mathrm{T}} \hat{A}_l \mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}} \hat{C}_l \mathbf{w}_l} \geq \frac{c_1}{(l+1)^2} \tag{5.2}$$

and

$$\tilde{\tau}_l^{\mathrm{GMRES/HB}} = \inf_{\mathbf{0} \neq \mathbf{w}_l \in \mathbb{R}^{N_l}} \frac{\mathbf{w}_l^{\mathrm{T}} \hat{A}_l^{-1} \mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}} \hat{C}_l^{-1} \mathbf{w}_l} \geq \frac{c_2}{1 + \tilde{\gamma}^2} \tag{5.3}$$

with constants c_1 and c_2 which are independent of l and $\tilde{\gamma}$ ($\tilde{\gamma}$ was defined in (2.3)*).*

Proof. The lower bound in (5.2) follows along the same lines as in the symmetric case from the results in [19, Theorem 1.2] and [20, Theorem 2.2].

In order to prove (5.3), we first observe that (2.3) implies the "weak" Cauchy-Schwarz inequality

$$\begin{aligned} a(v,w) &= a^{\mathrm{sy}}(v,w) + a^{\mathrm{sk}}(v,w) \\ &\leq a(v,v)^{1/2}\, a(w,w)^{1/2} + \tilde{\gamma} \|v\|_{1,\Omega}\, \|w\|_{0,\Omega} \\ &\leq (1 + \tilde{c}_1 \tilde{\gamma})\, a(v,v)^{1/2}\, a(w,w)^{1/2} . \end{aligned} \tag{5.4}$$

Setting $\mathbf{w}_l = \hat{A}_l \hat{T}_l^{-1} \mathbf{z}_l$, we obtain

$$\frac{\mathbf{w}_l^{\mathrm{T}} \hat{A}_l^{-1} \mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}} \hat{C}_l^{-1} \mathbf{w}_l} = \frac{\mathbf{w}_l^{\mathrm{T}} \hat{A}_l^{-1} \mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}} \hat{T}_l^{-1} \hat{T}_l^{-\mathrm{T}} \mathbf{w}_l} = \frac{\mathbf{z}_l^{\mathrm{T}} \hat{T}_l^{-\mathrm{T}} \hat{A}_l \hat{T}_l^{-1} \mathbf{z}_l}{\|\hat{T}_l^{-\mathrm{T}} \hat{A}_l \hat{T}_l^{-1} \mathbf{z}_l\|^2} = \frac{a(z_l, z_l)}{\|\hat{T}_l^{-\mathrm{T}} \hat{A}_l \hat{T}_l^{-1} \mathbf{z}_l\|^2}$$

where the function $z_l \in V_l$ is represented with respect to the hierarchical basis by $\mathbf{z}_l$. We denote by $s_l \in V_l$ the function with the hierarchical basis representation $\mathbf{s}_l = \hat{T}_l^{-\mathrm{T}} \hat{A}_l \hat{T}_l^{-1} \mathbf{z}_l$. (Note that $\mathbf{s}_l$ is originally the moment representation of a residual but we are certainly free to interpret this vector as hierarchical basis representation of a finite element function.) Using (5.4) and the relation $\|s_l\|_{1,\Omega} \le \tilde{c}_2\|\mathbf{s}_l\|$ established in [20, Theorem 2.2], we obtain

$$\begin{aligned}\|\hat{T}_l^{-\mathrm{T}}\hat{A}_l\hat{T}_l^{-1}\mathbf{z}_l\|^2 \;&=\; \mathbf{s}_l^{\mathrm{T}}\hat{T}_l^{-\mathrm{T}}\hat{A}_l\hat{T}_l^{-1}\mathbf{z}_l = a(z_l, s_l) \le (1+\tilde{c}_1\tilde{\gamma})\,a(z_l,z_l)^{1/2}\|s_l\|_{0,\Omega}\\ &\le\; \tilde{c}_2(1+\tilde{c}_1\tilde{\gamma})\,a(z_l,z_l)^{1/2}\,\|\hat{T}_l^{-\mathrm{T}}\hat{A}_l\hat{T}_l^{-1}\mathbf{z}_l\|\;.\end{aligned}$$

This completes the proof of (5.3). □

Multilevel preconditioning (see Bramble, Pasciak and Xu [5] and Yserentant [21]) is based on the decomposition of the residuals into $L^2(\Omega)$-orthogonal sections. The $L^2(\Omega)$ projection $Q_l : L^2(\Omega) \to V_l$ is defined by

$$(Q_l u, v_l)_{0,\Omega} = (u, v_l)_{0,\Omega} \text{ for all } v_l \in V_l\;. \tag{5.5}$$

This gives rise to the additive multilevel preconditioner (in abstract form)

$$C_l^{-1} = \sum_{j=0}^{l} M_j Q_j\;, \tag{5.6}$$

where the operator $M_l : V_l \to V_l$ is given by

$$M_l u_l = \sum_{\nu=1}^{N_l}(u_l, \Phi_l^{(\nu)})_{0,\Omega}\Phi_l^{(\nu)}$$

with respect to the nodal basis $\{\Phi_l^{(\nu)}\}_{\nu=1}^{N_L}$.

Suppose a residual $s_l \in V_l$ is given in the moment representation $\mathbf{s}_l = [(s_l, \Phi_l^{(\mu)})_{0,\Omega}]_{\mu=1}^{N_l}$, then the moment representation of $Q_{l-1}s_l \in V_{l-1}$ can be computed using the fact that each nodal basis function $\Phi_{l-1}^{(\mu)}$ on the coarser level is a linear combination of nodal basis functions $\Phi_l^{(\nu)}$. This defines the restriction operator I_l^{l-1} for computing $I_l^{l-1}\mathbf{s}_l = [(Q_{l-1}s_l, \Phi_{l-1}^{(\mu)})_{0,\Omega}]_{\mu=1}^{N_{l-1}}$. For the implementation of the abstract multilevel preconditioner (5.6) we have to compute the nodal basis representation of $C_l^{-1}s_l$ from the moment representation $\mathbf{s}_l = [(s_l, \Phi_l^{(\mu)})_{0,\Omega}]_{\mu=1}^{N_l}$. From

$$C_l^{-1}s_l = \sum_{j=0}^{l} M_j Q_j s_l = \sum_{j=0}^{l}\sum_{\nu=1}^{N_j}(Q_j s_l, \Phi_j^{(\nu)})\Phi_j^{(\nu)}$$

it is obvious that this is achieved by

$$\hat{C}_l^{-1}\mathbf{s}_l = \sum_{j=0}^{l} I_j^l I_l^j\,\mathbf{s}_l\;, \tag{5.7}$$

where I_l^j for $j < l-1$ is defined similarly as for $j = l-1$ and I_j^l is the interpolation operator for translating the nodal basis representation on level j into the one on level l.

Theorem 5.2. *For the GMRES algorithm with multilevel preconditioning, we have*

$$\tau_l^{\mathrm{GMRES/ML}} = \inf_{\mathbf{0} \neq \mathbf{w}_l \in \mathbb{R}^{N_l}} \frac{\mathbf{w}_l^{\mathrm{T}} \hat{A}_l \mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}} \hat{C}_l \mathbf{w}_l} \geq c_1 \tag{5.8}$$

and

$$\tilde{\tau}_l^{\mathrm{GMRES/ML}} = \inf_{\mathbf{0} \neq \mathbf{w}_l \in \mathbb{R}^{N_l}} \frac{\mathbf{w}_l^{\mathrm{T}} \hat{A}_l^{-1} \mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}} \hat{C}_l^{-1} \mathbf{w}_l} \geq \frac{c_2}{1 + \tilde{\gamma}^2} \tag{5.9}$$

with constants c_1 and c_2 which are independent of l and $\tilde{\gamma}$.

Proof. For quasi-uniform triangulations, it is well-known that

$$\tilde{c}_3 h_l^2 \|v_l\|_{0,\Omega}^2 \leq (M_l v_l, v_l)_{0,\Omega} \leq \tilde{c}_4 h_l^2 \|v_l\|_{0,\Omega}^2$$

with $\tilde{c}_3, \tilde{c}_4$ independent of l. This implies that

$$\tilde{c}_3 \sum_{j=0}^{l} h_j^2 \|Q_j v_l\|_{0,\Omega}^2 \leq (C_l^{-1} v_l, v_l)_{0,\Omega} \leq \tilde{c}_4 \sum_{j=0}^{l} h_j^2 \|Q_j v_l\|_{0,\Omega}^2$$

and

$$\tilde{c}_5 \sum_{j=0}^{l} \frac{1}{h_j^2} \|Q_j v_l\|_{0,\Omega}^2 \leq (C_l v_l, v_l)_{0,\Omega} \leq \tilde{c}_6 \sum_{j=0}^{l} \frac{1}{h_j^2} \|Q_j v_l\|_{0,\Omega}^2 .$$

Moreover, we will use the "multilevel norm equivalence"

$$\tilde{c}_7\, a(v_l, v_l) \leq \sum_{j=0}^{l} \frac{1}{h_j^2} \|Q_j v_l\|_{0,\Omega}^2 \leq \tilde{c}_8\, a(v_l, v_l) \tag{5.10}$$

which was established in a number of recent papers (see e.g. Oswald [13, Theorem 19], Bornemann and Yserentant [4], Zhang [23]).

This immediately leads to (5.8) since

$$\frac{\mathbf{w}_l^{\mathrm{T}} \hat{A}_l \mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}} \hat{C}_l \mathbf{w}_l} = \frac{a(w_l, w_l)}{(C_l w_l, w_l)_{0,\Omega}} \geq \frac{1}{\tilde{c}_6 \tilde{c}_8} .$$

For the second bound (5.9) note that, with $\mathbf{w}_l = \hat{A}_l \mathbf{z}_l$, we have

$$\frac{\mathbf{w}_l^{\mathrm{T}} \hat{A}_l^{-1} \mathbf{w}_l}{\mathbf{w}_l^{\mathrm{T}} \hat{C}_l^{-1} \mathbf{w}_l} = \frac{\mathbf{z}_l^{\mathrm{T}} \hat{A}_l \mathbf{z}_l}{\mathbf{w}_l^{\mathrm{T}} \hat{C}_l^{-1} \mathbf{w}_l} = \frac{a(z_l, z_l)}{(w_l, C_l^{-1} w_l)_{0,\Omega}} ,$$

where $\mathbf{z}_l$ is the nodal basis representation of $z_l \in V_l$ and $\mathbf{w}_l$ is the moment representation of $w_l \in V_l$. The proof then follows from

$$\frac{1}{\tilde{c}_4} (w_l, C_l^{-1} w_l)_{0,\Omega} \leq \sum_{j=0}^{l} h_j^2 \|Q_j w_l\|_{0,\Omega}^2 = \left(w_l, \sum_{j=0}^{l} h_j^2 Q_j w_l \right)_{0,\Omega}$$

$$a\left(z_l, \sum_{j=0}^{l} h_j^2 Q_j w_l\right) \leq (1+\tilde{c}_1\tilde{\gamma})\, a(z_l,z_l)^{1/2}\, a\left(\sum_{j=0}^{l} h_j^2 Q_j w_l, \sum_{j=0}^{l} h_j^2 Q_j w_l\right)^{1/2}$$

$$\leq \frac{1}{\tilde{c}_7}(1+\tilde{c}_1\tilde{\gamma})\, a(z_l,z_l)^{1/2} \left(\sum_{j=0}^{l} h_j^2 \|Q_j w_l\|_{0,\Omega}^2\right)^{1/2}$$

$$\leq \frac{1}{\tilde{c}_3\tilde{c}_7}(1+\tilde{c}_1\tilde{\gamma})\, a(z_l,z_l)^{1/2}\, (w_l, C_l^{-1} w_l)_{0,\Omega}^{1/2}$$

where we used the “weak” Cauchy-Schwarz inequality (5.4). □

6. Computational experiments

This section contains computational experiments carried out for an example of a nonsymmetric elliptic boundary value problems. We consider

$$-\Delta u + \sigma \cdot \nabla u \equiv f \tag{6.1}$$

$$\text{with} \quad f = \begin{cases} 1 \text{ in } (0,0.25)\times(0,0.25)\,, \\ 0 \text{ elsewhere}\,, \end{cases}$$

with homogeneous Dirichlet boundary conditions on the unit square $(0,1)\times(0,1)\subset\mathbb{R}^2$ and constant $\sigma=(\sigma_1,\sigma_2)$. This elliptic boundary value problem is discretized by finite elements using a uniform triangulation into squares of length h_l and piecewise bilinear trial functions. This leads to a 9-point stencil for the unknown values of u at the interior grid points. As initial guess, $u_l^{(0)}\equiv 0$ was used throughout our numerical examples. The coarsest level for both the hierarchical basis and the multilevel preconditioners consisted of just one point, i.e., $h_0=1/2$. In practice, the coarsest mesh-size will be chosen such that the solution on the coarsest mesh is still cheap compared to the overall work required for the evaluation of the preconditioner. In general, this allows for $h_0<1/2$ and the iteration counts in Tables 1 and 2 will be smaller for both preconditioners. The qualitative behavior, however, remains the same. Table 1 contains the computed estimates for $\tau_l^{\mathrm{GMRES/HB}}/\tilde{\tau}_l^{\mathrm{GMRES/HB}}$ and the number of GMRES iterations (in brackets) required to satisfy $|||\hat{C}_l^{-1}\mathbf{r}_l^{(n)}|||_l \leq 10^{-4}|||\hat{C}_l^{-1}\mathbf{r}_l^{(0)}|||_l$ (in brackets) using the hierarchical basis preconditioner (5.1). GMRES was restarted every 20 steps in order to account for the growing work and memory requirements.

Similarly, in Table 2 estimates for the quantities $\tau_l^{\mathrm{GMRES/ML}}/\tilde{\tau}_l^{\mathrm{GMRES/ML}}$ and GMRES iteration counts (in brackets) using the additive multilevel preconditioner (5.7) are listed.

The behavior predicted by Theorems 5.1 and 5.2 can essentially be observed in these two tables. In both cases, the estimates for $\tilde{\tau}_l$ remain bounded with decreasing h_l and the estimates for τ_l are independent on the size of σ (for

Table 1. Number of iterations for GMRES with hierarchical basis preconditioning

σ / h	(10, 10)	(50, 50)	(100, 100)
1/32	0.295 / 0.118 (23)	0.407 / 6.679 $\cdot 10^{-3}$ (72)	0.541 / 1.692 $\cdot 10^{-3}$ (156)
1/64	0.239 / 0.117 (28)	0.310 / 6.645 $\cdot 10^{-3}$ (82)	0.363 / 1.681 $\cdot 10^{-3}$ (193)
1/128	0.199 / 0.117 (32)	0.233 / 6.635 $\cdot 10^{-3}$ (99)	0.254 / 1.678 $\cdot 10^{-3}$ (217)
1/256	0.154 / 0.118 (34)	0.178 / 6.631 $\cdot 10^{-3}$ (106)	0.194 / 1.678 $\cdot 10^{-3}$ (244)

Table 2. Number of iterations for GMRES with multilevel preconditioning

σ / h	(10, 10)	(50, 50)	(100, 100)
1/32	2.131 / 8.321 $\cdot 10^{-2}$ (14)	2.277 / 4.232 $\cdot 10^{-3}$ (39)	2.342 / 1.065 $\cdot 10^{-3}$ (82)
1/64	2.100 / 8.153 $\cdot 10^{-2}$ (15)	2.245 / 4.236 $\cdot 10^{-3}$ (42)	2.251 / 1.065 $\cdot 10^{-3}$ (81)
1/128	2.124 / 8.077 $\cdot 10^{-2}$ (16)	2.194 / 4.243 $\cdot 10^{-3}$ (46)	2.198 / 1.064 $\cdot 10^{-3}$ (88)
1/256	2.177 / 8.191 $\cdot 10^{-2}$ (16)	2.166 / 4.248 $\cdot 10^{-3}$ (48)	2.238 / 1.067 $\cdot 10^{-3}$ (91)

the hierarchical basis preconditioner τ_l actually grows slightly with σ which seems to be caused by the fact that the Krylov subspace estimate becomes less accurate). The rate of decrease of $\tilde{\tau}_l$ with growing σ is also similar (and about the size predicted by the theory) for both preconditioners. While $\tau_l^{\mathrm{GMRES/ML}}$ remains constant as the mesh is refined, the numbers for $\tau_l^{\mathrm{GMRES/HB}}$ decrease slightly as expected. A consequence of these combined effects is that the performance of the hierarchical basis preconditioner becomes less favorable compared to the multilevel preconditioner as the convection term and the degrees of freedom increase. Of course, this trend can be expected to be much more pronounced for three-dimensional problems (see the remarks in [22, Sect. 6]).

7. Concluding remarks

We have studied the convergence behavior of preconditioned Krylov subspace methods for discretized nonsymmetric elliptic boundary value problems using bounds based on the field of values of the coefficient matrix and of its inverse. We have used these bounds to examine the convergence behavior of hierarchical basis and additive multilevel preconditioning for nonsymmetric problems. The quantities that occur in these convergence bounds, namely, the smallest real part of the field of values of the (preconditioned) matrix and of its inverse, can be estimated in the course of the Krylov subspace iteration. Our numerical experiments have shown that these estimates behave as expected from the theoretical results and that these estimates predict the increasing number of iterations required as the size of the skew-symmetric part grows. Finally, our theoretical analysis as well as our numerical experiments indicate that these standard hierarchical basis and multilevel preconditioners do not produce satisfactory results for more convection-dominated problems. We hope that the framework developed in this paper can be modified to deal with such problems and maybe even help in the construction of appropriate preconditioners.

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