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Optimal parameters in the HSS-like methods for saddle-point problems

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SUMMARY

For the Hermitian and skew-Hermitian splitting iteration method and its accelerated variant for solving the large sparse saddle-point problems, we compute their quasi-optimal iteration parameters and the corresponding quasi-optimal convergence factors for the more practical but more difficult case that the $(1,1)$-block of the saddle-point matrix is not algebraically equivalent to the identity matrix. In addition, the algebraic behaviors and the clustering properties of the eigenvalues of the preconditioned matrices with respect to these two iterations are investigated in detail, and the formulas for computing good iteration parameters are given under certain principle for optimizing the distribution of the eigenvalues.

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1. INTRODUCTION

We consider the iterative solutions of large sparse generalized saddle-point problems of the form

$$Ax \equiv \begin{bmatrix} B & E \\ -E^* & C \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv b \tag{1}$$

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where $B \in \mathbb{C}^{p\times p}$ is Hermitian positive definite, $C \in \mathbb{C}^{q\times q}$ is Hermitian positive semidefinite, $E \in \mathbb{C}^{p\times q}$ has full column rank, $p \geqslant q$, $f \in \mathbb{C}^p$, and $g \in \mathbb{C}^q$; see [1–5]. These assumptions guarantee the existence and uniqueness of the solution of the system of linear equations (1); see [6–8]. In particular, when $C=0$, the generalized saddle-point problem (1) reduces to the saddle-point problem

$$Ax \equiv \begin{bmatrix} B & E \\ -E^* & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv b \tag{2}$$

Recently, Benzi and Golub [6] discussed the convergence and the preconditioning property of the *Hermitian and skew-Hermitian splitting* (HSS) iteration method [9] when it is used to solve the generalized saddle-point problem (1). Then, Bai and Golub [10, 11] further proposed its two-parameter acceleration, called the *accelerated Hermitian and skew-Hermitian splitting* (AHSS) iteration method, and studied the convergence and the preconditioning properties of this iterative scheme. Both theory and experiments have shown that these two methods are very robust and efficient for solving the generalized saddle-point problem (1) when they are used as either solvers or preconditioners (for the Krylov subspace iteration methods). We also refer to [12–14] for related works of the HSS splitting.

To better reveal the theoretical properties of the HSS and the AHSS iteration methods for the saddle-point problems of the form (2), Bai *et al.* [15] and Bai and Golub [10, 11] computed their optimal iteration parameters and the corresponding optimal convergence factors for a special case of the saddle-point problem (2), which is algebraically equivalent to the choice of $B=I$. Theoretical analyses have shown that the AHSS iteration method is asymptotically much faster than the HSS iteration method, and numerical implementations have confirmed that the former is always superior to the latter. See also [16, 17] for some accurate or estimated formulas about the optimal iteration parameter of the HSS iteration method.

In this paper, we will focus on computing the optimal iteration parameters and the corresponding optimal convergence factor, as well as describing the algebraic behavior and the clustering property of the eigenvalues of the preconditioned matrix, for the AHSS iteration method about the more general, more practical, but more difficult case $B \neq I$ of the saddle-point problem (2). Following a similar approach we can obtain the corresponding results for the HSS iteration method. Theoretical analyses show that the asymptotic convergence rate of the optimal AHSS iteration method is much faster than that of the optimal HSS iteration method and the former can lead to tighter eigenvalue-clustering than the latter for suitable choices of the iteration parameters. Therefore, the AHSS iteration method will be more effective than the HSS iteration method in actual applications.

We address that the saddle-point problems of the form (1) or (2) with $B=I$ may arise in the unconstraint least-squares problems [18], while those with $B \neq I$ may arise in a wide variety of engineering and scientific applications [1, 19], e.g., the mixed finite element methods in engineering fields such as fluid and solid mechanics, and the interior point algorithms in both linear and nonlinear optimizations; see [20–24] and the references therein. Also, by making use of the Cholesky factorization of the matrix B, we can equivalently reformulate the saddle-point problems (1) and (2) as the ones with the $(1,1)$-blocks being the identity matrices; see [10, 11].

The organization of the paper is as follows: In Section 2, we review the AHSS and the HSS iteration methods and their unconditional convergence property. In Section 3, we compute the quasi-optimal iteration parameters and the corresponding quasi-optimal convergence factors of the AHSS and the HSS iteration methods. The properties of the AHSS and the HSS preconditioners are investigated in Section 4, and the numerical results are shown in Section 5. Finally, in Section 6, we discuss our conclusions.

Throughout the paper, we use $\eta_{\min}$ and $\eta_{\max}$ to denote the lower and the upper bounds of the eigenvalues of the matrix B, and $\mu_{\min}$ and $\mu_{\max}$ the lower and the upper bounds of the nonzero singular values of the matrix E. It then follows that $\eta_{\min} y^* y \leqslant y^* B y \leqslant \eta_{\max} y^* y$ and $\mu_{\min}^2 z^* z \leqslant z^* E^* E z \leqslant \mu_{\max}^2 z^* z$ hold for all $y \in \mathbb{C}^p$ and $z \in \mathbb{C}^q$. In addition, we define two constants

$$\tau_o = \frac{\mu_{\max} - \mu_{\min}}{\mu_{\max} + \mu_{\min}} \quad \text{and} \quad \kappa_o = \frac{\eta_{\max}}{\eta_{\min}}$$

2. THE AHSS ITERATION METHODS

Based on the HSS technique [9] and the two-parameter acceleration strategy, Bai and Golub [10, 11] recently established an AHSS iteration method for solving the generalized saddle-point problem (1). This AHSS iteration method is algorithmically described as follows.

The AHSS iteration method. Given an initial guess $x^{(0)} = (y^{(0)^*}, z^{(0)^*})^* \in \mathbb{C}^n$, and two positive constants α and β. For $k = 0, 1, 2, \ldots,$ until the iteration sequence $\{x^{(k)}\} = \{(y^{(k)^*}, z^{(k)^*})^*\} \subset \mathbb{C}^n$ converges, compute the next iterate $x^{(k+1)} = (y^{(k+1)^*}, z^{(k+1)^*})^*$ according to the following procedure:

Step 1: Compute the partial vectors $y^{(k+1/2)}$ and $z^{(k+1/2)}$ by solving the linear sub-systems

$$(\alpha I + B) y^{(k+1/2)} = \alpha y^{(k)} - E z^{(k)} + f$$
$$(\beta I + C) z^{(k+1/2)} = E^* y^{(k)} + \beta z^{(k)} + g$$

Step 2: Compute the partial vectors $y^{(k+1)}$ and $z^{(k+1)}$ by solving the linear sub-systems

$$\begin{bmatrix} \alpha I & E \\ -E^* & \beta I \end{bmatrix} \begin{bmatrix} y^{(k+1)} \\ z^{(k+1)} \end{bmatrix} = \begin{bmatrix} (\alpha I - B) y^{(k+1/2)} + f \\ (\beta I - C) z^{(k+1/2)} + g \end{bmatrix}$$

Evidently, at each of the AHSS iteration steps, we have to solve two sub-systems of linear equations with the coefficient matrices $\alpha I + B$ and $\beta I + C$, respectively, and one sub-system of linear equations with the coefficient matrix

$$\begin{bmatrix} \alpha I & E \\ -E^* & \beta I \end{bmatrix} \tag{3}$$

The linear system with the shifted skew-Hermitian coefficient matrix[‡] (3), i.e.,

$$\begin{bmatrix} \alpha I & E \\ -E^* & \beta I \end{bmatrix} \begin{bmatrix} \tilde{y} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} \tilde{f} \\ \tilde{g} \end{bmatrix}$$

[‡]The matrix in (3) is called a shifted skew-Hermitian matrix as it holds that

$$\begin{bmatrix} I & 0 \\ 0 & \sqrt{\frac{\alpha}{\beta}} I \end{bmatrix} \begin{bmatrix} \alpha I & E \\ -E^* & \beta I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \sqrt{\frac{\alpha}{\beta}} I \end{bmatrix} = \begin{bmatrix} \alpha I & \sqrt{\frac{\alpha}{\beta}} E \\ -\sqrt{\frac{\alpha}{\beta}} E^* & \alpha I \end{bmatrix} = \alpha I + \begin{bmatrix} 0 & \sqrt{\frac{\alpha}{\beta}} E \\ -\sqrt{\frac{\alpha}{\beta}} E^* & 0 \end{bmatrix}$$

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can be solved by first computing $\tilde{z}$ from

$$\left(\beta I+\frac{1}{\alpha}E^*E\right)\tilde{z}=\tilde{g}+\frac{1}{\alpha}E^*\tilde{f}$$

and then computing $\tilde{y}$ from

$$\tilde{y}=\frac{1}{\alpha}(\tilde{f}-E\tilde{z})$$

Here, we refer to [1, 25–28] for other efficient iterative solvers about this kind of structured linear systems. As the matrices $\alpha I+B$, $\beta I+C$ and $\beta I+(1/\alpha)E^*E$ are Hermitian positive definite, the linear systems with these coefficient matrices can be solved accurately and efficiently by either the Cholesky factorization or the preconditioned conjugate gradient method; see [18].

Evidently, the AHSS iteration method can be equivalently rewritten as

$$\begin{bmatrix} y^{(k+1)} \\ z^{(k+1)} \end{bmatrix}=\mathscr{L}(\alpha,\beta)\begin{bmatrix} y^{(k)} \\ z^{(k)} \end{bmatrix}+\mathscr{G}(\alpha,\beta)\begin{bmatrix} f \\ g \end{bmatrix} \tag{4}$$

where

$$\mathscr{L}(\alpha,\beta)=\begin{bmatrix} \alpha(\alpha I+B) & (\alpha I+B)E \\ -(\beta I+C)E^* & \beta(\beta I+C) \end{bmatrix}^{-1}\begin{bmatrix} \alpha(\alpha I-B) & -(\alpha I-B)E \\ (\beta I-C)E^* & \beta(\beta I-C) \end{bmatrix} \tag{5}$$

and

$$\mathscr{G}(\alpha,\beta)=2\begin{bmatrix} \alpha I+B & \frac{1}{\alpha}(\alpha I+B)E \\ -\frac{1}{\beta}(\beta I+C)E^* & \beta I+C \end{bmatrix}^{-1}$$

Here, $\mathscr{L}(\alpha,\beta)$ is the iteration matrix of the AHSS iteration method [29]. In fact, (4) may also result from the splitting

$$A=M(\alpha,\beta)-N(\alpha,\beta) \tag{6}$$

of the coefficient matrix A, with

$$M(\alpha,\beta)=\frac{1}{2}\begin{bmatrix} \alpha I+B & \frac{1}{\alpha}(\alpha I+B)E \\ -\frac{1}{\beta}(\beta I+C)E^* & \beta I+C \end{bmatrix} \tag{7}$$

and

$$N(\alpha,\beta)=\frac{1}{2}\begin{bmatrix} \alpha I-B & -\frac{1}{\alpha}(\alpha I-B)E \\ \frac{1}{\beta}(\beta I-C)E^* & \beta I-C \end{bmatrix}$$

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The matrix $M(\alpha,\beta)$ can be used as a preconditioner, called the AHSS preconditioner, for the generalized saddle-point matrix A.

We remark that when $\alpha=\beta$, the AHSS iteration method naturally reduces to the HSS iteration method in [6], with its iteration matrix being $\mathscr{L}(\alpha):=\mathscr{L}(\alpha,\alpha)$; see also [15, 30]. Analogously, the HSS iteration method may also result from the splitting $A=M(\alpha)-N(\alpha)$ of the coefficient matrix A, with $M(\alpha):=M(\alpha,\alpha)$ and $N(\alpha):=N(\alpha,\alpha)$; and the matrix $M(\alpha)$ can be used as a preconditioner, called the HSS preconditioner, for the generalized saddle-point matrix A. When $\alpha\neq\beta$, different choices of α and β can yield many new iteration methods for the generalized saddle-point problem (1).

From [31], we have the following result about the unconditional convergence of the AHSS and the HSS iteration methods; see also [6, 10, 11, 15].

Theorem 2.1 (*Bai et al.* [*31*])
Consider the generalized saddle-point problem (1). Let $B\in\mathbb{C}^{p\times p}$ be Hermitian positive definite, $C\in\mathbb{C}^{q\times q}$ be Hermitian positive semidefinite, $E\in\mathbb{C}^{p\times q}$ be of full column rank, and $p\geqslant q$. Let $\alpha,\beta>0$ be given iteration parameters. Then the AHSS iteration method converges unconditionally to the exact solution of the generalized saddle-point problem (1), i.e., $\rho(\mathscr{L}(\alpha,\beta))<1$, $\forall\alpha,\beta>0$.

In particular, the HSS iteration method converges unconditionally to the exact solution of the generalized saddle-point problem (1), too, i.e., $\rho(\mathscr{L}(\alpha))<1$, $\forall\alpha>0$.

In actual computations, the AHSS iteration method may be more tricky to be used than the HSS iteration method as the former requires to choose two arbitrary iteration parameters while the latter requires to choose only one. From both theoretical analyses and numerical experiments, however, we can see that the AHSS iteration method is faster than the HSS iteration method, in particular, when the quasi-optimal iteration parameter(s) are employed, and the former is also much less sensitive to the iteration parameters than the latter. Moreover, both HSS and AHSS iteration methods have the same computational costs at each iteration step. In this sense, the AHSS iteration method is superior to the HSS iteration method when they are used as linear solvers, or algebraic preconditioners for the Krylov subspace methods [32–34].

3. THE OPTIMAL ITERATION PARAMETERS

In this section, for the saddle-point problem (2) we describe the formula about the eigenvalues of the AHSS iteration matrix. Then, based on this formula, we compute the quasi-optimal iteration parameters and the corresponding quasi-optimal convergence factor. As a special case, we can technically obtain the corresponding results for the HSS iteration matrix.

Lemma 3.1
Consider the saddle-point problem (2). Let $B\in\mathbb{C}^{p\times p}$ be Hermitian positive definite, $E\in\mathbb{C}^{p\times q}$ be of full column rank, and $p\geqslant q$. Let $\alpha,\beta>0$ be given iteration parameters. Then λ is an eigenvalue of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ if and only if the matrix

$$\mathscr{Q}(\alpha,\beta;\lambda):=\lambda^2(\alpha I+B)-\lambda\cdot 2\alpha(\alpha\beta I+EE^*)^{-1}(\alpha\beta I-EE^*)+(\alpha I-B) \tag{8}$$

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is singular. As a result,

$$\sigma(\mathscr{L}(\alpha,\beta))\subseteq\left\{\lambda\,\middle|\,\frac{y^*\mathscr{Q}(\alpha,\beta;\lambda)y}{y^*y}=0,\ \forall y\in\mathbb{C}^p\setminus\{0\}\right\}\Big\backslash\{1\}$$

where $\sigma(\cdot)$ denotes the spectral set of the corresponding matrix. In particular, it holds that

$$\sigma(\mathscr{L}(\alpha))\subseteq\left\{\lambda\,\middle|\,\frac{y^*\mathscr{Q}(\alpha;\lambda)y}{y^*y}=0,\ \forall y\in\mathbb{C}^p\setminus\{0\}\right\}\Big\backslash\{1\}$$

where

$$\mathscr{Q}(\alpha;\lambda):=\lambda^2(\alpha I+B)-\lambda\cdot 2\alpha(\alpha^2 I+EE^*)^{-1}(\alpha^2 I-EE^*)+(\alpha I-B)$$

Proof
Suppose that $\lambda\in\mathbb{C}$ is an eigenvalue of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ and $x=\left[\begin{smallmatrix}y\\z\end{smallmatrix}\right]$ is the corresponding eigenvector, i.e.,

$$\mathscr{L}(\alpha,\beta)\begin{bmatrix}y\\z\end{bmatrix}=\lambda\begin{bmatrix}y\\z\end{bmatrix}$$

Then, by taking $C=0$ in (5) and substituting the resulted matrix into this equality we have

$$\begin{aligned}(\alpha I-B)y-\frac{1}{\alpha}(\alpha I-B)Ez&=\lambda\left[(\alpha I+B)y+\frac{1}{\alpha}(\alpha I+B)Ez\right]\\ E^*y+\beta z&=\lambda(-E^*y+\beta z)\end{aligned}\tag{9}$$

Obviously, if $y=0$, then (9) reduces to

$$-(\alpha I-B)Ez=\lambda(\alpha I+B)Ez\quad\text{and}\quad\beta z=\lambda\beta z$$

If $\lambda=1$, then the first equality immediately leads to $z=0$; and if $\lambda\neq 1$, then the second equality immediately results in $z=0$, too. Therefore, it must hold $y\neq 0$.

Now, suppose $y\neq 0$. Then we can rewrite (9) as

$$(\alpha I-B)(\alpha y-Ez)=\lambda(\alpha I+B)(\alpha y+Ez)\quad\text{and}\quad E^*y+\beta z=\lambda(-E^*y+\beta z)$$

According to Theorem 2.1 we know that $|\lambda|<1$. Hence, from the second of the above equalities we can easily get

$$z=\frac{\lambda+1}{\beta(\lambda-1)}E^*y$$

By substituting this relationship into the first of the above equalities we can obtain

$$(\alpha I-B)[\alpha\beta(\lambda-1)I-(\lambda+1)EE^*]y=\lambda(\alpha I+B)[\alpha\beta(\lambda-1)I+(\lambda+1)EE^*]y$$

Obviously, this happens if and only if the determinant of $\tilde{\mathscr{Q}}(\alpha,\beta;\lambda)$ is zero, i.e., $\det(\tilde{\mathscr{Q}}(\alpha,\beta;\lambda))=0$, with

$$\tilde{\mathscr{Q}}(\alpha,\beta;\lambda):=\lambda^2(\alpha I+B)(\alpha\beta I+EE^*)+\lambda\cdot 2\alpha(EE^*-\alpha\beta I)+(\alpha I-B)(\alpha\beta I+EE^*)$$

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This condition is evidently equivalent to $\det(\mathscr{Q}(\alpha,\beta;\lambda))=0$, where

$$\mathscr{Q}(\alpha,\beta;\lambda):=\lambda^2(\alpha I+B)-\lambda\cdot 2\alpha(\alpha\beta I+EE^*)^{-1}(\alpha\beta I-EE^*)+(\alpha I-B) \qquad \square$$

From Lemma 3.1 we easily see that for any $y\in\mathbb{C}^p$, with $\|y\|_2=1$, if we let $\eta=y^*By$ and $\tilde{\mu}=\sqrt{y^*EE^*y}$, then it holds that $\eta\in[\eta_{\min},\eta_{\max}]$, $\tilde{\mu}\in[\mu_{\min},\mu_{\max}]\cup\{0\}$, and

$$\sigma(\mathscr{L}(\alpha,\beta))\subseteq\{\lambda\,|\,\lambda=\varphi(\alpha,\beta;\eta,\tilde{\mu})\}\setminus\{1\}$$

with

$$\varphi(\alpha,\beta;\eta,\tilde{\mu}):=\frac{\alpha}{\alpha+\eta}\left(\frac{\alpha\beta-\tilde{\mu}^2}{\alpha\beta+\tilde{\mu}^2}\pm\sqrt{\left(\frac{\alpha\beta-\tilde{\mu}^2}{\alpha\beta+\tilde{\mu}^2}\right)^2+\frac{\eta^2}{\alpha^2}-1}\right)$$

Because

$$\varphi(\alpha,\beta;\eta,0)=\frac{\alpha\pm\eta}{\alpha+\eta}$$

and the nonzero eigenvalues of the matrix E^*E are the same as those of the matrix EE^*, we can further conclude that

$$\sigma(\mathscr{L}(\alpha,\beta))\subseteq\{\lambda\,|\,\lambda=\varphi(\alpha,\beta;\eta,\mu)\}\cup\left\{\frac{\alpha-\eta}{\alpha+\eta}\right\} \tag{10}$$

where

$$\varphi(\alpha,\beta;\eta,\mu):=\frac{\alpha}{\alpha+\eta}\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\pm\sqrt{\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2+\frac{\eta^2}{\alpha^2}-1}\right) \tag{11}$$

and $\mu=\sqrt{y^*E^*Ey}\in[\mu_{\min},\mu_{\max}]$, for any $y\in\mathbb{C}^p$ with $\|y\|_2=1$.

From the above investigations we easily see that an eigenvalue λ of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ is actually a function of α, β, and $\eta:=\eta(y)$, $\mu:=\mu(y)$, where $y\in\mathbb{C}^p$ is such that $x=(y^*,z^*)^*\in\mathbb{C}^{p+q}$ is the corresponding eigenvector. That is to say, in general, $\lambda\in\sigma(\mathscr{L}(\alpha,\beta))$ if for all $\eta\in[\eta_{\min},\eta_{\max}]$ it holds that

$$\lambda\in\Psi(\alpha,\beta;\eta,\mu):=\Phi(\alpha,\beta;\eta,\mu)\cup\left\{\frac{\alpha-\eta}{\alpha+\eta}\right\} \tag{12}$$

where

$$\Phi(\alpha,\beta;\eta,\mu):=\{\varphi(\alpha,\beta;\eta,\mu)\,|\,\mu\in[\mu_{\min},\mu_{\max}]\} \tag{13}$$

As a result, we can define the continuous spectral radius of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ as

$$\rho_c(\mathscr{L}(\alpha,\beta)):=\max_{y\in\mathbb{C}^p,\|y\|_2=1}\max\{|\psi(\alpha,\beta;\eta,\mu)|\,|\,\psi(\alpha,\beta;\eta,\mu)\in\Psi(\alpha,\beta;\eta,\mu)\}$$

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Obviously, it holds that

$$\rho(\mathscr{L}(\alpha,\beta))\leqslant\rho_{\rm c}(\mathscr{L}(\alpha,\beta)) \quad \forall\alpha,\beta>0$$

Unfortunately, computing $\rho_{\rm c}(\mathscr{L}(\alpha,\beta))$ is very complicated and almost impossible, as both η and μ are dependent on the variable y. Instead, we can consider η and μ as independent variables and define the quasi-spectral radius $\varrho(\mathscr{L}(\alpha,\beta))$ of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ as follows:

$$\varrho(\mathscr{L}(\alpha,\beta)):=\max\{|\psi(\alpha,\beta;\eta,\mu)|\,|\,\psi(\alpha,\beta;\eta,\mu)\in\Psi(\alpha,\beta;\eta,\mu)\} \tag{14}$$

Consequently, the quasi-optimal iteration parameters are defined by

$$\{\alpha_{\rm opt},\beta_{\rm opt}\}:=\operatorname*{argmin}_{\alpha,\beta>0}\varrho(\mathscr{L}(\alpha,\beta)) \tag{15}$$

and the corresponding quasi-optimal convergence factor is given by $\varrho(\mathscr{L}(\alpha_{\rm opt},\beta_{\rm opt}))$. Evidently, it holds that

$$\rho(\mathscr{L}(\alpha,\beta))\leqslant\rho_{\rm c}(\mathscr{L}(\alpha,\beta))\leqslant\varrho(\mathscr{L}(\alpha,\beta)) \quad \forall\alpha,\beta>0$$

Thus, the quasi-optimal iteration parameters actually minimize a best-possible upper bound of the exact spectral radius of the AHSS iteration matrix.

Our strategy for computing the quasi-spectral radius $\varrho(\mathscr{L}(\alpha,\beta))$ defined in (14) essentially consists of three steps: first, for any fixed $\eta\in[\eta_{\min},\eta_{\max}]$, we compute

$$\varphi^{(\eta)}(\alpha,\beta):=\max\{|\varphi(\alpha,\beta;\eta,\mu)|\,|\,\mu\in[\mu_{\min},\mu_{\max}]\} \tag{16}$$

then we compute

$$\varrho^{(\eta)}(\mathscr{L}(\alpha,\beta)):=\max\left\{\varphi^{(\eta)}(\alpha,\beta),\frac{|\alpha-\eta|}{\alpha+\eta}\right\} \tag{17}$$

and, finally, we compute

$$\varrho(\mathscr{L}(\alpha,\beta)):=\max\{\varrho^{(\eta)}(\mathscr{L}(\alpha,\beta))\,|\,\eta\in[\eta_{\min},\eta_{\max}]\} \tag{18}$$

Once $\varrho(\mathscr{L}(\alpha,\beta))$ is available, we can determine the quasi-optimal iteration parameters $\alpha_{\rm opt}$ and $\beta_{\rm opt}$ defined in (15), and then compute the corresponding quasi-optimal convergence factor $\varrho(\mathscr{L}(\alpha_{\rm opt},\beta_{\rm opt}))$ for the AHSS iteration method according to (16) and (17).

We remark that the quasi-optimal iteration parameter $\alpha_{\rm opt}$ and the corresponding quasi-optimal convergence factor $\varrho(\mathscr{L}(\alpha_{\rm opt}))$ for the HSS iteration method can be defined and computed in an analogous fashion.

From (11) we can easily obtain the following two facts:

(F_1) if $\alpha<\eta$, or if $\alpha\geqslant\eta$ and $(\alpha\beta+\mu^2)\eta\geqslant2\alpha\sqrt{\alpha\beta}\mu$, then the eigenvalue $\lambda=\varphi(\alpha,\beta;\eta,\mu)$ of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ is real and satisfies either

$$\begin{aligned}|\varphi(\alpha,\beta;\eta,\mu)| &= \frac{\alpha}{\alpha+\eta}\left(\frac{|\alpha\beta-\mu^2|}{\alpha\beta+\mu^2}+\sqrt{\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2+\frac{\eta^2}{\alpha^2}-1}\right)\\ &:=\theta(\alpha,\beta;\eta,\mu)\end{aligned} \tag{19}$$

or

$$|\varphi(\alpha,\beta;\eta,\mu)| = \frac{\alpha}{\alpha+\eta}\left|\frac{|\alpha\beta-\mu^2|}{\alpha\beta+\mu^2}-\sqrt{\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2+\frac{\eta^2}{\alpha^2}-1}\right|$$

$$:= \theta_-(\alpha,\beta;\eta,\mu)$$

(F_2) if $\alpha\geqslant\eta$ and $(\alpha\beta+\mu^2)\eta<2\alpha\sqrt{\alpha\beta}\mu$, then the eigenvalue $\lambda=\varphi(\alpha,\beta;\eta,\mu)$ of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ is complex and satisfies

$$|\varphi(\alpha,\beta;\eta,\mu)|=\sqrt{\frac{\alpha-\eta}{\alpha+\eta}}$$

Notice that

$$\theta_-(\alpha,\beta;\eta,\mu)\leqslant\theta(\alpha,\beta;\eta,\mu) \quad \text{and} \quad \theta(\alpha,\beta;\eta,\mu)\geqslant\sqrt{\frac{|\alpha-\eta|}{\alpha+\eta}}\geqslant\frac{|\alpha-\eta|}{\alpha+\eta}$$

hold within the definition domains of the functions θ_- and θ. Then, we have

$$\max_{\mu\in[\mu_{\min},\mu_{\max}]}\theta_-(\alpha,\beta;\eta,\mu)\leqslant\max_{\mu\in[\mu_{\min},\mu_{\max}]}\theta(\alpha,\beta;\eta,\mu)$$

and

$$\max_{\mu\in[\mu_{\min},\mu_{\max}]}\theta(\alpha,\beta;\eta,\mu)\geqslant\sqrt{\frac{|\alpha-\eta|}{\alpha+\eta}}\geqslant\frac{|\alpha-\eta|}{\alpha+\eta}$$

Define

$$\omega(t):=\sqrt{\frac{\alpha-t}{\alpha+t}}, \quad t\in(-\infty,\alpha] \tag{20}$$

and

$$\gamma(t):=\frac{|\alpha\beta-t|}{\alpha\beta+t}, \quad t\in[0,+\infty)$$

Then we can directly verify that $\omega(t)$ is monotonically decreasing in $(-\infty,\alpha]$, and $\gamma(t)$ is monotonically decreasing in $(0,\alpha\beta]$ and monotonically increasing in $(\alpha\beta,+\infty)$, respectively. Moreover, we can further verify that for any fixed μ the function $\theta(\alpha,\beta;\eta,\mu)$ is monotonically increasing with respect to η; see Lemma A1. It then follows that

$$\max_{\eta\in[\eta_{\min},\eta_{\max}]}\max_{\mu\in[\mu_{\min},\mu_{\max}]}\theta(\alpha,\beta;\eta,\mu)=\max\{\theta(\alpha,\beta;\eta_{\max},\mu_{\min}),\theta(\alpha,\beta;\eta_{\max},\mu_{\max})\}$$

when $(\alpha\beta+\mu^2)\eta_{\max}\geqslant2\alpha\sqrt{\alpha\beta}\mu$ with $\mu\in\{\mu_{\min},\mu_{\max}\}$; and

$$\max_{\eta\in[\eta_{\min},\eta_{\max}]}\omega(\eta)=\omega(\eta_{\min})=\sqrt{\frac{\alpha-\eta_{\min}}{\alpha+\eta_{\min}}}$$

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when $\alpha \geqslant \eta_{\min}$. Therefore, from (11)–(12) and (16)–(17) we have

$$\varrho(\mathscr{L}(\alpha,\beta)) = \begin{cases} \dfrac{\alpha}{\alpha+\eta_{\max}}\left(\dfrac{\mu_{\max}^2-\alpha\beta}{\mu_{\max}^2+\alpha\beta}+\sqrt{\left(\dfrac{\mu_{\max}^2-\alpha\beta}{\mu_{\max}^2+\alpha\beta}\right)^2+\dfrac{\eta_{\max}^2}{\alpha^2}-1}\right) \\ \quad \text{for } \alpha\beta<\mu_{\min}\mu_{\max} \\ \dfrac{\alpha}{\alpha+\eta_{\max}}\left(\dfrac{\alpha\beta-\mu_{\min}^2}{\alpha\beta+\mu_{\min}^2}+\sqrt{\left(\dfrac{\alpha\beta-\mu_{\min}^2}{\alpha\beta+\mu_{\min}^2}\right)^2+\dfrac{\eta_{\max}^2}{\alpha^2}-1}\right) \\ \quad \text{for } \mu_{\min}\mu_{\max}\leqslant\alpha\beta\leqslant\dfrac{\alpha\sqrt{\alpha^2-\eta_{\min}^2}+(\alpha^2-\eta_{\min}\eta_{\max})}{\alpha\sqrt{\alpha^2-\eta_{\min}^2}-(\alpha^2-\eta_{\min}\eta_{\max})}\mu_{\min}^2 \\ \sqrt{\dfrac{\alpha-\eta_{\min}}{\alpha+\eta_{\min}}} \\ \quad \text{for } \alpha\beta>\dfrac{\alpha\sqrt{\alpha^2-\eta_{\min}^2}+(\alpha^2-\eta_{\min}\eta_{\max})}{\alpha\sqrt{\alpha^2-\eta_{\min}^2}-(\alpha^2-\eta_{\min}\eta_{\max})}\mu_{\min}^2 \end{cases} \tag{21}$$

Note that both $\theta(\alpha,\beta;\eta_{\max},\mu_{\min})$ and $\theta(\alpha,\beta;\eta_{\max},\mu_{\max})$ are monotone functions with respect to β and $\omega(\eta_{\min})$ is a monotonically increasing function with respect to α. Hence, we can easily see that the quasi-spectral radius $\varrho(\mathscr{L}(\alpha,\beta))$, defined by (14) or (16)–(17), of the AHSS iteration method attains its minimum at the point $(\alpha_{\text{opt}},\beta_{\text{opt}})$ if and only if $(\alpha_{\text{opt}},\beta_{\text{opt}})$ is the unique positive root of the algebraic system of nonlinear equations

$$\theta(\alpha,\beta;\eta_{\max},\mu_{\min})=\theta(\alpha,\beta;\eta_{\max},\mu_{\max})=\omega(\eta_{\min})$$

After concretely solving this nonlinear system, we can obtain the quasi-optimal iteration parameters α_{opt} and β_{opt} that are the only positive root of the algebraic system of nonlinear equations

$$\begin{cases} \alpha\beta=\mu_{\min}\mu_{\max} \\ \alpha\beta=\dfrac{\alpha\sqrt{\alpha^2-\eta_{\min}^2}+(\alpha^2-\eta_{\min}\eta_{\max})}{\alpha\sqrt{\alpha^2-\eta_{\min}^2}-(\alpha^2-\eta_{\min}\eta_{\max})}\cdot\mu_{\min}^2 \end{cases}$$

and the corresponding quasi-optimal convergence factor $\varrho(\mathscr{L}(\alpha_{\text{opt}},\beta_{\text{opt}}))$ of the AHSS iteration method for solving the saddle-point problem (2).

The above analysis is precisely summarized in the following theorem:

Theorem 3.1
Consider the saddle-point problem (2). Let $B\in\mathbb{C}^{p\times p}$ be Hermitian positive definite, $E\in\mathbb{C}^{p\times q}$ be of full column rank, and $p\geqslant q$. Let $\alpha,\beta>0$ be given iteration parameters. Then, for the

AHSS iteration method of the saddle-point problem (2), the quasi-optimal values $\{\alpha_{\text{opt}}, \beta_{\text{opt}}\} = \text{argmin}_{\alpha,\beta>0}\varrho(\mathscr{L}(\alpha,\beta))$ of the iteration parameters $\{\alpha,\beta\}$ are given by

$$\alpha_{\text{opt}} = \frac{(\mu_{\min}+\mu_{\max})\sqrt{\eta_{\min}\eta_{\max}æ_o}}{2\sqrt{\mu_{\min}\mu_{\max}}}$$

and

$$\beta_{\text{opt}} = \frac{2(\mu_{\min}\mu_{\max})^{3/2}}{(\mu_{\min}+\mu_{\max})\sqrt{\eta_{\min}\eta_{\max}æ_o}}$$

and, correspondingly, the quasi-optimal convergence factor is given by

$$\varrho(\mathscr{L}(\alpha_{\text{opt}},\beta_{\text{opt}})) = \sqrt{\frac{(\mu_{\min}+\mu_{\max})\sqrt{\eta_{\max}æ_o}-2\sqrt{\eta_{\min}\mu_{\min}\mu_{\max}}}{(\mu_{\min}+\mu_{\max})\sqrt{\eta_{\max}æ_o}+2\sqrt{\eta_{\min}\mu_{\min}\mu_{\max}}}}$$

where

$$æ_o = 1-\frac{\tau_o^2}{2\kappa_o}+\frac{\tau_o}{2\kappa_o}\sqrt{\tau_o^2+4\kappa_o(\kappa_o-1)}$$

We remark that when $\eta_{\min}=\eta_{\max}=1$, the result of Theorem 3.1 naturally reduces to the one of Theorem 3.2 in [11]; see also [10, 16].

Recall that when $\alpha=\beta$, the AHSS iteration method reduces to the HSS iteration method for the saddle-point problem (2). Then from (10) we easily know that the eigenvalues of the HSS iteration matrix $\mathscr{L}(\alpha)$ satisfy

$$\sigma(\mathscr{L}(\alpha)) \subseteq \{\lambda \mid \lambda = \varphi(\alpha;\eta,\mu)\} \cup \left\{\frac{\alpha-\eta}{\alpha+\eta}\right\} \tag{22}$$

where

$$\begin{aligned}\varphi(\alpha;\eta,\mu) &:= \varphi(\alpha,\alpha;\eta,\mu)\\ &= \frac{\alpha}{\alpha+\eta}\left(\frac{\alpha^2-\mu^2}{\alpha^2+\mu^2} \pm \sqrt{\left(\frac{\alpha^2-\mu^2}{\alpha^2+\mu^2}\right)^2+\frac{\eta^2}{\alpha^2}-1}\right)\end{aligned} \tag{23}$$

with $\eta = y^*By$ and $\mu=\sqrt{y^*E^*Ey}$ for any $y\in\mathbb{C}^p$ satisfying $\|y\|_2=1$; see (11).

Furthermore, based on (22)–(23), after technical analysis and derivation analogous to (21), we can obtain the expression of the spectral radius $\varrho(\mathscr{L}(\alpha))$ of the HSS iteration matrix $\mathscr{L}(\alpha)$. To describe it precisely, we define three functions

$$\mathscr{F}_1(\alpha;\mu) = \frac{\alpha}{\alpha+\eta_{\max}}\left(\frac{\alpha^2-\mu^2}{\alpha^2+\mu^2}+\sqrt{\left(\frac{\alpha^2-\mu^2}{\alpha^2+\mu^2}\right)^2+\frac{\eta_{\max}^2}{\alpha^2}-1}\right)$$

for $\alpha<\eta_{\max}$

or for $\alpha\geqslant\eta_{\max}$, $\quad(\alpha^2+\mu^2)\eta_{\max}\geqslant 2\alpha^2\mu\quad$ and $\quad\alpha\geqslant\mu$

with $\mu\in\{\mu_{\min},\mu_{\max}\}$

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$$\mathscr{F}_2(\alpha;\mu)=\frac{\alpha}{\alpha+\eta_{\max}}\left(\frac{\mu^2-\alpha^2}{\mu^2+\alpha^2}+\sqrt{\left(\frac{\mu^2-\alpha^2}{\mu^2+\alpha^2}\right)^2+\frac{\eta_{\max}^2}{\alpha^2}-1}\right)$$
$$\text{for } \alpha<\eta_{\max}$$
$$\text{or for } \alpha\geqslant\eta_{\max},\quad (\alpha^2+\mu^2)\eta_{\max}\geqslant 2\alpha^2\mu \quad\text{and}\quad \alpha<\mu$$
$$\text{with } \mu\in\{\mu_{\min},\mu_{\max}\}$$
$$\Omega(\alpha;\eta)=\sqrt{\frac{\alpha-\eta}{\alpha+\eta}}\quad \text{for } \alpha\geqslant\eta$$

After straightforward calculations we have

$$\mathscr{F}_1(\alpha;\mu_{\min})=\mathscr{F}_2(\alpha;\mu_{\max})\quad \text{if } \alpha=\sqrt{\mu_{\min}\mu_{\max}}$$
$$\mathscr{F}_1(\alpha;\mu_{\min})=\Omega(\alpha;\eta_{\min})\quad \text{if } \alpha=\alpha_1^{(f)}$$
$$\mathscr{F}_2(\alpha;\mu_{\max})=\Omega(\alpha;\eta_{\min})\quad \text{if } \alpha=\alpha_2^{(f)}$$

where $\alpha_1^{(f)}$ and $\alpha_2^{(f)}$ are the real positive solutions of the nonlinear polynomial equations

$$\mathscr{T}_1(\mu_{\min})\alpha^6+\mathscr{T}_2(\mu_{\min})\alpha^4+\mathscr{T}_3(\mu_{\min})\alpha^2+\mathscr{T}_4(\mu_{\min})=0$$

and

$$\mathscr{T}_1(\mu_{\max})\alpha^6+\mathscr{T}_2(\mu_{\max})\alpha^4+\mathscr{T}_3(\mu_{\max})\alpha^2+\mathscr{T}_4(\mu_{\max})=0$$

respectively, with

$$\mathscr{T}_1(\mu)=-4\mu^2+\eta_{\min}(2\eta_{\max}-\eta_{\min})$$
$$\mathscr{T}_2(\mu)=2\eta_{\min}(2\eta_{\max}+\eta_{\min})\mu^2-\eta_{\min}^2\eta_{\max}^2$$
$$\mathscr{T}_3(\mu)=\eta_{\min}(2\eta_{\max}-\eta_{\min})\mu^4-2\eta_{\min}^2\eta_{\max}^2\mu^2$$
$$\mathscr{T}_4(\mu)=-\eta_{\min}^2\eta_{\max}^2\mu^4$$

Now, the curve $\varrho(\mathscr{L}(\alpha))$ with respect to the variable α can be given exactly in the following four cases:

(a) When $\eta_{\max}>2\mu_{\max}$, it holds that

$$\varrho(\mathscr{L}(\alpha))=\begin{cases}\mathscr{F}_2(\alpha;\mu_{\max}) & \text{for } \alpha<\sqrt{\mu_{\min}\mu_{\max}}\\ \mathscr{F}_1(\alpha;\mu_{\min}) & \text{for } \alpha\geqslant\sqrt{\mu_{\min}\mu_{\max}}\end{cases}$$

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(b) When $2\mu_{\min} < \eta_{\max} \leqslant 2\mu_{\max}$, it holds that

(i) if $\alpha_2^{(f)} \leqslant \sqrt{\mu_{\min}\mu_{\max}}$, then

$$\varrho(\mathscr{L}(\alpha)) = \begin{cases} \mathscr{F}_2(\alpha; \mu_{\max}) & \text{for } \alpha \leqslant \alpha_2^{(f)} \\ \Omega(\alpha; \eta_{\min}) & \text{for } \alpha_2^{(f)} < \alpha \leqslant \alpha_1^{(f)} \\ \mathscr{F}_1(\alpha; \mu_{\min}) & \text{for } \alpha > \alpha_1^{(f)} \end{cases}$$

(ii) if $\alpha_2^{(f)} > \sqrt{\mu_{\min}\mu_{\max}}$, then

$$\varrho(\mathscr{L}(\alpha)) = \begin{cases} \mathscr{F}_2(\alpha; \mu_{\max}) & \text{for } \alpha \leqslant \sqrt{\mu_{\min}\mu_{\max}} \\ \mathscr{F}_1(\alpha; \mu_{\min}) & \text{for } \sqrt{\mu_{\min}\mu_{\max}} < \alpha \leqslant \alpha_1^{(f)} \\ \Omega(\alpha; \eta_{\min}) & \text{for } \alpha > \alpha_1^{(f)} \end{cases}$$

(c) When $2\mu_{\max}\mu_{\min}/(\mu_{\max}+\mu_{\min}) < \eta_{\max} \leqslant 2\mu_{\min}$, it holds that

(i) if $\alpha_2^{(f)} \leqslant \sqrt{\mu_{\min}\mu_{\max}}$, then

$$\varrho(\mathscr{L}(\alpha)) = \begin{cases} \mathscr{F}_2(\alpha; \mu_{\max}) & \text{for } \alpha < \alpha_2^{(f)} \\ \Omega(\alpha; \eta_{\min}) & \text{for } \alpha \geqslant \alpha_2^{(f)} \end{cases}$$

(ii) if $\alpha_1^{(f)} \geqslant \sqrt{\mu_{\min}\mu_{\max}}$, then

$$\varrho(\mathscr{L}(\alpha)) = \begin{cases} \mathscr{F}_2(\alpha; \mu_{\max}) & \text{for } \alpha \leqslant \sqrt{\mu_{\min}\mu_{\max}} \\ \mathscr{F}_1(\alpha; \mu_{\min}) & \text{for } \sqrt{\mu_{\min}\mu_{\max}} < \alpha \leqslant \alpha_1^{(f)} \\ \Omega(\alpha; \eta_{\min}) & \text{for } \alpha > \alpha_1^{(f)} \end{cases}$$

(d) When $\eta_{\max} \leqslant 2\mu_{\max}\mu_{\min}/(\mu_{\max}+\mu_{\min})$, it holds that

$$\varrho(\mathscr{L}(\alpha)) = \begin{cases} \mathscr{F}_2(\alpha; \mu_{\max}) & \text{for } \alpha < \alpha_2^{(f)} \\ \Omega(\alpha; \eta_{\min}) & \text{for } \alpha \geqslant \alpha_2^{(f)} \end{cases}$$

Based on the above analysis, we can immediately obtain the following theorem:

Theorem 3.2

Consider the saddle-point problem (2). Let $B \in \mathbb{C}^{p\times p}$ be Hermitian positive definite, $E \in \mathbb{C}^{p\times q}$ be of full column rank, and $p \geqslant q$. Let $\alpha > 0$ be a given iteration parameter, and define the positive constant

$$\widetilde{\varrho}_{\text{opt}} := \frac{\sqrt{\mu_{\min}\mu_{\max}}}{\sqrt{\mu_{\min}\mu_{\max}+\eta_{\max}}} \left(\frac{\mu_{\max}-\mu_{\min}}{\mu_{\max}+\mu_{\min}} + \sqrt{\left(\frac{\mu_{\max}-\mu_{\min}}{\mu_{\max}+\mu_{\min}}\right)^2 + \frac{\eta_{\max}^2}{\mu_{\min}\mu_{\max}} - 1} \right)$$

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Then, for the HSS iteration method of the saddle-point problem (2), the quasi-optimal value α_{opt} of the iteration parameter α and the corresponding quasi-optimal convergence factor $\varrho(\mathscr{L}(\alpha_{opt}))$ are given, respectively, as follows:

(a) When $\eta_{max} > 2\mu_{max}$, $\alpha_{opt} = \sqrt{\mu_{min}\mu_{max}}$ and, correspondingly, the quasi-optimal convergence factor is given by

$$\varrho(\mathscr{L}(\alpha_{opt})) = \mathscr{F}_1(\alpha_{opt}; \mu_{min}) = \mathscr{F}_2(\alpha_{opt}; \mu_{max}) = \widetilde{\varrho}_{opt}$$

(b) When $2\mu_{min} < \eta_{max} \leqslant 2\mu_{max}$,

(i) if $\alpha_2^{(f)} \leqslant \sqrt{\mu_{min}\mu_{max}}$, then $\alpha_{opt} = \alpha_2^{(f)}$ and, correspondingly, the quasi-optimal convergence factor is given by

$$\varrho(\mathscr{L}(\alpha_{opt})) = \mathscr{F}_2(\alpha_{opt}; \mu_{max}) = \Omega(\alpha_{opt}; \eta_{min})$$

(ii) if $\alpha_2^{(f)} > \sqrt{\mu_{min}\mu_{max}}$, then

$$\alpha_{opt} = \begin{cases} \alpha_1^{(f)} & \text{for } \mathscr{F}_1(\alpha_1^{(f)}; \mu_{min}) < \mathscr{F}_1(\sqrt{\mu_{min}\mu_{max}}; \mu_{min}) \\ \sqrt{\mu_{min}\mu_{max}} & \text{for } \mathscr{F}_1(\alpha_1^{(f)}; \mu_{min}) \geqslant \mathscr{F}_1(\sqrt{\mu_{min}\mu_{max}}; \mu_{min}) \end{cases}$$

and, correspondingly, the quasi-optimal convergence factor is given by $\varrho(\mathscr{L}(\alpha_{opt})) = \mathscr{F}_1(\alpha_{opt}; \mu_{min})$, with $\mathscr{F}_1(\sqrt{\mu_{min}\mu_{max}}; \mu_{min}) = \widetilde{\varrho}_{opt}$.

(c) When $2\mu_{max}\mu_{min}/(\mu_{max}+\mu_{min}) < \eta_{max} \leqslant 2\mu_{min}$,

(i) if $\alpha_2^{(f)} \leqslant \sqrt{\mu_{min}\mu_{max}}$, then $\alpha_{opt} = \alpha_2^{(f)}$ and, correspondingly, the quasi-optimal convergence factor is given by $\varrho(\mathscr{L}(\alpha_{opt})) = \mathscr{F}_2(\alpha_{opt}; \mu_{max}) = \Omega(\alpha_{opt}; \eta_{min})$,

(ii) if $\alpha_1^{(f)} \geqslant \sqrt{\mu_{min}\mu_{max}}$, then

$$\alpha_{opt} = \begin{cases} \alpha_1^{(f)} & \text{for } \mathscr{F}_1(\alpha_1^{(f)}; \mu_{min}) < \mathscr{F}_1(\sqrt{\mu_{min}\mu_{max}}; \mu_{min}) \\ \sqrt{\mu_{min}\mu_{max}} & \text{for } \mathscr{F}_1(\alpha_1^{(f)}; \mu_{min}) \geqslant \mathscr{F}_1(\sqrt{\mu_{min}\mu_{max}}; \mu_{min}) \end{cases}$$

and, correspondingly, the quasi-optimal convergence factor is given by $\varrho(\mathscr{L}(\alpha_{opt})) = \mathscr{F}_1(\alpha_{opt}; \mu_{min})$, with $\mathscr{F}_1(\sqrt{\mu_{min}\mu_{max}}; \mu_{min}) = \widetilde{\varrho}_{opt}$.

(d) When $\eta_{max} \leqslant 2\mu_{max}\mu_{min}/(\mu_{max}+\mu_{min})$, $\alpha_{opt} = \alpha_2^{(f)}$ and, correspondingly, the quasi-optimal convergence factor is given by $\varrho(\mathscr{L}(\alpha_{opt})) = \mathscr{F}_1(\alpha_{opt}; \mu_{max}) = \Omega(\alpha_{opt}; \eta_{min})$.

We remark that when $\eta_{min} = \eta_{max} = 1$, the result of Theorem 3.2 naturally reduces to the one of Theorem 3.2 in [15]; see also [16].

4. PROPERTIES OF THE PRECONDITIONED MATRICES

In this section, we discuss the algebraic properties of the AHSS preconditioner $M(\alpha, \beta)$; see (7). As a special case, we can straightforwardly obtain the algebraic properties of the HSS preconditioner $M(\alpha)$. The readers are referred to [10, 11, 15, 35] for sufficient conditions about the HSS-preconditioned matrix to have real spectrum for all values of the iteration parameter α.

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The following result shows that the AHSS-preconditioned matrix $M(\alpha,\beta)^{-1}A$ and the HSS-preconditioned matrix $M(\alpha)^{-1}A$ are both positive stable:

Theorem 4.1
Consider the saddle-point problem (1). Let $B\in\mathbb{C}^{p\times p}$ be Hermitian positive definite, $C\in\mathbb{C}^{q\times q}$ be Hermitian positive semidefinite, $E\in\mathbb{C}^{p\times q}$ be of full column rank, and $p\geqslant q$. Let $\alpha,\beta>0$ be given iteration parameters. Then the AHSS-preconditioned matrix $M(\alpha,\beta)^{-1}A$ is positive stable for all $\alpha,\beta>0$. In particular, the HSS-preconditioned matrix $M(\alpha)^{-1}A$ is positive stable for all $\alpha>0$.

Proof
Because

$$M(\alpha,\beta)^{-1}A=M(\alpha,\beta)^{-1}(M(\alpha,\beta)-N(\alpha,\beta))=I-\mathscr{L}(\alpha,\beta)$$

and $\rho(\mathscr{L}(\alpha,\beta))<1$ for all $\alpha,\beta>0$, we easily know that the real parts of the eigenvalues of the matrix $M(\alpha,\beta)^{-1}A$ are all positive. In particular, when $\alpha=\beta$, the real parts of the eigenvalues of the matrix $M(\alpha)^{-1}A$ are all positive, too. $\square$

It is evident that to describe the behavior of the spectrum of $M(\alpha,\beta)^{-1}A$ we only need to discuss the eigenvalues of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$, as those of the matrix $M(\alpha,\beta)^{-1}A$ can follow directly from the relationship $M(\alpha,\beta)^{-1}A=I-\mathscr{L}(\alpha,\beta)$.

From (10) and (11) we know that the eigenvalues of $\mathscr{L}(\alpha,\beta)$ can be essentially categorized into two groups: one consists of eigenvalues of the form $\lambda=(\alpha-\eta)/(\alpha+\eta)$, which is always real; and another consists of eigenvalues of the form $\lambda=\varphi(\alpha,\beta;\eta,\mu)$, which is conditionally real or complex depending on the sign of the discriminant

$$\Delta(\alpha,\beta;\eta,\mu):=\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2+\frac{\eta^2}{\alpha^2}-1 \tag{24}$$

where $\eta=y^*By\in[\eta_{\min},\eta_{\max}]$ and $\mu=\sqrt{y^*E^*Ey}\in[\mu_{\min},\mu_{\max}]$, for any $y\in\mathbb{C}^p$, with $\|y\|_2=1$; see (11). After straightforward calculations, we easily see that the following facts hold true for $\eta\in[\eta_{\min},\eta_{\max}]$ and $\mu\in[\mu_{\min},\mu_{\max}]$:

(F_a) $\Delta(\alpha,\beta;\eta,\mu)\geqslant 0$, or $\lambda=\varphi(\alpha,\beta;\eta,\mu)$ is real, if and only if $(\alpha\beta+\mu^2)\eta\geqslant 2\alpha\sqrt{\alpha\beta}\mu$;
(F_b) $\Delta(\alpha,\beta;\eta,\mu)<0$, or $\lambda=\varphi(\alpha,\beta;\eta,\mu)$ is complex, if and only if $(\alpha\beta+\mu^2)\eta<2\alpha\sqrt{\alpha\beta}\mu$.

Theorem 4.2
Consider the saddle-point problem (2). Let $B\in\mathbb{C}^{p\times p}$ be Hermitian positive definite, $E\in\mathbb{C}^{p\times q}$ be of full column rank, and $p\geqslant q$. Let $\alpha,\beta>0$ be given iteration parameters. Then

(i) the real eigenvalues $\lambda=(\alpha-\eta)/(\alpha+\eta)$ of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ are bounded by

$$\frac{\alpha-\eta_{\max}}{\alpha+\eta_{\max}}\leqslant\lambda\leqslant\frac{\alpha-\eta_{\min}}{\alpha+\eta_{\min}}$$

(ii) when $\alpha\leqslant\eta_{\min}$, the eigenvalues $\lambda=\varphi(\alpha,\beta;\eta,\mu)$ of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ are real for all $\mu\in[\mu_{\min},\mu_{\max}]$ and all $\alpha,\beta>0$, and are bounded as $\lambda\in\mathscr{I}_-(\alpha,\beta)\cup\mathscr{I}_+(\alpha,\beta)$, with

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$\mathscr{I}_{-}(\alpha,\beta)=[-\mathscr{I}^{(\max)}(\alpha,\beta),-\mathscr{I}^{(\min)}(\alpha,\beta)]$ and $\mathscr{I}_{+}(\alpha,\beta)=[\mathscr{I}^{(\min)}(\alpha,\beta),\mathscr{I}^{(\max)}(\alpha,\beta)]$, where

$$\mathscr{I}^{(\min)}(\alpha,\beta)=\frac{\alpha}{\alpha+\eta_{\min}}\left(-\gamma^{(\max)}(\alpha,\beta)+\sqrt{[\gamma^{(\max)}(\alpha,\beta)]^2+\frac{\eta_{\min}^2}{\alpha^2}-1}\right)$$

$$\mathscr{I}^{(\max)}(\alpha,\beta)=\frac{\alpha}{\alpha+\eta_{\max}}\left(\gamma^{(\max)}(\alpha,\beta)+\sqrt{[\gamma^{(\max)}(\alpha,\beta)]^2+\frac{\eta_{\max}^2}{\alpha^2}-1}\right)$$

with

$$\gamma^{(\max)}(\alpha,\beta)=\begin{cases}\dfrac{\mu_{\max}^2-\alpha\beta}{\mu_{\max}^2+\alpha\beta} & \text{for } \alpha\beta\leqslant\mu_{\min}\mu_{\max}\\[2ex] \dfrac{\alpha\beta-\mu_{\min}^2}{\alpha\beta+\mu_{\min}^2} & \text{for } \alpha\beta>\mu_{\min}\mu_{\max}\end{cases}$$

(iii) when $\alpha>\eta_{\min}$, the eigenvalues $\lambda=\varphi(\alpha,\beta;\eta,\mu)$ of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ are real for all $\mu\in[\mu_{\min},\mu_{\max}]$ and all $\beta>0$ such that

$$\beta\leqslant\frac{\mu_{\min}^2}{\alpha\eta_{\min}^2}(\alpha-\sqrt{\alpha^2-\eta_{\min}^2})^2 \quad \text{or} \quad \beta\geqslant\frac{\mu_{\max}^2}{\alpha\eta_{\min}^2}(\alpha+\sqrt{\alpha^2-\eta_{\min}^2})^2 \tag{25}$$

and are bounded as

$$\lambda\in\mathscr{I}_{-}(\alpha,\beta)\cup\mathscr{I}_{+}(\alpha,\beta) \tag{26}$$

where $\mathscr{I}_{-}(\alpha,\beta)$ and $\mathscr{I}_{+}(\alpha,\beta)$ are defined as in (ii);

(iv) when $\alpha>\eta_{\max}$, the eigenvalues $\lambda=\varphi(\alpha,\beta;\eta,\mu)$ of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ are complex for all $\mu\in[\mu_{\min},\mu_{\max}]$ and for all $\beta>0$ such that

$$\frac{\mu_{\max}^2}{\alpha\eta_{\max}^2}(\alpha-\sqrt{\alpha^2-\eta_{\max}^2})^2<\beta<\frac{\mu_{\min}^2}{\alpha\eta_{\max}^2}(\alpha+\sqrt{\alpha^2-\eta_{\max}^2})^2 \tag{27}$$

and are bounded as

$$\begin{aligned}|\Re(\lambda)|&\leqslant\frac{\alpha\gamma^{(\max)}(\alpha,\beta)}{\alpha+\eta_{\min}}:=\mathscr{J}^{(\max)}(\alpha,\beta)\\ |\Im(\lambda)|&\leqslant\frac{1}{\alpha+\eta_{\min}}\sqrt{\alpha^2-\eta_{\min}^2-\alpha^2[\gamma^{(\min)}(\alpha,\beta)]^2}:=\mathscr{K}^{(\max)}(\alpha,\beta)\end{aligned} \tag{28}$$

where $\Re(\cdot)$ and $\Im(\cdot)$ denote the real and the imaginary parts of the corresponding complex, respectively, and

$$\gamma^{(\min)}(\alpha,\beta)=\begin{cases}\dfrac{|\alpha\beta-\mu_{\min}^2|}{\alpha\beta+\mu_{\min}^2} & \text{for } \alpha\beta\leqslant\mu_{\min}\mu_{\max}\\[2ex] \dfrac{|\alpha\beta-\mu_{\max}^2|}{\alpha\beta+\mu_{\max}^2} & \text{for } \alpha\beta>\mu_{\min}\mu_{\max}\end{cases}$$

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Proof
From (10)–(11) we know that the spectral set of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ consists of the following two types of eigenvalues:

$$\lambda=\frac{\alpha-\eta}{\alpha+\eta} \tag{29}$$

and

$$\lambda=\varphi(\alpha,\beta;\eta,\mu):=\frac{\alpha}{\alpha+\eta}\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\pm\sqrt{\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2+\frac{\eta^2}{\alpha^2}-1}\right) \tag{30}$$

where $\eta\in[\eta_{\min},\eta_{\max}]$ and $\mu\in[\mu_{\min},\mu_{\max}]$.

Obviously, $\lambda=(\alpha-\eta)/(\alpha+\eta)$ is real for all $\eta\in[\eta_{\min},\eta_{\max}]$. Moreover, it is a monotonically decreasing function with respect to η and is, hence, bounded as follows:

$$\frac{\alpha-\eta_{\max}}{\alpha+\eta_{\max}}\leqslant\frac{\alpha-\eta}{\alpha+\eta}\leqslant\frac{\alpha-\eta_{\min}}{\alpha+\eta_{\min}}\quad\forall\eta\in[\eta_{\min},\eta_{\max}]$$

This shows that (i) is true.

To prove (ii), we need to investigate the properties of the function $\varphi(\alpha,\beta;\eta,\mu)$ defined in (30). When $\alpha\leqslant\eta_{\min}$, we have

$$\frac{\eta^2}{\alpha^2}-1\geqslant\frac{\eta_{\min}^2}{\alpha^2}-1\geqslant 0\quad\forall\eta\in[\eta_{\min},\eta_{\max}]$$

Therefore, for all $\eta\in[\eta_{\min},\eta_{\max}]$ and $\mu\in[\mu_{\min},\mu_{\max}]$ the discriminant $\Delta(\alpha,\beta;\eta,\mu)$ defined in (24) is clearly nonnegative and the eigenvalues of the type $\lambda=\varphi(\alpha,\beta;\eta,\mu)$ are real. Moreover, it is easily seen that

$$\lambda_+\equiv\varphi_+(\alpha,\beta;\eta,\mu):=\frac{\alpha}{\alpha+\eta}\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}+\sqrt{\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2+\frac{\eta^2}{\alpha^2}-1}\right)>0$$

and

$$\lambda_-\equiv\varphi_-(\alpha,\beta;\eta,\mu):=\frac{\alpha}{\alpha+\eta}\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}-\sqrt{\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2+\frac{\eta^2}{\alpha^2}-1}\right)<0$$

hold true. Obviously, we have

$$\lambda_+\leqslant|\varphi_+(\alpha,\beta;\eta,\mu)|\leqslant\theta(\alpha,\beta;\eta,\mu)$$

where $\theta(\alpha,\beta;\eta,\mu)$ is defined by (19). According to Lemma A1, we know that $\theta(\alpha,\beta;\eta,\mu)$ is monotonically increasing with respect to η and, thereby,

$$\begin{aligned}|\varphi_+(\alpha,\beta;\eta,\mu)|&\leqslant\theta(\alpha,\beta;\eta,\mu)\leqslant\theta(\alpha,\beta;\eta_{\max},\mu)\\&=\frac{\alpha}{\alpha+\eta_{\max}}\left(\frac{|\alpha\beta-\mu^2|}{\alpha\beta+\mu^2}+\sqrt{\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2+\frac{\eta_{\max}^2}{\alpha^2}-1}\right)\end{aligned}$$

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Because

$$\frac{|\alpha\beta-\mu^2|}{\alpha\beta+\mu^2}\leqslant\gamma^{(\max)}(\alpha,\beta)\quad\forall\mu\in[\mu_{\min},\mu_{\max}]$$

we can further get

$$\theta(\alpha,\beta;\eta_{\max},\mu)\leqslant\frac{\alpha}{\alpha+\eta_{\max}}\left(\gamma^{(\max)}(\alpha,\beta)+\sqrt{[\gamma^{(\max)}(\alpha,\beta)]^2+\frac{\eta_{\max}^2}{\alpha^2}-1}\right)$$

which readily gives the upper bound for λ_+.

In addition, we have

$$\begin{aligned}
\lambda_+ &\geqslant \frac{\alpha}{\alpha+\eta}\left(-\frac{|\alpha\beta-\mu^2|}{\alpha\beta+\mu^2}+\sqrt{\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2+\frac{\eta^2}{\alpha^2}-1}\right)\\
&=\frac{\eta-\alpha}{\alpha}\frac{1}{\dfrac{|\alpha\beta-\mu^2|}{\alpha\beta+\mu^2}+\sqrt{\left(\dfrac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2+\dfrac{\eta^2}{\alpha^2}-1}}\\
&\geqslant\frac{\eta-\alpha}{\alpha}\frac{1}{\gamma^{(\max)}(\alpha,\beta)+\sqrt{[\gamma^{(\max)}(\alpha,\beta)]^2+\dfrac{\eta^2}{\alpha^2}-1}}\\
&=\frac{\alpha}{\alpha+\eta}\left(-\gamma^{(\max)}(\alpha,\beta)+\sqrt{[\gamma^{(\max)}(\alpha,\beta)]^2+\frac{\eta^2}{\alpha^2}-1}\right)\\
&\geqslant\frac{\alpha}{\alpha+\eta_{\min}}\left(-\gamma^{(\max)}(\alpha,\beta)+\sqrt{[\gamma^{(\max)}(\alpha,\beta)]^2+\frac{\eta_{\min}^2}{\alpha^2}-1}\right)
\end{aligned}$$

which gives the lower bound for λ_+. Here, we have used the fact that the function

$$\frac{\alpha}{\alpha+\eta}\left(-\gamma^{(\max)}(\alpha,\beta)+\sqrt{[\gamma^{(\max)}(\alpha,\beta)]^2+\frac{\eta^2}{\alpha^2}-1}\right)$$

is monotonically increasing with respect to η, which can be easily proved in a similar manner as Lemma A1.

Note that

$$\lambda_-=-\frac{\alpha}{\alpha+\eta}\left(-\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}+\sqrt{\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2+\frac{\eta^2}{\alpha^2}-1}\right)=-\tilde{\lambda}_+$$

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where

$$\tilde{\lambda}_+ := \frac{\alpha}{\alpha+\eta}\left(-\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}+\sqrt{\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2+\frac{\eta^2}{\alpha^2}-1}\right)$$

We can analogously obtain

$$\tilde{\lambda}_+ \in [\mathscr{I}^{(\min)}(\alpha,\beta),\ \mathscr{I}^{(\max)}(\alpha,\beta)]$$

and, hence,

$$\lambda_- \in [-\mathscr{I}^{(\max)}(\alpha,\beta),\ -\mathscr{I}^{(\min)}(\alpha,\beta)]$$

The above demonstration shows the validity of (ii).

We now turn to verify (iii). By straightforward calculations we know that when $\alpha \geqslant \eta$ the function $(\alpha-\sqrt{\alpha^2-\eta^2})/\eta$ is monotonically increasing and the function $(\alpha+\sqrt{\alpha^2-\eta^2})/\eta$ is monotonically decreasing with respect to η. It follows from Lemma A2 that for all $\alpha \geqslant \eta_{\min}$ and $\beta > 0$ satisfying (25) the discriminant $\Delta(\alpha,\beta;\eta,\mu)$ defined in (24) is nonnegative and, hence, $\lambda = \varphi(\alpha,\beta;\eta,\mu)$ is real. Now, the bounds of the function $\varphi(\alpha,\beta;\eta,\mu)$ can be obtained in a similar manner to (ii).

Finally, we demonstrate the validity of (iv). From Lemma A3 we know that when $\alpha \geqslant \eta_{\max}$ and $\beta > 0$ satisfying (27), the discriminant $\Delta(\alpha,\beta;\eta,\mu)$ defined in (24) is negative and, hence, $\lambda = \varphi(\alpha,\beta;\eta,\mu)$ is complex. Now, the real and the imaginary parts of $\varphi(\alpha,\beta;\eta,\mu)$ are given, respectively, by

$$\Re(\varphi(\alpha,\beta;\eta,\mu)) = \frac{\alpha}{\alpha+\eta}\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}$$

and

$$\begin{aligned}\Im(\varphi(\alpha,\beta;\eta,\mu)) &= \pm\frac{\alpha}{\alpha+\eta}\sqrt{1-\frac{\eta^2}{\alpha^2}-\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2}\\ &= \pm\frac{1}{\alpha+\eta}\sqrt{\alpha^2-\eta^2-\alpha^2\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2}\\ &= \pm\sqrt{\frac{\alpha-\eta}{\alpha+\eta}-\frac{\alpha^2}{(\alpha+\eta)^2}\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2}\end{aligned}$$

By straightforward calculations we know that the functions $1/(\alpha+\eta)$ and

$$\frac{\alpha-\eta}{\alpha+\eta}-\frac{\alpha^2}{(\alpha+\eta)^2}\left(\frac{\alpha\beta-\mu^2}{\alpha\beta+\mu^2}\right)^2$$

are monotonically decreasing with respect to η, and the bound

$$\frac{|\alpha\beta-\mu^2|}{\alpha\beta+\mu^2} \geqslant \gamma^{(\min)}(\alpha,\beta)$$

holds for $\forall \mu \in [\mu_{\min},\mu_{\max}]$. Therefore, the bounds given in (iv) can be easily obtained. □

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Theorem 4.2 also shows that for any $\eta\in[\eta_{\min},\eta_{\max}]$ and $\mu\in[\mu_{\min},\mu_{\max}]$, if $\lambda=\varphi(\alpha,\beta;\eta,\mu)$ is real then its bounds are given by (26), and if $\lambda=\varphi(\alpha,\beta;\eta,\mu)$ is complex then its bounds are given by (28). When $\alpha=\beta$, Theorem 4.2 and the above remark particularly result in bounds for the eigenvalues of the HSS-preconditioned matrix $M(\alpha)^{-1}A$. See [35, 36] for a different approach.

Theorem 4.3

Consider the saddle-point problem (2). Let $B\in\mathbb{C}^{p\times p}$ be Hermitian positive definite, $E\in\mathbb{C}^{p\times q}$ be of full column rank, and $p\geqslant q$. Let $\alpha>0$ be a given iteration parameter. Then

(i) the real eigenvalues $\lambda=(\alpha-\eta)/(\alpha+\eta)$ of the HSS iteration matrix $\mathscr{L}(\alpha)$ are bounded by

$$\frac{\alpha-\eta_{\max}}{\alpha+\eta_{\max}}\leqslant\lambda\leqslant\frac{\alpha-\eta_{\min}}{\alpha+\eta_{\min}}$$

(ii) when

$$\frac{\eta_{\min}}{\mu_{\min}}\leqslant 1 \quad\text{and}\quad 0<\alpha\leqslant\mu_{\min}\sqrt{\frac{\eta_{\min}}{2\mu_{\min}-\eta_{\min}}}$$

or when

$$\frac{\eta_{\min}}{\mu_{\max}}\geqslant 1 \quad\text{and}\quad 0<\alpha\leqslant\mu_{\max}\sqrt{\frac{\eta_{\min}}{2\mu_{\max}-\eta_{\min}}}$$

the eigenvalues $\lambda=\varphi(\alpha;\eta,\mu)$ of the HSS iteration matrix $\mathscr{L}(\alpha)$ are real for all $\mu\in[\mu_{\min},\mu_{\max}]$ and are bounded as $\lambda\in\mathscr{I}_-(\alpha)\cup\mathscr{I}_+(\alpha)$, with $\mathscr{I}_-(\alpha)=[-\mathscr{I}^{(\max)}(\alpha),-\mathscr{I}^{(\min)}(\alpha)]$ and $\mathscr{I}_+(\alpha)=[\mathscr{I}^{(\min)}(\alpha),\mathscr{I}^{(\max)}(\alpha)]$, where

$$\mathscr{I}^{(\min)}(\alpha)=\frac{\alpha}{\alpha+\eta_{\min}}\left(-\gamma^{(\max)}(\alpha)+\sqrt{[\gamma^{(\max)}(\alpha)]^2+\frac{\eta_{\min}^2}{\alpha^2}-1}\right)$$
$$\mathscr{I}^{(\max)}(\alpha)=\frac{\alpha}{\alpha+\eta_{\max}}\left(\gamma^{(\max)}(\alpha)+\sqrt{[\gamma^{(\max)}(\alpha)]^2+\frac{\eta_{\max}^2}{\alpha^2}-1}\right)$$

with

$$\gamma^{(\max)}(\alpha)=\begin{cases}\dfrac{\mu_{\max}^2-\alpha^2}{\mu_{\max}^2+\alpha^2} & \text{for } \alpha\leqslant\sqrt{\mu_{\min}\mu_{\max}}\\[2ex] \dfrac{\alpha^2-\mu_{\min}^2}{\alpha^2+\mu_{\min}^2} & \text{for } \alpha>\sqrt{\mu_{\min}\mu_{\max}}\end{cases}$$

(iii) when

$$\frac{\eta_{\max}}{\mu_{\max}}\geqslant 1,\quad \frac{\eta_{\max}}{\mu_{\min}}<2 \quad\text{and}\quad \alpha\geqslant\mu_{\min}\sqrt{\frac{\eta_{\max}}{2\mu_{\min}-\eta_{\max}}}$$

or when

$$\frac{\eta_{\max}}{\mu_{\max}}\leqslant 1,\quad 1\leqslant\frac{\eta_{\max}}{\mu_{\min}}<2$$

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and

$$\alpha \geqslant \begin{cases} \mu_{\max}\sqrt{\dfrac{\eta_{\max}}{2\mu_{\max}-\eta_{\max}}} & \text{for } \eta_{\max} \leqslant \dfrac{2\mu_{\min}\mu_{\max}}{\mu_{\min}+\mu_{\max}} \\ \mu_{\min}\sqrt{\dfrac{\eta_{\max}}{2\mu_{\min}-\eta_{\max}}} & \text{for } \eta_{\max} \geqslant \dfrac{2\mu_{\min}\mu_{\max}}{\mu_{\min}+\mu_{\max}} \end{cases}$$

or when

$$\frac{\eta_{\max}}{\mu_{\min}} \leqslant 1 \quad \text{and} \quad \alpha \geqslant \mu_{\max}\sqrt{\frac{\eta_{\max}}{2\mu_{\max}-\eta_{\max}}}$$

the eigenvalues $\lambda = \varphi(\alpha; \eta, \mu)$ of the HSS iteration matrix $\mathscr{L}(\alpha)$ are complex for all $\mu \in [\mu_{\min}, \mu_{\max}]$ and are bounded as

$$|\Re(\lambda)| \leqslant \frac{\alpha\gamma^{(\max)}(\alpha)}{\alpha+\eta_{\min}} := \mathscr{J}^{(\max)}(\alpha)$$

$$|\Im(\lambda)| \leqslant \frac{1}{\alpha+\eta_{\min}}\sqrt{\alpha^2-\eta_{\min}^2-\alpha^2[\gamma^{(\min)}(\alpha)]^2} := \mathscr{K}^{(\max)}(\alpha)$$

where

$$\gamma^{(\min)}(\alpha) = \begin{cases} \dfrac{|\alpha^2-\mu_{\min}^2|}{\alpha^2+\mu_{\min}^2} & \text{for } \alpha \leqslant \sqrt{\mu_{\min}\mu_{\max}} \\ \dfrac{|\alpha^2-\mu_{\max}^2|}{\alpha^2+\mu_{\max}^2} & \text{for } \alpha > \sqrt{\mu_{\min}\mu_{\max}} \end{cases}$$

Now, we choose the iteration parameters α and β such that the eigenvalues of the AHSS iteration matrix $\mathscr{L}(\alpha, \beta)$ and, hence, those of the AHSS-preconditioned matrix $M(\alpha, \beta)^{-1}A$ are tightly clustered. To this end, we mainly investigate the bounds of the eigenvalues $\lambda = \varphi(\alpha, \beta; \eta, \mu)$ given in Theorem 4.2. The following facts can be easily demonstrated by direct calculations:

(F_1) $\gamma^{(\min)}(\alpha, \beta)$ attains its local maximum and $\gamma^{(\max)}(\alpha, \beta)$ attains its global minimum at the same point $\alpha\beta = \mu_{\min}\mu_{\max}$ and it holds that

$$\gamma^{(\min)}(\alpha, \beta)\,|_{\alpha\beta=\mu_{\min}\mu_{\max}} = \gamma^{(\max)}(\alpha, \beta)\,|_{\alpha\beta=\mu_{\min}\mu_{\max}} = \frac{\mu_{\max}-\mu_{\min}}{\mu_{\max}+\mu_{\min}} := \tau_o$$

(F_2) $\mathscr{I}^{(\min)}(\alpha, \beta)$ is a monotonically decreasing function for $\alpha \leqslant \eta_{\min}$ and a monotonically increasing function for $\alpha > \eta_{\min}$, and $\mathscr{I}^{(\max)}(\alpha, \beta)$ and $\mathscr{J}^{(\max)}(\alpha, \beta)$ are monotonically increasing functions, with respect to the variable $\gamma := \gamma^{(\max)}(\alpha, \beta)$, respectively; and

(F_3) $\mathscr{K}^{(\max)}(\alpha, \beta)$ is a monotonically decreasing function with respect to the variable $\gamma := \gamma^{(\min)}(\alpha, \beta)$.

It follows straightforwardly from (F_1)–(F_3) that $\mathscr{I}^{(\min)}(\alpha, \beta)$ attains its maximum at $\alpha\beta = \mu_{\min}\mu_{\max}$ and

$$\widetilde{\mathscr{I}}^{(\min)}(\alpha) := \mathscr{I}^{(\min)}(\alpha, \beta)\,|_{\alpha\beta=\mu_{\min}\mu_{\max}} = \frac{\alpha}{\alpha+\eta_{\min}}\left(-\tau_o+\sqrt{\tau_o^2+\frac{\eta_{\min}^2}{\alpha^2}-1}\right)$$

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$\mathscr{I}^{(\max)}(\alpha,\beta)$ attains its minimum at $\alpha\beta=\mu_{\min}\mu_{\max}$ and

$$\widetilde{\mathscr{I}}^{(\max)}(\alpha):=\mathscr{I}^{(\max)}(\alpha,\beta)\,|_{\alpha\beta=\mu_{\min}\mu_{\max}}=\frac{\alpha}{\alpha+\eta_{\max}}\left(\tau_o+\sqrt{\tau_o^2+\frac{\eta_{\max}^2}{\alpha^2}-1}\right)$$

and both $\mathscr{J}^{(\max)}(\alpha,\beta)$ and $\mathscr{K}^{(\max)}(\alpha,\beta)$ attain their minimums at $\alpha\beta=\mu_{\min}\mu_{\max}$ and

$$\widetilde{\mathscr{J}}^{(\max)}(\alpha):=\mathscr{J}^{(\max)}(\alpha,\beta)\,|_{\alpha\beta=\mu_{\min}\mu_{\max}}=\frac{\alpha\tau_o}{\alpha+\eta_{\min}}$$

$$\widetilde{\mathscr{K}}^{(\max)}(\alpha):=\mathscr{K}^{(\max)}(\alpha,\beta)\,|_{\alpha\beta=\mu_{\min}\mu_{\max}}=\frac{1}{\alpha+\eta_{\min}}\sqrt{\alpha^2-\eta_{\min}^2-\alpha^2\tau_o^2}$$

Based on the above investigation and Theorem 4.2, we are ready to obtain the following result about the partially minimized bounds of the eigenvalues of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$.

Theorem 4.4

Consider the saddle-point problem (2). Let $B\in\mathbb{C}^{p\times p}$ be Hermitian positive definite, $E\in\mathbb{C}^{p\times q}$ be of full column rank, and $p\geqslant q$. Let $\alpha,\beta>0$ be given iteration parameters satisfying $\alpha\beta=\mu_{\min}\mu_{\max}$. Then

(i) the real eigenvalues $\lambda=(\alpha-\eta)/(\alpha+\eta)$ of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ are bounded by

$$\frac{\alpha-\eta_{\max}}{\alpha+\eta_{\max}}\leqslant\lambda\leqslant\frac{\alpha-\eta_{\min}}{\alpha+\eta_{\min}}$$

(ii) when $0<\alpha\leqslant\eta_{\min}$, the eigenvalues $\lambda=\varphi(\alpha,\beta;\eta,\mu)$ of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ are real for all $\mu\in[\mu_{\min},\mu_{\max}]$ and are bounded as $\lambda\in\widetilde{\mathscr{I}}_-(\alpha)\cup\widetilde{\mathscr{I}}_+(\alpha)$, with $\widetilde{\mathscr{I}}_-(\alpha)=[-\widetilde{\mathscr{I}}^{(\max)}(\alpha),-\widetilde{\mathscr{I}}^{(\min)}(\alpha)]$, and $\widetilde{\mathscr{I}}_+(\alpha)=[\widetilde{\mathscr{I}}^{(\min)}(\alpha),\widetilde{\mathscr{I}}^{(\max)}(\alpha)]$, where

$$\widetilde{\mathscr{I}}^{(\min)}(\alpha)=\frac{\alpha}{\alpha+\eta_{\min}}\left(-\tau_o+\sqrt{\tau_o^2+\frac{\eta_{\min}^2}{\alpha^2}-1}\right)$$

$$\widetilde{\mathscr{I}}^{(\max)}(\alpha)=\frac{\alpha}{\alpha+\eta_{\max}}\left(\tau_o+\sqrt{\tau_o^2+\frac{\eta_{\max}^2}{\alpha^2}-1}\right)$$

(iii) when $\alpha>\eta_{\max}/\sqrt{1-\tau_o^2}$, the eigenvalues $\lambda=\varphi(\alpha,\beta;\eta,\mu)$ of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ are complex for all $\mu\in[\mu_{\min},\mu_{\max}]$ and are bounded as

$$|\Re(\lambda)|\leqslant\widetilde{\mathscr{J}}^{(\max)}(\alpha),\quad |\Im(\lambda)|\leqslant\widetilde{\mathscr{K}}^{(\max)}(\alpha)$$

where

$$\widetilde{\mathscr{J}}^{(\max)}(\alpha)=\frac{\alpha\tau_o}{\alpha+\eta_{\min}},\quad \widetilde{\mathscr{K}}^{(\max)}(\alpha)=\frac{1}{\alpha+\eta_{\min}}\sqrt{\alpha^2-\eta_{\min}^2-\alpha^2\tau_o^2}$$

Proof

(i) is the same as that in Theorem 4.2.

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When the relationship $\alpha\beta=\mu_{\min}\mu_{\max}$ is substituted into the functions $\mathscr{I}^{(\min)}(\alpha,\beta)$ and $\mathscr{I}^{(\max)}(\alpha,\beta)$, from Theorem 4.2(ii) we know that (ii) holds for $0<\alpha<\eta_{\min}$. In addition, when the relationship $\alpha\beta=\mu_{\min}\mu_{\max}$ is substituted into the functions $\mathscr{I}^{(\min)}(\alpha,\beta)$ and $\mathscr{I}^{(\max)}(\alpha,\beta)$ as well as the inequality (25), we know that there is no feasible α such that (ii) holds true.

Finally, when the relationship $\alpha\beta=\mu_{\min}\mu_{\max}$ is substituted into the functions $\mathscr{J}^{(\min)}(\alpha,\beta)$ and $\mathscr{K}^{(\max)}(\alpha,\beta)$ as well as the inequality (27), we know that (iii) holds for $\alpha>\eta_{\max}/\sqrt{1-\tau_o^2}$. □

Through direct calculations, we can verify that with respect to α both $\widetilde{\mathscr{I}}^{(\min)}(\alpha)$ and $\widetilde{\mathscr{I}}^{(\max)}(\alpha)$ are monotonically decreasing functions for $0<\alpha\leqslant\eta_{\min}$ and both $\widetilde{\mathscr{J}}^{(\max)}(\alpha)$ and $\widetilde{\mathscr{K}}^{(\max)}(\alpha)$ are monotonically increasing functions for $\alpha>\eta_{\min}/\sqrt{1-\tau_o^2}$. It is also easily noticed that $\widetilde{\mathscr{K}}^{(\max)}(\alpha)=0$ holds at $\alpha=\eta_{\min}/\sqrt{1-\tau_o^2}$, which is, however, not within the definition interval $[\eta_{\max}/\sqrt{1-\tau_o^2},+\infty)$ of α.

Based on the above observations and Theorem 4.4, we can further choose the parameter α such that the eigenvalues of the AHSS iteration matrix $\mathscr{L}(\alpha,\beta)$ are real and tightly clustered as far as possible. This result is precisely summarized as the following theorem:

Theorem 4.5

Consider the saddle-point problem (2). Let $B\in\mathbb{C}^{p\times p}$ be Hermitian positive definite, $E\in\mathbb{C}^{p\times q}$ be of full column rank, and $p\geqslant q$. Then

(i) when $\alpha^\star=\eta_{\min}$ and $\beta^\star=\mu_{\min}\mu_{\max}/\eta_{\min}$, all eigenvalues of the AHSS iteration matrix $\mathscr{L}(\alpha^\star,\beta^\star)$ are real and located in the interval

$$\left[-\frac{\kappa_o-1}{\kappa_o+1},\frac{\tau_o+\sqrt{\tau_o^2+\kappa_o^2-1}}{\kappa_o+1}\right]$$

(ii) when $\alpha^\star=\eta_{\max}/\sqrt{1-\tau_o^2}$ and $\beta^\star=(\mu_{\min}\mu_{\max}/\eta_{\max})\sqrt{1-\tau_o^2}$, all real eigenvalues of the AHSS iteration matrix $\mathscr{L}(\alpha^\star,\beta^\star)$ are located in the interval

$$\left[\frac{1-\sqrt{1-\tau_o^2}}{1+\sqrt{1-\tau_o^2}},\frac{\kappa_o-\sqrt{1-\tau_o^2}}{\kappa_o+\sqrt{1-\tau_o^2}}\right]$$

and all complex eigenvalues of the AHSS iteration matrix $\mathscr{L}(\alpha^\star,\beta^\star)$ are located in the λ-domain satisfying

$$|\Re(\lambda)|\leqslant\frac{\kappa_o\tau_o}{\kappa_o+\sqrt{1-\tau_o^2}}\quad\text{and}\quad|\Im(\lambda)|\leqslant\frac{\sqrt{(\kappa_o^2-1)(1-\tau_o^2)}}{\kappa_o+\sqrt{1-\tau_o^2}}$$

Analogous to the derivations of Theorems 4.4 and 4.5, based on Theorem 4.3, we can further choose the parameter α such that the eigenvalues of the HSS iteration matrix $\mathscr{L}(\alpha)$ are real and tightly clustered as far as possible. This result is precisely summarized as the following theorem:

Theorem 4.6

Consider the saddle-point problem (2). Let $B\in\mathbb{C}^{p\times p}$ be Hermitian positive definite, $E\in\mathbb{C}^{p\times q}$ be of full column rank, and $p\geqslant q$. Define

$$\upsilon_o=\frac{\eta_{\min}}{\sqrt{\mu_{\min}\mu_{\max}}}\quad\text{and}\quad\Upsilon_o=\frac{\eta_{\max}}{\sqrt{\mu_{\min}\mu_{\max}}}$$

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and choose $\alpha^\star=\sqrt{\mu_{\min}\mu_{\max}}$. Then

(i) when

$$\frac{\eta_{\min}}{\mu_{\max}}\geqslant 1 \quad \text{and} \quad \eta_{\min}\geqslant\frac{2\mu_{\min}\mu_{\max}}{\mu_{\min}+\mu_{\max}}$$

all eigenvalues of the HSS iteration matrix $\mathscr{L}(\alpha^\star)$ are real and located in the union of the intervals

$$\left[\frac{1-\Upsilon_o}{1+\Upsilon_o},\frac{1-\upsilon_o}{1+\upsilon_o}\right]$$

and

$$\left[-\frac{\tau_o+\sqrt{\tau_o^2+\Upsilon_o^2-1}}{1+\Upsilon_o},\frac{\tau_o+\sqrt{\tau_o^2+\Upsilon_o^2-1}}{1+\Upsilon_o}\right]$$

(ii) when

$$\frac{\eta_{\max}}{\mu_{\max}}\geqslant 1,\quad \frac{\eta_{\max}}{\mu_{\min}}<2 \quad \text{and} \quad \eta_{\max}\geqslant\frac{2\mu_{\min}\mu_{\max}}{\mu_{\min}+\mu_{\max}}$$

or when

$$\frac{\eta_{\max}}{\mu_{\max}}\leqslant 1,\quad 1\leqslant\frac{\eta_{\max}}{\mu_{\min}}<2 \quad \text{and} \quad \eta_{\min}\leqslant\frac{2\mu_{\min}\mu_{\max}}{\mu_{\min}+\mu_{\max}}\leqslant\eta_{\max}$$

or when

$$\frac{\eta_{\max}}{\mu_{\min}}<1$$

the real eigenvalues of the HSS iteration matrix $\mathscr{L}(\alpha^\star)$ are located in the interval

$$\left[\frac{1-\Upsilon_o}{1+\Upsilon_o},\frac{1-\upsilon_o}{1+\upsilon_o}\right]$$

and all complex eigenvalues of the HSS iteration matrix $\mathscr{L}(\alpha^\star)$ are located in the λ-domain satisfying

$$|\Re(\lambda)|\leqslant\frac{\tau_o}{1+\upsilon_o} \quad \text{and} \quad |\Im(\lambda)|\leqslant\frac{\sqrt{1-\upsilon_o^2-\tau_o^2}}{1+\upsilon_o}$$

We should point out that in general the quasi-optimal iteration parameters used in the AHSS iteration method and the specified iteration parameters used in the AHSS iteration preconditioner are not easily computable as their formulas depend on the extreme eigenvalue bounds of the matrices B and E^*E. Because these two matrices are Hermitian positive definite, their extreme eigenvalue bounds may be roughly estimated by the norm estimators [37, Chapter 5] when they are well-conditioned, and may be more precisely computed by the norm estimators followed with the bisection iterations [38, Chapter 5] or even some simple iteration methods such as the shifted/inverse power method, the Rayleigh quotient iteration method or the Hermitian Lanczos algorithm [39] when they are ill-conditioned. If, in particular, these two matrices possess some special algebraic properties such as diagonal dominance, the Gershgorin-type eigenvalue inclusion theorems [29, 40, 41] may be first adopted to estimate bounds on their extreme eigenvalues, and the bisection iterations may be then followed to improve these estimated bounds.

5. NUMERICAL RESULTS

In this section, we use two examples to exhibit the superiority of AHSS to HSS, GMRES(ℓ) and BiCGSTAB when they are used as solvers, and show the advantage of AHSS over HSS when they are used as preconditioners to GMRES(ℓ) and BiCGSTAB for solving the saddle-point problem (2), from aspects of both the number of iteration steps and elapsed CPU time in seconds. Here, the integer ℓ in GMRES(ℓ) denotes the number of restarting steps.

In actual computations, we choose the right-hand-side vector $b=(f^{\mathrm{T}}, g^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^n$ such that the exact solution of the saddle-point problem (2) is

$$y_\star=(1,2,\ldots,p)^{\mathrm{T}} \in \mathbb{R}^p \quad \text{and} \quad z_\star=(1,2,\ldots,q)^{\mathrm{T}} \in \mathbb{R}^q$$

Besides, all runs are started from an initial vector $x^{(0)}=0$, terminated if the current iterates satisfy

$$\mathrm{RES} \equiv \frac{\|b-Ax^{(k)}\|_2}{\|b-Ax^{(0)}\|_2} \leqslant 10^{-6}$$

and performed in MATLAB 2007a on a personal computer with 2 GB of memory.

The first example is the following Stokes problem arising in fluid dynamics:

$$\begin{aligned} -\Delta u+\operatorname{grad} p &= \hat{f}, \\ \operatorname{div} u &= 0, \end{aligned} \qquad (x, y) \in [0,1]\times[0,1]$$

where the boundary conditions are $u_x=u_y=0$ on the three fixed walls $x=0$, $y=0$, and $x=1$, and $u_x=1$ and $u_y=0$ on the moving wall $y=1$; see [42]. When the *marker and cell* finite difference scheme based on $\mathrm{ne}\times\mathrm{ne}$ uniform grids of square meshes is used to discretize this problem, we obtain a saddle-point problem of the form (2), with $p=2\times\mathrm{ne}(\mathrm{ne}-1)$ and $q=\mathrm{ne}^2$; see [43].

In Tables I and II, we list the number of iteration steps with respect to different sizes of the discretization grids for the tested methods. From Table I, we see that when used as iterative solvers HSS, AHSS, GMRES(ℓ) ($\ell=5$ and 10) and BiCGSTAB methods can successfully compute

Table I. Number of iteration steps for HSS, AHSS, GMRES(ℓ), and BiCGSTAB.

ne	8	16	32	64
p	112	480	1984	8064
q	64	256	1024	4096
$n=p+q$	176	736	3008	12 160
HSS	147	263	518	997
AHSS	89	196	260	474
GMRES(5)	674	1389	2682	8928
GMRES(10)	513	1127	2697	6725
BiCGSTAB	133	859	874	852

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Table II. Number of iteration steps for GMRES(ℓ) and BiCGSTAB preconditioned by HSS and AHSS.

ne	8	16	32	64
p	112	480	1984	8064
q	64	256	1024	4096
$n=p+q$	176	736	3008	12 160
HSS–GMRES(5)	15	24	41	90
AHSS–GMRES(5)	13	19	25	49
HSS–GMRES(10)	15	21	35	63
AHSS–GMRES(10)	13	18	24	40
HSS–BiCGSTAB	9	14	21	36
AHSS–BiCGSTAB	7	11	15	27

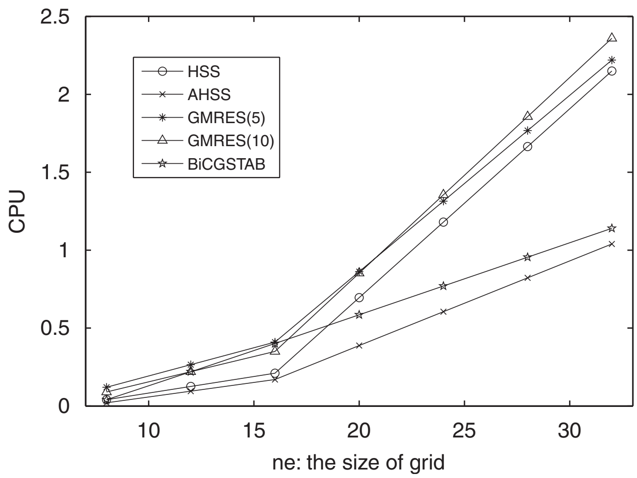

Figure 1. Curves of CPU times for HSS, AHSS, GMRES(ℓ), and BiCGSTAB.

approximate solutions for the saddle-point problems and, among them, AHSS is the fastest as it shows the least number of iteration steps. Moreover, for ne$=$16 and 32, HSS shows much less iteration steps than BiCGSTAB; while for ne$=$8 and 64, it shows slightly larger iteration steps than BiCGSTAB. However, HSS always outperforms GMRES(ℓ) ($\ell=5$ and 10) in iteration steps. From Table II, we see that as preconditioners both HSS and AHSS can drastically reduce the iteration steps of the preconditioned GMRES(ℓ) ($\ell=5$ and 10) and BiCGSTAB methods and, hence, they can considerably improve the numerical properties of GMRES(ℓ) ($\ell=5$ and 10) and BiCGSTAB. In addition, the AHSS preconditioner performs better than the HSS preconditioner.

In Figures 1 and 2, we list the CPU times with respect to different sizes of the discretization grids for the tested methods. From Figure 1, we see that when used as iterative solvers for the saddle-point problems, AHSS is the most effective method as it costs the least CPU times. Moreover, when ne is small, HSS costs less CPU times than BiCGSTAB; while when ne becomes larger, it costs more CPU times than BiCGSTAB. However, HSS always outperforms GMRES(ℓ) ($\ell=5$ and 10) in CPU times. From Figure 2, we see that as preconditioners both HSS and AHSS can largely reduce the CPU times of the preconditioned GMRES(ℓ) ($\ell=5$ and 10) and BiCGSTAB

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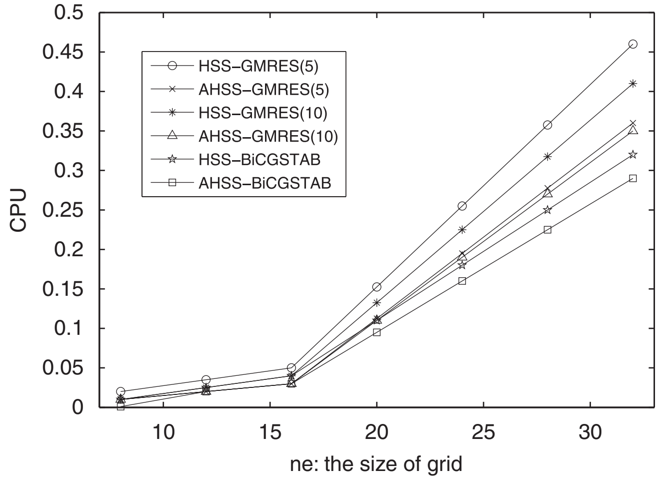

Figure 2. Curves of CPU times for GMRES(ℓ) and BiCGSTAB preconditioned by HSS and AHSS.

methods and, hence, they can considerably improve the numerical properties of GMRES(ℓ) ($\ell = 5$ and 10) and BiCGSTAB. In addition, the AHSS preconditioner is slightly more effective than the HSS preconditioner in computing timing.

We remark that for the Stokes problems there already exists optimal preconditioners, e.g., the block-diagonal and the block-tridiagonal ones, whose numerical behaviors are not heavily dependent on the mesh sizes; see [24, 42, 44, 45]. The HSS-like preconditioners are generally not optimal and their numerical behaviors are, however, dependent on the mesh sizes. Moreover, that the AHSS iteration method may be more effective than the HSS iteration method may give some additional insight to solve this class of saddle-point problems.

The second example is the saddle-point problem (2) whose coefficient matrix have the following blocks:

$$B = \mathrm{Diag}(2\widehat{B}^{\mathrm{T}}\widehat{B} + D_1, D_2, D_3) \in \mathbb{R}^{p\times p}$$

is a block-diagonal matrix and

$$E^{\mathrm{T}} = [\widehat{E}, -I, I] \in \mathbb{R}^{q\times p}$$

is a full row-rank matrix. Here $\widetilde{\mathrm{ne}} = \mathrm{ne}^2$; $\widehat{B} = (\widehat{b}_{ij}) \in \mathbb{R}^{\widehat{\mathrm{ne}}\times\widehat{\mathrm{ne}}}$, with $\widehat{b}_{ij} = \mathrm{e}^{-2((i/3)^2+(j/3)^2)}$; $D_1 = I \in \mathbb{R}^{\widehat{\mathrm{ne}}\times\widehat{\mathrm{ne}}}$ is the identity matrix; $D_i = \mathrm{diag}(d_j^{(i)}) \in \mathbb{R}^{2\widetilde{\mathrm{ne}}\times 2\widetilde{\mathrm{ne}}}$, $i = 2, 3$, are diagonal matrices, with

$$d_j^{(2)} = \begin{cases} 1 & \text{for } 1 \leqslant j \leqslant \widetilde{\mathrm{ne}} \\ 10^{-5}(j - \widetilde{\mathrm{ne}})^2 & \text{for } \widetilde{\mathrm{ne}} + 1 \leqslant j \leqslant 2\widetilde{\mathrm{ne}} \end{cases}$$

and

$$d_j^{(3)} = 10^{-5}(j + \widetilde{\mathrm{ne}})^2 \quad \text{for } 1 \leqslant j \leqslant 2\widetilde{\mathrm{ne}}$$

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$\widehat{E}$ is a blocked matrix given by

$$\widehat{E} = \begin{bmatrix} \widetilde{E} \otimes I_{\mathrm{ne}\times\mathrm{ne}} \\ I_{\mathrm{ne}\times\mathrm{ne}} \otimes \widetilde{E} \end{bmatrix}$$

with

$$\widetilde{E} = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix} \in \mathbb{R}^{\mathrm{ne}\times(\mathrm{ne}+1)} \tag{31}$$

or

$$\widetilde{E} = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \in \mathbb{R}^{\mathrm{ne}\times\mathrm{ne}} \tag{32}$$

Table III. Iteration steps and CPU times for HSS and AHSS preconditioners.

Matrix $\widetilde{E}$		The first-order difference			The second-order difference		
ne		16	32	48	16	32	48
p		1296	5152	11 568	1280	5120	11 520
q		512	2048	4608	512	2048	4608
$n=p+q$		1808	7200	16 176	1792	7168	16 128
GMRES(10)	IT	104	717	8760	158	1396	9352
	CPU	0.05	1.24	39.14	0.09	2.58	45.44
HSS–GMRES(10)	IT	17	57	110	17	54	118
	CPU	0.04	1.11	9.29	0.04	1.10	10.21
AHSS–GMRES(10)	IT	15	26	36	12	24	29
	CPU	0.03	0.52	3.04	0.02	0.49	2.51
GMRES(20)	IT	91	682	2947	133	975	4956
	CPU	0.06	1.66	18.87	0.10	2.37	33.63
HSS–GMRES(20)	IT	17	52	177	16	52	128
	CPU	0.04	1.05	15.26	0.04	1.09	11.29
AHSS–GMRES(20)	IT	14	25	35	12	18	27
	CPU	0.03	0.50	3.02	0.03	0.41	2.36
BiCGSTAB	IT	—	574	1950	—	1161	3997
	CPU	—	1.37	9.91	—	3.10	23.22
HSS–BiCGSTAB	IT	11	34	75	14	34	81
	CPU	0.03	0.68	6.42	0.03	0.71	7.09
AHSS–BiCGSTAB	IT	8	17	34	8	17	36
	CPU	0.02	0.34	2.87	0.02	0.35	3.10

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and

$$\widehat{\mathrm{ne}} = \begin{cases} \mathrm{ne}(\mathrm{ne}+1) & \text{if } \widetilde{E} \text{ is defined by (31)} \\ \mathrm{ne}^2 & \text{if } \widetilde{E} \text{ is defined by (32)} \end{cases}$$

This class of saddle-point problems, with $p=\widehat{\mathrm{ne}}+4\widetilde{\mathrm{ne}}$ and $q=2\widetilde{\mathrm{ne}}$, arises in computing the descent directions in the Newton steps involved in the modified primal–dual interior point method used to solve the nonsmooth and nonconvex minimization problems from restorations of piecewise constant images. The choice in (31) of the matrix $\widetilde{E}$ being the first-order difference operator corresponds to the circles image, and the choice in (32) of the matrix $\widetilde{E}$ being the second-order difference operator with the Neumann boundary condition corresponds to the diamonds image. See [46] for details.

In Table III we list the iteration steps and the computing times with respect to different matrix sizes and different difference operators for the tested methods. From Table III, we see that both HSS and AHSS preconditioners can considerably improve the numerical properties of GMRES(ℓ) ($\ell=10$ and 20) and BiCGSTAB, and the AHSS preconditioner performs much better than the HSS preconditioner.

6. CONCLUSION

For the large sparse saddle-point problems, we have further studied the numerical properties of the AHSS and the HSS iteration methods when they are used either as preconditioners for Krylov subspace methods or as linear solvers. In particular, the quasi-optimal iteration parameters that minimize the fields of values of the iteration matrices and the corresponding quasi-optimal convergence factors have been exactly computed, and the clustering property of the eigenvalues of the AHSS- and HSS-preconditioned matrices has been described in detail. Also, we have specified certain choices of the iteration parameters that may lead to good numerical behavior of the AHSS and the HSS preconditioners.

From both theoretical analysis and numerical implementations, we could claim that the AHSS iteration method may be used better as a preconditioner than as a solver, because the numerical property of a parameterized splitting preconditioner is usually less sensitive than that of a parameterized splitting solver so that the choice of the involved iteration parameters in the AHSS preconditioner is practically more convenient, although the AHSS iteration method also possesses good numerical property when it is used as a linear solver.

In general, for the large sparse generalized saddle-point problems, the optimal choices of the iteration parameters and the corresponding optimal convergence factors of the AHSS and the HSS iteration methods, even for the specially simplified case that the (1,1)-blocks in the generalized saddle-point matrices are identity, are still difficult problems, and they deserve in-depth study on both theory and computations. Some recent works along this line is given in Benzi and Ng [47] and Chan *et al.* [48]. Besides, the theoretical and numerical properties of the AHSS and the HSS preconditioners, in particular, the effects of the iteration parameters on the eigenvalue-clustering of the preconditioned matrices need to be further investigated.

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APPENDIX A

Lemma A1

Let α, β and μ be positive constants. Then the function $\theta(\alpha,\beta;\eta,\mu)$ defined by (19) is monotonically increasing with respect to η, if $(\alpha\beta+\mu^2)\eta\geqslant 2\alpha\sqrt{\alpha\beta}\mu$.

Proof

For convenience of our statements, we define the positive constant

$$\tau=\frac{|\alpha\beta-\mu^2|}{\alpha\beta+\mu^2}$$

Obviously, it holds that $\tau<1$ and $\alpha^2(\tau^2-1)+\eta^2\leqslant\eta^2$. Moreover, if $(\alpha\beta+\mu^2)\eta\geqslant 2\alpha\sqrt{\alpha\beta}\mu$, then it holds that $\alpha^2(\tau^2-1)+\eta^2\geqslant 0$. Now, we have

$$\theta(\alpha,\beta;\eta,\mu)=\frac{\alpha}{\alpha+\eta}\left(\tau+\frac{1}{\alpha}\sqrt{\alpha^2(\tau^2-1)+\eta^2}\right)$$

Because the derivative of $\theta(\alpha,\beta;\eta,\mu)$ with respect to η is given by

$$\frac{\mathrm{d}\theta(\alpha,\beta;\eta,\mu)}{\mathrm{d}\eta}=\frac{\alpha(\alpha+\eta-\alpha\tau^2-\tau\sqrt{\alpha^2(\tau^2-1)+\eta^2})}{(\alpha+\eta)^2\sqrt{\alpha^2(\tau^2-1)+\eta^2}}$$

we can obtain the estimate

$$\begin{aligned}\frac{\mathrm{d}\theta(\alpha,\beta;\eta,\mu)}{\mathrm{d}\eta}&\geqslant\frac{\alpha(\alpha+\eta-\alpha\tau^2-\tau\eta)}{(\alpha+\eta)^2\sqrt{\alpha^2(\tau^2-1)+\eta^2}}\\&=\frac{\alpha(1-\tau)[\alpha(1+\tau)+\eta]}{(\alpha+\eta)^2\sqrt{\alpha^2(\tau^2-1)+\eta^2}}\\&\geqslant 0\end{aligned}$$

This clearly shows that under the assumptions, the function $\theta(\alpha,\beta;\eta,\mu)$ is monotonically increasing with respect to η. □

We remark that if τ is replaced by $-\tau$ in $\theta(\alpha,\beta;\eta,\mu)$, the obtained function is still monotonically increasing with respect to η when $(\alpha\beta+\mu^2)\eta\geqslant 2\alpha\sqrt{\alpha\beta}\mu$.

Lemma A2

Let α,β and η,μ be positive constants satisfying $\alpha\geqslant\eta$ and $\mu\in[\mu_{\min},\mu_{\max}]$. Then the following three statements are equivalent:

(i) $(\alpha\beta+\mu^2)\eta\geqslant 2\alpha\sqrt{\alpha\beta}\mu$, for $\forall\mu\in[\mu_{\min},\mu_{\max}]$,
(ii) $(\alpha\beta+\mu^2)\eta\geqslant 2\alpha\sqrt{\alpha\beta}\mu$, for $\mu\in\{\mu_{\min},\mu_{\max}\}$,
(iii) $\beta\leqslant(\mu_{\min}^2/(\alpha\eta^2))(\alpha-\sqrt{\alpha^2-\eta^2})^2$ or $\beta\geqslant(\mu_{\max}^2/(\alpha\eta^2))(\alpha+\sqrt{\alpha^2-\eta^2})^2$.

Proof
Let $t:=\sqrt{\alpha\beta}$. Then the quadratic equation $(t^2+\mu^2)\eta=2\alpha\mu t$ obviously has two positive roots

$$t_-(\mu):=\frac{\mu}{\eta}(\alpha-\sqrt{\alpha^2-\eta^2}) \quad \text{and} \quad t_+(\mu):=\frac{\mu}{\eta}(\alpha+\sqrt{\alpha^2-\eta^2})$$

Evidently, $t_\pm(\mu)$ are nonnegative and monotonically increasing functions with respect to μ in the interval $[\mu_{\min},\mu_{\max}]$. Hence, (i) holds true if and only if either $t\in(0,t_-(\mu_{\min})]$ or $t\in[t_+(\mu_{\max}),+\infty)$, which is equivalent to (iii). Also, we easily see that (ii) holds true if and only if so does (iii). □

Lemma A3
Let α,β and η,μ be positive constants satisfying $\alpha\geqslant\eta$ and $\mu\in[\mu_{\min},\mu_{\max}]$. Then the following three statements are equivalent:

(i) $(\alpha\beta+\mu^2)\eta<2\alpha\sqrt{\alpha\beta}\mu$, for $\forall\mu\in[\mu_{\min},\mu_{\max}]$,
(ii) $(\alpha\beta+\mu^2)\eta<2\alpha\sqrt{\alpha\beta}\mu$, for $\mu\in\{\mu_{\min},\mu_{\max}\}$,
(iii) $(\mu_{\max}^2/(\alpha\eta^2))(\alpha-\sqrt{\alpha^2-\eta^2})^2\leqslant\beta\leqslant(\mu_{\min}^2/(\alpha\eta^2))(\alpha+\sqrt{\alpha^2-\eta^2})^2$.

Proof
Analogous to the proof of Lemma A2, we let $t:=\sqrt{\alpha\beta}$. Then the quadratic equation $(t^2+\mu^2)\eta=2\alpha\mu t$ obviously has two positive roots

$$t_-(\mu):=\frac{\mu}{\eta}(\alpha-\sqrt{\alpha^2-\eta^2}) \quad \text{and} \quad t_+(\mu):=\frac{\mu}{\eta}(\alpha+\sqrt{\alpha^2-\eta^2})$$

Again, it is easily seen that $t_\pm(\mu)$ are nonnegative and monotonically increasing functions with respect to μ in the interval $[\mu_{\min},\mu_{\max}]$. Hence, (i) holds true if and only if $t\in(t_-(\mu_{\max}),t_+(\mu_{\min}))$, which is equivalent to (iii). Also, it is obvious that (ii) holds true if and only if so does (iii). □

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