SOLVING ELLIPTIC PROBLEMS ON REGIONS PARTITIONED INTO SUBSTRUCTURES

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I. INTRODUCTION

In this paper, we will introduce and discuss some special iterative methods for the solution of finite element approximations for elliptic equations defined on regions regarded as unions of subregions. We are interested, in particular, in the design of methods for which the interaction between subproblems, i.e., discrete elliptic problems on the subregions, is computed by a conjugate gradient method while the subproblems are solved by a direct method. This approach differs from the standard one in industry in which a direct method is used throughout the entire solution process; see Section 2. As we will also see in that section, the whole problem can be reduced to finding accurate values of the solution on the interaces, i.e., the curves or surfaces which separate the subregions.

To simplify the discussion, we will only consider in detail the case of a plane region cut into two pieces. The region could equally well be subdivided into several subregions and the ideas can also be applied recursively. A fuller exposition will be given in Bjørstad and Widlund (1983), in

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which the optimality of the so-called good and excellent methods; see Section 3, will be rigorously established for second order elliptic problems in the plane. This theory is quite complicated. Its development requires the use of elliptic regularity theory on Lipschitz regions; see Grisvard (1980) and some apparently new finite element results. We regard the extension of these results to more general elliptic problems as a significant open problem.

The partition of elliptic problems into subproblems is a very natural idea that is much used in practice. In finite element computations the modelling of entire mechanical systems can profitably be organized by discretizing the given partial differential equations using finite elements on subregions into which the region is partitioned or from which it was originally assembled. These tasks and the factorization of the resulting stiffness matrices for the subproblems can obviously be assigned to different engineering groups and computer systems or processors with coordination required only at the interfaces between the substructures to assure matching of the finite element triangulations. These ideas are particularly attractive if some of the substructures are identical or have previously been analyzed, as in the case when a simulation is repeated after the redesign or damage of one or a few of the substructures.

We know of only a few previous efforts to develop iterative methods similar to ours. In a paper on generalized conjugate gradient methods, Concus, Golub, and O'Leary (1975) reported on numerical experiments with Poisson's equation on a T-shaped region. No results for very large problems are reported in their work which makes assessing its potential difficult. In Section IV of this paper, we will report on results obtained by three other methods which are described in Section III. Methods inspired by the Schwarz alternating method and by control theory are considered in Glowinski, Periaux and Dihn (1982), and by Glowinski in this volume. After comparing their numerical results with ours, we have come to believe that our methods are more powerful.

The pioneer in the mathematical analysis of the kind which we consider is Maksymilian Dryja; see Dryja (1982). He

has also contributed very important algorithmic ideas. We received his paper in March 1981 after our first series of numerical experiments. He there analyzes two of the methods for the five point formula for Laplace's equation on L-shaped regions. We also greatly benefited from his visit with the second author in October 1981. At that time, we were introduced to the so-called excellent method discussed in Sections III and IV.

We also note that both from algorithmic and theoretical points of view, our methods have much in common with capacitance matrix methods, see Bjørstad (1980), Dryja (1983), O'Leary and Widlund (1979) and Proskurowski and Widlund (1980), and the many references given in those papers.

II. SUBSTRUCTURED PROBLEMS AND BLOCK FORM OF THE STIFFNESS MATRICES

To simplify the discussion, we confine ourselves to Dirichlet problems defined on the union Ω of Ω_1, Ω_2 and Γ_3. Here Ω_1 and Ω_2 are plane, bounded, non-intersecting regions and Γ_3 the intersection of their closures. The boundaries of Ω_1 and Ω_2 are $\Gamma_1 \cup \Gamma_3$ and $\Gamma_2 \cup \Gamma_3$, respectively, and the boundary of Ω is $\Gamma_1 \cup \Gamma_2$. We assume that these boundaries are Lipschitz continuous; see Grisvard (1980). A linear, second order, positive definite, self adjoint elliptic operator is defined on Ω. Its symmetric bilinear form is denoted by $a_\Omega(u,v)$; see Ciarlet (1978) or Strang and Fix (1973). A simple example is given by the Laplace operator with homogeneous Dirichlet condition for which

$$a_\Omega(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad u,v \in H_0^1(\Omega) .$$

Here $H_0^1(\Omega)$ is the subspace of elements, with zero boundary values, of the Sobolev space of square integrable functions with square integrable first derivatives. Triangulations of Ω_1 and Ω_2 are introduced in such a way that the nodes on Γ_3 coincide. We assume that each degree of freedom of the finite element subspaces is associated with a node and a basis function in the finite element space and that the support of any basis function coincides with the triangles to which its node belongs. An element of the stiffness matrix has the form

$a_\Omega(\phi_i,\phi_j)$, where ϕ_i and ϕ_j are basis functions. It vanishes unless the two nodes belong to a common triangle, To simplify, we will only consider Lagrangian finite elements in this discussion.

It is easy to see that

$$a_{\Omega_1}(u,v) + a_{\Omega_2}(u,v) = a_\Omega(u,v) \tag{2.1}$$

and that therefore the stiffness matrix of a problem on Ω can be constructed from those of Ω_1 and Ω_2. This fact is frequently used in practice to construct stiffness matrices from the stiffness matrices on the individual triangles; see Strang and Fix (1973).

We write the stiffness matrices corresponding to Ω_1 and Ω_2 respectively as

$$A^{(1)} = \begin{bmatrix} A_{11} & A_{13} \\ A_{13}^T & A_{33}^{(1)} \end{bmatrix} \text{ and } A^{(2)} = \begin{bmatrix} A_{22} & A_{23} \\ A_{23}^T & A_{33}^{(2)} \end{bmatrix}, \tag{2.2}$$

where A_{11}, represents couplings between the pairs of nodes in Ω_1, A_{13} couplings between the pairs belonging to Ω_1 and Γ_3 respectively and $A_{33}^{(1)}$ couplings between nodes on Γ_3, etc. The stiffness matrix of the entire problem is of the form,

$$A = \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{bmatrix}$$

where, by a simple computation,

$$A_{33} = A_{33}^{(1)} + A_{33}^{(2)}. \tag{2.3}$$

We note that the degrees of freedom have been partitioned into three sets of which the third, the separator set, corresponds to the nodes on Γ_3. From the point of view of graph theory, the undirected graph of A becomes disconnected into two components if the nodes of the separator set and their incident edges are removed. If conforming finite elements are used, then it also follows from the assumptions on $a_\Omega(u,v)$, that A is positive definite and symmetric, and as a consequence so

are A_{11}, A_{22} and A_{33}; see Ciarlet (1978) or Strang and Fix (1973). In other cases we assume that A has these properties.

We consider the linear system of algebraic equations of the form

$$Ax = \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{2.4}$$

By using block-Gaussian elimination, we can reduce this system to the positive definite, symmetric system,

$$\begin{aligned} Sx_3 &= (A_{33} - A_{13}^T A_{11}^{-1} A_{13} - A_{23}^T A_{22}^{-1} A_{23})x_3 \\ &= b_3 - A_{13}^T A_{11}^{-1} b_1 - A_{23}^T A_{22}^{-1} b_2 = \tilde{b}_3 . \end{aligned} \tag{2.5}$$

It is common practice to complete the process by soling (2.5) by a direct method.

The right hand side $\tilde{b}_3$ can be obtained at the expense of solving the two subproblems on Ω_1 and Ω_2, multiplying the resulting vectors by the sparse matrices A_{13}^T and A_{23}^T respectively and by subtracting the resulting vectors from b_3. The matrix S, which is a so-called Schur complement (see Cottle (1974)) can be expensive to compute and store. However, we notice that Sy can be computed for a given vector y at the expense of solving the two subproblems with the sparse right hand sides $A_{13}y$ and $A_{23}y$, respectively, and certain sparse matrix and vector operations. In the next section, we will develop iterative methods which only require S in terms of such matrix-vector products.

The cost of computing Sy depends primarily on the efficiency of the solvers for the subproblems. It should also be noted that if a Gaussian elimination method is used, advantage can be taken of the sparsity of the vectors $A_{13}y$ and $A_{23}y$. Thus when the lower triangular systems of equations are solved, the computation can begin with the first equation which has a nonzero component in the right hand side. Similarly, the solution of the upper triangular systems can be stopped as soon as all the components of $A_{11}^{-1} A_{13}y$ and $A_{22}^{-1} A_{23}y$ necessary

for computing $A_{13}^T(A_{11}^{-1} A_{13}y)$ and $A_{23}^T(A_{22}^{-1} A_{23}y)$ have been found. This effectively reduces the size of the triangular systems necessary to carry out the iteration steps. It is particularly advantageous if all the variables at nodes adjacent to Γ_3 are ordered late. It should be noted, however, that such a constraint may be hard to impose on existing software or may lead to an increase in the time required to factor A_{11} and A_{22} into their triangular factors.

In the next section, we will need the Schur complements with respect to the matrices $A^{(1)}$ and $A^{(2)}$ defined in (2.2). They are,

$$S^{(1)} = A_{33}^{(1)} - A_{13}^T A_{11}^{-1} A_{13} \text{ and } S^{(2)} = A_{33}^{(2)} - A_{23}^T A_{33}^{-1} A_{23}, \quad (2.6)$$

Using equations (2.3) and (2.6), we find that

$$S = S^{(1)} + S^{(2)} . \quad (2.7)$$

III. CONJUGATE GRADIENT ALGORITHMS FOR SUBSTRUCTURED PROBLEMS AND AN INFORMAL THEORY

The general theory of conjugate methods is quite well known and will therefore be discussed only very briefly; see Concus, Golub and O'Leary (1975), Hestenes (1956) or Luenberger (1973).

Let Ax = b be a linear system of algebraic equations with a positive definite, symmetric matrix A. Let $x^{(0)}$ be an initial guess and $r^{(0)} = b - Ax^{(0)}$ the initial residual. The k-th iterate in the standard conjugate gradient method, $x^{(k)}$, can then be characterized as the minimizing element for the variational problem

$$\min[(\tfrac{1}{2})y^T Ay - y^T b]$$

where $y - x^{(0)}$ varies in the linear space spanned by $r^{(0)}$, $Ar^{(0)}, \ldots, A^{k-1}r^{(0)}$. By expanding in eigenvectors of A, it can be established that

$$(x^{(k)}-x)^T A(x^{(k)}-x)/(x^{(0)}-x)^T A(x^{(0)}-x)$$

is bounded from above by,

$$\min_{p \in P_{k-1}} \max_{\lambda \in \sigma(A)} (1 + \lambda p(\lambda))^2 , \tag{3.1}$$

see Luenberger (1973). Here x is the exact solution, P_{k-1} the space of all polynomial of degree k-1 and $\sigma(A)$ the spectrum of A. This bound can be used to establish that the convergence is rapid if A is well conditioned and that the rate of convergence can be bounded uniformly for entire families of operators if all the eigenvalues fall in a fixed interval.

Preconditioned conjugate gradient methods have been studied extensively in recent years. The idea goes back to the mid-fifties; see Hestenes (1956). Let A_0 be another positive definite, symmetric operator for which it is feasible to solve auxiliary systems of the form $A_0 y = c$ repeatedly for different right hand sides. In one of the versions of the method, the original problem $Ax = b$ is transformed into $AA_0^{-1}y = b$. The iterate $y^{(k)}$ is sought as the sum of an initial guess $y^{(0)}$ and a linear combination of $r^{(0)}$, $AA_0^{-1}r^{(0)}$, $\ldots,(AA_0^{-1})^{k-1}r^0$. If an appropriate inner product is used, a convenient recursion formula results, see e.g., Proskurowski and Widlund (1980). Each step of this algorithm requires the solution of an auxiliary linear system. It is important to note that the estimate (3.1) still holds, but that now the eigenvalues of AA_0^{-1}, i.e., those of the generalized eigenvalue problem $A\phi = \lambda A_0 \phi$, are of relevance rather than those of A. It is also worth noting that the estimate (3.1) can be used to show particularly rapid convergence if the eigenvalues are clustered.

For the problems at hand, we first consider the solution of equation (2.5) without preconditioning. The vector x_3 represents nodal values on Γ_3 and provide part of the Dirichlet data for two discrete subproblems. The restriction of Ax to the same set is computed when evaluating the residual We will now give an informal argument which shows that this mapping involves a loss of one derivative in $L^2(\Gamma_3)$. For a rigorous argument; see Bjørstad and Widlund (1983).

Let us recall that by (2.7), $S = S^{(1)} + S^{(2)}$ and show that the positive definite, symmetric operators $S^{(1)}$ and $S^{(2)}$ each have the property mentioned above. A discrete Dirichlet problem for the region Ω_1 can be written as

$$\begin{bmatrix} A_{11} & A_{13} \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ g_D \end{bmatrix}$$

if the right hand side f and the Dirichlet data on Γ_1 vanish. Similarly, a Neumann problem takes the form,

$$\begin{bmatrix} A_{11} & A_{13} \\ A_{13}^T & A_{33}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ g_N \end{bmatrix}$$

It is straightforward to show that

$$\begin{bmatrix} A_{11} & A_{13} \\ A_{13}^T & A_{33}^{(1)} \end{bmatrix} \begin{bmatrix} A_{11} & A_{13} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ g_D \end{bmatrix} = \begin{bmatrix} 0 \\ S^{(1)} g_D \end{bmatrix}$$

Thus we can regard $S^{(1)}$ as a discretization of a mapping which maps the boundary values of a solution of a homogeneous elliptic problem onto its normal derivate on the boundary. Such a mapping will have a condition number proportional to the number of nodes on Γ_3. We call this the mediocre method.

In view of what we just learned, it is natural to try to find a preconditioner which also involves the loss of a derivative in $L_2(\Gamma_3)$. A natural choice would be a tangential derivative but that is not a symmetric operator. Instead we use the square root of a discretization of the Laplacian on Γ_3. Such a method is practical for problems in the plane and has been tested; see Section IV. We call this the good method.

An even better method, referred to as the excellent method, involves the solution of a block lower triangular system,

$$\begin{bmatrix} A_{11} & A_{13} & 0 \\ A_{13}^T & A_{33}^{(1)} & 0 \\ 0 & A_{23} & A_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_3 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f_3 \\ 0 \end{bmatrix}$$

We obtain, straightforwardly, that

$$\begin{bmatrix} A_{11} & A_{13} & 0 \\ A_{13}^T & A_{33} & A_{23}^T \\ 0 & A_{23} & A_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_3 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ SS^{(1)^{-1}} f_3 \\ 0 \end{bmatrix}$$

It can be shown that $SS^{(1)^{-1}}$ has a very good spectrum. We note that this mapping involves solving a Neumann problem on Ω_1 and then a Dirichlet problem on Ω_2, using the values on Γ_3 just computed. The last step to find $SS^{(1)^{-1}} f_3$ is the computation of the residual on Γ_3.

IV. NUMERICAL EXPERIMENTS

We will only give a brief report on a few of the many experiments which we have carried out for the five point approximation of Poisson's equation on regions which are unions of two rectangles. The choice of such regions greatly simplifies the experiments and makes it feasible to conduct many experiments with very many degrees of freedom since fast Poisson solvers can be used to solve the subproblems. We note that this simple finite difference approximation can be viewed as a conforming finite element approximation on a mesh of right triangles using piecewise linear functions; see Strang and Fix (1973).

In the experiments considered here, we consider the union of two rectangles with corners at the points (0,0), (1,0), (1,1/2), (0,1/2) and (1/8,1/2), (5/8,1/2), (5/8,1), (1/8,1) respectively. The relevant parameters are the number of mesh points q on Γ_3, the interval between (1/8,1/2) and (5/8,1/2), and the total number of degrees of freedom. We have used data which are consistent with an exact solution $u(x,y) = x^2+y^2-xe^x \cos y$. We have found no real difference between the performance of our method for this and other cases.

We first show, in Table 1, how the number of iterations grow as a function of q using the excellent method. We stopped the iterations at the level of the truncation error. The initial guess was the zero function. We note that the overall number of degrees of freedom increases quadratically with q and equals 48641 for q = 127.

TABLE 1

q	Number of iterations	Maximum error on Ω
3	2	3.66×10^{-4}
7	3	9.59×10^{-5}
15	3	2.45×10^{-5}
31	4	6.09×10^{-6}
63	4	1.49×10^{-6}
127	5	3.02×10^{-7}

In Table 2, we compare the maximum error for the excellent and good methods for the case of q = 127. Ten to twelve iterations are required to decrease the maximum error by a factor of 10 if the mediocre method is used.

TABLE 2

Number of iterations	Excellent Method	Good Method
0	3.79×10^{-1}	3.79×10^{-1}
1	1.25×10^{-2}	3.22×10^{-2}
2	7.48×10^{-4}	4.01×10^{-3}
3	2.56×10^{-5}	5.26×10^{-4}
4	4.42×10^{-7}	8.74×10^{-5}
5	3.02×10^{-7}	1.05×10^{-5}
6	3.02×10^{-7}	1.33×10^{-6}
7	3.02×10^{-7}	3.08×10^{-7}
8	3.02×10^{-7}	3.03×10^{-7}

Finally, we give some information on the spectrum of $SS^{(1)-1}$ for q = 63, i.e., for the excellent method. The smallest eigenvalues were found to be $\lambda_1 = 1.713$, $\lambda_2 = 1.777$, ..., $\lambda_5 = 1.992$ with the others less than or equal to 2.000. The corresponding spectrum for the good method shows a somewhat less pronounced cluster with $\lambda_1 = 1.733$, $\lambda_2 = 1.804$, ... $\lambda_5 = 2.008$, ..., $\lambda_{63} = 2.828$.

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