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How bad are Hankel matrices?

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Summary. Considered are Hankel, Vandermonde, and Krylov basis matrices. It is proved that for any real positive definite Hankel matrix of order n, its spectral condition number is bounded from below by $3 \cdot 2^{n-6}$. Also proved is that the spectral condition number of a Krylov basis matrix is bounded from below by $3^{\frac{1}{2}} \cdot 2^{\frac{n}{2}-3}$. For $V = V(x_1, \ldots, x_n)$, a Vandermonde matrix with arbitrary but pairwise distinct nodes $x_1, \ldots, x_n$, we show that $\operatorname{cond}_2 V \geq 2^{n-2}/n^{\frac{1}{2}}$; if either $|x_j| \leq 1$ or $|x_j| \geq 1$ for all j, then $\operatorname{cond}_2 V \geq 2^{n-2}$.

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1. Introduction

This work originates from rather embarrassing results of testing a code which implements the well–known three–term recurrences for Hankel systems. Good accuracy was achieved only for very small, carefully chosen matrices, but in most experiments the accuracy fell far short of what can be called acceptable. However, it would be unjust to blame the code or the algorithm, because in all cases we in fact failed to appropriately pick up the very testing matrices.

To ascertain whether the algorithm is stable we should try it on well–conditioned matrices. Moreover, since the algorithm under examination successively finds the solutions to all truncated systems, all leading submatrices of the original Hankel matrix should be well–conditioned, too. The fact is that we failed to devise such matrices. Thus, perhaps the code is good, but there are no tasks that could be resolved by it. In this paper we intend to substantiate this observation by theorems.

In Sect. 2 we give necessary definitions and discuss the interconnection between Hankel and Vandermonde matrices. The principal result is Lemma 2.1; we have picked it up from [4], but here furnish a simpler proof.

Section 3 is a core of the paper. Here, some new results are given concerning the condition of the Krylov basis matrix. We show that the spectral condition number for a Krylov matrix of order n is lower bounded by $3^{\frac{1}{2}} \cdot 2^{\frac{n}{2}-3}$ (Lemma 3.4).

In Sect. 4 we present a byproduct of our research, a lower estimate on the spectral condition number of Vandermonde matrix $V = V(x_1, \dots, x_n)$ with arbitrary but distinct nodes $x_1, \dots, x_n$ (Theorem 4.1): $\mathrm{cond}_2\, V \geq 2^{n-2}/n^{\frac{1}{2}}$. We also show that if either $|x_j| \leq 1$ or $|x_j| \geq 1$ for all j, then $\mathrm{cond}_2\, V \geq 2^{n-2}$. These results complement and amplify those [1], where some particular configurations of nodes were examined.

In Sect. 5, we prove that for any real positive definite Hankel matrix H of order n, $\mathrm{cond}_2\, H \geq 3 \cdot 2^{n-6}$. This is our main result on Hankel matrices, but in our exposition here it appears a clear consequence of the preceding Krylov basis considerations. Our result can be juxtaposed with that of [8], where it was shown that for a family of Hankel matrices H_n which are the Gram matrices for the powers $\{1, x, x^2, \dots\}$, $\mathrm{cond}_2\, H_n$ asymptotically grows as 4^n. The estimate we propose is better in that it holds for arbitrary matrix of order n.

At last, we make some remarks on the indefinite Hankel matrices, finishing with some conjectures.

2. Hankel and Vandermonde matrices

We recall that matrix $H = [h_{ij}]$ of order n is called Hankel if $h_{ij} = h_{i+j-2}, 1 \leq i, j \leq n$, and matrix $V = V(x_1, \dots, x_n)$ is called Vandermonde if

$$V^{\mathrm{T}} = \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix}.$$

It is well known that Hankel and Vandermonde are connected in some way (for instance, see [3]): for arbitrary Vandermonde matrix V and arbitrary diagonal matrix D, the matrix

$$H = VDV^{\mathrm{T}} \tag{2.1}$$

is a Hankel one.

A certain reverse statement is also available. In [5] it is shown that decomposition (2.1) exists whenever H is nonsingular Hankel (see also [3, 10]). However, we want to deal here with a statement where all involved matrices are guaranteed to be real. The one that follows can be found in [4]; here we present it with a simpler proof.

Lemma 2.1. *For any positive definite Hankel matrix $H \in \mathbb{R}^{n \times n}$ there exist a Vandermonde matrix $V \in \mathbb{R}^{n \times n}$ and a diagonal $\Lambda \in \mathbb{R}^{n \times n}$ such that*

$$H = V\Lambda^2 V^{\mathrm{T}}. \tag{2.2}$$

Proof. Since H is positive definite, we may consider the Cholesky decomposition of it:

$$H = R^{\mathrm{T}} R, \tag{2.3}$$

where $R \in \mathbb{R}^{n \times n}$ is upper triangular. Following [4], we are now going to find an upper Hessenberg matrix T such that

$$r_{11}^{-1} R = [e_1, Te_1, \dots, T^{n-1} e_1], \tag{2.4}$$

where $e_1 = [1, 0, \dots, 0]^{\mathrm{T}}$. It is evident that all T's columns, save for the last one, are uniquely determined from (2.4). The last column can be arbitrary, and we take it such that

(2.5) $$Te_n = T^{\mathrm{T}}e_n,$$

where $e_n = [0, \ldots, 0, 1]^{\mathrm{T}}$. As usually, for $x, y \in \mathbb{R}^n$ we will write $(x, y) \equiv x^{\mathrm{T}}y$. Then, since H is Hankel,

$$(T^{i+1}e_1, T^j e_1) = (T^i e_1, T^{j+1}e_1), \quad 0 \le i, j \le n-2,$$

and hence

(2.6) $$(TT^i e_1, T^j e_1) = (T^{\mathrm{T}}T^i e_1, T^j e_1), \quad 0 \le i, j \le n-2.$$

Besides, on the strength of (2.5),

(2.7) $$(TT^i e_1, e_n) = (T^i e_1, T^{\mathrm{T}}e_n) = (T^i e_1, Te_n) = (T^{\mathrm{T}}T^i e_1, e_n).$$

Thus, for any vectors

$$x, y \in \{e_1, Te_1, \ldots, T^{n-2}e_1, e_n\}$$

we have

$$(Tx, y) = (T^{\mathrm{T}}x, y).$$

It is easy to check that

$$\operatorname{span}\{e_1, Te_1, \ldots, T^{n-2}e_1, e_n\} = \operatorname{span}\{e_1, Te_1, \ldots, T^{n-2}e_1, T^{n-1}e_1\},$$

and this means that $e_1, Te_1, \ldots, T^{n-2}e_1, e_n$ form a basis for $\mathbb{R}^n$. Therefore, $T = T^{\mathrm{T}}$ (and since T is Hessenberg, T is tridiagonal).

Further, we make use of the orthogonal diagonalization of T:

(2.8) $$T = Q^{\mathrm{T}}XQ,$$

where

$$Q^{\mathrm{T}}Q = I, X = \operatorname{diag}(x_1, \ldots, x_n); \quad Q, X \in \mathbb{R}^{n\times n}.$$

Then

$$\begin{aligned} r_{11}^{-1}R &= [Q^{\mathrm{T}}Qe_1, Q^{\mathrm{T}}XQe_1, \ldots, Q^{\mathrm{T}}X^{n-1}Qe_1] \\ &= Q^{\mathrm{T}}[q, Xq, \ldots, X^{n-1}q] = Q^{\mathrm{T}}\Omega V^{\mathrm{T}}, \end{aligned}$$

where

$$q \equiv [q_1, \ldots, q_n]^{\mathrm{T}} = Qe_1, \quad \Omega = \operatorname{diag}(q_1, \ldots, q_n), V = V(x_1, \ldots, x_n).$$

Allowing for (2.3) and (2.4), we obtain

$$H = r_{11}^2(V\Omega Q)(Q^{\mathrm{T}}\Omega V^{\mathrm{T}}) = V\Lambda^2 V^{\mathrm{T}}$$

where $\Lambda \equiv r_{11}\Omega$. $\square$

3. Krylov basis properties

As Lemma 2.1 shows, to study Hankel we can be better off if we first examine the scaled Vandermonde

(3.1) $$A = \Lambda V^{\mathrm{T}},$$

where

(3.2) $$\Lambda = \mathrm{diag}\,(q_1, \ldots, q_n), \quad V = V(x_1, \ldots, x_n).$$

In doing this we take advantage of viewing A as the Krylov basis matrix:

(3.3) $$A = [q, Xq, \ldots, X^{n-1}q],$$

where

(3.4) $$q = [q_1, \ldots, q_n]^{\mathrm{T}}, \quad X = \mathrm{diag}\,(x_1, \ldots, x_n),$$

and $x_1, \ldots, x_n$ are distinct.

Lemma 3.1. *Suppose that A is defined by (3.3), (3.4), where $q \in \mathbb{C}^n, x_1, \ldots, x_n \in [a, b]$, and denote by $\sigma_n \leq \sigma_{n-1} \leq \ldots \leq \sigma_1$ the singular values of A. Let*

(3.5) $$c = \frac{b-a}{2} < 2.$$

Then

(3.6) $$\left(\sum_{l=i}^{n} \sigma_l^2 \right)^{\frac{1}{2}} \leq \|q\|_2 \frac{c^{i-1}}{2^{i-2}} \left(\frac{1 - (\frac{c^2}{4})^{n+1-i}}{1 - \frac{c^2}{4}} \right)^{\frac{1}{2}}, i = 1, \ldots, n.$$

Proof. Consider the Krylov spaces

(3.7) $$\mathscr{K}_i = \mathrm{span}\,\{q, Xq, \ldots, X^{i-1}q\}, \quad i \geq 1; \quad \mathscr{K}_0 = \{0\};$$

and try to estimate the length of the perpendicular $\mathrm{ort}_{\mathscr{K}_i} X^i q$, dropped on $\mathscr{K}_i$ from $X^i q$. Obviously, for some $\alpha_1, \ldots, \alpha_i \in \mathbb{C}$

$$\mathrm{ort}_{\mathscr{K}_i} X^i q = X^i q + (\alpha_1 q + \ldots + \alpha_i X^{i-1} q),$$

and thence

$$\mathrm{ort}_{\mathscr{K}_i} X^i q = p_i(X) q$$

where $p_i(t) = t^i + \ldots \in \mathscr{P}_i$, and $\mathscr{P}_i$ means the set of all monic polynomials of degree i. Thus,

$$\|\mathrm{ort}_{\mathscr{K}_i} X^i q\|_2 = \min_{p_i(t) \in \mathscr{P}_i} \|p_i(X) q\|_2 =$$

$$\min_{p_i(t) \in \mathscr{P}_i} \left(\sum_{j=1}^{n} p_i^2(x_j) |q_j|^2 \right)^{1/2} \leq \min_{p_i(t) \in \mathscr{P}_i} \max_{a \leq x \leq b} |p_i(x)| \|q\|_2.$$

If $T_i(t)$ is the i-th Chebyshev polynomial for $[-1, 1]$, then

$$\begin{aligned} T_i \left(\frac{x - \frac{a+b}{2}}{\frac{b-a}{2}} \right) &= 2^{i-1} \left(\frac{2}{b-a} x - \frac{b+a}{b-a} \right)^i + \ldots \\ &= 2^{i-1} \left(\frac{2}{b-a} \right)^i x^i + \ldots \ . \end{aligned}$$

Now, the choice

$$p_i(x) = \frac{1}{2^{i-1}} \left(\frac{b-a}{2}\right)^i T_i\left(\frac{x - \frac{a+b}{2}}{\frac{b-a}{2}}\right)$$

immediately leads to

$$\|\mathrm{ort}_{\mathscr{K}_i} X^i q\|_2 \le \|q\|_2 \frac{c^i}{2^{i-1}}, \quad i = 1, \dots, n-1. \tag{3.8}$$

Note that bound (3.8) can be found in [8] (see also [6]).

To proceed further, we need to recall that the left–hand side of (3.6) is the Frobenius-norm distance between A and the subspace of $n \times n$ matrices whose rank is less then i. Let $Z_i = [z_1, \dots, z_n]$ where

$$z_j = \begin{cases} 0, & 1 \le j \le i-1, \\ \mathrm{ort}_{\mathscr{K}_{i-1}} X^{j-1} q, & i \le j \le n. \end{cases}$$

Then it is readily seen that

$$\mathrm{rank}\,(A - Z_i) = i - 1.$$

Therefore, the left–hand side of (3.6) is bounded from above by

$$\|Z_i\|_{\mathrm{F}} \le \left(\sum_{j=i}^{n} (\|q\|_2 \frac{c^{j-1}}{2^{j-2}})^2\right)^{\frac{1}{2}}. \tag{3.9}$$

It remains to verify that the right–hand side of (3.9) coincides with that of (3.6), and this completes the proof. □

Lemma 3.2. *Suppose that A is defined by (3.3), (3.4), where $q_1, \dots, q_n \in \mathbb{C}$ are nonzero, $x_1, \dots, x_n \in \mathbb{R}$ are distinct, and either*

$$|x_j| \le 1, \quad j = 1, \dots, n, \tag{3.10}$$

or

$$|x_j| \ge 1, \quad j = 1, \dots, n. \tag{3.11}$$

Then in both cases

$$\mathrm{cond}_2\, A \ge 2^{n-2}. \tag{3.12}$$

Proof. If (3.10) holds then $x_j \in [a, b] \equiv [-1, 1]$, and Lemma 3.1 yields

$$\|A^{-1}\|_2 = \sigma_n^{-1} \ge \frac{2^{n-2}}{\|q\|_2}.$$

On the other hand,

$$\|A\|_2 \ge \|q\|_2,$$

and so we arrive at (3.12).

If (3.11) holds, we set $\overline{q} = X^{n-1} q$, $\overline{X} = X^{-1}$, and consider the matrix $\overline{A}$ obtained from A by permuting columns in the reverse order. Clearly, $\mathrm{cond}_2\, A = \mathrm{cond}_2\, \overline{A}$, and at the same time

$$\overline{A} = [\overline{q}, \overline{X}\,\overline{q}, \dots, \overline{X}^{n-1} \overline{q}],$$

all $\overline{X}$'s elements belonging to $[-1, 1]$, and thus $\mathrm{cond}_2\, \overline{A} \ge 2^{n-2}$, as has been just established. □

Lemma 3.2 can be easily extended to a more general case.

Lemma 3.3. *Suppose A is defined by (3.3), (3.4), where $q_1, \dots, q_n \in \mathbb{C}$ are nonzero and nodes $x_1, \dots, x_n \in \mathbb{R}$ are distinct, and such that*

$$0 \le a \le |x_j| \le b, \quad j = 1, \dots, n. \tag{3.13}$$

Then

$$\operatorname{cond}_2 A \ge 2^{n-2} \max\{a^{n-1}, \frac{1}{b^{n-1}}\}. \tag{3.14}$$

Proof. Since $x_j \in [-b, b]$, by Lemma 3.1

$$\|A^{-1}\|_2 \ge \frac{2^{n-2}}{b^{n-1}\|q\|_2}.$$

Together with $\|A\|_2 \ge \|q\|_2$, this yields $\operatorname{cond}_2 A \ge 2^{n-2}/b^{n-1}$. Further, if $a \neq 0$, then for matrix $\bar{A}$, introduced in the previous proof, we have $\operatorname{cond}_2 \bar{A} \ge 2^{n-2}/(a^{-1})^{n-1}$. □

At last, we will present one more generalization of Lemma 3.2.

Lemma 3.4. *For any A of the form (3.3), (3.4), where $q_1, \dots, q_n \in \mathbb{C}$ are nonzero and nodes $x_1, \dots, x_n \in \mathbb{R}$ are pairwise distinct,*

$$\operatorname{cond}_2 A \ge 3^{\frac{1}{2}} 2^{\frac{n}{2}-3}. \tag{3.15}$$

Proof. Since any permutation of rows preserves singular values, we may assume that

$$|x_j| \le 1, \quad j = 1, \dots, m; \tag{3.16}$$

$$|x_j| \ge 1, \quad j = m+1, \dots, n. \tag{3.17}$$

Consider the partition

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, A_1 \in \mathbb{R}^{m \times n}, A_2 \in \mathbb{R}^{(n-m) \times n}, \tag{3.18}$$

and set

$$\bar{A} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times n}. \tag{3.19}$$

Then it is easy to check that

$$\lambda_n(AA^{\mathrm{T}}) \le \lambda_m(\bar{A}\bar{A}^{\mathrm{T}}); \tag{3.20}$$

here and below the eigenvalues $\lambda_1(\cdot) \ge \dots \ge \lambda_n(\cdot)$ for a symmetric matrix are always taken in non-increasing order. Now, let

$$\tilde{A} = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \in \mathbb{R}^{n \times n} \tag{3.21}$$

be any augmentation of A_1, such that

$$\tilde{A} = [q, Yq, \dots, Y^{n-1}q]$$

where $Y = \operatorname{diag}(y_1, \ldots, y_n), y_j = x_j$ for $1 \le j \le m$, and $y_1, \ldots, y_n$ are pairwise distinct and not greater in modulus than 1. We see that

$$\bar{A}^{\mathrm{T}} \bar{A} = \tilde{A}^{\mathrm{T}} \tilde{A} - B_1^{\mathrm{T}} B_1, \tag{3.22}$$

and relying on the well–known interlacing properties for a symmetric matrix when perturbed by a symmetric, nonpositive definite, rank–one matrix (see, for example, [2]; p. 412), we conclude that, after $n-m$ such perturbations,

$$\lambda_m(\bar{A}\bar{A}^{\mathrm{T}}) = \lambda_m(\bar{A}^{\mathrm{T}}\bar{A}) \le \lambda_m(\tilde{A}\tilde{A}^{\mathrm{T}}). \tag{3.23}$$

Applying Lemma 3.1 to $\tilde{A}$, we have $c = 1$, and hence

$$\begin{aligned} (\lambda_m(\tilde{A}\tilde{A}^{\mathrm{T}}))^{\frac{1}{2}} &\le \|q\|_2 \frac{1}{2^{m-2}} \left(\frac{1 - (\frac{1}{4})^{n+1-m}}{1 - \frac{1}{4}} \right)^{\frac{1}{2}} \\ &\le \|q\|_2 \frac{1}{2^{m-2}} \frac{2}{3^{\frac{1}{2}}}. \end{aligned}$$

Combining this with (3.20) and (3.23), we find

$$\sigma_n^{-1}(A) \ge 3^{\frac{1}{2}} 2^{m-3} \frac{1}{\|q\|_2},$$

and since $\sigma_1(A) \ge \|q\|_2$, we thus obtain

$$\operatorname{cond}_2 A \ge 3^{\frac{1}{2}} 2^{m-3}. \tag{3.24}$$

A similar construction, based on an augmentation of A_2, will result in one more inequality:

$$\operatorname{cond}_2 A \ge 3^{\frac{1}{2}} 2^{n-m-3}. \tag{3.25}$$

It remains to note that either $m \ge n/2$, or $n - m \ge n/2$, and that completes the proof. □

Remark. All the above lemmas remain valid if we assume that X in (3.3) is a symmetric matrix with distinct eigenvalues.

4. A lower bound for Vandermonde matrices

As a byproduct of our research, we obtain a lower estimate for the spectral condition number of Vandermonde matrices with arbitrary nodes.

Theorem 4.1. *For any real nonsingular Vandermonde matrix $V = V(x_1, \ldots, x_n)$ of order n,*

$$\operatorname{cond}_2 V \ge \frac{2^{n-2}}{n^{\frac{1}{2}}}. \tag{4.1}$$

If either $|x_j| \le 1$ or $|x_j| \ge 1$ for all j, then

$$\operatorname{cond}_2 V \ge 2^{n-2}. \tag{4.2}$$

Proof. Note that V^{T} is of the form (3.3) with $q = [1, \dots, 1]^{\mathrm{T}}$. If $c = \max_{1 \le j \le n} |x_j|$, then $x_1, \dots, x_n \in [-c, c]$, and by Lemma 3.1

$$\|V^{-\mathrm{T}}\|_2 \ge \frac{2^{n-2}}{n^{\frac{1}{2}} c^{n-1}}.$$

At the same time, $\|V^{\mathrm{T}}\|_2 \ge |(V^{\mathrm{T}} e_n, e_k)| = c^{n-1}$, where e_n and e_k are n-th and k-th columns of the identity matrix, and k is such that $|x_k| = c$. So we come to (4.1), for $\mathrm{cond}_2\, V = \mathrm{cond}_2\, V^{\mathrm{T}}$. Inequality (4.2) is a straightforward consequence of Lemma 3.2. □

This theorem complements and sharpens some estimates from [1]. In the cited work, two particular configurations of nodes $x_1, \dots, x_n$ were considered:

(1) $x_1 > \dots > x_n > 0$ (positive nodes)

(2) $x_j + x_{n+1-j} = 0, j = 1, \dots, n; x_1 > \dots > x_{[n/2]} > 0$ (symmetrical nodes).

For positive nodes it was proved that

$$\mathrm{cond}_\infty\, V > 2^{n-1}, \tag{4.3}$$

which is better than (4.1), while for symmetrical ones it was proved that

$$\mathrm{cond}_\infty\, V \ge 2^{\frac{n}{2}}. \tag{4.4}$$

Note that

$$\mathrm{cond}_\infty\, V \ge \mathrm{cond}_2\, V.$$

Therefore, our estimate (4.1) is tighter than (4.4) for $n \ge 7$, for symmetrical nodes.

5. Condition of Hankel matrices

Theorem 5.1. *For any real positive definite Hankel matrix H of order n,*

$$\mathrm{cond}_2\, H \ge 3 \cdot 2^{n-6}. \tag{5.1}$$

Proof. Thanks to Lemma 2.1 we have decomposition (2.2), which entails

$$\mathrm{cond}_2\, H = \mathrm{cond}_2^2\, \Lambda V^{\mathrm{T}}. \tag{5.2}$$

To estimate $\mathrm{cond}_2^2\, \Lambda V^{\mathrm{T}}$ we resort to Lemma 3.4. □

This theorem states that positive definite Hankel matrices are ill–conditioned even for not very high orders. It is plausible that they are indeed worse than suggested by (5.1). In particular, this is plausible from the results in [8]. In [8] a family of Hankel matrices

$$H_n = [h_{i+j-2}], \quad 1 \le i, j \le n, \tag{5.3}$$

was examined, which are generated by $\xi(x)$, a positive, non-decreasing, non-constant function of bounded variation on the interval $[a, b]$. Specifically,

$$h_{i+j-2} = \int_a^b x^{i+j-2}\, d\xi(x). \tag{5.4}$$

It was proved in [8] that

$$\lim_{n\to\infty} \inf(\mathrm{cond}_2 H_n)^{\frac{1}{n}} \geq 4. \tag{5.5}$$

So we may anticipate that $\mathrm{cond}_2\, H_n$ will grow as 4^n. However, (5.1) holds for arbitrary matrix of order n, whereas (5.5) describes the asymptotic behavior.

If we consider a family of indefinite Hankel matrices then it can be guaranteed that at least $\mathrm{cond}_2\, H_n \to \infty$ (private communication by G. Heinig). We suspect, though, that there should be some estimates on that growth.

Note that an indefinite Hankel matrix of order n can be perfectly conditioned. For instance, such is

$$J = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}.$$

Nevertheless, we believe that at any rate a leading submatrix should occur which is ill–conditioned, provided that all leading submatrices are nonsingular. This still needs to be proved.

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