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CONVERGENCE OF PSEUDOCONTRACTIONS AND APPLICATIONS TO TWO-STAGE AND ASYNCHRONOUS MULTISPLITTING FOR SINGULAR M-MATRICES*

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Abstract. Pseudocontractions, which are generalizations of paracontractions in the linear case, are introduced in this paper in order to study the convergence of nonstationary iterative methods for linear systems in which the coefficient matrices are singular M-matrices. A general convergence theorem for pseudocontractions is developed. This theorem is used to analyze the convergence of two nonstationary parallel iterations: two-stage multisplitting iterations and asynchronous multisplitting iterations for singular M-matrices, without other contractivity conditions on the iteration matrices.

Key words. pseudocontractions, paracontractions, two-stage iteration, asynchronous iteration, multisplitting, nonnegative matrices, singular M-matrices

AMS subject classifications. 15A48, 65F10, 65Y05

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1. Introduction.

Singular M-systems, which are linear systems of equations $Ax = b$, with A a singular M-matrix, may appear in many applications such as elliptic equations with periodic boundary conditions, finite Markov chains, etc. [6]. Stationary iterative methods to solve singular M-systems have been well studied; see, e.g., [6, 36]. For nonstationary iterative methods, in which the iteration operators change during the iterative procedure, for example, two-stage iterative methods and asynchronous iterative methods (see sections 3 and 4 below, respectively, for details), some assumptions on these operators have been made in the literature to guarantee the convergence. Bru, Elsner, and Neumann [11] assumed that the iterative operators are *paracontractive*, Migallón, Penadés, and Szyld [30] assumed that the operators are uniformly contractive.

In this paper, a new property of operators, called *pseudocontractivity*, is proposed. It is shown that pseudocontractivity is a generalization of the paracontractivity property in the linear case. A general convergence theorem for pseudocontractive iterations is proved in section 2. In section 3 it is proved that the product of iteration matrices associated with a class of stationary iterative methods is pseudocontractive and therefore that this class of methods is convergent. In sections 4 and 5, this theory is used to analyze two parallel nonstationary iterative methods for singular M-systems, specifically, nonstationary two-stage multisplitting methods and asynchronous multisplitting methods, and, under reasonable conditions, convergence results are obtained. No other contractivity condition on the iteration operators is required.

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2. Pseudocontractive operators and a general convergence theorem. Let X^* be a nonempty closed convex subset of $\mathbb{R}^n$, and let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. For any vector $x \in \mathbb{R}^n$, $y^* \in X^*$ is a *projection* vector of x onto X^* if

$$\|x - y^*\| = \min_{y \in X^*} \|x - y\|.$$

Remark. Since X^* is closed, the minimum is always attained. The projection vector may be not unique. For example, let $\|\cdot\| = \|\cdot\|_\infty$, $X^* = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$, $x = (2, 0)^T$, then all vectors $(1, a)^T$ with $a \in [-1, 1]$ are projection vectors of x onto X^*. We use $P(x)$ to denote an arbitrary but fixed projection vector of x and $\operatorname{dist}(x, X^*)$ to denote $\|x - P(x)\|$. Note that $\operatorname{dist}(x, X^*)$ is independent of the choice of $P(x)$.

Let T be an operator on $\mathbb{R}^n$. It is *nonexpansive* (with respect to $\|\cdot\|$ and X^*) if

$$\|Tx - x^*\| \leq \|x - x^*\| \quad \text{for all } x \in \mathbb{R}^n, x^* \in X^* \tag{2.1}$$

and *pseudocontractive* (with respect to $\|\cdot\|$ and X^*) if, in addition,

$$\operatorname{dist}(Tx, X^*) < \operatorname{dist}(x, X^*) \quad \text{for all } x \notin X^*. \tag{2.2}$$

We use $\mathcal{T}$ to denote the set of all pseudocontractive operators.

PROPOSITION 2.1. *If T is pseudocontractive with respect to $\|\cdot\|$ and X^*, then X^* is the set of all fixed points of T.*

Proof. For any fixed $x^* \in X^*$, in (2.1), let $x = x^*$, and thus we have $Tx^* = x^*$, which means that all points in X^* are fixed points of T. For any $x \notin X^*$, from (2.2), $Tx - P(Tx) \neq x - P(x)$, therefore, $Tx \neq x$, i.e., x is not a fixed point of T. □

Example 1. Let $T \in \mathbb{R}^{n \times n}$, $X^* = \{\alpha e \mid \alpha \in \mathbb{R}\}$, and let $e \in \mathbb{R}^n$ denote the vector with all components equal to 1. Then T is pseudocontractive with respect to X^* and the infinity norm $\|\cdot\|_\infty$ if and only if $Te = e$ and for any $x \in \mathbb{R}^n$ such that $\min_i x_i < \max_i x_i$, $\max_i (Tx)_i - \min_i (Tx)_i < \max_i x_i - \min_i x_i$.

Remark (paracontractivity versus pseudocontractivity). Paracontractive operators have been used mainly in the study of systems with multiple solutions; see, for example, [32, 14, 15, 11, 35, 8]. An operator is *paracontractive* if

$$\|Tx\| \leq \|x\| \quad \text{for all } x \in \mathbb{R}^n$$

and equality holds if and only if $Tx = x$. In the case that T is linear, if T is paracontractive, then

$$\|Tx - x^*\| = \|T(x - x^*)\| < \|x - x^*\| \quad \text{for all } x \notin X^*, x^* \in X^*,$$

where X^* is the subspace consisting of all fixed points of T. Thus, T is pseudocontractive. So in the linear case, pseudocontractive operators are generalizations of paracontractive ones. But the converse is not true. Consider the following inequalities:

$$\|Tx - P(Tx)\| \leq \|Tx - P(x)\| \leq \|x - P(x)\| \text{ for } x \notin X^*.$$

Paracontractivity requires the second inequality to be strict, while pseudocontractivity requires any one of these two inequalities to be strict.

Example 2. For the operator

$$T = \begin{pmatrix} .5 & .5 & 0 \\ .25 & .5 & .25 \\ 0 & .5 & .5 \end{pmatrix},$$

the norm, the vector e, and the set X^* are the same as in Example 1. For any x, $P(x) = 0.5(\max_i x_i + \min_i x_i)e$. For $x = (2, 2, 1)^T$, $Tx = (2, 1.75, 1.5)^T$, $P(x) = 1.5e$, and $P(Tx) = 1.75e$. Thus the first inequality in the equation above is strict while the second one is an equality. So this operator is pseudocontractive, not paracontractive.

Another simple property of pseudocontractions is given below.

PROPOSITION 2.2. *Let T_i be a set of nonexpansive or pseudocontractive operators (with respect to the same norm and the same set X^*). A product of any number of operators from this set that contains at least one pseudocontractive operator is pseudocontractive.*

Proof. Let T_1 be pseudocontractive and T_2 be nonexpansive. First consider the case that $T = T_1T_2$. For $x \notin X^*$, if $T_2x \in X^*$, then $\|Tx - P(Tx)\| = 0 < \|x - P(x)\|$. Suppose $T_2x \notin X^*$; from the definition of the operator P, it follows that

$$\begin{aligned} \|Tx - P(Tx)\| &= \|T_1T_2x - P(T_1T_2x)\| \\ &< \|T_2x - P(T_2x)\| \\ &\le \|T_2x - P(x)\| \\ &\le \|x - P(x)\|. \end{aligned}$$

Thus, T is pseudocontractive. Now consider the case that $T = T_2T_1$.

$$\|Tx - P(Tx)\| \le \|Tx - P(T_1x)\| \le \|T_1x - P(T_1x)\|.$$

Since T_1 is pseudocontractive, T is also pseudocontractive. □

Note that this proposition implies that if a product of operators is pseudocontractive, then only one of its factors need be pseudocontractive; the others may be nonexpansive.

In the following theorem, the operators are not necessarily linear, although in our later applications the operators are linear.

THEOREM 2.3. *Let $\{T_k\}$ be a sequence of nonexpansive operators (with respect to $\|\cdot\|$ and X^*), and let there exist a subsequence $\{T_{k_i}\}$ which converges to $T \in \mathcal{T}$. If T is pseudocontractive and uniformly Lipschitz continuous, then for any initial vector $x(0)$, the sequence of vectors*

$$x(k+1) = T_kx(k), \quad k = 0, 1, 2, \ldots,$$

converges to some $x^* \in X^*$.

Proof. Consider the subsequence of vectors $\{x(k_i)\}_{i=0}^{\infty}$ of the sequence $\{x(k)\}_{k=0}^{\infty}$. As T_k is nonexpansive, this subsequence is bounded, and it contains a convergent subsequence which, without loss of generality, can be taken to be $\{x(k_i)\}_{i=0}^{\infty}$ itself. Assume therefore that

$$\lim_{i\to\infty} x(k_i) = \xi.$$

If $\xi \in X^*$, as all T_i are nonexpansive,

$$\|x(k+1) - \xi\| \le \|x(k) - \xi\|, \quad k = 0, 1, 2 \ldots,$$

and we are done.

Suppose $\xi \notin X^*$, and therefore $\|\xi - P(\xi)\| > 0$. As T is pseudocontractive,

$$\beta := \frac{\|T\xi - P(T\xi)\|}{\|\xi - P(\xi)\|} < 1.$$

For arbitrary fixed $\varepsilon > 0$ small enough, there exists an integer k_ε such that

$$\|x(k_i) - \xi\| \le \varepsilon, \quad \|T_{k_i} - T\| \le \varepsilon \quad \text{for all } i \text{ such that } k_i \ge k_\varepsilon.$$

Consider i: $k_i \ge k_\epsilon$.

$$\begin{aligned}
&\|x(k_i+1) - P(x(k_i+1))\| \\
&\le \|x(k_i+1) - P(T\xi)\| \\
&= \|T_{k_i}x(k_i) - P(T\xi)\| \\
&\le \|T_{k_i}x(k_i) - Tx(k_i)\| + \|Tx(k_i) - T\xi\| + \|T\xi - P(T\xi)\| \qquad (2.3)\\
&\le \varepsilon\|x(k_i)\| + \varepsilon\|T\| + \beta\|\xi - P(\xi)\| \qquad (2.4)\\
&\le \beta\|\xi - P(\xi)\| + C\varepsilon,
\end{aligned}$$

where, from (2.3) to (2.4), we use the convergence of T_{k_i} for the first term, the uniform Lipschitz continuity of T for the second term ($\|T\|$ is used to denote the Lipschitz constant), the pseudocontractive property of T for the third term, and C is a positive constant scalar. As all T_k are nonexpansive, for $k \ge 0$,

$$\begin{aligned}
&\|x(k+1) - P(x(k+1))\| \le \|x(k+1) - P(x(k))\| \\
&= \|Tx(k) - P(x(k))\| \le \|x(k) - P(x(k))\|,
\end{aligned}$$

therefore,

$$\|x(k_{i+1}) - P(x(k_{i+1}))\| \le \|x(k_i+1) - P(x(k_i+1))\| \le \beta\|\xi - P(\xi)\| + C\varepsilon.$$

On the other hand,

$$\begin{aligned}
\|\xi - P(\xi)\| &\le \|\xi - P(x(k_{i+1}))\| \\
&\le \|\xi - x(k_{i+1})\| + \|x(k_{i+1}) - P(x(k_{i+1}))\| \\
&\le \varepsilon + \beta\|\xi - P(\xi)\| + C\varepsilon,
\end{aligned}$$

i.e., for any ε small enough,

$$\|\xi - P(\xi)\| \le \frac{C+1}{1-\beta}\,\varepsilon.$$

This contradicts $\|\xi - P(\xi)\| > 0$. ∎

This theorem is applied to prove the convergence of two-stage multisplitting iteration algorithms and asynchronous multisplitting iteration algorithms for singular M-matrices in the subsequent sections. The following corollary relates pseudocontractivity to the existence of limits of powers of a matrix.

COROLLARY 2.4. *If T is a pseudocontractive matrix, then $\lim_{n\to\infty} T^n$ exists.*

Proof. In the matrix case, $\lim_{n\to\infty} T^n$ exists if and only if for all x, $\lim_{n\to\infty} T^n x$ exists. The conclusion is now drawn from the above theorem. ∎

Theoretically, the convergence of the powers of a matrix is equivalent to the existence of a vector norm such that the matrix is paracontractive with respect to this norm. However, in practice, it is usually necessary to prove that a matrix has some contractive property with respect to a given norm, for example, with respect to the 2-norm in the case of symmetric matrices [32], or the weighted infinity norm in the case of nonnegative matrices. By the remark following Example 1, if a matrix is paracontractive with respect to a given norm, then it is always pseudocontractive with respect to this norm. On the other hand, if a matrix is not paracontractive with respect to some norm, it may still be pseudocontractive with respect to this norm. In this paper, it is proved that the product of a sufficiently large number (at most $n-1$, where n is the order of the matrix) of nonnegative matrices, which are induced from (possibly different) weak regular splittings of a singular M-matrix, is pseudocontractive. This is used to prove the convergence of some parallel iterative methods for singular M-matrices.

3. Pseudocontractive operators and weak regular splittings of an irreducible singular M-matrix. Let B be a nonnegative matrix (denoted $B \geq 0$), i.e., each element of B is nonnegative. From nonnegative matrix theory, see, e.g., [6, 42], $\rho(B)$, the spectral radius of B, is an eigenvalue of B, and there exists a nonnegative eigenvector, which is termed the Perron vector of B, associated to it. If B is irreducible, there is only one eigenvalue equal to $\rho(B)$ and the Perron vector is positive (componentwise, denoted as $\gg 0$) and unique (up to a scalar factor). A matrix $A \in \mathbb{R}^{n\times n}$ is a singular M-matrix if there exists a nonnegative matrix B such that $A = \rho(B)I - B$. Therefore, if A is an irreducible singular M-matrix, there exists a unique (up to a scalar factor) v which is positive such that $Av = 0$. In what follows, the following assumption will always be made.

Assumption. The vector v, referred to above, is equal to e, the vector with all components equal to 1.

Remark. This assumption makes our notation simpler and our demonstrations more intuitive: for example, $Av = 0$ means that all sums of elements in the same row of A are equal to zero. Furthermore, it entails no loss of generality. For, if $v \neq e$, then $A' = D^{-1}AD$ with $D = \mathrm{diag}(v)$ is a singular M-matrix as well, satisfying $A'e = 0$. If the same similarity transformation is applied to all the other matrices involved, then all the important properties assumed in this paper are preserved. For example, if $A = M - N$ is a weak regular or regular splitting (as defined below), then $A' = M' - N'$ is weak regular or regular as well. Consequently, the iterates $x(k)$ of the nonstationary two-stage multisplitting iteration described in section 4 (with splittings $A = M - N$, $M = F_l - N_l$) are related to those of the same algorithm with splittings $A' = M' - N'$, $M' = F'_l - G'_l$ through $x'(k) = D^{-1}x(k)$, provided that $x'(0) := D^{-1}x(0)$. Thus, the convergence proved in Theorem 4.2 for the special case $v = e$ actually also holds in the general case. Exactly the same situation arises for Theorem 5.1.

A splitting of the matrix $A : A = M - N$ is *weak regular* if M is nonsingular, $M^{-1} \geq 0$, and $M^{-1}N \geq 0$, and is *regular* if $M^{-1} \geq 0$ and $N \geq 0$.

PROPOSITION 3.1. *Let A be an irreducible singular M-matrix, the splitting $A = M - N$ be weak regular, and $T = M^{-1}N$. Then either T is irreducible, or there exists a permutation matrix P_e such that*

$$P_e{}^T T P_e = \begin{pmatrix} T_{11} & \\ T_{21} & T_{22} \end{pmatrix},$$

where T_{11} is irreducible, $\rho(T_{11}) = 1$, and $\rho(T_{22}) < 1$.

Proof. Suppose that T is reducible. For $v \gg 0$, $Av = 0$, we have $Tv = v$, therefore, from [29, p. 728], there exists a permutation matrix P_e such that

$$P_e{}^T T P_e = \begin{pmatrix} Q_{11} & & & \\ & \ddots & & \\ & & Q_{ii} & \\ Q_{i+1,1} & \cdots & \cdots & Q_{i+1,i+1} \end{pmatrix},$$

where $Q_{11}, \ldots, Q_{ii}$ are irreducible, $i \geq 1$, $\rho(Q_{11}) = \cdots = \rho(Q_{ii}) = 1$, $\rho(Q_{i+1,i+1}) < 1$, and $Q_{i+1,i+1}$ may be missing. If $i \geq 2$, let $u = P_e{}^T v = (u_1^T, \ldots, u_i^T, u_{i+1}^T)^T$, then any vector

$$\tilde{u} = \left(\alpha_1 u_1^T, \ldots, \alpha_i u_i^T, \left[(I - Q_{i+1,i+1})^{-1} \sum_{j=1}^{i} Q_{i+1,j} \alpha_j u_j \right]^T \right)^T, \quad \alpha_j > 0,$$

is also the Perron vector of $P_e{}^T T P_e$, which contradicts the fact that T has a unique (up to a scalar factor) Perron vector. Therefore $i = 1$. $\square$

We use $S(T)$ to denote the set of all row indices of T such that after the permutation in the above proposition, these rows become the rows of T_{11}. For any $x \in \mathbb{R}^n$, denote

$$\overline{x} \equiv \max_i x_i, \quad \underline{x} \equiv \min_i x_i.$$

PROPOSITION 3.2. *Let x be a vector such that $\underline{x} < \overline{x}$, A be an irreducible singular M-matrix such that $Ae = 0$ with $e = (1, 1, \ldots, 1)^T$, the splitting $A = M - N$ be weak regular, $T \equiv (t_{ij}) = M^{-1}N$ with $t_{ii} > 0$ for $1 \leq i \leq n$, $y = Tx$. Then we have the following:*

(i) *For all components of y,*

$$\underline{x} \leq y_i \leq \overline{x}, \quad 1 \leq i \leq n,$$

$$\{i \,|\, y_i = \underline{x}\} \subset \{i \,|\, x_i = \underline{x}\}, \tag{3.1}$$

$$\{i \,|\, y_i = \overline{x}\} \subset \{i \,|\, x_i = \overline{x}\}. \tag{3.2}$$

(ii) *Equation*

$$\{i \,|\, y_i = \underline{x}\} = \{i \,|\, x_i = \underline{x}\} \tag{3.3}$$

holds if and only if

$$\{i \,|\, y_i = \underline{x}\} = \{i \,|\, x_i = \underline{x}\} = S(T);$$

similarly, equation

$$\{i \,|\, y_i = \overline{x}\} = \{i \,|\, x_i = \overline{x}\} \tag{3.4}$$

holds if and only if

$$\{i \,|\, y_i = \overline{x}\} = \{i \,|\, x_i = \overline{x}\} = S(T).$$

(iii) *The number of the elements in set $\{i \,|\, y_i = \underline{x}$ or $y_i = \overline{x}\}$ is at least one less than the number of the elements in $\{i \,|\, x_i = \underline{x}$ or $x_i = \overline{x}\}$.*

Proof. Write y_i as

$$y_i = \sum_{x_j=\underline{x}} t_{ij}x_j + \sum_{x_j=\overline{x}} t_{ij}x_j + \sum_{\underline{x}<x_j<\overline{x}} t_{ij}x_j, \quad i = 1,\dots,n. \tag{3.5}$$

Using $t_{ii} > 0$ and $Te = e$, we have

$$\begin{array}{ll} \underline{x} \le y_i < \overline{x} & \text{for } i \text{ such that } x_i = \underline{x}; \\ \underline{x} < y_i < \overline{x} & \text{for } i \text{ such that } \underline{x} < x_i < \overline{x}; \\ \underline{x} < y_i \le \overline{x} & \text{for } i \text{ such that } x_i = \overline{x}. \end{array}$$

This is part (i) of this proposition.

From (3.5), $\{i \,|\, y_i = \underline{x}\} = \{i \,|\, x_i = \underline{x}\}$ if and only if

$$\sum_{j:\ x_j>\underline{x}} t_{ij} = 0 \quad \text{for all } i \text{ such that } x_i = \underline{x} \tag{3.6}$$

or equivalently,

$$\sum_{j:\ x_j=\underline{x}} t_{ij} = 1 \quad \text{for all } i \text{ such that } x_i = \underline{x}. \tag{3.7}$$

We use $\widetilde{T}_{11}$ to denote the principal submatrix of T consisting of those rows and columns whose indices belong to $\{i \,|\, y_i = \underline{x}\}$. From (3.7), e is a Perron vector of $\widetilde{T}_{11}$. Using Proposition 3.1, we have $\{i \,|\, x_i = \underline{x}\} = S(T)$; thus (3.3) holds. Similar arguments are valid for the necessary and sufficient condition of (3.4).

As $\underline{x} < \overline{x}$, equalities (3.3) and (3.4) cannot hold simultaneously, therefore, at least one of the inclusions (3.1), (3.2) is strict, and part (iii) is also proved. ∎

Remark. If $t_{ii} = 0$, then for any $\omega : 0 < \omega < 1$, all diagonal elements of $(1-\omega)I + \omega T$, the Jacobi extrapolating matrix of T, are positive. Let T be an irreducible nonnegative matrix with $\rho(T) = 1$. If at least one diagonal element of T is positive (thus T is primitive), then T is semiconvergent, i.e., $\lim_{k\to\infty} T^k$ exists, the Jacobi iterative method converges; see [6, Chapter 2]. If this is not satisfied, T may not be semiconvergent, for example,

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

PROPOSITION 3.3. *Let $A \in \mathbb{R}^{n\times n}$ be an irreducible singular M-matrix such that $Ae = 0$, let $A = M_k - N_k$, $k = 1,\dots,n-1$, be arbitrary weak regular splittings of A, and let $T_k = M_k^{-1}N_k$, with all diagonal elements of T_k positive. Then, with respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e \,|\, \alpha \in \mathbb{R}\}$,*

$$T = T_{n-1}T_{n-2}\cdots T_1$$

is pseudocontractive.

Proof. For any $x \notin X^*$, i.e., $\underline{x} < \overline{x}$, denote $y = Tx$, as $Te = e$, it is easy to prove that T is nonexpansive. Applying part (iii) of Proposition 3.2 repeatedly ($n-1$ times), we have that at least one set of $\{i \,|\, y_i = \underline{x}\}$ and $\{i \,|\, y_i = \overline{x}\}$ is empty. If, say, $\{i \,|\, y_i = \underline{x}\}$ is empty, then $y_i > \underline{x}$ for all $1 \le i \le n$, and furthermore

$$\|y - P(y)\| = \frac{\max_i y_i - \min_i y_i}{2} \le \frac{\overline{x} - \min_i y_i}{2} < \frac{\overline{x} - \underline{x}}{2} = \|x - P(x)\|;$$

therefore T is pseudocontractive. ∎

Remark. This T may be not paracontractive with respect to $\|\cdot\|_\infty$. For example, for $n = 3$, let

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \ M = \begin{pmatrix} 4 & -2 & -2 \\ 0 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix}, \ N = M - A,$$

$$T_1 = T_2 = M^{-1}N = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \end{pmatrix}, \ T = T_2T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0.75 & 0.25 & 0 \\ 0.75 & 0 & 0.25 \end{pmatrix}.$$

T_k and T satisfy all conditions in Proposition 3.3, so T is pseudocontractive. For $x = (2, 0, 0)^T \notin X^*$, $Tx = (2, 1.5, 1.5)^T \neq x$, the projection vectors $P(x) = (1, 1, 1)^T$, $P(Tx) = (1.75, 1.75, 1.75)^T$, thus

$$0.25 = \|Tx - P(Tx)\| < \|x - P(x)\| = 1, \quad \text{but } \|x\| = \|Tx\|.$$

Thus, T is pseudocontractive, but not paracontractive. In [11] it was shown that T positive guarantees that T is paracontractive. For the matrix T in Example 2, T^2 is positive. Here we do not require T to be positive or even to be nonnegative irreducible.

4. Two-stage multisplitting iterative methods. Since multisplitting iteration was first proposed by O'Leary and White [34] to solve systems of linear equations in a parallel computer, it has been studied for many types of systems, for example, nonsingular M-systems [1, 13, 33], H-systems [37], SPD-systems [31, 44], nonlinear systems [17, 16], linear or nonlinear complementarity problems [2, 4, 28], etc., combining with many kinds of methods, e.g., extrapolating methods [43, 18, 3], CG methods with preconditioning [9, 24, 23], two-stage methods [12, 25, 40], etc. However, multisplitting iteration should be viewed more as an analysis tool to study a variety of block iterative methods, including the Schwarz method, rather than as a competitive computational method. In this section, we discuss a nonstationary two-stage multisplitting method for solving singular M-systems and its convergence from this point of view. In the next section, another nonstationary multisplitting method called the asynchronous multisplitting method is discussed.

Let A be an singular M-matrix, $A = M - N$ be a weak regular splitting, $M = F_l - G_l$ be r splittings, E_l be r nonnegative diagonal matrices such that $\sum_{l=1}^r E_l = I$. $(F_l, G_l, E_l)_{l=1}^r$ is called a weak regular multisplitting of M if the r splittings $M = F_l - G_l$, $l = 1, \ldots, r$, are weak regular. The following nonstationary two-stage multisplitting method for solving singular M-systems was given in [30].

ALGORITHM (nonstationary two-stage multisplitting). Given the initial vector $x(0)$, and a sequence of numbers of inner iterations $q(l, k), 1 \le l \le r, k = 1, 2, \ldots,$

For $k = 1, 2, \ldots$, until convergence.
    For $l = 1$ to r % outer stage
        $y(k, 0) = x(k - 1)$
        For $j = 1$ to $q(l, k)$ % inner stage
            $F_l y(l, j) = G_l y(l, j - 1) + Nx(k - 1)$
    $x(k) = \sum_{l=1}^r E_l y(l, q(l, k))$.

The parallel implementation of this algorithm is obvious. Before we state our convergence theorem for it, we give a lemma.

LEMMA 4.1. *Let A be an irreducible singular M-matrix, $A = M - N$ be a weak regular splitting, and $(F_l, G_l, E_l)_{l=1}^r$ be a weak regular multisplitting of M. Then the sequence of iterative vectors $x(k)$ in the nonstationary two-stage multisplitting iteration satisfies*

$$x(k) = T_{k-1}x(k-1),$$

where

$$T_{k-1} = \sum_{l=1}^{r} E_l \left[R_l^{q(l,k)} + (I + R_l + \cdots + R_l^{q(l,k)-1}) F_l^{-1} N \right] \tag{4.1}$$

with

$$R_l = F_l^{-1} G_l.$$

Assume further that the diagonal elements of $M^{-1}N$ and $F_l^{-1}G_l$ are positive, and that $F_l^{-1}N \geq 0$, then there exists $\widetilde{M}_{k-1}$ and $\widetilde{N}_{k-1}$ such that $A = \widetilde{M}_{k-1} - \widetilde{N}_{k-1}$ is a weak regular splitting, $T_{k-1} = \widetilde{M}_{k-1}^{-1}\widetilde{N}_{k-1} \geq 0$, all diagonal elements of T_{k-1} are positive.

Proof. Equation (4.1) can be proved by induction. Since $R_l = F_l^{-1}G_l \geq 0$, $F_l N \geq 0$, we have $T_{k-1} \geq 0$. At the same time, since all diagonal elements of R_l are positive, all diagonal elements of T_{k-1} are also positive. Furthermore, as $M^{-1} \geq 0$, $M = F_l - G_l$ is weak regular, we know [42] that $\rho(R_l) < 1$ and

$$\begin{aligned} T_{k-1} &= \sum_{l=1}^{r} E_l \left[R_l^{q(l,k)} + (I - R_l)^{-1}(I - R_l^{q(l,k)}) F_l^{-1} N \right] \\ &= \sum_{l=1}^{r} E_l \left[R_l^{q(l,k)} + (I - R_l^{q(l,k)}) M^{-1} N \right]. \end{aligned} \tag{4.2}$$

Define

$$\widetilde{F}_{l,k} = M(I - R_l^{q(l,k)})^{-1},$$

$$\widetilde{G}_{l,k} = \widetilde{F}_{l,k} R_l^{q(l,k)};$$

we have $\widetilde{F}_{l,k} - \widetilde{G}_{l,k} = M$ and

$$\widetilde{F}_{l,k}^{-1} = (I + R_l + \cdots + R_l^{q(l,k)-1}) F_l^{-1} \geq 0$$

and

$$\widetilde{F}_{l,k}^{-1} \widetilde{G}_{l,k} = R_l^{q(l,k)} \geq 0,$$

thus $(\widetilde{F}_{l,k}, \widetilde{G}_{l,k}, E_l)_{l=1}^r$ is also a weak regular multisplitting of M. Since $M^{-1} \geq 0$, from [19, Theorem 2.1(ii)], we know that $\sum_{l=1}^r E_l \widetilde{F}_{l,k}^{-1}$ is nonsingular, thus we can define

$$\widetilde{M}_{k-1} = \left(\sum_{l=1}^{r} E_l \widetilde{F}_{l,k}^{-1} \right)^{-1}, \quad \widetilde{N}_{k-1} = \widetilde{M}_{k-1} - A.$$

It is easy to verify that $\widetilde{M}_{k-1}, \widetilde{N}_{k-1}$ satisfy the requirements of the lemma. ∎

Remark. If A is nonsingular, results similar to this lemma have appeared in, e.g., [40, 22]. If A is singular, the expression of $A = \widetilde{M}_{k-1} - \widetilde{N}_{k-1}$ is not unique; see [5].

THEOREM 4.2. *Suppose that the matrix A and all related splittings satisfy the conditions in Lemma 4.1. Then for any $x(0)$, the sequence of iterative vectors $x(k)$ converges to some x^* such that $Ax^* = 0$. More specifically, if $x(0)$ is positive, x^* is also positive.*

Proof. Without loss of generality, we assume that $Ae = 0$. Let $x(k) = T_{k-1}x(k-1)$, where T_{k-1} is defined by Lemma 4.1, and construct the following sequence of vectors iteratively:

$$y(0) = x(0);$$
$$y(m+1) = \widetilde{T}_m y(m), \quad m = 0, 1, 2, \ldots,$$

where

$$\widetilde{T}_m = T_{m(n-1)+n-2} \cdots T_{m(n-1)}.$$

From Lemma 4.1, T_k is induced by a weak regular splitting of A for $k = 0, 1, \ldots$. From the assumption of this theorem, all diagonals of T_k are positive, thus from Proposition 3.3, $\widetilde{T}_m$ is pseudocontractive with respect to $\| \, \|_\infty$ and $X^* = \{\alpha e \,|\, \alpha \in \mathbb{R}\}$. To apply Theorem 2.3, we need to look for a convergent subsequence $\{\widetilde{T}_{m_i}\}_{i=0}^\infty$ of $\{\widetilde{T}_m\}_{m=0}^\infty$. Once the splitting $A = M - N$ and the multisplitting of $M : (F_l, G_l, E_l)_{l=1}^r$ are defined, T_{k-1} is uniquely determined by an integer vector

$$\hat{q}(k) = (q(1,k), \ldots, q(l,k));$$

cf. (4.2). Note that if some component of $\hat{q}(k)$, say $q(1,k)$, is replaced by $+\infty$, then

$$T_{k-1} = E_1 M^{-1} N + \sum_{l=2}^{r} E_l \left[R_l^{q(l,k)} + (I - R_l^{q(l,k)}) M^{-1} N \right],$$

the operator T_{k-1} is well defined, and it has the same properties as the one which has finite parameters $q(1,k), \ldots, q(l,k)$. Similarly, each $\widetilde{T}_m$ is uniquely determined by an integer vector

$$\tilde{q}(m) = (\hat{q}((m+1)(n-1)-1), \ldots, \hat{q}(m(n-1))).$$

Now we choose a subsequence $\{\tilde{q}(m_i)\}_{i=0}^\infty$ of $\{\tilde{q}(m)\}_{m=0}^\infty$ such that for each component sequence of $\{\tilde{q}(m_i)\}_{i=0}^\infty$, either this component sequence has equal value for all i, or this component sequence tends to infinity as $i \to \infty$. From the above analysis, the subsequence $\{\widetilde{T}(m_i)\}_{i=0}^\infty$ of $\{\widetilde{T}(m)\}_{m=0}^\infty$ is convergent, and its limit is pseudocontractive.

So, by applying Theorem 2.3, the sequence of vectors $y(m)$ converges to some $x^* = \alpha^* e$. More specifically,

$$\min_i x_i(0) \leq \alpha^* \leq \max_i x_i(0).$$

As every T_k is nonexpansive, $\{y(m)\}$ is a subsequence of $\{x(k)\}$, the sequence of iterative vectors $x(k)$ converges to x^* also. ∎

Remark. The condition that $A = M - N$ is a regular splitting ($M^{-1} \geq 0$, $N \geq 0$) is not necessary, since only a weaker condition $F_l^{-1} N \geq 0$ is needed in this theorem (this has been observed earlier in [20]) . The condition that $q(l, k)$ is bounded for all k is not necessary either. The condition below was given in [30] for the convergence of the nonstationary iteration

$$\|T_k(I - T_k)(I - T_k)^{\#}\| \le \theta < 1, \quad k = 0, 1, 2, \ldots, \tag{4.3}$$

where T_k are the iteration matrices such that $x(k+1) = T_k x(k)$, and $^{\#}$ denotes the group inverse. The following example shows that there is a matrix T_k, which is pseudocontractive with respect to $\|\cdot\|_\infty$, but does not satisfy the above condition with respect to $\|\cdot\|_\infty$:

$$T_k = \frac{1}{100}\begin{pmatrix} 99 & 1 \\ 40 & 60 \end{pmatrix}, T_k^{\infty} = \frac{1}{41}\begin{pmatrix} 40 & 1 \\ 40 & 1 \end{pmatrix},$$

and $\|T_k(I-T_k)(I-T_k)^{\#}\|_\infty = \|T_k - T_k^{\infty}\|_\infty = \frac{236}{205} > 1$. We should note that the norm in (4.3) can be any norm, although the infinity norm is widely used in nonnegative matrix theory.

5. Asynchronous multisplitting iterations. Parallel multisplitting iterative methods can be implemented asynchronously, avoiding synchronization overhead and thus saving computational time. Asynchronous multisplitting iterations to solve nonsingular systems have been widely discussed; see, e.g., [10, 12, 39, 38]. The effectiveness of asynchronization was shown by Frommer, Schwandt, and Szyld [21] with numerical examples. To solve singular systems, Lubachevski and Mitra [27] proposed an asynchronous iterative algorithm in the case of a single splitting and Pott [35] gave a different approach to prove convergence.

Let $\{M_l, N_l, E_l\}_{l=1}^r$ be a multisplitting of A. The following asynchronous multisplitting iteration (AMI) was given by Bru, Elsner, and Neumann [10] to solve nonsingular systems; here we use it to solve singular systems. Its convergence will be proved under reasonable conditions.

ALGORITHM (AMI). Given the initial vectors $x(0), \ldots, x(-D)$, for $k = 0, 1, 2, \ldots,$

$$x(k+1) = (I - E_{l(k)})x(k) + E_{l(k)} M_{l(k)}^{-1} N_{l(k)} y(k), \tag{5.1}$$

with

$$y(k) = (x_1(k - d(k,1)), \ldots, x_n(k - d(k,n)))^T, \tag{5.2}$$

where $l(k) \in \{1, \ldots, r\}$, $0 \le d(k,i) \le k + D$ are integers less than or equal k.

Suppose we have a parallel computer consisting of a host and r slaves. There is a global approximation in the host. Every slave has a local approximation and does the following repeatedly: retrieves a global approximation y from the host, forms a local approximation, say $M_l^{-1} N_l y$, and sends it to the host. The host does the following repeatedly: receives a local approximation from some slave and forms a new global approximation as in (5.1). The terms $d(k,i)$ can be interpreted as follows: Suppose that the $l(k)$th slave retrieves a global approximation at the $(k-d)$th iteration and forms a new local approximation, the host uses this to form a new global approximation $x(k+1)$; during this period, other slave(s) may send their local approximations to the host and the host forms global approximations $x(k-d+1), \ldots, x(k-d)$; thus d is the iteration drift.

A more universal asynchronous model for a distributed parallel machine can be found in [21]. Here we use a simple model in order to show how to use the analysis technique developed above.

THEOREM 5.1. *Let A be an irreducible singular M-matrix, and $\{M_l, N_l, E_l\}_{l=1}^r$ be a weak regular multisplitting of A. If, in the AMI* (5.1) *and* (5.2),

(i) $\sum_{l=1}^r E_l M_l^{-1}$ *is nonsingular,*

(ii) *there exists some integer D such that*

$$\{l(k)\} \cup \{l(k+1)\} \cup \cdots \cup \{l(k+D)\} = \{1, \ldots, r\} \quad \text{for all } k = 0, 1, \ldots \tag{5.3}$$

and

$$0 \le d(k,i) \le D \quad \text{for all } k \ge 0, 1 \le i \le n, \tag{5.4}$$

(iii) *for each $1 \le i \le n$, either*

$$(E_l)_{ii} < 1 \tag{5.5}$$

or

$$(E_l)_{ii} = 1, \quad (T_l)_{ii} > 0, \quad \text{and } d(k,i) = 0 \text{ for } k : l(k) = l, \tag{5.6}$$

where $T_l = M_l^{-1} N_l$, then AMI converges.

Condition (i) is necessary even in the case of synchronous multisplitting iterations, see [26], to guarantee the consistency between the iteration and the system $Ax = 0$. If $A = M - N$ is a weak regular splitting of A and $(F_l, G_l, E_l)_l$ is a weak regular multisplitting of M (cf. section 4), then the multisplitting $(M_l, N_l, E_l)_l$ with

$$M_l = F_l, \quad N_l = G_l + N, \quad l = 1, \ldots, r,$$

satisfies condition (i). Condition (ii) is referred to as the condition that the sequence $l(k)$ be regulated in [10]. If condition (5.5) is satisfied, the AMI can be viewed as an extrapolating one; cf. [14, 15]. Condition (5.6) is referred to as a partial asynchronism condition; see [27, 7, 41]. In practice, this condition can be satisfied if the ith component is updated by only one processor in a distributed parallel computer system.

Proof. For an arbitrary fixed k', denote

$$\overline{\alpha} \equiv \max_{1 \le i \le n} \{x_i(k'-D), \ldots, x_i(k')\},$$

$$\underline{\alpha} \equiv \min_{1 \le i \le n} \{x_i(k'-D), \ldots, x_i(k')\}.$$

For any $k \ge k'$,

$$x_i(k+1) = (1 - (E_{l(k)})_{ii}) x_i(k) + (E_{l(k)})_{ii} \sum_{j=1}^n (T_{l(k)})_{ij} x_j(k - d(k,j)),$$

with condition (iii), by induction, (as in Proposition 3.2) we have

$$\begin{aligned} \underline{\alpha} < x_i(k) < \overline{\alpha} &\Rightarrow \underline{\alpha} < x_i(k+1) < \overline{\alpha}, \\ x_i(k) = \overline{\alpha} &\Rightarrow \underline{\alpha} < x_i(k+1) \le \overline{\alpha}, \\ x_i(k) = \underline{\alpha} &\Rightarrow \underline{\alpha} \le x_i(k+1) < \overline{\alpha}. \end{aligned} \tag{5.7}$$

LEMMA 5.2. *At least one of the sets* $\{i \mid x_i(k'+(n-1)(2D+1)) = \overline{\alpha}\}$ *and* $\{i \mid x_i(k'+(n-1)(2D+1)) = \underline{\alpha}\}$ *is empty.*

Proof. Without loss of generality, we suppose that both $\{i \mid x_i(k') = \overline{\alpha}\}$ and $\{i \mid x_i(k') = \overline{\alpha}\}$ are not empty, otherwise, from (5.7), we are done. From (5.7),

$$\{i \mid x_i(k'+(n-1)(2D+1)) = \underline{\alpha}\} \subset \cdots \subset \{i \mid x_i(k'+1) = \underline{\alpha}\} \subset \{i \mid x_i(k') = \underline{\alpha}\}$$

and

$$\{i \mid x_i(k'+(n-1)(2D+1)) = \overline{\alpha}\} \subset \cdots \subset \{i \mid x_i(k'+1) = \overline{\alpha}\} \subset \{i \mid x_i(k') = \overline{\alpha}\}.$$

We first prove that

$$\begin{aligned} &\{i \mid x_i(k'+2D+1) = \overline{\alpha} \text{ or } x_i(k'+2D+1) = \underline{\alpha}\} \\ &\neq \{i \mid x_i(k') = \overline{\alpha} \text{ or } x_i(k') = \underline{\alpha}\}. \end{aligned} \tag{5.8}$$

If this is not the case, by (5.7),

$$\{i \mid x_i(k') = \overline{\alpha}\} = \cdots = \{i \mid x_i(k'+2D+1) = \overline{\alpha}\},$$

$$\{i \mid x_i(k') = \underline{\alpha}\} = \cdots = \{i \mid x_i(k'+2D+1) = \underline{\alpha}\},$$

and

$$\underline{\alpha} < x_i(k) < \overline{\alpha} \quad \text{for all } i \text{ such that } \underline{\alpha} < x_i(k') < \overline{\alpha}, k = k', \ldots, k'+2D+1.$$

For $k : k'+D \le k \le k'+2D$, using (5.4), we have

$$k' \le k - d(k,j) \le k'+2D, \quad j = 1, \ldots, n,$$

for i such that $x_i(k') = \overline{\alpha}$, use $\sum_j (T_{l(k)})_{ij} = 1$, we can write $x_i(k+1)$ as

$$\begin{aligned} \overline{\alpha} &= x_i(k+1) \\ &= (1-(E_{l(k)})_{ii})\overline{\alpha} \\ &\quad + (E_{l(k)})_{ii} \times \left(\overline{\alpha} \sum_{j:\, x_j(k') = \overline{\alpha}} (T_{l(k)})_{ij} + \sum_{j:\, x_j(k') < \overline{\alpha}} (T_{l(k)})_{ij} x_j(k-d(k,j)) \right) \\ &= \overline{\alpha} + (E_{l(k)})_{ii} \times \sum_{j:\, x_j(k') < \overline{\alpha}} (T_{l(k)})_{ij} (x_j(k-d(i,j)) - \overline{\alpha}). \end{aligned}$$

This equation holds if and only if

$$(E_{l(k)})_{ii} (T_{l(k)})_{ij} = 0 \text{ for } j : x_j(k') < \overline{\alpha},$$

which means that

$$(E_{l(k)})_{ii} \left(1 - \sum_{j:\, x_j(k') = \overline{\alpha}} (T_{l(k)})_{ij} \right) = 0, \quad k = k'+D, \ldots, k'+2D,$$

i.e.,

$$(E_{l(k)})_{ii} \sum_{j:\, x_j(k') = \overline{\alpha}} (T_{l(k)})_{ij} = (E_{l(k)})_{ii}, \quad k = k'+D, \ldots, k'+2D.$$

From (5.3), we have

$$\bigcup_{k'+D\le k\le k'+2D}\{l(k)\}=\{1,\dots,r\},$$

so,

$$\sum_{l=1}^{r}(E_l)_{ii}\sum_{j:\,x_j(k')=\overline{\alpha}}(T_l)_{ij}=\sum_{l=1}^{r}(E_l)_{ii}=1,$$

which means that

$$\sum_{j:\,x_j(k')=\overline{\alpha}}T_{ij}=1\quad\text{for all } i \text{ such that } x_i(k')=\overline{\alpha}, \tag{5.9}$$

where

$$T=\sum_{l=1}^{r}E_lT_l.$$

By the same argument, we have also that

$$\sum_{j:\,x_j(k')=\underline{\alpha}}T_{ij}=1\quad\text{for all } i \text{ such that } x_i(k')=\underline{\alpha}. \tag{5.10}$$

Note that under the condition (i) of the theorem, T is also an iterative matrix induced by a weak regular splitting [26], so these two equalities (5.9) and (5.10) contradict Proposition 3.1. Therefore the number of elements in $\{i \mid x_i(k'+2D+1)=\overline{\alpha}$ or $x_i(k'+2D+1)=\underline{\alpha}\}$ is at least one less than the number of elements in $\{i \mid x_i(k')=\overline{\alpha}$ or $x_i(k')=\underline{\alpha}\}$.

Repeating the above proof, we have that either one of $\{i \mid x_i(k'+n'(2D+1))=\overline{\alpha}\}$ and $\{i \mid x_i(k'+n'(2D+1))=\underline{\alpha}\}$ is empty for some $1\le n'\le n-1$, in this case, the lemma has been proved, or the number of elements in $\{i \mid x_i(k'+n'(2D+1))=\overline{\alpha}$ or $x_i(k'+n'(2D+1))=\underline{\alpha}\}$ is at least one less than the number of elements in $\{i \mid x_i(k'+(n'-1)(2D+1))=\overline{\alpha}$ or $x_i(k'+(n'-1)(2D+1))=\underline{\alpha}\}$ for all $1\le n'\le n-1$. After at most $n-1$ steps, we get the conclusion. ∎

Proof of Theorem 5.1 *(continued).* Construct a big vector

$$\mathbf{x}(k)\equiv(x^T(k-D),\dots,x^T(k))^T\in\mathbb{R}^{(D+1)n},\quad k=0,1,2,\dots.$$

The asynchronous iteration (5.1) and (5.2) is equivalent to

$$\mathbf{x}(k+1)=\mathbf{T}_k\mathbf{x}(k),k=0,1,2,\dots,$$

where $\mathbf{T}_k$ has the form

$$\begin{pmatrix} I-E_{l(k)}+* & * & * & \cdots & * \\ I & 0 & 0 & \cdots & 0 \\ & I & 0 & \cdots & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & I & 0 \end{pmatrix}.$$

Consider

$$X^* = \{\alpha \mathbf{e} \mid \alpha \in \mathbb{R}\},$$

where $\mathbf{e} \in \mathbb{R}^{(D+1)n}$ is the vector with all components equal to 1. For any $\mathbf{x}(k') \notin X^*$, from the above lemma, we know that, say, $\{i \mid x_i(k' + (n-1)(2D+1)) = \overline{\alpha}\}$ is empty, and therefore $\{i \mid x_i(k) = \overline{\alpha}\}$ is empty for all $k \geq k' + (n-1)(2D+1)$, which means that

$$\|\mathbf{x}(k' + (n-1)(2D+1) + D) - P(\mathbf{x}(k' + (n-1)(2D+1) + D))\| < \|\mathbf{x}(k') - P(\mathbf{x}(k'))\|.$$

So for all $k' \geq 0$, the product

$$\mathbf{T}_{k'+(n-1)(2D+1)+D-1} \cdots \mathbf{T}_{k'+1}\mathbf{T}_{k'}$$

of $(n-1)(2D+1) + D$ operators is pseudocontractive with respect to $\|\cdot\|_\infty$ and X^*. Because there is only a finite number of operators, there is always a convergent subsequence in this sequence of operators, and its limit is also pseudocontractive, from Theorem 2.3, we obtain the convergence of $\mathbf{x}(k)$, which is equivalent to asserting that the sequence $x(k)$ converges. $\square$

6. Conclusions. This paper introduced a new property of operators called pseudocontractivity and showed that it is a generalization of the paracontractivity property. A general convergence theorem for pseudocontractive iterations was proved and it was shown that, under appropriate conditions, the product of at most $n-1$ (where n is the dimension of the matrix) iteration matrices induced from weak regular splittings is pseudocontractive. This analysis technique was applied to nonstationary iterative methods for solving singular M-systems, specifically, nonstationary two-stage multisplitting methods and asynchronous multisplitting methods, with no other contractivity condition on the iteration operators.

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