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Parallel asynchronous Schwarz and multisplitting methods for a nonlinear diffusion problem

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Parallel asynchronous subdomain algorithms with flexible communication for the numerical solution of nonlinear diffusion problems are presented. The discrete maximum principle is considered and the Schwarz alternating method and multisplitting methods are studied. A connection is made with M-functions for a classical nonlinear diffusion problem. Finally, computational experiments carried out on a shared memory multiprocessor are presented and analyzed.

Keywords: parallel iterative algorithms, asynchronous iterations, Schwarz alternating method, multisplitting method, subdomain methods, boundary value problems

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1. Introduction

Large scale algebraic systems occur typically in the numerical solution of boundary value problems. Due to the large amount of computation required, parallel numerical algorithms seem well adapted to an efficient solution of this class of problems. In particular, subdomain methods are naturally well suited to the parallel numerical solution of boundary value problems. Among subdomain methods, one can consider methods with overlap such as the Schwarz alternating method which has been studied in [11], or methods without overlap (see [10,15,16,21,24,25]). A new class of parallel asynchronous Schwarz alternating methods with flexible communication whereby the current value of the components of the iterate vector can be used at any time and without any fixed rule is presented in this paper. This class of algorithms was first proposed in [20]. We note that the theoretical context considered in this study for nonlinear problems is based on monotone convergence and is different from the one considered in [13], which uses essentially contraction techniques in the particular case of linear systems. Note that the monotone convergence of classical asynchronous iterations for the solution of mildly nonlinear equations has been studied by many authors, in particular, reference is made to [2,28]. The proposed methods present very good efficiency without making use of any load balancing techniques. The very good performance is due to the non synchronization of the computational tasks and to flexible communication. The concept presented in this paper generalizes classical asynchronous iterations studied in [5,6,8–10,12,14,17–19]. In this paper, we use the discrete maximum principle thanks to the M-function concept as defined by Rheinboldt in [23]. This permits us to show the monotone convergence of parallel asynchronous iterations with flexible communication. Note that the convergence of such parallel schemes of computation has been shown by using direct verification in [20] for various methods in the M-function context for problems of the form $\mathcal{A}(y) = 0$. In this paper, we study general fixed point methods such as for example Newton like methods and establish directly their monotone convergence by using a different approach. Then, we use this result in order to show that the fixed point mapping considered for the solution of the nonlinear diffusion problem is an approximate fixed point mapping, the so-called $\mathcal{A}$-supermapping defined in [20] and we derive a new convergence result for asynchronous iterations with flexible communication. In this context, nonlinear boundary value problems are solved via a combination of the asynchronous algorithms with flexible communication quoted above and the Schwarz alternating method applied to Newton's iterations. More precisely, we consider the following nonlinear diffusion problem

$$\begin{cases} -\Delta u + \phi(u) = f, & \text{everywhere in } \Omega, \\ u = 0, & \text{everywhere in } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\partial\Omega$ is the boundary of Ω, $f \in L^2(\Omega)$ is a given function and ϕ is a monotone increasing, convex and continuously differentiable nonlinear operator. For a convenient discretization, we show that an M-function and a suitable approximate fixed point mapping can be obtained from the system of equations and that the monotone convergence of asynchronous iterations with flexible communication can be derived. Finally, we present and analyze computational results for parallel synchronous and asynchronous iterations with flexible communication carried out on a shared memory multiprocessor IBM 3090-600 with 6 vector processors for the solution of the following problem

$$\begin{cases} -\Delta u + \mathrm{e}^{bu} = f, & \text{everywhere in } \Omega \subset \mathbb{R}^2, \\ u = 0, & \text{everywhere in } \partial\Omega, \end{cases} \tag{1.2}$$

where b is a given strictly positive real number.

Section 2 deals with preliminaries concerning M-functions and asynchronous iterations with flexible communication; a connection is also made with the Schwarz alternating method. In section 3, a general class of fixed point methods is proposed and application to the nonlinear diffusion problem (1.1) is considered. Convergence results are presented for asynchronous Schwarz and multisplitting methods with flexible communication which constitute an original contribution of the present work. Section 4 deals with the implementation of parallel algorithms on a shared memory multiprocessor and the analysis of experimental results.

2. Preliminaries

2.1. *Definitions and notations*

The n-dimensional linear space of column vectors will be denoted by $\mathbb{R}^n$ and $\mathcal{A}$ will denote a continuous mapping from $\mathbb{R}^n$ onto itself. Let $\mathcal{A}$ be a surjective M-function, then the mapping $\mathcal{A}$ is off-diagonally monotone decreasing and inverse monotone increasing (see [22,23]). We consider the solution of the following system of equations

$$\mathcal{A}(y) = 0. \tag{2.1}$$

Under the above assumptions, problem (2.1) has a unique solution y^* (see [22,23]). Let $\mathbb{R}^n$ be denoted also by $\mathbb{E}$ and consider the following decomposition of $\mathbb{E}$, $\mathbb{E} = \prod_{i=1}^{\alpha} \mathbb{E}_i$, where α is a positive integer, $\mathbb{E}_i = \mathbb{R}^{n_i}$, and $\sum_{i=1}^{\alpha} n_i = n$. Each subspace $\mathbb{E}_i$ is endowed with the natural partial ordering (i.e. component by component). Let $w \in \mathbb{E}$ and consider the following block-decomposition of w

$$w = \{w_1, \dots, w_i, \dots, w_\alpha\} \in \prod_{i=1}^{\alpha} \mathbb{E}_i,$$

and the corresponding block-decomposition of $\mathcal{A}$

$$\mathcal{A}(w) = \big\{\mathcal{A}_1(w), \dots, \mathcal{A}_i(w), \dots, \mathcal{A}_\alpha(w)\big\} \in \prod_{i=1}^{\alpha} \mathbb{E}_i.$$

For all $i \in \{1, \dots, \alpha\}$, we introduce the following mapping from $\mathbb{E}_i$ onto itself

$$y_i \to \mathcal{A}_i(y_i; w) = \mathcal{A}_i(w_1, \dots, w_{i-1}, y_i, w_{i+1}, \dots, w_\alpha).$$

Since $\mathcal{A}$ is a continuous, surjective M-function, it follows from [23, theorem 3.5] that for all $i \in \{1, \dots, \alpha\}$ and all $w \in \mathbb{E}$, the mapping $y_i \to \mathcal{A}_i(y_i; w)$ is a continuous, surjective M-function. Moreover, for all $i \in \{1, \dots, \alpha\}$ and $w \in \mathbb{E}$, the system

$$\mathcal{A}_i(z_i; w) = 0, \tag{2.2}$$

has a unique solution $\hat{z}_i$. Hence, we can define a fixed point mapping $F: E \to E$, associated with problem (2.1) such that

$$F(w) = \hat{z} = \{\hat{z}_1, \dots, \hat{z}_i, \dots, \hat{z}_\alpha\}, \tag{2.3}$$

the mapping F is well defined and monotone increasing on $\mathbb{E}$ (i.e. for all $x, y \in \mathbb{E}$ such that $x \leqslant y$, $F(x) \leqslant F(y)$) (see [19]). In order to solve problem (2.1), we will consider general fixed point iterative methods. The following concept permits one to define a set of starting points for the fixed point iterations.

Definition 2.1. A vector $y \in \mathbb{R}^n$ is an $\mathcal{A}$-supersolution if $\mathcal{A}(y) \geqslant 0$.

In the sequel, asynchronous sequences will be studied by using the order interval concept which is defined as follows.

Definition 2.2. Let $x_i, y_i \in \mathbb{E}_i$ such that $x_i \leqslant y_i$, then the order interval $\langle x_i, y_i \rangle_i$ is defined as $\{z_i \in \mathbb{E}_i \mid x_i \leqslant z_i \leqslant y_i\}$.

The definition is similar for $\langle x, y \rangle$, $x, y \in \mathbb{E}$. We introduce now a particular class of fixed point mappings which is very useful in the definition of asynchronous iterations with flexible communication.

Definition 2.3. Let $\mathcal{A}$ be an M-function. $F^{\mathcal{A}}$ is an $\mathcal{A}$-supermapping associated with F if for all $i \in \{1, \dots, \alpha\}$ and $y \in \mathbb{E}$ such that $\mathcal{A}_i(y) \geqslant 0$, there exists $F_i^{\mathcal{A}}(y) \in \mathbb{E}_i$, such that $F_i^{\mathcal{A}}(y) \leqslant y_i$, $\mathcal{A}_i(F_i^{\mathcal{A}}(y); y) \geqslant 0$ and $F_i^{\mathcal{A}}(y) \neq y_i$ if $F_i(y) \neq y_i$.

2.2. Asynchronous iterations with flexible communication

We consider now the parallel asynchronous fixed point iterations with flexible communication $\{y^p\}_{p \in \mathbb{N}}$ which are defined by using the following concepts.

Definition 2.4. A steering of block-components of the iterate vector is a sequence $\{s(p)\}$, $p \in \mathbb{N}$, such that $s(p) \in \{1, \dots, \alpha\}$, for all $p \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural integers. A sequence of delayed iteration numbers $\{\rho(p)\}$ of vectors $\rho(p) = (\rho_1(p), \dots, \rho_i(p), \dots, \rho_\alpha(p)) \in \mathbb{N}^\alpha$ is such that for all $p \in \mathbb{N}$ and $i \in \{1, \dots, \alpha\}$ we have $0 \leqslant \rho_i(p) \leqslant p$ and $\rho_i(p) = p$ if $i = s(p)$. For all $p \in \mathbb{N}$ and $i \in \{1, \dots, \alpha\}$, the set K_i^p of iteration numbers lower than p, which are relative to the computation of the ith block-component is such that $K_i^p = \{j \in \mathbb{N} \mid s(j) = i,\ 0 \leqslant j < p\}$.

Definition 2.5. The general class of asynchronous iterative methods with flexible communication is defined recursively as follows. For all $p \in \mathbb{N}$ and $i \in \{1, \dots, \alpha\}$, we have

$$\begin{cases} y_i^{p+1} = F_i^{\mathcal{A}}(\tilde{y}^p) & \text{if } i = s(p), \\ y_i^{p+1} = y_i^p & \text{if } i \neq s(p), \end{cases} \tag{2.4}$$

where

$$\tilde{y}^0 = y^0 \quad \text{is an } \mathcal{A}\text{-supersolution}, \tag{2.5}$$

$$\tilde{y}^p \in \left\langle y^p, \min\left(y^{\rho(p)}, \tilde{y}^q\right)\right\rangle \quad \text{if } p \geqslant 1, \tag{2.6}$$

the vector $y^{\rho(p)}$ denotes an element of $\mathbb{E}$ with block-components $y_i^{\rho_i(p)}$, $i \in \{1, \ldots, \alpha\}$ and $q = \mathrm{Max}\{j \in K^p_{s(p)}\}$ and $s(p)$, $\rho(p)$ and $K^p_{s(p)}$ are defined according to definition 2.4.

Remark 2.6. In the particular case where $K_i^p = \emptyset$, we have $\tilde{y}^p \in \langle y^p, y^{\rho(p)} \rangle$.

Remark 2.7. Asynchronous iterations with flexible communication defined recursively by (2.4)–(2.6) are general iterative methods whereby iterations are carried out in parallel by up to α processors without any order nor synchronization. The main feature of this class of iterative methods is to allow very flexible communication between the processors. In a typical update of the ith block-component of the iterate vector at iteration $p+1$, all the values $\tilde{y}_j^p$ of the block-components of the iterate vector can be taken anywhere in the order interval $\langle y_j^p, \min(y_j^{\rho_j(p)}, \tilde{y}_j^q) \rangle_j$, where $\tilde{y}_j^q$ was the value used in the last update of the ith block-component and $y_j^{\rho_j(p)}$ models the nondeterministic behavior of the iterative scheme and are not explicitly labelled by an iteration number. Thus, the values of the block-components of the iterate vector which are used in a computation may come from updates which are still in progress. It is important to note that the values of the components of the same block of the iterate vector which are used in such a typical update, can be relative to different iteration numbers as opposed to the classical case (see [5,6,8,17]). Practically, one will choose partial update corresponding to the last avaible value of each component.

We recall now an important result (see [20]).

Proposition 2.8. Let $\mathcal{A}$ be a continuous surjective M-function, F the fixed point mapping associated with $\mathcal{A}$ defined by (2.2) and (2.3), $F^{\mathcal{A}}$ an $\mathcal{A}$-supermapping associated with F, $y^0 \in \mathbb{E}$ an $\mathcal{A}$-supersolution. Then, the asynchronous iteration $\{y^p\}$ given by (2.4)–(2.6) is well defined and satisfies

$$y^p \downarrow \overline{y}, \quad p \to \infty, \tag{2.7}$$

where $\overline{y}$ is an $\mathcal{A}$-supersolution of problem (2.1) and (2.7) means that $\lim_{p\to\infty} y^p = \overline{y}$ and $\overline{y} \leqslant \cdots \leqslant y^{p+1} \leqslant y^p \leqslant \cdots \leqslant y^0$.

We introduce an order relation between $\mathcal{A}$-supermappings.

Definition 2.9. Two $\mathcal{A}$-supermappings $F^{\mathcal{A}}$ and $F^{\mathcal{B}}$ associated with F satisfy the relation $F^{\mathcal{A}} \prec F^{\mathcal{B}}$, if for all $i \in \{1, \ldots, \alpha\}$ and $y \in \mathbb{E}$ such that $\mathcal{A}_i(y) \geqslant 0$, we have: $F_i^{\mathcal{B}}(y) \in \langle F_i^{\mathcal{A}}(y), y_i \rangle_i$.

We concentrate now on a particular class of $\mathcal{A}$-supermappings.

Definition 2.10. $F^{\mathcal{A}}$ is an $\mathcal{M}$-continuous $\mathcal{A}$-supermapping associated with F if there exists an $\mathcal{A}$-supermapping $F^{\mathcal{B}}$ associated with F such that $F^{\mathcal{A}} \prec F^{\mathcal{B}}$ and

$$y^p \downarrow y^*, \quad p \to \infty, \quad \text{implies } F_i^{\mathcal{B}}(y^p) \downarrow F_i^{\mathcal{B}}(y^*), \quad p \to \infty \text{ for all } i \in \{1, \dots, \alpha\}. \tag{2.8}$$

Remark 2.11. The relation (2.8) can be interpreted as a property of continuity at the point y^* of the mapping $F^{\mathcal{B}}$ with respect to the partial ordering.

We recall now a global convergence result for asynchronous iterations with flexible communication associated with $\mathcal{M}$-continuous $\mathcal{A}$-supermappings (see [20]).

Proposition 2.12. Let assumptions of proposition 2.8 hold and $F^{\mathcal{A}}$ be an $\mathcal{M}$-continuous $\mathcal{A}$-supermapping associated with F. Assume that the steering satisfies

$$\{p \in \mathbb{N} \mid i \in s(p)\} \text{ is infinite}, \quad \text{for all } i \in \{1, \dots, \alpha\}, \tag{2.9}$$

and assume also that we have

$$\lim_{p \to \infty} \rho_i(p) = +\infty, \quad \text{for all } i \in \{1, \dots, \alpha\}. \tag{2.10}$$

Then, the sequence $\{y^p\}$ defined by (2.4)–(2.6) satisfies $y^p \downarrow y^*$, where y^* is the unique solution of problem (2.1).

2.3. Link with the Schwarz alternating method

We concentrate now on the Schwarz alternating method which is used for the solution of boundary value problems. This method is also well suited to parallel computing and represents an interesting domain of application for the above theoretical study. The mapping $\mathcal{A}$ being a surjective M-function, we study the solution of the following nonlinear simultaneous equations

$$\mathcal{A}(x) = 0, \tag{2.11}$$

via an asynchronous subdomain method derived from the Schwarz alternating method. We note that the augmentation process of the Schwarz alternating method transforms the nonlinear mapping $\mathcal{A}$ into the mapping $\tilde{\mathcal{A}}$ which is also a surjective M-function (see [20]). Thus the convergence results quoted above can be applied directly in the context of asynchronous Schwarz alternating methods with flexible communication. Reference is made to [4,28] for studies in the case of asynchronous Schwarz alternating methods for the solution of nonlinear systems.

3. Application to nonlinear partial differential equations

In this section, we present our original theoretical contribution to the study of the convergence of asynchronous iterations with flexible communication. A general approach for building approximate block iterative methods is proposed. Then, a monotone convergence result which is used for deriving A-supermappings is given. An original application of the general study to the nonlinear diffusion problem is also presented. Finally, a convergence result for asynchronous iterations with flexible communication applied to the discretized nonlinear diffusion problem is derived.

3.1. Results on monotone iterative methods

Let $\mathcal{A}$ be a continuous surjective M-function from $\mathbb{R}^n$ onto itself. Then, it follows from [23, theorem 3.3] that the system of equations $\mathcal{A}(y) = 0$ has a unique solution denoted by y^*. Moreover, let x be an initial point, we define the vectors $s, v, x^0, z^0 \in \mathbb{R}^n$ as follows

$$\begin{aligned} s_i &= \min\big(\mathcal{A}_i(x), 0\big), \qquad & v_i &= \max\big(\mathcal{A}_i(x), 0\big), \quad \forall i \in \{1, \dots, n\}, \\ x^0 &= \mathcal{A}^{-1}(s), & z^0 &= \mathcal{A}^{-1}(v). \end{aligned}$$

Thus, we have

$$\mathcal{A}\big(x^0\big) \leqslant \mathcal{A}(y^*) \leqslant \mathcal{A}\big(z^0\big), \tag{3.1}$$

and by the inverse isotonicity of $\mathcal{A}$, we have $x^0 \leqslant y^* \leqslant z^0$. Moreover, assume that there exists a mapping $C : \langle x^0, z^0\rangle \to \mathcal{L}(\mathbb{R}^n)$, such that for all $x, z \in \mathbb{R}^n$ satisfying $x^0 \leqslant x \leqslant z \leqslant z^0$, we have

$$\mathcal{A}(z) - \mathcal{A}(x) \leqslant C(z) \cdot (z - x), \tag{3.2}$$

$$x \leqslant z \quad \text{implies } C(x) \leqslant C(z), \tag{3.3}$$

$$C^{-1}(z) \geqslant 0, \quad \text{for all } z \in \big\langle x^0, z^0\big\rangle. \tag{3.4}$$

Remark 3.1. We note that the relation (3.4) implies the inequality (3.2) if we have $\mathcal{A}(z) - \mathcal{A}(x) = C(x + t(z - x)) \cdot (z - x)$, with $t \in]0, 1[$. The previous equality is satisfied in particular if $\mathcal{A}$ is G-differentiable and $C(x) = \mathcal{A}'(x)$.

Let the mapping ρ' from $\mathbb{N}$ into $\mathbb{N}$ such that

$$0 \leqslant \rho'(k) \leqslant k, \quad k = 0, 1, \dots. \tag{3.5}$$

Consider the following Newton-like algorithm

$$z^{k+1} = z^k - C^{-1}\big(z^{\rho'(k)}\big) \cdot \mathcal{A}\big(z^k\big), \quad k = 0, 1, \dots. \tag{3.6}$$

We can state the following convergence result from [26].

Proposition 3.2. Assume that the hypotheses (3.2)–(3.5) hold. Then, the sequence $\{z^k\}$ defined by (3.6) satisfies $z^k \downarrow y^*$, $k \to \infty$, where y^* is the solution of $\mathcal{A}(y) = 0$.

3.2. Example of $\mathcal{A}$-supermapping associated with Newton's method

We consider the following problem

$$\sum_{j=1}^{\alpha} A_{ij} x_j + \phi_i(x_i) = b_i, \quad \text{for all } i \in \{1, \dots, \alpha\}, \tag{3.7}$$

where $x_i, b_i \in \mathbb{E}_i$, $A = (A_{ij})$ according to the associated block decomposition and

$$A \text{ is an } M\text{-matrix}, \tag{3.8}$$

$$\begin{cases} \phi(y) \text{ is a diagonal, nonlinear operator with monotone increasing,} \\ \text{convex and continuously differentiable components } \phi_i(y_i). \end{cases} \tag{3.9}$$

The problem (3.7) is a problem of type (2.1) which satisfies the assumptions of section 2. In particular, this problem occurs in the discretization by finite difference of the nonlinear elliptic problem (1.1) modelling a nonlinear diffusion problem.

Remark 3.3. We note that a P_1-finite element discretization of the problem (1.1) does not lead to a monotone diagonal operator; however, the use of an appropriate numerical integration formula leads to a discrete problem belonging to the general framework of this study (i.e. the mapping $\mathcal{A} = A + \phi$ is an M-function).

We now define a fixed point mapping F associated with the decoupled problem (3.7), with components F_i such that

$$\hat{z}_i = F_i(w), \quad \forall i \in \{1, \dots, \alpha\}, \tag{3.10}$$

where $\hat{z}_i$ satisfies $A_{ii}\hat{z}_i + \phi_i(\hat{z}_i) = b_i - \sum_{j \neq i} A_{ij} w_j$, which corresponds to $\mathcal{A}_i(\hat{z}_i; w) = 0$. In the sequel, we will study $\mathcal{A}$-supermappings associated with F. $\mathcal{A}$-supermappings can be defined by means of an algorithm; we note in particular that we can perform one or several iterations of the algorithm (3.6), instead of relaxing exactly the components of each block. If w is such that

$$\mathcal{A}_i(w) = \sum_j A_{ij} w_j + \phi_i(w_i) - b_i \geqslant 0, \tag{3.11}$$

then, we can define an order interval $\langle x_i^0, z_i^0 \rangle_i$ by $x_i^0 = \hat{z}_i = F_i(w)$, $z_i^0 = w_i$. Thus, on the interval $\langle x_i^0, z_i^0 \rangle_i$, $\mathcal{A}_i$ is an M-function satisfying (3.1). Hence, we can introduce the mapping C_i derived from Newton's method which is defined by for all $z_i \in \langle x_i^0, z_i^0 \rangle_i \xrightarrow{C_i} C_i(z_i)$, where $C_i(z_i) \cdot y_i = A_{ii} y_i + \phi_i'(z_i) \cdot y_i$. It follows clearly from assumptions (3.8) and (3.9) that the mapping C_i satisfies (3.2) and (3.3). Moreover, by the convexity of the components of ϕ_i, it follows from (3.8) that $C_i(z_i)$ is an M-matrix for all $z_i \in \langle x_i^0, z_i^0 \rangle_i$, thus (3.4) is satisfied. We consider the mapping $F^{\mathcal{A}}$ from $\mathbb{E}$ into $\mathbb{E}$, with components $F_i^{\mathcal{A}}$ defined by $F_i^{\mathcal{A}}(w) = z_i^k$, $i \in \{1, \dots, \alpha\}$, where for all given value of k, z_i^k is the kth update generated by the algorithm (3.6) starting from $z_i^0 = w_i$.

Proposition 3.4. Let assumptions (3.8) and (3.9) hold. Then, $F^{\mathcal{A}}$ is an $\mathcal{M}$-continuous $\mathcal{A}$-supermapping associated with F.

Proof. If assumptions (3.8) and (3.9) hold and $\mathcal{A}_i(w) \geqslant 0$, then according to the above study, the mapping $\mathcal{A}_i$ is a continuous surjective M-function which satisfies (3.1) on $\langle x_i^0, z_i^0 \rangle_i = \langle F_i(w), w_i \rangle_i$ and C_i satisfies the assumptions (3.2)–(3.4). Thus, it results from proposition 3.2 and definition 2.3 that the mapping $F^{\mathcal{A}}$ is an $\mathcal{A}$-supermapping associated with F. We denote by $F^{\mathcal{B}}$ the $\mathcal{A}$-supermapping associated with F with components $F_i^{\mathcal{B}}(w) = z_i^1$, which is obtained by considering only the first step of the algorithm (3.6). If $\mathcal{A}_i(w) \geqslant 0$, then by proposition 3.2 and definition 2.9, we have $F^{\mathcal{A}} \alpha F^{\mathcal{B}}$. So, $F^{\mathcal{B}}(w) = z_i^1 = w_i - C_i^{-1}(w) \cdot \mathcal{A}_i(w)$, where $C(w)$ is a block-diagonal matrix with blocks $C_i(w) = A_{ii} + \phi_i'(w)$. It follows from (3.9) that the mapping $\mathcal{A}(w)$ given by $\mathcal{A}(w) = Aw + \phi(w) - b$, is continuous. Moreover, we have $(A_{ii} + \phi_i'(w_i))^{-1} \leqslant A_{ii}^{-1}$, since A is an M-matrix and ϕ' is positive (see [22]). It follows that the spectral radius of $C^{-1}(w)$ is lower than or equal to the spectral radius of the inverse of the block-diagonal matrix with blocks A_{ii} which is an M-matrix. Thus the linear application associated with the matrix $C^{-1}(w)$ is Lipschitz continuous and $C^{-1}(w)$ is uniformly Lipschitz continuous. By the continuity of $\mathcal{A}(w)$, $C(w)$ and the uniform Lipschitz continuity of $C^{-1}(w)$, the mapping $F^{\mathcal{B}}$ is continuous. Thus, the mapping $F^{\mathcal{A}}$ is an $\mathcal{M}$-continuous $\mathcal{A}$-supermapping associated with F. □

We combine now the parallel asynchronous scheme of computation with flexible communication presented in section 2 with the Newton like method presented in the previous subsection.

Proposition 3.5. Let the assumptions of proposition 3.4 hold and consider the solution of problem (3.7). Then, asynchronous Newton like iterations with flexible communication starting from an $\mathcal{A}$-supersolution converge to the solution y^* of the problem.

Proof. The result follows from proposition 2.12 combined with proposition 3.4. □

3.3. *Link with multisplitting methods*

We consider now the solution of the following problem:

$$Ax^* + \phi(x^*) = 0, \tag{3.12}$$

where $A \in L(\mathbb{R}^n)$ is an M-matrix, $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous diagonal maximal monotone operator. Let the following regular splittings of matrix A

$$A = M^l - N^l, \quad l = 1, \ldots, m, \tag{3.13}$$

where $(M^l)^{-1} \geqslant 0$ and $N^l \geqslant 0$. Let $F^l : \mathbb{R}^n \to \mathbb{R}^n$, $l = 1, \ldots, m$, be the fixed point mappings associated with problem (3.12) and defined by

$$F^l(x) = y \quad \text{such that} \quad M^l y = N^l x - \phi(x). \tag{3.14}$$

A formal multisplitting associated with problem (3.12) is defined by the collection of fixed point problems (see [1])

$$x^* - F^l(x^*) = 0, \quad l = 1, \ldots, m. \tag{3.15}$$

Let now $\mathbb{E} = (\mathbb{R}^n)^m$ and consider the following block-decomposition of $\mathbb{E}$

$$\mathbb{E} = \prod_{l=1}^{m} \mathbb{E}_l,$$

where $\mathbb{E}_l = \mathbb{R}^n$. Each subspace $\mathbb{E}_l$ is endowed with the natural (or componentwise) partial ordering associated with the cone $K_l = \mathbb{R}^n_+$ of vectors with nonnegative components in $\mathbb{R}^n$. Let $X \in \mathbb{E}$. We have the following block-decomposition of X

$$X = \{X_1, \ldots, X_l, \ldots, X_m\} \in \prod_{l=1}^{m} \mathbb{E}_l.$$

Definition 3.6. The extended fixed point mapping $T : \mathbb{E} \to \mathbb{E}$ associated with the formal multisplitting is given as follows

$$T(X) = Y, \quad \text{such that} \quad Y_l = F^l(Z_l) \quad \text{with } Z_l = \sum_{k=1}^{m} W_{lk} X_k,\ l = 1, \ldots, m,$$

where W_{lk} are nonnegative diagonal weighting matrices satisfying for all $l \in \{1, \ldots, m\}$

$$\sum_{k=1}^{m} W_{lk} = I,$$

I being is the identity matrix in $L(\mathbb{E}_l)$.

We note that for a particular choice of the weighting matrices W_{lk}, we can obtain the Schwarz method (see [1]). This last point establish a link between the Schwarz method and multisplitting methods. Let the following block-decomposition of the mapping T

$$T(X) = \big\{T_1(X), \ldots, T_l(X), \ldots, T_m(X)\big\} \in \prod_{l=1}^{m} \mathbb{E}_l.$$

The extended fixed point mapping T is associated with the following extended nonlinear problem

$$a\big(X^*\big) = 0, \tag{3.16}$$

where the mapping $a : \mathbb{E} \to \mathbb{E}$ is given by

$$a(X) = A^{\mathrm{e}} X + \phi^{\mathrm{e}}(X),$$

the function $\phi^{\mathrm{e}} : \mathbb{E} \to \mathbb{E}$ being the extended monotone perturbation operator and for all $l \in \{1, \ldots, m\}$

$$A_l^{\mathrm{e}} X = M^l X_l - N^l \sum_{k=1}^{m} W_{lk} X_k. \tag{3.17}$$

In the sequel $a_l(X_1, \ldots, X_{l-1}, Y_l, X_{l+1}, \ldots, X_m)$ will also be denoted by $a_l(Y_l\,;\ X)$.

Proposition 3.7. Let the above assumptions hold. The mapping a is a continuous surjective M-function.

Proof. We recall that an M-function is off-diagonally antitone and inverse isotone (see [22,23]). It follows from (3.17) and the definition of matrices W_{lk} that the off-diagonal entries of the matrix A^{e} are nonpositive since A is an M-matrix. Let $U = \{u, \ldots, u\}$ be the vector of $\mathbb{E}$ with block-components equal to the eigenvector u associated with the greatest eigenvalue of the Jacobi matrix $D^{-1}(D-A)$, where D denotes the diagonal part of matrix A and consider the Jacobi matrix J derived from the extended system (3.16). For all $l \in \{1, \ldots, m\}$, we have

$$\begin{aligned} J_l u &= \left(D^{\mathrm{e}}\right)_l^{-1} D_l^{\mathrm{e}} u - \left(D^{\mathrm{e}}\right)_l^{-1} M^l u + \left(D^{\mathrm{e}}\right)_l^{-1} N^l \sum_{k=1}^{m} W_{lk} u \\ &= \left(D^{\mathrm{e}}\right)_l^{-1} \left(D_l^{\mathrm{e}} - M^l + N^l\right) u = D^{-1}(D-A)u = \rho\left(D^{-1}(D-A)\right)u. \end{aligned}$$

It follows from [17] that the Jacobi matrix J is contracting relative to a weighted maximum norm and the result follows from [22, proposition 2.4.17]. The continuity and surjectivity of a follows from the continuity and maximal monotonicity of ϕ. $\square$

It follows from proposition 3.7 that we are in the theoretical framework of the study developped in this section. Thus, the monotone convergence of asynchronous iterations with flexible communication can be derived.

It follows from proposition 3.7 that

$$\begin{cases} \text{For all } l \in \{1, \ldots, m\} \text{ and all } X \in \mathbb{E}, \quad \text{the mapping: } Y_l \to a_l(Y_l, X) \\ \text{is a continuous surjective } M\text{-function of } \mathbb{E}_l \text{ onto } \mathbb{E}_l, \end{cases}$$

(see [23, theorem 3.5]). Moreover it follows from proposition 3.7 that

$$\begin{cases} \text{for all } l \in \{1, \ldots, m\} \text{ and } X \in \mathbb{E}, \\ \text{the problem: } a_l(Y_l\,;\ X) = 0 \text{ has a unique solution } Y_l. \end{cases}$$

It follows also from the above assumptions that T is isotone on $\mathbb{E}$ (see [19]).

The reader is refered to [3,28] for studies related to multisplitting methods combined with classical asynchronous iterations without any flexible communication.

4. Computational experiences and results

In this section, we present and analyze computational results for parallel asynchronous and synchronous Newton's methods applied to the discrete solution of the nonlinear diffusion problem (1.2), where Ω is the unit square. The computational experiments were carried out on a shared memory machine IBM 3090 with six vector processors. The parallelization uses macrotasking facilities of parallel Fortran. The vectorisation is performed automatically. The solution of linear systems is obtained via a band Gauss algorithm which preserves the monotonicity. The linearizing matrix being an M-matrix, all its principal submatrices are also M-matrices and all the principal minors are nonzero. Then, it is not necessary to perform row and column permutations since each pivot considered at each step is nonzero. Experimental results for parallel algorithms are given in table 1 for a problem with 25000 discretization points. In particular, table 1 displays the speedup and efficiency of parallel algorithms. We recall that the speedup corresponds to the ratio sequential time over parallel time and the efficiency is the ratio speedup over number of processors. The speedup and efficiency are computed according to an equivalent decomposition in the sequential and parallel cases. Indeed, the sequential time and average rate of convergence vary as a function of the number of subdomains. We report that a limitation of the synchronous parallel algorithms designed for the application studied in this paper, is that one cannot implement all possible decomposition strategies. In particular, one can not implement a synchronous method with α subdomains and α processors. An alternative solution would be to implement a different iterative scheme, such as the Jacobi method; however the convergence rate would be slower. We point out that asynchronous algorithms with flexible communication are faster than synchronous algorithms. For medium granularity, the number of relaxations of synchronous and asynchronous parallel algorithms is close to the number of relaxations of sequential algorithms. For a low granularity, the number of relaxations of synchronous and asynchronous algorithms is greater than the number of relaxations of sequential algorithms. Moreover, we note that the efficiency of synchronous algorithms decreases faster than the efficiency of asynchronous algorithms when the granularity decreases.

Table 1
Experimental results.

Processors	Subdomains	Mode	Relaxations	Time	Speedup	Efficiency
2	4	sync.	80	30.42	1.81	0.91
2	4	async.	78	27.60	2.0	1.0
3	6	sync.	138	18.96	2.61	0.87
3	6	async.	133	16.64	2.98	0.99
4	8	sync.	224	15.86	3.14	0.78
4	8	async.	209	12.89	3.87	0.97
5	10	sync.	340	13.75	3.93	0.78
5	10	async.	343	12.07	4.47	0.89
6	12	sync.	492	14.70	3.88	0.65
6	12	async.	506	11.81	4.83	0.81

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