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# Incomplete factorization-based preconditionings for solving the Helmholtz equation

Mardochée Magolu Monga Made*,†

*Université Libre de Bruxelles, Service des Milieux Continus (CP 194/5), 50, avenue F.D. Roosevelt, B-1050 Brussels, Belgium*

SUMMARY

Preconditioning techniques based on incomplete factorization of matrices are investigated, to solve highly indefinite complex-symmetric linear systems. A novel preconditioning is introduced. The real part of the matrix is made positive definite, or less indefinite, by adding properly defined perturbations to the diagonal entries, while the imaginary part is unaltered. The resulting preconditioning matrix, which is obtained by applying standard methods to the perturbed complex matrix, turns out to perform significantly better than classical incomplete factorization schemes. For realistic values of the GMRES restart parameter, spectacular reduction of iteration counts is observed. A theoretical spectral analysis is provided, in which the spectrum of the preconditioner applied to indefinite matrix is related to the spectrum of the same preconditioner applied to a Stieltjes matrix extracted from the indefinite matrix. Results of numerical experiments are reported, which display the efficiency of the new preconditioning. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: Helmholtz equations; finite elements; large sparse linear systems; incomplete factorizations; spectral bounds; Krylov subspace methods

## 1. INTRODUCTION

We are concerned with solving large linear systems that arise from finite element discretization of the Helmholtz equation, which governs the propagation of waves in structural acoustic analysis, see, e.g. References [1, 2]. As the size of the problem is in general fairly large, direct methods become excessively expensive, with respect to both time and memory. Hence the need for efficient iterative methods. Due to their relative robustness, Krylov subspace methods are intensively used [3–6]. As is well known, the convergence rate depends on the spectral properties of the (preconditioned) matrix [7–11].

*Correspondence to: Mardochée Magolu Monga Made, Faculté des Sciences Appliquées, Université Libre de Bruxelles, Service des Milieux Continus (CP 194/5), 50 Ave. F.D. Roosevelt, B-1050, Bruxelles, Belgium
†E-mail: magolu@ulb.ac.be

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In the case of interest to us, matrices involved are (complex-symmetric) indefinite and hard to solve as the frequency, or equivalently, the acoustic wave number increases. In such a case the eigenvalues tend to be scattered between both the right and the left half-plane [12, 13, 2]. In order to avoid the convergence of the Krylov subspace to dramatically deteriorate, it is of paramount importance to use a powerful preconditioning technique.

A great deal of effort has been invested in elaborating preconditionings for strongly indefinite problems. Nevertheless, their development has not yet reached the level of self-adjoint positive (semi)-definite problems, for which, well-established powerful preconditionings exist, with rigorous theoretical foundation. Indeed, optimal or quasi-optimal multigrid (multilevel) and domain decomposition methods, which work well for positive (semi)-definite systems, run into serious difficulties when applied to highly indefinite systems. For instance, if the coarse mesh size is not 'small enough', disappointing performances have been observed (see e.g. Reference [14]), to say nothing of the hard tasks of defining efficient restriction and/or prolongation operators, especially in the case of unstructured meshes.

In this work, we aim at adding another stone to the edifice. We introduce a new incomplete factorization-based preconditioning technique, which consists in adding small perturbations to the diagonal entries of the real part of the matrix. In doing so, the real part is made positive definite, or less indefinite. We then apply standard methods to the resulting (perturbed) complex matrix. Compared to classical incomplete factorization schemes, the new preconditioning matrix leads, in general, to a significant decrease of the number of iterations. Similar methods have been investigated in References [15, 16] for symmetric positive-definite linear systems, and in Reference [17] for unsymmetric linear systems. In the present paper, bounds for the spectrum of the preconditioned systems are provided, that allow to better understand the convergence of preconditioned Krylov subspace-type methods.

We organize the paper as follows. General terminology and notation we shall use are gathered in Section 2. In Section 3, we give a brief presentation of the generalized minimal residual (GMRES) method, which is used in our investigations. We also include a short review of generalized incomplete factorization algorithms. Section 4 is devoted to the description of the new preconditioning as well as to the analysis of the spectral properties of preconditioned systems. Results of numerical experiments are reported and commented on in Section 5.

## 2. GENERAL TERMINOLOGY AND NOTATION

### 2.1. Order relation

The *order relation* between real matrices and vectors is the usual componentwise order: if $A=(a_{ij})$ and $B=(b_{ij})$, then $A \leqslant B$ $(A<B)$ if $a_{ij} \leqslant b_{ij}$ $(a_{ij}<b_{ij})$ for all $i,j$; $A$ is called non-negative (positive) if $A \geqslant 0$ $(A>0)$. $A \neq 0$ means that at least one entry of $A$ is non-zero.

### 2.2. Miscellaneous symbols

$z \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n \times n}$ denote a complex vector with $n$ components and a complex square matrix of order $n$, respectively.

| | |
|---|---|
| i | the unit imaginary number, i.e. $\mathrm{i}=\sqrt{-1}$ |
| $\bar{A}$ | the conjugate of $A$ |

 

| | |
|---|---|
| $\|c\|$ | the module of the complex number $c$, i.e. $\sqrt{\bar{c}c}$ |
| $A^{\mathrm{t}}$ | the transpose of $A$ |
| $A^{\mathrm{H}}$ | the transpose conjugate of $A$, i.e., $\bar{A}^{\mathrm{t}}$ |
| $\mathrm{Re}(A)$ | the real part of $A$ |
| $\mathrm{Im}(A)$ | the imaginary part of $A$ |
| $\sigma(A)$ | the spectrum of $A$ |
| $\lambda(A)$ | any eigenvalue of $A$ |
| $\lambda_{\min}(A)$ | the eigenvalue of $A$ with smallest module |
| $\lambda_{\max}(A)$ | the eigenvalue of $A$ with largest module |
| $\|z\|$ | the two-norm of $z$, i.e., $\sqrt{z^{\mathrm{H}}z}$ |
| $\mathrm{diag}(A)$ | the pointwise diagonal matrix whose diagonal entries coincide with those of $A$ |
| $\mathrm{offdiag}(A)$ | $\mathrm{A}-\mathrm{diag}(A)$ |
| **e** | the vector with all components equal to 1 |

### 2.3. Stieltjes matrices

A real square matrix $A$ is called a *Stieltjes matrix* (or equivalently, a *symmetric M-matrix*) if it is symmetric positive definite and none of its offdiagonal entries is positive [18].

### 2.4. Exact LU-factorization

By the *standard LU-factorization* of a non-singular Stieltjes matrix $S$ we understand the factorization $S = L_s P_s^{-1} L_s^t$ where $P_s$ is a diagonal matrix and $L_s$ is lower triangular such that $\mathrm{diag}(L_s) = P_s$. The *normalized form* is defined as $S = \tilde{L}_s P_s \tilde{L}_s^{\mathrm{t}}$ where $\tilde{L}_s = L_s P_s^{-1}$. There holds $\mathrm{diag}(\tilde{L}_s) = I$.

## 3. BACKGROUND

As model problem, we consider solving large-scale linear system

$$Az = b \quad \text{with } A \in \mathbb{C}^{n\times n}, \quad b, z \in \mathbb{C}^n \tag{1}$$

which stems from finite element discretization of the Helmholtz boundary value problem

$$\begin{aligned} -\Delta u - k^2 u &= f && \text{in } \Omega \\ u &= g_0 && \text{on } \Gamma_0 \subset \partial\Omega \\ \frac{\partial u}{\partial n} + \alpha u &= g_\alpha && \text{on } \Gamma_\alpha \subset \partial\Omega \end{aligned} \tag{2}$$

where $k$ stands for the wave number, $\Omega$ and $\partial\Omega$ a bounded domain in $\mathbb{C}^d$ ($d = 2$ or 3) and its boundary, $n$ the normal external to $\Omega$ along $\partial\Omega$, while $f$, $g_0$ $g_\alpha$, and $\alpha$ denote given complex functions. We recall from Reference [2] that $k = \omega/c = 2\pi/\lambda$, where $\omega$, $c$, and $\lambda$ denote the angular frequency, the speed of the sound in the fluid medium, and the wavelength, respectively. For realistic values of the wave number $k$, $A$ is highly indefinite, which makes (1) hard to solve. A further difficulty is that the exact solution to (2) is oscillatory. With Galerkin approximation,

 

a fine mesh should be used, otherwise the solution to (1) would differ substantially from the solution to (2). Special discretization techniques have been worked out to cope with this inconvenience which is responsible for spurious noise in classical numerical solution methods, see, e.g. References [19–21, 2] for a survey.

### 3.1. The generalized minimal residual (GMRES) method

An approximate solution (to (1))

$$\mathbf{z}^{(m)} \in \mathbf{z}^{(0)} + \underbrace{\operatorname{span}\{r^{(0)}, Ar^{(0)}, A^2r^{(0)}, \ldots, A^{m-1}r^{(0)}\}}_{\mathscr{K}_{\mathbf{m}}(\mathbf{A},\mathbf{r}^{(0)})} \tag{3}$$

where $r^{(k)} = b - Az^{(k)}$ denotes the $k$th residual vector, while $\mathscr{K}_m(A, r^{(0)})$ stands for the $m$th Krylov subspace associated to $A$ and $r^{(0)}$, is sought so as to achieve [22]

$$\|\mathbf{b} - \mathbf{A}\mathbf{z}^{(m)}\| = \min_{\mathbf{z}^{(\mathbf{k})} \in \mathbf{z}^{(0)} + \mathscr{K}_{\mathbf{m}}(\mathbf{A},\mathbf{r}^{(0)})} \|\mathbf{b} - \mathbf{A}\mathbf{z}^{(\mathbf{k})}\| \tag{4}$$

which is equivalent to imposing

$$\mathbf{r}^{(\mathbf{m})} = \mathbf{b} - \mathbf{A}\mathbf{z}^{(\mathbf{m})} \perp \mathbf{A} \cdot \mathscr{K}_{\mathbf{m}}(\mathbf{A}, \mathbf{r}^{(0)}) \tag{5}$$

In this way, $z^{(m)}$ is the best approximate solution in $z^{(0)} + \mathscr{K}_m(A, r^{(0)})$. GMRES is robust but rather expensive; it requires storing the whole sequence of vectors to be orthogonalized. In practice, one resorts to *restarted* or *truncated* versions where the number of vectors is limited. The trouble with above-mentioned variants is that they could stagnate.

The convergence rate of GMRES is strongly influenced by the spectrum of the system matrix [22, 23]. It is therefore of paramount importance to use preconditioning which should transform the original system into another equivalent one

$$\mathbf{A}\mathbf{B}^{-1}\tilde{\mathbf{z}} = \mathbf{b} \quad \text{with } \mathbf{z} = \mathbf{B}^{-1}\tilde{\mathbf{z}} \tag{6}$$

which has nicer spectral properties. $B$ is the preconditioning matrix. In our experiments, we shall make use of GMRES($m$), the restarted GMRES, with right preconditioning, whose algorithm is given in Figure 1. Observe that with right preconditioning, GMRES minimizes the residual as opposed to the preconditioned residual in the case of left preconditioning ($B^{-1}Az = B^{-1}b$). When $B$ is ill-conditioned and/or strongly indefinite, minimizing the preconditioned residual could give rise to unpleasant surprises [11]. There is no general rule to choose the restart parameter $m$ ($m \ll n$). This is mostly a matter of experience.

### 3.2. Generalized incomplete factorizations

To solve the linear system (1) with a direct method, one factorizes $A$ as $A = \tilde{L}\tilde{L}^{\mathrm{t}}$, $\tilde{L}$ being lower triangular, and solves successively two triangular systems. Even if $A$ is sparse, $\tilde{L}$ is in general dense, which makes direct methods both memory and time consuming for $n$ fairly large, as in real-life scientific or industrial problems. A common remedy consists in ignoring certain fill-in entries, which yields an incomplete factorization preconditioning matrix $B = LL^{\mathrm{t}}$ [24]. There are two basic strategies for accepting or discarding fill-in:

 

1. $\mathbf{r^{(0)} = b - Az^{(0)}}$ , $\beta = \|\mathbf{r^{(0)}}\|$ , $\mathbf{v^{(1)}} = \frac{\mathbf{r^{(0)}}}{\beta}$
2. For $\mathbf{j = 1, 2, \ldots, m}$ Do :
3. $\quad \mathbf{w := AB^{-1}v^{(j)}}$
4. $\quad$ For $\mathbf{i = 1, 2, \ldots, j}$ Do :
5. $\quad\quad \mathbf{h_{i,j} := (w, v^{(i)})}$
6. $\quad\quad \mathbf{w := w - h_{i,j}v^{(i)}}$
7. $\quad$ EndDo
8. $\quad \mathbf{h_{j+1,j}} = \|\mathbf{w}\|$. If $\mathbf{h_{j+1,j}}$ small enough set $\mathbf{m := j}$ and GO TO 11
9. $\quad \mathbf{v^{(j+1)}} = \frac{\mathbf{w}}{\mathbf{h_{j+1,j}}}$
10. EndDo
11. $\quad$ Define $\mathbf{V_m := [v^{(1)}, v^{(2)}, \ldots, v^{(m)}]}$
$\quad \mathbf{\overline{H}_m = \{h_{i,j}\}_{1 \le i \le j+1; 1 \le j \le m}}$
12. $\mathbf{y^{(m)} = \mathrm{argmin}_y \|\beta e^{(1)} - \overline{H}_m y\|}$
$\mathbf{z^{(m)} = z^{(0)} + B^{-1}V_m y^{(m)}}$
13. If satisfied Stop, else set $\mathbf{z^{(0)} := z^{(m)}}$ and GO TO 1

Figure 1. GMRES($m$) with right preconditioning.

1. By level of fill-in (or by position). The level $\mathrm{lev}(l_{k,i})$ of the coefficient $l_{k,i}$ of $L$ is defined by Saad [11] (see Figure 2 for notation),

$$\textit{Initialization}: \ \mathrm{lev}(l_{k,i}) := \begin{cases} 0 & \text{if } l_{k,i} \neq 0 \text{ or } k = i \\ \infty & \text{otherwise} \end{cases}$$

$$\textit{Factorization}: \ \mathrm{lev}(l_{k,i}) := \min\{\mathrm{lev}(l_{k,i}),\ \mathrm{lev}(l_{i,j}) + \mathrm{lev}(l_{k,j}) + 1\}.$$

   The set $\mathscr{D}$ of fill-in entries to be discarded is taken as $\mathscr{D} = \{(k,i) \mid \mathrm{lev}(l_{k,i}) > \ell\}$, where the integer $\ell$ denotes a user specified maximal fill-in level.
2. By (numerical) value. Fill-in is ignored if it is 'too small' with respect to some prescribed tolerance.

Hybrid (or dual) approaches that combine ingredients from both structural and numerical strategies have been proposed in, e.g. Reference [11]. The choice of both $\ell$ and the drop tolerance depends, among other things, on the problem at hand and the workspace available. There is no panacea. Several variants of the basic incomplete factorization [24] have been designed, ranging from modified methods in which the discarded fill-in entries are added to the diagonal, to more sophisticated multilevel versions that use multigrid like (re)numbering strategies (see, e.g. Reference [7]). We give in Figure 2 the generalized relaxed incomplete factorization algorithm which encompasses most of standard schemes, through the relaxation parameters $\rho_j$.

 

Compute $L$ ( $B = LP^{-1}L^t$ with $P = diag(L)$ )

Initialization phase

$l_{i,j} = a_{i,j}$ , $i = 1, 2, \cdots, n$ , $j = 1, 2, \cdots, i$

Incomplete factorization process

do $j = 1, 2, \cdots, n-1$

compute parameter $\rho_j$

do $i = j+1, j+2, \cdots, n$

$l_{i,i} = l_{i,i} - \frac{l_{i,j}^2}{l_{j,j}}$

do $k = i+1, i+2, \cdots, n$

if $(i,k) \notin \mathcal{D}$ $l_{k,i} = l_{k,i} - \frac{l_{i,j}\, l_{k,j}}{l_{j,j}}$

otherwise

$$\begin{cases} l_{i,i} & = & l_{i,i} - \rho_j \frac{l_{i,j}\, l_{k,j}}{l_{j,j}} \\ l_{k,k} & = & l_{k,k} - \rho_j \frac{l_{i,j}\, l_{k,j}}{l_{j,j}} \end{cases}$$

end do

end do

end do

Figure 2. Generalized relaxed incomplete factorization.

- With $\rho_j = \rho$, $-\infty \leqslant \rho \leqslant 1$, one gets the relaxed method [25], which includes the basic incomplete LU-factorization ($\rho = 0$), as well as the modified variant without diagonal perturbations ($\rho = 1$) for which there holds $B\mathbf{e} = A\mathbf{e}$.
- Variable $\rho_j$ includes dynamically relaxed methods [26].

The algorithm may be safely applied whenever $A$ is (reducible to) a Stieltjes matrix [24, 27, 28]. For such matrices, spectral properties are analysed in References [25, 29–31, 26], and references gathered therein. The application of block variants of incomplete factorization preconditionings to the Helmholtz equation is discussed in References [32, 33].

## 4. PRECONDITIONING BY PERTURBING THE REAL PART

### 4.1. Description

In order to illustrate and motivate our approach, we assume for simplicity that, in (2), $\Omega = (0,1) \times (0,1)$ ($d = 2$), $\Gamma_\alpha = \Gamma_1 \cup \Gamma_2$ and that

$$\alpha = \begin{cases} 0 & \text{in } \Gamma_1 \\ \mathrm{i}k & \text{in } \Gamma_2 \end{cases} \quad \text{and} \quad g_\alpha = 0$$

 

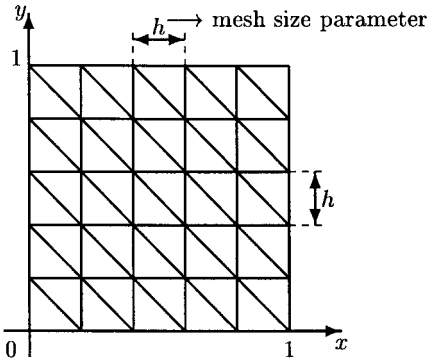


Figure 3. The finite element mesh for the use of linear basis functions.

Let $H^1(\Omega)$ denote some appropriate Sobolev space [19]. The variational formulation of (2) reads: find $u \in H^1(\Omega)$ such that, $\forall v \in H^1(\Omega)$,

$$\iint_\Omega \nabla u \nabla v \, \mathrm{d}\Omega - k^2 \iint_\Omega uv \, \mathrm{d}\Omega + \mathrm{i}k \int_{\Gamma_2} uv \, \mathrm{d}\Gamma_2 = \iint_\Omega fv \, \mathrm{d}\Omega \tag{7}$$

For simplicity, we assume that the Galerkin discretization with finite elements is performed by using piecewise linear basis functions over isosceles right triangles as illustrated in Figure 3; local nodes are ordered counterclockwise starting from the node at the right angle. In doing so, we use the lexicographic ordering in the $x-y$ plane to define the global node ordering. The resulting matrix may be written as

$$\mathbf{A} = \mathbf{K} - \mathbf{k}^2\mathbf{M} + \mathrm{i}\,\mathbf{kR} \tag{8}$$

where $K$ denotes the stiffness matrix, $M$ stands for the mass matrix, while $R$ results from the contribution of Robin boundary conditions. $A$ is complex-symmetric, non-Hermitian. For realistic values of the wave number $k$ and the mesh size parameter $h$, $\mathrm{Re}(A) = K - k^2 M$ is highly indefinite. For Krylov subspace-type methods, ideally, there should hold

1. either $\mathrm{Re}(\sigma(AB^{-1})) > 0$ or $\mathrm{Re}(\sigma(AB^{-1})) < 0$;
2. Clustering of $\lambda_i(AB^{-1})$ around some point(s) in the complex plane.

This suggests to extract from $A$ a matrix whose real part is definite, or less indefinite than $\mathrm{Re}(A)$. We need the following general results.

*Lemma 4.1* (*Berman and Plemmons* [18]). Let $S \neq 0$ be a real symmetric matrix and $v$ denote a positive real vector such that $\mathrm{offdiag}(S) \leqslant 0$ and $Sv \geqslant 0$, $Sv \neq 0$. Then $S$ is a non-singular Stieltjes matrix.

*Proposition 4.1.* Let $A$ be a complex-symmetric indefinite matrix such that $\mathrm{Re}(A) \leqslant 0$, $\gamma$ stand for a given parameter, and $\tilde{A} = (\tilde{a}_{ij})$ be the matrix defined by

$$\mathrm{Re}(\tilde{a}_{ij}) = \begin{cases} \mathrm{Re}(a_{ij}) & \text{if } i \neq j \\ \mathrm{Re}(a_{i,i}) - \gamma \min\{0, \mathrm{Re}((A\mathbf{e})_i)\} & \text{otherwise} \end{cases} \tag{9}$$

If $\gamma \geqslant 1$ then $\mathrm{Re}(\tilde{A})$ is a non-singular Stieltjes matrix.

 

*Proof:* One has obviously that $\mathrm{offdiag}(\mathrm{Re}(\tilde{A})) \leqslant 0$. One easily checks from (9) that $\mathrm{Re}((\tilde{A}\mathbf{e})_i) = \mathrm{Re}((A\mathbf{e})_i) - \gamma \min\{0, \mathrm{Re}((A\mathbf{e})_i)\} \geqslant 0$, which, together with Lemma 4.1, concludes the proof. □

*Remark.* Given that $\tilde{A}$ is a perturbation of the matrix $A$, the best would be to use the smallest possible value of, possibly variable, $\gamma$ $(0 \leqslant \gamma_i)$ that makes $\mathrm{Re}(\tilde{A})$ positive definite, or even slightly indefinite. Such an approach is beyond the goal of this work. In Proposition 4.1, there is no requirement on $\mathrm{Im}(\tilde{A})$. Since $A$ is complex, we advocate to take

$$\mathrm{Im}(\tilde{A}) = \mathrm{Im}(A) \tag{10}$$

in order to avoid approximating a complex matrix with a real one, as would be the case if $\mathrm{Im}(\tilde{A})$ is set to zero. In the latter case, $\tilde{A} = \mathrm{Re}(\tilde{A})$ is a non-singular Stieltjes matrix, whence it follows that the generalized incomplete factorization algorithm could be carried out with no trouble. Numerical experiments reveal that performances are better with (10). This will be made more clear in the next subsection.

### 4.2. Spectral analysis

The field of values associated to $A$ and $B^{-1}$ is defined by

$$\mathscr{F}(A, B^{-1}) = \left\{ \frac{(z, Az)}{(z, Bz)} : \ z \in \mathbb{C}^n \backslash \{0\} \right\} \tag{11}$$

We recall from References [34, 35] that $\sigma(AB^{-1}) = \sigma(B^{-1}A) \subset \mathscr{F}(A, B^{-1})$.

*Lemma 4.2.* For any complex-symmetric matrix $C$ one has that

$$\begin{aligned} \lambda_{\min}(\mathrm{Re}(C)) \leqslant \mathrm{Re}(\lambda(C)) \leqslant \lambda_{\max}(\mathrm{Re}(C)) \\ \lambda_{\min}(\mathrm{Im}(C)) \leqslant \mathrm{Im}(\lambda(C)) \leqslant \lambda_{\max}(\mathrm{Im}(C)) \end{aligned} \tag{12}$$

*Proof:* This readily follows from the well-known Bendixson's Theorem (see, e.g. Reference [34] or Reference [11, Theorem 1.20]) which states that, for any complex matrix $Z$ one has that $\lambda_{\min}(H) \leqslant \mathrm{Re}(\lambda(Z)) \leqslant \lambda_{\max}(H)$ and $\lambda_{\min}(G) \leqslant \mathrm{Im}(\lambda(Z)) \leqslant \lambda_{\max}(G)$, where $H$ and $G$ denote the Hermitian and the skew-Hermitian parts of $Z$, respectively. In other words, $H = \frac{1}{2}(Z + Z^{\mathrm{H}})$ and $G = \frac{1}{2\mathrm{i}}(Z - Z^{\mathrm{H}})$. □

*Lemma 4.3.* Let $A$ be a complex-symmetric indefinite matrix with $\mathrm{Re}(A) \leqslant 0$, $\tilde{A}$ be defined as in Proposition 4.1 with $\gamma \geqslant 1$, while $Q$ stand for the diagonal matrix whose diagonal entries $q_{ii}$ are defined by

$$q_{ii} = -\gamma \min\{0, \mathrm{Re}((A\mathbf{e})_i)\} \tag{13}$$

Set $A_0 = \mathrm{Re}(\tilde{A})$ and let $B_0$ denote the preconditioning matrix obtained by applying the generalized incomplete factorization algorithm (Figure 2) to $A_0$. Then, for any non-singular (complex-) symmetric matrix $B$ and for any $z \in \mathbb{C}^n \backslash \{0\}$ one has that

$$\frac{(z, Az)}{(z, Bz)} = \frac{(z, A_0 z)}{(z, B_0 z)} \left(1 - \frac{(z, Qz)}{(z, A_0 z)}\right) \left(1 + \mathrm{i}\frac{(z, \mathrm{Im}(A)z)}{(z, \mathrm{Re}(A)z)}\right) \frac{(z, B_0 z)}{(z, Bz)} \tag{14}$$

 

*Proof:* Obviously, there holds $A = A_0 - Q + \mathrm{i}\,\mathrm{Im}(A)$, whence it follows that,

$$\frac{(z,Az)}{(z,Bz)} = \frac{(z,A_0z) - (z,Qz) + \mathrm{i}(z,\mathrm{Im}(A)z)}{(z,B_0z)} \frac{(z,B_0z)}{(z,Bz)}$$
$$= \frac{(z,A_0z)}{(z,B_0z)} \left(1 - \frac{(z,Qz)}{(z,A_0z)}\right) \left(1 + \mathrm{i}\frac{(z,\mathrm{Im}(A)z)}{(z,\mathrm{Re}(A)z)}\right) \frac{(z,B_0z)}{(z,Bz)} \qquad \square$$

*Proposition 4.2.*

$$|\lambda(AB^{-1})| \leqslant \mu\beta_B\sqrt{1+\eta^2}\,\lambda_{\max}(A_0B_0^{-1}) \tag{15}$$

where

$$\mu = \max_{z\in\mathbb{C}^n\setminus\{0\}} \left|1 - \frac{(z,Qz)}{(z,A_0z)}\right|, \quad \eta = \max_{z\in\mathbb{C}^n\setminus\{0\}} \left|\frac{(z,\mathrm{Im}(A)z)}{(z,\mathrm{Re}(A)z)}\right|$$
$$\beta_B = \max_{z\in\mathbb{C}^n\setminus\{0\}} \left|\frac{(z,B_0z)}{(z,Bz)}\right|$$

Both above results display how the eigenvalues of $AB^{-1}$ are connected to those, positive, of $A_0B_0^{-1}$. Quantities $\mu$ and $\eta$ $(0 \leqslant \eta < 1)$ depend directly on the coefficients of the matrix $A$; $\beta_B$ depends on the preconditioning matrix $(B)$ involved. Equation (15) entails that $\sigma(AB^{-1})$ is contained in a circle centered at the origin whose radius is a multiplicative factor of $\lambda_{\max}(A_0B_0^{-1})$. Spectral bounds developed in References [7, 29–31, 26] (and references cited therein) apply to $\sigma(A_0B_0^{-1})$. $\beta_B$ is in general small when $B$ is constructed from a matrix closely related to $A_0$. With $B = B_0$ one has that $\beta_B = 1$. If $B = \tilde{B}$ is computed from $\tilde{A} = A_0 + \mathrm{i}\,\mathrm{Im}(A)$ then, given that $\mathrm{Im}(A)$ is positive semidefinite, and in light of the analysis performed in Reference [36], very likely $\beta_{\tilde{B}}$ might not be larger than 1. Therefore, the eigenvalues of $A\tilde{B}^{-1}$ are very likely more clustered than those of $AB_0^{-1}$ (a numerical illustration is provided by Figures 6 and 7 in Section 5). Observe that whatever the preconditioning used, there is risk of clustering of eigenvalues around the origin. In particular, when one increases the level of accepted fill-in, the eigenvalues of the preconditioned system are pushed towards 1, those with negative real part also tend to 1, by crossing the origin, which could slow down the convergence [37, pp. 198–199].

We know from observations made in Reference [4] that the convergence behaviours for complex and real (no complex Robin-like boundary condition) Helmholtz problems are not much different. An issue that is revealed by (14)–(15), that the distribution of eigenvalues is strongly influenced by the quantities $1 - (z,Qz)/(z,A_0z)$, which in turn, depend only on $\mathrm{Re}(A)$. We have in particular the following specific bounds.

*Proposition 4.3.* If $B = B_0$ then

$$\begin{aligned} \min_{\lambda\in\sigma(A_0B_0^{-1})} \min_{z\in\mathbb{C}^n\setminus\{0\}} \lambda\mu_z &\leqslant \mathrm{Re}(\lambda(AB_0^{-1})) \leqslant \lambda_{\max}(A_0B_0^{-1}) \\ \min_{\lambda\in\sigma(A_0B_0^{-1})} \min_{z\in\mathbb{C}^n\setminus\{0\}} \lambda\mu_z\eta_z &\leqslant \mathrm{Im}(\lambda(AB_0^{-1})) \leqslant \lambda_{\max}(A_0B_0^{-1}) \max_{z\in\mathbb{C}^n\setminus\{0\}} \mu_z\eta_z \end{aligned} \tag{16}$$

 

with

$$\eta_z = \frac{(z, \mathrm{Im}(A)z)}{(z, \mathrm{Re}(A)z)} \quad \text{and} \quad \mu_z = 1 - \frac{(z, Qz)}{(z, A_0 z)}$$

*Proof:* For any matrices $M$ and $C$ such that $C$ is (complex-)symmetric non-singular, one has that $\sigma(MC^{-1}) = \sigma(L^{-1}ML^{-\mathrm{t}})$, where $L$ is a non-singular lower triangular matrix such that $C = LL^{\mathrm{t}}$ [35]. Obviously, $L^{-1}ML^{-\mathrm{t}}$ is symmetric whenever $M$ is symmetric. Inequalities (16) then readily follow by application of Lemmas 4.2 and 4.3, through straightforward algebraic calculations. □

As long as the real part of the spectrum is concerned, the following more specific bounds hold:

$$\zeta_{\min} \leqslant \mathrm{Re}(\lambda(AB_0^{-1})) \leqslant \lambda_{\max}(A_0 B_0^{-1}) \tag{17}$$

where

$$\zeta_{\min} = \min\left\{ \left(1 - \frac{q_{\max}}{\lambda_{\min}(A_0)}\right) \lambda_{\min}(A_0 B_0^{-1}),\ \left(1 - \frac{q_{\max}}{\lambda_{\min}(A_0)}\right) \lambda_{\max}(A_0 B_0^{-1}) \right\}$$

with $q_{\max} = \max_i q_{ii}$.Moreover, if the generalized eigenvalue problems $A_0 z = \lambda B_0 z$ and $Az = \lambda B_0 z$ have approximately the same eigenvectors then, by (14), $\mathrm{Re}(\sigma(AB_0^{-1}))$ will reproduce essentially the distribution of the rightmost part of $\sigma(A_0 B_0^{-1})$. When it comes to the imaginary part of the spectrum, the situation is less clear. $\mathrm{Im}(A)$ is symmetric positive semidefinite, and it has only a few relatively small non-zero entries. $\mathrm{Re}(A)$ is symmetric and in general (strongly) indefinite. $\lambda_{\max}(A_0 B_0^{-1})\mu_z \eta_z$ might be large; in other words, preconditioning might amplify the imaginary part of the spectrum, in particular in the case of modified variants where $\lambda_{\max}(A_0 B_0^{-1})$ could be rather large. This should not necessarily slow down the convergence which mostly depends on the distribution of the eigenvalues. We conclude this subsection by noting that, since $A$ is complex-symmetric, one has obviously that $A_0 = H(A) + Q$ and $\tilde{A} = A + Q$, where $H(A)$ stands for the Hermitian part of $A$. In other words, $A_0$ is a diagonal perturbation of the Hermitian part of the system matrix, while with $\tilde{A}$, the same diagonal perturbation is added to the whole matrix.

## 5. NUMERICAL RESULTS

The preconditionings (to the complex-symmetric matrix $A$) include :

1. **IC**: the standard incomplete Cholesky applied to $A$.
2. **MIC**: the standard modified incomplete Cholesky applied to $A$.
3. **IC**$^\circ$: IC applied to $A_0 = \mathrm{Re}(A) + Q$ $(\gamma = 1)$.
4. $\widetilde{\mathbf{IC}}$ : IC applied to $\tilde{A} = A_0 + \mathrm{i}\,\mathrm{Im}(A)$.
5. MIC$^\circ$: MIC applied to $A_0 = \mathrm{Re}(A) + Q$ $(\gamma = 1)$.
6. $\widetilde{\mathbf{MIC}}$: MIC applied to $\tilde{A} = A_0 + \mathrm{i}\,\mathrm{Im}(A)$.

In order to assess our theoretical findings, we also consider the (positive) eigenvalues obtained by preconditioning the real Stieltjes matrix $A_0$ by IC and MIC, which we denote by **IC**($\mathbf{A}_0$) and **MIC**($\mathbf{A}_0$), respectively. For both illustration purpose and simplicity, we first simulate a complex-symmetric system as described below. This is mainly intended to easily illustrate our theoretical results.

 

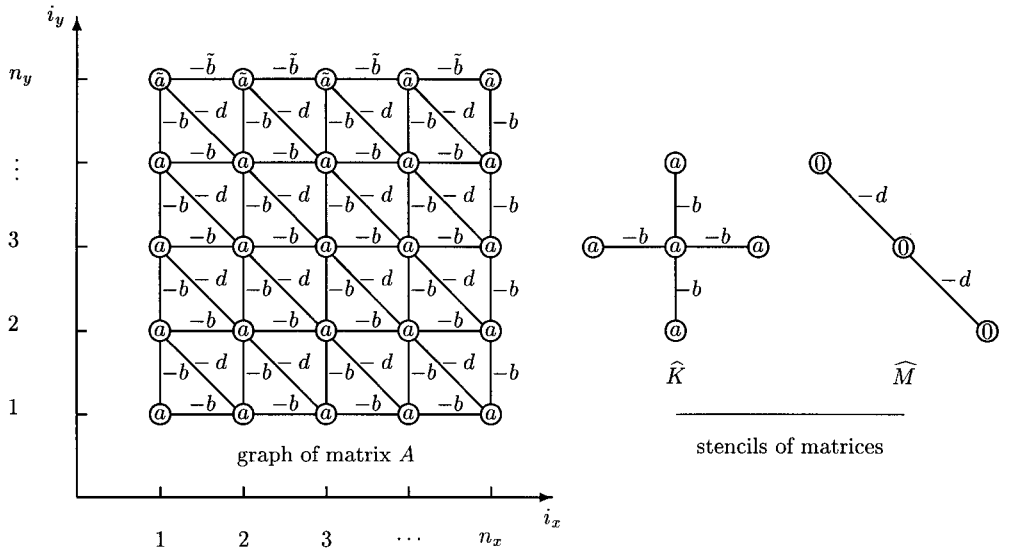


Figure 4. Problem 1. Graph of matrix $A$. The $i$th node is $i = i(i_x, i_y) = (i_y - 1)n_x + i_x$, $1 \leqslant i_x \leqslant n_x$, $1 \leqslant i_y \leqslant n_y$. $a = 4 - k^2h^2/2$, $\tilde{a} = a + \mathrm{i}2kh/3$, $b = 1 + k^2h^2/12$, $\tilde{b} = b - \mathrm{i}kh/6$, $d = k^2h^2/12$; $h = 1/(n_x + 1) = 1/(n_y + 1)$. Stencils for matrices $\hat{K}$ and $\hat{M}$.

*Problem 1.* We assume that Dirichlet boundary conditions are satisfied along the whole boundary. We next add purely imaginary $\mathcal{O}(kh)$-perturbation to the non-zero entries of the last main block diagonal, tridiagonal submatrix of the global matrix. We give in Figure 4 the matrix coefficients by using graph (stencil) notation [31]: numerical value of each diagonal entry $a_{ii}$ is written inside a circle representing node $i$ while the value of each offdiagonal entry $a_{ij} \neq 0$ is written along the edge $\{i, j\}$. The right-hand side $b$ in (1) is chosen as $b = A\tilde{z}$ where the components of $\tilde{z}$ are the discrete values of the function $u(x, y) = (1 + \mathrm{i})x(1 - x)y(1 - y)\mathrm{e}^{xy}$ on mesh nodes.

Observe that purely imaginary $\mathcal{O}(kh)$-terms typically arise from discretization of complex Robin boundary conditions (see Problem 2).

### 5.1. Analysis of the spectrum's real part

The matrix $\mathrm{Re}(A) = K - k^2M$ may be rewritten as $A = \hat{K} + \hat{M}$, where $\hat{K}$ and $\hat{M}$ are given by their stencils in Figure 4. $\hat{K}$ is a block Toeplitz matrix which has the same structure as the stiffness matrix that arises from discretization of the Laplace operator. Its eigenvalues are given by (see e.g. Reference [9, Section 4.1]),

$$\lambda_{i,j}(\hat{K}) = a - 2b(\cos \mathrm{i}\pi h + \cos j\pi h)$$
$$= 4\left(1 + \frac{k^2h^2}{12}\right)\left(\sin^2 \frac{\mathrm{i}\pi h}{2} + \sin^2 \frac{j\pi h}{2}\right) - \frac{5k^2h^2}{6}, \quad 1 \leqslant i, j \leqslant n_x = n_y$$

 

Therefore,

$$\lambda_{\min}(\widehat{K}) = 8\left(1+\frac{k^2h^2}{12}\right)\sin^2\frac{\pi h}{2} - \frac{5k^2h^2}{6} \approx 8\left(1+\frac{k^2h^2}{12}\right)\frac{\pi^2h^2}{4} - \frac{5k^2h^2}{6}$$

$$\lambda_{\max}(\widehat{K}) = 8\left(1+\frac{k^2h^2}{12}\right)\cos^2\frac{\pi h}{2} - \frac{5k^2h^2}{6} \approx 8\left(1+\frac{k^2h^2}{12}\right)\left(1-\frac{\pi^2h^2}{4}\right) - \frac{5k^2h^2}{6}$$

Now, the eigenvalues of $\widehat{M}$ satisfy [35]

$$-2\frac{k^2h^2}{12}\left(1-\frac{\pi^2h^2}{2}\right) \approx -2d\cos\pi h \leqslant \lambda(\widehat{M}) \leqslant 2d\cos\pi h \approx 2\frac{k^2h^2}{12}\left(1-\frac{\pi^2h^2}{2}\right)$$

whence it follows that

$$8\left(1+\frac{k^2h^2}{12}\right)\sin^2\frac{\pi h}{2} - \frac{5k^2h^2}{6} - 2\frac{k^2h^2}{12}\cos\pi h \leqslant \lambda(\mathrm{Re}(A)) \leqslant 8\left(1+\frac{k^2h^2}{12}\right)\cos^2\frac{\pi h}{2} - \frac{5k^2h^2}{6} + 2\frac{k^2h^2}{12}\cos\pi h \tag{18}$$

or, approximately,

$$\left(2\pi^2 - k^2\left(1-\frac{\pi^2h^2}{4}\right)\right)h^2 \lesssim \lambda(\mathrm{Re}(A)) \lesssim 8 - 2\pi^2h^2\left(1+\frac{k^2h^2}{8}\right) \tag{19}$$

It is an easy matter to check that the above lower spectral bound is non-negative whenever

$$k \leqslant \sqrt{\frac{2}{1-\pi^2h^2/4}}\pi \tag{20}$$

This upper bound coincides with the value computed, to the nearest 0.00001, from approximate eigenvalues as computed by the Arnoldi method, which is used in the GMRES process. For instance, with $h^{-1} = 51,101$, one gets respectively, 4.44499 and 4.44342. Note that

$$\lim_{h\to 0}\sqrt{\frac{2}{1-\pi^2h^2/4}}\ \pi = \sqrt{2}\,\pi = 4.44288$$

So, for realistic values of $k$, the matrix $A$ will be in general strongly indefinite. From (13) and Figure 4, one has that

$$q_{i,i} = \begin{cases} \gamma k^2h^2 & \text{if } 1 < i_x < n_x \text{ and } 1 < i_y < n_y \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

By applying the Gerschgorin disk theorem [35] to $A_0 = \mathrm{Re}(A) + Q$, one gets

$$k^2h^2(-1+\gamma) \leqslant \lambda(A_0)$$

whence it follows that $\gamma = 1$ yields approximately the smallest perturbations for which $A_0$ is positive definite. From (19), one would obtain the following estimate:

$$\gamma \approx s(h,k) = 1 - \pi^2\left(\frac{h^2}{4} + \frac{2}{k^2}\right) \tag{22}$$

 

It is of worth noting that

$$\lim_{h\to 0} s(h,k) = 1 - \frac{2\pi^2}{k^2} \quad \text{and} \quad \lim_{k\to\infty} \lim_{h\to 0} s(h,k) = 1$$

From our numerical experiments, performed on both real and complex cases, we have observed 'optimal' performances with $\gamma = 1$, as long as full GMRES is applied. Given that $A_0 = \mathrm{Re}(A) + Q$, it follows from (18) and (21) with $\gamma = 1$ that

$$\lambda_{\min}(A_0) \approx \lambda_{\min}(\mathrm{Re}(A)) + k^2h^2 = 8\left(1 + \frac{k^2h^2}{12}\right)\sin^2\frac{\pi h}{2} + \frac{k^2h^2}{6} - \frac{k^2h^2}{6}\cos\pi h \tag{23}$$

in which case there holds,

$$1 - \frac{q_{\max}}{\lambda_{\min}(A_0)} \approx \tau(h,k) = 1 - \frac{k^2h^2}{8(1 + k^2h^2/12)\sin^2\pi h/2 + k^2h^2/6 - (k^2h^2/6)\cos\pi h} \tag{24}$$

or, approximately,

$$1 - \frac{q_{\max}}{\lambda_{\min}(A_0)} \approx t(h,k) = 1 - \frac{k^2}{2\pi^2}\left(\frac{1}{1 + k^2h^2/8}\right) \tag{25}$$

We are now in position to apply (17), which yields

$$\min\{\tau(h,k)\lambda_{\min}(A_0B_0^{-1}), \tau(h,k)\lambda_{\max}(A_0B_0^{-1})\} \lesssim \mathrm{Re}(\lambda(AB_0^{-1})) \lesssim \lambda_{\max}(A_0B_0^{-1}) \tag{26}$$

Numerical experiments display that the above upper bound is sharp (see spectra on figures to come); the lower one is at most twice the actual values for small problems ($h^{-1} \lesssim 31$), otherwise it becomes somewhat pessimistic. As already mentioned, bounds for eigenvalues of $A_0B_0^{-1}$ have been extensively investigated in the literature, see, e.g. References [7, 31] and references gathered therein. In particular, for standard IC and MIC applied to our model problem with realistic values of $k$ and $h$, there are positive numbers $c_{\min}(k) > 2$, $c_{\max}(k) < 1$ and $c_M(k) \leqslant 1$, independent of $h$, such that

$$\begin{aligned} c_{\min}(k)\,\pi^2h^2 \;\leqslant\; \lambda^{(\mathrm{IC})}\left(A_0B_0^{-1}\right) \;\leqslant\; 1 + c_{\max}(k) < 2 \\ 1 \;\leqslant\; \lambda^{(\mathrm{MIC})}\left(A_0B_0^{-1}\right) \;\leqslant\; c_M(k)\,h^{-1} \end{aligned} \tag{27}$$

The larger $k$, the larger $c_{\max}(k)$ and $c_M(k)$, the smaller $c_{\min}(k)$. Note however that the diagonal dominance of the Stieltjes matrix $A_0$ is independent of $k$, except on 'boundary' nodes with respect

 

to Figure 4. Indeed, with $i = i(i_x, i_y) = (i_y - 1)n_x + i_x$ as in Figure 4, one has that

$$(A_0 e)_i = \begin{cases} 0 & \text{if } 1 < i_x < n_x \text{ and } 1 < i_y < n_y \\ 2 - \dfrac{2k^2h^2}{3} & \text{if } i_x = 1 \text{ and } i_y = 1 \text{ or, } i_x = n_x \text{ and } i_y = n_y \\ 2 - \dfrac{3k^2h^2}{4} & \text{if } i_x = 1 \text{ and } i_y = n_y \text{ or, } i_x = n_x \text{ and } i_y = 1 \\ 1 - \dfrac{5k^2h^2}{6} & \text{otherwise} \end{cases} \tag{28}$$

Now, it is well known and obvious that the stronger the diagonal dominance, the faster the convergence. For fixed and small enough $h$, $c_{\min}(k)$, $c_{\max}(k)$ and $c_M(k)$ will very likely weakly depend upon $k$. This does not imply that the eigenvalue distribution of $AB_0^{-1}$ will weakly depend upon $k$ too, because of the influence of $1 - q_{\max}/\lambda_{\min}(A_0)$ which strongly depends on $k$, see (24) or (25). Note also that $t(h,k)$ is negative whenever

$$k > \sqrt{\frac{2}{1 - \pi^2 h^2/4}}\pi \tag{29}$$

For two-dimensional problems, one usually requires that $h$ and $k$ satisfy $h \leqslant \pi/3k$ [19]; in which case $t(h,k)$ will be negative as soon as

$$k > \sqrt{2\left(1 + \frac{\pi^2}{72}\right)}\,\pi \approx 4.73762 \tag{30}$$

This, together with (26) and (27), implies that for realistic values of $h$ and $k$ ($k \gg 5$) the preconditioned system $AB_0^{-1}$ will very likely remain (possibly only slightly) indefinite. On the basis of (14), this might also hold for any incomplete factorization-based preconditioner $B$ computed directly from $A$, unless a huge amount of fill-in is allowed. In the latter event, one has that $B \approx A$, therefore quantities $\beta_z = (z, B_0 z)/(z, Bz)$ are very likely negative whenever the vectors $z$ are (approximately) eigenvectors associated to the smallest eigenvalues of $AB^{-1}$, whence it follows that quantities $\mu_z \beta_z$ are positive, and hence also $(z, Az)/(z, Bz)$.

Observe finally that,

$$(\mathrm{Re}(A)e)_i = \begin{cases} -k^2h^2 & \text{if } 1 < i_x < n_x \text{ and } 1 < i_y < n_y \\ (A_0 e)_i & \text{otherwise} \end{cases} \tag{31}$$

Therefore, for fixed wave number $k$, decreasing the mesh size $h$ amounts (in general) to

1. increasing the measures $(\mathrm{Re}(A)e)_i$ of 'diagonal dominance' of $\mathrm{Re}(A)$;
2. clustering some eigenvalues (of $\mathrm{Re}(A)$) with negative part;
3. moving the smallest negative eigenvalues of $\mathrm{Re}(A)$ towards 0;
4. increasing the number of distinct eigenvalues of $\mathrm{Re}(A)$.

Above four actions might have opposite effects on the convergence rate. The first two actions are beneficial. Whether this would improve the convergence is unpredictable, as it also depends on how the real parts of the smallest (in module) eigenvalues of the (preconditioned) matrix are close to the origin. As a conclusion, contrary to symmetric positive-definite cases, *a decrease of the mesh size would not necessarily lead to an increase of the number of iterations.*

 

### 5.2. *Analysis of the spectrum's imaginary part*

As mentioned at the end of Section 4, the analysis is not easy. To have an idea of the quality of the upper bound involved, we have computed the maximum of quantities $\mu_z\,\eta_z$, over a set of eigenvectors of $\widehat{K}$ (see Figure 4), say, (see, e.g., [9]),

$$z^{(i_x,i_y)}(x,y) = \sin\, i_x\pi x\, \sin\, i_y\pi y, \quad 1 \leqslant i_x, i_y \leqslant nx = ny \tag{32}$$

The above vectors are also good approximate eigenvectors of $\mathrm{Re}(A)$, a fact that has been exploited in the analysis of the spectrum's real part. Simple algebraic calculations yield

$$\mu_{z^{(i_x,i_y)}}\,\eta_{z^{(i_x,i_y)}} \approx \frac{(kh/3)(1+2\sin^2 i_x\pi h/2)}{4(1+k^2h^2/12)(\sin^2 i_x\pi h/2+\sin^2 i_y\pi h/2)+(k^2h^2/3)\sin^2 i_x\pi h/2}$$

whose maximum is attained for $i_x = i_y = 1$, whence it follows that

$$\max_{z\in\mathbb{C}^n\setminus\{0\}} \mu_z\,\eta_z \gtrsim \frac{k(1+\pi^2h^2/2)}{6\pi^2h(1+k^2h^2/8)} \tag{33}$$

The above estimate gives rise to pessimistic spectral bounds. As regards lower spectral bounds, similar observations hold. This should not come as a surprise, given that the bounds (16) on the imaginary parts of eigenvalues have been obtained by bounding three terms in the basic relation (14), in contrast to two terms in the case of real parts. We recall that the real part of the spectrum plays the main role in the convergence of the process.

The convergence rate of GMRES depends on the distribution (clusterings) of eigenvalues of the (preconditioned) system. For $h^{-1} = 11$ $(n = 100)$, we draw in Figures. 5–10 the estimated eigenvalues (ritz values) computed by the Arnoldi process. In most cases, we use $k = 10$ such that $kh \approx 1$, which may represent a hard situation. Except for Figure 5, for reasons explained below, all the spectra in the same figure are on the same scale for an easy comparison. We have the following comments on each figure.

- For all the preconditionings involved, there is a clustering of approximate eigenvalues around 1, as opposed to the unpreconditioned system.
- Figure 5: Even for so small problem, it was very hard to find a representative case to compare standard IC and MIC. Strongly isolated approximate eigenvalues are, in general, scattered in the complex plane. This makes it difficult to use the same scale. Remote approximate eigenvalues and approximate eigenvalues that are too close to the origin are mainly responsible for unpredictable behaviours.
- Figures 6 and 7: As expected from the theory, the real parts of the largest approximate eigenvalues of $\mathrm{IC}^0$ and $\mathrm{MIC}^0$, respectively, are bounded by the largest approximate eigenvalues of IC and MIC, respectively, applied to the Stieltjes matrix $A_0$. Approximate eigenvalues for $\widetilde{\mathrm{IC}}$ and $\widetilde{\mathrm{MIC}}$ are more clustered than for $\mathrm{IC}^0$ and $\mathrm{MIC}^0$, respectively. On the basis of the spectra, one may expect perturbed MIC-based methods to behave better than their IC counterparts. Observe that, even though approximate eigenvalues with negative real part are amplified by perturbed methods, in particular by *modified* variants, their number does not increase.
- Figure 8: It is evident from the spectra that, as the wave number increases, the system becomes more difficult to solve. There are more and more approximate eigenvalues with negative real part. A fact that is shared by all the preconditionings investigated.
- Figures 9 and 10: As one increases the fill level, approximate eigenvalues would normally tend to 1. This is indeed the case, but the 'convergence' is not monotonous, which makes it risky

 

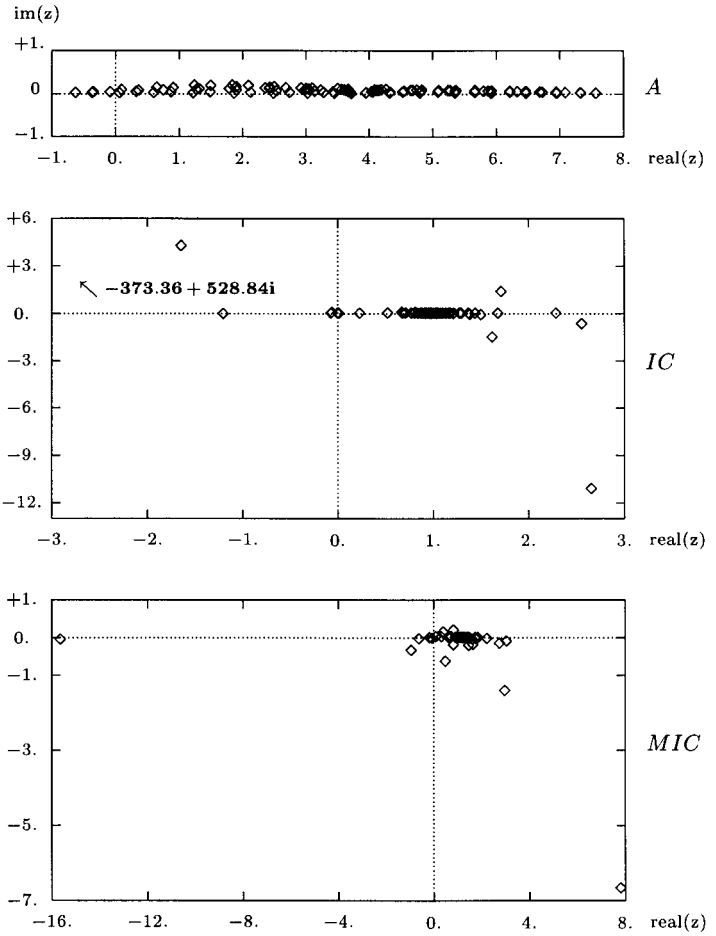


Figure 5. Ritz values for the matrix $A$ for standard preconditionings; fill level $= 0$. Problem 1. $h^{-1} = 11$ and $k = 10$. $\nwarrow -373.36 + 528.84i$ means that this ritz value observed is out of range (i.e. strongly isolated from the other ones).

to predict any gain when passing from a level to another one. As regards perturbed methods, the move of approximate eigenvalues with negative real parts towards 1 seems more regular, but not spectacular. We recall (see comments after (30)) that perturbed methods could hardly eliminate the negative part of the spectra. In other words, for very large fill level, one should use standard methods.

To save space, we do not report the number of iterations for the above rather academic example. The conclusions are the same as for the next problem, which consists of the following two-dimensional waveguide problem borrowed from Reference [12].

*Problem 2*. $\Omega = (0,1) \times (0,1)$. $u = 1$ on $\Gamma_{\text{left}}$, $\partial u/\partial n = 0$ on $\Gamma_{\text{bottom}} \bigcup \Gamma_{\text{top}}$, $\partial u/\partial n + \mathrm{i}ku = 0$ on $\Gamma_{\text{right}}$, where $\Gamma_{\text{bottom}} = \{(x,0); 0 \leqslant x \leqslant 1\}$, $\Gamma_{\text{top}} = \{(x,1); 0 \leqslant x \leqslant 1\}$, $\Gamma_{\text{left}} = \{(0,y); 0 \leqslant y \leqslant 1\}$, and $\Gamma_{\text{right}} = \{(1,y); 0 \leqslant y \leqslant 1\}$.

 

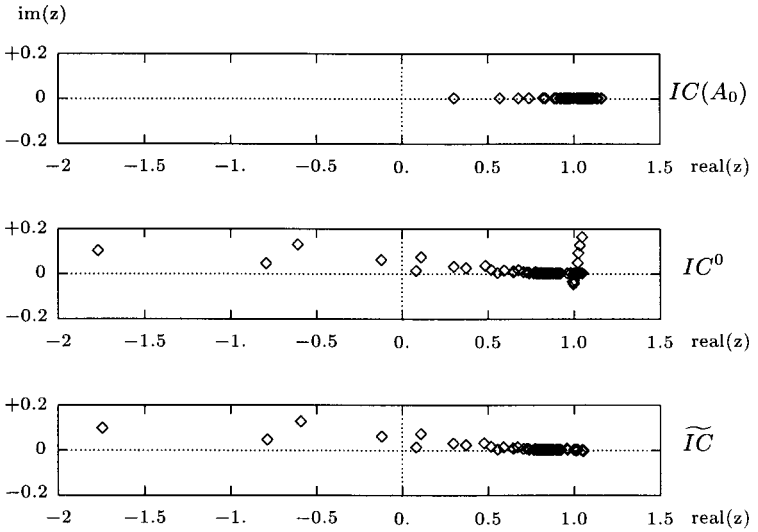


Figure 6. Ritz values for IC-type preconditionings; fill level $=0$. Problem 1, $h^{-1}=11$ and $k=10$.

We now present some results of numerical experiments we have performed. All computations were carried out in double precision on a Silicon Graphics Origin 2000 workstation. GMRES with right preconditioning is run with the zero vector as initial approximate solution and the residual error reduction $\|r^{(j)}\|/\|b\| \leqslant 10^{-7}$ as convergence criterion. Standard modified incomplete Cholesky factorizations are not reliable whenever all the rowsums of the matrix involved are not non-negative. For (strongly) indefinite matrices, all the known theories fail to provide any useful information on the spectral properties of the preconditioned matrices. We have observed that the performances are, in general, disastrous as long as the preconditioners are applied directly to the indefinite matrix $A$. That is the reason why this strategy has been discarded in our discussion. To see the effect of mesh size, wave number and fill level on the performance, we have first performed the following two experiments, with *full GMRES* ($m=\infty$ in Figure 1), whose results are collected in Tables I and II.

*Experiment 1*. We use $h=1/100$ ($n=10\,100$) and we let the fill level increase, taking different values from 0 to 30. The wave number varies as $k=2,4,10,30$.

*Experiment 2*. We take $fill\ level=8$, $k=10,20,30$ and we let the mesh size decrease by half from $1/50$ to $1/200$ ($n=2550,10\,100,40\,200$).

From the tables, the following observations are evident:

1. In contrast to the symmetric positive-definite case, iterations counts are weakly sensitive to mesh size variations. Sometimes, the number of iterations even decreases as the mesh size decreases. As argued at the end of the *analysis of the spectrum's real part*, this is mainly due to the increase of measures $(\mathrm{Re}(A)e)_i$ of 'diagonal dominance' of $\mathrm{Re}(A)$. Similar observations have been made in Reference [36].
2. The convergence rate is sensitive to the variation of the wave number. In agreement with previous works, see, e.g. References [12, 36, 5, 38], the larger $k$, the slower the convergence.

 

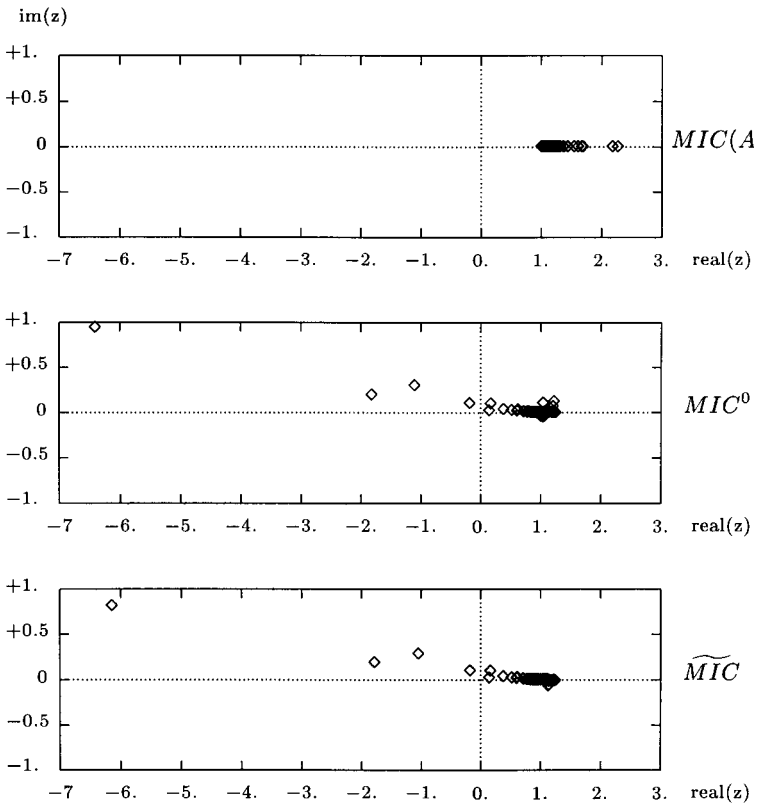


Figure 7. Ritz values for MIC-type preconditionings; fill level $=0$. Problem 1, $h^{-1}=11$ and $k=10$.

3. One is not repaid for allowing lot of fill-in entries. On the one hand, there is no guarantee that the performance will be improved, on account of the risk of clustering eigenvalues, previously with negative real part, around the origin. On the other hand, the reduction of the number of iterations, when there is some, is too small to compensate for the increase of the incomplete factorization costs. In our experiments, fill levels around 10 seem to be optimal for reducing the computation time.
4. For small wave number, $k<10$, the performances of standard IC and perturbed IC are essentially the same. From moderate to large wave numbers, perturbed IC should be preferred to standard IC. In general, perturbed modified versions perform better than the corresponding unmodified methods. As far as complex systems are concerned, keeping the imaginary part is better than dropping it prior to the incomplete factorization process.
5. At this stage, the best preconditioner is $\widetilde{\mathrm{MIC}}$. Given that $\mathrm{Im}(A)$ is positive semidefinite, $\tilde{A}=A_0+\mathrm{i}\,\mathrm{Im}(A)$ may be viewed as a move of $A_0$ along the imaginary axis. This issue will receive more attention in forthcoming work.

Storage requirement and computational cost of GMRES grow linearly and quadratically, respectively, with the iteration count. GMRES($m$) could be very slow to converge, or even stagnate, on account of loss of information at each restart phase. Purely modified incomplete factorizations

 

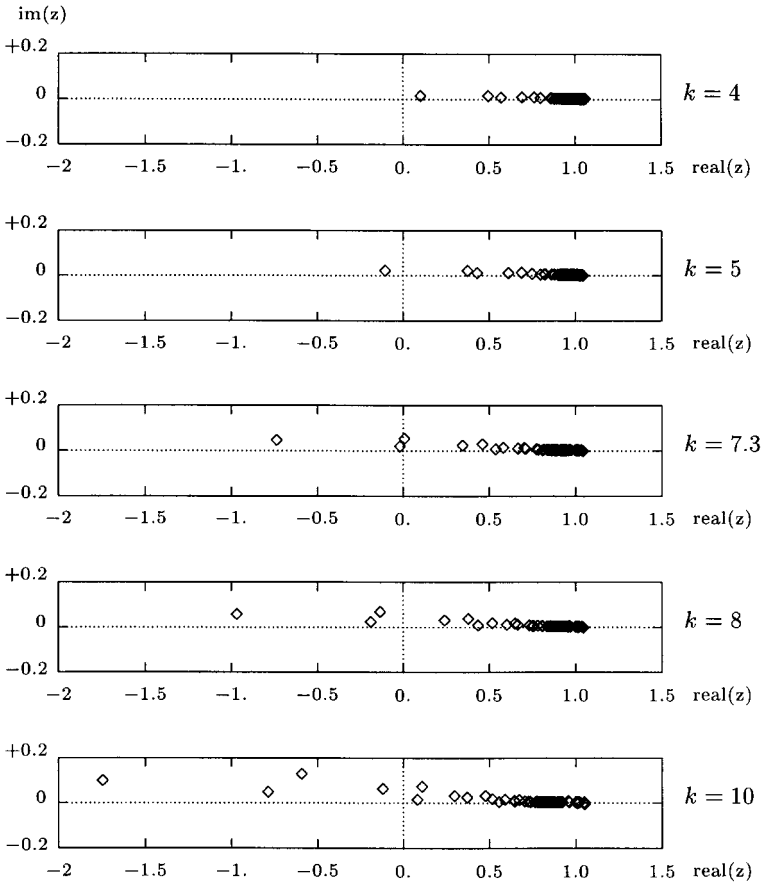


Figure 8. Ritz values for $\widetilde{\text{IC}}$; fill level $= 0$. Problem 1, $h^{-1} = 11$ and variable $k$.

are known to be very sensitive to loss of information in Krylov subspace methods, see, e.g., References [3, 37]. Moreover, as soon as one gets away from some ideal conditions covered by theories, almost anything can occur with modified incomplete factorizations without perturbations. With $\gamma = 1$, matrix $A_0 = \text{Re}(\tilde{A}) = \text{Re}(A + Q)$ has most of its rowsums equal to zero. Thus, there is some similarity with some *unperturbed* real Stieltjes PDE matrix. Now, it has been observed that perturbing real Stieltjes matrix improves the performance of modified methods, see, e.g., Reference [31]. On this basis, we strongly recommend to increase $\gamma$ in the case of $\widetilde{\text{MIC}}$. In the two experiments to follow, we use $m \leqslant 50$ to compare the performances of standard IC, $\widetilde{\text{IC}}$ with $\gamma = 1, 2$, and $\widetilde{\text{MIC}}$ with $\gamma = 2, 3$. As argued, the performance of $\widetilde{\text{MIC}}$ with $\gamma = 1$ was catastrophic.

*Experiment 3*. For $h = 1/200$ $(n = 40\,200)$, we run GMRES($m$) with $m = 20, 30, 50$, and take $k = 20, 30$ and *fill level* $= 8$.

*Experiment 4*. For $h = 1/200$ $(n = 40\,200)$, we take $m = 30, 50$, $k = 20, 30$ and we let the fill level vary from 8 to 16.

 

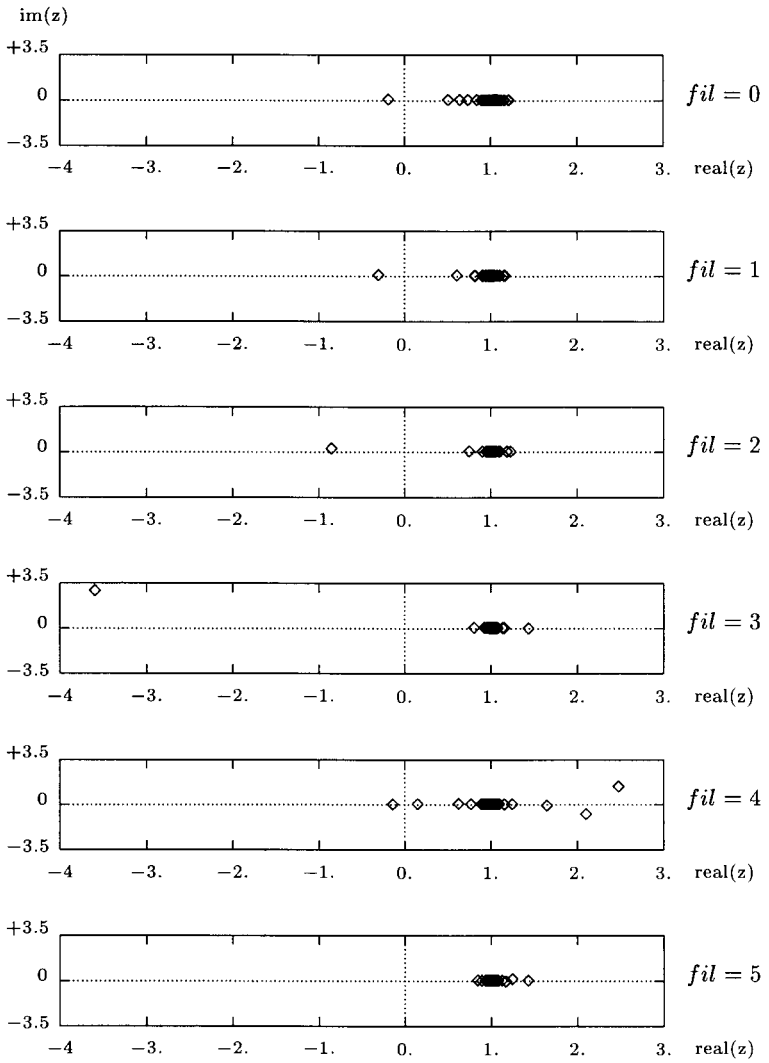


Figure 9. Ritz values computed by GMRES for $\widetilde{\text{IC}}$ with different fill levels. Problem 1, $h^{-1} = 11$ and $k = 5$.

The iteration counts that we report in Table III correspond to how many times step 3 (matrix vector multiplication and preconditioning step) of GMRES($m$) (Figure 1) was executed. The following trends may be stressed:

1. There is a spectacular gain in shifting from standard IC to $\widetilde{\text{IC}}$ and $\widetilde{\text{MIC}}$. The gap is in particular accentuated for high wave numbers ($k > 20$). Even for moderate wave number, $k \leqslant 20$, it is still advantageous to use perturbed methods.
2. The performances of $\widetilde{\text{IC}}$ and $\widetilde{\text{MIC}}$ are in general close, with however some advantage for $\widetilde{\text{IC}}$. Improvement could be achieved by 'optimizing' $\gamma$. Unfortunately, we have observed that

 

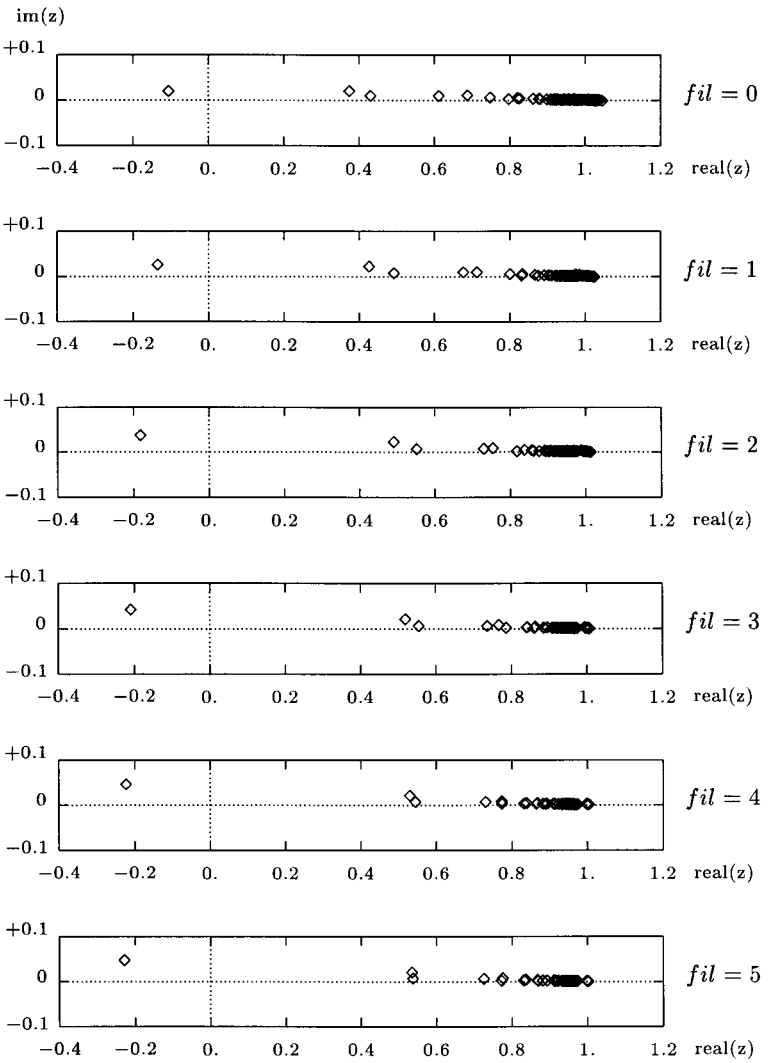


Figure 10. Ritz values for $\widetilde{\text{IC}}$ with different fill levels. Problem 1, $h^{-1} = 11$ and $k = 5$.

optimal values of $\gamma$ depends on both problem and method parameters ($k$, $h$ and $m$). This dependence is more pronounced for $\widetilde{\text{MIC}}$.

3. There is no substantial advantage to use high fill levels, for the same reasons as explained previously.
4. As would be expected, the longer the process run, the more severe the effects of loss of accumulated information on the Krylov subspace basis computed. Whenever possible, one should avoid $\varepsilon \leqslant 10^{-7}$.
5. Now, the best preconditioning strategy consists in applying $\widetilde{\text{IC}}$ with $\gamma$ around 1, and using $\varepsilon \geqslant 10^{-6}$.

 

Table I. Problem 2 ($h^{-1} = 100$). Experiment 1. Number of preconditioned GMRES iterations. $*$ means stagnation. Iteration numbers in bold correspond to the observed optimal for given fill level and wave number.

| **k** | Fill level / Prec | 0 | 1 | 2 | 4 | 8 | 12 | 16 | 20 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|
| | IC | 72 | 63 | 44 | 30 | 18 | 12 | 10 | 9 | 7 |
| | IC° | 72 | 63 | 44 | 30 | 18 | 13 | 11 | 10 | 9 |
| 2 | $\widetilde{\text{IC}}$ | 72 | 63 | 44 | 30 | 18 | **12** | **10** | 9 | 7 |
| | MIC° | 40 | 35 | 29 | 22 | 16 | 13 | 12 | 11 | 9 |
| | $\widetilde{\text{MIC}}$ | **40** | **35** | **29** | **22** | **16** | 13 | 11 | **9** | **7** |
| | IC | 82 | 67 | 49 | 33 | 20 | 15 | 12 | **10** | **8** |
| | IC° | 82 | 67 | 49 | 33 | 21 | 17 | 15 | 14 | 13 |
| 4 | $\widetilde{\text{IC}}$ | 82 | 67 | 49 | 33 | 20 | 15 | 12 | 11 | 10 |
| | MIC° | 46 | 40 | 34 | 26 | 20 | 17 | 15 | 14 | 13 |
| | $\widetilde{\text{MIC}}$ | **46** | **40** | **33** | **25** | **18** | **14** | **12** | 11 | 9 |
| | IC | 121 | 99 | 73 | 48 | **30** | 26 | 45 | 45 | 35 |
| | IC° | 123 | 100 | 76 | 55 | 40 | 35 | 34 | 33 | 32 |
| 10 | $\widetilde{\text{IC}}$ | 123 | 100 | 74 | 51 | 33 | 29 | 28 | 27 | 26 |
| | MIC° | 75 | **66** | 56 | 46 | 37 | 33 | 31 | 32 | 32 |
| | $\widetilde{\text{MIC}}$ | **75** | 67 | **56** | **43** | 31 | **26** | **25** | **26** | **26** |
| | IC | 270 | 223 | **168** | $*$ | $*$ | 153 | 202 | **137** | **120** |
| | IC° | 306 | 267 | 222 | 188 | 173 | 172 | 171 | 172 | 171 |
| 30 | $\widetilde{\text{IC}}$ | 292 | 249 | 200 | 167 | 154 | 152 | 152 | 152 | 151 |
| | MIC° | 235 | 211 | 186 | 160 | 163 | 167 | 169 | 171 | 172 |
| | $\widetilde{\text{MIC}}$ | **229** | **200** | 171 | **142** | **144** | **148** | **150** | 151 | 151 |

Table II. Problem 2. Experiment 2. Number of preconditioned GMRES iterations, for various preconditioners with fill level $= 8$. Iteration numbers in bold correspond to the observed optimal for given mesh size and wave number.

| **k** | | **10** | | | **20** | | | **30** | |
|---|---|---|---|---|---|---|---|---|---|
| $h^{-1}$ / Precond | 50 | 100 | 200 | 50 | 100 | 200 | 50 | 100 | 200 |
| IC | 55 | **30** | 52 | 114 | * | **85** | **148** | * | * |
| IC° | 35 | 40 | 57 | 88 | 93 | 97 | 168 | 173 | 192 |
| MIC° | 32 | 37 | 46 | 87 | 83 | 97 | 166 | 163 | 163 |
| $\widetilde{\text{IC}}$ | 28 | 33 | 54 | 77 | 80 | 90 | 151 | 154 | 164 |
| $\widetilde{\text{MIC}}$ | **26** | 31 | **43** | **75** | **70** | 87 | 150 | **144** | **145** |

*Means Stagnation.

 

Table III. Problem 2 ($h^{-1}=200$). Experiments 3 and 4. Number of matrix-vector products in GMRES, for various preconditioners with fill level = 8. $\|r^{(j)}\|/\|b\| \leqslant \varepsilon$. +1000 means iteration counts larger than 1000. Iteration numbers in bold correspond to the optimal within each column.

| m | | **20** | | | | **30** | | | | **50** | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **k** | | **20** | | **30** | | **20** | | **30** | | **20** | | **30** | |
| Precond | $\varepsilon$ | $10^{-5}$ | $10^{-6}$ | $10^{-5}$ | $10^{-6}$ | $10^{-5}$ | $10^{-6}$ | $10^{-5}$ | $10^{-6}$ | $10^{-5}$ | $10^{-6}$ | $10^{-5}$ | $10^{-6}$ |
| IC | | 72 | 155 | +1000 | +1000 | 63 | 138 | 988 | +1000 | 50 | 102 | 514 | +1000 |
| $\widetilde{\text{IC}}$ | ($\gamma=1$) | **61** | **112** | **127** | **214** | 49 | 102 | **86** | **198** | **45** | 90 | 70 | **183** |
| $\widetilde{\text{IC}}$ | ($\gamma=2$) | 80 | 116 | 158 | 225 | **48** | 93 | 97 | 206 | 45 | 88 | **63** | 188 |
| $\widetilde{\text{MIC}}$ | ($\gamma=2$) | 62 | 114 | 172 | 234 | 53 | 101 | 94 | 205 | 48 | 91 | 72 | 191 |
| $\widetilde{\text{MIC}}$ | ($\gamma=3$) | 88 | 118 | 208 | 279 | 48 | **89** | 158 | 237 | 46 | **85** | 67 | 209 |

↑ Experiment 3 Experiment 4 ↓

| | **k** | | | **20** | | | | | | **30** | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\gamma$ | | **1** | | | **2** | | | **1** | | | **2** | |
| m | $\varepsilon$ / fil | $10^{-5}$ | $10^{-6}$ | $10^{-7}$ | $10^{-5}$ | $10^{-6}$ | $10^{-7}$ | $10^{-5}$ | $10^{-6}$ | $10^{-7}$ | $10^{-5}$ | $10^{-6}$ | $10^{-7}$ |
| | 8 | 49 | 102 | 209 | 48 | 93 | 207 | 86 | 198 | 706 | 97 | 206 | 668 |
| | 10 | 46 | 92 | 205 | 44 | 83 | 178 | 90 | 213 | 750 | 99 | 205 | 658 |
| 30 | 12 | 45 | 89 | 213 | 41 | 79 | 167 | 116 | 257 | 824 | 105 | 208 | 667 |
| | 14 | 44 | 90 | 224 | 38 | 74 | 160 | 149 | 295 | 905 | 111 | 208 | 670 |
| | 16 | 46 | 89 | 237 | 37 | 68 | 159 | 191 | 343 | 981 | 112 | 209 | 667 |
| | | | | | | IC ↕ | | | | | | | |
| | 8 | 45 | 90 | 186 | 45 | 88 | 193 | 70 | 183 | 648 | 63 | 188 | 620 |
| | 10 | 41 | 83 | 179 | 41 | 79 | 164 | 70 | 192 | 669 | 59 | 188 | 627 |
| 50 | 12 | 39 | 79 | 176 | 38 | 75 | 148 | 79 | 215 | 716 | 58 | 190 | 630 |
| | 14 | 37 | 75 | 171 | 37 | 68 | 144 | 85 | 245 | 765 | 58 | 190 | 629 |
| | 16 | 37 | 74 | 179 | 36 | 63 | 141 | 96 | 273 | 781 | 58 | 192 | 641 |
| | $\gamma$ | | **2** | | | **3** | | | **2** | | | **3** | |
| | 8 | 53 | 101 | 196 | 48 | 89 | 204 | 94 | 205 | 668 | 158 | 237 | 720 |
| | 10 | 49 | 93 | 191 | 45 | 84 | 186 | 105 | 212 | 680 | 165 | 238 | 709 |
| 30 | 12 | 49 | 88 | 202 | 43 | 82 | 175 | 112 | 217 | 684 | 172 | 237 | 722 |
| | 14 | 47 | 85 | 206 | 44 | 83 | 175 | 116 | 225 | 687 | 158 | 239 | 731 |
| | 16 | 45 | 83 | 209 | 44 | 83 | 176 | 122 | 223 | 679 | 169 | 250 | 723 |
| | | | | | | $\widetilde{\text{MIC}}$ ↕ | | | | | | | |
| | 8 | 48 | 91 | 161 | 46 | 85 | 188 | 72 | 191 | 649 | 67 | 209 | 649 |
| | 10 | 45 | 86 | 162 | 44 | 80 | 165 | 68 | 197 | 655 | 64 | 210 | 656 |
| 50 | 12 | 44 | 81 | 181 | 41 | 78 | 155 | 67 | 199 | 646 | 62 | 209 | 670 |
| | 14 | 42 | 78 | 184 | 40 | 77 | 150 | 63 | 199 | 648 | 62 | 209 | 683 |
| | 16 | 40 | 76 | 185 | 39 | 76 | 155 | 61 | 198 | 649 | 61 | 209 | 676 |

## 6. CONCLUSIONS

A new preconditioning technique has been proposed. Standard incomplete factorization techniques are applied to complex-symmetric indefinite matrix, whose real part has been first made positive definite by adding appropriate diagonal perturbations. A significant savings of the number of GMRES steps has been obtained, with respect to standard methods, for realistic values of the restart parameter. The arguments still apply when the real parts of some offdiagonal entries of the original matrix are positive. Even though the real part of the perturbed matrix is no more a Stieltjes one, it is diagonally dominant, which allows the use of suitable incomplete factorizations. This needs further investigations.

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