Numerical Inversion of Laplace Transforms Using Laguerre Functions

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Abstract. A method is described for the numerical inversion of Laplace transforms, in which the inverse is obtained as an expansion in terms of orthonormal Laguerre functions. In order for this to be accomplished, the given Laplace transform is expanded in terms of the Laplace transforms of the orthonormal Laguerre functions. The latter expansion is then reduced to a cosine series whose approximate expansion coefficients are obtained by means of trigonometric interpolation. The computational steps have been arranged to facilitate automatic digital computation, and numerical illustrations have been given.

Introduction

The Laplace transformation provides a powerful method for analyzing linear systems. Unfortunately, many problems of physical interest lead to Laplace transforms whose inverses are not readily expressed in terms of tabulated functions. Although there exist extensive tables of transforms and their inverses, it is highly desirable to have methods for approximate numerical inversion.

Several such methods have been described in the literature. Schmittroth [1] has described a method in which the inverse transform is obtained from the complex inversion integral by use of numerical quadrature. This method gives good results, but may become time consuming if the inverse transform is required for a large number of values of the independent variable. The quadrature procedure must be repeated for each value of the independent variable. In cases where the inverse is required for many values of the independent variable, it is convenient to obtain the inverse as a series expansion in terms of a set of linearly independent functions. The inversion procedure then consists of determining the expansion coefficients once and for all from the given Laplace transform. The inverse can then be obtained at any value of the independent variable by means of a simple series summation.

Procedures based on this idea have been described by several authors. Norden [2] has described two such methods in which the expansion coefficients are calculated by solving a system of simultaneous equations. The problem of solving simultaneous linear equations can be reduced to solving a triangular system, or eliminated altogether, if one chooses to use orthogonal functions. Such a method, using orthogonal polynomials, has been described by Salzer [3, 4] and refined by Shirtliffe and Stephenson [5]. These authors attempt an approximate evaluation of the inversion integral using Gaussian quadrature in the complex plane. The chief disadvantage of this method is the necessity of finding all roots, real and complex, of a polynomial of high degree, and of the calculation of a set of complex Christoffel numbers.

A number of methods using orthogonal functions but involving only real quantities have been given by Lanczos [6] and by Papoulis [7]. Both authors describe methods in which the inverse transform is obtained as series expansions in terms of

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trigonometric functions, Legendre polynomials or Laguerre polynomials. For trigonometric functions and Legendre polynomials, it is necessary to map the interval $(0, \infty)$ into a finite interval by means of an exponential transformation. One arrives at expansion coefficients which range over many orders of magnitude and which must be calculated with great precision in order for the methods to work. A more natural procedure is the use of Laguerre functions. These functions form the most suitable orthogonal system for approximation on the interval $(0, \infty)$. Their importance to the theory of the Laplace transformation has been well established by the investigations of Widder [8, 9] and Shohat [10]. Numerical procedures using Laguerre functions have been described both by Lanczos [6] and Papoulis [7].

In the paper by Papoulis, the inverse transform is obtained as a series expansion in terms of Laguerre functions. The expansion coefficients are obtained from the coefficients of the Taylor series expansion of the Laplace transform by solving a triangular system of linear equations. The chief drawback to this method, however, is the necessity of obtaining the Taylor series expansion of the Laplace transform. In the method described by Lanczos [6, pp. 292–299] for finding the inverse transform as a series expansion in terms of Laguerre functions, one first applies a conformal mapping to the Laplace transform and then develops the resulting function in a Taylor series expansion.

Lanczos shows how to obtain the coefficients of the Taylor series expansion by synthetic division when the Laplace transform is the ratio of two polynomials. Lanczos also shows the connection between his Taylor series and a Fourier series, and mentions the possibility of using trigonometric interpolation as a possible way to obtain the expansion coefficients—although he does not develop the idea to any extent.

The present paper is essentially an elaboration and refinement of the ideas of Lanczos and Papoulis to meet the requirements of automatic computation. The form used in this paper for the expansion of the inverse transform is more closely related to the one used by Papoulis than to the one used by Lanczos. The expansion of the Laplace transform, however, is much more closely related to the expansion used by Lanczos than to that of Papoulis. The distinctive feature of the present work is that the coefficients in the expansion of the inverse transform are obtained directly by trigonometric interpolation applied to the Laplace transform. The resulting algorithm is quite suitable for automatic computation. It makes modest requirements of storage and can be expressed in a relatively compact code.

Formal Statement of the Problem

The problem under consideration is: Given the function $g(p)$, construct an approximation to the function $f(t)$ such that $g(p)$ is the Laplace transform of the function $f(t)$. It will be assumed that $f(t)$ satisfies the condition that there exists a number c_0 such that

$$\int_0^\infty e^{-ct} \mid f(t) \mid dt < \infty \tag{1a}$$

$$\int_0^\infty e^{-2ct} \mid f(t) \mid^2 dt < \infty \tag{1b}$$

whenever $c \geq c_0$. If these conditions are satisfied, then the integral

$$g(p) = \int_0^\infty f(t)e^{-pt}\,dt; \quad p = \sigma + i\omega \tag{2}$$

defines a function $g(p)$ which is analytic whenever $\sigma > c_0$. The function $g(p)$ is called the Laplace transform of $f(t)$, and $f(t)$ is called the inverse transform of $g(p)$. If the function $f(t)$ satisfies conditions (1a) and (1b), then $f(t)$ and $g(p)$ are further related by the Parseval theorem:

$$\int_0^\infty e^{-2ct}\,|f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^\infty |g(c+i\omega)|^2\,d\omega. \tag{3}$$

Proofs of the above assertions may be found in any of a number of texts on Laplace transforms (e.g., Doetsch [11, especially Ch. 1, Sec. 4; and Ch. 2, Sec. 9]).

Expansion of the Inverse Transform

In this section, a series expansion of the function $f(t)$ is obtained in terms of the orthonormal Laguerre functions:

$$\Phi_n(x) = e^{-x/2}L_n(x), \qquad n = 0, 1, 2, \cdots \tag{4}$$

where $L_n(x)$ is the Laguerre polynomial of degree n. These polynomials are determined by the requirements that

$$\int_0^\infty \Phi_n(x)\,\Phi_m(x)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m, \end{cases} \tag{5a}$$

$$\Phi_n(0) = 1. \tag{5b}$$

It follows from the completeness property of the orthonormal Laguerre functions that any function $f(t)$ satisfying conditions (1a) and (1b) can be approximated by a function $f_N(t)$:

$$f_N(t) = e^{ct}\sum_{n=0}^{N} a_n\,\Phi_n\left(\frac{t}{T}\right) \tag{6}$$

where $T > 0$ is a scale factor and

$$a_n = \frac{1}{T}\int_0^\infty e^{-ct}f(t)\Phi_n\left(\frac{t}{T}\right)dt. \tag{7}$$

The function $f_N(t)$ approximates $f(t)$ in the sense that for any $\epsilon > 0$, there exists an integer N_ϵ such that

$$\int_0^\infty e^{-2ct}|f(t) - f_N(t)|^2\,dt < \epsilon \tag{8}$$

whenever $N > N_\epsilon$.

Expansion of the Laplace Transform

Let $g_N(p)$ be the Laplace transform of $f_N(t)$. That is,

$$g_N(p) = \int_0^\infty f_N(t)e^{-pt}\,dt. \tag{9}$$

Putting eq. (6) into eq. (9) and evaluating the Laplace transform of $e^{ct}\Phi_n(t/T)$, one obtains

$$g_N(p) = \sum_{n=0}^{N} a_n \frac{\left(p - c - \frac{1}{2T}\right)^n}{\left(p - c + \frac{1}{2T}\right)^{n+1}}. \tag{10}$$

The Laplace transform of the Laguerre function can be obtained from the tables given by Magnus and Oberhettinger [12, especially the last entry on p. 129]. Of particular interest in this paper is the behavior of eq. (10) on the line $\sigma = c$ in the complex p-plane. It will be shown that $g_N(c + i\omega)$ converges in the mean to $g(c + i\omega)$ with increasing N. The proof is established by observing that the functions $\Phi_n(t/T)$ satisfy conditions (1a) and (1b) with any nonnegative c. Therefore, for any c such that $c > c_0$ and $c \geq 0$, the function $f(t) - f_N(t)$ is a linear combination of a finite number of functions which satisfy conditions (1a) and (1b), and consequently must itself satisfy these conditions. Therefore, it follows that the function $f(t) - f_N(t)$ must be related to its Laplace transform, $g(p) - g_N(p)$, by a Parseval theorem of the form of eq. (3). That is,

$$\int_0^{\infty} e^{-2ct} \,|\, f(t) - f_N(t) \,|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\, g(c + i\omega) - g_N(c + i\omega) \,|^2 \, d\omega \tag{11}$$

whenever $c > c_0$. Comparing eq. (11) with eq. (8), one sees that given any $\epsilon > 0$ there exists an integer N_ϵ such that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\, g(c + i\omega) - g_N(c + i\omega) \,|^2 \, d\omega < \epsilon \tag{12}$$

whenever $N > N_\epsilon$, which establishes the desired result.

To obtain a trigonometric expansion of the Laplace transform, first put eq. (10) with $p = c + i\omega$ into eq. (12) to obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| g(c + i\omega) - \sum_{n=0}^{N} a_n \frac{\left(i\omega - \frac{1}{2T}\right)^n}{\left(i\omega + \frac{1}{2T}\right)^{n+1}} \right|^2 d\omega < \epsilon \tag{13}$$

whenever $N > N_\epsilon$. Then make the change of variable

$$\omega = \frac{1}{2T} \cot \frac{\theta}{2} \tag{14}$$

to obtain

$$\frac{T}{2\pi} \int_{-\pi}^{+\pi} \left| \left(\frac{1}{2T} + \frac{i}{2T} \cot \frac{\theta}{2}\right) g\left(c + \frac{i}{2T} \cot \frac{\theta}{2}\right) - \sum_{n=0}^{N} a_n \, e^{in\theta} \right|^2 d\theta < \epsilon \tag{15}$$

whenever $N > N_\epsilon$. The significance of eq. (15) is that

$$\left(\frac{1}{2T} + \frac{i}{2T} \cot \frac{\theta}{2}\right) g\left(c + \frac{i}{2T} \cot \frac{\theta}{2}\right) \approx \sum_{n=0}^{N} a_n \, e^{in\theta} \tag{16}$$

in the sense that the right-hand side of eq. (16) converges in the mean to the left-hand side with increasing N.

If one now defines two functions $g_1(\sigma, \omega)$ and $g_2(\sigma, \omega)$ as the real and imaginary parts, respectively, of $g(\sigma + i\omega)$ so that

$$g(\sigma + i\omega) = g_1(\sigma, \omega) + ig_2(\sigma, \omega) \tag{17}$$

then eq. (16) can be reduced to an expansion involving only real quantities. If the notation of eq. (17) is introduced into eq. (16) and the real part of the latter is taken, one obtains the expansion

$$h(\theta) \approx \sum_{n=0}^{N} a_n \cos n\theta \tag{18}$$

where

$$h(\theta) = \frac{1}{2T} g_1\left(c, \frac{1}{2T}\cot\frac{\theta}{2}\right) - \frac{1}{2T}\cot\frac{\theta}{2}\, g_2\left(c, \frac{1}{2T}\cot\frac{\theta}{2}\right). \tag{19}$$

The numerical procedure for obtaining approximate values of the expansion coefficients a_n consists of applying to eq. (18), the well-known trigonometric interpolation formulas

$$a_0 = \frac{1}{N+1}\sum_{j=0}^{N} h(\theta_j) \tag{20a}$$

and

$$a_n = \frac{2}{N+1}\sum_{j=0}^{N} h(\theta_j)\ \cos n\theta_j\ ; \qquad n \neq 0 \tag{20b}$$

where

$$\theta_j = \left(\frac{2j+1}{N+1}\right)\frac{\pi}{2}; \qquad j = 0, 1, \cdots, N. \tag{21}$$

The interpolation formulas, eqs. (18)–(21), are derived in [13, pp. 389–391] and are closely related to the formulas for interpolation by means of Tchebychef polynomials. In fact, if one introduces the variable $z = \cos n\theta$, then $\cos n\theta = T_n(z)$ is the Tchebychef polynomial of degree n in the variable z and eqs. (18)–(21) become the usual formulas for Tchebychef interpolation.

Scale Factors T and c

The constant c was introduced in eqs. (1a) and (1b), and the scale factor T in eq. (6). Although the convergence in the mean of the series in eq. (6) was established for any positive value of T and for any value of c, such that $c > c_0$ and $c \geq 0$, one might suspect that the series will converge faster for some values of T and c than for other values. This is indeed borne out in practice. Some intuitive considerations will lead to a proper choice of these parameters.

First consider T. The function $\Phi_n(x)$ defined by eqs. (4), (5a) and (5b) is the product of a polynomial $L_n(x)$ of degree n and a decreasing exponential function. Szego [14] shows that the Laguerre polynomial $L_n(x)$ has n real, positive zeros. Furthermore, in [14, p. 127], it is shown that if x_n is the largest zero of $L_n(x)$, then x_n satisfies the inequality

$$x_n < 2n + 1 + \sqrt{(2n+1)^2 + \tfrac{1}{4}} \approx 4n. \tag{22}$$

Therefore, the function $\Phi_n(x)$ oscillates in the interval

$$0 < x < 4n \tag{23}$$

and approaches zero monotonically for $x > 4n$. Intuitively, it can be seen that the ability of a linear combination of the functions $\Phi_0(x)$, $\Phi_1(x)$, $\cdots$, $\Phi_N(x)$ to approximate an arbitrary function of x depends strongly on their oscillatory behavior. One would expect the approximation to break down for values of x so large that all these functions are exhibiting monotonically decaying behavior.

Hence, the function $f_N(t)$ defined by eq. (6) can be expected to yield a good approximation to the function $f(t)$ only in the interval $0 < t < t_{\max}$, where

$$\frac{t_{\max}}{T} < 4N. \tag{24}$$

For $t > t_{\max}$, all the functions $\Phi_0(x)$, $\Phi_1(x)$, $\cdots$, $\Phi_N(x)$ behave in a monotonically decaying manner and will not be especially useful for approximating an arbitrary $f(t)$. Empirically, it has been found that a satisfactory choice of the parameter T is obtained by taking:

$$T = \frac{t_{\max}}{N} \tag{25}$$

With this choice of T, eq. (6) gives a good approximation to $f(t)$ in the interval $0 < t < t_{\max}$, provided c is properly chosen and N is taken sufficiently large.

In principle, the constant c can be any positive number sufficiently large to guarantee the convergence of the integrals (1a) and (1b). The smallest number c_0 such that the integrals (1a) and (1b) converge whenever $c > c_0$ is the abscissa of convergence of the Laplace integral of eq. (2). The Laplace transform $g(p)$ is an analytic function of the complex variable $p = \sigma + i\omega$ whenever $\sigma > c_0$. The right-most singularity of $g(p)$ lies on the line $\sigma = c_0$. Empirically, it has been found that a satisfactory choice of c is obtained by taking

$$c = \left(c_0 + \frac{1}{t_{\max}}\right) u \left(c_0 + \frac{1}{t_{\max}}\right) \tag{26}$$

where c_0 is the abscissa of convergence of the Laplace integral and $u(x)$ is the unit step function:

$$u(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases} \tag{27}$$

The abscissa of convergence, c_0, is most readily obtained by examining $g(p)$ to find the position of the right-most singularity.

Computational Procedure

The first task in the numerical inversion of the Laplace transform $g(p)$ is to decide upon the largest value of t for which one wants to evaluate the function $f(t)$. Let this value be denoted by $t_{\max}$. The next step is to locate the right-most singularity of $g(p)$. If it is p_0, then $c_0 = \operatorname{Re} p_0$ is the abscissa of convergence of the Laplace integral. The parameter c can then be found from eq. (26). The next step is to choose the number of terms N to be used in the series expansion of $f(t)$. In most of the problems examined by the author, N takes on values between 20 and 50. Having chosen t_{max} and N, the scale factor T can be computed from eq. (25).

The calculation of the expansion coefficients a_n is the next step. It is assumed that a routine for the evaluation of the functions $g_1(\sigma, \omega)$ and $g_2(\sigma, \omega)$, defined by eq. (17), is available. Examination of eqs. (19)–(21) shows that $h(\theta)$ need be evaluated only for the $N+1$ values of θ_j given by eq. (21). Thus one needs to calculate

$$h(\theta_j) = \frac{1}{2T} g_1(c, \omega_j) - \omega_j\, g_2(c, \omega_j) \tag{28}$$

where

$$\omega_j = \frac{1}{2T} \cot \frac{\theta_j}{2}; \qquad j = 0, 1, \cdots, N, \tag{29}$$

θ_j being given by eq. (21). It follows from eqs. (21) and (29) and the well-known properties of trigonometric functions that the values of ω_j can be calculated recursively from the formula

$$\omega_j = \frac{\gamma\omega_{j-1} - \delta}{\omega_{j-1} + \gamma}; \qquad j = 1, 2, \cdots, N \tag{30}$$

where

$$\gamma = \frac{1}{2T} \cot \theta_0 = \frac{1}{2T}\left(\frac{\cos\theta_0}{\sin\theta_0}\right) \tag{31}$$

and

$$\delta = \frac{1}{4T^2}. \tag{32}$$

The starting value for the recursion formula, eq. (30), is given by

$$\omega_0 = \frac{1}{2T} \cot \frac{\theta_0}{2} = \frac{1}{2T}\left(\frac{1 + \cos\theta_0}{\sin\theta_0}\right). \tag{33}$$

One should note that once $\cos\theta_0$ and $\sin\theta_0$ are calculated, all other trigonometric functions required in the evaluation of the expansion coefficients a_n can be calculated recursively. For example, to calculate $\cos\theta_j$ for $j = 1, 2, \cdots, N$, the following formulas can be used:

$$\begin{aligned} \cos\theta_j &= \alpha \cos\theta_{j-1} - \beta \sin\theta_{j-1}, \\ \sin\theta_j &= \beta \cos\theta_{j-1} + \alpha \sin\theta_{j-1}; \end{aligned} \tag{34}$$

where

$$\begin{aligned} \alpha &= \cos 2\theta_0 = 2\cos^2\theta_0 - 1, \\ \theta &= \sin 2\theta_0 = 2 \sin\theta_0 \cos\theta_0 . \end{aligned} \tag{35}$$

Furthermore, given $\cos\theta_j$, one can calculate $\cos n\theta_j$ for $n = 2, 3, \cdots, N$ from the identity

$$\cos n\theta_j = 2\cos\theta_j \cos(n-1)\theta_j - \cos(n-2)\theta_j, \tag{36}$$

thus obtaining all the trigonometric functions required in the evaluation of a_n according to eqs. (20a) and (20b).

Given the expansion coefficients $a_0, a_1, \cdots, a_N$, the evaluation of the approximating function $f_N(t)$ from eq. (6) is most readily made by use of the recurrence relations

$$\begin{aligned} \Phi_0(x) &= e^{-x/2} \\ \Phi_1(x) &= (1-x)\Phi_0(x) \\ n\Phi_n(x) &= (2n-1-x)\Phi_{n-1}(x) - (n-1)\Phi_{n-2}(x); \quad n > 1. \end{aligned} \tag{37}$$

TABLE 1

(1) t	(2) Exact $f(t)$	Approximate $f_N(t)$: $c_0 = -0.5$ (3) $N = 30$	(4) $N = 20$	(5) $N = 10$	(6) Schmittroth [1]
.0	.0	.000000	.000011	−.00546	—
.5	.377345	.377345	.377343	.37857	.372
1.0	.533507	.533507	.533510	.532081	.534
1.5	.525424	.525424	.525422	.524683	.525
2.0	.419280	.419280	.419280	.419685	.419
2.5	.274110	.274110	.274111	.274808	.274
3.0	.133243	.133243	.133241	.133821	.133
3.5	.022128	.022128	.022128	.022663	.022
4.0	−.049530	−.049530	−.049528	−.048985	−.0496
4.5	−.083449	−.083449	−.083449	−.083101	−.0833
5.0	−.087942	−.087942	−.087945	−.088124	−.0877
5.5	−.073722	−.073722	−.073724	−.074617	−.0735
6.0	−.050892	−.050892	−.050891	−.052364	−.0508
6.5	−.027239	−.027239	−.027237	−.028853	−.0271
7.0	−.007644	−.007644	−.007644	−.008839	−.0076
7.5	.005714	.005714	.005712	.005401	.0057
8.0	.012715	.012715	.012713	.013479	.0127
8.5	.014511	.014511	.014511	.016222	.0144
9.0	.012805	.012805	.012807	.015055	.0127
9.5	.009302	.009302	.009304	.011543	.0092
10.0	.005385	.005385	.005386	.007077	.0053

TABLE 2

t	Exact $f(t)$	Approximate $f_N(t)$: $c_0 = 1$ $N = 50$	$N = 25$	$N = 10$	Norden [2] ($N = 18$)
0.201897	0.35738	0.35737	0.35739	0.35688	0.35737
.246597	.38255	.38256	.38248	.38116	.38254
.301194	.40851	.40850	.40853	.40710	.40849
.367897	.43511	.43509	.43509	.43461	.43508
.449329	.46221	.46220	.46220	.46272	.46218
.548812	.48967	.48966	.48967	.49012	.48963
.670320	.51731	.51729	.51730	.51667	.51725
.818731	.54495	.54494	.54493	.54403	.54489
1.000000	.57242	.57242	.57245	.57293	.57234
1.221403	.59953	.59954	.59951	.60029	.59945

TABLE 3

t	Exact $f(t)$	Approximate $f_N(t)$: $a_0 = 0$			Papoulis [7] ($N = 10$)
		$N = 30$	$N = 20$	$N = 10$	
0.01906	0.7353	0.7843	0.7811	0.9553	0.8133
.07654	.7842	.7847	.7871	.8698	.8739
.17334	.7795	.7797	.7804	.7715	.6958
.11012	.7665	.7660	.7621	.7026	.7787
.49188	.7386	.7390	.7352	.6837	.7896
.71921	.6871	.6872	.6865	.6975	.6363
.99743	.6019	.6015	.5983	.6901	.5977
1.33258	.4736	.4738	.4670	.5944	.5241
1.73287	.2976	.2980	.2967	.3708	.2612
2.20970	.0824	.0823	.0836	.0480	.0615
2.77032	−.1386	−.1390	−.1376	−.2613	−.0834
3.46574	−.2947	−.2950	−.2869	−.3989	−.3208
4.30643	−.2827	−.2830	−.2765	−.2647	−.3190
5.36443	−.0420	−.0425	−.0445	.0463	.0286
6.75813	.2278	.2276	.2187	.2046	.1748
8.75362	−.0211	−.0206	−.0122	−.0257	−.0011
12.20029	.0713	.0708	.0475	−.0283	−.0412

TABLE 4

t	Exact $f(t)$	Approximate $f_N(t)$: $a_0 = 0$			
		$N = 30$	$N = 30$	$N = 10$	
5.	.0000	−.0612	.2154	−.0216	—
10.	.0000	.0570	.1074	−.0219	—
15.	.0000	.1071	−.0573	.0032	—
20.	.0000	.0952	−.0359	.1929	—
25.	.5000	.5161	.4717	.4958	—
30.	1.0000	.9580	1.1117	.7022	—
35.	1.0000	1.0090	.9317	.9968	—
40.	1.0000	1.0206	1.0404	1.0873	—
45.	1.0000	.9816	.9867	1.0884	—
50.	1.0000	1.0030	.9821	1.0433	—
55.	1.0000	1.0030	1.0299	.9914	—
60.	1.0000	.9901	.9805	.9570	—
65.	1.0000	1.0076	.9870	.9483	—
70.	1.0000	.9802	1.0230	.9614	—
75.	1.0000	1.0050	.9952	.9862	—

These recurrence relations are obtained by multiplying the recurrence relation for the Laguerre polynomials $L_n(x)$ (as given in [12, p. 84; or 14, p. 100]) by the exponential factor $e^{-x/2}$.

Care must be taken to avoid loss of significant figures during the calculation of the expansion coefficients a_n. In the numerical examples discussed in the following section, the functions $g_1(c, \omega_j)$ and $g_2(c, \omega_j)$ were calculated using eight decimal positions. The operations indicated by eqs. (28)–(36), and (20a) and (20b), which proved most critical, were performed using sixteen decimal positions. The calculation of the Laguerre functions by means of eq. (37) and the evaluation of the sum indicated in eq. (6) were performed using eight decimal places. Ultimately, however, the quality of the approximation obtained depends largely on the behavior of the function $f(t)$ in the interval of interest.

Numerical Examples

The following four examples are representative of problems used to test the approximation method described above. The first three examples were chosen because they have been used by other authors to illustrate other methods for the numerical inversion of Laplace transforms. The fourth example was chosen because it gives a good illustration of the Gibbs phenomenon which arises in an attempt to approximate a function having a jump discontinuity.

The format of the examples is as follows (see Tables 1–4): The 1st column gives values of the independent variable t and the 2nd column, exact values of $f(t)$. The next three columns give approximate values of $f(t)$ for different numbers of terms N in the expansion of eq. (6). Where applicable, the 6th column gives approximate values obtained by other authors using different techniques for numerical inversion.

The peculiar choice of values of t in Examples 2 and 3 was made to obtain a comparison to the results obtained in [2, 7]. In these two references, both t and $f(t)$ are given parametrically in terms of a third independent variable. The accuracy of the method described in this paper appears comparable to the accuracy of the other methods described in the literature.

Example 1.

$$g(p) = \frac{1}{p^2 + p + 1}; \qquad f(t) = \frac{2}{3} e^{-t/2} \sin\left(\frac{3}{2} t\right)$$

Example 2.

$$g(p) = \frac{1}{p(\sqrt{p} + 1)}; \qquad f(t) = 1 - e^t \operatorname{erfc}(\sqrt{t})$$

Example 3.

$$g(p) = \frac{\pi}{4} \frac{1}{\sqrt{p^2 + 1}}; \qquad f(t) = \frac{\pi}{4} J_0(t)$$

Example 4.

$$g(p) = \frac{1}{p} e^{-25p}; \qquad f(t) = u(t - 25)$$

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