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THE NUMERICAL SOLUTION OF PARABOLIC AND ELLIPTIC DIFFERENTIAL EQUATIONS* †

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Introduction. Numerical approximations to solutions of the heat flow equation in two space dimensions may be obtained by the stepwise solution of an associated difference equation. Two types of difference equations have previously been studied: (1) explicit difference equations, which are simple to solve, but which require an uneconomically large number of time steps of limited size, and (2) implicit difference equations, which do not limit the time step but which require at each time step the solution by iteration of large sets of simultaneous equations.

In this paper, an alternating-direction implicit procedure is presented that requires the line-by-line solution of small sets of simultaneous equations that can be solved by a direct, non-iterative method. Analysis of the procedure shows it to be stable for any size time step and to require much less work than other methods that have been studied. As a practical test, the new procedure was used to solve the heat flow equation with boundary conditions for which the formal solution is known; the two solutions were in good agreement.

In addition, the alternating-direction implicit method is applicable to the iterative solution of two-dimensional steady-state problems. In a practical test, rapid convergence for the solution of Laplace's equation in a square was obtained by using a suitable set of iteration parameters which were easily calculated. An analysis is presented that shows the method to require about $(2 \log N)/N$ as many calculations as the best previously known iterative procedure for solving Laplace's equation, where N^2 is the number of points for which the solution is computed.

In the first part of the paper, the numerical solution of unsteady-state problems in two dimensions is discussed. For illustrative purposes, only the simplest type of problem is considered, that of unsteady-state heat flow in a square. In the second part of the paper, the numerical solution of steady-state problems in two dimensions is discussed. Again, only the simplest type problem is considered, namely, the solution of Laplace's equation in a square. In both parts of the paper, the analyses will be performed for special boundary conditions. These analyses can, in many cases, be extended in a straightforward manner to problems having different boundary conditions.

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The theoretical aspects of the procedures discussed here are treated in greater detail in a companion paper by Douglas [5].

SOLUTION OF THE HEAT FLOW EQUATION

The problem which is considered is that of unsteady-state heat flow in a square, wherein the boundaries are maintained at zero temperature and the square initially has a temperature of unity. The heat flow equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\partial T}{\partial t}, \tag{1}$$

where T is the temperature, x and y are the distances from the center of the square in the two perpendicular directions, and t is the time. The boundary conditions are

$$\begin{cases} x = 1, & T = 0. \\ x = -1, & T = 0. \\ y = 1, & T = 0. \\ y = -1, & T = 0. \end{cases} \tag{2}$$

and the initial condition is

$$t = 0, \qquad T = 1. \tag{3}$$

Because of symmetry, only a quarter of the square need be considered, in which case the boundary conditions become

$$\begin{cases} x = 0, & \dfrac{\partial T}{\partial x} = 0. \\ x = 1, & T = 0. \\ y = 0, & \dfrac{\partial T}{\partial y} = 0. \\ y = 1, & T = 0. \end{cases} \tag{4}$$

Background. In this section, numerical methods already developed for the solution of the heat flow equation are discussed in some detail in order to provide background for the description of the new method, and also in order to develop several equations that will be useful later.

Numerical procedures for the solution of equation (1) fall into two categories, explicit and implicit. One explicit scheme for the solution uses the following difference equation:

$$\frac{T_{i,j,n+1} - T_{i,j,n}}{\Delta t} = \frac{T_{i-1,j,n} - 2T_{i,j,n} + T_{i+1,j,n}}{(\Delta x^2)} + \frac{T_{i,j-1,n} - 2T_{i,j,n} + T_{i,j+1,n}}{(\Delta y)^2} \tag{5}$$

where i, j, and n are the indices in the x, y, and t directions, respectively. If we choose the mesh so that $\Delta x = \Delta y$, and define

$$\rho = \frac{(\Delta x)^2}{\Delta t} = \frac{(\Delta y)^2}{\Delta t}, \tag{6}$$

then the unknown temperatures at the $n + 1$ time step may be solved for explicitly by the equation

$$T_{i,j,n+1} = T_{i,j,n} + \frac{1}{\rho}[T_{i-1,j,n} + T_{i+1,j,n} + T_{i,j-1,n} + T_{i,j+1,n} - 4T_{i,j,n}]. \tag{7}$$

The stability of this scheme will be analyzed by a procedure very similar to that used by O'Brien, Hyman, and Kaplan [4]. Assume that there exists an error $\epsilon_{i,j,n}$ at each mesh point. It is easily shown for linear problems with constant coefficients that these errors obey the same difference equation that is used to obtain the numerical solution. Then, for the explicit scheme described above,

$$\epsilon_{i,j,n+1} = \epsilon_{i,j,n} + \frac{1}{\rho}[\epsilon_{i-1,j,n} + \epsilon_{i+1,j,n} + \epsilon_{i,j-1,n} + \epsilon_{i,j+1,n} - 4\epsilon_{i,j,n}]. \tag{8}$$

Because ϵ obeys both the difference equation (8) and the boundary conditions (4), it may be expanded in a finite double series of orthogonal functions that also satisfy both the difference equation (8) and the boundary conditions, namely

$$\epsilon_{i,j,n} = \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} A_{p,q,n} \cos \beta_p x \cos \beta_q y \tag{9}$$

where $\beta_p = (2p + 1)\pi/2$, $\beta_q = (2q + 1)\pi/2$, and N is the number of intervals in each of the x- and y-directions. Thus $N = 1/\Delta x = 1/\Delta y$. Substituting equation (9) into equation (8) and examining each term in the series separately shows that

$$A_{p,q,n+1} = A_{p,q,n}\left[1 + \frac{1}{\rho}(2 \cos \beta_p \Delta x + 2 \cos \beta_q \Delta y - 4)\right], \tag{10}$$

or

$$\frac{A_{p,q,n+1}}{A_{p,q,n}} = 1 - \frac{4}{\rho}\left(\sin^2 \frac{\beta_p \Delta x}{2} + \sin^2 \frac{\beta_q \Delta y}{2}\right). \tag{11}$$

For stability, this ratio must have an absolute value less than or equal to unity for all p and q. This requires that

$$-1 \leqq 1 - \frac{4}{\rho}\left(\sin^2 \frac{\beta_p \Delta x}{2} + \sin^2 \frac{\beta_q \Delta y}{2}\right) \leqq 1. \tag{12}$$

The right-hand inequality is trivial, while the left-hand inequality is satisfied if

$$\rho \geqq 4. \tag{13}$$

The restriction on the size of the time step is, then,

$$\Delta t \leqq \frac{(\Delta x)^2}{4} = \frac{1}{4N^2}. \tag{14}$$

Thus, a total of $4N^2t$ time steps are required to calculate the solution to time t. Since 7 arithmetic operations are required for each of the N^2 points at each time step, a total of 28 N^4t operations is necessary for the complete solution. For example, if it is desired to carry out the solution to time $t = 1.5$, when T is everywhere less than 0.001, 420,000 operations are required for a ten-by-ten net; the situation rapidly becomes worse for a finer mesh.

It has been shown [4] for the heat flow equation in one dimension that the restriction on the size of Δt arising from stability can be removed by the use of an implicit procedure in which the second derivatives are approximated by second differences evaluated in terms of the unknown temperatures, T_{n+1}. An extension of this idea to the problem in two dimensions yields the following equation:

$$\frac{T_{i,j,n+1} - T_{i,j,n}}{\Delta t} = \frac{T_{i-1,j,n+1} - 2T_{i,j,n+1} + T_{i+1,j,n+1}}{(\Delta x)^2} + \frac{T_{i,j-1,n+1} - 2T_{i,j,n+1} + T_{i,j+1,n+1}}{(\Delta y)^2}. \tag{15}$$

A derivation similar to the preceding one shows that

$$\frac{A_{p,q,n+1}}{A_{p,q,n}} = \frac{1}{1 + \frac{4}{\rho}\left(\sin^2 \frac{\beta_p \Delta x}{2} + \sin^2 \frac{\beta_q \Delta y}{2}\right)}. \tag{16}$$

For all values of ρ, this ratio has an absolute value less than unity, so that the procedure is stable for all size time steps.

Other stable implicit procedures similar to equation (15) might be proposed. However, they lead to large sets of N^2 linear simultaneous equations with N^2 unknowns. In particular, the simultaneous equations that arise from Equation (15) may be written as follows:

$$T_{i-1,j,n+1} + T_{i+1,j,n+1} + T_{i,j-1,n+1} + T_{i,j+1,n+1} - (4 + \rho)T_{i,j,n+1} = -\rho T_{i,j,n} \quad (0 \leqq n \leqq N - 1). \tag{17}$$

Solution of these equations by an elimination procedure is out of the question, since the labor involved is roughly proportional to $(N^2)^3$. Fortunately, iterative methods are quite successful for solving simultaneous equations of this type. Frankel [3] has examined the rates of convergence of several methods of iterating the Laplace difference equation (which may be obtained from equation (17) by setting $\rho = 0$) and found the best method to be the extrapolated Liebmann method (also known as the successive overrelaxation method). The authors, using Frankel's analysis of convergence, examined several methods of iterating equation (17) and found the extrapolated Liebmann method to be the best for these equations also. This process may be written:

$$T^{(m+1)}_{i,j,n+1} = T^{(m)}_{i,j,n+1} + \alpha[T^{(m+1)}_{i-1,j,n+1} + T^{(m)}_{i+1,j,n+1} + T^{(m+1)}_{i,j-1,n+1} + T^{(m)}_{i,j+1,n+1} + T^{(m)}_{i,j,n+1} - (4 + \rho)T^{(m)}_{i,j,n+1} + \rho T_{i,j,n}], \tag{18}$$

where the superscript, m, is the number of iteration stages already carried out, and α is the relaxation factor, chosen for the maximum rate of convergence. The analysis of convergence is the same as that given by Frankel for the Laplace difference equation, except that the boundary conditions (4) are not the same as those used by Frankel. If this difference in boundary conditions, as well as the presence of ρ, is taken into account, the optimum α is found to be the smaller root of

$$\alpha^2 u^2 - (4 + \rho)\alpha + 1 = 0 \tag{19}$$

where

$$u = 2 \cos \frac{\pi}{2N} \approx 2 - \frac{\pi^2}{4N^2}. \tag{20}$$

In each step of the iteration, the magnitude of every error component is decreased at worst by a factor, K^*, which is the largest eigenvalue of the difference operator (18). This eigenvalue is found to be

$$K^* = (4 + \rho)\alpha - 1. \tag{21}$$

Solving for α and K^*, we obtain

$$\alpha \approx \frac{4 + \rho - (8\rho + \rho^2 + 4\pi^2/N^2)^{1/2}}{8}, \tag{22}$$

$$K^* \approx 1 - \frac{(4 + \rho)(8\rho + \rho^2 + 4\pi^2/N^2)^{1/2} - (8\rho + \rho^2)}{8}. \tag{23}$$

If the iteration is continued until all errors are reduced by a factor of 10^{-v}, the number of cycles of iteration will be approximately

$$\eta \approx v/\log_{10} (1/K^*). \tag{24}$$

In each iteration, $7N^2$ arithmetic operations must be performed. Consequently,

$$7\nu N^2/\log_{10}(1/K^*)$$

operations are required for each time step.

In any case where an implicit procedure is used, ρ will be considerably less than 4, inasmuch as the explicit procedure may be used for $\rho \geqq 4$. As a minimum estimate of the number of iterations required, we may consider the case of $\rho = 1$, with N sufficiently large that the term containing it may be ignored. Then $K^* = 0.25$, and 5 cycles of iteration are required to reduce the errors by a factor of 10^{-3}. For larger time steps, corresponding to smaller values of ρ, more cycles of iteration are required.

Alternating-direction implicit method. As was shown in the preceding section, when the implicit procedure is set up with both of the second derivatives replaced by second differences evaluated in terms of the unknown values of T, large sets of simultaneous equations are formed, which can be solved, practically, only by iteration. If, however, only one of the second derivatives, say $\partial^2 T/\partial x^2$, is replaced by a second difference evaluated in terms of the unknown values of T, while the other derivative, $\partial^2 T/\partial y^2$, is replaced by a second difference evaluated in terms of known values of T, sets of simultaneous equations are formed that can be solved easily without iteration. These equations are implicit in the x-direction. If the procedure is then repeated for a second time step of equal size, with the difference equations implicit in the y-direction, the overall procedure for the two time steps is stable for any size time step.

Thus, two difference equations are used, one for the first time step, the other for the second time step:

$$\frac{T_{i,j,2n+1} - T_{i,j,2n}}{\Delta t} = \frac{T_{i-1,j,2n+1} - 2T_{i,j,2n+1} + T_{i+1,j,2n+1}}{(\Delta x)^2} + \frac{T_{i,j-1,2n} - 2T_{i,j,2n} + T_{i,j,+1,2n}}{(\Delta y)^2}, \tag{25}$$

$$\frac{T_{i,j,2n+2} - T_{i,j,2n+1}}{\Delta t} = \frac{T_{i-1,j,2n+1} - 2T_{i,j,2n+1} + T_{i+1,j,2n+1}}{(\Delta x)^2} + \frac{T_{i,j-1,2n+2} - 2T_{i,j,2n+2} + T_{i,j+1,2n+2}}{(\Delta y)^2}. \tag{26}$$

These equations may be arranged in the following form, more suitable for calculation.

$$T_{i-1,j,2n+1} - (2 + \rho)T_{i,j,2n+1} + T_{i+1,j,1n+1} = -T_{i,j-1,2n} + (2 - \rho)T_{i,j,2n} - T_{i,j+1,2n}, \tag{27}$$

$$T_{i,j-1,2n+2} - (2+\rho)T_{i,j,2n+2} + T_{i,j+1,2n+2} = -T_{i-1,j,2n+1} + (2-\rho)T_{i,j,2n+1} - T_{i+1,j,2n+1}. \tag{28}$$

Use of equation (27) or (28) at each time step leads to N sets of N simultaneous equations of the form

$$\begin{cases} B_0T_0 + C_0T_1 = D_0, \\ A_rT_{r-1} + B_rT_r + C_rT_{r+1} = D_r \qquad (1 \leqq r \leqq N-2), \\ A_{N-1}T_{N-2} + B_{N-1}T_{N-1} = D_{N-1}. \end{cases} \tag{29}$$

The solution of these equations may be obtained in a straightforward manner. Let

$$\begin{cases} w_0 = B_0, \\ w_r = B_r - A_r b_{r-1} \qquad (1 \leqq r \leqq N-1), \end{cases} \tag{30}$$

$$b_r = \frac{C_r}{w_r} \qquad (0 \leqq r \leqq N-2). \tag{31}$$

and

$$\begin{cases} g_0 = \dfrac{D_0}{w_0} \\ g_r = \dfrac{D_r - A_r g_{r-1}}{w_r} \qquad (1 \leqq r \leqq N-1). \end{cases} \tag{32}$$

The solution is

$$\begin{cases} T_{N-1} = g_{N-1} \\ T_r = g_r - b_r T_{r+1} \qquad (0 \leqq r \leqq N-2). \end{cases} \tag{33}$$

Thus, w, b, and g are computed in order of increasing r, and T is computed in order of decreasing r. Examples of the use of this method of solving equations (29) for the solution of one-dimensional flow problems, as well as a proof, by matrix algebra, of equations (30) to (33) are given in a recent article by the authors [1].

Stability. The stability of the procedure may be demonstrated by a derivation similar to that leading to equation (10). The following equations are obtained:

$$A_{p,q,2n+1}[2\cos\beta_p\Delta x - (2+\rho)] = A_{p,q,2n}[-2\cos\beta_q\Delta y + (2-\rho)] \tag{34}$$

$$A_{p,q,2n+2}[2\cos\beta_q\Delta y - (2+\rho)] = A_{p,q,2n+1}[-2\cos\beta_p\Delta x + (2-\rho)]. \tag{35}$$

If equation (27) were used for every time step, the amplification factor for each step would be

$$\frac{A_{p,q,2n+1}}{A_{p,q,2n}} = \frac{\rho - 4 \sin^2 (\beta_q \Delta y/2)}{\rho - 4 \sin^2 (\beta_p \Delta x/2)} \tag{36}$$

For some values of p, q, and ρ, this ratio has an absolute value considerably greater than unity. Hence, such a procedure is highly unstable. When equations (27) and (28) are used alternately, however, the amplification factor for the two-step procedure is

$$\frac{A_{p,q,2n+2}}{A_{p,q,2n}} = \frac{\rho - 4 \sin^2 (\beta_p \Delta x/2)}{\rho + 4 \sin^2 (\beta_p \Delta x/2)} \times \frac{\rho - 4 \sin^2 (\beta_q \Delta y/2)}{\rho + 4 \sin^2 (\beta_q \Delta y/2)}. \tag{37}$$

This ratio has an absolute value less than unity for all p, q, and ρ. Note that it is necessary that ρ and, therefore, Δt, be the same for two time steps.

Work requirement. For the N simultaneous equations resulting from the use of equation (27) or (28) on a single line, the solution by equations (30) to (33) require approximately $9N$ arithmetic operations. For each time step, $9N^2$ operations must be performed. We have seen that, for the extrapolated Liebmann method, with $\rho = 1$, at least five iterations are necessary, each requiring $7N^2$ operations. Therefore, for $\rho = 1$, the best iterative procedure known requires about four times as much work as the alternating-direction method. For smaller values of ρ, the iterative procedure compares even more unfavorably, as will be seen in the numerical example below. If ρ is kept constant, the number of operations is $9\rho N^4 t$. However, it is not necessary that a constant time step be used [6].

The alternating direction implicit method requires a relatively small increase in storage capacity over the extrapolated Liebmann method. The latter method requires approximately N^2 registers of storage, whereas the new method requires approximately $N^2 + N$ registers.

Numerical example. The alternating-direction implicit method was tested by carrying out a numerical solution of equations (1), (3) and (4) on an I.B.M. Card Programmed Calculator, wired to perform eight-digit, floating-decimal arithmetic. Because of storage limitations, fourteen increments were used in each of the x- and y-directions. Thirty-six time steps were used; small steps were used at the beginning and continually larger steps were used as the solution progressed. It was necessary, of course, that each size time step be used an even number of times. The schedule followed is shown in Table 1. Included in the table is the value of ρ associated with each Δt, as well as the value of η, the number of iterations that would have been required for each time step if the extrapolated Liebmann method had been used. The quantity η is calculated from equations (23) and (24) using $v = 3$.

Results of the numerical calculation at even time steps were compared with the exact solution of the problem at $t = 0.06$, 0.1, 0.2, 0.4, 0.8, and

1.5. The exact solution was calculated by the series given in Carslaw and Jaeger [2]. The maximum error found was 0.0042 at $t = 0.1$. It may be seen that this error is due to truncation by repeating the calculation with a smaller value of Δt and comparing the errors. First, the exact solution at $t = 0.06$ was taken for the initial condition, and the solution at $t = 0.1$ was calculated in two steps using $\Delta t = 0.02$. The maximum error was 0.0032. Then, with the same initial condition, the solution at $t = 0.1$ was calculated in ten steps, using $\Delta t = 0.004$. The maximum absolute error in this case was only 0.0004.

It is possible to make an exact comparison between the labor involved in using the alternating-direction implicit method for the solution of this problem and that which would have been involved if either the explicit method or the implicit method with iteration had been used. Since 36 time steps were used, the alternating-direction method required approximately $(36)(9)(14)^2 \approx 63{,}500$ arithmetic operations. The explicit procedure would have required about $28(14)^4(1.5) \approx 1{,}610{,}000$ operations, or about 25 times as much work. The implicit procedure, with extrapolated Liebmann iteration, would have required for each of the 36 time steps the number of iterations, η, shown in Table 1, or a total of 326 iterations. This implicit procedure, then, would have required about $(326)(7)(14)^2 \approx 447{,}000$ arithmetic operations, or about seven times as much work.

SOLUTION OF LAPLACE'S EQUATION

In this section, the numerical solution of steady-state heat flow and diffusion problems in two dimensions is discussed. The typical problem considered here is that of determining the temperature distribution in a square in which two opposite faces are at zero temperature while the remaining two faces have a temperature of unity. Because of symmetry, it is

TABLE 1.—Time Steps Used in the Solution of the Heat Flow Problem.

Δt	No. of times used	ρ	$\eta_{v=3}$
0.001	6	5.102	3
.002	4	2.551	3
.003	2	1.701	4
.005	4	1.020	5
.01	2	0.5102	7
.02	4	0.2551	9
.03	2	0.1701	11
.05	4	0.1020	13
.1	6	0.05102	17
.25	2	0.02041	21

again sufficient to consider only one-quarter of the square. The differential equation that applies is Laplace's equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{38}$$

The boundary conditions are

$$\begin{cases} x = 0, & \dfrac{\partial T}{\partial x} = 0 \\ x = 1, & T = 0 \\ y = 0, & \dfrac{\partial T}{\partial y} = 0 \\ y = 1, & T = 1. \end{cases} \tag{39}$$

Iteration by extrapolated Liebmann method. The conventional numerical solution of Laplace's equation consists in approximating it by the difference equation

$$T_{i-1,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1} - 4T_{i,j} = 0 \tag{40}$$

and iterating to the solution of the resulting simultaneous equations. As pointed out above, Frankel [3] has examined the rates of convergence of several methods of iterating the Laplace difference equation and found the extrapolated Liebmann method to be the best. The iteration process may be represented by equation (18) with $\rho = 0$, so that equations (22) and (23) may be used to calculate the optimum relaxation factor and the corresponding rate of convergence. These are:

$$\alpha \approx \frac{2 - \pi/N}{4} \tag{41}$$

$$K^* \approx 1 - \pi/N. \tag{42}$$

The number of cycles of iteration required to reduce all errors by a factor of 10^{-v} will be approximately

$$\eta = 2.3vN/\pi. \tag{43}$$

Since $7N^2$ arithmetic operations must be performed in each iteration, a total of approximately $5vN^3$ operations is necessary.

Iteration by alternating-direction implicit method. The alternating-direction implicit method may also be used to iterate to the solution of Laplace's equation. In this case, each stage of iteration may be regarded

as a time step of an unsteady-state problem, while the starting values used for the first iteration correspond to the initial condition. Equations (27) and (28) are used for alternate stages of the iteration, with n representing the number of pairs of stages already carried out and ρ serving as an iteration parameter.

Convergence. We now investigate the convergence of this iteration procedure in order to determine the optimum values of the iteration parameter, ρ. The rate of convergence is determined by studying how the errors decay at successive stages of the iteration. Since the starting values should satisfy the boundary conditions (39), the initial errors as well as the intermediate errors should satisfy the boundary conditions given by equation (4). Hence, these errors behave in precisely the same manner as the errors present in the application of equations (27) and (28) to the solution of the heat flow equation. Thus, the results obtained for the analysis of stability may also be used for the analysis of convergence. The iteration errors may be expanded in the double cosine series of equation (9), and the amplification factor for each component, $A_{p,q}$, of the error is again given by equation (37).

As may be seen from equation (37), the amplification factor may be reduced to zero for each component by suitable choices of ρ. If we use

$$\rho_p = 4 \sin^2 \frac{\beta_p \Delta x}{2} = 4 \sin^2 \frac{(2p + 1)\pi}{4N} \tag{44}$$

for each of the values of p: $0, 1, 2, \cdots, N - 1$, all of the components will be reduced to zero on some one of the iterations.

It would appear that all of the components could be reduced to zero by using only equation (27) with the values of ρ_p given by equation (44); that is, by iterating with the unknowns implicit in just the x-direction. However, equation (36) shows that such a process is highly unstable for all but the p^{th} component. It is, therefore, necessary to iterate using both equations (27) and (28) alternately. This permits the successive reduction of each component to zero without the simultaneous amplification of any of the other components.

From these considerations, it follows that $2N$ iterations are required to reduce all the error components to zero. However, it is not necessary to take $2N$ iterations to obtain a satisfactory convergence. If N is sufficiently large, an examination of the values of ρ_p shows that, at the larger values of p, the values of ρ_p lie fairly close together. Consequently, an average ρ may be used for a group of values of p. The amplification factors for the components corresponding to the values of p falling within that group, while not zero, are sufficiently small to reduce these components effectively.

When N is not too large, it is possible to calculate all the values of ρ_p by equation (44) and to set up the groups by inspection. For large values of N, however, it is desirable to have a formal procedure for setting up these groups, and to examine how the number of groups increases as N increases.

Let p_u be the lowest value of p in a particular group and p_{u+1} be the highest. It can be seen from equation (37) that every component, $A_{p,q}$ with $p_u \leqq p \leqq p_{u+1}$ or $p_u \leqq q \leqq p_{u+1}$, is acted upon at least once with an amplification factor whose magnitude is less than or equal to the greater of the two quantities

$$\left|\frac{\rho - 4\sin^2((2p_u+1)\pi/4N)}{\rho + 4\sin^2((2p_u+1)\pi/4N)}\right|, \qquad \left|\frac{\rho - 4\sin^2(2p_{u+1}+1)\pi/4N)}{\rho + 4\sin^2(2p_{u+1}+1)\pi/4N)}\right|,$$

where ρ is the iteration parameter associated with that group. It is thus possible to group the p's in such a way that every component is acted upon at least once by an amplification factor whose magnitude is less than or equal to some quantity, R. To have as few groups as possible, and therefore the minimum number of iterations, p_u, p_{u+1}, and ρ for each group must satisfy the following relationship:

$$\frac{\rho - 4\sin^2((2p_u+1)\pi/4N)}{\rho + 4\sin^2((2p_u+1)\pi/4N)} = -\frac{\rho - 4\sin^2((2p_{u+1}+1)\pi/4N)}{\rho + 4\sin^2((2p_{u+1}+1)\pi/4N)} = R. \tag{45}$$

It is assumed, for purposes of setting up the groups, that p_u may have non-integral values. By eliminating ρ from equation (45), we have for each group,

$$\sin^2\frac{(2p_{u+1}+1)\pi}{4N} = \left(\frac{1+R}{1-R}\right)^2 \sin^2\frac{(2p_u+1)\pi}{4N}. \tag{46}$$

For the group containing the highest values of p, $p_{u+1} = N - 1$, while for the group containing the lowest values, $p_u = 0$. Then, if s is the total number of groups,

$$\sin^2\frac{(2N-1)\pi}{4N} = \left(\frac{1+R}{1-R}\right)^{2s} \sin^2\frac{\pi}{4N}. \tag{47}$$

For large N,

$$\left(\frac{\pi}{4N}\right)^2 \approx \left(\frac{1-R}{1+R}\right)^{2s}, \tag{48}$$

$$s \approx \frac{\log N + \log(4/\pi)}{\log[(1+R)/(1-R)]} \approx \frac{\log N}{2R}. \tag{49}$$

TABLE 2.—Iteration Parameters for $N = 14$.

p	ρ	Δt_p
0	0.012576	0.40571
1	0.11223	.04546
2	0.30655	.01664
3	0.58579	.008710
4	0.93596	.005451
5	1.3394	.003809
6	1.7761	.00287
7	2.2239	.00229
8	2.6606	.00192
9	3.0642	.00166
10	3.4142	.00149
11	3.6935	.00138
12	3.8878	.00131
13	3.9874	.00128

Thus, the number of iterations increases approximately as the logarithm of N, and the number of arithmetic operations increases as $N^2 \log N$, as opposed to N^3 for the extrapolated Liebmann method.

Numerical example. Iteration of the Laplace difference equation by the alternating-direction implicit method was tested by carrying out the numerical solutions of equations (38) and (39) on the Card Programmed Calculator. Again, fourteen increments on a side were used. Using $N = 14$, the values of ρ_p were calculated from equation (44) and are listed in Table 2. The corresponding Δt_p, calculated from equation (6), are also listed.

For the calculation, five values of Δt were chosen by inspection: 0.0015 for $p = 8$ to 13; 0.003 for $p = 4$ to 7; 0.01 for $p = 2, 3$; 0.04546 for $p = 1$ and 0.40571 for $p = 0$. These values of Δt were used in order of increasing magnitude. Each Δt was used twice, once with the unknowns implicit in the x-direction, and once with the unknowns implicit in the y-direction. The following relationship for T was used for the first trial:

$$T = \frac{1 - x}{2 - x - y}. \tag{50}$$

In the ten stages of iteration, the solution converged to within 0.000014 of the exact solution of the difference equation. The initial guess had a maximum error of about 0.039, so that the ten steps reduced the errors by a factor of about 4×10^{-4}.

The work involved in reducing the errors by a factor of 10^{-3}, then, is about $(10)(9)(14)^2$ arithmetic operations. In the more general case, we may expect the number of iterations to be about

$$10 \frac{\log N}{\log 14} = 3.8 \log N,$$

and the total work to be about $(3.8 \log N)(9)N^2$ or $34N^2 \log N$ operations. The extrapolated Liebmann method requires $15N^3$ operations, so that the new method is better by a factor of about $N/(2 \log N)$ over the next best iterative method.

Since this paper was submitted, several examples involving more complex regions and less simple boundary conditions have been solved by means of the alternating-direction implicit method [7]; however, no proofs of the validity of the procedure in these cases have been obtained.

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