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ASYNCHRONOUS RELAXATIONS FOR THE NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS BY PARALLEL PROCESSORS*

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Abstract. We consider asynchronous relaxations for the numerical solution by parallel processors of initial value problems involving a class of ordinary differential equations. The differential equations may be nonlinear, nonautonomous and nonhomogeneous. The proposed algorithms have several advantages in the parallel processing context, and these apply irrespective of the number of processors and the architecture of the computing system. The form of the given equations is as a partitioned system which closely corresponds to the composition of physical systems from subsystems. The algorithms iterate asynchronously on functions. We prove that the computed functions uniformly converge at a geometric rate to the unique solution of the given equations. The assumption on the asynchronous aspects of the algorithm are that delays are uniformly bounded and that a nonstarvation condition applies. In practice these are not burdensome. The assumptions on the differential equations are dominance conditions. For linear autonomous equations the assumption is that a certain matrix is an M-matrix.

Key words. parallel processing, asynchronous relaxations, waveform relaxations, differential equations

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1. Introduction. We propose asynchronous relaxations for obtaining, by parallel processing, the solutions of initial value problems on ordinary differential equations. The differential equations may be nonlinear, nonautonomous and nonhomogeneous; the results of this paper apply when various restrictions are placed on the equations. Relaxation algorithms can be devised with various degrees of asynchronism, and an asynchronous computational model is proposed in this paper for their unified specification and analysis. The generality of the asynchronous computational model is derived from the presence of its various parameters. A particular limiting case, which is obtained by appropriate selections of the parameters, is synchronous relaxations. At the other extreme, we obtain a purely asynchronous algorithm of particular interest. This algorithm is characterized by the absence of essentially all synchronization or coordination on the processors. Other algorithms with intermediate degrees of asynchronism are also described by the model. The asynchronous computational model is precisely described in § 2.2.

Asynchronous algorithms have been previously proposed [1]–[5] and analyzed for use in parallel processors for the solution of various problems other than the solution of differential equations. Certain advantages are obvious: first, the reduction in synchronization between processors yields reduced idle times for the processors; contentions over communication recources and memory accesses are reduced; task management and programming are simplified. On the crucial question of algorithmic efficiency there is evidence that, in certain interesting cases, asynchronous algorithms outperform their synchronous counterparts on account of the different nature of information flow during computations. For instance, [5] gives experimental data which shows over an interesting range of parameters the improvement in performance gained from asynchronism. An important goal, then, is to be able to recognize problems which can be solved by asynchronous relaxations. The analysis in this paper is devoted to that end.

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We prove convergence results for the asynchronous computational model. Let $\|\mathbf{x}\|$ denote a norm of $\mathbf{x} \in \mathbb{R}^N$ and X the set of all N-vector functions which are continuous in the interval $[0, t']$. Define the norm of any function $\mathbf{x}$ in X to be

$$\|\mathbf{x}\| = \sup_{0 \leqq t \leqq t'} \|\mathbf{x}(t)\|.$$

We prove that $\|\mathbf{x}^{(i)} - \mathbf{x}\| \leqq cR^i$, $0 \leqq R < 1$, $c < \infty$, where $\mathbf{x}^{(i)}$ is the vector function available after the ith update in the asynchronous relaxations, and $\mathbf{x}$ is the unique vector function solution of the given initial value problem. Thus uniform convergence at a geometric rate is proven. See § 6.2 for a discussion and summary.

There are two sets of assumptions made to arrive at the above results. The first specifically concerns the asynchronous aspects and the second concerns the differential equations. Roughly, the first set (see § 2.3) requires that the delays in the implementation are bounded by a finite constant and that every type of task is guaranteed at least one completion in all sequences of task completions of an arbitrary, but fixed, length. These requirements are generally easy to satisfy. The assumptions on the differential equations (see §§ 4.1 and 5.1 for the linear and nonlinear equations, respectively) are all related to dominance conditions. In the special case where the homogeneous part of the equation is autonomous and linear, the only assumption is that a certain matrix, $\mathbf{M} - \mathbf{N}$, is an M-matrix [8]. In this case the rate of convergence is given in part by the spectral radius of $\mathbf{M}^{-1}\mathbf{N}$.

The iteration in the algorithm is on functions. From the point of view of this paper, the primitive tasks consist primarily of the computation of functions which are solutions of nonhomogeneous, initial value problems. The method of numerical integration used to solve the problem is not discussed in this paper.

The purely asynchronous algorithm, which we have mentioned before as a limiting case of the asynchronous computational model, is characterized by each processor successively executing the following cycle until termination, independently of the state of the other processors: read the functions of the task allocated to it, solve the initial-value problem associated with the task, and finally disseminate the solution functions by either writing into shared memory or by sending messages to the appropriate processors. Note that this form of asynchronous computations is compatible with the use of a scheduler to order the tasks. Typically such a scheduler is required to parallelize Gauss-Seidel schemes in which a single processor performs the tasks in some specific order [12].

The asynchronous properties of the relaxations are derived primarily from two sources. First, the times taken to perform tasks are allowed to vary unpredictably within large bounds, and, second, great latitude is allowed in the sequencing of tasks. The convergence result is for a computational model which accommodates major differences in the number and speed of processors, memory, communications and synchronization primitives. These differences are reflected in the values of parameters of the computational model, i.e. the delays and the compositions of the update sets.

The synchronous version of the algorithm, see § 2.2, has sometimes been called "waveform relaxations" [6], and these have been successfully applied to the solution of VLSI circuit-equations by uniprocessors. It is shown below that the synchronous algorithm is a special case of the asynchronous computational model described in § 2.2.

The paper is organized as follows. In § 2.1 we describe the equations, and in §§ 2.2 and 2.3 we give the computational model and the accompanying assumptions. Section 3 is mainly concerned with differential inequalities, which are used throughout the paper. Sections 4 and 5 analyze, in turn, linear and nonlinear equations. We feel that such a separation benefits the exposition. Section 6 picks up the qualitatively similar results obtained at the ends of §§ 4 and 5, to prove uniform convergence for both types of equations. Concluding remarks are contained in § 7.

2. Model of asynchronous relaxations. We give, first, a preliminary description of a class of differential equations and, second, a model of the asynchronous relaxations for numerically solving the equations.

2.1. A class of differential equations. The most general equations that are considered in this paper are nonlinear, nonautonomous and nonhomogeneous. The initial value is given and a unique solution is known to exist for the interval of interest, say $0 \leqq t < t'$. The value of t' is ignored here since our analysis applies for all t'; however, it is recognized that t' may be an important parameter in the implementation since storage requirements alone may cause the interval of interest to be partitioned into several smaller intervals [12].

The equations are of the form

$$\begin{aligned} &\frac{d}{dt}\mathbf{x}(t)+\mathbf{F}(\mathbf{x}, t)=\mathbf{G}(\mathbf{x}, t)+\mathbf{u}(t), \qquad t \geqq 0, \\ &\mathbf{x}(0)=\mathbf{x}_0, \end{aligned} \tag{2.1}$$

in which $\mathbf{x}$, $\mathbf{u}$, $\mathbf{F}$, and $\mathbf{G}$ are elements of $\mathbb{R}^N$ for each value of t. In addition, it is assumed that $\mathbf{F}$ has a special structure:

$$\mathbf{F}(\mathbf{x}, t)=\{\mathbf{F}_1(\mathbf{x}_1, t), \mathbf{F}_2(\mathbf{x}_2, t), \cdots, \mathbf{F}_n(\mathbf{x}_n, t)\}'. \tag{2.2}$$

$\mathbf{G}$ may not have this special structure. Thus (2.1) allows the following partitioning into n subsystems which are indexed by j.

$$\begin{aligned} &\frac{d}{dt}\mathbf{x}_j(t)+\mathbf{F}_j(\mathbf{x}_j, t)=\mathbf{G}_j(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n, t)+\mathbf{u}_j(t), \qquad t \geqq 0, \quad 1 \leqq j \leqq n, \\ &\mathbf{x}_j(0)=\mathbf{x}_{j,0}. \end{aligned} \tag{2.3}$$

In (2.2) and (2.3); $\mathbf{x}_j$, $\mathbf{u}_j$, $\mathbf{F}_j$ and $\mathbf{G}_j$ are elements of $\mathbb{R}^{N_j}$ for each value of t, and $\sum_{j=1}^n N_j = N$.

The assumed structure is naturally associated with physical systems composed of subsystems, in which case the internal dynamical relations of the jth subsystem are incorporated in the function $\mathbf{F}_j$, while the cross-couplings between variables of the jth subsystem and the variables in the other subsystems are reflected in the function $\mathbf{G}_j$. The function $\mathbf{G}_j$ is allowed to have $\mathbf{x}_j$ as an argument because in certain nonlinear physical systems a complete separation of variables may not be possible without defining very large subsystems.

The solution of the nonlinear equations (2.3) is considered in § 5, while in § 4 the simpler, linear counterpart of the equation and its solution are considered and the necessary techniques developed.

2.2. The model for asynchronous computations. In order to describe the computational model we need to first introduce a set of primitive tasks, $\{T_j\}$, of various types. The subscript gives the type, of which there are as many as there are subsystems in the system of differential equations, namely, n. As this suggests, there is a close connection between tasks T_j and the jth subsystem in (2.3).

For a task T_j to be completely defined, its arguments $(\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n)$ must be specified. These arguments are continuous vector functions of t, $t \geqq 0$, and $\mathbf{z}_j \in \mathbb{R}^{N_j}$,

$1 \leqq j \leqq n$, for each value of t. We say that $\mathbf{y} = T_j(\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n)$ if $\mathbf{y}(t)$, $t \geqq 0$, is the unique solution of the following initial value problem:

$$\begin{aligned} &\frac{d}{dt}\mathbf{y} + \mathbf{F}_j(\mathbf{y}, t) = \mathbf{G}_j(\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n, t) + \mathbf{u}_j(t), \qquad t \geqq 0, \\ &\mathbf{y}(0) = \mathbf{x}_{j,0}. \end{aligned} \tag{2.4}$$

Thus apart from its arguments, the task T_j is tied to the jth subsystem in (2.3). For this reason we sometimes refer to the completion of a task T_j as an update of the jth subsystem.

In the following description of the model for asynchronous computations, the symbol i is reserved for indexing the updates, i.e., task completions. Typically, but not always, the completion of only one task comprises an update. To allow the more general case, $U(i)$, a subset of $\{1, 2, \cdots, \text{n}\}$, denotes the types of tasks (or subsystem indices) whose completion comprise the ith update. If the ith update is due, perhaps in part only, to a task T_j being completed, i.e., $j \in U(i)$, then we denote the result of this completed task by $\mathbf{x}_j^{(i+1)}$.

The computational model is

$$\mathbf{x}_j^{(i+1)} = \begin{cases} T_j(\mathbf{x}_1^{(i-d(i,j,1))}, \mathbf{x}_2^{(i-d(i,j,2))}, \cdots, \mathbf{x}_n^{(i-d(i,j,n))}) & \text{if } j \in U(i), \\ \mathbf{x}_j^{(i)} & \text{if } j \notin U(i) \end{cases} \tag{2.5}$$

for $i = 0, 1, 2, \cdots$.

The terms $\{d(i, j, k)\}$ are referred to as delays. The delays are not assumed to be known in advance, and in fact, they will have many components, such as numerical integration time, communication delays, memory access delays and software delays. The substantial extent to which the implementation is allowed to be asynchronous in the model is due to the allowed dependence of the delays on their three arguments, and the minimal nature of the restriction placed on delay values. The only restriction that will be placed on the delay values, see A2 in § 2.3, is one of uniform boundedness.

Synchronous relaxations are obtained from (2.5) in the special case where $d(i, j, k) \equiv 0$ and $U(i) = \{1, 2, \cdots, n\}$ for each i. The synchronous relaxations have also been called "waveform relaxations" in the literature [6] on VLSI circuit-equations solution techniques. Also, the relaxations in which $U(i) = \{(i \bmod n) + 1\}$, $i \geqq 0$, and $d(i, j, k) \equiv 0$, may reasonably be called Gauss-Seidel; these relaxations may be implemented by only 1 processor.

To start the relaxations, we require the continuous vector functions $\mathbf{x}_j^{(0)}(t)$, $t \geqq 0$, $1 \leqq j \leqq n$, which are arbitrary except that their initial values match the given initial values for the subsystems, i.e. $\mathbf{x}_j^{(0)}(0) = \mathbf{x}_{j,0}$. We assume that

$$\mathbf{x}_j^{(0)}(t) = \mathbf{x}_j^{(i)}(t), \quad t \geqq 0, \quad 1 \leqq j \leqq n \quad \text{for } i < 0. \tag{2.6}$$

This convenient assumption places no additional restrictions on the implementation.

The tasks as defined here have not previously been considered in the literature on asynchronous algorithms. The model for asynchronous computations given in (2.5), augmented by appropriate definitions of the tasks, gives models closely related to those in [1]–[5].

2.3. Assumptions on the asynchronous relaxations. While additional assumptions will be made, in the course of the analysis on the differential equations given in § 2.1, there are only two assumptions on the model in (2.5) for asynchronous computations.

ASSUMPTION A1. There exists a finite d which uniformly bounds the delays, i.e.,

$$0 \leqq d(i, j, k) \leqq d < \infty, \quad i \geqq 0, \quad 1 \leqq j \leqq n, \quad 1 \leqq k \leqq n. \tag{2.7}$$

ASSUMPTION A2. There exists an integer s, $s < \infty$, such that every subsystem is updated at least once in every s consecutive updates.

In connection with Assumption A1 note that $d(i, j, k)$ is not greater than the total number of updates that occur between the ith update and the prior update which marks the beginning of the integration which results in the ith update. Therefore the bound d in A1 exists in the typical case where a given number of processors, working at possibly different positive speeds, implement in some arbitrary order the integrations of the subsystems, the associated workloads being possibly unequal. A naive value for d is obtained from the lower and upper bounds of the speeds and workloads.

Assumption A2 is a nonstarvation condition. Since each update concerns at least one subsystem, violations of A2 can only occur if the ratio of updates for some pair of subsystems is allowed to become unbounded.

Different versions of A1 and A2 have been assumed in previous work on asynchronous numeric algorithms [1]–[5].

3. Preliminaries.

3.1. Differential inequalities.

The subsequent analysis makes considerable use of differential inequalities. For this reason we collect here certain well-known results related to differential inequalities; for additional information the reader is referred to the excellent text by Coppel [7].

Observe that where $x(t)$ is continuously differentiable, the derivative of $|x(t)|$ may not exist at points where $x(t)=0$. However, the right derivative, $(d^+/dt)|x(t)|$, which is defined below, does exist,

$$\frac{d^+}{dt}|x(t)| \triangleq \lim_{h \to +0} \frac{|x(t+h)| - |x(t)|}{h}. \tag{3.1}$$

To calculate $(d^+/dt)|x(t)|$ it is helpful to have the function $\operatorname{sgn}(x)$ where

$$\operatorname{sgn}(x) = 1 \quad \text{if } x > 0; \quad = -1 \text{ if } x < 0; \quad = 0 \text{ if } x = 0. \tag{3.2}$$

It then follows that

$$\frac{d^+}{dt}|x(t)| = \operatorname{sgn}(x(t)) \frac{d}{dt} x(t) \quad \text{if } x(t) \neq 0, \tag{3.3i}$$

$$\frac{d^+}{dt}|x(t)| = \left|\frac{d}{dt} x(t)\right| \quad \text{if } x(t) = 0. \tag{3.3ii}$$

The main result on differential inequalities that we will need is a result from Coppel [7]. This theorem applies to differential inequalities in which the right-hand side is restricted to be of type K (after Kamke). A vector function $\mathbf{f} = (f_1, f_2, \cdots)$ of a vector variable $\mathbf{x} = (x_1, x_2, \cdots)$ is of type K in a set S if for any two vectors in S, $\mathbf{a}$ and $\mathbf{b}$,

$$\left.\begin{array}{l} a_p = b_p \\ a_q \leqq b_q, \ q \neq p \end{array}\right\} \quad \text{implies } f_p(\mathbf{a}) \leqq f_p(\mathbf{b}). \tag{3.4}$$

For the example

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{M}(t)\mathbf{x} + \mathbf{v}(t) \tag{3.5}$$

where $\mathbf{M}$ is a continuous, matrix function of t and $\mathbf{v}(t)$ is a continuous vector function of t, $\mathbf{f}(\mathbf{x}, t)$ is of type K for each fixed value of t if and only if all off-diagonal elements of $\mathbf{M}$ are nonnegative. In the analysis to follow we shall be concerned with differential inequalities involving vector functions of the above kind.

Coppel [7] proves, in particular, that the following holds. Let $\mathbf{z}(t)$ be the unique solution on an interval $[a, b]$ of the equation

$$\frac{d}{dt}\mathbf{z}(t)=\mathbf{f}(\mathbf{z}, t) \tag{3.6}$$

where $\mathbf{f}(\mathbf{z}, t)$ is continuous in an open set, satisfies the Lipschitz condition [7], and is of type K for each fixed value of t. If $\mathbf{x}(t)$ is continuous on $[a, b]$, satisfies the differential inequality

$$\frac{d^+}{dt}\mathbf{x}(t)\leqq \mathbf{f}(\mathbf{x}, t) \tag{3.7}$$

on (a, b) and $\mathbf{x}(a)\leqq \mathbf{z}(a)$, then $\mathbf{x}(t)\leqq \mathbf{z}(t)$ for $a\leqq t\leqq b$.

3.2. Notation. All vectors are assumed to be column vectors. Vector transposition is denoted by the superscript $'$. For $\mathbf{x}$ a vector with a typical element denoted by x_p, let $|\mathbf{x}|$ denote the vector in which $|\mathbf{x}|_p=|x_p|$ for each p. In particular

$$\frac{d^+}{dt}|\mathbf{x}(t)|=\left\{\frac{d^+}{dt}|x_1(t)|, \frac{d^+}{dt}|x_2(t)|, \cdots\right\}'.$$

We employ the inequality convention between vectors in which $\mathbf{x}\leqq \mathbf{y}$ is equivalent to $x_p\leqq y_p$ for each p.

The above conventions are extended to matrices in the natural way. Thus $|\mathbf{M}|$ denotes the matrix obtained from the matrix $\mathbf{M}$ by replacing elements by their absolute values. Also, we let $\mathbf{I}$ denote the identity matrix and $\mathbf{e}_q$ the qth column of $\mathbf{I}$. The vector $\mathbf{1}$ is defined to have unity for all its elements.

4. Linear differential equations. The problem that we will consider here is the numerical computation of the unique solution $\mathbf{x}(t)$, $t\geqq 0$, of a linear counterpart of the equations described in § 2.1. If we let $\mathbf{F}_j(\mathbf{x}_j, t)=\mathbf{D}(j; t)\mathbf{x}_j$, and $\mathbf{G}_j(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n, t)=\sum_{k=1}^n \mathbf{B}(j, k; t)\mathbf{x}_k(t)$ for $1\leqq j\leqq n$, then we obtain from (2.3) the following system of partitioned differential equations:

$$\begin{aligned}&\frac{d}{dt}\mathbf{x}_j(t)+\mathbf{D}(j; t)\mathbf{x}_j(t)=\sum_{k=1}^n \mathbf{B}(j, k; t)\mathbf{x}_k(t)+\mathbf{u}_j(t), \qquad t\geqq 0, \quad 1\leqq j\leqq n,\\ &\mathbf{x}_j(0)=\mathbf{x}_{j,0}.\end{aligned} \tag{4.1}$$

The constituents of the given quantities, $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{u}$, are continuous in t, $t\geqq 0$, and bounded. The matrices $\mathbf{D}(j; t)$ and $\mathbf{B}(j, k; t)$ are, respectively, N_j by N_j and N_j by N_k.

We let $D_{pq}(j; t)$, $1\leqq p\leqq N_j$, $1\leqq q\leqq N_j$ denote the components of $\mathbf{D}(j; t)$. Similarly, $B_{pq}(j, k; t)$, $1\leqq p\leqq N_j$, $1\leqq q\leqq N_k$ denote the components $\mathbf{B}(j, k; t)$.

4.1. Assumption. We let $\mathbf{M}(j; t)$ denote the N_j by N_j matrix derived from $\mathbf{D}(j; t)$ in the following manner:

$$M_{pq}(j; t)\triangleq\begin{cases}D_{pq}(j; t), & p=q,\\ -|D_{pq}(j; t)|, & p\neq q.\end{cases} \tag{4.2}$$

The N by N matrix $\mathbf{M}(t)$ is then defined in the natural way:

$$\mathbf{M}(t)=\mathbf{M}(1;t)\oplus\mathbf{M}(2;t)\oplus\cdots\oplus\mathbf{M}(n;t). \tag{4.3}$$

Thus $\mathbf{M}(t)$ is block-diagonal with nonpositive off-diagonal elements for each t.

The N_j by N_k matrix $\mathbf{N}(j,k;t)$ is derived from $\mathbf{B}(j,k;t)$: for $1\leqq j\leqq n$, $1\leqq k\leqq n$, $1\leqq p\leqq N_j$, $1\leqq q\leqq N_k$,

$$N_{pq}(j,k;t)=|B_{pq}(j,k;t)|. \tag{4.4}$$

Finally, the N by N matrix $\mathbf{N}(t)$ is defined to have $\mathbf{N}(j,k;t)$ as its (j,k)th block constituent.

In the analysis of linear differential equations we will need the following.

ASSUMPTION L1. There exists a constant N-vector $\boldsymbol{\pi}$ with all components finite and positive, and a positive constant ε such that

$$[\mathbf{M}(t)-\mathbf{N}(t)]\boldsymbol{\pi}\geqq\varepsilon\mathbf{1},\qquad t\geqq 0. \tag{4.5}$$

COROLLARY 1. *There exists a constant* r, $0\leqq r<1$, *such that*

$$\mathbf{N}(t)\boldsymbol{\pi}\leqq r\mathbf{M}(t)\boldsymbol{\pi},\qquad t\geqq 0. \tag{4.6}$$

Proof. From the prior assumption of boundedness of the constituent functions of the matrices appearing in (4.1), it follows that there exists some b, $b<\infty$, such that

$$\mathbf{M}(t)\boldsymbol{\pi}\leqq b\mathbf{1},\qquad t\geqq 0. \tag{4.7}$$

Let $r=1-\varepsilon/b$, so that $0\leqq r<1$. With this identification it is straightforward to verify (4.6). □

COROLLARY 2. *Suppose that in* (4.1), $\{\mathbf{D}(j;t)\}$ *and* $\{\mathbf{B}(j,k;t)\}$ *are constant. Then Assumption* L1 *is equivalent to the condition*

$$(\mathbf{M}-\mathbf{N})\ \textit{is an M-matrix.} \tag{4.8}$$

Proof. $(\mathbf{M}-\mathbf{N})$ is a matrix with all off-diagonal elements nonpositive. The assertion is a well-known property of such matrices [8]. (The reader will find in [9] a variety of other equivalent properties.) □

When (4.1) is autonomous, it is possible to obtain a sharp estimate of the important constant r which appears in (4.6).

COROLLARY 3. *If* $\{\mathbf{D}(j;t)\}$ *and* $\{\mathbf{B}(j,k;t)\}$ *in* (4.1) *are independent of t, then*

(4.9i) (i) $\rho<1$, *where* ρ *is the spectral radius of* $\mathbf{M}^{-1}\mathbf{N}$.

(ii) *For any* $r>\rho$, *there exists a positive vector* $\boldsymbol{\pi}$ *such that*

$$\mathbf{N}\boldsymbol{\pi}<r\mathbf{M}\boldsymbol{\pi}. \tag{4.9ii}$$

Proof. The proof of (i) is in [8]. The proof of (ii) follows from well-known properties of M-matrices: $\{r\mathbf{I}-\mathbf{M}^{-1}\mathbf{N}\}$ and $\mathbf{M}$ are M-matrices and consequently their inverses exist and are nonnegative; hence the inverse of $(r\mathbf{M}-\mathbf{N})$ exists and is nonnegative. Let $\boldsymbol{\pi}=(r\mathbf{M}-\mathbf{N})^{-1}\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is any positive vector. Clearly $\boldsymbol{\pi}$ is itself positive and also (4.9ii) holds. □

Note that in (ii) of the above corollary, r may be taken to be arbitrarily close to ρ.

4.2. Asynchronous relaxations. Recall that the symbol i is used to index updates, $U(i)$ denotes the set of indices of subsystems which are updated in the ith update, and $\{\mathbf{x}_j^{(i+1)}(t),\ t\geqq 0,\ 1\leqq j\leqq n\}$ are the computed function values on completion of the

ith update. From (2.4), (2.5) and (4.1) we see that these functions are the solutions of the equations

(4.10i)
$$\frac{d}{dt}\mathbf{x}_j^{(i+1)}(t)+\mathbf{D}(j;t)\mathbf{x}_j^{(i+1)}(t)=\sum_{k=1}^{n}\mathbf{B}(j,k;t)\mathbf{x}_j^{(i-d(i,j,k))}(t)+\mathbf{u}_j(t),\quad t\geqq 0\quad \text{if } j\in U(i),$$
$$\mathbf{x}_j^{(i+1)}(0)=\mathbf{x}_{j,0},$$

(4.10ii)
$$\mathbf{x}_j^{(i+1)}(t)\equiv\mathbf{x}_j^{(i)}(t),\quad t\geqq 0\quad \text{if } j\notin U(i).$$

It is easy to see that a unique solution exists to the initial value problem in (4.10i). The N_j-vector function $\mathbf{x}_j^{(i+1)}(t)$, $t\geqq 0$, is the estimate of $\mathbf{x}_j(t)$, as given by (4.1), after the ith update.

4.3. Analysis.

4.3.1. Comparison system. Define the N_j-vector $\mathbf{y}_j^{(i)}(t)$ thus:

$$\mathbf{y}_j^{(i)}(t)\triangleq\mathbf{x}_j^{(i)}(t)-\mathbf{x}_j(t),\qquad t\geqq 0,\quad 1\leqq j\leqq n,\quad i=0,1,\cdots. \tag{4.11}$$

From (4.1) and (4.10) it is clear that

$$\mathbf{y}_j^{(i)}(0)=\mathbf{0},\qquad 1\leqq j\leqq n,\quad i=0,1,\cdots. \tag{4.12}$$

Also, from (4.1) and (4.10),

(4.13i)
$$\frac{d}{dt}\mathbf{y}_j^{(i+1)}(t)+\mathbf{D}(j;t)\mathbf{y}_j^{(i+1)}(t)=\sum_{k=1}^{n}\mathbf{B}(j,k;t)\mathbf{y}_k^{(i-d(i,j,k))}(t)\quad \text{if } j\in U(i),$$

(4.13ii)
$$\mathbf{y}_j^{(i+1)}(t)\equiv\mathbf{y}_j^{(i)}(t),\quad t\geqq 0\quad \text{if } j\notin U(i).$$

4.3.2. Differential inequalities. Let $y_{j,p}^{(i+1)}(t)$ denote the pth constituent of the N_j-vector $\mathbf{y}_j^{(i+1)}(t)$. From (4.3i), for $j\in U(i)$,

$$\begin{aligned}&\{\operatorname{sgn} y_{j,p}^{(i+1)}(t)\}\frac{d}{dt}y_{j,p}^{(i+1)}(t)+D_{pp}(j;t)|y_{j,p}^{(i+1)}(t)|\\ &\qquad+\sum_{q:q\neq p}D_{pq}(j;t)[\{\operatorname{sgn} y_{j,p}^{(i+1)}(t)\}y_{j,q}^{(i+1)}(t)]\\ &\qquad=\sum_{k=1}^{n}\sum_{q=1}^{N_q}B_{pq}(j,k;t)[\{\operatorname{sgn} y_{j,p}^{(i+1)}(t)\}y_{k,q}^{(i-d(i,j,k))}(t)],\qquad t\geqq 0.\end{aligned} \tag{4.14}$$

It follows that

$$\begin{aligned}&\{\operatorname{sgn} y_{j,p}^{(i+1)}(t)\}\frac{d}{dt}y_{j,p}^{(i+1)}(t)+D_{pp}(j;t)|y_{j,p}^{(i+1)}(t)|-\sum_{q:q\neq p}|D_{pq}(j;t)||y_{j,q}^{(i+1)}(t)|\\ &\qquad\leqq\sum_{k=1}^{n}\sum_{q=1}^{N_q}|B_{pq}(j,k;t)||y_{k,q}^{(i-d(i,j,k))}(t)|,\qquad t\geqq 0.\end{aligned} \tag{4.15i}$$

Also, when $y_{j,p}^{(i+1)}(t)=0$ it follows directly from (4.13i) that

$$\left|\frac{d}{dt}y_{j,p}^{(i+1)}(t)\right|-\sum_{q:q\neq p}|D_{pq}(j;t)||y_{j,q}^{(i+1)}(t)|\leqq\sum_{k=1}^{n}\sum_{q=1}^{N_q}|B_{pq}(j,k;t)||y_{k,q}^{(i-d(i,j,k))}(t)|. \tag{4.15ii}$$

Hence, from the definition of the right derivative, see (3.3) in § 3.1, it follows from (4.2), (4.3) and (4.4) that

$$\frac{d^+}{dt}|\mathbf{y}_j^{(i+1)}(t)|+\mathbf{M}(j;t)|\mathbf{y}_j^{(i+1)}(t)|\leqq\sum_{k=1}^{n}\mathbf{N}(j,k;t)|\mathbf{y}_k^{(i-d(i,j,k))}(t)|,\qquad t\geqq 0. \tag{4.16}$$

In summary, if $j \in U(i)$ then (4.16) augmented by the initial condition $|\mathbf{y}_j^{(i+1)}(0)| = \mathbf{0}$ holds, and if $j \notin U(i)$ then

$$|\mathbf{y}_j^{(i+1)}(t)| = |\mathbf{y}_j^{(i)}(t)|, \qquad t \geqq 0. \tag{4.17}$$

4.3.3. Bounds. Making use of (4.6) in Corollary 1 to Assumption L1, the right-hand side of (4.16) may be upper bounded thus:

$$\sum_{k=1}^{n} \mathbf{N}(j, k; t)|\mathbf{y}_k^{(i-d(i,j,k))}(t)| \leqq [r\{\max_{1 \leqq k \leqq n} w_k^{(i-d(i,j,k))}\}]\mathbf{M}(j; t)\boldsymbol{\pi}_j, \tag{4.18}$$

for $t \geqq 0$, where

$$w_j^{(i)} \triangleq \sup_{t \in [0,\infty)} \max_{1 \leqq p \leqq N_j} \{|y_{j,p}^{(i)}(t)|/\pi_{j,p}\}, \qquad 1 \leqq j \leqq n, \quad i \geqq 0. \tag{4.19}$$

From (4.16) and (4.18) we then have, for $j \in U(i)$, $t \geqq 0$,

$$\begin{gathered} \frac{d^+}{dt}\{|\mathbf{y}_j^{(i+1)}(t)| - s_j^{(i+1)}\boldsymbol{\pi}_j\} + \mathbf{M}(j; t)\{|\mathbf{y}_j^{(i+1)}(t)| - s_j^{(i+1)}\boldsymbol{\pi}_j\} \leqq 0, \qquad t \geqq 0, \\ \{|\mathbf{y}_j^{(i+1)}(0)| - s_j^{(i+1)}\boldsymbol{\pi}_j\} = -s_j^{(i+1)}\boldsymbol{\pi}_j \end{gathered} \tag{4.20}$$

where

$$s_j^{(i+1)} \triangleq r\{\max_{1 \leqq k \leqq n} w_k^{(i-d(i,j,k))}\}. \tag{4.21}$$

Now, if the differential inequality in (4.20) is stated in the form

$$\frac{d^+}{dt}\mathbf{z}(t) \leqq \mathbf{f}(t, \mathbf{z}), \qquad t \geqq 0 \tag{4.22}$$

then $\mathbf{f}$ is of type K for each t, see § 3.1. The prime reason for this is that the off-diagonal terms of $\mathbf{M}(j; t)$ are nonpositive. An application of the main result on differential inequalities stated in § 3.1 then yields

$$|\mathbf{y}_j^{(i+1)}(t)| - s_j^{(i+1)}\boldsymbol{\pi}_j \leqq \mathbf{z}(t), \qquad t \geqq 0 \tag{4.23}$$

where $\mathbf{z}(t)$ is the unique solution of the following initial value problem:

$$\begin{gathered} \frac{d}{dt}\mathbf{z}(t) + \mathbf{M}(j; t)\mathbf{z}(t) = 0, \qquad t \geqq 0, \\ \mathbf{z}(0) = -s_j^{(i+1)}\boldsymbol{\pi}_j. \end{gathered} \tag{4.24}$$

Since $\mathbf{z}(0) \leqq \mathbf{0}$, it is easy to see that

$$\mathbf{z}(t) \leqq \mathbf{0}, \qquad t \geqq 0. \tag{4.25}$$

Hence, from (4.23) and (4.25),

$$|\mathbf{y}_j^{(i+1)}(t)| \leqq s_j^{(i+1)}\boldsymbol{\pi}_j, \qquad t \geqq 0.$$

From the definitions of $w_j^{(i+1)}$ and $s_j^{(i+1)}$ in (4.19) and (4.21), we obtain the result which is summarized now.

PROPOSITION 1. *For the linear differential equations in* (4.1) *subject to Assumption* L1, *the asynchronous relaxations in* (4.10) *satisfy the following*: *for* $i = 0, 1, 2, \cdots$

$$w_j^{(i+1)} \leqq r\{\max_{1 \leqq k \leqq n} w_k^{(i-d(i,j,k))}\} \quad \textit{if } j \in U(i), \tag{4.26i}$$

$$w_j^{(i+1)} = w_j^{(i)} \quad \textit{if } j \notin U(i) \tag{4.26ii}$$

where (*see* (4.6)) $0 \leqq r < 1$, *and*

$$w_j^{(i)} = \sup_{t \in [0,\infty]} \max_{1 \leqq p \leqq N_j} \{|y_{j,p}^{(i)}(t)|/\pi_{j,p}\},$$

and

$$y_{j,p}^{(i)}(t) = x_{j,p}^{(i)}(t) - x_{j,p}(t). \qquad \square$$

We will return to this result in § 6.

5. Nonlinear differential equations. We consider the equations given in (2.3), namely

$$\begin{aligned} &\frac{d}{dt}\mathbf{x}_j + \mathbf{F}_j(\mathbf{x}_j, t) = \mathbf{G}_j(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n, t) + \mathbf{u}_j(t), \qquad t \geqq 0, \quad 1 \leqq j \leqq n, \\ &\mathbf{x}_j(0) = \mathbf{x}_{j,0}, \end{aligned} \tag{5.1}$$

and analyze the convergence properties of the asynchronous relaxations described in § 2.2. In § 5.2 we give some examples of systems of equations satisfying the assumptions made on them.

5.1. Assumptions. In the notation we employ, $\mathbf{F}_j = (F_{j,1}, F_{j,2}, \cdots, F_{j,N})'$. The constituents functions in $\mathbf{G}_j$ are similarly represented.

ASSUMPTION N1. The function $\mathbf{F}(\mathbf{x}, t)$ is continuous for $t \geqq 0$ and for $\|\mathbf{x}\| < \infty$ ($\|\mathbf{x}\|$ any norm in $\mathbb{R}^N$). Furthermore, for $1 \leqq j \leqq n$, all $\mathbf{u} \in \mathbb{R}^{N_j}$ and all $\mathbf{v} \in \mathbb{R}^{N_j}$, there exist nonnegative, uniformly bounded functions $D_{pq}(j;\, t)$, $1 \leqq p \leqq N_j$, $1 \leqq q \leqq N_j$, continuous in t, $t \geqq 0$, such that, for $t \geqq 0$

$$\{\operatorname{sgn}(u_p - v_p)\}\{F_{j,p}(\mathbf{u}, t) - F_{j,p}(\mathbf{v}, t)\} \geqq D_{pp}(j;\, t)|u_p - v_p| - \sum_{q: q \neq p} D_{pq}(j;\, t)|u_q - v_q|, \tag{5.2i}$$

and, if $u_p = v_p$, then

$$|F_{j,p}(\mathbf{u}, t) - F_{j,p}(\mathbf{v}, t)| \leqq \sum_{q: q \neq p} D_{pq}(j;\, t)|u_q - v_q|. \tag{5.2ii}$$

Note that (5.2ii) is expected from (5.2i) if $(u_p - v_p)$ is perturbed about a small neighborhood of 0.

ASSUMPTION N2. The function $\mathbf{G}(\mathbf{x}, t)$ is continuous for $t \geqq 0$ and for $\|\mathbf{x}\| < \infty$. Furthermore, there exist nonnegative, continuous functions $\phi_{j,p}(t)$ such that for all $\mathbf{u} \in \mathbb{R}^N$ and all $\mathbf{v} \in \mathbb{R}^N$,

$$|G_{j,p}(\mathbf{u}, t) - G_{j,p}(\mathbf{v}, t)| \leqq \phi_{j,p}(t)\|\mathbf{u} - \mathbf{v}\|, \qquad t \geqq 0. \tag{5.3}$$

In (5.3), the norm is the weighted L_∞-norm in $\mathbb{R}^N$, i.e., for $\mathbf{x} \in \mathbb{R}^N$,

$$\|\mathbf{x}\| = \max_{1 \leqq j \leqq n} \max_{1 \leqq p \leqq N_j} |x_{j,p}|/\pi_{j,p} \tag{5.4}$$

where $\boldsymbol{\pi}$ is a fixed, positive, weight vector.

ASSUMPTION N3. There exists a constant ε, $\varepsilon > 0$, such that for $t \geqq 0$

$$\mathbf{M}(t)\boldsymbol{\pi} - \boldsymbol{\phi}(t) \geqq \varepsilon \mathbf{1}, \tag{5.5}$$

where the matrix function $\mathbf{M}(t)$ is constituted from the functions $D_{pq}(j, t)$ as specified before in (4.2) and (4.3), § 4.1.

This completes the assumptions that we will need in connection with the analysis of the asynchronous relaxations for nonlinear, nonautonomous differential equations.

Since $\mathbf{u}(t)$ is continuous in t, the continuity and the Lipschitz condition are satisfied in the initial-value problem in (5.1), and thus a unique solution exists [7].

Note that if we identify $\boldsymbol{\phi}(t)$ in (5.5) with $\mathbf{N}(t)\boldsymbol{\pi}$ in (4.5), the two inequalities are identical. We may therefore conclude the following, as in the first corollary to Assumption L1 in § 4.1.

COROLLARY. *There exists a constant* r, $0 \leqq r < 1$, *such that*

$$\boldsymbol{\phi}(t) \leqq r\mathbf{M}(t)\boldsymbol{\pi}, \qquad t \geqq 0. \tag{5.6}$$ □

5.2. Examples. We briefly consider some simple and typical examples of the N-vector function $\mathbf{F}(\mathbf{x}, t)$, and investigate the requirements for Assumption N1 to be satisfied and the specification of the functions $\{D_{pq}(j; t)\}$ which appear in the statement of the assumption.

5.2.1. A simple example is

$$\mathbf{F}(\mathbf{x}, t) = \mathbf{A}\mathbf{x} \tag{5.7}$$

where $\mathbf{A}$ is a constant matrix. In this case (5.2) requires that

$$A_{pp}(j) \geqq 0 \tag{5.8}$$

in which case we may satisfy (5.2) by taking

$$D_{pp}(j; t) = A_{pp}(j) \quad \text{and} \quad D_{pq}(j; t) = |A_{pq}(j)|, \quad p \neq q. \tag{5.9}$$

5.2.2. Another example is

$$\mathbf{F}_j(\mathbf{x}_j, t) = [f_{j,1}(x_{j,1}), f_{j,2}(x_{j,2}), \cdots, f_{j,N_j}(x_{j,N_j})]' \tag{5.10}$$

for $1 \leqq j \leqq n$, in which for each p, $f_{j,p}(x_{j,p})$ is continuously differentiable. In this case, in (5.2) we may take $D_{pq}(j; t) \equiv 0$ for $p \neq q$, and the requirement in (5.2) then translates to

$$\{\operatorname{sgn}(u - v)\}\{f_{j,p}(u) - f_{j,p}(v)\} \geqq D_{pp}(j; t)|u - v|, \tag{5.11}$$

for all real u and v, where $D_{pp}(j; t) \geqq 0$. Thus $f_{j,p}(u)$ must be monotonic, nondecreasing with increasing x, but note that it is not necessary that $f_{j,p}(0) = 0$. When the monotonicity condition holds, (5.2) is satisfied if we take

$$D_{pp}(j; t) = \inf_x f'_p(x) \geqq 0. \tag{5.12}$$

5.2.3. In the final example, for $1 \leqq j \leqq n$,

$$\mathbf{F}_j(\mathbf{x}_j, t) = \mathbf{A}(j)[f_{j,1}(x_{j,1}), f_{j,2}(x_{j,2}), \cdots, f_{j,N_j}(x_{j,N_j})]', \tag{5.13}$$

in which $f_{j,p}$ for each p is continuously differentiable and satisfies the Lipschitz condition, and $\mathbf{A}(j)$ is a constant matrix. It is easy to see that (5.2) is satisfied if

$$A_{pp}(j)\{\operatorname{sgn}(u - v)\}\{f_{j,p}(u) - f_{j,p}(v)\} \geqq D_{pp}(j; t)|u - v| \tag{5.14i}$$

and, for $p \neq q$,

$$|A_{pq}(j)||f_q(u) - f_q(v)| \leqq D_{pq}(j; t)|u - v| \tag{5.14ii}$$

for all real u and v.

In considering (5.14i) first, we note that $A_{pp}(j)$ may have either sign. However, necessarily, (a) if $A_{pp}(j) \geqq 0$ then $f_{j,p}(x)$ must be monotonic, nondecreasing with increasing x; (b) if $A_{pp}(j) \leqq 0$ then $f_{j,p}(x)$ must be monotonic, nonincreasing with increasing x. That is,

$$A_{pp}(j)f'_{j,p}(x) \geqq 0 \quad \text{for all } x \text{ and } p. \tag{5.15}$$

If this condition is satisfied then, it is quite easy to see that (5.14i) and (5.14ii), and therefore (5.2), are satisfied with the following choices:

$$\begin{aligned} &D_{pp}(j;\, t) = |A_{pp}(j)| S_{j,p} \quad \text{where } S_{j,p} \triangleq \inf_x |f'_{j,p}(x)|, \\ &\text{and, for } p \neq q, \\ &D_{pq}(j;\, t) = |A_{pq}(j)| L_{j,q} \quad \text{where } L_{j,q} \triangleq \sup_x |f'_{j,q}(x)|. \end{aligned} \tag{5.16}$$

5.3. Asynchronous relaxations. From (2.3), (2.4) and (2.5) we have, for $i = 0, 1, \cdots$ and $1 \leqq j \leqq n$,

$$\begin{aligned} &\frac{d}{dt}\mathbf{x}_j^{(i+1)}(t) + \mathbf{F}_j(\mathbf{x}_j^{(i+1)}, t) = \mathbf{G}_j(\mathbf{x}_1^{(i-d(i,j,1))}, \cdots, \mathbf{x}_n^{(i-d(i,j,n))}, t) + \boldsymbol{u}_j(t), \quad t \geqq 0, \\ &\mathbf{x}_j^{(i+1)}(0) = \mathbf{x}_{j,0} \quad \text{if } j \in U(i), \end{aligned} \tag{5.17i}$$

$$\mathbf{x}_j^{(i+1)}(t) = \mathbf{x}_j^{(i)}(t), \quad t \geqq 0 \quad \text{if } j \notin U(i). \tag{5.17ii}$$

Note that by our earlier assumptions, continuity and the Lipschitz condition are satisfied in the initial-value problem (5.17i) and thus a unique solution exists.

5.4. Analysis.

5.4.1. Comparison system. Define, as in (4.11)

$$\mathbf{y}_j^{(i)}(t) \triangleq \mathbf{x}_j^{(i)}(t) - \mathbf{x}_j(t), \qquad t \geqq 0, \quad 1 \leqq j \leqq n, \quad i = 1, 2, \cdots. \tag{5.18}$$

Clearly

$$\mathbf{y}_j^{(i)}(0) = \mathbf{0}.$$

From (5.1), (5.17) and (5.18),

$$\begin{aligned} &\frac{d}{dt} y_{j,p}^{(i+1)}(t) + [F_{j,p}(x_j^{(i+1)}, t) - F_{j,p}(x_j, t)] \\ &\qquad = [G_{j,p}(\mathbf{x}_1^{(i-d(i,j,1))}, \cdots, \mathbf{x}_n^{(i-d(i,j,n))}, t) - G_{j,p}(\mathbf{x}_1, \cdots, \mathbf{x}_n, t)], \\ &\qquad t \geqq 0, \quad y_{j,p}^{(i+1)}(0) = 0, \quad 1 \leqq p \leqq N_j \quad \text{if } j \in U(i) \end{aligned} \tag{5.19i}$$

$$\mathbf{y}_j^{(i+1)}(t) = \mathbf{y}_j^{(i)}(t), \qquad t \geqq 0 \quad \text{if } j \notin U(i). \tag{5.19ii}$$

We proceed to examine the case of $j \in U(i)$ and to obtain, with the aid of Assumptions N1 and N2, a differential inequality for $|\mathbf{y}_j^{(i+1)}(t)|$. The right-hand side of (5.19i) may be bounded by using Assumption N2:

$$\begin{aligned} &|G_{j,p}(\mathbf{x}_1^{(i-d(i,j,1))}, \cdots, \mathbf{x}_n^{(i-d(i,j,n))}, t) - G_{j,p}(\mathbf{x}_1, \cdots, \mathbf{x}_n, t)| \\ &\qquad \leqq \phi_{j,p}(t)[\max_{1 \leqq k \leqq n} w_k^{(i-d(i,j,k))}], \qquad t \geqq 0, \end{aligned} \tag{5.20}$$

where, as in (4.19),

$$w_j^{(i)} \triangleq \sup_{t \in [0,\infty]} \max_{1 \leqq p \leqq N_j} |y_{j,p}^{(i)}(t)| / \pi_{j,p}, \qquad 1 \leqq j \leqq n, \quad i \geqq 0. \tag{5.21}$$

By multiplying (5.19i) by {sgn $y_{j,p}^{(i+1)}(t)$] and taking note of Assumption N1, and also by evaluating $|(d/dt)y_{j,p}^{(i+1)}(t)|$ when $y_{j,p}^{(i+1)}(t)=0$, we obtain, analogously to (4.15),

$$\begin{aligned} \frac{d^+}{dt}|y_{j,p}^{(i+1)}(t)|+D_{pp}(j;\,t)|y_{j,p}^{(i+1)}(t)|-\sum_{q:q\neq p} D_{pq}(j;\,t)|y_{j,q}^{(i+1)}(t)| \\ \leqq \phi_{j,p}(t)[\max_{1\leqq k\leqq n} w_k^{(i-d(i,j,k))}], \qquad t\geqq 0. \end{aligned} \tag{5.22}$$

To recapitulate, if $j\in U(i)$ then we have (5.22), and if $j\notin U(i)$ then

$$|\mathbf{y}_j^{(i+1)}(t)|=|y_j^{(i)}(t)|, \qquad t\geqq 0.$$

5.4.2. Bounds. We may now make use of (5.6), a corollary to Assumption N3, which bounds $\boldsymbol{\phi}(t)$, to state the right-hand side of (5.22) in a more convenient form. Thus,

$$\phi_{j,p}(t)\left[\max_{1\leqq k\leqq n} w_k^{(i-d(i,j,k))}\right]\leqq s_j^{(i+1)}\left[D_{pp}(j;\,t)\pi_{j,p}-\sum_{q:q\neq p} D_{pq}(j;\,t)\pi_{j,q}\right] \tag{5.23}$$

where

$$s_j^{(i+1)}\triangleq r\{\max_{1\leqq k\leqq n} w_k^{(i-d(i,j,k))}\}. \tag{5.24}$$

From (5.22) and (5.23),

$$\begin{aligned} \frac{d^+}{dt}\{|y_{j,p}^{(i+1)}(t)|-s_j^{(i+1)}\pi_{j,p}\}+D_{pp}(j;\,t)\{|y_{j,p}^{(i+1)}(t)|-s_j^{(i+1)}\pi_{j,p}\} \\ -\sum_{q:q\neq p} D_{pq}(j;\,t)\{|y_{j,q}^{(i+1)}(t)|-s_j^{(i+1)}\pi_{j,q}\}\leqq 0, \qquad t\geqq 0, \end{aligned}$$

which in vector form is

$$\frac{d^+}{dt}\{|\mathbf{y}_j^{(i+1)}(t)|-s_j^{(i+1)}\boldsymbol{\pi}_j\}+\mathbf{M}(j;\,t)\{|y_j^{(i+1)}(t)|-s_j^{(i+1)}\boldsymbol{\pi}_j\}\leqq 0, \qquad t\geqq 0. \tag{5.25}$$

This differential inequality is identical to (4.20) and we may therefore conclude that

$$|\mathbf{y}_j^{(i+1)}(t)|\leqq s^{(i+1)}\boldsymbol{\pi}_j, \qquad t\geqq 0$$

and, thereby,

$$w_j^{(i+1)}\leqq r\{\max_{1\leqq k\leqq n} w_k^{(i-d(i,j,k))}\},$$

where $w_j^{(i+1)}$ is as defined in (5.21). In summary we have

PROPOSITION 2. *For the nonlinear differential equations in* (5.1) *subject to Assumptions* N1, N2 *and* N3, *the asynchronous relaxations in* (5.17) *satisfy the following*: *for* $i=0,1,2,\cdots$

$$w_j^{(i+1)}\leqq r\{\max_{1\leqq k\leqq n} w_k^{(i-d(i,j,k))}\} \quad \textit{if } j\in U(i),$$

$$w_j^{(i+1)}=w_j^{(i)} \quad \textit{if } j\notin U(i),$$

where $0\leqq r<1$ (*see* (5.6)), *and*

$$w_j^{(i)}=\sup_{t\in[0,\infty]}\max_{1\leqq p\leqq N_j}\{|y_{j,p}^{(i)}(t)|/\pi_{j,p}\}$$

and,

$$y_{j,p}^{(i)}(t)=x_{j,p}^{(i)}(t)-x_{j,p}(t). \qquad \square$$

6. Convergence of the asynchronous relaxations. The end results of the two preceding sections on, respectively, linear and nonlinear differential equations differ only in the values that may be attached to the parameter r. These results rely only on assumptions on the respective differential equations and not at all on those aspects of the asynchronous process which have to do with the relative frequencies of updates and the delays in the updates. Here we introduce these aspects and, by making use of Assumptions A1 and A2 given in § 2, give a unified proof of uniform convergence.

6.1. Convergence of the discrete, asynchronous process. From Propositions 1 and 2,

$$w_j^{(i+1)} \leqq r\{\max_{1\leqq k\leqq n} w_k^{(i-d(i,j,k))}\} \quad \text{if } j\in U(i),$$

$$w_j^{(i+1)} = w_j^{(i)} \quad \text{if } j\notin U(i).$$

Hence from Assumption A1,

$$w_j^{(i+1)} \leqq r[\max_{0\leqq\delta\leqq d}\max_{1\leqq k\leqq n} w_k^{(i-\delta)}] \quad \text{if } j\in U(i). \tag{6.1}$$

By recalling that $r<1$ we have, in particular,

$$\max_{0\leqq\delta\leqq d}\max_{1\leqq j\leqq n} w_j^{(i'-\delta)} \leqq \max_{0\leqq\delta\leqq d}\max_{1\leqq j\leqq n} w_j^{(i-\delta)}, \qquad 0\leqq i\leqq i'-1. \tag{6.2}$$

We also have

PROPOSITION 3.

$$\max_{1\leqq j\leqq n} w_j^{(i')} \leqq r[\max_{0\leqq\delta\leqq d}\max_{1\leqq j\leqq n} w_j^{(i-\delta)}], \qquad 0\leqq i\leqq i'-s. \tag{6.3}$$

Proof. From Assumption A2, subsystem j must be updated at least once in the series of consecutive updates with indices in $[i'-s,\ i'-1]$. Say that the last update of subsystem j in this series is indexed τ.

$$w_j^{(i')} = w_j^{(\tau+1)} \leqq r[\max_{0\leqq\delta\leqq d}\max_{1\leqq k\leqq n} w_k^{(\tau-\delta)}], \qquad \text{from (6.1)},$$

$$w_j^{(i')} = w_j^{(\tau+1)} \leqq r[\max_{0\leqq\delta\leqq d}\max_{1\leqq k\leqq n} w_k^{(i-\delta)}], \quad i\leqq\tau\leqq i'-s, \quad \text{from (6.2)}. \qquad \square$$

The above proposition has the following corollary in the form of a recursive inequality:

$$[\max_{0\leqq\delta\leqq d}\max_{1\leqq j\leqq n} w_j^{(i'-\delta)}] \leqq r[\max_{0\leqq\delta\leqq d}\max_{1\leqq j\leqq n} w_j^{(i-\delta)}], \qquad 0\leqq i\leqq i'-(d+s). \tag{6.4}$$

We now state the main result, in which the norm $\|\ \|$ in $\mathbb{R}^N$ is

$$\|\mathbf{x}\| = \max_{1\leqq j\leqq n}\max_{1\leqq p\leqq N_j} |x_{j,p}|/\pi_{j,p}.$$

PROPOSITION 4. *For* $i=0,1,2,\cdots$

$$\sup_{t\in[0,\infty]} \|\mathbf{x}^{(i)}(t)-\mathbf{x}(t)\| \leqq Cr^l, \tag{6.5}$$

where $0\leqq r<1$, $l=\lfloor i/(d+s)\rfloor$ *and* $C=\max_{1\leqq j\leqq n} w_j^{(0)}$.

Proof. Equation (6.4) yields

$$[\max_{0\leqq\delta\leqq d}\max_{1\leqq j\leqq n} w_j^{(i-\delta)}] \leqq Cr^l \tag{6.6}$$

where

$$C=\max_{0\leqq\delta\leqq d}\max_{1\leqq j\leqq n} w_j^{(-\delta)}=\max_{1\leqq j\leqq n} w_j^{(0)}.$$

The statement in the proposition is implied by (6.6). □

6.2. Discussion. We may conclude from the last proposition that the long-term average rate of geometric convergence (per update) is not less than $r^{1/(d+s)}$.

We have proven uniform convergence for the proposed asynchronous relaxations under different conditions for three variants of the differential equations in (2.1):

$$\frac{d}{dt}\mathbf{x}(t)+\mathbf{D}\mathbf{x}(t)=\mathbf{B}\mathbf{x}(t)+\mathbf{u}(t), \tag{6.7i}$$

$$\frac{d}{dt}\mathbf{x}(t)+\mathbf{D}(t)\mathbf{x}(t)=\mathbf{B}(t)\mathbf{x}(t)+\mathbf{u}(t), \tag{6.7ii}$$

$$\frac{d}{dt}\mathbf{x}(t)+\mathbf{F}(\mathbf{x},t)=\mathbf{G}(\mathbf{x},t)+\mathbf{u}(t). \tag{6.7iii}$$

We have assumed A1 and A2, see § 2.3, in all cases. In case (i) uniform convergence at a geometric rate occurs when $\mathbf{M}^{-1}\mathbf{N}$ is an M-matrix; in this case the parameter r in the bound on the rate of convergence may be chosen to be arbitrarily close to ρ, the spectral radius or $\mathbf{M}^{-1}\mathbf{N}$, where $\mathbf{M}$ and $\mathbf{N}$ are obtained from $\mathbf{D}$ and $\mathbf{B}$ as in (4.2) and (4.4). In particular, if $\mathbf{D}$ has nonpositive off-diagonal elements and $\mathbf{B}$ is nonnegative, the $\mathbf{M}=\mathbf{D}$ and $\mathbf{N}=\mathbf{B}$.

The uniform convergence for case (ii) requires Assumption L1, see § 4.1, and case (iii) requires Assumptions N1, N2 and N3, see § 5.1. The result for case (iii) subsumes the result for case (ii).

7. Concluding remarks. We have proven convergence results for an asynchronous computational model. This model has several parameters and, by appropriate selections of these parameters, it can be made to describe algorithms with various degrees of asynchronism. Similarly, architectural differences giving rise to varying rates of execution of the primitive tasks are also tolerated in the model.

Algorithmic efficiency, synchronization penalty, the software overhead for coordination and communication congestion are major factors which will contribute towards greater use of asynchronism in algorithms. These factors become more pronounced as the number of processors is increased. The Ultracomputer [10] and the CHOPP [11] are examples of designs in which ultimately thousands of processors are proposed to be used. The asynchronous computational model and results for it should therefore prove useful in the future.

We mention that [12] describes implementations of asynchronous relaxations on a parallel processor[1] for systems with up to 64,000 tightly coupled, linear differential equations which describe the evolution with time of the state probabilities of Markov processes. The Markov processes model manufacturing lines with random service times and finite buffers. The procedure employed in [12] consists of first obtaining a good Gauss-Seidel-like relaxation for uniprocessors and then obtaining parallel asynchronous versions of it. Also described are competing implementations and related experimental results on the scheduling of tasks, the use of "windows" related to finite integration intervals, distributed convergence tests and termination, and the distributed

[1] The BALANCE™ (trademark of Sequent Computer Systems, Inc.) 8000.

computation of statistics of the Markov process. Finally, a result is given which bounds the effect of independent error functions which are introduced in the execution of each of the primitive tasks.

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