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CONVERGENCE OF WAVEFORM RELAXATION METHODS FOR DIFFERENTIAL-ALGEBRAIC SYSTEMS*

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Abstract. This paper gives sufficient conditions for existence and uniqueness of solutions and for the convergence of Picard iterations and more general waveform relaxation methods for differential-algebraic systems of neutral type. The results are obtained by the contraction mapping principle on Banach spaces with weighted norms and by the use of the Perron–Frobenius theory of nonnegative and nonreducible matrices. It is demonstrated that waveform relaxation methods are convergent faster than the classical Picard iterations.

Key words. differential-algebraic system, waveform relaxation, Picard iterations

AMS subject classifications. 34A50, 65L05

1. Introduction. It is the purpose of this paper to investigate the convergence of the sequence of successive approximations to the solution of the differential-algebraic systems. These iterations are constructed by so-called waveform relaxation (WR) methods which were first proposed by Lelarasmee et al. [15] for analyzing large-scale integrated circuits. The large dimension of the systems of equations modeling such circuits and the progress in parallel computing have made the WR techniques competitive with the classical approaches to the numerical solution of differential equations based, for example, on discrete variable methods such as Runge–Kutta, linear multistep or predictor–corrector methods (see [1]–[3], [5], [7], [16]–[20], [23]).

WR methods are special variants of the modification of the Picard successive approximations (MPM method) which in a very abstract form were examined by Ważewski [25] (see also his short note [24]). Ważewski's approach is based on a comparison technique and was used in the 1960s to examine existence and uniqueness of various types of functional equations such as integral, delay, and functional differential equations (see, for example, [8]–[13]). For more details concerned with Ważewski's abstract approach, we refer to [11] and [13].

In contrast to the comparison technique employed by Ważewski [24], [25] the approach of the present paper is simpler, and it is based on the use of a classical tool such as the contraction mapping principle and a very efficient and elegant weighted norm technique due to Bielecki [4]. A similar technique for differential-algebraic equations without delays was used by Schneider [22]. We also refer to the recent paper [14] where a somewhat less general form of a system of differential-algebraic equations than the one considered in this paper was examined.

The organization of this paper is as follows. In the next section we give some remarks on the modified Picard method for mappings defined on complete metric spaces. In §3 we define the Picard iterations and WR iterations based on splitting of the differential-algebraic system under consideration. In §4 we investigate the existence and uniqueness of solutions to the system and the convergence of the Picard iterations. The conditions for convergence of WR iterations are examined in §5. Intuitively, the convergence of WR iterations is faster than that of Picard iterations and the precise relationship between these rates is established in §6. The paper concludes with §7 with some comments and examples.

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2. General remarks on the modified Picard methods. Before employing the MPM for the solution of differential-algebraic systems of equations we will first present this method in the general form for fixed point equations in metric spaces. Let a mapping $f : X \to X$ be given, where (X, d) is a complete metric space with the metric d. We can seek an approximation to the solution of the fixed point equation

$$x = f(x), \tag{2.1}$$

by the Picard method, i.e., by producing the sequence $\{x_k\}_{k=0}^{\infty}$ defined by

$$x_{k+1} = f(x_k), \tag{2.2}$$

$k = 0, 1, \ldots$, with an arbitrary initial value $x_0 \in X$. However, the merit of the MPM lies in choosing a splitting $F : X \times X \to X$ such that $F(x, x) = f(x)$ and producing the sequence $\{y_k\}_{k=0}^{\infty}$ defined by

$$y_{k+1} = F(y_{k+1}, y_k), \tag{2.3}$$

$k = 0, 1, \ldots$, where y_{k+1} is a solution of the equation $y = F(y, y_k)$. It is easy to see that the iterations $\{y_k\}_{k=0}^{\infty}$ are in fact Picard iterations for the equation

$$y = g(y),$$

with the mapping $g : X \to X$ defined implicitly by the relation

$$g(x) = F(g(x), x).$$

It follows from the Banach contraction principle that the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ defined by (2.2) and (2.3), respectively, are convergent if there are constants $\alpha, \beta \in [0, 1)$, $\alpha + \beta < 1$, such that

$$d(F(x, y), F(\bar{x}, \bar{y})) \leq \alpha d(x, \bar{x}) + \beta d(y, \bar{y}) \tag{2.4}$$

for any $x, \bar{x}, y, \bar{y} \in X$. These assumptions also imply that there exists a unique solution x^* to (2.1) and that $x_k \to x^*$ and $y_k \to x^*$ as $k \to \infty$. Moreover,

$$d(x^*, x_k) \leq (\alpha + \beta)^k d(x^*, x_0), \tag{2.5}$$

and

$$d(x^*, y_k) \leq \left(\frac{\beta}{1 - \alpha}\right)^k d(x^*, y_0) \tag{2.6}$$

for $k = 0, 1, \ldots$. Since $\beta/(1 - \alpha) < \alpha + \beta$ for all $\alpha \in (0, 1)$ the error bound (2.6) for the sequence $\{y_k\}_{k=0}^{\infty}$ is sharper than the corresponding error bound (2.5) for the sequence $\{x_k\}_{k=0}^{\infty}$.

For similar conclusions in the nonlinear case (i.e., when the right-hand side of (2.4) is nonlinear) and in the abstract case (i.e., when d is vector valued) we refer to the papers [10] and [13].

Observe also that MPM processes include a big variety of relaxation methods. For example, the well-known Newton method for nonlinear equations is also included in the family of MPM processes. In fact, if X is a Banach space and f has its derivative f', then we can take

$$F(y, x) = f(x) + f'(x)(y - x).$$

Further details related to the Newton-like processes concerned with ordinary differential equations can be found in [12].

3. Waveform relaxation methods for differential-algebraic systems. Let $T = [0, a]$, $a > 0$, and denote by $X_1 = C(T, B_1)$ the Banach space of all continuous functions from T into B_1 and by $X_2 = C^1(T, B_2)$ the Banach space of all functions of class C^1 from T into B_2. We will use the same symbol $||\cdot||$ to denote norms in both B_1 and B_2. Usually, $B_1 = R^{n_1}$ and $B_2 = R^{n_2}$. Let there be given continuous mappings $\tilde{f}_i : X_1^2 \times X_2 \to X_i$, $i = 1, 2$, and consider the differential-algebraic system of the form

$$x'(t) = \tilde{f}_1(x, x', z)(t), \qquad x(0) = \bar{x}_0, \tag{3.1}$$

$$z(t) = \tilde{f}_2(x, x', z)(t), \tag{3.2}$$

$t \in T$. Observe that (3.1) is a functional differential system of neutral type which includes as special cases delay differential equations or integrodifferential equations. In the cases when some or all of x, x', and z contain delays, say $\alpha(t)$, $\beta(t)$, and $\gamma(t)$, it will be assumed that $x(\alpha(t)) = \varphi(\alpha(t))$, $x'(\beta(t)) = \varphi'(\beta(t))$, and $z(\gamma(t)) = \psi(\gamma(t))$, whenever $\alpha(t) < 0$, $\beta(t) < 0$, and $\gamma(t) < 0$. Here, φ is a given continuously differentiable function defined to the left of zero and such that $\varphi(0) = \bar{x}_0$ and

$$\varphi'(0) = \tilde{f}_1(x, x', z)(0).$$

Similarly, ψ is a given continuous function defined to the left of zero and such that

$$\psi(0) = \tilde{f}_2(x, x', z)(0).$$

Using the arguments described above we can always incorporate such functions φ and ψ into the definition of the functions $\tilde{f}_1$ and $\tilde{f}_2$ to obtain the system (3.1)–(3.2) with initial condition defined only at one point.

Let there be given splitting functions

$$f_i : X_1^4 \times X_2^2 \to X_i, \qquad i = 1, 2,$$

which are continuous and such that

$$f_i(x, x, y, y, z, z)(t) = \tilde{f}_i(x, y, z)(t).$$

Then (3.1)–(3.2) can be rewritten as

$$x'(t) = f_1(x, x, x', x', z, z)(t), \qquad x(0) = \bar{x}_0, \tag{3.3}$$

$$z(t) = f_2(x, x, x', x', z, z)(t), \tag{3.4}$$

$t \in T$. We will always assume the following Lipschitz condition.

CONDITION L. *There exist nonnegative constants a_s and b_s, $s = 1, 2, \ldots, 6$ such that*

$$||f_1(u_1, \ldots, u_6)(t) - f_1(v_1, \ldots, v_6)(t)|| \le \sum_{s=1}^{6} a_s ||u_s - v_s||_t, \tag{3.5}$$

$$||f_2(u_1, \ldots, u_6)(t) - f_2(v_1, \ldots, v_6)(t)|| \le \sum_{s=1}^{6} b_s ||u_s - v_s||_t \tag{3.6}$$

for $t \in T$, $u_s, v_s \in X_1$, $s = 1, 2, 3, 4$, and $u_s, v_s \in X_2$, $s = 5, 6$. Here $||u||_t := \sup\{||u(s)|| : 0 \le s \le t\}$.

Observe that conditions (3.5) and (3.6) imply that the functions f_1 and f_2 are of Volterra type, i.e., the conditions $u_s(\xi) = v_s(\xi)$ for $0 \le \xi \le t$, $s = 1, 2, \ldots, 6$, imply that

$$f_i(u_1, \ldots, u_6)(t) = f_i(v_1, \ldots, v_6)(t),$$

$i = 1, 2$, $t \in T$. Observe also that (3.5)–(3.6) are global conditions, i.e., they are defined for all possible arguments u_s and v_s. It will be demonstrated in the next two sections that these conditions together with additional conditions specified later will guarantee the existence of the unique solution to (3.3)–(3.4) as well as the convergence of WR iterations (3.7)–(3.8) and classical Picard iterations (3.9)–(3.10) defined below to this solution. If one assumes the existence and uniqueness of solution to (3.3)–(3.4), then Condition L can be relaxed. In this case the iterations (3.7)–(3.8) and (3.9)–(3.10) are convergent if (3.5)–(3.6) are assumed to be satisfied only in some bounded region containing this solution.

The general form of the WR iteration (MPM iteration) (see [15], [8]–[13], [24]–[25]) for the system (3.3)–(3.4) reads

$$x'_{k+1}(t) = F_1(x_{k+1}, x_k, x'_{k+1}, x'_k, z_{k+1}, z_k)(t), \qquad x_{k+1}(0) = \bar{x}_0, \tag{3.7}$$

$$z_{k+1}(t) = F_2(x_{k+1}, x_k, x'_{k+1}, x'_k, z_{k+1}, z_k)(t), \tag{3.8}$$

with arbitrarily chosen $x_0 \in C^1(T, B_1)$ and $z_0 \in C(T, B_2)$. Here $C^1(T, B_1)$ is the space of functions from T into B_1 which are continuous together with their first derivatives. We will also consider the classical Picard iterations defined by

$$x'_{k+1}(t) = F_1(x_k, x_k, x'_k, x'_k, z_k, z_k)(t), \qquad x_{k+1}(0) = \bar{x}_0, \tag{3.9}$$

$$z_{k+1}(t) = F_2(x_k, x_k, x'_k, x'_k, z_k, z_k)(t), \tag{3.10}$$

where $x_0 \in C^1(T, B_1)$ and $z_0 \in C(T, B_2)$ are given. Observe that the right-hand sides of (3.9) and (3.10) could be replaced by $f_1(x_k, x'_k, z_k)(t)$ and $f_2(x_k, x'_k, z_k)(t)$, respectively. However, the somewhat longer notation adopted above will allow us later on to use some of the estimates derived for (3.9)–(3.10) in the convergence analysis of more general WR iterations defined by (3.7)–(3.8) (compare §5).

4. Existence and uniqueness of solution to differential-algebraic system and convergence of Picard iterations. As already mentioned in §3, to investigate the existence and uniqueness of solutions to (3.3)–(3.4) as well as the convergence of Picard iterations (3.9)–(3.10) to this solution, we could work with the simpler formulation (3.1)–(3.2) or its equivalent integral form. Having in mind, however, future applications of some of the estimates to the convergence analysis of WR iterations (3.7)–(3.8) (compare §5), it will be beneficial to work directly with a somewhat more complicated formulation (3.3)–(3.4) defined by the use of the splitting functions f_1 and f_2.

Let $y(t) = x'(t)$. Then $x(t) = (Jy)(t)$, where

$$(Jy)(t) = \bar{x}_0 + \int_0^t y(s)\, ds,$$

$t \in T$, and the system (3.3)–(3.4) can be rewritten as

$$y(t) = f_1(Jy, Jy, y, y, z, z)(t), \tag{4.1}$$

$$z(t) = f_2(Jy, Jy, y, y, z, z)(t). \tag{4.2}$$

Putting $w = [y, z]^T$, $w_i = [y_i, z_i]^T$, $i = 1, 2$, $F = [F_1, F_2]^T$, and

$$F_i(w_1, w_2)(t) = f_i(Jy_1, Jy_2, y_1, y_2, z_1, z_2)(t),$$

$i = 1, 2$, the system (4.1)–(4.2) can be rewritten as a fixed-point equation

$$w = F(w, w)$$

defined in the Banach space $W = C(T, B_1) \times C(T, B_2)$. We have used above the symbol "T" to denote the transposition operator.

Denote by A the matrix

$$A = \begin{bmatrix} a_3 + a_4, a_5 + a_6 \\ b_3 + b_4, b_5 + b_6 \end{bmatrix},$$

where a_s and b_s are Lipschitz constants appearing in (3.5)–(3.6). Assume that $A \geq 0$ (i.e., all entries of A are nonnegative) and that A is nonreducible. We refer to [21] for the definition of the nonreducible matrix. Then it follows from the Perron–Frobenius theorem (see [6]) that A has a positive eigenvalue ξ equal to the spectral radius $\rho(A)$ of A and there is a positive eigenvector $c = [c_1, c_2]^T$ (so-called Perron vector of A) associated with this eigenvalue ξ.

For $w = [y, z]^T \in W$, $\lambda > 0$, and c given above define the norm

$$||w||_{\lambda,c} = \max\left\{\frac{||y||_\lambda}{c_1}, \frac{||z||_\lambda}{c_2}\right\},$$

where for $x \in C(T, B_i)$

$$||x||_\lambda = \sup\{||x||_s e^{-\lambda s}, s \in T\}.$$

We have the following theorem.

THEOREM 1. *Assume that Condition* L *is satisfied, the matrix A is nonnegative and nonreducible, and $\rho(A) < 1$. Then the system* (4.1)–(4.2) *has a unique solution $[y^*, z^*]^T$ and the system* (3.3)–(3.4) *has a unique solution $[x^*, z^*]^T$, where $x^* = Jy^*$. Moreover, the sequence of iterations $[x_k, z_k]^T$, $k = 0, 1, \ldots$, defined by* (3.9)–(3.10) *is linearly convergent to $[x^*, z^*]^T$ with the rate $\xi = \rho(A)$.*

Proof. Since

$$||Ju - Jv||_t = \max_{0 \leq s \leq t} \left\| \int_0^s (u(r) - v(r))\, dr \right\| \leq \int_0^t ||u - v||_s\, ds$$

and $||u - v||_t = e^{\lambda t}||u - v||_t e^{-\lambda t} \leq e^{\lambda t}||u - v||_\lambda$ for $t \in T$ it follows that

$$\begin{aligned} ||F_1(w_1, w_2)(t) - F_1(\bar{w}_1, \bar{w}_2)(t)|| &= ||f_1(Jy_1, Jy_2, y_1, y_2, z_1, z_2)(t) \\ &\quad - f_2(J\bar{y}_1, J\bar{y}_2, \bar{y}_1, \bar{y}_2, \bar{z}_1, \bar{z}_2)(t)|| \\ &\leq a_1||Jy_1 - J\bar{y}_1||_t + a_2||Jy_2 - J\bar{y}_2||_t + a_3||y_1 - \bar{y}_1||_t \\ &\quad + a_4||y_2 - \bar{y}_2||_t + a_5||z_1 - \bar{z}_1||_t + a_6||z_2 - \bar{z}_2||_t \\ &\leq a_1 \int_0^t ||y_1 - \bar{y}_1||_s\, ds \\ &\quad + a_2 \int_0^t ||y_2 - \bar{y}_2||_s\, ds + a_3||y_1 - \bar{y}_1||_t \end{aligned}$$

$$
\begin{aligned}
&+ a_4||y_2 - \bar{y}_2||_t + a_5||z_1 - \bar{z}_1||_t + a_6||z_2 - \bar{z}_2||_t \\
&\le e^{\lambda t}[(\lambda^{-1}a_1 + a_3)||y_1 - \bar{y}_1||_\lambda \\
&\quad + (\lambda^{-1}a_2 + a_4)||y_2 - \bar{y}_2||_\lambda \\
&\quad + a_5||z_1 - \bar{z}_1||_\lambda + a_6||z_2 - \bar{z}_2||_\lambda],
\end{aligned}
$$

where $w_i = [y_i, z_i]^T$, and $\bar{w}_i = [\bar{y}_i, \bar{z}_i]^T$, $i = 1, 2$. Hence,

$$
\begin{aligned}
||F_1(w_1, w_2) - F_1(\bar{w}_1, \bar{w}_2)||_\lambda \le{}& (\lambda^{-1}a_1 + a_3)||y_1 - \bar{y}_1||_\lambda \\
&+ (\lambda^{-1}a_2 + a_4)||y_2 - \bar{y}_2||_\lambda \\
&+ a_5||z_1 - \bar{z}_1||_\lambda + a_6||z_2 - \bar{z}_2||_\lambda .
\end{aligned} \tag{4.3}
$$

Similarly, we obtain

$$
\begin{aligned}
||F_2(w_1, w_2) - F_2(\bar{w}_1, \bar{w}_2)||_\lambda \le{}& (\lambda^{-1}b_1 + b_3)||y_1 - \bar{y}_1||_\lambda \\
&+ (\lambda^{-1}b_2 + b_4)||y_2 - \bar{y}_2||_\lambda \\
&+ b_5||z_1 - \bar{z}_1||_\lambda + b_6||z_2 - \bar{z}_2||_\lambda .
\end{aligned} \tag{4.4}
$$

It follows from the definition of the norm $||\cdot||_{\lambda,c}$ that for $w = [y, z]^T$

$$
||y||_\lambda \le c_1||w||_{\lambda,c}, \qquad ||z||_\lambda \le c_2||w||_{\lambda,c},
$$

or

$$
[||y||_\lambda, ||z||_\lambda]^T \le ||w||_{\lambda,c}[c_1, c_2]^T, \tag{4.5}
$$

where the sign "$\le$" is understood componentwise. Using (4.3), (4.4), and (4.5) and recalling that $Ac = \xi c$, it follows that

$$
\begin{aligned}
&\begin{bmatrix} ||F_1(w_1, w_2) - F_1(\bar{w}_1, \bar{w}_2)||_\lambda \\ ||F_2(w_1, w_2) - F_2(\bar{w}_1, \bar{w}_2)||_\lambda \end{bmatrix} \le \begin{bmatrix} \lambda^{-1}a_1 + a_3, a_5 \\ \lambda^{-1}b_1 + b_3, b_5 \end{bmatrix} \begin{bmatrix} ||y_1 - \bar{y}_1||_\lambda \\ ||z_1 - \bar{z}_1||_\lambda \end{bmatrix} \\
&\quad + \begin{bmatrix} \lambda^{-1}a_2 + a_4, a_6 \\ \lambda^{-1}b_2 + b_4, b_6 \end{bmatrix} \begin{bmatrix} ||y_2 - \bar{y}_2||_\lambda \\ ||z_2 - \bar{z}_2||_\lambda \end{bmatrix} \\
&\quad \le ||w_1 - \bar{w}_1||_{\lambda,c} \begin{bmatrix} \lambda^{-1}a_1 + a_3, a_5 \\ \lambda^{-1}b_1 + b_3, b_5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&\qquad + ||w_2 - \bar{w}_2||_{\lambda,c} \begin{bmatrix} \lambda^{-1}a_2 + a_4, a_6 \\ \lambda^{-1}b_2 + b_4, b_6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&\quad \le \max\{||w_1 - \bar{w}_1||_{\lambda,c}, ||w_2 - \bar{w}_2||_{\lambda,c}\} \left(\begin{bmatrix} \lambda^{-1}(a_1 + a_2), 0 \\ \lambda^{-1}(b_1 + b_2), 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right. \\
&\qquad \left. + \begin{bmatrix} a_3 + a_4, & a_5 + a_6 \\ b_3 + b_4, & b_5 + b_6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right) \\
&= \max\{||w_1 - \bar{w}_1||_{\lambda,c}, ||w_2 - \bar{w}_2||_{\lambda,c}\} \left(\xi \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \lambda^{-1}c_1 \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix} \right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&||F(w_1, w_2) - F(\bar{w}_1, \bar{w}_2)||_{\lambda,c} \\
&\qquad = \max\left\{\frac{||F_1(w_1, w_2) - F_1(\bar{w}_1, \bar{w}_2)||_\lambda}{c_1}, \frac{||F_2(w_1, w_2) - F_2(\bar{w}_1, \bar{w}_2)||_\lambda}{c_2}\right\} \\
&\leq \max\{||w_1 - \bar{w}_1||_{\lambda,c}, ||w_2 - \bar{w}_2||_{\lambda,c}\} \max\{\xi + \lambda^{-1}(a_1 + a_2), \xi + \lambda^{-1}c_1c_2^{-1}(b_1 + b_2)\}.
\end{aligned}
$$

Since $\xi = \rho(A) < 1$ for any $\epsilon > 0$, $\xi + \epsilon < 1$, we can choose $\lambda > 0$ such that

$$\lambda^{-1} \max\{a_1 + a_2, c_1c_2^{-1}(b_1 + b_2)\} \leq \epsilon.$$

Hence

$$||F(w_1, w_2) - F(\bar{w}_1, \bar{w}_2)||_{\lambda,c} \leq (\xi + \epsilon) \max\{||w_1 - \bar{w}_1||_{\lambda,c}, ||w_2 - \bar{w}_2||_{\lambda,c}\}. \tag{4.6}$$

This inequality means that the mapping $G : W \to W$ defined by $G(w) = F(w, w)$ is a contraction. Therefore, it follows from the Banach contraction principle that there exists a unique solution $w^* = [y^*, z^*]^T$ to $w = G(w)$, or equivalently to $w = F(w, w)$. Moreover, w^* is the limit of Picard iterations $\{w_k\}_{k=0}^{\infty}$ defined by

$$w_{k+1} = F(w_k, w_k), \tag{4.7}$$

$k = 0, 1, \ldots$. Subtracting (4.7) from $w^* = F(w^*, w^*)$ and using (4.6) we get

$$||w^* - w_{k+1}||_{\lambda,c} \leq (\xi + \epsilon)||w^* - w_k||_{\lambda,c}$$

and it follows that the rate of linear convergence is $\xi = \rho(A)$. It is also clear that $[x^*, z^*]^T$, where $x^* = Jy^*$ is the unique solution to (3.3)–(3.4) and the sequence $[x_k, z_k]^T$ defined by (3.9)–(3.10) is linearly convergent to $[x, z]^T$ with the same rate ξ. This completes the proof. □

5. Convergence of WR iterations. In this section we will examine the convergence of WR iterations defined by (3.7)–(3.8). Putting $y_k(t) = x_k'(t)$ we have $x_k(t) = (Jy_k)(t)$, and these iterations can be represented in the form

$$y_{k+1}(t) = f_1(Jy_{k+1}, Jy_k, y_{k+1}, y_k, z_{k+1}, z_k)(t), \tag{5.1}$$

$$z_{k+1}(t) = f_2(Jy_{k+1}, Jy_k, y_{k+1}, y_k, z_{k+1}, z_k)(t), \tag{5.2}$$

$k = 0, 1, \ldots$. Using the notation introduced in §4 the system (5.1)–(5.2) can be rewritten as

$$w_{k+1} = F(w_{k+1}, w_k), \tag{5.3}$$

where $w_k = [y_k, z_k]^T$, and $w_{k+1} = [y_{k+1}, z_{k+1}]^T$.

For $\lambda > 0$, define

$$A_\lambda = A_{1,\lambda} + A_{2,\lambda},$$

where

$$A_{1,\lambda} = \begin{bmatrix} \lambda^{-1}a_1 + a_3, a_5 \\ \lambda^{-1}b_1 + b_3, b_5 \end{bmatrix}, \qquad A_{2,\lambda} = \begin{bmatrix} \lambda^{-1}a_2 + a_4, a_6 \\ \lambda^{-1}b_2 + b_4, b_6 \end{bmatrix}$$

and put

$$A_\infty = \lim_{\lambda\to\infty} A_\lambda, \qquad A_{1,\infty} = \lim_{\lambda\to\infty} A_{1,\lambda}, \qquad A_{2,\infty} = \lim_{\lambda\to\infty} A_{2,\lambda}.$$

Observe that $A_\infty = A$, where A is defined in §4. Define also the matrices $A_{0,\lambda}$ and A_0 by

$$A_{0,\lambda} = (I - A_{1,\lambda})^{-1} A_{2,\lambda}$$

and

$$A_0 = (I - A_{1,\infty})^{-1} A_{2,\infty},$$

if $I - A_{1,\lambda}$ and $I - A_{1,\infty}$ are nonsingular. We will need the following results.

LEMMA 2 (see §2.4.5 in [21]). *Let A be a square matrix and assume that $A \geq 0$. Then $(I - A)^{-1}$ exists and is nonnegative if and only if $\rho(A) < 1$.*

LEMMA 3 (compare §2.4.9 in [21]). *Let A and B be square matrices. Then $|B| \leq A$ implies that $\rho(B) \leq \rho(A)$. Here $|B| = [|b_{ij}|]_{i,j=1}^n$.*

LEMMA 4 (compare §2.4.17 in [21]). *Let A be a square matrix, and suppose that $A = B - C$ is a regular splitting. Then $\rho(B^{-1}C) < 1$ if and only if A^{-1} exists and is nonnegative.*

We recall that $A = B - C$ is a regular splitting of A if B is invertible, $B^{-1} \geq 0$, and $C \geq 0$ (compare Definition 2.4.15 in [21]).

These lemmas lead to the following results.

THEOREM 5. *Assume that $A_{1,\infty} \geq 0$, $A_{2,\infty} \geq 0$, and $\rho(A_\lambda) < 1$. Then $\rho(A_0) \leq \rho(A_{0,\lambda})$ for $\lambda > 0$.*

Proof. Since $A_{2,\lambda} \geq 0$, it follows that $A_{1,\lambda} \leq A_\lambda$ and in view of Lemma 3 $\rho(A_{1,\lambda}) \leq \rho(A_\lambda) < 1$. Expanding $(I - A_{1,\lambda})^{-1}$ into a Neumann series and taking into account that $A_{1,\lambda} \geq A_{1,\infty}$ and $A_{2,\lambda} \geq A_{2,\infty}$, we obtain

$$\begin{aligned} A_{0,\lambda} &= (I - A_{1,\lambda})^{-1} A_{2,\lambda} = (I + A_{1,\lambda} + A_{1,\lambda}^2 + \cdots) A_{2,\lambda} \\ &\geq (I + A_{1,\infty} + A_{1,\infty}^2 + \cdots) A_{2,\infty} = A_0. \end{aligned}$$

Now the conclusion follows from Lemma 3. □

THEOREM 6. *Assume that $A_{1,\infty} \geq 0$, $A_{2,\infty} \geq 0$,* $\det (I - A_{1,\infty}) \neq 0$, *and $(I - A_{1,\infty})^{-1} \geq 0$. Then $\rho(A_\infty) < 1$ if and only if $\rho(A_0) < 1$.*

Proof. Since $(I - A_{1,\infty})^{-1} \geq 0$ and $A_{2,\infty} \geq 0$ it follows that $(I - A_{1,\infty}) - A_{2,\infty}$ is a regular splitting of the matrix $I - A_\infty$. It follows from Lemma 4 that $\rho(A_0) < 1$ if and only if $(I - A_\infty)^{-1}$ exists and $(I - A_\infty)^{-1} \geq 0$. This in turn is equivalent to $\rho(A_\infty) < 1$ in view of Lemma 2. □

Observe that in view of Lemma 2 the assumptions of Theorem 6 imply that $\rho(A_{1,\infty}) < 1$.

We will demonstrate next that the sequence $\{w_k\}_{k=0}^\infty$ defined by (5.3) is convergent to the solution w^* of $w = F(w, w)$ and the rate of the convergence is $\rho(A_0)$. To be more precise, we have the following theorem.

THEOREM 7. *Assume that Condition* L *is satisfied,* $\det (I - A_{1,\infty}) \neq 0$, $(I - A_{1,\infty})^{-1} \geq 0$, *$A_0$ is irreducible, and $\rho(A_0) < 1$. Then the sequences of iterations $[y_k, z_k]^T$, $k = 0, 1, \ldots$, defined by* (5.1)–(5.2) *and $[x_k, z_k]^T$, $k = 0, 1, \ldots$, defined by* (3.7)–(3.8) *are linearly convergent to the solutions $[y^*, z^*]^T$ of* (4.1)–(4.2) *and $[x^*, z^*]^T$ of* (3.3)–(3.4), *respectively, with the same rate $\rho(A_0)$.*

Proof. Observe first that since $\rho(A_0) < 1$ we have also $\rho(A_\infty) = \rho(A) < 1$ in view of Theorem 6. Therefore, it follows from Theorem 1 that the system (4.1)–(4.2) has a unique solution $w^* = [y^*, z^*]^T$. It is also easy to verify that (5.3) has a unique solution w_{k+1} for all $k = 0, 1, \ldots$. Putting $w_1 = w_2 = w^*$, $\bar{w}_1 = w_{k+1}$, $\bar{w}_2 = w_k$ in (4.6) we get

$$||w^* - w_{k+1}||_{\lambda,c} \leq (\xi + \epsilon) \max\{||w^* - w_{k+1}||_{\lambda,c}, ||w^* - w_k||_{\lambda,c}\}.$$

Choosing $\epsilon > 0$ such that $\xi + \epsilon < 1$ we obtain

$$||w^* - w_{k+1}||_{\lambda,c} \leq (\xi + \epsilon)||w^* - w_k||_{\lambda,c}$$

which proves the convergence of the sequence $\{w_k\}_{k=0}^{\infty}$ to w^* with the rate $\xi = \rho(A)$.

To prove the convergence of $\{w_k\}_{k=0}^{\infty}$ to w^* with the rate $\eta = \rho(A_0)$ we rewrite the inequalities (4.3) and (4.4) in matrix form with w_1, w_2, $\bar{w}_1$ and $\bar{w}_2$ as defined above to get

$$\begin{bmatrix} ||y^* - y_{k+1}||_\lambda \\ ||z^* - z_{k+1}||_\lambda \end{bmatrix} \leq \begin{bmatrix} \lambda^{-1}a_1 + a_3, a_5 \\ \lambda^{-1}b_1 + b_3, b_5 \end{bmatrix} \begin{bmatrix} ||y^* - y_{k+1}||_\lambda \\ ||z^* - z_{k+1}||_\lambda \end{bmatrix} + \begin{bmatrix} \lambda^{-1}a_2 + a_4, a_6 \\ \lambda^{-1}b_2 + b_4, b_6 \end{bmatrix} \begin{bmatrix} ||y^* - y_k||_\lambda \\ ||z^* - z_k||_\lambda \end{bmatrix}.$$

The assumption of our theorem implies that, for some sufficiently large λ, $\det(I - A_{1,\lambda}) \neq 0$, $(I - A_{1,\lambda})^{-1} \geq 0$, and $\rho(A_{0,\lambda}) < 1$, where

$$A_{0,\lambda} = (I - A_{1,\lambda})^{-1} A_{2,\lambda}.$$

Hence,

$$\begin{bmatrix} ||y^* - y_{k+1}||_\lambda \\ ||z^* - z_{k+1}||_\lambda \end{bmatrix} \leq A_{0,\lambda} \begin{bmatrix} ||y^* - y_k||_\lambda \\ ||z^* - z_k||_\lambda \end{bmatrix}. \tag{5.4}$$

Let

$$A_{0,\lambda} = A_0 + B_\lambda,$$

where $B_\lambda \to 0$ as $\lambda \to \infty$, and denote by $d = [d_1, d_2]^T > 0$ the Perron vector of the matrix A_0. Then

$$A_0 d = \rho(A_0) d,$$

and the inequality (5.4) yields

$$\begin{aligned} \begin{bmatrix} ||y^* - y_{k+1}||_\lambda \\ ||z^* - z_{k+1}||_\lambda \end{bmatrix} &\leq ||w^* - w_k||_{\lambda,d}(A_0 + B_\lambda)d \\ &= ||w^* - w_k||_{\lambda,c}(\rho(A_0)d + B_\lambda d). \end{aligned}$$

Multiplying this inequality by $\operatorname{diag}(1/d_1, 1/d_2)$ we obtain

$$\begin{bmatrix} \dfrac{||y^* - y_{k+1}||_\lambda}{d_1} \\ \dfrac{||z^* - z_{k+1}||_\lambda}{d_2} \end{bmatrix} \leq ||w^* - w_k||_{\lambda,d} \left(\rho(A_0) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{d_1} & 0 \\ 0 & \dfrac{1}{d_2} \end{bmatrix} B_\lambda \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right),$$

and it follows that

$$||w^* - w_{k+1}||_{\lambda,d} \leq ||w^* - w_k||_{\lambda,d} \left(\rho(A_0) + \max\left\{ \frac{1}{d_1}, \frac{1}{d_2} \right\} \max\{d_1, d_2\} ||B_\lambda||_\infty \right).$$

Let $\epsilon > 0$ be such that $\rho(A_0) + \epsilon < 1$, and choose $\lambda > 0$ such that the second term in the parentheses is less than ϵ. Then

$$||w^* - w_{k+1}||_{\lambda,d} \leq (\rho(A_0) + \epsilon)||w^* - w_k||_{\lambda,d}$$

and it follows that $\{w_k\}_{k=0}^{\infty}$ is linearly convergent to w^* with the rate $\rho(A_0)$. It is also clear that the sequence $[x_k, z_k]^T$, $k = 0, 1, \ldots$, $x_k = Jy_k$, defined by (3.7)–(3.8), is convergent to the solution $[x^*, z^*]^T$ of (3.3)–(3.4) with the same rate $\rho(A_0)$. This completes the proof. $\square$

In the next section we investigate the relationship between the rates of convergence $\rho(A)$ and $\rho(A_0)$ of the Picard iterations (3.7)–(3.8) and WR iterations (3.9)–(3.10).

6. Relationship between rates of convergence of Picard and WR iterations. We will need the following result on nonnegative and nonreducible matrices of dimension $n \times n$.

THEOREM 8. *Assume that P and Q are nonnegative and nonreducible $n \times n$ matrices and that $\rho(P + Q) < 1$. Then*

$$\rho(P) < \rho(P + Q), \tag{6.1}$$

and

$$\rho((I - P)^{-1}Q) < \rho(P + Q). \tag{6.2}$$

Proof. The proof of the first inequality (6.1) can be found in [21]. This inequality implies that $\det(I - P) \neq 0$ and $(I - P)^{-1} \geq 0$ in view of Lemma 2.

Let $\xi = \rho(P + Q)$ and denote by $v > 0$ the Perron vector of the matrix $P + Q$, i.e.,

$$(P + Q)v = \xi v.$$

Since $\xi < 1$ we have $q = (1 - \xi)v > 0$ and

$$v = (P + Q)v + q.$$

Define the sequences $\{u_k\}_{k=0}^{\infty}$ and $\{w_k\}_{k=0}^{\infty}$ by the relations

$$u_{k+1} = (P + Q)u_k, \qquad u_0 = v,$$
$$w_{k+1} = Pw_{k+1} + Qw_k, \quad w_0 = v.$$

Then $u_k = \xi^k v \to 0$ as $k \to \infty$. We will show next that the sequence $\{w_k\}_{k=0}^{\infty}$ is positive and decreasing and $w_k \to 0$ as $k \to \infty$. Since Q is nonnegative and nonreducible, it follows that each row of Q has a positive element. Hence, $w_1 = (I + P + P^2 + \cdots)Qv > 0$. Observe also that $(I - P)^{-1} \geq 0$ implies that

$$\begin{aligned} w_1 &= (I - P)^{-1}Qv < (I - P)^{-1}Qv + (I - P)^{-1}q \\ &= (I - P)^{-1}(Qv + q) = (I - P)^{-1}(Qv + (1 - \xi)v) \\ &= (I - P)^{-1}(Qv + v - (P + Q)v) = v = w_0. \end{aligned}$$

Hence, $0 < w_1 < w_0$. Assume next that $0 < w_k < w_{k-1}$. Then

$$w_{k+1} = (I - P)^{-1}Qw_k < (I - P)^{-1}Qw_{k-1} = w_k$$

which proves that $0 < w_k < w_{k-1}$ for any $k = 1, 2, \ldots$. We have $w_0 = u_0$ and we will show that

$$w_k < u_k, \qquad k = 1, 2, \ldots . \tag{6.3}$$

Since P is nonnegative and nonreducible we have

$$w_1 = Pw_1 + Qw_0 < (P + Q)w_0 = u_1$$

which proves (6.3) for $k = 1$. Assuming (6.3) for k we get

$$w_{k+1} = Pw_{k+1} + Qw_k < (P + Q)w_k < (P + Q)u_k = u_{k+1}$$

which proves (6.3) for any $k = 1, 2, \ldots$.

Consider next the matrix $(I - P)^{-1}Q$. Since this matrix is nonnegative and nonreducible it follows from Perron–Frobenius theory that there exists an eigenvalue η equal to the spectral radius of $(I - P)^{-1}Q$ and a corresponding positive eigenvector z. Put

$$\alpha = \min\left\{\frac{v_i}{z_i}, \quad i = 1, 2, \ldots, n\right\},$$

$$\beta = \max\left\{\frac{v_i}{z_i}, \quad i = 1, 2, \ldots, n\right\},$$

where v_i and z_i denote the ith components of vectors v and z, respectively. Then

$$\alpha z \leq v \leq \beta z \tag{6.4}$$

and it follows that

$$\alpha((I - P)^{-1}Q)^k z \leq ((I - P)^{-1}Q)^k v \leq \beta((I - P)^{-1}Q)^k z.$$

This inequality can be rewritten as

$$\alpha\eta^k z \leq w_k \leq \beta\eta^k z. \tag{6.5}$$

Using (6.3), (6.4), and (6.5), we have also

$$\alpha\eta^k z \leq w_k < u_k = \xi^k v \leq \beta\xi^k z, \tag{6.6}$$

$k = 1, 2, \ldots$, and we may conclude that $\eta \leq \xi < 1$. Denote by r the index such that $\alpha = v_r/z_r$. Then it follows that

$$\frac{v_r\eta^k z_r}{z_r} < \xi^k v_r$$

which implies that $\eta < \xi$. This completes the proof. □

Remark. The theorem is also valid if we assume that P and Q are positive and $\rho(P + Q) < 1$. This follows from the fact that such matrices are obviously nonnegative and nonreducible.

Theorem 8 implies that

$$\rho(A_0) < \rho(A),$$

where the matrices A and A_0 are defined in §§4 and 5, respectively. This inequality means that the rate of convergence of the WR iterations defined by (3.7)–(3.8) is always better than the rate of convergence of classical Picard iterations defined by (3.9)–(3.10). The specific example is given in the next section.

7. Comments and examples. This paper deals with the theoretical analysis of the convergence of the classical Picard iterations and more general WR iterations for differential-algebraic systems of equations which may contain delay and/or neutral terms. It is proved that the Picard iterations are convergent if $\rho(A) < 1$ and WR iterations are convergent if

$\rho(A_0) < 1$ where A and A_0 are some nonnegative 2×2 matrices which depend on Lipschitz constants of the problem under consideration. The condition $\rho(A) < 1$ is strong; however, one should not be surprised by this fact. The reason for this is that the system (3.1)–(3.2) is general enough to include as special cases the linear systems of differential-algebraic equations for which this condition is also necessary. A simple example is given by

$$x' = 0, \qquad x(0) = \bar{x}_0,$$
$$z = az + 1,$$

where a and z are scalars. For this system,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & |a| \end{bmatrix}$$

and the condition $\rho(A) = |a| < 1$ is necessary for convergence of the Picard iterations

$$x'_{k+1} = 0, \qquad x_{k+1}(0) = \bar{x}_0,$$
$$z_{k+1} = az_k + 1,$$

$k = 0, 1, \ldots$. On the other hand, one can easily verify that the WR iterations

$$x'_{k+1} = 0, \qquad x_{k+1}(0) = \bar{x}_0,$$
$$z_{k+1} = bz_{k+1} + (a-b)z_k + 1,$$

$k = 0, 1, \ldots$, $b \neq 1$, are convergent if and only if

$$\left| \frac{a-b}{1-b} \right| < 1$$

which is equivalent to

$$\left(a < 1 \text{ and } b < \frac{1+a}{2} \right) \text{ or } \left(a > 1 \text{ and } b > \frac{1+a}{2} \right).$$

However, if we assume that $b \geq 0$ and $a - b \geq 0$, then it is easy to see that the condition $|a| < 1$ is also necessary in this case as well (this also follows from Theorem 6). This example illustrates that in the linear case we can exploit the signs of the coefficients of the system to extend the range of convergence of WR iterations. However, if the problem under consideration is nonlinear it is often necessary to work with a linear comparison system with nonnegative coefficients and such possibilities are lost. For specific examples we could possibly obtain less restrictive conditions by reformulating the problem in some way; however, it seems to be difficult, if not impossible, to define a general strategy for doing this.

It is also demonstrated that, under some conditions which are not too restrictive, $\rho(A) < 1$ if and only if $\rho(A_0) < 1$ (compare Theorem 6) and that $\rho(A_0) < \rho(A)$ (compare Theorem 8). This inequality means that the convergence of WR iterations is always faster than the convergence of Picard iterations.

To illustrate the different rates of convergence corresponding to $\rho(A)$ and $\rho(A_0)$, consider the following differential algebraic system

$$x'(t) = Mx(t) + Pt\sin(x(t)) + a\cos(x'(t)) + b\frac{z(t)}{1+z^2(t)}, \tag{7.1}$$

$$z(t) = \frac{Q\sqrt{t}}{1+x^2(t)} + Stx(t) + c\exp(-|x'(t)|) + d\sin(z(t)), \tag{7.2}$$

$x(0) = 0$, $t \in T$, where M, P, Q, S, a, b, c, and d are given constants. The Picard iterations for (7.1)–(7.2) take the form

$$x'_{k+1}(t) = Mx_k(t) + Pt\sin(x_k(t)) + a\cos(x'_k(t)) + b\frac{z_k(t)}{1+z_k^2(t)}, \tag{7.3}$$

$$z_{k+1}(t) = \frac{Q\sqrt{t}}{1+x_k^2(t)} + Stx_k(t) + c\exp(-|x'_k(t)|) + d\sin(z_k(t)), \tag{7.4}$$

$k = 0, 1, \ldots$, $x_0(t) = 0$, $z_0(t) = 0$, $x_k(0) = 0$, $t \in T$. It follows from Theorem 1 that (7.1)–(7.2) has a unique solution $[x, z]^T$ and the sequence $[x_k, z_k]^T$ defined by (7.3)–(7.4) is convergent to this solution if $\rho(A) < 1$, where the matrix A is defined by

$$A = \begin{bmatrix} |a|, |b| \\ |c|, |d| \end{bmatrix}.$$

It can be verified that this is equivalent to

$$|a| + |d| < 2 \quad \text{and} \quad |a| + |d| + |bc| < 1 + |ad|,$$

or

$$|a| < 1, \qquad |d| < 1, \quad \text{and} \quad |a| + \frac{|bc|}{1-|d|} < 1.$$

The last of these conditions can be obviously replaced by

$$|bc| < (1-|a|)(1-|d|),$$

which coincides with condition H_4 in [22].

Consider next the WR iterations defined by

$$\begin{aligned} x'_{k+1}(t) = {} & Mx_{k+1}(t) + Pt\sin(x_k(t)) + a_1\cos(x'_{k+1}(t)) \\ & + a_2\cos(x'_k(t)) + b_1\frac{z_{k+1}(t)}{1+z_{k+1}^2(t)} + b_2\frac{z_k(t)}{1+z_k(t)}, \end{aligned} \tag{7.5}$$

$$\begin{aligned} z_{k+1}(t) = {} & \frac{Q\sqrt{t}}{1+x_{k+1}^2(t)} + Stx_k(t) + c_1\exp(-|x'_{k+1}(t)|) \\ & + c_2\exp(-|x'_k(t)|) + d_1\sin(z_{k+1}(t)) + d_2\sin(z_k(t)), \end{aligned} \tag{7.6}$$

$k = 0, 1, \ldots$, $x_0(t) = 0$, $z_0(t) = 0$, $x_k(0) = 0$, $t \in T$, where a_i, b_i, c_i, d_i, $i = 1, 2$, have the same sign as a, b, c, and d, respectively, and $a = a_1 + a_2$, $b = b_1 + b_2$, $c = c_1 + c_2$, and $d = d_1 + d_2$. Then it follows from Theorem 7 that (7.5)–(7.6) are convergent to the solution $[x, z]^T$ of (7.1)–(7.2) if $\det(I - A_1) \neq 0$, $(I - A_1)^{-1} \geq 0$, and $\rho(A_0) < 1$, where

$$A_0 = (I - A_1)^{-1}A_2,$$

with

$$A_1 = \begin{bmatrix} |a_1|, |b_1| \\ |c_1|, |d_1| \end{bmatrix}, \qquad A_2 = \begin{bmatrix} |a_2|, |b_2| \\ |c_2|, |d_2| \end{bmatrix}.$$

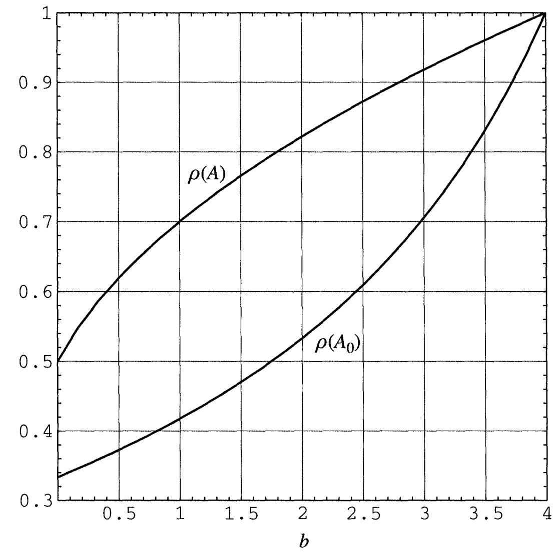

FIG. 1. *Spectral radius of A and A_0 versus b.*

To illustrate the difference between different rates of convergence of the iterations (7.3)–(7.4) and (7.5)–(7.6) we have chosen $a = 0.2$, $c = 0.1$, $d = 0.5$, $a_1 = a_2 = 0.1$, $c_1 = 0.1$, $c_2 = 0$, $d_1 = d_2 = 0.25$, $b_1 = b$, $b_2 = 0$, where b is a free parameter. Then

$$A = \begin{bmatrix} 0.2 & b \\ 0.1 & 0.5 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0.1 & b \\ 0.1 & 0.25 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.25 \end{bmatrix},$$

and it is easy to verify that $(I - A_1)^{-1} \geq 0$, $\rho(A) < 1$, and $\rho(A_0) < 1$ if and only if $0 \leq b < 4$. We have plotted in Figure 1 $\rho(A)$ and $\rho(A_0)$ versus b for $0 \leq b \leq 4$. For example for $b = 3$, $\rho(A) \approx 0.9179$ while $\rho(A_0) \approx 0.7055$. Observe also that in this case the standard norm condition $||A|| < 1$, where

$$||A|| := \max \left\{ \sum_{j=1}^{n} |a_{ij}| : 1 \leq i \leq n \right\}$$

for convergence of Picard iterations, does not hold since we have $||A|| = 3.2$.

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