SIAM J. SCI. COMPUT.
Vol. 15, No. 3, pp. 604–620, May 1994

008

DOMAIN DECOMPOSITION ALGORITHMS WITH SMALL OVERLAP*

MAKSYMILIAN DRYJA† AND OLOF B. WIDLUND‡

Abstract. Numerical experiments have shown that two-level Schwarz methods often perform very well even if the overlap between neighboring subregions is quite small. This is true to an even greater extent for a related algorithm, due to Barry Smith, where a Schwarz algorithm is applied to the reduced linear system of equations that remains after the variables interior to the subregions have been eliminated. In this paper, a supporting theory is developed.

Key words. domain decomposition, elliptic finite element problems, preconditioned conjugate gradients, Schwarz methods

AMS subject classifications. 65F10, 65N30, 65N55

1. Introduction. Over the last decade, considerable interest has developed in Schwarz methods and other domain decomposition methods for partial differential equations; see, e.g., the proceedings of five international symposia [17]–[19], [34], [35]. A general theory has evolved and a substantial number of new algorithms has been designed, analyzed and tested numerically. Among them are *two-level, additive Schwarz* methods first introduced in 1987; see Dryja and Widlund [25], [28]–[30], [55]. For related work see also Bjørstad, Moe, and Skogen [1]–[3], Cai [10]–[12], Mathew [40]–[42], Matsokin and Nepomnyaschikh [43], Nepomnyaschikh [44], Skogen [47], Smith [48]–[52], and Zhang [58], [59]. As shown in Dryja and Widlund [30], a number of other domain decomposition methods, in particular those of Bramble, Pasciak, and Schatz [5], [6], can also be derived and analyzed using the same framework. Recent efforts by Bramble et al. [7] and Xu [56] have extended the general framework making a systematic study of multiplicative Schwarz methods possible. The multiplicative algorithms are direct generalizations of the original alternating method discovered more than 120 years ago by H. A. Schwarz [46]. For other recent projects, that also use the Schwarz framework, see Dryja, Smith, and Widlund [27] and Dryja and Widlund [32], [33].

When a two-level method is used, the restrictions of the discrete elliptic problem to overlapping subregions, into which the given region has been decomposed, are solved exactly or approximately. These local solvers form an important part of a preconditioner for a conjugate gradient method. In addition, to enhance the convergence rate, the preconditioner includes a global problem of relatively modest dimension.

Generalizations to more than two levels have also been developed; see, e.g., Bramble, Pasciak, and Xu [8], Dryja and Widlund [31], and Zhang [58]–[60]; here the families of domain decomposition methods and multigrid algorithms merge. Recently there has also been a considerable interest in nonsymmetric and indefinite problems; cf., e.g., Bramble, Leyk, and Pasciak [4], Cai [10]–[12], Cai, Gropp, and Keyes [13], Cai and Widlund [14], [15], Cai and Xu [16], and Xu [57]. However, in this paper, we work exclusively with two-level methods for positive definite, symmetric problems.

*Received by the editors May 26, 1992; accepted for publication (in revised form) February 1, 1993.

†Department of Mathematics, Warsaw University, 2 Banach, 02-097 Warsaw, Poland (dryja@mimuw.edu.pl). The work of this author was supported in part by National Science Foundation grant NSF-CCR-8903003, by Polish Scientific grant 211669101, and by the Center for Computational Sciences of the University of Kentucky at Lexington.

‡Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, New York 10012 (widlund@cs.nyu.edu). This work was supported in part by National Science Foundation grant NSF-CCR-8903003 and in part by the U.S. Department of Energy contract DE-FG02-88ER25053 at the Courant Mathematics and Computing Laboratory.

The main result of our early study of two-level Schwarz methods shows that the condition number of the operator, which is relevant for the conjugate gradient iteration, is uniformly bounded if the overlap between neighboring subregions is sufficiently generous in proportion to the diameters of the subregions.

Our current work has been inspired very directly by several series of numerical experiments that indicate that the rate of convergence is quite satisfactory even for a small overlap, and that the running time of the programs is often the smallest when the overlap is at a minimum. The number of conjugate gradient iterations is typically higher in such a case, but this can be compensated for by the fact that the local problems are smaller and therefore cheaper to solve; see, in particular, Bjørstad, Moe, and Skogen [2], Bjørstad and Skogen [3], Cai [10], [11], Cai, Gropp, and Keyes [13], and Skogen [47]. If the local problems are themselves solved by an iterative method, then a smaller overlap will give better conditioned local problems and therefore a higher rate of convergence; see Skogen [47] for a detailed discussion of this effect. All this work also shows that these algorithms are relatively easy to implement. Recent experiments by Gropp and Smith [37] for problems of linear elasticity provide strong evidence that these methods can be quite effective even for very ill-conditioned problems. In this paper, we show that the condition number of the preconditioned operator for the algorithm, introduced in 1987 by Dryja and Widlund [28], is bounded from above by const.$(1 + (H/\delta))$. Here H measures the diameter of a subregion and δ the overlap between neighboring subregions. We note that H/δ is a measure of the aspect ratio of the subregion common to two overlapping neighboring subregions.

We then turn our attention to a very interesting method, introduced in 1989 by Barry Smith [48], [51]. It is known as the *vertex space* (or *Copper Mountain*) algorithm. Numerical experiments, for problems in the plane, have shown that this method converges quite rapidly even for problems, which were originally very ill conditioned, even if the overlap is very modest; see Smith [48]. For additional work on variants of this method, see Chan and Mathew [20], [21], and Chan, Mathew, and Shao [22].

When Smith's algorithm is used, the given large linear system of algebraic equations, resulting from a finite element discretization of an elliptic problem, is first reduced in size by eliminating all variables associated with the interiors of the nonoverlapping *substructures* $\{\Omega_i\}$ into which the region has been subdivided. The reduced problem is known as the *Schur complement* system and the remaining degrees of freedom are associated with the set $\{\partial\Omega_i\}$ of substructure boundaries that form the interface Γ between the substructures. The preconditioner of this domain decomposition method, classified as a *Schwarz method on the interface* in Dryja and Widlund [30], is constructed from a coarse mesh problem, with the substructures serving as elements, and a potentially large number of local problems. The latter correspond to an overlapping covering of Γ, with each subset corresponding to a set of adjacent interface variables.

Smith's main theoretical result, given in [48], [51], is quite similar to that for the original two-level Schwarz method; the condition number of this domain decomposition algorithm is uniformly bounded for a class of second-order elliptic problems provided that there is a relatively generous overlap between neighboring subregions that define the subdivision of the domain decomposition method. In this paper, we show that the condition number of the iteration operator grows only in proportion to $(1 + \log(H/\delta))^2$. We note that even for a minimal overlap of just one mesh width h, this bound is as strong as those for the well-known iterative substructuring methods considered by Bramble, Pasciak, and Schatz [5], [6], Dryja [24], Dryja, Proskurowski, and Widlund [26], Smith [49], and Widlund [54]; cf. also Dryja, Smith, and Widlund [27]. We also note that the successful iterative substructuring methods for problems in three dimensions require the use of more complicated coarse subspaces and that therefore Smith's method, considered in this paper, seems to offer an advantage.

2. Some Schwarz methods for finite element problems. As usual, we write our continuous and finite element elliptic problems as: Find $u \in V$, such that

$$a(u, v) = f(v) \quad \forall\, v \in V,$$

and, find $u_h \in V^h$, such that

$$a(u_h, v_h) = f(v_h) \quad \forall\, v_h \in V^h, \tag{1}$$

respectively. We assume that the bilinear form $a(u, v)$ is selfadjoint and elliptic and that it is bounded in $V \times V$. In the case of Poisson's equation, the bilinear form is defined by

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx. \tag{2}$$

We assume that Ω is a Lipschitz region in R^n, $n = 2, 3$, and that its diameter is on the order of 1. (We will follow Nečas [45] when defining Lipschitz regions and Sobolev spaces on Ω.) The bilinear form $a(u, v)$ is directly related to the Sobolev space $H^1(\Omega)$ that is defined by the seminorm and norm

$$|u|^2_{H^1(\Omega)} = a(u, u) \quad \text{and} \quad \|u\|^2_{H^1(\Omega)} = |u|^2_{H^1(\Omega)} + \|u\|^2_{L^2(\Omega)},$$

respectively. When we are considering subregions $\Omega_i \subset \Omega$, of diameter H, we use a different relative weight, obtained by dilation,

$$\|u\|^2_{H^1(\Omega_i)} = |u|^2_{H^1(\Omega_i)} + \frac{1}{H^2}\|u\|^2_{L^2(\Omega_i)}.$$

Whenever appropriate, we tacitly assume that the elements of $H_0^1(\Omega_i)$, the subspace of $H^1(\Omega_i)$ with zero trace on the boundary $\partial\Omega_i$, are extended by zero to $\Omega \setminus \Omega_i$.

To avoid unnecessary complications, we confine our discussion to Poisson's equation, to homogeneous Dirichlet conditions, and to continuous, piecewise linear finite elements and a polygonal region Ω triangulated using triangles or tetrahedra. It is well known that the resulting space $V^h \subset H_0^1(\Omega)$, i.e., it is conforming.

For the problem considered in an 1870 paper by H.A. Schwarz [46], two overlapping subregions Ω_1' and Ω_2' are used; the union of the two is Ω. There are two sequential, fractional steps of the iteration in which the approximate solution of the elliptic equation on Ω is updated by solving the given problem restricted to the subregions one at a time. The most recent values of the solution are used as boundary values on the part of $\partial\Omega_i'$ that is not a part of $\partial\Omega$.

The finite element version of the algorithm can conveniently be described in terms of projections $P_i : V^h \to V_i^h = H_0^1(\Omega_i') \bigcap V^h$, defined by

$$a(P_i v_h, \phi_h) = a(v_h, \phi_h) \quad \forall \phi_h \in V_i^h. \tag{3}$$

It is easy to show that the error propagation operator of this multiplicative Schwarz method is

$$(I - P_2)(I - P_1).$$

This algorithm can therefore be viewed as a simple iterative method for solving

$$(P_1 + P_2 - P_2 P_1) u_h = g_h,$$

with an appropriate right-hand side g_h.

This operator is a polynomial of degree two and therefore not ideal for parallel computing since two sequential steps are involved. This effect is further pronounced if more than two subspaces are used. Therefore, it is often advantageous to collect subregions, which do not intersect, into groups; the subspaces of each group can then be regarded as one. The number of subspaces is thus reduced, and the algorithm becomes easier to parallelize. Numerical experiments with multiplicative Schwarz methods have also shown that the convergence rate often is enhanced if such a strategy is pursued; this approach is similar to a red-black or multicolor ordering in the context of classical iterative methods. In the case of an additive Schwarz method, this ordering only serves as a device to facilitate the analysis.

In the additive form of the algorithm, we work with the simplest possible polynomial of the projections: The equation

$$Pu_h = (P_1 + P_2 + \cdots + P_N)u_h = g_h' \tag{4}$$

is solved by an iterative method. Here $P_i : V \to V_i$, and $V = V_1 + \cdots + V_N$. Since the operator P can be shown to be positive definite, symmetric with respect to $a(\cdot,\cdot)$, the iterative method of choice is the conjugate gradient method. Equation (4) must also have the same solution as (1), i.e., the correct right-hand side must be found. This can easily be arranged; see, e.g., [29], [30], [55]. Much of the work, in particular that which involves the individual projections, can be carried out in parallel.

2.1. The Dryja–Widlund algorithm. We now describe the special additive Schwarz method introduced in Dryja and Widlund [28]; see also Dryja [25] and Dryja and Widlund [29]. We start by introducing two triangulations of Ω into nonoverlapping triangular or tetrahedral substructures Ω_i and into triangular or tetrahedral elements. We obtain the elements by subdividing the substructures. We always assume that the two triangulations are shape regular, see, e.g., Ciarlet [23], and, to simplify our arguments, that the diameters of all the substructures are on the order of H. In this algorithm, we use overlapping subregions obtained by extending each substructure Ω_i to a larger region Ω_i'. The overlap is said to be generous if the distance between the boundaries $\partial\Omega_i$ and $\partial\Omega_i'$ is bounded from below by a fixed fraction of H. We always assume that $\partial\Omega_i'$ does not cut through any element. We carry out the same construction for the substructures that meet the boundary except that we cut off the part of Ω_i' that is outside of Ω.

We remark that other decompositions are also of interest. In particular, the analysis in §4 extends immediately to the case when no degrees of freedom are shared between neighboring subregions. In this case the distance between $\partial\Omega_i'$ and $\partial\Omega_j'$ is just h for neighboring subregions. This additive Schwarz method corresponds to a block Jacobi preconditioner augmented by a coarse solver.

For this Schwarz method, the finite element space is represented as the sum of N+1 subspaces

$$V^h = V_0^h + V_1^h + \cdots + V_N^h.$$

The first subspace V_0^h is equal to V^H, the space of continuous, piecewise linear functions on the coarse mesh defined by the substructures Ω_i. The other subspaces are related to the subdomains in the same way as in the original Schwarz algorithm, i.e., $V_i^h = H_0^1(\Omega_i') \bigcap V^h$.

It is often more economical to use approximate rather than exact solvers for the problems on the subspaces. The approximate solvers are described in the following terms: Let $b_i(u, v)$ be an inner product defined on $V_i^h \times V_i^h$ and assume that there exists a constant ω such that

$$a(u,u) \le \omega b_i(u,u) \quad \forall u \in V_i^h. \tag{5}$$

In terms of matrices, this inequality becomes a one-sided bound of the stiffness matrix corresponding to $a(\cdot,\cdot)$ and V_i^h, in terms of the matrix corresponding to the bilinear form $b_i(u,u)$.

An operator $T_i : V^h \to V_i^h$, which replaces P_i, is now defined by

$$b_i(T_i u, \phi_h) = a(u, \phi_h) \quad \forall \phi_h \in V_i^h. \tag{6}$$

It is easy to show that the operator T_i is positive semidefinite and symmetric with respect to $a(\cdot,\cdot)$ and that the minimal constant ω in (5) is $\|T_i\|_a$. Additive and multiplicative Schwarz methods can now be defined straightforwardly in terms of polynomials of the operators T_i. We note that if exact solvers, and thus the projections P_i, are used, then $\omega = 1$.

2.2. The Smith algorithm. Smith's method has previously been described in Smith [48], [51]. Let K be the stiffness matrix given by the bilinear form of (2). In the first step of this and many other domain decomposition methods, the unknowns of the linear system of equations

$$Kx = b$$

that correspond to the interiors of the substructures are eliminated. We now describe this procedure in some detail.

Let $K^{(i)}$ be the stiffness matrix corresponding to the bilinear form $a_{\Omega_i}(u_h, v_h)$, which represents the contribution of the substructure Ω_i to the integral $a_\Omega(u_h, v_h) = a(u_h, v_h)$. Let x and y be the vectors of nodal values that correspond to the finite element functions u_h and v_h, respectively. Then the stiffness matrix K of the entire problem can be obtained by using the *method of subassembly* defined by the formula

$$x^T K y = \sum_i x^{(i)^T} K^{(i)} y^{(i)}.$$

Here $x^{(i)}$ is the subvector of nodal parameters associated with $\overline{\Omega}_i$, the closure of Ω_i. We represent $K^{(i)}$ as

$$\begin{pmatrix} K_{II}^{(i)} & K_{IB}^{(i)} \\ K_{IB}^{(i)T} & K_{BB}^{(i)} \end{pmatrix}.$$

Here we have divided the subvector $x^{(i)}$ into two, $x_I^{(i)}$ and $x_B^{(i)}$, corresponding to the variables that are interior to the substructure and those that are shared with other substructures, i.e., those associated with the nodal points of $\partial\Omega_i$. Since the interior variables of Ω_i are coupled only to other variables of the same substructure, they can be eliminated locally and in parallel. The resulting reduced matrix is a Schur complement and is of the form

$$S^{(i)} = K_{BB}^{(i)} - K_{IB}^{(i)T} K_{II}^{(i)-1} K_{IB}^{(i)}. \tag{7}$$

From this follows that the Schur complement, corresponding to the global stiffness matrix K, is given by S where

$$x_B^T S y_B = \sum_i x_B^{(i)T} S^{(i)} y_B^{(i)}. \tag{8}$$

If the local problems are solved exactly, what remains is to find a sufficiently accurate approximation of the solution of the linear system

$$S x_B = g_B. \tag{9}$$

It is convenient to rewrite (9) in variational form. Let $s_i(u_h, v_h)$ and $s(u_h, v_h)$ denote the forms defined by (7) and (8), respectively, i.e.,

$$s_i(u_h, v_h) = x_B^{(i)} S^{(i)} y_B^{(i)} \quad \text{and} \quad s(u_h, v_h) = x_B^T S y_B.$$

Equation (9) can then be rewritten as

$$s(u_h, v_h) = (g, v_h)_{L^2(\Gamma)} \quad \forall v_h \in V^h(\Gamma). \tag{10}$$

Here $\Gamma = \bigcup \partial\Omega_i \setminus \partial\Omega$.

Problem (10) will be solved by an iterative method of additive Schwarz type. The most important difference between this algorithm and that of the previous subsection is that we are now working with the trace space $H^{1/2}(\Gamma)$ instead of $H^1(\Omega)$; see §3 for a definition of $H^{1/2}(\Gamma)$.

It is well known that

$$x_B^{(i)T} S^{(i)} x_B^{(i)} = \min_{x_I^{(i)}} x^{(i)T} K^{(i)} x^{(i)}. \tag{11}$$

Therefore, if u_h is the minimal, *the discrete harmonic* extension of the boundary data represented by x_B, then

$$x_B^{(i)T} S^{(i)} x_B^{(i)} = |u_h|^2_{H^1(\Omega_i)}.$$

Smith's algorithm can now be described in terms of a subspace decomposition. We use the same coarse space as in the previous subsection, i.e., V^H, but we restrict its values to Γ. In the case when the original problem is two dimensional, we introduce one subspace for each interior edge and one for each vertex of the substructures. An edge space is defined by setting all nodal values, except those associated with the interior of the edge in question, to zero. Similarly, a vertex space is obtained by setting to zero all values at the nodes on Γ, which are at a distance greater than δ. For many more details and a discussion of implementation details, see Smith [48], [51].

In the case when the original problem is three dimensional, we introduce one subspace for each interior face, edge, and vertex. The elements of a face subspace vanish at all nodes on Γ that do not belong to the interior of the face. Similarly, an edge space is supported in the strips of width δ, which belong to the faces, that have this edge in common. Finally, a vertex space is defined in terms of the nodes on Γ that are within a distance δ of the vertex. We also again use the standard coarse space V^H.

2.3. Basic theory. To estimate the rate of convergence of our special, or any other, additive Schwarz methods, we need upper and lower bounds for the spectrum of the operator relevant in the conjugate gradient iteration. A lower bound can be obtained by using the following lemma; see, e.g., Dryja and Widlund [29], [33] or Zhang [60].

LEMMA 2.1. *Let T_i be the operators defined in* (6) *and let $T = T_0 + T_1 + \cdots + T_N$. Then*

$$a(T^{-1}u, u) = \min_{u=\sum u_i} \sum b_i(u_i, u_i), \qquad u_i \in V_i.$$

Therefore, if a representation, $u = \sum u_i$, can be found, such that

$$\sum b_i(u_i, u_i) \le C_0^2 a(u, u) \quad \forall u \in V^h,$$

then

$$\lambda_{\min}(T) \ge C_0^{-2}.$$

An upper bound for the spectrum of T is often obtained in terms of strengthened Cauchy–Schwarz inequalities between the different subspaces. Note that we now exclude the index 0; the coarse subspace is treated separately.

DEFINITION 1. *The matrix $\mathcal{E} = \{\varepsilon_{ij}\}$ is the matrix of strengthened Cauchy–Schwarz constants, i.e., ε_{ij} is the smallest constant for which*

$$|a(v_i, v_j)| \leq \varepsilon_{ij} \|v_i\|_a \|v_j\|_a \quad \forall v_i \in V_i \quad \forall v_j \in V_j \qquad i, j \geq 1 \tag{12}$$

holds.

The following lemma is easy to prove; see Dryja and Widlund [33].

LEMMA 2.2. *Let $\rho(\mathcal{E})$ be the spectral radius of the matrix $\mathcal{E}$. Then, the operator T satisfies*

$$T \leq \omega(\rho(\mathcal{E}) + 1)I.$$

For the particular algorithms considered in this paper, it is very easy to show that there is a uniform upper bound. In fact, by collecting and merging local subspaces that belong to nonoverlapping subregions, the number of subspaces, and $\rho(\mathcal{E})$, can be made uniformly bounded; see §4 for an alternative argument.

By combining Lemmas 2.1 and 2.2, we obtain the following theorem.

THEOREM 2.3. *The condition number $\kappa(T)$ of the operator T of the additive Schwarz method satisfies*

$$\kappa(T) \equiv \lambda_{\max}(T)/\lambda_{\min}(T) \leq \omega(\rho(\mathcal{E}) + 1)C_0^2.$$

In the multiplicative case, we need to provide an upper bound for the spectral radius, or norm, of the error propagation operator

$$E_J = (I - T_J)\dots(I - T_0). \tag{13}$$

The following theorem is a variant of a result of Bramble, Pasciak, Wang, and Xu [7]; see also Cai and Widlund [15], Xu [56], or Zhang [60]. Note that this bound is also given in terms of the same three parameters that appear in Theorem 2.3.

THEOREM 2.4. *In the symmetric, positive definite case*

$$\|E_J\|_a \leq \sqrt{1 - \frac{(2-\omega)}{(2\omega^2\rho(\mathcal{E})^2 + 1)C_0^2}}\,.$$

We note that this formula is useless if $\omega \geq 2$; since $\|I - T_i\|_a > 1$ if $\|T_i\|_a > 2$, the assumption that $\omega < 2$ is most natural. If we wish to use a multiplicative algorithm and ω is too large, we can scale the bilinear forms $b_i(\cdot,\cdot)$ suitably.

In this paper, our results are only formulated for additive algorithms. The corresponding bounds for the multiplicative variants can easily be worked out as applications of Theorem 2.4.

3. Technical tools.

In this section, we collect a number of technical tools that are used to prove our main results. Some of these tools are quite familiar to specialists of the field. Others, to our knowledge, have not previously been used in the analysis of domain decomposition; see Il'in [38] for some similar inequalities.

As before, $\Omega \subset R^n$, n =2 or 3, is a bounded, polygonal region and $\{\Omega_i\}$ a nonoverlapping decomposition of Ω into substructures. To simplify our considerations, we now assume that the substructures are squares or cubes; see, e.g., Nečas [45] where simple maps and partions of unity are used to derive bounds for Lipschitz regions from bounds for such special regions;

if we can handle a corner of a square or cube, then we can analyze the general polygonal case. Our estimates, given in the next two sections, are developed for one substructure at a time; our arguments can be modified to make them valid for any shape regular substructure with a boundary consisting of a finite number of smooth curves or surfaces.

As before, let $\Gamma = \bigcup \partial\Omega_i \setminus \partial\Omega$. Let $\Gamma_{\delta,i} \subset \Omega_i$ be the set of points that is within a distance δ of Γ.

LEMMA 3.1. *Let u be an arbitrary element of $H^1(\Omega_i)$. Then*

$$\|u\|^2_{L^2(\Gamma_{\delta,i})} \leq C\,\delta^2\big((1+H/\delta)|u|^2_{H^1(\Omega_i)} + 1/(H\delta)\|u\|^2_{L^2(\Omega_i)}\big).$$

Proof. We first consider a square region $(0, H) \times (0, H)$ in detail; the extension of the proof to the case of three dimensions is straightforward. Since

$$u(x,0) = u(x,y) - \int_0^y \frac{\partial u(x,\tau)}{\partial y} d\tau,$$

we find, by elementary arguments, that

$$H\int_0^H |u(x,0)|^2 dx \leq 2\int_0^H\int_0^H |u(x,y)|^2 dxdy + H^2\int_0^H\int_0^H \left|\frac{\partial u}{\partial y}\right|^2 dxdy.$$

Therefore,

$$H\int_0^H |u(x,0)|^2 dx \leq 2\|u\|^2_{L^2(\Omega_i)} + H^2|u|^2_{H^1(\Omega_i)}.$$

Now consider the integral over a narrow subregion next to one of the sides of the square. Using similar arguments, we obtain

$$\int_0^H\int_0^\delta |u(x,y)|^2 dxdy \leq \delta^2|u|^2_{H^1(\Omega_i)} + 2\delta\int_0^H |u(x,0)|^2 dx. \tag{14}$$

By combining this and the previous inequality, we obtain

$$\int_0^H\int_0^\delta |u(x,y)|^2 dxdy \leq \delta^2|u|^2_{H^1(\Omega_i)} + 2\delta\left(\frac{2}{H}\|u\|^2_{L^2(\Omega_i)} + H|u|^2_{H^1(\Omega_i)}\right)$$

as required.

The modifications necessary for the case of an arbitrary, shape regular substructure and the extension of the proof to the case of three dimensions are routine. □

LEMMA 3.2. *Let u_h be a continuous, piecewise quadratic function defined on the finite element triangulation and let $I_h u_h \in V^h$ be its piecewise linear interpolant on the same mesh. Then there exists a constant C, independent of h and H, such that*

$$|I_h u_h|_{H^1(\Omega_i)} \leq C|u_h|_{H^1(\Omega_i)}.$$

The same type of bounds hold for the L_2 and $H^{1/2}_{00}$ norms and it can also be extended, with different constants, to the case of piecewise cubic functions, etc.

Proof. It is elementary to show that,

$$|I_h u_h|^2_{H^1(\Omega_i)} \leq 2\big(|I_h u_h - u_h|^2_{H^1(\Omega_i)} + |u_h|^2_{H^1(\Omega_i)}\big).$$

Consider the contribution to the first term on the right-hand side from an individual element K. We obtain

$$|I_h u_h - u_h|^2_{H^1(K)} \leq Ch^2 |u_h|^2_{H^2(K)} \leq C|u_h|^2_{H^1(K)}$$

by using a standard error bound and an elementary inverse inequality for quadratic polynomials. The bound in L_2 follows from the linear independence of the standard finite element basis for the space of quadratic polynomials, see Ciarlet [23]. The bounds for the $H^{1/2}_{00}$ norm, which is further discussed later in this section, can be obtained by interpolation in Sobolev spaces; cf., e.g., Lions and Magenes [39, pp. 98–99]. We use the K-method and a norm, which is equivalent to the one given below,

$$\|u\|^2_{H^{1/2}_{00}} = \|u\|^2_{L_2} + \int_0^\infty t^{-2} K(t,u)^2 dt,$$

where

$$K(t,u) = \inf_{u=u_0+u_1} (\|u_0\|^2_{L^2} + t^2 \|u_1\|^2_{H^1_0})^{1/2}.$$

The crucial observation is that

$$K(t, I_h u_h) \leq CK(t, u_h),$$

To prove this estimate, consider an arbitrary decomposition $u_h = u_0 + u_1$ of a piecewise quadratic u_h. Let Q be the L_2-projection onto the space of piecewise quadratic finite elements. It is well known that Q is bounded in H^1 as well as L_2; see, e.g., Bramble and Xu [9]. Using the H^1 and L_2 bounds of I_h derived in this proof, we find that

$$K(t, I_h u_h)^2 \leq \|I_h Q u_0\|^2_{L^2} + t^2 \|I_h Q u_1\|^2_{H^1_0} \leq C(\|u_0\|^2_{L^2} + t^2 \|u_1\|^2_{H^1_0}).$$

The proof now follows easily. □

We now turn to the other auxiliary results needed in the analysis of Smith's algorithm. We begin by writing down a standard expression for the norm of $H^{1/2}$; see Chapters 1.3.2 and 1.5 of Grisvard [36] for a detailed discussion.

Let $I \subset R^1$ be an open interval of diameter H. Then

$$\|u\|^2_{H^{1/2}(I)} = \int_I \int_I \frac{|u(s)-u(t)|^2}{|s-t|^2} ds dt + \frac{1}{H} \|u\|^2_{L_2(I)}. \tag{15}$$

The relative weight of the two terms is obtained, by dilation, from the norm defined on a region of diameter 1.

It is well known that the extension by zero of the elements of $H^{1/2}(I)$ does not define a continuous map into $H^{1/2}(R^1)$; see Lemma 1.3.2.6 of Grisvard [36] or Lions and Magenes [39]. The largest subspace for which this extension operator is continuous is $H^{1/2}_{00}(I)$, which is defined in terms of the norm obtained by replacing the last term of (15) by

$$\int_I \frac{|u(s)|^2}{d(s)} ds. \tag{16}$$

Here $d(s)$ is the distance to the nearest end point of I.

In the case of a subset Ψ of the boundary of a three-dimensional region, the formula (15) is valid after replacing $|s-t|^2$ by $|s-t|^3$ and I by Ψ. However, for our purposes, it is more convenient to use an alternative formula; see Dryja [24] and Nečas [45, Lemma 5.3, Chap. 2]. In the special case of a square with side H, the seminorm is defined by

$$\int_0^H \int_0^H \frac{\|u(s_1,.) - u(t_1,.)\|_{L_2}^2}{|s_1 - t_1|^2} ds_1 dt_1 + \int_0^H \int_0^H \frac{\|u(.,s_2) - u(.,t_2)\|_{L_2}^2}{|s_2 - t_2|^2} ds_2 dt_2. \tag{17}$$

To obtain the norm for the subspace $H_{00}^{1/2}((0, H)^2)$, we add a weighted norm

$$\int_0^H \int_0^H \frac{|u(s_1, s_2)|^2}{d(s)} ds_1 ds_2 \tag{18}$$

just as in (16).

In addition to the space V^h, we will also use a coarser space V^δ, defined on a mesh with mesh size δ, in our proofs. We now formulate results that have been used extensively in work of this kind; see Dryja [24] or Bramble and Xu [9]. The first inequality of the lemma is given as Lemma 1 in Dryja [24]. The second is part of the proof of Lemma 4 in the same paper.

LEMMA 3.3. *Let I be an interval of length H. Then*

$$\|u_\delta\|_{L_\infty(I)}^2 \le C(1 + \log(H/\delta))\|u_\delta\|_{H^{1/2}(I)}^2 \quad \forall u_\delta \in V^\delta.$$

Let I be an edge of a face Ψ of diameter H of a cube. Then

$$\|u_\delta\|_{L^2(I)}^2 \le C(1 + \log(H/\delta))\|u_\delta\|_{H^{1/2}(\Psi)}^2 \quad \forall u_\delta \in V^\delta.$$

The next result gives a bound that is similar to the second formula of Lemma 3.3. However, the bound holds for all of $H^{1/2}$.

LEMMA 3.4. *Let $\Psi = (0, H)^2$ and let $\Psi_\delta = (0, H) \times (0, \delta)$. Then*

$$\|u\|_{L_2(\Psi_\delta)}^2 \le C\delta(1 + \log(H/\delta))\|u\|_{H^{1/2}(\Psi)}^2 \quad \forall u \in H^{1/2}(\Psi).$$

The same result holds if we replace Ψ and Ψ_δ by $(0, H)$ and $(0, \delta)$, respectively.

Proof. We only provide a proof for the first of the two cases; the proof in the other case is completely analogous. Let $Q_\delta : L_2(\Psi) \to V^\delta$, be the L_2- projection. It is well known that $\|Q_\delta\|_{H^1(\Psi)}$ is bounded; see, e.g., Bramble and Xu [9]. Since, trivially, this operator is also bounded in L_2, it follows that $\|Q_\delta\|_{H^{1/2}(\Psi)}$ is bounded. By a standard argument,

$$\|u - Q_\delta u\|_{L_2(\Psi)}^2 \le \delta C|u|_{H^{1/2}(\Psi)}^2. \tag{19}$$

We now only need to show that

$$\|Q_\delta u\|_{L_2(\Psi_\delta)}^2 \le C\delta(1 + \log(H/\delta))\|Q_\delta u\|_{H^{1/2}(\Psi)}^2. \tag{20}$$

To prove (20), we use the bound (14) derived in the proof of Lemma 3.1. Thus

$$\|Q_\delta u\|_{L_2(\Psi_\delta)}^2 \le \delta^2|Q_\delta u|_{H^1(\Psi)}^2 + 2\delta \int_0^H |Q_\delta u(x, 0)|^2 dx.$$

By using an inverse inequality, the first term can be replaced by $\delta|Q_\delta u|_{H^{1/2}(\Psi)}^2$. The second term is estimated using Lemma 3.3. $\square$

The final lemma will be used to estimate the weighted L_2 term in the $H_{00}^{1/2}$ norm.

LEMMA 3.5. *Let $u \in H^{1/2}(0, H)$. Then there exists a constant C, such that*

$$\int_\delta^H \frac{|u(s)|^2}{s} ds \le C(1 + \log(H/\delta))^2\|u\|_{H^{1/2}(0,H)}^2.$$

Similarly, let $u \in H^{1/2}((0,H)^2)$. Then there exists a constant C, such that

$$\int_0^H \left(\int_\delta^H \frac{|u(s,t)|^2}{s} ds\right) dt \le C(1+\log(H/\delta))^2 \|u\|^2_{H^{1/2}((0,H)^2)}.$$

Proof. We begin by considering the first inequality. Let $Q_\delta : L_2(0,H) \to V^\delta(0,H)$, be the L_2–projection onto the finite element space with mesh size δ. We write $u = (u - Q_\delta u) + Q_\delta u$ and estimate each term separately. We first note that by a standard estimate,

$$\|u - Q_\delta u\|^2_{L_2(0,H)} \le C\delta |u|^2_{H^{1/2}(0,H)}.$$

The bound for the first term is therefore obtained by noting that $s \ge \delta$ over the interval of integration.

The other term can be estimated by using the first bound of Lemma 3.3, which results in one logarithmic factor, and the observation that

$$\int_\delta^H \frac{|Q_\delta u|^2}{s} ds \le \|Q_\delta u\|^2_{L_\infty} \int_\delta^H \frac{ds}{s},$$

from which the second logarithmic factor arises.

The second inequality follows by considering $u(s,t)$, a function of two variables, and applying the first inequality with respect to s. We then integrate with respect to t, change the order of integration, and use formula (17) to complete the proof. □

4. Analysis of the Dryja-Widlund algorithm. We now use the set $\Gamma_{\delta,i}$, previously introduced, to characterize the extent of the overlap. We assume that all $x \in \Omega_i$, which belong to at least one additional overlapping subregion Ω'_j, lie in $\Gamma_{\delta,i}$.

THEOREM 4.1. *In the case when exact solvers are used for the subproblems, the condition number of the additive Schwarz method satisfies*

$$\kappa(P) \le C(1 + H/\delta).$$

The constant is independent of the parameters H, h, and δ.

We note that for the case of two subregions, it is easy to show that this result is sharp. It is routine to modify Theorem 4.1 to cover cases where inexact solvers are used.

Proof. The proof is a refinement of a result first given in Dryja and Widlund [28]; cf. [29] for a better discussion. The proof is equally valid for two and three dimensions. We first show that a constant upper bound for the spectrum of P can be obtained without the use of Lemma 2.2.

We note that P_i is also an orthogonal projection of $H^1(\Omega'_i) \bigcap V^h$ onto V_i. Therefore,

$$a(P_i u_h, u_h) \le a_{\Omega'_i}(u_h, u_h).$$

Since, by construction, there is an upper bound, N_c, on the number of subregions to which any $x \in \Omega$ can belong, we have

$$\sum_{i=1}^N a_{\Omega'_i}(u_h, u_h) \le N_c\, a(u_h, u_h).$$

In addition, we use the fact that the norm of P_0 is equal to one and obtain

$$\lambda_{\max}(P) \le (N_c + 1).$$

The lower bound is obtained by using Lemma 2.1. A natural choice of u_0 is the L_2-projection $Q_H u_h$ of u_h onto V^H. As previously pointed out, this projection is bounded in L_2 as well as H^1 and there exists a constant, independent of h and H, such that

$$\|u_h - Q_H u_h\|_{L_2} \leq C\, H\|u_h\|_a. \tag{21}$$

Let $w_h = u_h - Q_H u_h$ and let $u_i = I_h(\theta_i w_h)$, $i = 1, \ldots, N$. Here I_h is the interpolation operator onto the space V^h and the $\theta_i(x)$ define a partition of unity, i.e., $\sum_i \theta_i(x) \equiv 1$. These functions are chosen as nonnegative elements of V^h. It is easy to see that

$$u_h = Q_H u_h + \sum u_i.$$

In the interior part of Ω_i, which does not belong to $\Gamma_{\delta,i}$, $\theta_i \equiv 1$. This function must decrease to 0 over a distance on the order of δ. It is easy to construct a partition of unity with $0 \leq \theta_i \leq 1$ and such that

$$|\nabla\theta_i| \leq \frac{C}{\delta}\ .$$

To use Lemma 2.1, we first estimate $a(u_i, u_i)$ in terms of $a(w_h, w_h)$. We consider the contribution from one substructure at a time and note that, trivially,

$$a_{\Omega_i\setminus\Gamma_{\delta,i}}(u_i, u_i) = a_{\Omega_i\setminus\Gamma_{\delta,i}}(w_h, w_h).$$

Let K be an element in $\Gamma_{\delta,i}$. Then, using the definition of u_i,

$$a_K(u_i, u_i) \leq 2a_K(\bar{\theta}_i w_h, \bar{\theta}_i w_h) + 2a_K(I_h((\theta_i - \bar{\theta}_i)w_h), I_h((\theta_i - \bar{\theta}_i)w_h)),$$

where $\bar{\theta}_i$ is the average of θ_i over the element K. Using the fact that the diameter of K is on the order of h and the bound on $\nabla\theta_i$, we obtain, after adding over all the relevant elements,

$$a_{\Gamma_{\delta,i}}(u_i, u_i) \leq 2a_{\Omega_i}(w_h, w_h) + \frac{C}{\delta^2}\|w_h\|^2_{L^2(\Gamma_{\delta,i})}.$$

We also need to estimate $a_{\Gamma_{\delta,i}}(u_j, u_j)$ for the j that correspond to neighboring substructures. This presents no new difficulties.

To complete the proof, we need to estimate $\|w_h\|^2_{L^2(\Gamma_{\delta,i})}$. We note that each $x \in \Omega$ is covered only a finite number of times by the subregions. We apply Lemma 3.1 to the function w_h, sum over i and use inequality (21) to complete the estimate of the parameter C_0^2 of Lemma 2.1. □

5. Analysis of the Smith method. A description of the reduction of the original linear system to one for the degrees of freedom on Γ, and of the algorithm, have been given in §2. We will now work in the $H^{1/2}(\Gamma)$ norm. The fact that this is a weaker norm than H^1 is reflected in a stronger bound than that of the previous section; better bounds of the components in the different subspaces can be obtained. There are many similarities between the two cases. Much of the analysis is again carried out one substructure at a time.

To show that we can work with $|u_h|^2_{H^{1/2}(\partial\Omega_i)}$ instead of $x_B^{(i)T} S^{(i)} x_B^{(i)}$, we must show that these norms are equivalent. We use (11) and the standard trace theorem to bound $|u_h|^2_{H^{1/2}(\partial\Omega_i)}$ from above by $x_B^{(i)T} S^{(i)} x_B^{(i)}$. The proof of the reverse inequality requires an extension theorem for finite element spaces given in Widlund [53]; see further discussion in Smith [51].

THEOREM 5.1. *In the case when exact solvers are used for the subproblems, the condition number of the vertex space method satisfies*

$$\kappa(P) \leq C(1 + \log(H/\delta))^2.$$

The constant is independent of the parameters H, h, and δ.

Proof. As in the proof of Theorem 4.1, there is no difficulty in establishing a uniform upper bound on the spectrum of P.

We now turn to the lower bound in the case where the original problem is two dimensional and thus the interface is of dimension one. To use Lemma 2.1, we have to decompose functions defined on Γ.

We use the $L_2(\Omega)$ projection onto V^H of the discrete harmonic function u_h, introduced in §2.2, to define the component of the coarse space. We only use the values on Γ. In addition, we use a partition of unity to represent the local space components. In the study of the local spaces, it is sufficient to consider one substructure Ω_i at a time. The partition of unity is based on simple, piecewise linear functions. Let $0 < t < H$ represent one of the edges of the boundary of this substructure and let $\theta_e(t)$ be a piecewise linear function that vanishes for t outside $(0, H)$, grows linearly to 1 at $t = \delta$, is equal to 1 for $\delta \le t \le H - \delta$, and drops to zero linearly over the interval $(H - \delta, H)$. In the decomposition, we choose $I_h(\theta_e w_h)$ as the component corresponding to this edge. As in the previous section, $u_0 = Q_H u_h$ and $w_h = u_h - Q_H u_h$ is the error of the L_2- projection. It follows from Lemma 3.2 that it is sufficient to estimate $\|v_e\|_{H_{00}^{1/2}(0,H)}$, where $v_e(t) = \theta_e(t) w_h(t)$. We note that we cannot use the weaker norm of $H^{1/2}(0, H)$ here; we must estimate the $H^{1/2}(\partial\Omega_i)$ norm of v_e extended by zero to the rest of the boundary, i.e., $\|v_e\|_{H_{00}^{1/2}(0,H)}$.

We first consider

$$\int_0^H \int_0^H \frac{|v_e(s) - v_e(t)|^2}{|s-t|^2} ds dt, \tag{22}$$

and then the additional term (16), which completes the definition of the relevant norm. We divide the interval $[0, H]$ into three parts, $[0, \delta]$, $[\delta, H - \delta]$, and $[H - \delta, H]$, and take the tensor product of $[0, H]$ with itself. The double integral (22) is then split into the sum of nine. By symmetry, only six different cases need to be considered. The integral over $[\delta, H - \delta] \times [\delta, H - \delta]$ is completely harmless. We now consider the diagonal term corresponding to the set $[0, \delta] \times [0, \delta]$ and use the identity

$$v_e(s) - v_e(t) \equiv \frac{s w_h(s) - t w_h(t)}{\delta} = \frac{(s+t)(w_h(s) - w_h(t))}{2\delta} + \frac{(s-t)(w_h(s) + w_h(t))}{2\delta}.$$

The integral corresponding to the first term is estimated by $\|w_h\|_{H^{1/2}}$ after noting that, for the relevant values of s and t, $|s+t|/2\delta \le 1$.

The integral, corresponding to the second term, can be estimated by

$$1/\delta^2 \int_0^\delta \int_0^\delta |w_h(t)|^2 ds dt = 1/\delta \|w_h\|^2_{L^2(0,\delta)},$$

which, in turn, can be estimated appropriately by using Lemma 3.4.

The third diagonal double integral is estimated in exactly the same way.

We next estimate the off-diagonal double integrals. We note that for $0 \le t \le \delta$ and $\delta \le s \le H - \delta$,

$$v_e(s) - v_e(t) = w_h(s) - \frac{t}{\delta} w_h(t) = (w_h(s) - w_h(t)) + \frac{(\delta - t)}{\delta} w_h(t).$$

The first term gives an integral that can be estimated straightforwardly in terms of $|w_h|^2_{H^{1/2}(0,H)}$. What remains is the integral

$$\int_0^\delta \left(\int_\delta^H \frac{(\delta - t)^2}{\delta^2 (t-s)^2} ds \right) |w_h(t)|^2 dt.$$

We integrate with respect to s and find that the inner integral is bounded by $1/\delta$ and the double integral by

$$1/\delta \|w_h\|^2_{L^2(0,\delta)}. \tag{23}$$

The estimates of the other integrals can be carried out quite similarly. To complete the estimate of the $\|v_e\|_{H^{1/2}_{00}(0,H)}$, we consider

$$\int_0^H \left(\frac{|v_e(s)|^2}{s} + \frac{|v_e(s)|^2}{H-s} \right) ds.$$

For $s \in (0,\delta)$, and $s \in (H-\delta, H)$, we obtain contributions that can be estimated by the expression given in (23). For the integral over $(\delta, H-\delta)$, we use Lemma 3.5.

We next turn to the space associated with one of the vertices of Ω_i. We now use $\theta_v(t) = (\delta - |t|)/\delta$ to complete the partition of unity, i.e., we use $I_h(\theta_v w_h) = I_h(v_v)$. It follows from Lemma 3.2 that we can again ignore the interpolation operator I_h. We need to estimate

$$\int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{|v_v(s) - v_v(t)|^2}{|s-t|^2} ds dt$$

and

$$\int_{-\delta}^{\delta} \left(\frac{|v_v(s)|^2}{\delta - s} + \frac{|v_v(s)|^2}{s+\delta} \right) ds.$$

Considering the double integral, we note that

$$\frac{(\delta - |s|)w_h(s) - (\delta - |t|)w_h(t)}{\delta(s-t)} = \frac{w_h(s) - w_h(t)}{s-t} \left(1 - \frac{|s| + |t|}{2\delta}\right) - \frac{w_h(s) + w_h(t)}{2\delta} \frac{|s| - |t|}{s-t}.$$

Since $|1 - \frac{|s|+|t|}{2\delta}| \le 1$ and $|\frac{|s|-|t|}{s-t}| \le 1$, for relevant values of s and t, the two contributions to the integral can be estimated in terms of

$$|w_h|^2_{H^{1/2}(0,H)} \quad \text{and} \quad 1/\delta \|w_h\|^2_{L^2(0,\delta)},$$

respectively. Arguments, quite similar to those given above, complete the proof for the case of two dimensions.

We now turn to problems in three dimensions, i.e., the case where the interface Γ is two dimensional. In addition to the coarse space, we use three types of local subspaces associated with faces, and neighborhoods of edges and vertices, respectively. The diameter of the point set associated with a vertex subspace is on the order of δ. Similarly, the edge spaces include the degrees of freedom on Γ that are within a distance δ of the edge in question. We again construct a partition of unity associated with these sets. As before these functions are continuous, piecewise linear functions and their gradients are bounded by C/δ. The proof proceeds as in the case of two dimensions. We give only a few details. We use $\theta_e(t_1)\theta_e(t_2)$ to construct the contribution to the decomposition related to a face. Similarly, we use $\theta_e(t_1)\theta_v(t_2)$ as the part of the partition of unity associated with an edge. Using our formulas for θ_e and θ_v, we then show that the partition of unity is completed by adding functions that differ from zero only in small neighborhoods of the vertices. The estimates necessary for the use of Lemma 2.1 and the completion of this proof are then carried out as in the two-dimensional case. □

REFERENCES

[1] P. E. BJØRSTAD, *Multiplicative and additive Schwarz methods: Convergence in the two domain case*, in Domain Decomposition Methods, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1989.

[2] P. E. BJØRSTAD, R. MOE, AND M. SKOGEN, *Parallel domain decomposition and iterative refinement algorithms*, in Parallel Algorithms for PDEs, Proceedings of the 6th GAMM-Seminar held in Kiel, Germany, Jan. 19–21, 1990, W. Hackbusch, ed., Vieweg & Son, Braunschweig, Wiesbaden, 1990.

[3] P. E. BJØRSTAD AND M. SKOGEN, *Domain decomposition algorithms of Schwarz type, designed for massively parallel computers*, in Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, D. E. Keyes, G. A. Meurant, J. S. Scroggs, and R. G. Voigt, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[4] J. H. BRAMBLE, Z. LEYK, AND J. E. PASCIAK, *Iterative schemes for non-symmetric and indefinite elliptic boundary value problems*, Math. Comp., 60 (1993), pp. 1–22.

[5] J. H. BRAMBLE, J. E. PASCIAK, AND A. H. SCHATZ, *The construction of preconditioners for elliptic problems by substructuring*, I, Math. Comp., 47 (1986), pp. 103–134.

[6] ———, *The construction of preconditioners for elliptic problems by substructuring*, IV, Math. Comp., 53 (1989), pp. 1–24.

[7] J. H. BRAMBLE, J. E. PASCIAK, J. WANG, AND J. XU, *Convergence estimates for product iterative methods with applications to domain decomposition*, Math. Comp., 57 (1991), pp. 1–21.

[8] J. H. BRAMBLE, J. E. PASCIAK, AND J. XU, *Parallel multilevel preconditioners*, Math. Comp., 55 (1990), pp. 1–22.

[9] J. H. BRAMBLE AND J. XU, *Some estimates for a weighted L^2 projection*, Math. Comp., 56 (1991), pp. 463–476. Ph.D. thesis compare [58].

[10] X.-C. CAI, *Some Domain Decomposition Algorithms for Nonselfadjoint Elliptic and Parabolic Partial Differential Equations*, Ph.D. thesis, Mathematics Dept., Courant Institute of Mathematical Sciences, New York University, Sept. 1989; Tech. Report 461, Dept. of Computer Science, Courant Institute of Mathematical Sciences, New York University, Sept. 1989.

[11] ———, *An additive Schwarz algorithm for nonselfadjoint elliptic equations*, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1990.

[12] ———, *Additive Schwarz algorithms for parabolic convection-diffusion equations*, Numer. Math., 60 (1991), pp. 41–61.

[13] X.-C. CAI, W. D. GROPP, AND D. E. KEYES, *A comparison of some domain decomposition algorithms for nonsymmetric elliptic problems*, in Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, D. E. Keyes, G. A. Meurant, J. S. Scroggs, and R. G. Voigt, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[14] X.-C. CAI AND O. WIDLUND, *Domain decomposition algorithms for indefinite elliptic problems*, SIAM J. Sci. Statist. Comput., 13 (1992), pp. 243–258.

[15] ———, *Multiplicative Schwarz algorithms for some nonsymmetric and indefinite problems*, SIAM J. Numer. Anal., 30 (1993), pp. 936–952.

[16] X.-C. CAI AND J. XU, *A preconditioned GMRES method for nonsymmetric or indefinite problems*, Math. Comp., 59 (1992), pp. 311–319.

[17] T. CHAN, R. GLOWINSKI, J. PÉRIAUX, AND O. WIDLUND, EDS., *Domain Decomposition Methods*, in Second International Symposium on Domain Decomposition Methods, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1989.

[18] ———, EDS., *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1990.

[19] T. F. CHAN, D. E. KEYES, G. A. MEURANT, J. S. SCROGGS, AND R. G. VOIGT, EDS., *Fifth Conference on Domain Decomposition Methods for Partial Differential Equations*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[20] T. F. CHAN AND T. P. MATHEW, *An application of the probing technique to the vertex space method in domain decomposition*, in Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, Y. A. Kuznetsov, G. A. Meurant, J. Périaux, and O. Widlund, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1991.

[21] ———, *The interface probing technique in domain decomposition*, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 212–238.

[22] T. F. CHAN, T. P. MATHEW, AND J.-P. SHAO, *Efficient variants of the vertex space domain decomposition algorithm*, Tech. Report CAM 92-07, Department of Mathematics, Univ. of California Los Angeles, Jan. 1992. SIAM J. Sci. Comput., 15 (1994).

[23] P. G. CIARLET, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[24] M. DRYJA, *A method of domain decomposition for* 3-D *finite element problems*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1988.

[25] ———, *An additive Schwarz algorithm for two- and three-dimensional finite element elliptic problems*, in Domain Decomposition Methods, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1989.

[26] M. DRYJA, W. PROSKUROWSKI, AND O. WIDLUND, *A method of domain decomposition with crosspoints for elliptic finite element problems*, in Optimal Algorithms, B. Sendov, ed., Bulgarian Academy of Sciences, Sofia, Bulgaria, 1986, pp. 97–111.

[27] M. DRYJA, B. F. SMITH, AND O. B. WIDLUND, *Schwarz analysis of iterative substructuring algorithms for problems in three dimensions*, Tech. Report 638, Dept. of Computer Science, Courant Institute of Mathematical Sciences, New York University, May 1993.

[28] M. DRYJA AND O. B. WIDLUND, *An additive variant of the Schwarz alternating method for the case of many subregions*, Tech. Report 339; also, Ultracomputer Note 131, Department of Computer Science, Courant Institute of Mathematical Sciences, New York University, 1987.

[29] ———, *Some domain decomposition algorithms for elliptic problems*, in Iterative Methods for Large Linear Systems, L. Hayes and D. Kincaid, eds., Academic Press, San Diego, CA, 1989, pp. 273–291.

[30] ———, *Towards a unified theory of domain decomposition algorithms for elliptic problems*, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1990.

[31] ———, *Multilevel additive methods for elliptic finite element problems*, in Parallel Algorithms for Partial Differential Equations, W. Hackbusch, ed., Proceedings of the Sixth GAMM-Seminar, Kiel, Jan. 19–21, 1990, Vieweg & Son, Braunschweig, Germany, 1991.

[32] ———, *Additive Schwarz methods for elliptic finite element problems in three dimensions*, in Fifth Conference on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, D. E. Keyes, G. A. Meurant, J. S. Scroggs, and R. G. Voigt, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[33] ———, *Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems*, Tech. Report 626, Department of Computer Science, Courant Institute of Mathematical Sciences, New York University, March 1993.

[34] R. GLOWINSKI, G. H. GOLUB, G. A. MEURANT, AND J. PÉRIAUX, EDS., *Domain Decomposition Methods for Partial Differential Equations*, Proceedings of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1988.

[35] R. GLOWINSKI, Y. A. KUZNETSOV, G. A. MEURANT, J. PÉRIAUX, AND O. WIDLUND, EDS., *Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1991.

[36] P. GRISVARD, *Elliptic problems in nonsmooth domains*, Pitman, Boston, 1985.

[37] W. D. GROPP AND B. F. SMITH, *Experiences with domain decomposition in three dimensions: Overlapping Schwarz methods*, Tech. Report, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL, 1992; Proceedings of the Sixth International Symposium on Domain Decomposition Methods, to appear.

[38] V. P. IL'IN, *The Properties of Some Classes of Differentiable Functions of Several Variables Defined in an n-dimensional Region*, Amer. Math. Soc., 81 (1969), pp. 91–256. Originally in Trudy Mat. Inst. Steklov., 66 (1962), pp. 227–363.

[39] J.-L. LIONS AND E. MAGENES, *Nonhomogeneous Boundary Value Problems and Applications*, Vol. I, Springer-Verlag, New York, Heidelberg, Berlin, 1972.

[40] T. P. MATHEW, *Domain Decomposition and Iterative Refinement Methods for Mixed Finite Element Discretisations of Elliptic Problems*, Ph.D. thesis, Mathematics Dept., Courant Institute of Mathematical Sciences, New York University, Sept. 1989; Tech. Report 463, Dept. of Computer Science, Courant Institute of Mathematical Sciences, New York, 1989.

[41] ———, *Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems, part* I: *Algorithms and numerical results*, Numer. Math., 65 (1993), pp. 445–468.

[42] ———, *Schwarz alternating, iterative refinement and Schur complement based methods for mixed formulations of elliptic problems, part* II: *Convergence theory*, Numer. Math., 65 (1993), pp. 469–492.

[43] A. M. MATSOKIN AND S. V. NEPOMNYASCHIKH, *A Schwarz alternating method in a subspace*, Soviet Math., 29 (1985), pp. 78–84.

[44] S. V. Nepomnyaschikh, *Domain Decomposition and Schwarz Methods in a Subspace for the Approximate Solution of Elliptic Boundary Value Problems*, Ph.D. thesis, Computing Center of the Siberian Branch of the USSR Academy of Sciences, Novosibirsk, USSR, 1986.

[45] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Academia, Prague, 1967.

[46] H. A. Schwarz, *Gesammelte Mathematische Abhandlungen*, Vol. 2, Springer-Verlag, Berlin, 1890, pp. 133–143. First published in Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, 15 (1870), pp. 272–286.

[47] M. D. Skogen, *Schwarz Methods and Parallelism*, Ph.D. thesis, Department of Informatics, University of Bergen, Norway, Feb. 1992.

[48] B. F. Smith, *Domain Decomposition Algorithms for the Partial Differential Equations of Linear Elasticity*, Ph.D. thesis, Mathematics Dept., Courant Institute of Mathematical Sciences, New York University, Sept. 1990; Tech. Report 517, Dept. of Computer Science, Courant Institute of Mathematical Sciences, New York University, 1990.

[49] ———, *A domain decomposition algorithm for elliptic problems in three dimensions*, Numer. Math., 60 (1991), pp. 219–234.

[50] ———, *An iterative substructuring algorithm for problems in three dimensions*, in Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, D. E. Keyes, G. A. Meurant, J. S. Scroggs, and R. G. Voigt, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[51] ———, *An optimal domain decomposition preconditioner for the finite element solution of linear elasticity problems*, SIAM J. Sci. Stat. Comput., 13 (1992), pp. 364–378.

[52] ———, *A parallel implementation of an iterative substructuring algorithm for problems in three dimensions*, SIAM J. Sci. Comput., 14 (1993), pp. 406–423.

[53] O. B. Widlund, *An extension theorem for finite element spaces with three applications*, in Numerical Techniques in Continuum Mechanics, W. Hackbusch and K. Witsch, eds., Braunschweig/Wiesbaden, 1987, Notes on Numerical Fluid Mechanics, V. 16, Friedr. Vieweg und Sohn, pp. 110–122.

[54] ———, *Iterative substructuring methods: Algorithms and theory for elliptic problems in the plane*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1988.

[55] ———, *Some Schwarz methods for symmetric and nonsymmetric elliptic problems*, in Fifth Conference on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, D. E. Keyes, G. A. Meurant, J. S. Scroggs, and R. G. Voigt, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[56] J. Xu, *Iterative methods by space decomposition and subspace correction*, SIAM Rev., 34 (1992), pp. 581–613.

[57] ———, *A new class of iterative methods for nonselfadjoint or indefinite problems*, SIAM J. Numer. Anal., 29 (1992), pp. 303–319.

[58] X. Zhang, *Studies in Domain Decomposition: Multilevel Methods and the Biharmonic Dirichlet Problem*, Ph.D. thesis, Mathematics Dept., Courant Institute of Mathematical Sciences, New York University, Sept. 1991; Tech. Report 584, Dept. of Computer Science, Courant Institute of Mathematical Sciences, New York University, 1991.

[59] ———, *Domain decomposition algorithms for the biharmonic Dirichlet problem*, in Fifth Conference on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, D. E. Keyes, G. A. Meurant, J. S. Scroggs, and R. G. Voigt, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[60] ———, *Multilevel Schwarz methods*, Numer. Math., 63 (1992), pp. 521–539.