



**NORTH-HOLLAND**

## **A Convergence Theorem for Chaotic Asynchronous Relaxation**

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### **ABSTRACT**

We present a convergence result for chaotic asynchronous relaxation that is a modification of the result of Chazan and Miranker. The modification is a restriction to the case of global memory or fast communication. The extra restriction is that each update is based on a prior state of the system, rather than on prior substates. © Elsevier Science Inc., 1997

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### **1. INTRODUCTION**

In their seminal paper [5] Chazan and Miranker studied chaotic relaxation, now usually called asynchronous relaxation, for the solution of linear systems. In chaotic relaxation the order in which components of the solution are updated is arbitrary and the past values of components that are used in the updates are also selected arbitrarily. This is meant to be a model for parallel computation in which different processors work independently and have access to data values in local memory.

The chaotic model of Chazan and Miranker is interesting because of the great generality allowed and because there is a simple necessary and sufficient condition for the system to converge for chaotic updates. It is an extreme model in that many real systems place more restrictions on the process than Chazan and Miranker do.

Much of the paper [5] deals with the special case of periodic relaxation, in which there is a well-defined order to the process of updating. Much of the

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work that has followed that of Chazan and Miranker extends the analysis of periodic relaxation; see, for example, [4] and [7]. At the end of [5] Chazan and Miranker prove an interesting very general theorem on convergence for chaotic iteration. In this paper we present, we believe, the first theorem that modifies their general theorem for chaotic relaxation. Our version of chaotic relaxation is more restrictive than is that of Chazan and Miranker.

The structure of this paper is as follows. Section 2 presents the chaotic relaxation of Chazan and Miranker with a discussion of our modification. Both the general theorem of Chazan and Miranker and our new result are stated. Section 3 presents the proof of our theorem, and Section 4 presents some conclusions.

## 2. CHAOTIC ASYNCHRONOUS RELAXATION

We consider the solution of linear systems of the form

$$(I - B)\vec{x} = \vec{d}, \quad (2.1)$$

where  $I$  is the  $N$  by  $N$  identity matrix,  $B$  is an  $N$  by  $N$  matrix,  $\vec{d}$  is the data vector, and  $\vec{x}$  is the vector of unknowns. Both  $\vec{x}$  and  $\vec{d}$  have  $N$  components. The iteration begins with an initial vector  $\vec{x}^0$  and proceeds by computing successive iterates  $\vec{x}^\nu$ .

We first establish some notation. The spectral radius of a matrix  $A$  will be denoted by  $\rho(A)$ . We define the absolute value of a vector  $\vec{x}$  to be the vector whose components are the absolute values of  $\vec{x}$ , and we denote the absolute value of  $x$  as  $|\vec{x}|$ . The absolute value of a matrix  $A$  is defined similarly and will be written  $|A|$ . The Euclidean norm of a vector  $\vec{x}$  will be written as  $\|\vec{x}\|$ .

Chaotic relaxation as defined by Chazan and Miranker can be specified in terms of two functions, an update function  $u(\cdot)$  and a shift function  $s(\cdot, \cdot)$ . For each nonnegative integer  $\nu$  the component to be updated at step  $\nu$  is given by  $u(\nu)$ . For the update at step  $\nu$  the value of the  $m$ th component used in the update is the value at  $s(\nu, m)$  steps back. This can be expressed as

$$x_l^{\nu+1} = \begin{cases} \sum_{m=1}^N b_{l,m} x_m^{\nu-s(\nu,m)} + d_l & \text{for } l = u(\nu), \\ x_l^\nu & \text{for } l \neq u(\nu). \end{cases}$$

The standard Jacobi and Gauss-Seidel methods of solution are particular cases in which there is a repeating pattern to the updating of components. For example, the Jacobi method can be described by

$$\begin{aligned} u(\nu) &= (\nu \bmod N) + 1, \\ s(\nu, m) &= \nu \bmod N, \end{aligned}$$

and similarly, the Gauss-Seidel method can be described by

$$\begin{aligned} u(\nu) &= (\nu \bmod N) + 1, \\ s(\nu, m) &= 0. \end{aligned}$$

As is well known, a necessary and sufficient condition for the convergence of the Jacobi method is that  $\rho(B)$  is less than 1.

For other methods of splitting the matrix computations between processors in a multiprocessor environment see [6], [9], and [10].

Of interest here are cases where there is no particular pattern to the order of the updates. Bertsekas and Tsitsiklis [2, p. 435] consider a seemingly more general form of asynchronous relaxation in which the function  $u(\cdot)$  takes sets as values. All the components in the set  $u(\nu)$  are updated at step  $\nu$ . However, it is easily seen that this is equivalent to the chaotic relaxation of Chazan and Miranker under a renumbering of the steps.

The update and shift functions are required to satisfy the following conditions.

CONDITION 1. The update function  $u(\cdot)$  takes on each value  $l$  for  $1 \leq l \leq N$  infinitely often.

CONDITION 2. The shift function is bounded, that is, there is an integer  $\bar{s}$  such that  $0 \leq s(\nu, m) < \bar{s}$ . For the initial steps to be well defined, we require  $s(\nu, m) \leq \nu$  as well.

Condition 1 is obviously necessary for the process to converge, and Bertsekas and Tsitsiklis [3] have an example of a nonconverging process in the case that Condition 2 fails to hold.

For the general chaotic relaxation as defined here, Chazan and Miranker have proved the following stability theorem. (See also Bertsekas and Tsitsiklis [2] for a proof.)

**THEOREM 2.1.** *A necessary and sufficient condition for the chaotic iteration to converge for all update functions and shift functions satisfying Conditions 1 and 2 is that  $\rho(|B|) < 1$ .*

An interesting aspect of the definition of Chazan and Miranker is the extreme generality allowed. We emphasize two features especially. First, if a component is updated twice, say  $u(\nu_1) = u(\nu_2) = l$  with  $\nu_1 < \nu_2$ , there is no constraint on  $s(\nu_1, m)$  compared with  $s(\nu_2, m)$ . A reasonable restriction would be to require more recent data in the update at  $\nu_2$  than were used at  $\nu_1$ . In particular, one might require

$$\nu_1 - s(\nu_1, m) \leq \nu_2 - s(\nu_2, m) \quad \text{when} \quad u(\nu_1) = u(\nu_2) \quad \text{for} \quad \nu_1 < \nu_2. \quad (2.2)$$

Secondly, for processes in which there is a global common memory or in which there is some control of the consistency of the data, the update of any particular component would be required to use the data at a given state. That is, the function  $s(\cdot, \cdot)$  would be independent of  $m$ . In this case the update of component  $u(\nu)$  depends only on the vector  $\bar{x}^{\nu - s(\nu)}$ . This condition can be stated as:

**CONDITION 3.** The shift function is independent of  $m$ .

Here we are interested in considering a modification of the processes allowed by Chazan and Miranker by considering Condition 3 along with Conditions 1 and 2. In the notation of Bertsekas and Tsitsiklis [2] this condition is that  $\tau_j^i(t)$  is independent of  $j$ .

The extra constraint in Condition 3 is reasonable for computations with a common memory or for processes with distributed memory but with some control of the processes. For example, if data are transmitted between processors via a master processor, then the arrival of new data at this master processor corresponds to a new state of the system.

Our main result is the following theorem.

**THEOREM 2.2.** *A necessary and sufficient condition for the chaotic iteration to converge for all update functions and shift functions satisfying Conditions 1, 2, and 3 is that  $\rho(|B|) < 1$ .*

The proof of this result does not follow from Chazan and Miranker's proof, because the construction of the nonconverging sequence in [5] depends heavily on the function  $s(\cdot, \cdot)$ , which depends on the component of the vector. In fact, the construction of the nonconverging sequence for the restricted case of Theorem 2.2 is more involved than in the general case of Theorem 2.1.

### 3. PROOF OF THEOREM 2.2

The proof that the condition  $\rho(|B|) < 1$  implies that the restricted processes will converge follows from the theorem of Chazan and Miranker. However, we include it here for completeness. The proof of the following lemma is due to Baudet [1].

**LEMMA 3.1.** *The condition  $\rho(|B|) < 1$  implies that there is a value  $\alpha$  with  $0 < \alpha < 1$  and a vector  $\vec{p}$  with positive components such that  $|B|\vec{p} \leq \alpha\vec{p}$ .*

*Proof.* Suppose first that  $|b_{l,m}| > 0$  for all components of  $|B|$ . Then by Perron's theorem [8, Vol. 2, p. 53] on positive matrices, the largest eigenvalue of  $|B|$  is simple and the eigenvector has positive values. If  $\vec{p}$  is this eigenvector and  $\alpha$  is the eigenvalue, then  $|B|\vec{p} = \alpha\vec{p}$ , and the theorem is proved in this special case.

If  $b_{l,m} = 0$  for some entries of  $B$ , consider the matrix  $B_\varepsilon$  in which all zero entries in  $B$  are replaced by a small positive value  $\varepsilon$ . Since the spectral radius of a matrix is a continuous function of the matrix entries, we can choose  $\varepsilon$  small enough so that  $\rho(|B_\varepsilon|)$  is less than 1. As in the special case above, there is an  $\alpha$  and  $\vec{p}$  satisfying  $|B_\varepsilon|\vec{p} = \alpha\vec{p}$ . However, we also have  $|B|\vec{p} \leq |B_\varepsilon|\vec{p} = \alpha\vec{p}$ . This proves the lemma. ■

*Proof of Theorem 2.2.* We first prove that if  $\rho(|B|) < 1$ , then the process is convergent. We consider the difference between  $x^*$ , the solution of the system (2.1), and the iterates. Let  $\vec{y}^n = \vec{x}^* - \vec{x}^n$ . Using the vector  $\vec{p}$  and value  $\alpha$  discussed in Lemma 3.1, we consider the first  $\bar{s}$  vectors in the process. Since all components of  $\vec{p}$  are positive, there is a positive number  $M$

such that  $|\vec{y}^\nu| \leq M\vec{p}$  for  $0 \leq \nu < \bar{s}$ . Consider any component updated using any of these  $\bar{s}$  vectors, the update satisfies the estimate

$$|y_l^{\nu+1}| \leq \sum_{m=1}^N |b_{l,m}| |y_m^{\nu-s(\nu)}| \leq \alpha M v_l.$$

If  $\nu_1$  is the first instance after  $\bar{s}$  for which all the components have been updated, then  $|\vec{y}^{\nu_1}| \leq \alpha M\vec{p}$ . Moreover,  $|\vec{y}^\nu| \leq \alpha M\vec{p}$  for all  $\nu$  greater than  $\nu_1$ . Similarly, if  $\nu_2$  is the next instance after  $\nu_1$  for which all components have again been updated, then  $|\vec{y}^\nu| \leq \alpha^2 M\vec{p}$  for all  $\nu$  greater than  $\nu_2$ . In this way we see that  $\vec{y}^\nu$  converges to 0.

We now consider the proof that if  $\rho(|B|) \geq 1$ , then there is an update function  $u(\cdot)$  and a shift function  $s(\cdot)$  such that the process does not converge. We may assume that  $\rho(B) < 1$ , since if this is not the case, the Jacobi method does not converge.

Our construction relies on the following simple lemma.

**LEMMA 3.2.** *Let  $\mathcal{B}$  be a matrix with  $\rho(\mathcal{B}) \geq 1$ , and let  $Z$  be a nonzero vector such that  $\mathcal{B}^T Z = \lambda Z$  with  $|\lambda| \geq 1$ . If  $X$  is a vector such that  $X^T Z \neq 0$ , then  $\|\mathcal{B}^n X\|$  is bounded away from 0.*

The proof follows immediately from the relations

$$0 < |\lambda^n Z^T X| = \left| \left[ (\mathcal{B}^T)^n Z \right]^T X \right| = |Z^T (\mathcal{B}^n X)| \leq \|Z\| \|\mathcal{B}^n X\|.$$

The construction of a process that does not converge involves the construction of an auxiliary process that starts from a large set of initial vectors and can be shown to be nonconvergent. Then we show that there exist nonconvergent processes that start with a single vector. The auxiliary process starts with a set of  $P + 1$  vectors  $w^\nu$  for  $0 \leq \nu \leq P$ . We construct a process that diverges, starting with these vectors.

Let  $\vec{v}$  be an eigenvector of  $|B|$  with eigenvalue  $\lambda$ , i.e.,  $|B|\vec{v} = \lambda\vec{v}$ , and let  $\vec{v} \geq 0$  and  $\lambda \geq 1$ .

We begin with a sequence of vectors  $\vec{w}^\nu$  for  $0 \leq \nu \leq P$  for some integer  $P$  satisfying the following three criteria:

1.  $|\vec{w}^\nu| = \vec{v}$ .
2. Between successive vectors  $\vec{w}^i$  and  $\vec{w}^{i+1}$  there is precisely one component that changes sign.

3. For each row of  $B$ , there is a  $\vec{w}^\nu$  with the same pattern of signs and a  $\vec{w}^\mu$  with the opposite pattern of signs. That is, for each  $l$  there is some  $\nu$  and some  $\mu$  such that

$$\sum_{m=1}^N b_{l,m} w_m^\nu = \sum_{m=1}^N |b_{l,m}| v_m = \lambda v_l$$

and

$$\sum_{m=1}^N b_{l,m} w_m^\mu = - \sum_{m=1}^N |b_{l,m}| v_m = -\lambda v_l.$$

Notice that  $P$  must be at least  $2N$  and can be taken to be less than  $N^2$ . We conjecture that  $P$  can always be chosen proportional to  $N$ .

For each  $m$  with  $0 \leq m \leq N$ , let  $\sigma_m$  be the sign of  $\vec{w}_m^0$ . We first construct vectors  $\vec{w}^{\nu+1}$  for  $P \leq \nu < P+N$ . For now we define the update function  $u(\nu)$  and the shift function  $s(\nu)$  only for  $\nu$  greater than and equal to  $P$ . Let  $l = 1 + (\nu - P) \bmod N$ , and then determine  $\mu$  with  $0 \leq \mu \leq P$  such that  $\vec{w}^\mu$  has the sign sequence of the  $l$ th row of  $B$  if  $\sigma_l = 1$  or where  $\vec{w}^\mu$  has the opposite sign sequence of the  $l$ th row of  $B$  if  $\sigma_l = -1$ . Then for  $P \leq \nu < P+N$  set

$$u(\nu) = l \quad \text{and} \quad s(\nu) = \nu - \mu.$$

With this choice of  $u(\cdot)$  and  $s(\cdot)$  we have for  $l = u(\nu)$

$$\sum_{m=1}^N b_{l,m} w_m^{\nu-s(\nu)} = \sum_{m=1}^N b_{l,m} w_m^\mu = \sigma_l \sum_{m=1}^N |b_{l,m}| v_m = \sigma_l \lambda v_l = \lambda w_l^0.$$

In this way, the vector  $\vec{w}^{P+N}$  is  $\lambda \vec{w}^0$ .

For  $\nu$  from  $P+N$  up to  $2P+N$ , the update and shift functions are defined as follows. For  $\nu$  in this range, define  $\mu$  as  $\nu - P - N$ . The vectors  $\vec{w}^\mu$  and  $\vec{w}^{\mu+1}$  differ only in the sign of one component. Let this component be  $l$ , we define  $u(\nu) = l$ . We define  $s(\nu)$  in this way. If the sign of  $\vec{w}_l^{\mu+1}$  is positive, let  $\tau$  with  $0 \leq \tau \leq P$  be the index of the vector  $\vec{w}^\tau$  that has the same sign sequence as the  $l$ th row of  $B$ . If the sign of  $\vec{w}_l^{\mu+1}$  is negative, let  $\tau$  with  $0 \leq \tau \leq P$  be the index of the vector  $\vec{w}^\tau$  that has the opposite sign sequence to the  $l$ th row of  $B$ . Then  $s(\nu) = \nu - \tau$ .

With this choice of  $u(\nu)$  and  $s(\nu)$ , the vector  $\vec{w}^{\nu+1}$  has the same sign sequence as  $\vec{w}^{\nu+1-P-N}$  and is, in fact,  $\lambda \vec{w}^{\nu+1-P-N}$ . In this way the vectors

$\vec{w}^\nu$  for  $P + N \leq \nu \leq 2P + N$  are equal to  $\lambda$  times the sequence  $\vec{w}^\nu$  for  $0 \leq \nu \leq P$ .

For  $\nu$  greater than or equal to  $2P + N$ , the functions  $u(\cdot)$  and  $s(\cdot)$  are defined by periodicity with period  $P + N$ .

We now construct the matrix  $\mathcal{B}$  and vector  $Y$ . The vector  $Y$  is formed by using the direct sum of the vectors  $\vec{w}^\nu$  for  $\nu = 0, \dots, P$ . [The dimension of  $Y$  is  $N(P + 1)$ .] The rows and columns of  $\mathcal{B}$  are indexed by the ordered pairs  $(\mu, m)$  where  $0 \leq \mu \leq P$  and  $1 \leq m \leq N$ . The first  $N$  rows of  $\mathcal{B}$  are defined using the steps to construct  $w^\nu$  for  $P \leq \nu < P + N$ :

$$\mathcal{B}_{(0, l) \times (\beta, m)} = \begin{cases} b_{l, m} & \text{if } \beta = P + l - 1 - s(P + l - 1), \\ 0 & \text{otherwise.} \end{cases}$$

The elements  $\mathcal{B}_{(\alpha, l) \times (\beta, m)}$  for  $\alpha > 0$  are constructed using the update and shift functions for  $P + N \leq \nu < 2P + N$ :

$$\mathcal{B}_{(\alpha, l) \times (\beta, m)} = \begin{cases} b_{l, m} & \text{if } \beta = \alpha + P + N - 1 - s(\alpha + P + N - 1) \\ & \text{and } l = u(\alpha + P + N - 1), \\ \mathcal{B}_{(\alpha-1, l) \times (\beta, m)} & \text{otherwise.} \end{cases}$$

The vector  $\mathcal{B}Y$  is the vector formed from the  $\vec{w}^\nu$  for  $P + N \leq \nu \leq 2P + N$ . In general, the vector  $\mathcal{B}^m Y$  is the vector formed from the  $\vec{w}^\nu$  for  $m(P + N) \leq \nu \leq (m + 1)P + mN$ . By the construction of  $\mathcal{B}$ , we have that  $\mathcal{B}Y = \lambda Y$ .

We now construct the sequence starting from an initial vector  $\vec{x}^0$ . Let  $Z$  be a vector such that  $Z^T Y = 1$  and  $\mathcal{B}^T Z = \lambda Z$ . From  $\vec{x}^0$  we will construct a vector  $X$  such that  $Z^T X \neq 0$ , and by Lemma 3.2, this will imply that the process starting with  $\vec{x}^0$  is nonconvergent.

Partition the components of  $Z$  in the same way as those of  $Y$ . We index the components as  $z_i^\nu$ . Some of the components of  $Z$  must be nonzero, choose a component  $i^*$  such that for some  $\nu$  the product  $v_{i^*} z_{i^*}^\nu$  is nonzero. Define

$$\mu^* = \max_{0 \leq \nu \leq P} \{ \nu : v_{i^*} z_{i^*}^\nu \neq 0 \}.$$

Let  $j^*$  be any index other than  $i^*$ . By the construction of  $\mathcal{B}$  and because  $Z$  is a left eigenvector,  $\mu^*$  is not zero.

Since we assume that  $\rho(B) < 1$ , the matrix  $I - B$  is nonsingular, and we define  $\vec{x}^0$  by  $(I - B)\vec{x}^0 = -\vec{v}$ . Note that  $B\vec{x}^0 = \vec{x}^0 + \vec{v}$ . Define  $u(\nu) = j^*$  for  $0 \leq \nu < P$ , and define  $s(\nu) = \nu$  for these same values. Thus the vectors



$\bar{x}^\nu$  are equal to  $\bar{x}^0$  except in the  $j^*$ th component, where the value is  $x_{j^*}^0 + v_{j^*}$ . The vector  $X$  of dimension  $N(P + 1)$  is formed from the components of the  $\bar{x}^\nu$  in order.

To use Lemma 3.2 we need to consider the dot product of  $X$  and  $Z$ . This is the sum of the dot products of  $\bar{x}^\nu$  with  $\bar{z}^\nu$ . That is,

$$X^T Z = \bar{x}^0 \cdot \sum_{\nu=0}^P \bar{z}^\nu + v_{j^*} \sum_{\nu=1}^P z_{j^*}^\nu.$$

If  $X^T Z$  is nonzero, then by Lemma 3.2, the sequence  $\|\mathcal{B}^n X\|$  is nonconvergent. This implies that the chaotic process is nonconvergent.

If  $X^T Z$  is zero, we modify the above process to have  $u(\mu^* - 1) = i^*$  instead of  $u(\mu^* - 1) = j^*$ . In this case the dot product is

$$X^T Z = \bar{x}^0 \cdot \sum_{\nu=0}^P \bar{z}^\nu + v_{j^*} \sum_{\nu=1}^P z_{j^*}^\nu + v_{i^*} z_{i^*}^{\mu^*} = v_{i^*} z_{i^*}^{\mu^*} \neq 0,$$

and therefore, as above, this process is nonconvergent. This proves Theorem 2.2. ■

#### 4. CONCLUSIONS AND DISCUSSION

We have given a necessary and sufficient condition for the convergence of chaotic relaxation under the additional constraint that there must be a definite state to the data set at each step of the chaotic iteration.

A interesting topic for further study would be the analysis of systems that are less chaotic than the general system of Chazan and Miranker [5], but allow for behavior that is not deterministic. The situation in which there is some constraint on older data not being available once newer data are available [see (2.2)] appears to be much more difficult than the case studied here. We have not been able to use the methods of this paper to approach this question.

From the proof of Theorem 2.2, it appears that the addition of Condition 3 requires that the nonconvergent sequence have greater value of  $\bar{s}$  than did the construction of Chazan and Miranker. It would be of interest to see how the results of Theorem 2.2 would be modified under restrictions on  $\bar{s}$ .

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