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ITERATIVE METHODS FOR THE SOLUTION OF ELLIPTIC PROBLEMS ON REGIONS PARTITIONED INTO SUBSTRUCTURES*

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Abstract. Finite element problems can often naturally be divided into subproblems which correspond to subregions into which the region has been partitioned or from which it was originally assembled. A class of iterative methods is discussed in which these subproblems are solved by direct methods, while the interaction across the curves or surfaces which divide the region is handled by a conjugate gradient method. A mathematical framework for this work is provided by regularity theory for elliptic finite element problems and by block Gaussian elimination. A full development of the theory, which shows that certain of these methods are optimal, is given for Lagrangian finite element approximations of second order linear elliptic problems in the plane. Results from numerical experiments are also reported.

Key words. iterative methods, elliptic problems, substructures, finite elements

AMS(MOS) subject classifications. 65N20, 65F10, 65N30

1. Introduction. In this paper, which is an expanded version of Bjørstad and Widlund [4], we develop and analyze some iterative methods for the solution of elliptic problems on a region regarded as a union of subregions. We are, in particular, interested in the design of methods for which the interaction between the subproblems, i.e. the discrete elliptic problems on the subregions, is computed by a conjugate gradient method, but the subproblems are solved using a direct method.

The partition of elliptic problems into subproblems is a very natural idea and is very much in use in practice. In finite element computations, the modeling of an entire structure can profitably be organized by discretizing the given partial differential equations by finite elements on subregions into which region is partitioned or from which it has originally been assembled. The Cholesky triangular factorization of the stiffness matrices of the subregions is then computed. These tasks can obviously be assigned to different engineering groups, computer systems or processors with coordination required only at the interfaces between the substructures to assure a matching of the finite element triangularizations. The solution of the entire problem is completed by taking the interaction between the substructures into account, see Przemieniecki [28], [29]. These ideas are also often used recursively and they are particularly attractive if several substructures are identical or if some of them have been analyzed previously. Such a situation arises for example if an analysis is repeated after the redesign or damage of one or relatively few of the substructures, see Bell et al. [1] and Hatlestad and Mellingen [18].

It is normal practice to conclude the solution process by computing the matrix which describes the interaction between the substructures and its triangular factorization and by using the triangular factors to solve the linear system. There are many

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interesting sparse matrix and software issues involved which are being studied actively, see e.g. Duff and Reid [13]. We also note that the important work by George [14] and others on nested dissection of finite element and related systems of linear algebraic equations can be regarded as a recursive application of similar ideas. The results of George have inspired very interesting work on Gaussian elimination and graph algorithms, see Lipton, Rose and Tarjan [21], Gilbert [15] and Rose and Whitten [30]. In our study, we use the language of block Gaussian elimination extensively, but we focus exclusively on iterative methods which do not require that the last stage of the Gaussian factorization process be carried out. In each iteration it is required that all the subproblems be solved with somewhat special data. Much of this work can be carried out in parallel. See further discussion in § 3.

In the second section, we consider the basic structure of the linear systems of equations arising in a finite element discretization. To simplify our notation, we restrict our discussion, throughout this paper, to the case when a region is partitioned into two substructures. The extension of our methods and results to cases with a number of cuts, which do not intersect in the interior of the region, is immediate; see Fig. 1a and Fig. 1b, respectively.

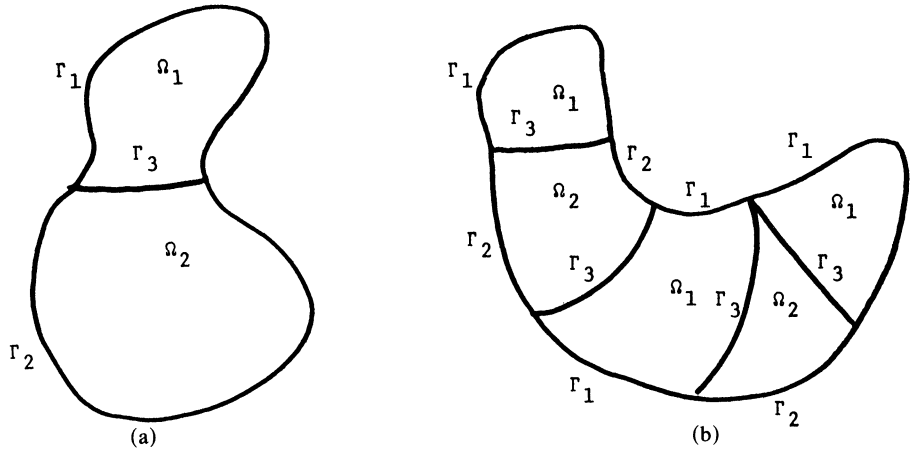

FIG. 1. (a) *A region partitioned into two substructures.* (b) *Another case for which our methods and results apply.*

In the third section, we discuss preconditioned conjugate gradient methods in general and introduce three iterative methods. We show informally that two of these can be expected to be optimal, i.e. that the number of iterations required for a given tolerance will remain constant when the finite element model is refined.

In the fourth section, a detailed analysis, including specific bounds on the spectrum of the iteration operators, is given for a model problem, namely Poisson's equation on L- and T-shaped regions. In the fifth section, this analysis is extended, using quite different mathematical tools, to general plane regions and arbitrary conforming Lagrangian finite element approximations of second order linear elliptic problems.

Earlier work on algorithms of this general nature is reported in Concus, Golub and O'Leary [7], who gave an algorithm and numerical results for Poisson's equation on T-shaped regions. Two additional preconditioners were recently introduced in Golub and Meyers [16]. Methods inspired by the Schwarz alternating principle and by control theory are considered in Dihn, Glowinski, and Périaux [9]. Some of these methods are described and analyzed in § 6. We also note that we have learned much from the papers by Dryja [11], [12] and from his October 1981 visit with the second

author. At that time, we were first introduced to the method which has proven most successful in our numerical experiments.

The authors are currently actively extending their work to more general problems. We have begun a series of experiments with actual engineering structures and have also made progress towards the extension of the theory to a larger class of elliptic problems, to cases in dimensions higher than two and to non-Lagrangian finite elements. The use of preconditioners rather than exact solvers for the subproblems has recently been considered by Bramble, Pasciak and Schatz [5]. Their work has been reinterpreted in block Gaussian elimination terms in Widlund [35], where a relationship between their algorithms and those of this paper is also established.

We note that both from an algorithmic and analytic point of view, our methods have much in common with capacitance matrix methods, see Bjørstad [2], [3], Dryja [10], O'Leary and Widlund [24], [25], Proskurowski and Widlund [26], [27] and the references given in those papers. See also Widlund [34], in which related algorithms for mixed finite element methods for the biharmonic and Stokes' problems are also discussed.

2. Substructured finite element problems and block forms of the stiffness matrices. To simplify the discussion, we confine ourselves to problems defined on a connected region, Ω, which is a union of two connected regions Ω_1, Ω_2 and Γ_3 and to the Dirichlet case; see Fig. 1a. Here Ω_1 and Ω_2 are plane, bounded, nonintersecting regions and $\bar{\Gamma}_3$ the intersection of their closures. The boundaries of Ω_1 and Ω_2 are $\bar{\Gamma}_1 \cup \bar{\Gamma}_3$ and $\bar{\Gamma}_2 \cup \bar{\Gamma}_3$, respectively, and the boundary of Ω is $\bar{\Gamma}_1 \cup \bar{\Gamma}_2$. We assume that these subregions are curvilinear polygons, i.e. in particular they are Lipschitz regions; see § 5 or Grisvard [17].

A linear, second order, positive definite, selfadjoint elliptic operator is defined on Ω. Its symmetric bilinear form is denoted by $a_\Omega(u, v)$. A simple example is given by the Laplace operator with a homogeneous Dirichlet condition for which

$$a_\Omega(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad u, v \in H_0^1(\Omega).$$

Here $H_0^1(\Omega)$ is the subspace of elements with zero boundary values of the Sobolev space $H^1(\Omega)$ of square integrable functions with square integrable first derivatives. Triangulations of Ω_1 and Ω_2 are introduced in such a way that the nodes on Γ_3 coincide and Γ_3 follows element boundaries. We assume that each degree of freedom of the finite element subspaces is associated with a node and a basis function in the finite element space and that the support of any basis function coincides with the triangles to which its node belongs. An element of the stiffness matrix has the form $a_\Omega(\phi_i, \phi_j)$, where ϕ_i and ϕ_j are basis functions, and it therefore vanishes unless the two nodes belong to a common triangle.

In the case of a general curved boundary, the region Ω will normally be approximated by a union of straight or isoparametric elements. We will discuss neither the modifications of the bilinear form which are required in such a case nor the related issues of numerical quadrature and nonconforming finite elements; see e.g. Ciarlet [6]. For our purposes uniform V_h-ellipticity of the bilinear form or the modified bilinear form will be sufficient. We will similarly assume that the triangulation of the region is regular enough.

It is easy to see from the definition of the bilinear form that

$$a_\Omega(u, v) = a_{\Omega_1}(u, v) + a_{\Omega_2}(u, v) \tag{2.1}$$

and that therefore the stiffness matrix of a problem on Ω can be constructed from those of Ω_1 and Ω_2. The same relation holds for any pair of nonoverlapping subregions. This fact is frequently used in practice to construct stiffness matrices from the stiffness matrices of smaller substructures. A special case is the common process in which the stiffness matrix is assembled from the contributions from the individual elements.

The stiffness matrix of the entire problem is of the form,

$$K = \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & K_{33} \end{pmatrix}$$

where K_{11} represents couplings between the pairs of nodes in Ω_1, K_{13} couplings between the pairs belonging to Ω_1 and Γ_3 respectively and K_{33} couplings between nodes on Γ_3, etc. In a case with more than two substructures, cf. Fig. 1b, the diagonal blocks K_{11} and K_{22} can be represented as direct sums of stiffness matrices corresponding to the different substructures of Ω_1 and Ω_2. By (2.1) this matrix can be assembled from the stiffness matrices corresponding to Ω_1 and Ω_2 respectively,

$$K^{(1)} = \begin{pmatrix} K_{11} & K_{13} \\ K_{13}^T & K_{33}^{(1)} \end{pmatrix} \quad \text{and} \quad K^{(2)} = \begin{pmatrix} K_{22} & K_{23} \\ K_{23}^T & K_{33}^{(2)} \end{pmatrix}, \tag{2.2}$$

where $K_{33}^{(1)}$ and $K_{33}^{(2)}$ contain only the contributions from the integrals over Ω_1 and Ω_2, respectively. By (2.1), we see that

$$K_{33} = K_{33}^{(1)} + K_{33}^{(2)}. \tag{2.3}$$

We note that the degrees of freedom have been partitioned into three sets of which the third, the separator set, corresponds to the nodes of $\bar{\Gamma}_3$. From the point of view of graph theory, the undirected graph of K becomes disconnected into two components if the nodes of the separator set and their incident edges are removed. Since conforming finite elements are used, i.e. the finite element space $V_h \subset H^1$, it also follows from the assumptions on $a_\Omega(u, v)$ that K is positive definite, symmetric and as a consequence so are K_{11}, K_{22} and K_{33}.

The matrix $K^{(1)}$, defined in (2.2), is the matrix of the elliptic problem on Ω_1 with a natural boundary condition on Γ_1 and a Dirichlet condition on Γ_1, i.e., the problem

$$\begin{aligned} a_{\Omega_1}(u_h, v_h) &= \int_{\Omega_1} f v_h \, dx + \int_{\Gamma_3} g_N v_h \, ds \quad \forall v_h \in V_h \cap H_0^1(\Omega_1, \Gamma_1), \\ u_h &\in V_h, \quad \gamma_1 u_h = g_D. \end{aligned} \tag{2.4}$$

Here $\gamma_1 u_h$ is the trace on Γ_1, i.e. the restriction of u_h to Γ_1, and $H_0^1(\Omega_1, \Gamma_1)$ the subspace of $H^1(\Omega_1)$ with vanishing trace on Γ_1. In this case x_3 is a vector of unknowns and the linear system is

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{13}^T & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \end{pmatrix}. \tag{2.5}$$

We note that the vector b_1 vanishes if f and g_D are zero in which case the vector b_3 represents the Neumann data on Γ_3.

Written in variational form, a Dirichlet problem on Ω_1 has the form

$$a_{\Omega_1}(u_h, v_h) = \int_{\Omega_1} f v_h \, dx \quad \forall v_h \in V_h \cap H_0^1(\Omega_1), \quad u_h \in V_h, \quad \gamma u_h = g_D, \tag{2.6}$$

where γu_h is the trace on $\bar{\Gamma}_1 \cup \bar{\Gamma}_3$. In matrix form, this problem can be written as

$$\begin{pmatrix} K_{11} & K_{13} \\ 0 & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \end{pmatrix}. \tag{2.7}$$

We consider the linear system of algebraic equations of the form

$$Kx = \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & K_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

By using block-Gaussian elimination, we can reduce this system to the positive definite, symmetric system,

$$\begin{aligned} Sx_3 &= (K_{33} - K_{13}^T K_{11}^{-1} K_{13} - K_{23}^T K_{22}^{-1} K_{23}) x_3 \\ &= b_3 - K_{13}^T K_{11}^{-1} b_1 - K_{23}^T K_{22}^{-1} b_2 = \tilde{b}_3. \end{aligned} \tag{2.8}$$

It is common practice to complete the process by solving (2.8) by a direct method.

The right-hand side $\tilde{b}_3$ can be obtained at the expense of solving the two subproblems on Ω_1 and Ω_2, multiplying the resulting vectors by the sparse matrices K_{13}^T and K_{23}^T, respectively, and subtracting the resulting vectors from b_3. From now on, we will consider only the case when b_1 and b_2 are zero. Such a reduction can of course easily be accomplished.

The matrix S, which is a so-called Schur complement, see Cottle [8], can be expensive to compute and store. However, we notice that Sy can be computed for a given vector y at the expense of solving the two subproblems with the sparse right-hand sides $K_{13}y$ and $K_{23}y$, respectively, and certain sparse matrix and vector operations. In the next section, we will develop iterative methods which only require S in terms of such matrix-vector products.

The cost of computing Sy depends primarily on the efficiency of the solvers for the subproblems. It should also be noted that if a Gaussian elimination method is used, advantage can be taken of the sparsity of the vectors $K_{13}y$ and $K_{23}y$. Thus when the lower triangular systems of equations are solved, the computation can begin with the first equation which has a nonzero right-hand side. Similarly, the solution of the upper triangular systems can be stopped as soon as all the components of $K_{11}^{-1}K_{13}y$ and $K_{22}^{-1}K_{23}y$, necessary for computing $K_{13}^T(K_{11}^{-1}K_{13}y)$ and $K_{23}^T(K_{22}^{-1}K_{23}y)$, have been found. This can effectively reduce the size of the triangular systems necessary to carry out the iteration steps. It is thus particularly advantageous if all the variables at nodes adjacent to Γ_3 are ordered late. It should be noted, however, that such a constraint may be hard to impose on existing software or that it may lead to an increase in the time and space required to factor K_{11} and K_{22} into their triangular factors.

We will also need the Schur complements with respect to the matrices $K^{(1)}$ and $K^{(2)}$ defined in (2.2). They are,

$$S^{(1)} = K_{33}^{(1)} - K_{13}^T K_{11}^{-1} K_{13} \quad \text{and} \quad S^{(2)} = K_{33}^{(2)} - K_{23}^T K_{22}^{-1} K_{23}. \tag{2.9}$$

Using (2.3), (2.8) and (2.9), we find that

$$S = S^{(1)} + S^{(2)}. \tag{2.10}$$

The mappings S and $S^{(1)}$ play an important role in what follows. The vector $S^{(1)}y$ can be computed by solving the Dirichlet problem (2.7) and then applying the matrix of (2.5), which corresponds to a Neumann case, to the solution vector. This can be seen by a straightforward computation:

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{13}^T & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} K_{11} & K_{13} \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ S^{(1)}y \end{pmatrix}. \tag{2.11}$$

This map thus takes the discrete Dirichlet data y on Γ_3 into the discrete Neumann data $S^{(1)}y$ on the same set. For a discussion of the continuous case, see the next section.

In § 4, we return to a discussion of the structure of this problem in the special case when Ω_1 and Ω_2 are rectangles and Γ_3 is an interval. We have carried out numerical experiments for this particular geometry using uniform meshes of right triangles and piecewise linear continuous finite element functions.

3. Conjugate gradient algorithms for substructured problems and an informal theory. In this section, we will introduce three iterative methods of conjugate gradient type. The general theory of such methods is quite well known and it will therefore be discussed only briefly; see e.g. Concus, Golub and O'Leary [7], Hestenes [19] or Luenberger [22].

Let $Ax = b$ be a linear system of algebraic equations with a positive definite, symmetric matrix A. Let $x^{(0)}$ be an initial guess and $r^{(0)} = b - Ax^{(0)}$ be the initial residual. The kth iterate in the standard conjugate gradient method, $x^{(k)}$, can then be characterized as the minimizing element for the variational problem

$$\min (1/2)y^T Ay - y^T b,$$

where $y - x^{(0)}$ varies in the linear space spanned by $r^{(0)}, Ar^{(0)}, \cdots, A^{k-1}r^{(0)}$. By expanding in eigenvectors of A, it can be established that

$$(x^{(k)} - x)^T A(x^{(k)} - x)/(x^{(0)} - x)^T A(x^{(0)} - x)$$

is bounded from above by

$$\min_{p \in P_{k-1}} \max_{\lambda \in \sigma(A)} (1 - \lambda p(\lambda))^2, \tag{3.1}$$

see Luenberger [22]. Here x is the exact solution, P_{k-1} the space of all polynomials of degree $k-1$ and $\sigma(A)$ the spectrum of A. This bound can be used to establish that the convergence is rapid if A is well conditioned and that the rate of convergence can be bounded uniformly for entire families of operators if all the eigenvalues fall in a fixed interval.

Preconditioned conjugate gradient methods have been studied extensively in recent years. The idea goes back to the mid-1950s; see Hestenes [19]. Let A_0 be another positive definite, symmetric operator for which it is feasible to solve auxiliary systems of the form $A_0 y = c$ repeatedly for different right-hand sides. In one of the versions of the method, the original problem $Ax = b$ is transformed into $AA_0^{-1} = b$. The iterate $y^{(k)}$ is sought as the sum of an initial guess $y^{(0)}$ and a linear combination of $r^{(0)}$, $AA_0^{-1}r^{(0)}, \cdots, (AA_0^{-1})^{k-1}r^{(0)}$. If an appropriate inner product is used, a convenient recursion formula results, see e.g. Proskurowski and Widlund [27]. Each step of this algorithm requires the solution of an auxiliary linear system. It is important to note that the estimate (3.1) still holds, but that now the eigenvalues of AA_0^{-1}, i.e. those of the symmetric generalized eigenvalue problem $A\phi = \lambda A_0 \phi$, are of relevance rather than those of A. It is also worth noting that the estimate (3.1) can be used to show particularly rapid convergence if the eigenvalues are clustered.

In problems such as those considered in this paper, it is convenient to use a version of the algorithm in which the operator A only appears in a subroutine which provides the product of $A - A_0$ times a vector.

For the problem at hand, we first consider the solution of equation (2.8) without preconditioning. From (2.10), we see that $Sy = S^{(1)}y + S^{(2)}y$. It is therefore at least plausible that S will be ill conditioned if $S^{(1)}$ and $S^{(2)}$ are. As shown in § 2, $S^{(1)}$ represents a Dirichlet-Neumann map and therefore involves a loss of a derivative in $L_2(\Gamma_3)$. Such a map will have a spectral condition number proportional to the number of nodes on Γ_3. In order to clarify this point, we consider the continuous case, leaving the details on the finite element case to §§ 4 and 5.

Thus consider two harmonic functions u_1 and u_2 defined on Ω_1 and Ω_2 respectively. These functions vanish on Γ_1 and Γ_2 and have the same trace on Γ_3. They can therefore be combined to form $u \in H_0^1(\Omega)$. This function satisfies

$$a_\Omega(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx = f(v) \quad \forall v \in H_0^1(\Omega), \tag{3.2}$$

where the linear functional f has its support on Γ_3. It is easy to show that

$$\begin{aligned} a_\Omega(u, v) &= \int_{\Omega_1} \nabla u_1 \cdot \nabla v \, dx + \int_{\Omega_2} \nabla u_2 \cdot \nabla v \, dx \\ &= \int_{\Gamma_3} \frac{\partial u_1}{\partial \nu} v \, ds - \int_{\Gamma_3} \frac{\partial u_2}{\partial \nu} v \, ds = \int_{\Gamma_3} \left[\frac{\partial u}{\partial \nu}\right] v \, ds, \end{aligned}$$

where ν is the normal outward with respect to Ω_1. We can therefore rewrite (3.2) as

$$\left[\frac{\partial u}{\partial \nu}\right] = f \quad \text{on } \Gamma_3,$$

where $[\partial u/\partial \nu]$ corresponds to Sy. Similarly $\partial u_1/\partial \nu$ and $-\partial u_2/\partial \nu$ correspond to $S^{(1)}y$ and $S^{(2)}y$, respectively. Following Lions and Magenes [20] and Grisvard [17], we see that for $u \in H_0^1(\Omega)$, $\gamma_3 u \in H_{00}^{1/2}(\Gamma_3)$, where $H_{00}^{1/2}(\Gamma_3)$ is a subspace of $H^{1/2}(\Gamma_3)$; see § 5.

It then follows, from a standard variational argument, that $\partial u_1/\partial \nu$, $\partial u_2/\partial \nu$ and $[\partial u/\partial \nu]$ belong to the dual space of $H_{00}^{1/2}(\Gamma_3)$; i.e. a derivative is lost in comparison with $\gamma_3 u$.

In view of what we have just learned, it is natural to try to find a preconditioner which also involves the loss of a derivative in $L_2(\Gamma_3)$. A natural choice would be a tangential derivative but that is not a symmetric operator. Instead we can use the square root of the negative of a discretization of the Laplacian on Γ_3. Such a method is practical at least for problems in the plane and has been tested; see §§ 4 and 7 for details. We denote this operator by J.

In our experience, an even better method involves the solution of a system,

$$\begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & 0 & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix}.$$

This solution can be obtained by solving (2.5) with the right-hand side of $(0, y)^T$ and then the discrete Dirichlet problem on Ω_2, using x_3 as data, cf. (2.7). The relevant

mapping is now $SS^{(1)^{-1}}$ since

$$\begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & K_{33} \end{pmatrix} \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & 0 & K_{33}^{(1)} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ SS^{(1)^{-1}} y \end{pmatrix}.$$

For an analysis of the spectrum of $SS^{(1)^{-1}}$, see §§ 4 and 5.

We note that we need not construct any auxiliary operator when this preconditioner is used. It is also interesting to note that if a symmetric region is cut in half, and treated fully symmetrically, then $S = 2S^{(1)}$ and the conjugate gradient iteration converges in one step.

It is well known that it is difficult to make accurate predictions of the relative performance of different methods on the parallel computers of the future, but some remarks can nevertheless be offered. In each iteration step, the first method requires less synchronization than the others. Thus in the second method there is an additional, separate preconditioning step and in the third the solution of the Neumann problem on Ω_1 must be completed before the solution of the Dirichlet problem on Ω_2 can begin. On sequential computers the cost per iteration for the first and third methods is typically quite close. As is demonstrated in this paper there is of course an appreciable difference in the rate of convergence. For all the methods considered, parallelism can be exploited when factoring the different diagonal blocks of the stiffness matrix which are associated with the different substructures and when the corresponding linear systems are solved.

4. Analysis of a model problem. In this section, we consider a model problem where Ω_1 and Ω_2 are rectangles and Ω is L- or T-shaped (see Fig. 2). The domain is triangulated using right triangles of equal size. The number of interior nodes in the horizontal and vertical directions are q and r, respectively, for Ω_1 and m and n for Ω_2. There are p internal nodes in the horizontal direction before a node in Ω_2 aligns with the first vertical column of nodes in Ω_1. We assume that $p+q \leqq m$.

With piecewise linear, continuous elements, equal mesh spacing in the coordinate directions and Poisson's equation, we obtain

$$K_{11} = (I_r \otimes R_q) + (R_r \otimes I_q), \qquad K_{22} = (I_n \otimes R_m) + (R_n \otimes I_m),$$

where the $q \times q$ matrix $R_q = \text{tridiag}\,(-1, 2, -1)$. Furthermore

$$K_{13} = (0, -I_q)^T, \qquad K_{23} = (0, -I_q, 0)^T$$

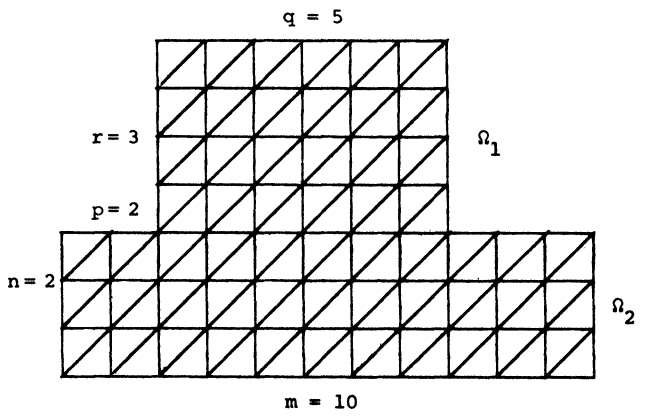

FIG. 2. *Model problem.*

and

$$K_{33}^{(1)} = K_{33}^{(2)} = I_q + 1/2 R_q.$$

We will also use the notation $J_q = R_q^{T/2}$. This is a positive definite symmetric matrix.

The orthogonal matrix Q_q defined by

$$(Q_q)_{ij} = (2/(q+1))^{1/2} \sin(ij\pi/(q+1))$$

provides the eigenvectors of R_q and J_q and $Q_q R_q Q_q = \Lambda_q = \text{diag}(\lambda_j^{(q)})$ with

$$\lambda_j^{(q)} = 4 \sin^2(j\pi/2(q+1)), \qquad j = 1, 2, \cdots, q.$$

The eigenvectors of K_{11} and K_{22} can be constructed as tensor products. Thus

$$(Q_r \otimes Q_q) K_{11} (Q_r \otimes Q_q) = (I_r \otimes \Lambda_q) + (\Lambda_r \otimes I_q) = \psi_{qr}$$

and similarly

$$(Q_n \otimes Q_m) K_{22} (Q_n \otimes Q_m) = (I_n \otimes \Lambda_m) + (\Lambda_n \otimes I_m) = \psi_{mn}.$$

This spectral decomposition of the discrete Laplacian is well known; see e.g. Stoer and Burlisch [32].

With the Schur complement $S^{(1)}$ defined as in (2.9), we find that

$$\begin{aligned} Q_q S^{(1)} Q_q &= I_q + \tfrac{1}{2}\Lambda_q - Q_q K_{13}^T (Q_r \otimes Q_q) \psi_{rq}^{-1} (Q_r \otimes Q_q) K_{13} Q_q \\ &= I_q + \tfrac{1}{2}\Lambda_q - (2/(r+1)) \sum_{k=1}^{r} \sin^2(k\pi/(r+1)) (\Lambda_q + \lambda_k^{(r)} I)^{-1} \\ &= I_q + \tfrac{1}{2}\Lambda_q - D_{qr}. \end{aligned}$$

LEMMA 4.1. *The elements of the diagonal matrix D_{qr}, defined above, are given by*

$$d_j^{(q,r)} = a_j^{(q)} (1 - (a_j^{(q)})^{2r}) / (1 - (a_j^{(q)})^{2(r+1)}), \qquad j = 1, 2, \cdots, q, \tag{4.1}$$

where

$$a_j^{(q)} = 1 + \lambda_j^{(q)}/2 - (\lambda_j^{(q)} (1 + \lambda_j^{(q)}/4))^{1/2}. \tag{4.2}$$

Proof. The matrix D_{qr} is diagonal and its elements are given by

$$\begin{aligned} d_j^{(q,r)} &= 2/(r+1) \sum_{k=1}^{r} \sin^2(k\pi/(r+1)) / (\lambda_j^{(q)} + \lambda_k^{(r)}) \\ &= (2a_j^{(q)}/(r+1)) \sum_{k=1}^{r} \sin^2(k\pi/(r+1)) / (1 - 2a_j^{(q)} \cos(k\pi/(r+1)) + (a_j^{(q)})^2). \end{aligned}$$

This sum can be evaluated by using Poisson's summation formula. Thus

$$\begin{aligned} d_j^{(q,r)} &= (2a_j^{(q)}/(r+1)) [(\tfrac{1}{2}) f(0) + f(\pi/(r+1)) + f(2\pi/(r+1)) + \cdots + (\tfrac{1}{2}) f(\pi)] \\ &= (2a_j^{(q)}/\pi) \left[F_0 + 2 \sum_{k=1}^{\infty} F_{2k(r+1)} \right], \end{aligned}$$

where

$$f(x) = \sin^2 x / (1 - 2a_j^{(q)} \cos x + a_j^{(q)^2})$$

and

$$F_m = \int_0^{\pi} f(x) \cos mx \, dx.$$

By using Poisson's integration formula, we obtain

$$F_0 = \pi/2,$$
$$F_1 = \pi a_j^{(q)}/4,$$
$$F_m = -\pi a_j^{(q)^{(m-2)}}(1-a_j^{(q)^2})/4, \quad m = 2, 3, \cdots.$$

Therefore

$$d_j^{(q,r)} = a_j^{(q)}\left(1-(a_j^{(q)^{-2}}-1)\sum_{k=1}^{\infty}(a_j^{(q)})^{2(r+1)k}\right)$$
$$= a_j^{(q)}(1-(a_j^{(q)})^{2r})/(1-(a_j^{(q)})^{2(r+1)}).$$

The parameter $a_j^{(q)}$, defined by (4.2), decreases monotonically from 1 to $3-2\sqrt{2}$ as $\lambda_j^{(q)}$ varies from 0 to 4. The ratio $d_j^{(q,r)}/a_j^{(q)}$ is monotonely increasing as a function of $a_j^{(q)}$ and

$$(r/(r+1))a_j^{(q)} < d_j^{(q,r)} < a_j^{(q)} < 1. \tag{4.3}$$

Note that $\lambda_j^{(q)} = (1-a_j^{(q)})^2/a_j^{(q)}$. From this it easily follows that

$$Q_q S^{(1)} Q_q = F_{qr}^{1/2}(\Lambda_q + (\tfrac{1}{4})\Lambda_q^2)^{1/2}F_{qr}^{1/2} \tag{4.4}$$

where the matrix $F_{qr}^{1/2}$ is diagonal with the elements

$$(F_{qr})_{jj} = (1+a_j^{(q)^{2r+2}})/(1-a_j^{(q)^{2r+2}}), \quad j = 1, 2, \cdots, q. \tag{4.5}$$

We note that these elements are $\geqq 1$ and that they approach 1 exponentially fast with increasing r for a fixed $a_j^{(q)} < 1$. It is also easy to see that $J_q \leqq S^{(1)}$.

Denote by E_m the $m \times q$ matrix

$$E_m = (0, I_q, 0)^T$$

where the leading zero block is $p \times q$. Similarly as in our computation of the eigenvalues of $S^{(1)}$, we obtain, since $K_{23} = -E_m$,

$$Q_q S^{(2)} Q_q = Q_q E_m^T Q_m F_{mr}^{1/2}(\Lambda_m + (\tfrac{1}{4})\Lambda_m^2)^{1/2}F_{mr}^{1/2}Q_m E_m Q_q.$$

LEMMA 4.2. *For all* x

$$x^T E_m^T J_m E_m x \leqq x^T J_q x.$$

Proof. We will show that a principal minor of the square root of a positive definite matrix is dominated by the square root of the corresponding minor.

Let

$$B = A^{1/2} = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{pmatrix}.$$

Then $A_{11} = B_{11}^2 + B_{12}B_{12}^T$ and it is easy to see that $x^T B_{11} x \leqq x^T (B_{11}^2 + B_{12}B_{12}^T)^{1/2} x$. Since $E_m^T R_m E_m = R_q$, the lemma follows.

We will now give upper and lower bounds for the quadratic form $x^T S x$ in terms of $x^T S^{(1)} x$ and $x^T J_q x$, which will provide bounds on the performance of the preconditioned conjugate gradient methods. We first consider the case when J_q is used. Since $J_q \leqq S^{(1)}$ and $S^{(2)} \geqq 0$, we find that

$$1 = x^T J_q x / x^T J_q x \leqq x^T S^{(1)} x / x^T J_q x \leqq x^T S^{(1)} x / x^T J_q x + x^T S^{(2)} x / x^T J_q x.$$

In order to derive an upper bound, we consider the two expressions separately. With $y = Q_q x$, it follows from (4.4) that

$$x^T S^{(1)} x / x^T J_q x = y^T F_{qr}^{1/2} (\Lambda_q + (\tfrac{1}{4})\Lambda_q^2)^{1/2} F_{qr}^{1/2} y / y^T \Lambda_q^{1/2} y \leqq \sqrt{2}\, y^T F_{qr}^{1/2} \Lambda_q^{1/2} F_{qr}^{1/2} y / y^T \Lambda_q^{1/2} y. \tag{4.6}$$

Using Lemma 4.2, we similarly obtain

$$x^T S^{(2)} x / x^T J_q x \leqq \sqrt{2}\, v^T F_{mn}^{1/2} \Lambda_m^{1/2} F_{mn}^{1/2} v / v^T \Lambda_m^{1/2} v, \tag{4.7}$$

where $v = Q_m E_m x$.

From these estimates and using properties of diagonal matrices F_{qr} and F_{mn}, we will now show the following.

THEOREM 4.1. *If J_q is used as a preconditioner, then for any small $\varepsilon > 0$,*

$$x^T S x / x^T J_q x < 2\sqrt{2}\,(1+\varepsilon) \quad \forall x \in T.$$

Here T is a large subspace and its orthogonal complement $T_\perp$ satisfies

$$\dim(T_\perp) \leqq \frac{1}{2\pi}\left(\frac{q+1}{r+1} + \frac{m+1}{n+1}\right) \log(2/\varepsilon).$$

Proof. Once we have shown that sufficiently many of the elements $(F_{qr})_{jj}$ are less than $1+\varepsilon$, the inequality follows from (4.6) and (4.7) and a Courant-Fischer argument.

By the definition (4.5) of the elements of F_{qr}, we are led to the inequality

$$(1 + (a_j^{(q)})^{2r+2}) / (1 - (a_j^{(q)})^{2r+2}) \leqq 1 + \varepsilon.$$

Ignoring $O(\varepsilon^2)$ terms, this holds if

$$(a_j^{(q)})^{2r+2} \leqq \varepsilon/2$$

or

$$-\log a_j^{(q)} \geqq \log(2/\varepsilon)/(2r+2).$$

Since $a_j^{(q)} = e^{-\pi j/(q+1)} + O((j/(q+1))^3)$ and only small values of $j/(q+1)$ are relevant, we find that

$$(F_{qr})_{jj} < 1 + \varepsilon$$

for

$$j > \frac{1}{2\pi}\left(\frac{q+1}{r+1}\right) \log(2/\varepsilon).$$

With the techniques just used, we can also obtain explicit bounds for the largest few eigenvalues of SJ_q^{-1}. However it is well known that a fixed number of separate eigenvalues outside a basic interval does not affect the asymptotic rate of convergence of the conjugate gradient method.

We next consider the use of $S^{(1)}$ as a preconditioner. Then

$$\begin{aligned} 1 \leqq x^T S x^T / x^T S^{(1)} x &= 1 + x^T S^{(2)} x / x^T S^{(1)} x \\ &= 1 + (v^T F_{mn}^{1/2} (\Lambda_m + (\tfrac{1}{4})\Lambda_m^2)^{1/2} F_{mn}^{1/2} v / v^T (\Lambda_m + (\tfrac{1}{4})\Lambda_m^2)^{1/2} v) \\ &\quad \cdot (x^T E_m^T J_m (I_m + (\tfrac{1}{4})J_m^2)^{1/2} E_m x / x^T J_q (I_q + (\tfrac{1}{4})J_q^2)^{1/2} x), \end{aligned}$$

where $v = Q_m E_m x$. The first factor can be bounded by $1+\varepsilon$ for a large subspace as in Theorem 4.1. Based on numerical experiments, we conjecture that the second factor can be bounded by one. We have not been able to prove the required counterpart of Lemma 4.2. It is however easy to bound the second factor by $\sqrt{2}$ and we then obtain

THEOREM 4.2. *If $S^{(1)}$ is used as a preconditioner, then for any small $\varepsilon > 0$,*

$$x^T S x / x^T S^{(1)} x < 1 + \sqrt{2}\,(1+\varepsilon) \quad \forall x \in T.$$

Here T is a large subspace and its orthogonal complement T satisfies

$$\dim(T_\perp) \leqq \frac{1}{2\pi}\left(\frac{m+1}{n+1}\right)\log(2/\varepsilon).$$

We will compare the results of Theorems 4.1 and 4.2 with numerical experiments in § 7. These results confirm that the effective upper bound when S^1 is used as a preconditioner should be $2+\varepsilon$. In all the cases which we have tried, the lower bounds have exceeded 1.6. Our preconditioners therefore lead to extremely well conditioned problems.

5. Bounds for the spectra in the general case. In this section, we will formulate and rigorously prove our results on the spectra of the operators S, SJ^{-1} and $S(S^{(1)})^{-1}$ which correspond to the three methods introduced in § 3. Almost the entire section will be devoted to the proof of the following.

THEOREM 5.1. *The condition number of S grows linearly with the number of degrees of freedom associated with Γ_3. Those of SJ^{-1} and $S(S^{(1)})^{-1}$ are uniformly bounded and the corresponding methods are therefore optimal.*

Our results will be established for conforming, Lagrangian finite element approximations of Dirichlet problems for self-adjoint, second order elliptic problems in plane regions,

$$Lu = -\sum_{i,j}\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial u}{\partial x_j} + a_0(x)u = f(x), \quad x \in \Omega,$$
$$u(x) = g(x), \quad x \in \bar{\Gamma}_1 \cup \bar{\Gamma}_2.$$

The operator L has real, sufficiently smooth coefficients, $a_{ji}(x) = a_{ij}(x)$ and the bilinear form

$$a(u, v) = \int_\Omega \sum_{i,j} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + a_0(x)uv\,dx$$

satisfies

$$\frac{1}{C}\|u\|^2_{H^1(\Omega)} \leqq a(u, u) \quad \forall u \in H_0^1(\Omega) \tag{5.1}$$

and

$$|a(u, v)| \leqq C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}. \tag{5.2}$$

We adopt the common practice of using C to denote a generic constant, strictly positive and independent of the dimension of the finite dimensional problems considered, with the value of C not necessarily the same in different instances.

By assumption, the finite element problems of the two subregions are solved exactly. We therefore concentrate our study on the solution of equation (2.8), or what is the same, the discrete problem with zero boundary conditions on $\bar{\Gamma}_1 \cup \bar{\Gamma}_2$ and a right-hand side different from zero on Γ_3 only. Any problem can be reduced to such a problem by solving two subproblems.

Even if the original problem has a very smooth boundary, its partition into subregions makes it necessary to consider regions with corners. Our main technical tools will be borrowed from elliptic regularity theory which is very well developed for the case of sufficiently smooth boundaries; see e.g. Lions and Magenes [20]. Many regions arising in practice have corners, but in spite of this, the extension of the theory to such cases has been relatively neglected; see however Grisvard [17]. Following Grisvard [17], we concentrate our attention on curvilinear polygons. The boundary Γ of such a region is $\cup_{\text{finite}} \bar{\Gamma}_j'$ where $\bar{\Gamma}_j'$ is the closure of an open sufficiently smooth curve Γ_j'. Locally such a region is the image under a smooth mapping of a region with a corner with an angle equal to π, $\pi/2$ or $3\pi/2$. Denote by C^l the class of l times continuously differentiable functions.

DEFINITION. Let Ω be a bounded open subset of R^2. Its boundary Γ is a curvilinear polygon of class C^l, l an integer $\geqq 1$, if for every $x \in \Gamma$ there exists a neighborhood V of x in R^2 and a mapping ψ from V into R^2 such that

(i) $\psi(y) = (\psi_1(y), \psi_2(y))$ is injective,

(ii) ψ and $\psi^{-1} \in C^l$,

(iii) $\Omega \cap V$ is either $\{y \in \Omega \,|\, \psi_2(y) < 0\}$, $\{y \in \Omega \,|\, \psi_1(y) < 0 \text{ and } \psi_2(y) < 0\}$ or $\{y \in \Omega \,|\, \psi_1(y) < 0 \text{ or } \psi_2(y) < 0\}$.

We will assume that l is large enough and that the separator curve Γ_3 is sufficiently smooth. Our arguments easily extend to the case where Γ_3 is only piecewise in C^l.

We will need a number of Sobolev spaces and some of their properties. By $L_2(\Omega)$, we denote the space of square integrable functions on Ω. For integer $m \geqq 0$, $H^m(\Omega)$ is the subspace of $L_2(\Omega)$ for which,

$$\|u\|_{H^m(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \leqq m} \left| \left(\frac{\partial}{\partial x} \right)^\alpha u(x) \right|^2 dx \right)^{1/2} < \infty. \tag{5.3}$$

Here

$$\left(\frac{\partial}{\partial x} \right)^\alpha = \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial x_2} \right)^{\alpha_2}, \qquad |\alpha| = \alpha_1 + \alpha_2.$$

For $s = m + \sigma > 0$, $0 < \sigma < 1$, we define $H^s(\Omega)$ in terms of the norm,

$$\|u\|_{H^s(\Omega)} = \Bigg(\|u\|^2_{H^m(\Omega)} + \int_\Omega \int_\Omega \sum_{|\alpha| = m} \left| \left(\left(\frac{\partial}{\partial x} \right)^\alpha u(x) - \left(\frac{\partial}{\partial y} \right)^\alpha u(y) \right) (x-y)^{-(1+\sigma)} \right|^2 dx\, dy \Bigg)^{1/2}. \tag{5.4}$$

We will also need the seminorms $|u|_{H^s(\Omega)}$, which are obtained by dropping all terms with derivatives of order less than m in (5.3) and the first term in (5.4). We can of course also define Sobolev spaces on the curves Γ_j.

For Dirichlet problems, we use $H_0^s(\Omega)$, $s > 0$, which is a subspace of $H^s(\Omega)$ defined as the closure in $H^s(\Omega)$ of $D(\Omega)$, the space of C^∞ functions with compact support in Ω. The dual space of $H_0^s(\Omega)$ is denoted by $H^{-s}(\Omega)$.

Let γ_j be the trace operator on Γ_j, defined for continuous functions as the restriction of the function to Γ_j. This set of operators can be extended to an operator from $H^1(\Omega)$ onto a subspace of $\prod_j H^{1/2}(\Gamma_j)$. This subspace is characterized in detail in Theorem 1.5.2.3 in Grisvard [17]. Here we are primarily interested in the trace of functions in

the spaces $H_0^1(\Omega_i, \Gamma_i)$, $i = 1, 2$, defined as the subspaces of $H^1(\Omega_i)$ with zero trace on Γ_i. In this special case it follows that γ_3 maps $H_0^1(\Omega_i, \Gamma_i)$ onto

$$H_{00}^{1/2}(\Gamma_3) = \{u \in H_0^{1/2}(\Gamma_3) \mid \|u\|_{H_{00}^{1/2}(\Gamma_3)} = (\|u\|^2_{H^{1/2}(\Gamma_3)} + \|\rho^{-1/2}u\|^2_{L_2(\Gamma_3)})^{1/2} < \infty\}.$$

Here ρ is the distance to the boundary, i.e. to the end points of Γ_3.

The space $H_{00}^{1/2}(\Gamma_3)$ is strictly contained in $H_0^{1/2}(\Gamma_3)$; see Lions and Magenes [20, p. 66]. It can also be defined by interpolation. Thus,

$$H_{00}^{1/2}(\Gamma_3) = [H_0^1(\Gamma_3), L_2(\Gamma_3)]_{1/2}.$$

We also need

$$H_0^{3/4}(\Gamma_3) = [H_0^1(\Gamma_3), L_2(\Gamma_3)]_{3/4};$$

see Lions and Magenes [20, pp. 64–66]. We can use the K-method, see Lions and Magenes [20, pp. 98–99], to find useful formulas for the $H_{00}^{1/2}(\Gamma_3)$ and $H_0^{3/4}(\Gamma_3)$ norms of finite element functions. Let $\omega \in H_0^1(\Gamma_3) + L_2(\Gamma_3)$, i.e. $\omega = \omega_0 + \omega_1$, where $\omega_0 \in H_0^1(\Gamma_3)$ and $\omega_1 \in L_2(\Gamma_3)$. A functional $K(t, \omega)$ is defined by

$$K(t, \omega) = \inf_{\omega = \omega_0 + \omega_1} (\|\omega_0\|^2_{H_0^1(\Gamma_3)} + t^2\|\omega_1\|^2_{L_2(\Gamma_3)})^{1/2}.$$

The equivalent norms are then given by the expression

$$\left(\|\omega\|^2_{L_2(\Gamma_3)} + \int_0^\infty t^{-(1+2\theta)}(K(t, \omega))^2\, dt\right)^{1/2}$$

with $\theta = \frac{1}{2}$ and $\frac{3}{4}$, respectively. By using the same arguments as in Lions and Magenes [20], we can now compute the $H_{00}^{1/2}(\Gamma_3)$ and $H_0^{3/4}(\Gamma_3)$ norms of ω_h, the restriction to Γ_3 of a finite element function defined on the triangulation of Ω. The $H_0^1(\Gamma_3)$ norm of ω_h equals $(a_{\Gamma_3}(\omega_h, \omega_h))^{1/2}$, where the bilinear form $a_{\Gamma_3}(u, v)$ is defined by

$$a_{\Gamma_3}(u, v) = \int_{\Gamma_3} \left(\frac{du}{ds}\frac{dv}{ds} + uv\right) ds, \quad u, v \in H_0^1(\Gamma_3).$$

For a given finite element space, we can compute the stiffness and mass matrices K_Γ and M_Γ associated with this Dirichlet problem on Γ_3. If the coordinate vector of ω_h with respect to the finite element basis is α, then it follows from the computation that

$$\|\omega_h\|^2_{H_{00}^{1/2}(\Gamma_3)} = \alpha^T J \alpha,$$

where $J = M_\Gamma^{1/2}(M_\Gamma^{-1/2}K_\Gamma M_\Gamma^{-1/2})^{1/2}M_\Gamma^{1/2}$ and that

$$\|\omega_h\|^2_{H_0^{3/4}(\Gamma_3)} = \alpha^T M_\Gamma^{1/2}(M_\Gamma^{-1/2}K_\Gamma M_\Gamma^{-1/2})^{3/4}M_\Gamma^{1/2}\alpha.$$

Let Q be the M_Γ-orthonormal matrix of eigenvectors of the generalized eigenvalue problem $JQ = M_\Gamma Q\Lambda$, and Λ be the corresponding diagonal matrix of eigenvalues. It is then easy to see that the pair Q, Λ^2 solves the generalized eigenvalue problem.

$$K_\Gamma Q = M_\Gamma Q \Lambda^2.$$

In what follows, we also need to know that the trace class on Γ_3 of $H^{5/4}(\Omega_i) \cap H_0^1(\Omega_i, \Gamma_i)$, $i = 1, 2$, is given by $H_0^{3/4}(\Gamma_3)$ and that

$$\|\gamma_3 u\|_{H_0^{3/4}(\Gamma_3)} \leqq C\|u\|_{H^{5/4}(\Omega_i)}, \qquad i = 1, 2, \tag{5.5}$$

where the constant depends on the region only.

One of the principal results in elliptic theory is sometimes called the shift theorem. For Dirichlet's problem and an elliptic operator L of order $2k$ it states that

$$\|u\|_{H^{s+k}(\Omega)} \leqq C(\|f\|_{H^{s-k}(\Omega)} + \|u_0\|_{H^{s+k}(\Omega)}). \tag{5.6}$$

Here $Lu = f$ and u_0 is any extension of the Dirichlet data such that $u - u_0 \in H_0^{s+k}(\Omega)$. It is easy to establish this result for $s = 0$ for our family of second order elliptic problems by using a standard variational argument. These bounds can also be derived for a wide range of values of s for problems with sufficiently smooth boundaries, but this is no longer true for Lipschitz regions. However, Nečas [23] has shown that the estimate (5.6) is true for $|s| < 1/2$. In the estimate, the term $\|u_0\|_{H^{s+k}(\Omega)}$ can be replaced by a suitable trace term, by using an extension theorem which states that there exists a continuous operator which extends any element in the proper trace class to an element in $H^{s+k}(\Omega)$; see Grisvard [17] or Stein [31].

Before we turn to the proof of Theorem 5.1, we will show that the continuous operator corresponding to $SS^{(1)^{-1}}$ is bounded in an appropriate norm. For simplicity, we will use the same notation for the operators in the continuous case. We note that a bound for $S^{(1)}S^{-1}$ is easy to obtain since $S = S^{(1)} + S^{(2)}$ and $S^{(1)}$ and $S^{(2)}$ are positive definite and selfadjoint.

The natural domain of definition for $(S^{(1)})^{-1}$ is $(H_{00}^{1/2}(\Gamma_3))'$, the dual of the trace class $H_{00}^{1/2}(\Gamma_3)$. Consider the variational problem

$$a_{\Omega_1}(u_1, v) = \int_{\Omega_1} \left(\sum a_{ij} \frac{\partial u_1}{\partial x_j} \frac{\partial v}{\partial x_i} + a_0 u_1 v \right) dx = \int_{\Gamma_3} g_N v \, ds,$$

$$u_1 \in H_0^1(\Omega_1, \Gamma_1) \quad \forall v \in H_0^1(\Omega_1, \Gamma_1).$$

Here $g_N \in (H_{00}^{1/2}(\Gamma_3))'$ is the Neumann data given on Γ_3. From the ellipticity assumption (5.1), we obtain

$$\frac{1}{C}\|u_1\|_{H^1(\Omega_1)}^2 \leqq a_{\Omega_1}(u_1, u_1) = \int_{\Gamma_3} g_N u_1 \, ds \leqq \|g_N\|_{(H_{00}^{1/2}(\Gamma_3))'} \|\gamma_3 u_1\|_{H_{00}^{1/2}(\Gamma_3)}.$$

By the trace theorem

$$\|\gamma_3 v\|_{H_{00}^{1/2}(\Gamma_3)} \leqq C \|v\|_{H^1(\Omega_1)} \quad \forall v \in H_0^1(\Omega_1, \Gamma_1).$$

Therefore

$$\|\gamma_3 u_1\|_{H_{00}^{1/2}(\Gamma_3)} \leqq C \|g_N\|_{(H_{00}^{1/2}(\Gamma_3))'}.$$

By the characterization of the trace class of $H_0^1(\Omega_2, \Gamma_2)$, we see that $\gamma_3 u_1$, extended by zero on Γ_2, is the trace of some $\tilde{u}_2 \in H_0^1(\Omega_2, \Gamma_2)$, with

$$\|\tilde{u}_2\|_{H^1(\Omega_2)} \leqq C \|\gamma_3 u_1\|_{H_{00}^{1/2}(\Gamma_3)}.$$

By using the shift theorem in the simple case of $s = 0$, we can solve $Lu_2 = 0$ on Ω_2 with $u_2 - \tilde{u}_2 \in H_0^1(\Omega_2)$. We note that

$$L(u_2 - \tilde{u}_2) = -L\tilde{u}_2, \qquad u_2 - \tilde{u}_2 \in H_0^1(\Omega_2),$$

and that

$$\|L\tilde{u}_2\|_{H^{-1}(\Omega_2)} \leqq C \|\tilde{u}_2\|_{H^1(\Omega_2)}.$$

Thus we obtain $u_1 \in H_0^1(\Omega_1, \Gamma_1)$ and $u_2 \in H_0^1(\Omega_2, \Gamma_2)$ which, by construction, have the same trace on Γ_3. Together they therefore form $u \in H_0^1(\Omega)$ with

$$\|u\|_{H_0^1(\Omega)} \leqq C \|g_N\|_{(H_{00}^{1/2}(\Gamma_3))'}.$$

The operator L maps $H_0^1(\Omega)$ continuously onto $H^{-1}(\Omega)$. In this case Lu is a distribution with a support on Γ_3 and can therefore be regarded as an element of $(H_{00}^{1/2}(\Gamma_3))'$. The boundedness of $S(S^{(1)})^{-1}$ is thus established.

The proof in the discrete case proceeds somewhat differently, since we want to obtain bounds for SJ^{-1} as well. We also need to establish an extension theorem in the finite element case. The proof of Theorem 5.1 follows from

$$\frac{1}{C}J \leqq S^{(i)} \leqq CJ, \qquad i=1,2. \tag{5.7}$$

To establish the lower bound, we can argue as in the continuous case and consider

$$a_{\Omega_1}(u_{1,h}, v_h) = \int_{\Gamma_3} g_N v_h \, ds \quad \forall v_h \in V_h \cap H_0^1(\Omega_1, \Gamma_1),$$

$$u_{1,h} \in V_h \cap H_0^1(\Omega_1, \Gamma_1).$$

As above we find

$$\|\gamma_3 u_{1,h}\|^2_{H_{00}^{1/2}(\Gamma_3)} \leqq C a_{\Omega_1}(u_{1,h}, u_{1,h}) = C \int_{\Gamma_3} g_N u_{1,h} \, ds.$$

By using the definition of the dual norm and the formula for $\|\gamma_3 u_{1,h}\|_{H_{00}^{1/3}(\Gamma_3)}$, we obtain

$$\|\gamma_3 u_{1,h}\|^2_{H_{00}^{1/2}(\Gamma_3)} \leqq C\delta^T J^{-1}\delta, \tag{5.8}$$

where the components of δ are $\int_{\Gamma_3} g_N \phi_i \, ds$, where ϕ_i is a basis function for the finite element space on Γ_3. The coordinate vector for $\gamma_3 u_{1,h}$ is $(S^{(1)})^{-1}\delta$ and the inequality (5.8) can therefore be written as

$$\delta^T (S^{(1)})^{-1} J (S^{(1)})^{-1} \delta \leqq C\delta^T J^{-1}\delta$$

from which the lower bound of (5.7) follows by using that $S^{(1)}$ and J are positive definite and symmetric matrices.

The main difficulty in establishing the upper bound is the proof of the following lemma.

LEMMA 5.1. *There exists an extension mapping from Γ_3 to Ω_1 which extends a finite element function g_h on Γ_3 to an element $v_h \in V_h \cap H_0^1(\Omega_1, \Gamma_1)$ such that*

$$\|v_h\|_{H^1(\Omega_1)} \leqq C \|g_h\|_{H_{00}^{1/2}(\Gamma_3)}. \tag{5.9}$$

Proof. We first construct $v \in H_0^1(\Omega_1, \Gamma_1)$ by solving the continuous homogeneous Dirichlet problem with the prescribed boundary values. Similarly, we define v_h as the solution of the finite element approximation to the same problem. By the triangle inequality

$$\|v_h\|_{H^1(\Omega_1)} \leqq \|v\|_{H^1(\Omega_1)} + \|v_h - v\|_{H^1(\Omega_1)}.$$

The solution of the continuous problem can be estimated as follows,

$$\|v\|_{H^1(\Omega_1)} \leqq C \|g_h\|_{H_{00}^{1/2}(\Gamma_3)}$$

and

$$\|v\|_{H^{5/4}(\Omega_1)} \leqq C \|g_h\|_{H_0^{3/4}(\Gamma_3)};$$

see Grisvard [17, Chap. 1.5.2]. From the variational formulation of the discrete problem and (5.2), we know that

$$\frac{1}{C}\|v_h - v\|^2_{H^1(\Omega_1)} \leqq a_{\Omega_1}(v_h - v, v_h - v) = a_{\Omega_1}(v_h - v, \Pi_h v - v)$$

$$\leqq C \|v_h - v\|_{H^1(\Omega_1)} \|\Pi_h v - v\|_{H^1(\Omega_1)},$$

where $\Pi_h v$ is the interpolant of v in V_h. It is important to note that $\Pi_h v - v = 0$ on $\bar{\Gamma}_1 \cup \bar{\Gamma}_3$. We use the following lemma to complete our proof of Lemma 5.1.

LEMMA 5.2. *The interpolant* $\Pi_h v$ *satisfies the bound*

$$\|\Pi_h v - v\|_{H^1(\Omega_1)} \leqq Ch^{1/4} |v|_{H^{5/4}(\Omega_1)}.$$

With the aid of Lemma 5.2, we conclude that

$$\begin{aligned}\|v_h - v\|^2_{H^1(\Omega_1)} &\leqq Ch^{1/2} |v|^2_{H^{5/4}(\Omega_1)} \\ &\leqq Ch^{1/2} \|g_h\|^2_{H_0^{3/4}(\Gamma_3)} \leqq Ch^{1/2} \delta^T M_\Gamma^{1/2} (M_\Gamma^{-1/2} K_\Gamma M_\Gamma^{-1/2})^{3/4} M_\Gamma^{1/2} \delta.\end{aligned}$$

It is easy to establish, under our assumptions on the triangulation, that

$$M_\Gamma^{-1/2} K_\Gamma M_\Gamma^{-1/2} \leqq Ch^{-2}.$$

Therefore

$$\|g_h\|^2_{H_0^{3/4}(\Gamma_3)} \leqq Ch^{-1/2} \delta^T M_\Gamma^{1/2} (M_\Gamma^{-1/2} K_\Gamma M_\Gamma^{-1/2})^{1/2} M_\Gamma^{1/2} \delta = Ch^{-1/2} \delta^T J \delta,$$

which completes the proof of Lemma 5.1.

Proof of Lemma 5.2. The proof is a modification of the Bramble-Hilbert lemma. Let $F(u)$ be a bounded functional on $H^{5/4}(\Delta)$, where Δ is a triangle, such that $F(p) = 0$ for all polynomials of degree one. Then

$$|F(u)| \leqq C |u|_{H^{5/4}(\Delta)}.$$

Since the modification of the standard version of the Bramble-Hilbert lemma is relatively minor, see Ciarlet [6], details will not be given. We only note that the result is obtained by scaling and adding the contributions from all the triangles in Ω_1. We also note that it is easy to prove, by Sobolev's lemma, that Π_h is a bounded operator in $H^{5/4}$ since, by assumption, we work with Lagrangian finite elements only and thus use only values of v and no values of derivatives.

With the help of Lemma 5.1, we can reduce a discrete Dirichlet problem to one with homogeneous boundary values for which a simple variational argument is sufficient. We can thus establish the estimate (5.9) for the approximate solution $u_{1,h}$ of $Lu = 0$. For given Dirichlet data $g_D \in H_{00}^{1/2}(\Gamma_3)$, we can thus find $u_{1,h} \in H_0^1(\Omega_1, \Gamma_1)$ such that

$$\|u_{1,h}\|_{H^1(\Omega_1)} \leqq C \|g_D\|_{H_{00}^{1/2}(\Gamma_3)}.$$

Translating the results of § 3, we find that

$$a_{\Omega_1}(u_{1,h}, v_h) = \int_{\Gamma_3} S^{(1)} g_D v_h \, ds \qquad \forall v_h \in V_h \cap H_0^1(\Omega_1, \Gamma_1),$$

from which follows that

$$\|S^{(1)} g_D\|_{H_{00}^{1/2}(\Gamma_3)'} \leqq C \|g_D\|_{H_{00}^{1/2}(\Gamma_3)}.$$

By using the formulas for the norms of finite element functions, we obtain,

$$S^{(1)} \leqq CJ.$$

The proof of the theorem can now be completed since the bounds for SJ^{-1} and $SS^{(1)^{-1}}$ follow immediately. It is also easily seen that the condition number of S grows like that of J, i.e. linearly with the number of nodes on Γ_3.

6. A discussion of certain other methods. In this section, we will describe and analyze some algorithms introduced in Concus, Golub and O'Leary [7], Dihn, Glowinski and Périaux [9] and Golub and Meyers [16].

In a well-known paper on generalized conjugate gradient methods, Concus, Golub and O'Leary [7] considered, among other things, the five point difference approximation of Poisson's equation on a T-shaped region, i.e. the same problem as in §§ 4 and 7 of this paper. To describe their approach, we modify our notation slightly. We split off the q mesh points in Ω_2, immediately adjacent to the horizontal separator set, and associate the index 4 with this new set. The resulting finite difference problem is then of the form

$$\begin{pmatrix} K_{11} & 0 & K_{13} & 0 \\ 0 & K_{22} & 0 & K_{24} \\ K_{13}^T & 0 & K_{33} & K_{34} \\ 0 & K_{24}^T & K_{34}^T & K_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}. \tag{6.1}$$

A symmetric, positive definite preconditioner is obtained by replacing the matrices K_{34} and K_{34}^T by zero. After a symmetric permutation, the preconditioner is seen to be a direct sum of two discrete Dirichlet problems on the rectangles Ω_1 and Ω_2. Without loss of generality, we can set the subvectors b_1 and b_2 equal to zero, since we can reduce the system (6.1) to that form at the expense of solving two discrete problems on the rectangular subregions.

The rate of convergence of this generalized conjugate gradient method is determined by the eigenvalues of a $2q \times 2q$ matrix.

$$\hat{S} = \begin{pmatrix} I & -(I+(1/2)R_q+S^{(2)})^{-1} \\ -(I+(1/2)R_q+S^{(1)})^{-1} & I \end{pmatrix} = \begin{pmatrix} I & \hat{S}_{34} \\ \hat{S}_{43} & I \end{pmatrix}.$$

Here $S^{(i)}$ and R_q are matrices defined in §§ 3 and 4, respectively. The matrix $\hat{S}$ is relevant, because by a calculation quite similar to that which led to equation (2.11),

$$\begin{pmatrix} K_{11} & 0 & K_{13} & 0 \\ 0 & K_{22} & 0 & K_{24} \\ K_{13}^T & 0 & K_{33} & K_{34} \\ 0 & K_{24}^T & K_{34}^T & K_{44} \end{pmatrix} \begin{pmatrix} K_{11} & 0 & K_{13} & 0 \\ 0 & K_{22} & 0 & K_{24} \\ K_{13}^T & 0 & K_{33} & 0 \\ 0 & K_{24}^T & 0 & K_{44} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ y_3+\hat{S}_{34}y_4 \\ \hat{S}_{43}y_3+y_4 \end{pmatrix}.$$

Let λ be an eigenvalue of $\hat{S}$. Then for $\lambda \neq 1$,

$$\begin{aligned} \det \begin{pmatrix} (1-\lambda)I & \hat{S}_{34} \\ \hat{S}_{43} & (1-\lambda)I \end{pmatrix} &= \det \begin{pmatrix} I & \hat{S}_{34} \\ 0 & (1-\lambda)^2 I - \hat{S}_{43}\hat{S}_{34} \end{pmatrix} \\ &= \det ((1-\lambda)^2 I - \hat{S}_{43}\hat{S}_{34}). \end{aligned}$$

We note that $\lambda = 1$ is not an eigenvalue since $\hat{S}_{34}$ and $\hat{S}_{43}$ are both symmetric, negative definite matrices.

To see that this method is not optimal, we can consider the case when Ω is a rectangle cut in half. In this case $S^{(1)} = S^{(2)}$ and $\hat{S}_{34} = \hat{S}_{43}$.

The eigenvalues of $\hat{S}$ are then of the form

$$1 \pm 1/\lambda_j(I+1/2R_q+S^{(1)}).$$

By a straightforward calculation they are equal to $1 \pm 1/(2+\lambda_j^{(q)} - d_j^{(q,r)})$. By using (4.2) and (4.3) one easily sees that the eigenvalues vary between $O(1/q+1/r)$ and 2.

The paper by Dihn, Glowinski and Périaux [9] is a report on a project to develop decomposition methods for nonlinear problems in fluid mechanics e.g. transonic flow on large computational domains. Here we will only discuss the two methods for elliptic problems which were introduced in § IV of their paper. In § IV.1, the authors consider

a method in which the unknown λ is the common value of $\partial u/\partial n$ on Γ_3. The traces of the solutions on Ω_1 and Ω_2 are then compared. The corresponding mapping is, in our notation, given by

$$(S^{(1)^{-1}}+S^{(2)^{-1}})\lambda$$

and by the results of § 5, this operator has a condition number which grows linearly with the number of degrees of freedom associated with Γ_3.

In § IV.2, the authors consider the method which corresponds to solving equation (2.8) without preconditioning.

In a recent paper, Golub and Meyers [16] consider primarily three good methods for the model problem of § 4 of this paper. Having observed experimentally that the Schur complement S is quite close to a Toeplitz matrix, they derive an approximation of S by solving a simplified problem in which the boundaries of the rectangles Ω_1 and Ω_2 are moved to infinity. They also study experimentally the use of the preconditioners J_q and $J_q(I+J_q^2/4)^{1/2}$; cf. § 4 of this paper. They find that the use of the latter gives the best rate of convergence.

The preconditioners $S^{(1)}$, considered in this paper, and $J_q(I+J_q^2/4)^{1/2}$ are in fact quite close. In the case considered, J_q and $S^{(1)}$ commute and the same calculation as in § 4 shows that the jth eigenvalue of $S^{(1)}$ exceeds the jth eigenvalue of $J_q(I+J_q^2/4)^{1/2}$ by the factor

$$(F_{qr})_{jj}=(1+(a_j^{(q)})^{2r+2})/(1-(a_j^{(q)})^{2r+2}).$$

We have already shown that this factor approaches one very rapidly with increasing values of j. These matrices therefore differ appreciably only on a low dimension subspace. The performance of the algorithms is in practice almost identical.

7. Numerical experiments with a substructured model problem. We report on experiments with the five point approximation of Poisson's equation on unions of two rectangles, cf. § 4 and Fig. 2. The choice of such regions makes it feasible to conduct many experiments with very many degrees of freedom since fast Poisson solvers can be used to solve the subproblems. We note that the five point finite difference approximation results from the use of a conforming finite element approximation with piecewise linear basis functions on a mesh of right triangles; see e.g. Strang and Fix [33]. Using the results of § 4, we also see that, on these regions, an iteration step of our algorithms can be carried out at a speed which is an order of magnitude faster than a fast Poisson solver with general data. This follows from the fact that the operation $y=Sx$, as well as the preconditioning steps, can be carried out using a few FFTs. The complexity of the whole problem is therefore only about twice that of the two rectangular Poisson subproblems.

We note that in a more general case, the cost of an iteration step will be much more appreciable. The relative cost of using the preconditioner J_q and the solution of the problem on a subregion should also be evaluated in each case. We therefore believe that a detailed comparison of the computer time for the methods tried in our model case experiments is less relevant than the study of the convergence rates themselves.

In the experiments reported here, we consider the union of two rectangles with corners at the points $(0,0),(1,0),(1,1/2),(0,1/2)$ and $(1/8,1/2),(k/8,1/2),(k/8,l/8),(l/8,l/8)$, respectively. In Tables 1-5, we consider $k=3$ and 5, with $l=6$, 8 or 12. The mesh is always uniform in both coordinate directions and the number of points, q, on Γ_3 will therefore determine the discretization uniquely. We have used data which are

consistent with an exact solution $u(x, y) = x^2 + y^2 - xe^x \cos y$. We have found no real difference between the performance of our method for this and other cases.

We first show, in Table 1, how five different methods compare when solving the problem with parameters $k = 5$, $l = 8$ and using $q = 63$ unknowns on the interface Γ_3. We report on the maximum error on the entire region for each method for the iteration at which one of the methods reaches the level of the truncation error. The first method uses $S^{(1)}$ as a preconditioner, the second J while the third method considered solves a normal-equation formulation of the problem, denoted by R_N, allowing the use of $R = J^2$ as a preconditioner. The fourth column in the table displays the results when R is used. This case essentially shows the same behavior as some of the nonoptimal methods analyzed in § 6. The last column shows the very slow convergence when no preconditioning is used.

Table 2 shows how the number of iterations depends on q, using the two optimal preconditioners $S^{(1)}$ and J. The iterations were stopped at the level of the truncation error and the initial guess was the zero function. We note that the overall number of degrees of freedom increases quadratically with q and equals 48,641 for $q = 127$. It is clearly demonstrated that both methods converge at a rate which is independent of the mesh size. The modest increase in the number of iterations reflects the increased accuracy requirement as the truncation error decreases. We employed the same initial guess in all these runs.

TABLE 1
Comparison of 5 different preconditioners.

Iteration	$S^{(1)}$	J	R_N	R	I
0	3.73×10^{-1}	3.73×10^{-1}	3.73×10^{-1}	3.73×10^{-1}	3.73×10^{-1}
4	1.49×10^{-6}	7.82×10^{-5}	3.63×10^{-3}	3.95×10^{-2}	1.55×10^{-1}
6	1.48×10^{-6}	1.52×10^{-6}	2.13×10^{-4}	1.17×10^{-2}	9.60×10^{-2}
10	1.48×10^{-6}	1.48×10^{-6}	1.49×10^{-6}	3.28×10^{-4}	3.78×10^{-2}
14	1.48×10^{-6}	1.48×10^{-6}	1.48×10^{-6}	3.17×10^{-6}	1.85×10^{-2}

TABLE 2
Iterations as a function of mesh size.

	Iterations		Max. error on Ω	
q	$S^{(1)}$	J	$S^{(1)}$	J
3	2	3	3.66×10^{-4}	3.66×10^{-4}
7	3	4	9.59×10^{-5}	9.55×10^{-5}
15	3	5	2.45×10^{-5}	2.43×10^{-5}
31	4	6	6.09×10^{-6}	6.11×10^{-6}
63	4	6	1.49×10^{-6}	1.52×10^{-6}
127	5	7	3.02×10^{-7}	3.08×10^{-7}

Table 3 shows a more detailed comparison between the two optimal methods, giving the maximum error for each method, at every iteration for the case when $q = 127$.

We next give some information on the spectra of the iteration operators $S(S^{(1)})^{-1}$ and SJ^{-1}. In Tables 4 and 5, we show the two smallest, the fifth and the two largest eigenvalues for different values of q and for three different geometries. The upper bounds for the spectra are 2 and $2\sqrt{2}$ in close agreement with the results of § 4. We also note that the eigenvalues of $S(S^{(1)})^{-1}$ lie in a smaller interval than those of SJ^{-1} and, just as important, that they are more clustered. This explains the faster convergence observed when $S^{(1)}$ is used as a preconditioner.

The theoretical results of § 4 do not preclude that there are eigenvalues larger than 2 and $2\sqrt{2}$, respectively. In Tables 6–9, we show that there indeed exist eigenvalues significantly above these bounds if one of the rectangles is very flat, e.g. if $(m+1)/(n+1)$ is large. Tables 6 and 7 display the interval containing the six largest eigenvalues of $S(S^{(1)})^{-1}$ for different aspect ratios. Table 6 shows that even for large aspect ratios for

TABLE 3
A comparison of $S^{(1)}$ and J.

Iterations	$S^{(1)}$	J
0	3.79×10^{-1}	3.79×10^{-1}
1	1.25×10^{-2}	3.22×10^{-2}
2	7.48×10^{-4}	4.01×10^{-3}
3	2.56×10^{-5}	5.26×10^{-4}
4	4.42×10^{-7}	8.74×10^{-5}
5	3.02×10^{-7}	1.05×10^{-5}
6	3.02×10^{-7}	1.33×10^{-6}
7	3.02×10^{-7}	3.08×10^{-7}
8	3.02×10^{-7}	3.03×10^{-7}

TABLE 4
Selected eigenvalues of the iteration operator using $S^{(1)}$.

	$k=5, l=6$		$k=5, l=8$		$k=3, l=12$	
λ	$q=31$	$q=63$	$q=31$	$q=63$	$q=31$	$q=63$
λ_1	1.714	1.684	1.751	1.713	1.712	1.679
λ_2	1.824	1.776	1.826	1.777	1.820	1.772
λ_5	1.994	1.985	1.997	1.992	1.996	1.990
λ_{q-1}	2.000	2.000	2.000	2.000	2.000	2.000
λ_q	2.000	2.000	2.000	2.000	2.000	2.000

TABLE 5
Selected eigenvalues of the iteration operator using J.

	$k=5, l=6$		$k=5, l=8$		$k=3, l=12$	
λ	$q=31$	$q=63$	$q=31$	$q=63$	$q=31$	$q=63$
λ_1	1.825	1.768	1.778	1.733	1.730	1.692
λ_2	1.868	1.806	1.865	1.804	1.859	1.799
λ_5	2.050	2.014	2.046	2.008	2.046	2.008
λ_{q-1}	2.822	2.827	2.822	2.827	2.822	2.827
λ_q	2.827	2.828	2.827	2.828	2.827	2.828

TABLE 6
The interval $[\lambda_{q-5}, \lambda_q]$ containing the six largest eigenvalues as a function of the aspect ratios of the rectangles. Preconditioning with $S^{(1)}$, $q=63$, $m=127$ and $p=32$.

$\frac{q+1}{r+1}$ \ $\frac{m+1}{n+1}$	8	16	32	64
0.5	2.00-2.50	2.02-3.67	2.21-6.16	2.90-11.2
1	2.00-2.50	2.02-3.66	2.21-6.14	2.90-11.2
2	2.00-2.38	2.02-3.45	2.21-5.73	2.90-10.4
4	2.00-2.00	2.02-2.75	2.21-4.38	2.90-7.70
8	2.00-2.00	2.00-2.00	2.19-2.93	2.86-4.82
16	2.00-2.00	2.00-2.00	2.00-2.00	2.57-2.98
32	2.00-2.00	2.00-2.00	2.00-2.00	2.00-2.00

TABLE 7
The interval $[\lambda_{q-5}, \lambda_q]$ containing the six largest eigenvalues as a function of the aspect ratios of the rectangles. Preconditioning with $S^{(1)}$, $q=127$, $m=255$ and $p=64$.

$\frac{q+1}{r+1}$ \ $\frac{m+1}{n+1}$	8	16	32	64
1	2.00-2.50	2.01-3.66	2.21-6.14	2.89-11.2
2	2.00-2.38	2.01-3.45	2.21-5.73	2.89-10.4
4	2.00-2.00	2.01-2.75	2.21-4.38	2.89-7.70
8	2.00-2.00	2.00-2.00	2.18-2.93	2.86-4.82
16	2.00-2.00	2.00-2.00	2.00-2.00	2.56-2.98
32	2.00-2.00	2.00-2.00	2.00-2.00	2.00-2.00
64	2.00-2.00	2.00-2.00	2.00-2.00	2.00-2.00

the rectangle Ω_2, only a relatively few eigenvalues are significantly larger than 2 and that, as predicted by the theory, the geometry of Ω_1 has only a relatively minor influence on the larger eigenvalues. Table 7 shows that this part of the spectrum changes very little when the mesh size is decreased by a factor 2.

In Tables 8 and 9, we give the same data for the operators SJ^{-1}. In agreement with the theory developed in § 4, the largest eigenvalues increase with $(m+1)/(n+1)$ as well as $(q+1)/(r+1)$. There is no significant change when the discretization is refined; cf. Table 9.

TABLE 8

The interval $[\lambda_{q-5}, \lambda_q]$ containing the six largest eigenvalues as a function of the aspect ratio of the rectangles. Preconditioning J, with $q=63$, $m=127$ and $p=32$.

$\frac{q+1}{r+1}$ \ $\frac{m+1}{n+1}$	8	16	32	64
8	2.82-4.18	2.82-5.35	2.83-7.84	2.95-12.9
16	2.83-6.66	2.83-7.83	2.83-10.3	3.15-15.4
32	2.91-11.7	2.93-12.9	3.14-15.4	3.84-20.5

TABLE 9

The interval $[\lambda_{q-5}, \lambda_q]$ containing the six largest eigenvalues as a function of the aspect ratio of the rectangles. Preconditioning J, with $1=127$, $m=255$, and $p=64$.

$\frac{q+1}{r+1}$ \ $\frac{m+1}{n+1}$	8	16	32	64
8	2.83-4.18	2.83-5.35	2.83-7.83	2.92-12.9
16	2.83-6.66	2.83-7.83	2.83-10.3	3.11-15.4
32	2.87-11.7	2.90-12.9	3.10-15.4	3.79-20.4
64	4.48-21.9	4.51-23.1	4.71-25.6	5.40-30.6

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