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# A survey of finite element methods for time-harmonic acoustics

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**Abstract**

Many of the current issues and methodologies related to finite element methods for time-harmonic acoustics are reviewed. The effective treatment of unbounded domains is a major challenge. Most prominent among the approaches that have been developed for this purpose are absorbing boundary conditions, infinite elements, and absorbing layers. Standard computational methods are unable to cope with wave phenomena at short wave lengths due to resolutions required to control dispersion and pollution errors, leading to prohibitive computational demands. Since computation naturally separates the scales of a problem according to the mesh size, multiscale considerations provide a useful framework for viewing these difficulties and developing methods to counter them. Other issues addressed are related to the efficient solution of systems of specialized algebraic equations, and inverse problems of acoustics. The tremendous progress that has been made in all of the above areas in recent years will surely continue, leading to many more exciting developments.
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## 1. Introduction

This paper reviews many of the current issues and methodologies related to finite element methods for time-harmonic acoustics. Computational acoustics has been an area of active research for almost half a century, also related to other fields of application, such as geophysics, meteorology, electromagnetics, etc.

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Numerous recent conference sessions and special journal issues are dedicated to this area [1–5]. Today the field is regarded as one of the most challenging in scientific computation.

The challenge of efficient computation at high wave numbers, in particular, has been designated as one of the problems still unsolved by current numerical techniques [6]. Standard computational methods are unable to cope with wave phenomena at short wave lengths because they require a prohibitive computational effort in order to resolve the waves and control numerical dispersion errors. The failure to adequately represent sub-grid scales misses not only the fine-scale part of the solution, but often causes severe pollution of the solution on the resolved scale as well. This phenomenon is related to the deterioration of numerical stability due to accumulation of dispersion error. Many current discretization techniques are being developed in response to the challenge of controlling such errors effectively.

The second major current difficulty lies in the effective treatment of unbounded domains by standard domain-based methods. Finite element methods cannot directly handle such configurations in an effective way. An artificial boundary that truncates the unbounded domain is used to form a bounded computational domain. Special techniques are then required to reduce spurious reflection of waves that impinge on this artificial boundary. Among the desirable features of such techniques are accuracy and computational efficiency, simplicity of concept and implementation, and robustness and geometric flexibility. Different schemes offer some of these features and lack others. The challenge of developing methods that properly balance the various features is by no means a trivial matter.

The bulk of this paper is devoted to surveying issues and techniques that are associated with these two difficulties. Section 2 describes exterior boundary-value problems, and methods that enable finite element computation in unbounded domains. Finite element discretization methods are presented in Section 3. Other topics that are described more briefly are related to the efficient solution of the systems of algebraic equations that arise in computational acoustics in Section 4 and inverse problems of acoustics, primarily inverse obstacle scattering, in Section 5. The methods that are discussed are those that appear to hold the most prospects. The descriptions are generally concise and accompanied by relevant citations for reference to more lengthy expositions. Somewhat extended presentations are provided for more promising techniques. Naturally, a complete record of all issues, methods, and references in the field cannot be given.

## 2. Exterior problems for time-harmonic acoustics

Let $\mathscr{R} \subset \mathbb{R}^d$ be a $d$-dimensional unbounded region. The boundary of $\mathscr{R}$, denoted by $\Gamma$, is internal and assumed piecewise smooth (Fig. 1, left). The outward unit vector normal to $\Gamma$ is denoted by $\boldsymbol{n}$. We assume that $\Gamma$ admits the partition $\Gamma = \overline{\Gamma_p \cup \Gamma_v}$, where $\Gamma_p \cap \Gamma_v = \emptyset$.

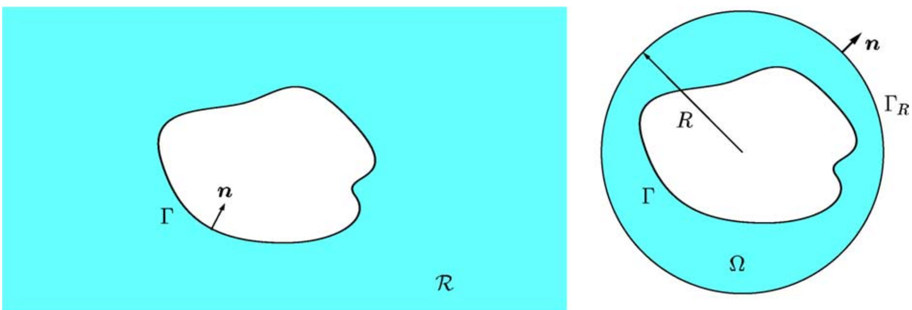


Fig. 1. An unbounded region (left), and a bounded computational domain.

We consider a boundary-value problem related to acoustic radiation and scattering governed by the Helmholtz equation: find $u : \overline{\mathscr{R}} \to \mathbb{C}$, the spatial component of the acoustic pressure or velocity potential, such that

$$\mathscr{L}u = f \quad \text{in } \mathscr{R}, \tag{1}$$

$$u = p \quad \text{on } \Gamma_p, \tag{2}$$

$$\nabla u \cdot \boldsymbol{n} = \mathrm{i}kv \quad \text{on } \Gamma_v, \tag{3}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) = 0. \tag{4}$$

Here, $\mathscr{L}u = -\Delta u - k^2 u$ is the indefinite Helmholtz operator, $\Delta$ is the Laplace operator and $k \in \mathbb{C}$ is the wave number, $\operatorname{Im} k \geqslant 0$ (taken to be positive, real-valued in the following); $\nabla$ is the gradient operator and $\mathrm{i} = \sqrt{-1}$ is the imaginary unit; $r = |\boldsymbol{x}|$ is the distance from the origin; and $f : \mathscr{R} \to \mathbb{C}$, $p : \Gamma_p \to \mathbb{C}$, and $v : \Gamma_v \to \mathbb{C}$ are the prescribed data. Derivation and discussions of the boundary-value problems (1)–(4) are available [7–10].

Eq. (4) is the *Sommerfeld radiation condition* and allows only outgoing waves proportional to $\exp(\mathrm{i}kr)$ at infinity. The radiation condition requires that energy flux at infinity be positive, thereby guaranteeing that the solution to the boundary-value problems (1)–(4) is unique [11,12]. Appropriate representation of this condition is crucial to the reliability of any numerical procedure.

In scattering problems the acoustic field is decomposed into a known incident field $u^{\mathrm{i}}$ and a scattered field $u$, i.e., $u^{\mathrm{i}} + u$. The scattered field $u$ satisfies the boundary-value problems (1)–(4). The homogeneous Dirichlet problem ($\Gamma = \Gamma_p$, $p = 0$) is called acoustically soft, and the Neumann problem ($\Gamma = \Gamma_v$, $v = 0$) is called hard. Extensive results on properties of solutions to the boundary-value problems (1)–(4) with particular reference to scattering are available [7,10–13].

In order to employ domain-based computation (e.g., with the finite element method), the unbounded domain $\mathscr{R}$ is truncated by an artificial boundary $\Gamma_R$ (characterized by curvature $1/R$), yielding a bounded domain $\Omega$ that is suitable for direct discretization (Fig. 1, right). A circular artificial boundary $\Gamma_R$ is shown, although at times other shapes may be preferred. Reducing the size of the bounded domain decreases the computational cost of the domain-based discretization.

A profusion of methodologies are available for completing the definition of the boundary-value problem in $\Omega$ by analyzing the problem in the unbounded complement $\mathscr{R} \setminus \Omega$ (see, e.g., the books [8,9,14] and the review papers [15–19], and references therein). Promising methodologies among the three main approaches for truncating the domain are outlined briefly in the following. Broadly speaking, the perfectly matched layer (PML) described in Section 2.3 offers the simplest implementation, accommodating standard finite element programming techniques and efficient solution algorithms, easily fitting around slender bodies, yet its theoretical framework and optimal setting are still lacking. On the other hand, high-order local boundary conditions with auxiliary variables (Section 2.1) are highly accurate and relatively easy to implement.

### 2.1. Absorbing boundary conditions

In this approach, boundary conditions involving a relation of the unknown solution and its derivatives are specified on $\Gamma_R$, with the goal of eliminating spurious reflection from the boundary. Absorbing boundary conditions are usually either global or local. The accuracy of global 'exact' boundary conditions can be easily increased to a desired level (in theory), but they are often restricted to artificial boundaries with relatively simple shapes. Furthermore, all degrees of freedom on the artificial boundary are coupled, requiring special procedures for implementation and parallelization, as well as usually affecting computational cost and storage. The DtN method [20,21] falls in this category. The number of terms taken in the series representation of the DtN boundary condition determines the accuracy of the procedure, as well as its well-posedness in terms of uniqueness of the solution [22]. The number of terms required to guarantee uniqueness may exceed the number desired for accuracy. The modified DtN operator circumvents this difficulty [23].

Local boundary conditions traditionally have difficulty in attaining high-order accuracy, but are often simple and applicable to more general shapes of the artificial boundary, and usually guarantee uniqueness of the solution. Low-order local boundary conditions are relatively easy to implement and parallelize, and retain the local structure of the computational scheme. High-order terms of the Engquist–Majda [24] and Bayliss–Gunzburger–Turkel [25] sequences of increasing-order local boundary conditions are difficult to implement due to the presence of high-order derivatives, but the second-order schemes are still widely employed [15,26–28].

The development of high-order local boundary conditions for which the order can be easily increased to a desired level is relatively recent (see the review [29] and references therein). This approach, using auxiliary variables to eliminate higher-order derivatives, seems particularly promising.

### 2.2. Infinite elements

Infinite element schemes use complex-valued basis functions with outwardly propagating wave-like behavior to represent the unbounded complement. The development of the method from inception [30] to its state a decade ago is described in the monograph [14], and subsequent development in the reviews [15,31], and references therein.

The two leading approaches that have emerged differ primarily in the treatment of weighting functions. The original infinite element, based on an unconjugated approach, evolved into what is currently termed the 'Burnett' element, noteworthy in the use of basis functions that are separable into radial and transverse parts, based on multipole expansions in an ellipsoidal system [32,33]. In the alternative formulation the weighting functions are conjugated and a geometric weighting factor is currently included [34]. This approach, noted with hindsight to fall within a known variational framework [35], is termed the 'Astley–Leis' element.

The information currently available on the performance of the various schemes is incomplete. Convergence behavior of infinite element formulations is investigated, based on separation of variables in exterior spherical domains [36]. The unconjugated Burnett formulation is considered to provide the best near-field accuracy, whereas the Astley–Leis formulation is the most effective in the far-field [37,38]. Ill-conditioning is a concern, particularly for unconjugated formulations. The use of special radial basis functions improves the conditioning of Astley–Leis elements with spherical artificial boundaries [39]. A recent study shows that the performance of both formulations deteriorates at high frequencies and high aspect ratios of the artificial boundary [40].

### 2.3. Absorbing layers

An absorbing layer of finite thickness replaces the unbounded complement $\mathscr{R} \setminus \Omega$ with properties that cause waves to evanesce, considerably reducing reflection. This is a classical approach that was greatly refined by the perfectly matched layer (PML) formulation [41] developed in the context of finite difference computation of time-dependent electromagnetic waves. By splitting a scalar field, the original PML equations describe decaying waves. Proper selection of the PML coefficients eliminates reflection of plane waves at any angle of incidence at the layer interface.

The PML method rapidly gained immense popularity among practitioners in computational electromagnetics. A large body of literature contains alternative PML formulations and extensions of this concept to additional geometries and applications. PML equations are also derived by the related approaches of a complex coordinate transformation [42] or introducing complex-valued anisotropic material properties [43], instead of field splitting as in the original concept. This latter approach has been extended to time-harmonic acoustics [44], leading to a symmetric formulation that is suitable for finite element computation [45].

While a great deal of numerical experience has accumulated, the understanding of the mathematical framework underlying this methodology is incomplete. On one hand, the PML enforces the Sommerfeld condition and provides a solution that converges to the original solution as the thickness of the layer is increased [46]. On the other hand, there are indications that some stability issues remain unresolved [47,48], although little numerical evidence has been reported. Practical implementation of PML requires setting several numerical parameters, namely the width of the layer and the number of divisions, as well as the variation of the PML coefficients and their maximal values. Optimization of these parameters [49,50] is ongoing.

## 3. Finite element methods

Domain-based methods such as finite elements are suitable for solving interior problems as well as exterior radiation and scattering problems in bounded domains that have been truncated by any of the methodologies outlined in Section 2 (see, e.g., the book [9]). Historically, boundary element schemes based on integral equations [51–53], which do not require special treatment of the unbounded domain, were the preferred computational method in acoustics due to the reduced dimensionality of the domain leading to fewer degrees of freedom. About a decade ago it became apparent that finite elements can be more efficient on large-scale problems because of the structure of their matrices in comparison to the global nature for boundary element discretization [32,54]. While this conclusion becomes less obvious with the recent incorporation of fast multipole methods [55,56], finite element methods retain the advantages of robustness and ease of integration with other discrete models in coupled problems.

For simplicity, consider the (homogeneous Dirichlet) problem

$$\mathscr{L}u = f \quad \text{in } \Omega, \tag{5}$$

$$u = 0 \quad \text{on } \Gamma. \tag{6}$$

Generalization of the following presentation to other types of boundary conditions, including radiation conditions representing unbounded domains, is straightforward.

### 3.1. Continuous Galerkin approximation

Partition $\Omega$ into non-overlapping regions (element domains) in the usual way. The Galerkin approximation is stated in terms of the set of functions $\mathscr{V}^h \subset H_0^1(\Omega)$. The standard finite element method is: find $u^h \in \mathscr{V}^h$ such that

$$a(v^h, u^h) = (v^h, f), \quad \forall v^h \in \mathscr{V}^h. \tag{7}$$

The weak operator is $a(v,u) = (\nabla v, \nabla u) - (v, k^2 u)$ where $(\cdot\,,\cdot)$ is the $L_2(\Omega)$ inner product.

Standard Galerkin finite element solutions with low-order piecewise polynomials differ significantly from the best approximation, due to spurious dispersion in the computation, unless the mesh is sufficiently refined. This phenomenon, related to the indefiniteness of the Helmholtz operator and originally derived by Garding's inequality [57], is known as the pollution effect [58–60]. In practical terms, this leads to an increase in the cost of the finite element solution of the Helmholtz equation at higher wave numbers.

### 3.2. Variational multiscale framework

Numerous approaches to alleviating the above deficiency have been proposed. Several such related methods can be derived by the Variational Multiscale (VMS) approach [61,62]. Higher-order Galerkin [63] and wavelet methods [64] are also used.

By the VMS method we consider an overlapping sum decomposition of the solution. The separation of scales is within the continuum description, in reference to the numerical mesh employed. In finite element computation we have

$$u^h = u^{\mathrm{P}} + u^{\mathrm{E}}. \tag{8}$$

Here, $u^{\mathrm{P}} \in \mathscr{V}^{\mathrm{P}}$ is based on standard, finite element polynomials, representing coarse scales that are resolved by a given mesh, and $u^{\mathrm{E}} \in \mathscr{V}^{\mathrm{E}}$ is an enhancement or enrichment, representing fine or sub-grid scales, satisfying the direct sum relationship

$$\mathscr{V}^h = \mathscr{V}^{\mathrm{P}} \oplus \mathscr{V}^{\mathrm{E}}. \tag{9}$$

Such a decomposition of the solution into a linear part and a bubble was already considered [65]. The determination of the fine scales is key to the multiscale representation. Various implementations approach this issue in different ways (see Section 3.3).

Substituting the overlapping sum (8) (and its weighting function counterpart) into the variational formulation (7), leads to a decomposed form [62]

$$a(v^{\mathrm{P}}, u^{\mathrm{P}}) + (\mathscr{L}^* v^{\mathrm{P}}, u^{\mathrm{E}}) = (v^{\mathrm{P}}, f), \tag{10}$$

$$a(v^{\mathrm{E}}, u^{\mathrm{P}}) + a(v^{\mathrm{E}}, u^{\mathrm{E}}) = (v^{\mathrm{E}}, f). \tag{11}$$

The second term in the left-hand side of Eq. (10) is integrated by parts, leading to an interpretation of $\mathscr{L}^* v^{\mathrm{P}}$ as a Dirac distribution on the entire domain, with integrals over element interiors and jump terms integrated across element boundaries [62].

Eq. (11) provides a formula for the unresolved, fine scales

$$u^{\mathrm{E}} = M^{\mathrm{E}}(\mathscr{L} u^{\mathrm{P}} - f), \tag{12}$$

in terms of the integral, generally nonlocal, operator $M^{\mathrm{E}}$ which depends on the space of fine scales [62]. This formula is substituted into Eq. (10) to eliminate the fine scales

$$a(v^{\mathrm{P}}, u^{\mathrm{P}}) + (\mathscr{L}^* v^{\mathrm{P}}, M^{\mathrm{E}} \mathscr{L} u^{\mathrm{P}}) = (v^{\mathrm{P}}, f) + (\mathscr{L}^* v^{\mathrm{P}}, M^{\mathrm{E}} f). \tag{13}$$

This equation for the coarse scales includes the nonlocal effect of the fine scales. Various approximations arise from different treatments of the fine scales.

### 3.3. Fine scales

A simple approach is to employ a bubble representation of the fine scales (12), thereby localizing the effect of the fine scales. Solving a homogeneous Dirichlet, element-level, problem for the fine scales is the approach of residual-free bubbles (RFB) [66], with the variational equation

$$a(v^{\mathrm{P}}, u^{\mathrm{P}}) + (\mathscr{L}^* v^{\mathrm{P}}, u^{\mathrm{E}})_{\tilde{\Omega}} = (v^{\mathrm{P}}, f) \tag{14}$$

(here $u^{\mathrm{E}}$ is the bubble-based enrichment and $\tilde{\Omega}$ denotes the union of element interiors such that $\overline{\Omega} = \overline{\tilde{\Omega}}$). A related bubble-based method is nearly optimal Petrov–Galerkin [67]. The explicit integration over element interiors supersedes the distributional interpretation in this case. Employing an element Green's function leads to a similar result [61], related to RFB [68]. The deficiency of the loss of global effects inherent in local approaches may be overcome by employing nonconforming methods [69].

Stabilized methods of adjoint type, also called 'unusual stabilized finite element methods' [70], may be derived in the VMS framework as well, and are related to RFB

$$a(v^{\mathrm{P}}, u^{\mathrm{P}}) - (\mathscr{L}^* v^{\mathrm{P}}, \tau \mathscr{L} u^{\mathrm{P}})_{\tilde{\Omega}} = (v^{\mathrm{P}}, f) - (\mathscr{L}^* v^{\mathrm{P}}, \tau f)_{\tilde{\Omega}}. \tag{15}$$

The structure of the second term on the left-hand side of Eq. (13) indicates that the mesh-dependent stability parameter $\tau$ provides an algebraic approximation of the integral operator $M^{\mathrm{E}}$.

In practice, for the self-adjoint Helmholtz operator considered herein, this method is form-identical to the Galerkin/least-squares (GLS) method [71]

$$a(v^{\mathrm{P}}, u^{\mathrm{P}}) + (\mathscr{L} v^{\mathrm{P}}, \tau \mathscr{L} u^{\mathrm{P}})_{\tilde{\Omega}} = (v^{\mathrm{P}}, f) + (\mathscr{L} v^{\mathrm{P}}, \tau f)_{\tilde{\Omega}} \tag{16}$$

(the only difference is in the sign of the stability parameter). Stabilized methods stand out among the numerous improved approaches, by combining substantial improvement in performance with extremely simple implementation. The stability parameter is usually defined by dispersion considerations [71–73], which don't account for unstructured meshes, although improved performance in computation is not limited to structured meshes [72,74]. There is recent progress in the definition of the stability parameter for distorted elements [75]. The VMS distributional interpretation motivated the development of a stabilized method that includes the inter-element jump terms [76], that are usually omitted in the local approach.

The related method of Galerkin-gradient/least-squares (GGLS)

$$a(v^{\mathrm{P}}, u^{\mathrm{P}}) + (\nabla \mathscr{L} v^{\mathrm{P}}, \tau^{\mathrm{G}} \nabla \mathscr{L} u^{\mathrm{P}})_{\tilde{\Omega}} = (v^{\mathrm{P}}, f) + (\nabla \mathscr{L} v^{\mathrm{P}}, \tau^{\mathrm{G}} \nabla f)_{\tilde{\Omega}} \tag{17}$$

was originally developed in order to stabilize problems governed by the modified Helmholtz equation [77], and was later shown to be effective on the Helmholtz equation as well [78]. The GLS and GGLS methods are quite similar for linear finite elements. In fact, both produce *identical* solutions on structured meshes of linear elements (for constant-coefficient Dirichlet problems with uniform source distributions) [78]. Numerical comparisons of the two methods in more elaborate configurations show that their performance is similar [72].

An alternative approach that has appeared predominantly in time-harmonic acoustic applications is to base the fine scales on free-space solutions of the homogeneous differential equation (for example, plane waves in the case of the Helmholtz equation). These functions are often readily available, but typically global and hence require specialized treatment in practice. The generalized finite element method (GFEM) [79] is a recent extension of the partition of unity method (PUM) [80], applied to acoustics [81], in which the free-space homogeneous solutions are multiplied by conventional finite element shape functions. The piecewise polynomial shape functions localize the free-space homogeneous solutions and provide inter-element continuity. In PUM, the product of free-space homogeneous solutions and finite element shape functions constitutes the entire approximation, whereas in GFEM only the fine scales are based on this product, together with conventional finite element functions for the coarse scales, thus alleviating the severe ill-conditioning to which PUM is susceptible.

Similar ideas for incorporating features of the differential equation in the approximation, but in discontinuous frameworks with specialized treatment for inter-element continuity, go back to the weak element method [82], as well as the recent ultra weak variational formulation [83] and least-squares method [84]. As in PUM, the special basis functions in these methods *replace* the standard finite element polynomials.

In the discontinuous enrichment method (DEM), standard finite element polynomials are retained for the coarse scales, and enriched within each element by nonconforming free-space homogeneous solutions representing fine scales, with continuity enforced in the variational formulation [85]. The strategy that underlies DEM is based on the assumption that particular solutions are usually well resolved, and thus may be considered coarse scales. The fine scales should therefore contain solutions of the homogeneous partial differential equation. This interpretation of the fine scales differs somewhat from that of conventional multiscale numerical representations. Weak enforcement of inter-element continuity permits the use of free-space solutions, i.e., $\mathscr{V}^{\mathrm{E}}$ is spanned by solutions of

$$\mathscr{L}u^{\mathrm{E}} = 0 \quad \text{in } \mathbb{R}^d, \tag{18}$$

that are not already represented in the polynomial basis, leading to relative ease of implementation, yet retaining global, fine-scale effects.

The discontinuous Galerkin approximation is stated in terms of the set of functions $\mathscr{V}^h \subset L_2(\Omega) \cap H^1(\tilde{\Omega})$, with Lagrange multiplier approximations $\lambda^h \in \mathscr{W}^h \subset H^{-1/2}(\tilde{\Gamma})$ defined on the union of element interiors $\tilde{\Gamma}$ (and corresponding weights $\mu^h$). The hybrid variational formulation that underlies DEM, may be decomposed as

$$a(v^{\mathrm{P}}, u^{\mathrm{P}}) + a(v^{\mathrm{P}}, u^{\mathrm{E}}) - \langle \lambda^h, v^{\mathrm{P}} \rangle_{\tilde{\Gamma}} = (v^{\mathrm{P}}, f), \tag{19}$$

$$a(v^{\mathrm{E}}, u^{\mathrm{P}}) + a(v^{\mathrm{E}}, u^{\mathrm{E}}) - \langle \lambda^h, v^{\mathrm{E}} \rangle_{\tilde{\Gamma}} = (v^{\mathrm{E}}, f), \tag{20}$$

$$- \langle \mu^h, u^{\mathrm{P}} \rangle - \langle \mu^h, u^{\mathrm{E}} \rangle_{\tilde{\Gamma}} - \langle \lambda^h, v^{\mathrm{E}} \rangle_{\tilde{\Gamma}} = 0. \tag{21}$$

Here, $\langle \cdot , \cdot \rangle$ is the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Allowing for discontinuities, the weak operator in this case is $a(v, u) = (\nabla v, \nabla u)_{\tilde{\Omega}} - (v, k^2 u)$. Element-level basis functions for $u^{\mathrm{E}}$ that satisfy (18) for constant $k$ are plane waves of the form $\exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x})$, where $|\boldsymbol{k}| = k$. For a plane wave propagating in the $\theta$-direction in two dimensions, $\boldsymbol{k}^{\mathrm{T}} = k[\cos\theta, \sin\theta]$. Early numerical testing of DEM exhibits little pollution [85] (the error depends primarily on resolution), perhaps due to the discontinuous nature of the approximation. This striking result could have significant consequences regarding the performance of the method.

### 3.4. Nonoverlapping decompositions

A different perspective to the multiscale problem, still within the VMS framework [61], leads to decomposition of the domain into one part that contains features which are represented well by a global, geometric theory, perhaps short-wavelength asymptotics, often associated with the far field. This part, naturally limited to relatively simple configurations, may include treatment of the radiation condition (4) in unbounded domains [86]. The finite element method is applied in the other part, which contains more elaborate configurations often associated with the near field. Various implementations of such hybrid approaches have been suggested [87,88].

## 4. Algebraic solvers

Discretization of problems for the indefinite Helmholtz operator leads to algebraic equations which are increasingly indefinite as the wave number grows. Nonetheless, for exterior problems with suitable representation of the radiation condition (4), the symmetric, complex-valued (nonHermitian) coefficient matrix is nonsingular. Direct solvers may be employed in straightforward fashion, although the use of global absorbing boundary conditions usually increases the bandwidth.

Larger systems of equations are usually solved with iterative methods (see the review [89] and the recent monograph [90]). The structure of the coefficient matrix necessitates the use of more general Krylov space approaches such as BiCG-Stab [91], GMRES [92], and QMR [93], rather than the common conjugate gradient method. Specialized techniques overcome the difficulties presented by higher-order absorbing boundary conditions [94–96]. Several approaches are available for the efficient computation of the response in a range of frequencies [97–99].

Preconditioning is essential for efficient performance of iterative solvers. Either general-purpose preconditioners [90], such as approximate inverses and incomplete factorizations (e.g., ILU [100]), or preconditioners that are related to the Helmholtz operator [101,102] may be used.

Standard domain decomposition methods are not well suited for problems of acoustics. The application of the widely employed nonoverlapping domain decomposition method based on Lagrange multipliers to the Helmholtz equation (FETI-H) [103], employs local transmission conditions to guarantee uniqueness on interior sub-domains (see also related approaches [104,105]).

Resolution requirements pose a particular challenge to the multigrid approach, exacerbated by the pollution effect [106]. Multigrid smoothers are frequently local and hence also susceptible to difficulties associated with loss of uniqueness. Some of these problems may be circumvented by using ideas from geometric optics [107].

## 5. Inverse acoustic problems

Inverse problems arise in many fields of science and engineering [108]. Direct inverse problems often involve determining the shape of a body from acoustic patterns [109]. Computational methods address this nonlinear [110] and ill-posed [111] problem either by Newton-like approaches [112,113], or explicitly as a nonlinear minimization problem [114,115], both requiring iterative solution strategies. Traditionally the latter approach has been more successful for three-dimensional configurations since it avoids the solution of the corresponding direct problem and stabilizes the ill-posed inverse problem [116].

Recent techniques, such as the linear sampling method [7], avoid repeated solution of direct problems, yet involve smaller-scale optimization problems by requiring measurements for a large number of incident waves. However, since this approach seems to perform poorly on nonconvex bodies, its best use may be to provide a good initial guess of the shape. A related method represents the shape of the body as a superposition of a smooth deformation on an underlying geometry for which the direct problem can be solved analytically [117]. The range of applications is thus restricted by the smoothness and magnitude of the required deformation.

While numerical results often indicate that Newton-like methods produce more accurate solutions, their extension to three-dimensional configurations had been held up by practical considerations such as the necessity for efficient solution of large-scale direct problems. This difficulty has been overcome by a promising computational methodology that incorporates exact sensitivities of far-field patterns with a domain decomposition method for the fast solution of direct acoustic problems [118].

## 6. Conclusions

The study of waves in the effort to predict acoustic radiation and scattering by submerged objects has driven the development of finite element methods for wave phenomena that appear naturally in many areas of engineering and physics. The main difficulty in solving exterior problems arises from the unboundedness of the domain, which cannot be discretized completely with standard finite elements based on polynomial shape functions. Most prominent among the approaches that have been developed are absorbing boundary conditions, infinite elements, and absorbing layers. The development of efficient discretization schemes is another important topic due to the numerical difficulties that arise in the solution of wave problems, particularly at high wave numbers. Since computation naturally separates the scales of a problem according to the mesh size, multiscale considerations provide a useful framework for viewing these difficulties and developing methods to counter them. Other challenges that arise in this field are related to the efficient solution of systems of specialized algebraic equations, and inverse problems of acoustics.

In recent years, tremendous progress has been made in all of the above areas. The interest in these topics and the progress that has been made are readily confirmed by the number of published papers, workshops and conference sessions dedicated to such research. The diversity of these contributions demonstrates both the breadth of the numerical methodology which is now applied to acoustic problems, and the many possibilities that exist for future research in this area. This interest will surely increase, leading to many more exciting developments.

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