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On the convergence of a dual-primal substructuring method*

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Summary. In the Dual-Primal FETI method, introduced by Farhat et al. [5], the domain is decomposed into non-overlapping subdomains, but the degrees of freedom on crosspoints remain common to all subdomains adjacent to the crosspoint. The continuity of the remaining degrees of freedom on subdomain interfaces is enforced by Lagrange multipliers and all degrees of freedom are eliminated. The resulting dual problem is solved by preconditioned conjugate gradients. We give an algebraic bound on the condition number, assuming only a single inequality in discrete norms, and use the algebraic bound to show that the condition number is bounded by $C(1+\log^2(H/h))$ for both second and fourth order elliptic selfadjoint problems discretized by conforming finite elements, as well as for a wide class of finite elements for the Reissner-Mindlin plate model.

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1. Introduction

This article is concerned with convergence bounds for an iterative method for the parallel solution of symmetric, positive definite systems of linear equations that arise from elliptic boundary value problems discretized by finite elements. The original Finite Element Tearing and Interconnecting

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method (FETI) was proposed by Farhat and Roux [9]. The FETI method consists of decomposing the domain into non-overlapping subdomains, enforcing that the corresponding degrees of freedom on subdomain interfaces coincide by using Lagrange multipliers, and eliminating all degrees of freedom, leaving a dual system for the Lagrange multipliers. The dual system is solved by preconditioned conjugate gradients with a diagonal preconditioner. Evaluation of the dual operator involves the solution of independent Neumann problems in all subdomains, and of a small system of equations for the nullspace component.

Farhat, Mandel, and Roux [8] recognized that this system for the nullspace components plays the role of a coarse problem that facilitates global exchange of information between the subdomains, causing the condition to be bounded as the number of subdomains increases. They also replaced the diagonal preconditioner by a block preconditioner with the solution of independent Dirichlet problems in each subdomain and observed numerically that this Dirichlet preconditioner results in a very slow growth of the condition number with subdomain size. Mandel and Tezaur [14] proved that the condition number grows at most as $C(1 + \log(H/h))^3$, where H is subdomain size and h is element size, both in 2D and 3D. Tezaur [20] proved that a method by Park, Justino, and Felippa [17] is equivalent to the method of [9] with a special choice of the constraint matrices, and proved a $C(1+\log(H/h))^2$ bound for this variant. For further comparison, see Rixen et al. [19].

Klawonn and Widlund [10] have used preconditioned conjugate residuals to solve a saddle problem keeping both the original degrees of freedom and the Lagrange multipliers, and obtained the asymptotic bound $C(1 + \log(H/h))^2$ using an extension of the theory of [14]. The saddle point approach has the advantage that approximate solvers can be used for both the Neumann and the Dirichlet subdomain problems, at a cost of solving a larger indefinite problem instead of a small positive definite problem. In [11], Klawonn and Widlund have proposed new preconditioners, proved uniform bounds for modifications of the method for the case of coefficient jumps, which include an earlier algorithm of Rixen and Farhat [18], and provided further theoretical insights.

The original method of [9, 8] does not converge well for plate and shell problems, and the existence and the form of the coarse space depend on the singularity of the subdomain matrices. Therefore, Mandel, Tezaur, and Farhat [15] and Farhat, Mandel, and Chen [7] proposed to project the Lagrange multipliers in each iteration onto an auxiliary space. With the auxiliary space chosen so that the corresponding primal solutions are continuous at the crosspoints, it is possible to prove that the condition number does not grow faster than $C(1 + \log(H/h))^3$ for plate problems [15], and fast convergence was observed for plate [7] as well as shell problems [4]. This method is now called FETI-2. For related results for symmetric positive definite problems, see [6,19] and references therein.

The subject of this paper is the Dual-Primal FETI method (FETI-DP), introduced by Farhat et al. [5]. This method enforces the continuity of the primal solution at the crosspoints directly in the formulation of the dual problem: the degrees of freedom at a crosspoint remain common to all subdomains sharing the crosspoint and the continuity of the remaining degrees of freedom on the interfaces in enforced by Lagrange multipliers. The degrees of freedom are then eliminated and the resulting dual problem for the Lagrange multipliers is solved by preconditioned conjugate gradients with a Dirichlet preconditioner. Evaluating the dual operator involves the solution of independent subdomain problems with nonsingular matrices and of a coarse problem based on the subdomain corners. The advantage of this method is a simpler formulation than those of [15,7]; there is also no need to solve problems with singular matrices, and the method has been observed to be significantly faster in practice for 2D problems. However, the design of a good 3D variant of the method is an open problem [5].

In this paper, we prove that the condition number of the FETI-DP method with the Dirichlet preconditioner does not grow faster than $C(1 + \log(H/h))^2$ for both second order and fourth order problems in 2D. By spectral equivalence, the result for fourth order problems extends to a large class of Reissner-Mindlin elements for plate bending as in [12,13,15]. After few initial definitions, which are substantially different, the analysis is related to that in [14,15]. Just as the formulation of the present method is simpler, the analysis is simpler and more elegant than in [14,15].

The paper is organized as follows. Notations and assumptions are introduced in Sect. 2. In Sect. 3, we review the algorithm from [5]. Section 4 gives algebraic condition number estimates; these estimates apply to partitioning of any symmetric positive definite system, not necessarily originating from partial differential equations. Finally, in Sect. 5, we prove polylogarithmic condition number bounds for two model problems of second and fourth order.

2. Domain partitioning, notations, and assumptions

We are concerned with iterative solution of symmetric, positive definite linear algebraic systems. Partitioning of the system is motivated as follows. Let Ω be a domain in $\Re^2$ decomposed into N_s non-overlapping subdomains $\Omega^1, \Omega^2, \ldots, \Omega^{N_s}$, and where each of the subdomains be a union of some of the elements. Let u^s be the vector of degrees of freedom for the subdomain Ω^s corresponding to a conforming finite element discretization of a second order elliptic problem or a fourth order plate bending problem defined on Ω. Let K^s and f^s be the local stiffness matrix and the load vector associated with the subdomain Ω^s. We denote the edges of the subdomains by $\Gamma^{st} = \partial\Omega^s \cap \partial\Omega^t$. Corners are endpoints of edges.

The subdomain vectors are partitioned as

$$u^s = \begin{bmatrix} u_i^s \\ u_r^s \\ u_c^s \end{bmatrix}, \tag{2.1}$$

where u_i^s are the values of the degrees of freedom in the subdomain interior, u_c^s the values of the degrees of freedom at the corners of the subdomain, and u_r^s are the remaining values of the degrees of freedom, i.e. those located on the edges of the subdomains between the corners. The subdomain matrices are partitioned accordingly,

$$K^s = \begin{bmatrix} K_{ii}^s & K_{ir}^s & K_{ic}^s \\ K_{ri}^s & K_{rr}^s & K_{rc}^s \\ K_{ci}^s & K_{cr}^s & K_{cc}^s \end{bmatrix}. \tag{2.2}$$

We use the block notation

$$u = \begin{bmatrix} u^1 \\ \vdots \\ u^{N_s} \end{bmatrix} \quad \text{and} \quad K = \operatorname{diag}(K^s) = \begin{bmatrix} K^1 & & \\ & \ddots & \\ & & K^{N_s} \end{bmatrix}.$$

The block vectors u_i, u_c and u_r of all internal, all corner, and all remaining degrees of freedom, respectively, are then defined similarly,

$$u_i = \begin{bmatrix} u_i^1 \\ \vdots \\ u_i^{N_s} \end{bmatrix}, \quad u_c = \begin{bmatrix} u_c^1 \\ \vdots \\ u_c^{N_s} \end{bmatrix}, \text{ and } u_r = \begin{bmatrix} u_r^1 \\ \vdots \\ u_r^{N_s} \end{bmatrix}.$$

Vectors of values of degrees of freedom on the whole $\partial\Omega^s$ and the corresponding block vectors will be written as

$$v_{r,c}^s = \begin{bmatrix} v_r^s \\ v_c^s \end{bmatrix}, \quad v_{r,c} = \begin{bmatrix} v_{r,c}^1 \\ \vdots \\ v_{r,c}^{N_s} \end{bmatrix}.$$

Let the global to local map be a $0-1$ block matrix

$$L = \begin{bmatrix} L^1 \\ \vdots \\ L^{N_s} \end{bmatrix}.$$

For a global vector of degrees of freedom u^g, the vector of degrees of freedom correspoding to Ω^s is $L^s u^g$. The problem to be solved is then given by subassembly,

$$L^{\mathrm{T}} K L u^g = L^{\mathrm{T}} f. \tag{2.3}$$

Note that $\mathrm{Im} L$ is the space of all vectors u that are continuous across the subdomain interfaces and that L is of full column rank.

We assume that each matrix K^s is symmetric positive semidefinite and that K^s is positive definite on the subspace of vectors that are zero at the subdomain corners, $\{u^s | u_c^s = 0\}$. We also assume that the global stiffness matrix $L^{\mathrm{T}} K L$ is positive definite, or, equivalently, that K is positive definite on $\mathrm{Im} L$. These assumptions are satisfied in the intended finite element applications.

3. Formulation of the algorithm

In this section, we review the algorithm proposed in [5] in a form suitable for our purposes.

The degrees of freedom from both sides of each edge Γ^{st} should coincide,

$$u_r^s|_{\Gamma^{st}} - u_r^t|_{\Gamma^{st}} = 0. \tag{3.1}$$

In (3.1), each pair of subdomains $\{s, t\}$ is taken only once, with the order (s, t) chosen arbitrarily. We write the constraints (3.1) as

$$B_r u_r = 0, \quad B_r = [B_r^1, \ldots, B_r^{N_s}].$$

Note that it follows immediately from the definition of B_r that

$$B_r B_r^{\mathrm{T}} = 2I \tag{3.2}$$

and, for any edge Γ^{st},

$$(B_r^{\mathrm{T}} B_r u_r)^s|_{\Gamma^{st}} = \pm(u_r^s|_{\Gamma^{st}} - u_r^t|_{\Gamma^{st}}). \tag{3.3}$$

Let L_c be a matrix with $0, 1$ entries implementing the global-to-local map at the subdomain corners. That is, the equation

$$u_c = L_c u_c^g, \quad L_c = \begin{bmatrix} L_c^1 \\ \vdots \\ L_c^{N_s} \end{bmatrix},$$

determines the common values of the degrees of freedom at the subdomain corners from a global vector u_c^g.

From the construction, the space of all vectors of degrees of freedom continuous across the interfaces can be written as

$$\mathrm{Im}L = \{u | B_r u_r = 0, u_c \in \mathrm{Im}L_c\}.$$

The problem (2.3) is reformulated as the equivalent constrained minimization problem

$$\begin{array}{l}\frac{1}{2}u^{\mathrm{T}}Ku - u^{\mathrm{T}}f \to \min, \\ \text{subject to } B_r u_r = 0 \text{ and } u_c = L_c u_c^g \text{ for some } u_c^g,\end{array}$$

which is in turn equivalent to finding the stationary point of the Lagrangean

$$\mathcal{L}(u_i, u_r, u_c^g, \lambda) = \frac{1}{2}v^{\mathrm{T}}Kv - v^{\mathrm{T}}f + u_r^{\mathrm{T}}B_r^{\mathrm{T}}\lambda,\ v = [v^s],\ v^s = \begin{bmatrix} u_i^s \\ u_r^s \\ L_c^s u_c^g \end{bmatrix}.$$

Eliminating u_i^s, u_r^s, and u_c^g from the Euler equations, we obtain a dual system of the form, cf. [5, Eq. (14)],

$$F\lambda = g, \tag{3.4}$$

and solve it using the preconditioned conjugate gradients method with the preconditioner

$$M = B_r S_{rr} B_r^{\mathrm{T}}, \tag{3.5}$$

where

$$S_{rr} = \mathrm{diag}(S_{rr}^s), \quad S_{rr}^s = K_{rr}^s - K_{ri}^s K_{ii}^{s\,-1} K_{ir}^s. \tag{3.6}$$

For details of the implementation and numerical results, see [5].

4. Algebraic bounds

In this section, we prove bounds on the condition number of the iterative method defined by Eqs. (3.4) and (3.5). Denote $\|u\| = \sqrt{u^{\mathrm{T}}u}$ and, for a symmetric positive semidefinite matrix A, denote the induced matrix seminorm $|u|_A = \sqrt{u^{\mathrm{T}}Au} = \|A^{1/2}u\|$. If A is known to be positive definite, we write $\|u\|_A$ instead or $|u|_A$, because the seminorm is then known to be a norm.

From the minimization property of the Schur complement, we immediately obtain the following lemma, which characterizes the bilinear form associated with the preconditioner.

Lemma 4.1. *The matrix S_{rr} satisfies* $u_r^{\mathrm{T}}S_{rr}u_r = \min\{v^{\mathrm{T}}Kv | v_r = u_r, v_c = 0\}$.

The next lemma gives a more specific description of the matrix of the dual equation (3.4).

Lemma 4.2. *The system matrix from (3.4) is given by* $F = B_r\tilde{S}^{-1}B_r^{\mathrm{T}}$*, where the positive definite matrix* $\tilde{S}$ *is defined by*

$$u_r^{\mathrm{T}}\tilde{S}u_r = \min\{v^{\mathrm{T}}Kv | v_r = u_r, v_c \in \mathrm{Im}L_c\}. \tag{4.1}$$

Proof. Let

$$\tilde{\mathcal{L}}(u_r, \lambda) = \min_{u_i, u_c^g} \mathcal{L}(u_i, u_r, u_c^g, \lambda)$$

Then,

$$\tilde{\mathcal{L}}(u_r, \lambda) = \frac{1}{2}u_r^{\mathrm{T}}\tilde{S}u_r + u_r^{\mathrm{T}}B_r^{\mathrm{T}}\lambda - u_r^{\mathrm{T}}h_r, \tag{4.2}$$

with some h_r. Minimizing over u_r, we get $u_r = \tilde{S}^{-1}(h_r - B_r^{\mathrm{T}}\lambda)$. Substituting u_r into (4.2) and varying λ gives (3.4) with $F = B_r\tilde{S}^{-1}B_r^{\mathrm{T}}$.

We can now characterize the norm induced by the dual matrix F.

Lemma 4.3. *The matrix* F *satisfies the property*

$$\lambda^{\mathrm{T}}F\lambda = \max_{v_r \neq 0} \frac{|v_r^{\mathrm{T}}B_r^{\mathrm{T}}\lambda|^2}{\|v_r\|_{\tilde{S}}^2}. \tag{4.3}$$

Proof. From Lemma 4.2,

$$\lambda^{\mathrm{T}}F\lambda = \|\tilde{S}^{-1/2}B_r^{\mathrm{T}}\lambda\|^2 = \max_{w_r \neq 0} \frac{|w_r^{\mathrm{T}}\tilde{S}^{-1/2}B_r^{\mathrm{T}}\lambda|^2}{\|w_r\|^2}.$$

The substitution $w_r = \tilde{S}^{1/2}v_r$ yields (4.3).

Let $V = \Re^p$ be the space of Lagrange multipliers. In this space, define the norm

$$\|\mu\|_V = \|B_r^{\mathrm{T}}\mu\|_{S_{rr}}$$

and also the dual norm,

$$\|\lambda\|_{V'} = \max_{\mu \neq 0} \frac{|\mu^{\mathrm{T}}\lambda|}{\|\mu\|_V}. \tag{4.4}$$

Since $V = \mathrm{Im}B_r$, substituting $\mu = B_r w_r$, we can rewrite (4.4) as

$$\|\lambda\|_{V'} = \max_{B_r w_r \neq 0} \frac{|w_r^{\mathrm{T}}B_r^{\mathrm{T}}\lambda|}{\|B_r^{\mathrm{T}}B_r w_r\|_{S_{rr}}}. \tag{4.5}$$

The main result of this section is the following theorem, which gives a bound on the minimal and maximal eigenvalues of the preconditioned operator MF.

Theorem 4.4. *If there exists a constant c_1 such that for all w_r,*

$$\|B_r^{\mathrm{T}} B_r w_r\|_{S_{rr}}^2 \leq c_1 \|w_r\|_{\tilde{S}}^2,$$

then

$$\frac{\lambda_{max}(MF)}{\lambda_{min}(MF)} \leq \frac{c_1}{4}.$$

Proof. The proof is based on a comparison of (4.5) and (4.3). Using Lemma 4.3, the substitution $v_r = B_r^{\mathrm{T}} B_r w_r$, and the property (3.2), we find that

$$\begin{aligned}\lambda^{\mathrm{T}} F \lambda &= \max_{v_r \neq 0} \frac{|v_r^{\mathrm{T}} B_r^{\mathrm{T}} \lambda|^2}{\|v_r\|_{\tilde{S}}^2} \\ &\geq \max_{0 \neq v_r \in \mathrm{Im} B_r^{\mathrm{T}} B_r} \frac{|v_r^{\mathrm{T}} B_r^{\mathrm{T}} \lambda|^2}{\|v_r\|_{\tilde{S}}^2} = 4 \max_{B_r w_r \neq 0} \frac{|w_r^{\mathrm{T}} B_r^{\mathrm{T}} \lambda|^2}{\|B_r^{\mathrm{T}} B_r w_r\|_{\tilde{S}}^2}.\end{aligned}$$

Since, by definition, $\|B_r^{\mathrm{T}} B_r w_r\|_{\tilde{S}}^2 \leq \|B_r^{\mathrm{T}} B_r w_r\|_{S_{rr}}^2$, we conclude, comparing with (4.5), that

$$\lambda^{\mathrm{T}} F \lambda \geq 4 \|\lambda\|_{V'}^2.$$

On the other hand, from the assumption, we obtain

$$\lambda^{\mathrm{T}} F \lambda = \max_{w_r \neq 0} \frac{|w_r^{\mathrm{T}} B_r^{\mathrm{T}} \lambda|^2}{\|w_r\|_{\tilde{S}}^2} \leq c_1 \max_{B_r w_r \neq 0} \frac{|w_r^{\mathrm{T}} B_r^{\mathrm{T}} \lambda|}{\|B_r^{\mathrm{T}} B_r w_r\|_{S_{rr}}} = c_1 \|\lambda\|_{V'}^2.$$

We can conclude that

$$c_2 \|\lambda\|_{V'}^2 \leq \lambda^{\mathrm{T}} F \lambda \leq c_1 \|\lambda\|_{V'}^2, \qquad c_2 = 4. \tag{4.6}$$

Trivially from the definition of M and the norm in V, we have $\|\mu\|_V^2 = \mu^{\mathrm{T}} M \mu$, hence

$$c_4 \|\mu\|_V^2 \leq \mu^{\mathrm{T}} M \mu \leq c_3 \|\mu\|_V^2, \qquad c_3 = c_4 = 1. \tag{4.7}$$

By [14, Lemma 3.1], it follows from the inequalities (4.6) and (4.7) that

$$\frac{\lambda_{max}(MF)}{\lambda_{min}(MF)} \leq \frac{c_1 c_3}{c_2 c_4} = \frac{c_1}{4}.$$

We now show how to verify the assumption of Theorem 4.4 from inequalities which are of a form more common in iterative substructuring studies and which are easier to obtain for boundary value problems.

Denote by $E^{s,t}$ the operator that extends the vector of values of degrees of freedom on Γ^{st}, excluding the corners, by zero entries to a vector of values of degrees of freedom on all of $\partial\Omega^s$, and let $\mathcal{E}^s$ be the set of all indices of neighbors Ω^t of the domain Ω^s, with a common edge Γ^{st}. Denote by S^s

the Schur complement on $\partial\Omega^s$ obtained by eliminating the interior degrees of freedom of Ω^s, i.e.,

$$u^{s\mathrm{T}} S^s u^s = \min\{v^{s\mathrm{T}} K^s v^s | v^s_{r,c} = u^s_{r,c}\}.$$

Our estimate is based on an a-priori bound of the error of approximating a vector of interface degrees of freedom that is continuous at the corners by a vector that is also continuous across the edges. In the applications in Sect. 5, the approximating vector will be chosen as the natural interpolation from the corners to the edges.

Theorem 4.5. *Suppose there is a constant c_2 such that for every $w_{r,c}$, such that $w_c \in \mathrm{Im} L_c$, there exists a vector u_r such that $B_r u_r = 0$ and, for all s and all $t \in \mathcal{E}^s$,*

$$|E^{s,t}(w^i_r - u^i_r)|^2_{S^s} \leq c_2 |w^i_{r,c}|^2_{S^i}, \quad i = s, t. \tag{4.8}$$

Then,

$$\frac{\lambda_{max}(MF)}{\lambda_{min}(MF)} \leq c_2 n_e,$$

where n_e is the maximum number of the edges of any subdomain.

Proof. Let w_r be given and define w_c as the optimal corner degrees of freedom given by the definition of $\tilde{S}$, cf. (4.1). Then

$$\|w_r\|^2_{\tilde{S}} = \sum_{s=1}^{N_s} |w^s_{r,c}|^2_{S^s}. \tag{4.9}$$

Let u^s_r be as in the assumption of the theorem. Then $B_r u_r = 0$, and, consequently,

$$B_r^{\mathrm{T}} B_r w_r = B_r^{\mathrm{T}} B_r (w_r - u_r).$$

Extending $B_r^{\mathrm{T}} B_r w_r$ by zero values at all corner degrees of freedom, we get using the definition of S_{rr}, cf., (3.6), that

$$\|B_r^{\mathrm{T}} B_r w_r\|^2_{S_{rr}} = \sum_{s=1}^{N_s} |v^s_{r,c}|^2_{S^s}, \tag{4.10}$$

where

$$v^s_r = \left(B_r^{\mathrm{T}} B_r (w_r - u_r)\right)^s, \quad v^s_c = 0.$$

Using the definition of $E^{s,t}$, we have

$$v^s_{r,c} = \sum_{t \in \mathcal{E}^s} E^{s,t} v^s_r.$$

Hence, from the triangle inequality, and then using the property (3.3) of B_r and the triangle inequality again, it follows that

$$\begin{aligned} |v^s_{r,c}|_{S^s} &\le \sum_{t\in\mathcal{E}^s} |E^{s,t} v^s_{r,c}|_{S^s} \\ &\le \sum_{t\in\mathcal{E}^s} \left(|E^{s,t}(w^s_r - u^s_r)|_{S^s} + |E^{s,t}(w^t_r - u^t_r)|_{S^s}\right). \end{aligned}$$

Squaring the last inequality, using the inequality $(a+b)^2 \le 2(a^2+b^2)$, and the a-priori bound (4.8) yields

$$|v^s_{r,c}|^2_{S^s} \le 2c_2 \sum_{t\in\mathcal{E}^s} \left(|w^s_{r,c}|^2_{S^s} + |w^t_{r,c}|^2_{S^t}\right).$$

By summation over the subdomains and using (4.10) and (4.9), we can conclude that

$$\|B_r^{\mathrm{T}} B_r w_r\|^2_{S_{rr}} = \sum_{s=1}^{N_s} |v^s_{r,c}|^2_{S^s} \le 4c_2 n_e \sum_{s=1}^{N_s} |w^s_{r,c}|^2_{S^s} = 4c_2 n_e \|w_r\|^2_{\tilde{S}}.$$

Finally, we use Theorem 4.4.

5. Applications

In this section, we verify the assumption of Theorem 4.5 for two model problems. The Sobolev seminorm, denoted by $|u|_{m,p,X}$, is the $L^p(X)$ norm of the generalized derivatives of order $m \ge 0$ of the function u. The Sobolev norm is then defined by $\|u\|_{m,p,X} = \|\,[|u|_{0,p,X}, \ldots, |u|_{m,p,X}]\,\|_{\ell^p}$. Sobolev norms for noninteger m are defined by interpolation; cf., e.g., [2,16] for details and references. We will be using norms with $p=2$ and $p=+\infty$. In particular, $|u|_{0,p,X} = \|u\|_{0,p,X} = \|u\|_{L^p(X)}$, and, on the boundary Γ of a domain in $\Re^2$,

$$|u|^2_{\frac{1}{2},2,\Gamma} = \int_\Gamma \int_\Gamma \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy.$$

5.1. A second order elliptic problem

Consider the boundary value problem

$$\mathcal{A}u = g \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{5.1}$$

where

$$\mathcal{A}v = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(\alpha(x)\frac{\partial v(x)}{\partial x_j}\right),$$

with $\alpha(x)$ a measurable function such that $0 < \alpha_0 \leq \alpha(x) \leq \alpha_1$ a.e. in Ω.

We assume that the model problem (5.1) is discretized using conforming P1 or Q1 elements and denote by $V_h^{P1}(\Omega)$ the corresponding finite element space that satisfies the usual regularity and inverse properties [2]. Let h denote the characteristic element size. We assume that all functions in $V_h^{P1}(\Omega)$ vanish on the boundary of Ω and also assume that each subdomain is the union of some of the elements. Denote the space of restrictions of functions from $V_h^{P1}(\Omega)$ to the subdomains by $V_h^{P1}(\Omega^s)$. For every vector of degrees of freedom u^s, denote by $I_{P1}u^s$ the corresponding finite element function. The trace of this function is determined by degrees of freedom on the boundary only and we often abuse the notation to define the trace from the boundary values only.

For simplicity, we assume that the subdomains Ω^s, $s = 1, ..., N_s$ form a quasi-regular triangulation of the domain Ω. Denote the characteristic size of the subdomains by H. Finally, let C denote a generic constant independent of h and H. The symbol $\approx$ denotes equivalence of norms or seminorms, with constants independent of h and H.

We start by writing a well known estimate by Bramble, Pasciak, and Schatz [1, Lemma 3.5] in a form suitable for our purposes; cf., also Widlund [21, Lemma 3.3].

Lemma 5.1. *Let $w \in V_h^{P1}(\Omega^s)$ such that $w = 0$ at the corners of Ω^s, let $w_L \in V_h^{P1}(\Omega^s)$ be linear on all edges $\Gamma^{st} \subset \Omega^s$, and, for each $t \in \mathcal{E}^s$, let w^{st} be defined by $w^{st} = w$ on Γ^{st} and $w^{st} = 0$ on $\partial\Omega^s \setminus \Gamma^{st}$. Then,*

$$\sum_{t \in \mathcal{E}^s} |w^{st}|^2_{\frac{1}{2},2,\partial\Omega^s} \leq C \left(1 + \log \frac{H}{h}\right)^2 |w + w_L|^2_{\frac{1}{2},2,\partial\Omega^s}.$$

We are now ready for the main result.

Theorem 5.2. *For the model second order problem, it holds that*

$$\frac{\lambda_{max}(MF)}{\lambda_{min}(MF)} \leq C \left(1 + \log \frac{H}{h}\right)^2,$$

where the C constant does not depend on H and h.

Proof. We only need to verify the assumptions of Theorem 4.5. For a given $w_{r,c}$, define $u_{r,c}$ by linear interpolation from w_c, i.e., $I_{P1}u^s_{r,c}$ is linear on all edges of Ω^s, and $u_c = w_c$. Writing $w_{r,c} = (w_{r,c} - u_{r,c}) + u_{r,c}$, it follows from Lemma 5.1 that

$$|I_{P1}E^{s,t}(w^s_r - u^s_r)|^2_{\frac{1}{2},2,\partial\Omega^s} \leq C \left(1 + \log \frac{H}{h}\right)^2 |I_{P1}w^s_{r,c}|^2_{\frac{1}{2},2,\partial\Omega^s}.$$

Using the uniform equivalence of the seminorms $|I_{P1}v^s_{r,c}|_{\frac{1}{2},2,\partial\Omega^s} \approx |v^s_{r,c}|_{S^s}$, cf., e.g., [1], and the uniform equivalence of the seminorms,

$$|v|_{\frac{1}{2},2,\partial\Omega^s} \approx |v|_{\frac{1}{2},2,\partial\Omega^t} \text{ if } v = 0 \text{ on } \partial\Omega^s \cup \partial\Omega^t \setminus \Gamma^{st}, \tag{5.2}$$

we obtain (4.8) with $c_2 = C\left(1 + \log\frac{H}{h}\right)^2$. The bound (5.2) follows from the fact that the subdomains are shape regular.

5.2. A fourth order problem

Consider a biharmonic boundary value problem in a variational form: Find $u \in H^2_0(\Omega)$ such that

$$a(u,v) = f(v), \quad \forall v \in H^2_0(\Omega), \tag{5.3}$$

where

$$a(u,v) = \int_\Omega \partial_{11}u\partial_{11}v + 2\partial_{12}u\partial_{12}v + \partial_{22}u\partial_{22}v, \qquad \forall u,v \in H^2_0(\Omega).$$

Let the model problem (5.3) be discretized by reduced HCT elements [2]. We use the same assumptions on the decomposition as in Sect. 5.1; in particular, the subdomain edges are straight. Let $V_h^{HCT}(\Omega)$ be the finite element space of reduced HCT elements satisfying the usual regularity and inverse properties and the essential boundary conditions. Note that $V_h^{HCT}(\Omega) \subset C^1(\Omega) \cap H^2_0(\Omega)$. On each element, a function v in $V_h^{HCT}(\Omega)$ is determined by the values $v(a_k)$ and the values of its derivatives $\frac{\partial v}{\partial x_j}(a_k)$, $j = 1,2$, at the vertices a_k of the element. Denote by t, n, ∂_t, and ∂_n the tangential and normal directions, and the tangential and normal derivative, respectively. The traces of functions from $V_h^{HCT}(\Omega)$ on $\partial\Omega$ are pairs of functions $(u, \nabla v)$ such that u is piecewise cubic, $n \cdot \nabla v$ is piecewise linear, and u and ∇u are consistent, i.e., $\partial_t u = t \cdot \nabla v$. We will abuse notation and write the trace functions simply as $(u, \nabla u)$ or just u. The space of HCT trace functions is denoted by $V_h^{HCT}(\partial\Omega)$. Denote the finite element interpolation operator by I_{HCT}. As in Sect. 5.1, we abuse the notation by defining the trace of the interpolation from the boundary degrees of freedom and write, e.g., $I_{HCT}u^s_{r,c} \in V_h^{HCT}(\partial\Omega^s)$. We adopt the convention that all functions are understood extended by zero outside of their stated domain; e.g., $u|_\Gamma$ is zero outside of Γ.

The following lemma gives a bound on the interpolation from subdomain corners to edges.

Lemma 5.3. *For every $w \in V_h^{HCT}(\partial\Omega^s)$, define $u \in V_h^{HCT}(\partial\Omega^s)$ by HCT interpolation from the corners of Ω^s to the edges, i.e., $u = v$ and $\nabla u = \nabla w$*

at the corners of Ω^s, and, on each edge Γ^{st}, u is a cubic polynomial and $n \cdot \nabla u$ is linear. Then

$$|(\nabla(w-u)|_{\Gamma^{st}})|_{\frac{1}{2},2,\partial\Omega^s} \le C\left(1+\log\frac{H}{h}\right)|\nabla w|_{\frac{1}{2},2,\partial\Omega^s}. \tag{5.4}$$

Proof. Denote $v = w - u \in V_h^{HCT}(\partial\Omega)$. Then $v = 0$ and $\nabla v = 0$ at the corners of Ω^s, so from [13, Lemma 4.1], it follows that

$$|(\nabla v|_{\Gamma^{st}})|^2_{\frac{1}{2},2,\partial\Omega^s} \le |\nabla v|^2_{\frac{1}{2},2,\Gamma^{st}} + C\left(1+\log\frac{H}{h}\right)\|\nabla v\|^2_{0,\infty,\Gamma^{st}}. \tag{5.5}$$

Since $\nabla u|_{\Gamma^{st}}$ is from a space of dimension 6 and all norms on a finite dimensional space are equivalent, we have in the case when the length of Γ^{st} is one that

$$|\nabla u|_{\frac{1}{2},2,\Gamma^{st}} \le \|\nabla u\|_{\frac{1}{2},2,\Gamma^{st}} \approx \|\nabla u\|_{0,\infty,\Gamma^{st}}.$$

Since $|\nabla u|_{\frac{1}{2},2,\Gamma^{st}}$ and $\|\nabla u\|_{0,\infty,\Gamma^{st}}$ are invariant to stretching of the edge, we get

$$|\nabla u|_{\frac{1}{2},2,\Gamma^{st}} \le C\|\nabla u\|_{0,\infty,\Gamma^{st}} \tag{5.6}$$

in the general case by scaling. We will show at the end of the proof that

$$\|\nabla u\|_{0,\infty,\Gamma^{st}} \le 5\|\nabla w\|_{0,\infty,\Gamma^{st}}. \tag{5.7}$$

From the discrete Sobolev inequality as generalized by [13, Lemma 4.2], we have

$$\|\nabla w\|^2_{0,\infty,\Gamma^{st}} \le C\left(1+\log\frac{H}{h}\right)\left(|\nabla w|^2_{\frac{1}{2},2,\Gamma^{st}} + \frac{1}{H}\|\nabla w\|^2_{0,2,\Gamma^{st}}\right) \tag{5.8}$$

Combining (5.5), (5.7), and (5.8), we obtain

$$|\nabla v|^2_{\frac{1}{2},2,\partial\Omega^s} \le C\left(1+\log\frac{H}{h}\right)\left(|\nabla w|^2_{\frac{1}{2},2,\partial\Omega^s} + \frac{1}{H}\|\nabla w\|^2_{0,2,\partial\Omega^s}\right) \tag{5.9}$$

First consider the case when $\partial\Omega^s \cap \partial\Omega = \emptyset$. Then (5.4) is invariant to adding a linear function to w because v is the error of linear interpolation of w and because this adds only a constant to ∇w. So, without loss of generality, let $\int_{\partial\Omega^s} \nabla w = 0$. Then by the Poincaré-Friedrichs inequality

$$\frac{1}{H}\|y\|^2_{0,2,\partial\Omega^s} \le C|y|^2_{\frac{1}{2},2,\partial\Omega^s}, \tag{5.10}$$

proved first on a reference domain and then scaled: cf., [3,21]. Now (5.10) and (5.9) give (5.4). In the case when $\partial\Omega^s \cap \partial\Omega \neq \emptyset$, there are some essential boundary conditions on $\partial\Omega^s$. Since (5.4) has already been proved in the absence of essential boundary conditions on $\partial\Omega^s$, it is sufficient to restrict (5.4) to the subspace defined by the boundary conditions, noting that u satisfies the boundary conditions as well.

It remains to prove (5.7). The inequality is trivial for $n \cdot \nabla u$ because the normal derivative is interpolated linearly between the corners. Let u_L be the linear function on the edge Γ^{st} defined by the values of u on the corners. Then, using the triangle inequality,

$$\|\partial_t u\|_{0,\infty,\Gamma^{st}} \leq \|\partial_t (u - u_L)\|_{0,\infty,\Gamma^{st}} + \|\partial_t u_L\|_{0,\infty,\Gamma^{st}}. \tag{5.11}$$

By a simple computation, we see that the Hermite basis function ϕ, $\phi(0) = 0$, $\phi'(0) = 1$, $\phi(1) = \phi'(1) = 0$, ϕ a polynomial of order 3, attains the maximum of $|\phi'|$ at 0. Mapping the interval $(0, 1)$ on the edge Γ^{st} and noting that $u - u_L$ is zero at the endpoints x_1 and x_2 of Γ^{st}, we get by using the triangle inequality that

$$\begin{aligned}\|\partial_t (u - u_L)\|_{0,\infty,\Gamma^{st}} &\leq |\partial_t (u - u_L)(x_1)| + |\partial_t (u - u_L)(x_2)| \\ &\leq 2\|\partial_t w\|_{0,\infty,\Gamma^{st}} + 2\|\partial_t u_L\|_{0,\infty,\Gamma^{st}},\end{aligned} \tag{5.12}$$

because $\partial_t u(x_i) = \partial_t w(x_i)$, $i = 1, 2$. From the mean value theorem and the fact that $u_L(x_i) = w(x_i)$, $i = 1, 2$, it follows that

$$\|\partial_t u_L\|_{0,\infty,\Gamma^{st}} \leq \|\partial_t w\|_{0,\infty,\Gamma^{st}}, \tag{5.13}$$

hence (5.12) gives

$$\|\partial_t (u - u_L)\|_{0,\infty,\Gamma^{st}} \leq 4\|\partial_t w\|_{0,\infty,\Gamma^{st}}. \tag{5.14}$$

Now (5.11), (5.13), and (5.14) give (5.7).

Theorem 5.4. *For the fourth order model problem,*

$$\frac{\lambda_{max}(MF)}{\lambda_{min}(MF)} \leq C \left(1 + \log \frac{H}{h}\right)^2,$$

where the constant does not depend on H and h.

Proof. The proof follows immediately from Theorem 4.5, with (4.8) being a consequence of Lemma 5.3 and the uniform equivalence of seminorms $|\nabla I_{HCT} w^s_{r,c}|_{\frac{1}{2},2,\partial\Omega^s} \approx |w^s_{r,c}|_{S^s}$, cf., [13].

Remark 5.5. The result extends, by spectral equivalence, to DKT elements and a certain class of non-locking elements for the Reissner-Mindlin plate model as in [13].

References

1. J. H. Bramble, J. E. Pasciak, and A. H. Schatz, The construction of preconditioners for elliptic problems by substructuring, I. Math. Comp. **47**, 103–134 (1986)
2. P. G. Ciarlet, Basic error estimates for elliptic problems, in Handbook of Numerical Analysis, P. Ciarlet and J. L. Lions, eds., vol. II, North-Holland, Amsterdam, 1989, pp. 17–352
3. M. Dryja, A method of domain decomposition for 3-D finite element problems, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., Philadelphia, PA, 1988. SIAM, pp. 43–61
4. C. Farhat, P.-S. Chen, J. Mandel, and F.-X. Roux, The two-level FETI method - Part II: Extension to shell problems, parallel implementation and performance results. Comp. Meth. Appl. Mech. Engrg. **155**, 153–180 (1998)
5. C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, and D. Rixen, FETI-DP: A dual-primal unified FETI method. Part I: A faster alternative to the two-level FETI method. Int. J. Numer. Meth. Engrg. (in print)
6. C. Farhat and J. Mandel, Scalable substructuring by Lagrange multipliers in theory and practice, in Proceedings of the 9th International Conference on Domain Decomposition, Bergen, Norway, June 1996, DDM.ORG, 1998, pp. 20–30
7. C. Farhat and J. Mandel, The two-level FETI method for static and dynamic plate problems - Part I: An optimal iterative solver for biharmonic systems. Comp. Meth. Appl. Mech. Engrg. **155**, 129–152 (1998)
8. C. Farhat, J. Mandel, and F.-X. Roux, Optimal convergence properties of the FETI domain decomposition method. Comput. Methods Appl. Mech. Engrg. **115**, 367–388 (1994)
9. C. Farhat and F.-X. Roux, An unconventional domain decomposition method for an efficient parallel solution of large-scale finite element systems. SIAM J. Sci. Stat. Comput. **13**, 379–396 (1992)
10. A. Klawonn and O. B. Widlund, A domain decomposition method with Lagrange multipliers and inexact solvers for linear elasticity. SIAM J. Sci. Comput. **22**, 1199–1219 (2000)
11. A. Klawonn and O. B. Widlund, FETI and Neumann-Neumann iterative substructuring methods: Connections and new results. Comm. Pure Appl. Math. **54**, 57–90 (2001)
12. P. Le Tallec, J. Mandel, and M. Vidrascu, Balancing domain decomposition for plates, Contemporary Mathematics, 180 (1994), pp. 515–524. Proceedings of the 7th International Symposium on Domain Decomposition Methods, Penn State, November 1993
13. P. Le Tallec, J. Mandel, and M. Vidrascu, A Neumann-Neumann domain decomposition algorithm for solving plate and shell problems. SIAM J. Numer. Anal. **35**, 836–867 (1998)
14. J. Mandel and R. Tezaur, Convergence of a substructuring method with Lagrange multipliers. Numerische Mathematik **73**, 473–487 (1996)
15. J. Mandel, R. Tezaur, and C. Farhat, A scalable substructuring method by Lagrange multipliers for plate bending problems. SIAM J. Numer. Anal. **36**, 1370–1391 (1999)
16. J. Nečas, Les méthodes directes en théorie des équations elliptiques, Academia, Prague, 1967
17. K. C. Park, M. R. Justino, Jr., and C. A. Felippa, An algebraically partitioned FETI method for parallel structural analysis: Algorithm description. Int. J. Numer. Meth. Engrg. **40**, 2717–2737 (1997)

18. D. Rixen and C. Farhat, A simple and efficient extension of a class of substructure based preconditioners to heterogeneous structural mechanics problems. Int. J. Numer. Meth. Engrg. **46**, 489–516 (1999)
19. D. J. Rixen, C. Farhat, R. Tezaur, and J. Mandel, Theoretical comparison of the FETI and algebraically partitioned FETI methods, and performance comparisons with a direct sparse solver. Intl. Journal for Numerical Methods in Engineering **46**, 501–534 (1999)
20. R. Tezaur, Analysis of Lagrange Multiplier Based Domain Decomposition, PhD thesis, University of Colorado at Denver, 1998.
http://www-math.cudenver.edu/graduate/thesis/rtezaur.ps.gz
21. O. B. Widlund, Iterative substructuring methods: Algorithms and theory for elliptic problems in the plane, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., Philadelphia, PA, 1988. SIAM, pp. 113–128