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Total FETI—an easier implementable variant of the FETI method for numerical solution of elliptic PDE

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SUMMARY

A new variant of the FETI method for numerical solution of elliptic PDE is presented. The basic idea is to simplify inversion of the stiffness matrices of subdomains by using Lagrange multipliers not only for gluing the subdomains along the auxiliary interfaces, but also for implementation of the Dirichlet boundary conditions. Results of numerical experiments are presented which indicate that the new method may be even more efficient then the original FETI. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The FETI (finite element tearing and interconnecting) domain decomposition method by Farhat and Roux [1, 2] turned out to be one of the most successful methods for parallel solution of linear problems described by elliptic partial differential equations. Its key ingredient is decomposition of the spatial domain into non-overlapping subdomains that are 'glued' by Lagrange multipliers, so that, after eliminating the primal variables, the original problem is reduced to a small, relatively well conditioned, typically equality constrained quadratic programming problem that is solved iteratively. The time that is necessary for both the elimination and iterations can be reduced nearly

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proportionally to the number of the subdomains, so that the algorithm enjoys parallel scalability. Observing that the equality constraints may be used to define so-called 'natural coarse grid', Farhat *et al.* [3] modified the basic FETI algorithm so that it was possible to prove its numerical scalability, i.e. the asymptotically linear complexity.

Implementation of the FETI method into general-purpose packages requires an effective method for automatic identification of the kernels of the stiffness matrices of the subdomains as these kernels are used both in elimination of the primal variables and in definition of the natural coarse grid projectors. An effective method based on combination of the Cholesky factorization and the singular value decomposition was proposed by Farhat and Gérardin [4]. However, it turns out that it is still quite difficult to determine the kernels reliably in the presence of rounding errors. This was one of the motivations that led to development of the FETI-DP (dual-primal) method introduced by Farhat *et al.* [5]. The FETI-DP method is very similar to the original FETI, the only difference is that it enforces the continuity of the displacements at the corners on primal level so that the stiffness matrices of the subdomains of the FETI-DP method are invertible. However, even though FETI-DP may be efficiently preconditioned so that it scales better than the original FETI for plates and shells, the coarse grid defined by the corners without additional preconditioning is less efficient [6, 7] than that defined by the rigid body motions, which is important for some applications [6–8] and the FETI-DP method is more difficult to implement as it requires special treatment of the corners. More discussions concerning the FETI and FETI-DP methods may be found, e.g. in Reference [9].

In this paper, we propose a new variant of the FETI method for numerical solution of elliptic PDE which is easier to implement and which preserves efficiency of the coarse grid of the classical FETI. The basic idea is to simplify inversion of the stiffness matrices of subdomains by using the Lagrange multipliers not only for gluing of the subdomains along the auxiliary interfaces, but also for implementation of the Dirichlet boundary conditions. We give heuristic arguments and results of numerical experiments which indicate that the new method may be not only easier to implement, but also more efficient then the original FETI.

2. FETI AND TOTAL FETI (TFETI)

Let us consider a boundary value problem for Lamé equations describing the equilibrium of an elastic body which occupies in the reference configuration the domain $\Omega \subset \mathbb{R}^d$, $d=2$ or $d=3$, with the boundary Γ. We shall assume that the boundary Γ is decomposed into two disjoint parts Γ_u and Γ_f, $\Gamma=\overline{\Gamma}_u \cup \overline{\Gamma}_f$ and that there are prescribed Dirichlet boundary conditions on Γ_u and Neumann boundary conditions on Γ_f as in Figure 1. The Dirichlet boundary conditions represent the prescribed displacements and the Neumann conditions represent the surface tractions. We shall assume that our problem is well posed. Let us mention that our reasoning remains valid for any well-posed elliptic boundary value problem.

To apply the FETI-based domain decomposition [3], we partition Ω into N_s subdomains Ω^s as in Figure 2 and we denote by K_s, f_s, u_s and B_s, respectively, the subdomain stiffness matrix, the subdomain force and displacement vectors and the signed matrix with entries $-1, 0, 1$ describing the subdomain interconnectivity. We shall get the discretized problem

$$\min \tfrac{1}{2}u^\top K u - u^\top f \quad \text{s.t. } Bu=0 \tag{1}$$

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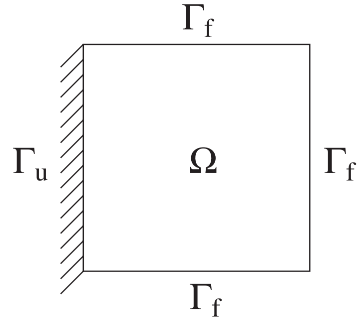

Figure 1. Model problem.

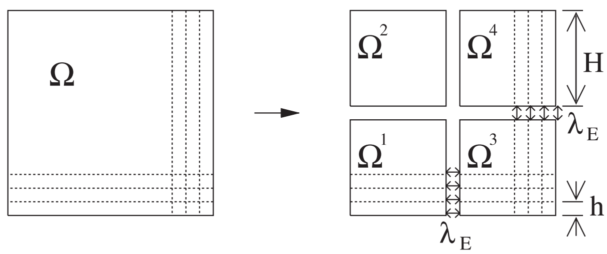

Figure 2. Decomposition and discretization of model problem.

with the block-diagonal stiffness matrix

$$K = \begin{bmatrix} K_1 & & \\ & \ddots & \\ & & K_{N_s} \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ \vdots \\ f_{N_s} \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_{N_s} \end{bmatrix}, \quad B = [B_1, \ldots, B_{N_s}] \tag{2}$$

The original FETI method assumes that the boundary subdomains inherit the Dirichlet conditions from the original problem as in Figure 3(a), so that *the defect of the stiffness matrices K_s may vary* from zero corresponding to the boundary subdomains with sufficient Dirichlet data to the maximum corresponding to the interior floating subdomains. For 3D linear elasticity, this maximum is six corresponding to the number of independent rigid body motions.

In this paper, we shall use the other option which we shall coin TFETI (total FETI). The basic idea is to keep all the subdomain stiffness matrices K_s as if there were no prescribed displacements and to *enhance the prescribed displacements into the matrix of constraints B*. To enhance the boundary conditions like $u_i = 0$, just append the row b with all the entries equal to zero except $b_i = 1$. The prescribed displacements will be enforced by the Lagrange multipliers which may be interpreted as forces as in Figure 3(b). An immediate result of this procedure is that all the subdomain stiffness matrices will have known and typically the same defect. For example, the defect of all K_s for 2D and 3D elasticity will be equal to three and six, respectively.

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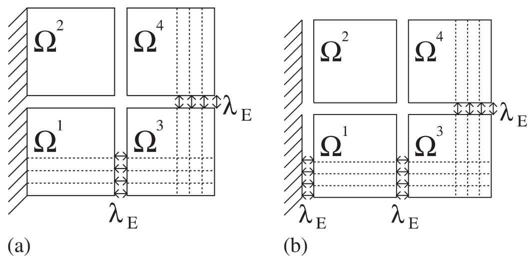

Figure 3. Two possibilities how to prescribe Dirichlet boundary conditions.

The remaining procedure is exactly the same as in the paper by Farhat *et al.* [3]. We shall introduce the Lagrangian associated with problem (1) by

$$L(u, \lambda) = \tfrac{1}{2} u^\top K u - f^\top u + \lambda^\top B u$$

It is well known [10] that (1) is equivalent to the saddle point problem

$$\text{Find } (\overline{u}, \overline{\lambda}) \quad \text{so that } L(\overline{u}, \overline{\lambda}) = \sup_{\lambda} \inf_{u} L(u, \lambda) \tag{3}$$

For fixed λ, the Lagrange function $L(\cdot, \lambda)$ is convex in the first variable and the minimizer u of $L(\cdot, \lambda)$ satisfies

$$Ku - f + B^\top \lambda = 0 \tag{4}$$

Equation (4) has a solution iff

$$f - B^\top \lambda \in \operatorname{Im} K \tag{5}$$

which can be expressed more conveniently by means of a matrix R whose columns span the null space of K as

$$R^\top (f - B^\top \lambda) = 0 \tag{6}$$

The key point is that *the kernels R_s of the local stiffness matrices K_s are often known* and can be formed directly. For example, if the subdomain Ω^s of a 2D elasticity problem is discretized by means of n_s nodes with the co-ordinates (x_i, y_i), $i = 1, \ldots, n_s$, then

$$R_s^\top = [(R_s^1)^\top, \ldots, (R_s^{n_s})^\top]^\top, \quad R_s^i = \begin{bmatrix} 1 & 0 & -y_i \\ 0 & 1 & x_i \end{bmatrix}, \quad i = 1, \ldots, n_s \tag{7}$$

Using R_s, we can easily assemble the block-diagonal basis R of the kernel of K as

$$R = \begin{bmatrix} R_1 & & \\ & \ddots & \\ & & R_{N_s} \end{bmatrix} \tag{8}$$

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To eliminate the primal variables u from (4), assume that λ satisfies (5) and denote by $K^\dagger$ any matrix that satisfies

$$KK^\dagger K = K \tag{9}$$

Let us note that the generalized inverse $K^\dagger$ may be chosen in such a way that it is symmetric and that for any vector x, $K^\dagger x$ may be conveniently evaluated at the cost comparable with the Cholesky decomposition of regularized K by using either modified Choleski factorization or a clever combination of the singular value decomposition and Cholesky factorization [4]. The critical point of these procedure, the determination of the ranks of the subdomain stiffness matrices K_s, is trivial when the TFETI procedure is applied.

It may be verified directly that if u solves (4), then there is a vector α such that

$$u = K^\dagger(f - B^\top\lambda) + R\alpha \tag{10}$$

After substituting expression (10) into problem (3) and changing signs, we shall get the minimization problem to find

$$\min\, \Theta(\lambda) \quad \text{s.t. } R^\top(f - B^\top\lambda) = 0 \tag{11}$$

where

$$\Theta(\lambda) = \tfrac{1}{2}\lambda^\top BK^\dagger B^\top\lambda - \lambda^\top BK^\dagger f \tag{12}$$

Let us denote

$$F = BK^\dagger B^\top, \quad d = BK^\dagger f$$
$$G = R^\top B^\top, \qquad e = R^\top f$$

so that problem (12) reads

$$\min\, \tfrac{1}{2}\lambda^\top F\lambda - \lambda^\top d \quad \text{s.t. } G\lambda = 0 \tag{13}$$

Our final step is based on observation that problem (13) is equivalent to

$$\min\, \tfrac{1}{2}\lambda^\top PFP\lambda - \lambda^\top Pd \quad \text{s.t. } G\lambda = 0 \tag{14}$$

where

$$Q = G^\top(GG^\top)^{-1}G \quad \text{and} \quad P = I - Q$$

denote the orthogonal projectors on the image space of $G^\top$ and on the kernel of G, respectively. Problem (14) may be solved effectively by the conjugate gradient method as the proof of the classical estimate by Farhat *et al.* [3, Theorem 3.2]

$$\kappa(PFP|\mathrm{Im}\, P) \leqslant \mathrm{const}\frac{H}{h} \tag{15}$$

of the spectral condition number κ of the restriction of PFP to the range of P by the ratio of the decomposition parameter H and the discretization parameter h remains valid for TFETI.

Closer inspection of the proof [3] of (15) reveals that the constants in the bound (15) may be in many cases more favourable for TFETI than for classical FETI. The reason is that the proof of (15)

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is based on estimates of bounds of the eigenvalues of the regular parts of the subdomain matrices K_s that will be in many cases more favourable than those generated by FETI. For example, if Ω is decomposed into identical subdomains, introduction of the Dirichlet boundary conditions into the boundary subdomains can only increase these local estimates. From a different point of view, the *slightly faster rate of convergence of TFETI than that of FETI should not be surprising* as TFETI generates larger kernel of the global stiffness matrix K that have similar role as the coarse grid in multigrid methods. A certain drawback of the procedure is that the dimension of the coarse problem is larger, so that it may happen that the number of the iterations may be slightly larger then that required by the classical FETI.

3. NUMERICAL EXPERIMENTS

We have implemented FETI as described in Reference [3] and TFETI in Matlab 7 and applied both methods to the solution of two simple problems.

Problem 1
Our first benchmark is a scalar boundary value problem to find u defined on $\Omega = (0, 1) \times (0, 1)$ such that

$$\Delta u = 1 \quad \text{for } (x, y) \in \Omega$$

with the boundary conditions as in Figure 1, i.e. $u(0, y) = 0$, $\delta u(1, y)/\delta x = 0$ for $y \in [0, 1]$ and $\delta u(x, 0)/\delta y = 0$, $\delta u(x, 1)/\delta y = 0$ for $x \in [0, 1]$. The solution of the model problem is in Figure 4

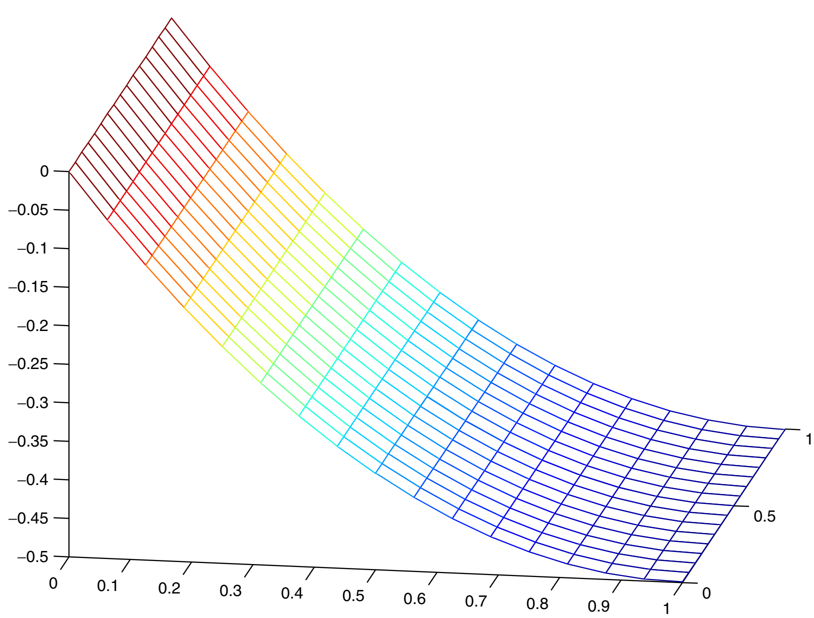

Figure 4. Solution of Problem 1.

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Table I. Numerical scalability for the 2D model problem.

H/h	H	h	Prim.	Dual FETI	Dual TFETI	CG/time FETI	CG/time TFETI
8	1/2	1/16	324	35	53	6/0.02	15/0.02
8	1/4	1/32	1296	207	243	32/0.14	29/0.13
8	1/8	1/64	5184	959	1031	42/2.43	39/2.34
32	1/2	1/64	4356	131	197	7/0.22	24/0.33
32	1/4	1/128	17424	783	915	54/2.55	47/2.26
32	1/8	1/256	69696	3647	3911	74/38.52	72/37.85
128	1/2	1/256	66564	515	773	7/25.49	36/30.53
128	1/4	1/512	266256	3087	3603	99/235.4	81/189.2
128	1/8	1/1024	1 065 024	14 399	15 431	133/1701.1	123/1569.3

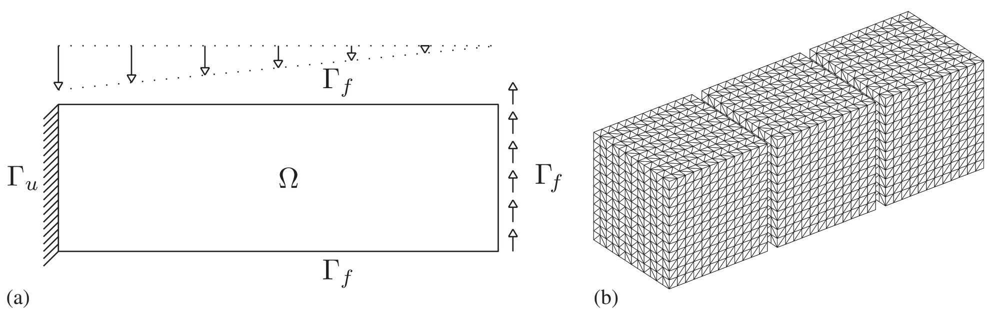

Figure 5. (a) The cross-section of the 3D brick Ω; and (b) domain decomposition and discretization.

and may be interpreted as the displacement of the membrane under the unit traction. We carried out our computations with $H \in \{1/2, 1/4, 1/8, 1/16\}$ and $H/h \in \{2, 4, 8\}$. The results are in Table I. We can observe that the performance of both algorithms is very similar as predicted by the theory. The slightly better results achieved by TFETI may be explained by the fact that it uses richer natural coarse grid of the rigid body motions.

Problem 2

The 3D model problem is described by a model brick $\Omega = (0, 3) \times (0, 1) \times (0, 1)$ made of an elastic isotropic, homogeneous material characterized by the Young modulus $E = 21.2 \times 10^{10}$ and the Poisson's ratio $\sigma = 0.277$ (steel). The applied surface tractions are seen in Figure 5(a). The volume forces were assumed to vanish. The brick Ω is decomposed into three parts as in Figure 5(b), so that the resulting problem has 12 or 18 rigid modes when we use FETI or TFETI, respectively. The performance of both FETI and TFETI for varying regular grids is in Table II. It may be observed that in this case FETI performs better, which may be caused by the increase of the dimension of the dual problem by 50%.

We conclude that the results of numerical experiments are in agreement with the theory, so that TFETI is a reasonable variant of FETI to be considered for the solution of realistic problems.

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Table II. Numerical scalability for the 3D model problem.

Prim.	Dual FETI	Dual TFETI	CG steps FETI	CG steps TFETI
243	54	81	17	22
1125	150	225	22	31
3087	294	441	25	36
6561	489	729	29	41
11 979	726	1089	31	43
19 773	1014	1521	34	47

4. COMMENTS AND CONCLUSION

We have presented a new variant of the classical FETI method called TFETI with preconditioning by the natural coarse grid formed by the kernels of the local operators. The new method is easier to implement and both the theory and results of numerical experiments indicate that it should enjoy in many cases faster convergence then the original method at the cost of larger dual problem. Similar to the classical FETI method, the performance of the method may be improved by standard FETI preconditioners [9, 11]. The results are useful also for solving contact problems by FETI-based methods [8]. We shall discuss these topics elsewhere.

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