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Modified HSS iteration methods for a class of non-Hermitian positive-definite linear systems ☆

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ABSTRACT

We consider the numerical solution of a class of non-Hermitian positive-definite linear systems by the modified Hermitian and skew-Hermitian splitting (MHSS) iteration method. We show that the MHSS iteration method converges unconditionally even when the real and the imaginary parts of the coefficient matrix are nonsymmetric and positive semidefinite and, at least, one of them is positive definite. At each step the MHSS iteration method requires to solve two linear sub-systems with real nonsymmetric positive definite coefficient matrices. We propose to use inner iteration methods to compute approximate solutions of these linear sub-systems. We illustrate the performance of the MHSS method and its inexact variant by two numerical examples.

1. Introduction

Many problems in scientific computing require to solve the linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a large sparse non-Hermitian positive definite matrix and $x, b \in \mathbb{C}^n$. Here we use A^H to denote the conjugate transpose of the matrix A, and we call a non-Hermitian matrix $A \in \mathbb{C}^{n\times n}$ positive definite if its Hermitian part $\frac{1}{2}(A + A^H)$ is positive definite; see [1]. For solving this class of linear systems, a sequence of splitting iteration methods has been developed, e.g., *Hermitian and skew-Hermitian splitting* (HSS) iteration [8], *preconditioned Hermitian and skew-Hermitian splitting* (PHSS) iteration [6], and *block triangular and skew-Hermitian splitting* (BTSS) iteration [7]; see [4,5,10–17] for other developments and [2] for an overview.

To solve the linear system (1.1) iteratively, Bai et al. [8] used the *Hermitian and skew-Hermitian splitting* (HSS) of the coefficient matrix A, i.e.,

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^H) \quad \text{and} \quad S = \frac{1}{2}(A - A^H)$$

and established the following HSS iteration method:

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Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \ldots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases}$$

where α is a given positive constant.

In [8] Bai et al. also proved that for any positive constant α the HSS iteration method converges unconditionally to the unique solution of the linear system (1.1). In [9] Bai, Golub and Ng presented the *inexact Hermitian and skew-Hermitian splitting* (IHSS) scheme to avoid the exact inverses of the n-by-n matrices $\alpha I + H$ and $\alpha I + S$. They also proved that the asymptotic convergence rate of the IHSS iteration approach to that of the HSS iteration when the tolerances of the inner iterations tend to zero as the number of outer iteration steps increases.

Denote by $\imath$ the imaginary unit. When $A \in \mathbb{C}^{n\times n}$ has the form $A = W + \imath T$, where $W, T \in \mathbb{R}^{n\times n}$ are real symmetric matrices, with W positive definite and T positive semidefinite, we can see that the Hermitian part H and the skew-Hermitian part S of A become

$$H = W \quad \text{and} \quad S = \imath T$$

and the HSS iteration scheme becomes

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - \imath T)x^{(k)} + b, \\ (\alpha I + \imath T)x^{(k+1)} = (\alpha I - W)x^{(k+\frac{1}{2})} + b. \end{cases}$$

To avoid the complex arithmetic, Bai et al. [3] skillfully modified the above HSS iteration scheme, and presented the following *modified Hermitian and skew-Hermitian splitting* (MHSS) iteration method:

Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \ldots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - \imath T)x^{(k)} + b, \\ (\alpha I + T)x^{(k+1)} = (\alpha I + \imath W)x^{(k+\frac{1}{2})} - \imath b, \end{cases} \tag{1.2}$$

where α is a given positive constant.

In [3] Bai et al. proved that the MHSS iteration method converges unconditionally to the unique solution of the linear system for any positive constant α. Also, they showed that when

$$\alpha = \sqrt{\gamma_{min}\gamma_{max}},$$

an upper bound of the spectral radius of the MHSS iteration matrix can be minimized, and the corresponding minimum upper bound is given by $\frac{\sqrt{\kappa(W)+1}}{\sqrt{\kappa(W)}+1}$, where γ_{min} and γ_{max} are the minimum and the maximum eigenvalues of the matrix W, respectively, and $\kappa(W) = \gamma_{max}/\gamma_{min}$ is the spectral condition number of W.

In this paper, we consider the MHSS iteration method for solving the linear system (1.1) with $A = W + \imath T$, where $W, T \in \mathbb{R}^{n\times n}$ are real nonsymmetric matrices. We prove that the MHSS iteration method also converges to the unique solution of the linear system mentioned above for any positive parameter α.

The paper is organized as follows. In Section 2, we discuss the convergence property of the MHSS iteration method for non-Hermitian positive definite linear systems. In Section 3, we present the IMHSS iteration method. Finally, in Section 4 we use numerical examples to illustrate the effectiveness of our methods.

2. Convergence analysis of MHSS iteration

In this section, we study the convergence property of the MHSS iteration method under certain conditions on the matrices W and T.

The MHSS iteration scheme (1.2) can be equivalently rewritten as

$$x^{(k+1)} = M(\alpha)x^{(k)} + G(\alpha)b, \quad k = 0, 1, 2, \ldots,$$

where

$$M(\alpha) = (\alpha I + T)^{-1}(\alpha I + \imath W)(\alpha I + W)^{-1}(\alpha I - \imath T)$$

and

$$G(\alpha) = (1 - \imath)\alpha(\alpha I + T)^{-1}(\alpha I + W)^{-1}.$$

Thus, $M(\alpha)$ is the iteration matrix of the MHSS iteration method. Actually, we can split A into

$$A = B(\alpha) - C(\alpha),$$

with

$$\begin{cases} B(\alpha) = \frac{1+\imath}{2\alpha}(\alpha I + W)(\alpha I + T), \\ C(\alpha) = \frac{1+\imath}{2\alpha}(\alpha I + \imath W)(\alpha I - \imath T). \end{cases}$$

Then it holds that

$$M(\alpha) = B(\alpha)^{-1}C(\alpha).$$

In order to present the convergent property of the MHSS iteration, we first give two lemmas.

Lemma 2.1. *The matrix $W \in \mathbb{R}^{n\times n}$ is a real nonsymmetric matrix. If the matrix $(1-\imath)W$ is positive definite (or positive semidefinite), then the matrix W is also positive definite (or positive semidefinite).*

Proof. As is known if a real matrix has complex eigenvalues, they must appear in pairs. Let $s \pm \imath t$ be one pair of the complex eigenvalues of the matrix W, then

$$(1-\imath)(s+\imath t) = (s+t) + \imath(t-s)$$

and

$$(1-\imath)(s-\imath t) = (s-t) - \imath(s+t),$$

are two eigenvalues of the matrix $(1-\imath)W$.

Since the matrix $(1-\imath)W$ is positive definite (or positive semidefinite), the real parts $s \pm t$ of the two eigenvalues $(s+t)+\imath(t-s)$ and $(s-t)-\imath(s+t)$ satisfy

$$s \pm t > 0 \quad (\text{or} \quad s \pm t \geqslant 0).$$

So it holds that

$$s > 0 \quad (\text{or} \quad s \geqslant 0),$$

i.e. the matrix W is positive definite (or positive semidefinite). □

Lemma 2.2. *The matrix $T \in \mathbb{R}^{n\times n}$ is a real nonsymmetric matrix. If the matrix $(1+\imath)T$ is positive definite (or positive semidefinite), then the matrix T is also positive definite (or positive semidefinite).*

Proof. The proof here proceeds very much like that of Lemma 2.1, we omit the details. □

We give the convergence property of the MHSS iteration method in the following theorem.

Theorem 2.1. *Let $A = W + \imath T \in \mathbb{C}^{n\times n}$, with $W, T \in \mathbb{R}^{n\times n}$ being nonsymmetric. Let $(1-\imath)W$ be positive definite, $(1+\imath)T$ be positive semidefinite, and α be a given positive constant. Then the MHSS iteration matrix $M(\alpha)$ given by*

$$M(\alpha) = (\alpha I + T)^{-1}(\alpha I + \imath W)(\alpha I + W)^{-1}(\alpha I - \imath T)$$

has spectral radius $\rho(M(\alpha))$ less than 1, i.e.,

$$\rho(M(\alpha)) < 1, \quad \forall \alpha > 0,$$

i.e., the MHSS iteration method converges unconditionally to the unique solution of the linear system (1.1).

Proof. By direct computations we have

$$\begin{aligned}\rho(M(\alpha)) &= \rho((\alpha I + T)^{-1}(\alpha I + \imath W)(\alpha I + W)^{-1}(\alpha I - \imath T)) = \rho((\alpha I + \imath W)(\alpha I + W)^{-1}(\alpha I - \imath T)(\alpha I + T)^{-1}) \\ &\leqslant \|(\alpha I + \imath W)(\alpha I + W)^{-1}(\alpha I - \imath T)(\alpha I + T)^{-1}\|_2 \leqslant \|(\alpha I + \imath W)(\alpha I + W)^{-1}\|_2 \|(\alpha I - \imath T)(\alpha I + T)^{-1}\|_2.\end{aligned}$$

Because $(1-\imath)W$ is positive definite, we know that

$$\frac{1}{2}((1-\imath)W + ((1-\imath)W)^H) = \frac{1}{2}(W + W^T - \imath(W - W^T))$$

is positive definite. Then for any $y \in \mathbb{C}^n \setminus \{0\}$, the inequality

$$\langle (W + W^T - \imath(W - W^T))y, y\rangle > 0$$

holds true. Here $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{C}^n$. It straightforwardly follows that

$$\langle(\alpha I + \imath W)y, (\alpha I + \imath W)y\rangle < \langle(\alpha I + W)y, (\alpha I + W)y\rangle, \quad \forall y \in \mathbb{C}^n \setminus \{0\}. \tag{2.1}$$

Note that $(1-\imath)W$ is positive definite. According to Lemma 2.1, we easily know that W is also positive definite. So the matrix $\alpha I + W$ is nonsingular. Let $x = (\alpha I + W)y$. Then $x \neq 0$ if and only if $y \neq 0$. So the inequality (2.1) can be equivalently written as

$$\langle(\alpha I + \imath W)(\alpha I + W)^{-1}x, (\alpha I + \imath W)(\alpha I + W)^{-1}x\rangle < \langle x, x\rangle, \quad \forall x \in \mathbb{C}^n \setminus \{0\}.$$

That is to say, it holds that

$$\frac{\|(\alpha I + \imath W)(\alpha I + W)^{-1}x\|_2}{\|x\|_2} < 1, \quad \forall x \in \mathbb{C}^n \setminus \{0\},$$

i.e.,

$$\|(\alpha I + \imath W)(\alpha I + W)^{-1}\|_2 < 1.$$

Similarly, we can get

$$\|(\alpha I - \imath T)(\alpha I + T)^{-1}\|_2 \leqslant 1.$$

Therefore,

$$\rho(M(\alpha)) < 1, \quad \forall \alpha > 0. \qquad \square$$

From Theorem 2.1, we know that the MHSS iteration method converges unconditionally to the unique solution of the linear system.

Remark 1. When W is a real symmetric matrix, Theorem 2.1 reduces to the same conclusion as that in [3]. But when W is nonsymmetric, it is difficult to find the bounds for the convergence rate theoretically. We can only give the better values of the parameter α by numerical experiments as done in Section 4.

3. The IMHSS iteration method

In the process of the MHSS iteration, we need to solve two linear sub-systems whose coefficient matrices are $\alpha I + W$ and $\alpha I + T$, respectively. Since these two matrices are nonsymmetric positive definite, to inverse them exactly is costly and even impractical in actual implementations. In order to implement the MHSS iteration more efficiently, we employ inexact MHSS (IMHSS) iteration, which solves the two linear sub-systems iteratively. We solve the linear sub-systems by the preconditioned GMRES (PGMRES) method; see [3,8,9]. The IMHSS iteration scheme is described in the following.

The IMHSS iteration method. Input an initial guess $x^{(0)} \in \mathbb{C}^n$, a stopping tolerance ε for the outer iteration, a largest admissible number k_{max} of the outer iteration steps, a stopping tolerance ε_{pgmres} for the inner PGMRES iteration, and a positive integer sequence $\{\mu_k\}$ of the largest admissible inner PGMRES iteration steps.

1. $k := 0$.
2. $r^{(k)} = b - Ax^{(k)}$ and $\rho^{(k)} = \|r^{(k)}\|_2^2$.
3. If $\sqrt{\rho^{(k)}} \leqslant \varepsilon\|b\|_2$ or $k > k_{max}$ then goto 10.
4. Call pgmres($W, \alpha, r^{(k)}, \rho^{(k)}, \mu_k$, $\varepsilon_{pgmres}, y^{(\mu_k)}$).
5. $x^{(k+\frac{1}{2})} = x^{(k)} + y^{(\mu_k)}$.
6. $r^{(k+\frac{1}{2})} = -\imath\alpha y^{(\mu_k)} - Ty^{(\mu_k)}$ and $\rho^{(k+\frac{1}{2})} = \|r^{(k+\frac{1}{2})}\|_2^2$.
7. Call pgmres($T, \alpha, r^{(k+\frac{1}{2})}, \rho^{(k+\frac{1}{2})}, \mu_k, \varepsilon_{pgmres}, y^{(\mu_k)}$).
8. $x^{(k+1)} = x^{(k+\frac{1}{2})} + y^{(\mu_k)}$.
9. Set $k := k + 1$ and goto 2.
10. Set $x := x^{(k)}$ and output x.

Subroutine pgmres(B, α, r, ρ, μ, ε_{pgmres}, y)   % $L_B U_B$ is the preconditioner.

1. Sparse the matrix $\alpha I + B$.
2. Factorize the matrix $\alpha I + B$ by inexact LU factorization, and get the factors L_B and U_B.
3. Call gmres(B, r,restart,$\varepsilon_{pgmres}, \mu, L_B, U_B$).

4. Numerical experiments

We consider the examples that analogous to Examples 4.1 and 4.2 in [3]. In order to make W and T be nonsymmetric, we add a gradient term to the original equation and discretize the resulting problem by finite difference scheme. In actual implementations, we set the initial guess $x^{(0)}$ to be zero, and stop the iteration once the current iterate $x^{(k)}$ satisfies

$$\frac{\|b - Ax^{(k)}\|_2}{\|b\|_2} \leqslant 10^{-6}.$$

For the IMHSS iteration method, the stopping criteria for the inner preconditioned GMRES iterations is set to be

$$\frac{\|r^{(k,\mu_k)}\|_2}{\|b - Ax^{(k)}\|_2} \leqslant 10^{-2},$$

where $r^{(k,\mu_k)}$ represents the residual of the μ_k th inner iterate in the kth outer iterate.

Example 4.1. The linear system is of the form

$$\left[\left(K + G + \frac{3-\sqrt{3}}{\tau} I\right) + \imath \left(K + G + \frac{3+\sqrt{3}}{\tau} I\right)\right] x = b,$$

where τ is the time step-size, K is the five-point centered difference matrix approximating the negative Laplacian operator $L = -\Delta$ with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0,1] \times [0,1]$ with the mesh-size $h = \frac{1}{m+1}$, and G is the centered difference matrix approximating the gradient operator ∇ with homogeneous Dirichlet boundary conditions, on the same uniform mesh with the same mesh-size. The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$, with $V_m = h^{-2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}$, and the matrix $G \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = I \otimes U_m + U_m \otimes I$, with $U_m = \frac{1}{2h} tridiag(-1, 0, 1) \in \mathbb{R}^{m\times m}$. Hence, K and G are $n \times n$ block-tridiagonal matrices, with $n = m^2$.

We take

$$W = K + G + \frac{3-\sqrt{3}}{\tau} I$$

and

$$T = K + G + \frac{3+\sqrt{3}}{\tau} I$$

and the right-hand side vector b to be a complex vector with its jth entry b_j being given by

$$b_j = \frac{(1-i)j}{\tau(j+1)^2}, \quad j = 1, 2, \ldots, n.$$

In our tests, we take $\tau = h$. Furthermore, we normalize coefficient matrix and right-hand side by multiplying both by h^2.

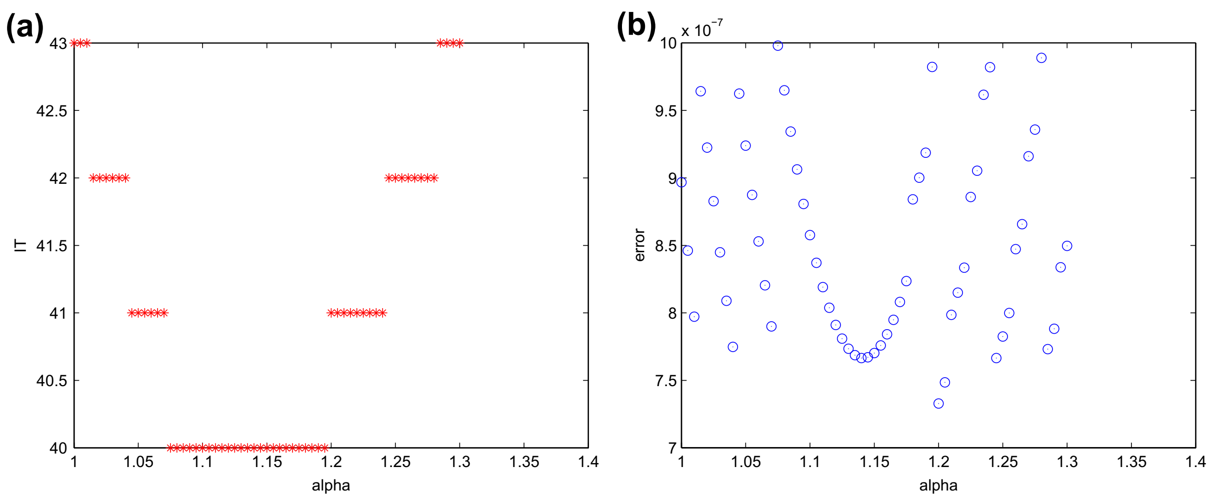

Fig. 4.1. The number of iteration steps and the relative residual error with respect to α.

Table 1
Numerical results of the MHSS method for Example 4.1.

m	8	16	32	64
α^*	1.57	1.14	0.81	0.576
IT	30	40	54	74
CPU	0.1872	0.7020	6.6457	106.7047
RES	9.82e−7	7.67e−7	9.32e−7	9.28e−7

Table 2
Numerical results of the MHSS method for Example 4.2.

m	8	16	32	64
α^*	0.59	0.205	0.087	0.039
IT	29	34	37	50
CPU	0.2028	0.6552	5.0076	75.3017
RES	6.75e−7	8.87e−7	9.70e−7	9.11e−7

We use the optimal values of the parameter α (denoted by α^*) for the MHSS iteration method. Note that α^* is obtained by minimizing both number of iteration steps and the 2-norm of relative residual error. For $m = 16$, Fig. 4.1a and b depict the number of iteration steps and the errors with respect to α. From Fig. 4.1a we find that when $\alpha \in [1.075, 1.195]$, the number of iteration steps are all equal to 40. From Fig. 4.1b, we see that when $\alpha \in [1.075, 1.195]$, the error is minimized at $\alpha = 1.14$. Considering both iteration step and the error, we choose $\alpha^* = 1.14$ as the optimal value of the parameter α for $m = 16$. The strategy is the same for choosing α^* for other cases of m.

The numerical results are listed in Table 1. From this table, we see that the optimal α^* decreases approximately by a factor $\sqrt{2}$ each time when m is doubled.

Example 4.2. The system of linear equations is of the form

$$[(-\omega^2 M + K + G) + \imath(\omega C_V + C_H + \mu G)]x = b,$$

where M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, respectively, and ω is the driving circular frequency. We set K and G to be the matrices as in Example 4.1. We take $C_H = \mu K$ with μ a damping coefficient, $M = I$ and $C_V = 10I$. In our experiments we set $\omega = \pi,\ \mu = 0.02$, and the right-hand side vector b to be $b = (1 + \imath)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1.

We take

$$W = -\omega^2 M + K + G$$

and

$$T = \omega C_V + C_H + \mu G.$$

As before, we normalize the linear system by multiplying both sides through by h^2.

In our tests we also use the optimal value α^* of the parameter α for the MHSS iteration method. The numerical results are listed in Table 2.

From Tables 1 and 2, we see that the linear systems in Examples 4.1 and 4.2 can be solved efficiently by the MHSS iteration method.

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