New Techniques for the Analysis of Linear Interval Equations

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ABSTRACT

The basic properties of interval matrix multiplication and several well-known solution algorithms for linear interval equations are abstracted by the concept of a sublinear map. The new concept, coupled with a systematic use of Ostrowski's comparison operator (in a form generalized to interval matrices), is used to derive quantitative information about the result of interval Gauss elimination and the limit of various iterative schemes for the solution of linear interval equations. Moreover, optimality results are proved for (1) the use of the midpoint inverse as a preconditioning matrix, and (2) Gauss-Seidel iteration with componentwise intersection. This extends and improves results by Scheu, Krawczyk, and Alefeld and Herzberger.

INTRODUCTION

In this paper new methods are introduced for the study of direct and iterative solution algorithms for linear interval equations. The basic problem in linear interval equations is to find good interval enclosures for the set

$$\{\tilde{A}^{-1}\tilde{x} | \tilde{A} \in A, \tilde{x} \in x\} \tag{1}$$

of solutions $\tilde{y}$ of $\tilde{A}\tilde{y} = \tilde{x}$, where $\tilde{A}$ ranges over a matrix interval A, and $\tilde{x}$ ranges over a vector interval x. We would like to compute the hull $A^H x$ of (1), i.e. the intersection of all intervals containing (1). By imitating the Gauss elimination algorithm for the solution of ordinary linear equations in interval arithmetic, an enclosure $A^G x$ for $A^H x$ is obtained (Moore [17]). But it is well known (see e.g. Hansen and Smith [11], Wongwises [26]) that in many cases, $A^G x$ does not exist (due to division by an interval containing zero) or gives

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pessimistic bounds (due to "dependence"). On the other hand, $A^G x = A^H x$ if A is an interval M-matrix and $x \geqslant 0$, $x \leqslant 0$, or $x \ni 0$ (Barth and Nuding [4], Beeck [5]). If A is an H-matrix, $A^G x$ exists (Alefeld [1]), but nothing has been known about the quality of $A^G x$.

Similarly, interval imitation of Jacobi or Gauss-Seidel iteration produces (in case of convergence) a limiting interval vector $A^F x$, which again is an enclosure of $A^H x$. If A is an interval M-matrix then $A^F x = A^H x$ for all interval vectors x (Barth and Nuding [4]). For other matrices A, these schemes were considered by Gay [9] in 1982. Already, Apostolatos and Kulisch [3], O. Mayer [16], and Alefeld and Herzberger [2] had considered a whole-step and single-step iteration for the related problem $y = x + By$, and for $B = I - A$, its solution y (if it exists) is an enclosure of $A^H x$. As trivial one-dimensional examples show, Jacobi or Gauss-Seidel iteration has a wider convergence domain than whole-step and single-step iteration, a fact which finds its natural explanation in Section 8. Again, apart from the case of nonnegative B (corresponding to M-matrices), no investigation of the quality of the enclosure has been made before.

In order to reduce the amount of overestimation in $A^G x$, Hansen and Smith [11] suggest preconditioning the interval matrix A by left multiplication with an approximate inverse of the midpoint. The resulting problem usually has a diagonally dominant matrix (almost the identity) and is processed by Gauss elimination. In practice, this gives bounds of the same order as bounds obtainable from perturbation theory; see the extensive tests and experimental comparison with other methods in Fitzgerald [8, especially p. 306]. In connection with iterative procedures, preconditioning by an approximate midpoint inverse has been considered by Krawczyk [13], Wongwises [28], and Gay [9]; Gay gives a proof that under a mild hypothesis on A, the radius of the computed enclosure is not much bigger than that of $A^H x$.

In this paper we develop new tools for the study of the above enclosures: an interval version of Ostrowski's [21] comparison operator which we write—in (justified) analogy to the absolute value—as $\langle \cdot \rangle$, and the concept of a sublinear map. Sublinear maps are basic in the sense that the maps A^H, A^G, A^F defined above share the sublinearity axioms with the left multiplication by an interval matrix (Section 3). The Ostrowski operator is tightly connected to estimates for the absolute values; indeed $|A^H x| \leqslant |A^G x| \leqslant \langle A \rangle^{-1} |x|$ (cf. Theorem 4) and the same holds with A^F in place of A^G (cf. Theorem 9). Since $|A^H x|$ is generally of the order of $|A^{-1}|\,|x|$ (with equality if the midpoint of x is zero), where A^{-1} is the hull of the inverses of elements of A, the quality of the enclosures $A^G x$ and $A^F x$ is determined by the ratio $\|\langle A \rangle^{-1}\| / \|A^{-1}\|$.

The contents of the paper is organized as follows. In Section 1, basic definitions and properties of interval arithmetic are reviewed. Since interval arithmetic does not satisfy the standard laws (of fields and vector spaces) and since a lot of nonstandard formulae are used in later sections without reference, it seemed best to list all formulae used.

In Section 2, technical definitions and results are collected about the Ostrowski operator, norms, the spectral radius, M- and H-matrices, and triangular splittings, and some basic properties of regular interval matrices and their inverses are discussed. Sections 3 and 4 then treat the abstract theory of sublinear maps, complemented by a discussion of the motivating examples. As a corollary of the discussion we find in Lemma 13 a very simple (but somewhat crude) enclosure for $A^H x$ in the case that A is an H-matrix.

Section 5 discusses Gauss elimination, aiming at the upper bound for $|A^G x|$ mentioned above. Instead of the conventional algorithmic way we chose an inductive definition of the triangular decomposition in terms of Schur complements, which is more suitable for theoretical purposes; cf. e.g. Newman [20].

In Section 6 preconditioning is considered. We prove two theorems which show the optimality of the midpoint inverse as a preconditioning matrix. On the other hand, we show by an example that preconditioning with the midpoint inverse may destroy the regularity of the matrix.

Sections 7 and 8 are concerned with the iterative determination of an enclosure of (1) by "global" solvers, or an enclosure of those vectors of (1) which lie inside a given interval vector by "local" solvers. Among other things, we show that the Gauss-Seidel iteration (with componentwise intersection) is optimal within the class of all iterations defined by triangular splittings (Theorems 10 and 11). This is in the spirit of Alefeld and Herzberger's [2] results on the single-step iteration; but it also shows that a single-step cycle is inferior to a Gauss-Seidel step. It is also considered when a local solver in fact encloses the global solution $A^H x$; this question is mainly of interest for applications to nonlinear systems, which will be discussed elsewhere.

In the following, I denotes the identity matrix (of any size), and $e^{(i)}$ denotes the ith column of I. All other notation used is defined in Sections 1, 2, or 3.

1. INTERVAL ARITHMETIC

For easy reference, this section contains a summary of well-known properties of interval arithmetic. No proofs are given; they are either trivial, or can be found in the book by Alefeld and Herzberger [2].

1.1. *Partial Ordering and Intervals*

We denote by $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n\times m}$ the set of real numbers, real n-dimensional (column) vectors, and real $n\times m$ matrices, respectively. We shall identify $n\times 1$ matrices with vectors and 1×1 matrices with real numbers, so that

$\mathbb{R} = \mathbb{R}^1 = \mathbb{R}^{1\times 1}$, $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. This allows us a certain economy in definitions and statements.

We equip $\mathbb{R}^{n\times m}$ with the componentwise defined relations $\leqslant$, $<$, $\geqslant$, $>$. Associated to the partial ordering $\leqslant$ is the set $\Pi\mathbb{R}^{n\times m}$ of all *interval matrices*, i.e. subsets of $\mathbb{R}^{n\times m}$ of the form

$$A := [\underline{A}, \overline{A}] := \{\tilde{A} \in \mathbb{R}^{n\times m} | \underline{A} \leqslant \tilde{A} \leqslant \overline{A}\},$$

where $\underline{A}, \overline{A} \in \mathbb{R}^{n\times m}$, $\underline{A} \leqslant \overline{A}$. We shall identify a degenerate interval matrix A with $\underline{A} = \overline{A}$ and its unique endpoint $\underline{A} = \overline{A}$; in this way, $\mathbb{R}^{n\times m}$ is naturally embedded into $\Pi\mathbb{R}^{n\times m}$. In this paper, the term *matrix* will refer to either an interval matrix or a real matrix; what is meant will be always clear from the context.

To each matrix $A \in \Pi\mathbb{R}^{n\times m}$ we associate

the *endpoints* $\underline{A}$, $\overline{A}$ such that $A = [\underline{A}, \overline{A}]$,
the *midpoint* $\check{A} := \frac{1}{2}(\overline{A} + \underline{A})$,
the *radius* $\rho(A) := \frac{1}{2}(\overline{A} - \underline{A})$,
the *radial part* $\mathring{A} := [-\rho(A), \rho(A)]$,
the *interior* $\mathrm{int}(A) := \{\tilde{A} \in \mathbb{R}^{n\times m} | \underline{A} < \tilde{A} < \overline{A}\}$, and
the *absolute value* $|A| := \sup\{|\tilde{A}| | \tilde{A} \in A\}$,

where the absolute value of a real matrix is taken componentwise. Note that $\underline{A}, \overline{A}, \check{A}, \rho(A), |A|$ are real matrices and $\mathring{A}$ is an interval matrix, whereas $\mathrm{int}(A)$ is *not* an interval matrix. We have

$$\check{A} \in A, \qquad \rho(A) \geqslant 0,$$

$$\underline{A} = \check{A} - \rho(A), \qquad \overline{A} = \check{A} + \rho(A),$$

$$|A| = \sup(|\underline{A}|, |\overline{A}|) = |\check{A}| + \rho(A).$$

We call a matrix $A \in \Pi\mathbb{R}^{n\times m}$

thin if $\rho(A) = 0$ ($\Leftrightarrow$ $\underline{A} = \overline{A}$ $\Leftrightarrow$ $A = \check{A}$),
thick if $\rho(A) > 0$ ($\Leftrightarrow$ $\mathrm{int}(A) \neq \varnothing$), and
symmetric if $\check{A} = 0$ ($\Leftrightarrow$ $A = \mathring{A}$).

We extend the relations $\leqslant$, $<$ to interval matrices by the definition

$$A \leqslant B \quad :\Leftrightarrow \quad \tilde{A} \leqslant \tilde{B} \quad \text{for all } \tilde{A} \in A,\ \tilde{B} \in B,$$

$$A < B \quad :\Leftrightarrow \quad \tilde{A} < \tilde{B} \quad \text{for all } \tilde{A} \in A,\ \tilde{B} \in B.$$

Then

$$A \leqslant B \quad \Leftrightarrow \quad \bar{A} \leqslant \underline{B},$$
$$A < B \quad \Leftrightarrow \quad \bar{A} < \underline{B},$$

and for the set intersection $\cap$ we have

$$A \cap B = \begin{cases} [\sup(\underline{A}, \underline{B}), \inf(\bar{A}, \bar{B})] & \text{if } \underline{A} \leqslant \bar{B}, \quad \underline{B} \leqslant \bar{A}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Set inclusion and nonempty intersection can be characterized in terms of midpoint and radius:

(I1) $B \subseteq A \Leftrightarrow |\check{B} - \check{A}| + \rho(B) \leqslant \rho(A)$,
(I2) $B \subseteq \operatorname{int}(A) \Leftrightarrow |\check{B} - \check{A}| + \rho(B) < \rho(A)$,
(I3) $A \cap B \neq \varnothing \Leftrightarrow |\check{B} - \check{A}| \leqslant \rho(A) + \rho(B)$.

For real matrices $A \in \mathbb{R}^{n \times m}$, we denote the (i, k) entry of A by A_{ik}. For interval matrices $A = [\underline{A}, \bar{A}]$ we extend this notion by defining the (i, k) *entry* of A as $A_{ik} := [\underline{A}_{ik}, \bar{A}_{ik}]$ (this is consistent with the identification of thin interval matrices and their unique endpoint). Thus we may regard an interval matrix as a matrix with interval entries.

For $m = 1$ and for $n = m = 1$, the above concepts immediately apply respectively to *interval vectors*, i.e. elements $x \in \Pi\mathbb{R}^n := \Pi\mathbb{R}^{n \times 1}$, and to (ordinary real) *intervals*, i.e. elements $a \in \Pi\mathbb{R} := \Pi\mathbb{R}^{1 \times 1}$. Again, the term *vector* may refer to either interval vectors or real vectors, depending on the context. The ith entry of a real vector $x \in \mathbb{R}^n$ is denoted by x_i, and the *i th entry* of an interval vector $x = [\underline{x}, \bar{x}]$ is defined as $x_i := [\underline{x}_i, \bar{x}_i]$.

1.2. *Operations*

If Σ is a bounded subset of $\mathbb{R}^{n \times m}$, we denote by

$$\square\Sigma := [\inf \Sigma, \sup \Sigma]$$

the *interval hull* of Σ, i.e. the intersection of all interval matrices containing Σ. The hull has the properties

$$\Sigma \subseteq A \in \Pi\mathbb{R}^{n \times m} \quad \Rightarrow \quad \square\Sigma \subseteq A$$
$$\Sigma' \subseteq \Sigma \quad \Rightarrow \quad \square\Sigma' \subseteq \square\Sigma,$$

and

$$|\square\Sigma| = \sup\{|\tilde{A}| \mid \tilde{A} \in \Sigma\}.$$

If $A, B \in \Pi\mathbb{R}^{n\times m}$, then the *sum* and *difference* of A and B are defined

$$A \pm B := \square\{\tilde{A} \pm \tilde{B} | \tilde{A} \in A, \tilde{B} \in B\}. \tag{1}$$

If $A \in \Pi\mathbb{R}^{n\times m}$, $B \in \Pi\mathbb{R}^{m\times p}$ then the *product* of A and B is defined as

$$AB := A \cdot B := \square\{\tilde{A}\tilde{B} | \tilde{A} \in A, \tilde{B} \in B\}. \tag{2}$$

In particular, this definition applies to the multiplication of an interval vector $A \in \Pi\mathbb{R}^n$ by an interval $B \in \Pi\mathbb{R}$.

If $A, B \in \Pi\mathbb{R}$ and $0 \notin B$ then the quotient of A and B is defined as

$$A/B := \square\{\tilde{A}/\tilde{B} | \tilde{A} \in A, \tilde{B} \in B\}. \tag{3}$$

The expressions (1), (2), (3) can be calculated from the endpoints of A and B as follows: For $A, B \in \Pi\mathbb{R}^{n\times m}$ we have

$$A + B = [\underline{A} + \underline{B}, \overline{A} + \overline{B}], \qquad A - B = [\underline{A} - \overline{B}, \overline{A} - \underline{B}];$$

in particular

$$-A := 0 - A = [-\overline{A}, -\underline{A}].$$

For $A, B \in \Pi\mathbb{R}$ we have

$$AB = \square\{\underline{A}\underline{B}, \underline{A}\overline{B}, \overline{A}\underline{B}, \overline{A}\overline{B}\},$$

and

$$A/B = \square\{\underline{A}/\underline{B}, \underline{A}/\overline{B}, \overline{A}/\underline{B}, \overline{A}/\overline{B}\} \qquad \text{if} \quad 0 \notin B.$$

Finally, the product of $A \in \Pi\mathbb{R}^{n\times m}$ and $B \in \Pi\mathbb{R}^{m\times p}$ can be calculated componentwise by

$$(AB)_{ik} = \sum_{j=1}^{m} A_{ij}B_{jk}, \tag{4}$$

where the operations on the right-hand side are interval operations.

LEMMA 1 (Inclusion properties).

(i) *If* $A' \subseteq A$ *then*

$$\rho(A') \leqslant \rho(A), \qquad |A'| \leqslant |A|.$$

(ii) *For* $* = +, -, \cdot,$ *or* $/$, *if* $A * B$ *is defined and* $A' \subseteq A$, $B' \subseteq B$, *then* $A' * B'$ *is defined and satisfies*

$$A' * B' \subseteq A * B \qquad (\textit{inclusion isotonicity}).$$

(iii) *For* $A, B \in \Pi\mathbb{R}^{n\times m}$, $\tilde{C} \in \mathbb{R}^{n\times m}$,

$$\tilde{C} \in A \pm B \quad \Rightarrow \quad \exists \tilde{A} \in A, \tilde{B} \in B: \quad \tilde{C} = \tilde{A} \pm \tilde{B}.$$

(iv) *For* $A \in \Pi\mathbb{R}^{n\times m}$, $\tilde{x} \in \mathbb{R}^m$, $\tilde{y} \in \mathbb{R}^n$,

$$\tilde{y} \in A\tilde{x} \quad \Rightarrow \quad \exists \tilde{A} \in A: \quad \tilde{y} = \tilde{A}\tilde{x}.$$

1.3. *The Basic Laws*

Interval arithmetic has some anomalies which distinguish it from ordinary real arithmetic. For example,

$$A - A = 0 \quad \Rightarrow \quad A \text{ thin}.$$

Other familiar laws, like the distributive law and the associative law for matrix multiplication, also fail to hold generally; for counterexamples see Alefeld and Herzberger [2]. Therefore some care is needed in proofs involving interval-arithmetic calculations. In the following collection of rules, all variables are interval matrices, and we assume that their sizes are such that the occurring expressions are defined.

Rules for the Absolute Value.

(B1) $\exists \tilde{A} \in A$: $|A| = |\tilde{A}|$,
(B2) $|A| = |\check{A}| + \rho(A)$,
(B3) $|A| - |B| \leqslant |A \pm B| \leqslant |A| + |B|$,
(B4) $|AB| \leqslant |A|\,|B|$, with equality if $A, B \in \Pi\mathbb{R}$.

Moreover, if A is symmetric then

(B5) $|AB| = |A|\,|B|$, $|BA| = |B|\,|A|$,
(B6) $AB = A|B|$, $BA = |B|A$.

Rules for the Midpoint.

(B7) $A = \check{A} + \mathring{A},\ \mathring{A} = A - \check{A} = \check{A} - A$,
(B8) $C = A \pm B \ \Rightarrow\ \check{C} = \check{A} \pm \check{B}$,

and if one of A, B is thin or symmetric then

(B9) $C = AB \ \Rightarrow\ \check{C} = \check{A}\check{B}$.

Rules for the Radius.

(B10) $\rho(A) = \rho(\mathring{A}) = |\mathring{A}| \leqslant |A|$,
(B11) $\rho(A \pm B) = \rho(A) + \rho(B)$,
(B12) $\rho(A)|B| \leqslant \rho(AB) \leqslant \rho(A)|B| + |\check{A}|\rho(B)$,
(B13) $|A|\rho(B) \leqslant \rho(AB) \leqslant |A|\rho(B) + \rho(A)|\check{B}|$.

In particular, if A is thin or B is symmetric then

(B14) $\rho(AB) = |A|\rho(B),\ \rho(BA) = \rho(A)|B|$.

Algebraic Properties.

(B15) $A + B = B + A$,
(B16) $-(A - B) = B - A$,
(B17) $A + (B \pm C) = (A + B) \pm C$,
(B18) $A(B \pm C) \subseteq AB \pm AC$,
(B19) $(A \pm B)C \subseteq AC \pm BC$.

Moreover, if A is thin then

(B20) $A(B \pm C) = AB \pm AC$,
(B21) $(B \pm C)A = BA \pm CA$,
(B22) $(AB)C \subseteq A(BC)$,
(B23) $B(CA) \subseteq (BC)A$.

Direct Sums. Let $A, B \in \Pi\mathbb{R}^{n \times m}$. If $A_{ik} \neq 0 \ \Rightarrow\ B_{ik} = 0$, we write

(B24) $A \textcircled{\scriptsize$\pm$} B := A \pm B$.

The following rules hold:

(B25) $|A \textcircled{\scriptsize$\pm$} B| = |A| + |B|$,
(B26) $C(A \textcircled{\scriptsize$\pm$} B) = CA \pm CB$,
(B27) $(A \textcircled{\scriptsize$\pm$} B)C = AC \pm BC$.

Endpoints of Products. If $A \geqslant 0$ then

(B28) $AB = [\underline{A}\underline{B}, \overline{A}\overline{B}],\ BA = [\underline{B}\underline{A}, \overline{B}\overline{A}]$ for $B \geqslant 0$,
(B29) $AB = [\overline{A}\underline{B}, \overline{A}\overline{B}],\ BA = [\underline{B}\overline{A}, \overline{B}\overline{A}]$ for $B \ni 0$,
(B30) $AB = [\overline{A}\underline{B}, \underline{A}\overline{B}],\ BA = [\underline{B}\overline{A}, \overline{B}\underline{A}]$ for $B \leqslant 0$.

1.4. *Topology*

As a measure of closeness of two matrices $A, B \in \Pi\mathbb{R}^{n\times m}$ we introduce the *distance*

(D1) $q(A, B) := |\check{A} - \check{B}| + |\rho(A) - \rho(B)|$.

Note that the distance is a real *matrix*. But for a monotone matrix norm, $\|q(A, B)\|$ defines a metric on $\Pi\mathbb{R}^{n\times m}$. In such a metric, a sequence $A^{(l)}$ converges to A iff $\check{A}^{(l)} \to \check{A}$ and $\rho(A^{(l)}) \to \rho(A)$, iff $\underline{A}^{(l)} \to \underline{A}$ and $\overline{A}^{(l)} \to \overline{A}$. In particular, $\Pi\mathbb{R}^{n\times m}$ with such a metric is a locally compact metric space. Moreover, all operations $+, -, \cdot, /, \rho, \check{\ }, \underline{\ }, \overline{\ }, |\ |$ are continuous. Finally, a sequence $A^{(0)} \supseteq A^{(1)} \supseteq A^{(2)} \supseteq \cdots$ of nested intervals always converges to the limit $A = \bigcap_{l\geqslant 0} A^{(l)}$.

Rules for the Distance.

(D2) $q(A, B) \geqslant 0$, with equality iff $A = B$,
(D3) $q(A + C, B + C) = q(A, B)$,
(D4) $q(A, C) \leqslant q(A, B) + q(B, C)$,
(D5) $q(A + C, B + D) \leqslant q(A, B) + q(C, D)$,
(D6) $q(AC, BC) \leqslant q(A, B)|C|$,
(D7) $q(AB, AC) \leqslant |A| q(B, C)$.

2. TOOLS FOR THE ANALYSIS OF LINEAR EQUATION SOLVERS

In this section we consider several notions and tools which are of importance for the development of later sections. Specifically we give natural extensions of the definitions of regular matrices, *M*-matrices, and *H*-matrices to interval matrices, and discuss extensions of norms and Ostrowski's comparison operator to interval arguments. Then we prove some basic results on triangular splittings of an interval matrix, thus preparing for the iterative solution of linear interval equations in Sections 7 and 8. Finally we define the inverse of a regular interval matrix and look at some easy special cases.

2.1. *Regularity*

A square matrix $A \in \Pi\mathbb{R}^{n\times n}$ is called *regular* if all $\tilde{A} \in A$ are regular, i.e. if $\tilde{A}\tilde{x} = 0 \Rightarrow \tilde{x} = 0$. This can be rewritten in several equivalent ways:

LEMMA 2. *For $A \in \Pi\mathbb{R}^{n\times n}$, the following statements are equivalent:*

(i) *A is regular,*
(ii) $0 \in A\tilde{x} \Rightarrow \tilde{x} = 0$,
(iii) $|\check{A}\tilde{x}| \leqslant \rho(A)|\tilde{x}| \Rightarrow \tilde{x} = 0$,

(iv) $\check{A}$ *is regular, and*

$$\rho(A)|\check{A}^{-1}\tilde{x}| \geqslant |\tilde{x}| \quad \Rightarrow \quad \tilde{x} = 0.$$

Proof. By Lemma 1(iv), the first two statements are equivalent, (iii) is a restatement of (ii) [use (B14)], and the substitution of $\check{A}^{-1}\tilde{x}$ for $\tilde{x}$ in (iii) gives (iv). ∎

2.2. *The Ostrowski Operator*

In 1938, Ostrowski [21] defined a comparison operator for square real matrices, relevant for the discussion of diagonal dominant matrices and generalizations. Here we extend his definition to square interval matrices and show that the resulting operator $\langle\cdot\rangle: \Pi\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ has properties which closely resemble the absolute value. In the one-dimensional case, the Ostrowski operator is simply the map $\langle\cdot\rangle: \Pi\mathbb{R} \to \mathbb{R}$ defined by

$$\langle x\rangle := \inf\{|\tilde{x}|\,|\,\tilde{x} \in x\},$$

or, explicitly, $\langle x\rangle = \min(|\underline{x}|, |\bar{x}|)$ if $0 \notin x$, $\langle x\rangle = 0$ otherwise. In the matrix case, the Ostrowski operator is defined in terms of the components as the map $\langle\cdot\rangle: \Pi\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ such that

$$\langle A\rangle = (\alpha_{ik}), \qquad \alpha_{ii} = \langle A_{ii}\rangle, \quad \alpha_{ik} = -|A_{ik}| \text{ if } i \neq k.$$

Rules for the Ostrowski Operator. If $A, B \in \Pi\mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times m}$ then

(O1) $\langle A \pm B\rangle \geqslant \langle A\rangle - |B|$,
(O2) $|AC| \geqslant \langle A\rangle|C|$,
(O3) $B \subseteq A \Rightarrow \langle B\rangle \geqslant \langle A\rangle$,
(O4) $\langle A\rangle \geqslant \langle\check{A}\rangle - \rho(A)$,
(O5) $\exists\tilde{A} \in A$: $\langle A\rangle = \langle\tilde{A}\rangle$.

Moreover, if $0 \notin A_{ii}$ for $i = 1,\dots,n$, then equality holds in (O4).

Proof. (O1): $\tilde{C} \in A \pm B \Rightarrow \exists\tilde{A} \in A$, $\tilde{B} \in B$ with $\tilde{C} = \tilde{A} \pm \tilde{B}$ [Lemma 1(iii)]. So $|\tilde{C}| = |\tilde{A} \pm \tilde{B}| \geqslant |\tilde{A}| - |\tilde{B}| \geqslant \langle A\rangle - |B|$, whence also $\langle A \pm B\rangle = \inf\{|\tilde{C}|\,|\,\tilde{C} \in A \pm B\} \geqslant \langle A\rangle - |B|$.

(O2): $|AC|_{ij} = |\Sigma A_{ik}C_{kj}| \geqslant |A_{ii}C_{ij}| - \Sigma_{i\neq k}|A_{ik}C_{kj}| = |A_{ii}|\,|C_{ij}| - \Sigma_{i\neq k}|A_{ik}|\,|C_{kj}| \geqslant \langle A_{ii}\rangle|C_{ij}| - \Sigma_{i\neq k}|A_{ik}|\,|C_{kj}| = (\langle A\rangle|C|)_{ij}$.

(O3): Immediate from the definition.

(O4): If $0 < A$ then $\langle A \rangle = \underline{A} = \check{A} - \rho(A) = \langle \check{A} \rangle - \rho(A)$. If $0 > A$ then $\langle A \rangle = -\bar{A} = -\check{A} - \rho(A) = \langle \check{A} \rangle - \rho(A)$. But if $0 \in A$ then $\langle \check{A} \rangle \leqslant \rho(A)$, whence $\langle \check{A} \rangle - \rho(A) \leqslant 0 = \langle A \rangle$. This proves (O4) in the case $m = n = 1$. In general we now have $\langle A \rangle_{ii} = \langle A_{ii} \rangle \geqslant \langle \check{A}_{ii} \rangle - \rho(A_{ii})$ and $\langle A \rangle_{ik} = -|A_{ik}| = -|\check{A}_{ik}| - \rho(A_{ik})$ for $i \neq k$, whence $\langle A \rangle \geqslant \langle \check{A} \rangle - \rho(A)$. If $0 \notin A_{ii}$ for $i = 1, \dots, n$ then equality holds throughout the argument.

(O5): Define $\tilde{A}$ on the diagonal by $\tilde{A}_{ii} = \underline{A}_{ii}$ if $0 < A_{ii}$, $\tilde{A}_{ii} = \bar{A}_{ii}$ if $0 > A_{ii}$, $\tilde{A}_{ii} = 0$ if $0 \in A_{ii}$, and off the diagonal by $\tilde{A}_{ik} = \bar{A}_{ik}$ if $0 \leqslant \check{A}_{ik}$, $\tilde{A}_{ik} = \underline{A}_{ik}$ if $0 > \check{A}_{ik}$. Then $\langle \tilde{A} \rangle = \langle A \rangle$. ■

2.3. *Norms and Spectral Radius*

Let $\|\cdot\|$ denote a monotone vector norm in $\mathbb{R}^n$, i.e. a map into $\mathbb{R}$ such that the rules

(N1) $\|x\| = \||x|\| > 0$ if $x \neq 0$,
(N2) $\|x\alpha\| = \|x\| \cdot |\alpha|$,
(N3) $\|x \pm y\| \leqslant \|x\| + \|y\|$,

which imply (cf. Householder [12]) the rule

(N4) $|y| \leqslant x \Rightarrow \|y\| \leqslant \|x\|$,

are valid for all $x, y \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. Then for $A \in \mathbb{R}^{n \times n}$ we denote by $\|A\|$ the (nonstandard) associated matrix norm

$$\|A\| := \left\| \sup_{\|x\|=1} |Ax| \right\|;$$

by Neumaier [19], the rules

(N5) $\|A\| = \||A|\| > 0$ if $A \neq 0$,
(N6) $\|Ax\| \leqslant \|A\| \, \|x\|$,
(N7) $|B| \leqslant A \Rightarrow \|B\| \leqslant \|A\|$

hold for $x \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n \times n}$. In this paper we mainly use *scaled maximum norms*, defined by

(N8) $\|x\|_u := \max_i \dfrac{|x_i|}{u_i} = \inf\{\alpha \geqslant 0 \mid |x| \leqslant \alpha u\}$ for some fixed *positive* vector $u \in \mathbb{R}^n$.

$\|\cdot\|_u$ is a monotone vector norm, and the associated matrix norm satisfies

(N9) $\|A\|_u = \||A|u\| = \max_i (\Sigma_k |A_{ik}| u_k)/u_i$;

cf. Schröder [26, p. 38]. In particular, we mention the formula

(N10) $|A|u \leqslant \alpha u \Rightarrow \|A\|_u \leqslant \alpha$.

We extend monotone norms to interval vectors $x \in \Pi\mathbb{R}^n$ and interval matrices $A \in \Pi\mathbb{R}^{n\times n}$ by the definition

$$\|x\| := \||x|\|, \qquad \|A\| := \||A|\|.$$

By (N1) and (N5), this definition is consistent with the identification of a thin interval vector (matrix) and the corresponding real vector (matrix). It is easily checked that the rules (N1)–(N10) remain valid for all $x, y \in \Pi\mathbb{R}^n$, $A, B \in \Pi\mathbb{R}^{n\times n}$, $\alpha \in \mathbb{R}$. Moreover, we have the inclusion rules

(N11) $y \subseteq x \Rightarrow \|y\| \leqslant \|x\|$,
(N12) $B \subseteq A \Rightarrow \|B\| \leqslant \|A\|$.

Related to matrix norms is the *spectral radius* of matrices, defined by

$$\sigma(A) := \max\{\, |\lambda| \,|\, \lambda \text{ is an eigenvalue of } A\,\}.$$

We list here a number of well-known properties of the spectral radius, which are either trivial or consequences of the Perron-Frobenius theory of nonnegative matrices. For details, see e.g. Bermann and Plemmons [6], Schröder [26], Varga [27].

Properties of the Spectral Radius. Let $A, B \in \mathbb{R}^{n\times n}$, $\alpha \in \mathbb{R}$. Then

(P1) For all matrix norms, $\sigma(A) \leqslant \|A\|$,
(P2) $|A| \leqslant B \Rightarrow \sigma(A) \leqslant \sigma(B)$,
(P3) $\sigma(AB) = \sigma(BA)$,
(P4) $\sigma(A) < 1 \Rightarrow I - A$ is regular,
(P5) $\sigma(|A|) < 1 \Rightarrow |(I-A)^{-1}| \leqslant (I - |A|)^{-1}$.

Moreover, if $A \geqslant 0$ then

(P6) $\sigma(A) = \inf_{u>0} \|A\|_u$
(P7) $\sigma(A) < \alpha \Leftrightarrow \exists u > 0$: $Au < \alpha u$,
(P8) $\alpha > 0 \Rightarrow [\sigma(A) \geqslant \alpha \Leftrightarrow \exists u \geqslant 0$: $Au \geqslant \alpha u \neq 0]$.

2.4. *M-matrices*

We call a square interval matrix $A \in \Pi\mathbb{R}^{n\times n}$ an *M-matrix* if $\overline{A}_{ik} \leqslant 0$ for $i \neq k$ and $Au > 0$ for some positive vector $u \in \mathbb{R}^n$. Since $Au = [\underline{A}u, \overline{A}u]$ for $u > 0$, an interval matrix A is an M-matrix iff $\underline{A}$ is a real M-matrix and $\overline{A}$ has no positive off-diagonal entries; therefore, our definition is equivalent with that used e.g. in Barth and Nuding [4]. For thin matrices A, our definition reduces to one of the many known equivalent definitions of a (nonsingular) real M-matrix; cf. Berman and Plemmons [6]. There one can also find a proof of the following lemma.

LEMMA 3. *Let $A, B \in \mathbb{R}^{n\times n}$, and suppose that A is an M-matrix, $B \geqslant 0$. Then*

(i) *A is regular, $A^{-1} \geqslant 0$, and $\langle A \rangle = A$,*
(ii) *$x > 0 \Rightarrow A^{-1}x > 0$,*
(iii) *$A - B$ is an M-matrix iff $\sigma(A^{-1}B) < 1$.*

LEMMA 4. *Let $A \in \Pi\mathbb{R}^{n\times n}$ be an M-matrix, $B \subseteq A$. Then B is an M-matrix. In particular, all $\tilde{A} \in A$ are M-matrices.*

Proof. Immediate from the definition, since $B_{ik} \subseteq A_{ik}$, $Bu \subseteq Au$. ∎

In the next lemma (whose first part is well known) we show a simple way of bounding the inverse of a real M-matrix.

LEMMA 5. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix, and let $u, v \in \mathbb{R}^n$ be positive vectors such that $Au \geqslant v$. Then for $\alpha \in \mathbb{R}$,*

$$|x| \leqslant \alpha v \quad \Rightarrow \quad |A^{-1}x| \leqslant \alpha u; \tag{1}$$

in particular

$$0 \leqslant A^{-1} \leqslant uw^T \tag{2}$$

for the vector w whose components are $w_i = v_i^{-1}$.

Proof. Since A is an M-matrix, $A^{-1} \geqslant 0$; hence $A^{-1}v \leqslant u$. Therefore $|x| \leqslant \alpha v$ implies $|A^{-1}x| \leqslant A^{-1}|x| \leqslant A^{-1}\alpha v \leqslant \alpha u$, i.e. (1). In particular, $|e^{(k)}| \leqslant v_k^{-1}v$, whence $(A^{-1})_{ik} = (A^{-1}e^{(k)})_i \leqslant (v_k^{-1}u)_i = u_i v_k^{-1} = (uw^T)_{ik}$, which implies (2). ∎

Note that the lemma holds more generally for inverse positive matrices (regular matrices A with $A^{-1} \geqslant 0$). Vectors $u, v > 0$ with $Au \geqslant v$ can be conveniently found as follows. The equation $Ax = e = (1,\dots,1)^T$ has as solution $x = A^{-1}e > 0$. Hence $u = x$ or sufficiently good approximations u of x are positive and have positive $v = Au \approx Ax = e$. In practice, the approximation u can often be quite crude, e.g. $u = e$ or $u = \mathrm{Diag}(A)^{-1}e$.

2.5. *H-matrices*

We call a square interval matrix $A \in \Pi\mathbb{R}^{n\times n}$ an *H-matrix* if $\langle A\rangle u > 0$ for some positive vector $u \in \mathbb{R}^n$—equivalently, if $\langle A\rangle$ is a real *M*-matrix (cf. Alefeld [1]). For thin A, this is consistent with the standard definition. For $u = e = (1,\dots,1)^T$ the condition $\langle A\rangle u > 0$ is equivalent to the diagonal-dominance definition

$$\sum_{k\neq i} |A_{ik}| < \langle A_{ii}\rangle \qquad \text{for} \quad i = 1,\dots,n.$$

LEMMA 6. *Every regular (lower or upper) triangular matrix $A \in \Pi\mathbb{R}^{n\times n}$ is an H-matrix.*

Proof. $0 \notin A_{ii}$, since A is regular; hence $\langle A\rangle$ has a positive diagonal, and classical forward or back substitution shows that the solution u of $\langle A\rangle u = v$ is positive for every $v > 0$. ∎

LEMMA 7.

(i) *If $A \in \Pi\mathbb{R}^{n\times n}$ is an H-matrix and $B \subseteq A$, then B is an H-matrix too.*
(ii) *Every H-matrix $A \in \Pi\mathbb{R}^{n\times n}$ is regular.*

Proof. (i): If $u > 0$, $\langle A\rangle u > 0$, and $B \subseteq A$, then $\langle B\rangle u \geqslant \langle A\rangle u > 0$ by (O3), whence B is an *H*-matrix.

(ii): If $\tilde{A} \in A$, then $\tilde{A}$ is a real *H*-matrix by (i), and hence nonsingular by a classical result of Ostrowski [21]. Hence A is regular. ∎

2.6. *Triangular Splittings*

In the context of iterative solution of linear interval equations we shall use the fact that the coefficient matrix $A \in \Pi\mathbb{R}^{n\times n}$ can be written as a difference of two other matrices, $A = L - E$. We call such a representation a *splitting* of A. The splitting $A = L - E$ is called *triangular* if L is a lower triangular matrix, *direct* if $A = L \ominus E$ and E has zero diagonal, and *strong* if $\langle L\rangle - |E|$ is an *M*-matrix.

In particular, we use in the following the *Richardson splitting*

$$A = I - E \qquad (E := I - A),$$

the *Jacobi splitting*

$$A = D - E,$$

where $D =: \mathrm{Diag}(A)$ and E are defined by $D_{ii} = A_{ii}$, $E_{ii} = 0$, and $D_{ik} = 0$, $E_{ik} = -A_{ik}$ if $i \neq k$, and the *Gauss-Seidel splitting*

$$A = L - R,$$

where L and R are defined by $L_{ik} = 0$, $R_{ik} = -A_{ik}$ if $i < k$, and $L_{ik} = A_{ik}$, $R_{ik} = 0$ if $i \geqslant k$. All three splittings are triangular; moreover the Jacobi and Gauss-Seidel splitting are direct splittings.

LEMMA 8.

(i) *If $A = L - E$ is a direct splitting then $\langle A \rangle = \langle L \rangle - |E|$.*
(ii) *A direct splitting of a matrix $A \in \Pi\mathbb{R}^{n\times n}$ is strong if and only if A is an H-matrix.*

Proof. Since we have a direct splitting, $L_{ii} = A_{ii}$, $E_{ii} = 0$, whence $\langle A \rangle_{ii} = \langle L \rangle_{ii} = (\langle L \rangle - |E|)_{ii}$. Similarly, for $i \neq k$ we have either $L_{ik} = 0$, $E_{ik} = -A_{ik}$, and then $\langle A \rangle_{ik} = -|A_{ik}| = -|E_{ik}| = (\langle L \rangle - |E|)_{ik}$; or $L_{ik} = A_{ik}$, $E_{ik} = 0$, and then $\langle A \rangle_{ik} = -|A_{ik}| = \langle L \rangle_{ik} = (\langle L \rangle - |E|)_{ik}$. This proves (i). The second part follows then immediately from (i) and the definitions. ■

LEMMA 9. *If the matrix $A \in \Pi\mathbb{R}^{n\times n}$ has a strong splitting $A = L - E$ then A and L are H-matrices.*

Proof. If $A = L - E$ is a strong splitting then $\langle L \rangle - |E|$ is an M-matrix, whence $(\langle L \rangle - |E|)u > 0$ for some $u > 0$. But then $\langle A \rangle u = \langle L - E \rangle u \geqslant (\langle L \rangle - |E|)u > 0$ by (O1), so that A is an H-matrix, and $\langle L \rangle u > |E|u \geqslant 0$, so that L is an H-matrix. ■

LEMMA 10. *The splitting $A = L - E$ of a matrix $A \in \Pi\mathbb{R}^{n\times n}$ is strong if and only if L is an H-matrix and*

$$\sigma(\langle L \rangle^{-1}|E|) < 1. \tag{3}$$

Proof. By the previous lemma, $A = L - E$ is a strong splitting iff $\langle L \rangle$ and $\langle L \rangle - |E|$ are M-matrices. By Lemma 3(iii), this holds iff L is an H-matrix and (3) holds. ■

2.7. The Inverse of an Interval Matrix

For a regular matrix $A \in \mathbb{IR}^{n\times n}$, we define

$$A^{-1} := \square\{\tilde{A}^{-1} \mid \tilde{A} \in A\}. \tag{4}$$

Clearly, this is consistent with the identification of thin matrices and real matrices. For $n = 1$ we have

$$a^{-1} = 1/a = [\bar{a}^{-1}, \underline{a}^{-1}] \qquad \text{if} \quad a \in \mathbb{IR},$$

and we therefore have

$$A^{-1} = [\bar{A}^{-1}, \underline{A}^{-1}] \qquad \text{if} \quad A \in \mathbb{IR}^{n\times n} \text{ is diagonal.}$$

From this we immediately find the relations

$$(A^{-1})^{-1} = A, \qquad |A^{-1}| = \langle A\rangle^{-1}$$

for regular diagonal matrices. This relation with the Ostrowski-operator generalizes to M-matrices and H-matrices as follows.

LEMMA 11. *If $A \in \mathbb{IR}^{n\times n}$ is an H-matrix, then*

$$|A^{-1}| \leqslant \langle A\rangle^{-1}. \tag{5}$$

Equality holds e.g. if A is an M-matrix.

Proof. Each $\tilde{A} \in A$ is itself an H-matrix and satisfies $\langle\tilde{A}\rangle \geqslant \langle A\rangle$. By (O2) we have $I = |\tilde{A}\tilde{A}^{-1}| \geqslant \langle\tilde{A}\rangle|\tilde{A}^{-1}| \geqslant \langle A\rangle|\tilde{A}^{-1}|$, and since $\langle A\rangle$ is an M-matrix, multiplication by $\langle A\rangle^{-1} \geqslant 0$ gives $|\tilde{A}^{-1}| \leqslant \langle A\rangle^{-1}$. Therefore, $|A^{-1}| = |\square\{\tilde{A}^{-1} \mid \tilde{A} \in A\}| = \sup\{|\tilde{A}^{-1}| \mid \tilde{A} \in A\} \leqslant \langle A\rangle^{-1}$, and (5) holds. If A is an M-matrix then $\langle A\rangle = \underline{A}$, whence $|\underline{A}^{-1}| = \underline{A}^{-1} = \langle A\rangle^{-1}$, so that the bound is attained. ∎

In general, the computation of A^{-1} for a regular interval matrix A—and for certain A even the estimation of $|A^{-1}|$—is a difficult problem; the known methods all require a time exponential in n (cf. Rohn [23]). But in an important special case, the following explicit result is available; it is a consequence of Exercises 2.4 and 2.6 in Schröder [26, Chapter II].

LEMMA 12. *Suppose that for* $A \in \Pi\mathbb{R}^{n\times n}$, *the matrices* $\underline{A}$ *and* $\overline{A}$ *are regular, and* $\underline{A}^{-1} \geqslant 0$, $\overline{A}^{-1} \geqslant 0$. *Then* A *is regular, and*

$$A^{-1} = \left[\overline{A}^{-1}, \underline{A}^{-1}\right]. \tag{6}$$

In particular, this holds when A *is an M-matrix.*

Proof. Pick $v > 0$. Then $u := \underline{A}^{-1}v > 0$, since $\underline{A}^{-1} \geqslant 0$ and $\underline{A}^{-1}$ is regular. Now if $\tilde{A} \in A$ then $\underline{A} \leqslant \tilde{A} \leqslant \overline{A}$, whence

$$\overline{A}^{-1}\tilde{A} \leqslant I \leqslant \underline{A}^{-1}\tilde{A}. \tag{7}$$

Hence $\tilde{B} := \overline{A}^{-1}\tilde{A}$ satisfies $\tilde{B} \leqslant I$ and $\tilde{B}u = \overline{A}^{-1}\tilde{A}u \geqslant \overline{A}^{-1}\underline{A}u = \overline{A}^{-1}v > 0$. Therefore, $\tilde{B}$ is an M-matrix. In particular, $\tilde{B}$ and $\tilde{A} = \overline{A}\tilde{B}$ are regular. Now $\tilde{A}^{-1} = \tilde{B}^{-1}\overline{A}^{-1} \geqslant 0$; hence (7) implies $\overline{A}^{-1} \leqslant \tilde{A}^{-1} \leqslant \underline{A}^{-1}$, and the bounds are attained for $\tilde{A} = \overline{A}$ and $\tilde{A} = \underline{A}$. Therefore (6) holds. If A is an M-matrix, then $\underline{A}$ and $\overline{A}$ are M-matrices and the hypothesis is satisfied. ■

Finally we mention that if D_1 and D_2 are thin regular *diagonal* matrices and $A = D_1BD_2$, B regular, then $A^{-1} = D_2^{-1}B^{-1}D_1^{-1}$. In particular, this applies when D_1 and D_2 are *signature matrices*, i.e. diagonal matrices with diagonal entries $+1$ or -1.

3. SUBLINEAR MAPS

We call a map $S: \Pi\mathbb{R}^n \to \Pi\mathbb{R}^n$ *sublinear* if the axioms

(S1) $x \subseteq y \Rightarrow Sx \subseteq Sy$ (inclusion isotonicity),
(S2) $\alpha \in \mathbb{R} \Rightarrow S(x\alpha) = (Sx)\alpha$ (homogeneity),
(S3) $S(x \pm y) \subseteq Sx \pm Sy$ (subadditivity)

are valid for all $x, y \in \Pi\mathbb{R}^n$, and *linear* if (S1), (S2), and

(S3a) $S(x \pm y) = Sx \pm Sy$ (additivity)

hold for all $x, y \in \Pi\mathbb{R}^n$. We emphasize that (S2) need *not* be valid for scalar intervals $\alpha \in \Pi\mathbb{R}$.

As an easy consequence of (S2) we note

(S2a) $y = Sx,\ \check{x} = 0 \Rightarrow \check{y} = 0$;

indeed, if $\check{x} = 0$ then $y = Sx = S(-x) = -Sx = -y$, whence $\check{y} = -\check{y} = 0$.

Two important matrices are related to each sublinear map S. The *kernel* of S is the unique interval matrix $\kappa(S) \in \Pi\mathbb{R}^{n\times n}$ satisfying

$$Sx = \kappa(S)x \qquad \text{for} \quad x = e^{(i)} \quad (i = 1,\ldots,n); \tag{1}$$

explicitly, the kernel is given column by column as

$$\kappa(S) = \left(Se^{(1)},\ldots,Se^{(n)}\right).$$

The *absolute value* of S is the unique nonnegative matrix $|S| \in \mathbb{R}^{n\times n}$ satisfying

$$Sx = |S|x \qquad \text{for} \quad x = f^{(i)} := \left[-e^{(i)}, e^{(i)}\right] \quad (i = 1,\ldots,n); \tag{2}$$

explicitly,

$$|S| = \left(|Sf^{(1)}|,\ldots,|Sf^{(n)}|\right).$$

EXAMPLE 1. For $A \in \Pi\mathbb{R}^{n\times n}$, the map

$$A^M : x \to Ax \tag{3}$$

is sublinear; moreover, if A is thin, then A^M is linear. Obviously

$$\kappa(A^M) = A, \qquad |A^M| = |A|. \tag{4}$$

(The letter M stands for "multiplication.")

EXAMPLE 2. For $A \in \Pi\mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ with $|A| \leqslant B$, the map

$$S := S_{A,B} : x \to A\check{x} + B\mathring{x} \tag{5}$$

is sublinear, and linear if A is thin. We verify axiom (S1), which is not immediate. With $x' = Sx$, $y' = Sy$, (B11) and (B14) imply $\check{x}' = \check{A}\check{x}$, $\rho(x') = \rho(A)|\check{x}| + B\rho(x)$, $\check{y}' = \check{A}\check{y}$, $\rho(y') = \rho(A)|\check{y}| + B\rho(y)$; hence

$$\begin{aligned}|\check{x}' - \check{y}'| + \rho(x') &= |\check{A}\check{x} - \check{A}\check{y}| + \rho(A)|\check{x}| + B\rho(x)\\ &\leqslant |\check{A}||\check{x} - \check{y}| + \rho(A)\left(|\check{x} - \check{y}| + |\check{y}|\right) + B\rho(x)\end{aligned}$$

$$\begin{aligned}
&= \rho(A)|\check{y}| + |A||\check{x} - \check{y}| + B\rho(x) \qquad \text{by (B2)} \\
&\leqslant \rho(A)|\check{y}| + B|\check{x} - \check{y}| + B\rho(x) \\
&\leqslant \rho(A)|\check{y}| + B\rho(y) = \rho(y') \qquad \text{by (I1)}
\end{aligned}$$

if $x \subseteq y$. Hence in this case $x' \subseteq y'$ by (I1), and (S1) holds. The kernel and absolute value of $S_{A,B}$ are given by

$$\kappa(S_{A,B}) = A, \qquad |S_{A,B}| = B. \tag{6}$$

We now show that every sublinear map is enclosed in one of the form (5).

THEOREM 1. *Let $S: \Pi\mathbb{R}^n \to \Pi\mathbb{R}^n$ be a sublinear map. Then*

$$|\kappa(S)| \leqslant |S|, \tag{7}$$

and the inclusion relation

$$Sx \subseteq \kappa(S)\check{x} + |S|\mathring{x} \tag{8}$$

holds for all $x \in \Pi\mathbb{R}^n$. Moreover, if S is linear, then the kernel $\kappa(S)$ is thin, and (8) *holds with equality.*

Proof. (i): We have $\kappa(S)e^{(i)} = Se^{(i)} \subseteq Sf^{(i)} = |S|f^{(i)} = [-|S|e^{(i)}, |S|e^{(i)}]$; hence $\kappa(S) \subseteq [-|S|, |S|]$, which implies (7).

(ii): Since S is sublinear,

$$\begin{aligned}
Sx &= S(\check{x} + \mathring{x}) = S\left(\sum e^{(k)}\check{x}_k + f^{(k)}\rho(x)_k\right) \\
&\subseteq \sum Se^{(k)}\check{x}_k + \sum Sf^{(k)}\rho(x)_k \\
&= \sum \kappa(S)e^{(k)}\check{x}_k + \sum |S|f^{(k)}\rho(x)_k \\
&= \sum \kappa(S)e^{(k)}\check{x}_k + \sum |S|e^{(k)}\mathring{x}_k = \kappa(S)\check{x} + |S|\mathring{x}.
\end{aligned}$$

Hence (8) holds, and equality holds in (8) if S is linear.

(iii): If S is linear, then $0 = S(e^{(i)} - e^{(i)}) = Se^{(i)} - Se^{(i)} = \kappa(S)e^{(i)} - \kappa(S)e^{(i)}$, whence $\kappa(S) - \kappa(S) = 0$. This implies that $\kappa(S)$ is thin. ∎

Note that a sublinear map is not necessarily determined by its kernel and absolute value. For example ($n = 1$), $S_1 = [-1,3]^M$ and $S_2 = S_{[-1,3],3}$ both have kernel $[-1,3]$ and absolute value 3, but $S_1[-1,3]=[-3,9]\neq[-7,9]$ $=S_2[-1,3]$.

EXAMPLE 3. For nonnegative matrices B_1, $B_2 \in \mathbb{R}^{n\times n}$, the map

$$S: x \to [B_1\underline{x} - B_2\bar{x}, B_1\bar{x} - B_2\underline{x}] \tag{9}$$

is linear; indeed, $Sx=(B_1-B_2)\check{x}+(B_1+B_2)\mathring{x}$ for all $x\in \Pi\mathbb{R}^n$, whence we have a particular case of Example 2. These maps arise if the technique of *monotone splittings* (cf. Collatz [7]) is used; indeed

$$(B_1-B_2)\tilde{x}\in[B_1\underline{x} - B_2\bar{x}, B_1\bar{x} - B_2\underline{x}] \qquad \text{for all} \quad \tilde{x}\in x. \tag{9a}$$

Conversely, by Theorem 1, a linear map S satisfies the relation

$$\begin{aligned} Sx &= \kappa(S)\check{x}+|S|\mathring{x} \\ &= (B_1-B_2)\check{x}+(B_1+B_2)\mathring{x} \\ &= [B_1\underline{x} - B_2\bar{x}, B_1\bar{x} - B_2\underline{x}] \end{aligned}$$

where

$$B_1=\tfrac{1}{2}[|S|+\kappa(S)], \qquad B_2=\tfrac{1}{2}[|S|-\kappa(S)] \tag{10}$$

are nonnegative matrices [by (7)]. Hence we have

THEOREM 2. *Every linear map is induced by some monotone splitting, uniquely determined by its kernel and absolute value.*

EXAMPLE 4. *Let $A\in\Pi\mathbb{R}^{n\times n}$ be regular. Then the map*

$$A^H: x\to \square\{\tilde{A}^{-1}\tilde{x}\,|\,\tilde{A}\in A, \tilde{x}\in x\}, \tag{11}$$

called the hull inverse of A, is sublinear. The slightly nontrivial subadditivity (S3) *follows from Lemma* 1(*iii*).

PROPERTIES OF THE HULL-INVERSE. *Let $A, B\in\Pi\mathbb{R}^n$ be regular. Then for all $x\in\Pi\mathbb{R}^n$,*

$$A\subseteq B \quad\Rightarrow\quad A^Hx\subseteq B^Hx, \tag{12}$$

$$A^Hx\subseteq A^{-1}x, \tag{13}$$

$$\kappa(A^H)=A^{-1}, \qquad |A^H|=|A^{-1}|. \tag{14}$$

Moreover, if one of the conditions

(i) *A thin,*
(ii) *A diagonal,*
(iii) $A^{-1} \geqslant 0$ *and* $x \geqslant 0$, $x \leqslant 0$, *or* $x \ni 0$,
(iv) $\check{x} = 0$

is satisfied, then (13) *holds with equality.*

Proof. (12) and (13) follow immediately from (11). Equality in (13) is obvious from the definitions in cases (i) and (ii), and follows in case (iii) from Lemma 12 and the endpoint rules of Section 1.3. In case (iv) we have for $\tilde{A} \in A$ the inclusion $|\tilde{A}^{-1}|x = \tilde{A}^{-1}x = \tilde{A}^{H}x \subseteq A^{H}x$ by (B6) and (12), hence, by (B6) again, $A^{-1}x = |A^{-1}|x \subseteq A^{H}x$, whence again (13) holds with equality. This also implies $|A^{H}| = |A^{-1}|$, and the formula for the kernel is obvious. ∎

EXAMPLE 5. For a symmetric vector $[-u, u]$ and an arbitrary monotone norm, the map

$$S: x \to [-u, u]\|x\|$$

is sublinear; kernel and absolute value are given by

$$\kappa(S) = [-uw^{T}, uw^{T}], \qquad |S| = uw^{T},$$

where w is the vector whose ith component is $w_i = \|e^{(i)}\|$. For a positive vector $v \in \mathbb{R}^n$, and an arbitrary sublinear map S, the inclusion relation

$$Sx \subseteq [-u, u]\|x\|_{v} \tag{15}$$

holds for every vector $u \geqslant |S|v$. Indeed, by rule (R7) below, $|Sx| \leqslant |S||x| \leqslant |S|v\|x\|_{v} \leqslant u\|x\|_{v}$, which implies (15). (An attempt to extend this argument to arbitrary monotone norms leads to the concept of hybrid norms; see Neumaier [19].) In particular, we obtain the following generalization of Lemma 5 to H-matrices:

LEMMA 13. *Let $A \in \Pi\mathbb{R}^{n\times n}$ be an H-matrix, and let $u, v \in \mathbb{R}^n$ be positive vectors with*

$$\langle A\rangle u \geqslant v > 0. \tag{16}$$

Then

$$A^H x \in [-u, u]\|x\|_v. \tag{17}$$

Proof. Since A is an H-matrix, $\langle A\rangle^{-1} \geqslant 0$, so that $|A^H|v = |A^{-1}|v \leqslant \langle A\rangle^{-1}v \leqslant u$ by (14), Lemma 11, and the assumption (16). Hence the preceding observation applies. ∎

REMARK. This somewhat crude enclosure can be improved by local solvers considered in Section 8; see the discussion after Theorem 13.

The inclusion phenomenon in Theorem 1 suggests that we define *inclusion* for sublinear maps S and T by

$$S \subseteq T \quad :\Leftrightarrow \quad Sx \subseteq Tx \quad \text{for all } x \in \Pi\mathbb{R}^n.$$

We also define sum, difference, and product in the usual way by

$$S \pm T : x \to Sx \pm Tx,$$

$$ST : x \to S(Tx).$$

Clearly, $S \pm T$ and ST are sublinear (linear) if S and T are. Moreover, in contrast to interval matrix multiplication, the multiplication of sublinear maps is associative. But other familiar laws fail, a conspicuous nonstandard law being

$$S - S = 0 \quad \Rightarrow \quad S = 0.$$

Rules for Sublinear Maps. Let $S, T : \Pi\mathbb{R}^n \to \Pi\mathbb{R}^n$ be sublinear. Then:

(R1) $S \subseteq T \Rightarrow \kappa(S) \subseteq \kappa(T)$,
(R2) $\kappa(S \pm T) = \kappa(S) \pm \kappa(T)$,
(R3) T linear $\Rightarrow \kappa(ST) \subseteq \kappa(S)\kappa(T)$,
(R4) $S \subseteq T \Rightarrow |S| \leqslant |T|$,
(R5) $|S \pm T| = |S| + |T|$,
(R6) $|ST| \leqslant |S|\,|T|$,
(R7) $|Sx| \leqslant |S|\,|x|$ for all $x \in \Pi\mathbb{R}^n$.

Note the equality sign in (R5).

Proof. (R1): If $S \subseteq T$ then $\kappa(S)e^{(i)} = Se^{(i)} \subseteq Te^{(i)} = \kappa(T)e^{(i)}$, whence $\kappa(S) \subseteq \kappa(T)$.

(R2): $\kappa(S \pm T)e^{(i)} = (S \pm T)e^{(i)} = Se^{(i)} \pm Te^{(i)} = \kappa(S)e^{(i)} \pm \kappa(T)e^{(i)}$, whence $\kappa(S \pm T) = \kappa(S) \pm \kappa(T)$.

(R3): For linear T the kernel $\kappa(T)$ is thin. Hence $\kappa(ST)e^{(i)} = STe^{(i)} = S[\kappa(T)e^{(i)}] \subseteq \kappa(S)[\kappa(T)e^{(i)}] \subseteq [\kappa(S)\kappa(T)]e^{(i)}$ by (B23), whence $\kappa(ST) \subseteq \kappa(S)\kappa(T)$.

(R4): If $S \subseteq T$ then $|S|e^{(i)} = |Sf^{(i)}| \leqslant |Tf^{(i)}| = |T|e^{(i)}$, whence $|S| \leqslant |T|$.

(R5): Since $\check{f}^{(i)} = 0$, (S2a) implies $|S \pm T|e^{(i)} = |(S \pm T)f^{(i)}| = \rho((S \pm T)f^{(i)}) = \rho(Sf^{(i)}) + \rho(Tf^{(i)}) = |Sf^{(i)}| + |Tf^{(i)}| = |S|e^{(i)} + |T|e^{(i)}$, whence $|S \pm T| = |S| + |T|$.

(R6): Again by (S2a) and Theorem 1, we have $|ST|e^{(i)} = |STf^{(i)}| = |S(|T|f^{(i)})| \leqslant |\kappa(S)\cdot 0 + |S|\,|T|f^{(i)}| = |S|\,|T|e^{(i)}$, whence $|ST| \leqslant |S|\,|T|$.

(R7): $|Sx| = |\Sigma Se^{(k)}x_k| \leqslant \Sigma |Se^{(k)}|\,|x_k| = \Sigma |\kappa(S)e^{(k)}|\,|x_k| = |\kappa(S)|\,|x| \leqslant |S|\,|x|$. ∎

4. NORMAL MAPS

We call a sublinear map $S: \Pi\mathbb{R}^n \to \Pi\mathbb{R}^n$ *normal* if the following two submultiplicative axioms hold for all $x, y \in \Pi\mathbb{R}^n$:

(S4) $\rho(Sx) \geqslant |S|\rho(x)$,
(S5) $q(Sx, Sy) \leqslant |S|q(x, y)$.

EXAMPLE 1. By the discussion in Section 1, the multiplication maps A^M ($A \in \Pi\mathbb{R}^{n\times n}$) are normal, since $|A^M| = |A|$.

EXAMPLE 2. The maps $S = S_{A,B}$, $|A| \leqslant B$, discussed in Section 3, Example 2, are normal. Indeed, by the corresponding rules for interval matrices,

$$\begin{aligned}
\rho(Sx) &= \rho(A\check{x} + B\mathring{x}) = \rho(A\check{x}) + \rho(B\mathring{x}) \qquad \text{by (B10)}\\
&\geqslant \rho(B\mathring{x}) = |B|\rho(\mathring{x}) = |S|\rho(x) \qquad \text{by (B14)},\\
q(Sx, Sy) &= q(A\check{x} + B\mathring{x}, A\check{y} + B\mathring{y})\\
&\leqslant q(A\check{x}, A\check{y}) + q(B\mathring{x}, B\mathring{y}) \qquad \text{by (D5)}\\
&\leqslant |A|q(\check{x}, \check{y}) + |B|q(\mathring{x}, \mathring{y}) \qquad \text{by (D7)}\\
&\leqslant |S|\,|\check{x} - \check{y}| + |S|\,|\rho(x) - \rho(y)| = |S|q(x, y) \qquad \text{by (D1)}.
\end{aligned}$$

As a simple consequence we have

THEOREM 2a. *Every linear map S is normal, and* (S4) *holds with equality.*

Proof. Let S be linear. By Theorem 1 (8), $S = S_{A,B}$ with $A = \kappa(S)$, $B = |S|$. Hence S is normal. Moreover, $\kappa(S)$ is thin, whence property (S2a) and (B10) imply

$$\rho(Sx) = \rho(\kappa(S)\check{x} + |S|\mathring{x}) = ||S|\mathring{x}| = |S|\rho(x). \qquad \blacksquare$$

Further Rules for Normal Maps. Let S, T be sublinear, S normal. Then

(R6a) $|ST| = |S|\,|T|$,
(R8) $Sx = |S|x$ if $\check{x} = 0$.

Proof. (R6a): $|ST|e^{(i)} = \rho(|ST|f^{(i)}) = \rho(STf^{(i)}) \geqslant |S\rho(Tf^{(i)}) \geqslant |S|\,|T|\rho(f^{(i)}) = |S|\,|T|e^{(i)}$ by (B6) and (S4). Hence $|ST| \geqslant |S|\,|T|$, and by (R6), equality holds.

(R8): If $\check{x} = 0$ then by (S2a) and (S4) we have $Sx = [-\rho(Sx), \rho(Sx)] \supseteq [-|S|\rho(x), |S|\rho(x)] = |S|x$; hence $|S|x \subseteq Sx$. But by Theorem 1, the converse relation holds, whence we have equality. $\blacksquare$

PROPOSITION 2. *If S, T are normal, then $S \pm T$ and ST are normal.*

Proof. Let S, T be normal. Then $\rho((S \pm T)x) = \rho(Sx \pm Tx) = \rho(Sx) + \rho(Tx) \geqslant |S|\rho(x) + |T|\rho(x) = (|S| + |T|)\rho(x) = |S \pm T|\rho(x)$ by (B11), (S4), and (R5). Similarly by (D5), (S5), and (R5), we have

$$\begin{aligned} q((S \pm T)x, (S \pm T)y) &= q(Sx \pm Tx, Sy \pm Ty) \\ &\leqslant q(Sx, Sy) + q(Tx, Ty) \\ &\leqslant |S|q(x,y) + |T|q(x,y) \\ &= |S \pm T|q(x,y). \end{aligned}$$

Hence $S \pm T$ are normal. The normality of ST follows in the same way from (R6a). $\blacksquare$

PROPOSITION 3. *Let R, S, T be sublinear maps such that*

$$Rx = S(x + TRx) \qquad \textit{for all} \quad x \in \Pi\mathbb{R}^n.$$

If S, T are normal, then

$$(I-|S||T|)|R|=|S|.$$

If, in addition, $\sigma(|S||T|)<1$ then R is normal, too.

Proof. We have

$$\begin{aligned} q(Rx, Ry) &= q(S(x+TRx), S(y+TRy)) \\ &\leqslant |S| q(x+TRx, y+TRy) \qquad \text{by (S5)} \\ &\leqslant |S|[q(x,y)+q(TRx, TRy)] \qquad \text{by (D5)} \\ &\leqslant |S|[q(x,y)+|T| q(Rx, Ry)] \qquad \text{by (S5)}, \\ \rho(Rx) &= \rho(S(x+TRx)) \geqslant |S| \rho(x+TRx) \qquad \text{by (S4)} \\ &= |S|[\rho(x)+\rho(TRx)] \qquad \text{by (B11)} \\ &\geqslant |S|[\rho(x)+|T| \rho(Rx)] \qquad \text{by (S4)}; \end{aligned}$$

hence

$$\begin{aligned} (I-|S||T|) q(Rx, Ry) &\leqslant |S| q(x,y), \\ (I-|S||T|) \rho(Rx) &\leqslant |S| \rho(x). \end{aligned}$$

If we insert $x=f^{(i)}$, $y=0$, we find that $(I-|S||T|)|R|e^{(i)} \leqslant |S|e^{(i)}$ and $\geqslant |S|e^{(i)}$; hence $=|S|e^{(i)}$, which implies

$$(I-|S||T|)|R|=|S|.$$

If, in addition, $\sigma(|S||T|)<1$, then $I-|S||T|$ is nonsingular and multiplication by its inverse gives normality. ∎

Finally we mention that not every sublinear map is normal. Simple counterexamples are the maps $x \to [-u, u]\|x\|_v$ considered in Section 3, Example 5—they violate (R8).

5. GAUSS ELIMINATION

Gauss elimination can be separated into three different stages: triangular decomposition, forward substitution, and back substitution. We begin with the analysis of forward and back substitution.

Let $L \in \Pi\mathbb{R}^{n\times n}$ be a regular lower triangular matrix, i.e. $L_{ii} \not\ni 0$, $L_{ik} = 0$ if $k > i$. We denote by L^F the map

$$L^F : x \to y$$

which maps a vector x to the result y of the *forward substitution* defined by

$$y_i := \frac{x_i - L_{i1}y_1 - \cdots - L_{i,i-1}y_{i-1}}{L_{ii}} \qquad (i = 1,\ldots,n). \tag{1}$$

Similarly, for a regular upper triangular matrix $R \in \Pi\mathbb{R}^{n\times n}$, i.e. $R_{ii} \not\ni 0$, $R_{ik} = 0$ if $k < i$, we denote by R^F the map

$$R^F : x \to y$$

which maps a vector x to the result y of the *back substitution* defined by

$$y_i := \frac{x_i - R_{in}y_n - \cdots - R_{i,i+1}y_{i+1}}{R_{ii}} \qquad i = n, n-1,\ldots,1. \tag{2}$$

PROPOSITION 4. *Let $A \in \Pi\mathbb{R}^{n\times n}$ be a regular (lower or upper) triangular matrix. Then the map A^F defined by* (1) *or* (2), *respectively, is a normal sublinear map such that*

$$A^H \subseteq A^F, \tag{3}$$

$$|A^F| = \langle A\rangle^{-1}. \tag{4}$$

Moreover, if A is thin, then A^F is linear, and

$$\kappa(A^F) = A^{-1} \qquad \textit{for thin } A. \tag{5}$$

Proof. W.l.o.g. we treat the case when $A = L$ is lower triangular. Clearly, L^F is sublinear. With the Jacobi splitting $L = D - E$ of L (see Section 2), we may rewrite (1) as $y = D^{-1}(x + Ey)$. Therefore we have

$$L^F x = D^{-1}(x + EL^F x) \qquad \text{for all} \quad x \in \Pi\mathbb{R}^n, \tag{6}$$

and Proposition 3 applies with $R = L^F$, $S = (D^{-1})^M$, $T = E^M$. Now $\sigma(|S||T|)$

$=\sigma(\langle D\rangle^{-1}|E|)=0$, since $\langle D\rangle^{-1}|E|$ is strictly lower triangular. Hence L^F is normal, and $|L^F|=|R|=(I-|S|\,|T|)^{-1}|S|=(I-\langle D\rangle^{-1}|E|)^{-1}\langle D\rangle^{-1}=(\langle D\rangle-|E|)^{-1}=\langle L\rangle^{-1}$. This proves (4).

The inclusion isotonicity of interval arithmetic implies that $\tilde{L}^{-1}\tilde{x}\in L^F x$ for all $\tilde{L}\in L$, $\tilde{x}\in x$. Hence $L^H x=\square\{\tilde{L}^{-1}\tilde{x}\mid \tilde{L}\in L,\ \tilde{x}\in x\}\subseteq L^F x$ for all $x\in \Pi\mathbb{R}^n$. This proves (3).

Finally, the assertions when L is thin are obvious. ■

EXAMPLE. For the thin $n\times n$ matrix

$$L=\begin{pmatrix}1 & & & 0\\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 1\end{pmatrix},$$

we have

$$\begin{pmatrix}1 & & & 0\\ -1 & 1 & & \\ & \ddots & \ddots & \\ 0 & & -1 & 1\end{pmatrix},\qquad \langle L\rangle^{-1}=\begin{pmatrix}1 & & & & 0\\ 1 & 1 & & & \\ 2 & 1 & 1 & & \\ \vdots & & \ddots & \ddots & \\ 2^{n-2} & \cdots & 2 & 1 & 1\end{pmatrix}.$$

The exponential growth of the elements of $|L^F|=\langle L\rangle^{-1}$, which does not occur for $|L^H|=|L^{-1}|$, shows that—in contrast to the noninterval case—forward substitution can be an unstable method of enclosing $L^H x$. This is well known, of course, but the present techniques allow a closer investigation of the amount of overestimation. In particular, no overestimation occurs when $|L^{-1}|=\langle L\rangle^{-1}$, which holds e.g. if $L=\langle L\rangle$. Note that the first two components are always computed optimally.

A formal discussion of triangular factorization of interval matrices presents some difficulties due to the fact that the "factorization" (L,R) of an interval matrix A generally does *not* satisfy the equation $A=LR$. We therefore adopt a recursive definition involving Schur complements; it is left to the reader to verify that our approach is equivalent to the algorithmic approach of Hansen and Smith [11] or Alefeld and Herzberger [2].

For a matrix $A\in\Pi\mathbb{R}^{n\times n}$, $n>1$, written with explicit first row and column as

$$A=\begin{pmatrix}\alpha & a^T\\ b & B\end{pmatrix},\tag{7}$$

the *Schur complement* [of the (1,1) entry] is defined as

$$\Sigma(A) := B - b\alpha^{-1}a^T \qquad \text{if} \quad \alpha \not\ni 0, \tag{8}$$

and is not defined if $\alpha \ni 0$.

DEFINITION. We say that $A \in \Pi\mathbb{R}^{n\times n}$ has the triangular decomposition (L_A, R_A) if either $n = 1$ and $L_A = 1$, $R_A = A \not\ni 0$, or $n > 1$, and

$$L_A = \begin{pmatrix} 1 & 0 \\ b\alpha^{-1} & L \end{pmatrix}, \qquad R_A = \begin{pmatrix} \alpha & a^T \\ 0 & R \end{pmatrix}, \tag{9}$$

where α, a, b, B are defined by (7), $0 \notin \alpha$, and (L, R) is the triangular decomposition of the Schur complement $\Sigma(A)$ of A.

PROPOSITION 5. *If $A \in \Pi\mathbb{R}^{n\times n}$ has the triangular decomposition (L_A, R_A), then A is regular, and for every $\tilde{A} \in A$ there are matrices $\tilde{L} \in L_A$, $\tilde{R} \in R_A$ such that $\tilde{A} = \tilde{L}\tilde{R}$. Moreover, if A is thin, then L_A, R_A are thin and $A = L_A R_A$.*

Proof. Suppose that

$$\tilde{A} = \begin{pmatrix} \tilde{\alpha} & \tilde{a}^T \\ \tilde{b} & \tilde{B} \end{pmatrix} \in A.$$

Then $\tilde{\alpha} \neq 0$, and $\Sigma(\tilde{A}) = \tilde{B} - \tilde{b}\tilde{\alpha}^{-1}\tilde{a}^T \in B - b\alpha^{-1}a^T = \Sigma(A)$. Hence we may proceed by induction on n (the case $n = 1$ is obvious), and assume that there are matrices $\tilde{L}_0 \in L$, $\tilde{R}_0 \in R$, where (L, R) is the triangular decomposition of $\Sigma(A)$, with $\Sigma(\tilde{A}) = \tilde{L}_0\tilde{R}_0$. But then

$$\tilde{L} := \begin{pmatrix} 1 & 0 \\ \tilde{b}\tilde{\alpha}^{-1} & \tilde{L}_0 \end{pmatrix} \in L_A, \qquad \tilde{R} := \begin{pmatrix} \tilde{\alpha} & \tilde{a}^T \\ 0 & \tilde{R}_0 \end{pmatrix} \in R_A,$$

and

$$\tilde{L}\tilde{R} = \begin{pmatrix} \tilde{\alpha} & \tilde{a}^T \\ \tilde{b}\tilde{\alpha}^{-1}\tilde{\alpha} & \tilde{b}\tilde{\alpha}^{-1}\tilde{a}^T + \tilde{L}_0\tilde{R}_0 \end{pmatrix} = \begin{pmatrix} \tilde{\alpha} & \tilde{a}^T \\ \tilde{b} & \tilde{b}\tilde{\alpha}^{-1}\tilde{a}^T + \Sigma(\tilde{A}) \end{pmatrix} = \tilde{A}.$$

This completes the induction proof. Now each $\tilde{L} \in L_A$, $\tilde{R} \in R_A$ is regular, whence $\tilde{A} = \tilde{L}\tilde{R}$ is regular, too. Since $\tilde{A} \in A$ was arbitrary, A is regular. If A is thin, then, again by induction, L_A and R_A are thin, whence $A = \tilde{A} = \tilde{L}\tilde{R} = L_A R_A$. ■

If $A \in \Pi\mathbb{R}^{n\times n}$ has a triangular decomposition (L_A, R_A), then L_A and R_A are (by definition) regular, and the map $A^G := R_A^F L_A^F$ with

$$A^G x = R_A^F L_A^F x \qquad \text{for} \quad x \in \Pi\mathbb{R}^n \tag{10}$$

is defined. We call A^G the *Gauss inverse* of A. Thus, $z = A^G x$ is obtained from x by triangular decomposition of A, followed by forward substitution $y = L_A^F x$ and back substitution $z = R_A^F y$. By combining Propositions 4 and 5 we now get:

THEOREM 3. *Suppose that the Gauss inverse A^G of $A \in \Pi\mathbb{R}^{n\times n}$ exists. Then A^G is a normal sublinear map, A is regular, and*

$$A^H \subseteq A^G. \tag{11}$$

Moreover, if A is thin, then A^G is linear and

$$\kappa(A^G) = A^{-1}. \tag{12}$$

Proof. By Proposition 4, L_A^F and R_A^F are sublinear and normal, hence their product A^G is. By Proposition 5, A is regular, and for $\tilde{A} \in A$ there are $\tilde{L} \in L_A$, $\tilde{R} \in R_A$ such that $\tilde{A} = \tilde{L}\tilde{R}$; hence $\tilde{A}^{-1}x = \tilde{R}^{-1}\tilde{L}^{-1}\tilde{x} \in R_A^F L_A^F x = A^G x$ for all $\tilde{x} \in x$. This implies (11). If A is thin, then by Proposition 4, L_A^F and R_A^F are linear; hence A^G is linear, and $\kappa(A^G) = \kappa(R_A^F)\kappa(L_A^F) = R_A^{-1}L_A^{-1} = (L_A R_A)^{-1} = A^{-1}$ by Proposition 5. ■

REMARK 1. Some authors, e.g. Alefeld and Platzöder [2], use the notation IGA(A, x) for $A^G x$.

REMARK 2. There is some arbitrariness in the definition of the triangular decomposition, namely in the choice of the diagonal elements of L_A and R_A. A different choice leads to distinct decompositions which (in contrast to the noninterval case) may yield different results in $A^G x$.

REMARK 3. Not every regular matrix A has a Gauss inverse. An example (a variant of Reichmann [22]) is the matrix

$$A = \begin{pmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{pmatrix}, \qquad x = [0, \tfrac{2}{3}],$$

which is regular, since $\det \tilde{A} \geqslant \frac{1}{9}$ for all $\tilde{A} \in A$, whereas the Schur complement $\Sigma(A)$ contains the singular matrix

$$\begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

The same example shows that (in contrast to the noninterval case) permutations of rows or columns do not help.

The absolute value of the Gauss inverse can be readily deduced from Proposition 4 and (R6a). We obtain

$$|A^G| = \langle R_A \rangle^{-1} \langle L_A \rangle^{-1}. \tag{13}$$

In order to eliminate the triangular factors from (13) we restrict ourselves to the case when A is an H-matrix. By refining the proof of Alefeld [1], who shows that the triangular decomposition exists for all H-matrices, we are able to bound $|A^G|$ by $\langle A \rangle^{-1}$. As a preparation we prove a preliminary result.

PROPOSITION 6. *Suppose $A \in \Pi\mathbb{R}^{n\times n}$, $n > 1$, and $0 \notin A_{11}$.*

(i) *We always have*

$$\langle \Sigma(A) \rangle \geqslant \Sigma(\langle A \rangle). \tag{14}$$

(ii) *If A is an M-matrix then $\Sigma(A)$ is an M-matrix, too; moreover,*

$$\Sigma(A) = \left[\Sigma(\underline{A}), \Sigma(\overline{A})\right], \tag{15}$$

and (14) *holds with equality.*

(iii) *If A is an H-matrix then $\Sigma(A)$ is an H-matrix, too.*

Proof. (i): Using the partition (7) of A, we have

$$\langle \Sigma(A) \rangle = \langle B - b\alpha^{-1}a^T \rangle \geqslant \langle B \rangle - |b||\alpha^{-1}||a|^T \qquad \text{by (O1), (O2)}$$

$$= \langle B \rangle - |b| \langle \alpha \rangle^{-1} |a|^T = \Sigma(\langle A \rangle) \qquad \text{by Lemma 11.}$$

(ii): If A is an M-matrix then $\alpha > 0$, $b \leqslant 0$, $a \leqslant 0$ whence by (B30)

$$\begin{aligned}\Sigma(A) &= [\underline{B}, \overline{B}] - [\underline{b}, \overline{b}][\underline{\alpha}, \overline{\alpha}]^{-1}[\underline{a}^T, \overline{a}^T] \\ &= [\underline{B}, \overline{B}] - [\underline{b}, \overline{b}][\underline{\alpha}^{-1}\underline{a}^T, \overline{\alpha}^{-1}\overline{a}^T] \\ &= [\underline{B}, \overline{B}] - [\overline{b}\overline{\alpha}^{-1}\overline{a}^T, \underline{b}\underline{\alpha}^{-1}\underline{a}^T] \\ &= [\underline{B} - \underline{b}\underline{\alpha}^{-1}\underline{a}^T, \overline{B} - \overline{b}\overline{\alpha}^{-1}\overline{a}^T] = [\Sigma(\underline{A}), \Sigma(\overline{A})].\end{aligned}$$

Moreover, $\Sigma(A)_{ik} \leqslant \overline{B}_{ik} \leqslant 0$ for $i \neq k$. Finally, since A is an M-matrix, there is a vector

$$\begin{pmatrix}\omega\\ u\end{pmatrix} > 0$$

such that

$$0 < \underline{A}\begin{pmatrix}\omega\\ u\end{pmatrix} = \begin{pmatrix}\underline{\alpha}\omega + \underline{a}^T u\\ \underline{b}\omega + \underline{B}u\end{pmatrix}.$$

Therefore, $\Sigma(A)u \geqslant \Sigma(\underline{A})u = (\underline{B} - \underline{b}\underline{\alpha}^{-1}\underline{a}^T)u = \underline{B}u - \underline{b}\underline{\alpha}^{-1}\underline{a}^T u \geqslant \underline{B}u + \underline{b}\underline{\alpha}^{-1}(\underline{\alpha}\omega) = \underline{B}u + \underline{b}\omega > 0$, whence $\Sigma(A)$ is an M-matrix.

(iii): If A is an H-matrix, then $\langle A\rangle$ is a thin M-matrix, and by the above, there is $u > 0$ such that $\Sigma(\langle A\rangle)u > 0$. But then (14) implies $\langle\Sigma(A)\rangle u > 0$, whence $\Sigma(A)$ is an H-matrix. ∎

REMARK. The proof also shows that if A is diagonally dominant, then so is $\Sigma(A)$ (cf. Alefeld and Herzberger [2, Chapter 15]).

PROPOSITION 7. *If A is an H-matrix, then the triangular decomposition (L_A, R_A) of A exists and satisfies*

$$\langle L_A\rangle\langle R_A\rangle \geqslant \langle A\rangle. \tag{16}$$

Proof. We proceed by induction on the size n of the matrix A. For $n = 1$, the assertion is obvious. For $n > 1$, we assume that the proposition holds for matrices of size $< n$. By Proposition 6, the Schur complement $\Sigma(A)$ of A is an H-matrix of size $n-1$; hence $\Sigma(A)$ has a triangular decomposition (L, R) satisfying $\langle L\rangle\langle R\rangle \geqslant \langle\Sigma(A)\rangle \geqslant \Sigma(\langle A\rangle)$. Using the partition (7) of A, we have $\alpha \not\ni 0$ since A is an H-matrix. Hence the triangular decomposition

(L_A, R_A) of A exists and is given by (9). Therefore

$$\langle L_A\rangle\langle R_A\rangle = \begin{pmatrix} 1 & 0 \\ -|b|\langle\alpha\rangle^{-1} & \langle L\rangle \end{pmatrix}\begin{pmatrix} \langle\alpha\rangle & -|a|^T \\ 0 & \langle R\rangle \end{pmatrix}$$
$$= \begin{pmatrix} \langle\alpha\rangle & -|a|^T \\ -|b| & |b|\langle\alpha\rangle^{-1}|a|^T + \langle L\rangle\langle R\rangle \end{pmatrix}$$
$$\geqslant \begin{pmatrix} \langle\alpha\rangle & -|a|^T \\ -|b| & |b|\langle\alpha\rangle^{-1}|a|^T + \Sigma(\langle A\rangle) \end{pmatrix} = \langle A\rangle.$$

This completes the induction. ∎

THEOREM 4. *If A is an H-matrix, then A^G exists and satisfies*

$$|A^G| = \langle R_A\rangle^{-1}\langle L_A\rangle^{-1} \leqslant \langle A\rangle^{-1}. \tag{17}$$

Proof. By Proposition 7, the triangular decomposition (L_A, R_A) of A exists; hence $A^G = R_A^F L_A^F$ exists. Moreover, $\langle L_A\rangle\langle R_A\rangle \geqslant \langle A\rangle$ whence by Proposition 4,

$$|A^G| = |R_A^F||L_A^F| = \langle R_A\rangle^{-1}\langle L_A\rangle^{-1} = (\langle L_A\rangle\langle R_A\rangle)^{-1} \leqslant \langle A\rangle^{-1};$$

the inequality is valid because $\langle A\rangle^{-1}, \langle L_A\rangle^{-1}, \langle R_A\rangle^{-1}$ are nonnegative. ∎

6. PRECONDITIONING

If A^G does not exist, it is often possible to *precondition* A by pre- and/or postmultiplication of A with thin nonsingular matrices C_1, C_2 in order to obtain a new matrix B for which B^G exists. The hull inverses of A and $B = C_1AC_2$ are related by the following proposition, which allows one to enclose A^H if an enclosure for B^H (like B^G) is known.

PROPOSITION 8. *Let $A \in \Pi\mathbb{R}^{n\times n}$, $C_1, C_2 \in \mathbb{R}^{n\times n}$. If C_1AC_2 is regular, then A is regular, and*

$$A^H \subseteq C_2(C_1AC_2)^H C_1. \tag{1}$$

Proof. If C_1AC_2 is regular, then clearly C_1, A, and C_2 are regular. Hence if $\tilde{A} \in A$, $\tilde{x} \in x$, then $\tilde{A}^{-1}\tilde{x} = C_2(C_1\tilde{A}C_2)^{-1}C_1\tilde{x} \in C_2(C_1\tilde{A}C_2)^H C_1 x$. This implies $A^H x \subseteq C_2(C_1\tilde{A}C_2)^H C_1 x$, whence (1) holds. ∎

Since we know that B^G exists for H-matrices B, a natural question is how to find preconditioning matrices C_1 and C_2 for A such that $B = C_1AC_2$ is an H-matrix. The special choice $C_1 = \check{A}^{-1}$, $C_2 = I$ (or $C_1 = I$, $C_2 = \tilde{A}^{-1}$) for some $\tilde{A} \in A$, in particular for $\tilde{A} = \check{A}$, has been recommended in the literature (Hansen and Smith [11]). The justification is that then B contains the unit matrix; hence, if the radii of the entries of A are sufficiently small, B will certainly be diagonally dominant. In the following we shall give stronger justifications for this choice.

As a preparation we prove the following technical result.

PROPOSITION 9. *Let $A \in \Pi\mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times n}$, and suppose that CA is an H-matrix. Then $\check{A}$ is regular, and the following relations hold:*

$$|C| \geqslant \langle C\check{A}\rangle|\check{A}^{-1}|, \tag{2}$$

$$I - |\check{A}^{-1}|\rho(A) \geqslant \langle C\check{A}\rangle^{-1}\langle CA\rangle. \tag{3}$$

Proof. Since every H-matrix is regular, CA and therefore A and $\check{A} \in A$ are regular. To prove (2) we use the Jacobi decomposition $C\check{A} = D - E$ of $C\check{A}$. Since $C\check{A}$ is thin, $\langle D\rangle = |D|$, whence (D being diagonal) $\langle D\rangle|\check{A}^{-1}| = |D\check{A}^{-1}| = |C + E\check{A}^{-1}| \leqslant |C| + |E|\,|\check{A}^{-1}|$. Hence we have $\langle C\check{A}\rangle|\check{A}^{-1}| = (\langle D\rangle - |E|)|\check{A}^{-1}| \leqslant |C|$, and (2) holds.

Since C is thin, $CA = C\check{A} + C\mathring{A} = C\check{A} + [-1,1]|C|\rho(A)$ by (B20), and since $0 \notin (CA)_{ii}$ we have, by (O4),

$$\langle CA\rangle = \langle C\check{A}\rangle - |C|\rho(A). \tag{4}$$

Now $C\check{A} \in CA$ is H-matrix, hence $\langle C\check{A}\rangle^{-1} \geqslant 0$, and

$$\begin{aligned}
\big[I - |\check{A}^{-1}|\rho(A)\big] &- \langle C\check{A}\rangle^{-1}\langle CA\rangle \\
&= \langle C\check{A}\rangle^{-1}\big[\langle C\check{A}\rangle - \langle CA\rangle - \langle C\check{A}\rangle|\check{A}^{-1}|\rho(A)\big] \\
&= \langle C\check{A}\rangle^{-1}\big(|C| - \langle C\check{A}\rangle|\check{A}^{-1}|\big)\rho(A) \geqslant 0
\end{aligned}$$

by (4) and (2). Therefore (3) holds. ∎

THEOREM 5. *For $A \in \Pi\mathbb{R}^{n\times n}$, the following are equivalent:*

(i) *$\check{A}$ is regular, and $\check{A}^{-1}A$ is an H-matrix.*
(ii) *There are matrices $C_1, C_2 \in \mathbb{R}^{n\times n}$ such that C_1AC_2 is an H-matrix.*
(iii) *$\check{A}$ is regular, and $\sigma(|\check{A}^{-1}|\rho(A)) < 1$.*

Proof. (i) → (ii) is immediate. Suppose now that (ii) holds. Put $B = AC_2$.

Then $\check{B}=\check{A}C_2$ and $C_1\check{B}=C_1\check{A}C_2\in C_1AC_2$ is an H-matrix. Hence C_1 and $C_1\check{B}$ are regular, and $\langle C_1\check{B}\rangle^{-1}\geqslant 0$. Since $C_1B=C_1AC_2$ is an H-matrix, there are positive vectors u,v such that $\langle C_1B\rangle u=v$. Hence by (3), $[I-|\check{B}^{-1}|\rho(B)]u \geqslant \langle C_1\check{B}\rangle^{-1}\langle C_1B\rangle u=\langle C_1\check{B}\rangle^{-1}v>0$, since $\langle C_1\check{B}\rangle^{-1}$ is nonnegative and regular. So $|B^{-1}|\rho(B)u<u$, whence $\sigma(|\check{B}^{-1}|\rho(B))<1$. By (P3) and (B12) we have

$$\begin{aligned}\sigma\big(|\check{A}^{-1}|\rho(A)\big)&=\sigma\big(|C_2\check{B}^{-1}|\rho(A)\big)\leqslant\sigma\big(|C_2||\check{B}^{-1}|\rho(A)\big)\\&=\sigma\big(|\check{B}^{-1}|\rho(A)|C_2|\big)\leqslant\sigma\big(|\check{B}^{-1}|\rho(AC_2)\big)\\&=\sigma\big(|\check{B}^{-1}|\rho(B)\big)<1,\end{aligned}$$

and (iii) holds. Finally, if (iii) holds, then there is $u>0$ with $|\check{A}^{-1}|\rho(A)u<u$. Hence $\langle\check{A}^{-1}A\rangle u=\langle I-\check{A}^{-1}\mathring{A}\rangle u\geqslant(I-|\check{A}^{-1}\mathring{A}|)u=u-|\check{A}^{-1}|\rho(A)u>0$, by (B7), (O1), and (B5). Therefore $\check{A}^{-1}A$ is an H-matrix and (i) holds. ∎

Among the one-sided preconditioners, the midpoint inverse has the following extremal property.

THEOREM 6. *Let $A\in\Pi\mathbb{R}^{n\times n}$. If, for some $C\in\mathbb{R}^{n\times n}$ and some scaled maximum norm, $\|I-CA\|<1$, then $\check{A}$ is regular and*

$$\|I-\check{A}^{-1}A\|\leqslant\|I-CA\|. \tag{5}$$

In particular if $\check{A}^{-1}A$ is an H-matrix, then

$$\sigma\big(|I-\check{A}^{-1}A|\big)\leqslant\sigma\big(|I-CA|\big)\qquad\textit{for all}\quad C\in\mathbb{R}^{n\times n}. \tag{6}$$

Proof. We put

$$S=I-CA,\qquad T=I-\check{A}^{-1}A.$$

Then $\check{S}=I-C\check{A}$, $\check{T}=0$, whence

$$\begin{aligned}|C|&=|(I-\check{S})\check{A}^{-1}|=|\check{A}^{-1}-\check{S}\check{A}^{-1}|\\&\geqslant|\check{A}^{-1}|-|\check{S}||\check{A}^{-1}|=(I-|\check{S}|)|\check{A}^{-1}|,\\|S|-|\check{S}|&=\rho(S)=|C|\rho(A)\geqslant(I-|\check{S}|)|\check{A}^{-1}|\rho(A)\\&=(I-|\check{S}|)\rho(T)=(I-|\check{S}|)|T|.\end{aligned}$$

Therefore

$$(I-|\check{S}|)(|S|-|T|)\geqslant|\check{S}|(I-|S|). \tag{7}$$

But by hypothesis, $\|\cdot\|=\|\cdot\|_u$ for some $u>0$, and $|S|u=|I-CA|u=v<u$. In particular, $|\check{S}|u\leqslant|S|u<u$, whence $\sigma(|\check{S}|)<1$ and $(I-|\check{S}|)^{-1}\geqslant 0$. Therefore (7) implies

$$(|S|-|T|)u\geqslant(I-|\check{S}|)^{-1}|\check{S}|(I-|S|)u\geqslant 0,$$

whence $\|T\|=\||T|u\|\leqslant\||S|u\|=\|S\|$. This implies (5). If $\check{A}^{-1}A$ is an H-matrix, then $\sigma(|I-\check{A}^{-1}A|)=\sigma(|\check{A}^{-1}|\rho(A))<1$, and since the spectral radius of $|S|$ ($|T|$) is the infimum of $\|S\|$ ($\|T\|$) over all scaled maximum norms, (6) follows. ∎

REMARK 1. In different context, Scheu [24] shows a similar extremal property $\||\check{A}^{-1}||\check{A}-A|\|_\infty\leqslant\||B^{-1}||B-A|\|_\infty$ for $B\in\mathbb{R}^{n\times n}$.

REMARK 2. The relation (6) has been proved by Krawczyk [14] for a restricted choice of C.

As a corollary of Theorems 5 and 6, we improve a result of Krawczyk and Selsmark [15, Theorem 5].

PROPOSITION 10. *Suppose that for $A\in\Pi\mathbb{R}^{n\times n}$, the matrix $\check{A}^{-1}A$ is an H-matrix. Then for every $B\subseteq A$, the matrix $\check{B}^{-1}B$ is again an H-matrix. Moreover*

$$\|I-\check{B}^{-1}B\|\leqslant\|I-\check{A}^{-1}A\| \tag{8}$$

for every scaled maximum norm with $\|I-\check{A}^{-1}A\|<1$; in particular

$$\sigma(|I-\check{B}^{-1}B|)\leqslant\sigma(|I-\check{A}^{-1}A|). \tag{9}$$

Proof. $\check{A}^{-1}B\subseteq\check{A}^{-1}A$, whence $\check{A}^{-1}B$ is an H-matrix. Hence by Theorem 5, $\check{B}^{-1}B$ is an H-matrix, and by Theorem 6, $\|I-\check{B}^{-1}B\|\leqslant\|I-\check{A}^{-1}B\|\leqslant\|I-\check{A}^{-1}A\|$. Hence (8) holds, and (9) follows as before. ∎

Theorem 5 leaves the question unanswered whether every regular matrix A can be preconditioned to an H-matrix. The regular matrix A considered in

Remark 3 after Theorem 3 shows that the answer is negative. In fact, with $y=[0.8,1.2]$, $z=[-0.5,0.5]$ we have

$$\check{A}^{-1}=\begin{pmatrix} 1.2 & -0.1 & -0.1 \\ -0.1 & 1.2 & -0.1 \\ -0.1 & -0.1 & 1.2 \end{pmatrix}, \qquad \check{A}^{-1}A=\begin{pmatrix} y & z & z \\ z & y & z \\ z & z & y \end{pmatrix}.$$

Hence $(\check{A}^{-1}A)u=[-0.2,2.2]u \ni 0$ for $u=(1,1,1)^T$, so that by Lemma 2 and Lemma 7, $\check{A}^{-1}A$ is not an H-matrix. By Theorem 5, no choice of C_1 and C_2 makes C_1AC_2 an H-matrix. It is an open question whether C_1 and C_2 can be chosen such that $(C_1AC_2)^G$ exists.

In contrast to this, we have the following interesting result.

THEOREM 7. *If $A \in \Pi\mathbb{R}^{n\times n}$ is regular and $\check{A}^{-1} \geqslant 0$, then $\check{A}^{-1}A$ is an H-matrix.*

Proof. By contradiction. If $\check{A}^{-1}A$ is not an H-matrix, then $\sigma(\rho(A)\check{A}^{-1}) = \sigma(\check{A}^{-1}\rho(A)) = \sigma(|\check{A}^{-1}|\rho(A)) \geqslant 1$ by Theorem 5. Hence there is a nonzero vector $u \geqslant 0$ such that $\rho(A)\check{A}^{-1}u \geqslant u$. By Lemma 2(iv), this contradicts the fact that A is regular. ∎

7. FIXPOINT ITERATION

In this section we study the iteration $y^{(l+1)}=L^F(x+Ey^{(l)})$ for a strong splitting $A=L-E$ (L lower triangular) of an H-matrix A and arbitrary starting vectors $y^{(0)}$. This will lead us to a further sublinear map enclosing A^H: the fixpoint inverse A^F. The restriction to H-matrices is essential (cf. Theorem 12 below).

THEOREM 8. *Let A be an H-matrix, and suppose that $A=L-E$ is a strong triangular splitting of A. Then for arbitrary $x \in \Pi\mathbb{R}^n$, the following statements hold:*

(i) *The equation*

$$y=L^F(x+Ey) \tag{1}$$

has a unique solution $y \in \Pi\mathbb{R}^n$.

(ii) *The iteration*

$$y^{(l+1)}=L^F\left(x+Ey^{(l)}\right) \qquad (l=0,1,2,\dots) \tag{2}$$

converges to y for all starting vectors $y^{(0)} \in \Pi\mathbb{R}^n$, with

$$\|q(y^{(l+1)}, y)\| \leqslant \beta\|q(y^{(l)}, y)\| \tag{3}$$

for any monotone norm satisfying

$$\|\langle L\rangle^{-1}|E|\| = \beta < 1. \tag{4}$$

(iii) *If $y^{(1)} \subseteq y^{(0)}$, then for $i = 0,1,\ldots,$*

$$y \subseteq y^{(i)} \subseteq y^{(i-1)} \subseteq \cdots \subseteq y^{(0)}. \tag{5}$$

(iv) *If $y^{(0)} \subseteq y^{(1)}$, then for $i = 0,1,\ldots,$*

$$y^{(0)} \subseteq \cdots \subseteq y^{(i-1)} \subseteq y^{(i)} \subseteq y. \tag{6}$$

Proof. By Lemma 3(iii), the matrix $\langle L\rangle^{-1}|E|$ has spectral radius less than one. Therefore there exists a monotone norm such that (4) holds. Now the map $\Psi : \Pi\mathbb{R}^n \to \Pi\mathbb{R}^n$ defined by

$$\Psi y := L^F(x + Ey)$$

satisfies

$$\begin{aligned} q(\Psi y, \Psi z) &= q\big(L^F(x + Ey), L^F(x + Ez)\big) \\ &\leqslant |L^F| q(x + Ey, x + Ez) \qquad \text{by (5)} \\ &= |L^F| q(Ey, Ez) \\ &\leqslant |L^F||E| q(y, z) \qquad \text{by (D7)} \\ &= \langle L\rangle^{-1}|E| q(y, z) \end{aligned}$$

by Proposition 4, and (4) implies

$$\|q(\Psi y, \Psi z)\| \leqslant \beta\|q(y, z)\| \tag{7}$$

for all $y, z \in \Pi\mathbb{R}^n$. By Schröder's [25] generalization of the Banach fixpoint theorem to locally complete metric spaces, it follows that Ψ has a unique fixpoint $y \in \Pi\mathbb{R}^n$, and for arbitrary $y^{(0)}$, the iteration $y^{(l+1)} = \Psi y^{(l)}$, i.e. (2), converges to y with speed determined by (3). This proves (i) and (ii).

For the proof of (iii) we first note that Ψ is inclusion isotonic. Hence if $y^{(l)} \subseteq y^{(l-1)}$ then $y^{(l+1)} = \Psi y^{(l)} \subseteq \Psi y^{(l-1)} = y^{(l)}$. So if $y^{(1)} \subseteq y^{(0)}$, then $y^{(l)} \subseteq y^{(k)}$ for all $l \geqslant k$, and for $l \to \infty$ we find $y \subseteq y^{(k)}$. This implies (iii). (iv) follows in the same way by reversing the inclusion signs. ■

PROPOSITION 11. *Let A be an H-matrix. Then there is a unique map $A^F : \Pi\mathbb{R}^n \to \Pi\mathbb{R}^n$ such that for all direct triangular splittings $A = L \ominus E$ and all $x, y \in \Pi\mathbb{R}^n$ we have*

$$A^F x = L^F(x + EA^F x), \tag{8}$$

$$L^F(x + Ey) \subseteq y \quad \Rightarrow \quad A^F x \subseteq y, \tag{9}$$

$$y \subseteq L^F(x + Ey) \quad \Rightarrow \quad y \subseteq A^F x. \tag{10}$$

Moreover, A^F encloses the hull inverse of A,

$$A^H \subseteq A^F. \tag{11}$$

Proof. Let $L \ominus E$ be a direct triangular splitting of A. By Lemma 8(ii), $L - E$ is a strong splitting, whence, by the last theorem, for every $x \in \Pi\mathbb{R}^n$ there is a unique vector y with (1). Let $L = D - L_0$ be the Jacobi splitting of L. Since $L - E$ is a direct splitting, $E_0 = E + L_0$ is a direct sum and $D - E_0$ is the Jacobi splitting of A. By (1), Proposition 4, and (B27), we now have

$$\begin{aligned} y &= L^F(x + Ey) = D^F(x + Ey + L_0 y) \\ &= D^F\left[x + (E \oplus L_0)y\right] = D^F(x + E_0 y). \end{aligned}$$

Hence y is independent of the particular direct splitting of A. Therefore the map A^F which maps $x \in \Pi\mathbb{R}^n$ to the unique $y \in \Pi\mathbb{R}^n$ with (1) is well defined and satisfies (8). The implications (9) and (10) are immediate consequences of Theorem 8(iii),(iv). To show (11) we observe that for every $\tilde{A} \in A$ there are $\tilde{L} \in L$, $\tilde{E} \in E$ such that $\tilde{A} = \tilde{L} - \tilde{E}$. Hence if $\tilde{x} \in x$, $y = \tilde{A}^{-1}\tilde{x}$, then $\tilde{x} = \tilde{A}y = \tilde{L}y - \tilde{E}y$, whence $y = \tilde{L}^{-1}(x + \tilde{E}y) \in L^F(x + Ey)$, and (10) implies $y \in A^F x$. Therefore $A^F x$ also contains the hull $A^H x$ of all $\tilde{A}^{-1}\tilde{x}$, $\tilde{A} \in A$, $\tilde{x} \in x$. ■

We call A^F the *F-inverse* (*fixpoint inverse*) of the H-matrix A. Note that by Proposition 4, A^F agrees with the previously defined A^F for triangular matrices A (which always are H-matrices).

THEOREM 9. *Let A be an H-matrix. Then*

(i) *A^F is a sublinear and normal map with*

$$|A^F| = \langle A \rangle^{-1}. \tag{12}$$

(ii) *If A is thin then A^F is linear and*

$$\kappa(A) = A^{-1}. \tag{13}$$

(iii) *If $B \subseteq A$ then B is an H-matrix and $B^F \subseteq A^F$.*

Proof. We fix a direct triangular splitting $A = L \ominus E$. If $x \subseteq y$ then $L^F(x + EA^F y) \subseteq L^F(y + EA^F y) = A^F y$, whence $A^F x \subseteq A^F y$ by (9). Hence A^F is inclusion isotonic. Homogeneity is immediate. To show subadditivity we put $z := A^F x + A^F y$, so that

$$\begin{aligned} L^F(x + y + Ez) &= L^F\left[x + y + E(A^F x + A^F y)\right] \\ &\subseteq L^F(x + y + EA^F x + EA^F y) \\ &\subseteq L^F(x + EA^F x) + L^F(y + EA^F y) \\ &= A^F x + A^F y = z. \end{aligned} \tag{14}$$

Hence $A^F(x + y) \subseteq z = A^F x + A^F y$ by (9). Therefore A^F is sublinear. Moreover, if A is thin, then L, E are thin, L^F is linear, and (14) holds with equality throughout. Since the fixpoint is unique, $A^F(x + y) = z = A^F x + A^F y$ and A^F is linear.

To show (12) we apply Proposition 3 and obtain

$$\begin{aligned} |A^F| &= (I - |L^F||E^M|)^{-1}|L^F| \\ &= (I - \langle L \rangle^{-1}|E|)^{-1}\langle L \rangle^{-1} \\ &= (\langle L \rangle - |E|)^{-1} = \langle A \rangle^{-1}, \end{aligned}$$

since $L - E$ is a direct splitting. Moreover, Proposition 3 also show that A^F is normal.

To show (13) we observe that for thin A (and hence L, E) and thin x the vector $y = A^{-1}x$ satisfies $x + Ey = Ay + Ey = Ly$; hence $y = L^{-1}(x + Ey) = L^F(x + Ey)$, whence $A^F x = y = A^{-1}x$. Substitution of $x = e^{(i)}$ for $i = 1,\dots,n$ provides (13).

Finally, if $B \subseteq A$, then B is an H-matrix [Lemma 7(i)]. If $A = L - E$ and $B = L' - E'$ are Jacobi splittings, then $L' \subseteq L$, $E' \subseteq E$, and $L'^F(x + E'A^F x) \subseteq L^F(x + EA^F x) = A^F x$, whence by (9) (adapted to B) we have $B^F x \subseteq A^F x$ for all $x \in \Pi\mathbb{R}^n$. This implies (iii). ∎

We end this section with an optimality result for the convergence speed of the iteration (2).

THEOREM 10. *Let A be an H-matrix, and let $\|\cdot\|$ be a scaled maximum norm. Then among all strong triangular splittings $A = L - E$, the spectral radius $\sigma(\langle L\rangle^{-1}|E|)$ and the guaranteed upper bound $\|\langle L\rangle^{-1}|E|\,\|$ for the convergence factor of the iteration*

$$y^{(l+1)} = L^F\big(x + Ey^{(l)}\big)$$

assume their minimal value for the Gauss-Seidel splitting.

Proof. Let $A = L - E$ be a strong triangular splitting. Let $A = L^* - R^*$ be the Gauss-Seidel splitting of A, and let $E = L_0 - R_0$ be the Gauss-Seidel splitting of E. Then

$$R_0 = -R^*, \qquad L^* = L - L_0, \tag{15}$$

whence in particular

$$|E| = |L_0| + |R^*|. \tag{16}$$

Now if $\|\langle L\rangle^{-1}|E|\,\| < 1$ and $\|\cdot\| = \|\cdot\|_u$, then

$$v := \langle L\rangle^{-1}|E|u < u. \tag{17}$$

Therefore by (O1),

$$\begin{aligned}\langle L^*\rangle v = \langle L - L_0\rangle v &\geqslant \langle L\rangle v - |L_0|v = |E|u - |L_0|v\\ &= |L_0|(u - v) + |R^*|u \geqslant |R^*|u,\end{aligned}$$

and since $\langle L^* \rangle^{-1} \geqslant 0$, this implies

$$\langle L^* \rangle^{-1} |R^*| u \leqslant v. \tag{18}$$

Clearly (18) implies $\| \langle L^* \rangle^{-1} |R^*| \|_u \leqslant \|v\|_u = \| \langle L \rangle^{-1} |E| \|_u$. By property (P6) of the spectral radius, the Gauss-Seidel splitting also minimizes $\sigma(\langle L \rangle^{-1} |E|)$. ■

8. LOCAL SOLUTIONS OF LINEAR EQUATIONS

In the context of solving nonlinear equations an important auxiliary problem is the efficient enclosure of those solutions of the equation

$$\tilde{A}\tilde{y} = \tilde{x} \qquad (\tilde{A} \in A, \quad \tilde{x} \in x) \tag{1}$$

which lie in a given interval vector z ("local solutions"). Several algorithms for doing this have been suggested, and an important class of them is discussed below. We call a function Φ a local solver for the matrix A if for every $x, z \in \Pi\mathbb{R}^n$ (or $z = \varnothing$), $\Phi(x, z)$ is an interval vector (or the empty set) with

$$\Phi(x, z) \supseteq \{ \tilde{y} \in z | \tilde{A}\tilde{y} = \tilde{x} \text{ for some } \tilde{A} \in A, \tilde{x} \in x \} \tag{2}$$

which is inclusion isotonic with respect to z:

$$z \subseteq z' \quad \Rightarrow \quad \Phi(x, z) \subseteq \Phi(x, z').$$

EXAMPLE 1. The definitions $\Phi(x, z) := A^{-1}x$ (if $A^{-1} \geqslant 0$) and $\Phi(x, z) := A^G x$ (if the triangular decomposition of A exists) define local solvers for A which are independent of z. We may call them *global solvers* for A.

EXAMPLE 2. Suppose that

$$A = L - E \qquad (L \text{ regular lower triangular}) \tag{3}$$

is a triangular splitting of A. We define

$$\Phi(x, z) := L^F(x + Ez). \tag{4}$$

If (1) holds for some $\tilde{y} \in z$, then there are $\tilde{L} \in L$, $\tilde{E} \in E$ such that $\tilde{A} = \tilde{L} - \tilde{E}$, whence $\tilde{L}\tilde{y} = \tilde{x} + \tilde{E}\tilde{y}$ and $\tilde{y} = \tilde{L}^{-1}(\tilde{x} + \tilde{E}\tilde{y}) \in L^F(x + Ez) = \Phi(x, z)$. Hence (2) holds. Clearly Φ is inclusion isotonic with respect to z. Therefore, Φ is a local solver for A.

EXAMPLE 3. In Example 2, the vector $y = \Phi(x, z)$ defined by (4) can be computed componentwise as

$$y_i = \frac{x_i - \sum\limits_{k<i} L_{ik} y_k + \sum\limits_{k} E_{ik} z_k}{L_{ii}} \qquad (i = 1, \ldots, n).$$

Since both y_k and z_k enclose the kth component of the desired local solution set, a sharper enclosure can be obtained by using in place of y_k the intersection z_k' of y_k and z_k. If we do this in each step we obtain

$$\left.\begin{aligned} y_i &= \frac{x_i - \sum\limits_{k<i} L_{ik} z_k' + \sum\limits_{k} E_{ik} z_k}{L_{ii}} \\ z_i' &= y_i \cap z_i \end{aligned}\right\} \qquad (i = 1, \ldots, n). \tag{5}$$

We denote the resulting vector z' by $\Gamma_L(A, x, z)$. Then $\Gamma_L(A, \cdot, \cdot)$ is a local solver and has the obvious property

$$\Gamma_L(A, x, z) \subseteq L^F(x + Ez) \cap z \qquad \text{for all} \quad x, z \in \Pi\mathbb{R}^n. \tag{6}$$

As we shall see below, the most important case is when (3) is the Gauss-Seidel splitting; then (5) becomes

$$\left.\begin{aligned} y_i &= \frac{x_i - \sum\limits_{k<i} A_{ik} z_k' - \sum\limits_{k>i} A_{ik} z_k}{A_{ii}} \\ z_i' &= y_i \cap z_i \end{aligned}\right\} \qquad (i = 1, \ldots, n), \tag{5a}$$

and we denote the resulting vector z' by $\Gamma(A, x, z)$. We call $\Gamma(A, \cdot, \cdot)$ the *Gauss-Seidel solver* for A; Γ is inclusion isotonic with respect to A, x, and z.

Before returning to the solution of (1) we discuss some properties of local solvers. Our first result shows the optimality of the Gauss-Seidel solver within the class of local solvers defined by triangular splittings.

THEOREM 11. *Let* $A \in \Pi\mathbb{R}^{n \times n}$, *and suppose that* $0 \notin A_{ii}$ *for* $i = 1, \dots, n$. *Then the inclusion*

$$\Gamma(A, x, z) \subseteq \Gamma_L(A, x, z) \subseteq L^F(x + Ez) \cap z$$

holds for all triangular splittings $A = L - E$ *of* A *with regular* L.

For the *proof* we need the following auxiliary result.

LEMMA 14. *For* $x, y, a, b, c \in \Pi\mathbb{R}$ *with* $0 \notin a, b$, *suppose that*

$$y \subseteq x/a, \qquad a = b - c.$$

Then

$$y \subseteq (x + cy)/b.$$

Proof. For every $\tilde{y} \in y$ there are $\tilde{x} \in x$, $\tilde{a} \in a$ and $\tilde{b} \in b$, $\tilde{c} \in c$ such that $\tilde{y} = \tilde{x}/\tilde{a}$, $\tilde{a} = \tilde{b} - \tilde{c}$. Hence $\tilde{x} = \tilde{a}\tilde{y} = \tilde{b}\tilde{y} - \tilde{c}\tilde{y}$ and $\tilde{y} = (\tilde{x} + \tilde{c}\tilde{y})/\tilde{b} \in (x + cy)/b$. Since $\tilde{y} \in y$ is arbitrary, $y \subseteq (x + cy)/b$. ∎

Proof of Theorem 11. Let $A = L - E$ be a triangular splitting with regular L. Then

$$A_{ik} = \begin{cases} L_{ik} - E_{ik} & \text{if} \quad k \leqslant i, \\ -E_{ik} & \text{if} \quad k > i. \end{cases} \tag{7}$$

By (5), the ith component of the vector $z^* := \Gamma_L(A, x, z)$ satisfies

$$z_i^* = z_i \cap y_i^*/L_{ii}, \tag{8}$$

where

$$y_i^* = x_i - \sum_{k<i} L_{ik} z_k^* + \sum_k E_{ik} z_k.$$

By (5a), the ith component of the vector $z' := \Gamma(A, x, z)$ satisfies

$$z_i' \subseteq y_i = y_i'/A_{ii}, \tag{9}$$

where by (7),

$$y_i' = x_i - \sum_{k<i} A_{ik} z_k' - \sum_{k>i} A_{ik} z_k$$
$$\subseteq x_i - \sum_{k<i} L_{ik} z_k' + \sum_{k<i} E_{ik} z_k' + \sum_{k>i} E_{ik} z_k.$$

Suppose now that $z_k' \subseteq z_k^*$ for all $k < i$ (this is certainly true for $i = 1$). Since $z_k' \subseteq z_k$, we then have

$$y_i' + E_{ii} z_i' \subseteq y_i^*. \tag{10}$$

Now apply Lemma 14 to (9) with $A_{ii} = L_{ii} - E_{ii}$. We find using (10), that

$$z_i' \subseteq \frac{y_i' + E_{ii} z_i'}{L_{ii}} \subseteq \frac{y_i^*}{L_{ii}}. \tag{11}$$

Since $z_i' \subseteq z_i$, (11) and (8) show that $z_i' \subseteq z_i^*$. By induction, $z_i' \subseteq z_i^*$ holds for all i, whence $z' \subseteq z^*$. This proves $\Gamma(A, x, z) \subseteq \Gamma_L(A, x, z)$; the other inclusion is in (6). ∎

As a corollary we have the following complement to Theorem 8.

PROPOSITION 12. *Let A be an H-matrix, and let $A = L - E$ be a strong triangular splitting of A. Then for $x, z \in \Pi\mathbb{R}^n$,*

$$L^F(x + Ez) \subseteq z \quad \Rightarrow \quad A^F x \subseteq z.$$

In particular, $A^F x$ is contained in the solution y of the equation $y = L^F(x + Ey)$.

Proof. $z^* := A^F x$ satisfies $z^* = \Gamma(A, x, z^*)$, whence $z^* \subseteq L^F(x + Ez^*)$ by Theorem 11. Hence by Theorem 8(iv), z^* is contained in the solution y of $y = L^F(x + Ey)$. Now if $L^F(x + Ez) \subseteq z$, then by Theorem 8(iii), $y \subseteq z$, so that $A^F x = z^* \subseteq y \subseteq z$. ∎

Our next result is concerned with the iterated use of the local solvers defined by triangular splittings. It gives an *a posteriori* sufficient condition for *all* solutions of (1) to be enclosed in a given vector z. It also indicates that this class of local solvers is of limited value for matrices which are not H-matrices.

THEOREM 12. *Let* $A = L - E$ *be a triangular splitting of* A *with regular* L, *and suppose that for the sequence defined by*

$$z^{(0)} := z, \qquad z^{(l+1)} := \Gamma_L(A, x, z^{(l)}), \quad l = 0, 1, \ldots, \tag{12}$$

there is an index $k > 0$ *such that*

$$\emptyset \neq z^{(k)} \subseteq \operatorname{int}(z). \tag{13}$$

Then A is an H-matrix, $L - E$ is a strong splitting of A, and

$$A^H x \subseteq A^F x \subseteq z^{(k)} \subseteq \operatorname{int}(z). \tag{14}$$

As a preparation of the *proof* we show:

PROPOSITION 13. *If the assumptions of Theorem* 12 *are satisfied, then*

$$\emptyset \neq z^{(l+1)} = L^F(x + Ez^{(l)}) \qquad \textit{for all} \quad l \geqslant k,$$

i.e., the intersection in (5) *is superfluous for* $l \geqslant k$.

Proof. Define

$$y_i^{(l)} := \frac{x_i - \sum_{k<i} L_{ik} z_k^{(l+1)} + \sum_k E_{ik} z_k^{(l)}}{L_{ii}}. \tag{15}$$

We first show that

$$z_i^{(l+1)} = y_i^{(l)} \cap z_i. \tag{16}$$

Indeed, since $z^{(0)} = z$, this holds for $l = 0$ and all i. Suppose now that (16) holds for all pairs (l, i) lexicographically smaller than (m, j). Since $z^{(m+1)} \subseteq z^{(m)} \subseteq z^{(m-1)}$, inclusion isotonicity implies

$$y_j^{(m)} \subseteq y_j^{(m-1)}. \tag{17}$$

Hence, by definition of Γ_L, (16), and (17),

$$z_j^{(m+1)} = y_j^{(m)} \cap z_j^{(m)} = y_j^{(m)} \cap y_j^{(m-1)} \cap z_j = y_j^{(m)} \cap z_j.$$

By induction, (16) holds generally.

Now if (13) holds, then by (12) and (6) we have $z^{(l+1)} \subseteq \text{int}(z)$ for all $l \geqslant k$, whence by (16) we must have $y_i^{(l)} = z_i^{(l+1)}$ (otherwise, $z_i^{(l+1)}$ would contain some endpoint of z_i). Insertion into (15) gives $z^{(l+1)} = L^F(x + Ez^{(l)})$; in particular $z^{(l+1)} \neq \varnothing$ if $z^{(l)} \neq \varnothing$, so that all $z^{(l)}$ are nonempty. ∎

Proof of Theorem 12. We proceed by contraposition and assume that $L - E$ is not a strong splitting of A. By Lemma 3(iii) we have $\sigma(\langle L \rangle^{-1}|E|) \geqslant 1$, and by (P8), there is a vector $v \geqslant 0$ such that $\langle L \rangle^{-1}|E|v \geqslant v \neq 0$. Choose $\tilde{L} \in L$, $\tilde{E} \in E$ such that $\langle \tilde{L} \rangle = \langle L \rangle$, $|\tilde{E}| = |E|$; then

$$\langle \tilde{L} \rangle^{-1}|\tilde{E}|v \geqslant v \neq 0. \tag{18}$$

By Proposition 13, the vector $y := \bigcap_{l \geqslant k} z^{(l)} = \lim_{l \to \infty} z^{(l)}$ is nonempty and satisfies the equation $L^F(x + Ey) = y$. Hence the iteration defined by

$$w^{(0)} := y, \qquad w^{(l+1)} := \tilde{L}^F(x + \tilde{E}w^{(l)}), \quad l = 0, 1, \ldots,$$

satisfies $w^{(1)} = \tilde{L}^F(x + \tilde{E}y) \subseteq L^F(x + Ey) = y = w^{(0)}$, and inductively $w^{(l+1)} = \tilde{L}^F(x + \tilde{E}w^{(l)}) \subseteq \tilde{L}^F(x + \tilde{E}w^{(l-1)}) = w^{(l)}$. So the $w^{(l)}$ form a nested sequence of intervals, whence $w := \bigcap_{l \geqslant 0} w^{(l)} = \lim_{l \to \infty} w^{(l)}$ exists and satisfies

$$\tilde{L}^F(x + \tilde{E}w) = w. \tag{19}$$

Moreover, since $w \subseteq y \subseteq z^{(k)} \subseteq \text{int}(z)$, we can find a number $\alpha > 0$ such that

$$b := w + v[-\alpha, \alpha] \subseteq z. \tag{20}$$

We now show that

$$b \subseteq c := \tilde{L}^F(x + \tilde{E}b). \tag{21}$$

Indeed,

$$\begin{aligned}
c &= \tilde{L}^F\big(x + \tilde{E}w + |\tilde{E}|v[-\alpha, \alpha]\big) && \text{by (B20) and (B6),} \\
&= \tilde{L}^F(x + \tilde{E}w) + \langle \tilde{L} \rangle^{-1}|\tilde{E}|v[-\alpha, \alpha] && \text{by Proposition 4 and (R8),} \\
&\supseteq \tilde{L}^F(x + \tilde{E}w) + v[-\alpha, \alpha] && \text{by (18),} \\
&= w + v[-\alpha, \alpha] = b && \text{by (19) and (20).}
\end{aligned}$$

Next we show that

$$b_i \subseteq z_i^{(l)} \tag{22}$$

for all $l \geqslant 0$ and $i = 1,\ldots,n$. Indeed, (22) holds for $l = 0$ by (20), and if it holds for all pairs (l, i) lexicographically smaller than $(m+1, j)$ then

$$\begin{aligned} y_j^{(m)} :&= \frac{x_j - \sum_{k<j} L_{jk} z_k^{(m+1)} + \sum_k E_{jk} z_k^{(m)}}{L_{jj}} \\ &\supseteq \frac{x_j - \sum_{k<j} \tilde{L}_{jk} b_k + \sum_k \tilde{E}_{jk} b_k}{L_{jj}} \\ &= c_j \supseteq b_j. \end{aligned}$$

Hence $b_j \subseteq z_j^{(m)} \cap y_j^{(m)} = z_j^{(m+1)}$. By induction, (22) holds for all l, i, whence in the limit, $b \subseteq y = w^{(0)}$. If we know $b \subseteq w^{(l)}$ for some l, then $w^{(l+1)} = \tilde{L}^F(x + \tilde{E}w^{(l)}) \supseteq \tilde{L}^F(x + \tilde{E}b) = c \supseteq b$; therefore again $b \subseteq w^{(l)}$ for all l, and in the limit $b \subseteq w$, which contradicts (20), since $v \neq 0$. Therefore, $L - E$ must be a strong splitting of A, and by Lemma 9 , A is an H-matrix. Assertion (14) now follows immediately from Propositions 12 and 13. ∎

As an alternative to the componentwise intersection used in (5) and (6) we may consider intersection formation after the complete evaluation of a local solver. In analogy to Proposition 13, an easy condition can be given such that the intersections are superfluous again.

PROPOSITION 14. *Let f be a local solver, and suppose that for the sequence defined by*

$$z^{(0)} := z, \qquad z^{(l+1)} := \phi(x, z^{(l)}) \cap z^{(l)}, \quad l = 0, 1, \ldots, \tag{23}$$

there is an index $k \geqslant 0$ such that

$$\varnothing \neq \phi(x, z^{(k)}) \subseteq z. \tag{24}$$

Then for all $l \geqslant k$,

$$z^{(l+1)} = \phi(x, z^{(l)}) \subseteq z^{(l)}. \tag{25}$$

Proof. Put $y^{(l)} := \phi(x, z^{(l)})$, so that by (23),

$$z^{(l+1)} = y^{(l)} \cap z \tag{26}$$

holds for $l = 0$. If (26) holds for some l, then, since ϕ is inclusion isotone,

$$\begin{aligned} y^{(l+1)} &= \phi(x, z^{(l+1)}) \subseteq \phi(x, z^{(l)}) = y^{(l)}, \\ z^{(l+2)} &= y^{(l+1)} \cap z^{(l+1)} = y^{(l+1)} \cap y^{(l)} \cap z \\ &= y^{(l+1)} \cap z, \end{aligned} \tag{27}$$

so that by induction, (26), (27) hold generally. In particular, if $\varnothing \neq y^{(k)} \subseteq z$, then by (27), $y^{(l)} \subseteq z$ for all $l \geqslant k$, so that $z^{(l+1)} = y^{(l)}$ by (26). This proves (25). ∎

With Proposition 14 in place of Proposition 13, the proof of Theorem 12 shows that Theorem 12 remains true if (12) and (13) are replaced by (23) and (24), with the local solver ϕ defined by (4). On the other hand, for general local solvers, only the following weak substitute of Theorem 12 is available; unfortunately, its hypothesis cannot be tested (as yet) in a practical way.

THEOREM 13. *Let $\phi(x, z)$ be a local solver for A, and suppose that there are $A^* \in A$, $x^* \in x$, $z^* \in z$ such that $A^* z^* = x^*$. If*

$$\phi(x, z) \subseteq \operatorname{int}(z),$$

then A is regular and

$$A^H x \subseteq \operatorname{int}(z).$$

The *proof* involves the following preliminary result.

LEMMA 15. *Under the assumption of Theorem 13, let $\tilde{z} \in z$, $\tilde{A} \in A$, $\tilde{x} \in x$ satisfy $\tilde{A}\tilde{z} = \tilde{x}$. Then $\tilde{z} \in \operatorname{int}(z)$ and $\tilde{A}$ is regular.*

Proof. Since ϕ is a local solver, $\tilde{z} \in \phi(x, z) \subseteq \operatorname{int}(z)$, by assumption. If $\tilde{A}$ is singular, then there is $\tilde{y} \neq 0$ such that $\tilde{A}\tilde{y} = 0$. Now $\tau := \sup\{t \geqslant 0 \mid \tilde{z} + \tilde{t}\tilde{y} \in z\}$ exists, since z is bounded, and $z^* := \tilde{z} + \tau\tilde{y} \in z \setminus \operatorname{int}(z)$. But since $\tilde{A}z^* = \tilde{x}$, this contradicts the first part. Hence $\tilde{A}$ is regular. ∎

Proof of Theorem 13. Since ϕ is a local solver, $z^* \in \phi(x, z) \subseteq \mathrm{int}(z)$. We proceed in two stages.

(i) Let $\tilde{A} \in A$. We want to show that $\tilde{A}$ is regular. Put $\tilde{A}(t) := A^* + t(\tilde{A} - A^*)$, so that $\tilde{A}(0) = A^*$, $\tilde{A}(t) \in A$ for $t \in [0,1]$. By Lemma 15, $\tilde{A}(t)$ is regular for every t in the set $T := \{t \in [0,1] \mid \exists \tilde{z}(t) \in \mathrm{int}(z): \tilde{A}(t)\tilde{z}(t) = x^*\}$. Now T is closed, since $\tilde{A}(t)$ is continuous and z is closed. Moreover $0 \in T$ by assumption. Therefore $\tau := \sup T \in T$, and there is a vector $\tilde{z}(\tau) \in z$ with $\tilde{A}(\tau)\tilde{z}(\tau) = x^*$. By Lemma 15, $\det \tilde{A}(\tau) \neq 0$, and $\tilde{A}(\tau)^{-1}x^* = \tilde{z}(\tau) \in \mathrm{int}(z)$. Hence $\tilde{z}(t) := \tilde{A}(t)^{-1}x^*$ is defined and is in $\mathrm{int}(z)$ for all $t > \tau$ sufficiently close to τ. By construction of τ this implies $\tau = 1$. Hence $1 \in T$ and $\tilde{A} = \tilde{A}(1)$ is regular. Since $\tilde{A} \in A$ was arbitrary, A is regular.

(ii) Let $\tilde{A} \in A$, $\tilde{x} \in x$, and put $\tilde{A}(t) := A^* + t(\tilde{A} - A^*)$, $\tilde{x}(t) := x^* + t(\tilde{x} - x^*) \in x$. If $t \in [0,1]$, then $\tilde{x}(t) \in x$ and $\tilde{A}(t)$ is regular. Hence $\tilde{z}(t) := \tilde{A}(t)^{-1}\tilde{x}(t)$ describes a path with $\tilde{z}(0) = A^{*-1}x^* = z^* \in \mathrm{int}(z)$. By Lemma 15, this path can nowhere cross the boundary of z. Hence $\tilde{z}(t) \in \mathrm{int}(z)$ for all $t \in [0,1]$. In particular, $\tilde{A}^{-1}\tilde{x} = \tilde{z}(1) \in \mathrm{int}(z)$. Since $\tilde{A} \in A$, $\tilde{x} \in x$ were arbitrary, this implies that $A^H x \subseteq \mathrm{int}(z)$. ∎

We now return to the solution of (1) for an H-matrix $A \in \Pi\mathbb{R}^{n\times n}$ and $x \in \Pi\mathbb{R}^n$. To determine an enclosure of the local solutions of (1) in $z \in \Pi\mathbb{R}^n$ we may iterate with (12) or (23). By Theorems 10 and 11, it is sensible to choose (12) with the Gauss-Seidel solver $\Gamma_L = \Gamma$. Clearly, all $z^{(l)}$ contain the local solutions; moreover $z^{(l+1)} \subseteq z^{(l)}$ so that the sequence converges. By Theorem 11, Theorem 8, and Proposition 11, the limit z^* satisfies

$$z^* \subseteq A^F x \cap z, \tag{28}$$

and by Theorem 12 we cannot expect anything much better than this.

If we are interested in an enclosure of all solutions of (1), then we must choose the starting vector z of (12) sufficiently large, and the condition (13) is a sufficient *a posteriori* condition for us to have succeeded. A more satisfactory choice of z is provided by Lemma 13 of Section 3: The knowledge of a pair u, v of positive real vectors with

$$\langle A \rangle u \geqslant v > 0 \tag{29}$$

immediately gives the initial enclosure $z := [-u, u]\|x\|_v$ of $A^H x$, and by definition of a local solver, the iterates $z^{(l)}$ then contain all solution of (1). We show that the $z^{(l)}$ converge to $A^F x$. Indeed, with

$$\alpha := \|x\|_v, \qquad z' := L^F(x + Ez)$$

for some direct splitting $A = L - E$, we have $|x| \leqslant \alpha v$; hence

$$|z'| \leqslant \langle L \rangle^{-1}|x + Ez| \leqslant \langle L \rangle^{-1}(\alpha v + |E||z|)$$
$$\leqslant \langle L \rangle^{-1}(\alpha\langle A \rangle u + |E|\alpha u) = \langle L \rangle^{-1}(\alpha\langle L \rangle u) = \alpha u$$

by (29) and Lemma 8, so that $z' \subseteq z$ and $A^F x \subseteq z$ by Proposition 11 (9). But it is easy to see that $A^F x \subseteq z^{(l)}$ implies $A^F x \subseteq \Gamma(A, x, z^{(l)}) = z^{(l+1)}$; therefore $A^F x$ is contained in the limit z^*, and by (28) $z^* = A^F x$.

For the construction of positive vectors u, v with (29), if they are not readily available, see the remark after Lemma 5.

Finally, we relate the results of this section to some iteration procedures considered by Alefeld and Herzberger [2] (see also [3], [16]). They associate with the fixpoint problem

$$y = x + By \tag{30}$$

the *whole-step iteration*,

$$z^{(l+1)} = x + Bz^{(l)}, \tag{31}$$

and the *single-step iteration*,

$$z_i^{(l+1)} = x_i + \sum_{k<i} B_{ik} z_k^{(l+1)} + \sum_{k \geqslant i} B_{ik} z_k^{(l)}. \tag{32}$$

We recognize (31) and (32) as the iterated use of the local solver (4) for the matrix

$$A := I - B \tag{33}$$

with the Richardson splitting in (31), and the splitting

$$A = L - E, \qquad \begin{cases} L_{ik} = -B_{ik}, & E_{ik} = 0 \quad \text{for } k < i, \\ L_{ii} = 1, & E_{ii} = B_{ii}, \\ L_{ik} = 0, & E_{ik} = B_{ik} \quad \text{for } k > i \end{cases} \tag{34}$$

in (32). Intersection forming is also treated in [2], resulting in the *whole-step* *iteration with intersection,*

$$z^{(l+1)} = \left(x + Bz^{(l)}\right) \cap z^{(l)}, \tag{35}$$

the *single-step iteration with intersection,*

$$\begin{aligned} y_i^{(l)} &= x_i + \sum_{k<i} B_{ik} y_k^{(l)} + \sum_{k \geqslant i} B_{ik} z_k^{(l)}, \\ z^{(l+1)} &= y^{(l)} \cap z^{(l)}, \end{aligned} \tag{36}$$

and the single-step iteration with componentwise intersection,

$$\begin{aligned} y_i^{(l)} &= x_i + \sum_{k<i} B_{ik} z_k^{(l+1)} + \sum_{k \geqslant i} B_{ik} z_k^{(l)}, \\ z_i^{(l+1)} &= y_i^{(l)} \cap z_i^{(l)}. \end{aligned} \tag{37}$$

(35) and (36) are the iteration (23), applied to the local solvers (4) for the splittings (33) and (34), and (37) is the iteration (12) for the splitting (34). By Theorems 10 and 11 and Proposition 12, these iterations are inferior to those derived from the Gauss-Seidel splitting with respect to asymptotic convergence speed, attainable limit radius, and the radius of the lth iterate.

Moreover, it is shown in [2] that whole-step and/or single-step iteration converge for all starting vectors precisely when $\sigma|B| < 1$. This implies that A is an H-matrix; but as even simple one-dimensional examples show, the converse is not the case. Therefore, whole-step and single-step iteration are not as widely applicable as the iterations defined by direct splittings.

Overrelaxation, also considered in [2], again leads to local solvers defined by triangular splittings, and hence does not give any improvement over the Gauss-Seidel solver.

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