Convergence of Relaxed Parallel Multisplitting Methods

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ABSTRACT

Multisplitting methods are parallel methods for the solution of a linear system $Ax = b$. It has been observed that the convergence of multisplitting methods is often improved if some kind of relaxation is used. We investigate two different variants of relaxed multisplitting methods. If A is an H-matrix, these methods converge if the relaxation parameter is from an interval $(0, \omega_0)$ with $\omega_0 > 1$.

1. INTRODUCTION

Multisplitting methods for the solution of a linear system of equations $Ax = b$ are genuine parallel iterative methods which are based on several splittings of the coefficient matrix A. They were introduced by O'Leary and White in [9], where several basic convergence results may be found. Recently, Neumann and Plemmons [7] developed some more refined convergence results for one of the cases considered in [9]. It has already been observed in [9] that the introduction of a relaxation parameter may considerably improve the multisplitting methods, but until now no convergence results have been published for these modifications of multisplitting methods.

In the present paper we will consider two relaxed variants of multisplitting methods and we will establish the convergence of these methods under certain restrictions on the relaxation parameter and on the underlying multisplittings. In this manner we obtain convergence results not only for relaxed multisplitting methods but also for unrelaxed methods which have not been considered in the literature before.

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We introduce the relaxed multisplitting methods in Section 2. After some auxiliary results on H-matrices in Section 3, we develop our main results in Section 4. In particular, we will establish the convergence of certain relaxed multisplitting methods, provided A is an H-matrix and the relaxation parameter is from an interval $(0, \omega_0)$ with $\omega_0 > 1$.

2. RELAXED MULTISPLITTINGS

Recall the following definition of a multisplitting given in [9].

DEFINITION 2.1. Let A be a nonsingular real $n \times n$ matrix, and suppose that for some $K \in \mathbb{N}$ we are given matrices $M_k, N_k, E_k \in \mathbb{R}^{n\times n}$, $k = 1,\dots,K$, satisfying

(i) $A = M_k - N_k$ for $k = 1,\dots,K$,
(ii) M_k is nonsingular for $k = 1,\dots,K$,
(iii) E_k is a diagonal matrix with nonnegative entries for $k = 1,\dots,K$, and $\sum_{k=1}^{K} E_k = I$ ($n \times n$ identity matrix).

Then the collection of triples (M_k, N_k, E_k), $k = 1,\dots,K$, is called a *multisplitting* of A. The corresponding *multisplitting method* to solve $Ax = b$ (with $b \in \mathbb{R}^n$) is defined by the iteration

$$x^{m+1} = \sum_{k=1}^{K} E_k y^{m,k}, \qquad m = 0,1,\dots, \tag{1a}$$

where

$$M_k y^{m,k} = N_k x^m + b, \qquad k = 1,\dots,K. \tag{1b}$$

The iteration (1) may be written in the form $x^{m+1} = Hx^m + c$ with $c \in \mathbb{R}^n$ and $H = \sum_{k=1}^{K} E_k M_k^{-1} N_k$. Let $\rho(B)$ denote the spectral radius of an $n \times n$ matrix B. Then (1) converges for any starting vector x^0 if and only if $\rho(H) < 1$.

Clearly, the calculations of $y^{m,k}$ for various k are independent and can therefore be performed in parallel. Moreover, considerable savings in computational work may be possible, since a component of y^{mk} needs not be computed if the corresponding diagonal entry of E_k is zero. The role of the matrices E_k may be regarded as determining the distribution of the computational work to the individual processors. For details we refer to [9]. Note that the above idea of breaking up a problem into several parallel parts has meanwhile given rise to multisplitting methods for systems of nonlinear equations (White [11, 12], Frommer [4]) and for systems of linear equations with coefficients which vary in intervals (Frommer and Mayer [5]).

In [9] O'Leary and White also considered a relaxed multisplitting method in which a positive relaxation parameter ω is used in the same manner as in the relaxed Jacobi method. This method is thus given by the iteration

$$x^{m+1} = \omega \sum_{k=1}^{K} E_k M_k^{-1}(N_k x^m + b) + (1-\omega)x^m, \qquad m = 0,1,\dots . \quad (2)$$

Clearly, the iteration (2) may be written in the form $x^{m+1} = Hx^m + c$ with

$$H = \omega \sum_{k=1}^{K} E_k M_k^{-1} N_k + (1-\omega)I = \sum_{k=1}^{K} E_k M_k^{-1}\left[(1-\omega)M_k + \omega N_k\right]. \quad (3)$$

Another possibility to introduce a relaxation parameter arises if all the matrices M_k, $k = 1,\dots, K$, are of a particular lower triangular form. It is convenient to use a different notation in this case. Suppose that we have $A = D - L_k - V_k$ for $k = 1,\dots, K$, where D is the diagonal part of A, L_k is strictly lower triangular, and V_k is such that $A = D - L_k - V_k$. (Hence V_k is in general not upper triangular.) Assume that D is nonsingular and denote the entries of A, D, L_k and V_k by a_{ij}, d_{ij}, l_{ij}^k, and v_{ij}^k, respectively. Consider the multisplitting $(D - L_k, V_k, E_k)$, $k = 1,\dots, K$. Then the calculation of the intermediate results $y^{m,k}$ will be done by solving the lower triangular systems

$$(D - L_k)y^{m,k} = V_k x^m + b, \qquad k = 1,\dots, K.$$

We may thus use a positive relaxation parameter in the same manner as in the Gauss-Seidel method, i.e., we may perform the iteration

$$x^{m+1} = \sum_{k=1}^{K} E_k \bar{y}^{m,k}, \qquad m = 0,1,\dots, \quad (4a)$$

where for $k = 1,\dots, K$ the components of $\bar{y}^{m,k}$ are computed successively by

$$\bar{y}_i^{m,k} = \frac{\omega}{d_{ii}}\left(\sum_{j=1}^{i-1} l_{ij}^k \bar{y}_j^{m,k} + \sum_{j=1}^{n} v_{ij}^k x_j^m + b_i\right) + (1-\omega)x_i^m, \qquad i = 1,\dots, n. \quad (4b)$$

Here, the iteration (4) may be written in the form $x^{m+1} = Hx^m + c$ with

$$H = \sum_{k=1}^{K} E_k(D - \omega L_k)^{-1}[(1-\omega)D + \omega V_k]. \tag{5}$$

The two relaxed multisplitting methods (2) and (4) are the methods we will deal with in the present paper. Of course, convergence of (2) and (4) is equivalent to $\rho(H) < 1$, where the matrix H is defined by (3) and (5), respectively.

We finish this section by giving an example of multisplittings arising from an (overlapping) block decomposition of the matrix A. To this purpose suppose that for $k = 1,\dots,K$ there are subsets $S_k \subseteq \{1,\dots,n\}$ with $\bigcup_{k=1}^{K} S_k = \{1,\dots,n\}$. (The sets S_k need not be pairwise disjoint.) In the case of (2) the entries of $M_k = (m_{ij}^k)$ are given by

$$m_{ij}^k = \begin{cases} a_{ij} & \text{if} \quad i = j \text{ or } i, j \in S_k, \\ 0 & \text{else,} \end{cases}$$

and M_k is assumed to be nonsingular. The ith diagonal entry of E_k is chosen to be zero if $i \notin S_k$ and positive if $i \in S_k$ such that (iii) of Definition (2.1) holds. In the case of (4) the matrices L_k are given by the negative strictly lower triangular part of M_k, where now D, i.e. the diagonal part of A, is assumed to be nonsingular. For these multisplittings one immediately sees that we actually need only those components $\bar{y}_i^{m,k}$ of $\bar{y}^{m,k}$ for which $i \in S_k$, and this, in turn, means that the calculations in (4b) need only be done for $i \in S_k$. Similarly, in the case of (2) the calculation of $M_k^{-1}(N_k x^m + b)$ actually reduces to a linear system of smaller size, since we do not need the ith component if $i \notin S_k$.

3. NOTATION AND AUXILIARY RESULTS

In this section we review some known results needed in Section 4. To formulate them we begin with some basic notation used throughout the remaining part of this paper.

Let C be an $n \times n$ matrix. By diag(C) we denote the $n \times n$ diagonal matrix coinciding in its diagonal with C.

We write $A \leqslant B$ if $a_{ij} \leqslant b_{ij}$ holds for all entries of $A = (a_{ij})$ and $B = (b_{ij}) \in \mathbb{R}^{m \times n}$, calling A nonnegative if $A \geqslant 0$. This definition carries immediately over to vectors by identifying them with $n \times 1$ matrices. In particular, we call the vector $x \in \mathbb{R}^n$ positive (writing $x > 0$) if all its entries are positive.

By $|A| = (|a_{ij}|)$ we define the absolute value of $A \in \mathbb{R}^{m \times n}$; it is a nonnegative $m \times n$ matrix satisfying $|AB| \leqslant |A||B|$ for $B \in \mathbb{R}^{n \times p}$. As in [6], we denote by $\langle A \rangle = (\alpha_{ij})$ the $n \times n$ comparison matrix of $A \in \mathbb{R}^{n \times n}$ where

$$\alpha_{ij} = \begin{cases} |a_{ii}| & \text{if} \quad i = j, \\ -|a_{ij}| & \text{if} \quad i \neq j. \end{cases}$$

We call $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ an M-matrix if it is nonsingular with $a_{ij} \leqslant 0$ for $i \neq j$ and with $A^{-1} \geqslant 0$. We call it an H-matrix if $\langle A \rangle$ is an M-matrix. (Cf. [2] or [13].)

Fan proved in [3] the following necessary and sufficient criterion on M-matrices.

Lemma 3.1 [3]. *Let $A \in \mathbb{R}^{n \times n}$ have nonpositive off-diagonal entries. Then A is an M-matrix if and only if there exists a positive vector $u \in \mathbb{R}^n$ such that $Au > 0$.*

Lemma 3.1 implies many well-known results on M-matrices. We list some of them in the following lemma.

Lemma 3.2 [10, 2.4.8 and 2.4.19]. *Let A, B be $n \times n$ M-matrices, $D =$ diag(A), and $C \in \mathbb{R}^{n \times n}$. Then:*

(a) *D is nonsingular; D, D^{-1} are nonnegative matrices with positive diagonal entries.*
(b) $A \leqslant B \Rightarrow B^{-1} \leqslant A^{-1}$.
(c) *$A \leqslant C \leqslant D \Rightarrow C$ is an M-matrix.*

We now cite another lemma which is proved in [10, 2.4.9] and which will frequently be used in the sequel.

Lemma 3.3. *Let $A, B \in \mathbb{R}^{n \times n}$ such that $|A| \leqslant B$. Then $\rho(A) \leqslant \rho(B)$.*

We end this section by a lemma on H-matrices.

Lemma 3.4. *Let $A \in \mathbb{R}^{n \times n}$ be an H-matrix, $D =$ diag(A), and $A = D - B$. Then:*

(a) *A is nonsingular.*
(b) $|A^{-1}| \leqslant \langle A \rangle^{-1}$.
(c) *$|D|$ is nonsingular and* $\rho(|D|^{-1}|B|) < 1$.

Proof. $|D|$ is nonsingular by Lemma 3.2(a). $\langle A\rangle = |D| - |B|$ is a regular splitting of the M-matrix $\langle A\rangle$; hence $\rho(|D|^{-1}|B|) < 1$ (cf. [13, Theorem 3.13]). This proves (c). By Lemma 3.3 we have $\rho(D^{-1}B) < 1$. To prove (a) and (b) we now simply use Neumann's series to see that $A^{-1} = (I - D^{-1}B)^{-1}D^{-1}$ exists and satisfies

$$|A^{-1}| = \left|\sum_{k=0}^{\infty} (D^{-1}B)^k D^{-1}\right| \leqslant \left[\sum_{k=0}^{\infty} (|D|^{-1}|B|)^k\right]|D|^{-1} = \langle A\rangle^{-1}. \qquad \blacksquare$$

4. CONVERGENCE RESULTS

Let A be an $n \times n$ matrix; let $D = \operatorname{diag}(A)$, $A = D - B$; and if $|D|$ is nonsingular, let $J = |D|^{-1}|B|$. In addition, let us write ρ instead of $\rho(J)$. If A is an H-matrix, we have $\rho < 1$ by Lemma 3.4(c). We first establish a convergence result for the relaxed multisplitting method (2).

THEOREM 4.1. *Let A be an H-matrix, and let (M_k, N_k, E_k), $k = 1,\dots,K$, be a multisplitting of A with* $\operatorname{diag}(M_k) = D$, $\langle A\rangle = \langle M_k\rangle - |N_k|$ *for* $k = 1,\dots,K$. *Then the relaxed multisplitting method* (2) *converges for any starting vector x^0, provided $\omega \in (0, 2/(1+\rho))$.*

Proof. We will show that $\rho(|H|) < 1$, where H is the matrix given by (3). The theorem is then proved, since $\rho(H) \leqslant \rho(|H|)$ by Lemma 3.3.

By Lemma 3.2(c) the matrices $\langle M_k\rangle$ are M-matrices. Using the relations $|M_k^{-1}| \leqslant \langle M_k\rangle^{-1}$ [cf. Lemma 3.4(b)], $|N_k| = \langle M_k\rangle - \langle A\rangle$, and $\langle A\rangle = |D|(I - J)$, we obtain

$$\begin{aligned}
|H| &\leqslant \omega \sum_{k=1}^{K} E_k|M_k^{-1}||N_k| + |1-\omega|I \leqslant \omega \sum_{k=1}^{K} E_k\langle M_k\rangle^{-1}|N_k| + |1-\omega|I \\
&= \omega \sum_{k=1}^{K} E_k\langle M_k\rangle^{-1}(\langle M_k\rangle - \langle A\rangle) + |1-\omega|I \\
&= (\omega + |1-\omega|)I - \omega \sum_{k=1}^{K} E_k\langle M_k\rangle^{-1}|D|(I-J).
\end{aligned} \tag{6}$$

Let e denote the vector $e=(1,1,\dots,1)^T \in \mathbb{R}^n$. Since J is nonnegative, the matrix $J+\varepsilon ee^T$ has only positive entries and is irreducible for any $\varepsilon>0$. By the theorem of Perron and Frobenius (see [13]), for any $\varepsilon>0$ there is a vector $x_\varepsilon>0$ such that $(J+\varepsilon ee^T)x_\varepsilon=\rho_\varepsilon x_\varepsilon$, where $\rho_\varepsilon=\rho(J+\varepsilon ee^T)$. Moreover, since $\omega\in(0,2/(1+\rho))$ we have $|1-\omega|+\omega\rho<1$. By continuity of the spectral radius we also get

$$|1-\omega|+\omega\rho_\varepsilon<1 \tag{7}$$

if $\varepsilon>0$ is sufficiently small. In this case we first obtain from (6)

$$|H|\leqslant(\omega+|1-\omega|)I-\omega\sum_{k=1}^{K}E_k\langle M_k\rangle^{-1}|D|\left[I-(J+\varepsilon ee^T)\right]$$

and then, multiplying by x_ε,

$$\begin{aligned}|H|x_\varepsilon &\leqslant(\omega+|1-\omega|)x_\varepsilon-\omega\sum_{k=1}^{K}E_k\langle M_k\rangle^{-1}|D|\left[I-(J+\varepsilon ee^T)\right]x_\varepsilon\\ &=(\omega+|1-\omega|)x_\varepsilon-\omega(1-\rho_\varepsilon)\sum_{k=1}^{K}E_k\langle M_k\rangle^{-1}|D|x_\varepsilon.\end{aligned} \tag{8}$$

Making use of the inequality $|D|^{-1}\leqslant\langle M_k\rangle^{-1}$ [cf. Lemma 3.2(b)], we deduce from (8) together with (7)

$$\begin{aligned}|H|x_\varepsilon &\leqslant(\omega+|1-\omega|)x_\varepsilon-\omega(1-\rho_\varepsilon)\sum_{k=1}^{K}E_k|D|^{-1}|D|x_\varepsilon\\ &=\left[\omega+|1-\omega|-\omega(1-\rho_\varepsilon)\right]x_\varepsilon\\ &=(|1-\omega|+\omega\rho_\varepsilon)x_\varepsilon<x_\varepsilon.\end{aligned}$$

Therefore, $(|H|x_\varepsilon)_i/(x_\varepsilon)_i<1$ for $i=1,\dots,n$, and Exercise 2 in [13, p. 47] guarantees $\rho(|H|)<1$. ■

The content of Theorem 4.1 will be discussed later in this section. First, let us turn to a theorem on the convergence of the relaxed multisplitting method (4).

THEOREM 4.2. *Let A be an H-matrix, and for $k = 1,\dots,K$ let L_k be a strictly lower triangular matrix. Define the matrices V_k such that the equalities $A = D - L_k - V_k$ hold. Assume that we have $\langle A\rangle = |D| - |L_k| - |V_k|$ for $k = 1,\dots,K$. Then the relaxed multisplitting method* (4) *converges for any starting vector x^0, provided $\omega \in (0, 2/(1+\rho))$.*

Proof. We will show that $\rho(|H|) < 1$, where H is the matrix given by (5). As before, the theorem then follows from the inequality $\rho(H) \leqslant \rho(|H|)$.

We first observe that the matrices $D - \omega L_k$ are H-matrices for $k = 1,\dots,K$. Indeed, $\langle D - \omega L_k\rangle$ is a nonsingular lower triangular matrix which has the sign pattern of an M-matrix. Moreover, all principal minors of $\langle D - \omega L_k\rangle$ are equal to those of $|D|$ and are therefore positive. Thus by [2, Theorem 6.2.3], $\langle D - \omega L_k\rangle$ is an M-matrix, and by Lemma 3.4(b) the inequality $|(D - \omega L_k)^{-1}| \leqslant \langle D - \omega L_k\rangle^{-1}$ holds. Hence

$$
\begin{aligned}
|H| &\leqslant \sum_{k=1}^{K} E_k\left|(D - \omega L_k)^{-1}\right|\left|(1-\omega)D + \omega V_k\right| \\
&\leqslant \sum_{k=1}^{K} E_k\langle D - \omega L_k\rangle^{-1}\left|(1-\omega)D + \omega V_k\right| \\
&= \sum_{k=1}^{K} E_k\langle D - \omega L_k\rangle^{-1}\left(|1-\omega|\,|D| + \omega|V_k|\right). \qquad (9)
\end{aligned}
$$

Making use of the equalities $|D|J = |L_k| + |V_k|$ and $\langle D - \omega L_k\rangle = |D| - \omega|L_k|$, we obtain from (9)

$$
\begin{aligned}
|H| &\leqslant \sum_{k=1}^{K} E_k\langle D - \omega L_k\rangle^{-1}\left[\langle D - \omega L_k\rangle + (|1-\omega| - 1)|D| + \omega|D|J\right] \\
&= I - \sum_{k=1}^{K} E_k\langle D - \omega L_k\rangle^{-1}|D|\left[(1 - |1-\omega|)I - \omega J\right]. \qquad (10)
\end{aligned}
$$

As in the proof of Theorem 4.1, let e denote the vector $e = (1,1,\dots,1)^T \in \mathbb{R}^n$ and let $x_\varepsilon > 0$ denote a vector satisfying $(J + \varepsilon ee^T)x_\varepsilon = \rho_\varepsilon x_\varepsilon$, where $\varepsilon > 0$ is sufficiently small that $|1-\omega| + \omega\rho_\varepsilon < 1$.

From (10) we first get

$$|H| \leqslant I - \sum_{k=1}^{K} E_k \langle D - \omega L_k \rangle^{-1} |D| \left[(1 - |1 - \omega|) I - \omega (J + \varepsilon e e^T) \right],$$

and then, multiplying by x_ε,

$$|H| x_\varepsilon \leqslant x_\varepsilon - \sum_{k=1}^{K} E_k \langle D - \omega L_k \rangle^{-1} |D| (1 - |1 - \omega| - \omega \rho_\varepsilon) x_\varepsilon.$$

By Lemma 3.2(b) we have $|D|^{-1} \leqslant \langle D - \omega L_k \rangle^{-1}$ and hence

$$|H| x_\varepsilon \leqslant x_\varepsilon - \sum_{k=1}^{K} E_k |D|^{-1} |D| (1 - |1 - \omega| - \omega \rho_\varepsilon) x_\varepsilon$$

$$= \left(1 - [1 - |1 - \omega| - \omega \rho_\varepsilon]\right) x_\varepsilon = (|1 - \omega| + \omega \rho_\varepsilon) x_\varepsilon < x_\varepsilon.$$

Thus $\rho(|H|) < 1$ follows again by [13, p. 47]. ∎

We first remark that $(0, 2/(1+\rho)) \supset (0, 1]$, since $\rho < 1$. Therefore, our theorems guarantee in particular the convergence of (2) and (4) for $\omega \in [1, 2/(1+\rho))$. Next, observe that the equality $\langle A \rangle = \langle M_k \rangle - |N_k|$ occurring in Theorem 4.1 essentially means that the (off-diagonal) entries of M_k are between the corresponding entries of A and zero. The condition $\langle A \rangle = |D| - |L_k| - |V_k|$ in Theorem 4.2 may be interpreted similarly. Thus Theorem 4.1 and Theorem 4.2 hold in the important case where the entries of the matrices M_k and $-L_k$ are either equal to the corresponding entries of A or zero. Hence our two theorems apply, for example, to the multisplittings based on an (overlapping) block decomposition as described at the end of Section 2. Note that if A is an H-matrix, the nonsingularity of the matrices M_k belonging to a multisplitting based on a block decomposition is no additional requirement, but follows directly by Lemma 3.2(c).

Let us also note that for $\omega = 1$, i.e., for the unrelaxed multisplitting methods, Theorem 4.1 and Theorem 4.2 give convergence criteria which have not been considered in the literature before.

The condition of A being an H-matrix covers several interesting cases. We consider some of them in the next corollary.

COROLLARY 4.3. *Suppose that the matrix A satisfies one of the following conditions*

(i) *A is an M-matrix.*
(ii) *A is strictly or irreducibly diagonally dominant* (*cf.* [10, 2.3.7]).
(iii) *$\langle A \rangle$ is symmetric and positive definite.*

Then A is an H-matrix and therefore Theorem 4.1 and Theorem 4.2 hold. In particular, the relaxed multisplitting methods based on an (overlapping) block decomposition as described at the end of Section 2 converge, provided $\omega \in (0, 2/(1+\rho))$.

Proof. For (i) it is obvious that A is an H-matrix. In the case of (ii) the matrix $\langle A \rangle$ is strictly or irreducibly diagonally dominant. This implies that $\langle A \rangle$ is an M-matrix (see [10, 2.4.14]) and thus A is an H-matrix. The assumptions in (iii) guarantee that $\langle A \rangle$ is a symmetric and positive definite matrix with the sign pattern of an M-matrix. Therefore, $\langle A \rangle$ is an M-matrix (cf. [2, Exercise 6.2.6]) and A is again an H-matrix. ∎

Let us now briefly consider decompositions of the form $A = D - L_k - U_k - W_k$, where L_k is strictly lower and U_k is strictly upper triangular for $k = 1, \dots, K$. A relaxed multisplitting method based on these decompositions is given by

$$x^{m+1} = \sum_{k=1}^{K} E_k y^{m,k}, \qquad m = 0, 1, \dots, \tag{11a}$$

where the iterates $y^{m,k}$ are calculated in two steps by solving the triangular systems

$$\begin{aligned} (D - \omega L_k) z^{m,k} &= \left[(1-\omega)D + \omega U_k\right] x^m + \omega W_k x^m + \omega b, \\ (D - \omega U_k) y^{m,k} &= \left[(1-\omega)D + \omega L_k\right] z^{m,k} + \omega W_k x^m + \omega b \end{aligned} \tag{11b}$$

for $k = 1, \dots, K$. This method may be regarded as a (relaxed) multisplitting variant of the symmetric Gauss-Seidel method which was considered in Neumann and Plemmons [7]. Note that $W_k x^m$ need only be computed once in (11b). However, no additional savings will in general be possible when performing the method (11). This is in contrast to the usual symmetric Gauss-Seidel method, which can be implemented in such a manner that apart from the first half step it requires exactly the same computational work as the

Gauss-Seidel method (see Niethammer [8]). The method (11) may be written in the form $x^{m+1} = Hx^m + c$ with

$$H = \sum_{k=1}^{K} E_k(D - \omega U_k)^{-1}\big\{\big[(1-\omega)D + \omega L_k\big](D - \omega L_k)^{-1} \times \big[(1-\omega)D + \omega U_k + \omega W_k\big] + \omega W_k\big\}.$$

Now we just mention that Theorem 4.2 carries over to the method (11), i.e., one may show in an analogous manner to the proof of Theorem 4.2 that $\rho(H) \leqslant \rho(|H|) < 1$ for $\omega \in (0, 2/(1+\rho))$, provided that A is an H-matrix and that $\langle A \rangle = |D| - |L_k| - |U_k| - |W_k|$ holds for $k = 1, \dots, K$.

We finish this section by remarking that the usual relaxed Jacobi method, the relaxed Gauss-Seidel method, and the relaxed symmetric Gauss-Seidel method may be regarded as special cases of the multisplittings (2), (4), and (11), respectively, with $K = 1$. If A is an H-matrix and if $\omega \in (0, 2/(1+\rho))$, it is known that the relaxed Jacobi and the relaxed Gauss-Seidel method are convergent (see Varga [14]). This is also true for the symmetric Gauss-Seidel method (see Alefeld and Varga [1]). Our theorems generalize these results to relaxed multisplitting methods.

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REFERENCES

1 G. Alefeld and R. S. Varga, Zur Konvergenz des symmetrischen Relaxationsverfahrens, *Numer. Math.* 25:291–295 (1976).
2 A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic, New York, 1979.
3 Ky Fan, Topological proofs for certain theorems on matrices with non-negative elements, *Monatsh. Math.* 62:219–237 (1958).
4 A. Frommer, Parallel nonlinear multisplitting methods, to appear in Numer. Math.
5 A. Frommer and G. Mayer, Parallel interval multisplittings, submitted for publication.
6 A. Neumaier, New techniques for the analysis of linear interval equations, *Linear Algebra Appl.* 58:273–325 (1984).
7 M. Neumann and R. J. Plemmons, Convergence of parallel multisplitting iterative methods for M-matrices, *Linear Algebra Appl.* 88/89:559–573 (1987).
8 W. Niethammer, Relaxation bei komplexen Matrizen, *Math. Z.*, 86:34–40 (1964).

9 D. P. O'Leary and R. E. White, Multi-splittings of matrices and parallel solution of linear systems, *SIAM J. Algebraic Discrete Methods* 6:630–640 (1985).
10 J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic, New York, 1970.
11 R. E. White, A nonlinear parallel algorithm with applications to the Stefan problem, *SIAM J. Numer. Anal.* 23:639–652 (1986).
12 R. E. White, Parallel algorithms for nonlinear problems, *SIAM J. Algebraic Discrete Methods* 7:137–149 (1986).
13 R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
14 R. S. Varga, On recurring theorems on diagonal dominance, *Linear Algebra Appl.* 13:1–9 (1976).

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