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NEW RESULTS ON NARROWING THE DUALITY GAP OF THE EXTENDED CELIS–DENNIS–TAPIA PROBLEM*

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Abstract. In this paper, we consider the extended Celis–Dennis–Tapia (CDT) problem that has a positive duality gap. It is presented in theory that this positive duality gap can be narrowed by adding an appropriate second-order-cone (SOC) constraint, which may lead to dividing the problem into two separate subproblems. More concretely, for any extended CDT problem with a positive duality gap, we prove that one SOC constraint is valid to narrow the positive duality gap if and only if the corresponding hyperplane intersects the "open optimal line segment." Especially when the second constraint function consists of the product of two linear functions, we prove that the positive duality gap can be eliminated thoroughly by solving two subproblems with SOC constraints. For any classical CDT problem with a positive duality gap, a new model with two SOC constraints is proposed, and a sufficient condition is presented under which this positive duality gap can be eliminated thoroughly. In particular, based on the sufficient condition, it is proved that the positive duality gaps of any two-dimensional classical CDT problem and a class of three-dimensional classical CDT problems can be eliminated thoroughly. Numerical results of some gap-existing examples coming from other papers show that their positive duality gaps are indeed eliminated by our SOC reformulation technique.

Key words. quadratically constrained quadratic programming, CDT problem, second-order cone, global solutions, SDP relaxation

AMS subject classifications. 90C20, 90C22, 90C25, 90C26, 90C30

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1. Introduction.

In this paper we consider the extended Celis–Dennis–Tapia (CDT) [8] problem as follows:

$$\begin{array}{lll} (QP) & \text{minimize} & q_0(d) = d^T Q_0 d + 2b_0^T d \\ & \text{subject to} & q_1(d) = \|d\|^2 - 1 \le 0, \\ & & q_2(d) = d^T Q_1 d + 2b_1^T d + c_1 \le 0, \end{array} \tag{1.1}$$

where $Q_0, Q_1 \in \mathcal{S}^{n\times n}$, $b_0, b_1, d \in \mathcal{R}^n$, and $c_1 \in \mathcal{R}$. The classical CDT problem (in which $Q_1 \succeq 0$) was first proposed by Celis, Dennis, and Tapia [8] to solve a nonlinear constrained optimization problem by using the trust region method in 1985, and (QP) played the role as a model for validating a trust region step. From then on, a number of papers have involved studying the structure and the solution algorithms of the CDT problem [1, 4, 5, 6, 9, 10, 11, 12, 14, 13, 15, 16, 17, 19, 20, 22]. A brief introduction of the CDT problem was given by Yuan [21] in 2015.

A remarkable property which makes the CDT problem interesting and intriguing is that at a global optimal solution, the Hessian matrix of the Lagrangian function may not necessarily be positive semidefinite; however, it may have at most one negative eigenvalue (see Yuan [19]). Yuan [20] also suggested an algorithm for the classical CDT problem with a convex objective function. Shortly afterward, Zhang [22]

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proposed an algorithm for the classical CDT problem with a positive semidefinite optimal Lagrangian Hessian matrix. After almost 10 years, Chen and Yuan [10] in 2001 presented a sufficient condition (termed Property $\mathcal{J}$ in [10]) under which the classical CDT problem would have the strong duality. In 2006, Beck and Eldar [3] used the complex valued approach to come up with a similar sufficient condition to Chen and Yuan's for the extended CDT problem. Moreover, Ai and Zhang [1] in 2009 presented a necessary and sufficient condition (termed Property $\mathcal{I}'$ in [1]) to characterize when the extended CDT problem admitted the strong duality. This condition was also stated as the semidefinite programming (SDP) relaxation version (termed Property $\mathcal{I}$ in [1]).

Some new results have appeared in recent years. In 2014, Bomze and Overton [5] gave a new sufficient condition of the global optimality for the classical CDT problem. They presented numerically that there exist examples that satisfy this sufficient condition but violate the strong duality. Recently, Bienstock [4] proved that the extended CDT problem can be solved in polynomial time. It first confirms that the extended CDT problem is a polynomial-solvable problem, although the presented algorithm seems not to be practicable. Meanwhile, Sakaue et al. [14] presented another algorithm for the extended CDT problem. Recently, Yang and Burer [16] also proposed an interesting method to try to solve the classical CDT problem.

On the other hand, a second-order-cone (SOC) reformulation technique has been studied by many researchers [6, 7, 15, 18]. To our knowledge, Sturm and Zhang [15] first used an SOC to reformulate an optimization problem that minimizes a quadratic function over a ball and a linear inequality. They proved that the SDP relaxation of this reformulation is an exact relaxation, that is, the reformulation is an implicit convex optimization problem. Recently, Burer, Anstreicher, and Yang [6, 7] added some valid SOC constraints to a quadratic minimization problem with a ball constraint and several linear inequality constraints to tighten its SDP relaxation. They proved that the tightness holds if only the linear inequality constraints are nonintersecting. For the "intersecting" case with two linear inequality constraints, Yuan et al. [18] presented a sufficient and necessary condition of the tightness. For the classical CDT problem, Burer and Anstreicher in [6] presented some numerical results, which showed that the duality gap of the CDT problem may be narrowed by using a lot of SOC constraints.

In this paper, we focus on those extended CDT problems that have positive duality gaps, that is, we consider only the problems whose Lagrangian Hessian matrices at the global optimal solutions have negative eigenvalues. We shall discuss how to theoretically judge whether an SOC constraint is valid to narrow the duality gap. First, we shall prove that the SDP relaxation of any extended CDT problem with a positive duality gap has one unique optimal solution. Second, based on the above "uniqueness," we shall present an easily verifiable sufficient and necessary condition to characterize when an SOC constraint is valid to narrow the duality gap, in which the primal problem may be divided into two separate subproblems. Following this sufficient and necessary condition, several exact relaxation cases using the SOC reformulation are found.

This paper is organized as follows. In section 2, some preliminary knowledge is discussed. In section 3 we focus on narrowing the duality gap of the extended CDT problem by providing a valid SOC constraint. Section 4 concerns a special extended CDT problem, whose duality gap can be eliminated by solving two subproblems with SOC constraints. In section 5 the classical CDT problem is considered. We give a sufficient condition under which one can get rid of the duality gap by using the SOC reformulation. Especially, any two-dimensional classical CDT problem with a positive duality gap can always satisfy the condition. Also, some numerical examples are listed. A special type of three-dimensional classical CDT problem is also verified to satisfy the condition.

Throughout this paper, $\mathcal{S}^{n\times n}$ and $\mathcal{S}_+^{n\times n}$ denote the set of all real $n\times n$ symmetric matrices and the set of all real $n \times n$ positive semidefinite matrices, respectively. For $A, B \in \mathcal{S}^{n\times n}$, the notation $A \bullet B := \operatorname{tr} AB$ denotes the matrix inner-product between A and B. The notation $\partial(*)$, $\operatorname{int}(*)$, and $card(*)$ denote the boundary, the interior, and the cardinal number of a set "$*$," respectively. $v(*)$ denotes the optimal objective value of a problem $(*)$. Sometimes we may use $(x)_1$ to denote the first component of a vector x.

2. Some preliminary knowledge.

In this section, we review and discuss some important properties that involve the duality gap of the problem (QP). These properties will be fundamentals of further discussions.

The Lagrangian function of (QP) is

$$L(y_1, y_2, d) := q_0(d) + y_1 q_1(d) + y_2 q_2(d). \tag{2.1}$$

And its Lagrangian dual problem is

$$(QD) \quad \max_{y_1\geq 0, y_2 \geq 0} \min_{d\in\mathcal{R}^n} L(y_1, y_2, d). \tag{2.2}$$

The following definition and theorem come from [1].

Definition 2.1 (Definition 5.1 of [1]). *For given Lagrangian multipliers $\hat{y}_1$ and $\hat{y}_2$ for the quadratic program (QP), we say that they have Property $\mathcal{I}'$ if*

(1) $\hat{y}_1\hat{y}_2 > 0$*;*
(2) $H(\hat{y}_1, \hat{y}_2) := Q_0 + \hat{y}_1 I + \hat{y}_2 Q_1 \succeq 0$ *and* $\operatorname{rank}(H(\hat{y}_1, \hat{y}_2)) = n - 1$*;*
(3) *the system of linear equations $H(\hat{y}_1, \hat{y}_2)d + b_0 + \hat{y}_2 b_1 = 0$ has two solutions $\hat{d}_1$ and $\hat{d}_2$ satisfying $q_1(\hat{d}_1) = q_1(\hat{d}_2) = 0$, and $q_2(\hat{d}_1) < 0, q_2(\hat{d}_2) > 0$.*

Theorem 2.2 (Theorem 5.2 of [1]). *Suppose that (QP) satisfies the Slater condition. Then, (QP) has a positive duality gap if and only if there exist multipliers $\hat{y}_1$ and $\hat{y}_2$ such that Property $\mathcal{I}'$ holds.*

It can be easily verified that the pair $(\hat{y}_1, \hat{y}_2)$ satisfying Property $\mathcal{I}'$ is just the optimal solution to the dual problem (QD). Furthermore, the solutions to the linear system $H(\hat{y}_1, \hat{y}_2)d + b_0 + \hat{y}_2 b_1 = 0$, which consist of all points on the straight line $\mathcal{L}_\lambda$ connecting $\hat{d}_1$ and $\hat{d}_2$,

$$\mathcal{L}_\lambda := \{(1-\lambda)\hat{d}_1 + \lambda\hat{d}_2 \,|\, \lambda \in \mathcal{R}\}, \tag{2.3}$$

are exactly the optimal solutions to the problem $\min_{d\in\mathcal{R}^n} L(\hat{y}_1, \hat{y}_2, d)$. So the straight line $\mathcal{L}_\lambda$ is called *the optimal line of the dual problem.* Let l_λ denote the *"open optimal line segment"* connecting $\hat{d}_1$ and $\hat{d}_2$,

$$l_\lambda := \{(1-\lambda)\hat{d}_1 + \lambda\hat{d}_2 \,|\, 0 < \lambda < 1\}. \tag{2.4}$$

This "open optimal line segment" will play an important role in later discussions.

Since we need to utilize the SDP-relaxation technique, the SDP-relaxation versions of Property $\mathcal{I}'$ and Theorem 2.2 are also stated as follows. Consider the SDP relaxation model of (QP),

$$
\begin{array}{llll}
(SP) & \text{minimize} & M_0 \bullet X & \\
 & \text{subject to} & M_1 \bullet X \le 0, & \\
 & & M_2 \bullet X \le 0, & (2.5)\\
 & & E_{00} \bullet X = 1, & \\
 & & X \succeq 0, &
\end{array}
$$

and its dual model

$$
\begin{array}{llll}
(SD) & \text{maximize} & y_0 & \\
 & \text{subject to} & M_0 - y_0 E_{00} + y_1 M_1 + y_2 M_2 \succeq 0, & (2.6)\\
 & & y_1 \ge 0,\ y_2 \ge 0, &
\end{array}
$$

where

$$
M_0 := \begin{bmatrix} 0 & b_0^T \\ b_0 & Q_0 \end{bmatrix},\ M_1 := \begin{bmatrix} -1 & 0^T \\ 0 & I \end{bmatrix},\ M_2 := \begin{bmatrix} c_1 & b_1^T \\ b_1 & Q_1 \end{bmatrix},\ E_{00} := \begin{bmatrix} 1 & 0^T \\ 0 & O \end{bmatrix}. \tag{2.7}
$$

The following proposition is straightforward.

PROPOSITION 2.3. *If (QP) satisfies the Slater condition, then both (SP) and (SD) have interior feasible points; furthermore, they both have optimal solutions and the strong duality holds.*

Due to the fact that (QD) is identical to (SD), Theorem 2.2 can be restated as follows.

THEOREM 2.4 (Theorem 4.2 of [1]). *Let (QP) satisfy the Slater condition, and let $\hat{X}$ and $(\hat{y}_0, \hat{y}_1, \hat{y}_2)$ be any pair of optimal solutions to (SP) and (SD). Then $v(QP) > v(SP)$ holds if and only if the optimal pair satisfies Property $\mathcal{I}$ as follows:*

(1) $\hat{y}_1 \hat{y}_2 > 0$;

(2) $\hat{Z} := M_0 - \hat{y}_0 E_{00} + \hat{y}_1 M_1 + \hat{y}_2 M_2 \succeq 0$ *and* $\operatorname{rank}(\hat{Z}) = n - 1$;

(3) $\operatorname{rank}(\hat{X}) = 2$, *and there is a rank-one decomposition of $\hat{X}$, $\hat{X} = \hat{x}_1 \hat{x}_1^T + \hat{x}_2 \hat{x}_2^T$, such that $M_1 \bullet \hat{x}_i \hat{x}_i^T = 0$ $(i = 1, 2)$ and $M_2 \bullet \hat{x}_1 \hat{x}_1^T < 0$, $M_2 \bullet \hat{x}_2 \hat{x}_2^T > 0$.*

Note 2.1. For the above rank-one decomposition, $\hat{X} = \hat{x}_1 \hat{x}_1^T + \hat{x}_2 \hat{x}_2^T$, we always assume that $(\hat{x}_1)_1 \ge 0$ and $(\hat{x}_2)_1 \ge 0$ (otherwise, replace $\hat{x}_i$ with $-\hat{x}_i$, $i = 1, 2$) in this paper. In this sense, the equalities $M_1 \bullet \hat{x}_i \hat{x}_i^T = 0$ $(i = 1, 2)$ are equivalent to $\hat{x}_1, \hat{x}_2 \in \partial(SOC)$.

It is straightforward that $\hat{y}_0 = v(SD) = v(SP)$, $(\hat{y}_1, \hat{y}_2)$ is the same as that of Property $\mathcal{I}'$, and the vectors $\hat{x}_1$ and $\hat{x}_2$ satisfy the following relations:

$$
\hat{x}_1 = t_1 \begin{bmatrix} 1 \\ \hat{d}_1 \end{bmatrix},\ \hat{x}_2 = t_2 \begin{bmatrix} 1 \\ \hat{d}_2 \end{bmatrix},\ t_1 = \sqrt{\frac{q_2(\hat{d}_2)}{q_2(\hat{d}_2) - q_2(\hat{d}_1)}},\ \text{and } t_2 = \sqrt{\frac{-q_2(\hat{d}_1)}{q_2(\hat{d}_2) - q_2(\hat{d}_1)}}, \tag{2.8}
$$

where $\hat{d}_1$ and $\hat{d}_2$ are defined in Property $\mathcal{I}'$.

An interesting question is whether the optimal pairs to the (SP) and (SD) satisfying Property $\mathcal{I}$ are unique. To answer this question, let us first state a property of the $(n+1)$-dimensional SOC, which is straightforward.

PROPOSITION 2.5. *If x and y are two independent boundary points of SOC, then for any constants k_1 and k_2, the vector $k_1x + k_2y \in SOC \iff k_1 \geq 0,\ k_2 \geq 0$. Moreover, $k_1x + k_2y \in \partial(SOC) \iff k_1k_2 = 0$.*

Now we are ready to answer the above question.

THEOREM 2.6. *Suppose that (QP) satisfies the Slater condition and has a positive duality gap. Then each of both (SP) and (SD) has one and only one optimal solution.*

Proof. Proposition 2.3 and the assumption tell us that (SP) and (SD) both have optimal solutions and $v(SP) = v(SD)$. Let $\hat{X}$ and $(\hat{y}_0, \hat{y}_1, \hat{y}_2)$ be a pair of optimal solutions to (SP) and (SD). Set

$$\hat{Z} := M_0 - \hat{y}_0 E_{00} + \hat{y}_1 M_1 + \hat{y}_2 M_2.$$

By the assumption and Theorem 2.4, the pair $\hat{X}$ and $(\hat{y}_0, \hat{y}_1, \hat{y}_2)$ must satisfy Property $\mathcal{I}$. Let

$$\hat{X} = \hat{x}_1\hat{x}_1^T + \hat{x}_2\hat{x}_2^T$$

be a rank-one decomposition that satisfies

$$\hat{x}_1, \hat{x}_2 \in \partial(SOC),\ \hat{x}_1^T M_2 \hat{x}_1 < 0,\ \text{and } \hat{x}_2^T M_2 \hat{x}_2 > 0, \tag{2.9}$$

where $\hat{x}_1, \hat{x}_2 \in \partial(SOC)$ is from Note 2.1.

As $\hat{Z} \bullet \hat{X} = 0$, $\operatorname{rank}(\hat{Z}) = n - 1$, and $\operatorname{rank}(\hat{X}) = 2$, there must be

$$L(\hat{x}_1, \hat{x}_2) = \text{Null}(\hat{Z}), \tag{2.10}$$

where $L(\hat{x}_1, \hat{x}_2)$ is a space spanned by both vectors $\hat{x}_1$ and $\hat{x}_2$, and $\text{Null}(\hat{Z})$ denotes the null space of the matrix $\hat{Z}$.

First, we manage to show that (SD) has the unique optimal solution. Suppose that (SD) has another optimal solution $(\bar{y}_0, \bar{y}_1, \bar{y}_2)$. Obviously $\bar{y}_0 = v(SD) = \hat{y}_0$. Let

$$\bar{Z} := M_0 - \bar{y}_0 E_{00} + \bar{y}_1 M_1 + \bar{y}_2 M_2.$$

As $\hat{X}$ and $(\bar{y}_0, \bar{y}_1, \bar{y}_2)$ are a pair of optimal solutions to (SP) and (SD), by Theorem 2.4, the pair has Property $\mathcal{I}$ also. Similarly to (2.10), it holds that

$$L(\hat{x}_1, \hat{x}_2) = \text{Null}(\bar{Z}). \tag{2.11}$$

Then (2.10) and (2.11), together with $\bar{y}_0 = \hat{y}_0$ and $\hat{x}_1^T M_1 \hat{x}_1 = 0$, yield that

$$0 = \hat{x}_1^T(\bar{Z} - \hat{Z})\hat{x}_1 = (\bar{y}_2 - \hat{y}_2)\hat{x}_1^T M_2 \hat{x}_1 \Longrightarrow \bar{y}_2 = \hat{y}_2.$$

Furthermore, due to

$$\hat{y}_0 = \bar{y}_0,\ \hat{y}_2 = \bar{y}_2,\ \text{and } \hat{x}_1 + \hat{x}_2 \in \text{int}(SOC),\ \text{i.e.,}\ \left(\hat{x}_1 + \hat{x}_2\right)^T M_1 \left(\hat{x}_1 + \hat{x}_2\right) < 0,$$

one obtains

$$0 = \left(\hat{x}_1 + \hat{x}_2\right)^T \left(\bar{Z} - \hat{Z}\right)\left(\hat{x}_1 + \hat{x}_2\right) = \left(\bar{y}_1 - \hat{y}_1\right)\left(\hat{x}_1 + \hat{x}_2\right)^T M_1 \left(\hat{x}_1 + \hat{x}_2\right) \Longrightarrow \bar{y}_1 = \hat{y}_1.$$

Thus it has been proved that $(\bar{y}_0, \bar{y}_1, \bar{y}_2) = (\hat{y}_0, \hat{y}_1, \hat{y}_2)$.

Second, we try to verify the uniqueness of the optimal solutions to (SP). Suppose that (SP) has another optimal solution $\bar{X}$. Then the pair of optimal solutions $\bar{X}$

and $(\hat{y}_0, \hat{y}_1, \hat{y}_2)$ satisfies Property $\mathcal{I}$. Similarly to (2.9) and (2.10), $\bar{X}$ has a rank-one decomposition $\bar{X} = \bar{x}_1\bar{x}_1^T + \bar{x}_2\bar{x}_2^T$ that satisfies

$$\bar{x}_1, \bar{x}_2 \in \partial(SOC),\ \bar{x}_1^T M_2 \bar{x}_1 < 0,\ \bar{x}_2^T M_2 \bar{x}_2 > 0,\ \text{and } L(\bar{x}_1, \bar{x}_2) = \mathrm{Null}(\hat{Z}). \tag{2.12}$$

Therefore $L(\bar{x}_1, \bar{x}_2) = L(\hat{x}_1, \hat{x}_2)$, which implies that the vectors $\bar{x}_1$ and $\bar{x}_2$ can be linearly expressed by the vectors $\hat{x}_1, \hat{x}_2$, that is, there exist four constants k_1, k_2, k_3, and k_4 such that

$$\bar{x}_1 = k_1\hat{x}_1 + k_2\hat{x}_2 \text{ and } \bar{x}_2 = k_3\hat{x}_1 + k_4\hat{x}_2.$$

From Proposition 2.5, one has $k_1k_2 = k_3k_4 = 0$, $k_i \geq 0\,(i = 1, 2, 3, 4)$. Furthermore, as $\bar{x}_1^T M_2\bar{x}_1 < 0, \hat{x}_1^T M_2\hat{x}_1 < 0,\ \bar{x}_2^T M_2\bar{x}_2 > 0, \hat{x}_2^T M_2\hat{x}_2 > 0$, there are $k_2 = k_3 = 0$ and $k_1 > 0, k_4 > 0$. To complete the proof, one needs only to show that $k_1 = k_4 = 1$. In fact, one can get the following relations:

$$\begin{cases} 0 = M_2 \bullet \hat{X} = M_2 \bullet \hat{x}_1\hat{x}_1^T + M_2 \bullet \hat{x}_2\hat{x}_2^T \Longrightarrow M_2 \bullet \hat{x}_2\hat{x}_2^T = -M_2 \bullet \hat{x}_1\hat{x}_1^T, \\ 0 = M_2 \bullet \bar{X} = k_1^2 M_2 \bullet \hat{x}_1\hat{x}_1^T + k_4^2 M_2 \bullet \hat{x}_2\hat{x}_2^T = (k_1^2 - k_4^2) M_2 \bullet \hat{x}_1\hat{x}_1^T \Longrightarrow k_1 = k_4, \\ 1 = \hat{X}_{1,1} = (\hat{x}_1)_1^2 + (\hat{x}_2)_1^2, \\ 1 = \bar{X}_{1,1} = (\bar{x}_1)_1^2 + (\bar{x}_2)_1^2 = k_1^2(\hat{x}_1)_1^2 + k_4^2(\hat{x}_2)_1^2 = k_1^2\left((\hat{x}_1)_1^2 + (\hat{x}_2)_1^2\right) = k_1^2 \\ \Longrightarrow k_1 = k_4 = 1, \end{cases}$$

where $\hat{X}_{1,1}$ and $\bar{X}_{1,1}$ denote the entries at the first row and the first column of the matrices $\hat{X}$ and $\bar{X}$, respectively. ∎

The following result is essentially identical to Corollary 4 of [15] because if $G \bullet X < 0$, one needs only to replace G with $-G$. It announces a rank-one decomposition property of a semidefinite positive matrix, and we shall use it for later discussions.

Lemma 2.7 (Corollary 4 of [15]). *Let $G \in \mathcal{S}^{n\times n}$, $X \in \mathcal{S}_+^{n\times n}$, and* $\mathrm{rank}\,(X) = r$. *Then there exists a rank-one decomposition for X such that*

$$X = x_1x_1^T + x_2x_2^T + \cdots + x_rx_r^T$$

and

$$G \bullet x_ix_i^T = \frac{G \bullet X}{r} \quad \text{for } i = 1, 2, \ldots, r.$$

3. Narrowing the duality gap of the extended CDT problem.

Let

$$\Omega := \{d \in \mathcal{R}^n | q_1(d) \leq 0, q_2(d) \leq 0\} \tag{3.1}$$

denote the feasible region of the problem (QP). In this section, we focus on the case when (QP) admits a positive duality gap, and we try to clarify whether a given SOC constraint is valid to narrow the duality gap of (QP).

For any given $(n+1)$-dimensional vector $a = [c_2,\ b_2^T]^T$, where $0 \neq b_2 \in \mathcal{R}^n$, consider a half space $D(a)$,

$$D(a) := \left\{d \in \mathcal{R}^n \mid b_2^T d + c_2 \geq 0\right\}, \tag{3.2}$$

and a corresponding new optimization problem $(QP(a))$,

$$(QP(a)) \quad \min\ q_0(d) \quad \text{s.t. } d \in \Omega(a) := \Omega \cap D(a). \tag{3.3}$$

This problem can be equivalently reformulated into the following form:

$$(3.4)\qquad \begin{array}{lll} (QP(a)) & \min\limits_{d\in\mathcal{R}^n} & q_0(d) = d^TQ_0d + 2b_0^Td \\ & \text{subject to} & q_1(d) = \|d\|^2 - 1 \le 0, \\ & & q_2(d) = d^TQ_1d + 2b_1^Td + c_1 \le 0, \\ & & \|(b_2^Td + c_2)d\| \le b_2^Td + c_2. \end{array}$$

The SDP relaxation model of $(QP(a))$ is

$$(3.5)\qquad \begin{array}{lll} (SOCP(a)) & \text{minimize} & M_0 \bullet X \\ & \text{subject to} & M_1 \bullet X \le 0, \\ & & M_2 \bullet X \le 0, \\ & & Xa \in SOC, \\ & & E_{00} \bullet X = 1, \\ & & X \succeq 0, \end{array}$$

and its dual model is

$$\begin{array}{lll} (SOCD(a)) & \text{maximize} & y_0 \\ & \text{subject to} & M_0 - y_0E_{00} + y_1M_1 + y_2M_2 - \dfrac{1}{2}(ua^T + au^T) \succeq 0, \\ & & y_1 \ge 0,\ y_2 \ge 0,\ u \in SOC, \end{array}$$

where M_0, M_1, M_2, E_{00} are defined the same as those in (2.7).

Similarly to Proposition 2.3, we have the following proposition for $(SOCP(a))$ and $(SOCD(a))$, which is straightforward.

PROPOSITION 3.1. *If $(QP(a))$ satisfies the Slater condition, then both $(SOCP(a))$ and $(SOCD(a))$ have interior feasible points; furthermore, they both have optimal solutions and the strong duality holds.*

LEMMA 3.2. *Suppose that (QP) satisfies the Slater condition and has a positive duality gap. If $\hat{d}_1, \hat{d}_2 \in D(a)$, then $(QP(a))$ has interior feasible points, where $\hat{d}_1$ and $\hat{d}_2$ are defined in Property $\mathcal{I}'$.*

Proof. The assumption guarantees that Property $\mathcal{I}'$ holds for (QP), which means the vectors $\hat{d}_1$ and $\hat{d}_2$ are well defined. One can easily verify that the vector

$$(1 - \varepsilon_1)\hat{d}_1 + \varepsilon_1\hat{d}_2 + \varepsilon_2b_2$$

is just an interior feasible solution to $(QP(a))$, where ε_1 and ε_2 are two sufficiently small positive numbers. ∎

LEMMA 3.3. *Suppose that $(QP(a))$ satisfies the Slater condition. Let $(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \tilde{u})$ be an optimal solution to $(SOCD(a))$. If $v(SOCP(a)) > v(SP)$, then $\tilde{u} \ne 0$.*

Proof. From Propositions 2.3 and 3.1, (SP), (SD), $(SOCP(a))$, and $(SOCD(a))$ have optimal solutions and satisfy $v(SP) = v(SD)$ and $v(SOCP(a)) = v(SOCD(a))$. Suppose that $\tilde{u} = 0$. Then $(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2)$ is also an optimal solution to (SD). Therefore one obtains $v(SOCP(a)) = v(SOCD(a)) = v(SD) = v(SP)$, which is a contradiction. The proof is complete. ∎

LEMMA 3.4. *Suppose that $(QP(a))$ satisfies the Slater condition and (QP) has a positive duality gap. Then $v(SOCP(a)) = v(SP)$ holds if and only if $\hat{d}_1, \hat{d}_2 \in D(a)$, where $\hat{d}_1, \hat{d}_2$ are defined in Property $\mathcal{I}'$.*

Proof. From Theorem 2.6 and (2.8), (SP) has one unique optimal solution $\hat{X} = \hat{x}_1\hat{x}_1^T + \hat{x}_2\hat{x}_2^T$ such that

$$\hat{x}_1 = t_1 \begin{bmatrix} 1 \\ \hat{d}_1 \end{bmatrix},\ \hat{x}_2 = t_2 \begin{bmatrix} 1 \\ \hat{d}_2 \end{bmatrix},\ t_1 = \sqrt{\frac{q_2(\hat{d}_2)}{q_2(\hat{d}_2) - q_2(\hat{d}_1)}},\ \text{and } t_2 = \sqrt{\frac{-q_2(\hat{d}_1)}{q_2(\hat{d}_2) - q_2(\hat{d}_1)}}. \tag{3.6}$$

$\Longleftarrow$. Since $\hat{d}_1 \in D(a)$ and $\hat{d}_2 \in D(a)$, one has

$$\hat{x}_1^T a = t_1(b_2^T \hat{d}_1 + c_2) \geq 0 \text{ and } \hat{x}_2^T a = t_2(b_2^T \hat{d}_2 + c_2) \geq 0.$$

By Proposition 2.5, there is

$$\hat{X}a = \hat{x}_1\hat{x}_1^T a + \hat{x}_2\hat{x}_2^T a \in SOC.$$

It means the matrix $\hat{X}$ is a feasible solution to $(SOCP(a))$. Notice that $v(SP) \leq v(SOCP(a))$. Thus one has $v(SOCP(a)) = v(SP) = M_0 \bullet \hat{X}$.

$\Longrightarrow$. From Proposition 3.1, $(SOCP(a))$ has an optimal solution, say, $\tilde{X}$. As $v(SOCP(a)) = v(SP)$, $\tilde{X}$ is also an optimal solution to (SP). By Theorem 2.6, there is $\tilde{X} = \hat{X} = \hat{x}_1\hat{x}_1^T + \hat{x}_2\hat{x}_2^T$. Then one has

$$\tilde{X}a = \hat{X}a = \hat{x}_1\hat{x}_1^T a + \hat{x}_2\hat{x}_2^T a = (\hat{x}_1^T a)\hat{x}_1 + (\hat{x}_2^T a)\hat{x}_2 \in SOC. \tag{3.7}$$

By Proposition 2.5, (3.7) results in $\hat{x}_1^T a = t_1(b_2^T \hat{d}_1 + c_2) \geq 0$ and $\hat{x}_2^T a = t_2(b_2^T \hat{d}_2 + c_2) \geq 0$, which imply $b_2^T \hat{d}_1 + c_2 \geq 0$ and $b_2^T \hat{d}_2 + c_2 \geq 0$. Thus $\hat{d}_1 \in D(a)$ and $\hat{d}_2 \in D(a)$. ◻

As $\Omega = \Omega(a) \cup \Omega(-a)$, one gets

$$v(QP) = \min\{v(QP(a)), v(QP(-a))\}.$$

Here, if one of $\Omega(a)$ and $\Omega(-a)$ is an empty set, say, $\Omega(a) = \emptyset$, we define $v(QP(a)) = +\infty$. Note that if (QP) has interior feasible points, at least one of $\Omega(a)$ and $\Omega(-a)$ has interior feasible points. Then, under the (QP)'s Slater condition, one can define

$$\begin{aligned} &\min\{v(SOCP(a)), v(SOCP(-a))\} \\ &:= \begin{cases} \min\{v(QP(a)), v(SOCP(-a))\} & \text{if } \mathrm{int}(\Omega(a)) = \emptyset, \mathrm{int}(\Omega(-a)) \neq \emptyset; \\ \min\{v(SOCP(a)), v(QP(-a))\} & \text{if } \mathrm{int}(\Omega(-a)) = \emptyset, \mathrm{int}(\Omega(a)) \neq \emptyset; \\ \min\{v(SOCP(a)), v(SOCP(-a))\} & \text{else, i.e., } \mathrm{int}(\Omega(a)) \neq \emptyset, \mathrm{int}(\Omega(-a)) \neq \emptyset. \end{cases} \end{aligned} \tag{3.8}$$

Note 3.1.

(1) Even if $\mathrm{int}(\Omega(a)) = \emptyset$ there may be $\Omega(a) \nsubseteq \Omega(-a)$ because $\Omega(a)$ may include some "separated points" that are far beyond $\Omega(-a)$.

(2) In the above definition, if $\mathrm{int}(\Omega(a)) = \emptyset$, then $v(SOCP(a))$ is replaced with $v(QP(a))$ because for this case the problem $(QP(a))$ is easy to solve, while its relaxation $(SOCP(a))$ may become complicated.

THEOREM 3.5. *Suppose that (QP) satisfies the Slater condition and has a positive duality gap. Then, for any given $(n+1)$-dimensional vector $a = [c_2, b_2^T]^T$ $(b_2 \neq 0)$, $\min\{v(SOCP(a)), v(SOCP(-a))\} > v(SP)$ holds if and only if the corresponding* *hyperplane* $b_2^T d + c_2 = 0$ *intersects the open optimal line segment* l_λ *defined in* (2.4), *where* $\min\{v(SOCP(a)), v(SOCP(-a))\}$ *is defined in* (3.8).

Proof. The assumption guarantees that the l_λ is well defined and at least one of both sets $\Omega(a)$ and $\Omega(-a)$ has interior points.

$\Longrightarrow$. Suppose that the hyperplane $b_2^T d + c_2 = 0$ does not intersect the open optimal line segment l_λ. Then there is either $\hat{d}_1, \hat{d}_2 \in D(a)$ or $\hat{d}_1, \hat{d}_2 \in D(-a)$, say, $\hat{d}_1, \hat{d}_2 \in D(a)$. Then by Lemma 3.2, the problem $(QP(a))$ satisfies the Slater condition. From Lemma 3.4, one has $v(SOCP(a)) = v(SP)$, which is a contradiction.

$\Longleftarrow$. Note that the assumption "(QP) has a positive duality gap" ensures

$$\min\{v(QP(a)), v(QP(-a))\} > v(SP).$$

By the definition (3.8), one needs only to prove that if $\mathrm{int}(\Omega(a))$ (or $\mathrm{int}(\Omega(-a))) \neq \emptyset$, then it holds that $v(SOCP(a))$ (or $v(SOCP(-a))) > v(SP)$. In fact, "the hyperplane $b_2^T d + c_2 = 0$ intersects the l_λ" means that $(b_2^T \hat{d}_1 + c_2)(b_2^T \hat{d}_2 + c_2) < 0$, say $b_2^T \hat{d}_1 + c_2 > 0$ and $b_2^T \hat{d}_2 + c_2 < 0$, which implies that $\hat{d}_1 \in \mathrm{int}(D(a))$ and $\hat{d}_2 \in \mathrm{int}(D(-a))$. So there are neither $\hat{d}_1, \hat{d}_2 \in D(a)$ nor $\hat{d}_1, \hat{d}_2 \in D(-a)$. According to Lemma 3.4, it holds that $v(SOCP(a))$ (or $v(SOCP(-a))) > v(SP)$ if only $\mathrm{int}(\Omega(a))$ (or $\mathrm{int}(\Omega(-a))) \neq \emptyset$. $\square$

Now consider all the hyperplanes that intersect with the open line segment l_λ defined in (2.4), that is, define a set

$$P_l := \left\{ a = \begin{bmatrix} c_2 \\ b_2 \end{bmatrix} \middle| c_2 \in \mathcal{R}, 0 \neq b_2 \in \mathcal{R}^n, \{d \in \mathcal{R}^n | b_2^T d + c_2 = 0\} \cap l_\lambda \neq \emptyset \right\}. \tag{3.9}$$

Obviously,

$$a \in P_l \iff (b_2^T \hat{d}_1 + c_2)(b_2^T \hat{d}_2 + c_2) < 0 \iff (a^T \hat{x}_1)(a^T \hat{x}_2) < 0,$$

where $\hat{x}_1$ and $\hat{x}_2$ are defined in Property $\mathcal{I}$.

4. Exact relaxation for a class of extended CDT problems.

Theorem 3.5 in the last section showed that any vector $a \in P_l$ can narrow the duality gap of (QP) by utilizing the SOC constraint $Xa \in SOC$ (or $-Xa \in SOC$). A natural question is, What is the best vector in the set P_l? In other words, is there one vector $a \in P_l$ such that the duality gap vanishes thoroughly by using the vector a?

In this section, consider a special class of the problem (1.1) as follows:

$$\begin{array}{llll} (QP_L) & \text{minimize} & q_0(d) = d^T Q_0 d + 2b_0^T d \\ & \text{subject to} & q_1(d) = \|d\|^2 - 1 \leq 0, \\ & & q_2(d) = -(b_3^T d + c_3)(b_4^T d + c_4) \leq 0, \end{array} \tag{4.1}$$

where $b_3,\ b_4 \in \mathcal{R}^n,\ c_3,\ c_4 \in \mathcal{R}$. $q_2(d)$ is a product of two linear functions. For (QP_L), put

$$a_3 := \begin{bmatrix} c_3 \\ b_3 \end{bmatrix}, a_4 := \begin{bmatrix} c_4 \\ b_4 \end{bmatrix}, \quad \text{and } M_2 = -\frac{1}{2}\left(a_3 a_4^T + a_4 a_3^T\right). \tag{4.2}$$

Corresponding to (QP_L), the problems (SP), (SD), $(QP(a))$, $(SOCP(a))$, and $(SOCD(a))$ are defined the same as those in the last section.

LEMMA 4.1. *If* (QP_L) *satisfies the Slater condition and has a positive duality gap, then an alternative of either* $a_3 \in P_l$ *or* $a_4 \in P_l$ *holds.*

Proof. By Proposition 2.3, Theorem 2.4, and Theorem 2.6, (SP) has the unique optimal solution, denoted by $\hat{X}$, and satisfies Property $\mathcal{I}$. Then $\hat{X}$ has a rank-one decomposition $\hat{X} = \hat{x}_1\hat{x}_1^T + \hat{x}_2\hat{x}_2^T$ such that

$$\begin{aligned}\hat{x}_1,\ \hat{x}_2 \in \partial(SOC) \text{ and } (M_2 \bullet \hat{x}_1\hat{x}_1^T)(M_2 \bullet \hat{x}_2\hat{x}_2^T) &= \left(a_3^T\hat{x}_1\hat{x}_1^T a_4\right)\left(a_3^T\hat{x}_2\hat{x}_2^T a_4\right)\\ &= \left(a_3^T\hat{x}_1 a_3^T\hat{x}_2\right)\left(a_4^T\hat{x}_1 a_4^T\hat{x}_2\right) < 0.\end{aligned}$$

This implies an alternative of either $a_3^T\hat{x}_1 a_3^T\hat{x}_2 < 0$ or $a_4^T\hat{x}_1 a_4^T\hat{x}_2 < 0$ holds. Thus there is an alternative of either $a_3 \in P_l$ or $a_4 \in P_l$. ∎

THEOREM 4.2. *Suppose that (QP_L) satisfies the Slater condition and has a positive duality gap. Set $a = a_3$ or $a_4 \in P_l$. Then,*

(i) *if $(QP(a))$ (or $(QP(-a))$) satisfies the Slater condition, there is $v(QP(a)) = v(SOCP(a))$ (or $v(QP(-a)) = v(SOCP(-a))$);*
(ii) $v(QP_L) = \min\{v(SOCP(a)), v(SOCP(-a))\}$.

Proof. By Lemma 4.1, either $a_3 \in P_l$ or $a_4 \in P_l$ holds, say, $a = a_3 \in P_l$.

(i) Without loss of generality, we assume that $(QP(a))$ satisfies the Slater condition and manage to show $v(QP(a)) = v(SOCP(a))$.
From Proposition 3.1, both $(SOCP(a))$ and $(SOCD(a))$ have optimal solutions and $v(SOCP(a)) = v(SOCD(a))$. Let $\tilde{X}$ and $(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \tilde{u})$ be a pair of optimal solutions to $(SOCP(a))$ and $(SOCD(a))$, and they must satisfy the KKT complementary conditions as follows:

$$\left\{\begin{array}{l}\tilde{y}_1 M_1 \bullet \tilde{X} = 0,\\ \tilde{y}_2 M_2 \bullet \tilde{X} = 0,\\ \tilde{u}^T\tilde{X}a_3 = 0,\\ \tilde{Z} \bullet \tilde{X} = 0,\end{array}\right. \tag{4.3}$$

where $\tilde{Z} := M_0 - \tilde{y}_0 E_{00} + \tilde{y}_1 M_1 + \tilde{y}_2 M_2 - \frac{1}{2}(\tilde{u}a_3^T + a_3\tilde{u}^T)$. Denote $r := \operatorname{rank}(\tilde{X})$. To show $v(QP(a)) = v(SOCP(a))$, one needs only to find a rank-one feasible solution to $(SOCP(a))$ that satisfies the complementary conditions (4.3). As $a = a_3 \in P_l$, by Theorem 3.5 and Lemma 3.3, there is $\tilde{u} \neq 0$. Then the complementary condition "$\tilde{u}^T\tilde{X}a_3 = 0$" implies $\tilde{X}a_3 \in \partial(SOC)$. We proceed with the two following cases.

Case 1. $\tilde{X}a_3 = 0$. By Lemma 2.7, one can find a rank-one decomposition of $\tilde{X}$,

$$\tilde{X} = \tilde{x}_1\tilde{x}_1^T + \tilde{x}_2\tilde{x}_2^T + \cdots + \tilde{x}_r\tilde{x}_r^T,$$

such that

$$M_1 \bullet \tilde{x}_i\tilde{x}_i^T = M_1 \bullet \tilde{X}/r \le 0 \quad \text{for } i = 1, 2, \ldots, r.$$

Note that $\tilde{X}a_3 = 0$ means

$$\tilde{x}_i^T a_3 = 0 \implies M_2 \bullet \tilde{x}_i\tilde{x}_i^T = -a_3^T\tilde{x}_i\tilde{x}_i^T a_4 = 0 \;\text{ for } i = 1, 2, \ldots, r.$$

Thus one can easily verify that the rank-one matrix $\tilde{x}_i\tilde{x}_i^T/(\tilde{x}_i)_1^2$ $(i = 1, 2, \ldots, r)$ is a feasible solution to $(SOCP(a))$ and satisfies the complementary conditions (4.3). Here $M_1 \bullet \tilde{x}_i\tilde{x}_i^T \le 0$ and $\tilde{x}_i \neq 0$ guarantee that $(\tilde{x}_i)_1 \neq 0$.

Case 2. $\tilde{X}a_3 \neq 0$. For this case, $\tilde{u}$ and $\tilde{X}a_3$ are both nonzero boundary vectors of the SOC. This implies $M_1 \bullet (\tilde{X}a_3)(\tilde{X}a_3)^T = 0$ and $(\tilde{X}a_3)_1 \neq 0$. As

$a_3^T\tilde{X}a_3 \geq 0$, there are

$$M_2 \bullet (\tilde{X}a_3)(\tilde{X}a_3)^T = -a_3^T(\tilde{X}a_3)(\tilde{X}a_3)^T a_4 = (a_3^T\tilde{X}a_3)M_2 \bullet \tilde{X} \leq 0,$$
$$\tilde{y}_2 M_2 \bullet (\tilde{X}a_3)(\tilde{X}a_3)^T = (a_3^T\tilde{X}a_3)\tilde{y}_2 M_2 \bullet \tilde{X} = 0.$$

So the rank-one matrix $(\tilde{X}a_3)(\tilde{X}a_3)^T/(\tilde{X}a_3)_1^2$ is a feasible solution to $(SOCP(a))$ and satisfies the complementary conditions (4.3).

(ii) Note that $v(QP_L) = \min\{v(QP(a)), v(QP(-a))\}$. By the definition (3.8) and the conclusion (i), one immediately obtains the conclusion (ii). ∎

The above theorem shows that the duality gap of (QP_L) can be eliminated by selecting $a = a_3$ or $a_4 \in P_l$. Define

$$\Omega^+ := \{d \in \mathcal{R}^n | q_1(d) \leq 0, b_3^T d + c_3 \geq 0, b_4^T d + c_4 \geq 0\},$$
$$\Omega^- := \{d \in \mathcal{R}^n | q_1(d) \leq 0, -(b_3^T d + c_3) \geq 0, -(b_4^T d + c_4) \geq 0\}.$$

As $\Omega = \Omega^+ \cup \Omega^-$, the problem (QP_L) may consist of two problems (QP_L^+) and (QP_L^-) as follows:

$$(QP_L^+) \quad \min_{d\in\mathcal{R}^n} q_0(d) \quad \text{s.t. } d \in \Omega^+ \quad \text{and} \quad (QP_L^-) \quad \min_{d\in\mathcal{R}^n} q_0(d) \quad \text{s.t. } d \in \Omega^-.$$

(QP_L^+) can be exactly reformulated into the following form:

$$(4.4) \qquad \begin{array}{lll} (QP_L^+) & \displaystyle\min_{d\in\mathcal{R}^n} & q_0(d) = d^T Q_0 d + 2b_0^T d \\ & \text{subject to} & q_1(d) = \|d\|^2 \leq 1, \\ & & q_2(d) = -(b_3^T d + c_3)(b_4^T d + c_4) \leq 0, \\ & & \|(b_3^T d + c_3)d\| \leq b_3^T d + c_3, \\ & & \|(b_4^T d + c_4)d\| \leq b_4^T d + c_4. \end{array}$$

And the SDP relaxation model of (QP_L^+) is

$$(4.5) \qquad \begin{array}{lll} (SOCP_L^+) & \text{minimize} & M_0 \bullet X \\ & \text{subject to} & M_1 \bullet X \leq 0, \\ & & M_2 \bullet X \leq 0, \\ & & Xa_3 \in SOC, \\ & & Xa_4 \in SOC, \\ & & E_{00} \bullet X = 1, \\ & & X \succeq 0. \end{array}$$

Similarly, (QP_L^-) and $(SOCP_L^-)$ also can be reformulated and defined, respectively.

Such a problem as (QP_L^+) (or (QP_L^-)) has been researched recently by Burer et al. in [6] and [7]. It has been proved in [7] that if the hyperplanes $b_3^T d + c_3 = 0$ and $b_4^T d + c_4 = 0$ have no *intersection* in the open unit ball $\{d \in \mathcal{R}^n \mid \|d\|^2 < 1\}$, there is $v(QP_L^+) = v(SOCP_L^+)$ (or $v(QP_L^-) = v(SOCP_L^-)$). Otherwise, in the *intersecting* case, a counterexample presented in [6] shows that there may be $v(QP_L^+) > v(SOCP_L^+)$ (or $v(QP_L^-) > v(SOCP_L^-)$).

Notice that for the *intersecting* case, there is either $\text{int}(\Omega^+) \neq \emptyset$, $\text{int}(\Omega^-) \neq \emptyset$, or

$$(4.6) \qquad \Omega^+ = \Omega^- = \Omega = \left\{d \,\middle|\, \|d\|^2 - 1 \leq 0,\ b_3^T d + c_3 = 0\right\}.$$

Similarly to the definition (3.8), one can define

$$\min\{v(SOCP_L^+), v(SOCP_L^-)\} := \begin{cases} \min\{v(QP_L^+), v(SOCP_L^-)\} & \text{if } \operatorname{int}(\Omega^+) = \emptyset; \\ \min\{v(SOCP_L^+), v(QP_L^-)\} & \text{if } \operatorname{int}(\Omega^-) = \emptyset; \\ \min\{v(SOCP_L^+), v(SOCP_L^-)\} & \text{if } \operatorname{int}(\Omega^+) \neq \emptyset, \operatorname{int}(\Omega^-) \neq \emptyset. \end{cases} \tag{4.7}$$

The following theorem shows that the problem (QP_L) can always have an exact SDP relaxation by using $(SOCP_L^+)$ and $(SOCP_L^-)$.

THEOREM 4.3. *For any problem (QP_L), one has*

$$v(QP_L) = \min\{v(SOCP_L^+), v(SOCP_L^-)\}.$$

Proof. By the papers [6], [7] and the formula (4.6), one needs only to consider the following hard case: $\operatorname{int}(\Omega^+) \neq \emptyset$, $\operatorname{int}(\Omega^-) \neq \emptyset$, *and* (QP_L) *has a positive duality gap.*

For this hard case, by Lemma 4.1 and Theorem 4.2, one can choose $a = a_3$ or $a_4 \in P_l$ such that

$$v(QP_L) = \min\{v(SOCP(a)), v(SOCP(-a))\}.$$

On the other hand,

$$\begin{aligned} &\min\{v(SOCP(a)), v(SOCP(-a))\} = v(QP_L) = \min\{v(QP_L^+), v(QP_L^-)\} \\ &\geq \min\{v(SOCP_L^+), v(SOCP_L^-)\} \geq \min\{v(SOCP(a)), v(SOCP(-a))\}, \end{aligned}$$

which means $v(QP_L) = \min\{v(SOCP_L^+), v(SOCP_L^-)\}$. ∎

Example 4.4. The following instance of (QP_L) is of the "intersecting" case, which is presented by Burer and Anstreicher in section 4 of [6]. It is defined as follows:

$$n = 3,\ Q_0 = \begin{bmatrix} 2 & 3 & 12 \\ 3 & -19 & 6 \\ 12 & 6 & 0 \end{bmatrix},\ b_0 = \begin{bmatrix} 7 \\ 7 \\ \frac{9}{2} \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ \frac{6}{5} \\ 0 \end{bmatrix},\ b_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},$$

$c_3 = \frac{1}{2}$ and $c_4 = 0$.

From [6], $v(QP_L^+) \approx -12.9419$ with the optimal solution

$$d^{+*} \approx [-0.8536,\ 0.2947,\ 0.4294]^T$$

and $v(SOCP_L^+) \approx -13.8410$ with the optimal solution

$$X_{SOCP}^{+*} \approx \begin{bmatrix} 1 & -0.3552 & 0.3881 & -0.2119 \\ -0.3552 & 0.2595 & -0.2248 & -0.0913 \\ 0.3881 & -0.2248 & 0.4495 & -0.0694 \\ -0.2119 & -0.0913 & -0.0694 & 0.2911 \end{bmatrix}.$$

Then we solve (QP_L^-) by using the MATLAB symbolic math toolbox "solve" and find $v(QP_L^-) \approx -33.0854$ with the optimal solution $d^{-*} \approx [0,\ -0.9984,\ 0.0566]^T$.

Furthermore, we solve (SP) and $(SOCP_L^-)$ by using the CVX package and obtain $v(SP) = v(SOCP_L^-) \approx -33.0854$ with the optimal solutions

$$X_{SP}^* = X_{SOCP}^{-*} \approx \begin{bmatrix} 1 & 0 & -0.9984 & 0.0566 \\ 0 & 0 & 0 & 0 \\ -0.9984 & 0 & 0.9968 & -0.0565 \\ 0.0566 & 0 & -0.0565 & 0.0032 \end{bmatrix}.$$

So $v(QP_L) = v(QP_L^-) = v(SOCP_L^-) = v(SP) = -33.0854$. The numerical results are in accord with Theorem 4.3.

5. Exact relaxation for classical CDT problems. In this section, we consider a classical n-dimensional CDT problem as follows:

$$(5.1)\qquad \begin{array}{lll} (QP_C) & \text{minimize} & q_0(d) = d^TQ_0d + 2b_0^Td \\ & \text{subject to} & q_1(d) = \|d\|^2 - 1 \le 0, \\ & & q_2(d) = \|A^Td + f\|^2 - 1 \le 0, \end{array}$$

where $Q_0 \in \mathcal{S}^{n\times n}$, $b_0, d \in \mathcal{R}^n$, $A \in \mathcal{R}^{n\times m}$, and $f \in \mathcal{R}^m$.

Corresponding to (QP_C), the problems (SP), (SD), $(QP(a))$, $(SOCP(a))$, and $(SOCD(a))$ are defined the same as those in section 3. However, since the constraint $q_2(d) = \|A^Td + f\|^2 - 1 \le 0$ is also an $(m+1)$-dimensional SOC, one can add a redundant constraint $\|(b_2^Td + c_2)(A^Td + f)\| \le b_2^Td + c_2$ to the problem $(QP(a))$. Therefore the problem $(QP(a))$ can be further reformulated into the following form:

$$(5.2)\qquad \begin{array}{lll} (QP(a)) & \text{minimize} & d^TQ_0d + 2b_0^Td \\ & \text{subject to} & \|d\|^2 \le 1, \\ & & \|A^Td + f\|^2 \le 1, \\ & & \|(b_2^Td + c_2)d\| \le b_2^Td + c_2, \\ & & \|(b_2^Td + c_2)(A^Td + f)\| \le b_2^Td + c_2. \end{array}$$

The redundant constraint "$\|(b_2^Td+c_2)(A^Td+f)\| \le b_2^Td+c_2$" will play an important role in the proof of later theorems. The SDP relaxation model of the above $(QP(a))$ is

$$(5.3)\qquad \begin{array}{lll} (CDTSP(a)) & \text{minimize} & M_0 \bullet X \\ & \text{subject to} & M_1 \bullet X \le 0, \\ & & M_2 \bullet X \le 0, \\ & & Xa \in SOC(n+1), \\ & & BXa \in SOC(m+1), \\ & & E_{00} \bullet X = 1, \\ & & X \succeq 0, \end{array}$$

and the corresponding dual model is

$$\begin{array}{lll} (CDTSD(a)) & \text{maximize} & y_0 \\ & \text{subject to} & M_0 - y_0E_{00} + y_1M_1 + y_2M_2 \\ & & -\dfrac{1}{2}(ua^T + au^T) - \dfrac{1}{2}(B^Tva^T + av^TB) \succeq 0, \\ & & y_1 \ge 0,\ y_2 \ge 0,\ u \in SOC(n+1),\ v \in SOC(m+1), \end{array}$$

where M_0, M_1, E_{00} are defined the same as those in (2.7), and

$$M_2 = \begin{bmatrix} f^Tf-1 & f^TA^T \\ Af & AA^T \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 0^T \\ f & A^T \end{bmatrix}. \tag{5.4}$$

The following result is similar to Lemma 3.3.

LEMMA 5.1. *Suppose that $(QP(a))$ satisfies the Slater condition and (QP_C) has a positive duality gap. Let $\tilde{X}$ and $(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \tilde{u}, \tilde{v})$ be a pair of optimal solutions to $(CDTSP(a))$ and $(CDTSD(a))$. If $a \in P_l$, then there is $\tilde{u} \neq 0$ or $\tilde{v} \neq 0$.*

Proof. By the assumption, all problems (SP), (SD), $(SOCP(a))$, $(SOCD(a))$, $(CDTSP(a))$, and $(CDTSD(a))$ have optimal solutions and satisfy the strong duality. Moreover, the assumption "(QP_C) has a positive duality gap" means the set P_l is well defined. As $a \in P_l$, by Theorem 3.5, there is

$$v(CDTSP(a)) \geq v(SOCP(a)) > v(SP).$$

Now we assume that $\tilde{u} = \tilde{v} = 0$; then $(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2)$ is also an optimal solution to (SD). Therefore we obtain

$$v(CDTSP(a)) = v(CDTSD(a)) = v(SD) = v(SP),$$

which is a contradiction. The proof is complete. ∎

Notice that the feasible domain Ω of (QP_C) is a convex region. So Ω contains no separated point. Therefore, if only (QP_C) satisfies the Slater condition, then $\mathrm{int}(\Omega(a)) = \emptyset$ (or $\mathrm{int}(\Omega(-a)) = \emptyset$) implies $\Omega(a) \subseteq \Omega(-a)$ (or $\Omega(-a) \subseteq \Omega(a)$). So, similarly to the definition (3.8), if $\mathrm{int}(\Omega) \neq \emptyset$ one can define

$$\begin{aligned} &\min\{v(CDTSP(a)), v(CDTSP(-a))\} \\ &\quad := \begin{cases} \min\{v(QP(a)), v(CDTSP(-a))\} \\ \quad = v(CDTSP(-a)) & \text{if } \mathrm{int}(\Omega(a)) = \emptyset, \mathrm{int}(\Omega(-a)) \neq \emptyset; \\ \min\{v(CDTSP(a)), v(QP(-a))\} \\ \quad = v(CDTSP(a)) & \text{if } \mathrm{int}(\Omega(-a)) = \emptyset, \mathrm{int}(\Omega(a)) \neq \emptyset; \\ \min\{v(CDTSP(a)), v(CDTSP(-a))\} & \text{else.} \end{cases} \end{aligned} \tag{5.5}$$

THEOREM 5.2. *Suppose that (QP_C) satisfies the Slater condition and has a positive duality gap, and suppose that one vector $a = [c_2,\ b_2^T]^T \in P_l$ satisfies the following relation:*

$$\{d \in \mathcal{R}^n | q_1(d) \leq 0, b_2^T d + c_2 = 0\} = \{d \in \mathcal{R}^n | q_2(d) \leq 0, b_2^T d + c_2 = 0\}. \tag{5.6}$$

Then,

(i) *if $(QP(a))$ (or $(QP(-a))$) satisfies the Slater condition, there is $v(QP(a)) = v(CDTSP(a))$ (or $v(QP(-a)) = v(CDTSP(-a))$);*
(ii) *$v(QP_C) = \min\{v(CDTSP(a)), v(CDTSP(-a))\}$.*

Note 5.1. The relation (5.6) holds if and only if both sets $\{d \in \mathcal{R}^n | q_1(d) \leq 0, b_2^T d + c_2 = 0\}$ and $\{d \in \mathcal{R}^n | q_2(d) \leq 0, b_2^T d + c_2 = 0\}$ are contained in the feasible domain of the primal problem (QP_C).

Proof.

(i) Without loss of generality, we assume that $(QP(a))$ satisfies the Slater condition and manage to show $v(QP(a)) = v(CDTSP(a))$. Let $\tilde{X}$ and $(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \tilde{u}, \tilde{v})$ be a pair of optimal solutions to $(CDTSP(a))$ and $(CDTSD(a))$, and they must satisfy the KKT complementary conditions as follows:

$$\left\{\begin{array}{l} \tilde{y}_1 M_1 \bullet \tilde{X} = 0, \\ \tilde{y}_2 M_2 \bullet \tilde{X} = 0, \\ \tilde{u}^T \tilde{X} a = 0, \\ \tilde{v}^T B \tilde{X} a = 0, \\ \tilde{Z} \bullet \tilde{X} = 0, \end{array}\right. \tag{5.7}$$

where $\tilde{Z} := M_0 - \tilde{y}_0 E_{00} + \tilde{y}_1 M_1 + \tilde{y}_2 M_2 - \frac{1}{2}(\tilde{u}a^T + a\tilde{u}^T) - \frac{1}{2}(B^T \tilde{v} a^T + a \tilde{v}^T B)$. To show $v(QP(a)) = v(CDTSP(a))$, one needs only to find a rank-one feasible solution to $(CDTSP(a))$ that satisfies the complementary conditions (5.7). Define $r := \operatorname{rank}(\tilde{X})$. If $r = 1$, then the rank-one matrix $\tilde{X}$ is the one we want to find. Now we assume $r \geq 2$ and proceed with two following cases.

Case 1. $\tilde{X}a = 0$. By Lemma 2.7 and the assumption $\tilde{X}a = 0$, one can find a rank-one decomposition of $\tilde{X}$, $\tilde{X} = \tilde{x}_1\tilde{x}_1^T + \tilde{x}_2\tilde{x}_2^T + \cdots + \tilde{x}_r\tilde{x}_r^T$, such that

$$M_1 \bullet \tilde{x}_i\tilde{x}_i^T = M_1 \bullet \tilde{X}/r \leq 0 \text{ and } \tilde{x}_i^T a = 0 \text{ for } i = 1, 2, \ldots, r. \tag{5.8}$$

So (5.8) and (5.6) lead to $M_2 \bullet \tilde{x}_i\tilde{x}_i^T \leq 0$ $(i = 1, 2, \ldots, r)$. Therefore

$$\sum_{i=1}^{r} \tilde{y}_2 M_2 \bullet \tilde{x}_i\tilde{x}_i^T = \tilde{y}_2 M_2 \bullet \tilde{X} = 0 \implies \tilde{y}_2 M_2 \bullet \tilde{x}_i\tilde{x}_i^T = 0, \quad i = 1, 2, \ldots, r.$$

All the above results imply that the rank-one matrix $\tilde{x}_i\tilde{x}_i^T/(\tilde{x}_i)_1^2$ $(i = 1, 2, \ldots, r)$ is a feasible solution to $(CDTSP(a))$ and satisfies the complementary conditions (5.7), where the inequality $M_1 \bullet \tilde{x}_i\tilde{x}_i^T \leq 0$ guarantees $(\tilde{x}_i)_1 \neq 0$.

Case 2. $\tilde{X}a \neq 0$. From Lemma 5.1, there is $\tilde{u} \neq 0$ or $\tilde{v} \neq 0$. Without loss of generality, we assume $\tilde{v} \neq 0$. The complementary condition "$\tilde{v}^T B \tilde{X} a = 0$" implies $B\tilde{X}a \in \partial(SOC)$. Put

$$\tilde{x}_1 := \frac{\tilde{X}a}{\sqrt{a^T \tilde{X} a}} \text{ and } X_1 := \tilde{X} - \tilde{x}_1\tilde{x}_1^T.$$

One obtains

$$\begin{aligned} & M_2 \bullet \tilde{x}_1\tilde{x}_1^T = 0,\ M_2 \bullet X_1 = M_2 \bullet \tilde{X}; \\ & X_1 a = \tilde{X}a - \tilde{x}_1\tilde{x}_1^T a = 0,\ \operatorname{rank}(X_1) = r - 1. \end{aligned} \tag{5.9}$$

Note that $\tilde{X}a \in SOC \implies M_1 \bullet \tilde{x}_1\tilde{x}_1^T \leq 0$. If $\tilde{y}_1 M_1 \bullet \tilde{x}_1\tilde{x}_1^T = 0$, then the rank-one matrix $\tilde{x}_1\tilde{x}_1^T/(\tilde{x}_1)_1^2$ is a feasible solution to $(SOCP(a))$ and satisfies the complementary conditions (5.7).

Now we assume $\tilde{y}_1 M_1 \bullet \tilde{x}_1\tilde{x}_1^T \neq 0$, which means

(5.10)

$$\begin{array}{ll} \tilde{y}_1 > 0,\ \tilde{y}_1 M_1 \bullet \tilde{X} = 0,\ M_1 \bullet \tilde{x}_1\tilde{x}_1^T < 0 & \implies M_1 \bullet \tilde{X} = 0,\ \tilde{x}_1 \in \operatorname{int}(SOC); \\ \tilde{x}_1 \in \operatorname{int}(SOC), \tilde{u} \in SOC, \tilde{u}^T \tilde{x}_1 = \frac{\tilde{u}^T \tilde{X} a}{\sqrt{a^T \tilde{X} a}} = 0 & \implies \tilde{u} = 0. \end{array}$$

By Lemma 2.7 and the equalities $M_2 \bullet X_1 = M_2 \bullet \tilde{X}$ and $X_1 a = 0$ from (5.9), one can find a rank-one decomposition of X_1,

$$X_1 = \tilde{x}_2\tilde{x}_2^T + \tilde{x}_3\tilde{x}_3^T + \cdots + \tilde{x}_r\tilde{x}_r^T,$$

such that

$$M_2 \bullet \tilde{x}_i\tilde{x}_i^T = \frac{M_2 \bullet X_1}{r-1} = \frac{M_2 \bullet \tilde{X}}{r-1} \leq 0 \text{ and } \tilde{x}_i^T a = 0 \quad \text{for } i = 2, 3, \ldots, r. \tag{5.11}$$

We assert that there exists an index $i_0 \in \{2, 3, \ldots, r\}$ such that $(\tilde{x}_{i_0})_1 = 0$ because if all $(\tilde{x}_i)_1 \neq 0$ $(i = 2, 3, \ldots, r)$, then (5.11) and (5.6) yield $M_1 \bullet \tilde{x}_i\tilde{x}_i^T \leq 0$ $(i = 2, 3, \ldots, r)$, which induces

$$M_1 \bullet \tilde{X} = M_1 \bullet \tilde{x}_1\tilde{x}_1^T + \sum_{i=2}^{r} M_1 \bullet \tilde{x}_i\tilde{x}_i^T < 0.$$

It contradicts (5.10). So there exists an $i_0 \in \{2, 3, \ldots, r\}$ such that $(\tilde{x}_{i_0})_1 = 0$. Denote

$$\tilde{x}_{i_0} := \begin{bmatrix} 0 \\ \tilde{d} \end{bmatrix} \text{ and } z(t) := \tilde{x}_1 + t\tilde{x}_{i_0}, t \in \mathcal{R}.$$

They satisfy

$$\begin{cases} M_1 \bullet \tilde{x}_{i_0}\tilde{x}_{i_0}^T = \|\tilde{d}\|^2 > 0; \\ M_2 \bullet \tilde{x}_{i_0}\tilde{x}_{i_0}^T = \|A^T\tilde{d}\|^2 \leq 0 \implies A^T\tilde{d} = 0 \implies B\tilde{x}_{i_0} = 0, M_2\tilde{x}_{i_0} = 0; \\ M_2 \bullet z(t)z(t)^T = \tilde{x}_1^T M_2\tilde{x}_1 + 2t\tilde{x}_1^T M_2\tilde{x}_{i_0} + t^2\tilde{x}_{i_0}^T M_2\tilde{x}_{i_0} = 0; \\ \tilde{v}^T Bz(t) = \tilde{v}^T B\tilde{x}_1 + t\tilde{v}^T B\tilde{x}_{i_0} = \tilde{v}^T B\tilde{x}_1 = \frac{\tilde{v}^T B\tilde{X}a}{\sqrt{a^T\tilde{X}a}} = 0; \\ a^T z(t) = a^T\tilde{x}_1 + ta^T\tilde{x}_{i_0} = a^T\tilde{x}_1 = \sqrt{a^T\tilde{X}a} > 0; \\ (z(t))_1 = (\tilde{x}_1)_1 = \frac{(\tilde{X}a)_1}{\sqrt{a^T\tilde{X}a}} > 0. \end{cases} \tag{5.12}$$

Consider the following quadratic equation with respect to t:

$$M_1 \bullet z(t)z(t)^T = \tilde{x}_1^T M_1\tilde{x}_1 + 2t\tilde{x}_1^T M_1\tilde{x}_{i_0} + t^2\tilde{x}_{i_0}^T M_1\tilde{x}_{i_0} = 0. \tag{5.13}$$

Due to $\tilde{x}_1^T M_1\tilde{x}_1 < 0$ and $\tilde{x}_{i_0}^T M_1\tilde{x}_{i_0} > 0$, the quadratic equation (5.13) has two real roots t_1^* and t_2^*. Set $z_i^* = \tilde{x}_1 + t_i^*\tilde{x}_{i_0}$ $(i = 1, 2)$. Then one can easily verify that $z_i^*(z_i^*)^T/(z_i^*)_1^2$ $(i = 1, 2)$ is a feasible solution to $(CDTSP(a))$ and satisfies the complementary conditions (5.7).

(ii) As $v(QP_C) = \min\{v(QP(a)), v(QP(-a))\}$, the definition (5.5) and the conclusion (i) immediately induce the conclusion (ii). ∎

Corollary 5.3. *Suppose that (QP_C) is a two-dimensional classical CDT problem, satisfies the Slater condition, and has a positive duality gap. Then one can find a three-dimensional vector $a = [c_2,\ b_2^T]^T$ $(0 \neq b_2 \in \mathcal{R}^2)$ such that*

$$v(QP_C) = \min\{v(CDTSP(a)), v(CDTSP(-a))\}.$$

Proof. As the strong duality is violated, Property $\mathcal{I}'$ holds and the set P_l in (3.9) is well defined. By the Schur matrix-decomposition theorem it can be proved that, for $n = 2$, the curve $q_2(d) = 0$ is either two parallel straight lines (a degenerate ellipse) or an ellipse.

If the curve $q_2(d) = 0$ is two parallel straight lines, the problem (QP_C) degenerates into a problem (QP_L) discussed in section 4. Then by Theorem 4.2 one can find a vector $a = [c_2,\ b_2^T]^T \in P_l$ such that

$$\begin{aligned}&\min\{v(SOCP(a)), v(SOCP(-a)) = v(QP_C)\\ &\geq \min\{v(CDTSP(a)), v(CDTSP(-a)) \geq \min\{v(SOCP(a)), v(SOCP(-a)),\end{aligned}$$

which implies the corollary holds.

Now we assume that $q_2(d)$ is an elliptical function. Denote

$$\partial(q_1) := \{d \in \mathcal{R}^2 | q_1(d) = 0\},\ \partial(q_2) := \{d \in \mathcal{R}^2 | q_2(d) = 0\}.$$

Since $\mathrm{int}(\Omega) \neq \emptyset$ and the strong duality is violated, the set $\partial(q_1) \cap \partial(q_2)$ may contain two or three or four points (note that the strong duality must hold for $\partial(q_1) \cap \partial(q_2) \neq \emptyset$ or $\partial(q_1) \cap \partial(q_2) = \partial(q_1)$ or $card(\partial(q_1) \cap \partial(q_2))$=1). Without loss of generality, we consider the most complicated case: $\partial(q_1) \cap \partial(q_2)$ contains four points (the other two cases can be proved similarly). Denote $\partial(q_1) \cap \partial(q_2) = \{d_1^+, d_2^+, d_3^+, d_4^+\}$, as shown in Figure 5.1. The four points d_i^+ $(i = 1, 2, 3, 4)$ partition the circle $\partial(q_1)$ into four arcs, denoted by Γ_1, Γ_2, Γ_3, and Γ_4, respectively, as shown in Figure 5.1. Here all the arcs Γ_i $(i = 1, 2, 3, 4)$ do not include their endpoints. Let the vectors $\hat{d}_1$ and $\hat{d}_2$ be defined by Property $\mathcal{I}'$ in Definition 2.1. Obviously, it holds that

$$\hat{d}_1 \in \Gamma_1 \cup \Gamma_3 \text{ and } \hat{d}_2 \in \Gamma_2 \cup \Gamma_4.$$

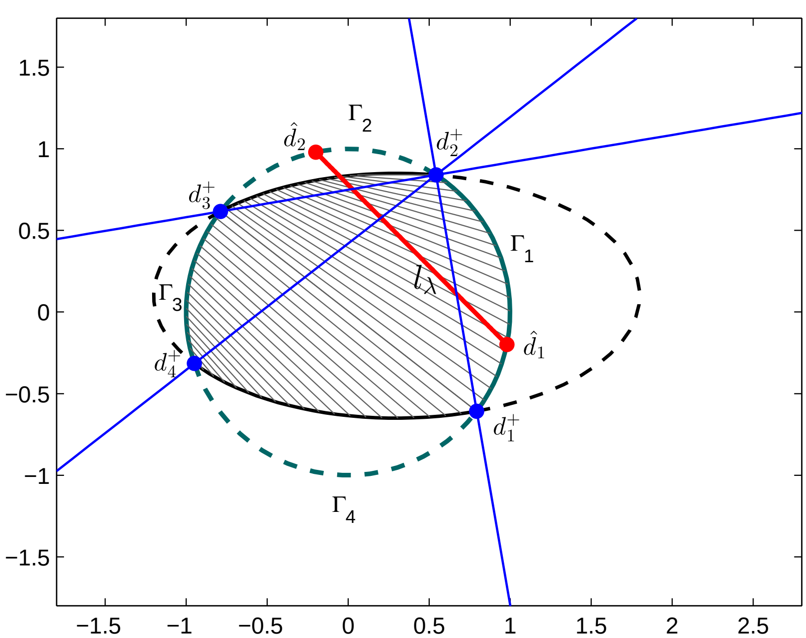

FIG. 5.1. $\partial(q_1) \cap \partial(q_2)$ *contains four points.*

TABLE 5.1
Numerical results of the gap-existing examples.

Examples	$v(QP_C)$	D-gap	$v(CDTSP)$	X^*
Example 6.1 of [1]	-0.1453	0.5606	-0.1453	X_1^*
Example from section 5.2 of [6]	-4	0.25	-4.0000	X_2^*
Example 7.3 of [9]	5.5	0.1667	5.5000	X_3^*
Example from section 5.4 of [16]	-1.4608	0.1642	-1.4608	X_4^*
Example (EX_1) of [17]	-2	1	-2.0000	X_5^*
Example (EX_2) of [17]	-2	0.3852	-2.0000	X_6^*
Example from Lemma 2.2 of [19]	0	0.5	0.0000	X_7^*

Without loss of generality, one assumes $\hat{d}_1 \in \Gamma_1$ and $\hat{d}_2 \in \Gamma_2$. Then each of three straight lines $\mathcal{L}_{d_2^+ d_i^+}$ $(i = 1, 3, 4)$ intersects the open line segment l_λ defined in (2.4), where $\mathcal{L}_{d_2^+ d_i^+}$ denotes the straight line connecting both points d_2^+ and d_i^+. Note that the equation of the line $\mathcal{L}_{d_2^+ d_i^+}$ $(i = 1, 3, 4)$ is

$$(d_{iy}^+ - d_{2y}^+)d_x + (d_{2x}^+ - d_{ix}^+)d_y + (d_{2y}^+ - d_{iy}^+)d_{2x}^+ + (d_{ix}^+ - d_{2x}^+)d_{2y}^+ = 0,$$

where $d = [d_x, d_y]^T$, $d_2^+ = [d_{2x}^+, d_{2y}^+]^T$, and $d_i^+ = [d_{ix}^+, d_{iy}^+]^T$. Then the vector $a = [(d_{2y}^+ - d_{iy}^+)d_{2x}^+ + (d_{ix}^+ - d_{2x}^+)d_{2y}^+,\ d_{iy}^+ - d_{2y}^+,\ d_{2x}^+ - d_{ix}^+]^T \in P_l$ and satisfies (5.6). By Theorem 5.2, the conclusion of the corollary is true. ∎

Since the proof of Corollary 5.3 presents a constructive procedure, an algorithm is available to compute the vector $a = [c_2,\ b_2^T]^T \in P_l$ satisfying (5.6). Then some gap-existing examples from [1, 6, 9, 16, 17, 19] are listed to show the performance of the corollary. By a scalar multiplication of the decision variable vector, the ball constraint of each example is reformulated into the unit ball constraint such that the example accords with the model (5.1) of (QP_C). The numerical results are shown in Table 5.1. In the table, "$v(QP_C)$" and "D-gap" denote the optimal value of each example and its duality gap, respectively; "$v(CDTSP)$" and "X^*" denote $\min\{v(CDTSP(a)), v(CDTSP(-a))\}$ and the corresponding optimal solution, respectively.

The X_i^* $(i = 1, 2, \ldots, 7)$ are displayed as follows:

$$X_1^* \approx \begin{bmatrix} 1 & -0.1318 & 0.9913 \\ -0.1318 & 0.0174 & -0.1307 \\ 0.9913 & -0.1307 & 0.9826 \end{bmatrix}, X_2^* \approx \begin{bmatrix} 1 & -0.7071 & 0.7071 \\ -0.7071 & 0.5 & -0.5 \\ 0.7071 & -0.5 & 0.5 \end{bmatrix},$$
$$X_3^* \approx \begin{bmatrix} 1 & -0.7071 & -0.7071 \\ -0.7071 & 0.5 & 0.5 \\ -0.7071 & 0.5 & 0.5 \end{bmatrix}, X_4^* \approx \begin{bmatrix} 1 & 0.7071 & 0.7071 \\ 0.7071 & 0.5 & 0.5 \\ 0.7071 & 0.5 & 0.5 \end{bmatrix},$$
$$X_5^* = X_6^* = X_7^* \approx \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

One can easily verify that each X_i^* $(i = 1, 2, \dots, 7)$ is a rank-one matrix. As shown in Table 5.1, the duality gaps of all the above examples are eliminated by using Corollary 5.3.

COROLLARY 5.4. *Suppose that* (QP_C) *is a three-dimensional classical CDT problem, satisfies the Slater condition, and has a positive duality gap. If*

$$q_2(d) = \lambda_1 d_x^2 + \lambda_2 d_y^2 + d_z^2 - 1, \quad \lambda_1 > 1 > \lambda_2, \quad d = [d_x, d_y, d_z]^T,$$

then one finds a four-dimensional vector $a = [c_2,\ b_2^T]^T$ $(0 \neq b_2 \in \mathcal{R}^3)$ *such that*

$$v(QP_C) = \min\{v(CDTSP(a)), v(CDTSP(-a))\}.$$

Proof. The assumption guarantees that the vectors $\hat{d}_1$, $\hat{d}_2$ of Property $\mathcal{I}'$ and the set P_l are well defined. Define

$$\begin{aligned} a_3 &= [c_3, b_3^T]^T = [0,\ \sqrt{\lambda_1 - 1},\ \sqrt{1-\lambda_2},\ 0]^T, \\ a_4 &= [c_4, b_4^T]^T = [0,\ \sqrt{\lambda_1 - 1},\ -\sqrt{1-\lambda_2},\ 0]^T, \end{aligned}$$

and

$$x = [1, d_x, d_y, d_z]^T.$$

We assert that one of both a_3 and a_4 belongs to P_l and satisfies (5.6). In fact, one has the following relation:

$$\begin{aligned} a_3^T x a_4^T x &= \left(\sqrt{\lambda_1 - 1} d_x + \sqrt{1-\lambda_2} d_y\right)\left(\sqrt{\lambda_1 - 1} d_x - \sqrt{1-\lambda_2} d_y\right) \\ &= \lambda_1 d_x^2 + \lambda_2 d_y^2 - (d_x^2 + d_y^2) = q_2(d) - q_1(d). \end{aligned} \tag{5.14}$$

Therefore

$$\begin{aligned} & (c_3 + b_3^T \hat{d}_1)(c_3 + b_3^T \hat{d}_2)(c_4 + b_4^T \hat{d}_1)(c_4 + b_4^T \hat{d}_2) \\ = \ & (q_2(\hat{d}_1) - q_1(\hat{d}_1))(q_2(\hat{d}_2) - q_1(\hat{d}_2)) = q_2(\hat{d}_1) q_2(\hat{d}_2) < 0, \end{aligned}$$

which implies that either $(c_3 + b_3^T \hat{d}_1)(c_3 + b_3^T \hat{d}_2) < 0$ or $(c_4 + b_4^T \hat{d}_1)(c_4 + b_4^T \hat{d}_2) < 0$, that is,

$$\text{either } a_3 \in P_l \text{ or } a_4 \in P_l. \tag{5.15}$$

For any $d \in \mathcal{R}^3$ satisfying either $c_3 + b_3^T d = 0$ or $c_4 + b_4^T d = 0$, the relation (5.14) always yields

$$q_2(d) - q_1(d) = 0 \implies q_1(d) \le 0 \text{ if and only if } q_2(d) \le 0 \implies \text{(5.6) holds.} \tag{5.16}$$

Both (5.15) and (5.16) ensure the assertion is true. Then by Theorem 5.2, the conclusion of the corollary holds. ∎

Note that the model studied in Corollary 5.4 contains linear terms in the objective function and the strong duality may not hold, while the model studied by Ye and Zhang in section 2.2 of [17] is homogeneous and the strong duality holds.

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