Periodic Chaotic Relaxation

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ABSTRACT

A recent paper considered the subject of chaotic relaxation. The present paper generalizes two results of that paper concerning periodic chaotic relaxation in which overrelaxation is allowed. One of these generalizations is illustrated with an example in which the optimum rate of convergence is shown to be $0(h^2)$, where h is the mesh length, as against $0(h)$ for standard relaxation.

1. INTRODUCTION

In a recent paper, Chazan and Miranker [1] have discussed a method of solving the linear system

$$Ay = d, \tag{1.1}$$

which they term chaotic relaxation. This is a generalization of the ordinary method of relaxation in which several processors perform relaxations in parallel. Each processor takes from a central store components of the approximate solution and at the end of each relaxation overwrites the component or components which it has calculated. In the meantime several other processors will have finished and the solution will have been updated several times.

Since we are interested only in the error vector $x = y - A^{-1}d$, we shall replace (1.1) by

$$x = Bx, \tag{1.2}$$

where $B = I - A$ is the Jacobi iteration matrix. We shall consider the iterations as they are performed on (1.2). We shall depart slightly from the notation of Chazan and Miranker [1] and write the nth iteration,

$$\begin{aligned} x_i^{m+1} &= \sum_{j=1}^{n} b_{ij} x_j^{m-k(m,j)}, & i &= h(m), \\ &= x_i^m, & i &\neq h(m). \end{aligned} \tag{1.3}$$

Here $h(m)$ is the component altered in the mth updating, and the value of x_j used to calculate it is taken from the $m - k(m, j)$ iterate.

Let us assume that there is a fixed $s > 0$ for which $0 \leqslant k(m, j) < s$, all m, j, and that $h(m)$ takes each value between 1 and n infinitely often. The conditions are obviously necessary for convergence. For general matrices, Chazan and Miranker [1] have shown that, with these assumptions, the condition $\rho(|B|) < 1$ is necessary and sufficient for convergence, where $\rho(|B|)$ is the spectral radius of $|B|$.

In this paper we shall be concerned only with positive definite matrices. We shall further restrict ourselves to periodic schemes, which are schemes in which $h(m) = m(\text{mod}\, n)$ and

$$k(m, j) = \min(m - 1, r - 1), \qquad \text{all } m, j,$$

for some r, $1 \leqslant r \leqslant n$.

We shall prove two theorems which generalize results given in [1] and discuss the application of periodic chaotic, or (as we shall call it) delayed, relaxation to a model problem, namely, the simplest discrete approximation to the three-dimensional Laplace's operator on a unit cube. The second theorem will in fact be shown to apply to schemes which are not periodic.

2. A THEOREM

We adopt the notation of Section 3 and generalize Theorem 3.2 of [1]. We notice that in periodic schemes the complete vector is altered every n iterations. Let $z^p = x^{n(p-1)+1}$. Then z^{p+1} is defined in terms of z^p and z^{p-1}. We shall assume that A is positive definite and has the form

$$A = D - G - F - E - E^* - F^* - G^*, \tag{2.1}$$

where E, F, and G are band matrices or, more generally, block band matrices. The structure of $D - A$ is shown schematically in Fig. 1. If $2r > n$, then $-F$ is the band matrix $E \cap G$. The iteration process can be written

$$Dz^{p+1} = Gz^{p+1} + Fz^{p+1} + Ez^p + E^*z^p + F^*z^p + G^*z^{p-1}. \tag{2.2}$$

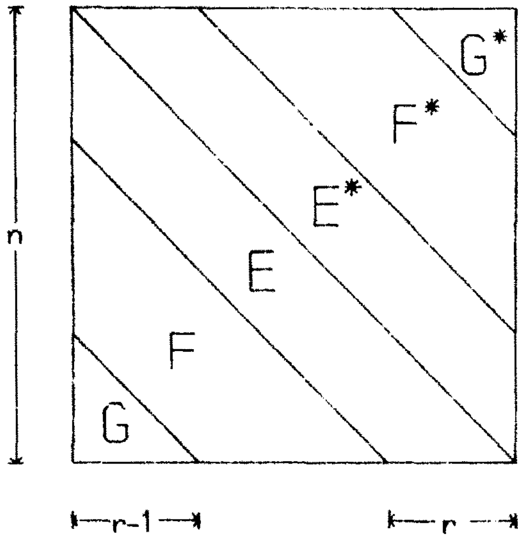

FIG. 1. Structure of the matrix A.

Our results will be deduced from (2.2) regardless of the particular structure of the matrices occurring in it. If we allow overrelaxation, we have

$$(D-\omega F-\omega G)z^{p+1} = (1-\omega)Dz^p + \omega(E+E^*+F^*)z^p + \omega G^* z^{p-1}. \quad (2.3)$$

The following lemma ensures that z^{p+1} is always uniquely determined when $((E+E^*)z, z)$ is bounded below with respect to (Dz, z).

LEMMA 2.1. *Let A be as in* (2.1) *and let A and D be positive definite. If, for some $\alpha(>-1)$, $\alpha D+E+E^*$ is positive definite, then $D-\omega F-\omega G$ is nonsingular for $0 \leq \omega \leq 2/(1+\alpha)$.*

Proof. The case $\omega = 0$ is obvious. Suppose $\omega \neq 0$ and $(D-\omega F-\omega G)z = 0$. Then

$$(Dz, z) = \omega((F+G)z, z) = \omega((F^*+G^*)z, z),$$

so that

$$(Az, z) = \left(\left(1+\alpha-\frac{2}{\omega}\right)Dz, z\right) - ((\alpha D+E+E^*)z, z).$$

For $0 < \omega \leq 2/(1+\alpha)$ this implies $(Az, z) \leq 0$ or $z = 0$.

Remark. The most comprehensive value of α is the infimum of all $\beta > -1$ such that $\beta D+E+E^*$ is positive definite.

The following theorem is modeled on the version of the Ostrowski-Reich theorem given in Varga [2].

THEOREM 2.1. *Let A be as in (2.1) with A and D positive definite. If there exists a positive definite matrix S and an $\alpha > -1$ such that*

$$\begin{bmatrix} S & G \\ G^* & \alpha D + E + E^* - S \end{bmatrix} \tag{2.4}$$

is positive semidefinite, then, for $0 < \omega < 2/(1+\alpha)$, the iteration (2.3) converges.

*Proof.** By Lemma 2.1, z^p is uniquely determined. Set $w^p = z^{p+1} - z^p$. Then

$$\begin{aligned}(D - \omega F - \omega G)w^p &= -\omega D z^p + \omega(G + F + E + E^* + F^*)z^p + \omega G^* z^{p-1} \\ &= -\omega A z^p - \omega G^* w^{p-1}. \end{aligned} \tag{2.5}$$

Now

$$\begin{aligned}&(Az^{p+1}, z^{p+1}) - (Az^p, z^p) \\ &= (Az^p, w^p) + (Aw^p, z^p) + (Aw^p, w^p) \\ &= \left(\left(-\frac{1}{\omega}D + G + F\right)w^p, w^p\right) - (G^* w^{p-1}, w^p) \\ &\quad + \left(\left(-\frac{1}{\omega}D + G^* + F^*\right)w^p, w^p\right) - (Gw^p, w^{p-1}) + (Aw^p, w^p) \\ &= -\left(\left(\left(\frac{2}{\omega} - 1\right)D + E + E^*\right)w^p, w^p\right) - (G^* w^{p-1}, w^p) - (Gw^p, w^{p-1}). \end{aligned} \tag{2.6}$$

Let

$$q_p = (Az^p, z^p) + (Sw^{p-1}, w^{p-1}).$$

Then

$$\begin{aligned} q_p - q_{p+1} = (Sw^{p-1}, w^{p-1}) &+ (Gw^{p-1}, w^p) + (G^* w^p, w^{p-1}) \\ &+ \left(\left(\left(\frac{2}{\omega} - 1\right)D + E + E^* - S\right)w^p, w^p\right). \end{aligned}$$

For $0 < \omega < 2/(1+\alpha)$ we have $q_p \geqslant q_{p+1} \geqslant 0$, so that q_p tends to a limit as p increases and $q_p - q_{p+1}$ tends to zero. This implies that w^p

* This proof is a generalization of the original proof in the case of ordinary relaxation outlined by Black and Southwell [3] and given in detail by Temple [4].

and w^{p-1} tend to zero and hence that z^p tends to z. From (2.3) it follows that $Az = 0$ and thus $z = 0$.

Remarks. It is necessary for A to be positive definite, since otherwise we would take $z^0 = z^1$ with $(Az^1, z^1) \leqslant 0$, and convergence of z^p to zero is impossible. The condition that the matrix (2.4) should be positive semidefinite can be shown to be equivalent to the condition that

$$\alpha D + E + E^* - S - G^* S^{-1} G$$

should be positive semidefinite.

COROLLARY 2.1. *Let $G = 0$. If, for some $\alpha > -1$, $\alpha D + E + E^*$ is positive semidefinite, then for $0 < \omega < 2/(1 + \alpha)$ the iteration (2.3) converges.*

Proof. We see from (2.6) that, for $G = 0$, (Az, z) itself is strictly decreasing. Effectively we take $S = 0$ in the proof of the theorem.

Remark. The value of α suggested in the Remark on Lemma 2.1 gives the widest possible range for ω.

3. A MODEL PROBLEM

Let us examine how Theorem 2.1 can be applied to the simple finite difference approximation to the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ on the unit cube $0 \leqslant x, y, z \leqslant 1$. The corresponding $(n-1)^3$ by $(n-1)^3$ matrix A has $a_{ii} = 6$, $a_{ij} = -1$ if i, j correspond to mesh points distance $h(= 1/n)$ apart and $a_{ij} = 0$ otherwise. Suppose that the iterative method used is single-line overrelaxation. The lines are to be arranged in planes, and the relaxation is to proceed by lines in the same direction across each plane and from plane to plane. Let $2 \leqslant r \leqslant n$ so that D is block diagonal, each block corresponding to a line and having 6's on the diagonal, -1's immediately above and below, and zeros elsewhere. E and F are made up of blocks of unit matrices corresponding to the linkage between, respectively, lines in the same plane and lines in neighboring planes and zero matrices. $G = 0$.

Now $\alpha D + E + E^*$ consists of blocks corresponding to each plane, and the nonzero elements in this part of the finite difference scheme can be represented schematically:

$$\begin{array}{ccc} & -1 & \\ -\alpha & 6\alpha & -\alpha \\ & -1 & \end{array}$$

from which we may deduce that $\alpha D + E + E^*$ is positive definite for

$$6\alpha - 2\alpha\cos\pi h - 2\cos\pi h > 0.$$

If $\alpha = \frac{1}{2}$, this is true for all h. We deduce that single-line overrelaxation converges for $0 < \omega \leqslant \frac{4}{3}$, where we include the equality since $h > 0$.

To calculate an optimum value for ω we need a closer analysis. We shall assume that A is any matrix which satisfies (2.1) with $G = 0$. If z is an eigenvector of the overrelaxation process (2.3) and λ the eigenvalue, we have

$$\lambda(D - \omega F)z = (1 - \omega)Dz + \omega(E + E^* + F^*)z$$

or

$$(\lambda\omega F + \omega F^* - (\lambda + \omega - 1)D + \omega(E + E^*))z = 0. \qquad (3.1)$$

Let us assume (as is the case for the model problem) that the splitting of A into $D - E - E^*, F$, and F^* has property A (is consistently ordered, weakly cyclic of index 2). For $\lambda \neq 0$, this allows us to replace (3.1) by

$$(\lambda^{1/2}\omega(F + F^*) - (\lambda + \omega - 1)D + \omega(E + E^*))w = 0. \qquad (3.2)$$

Now let us assume that the eigenvector w is expressible as a direct sum of the eigenvectors w' and w'' which satisfy

$$(F + F^*)w' = \sigma Dw'$$

and

$$(E + E^*)w'' = \tau Dw''.$$

This in effect limits us to the model problem with a different mesh length in each coordinate direction. Then λ is given by the roots of

$$\lambda - \omega\sigma\lambda^{1/2} + \omega(1 - \tau) - 1 = 0.$$

Suppose that $-S \leqslant \sigma \leqslant S$ and $-T \leqslant \tau \leqslant T$, which is again the case for the model problem. Since A is positive definite we must have $S + T < 1$.

We wish to choose a value of ω which gives the least value to $\max|\lambda|$. Now the level curves in the (a, b) plane of $\max|x| = k$ for roots of $x^2 -$

$ax + b = 0$ are the triangles with vertices $(0, -k^2)$, $(2k, k^2)$, $(-2k, k^2)$ as shown in Fig. 2.

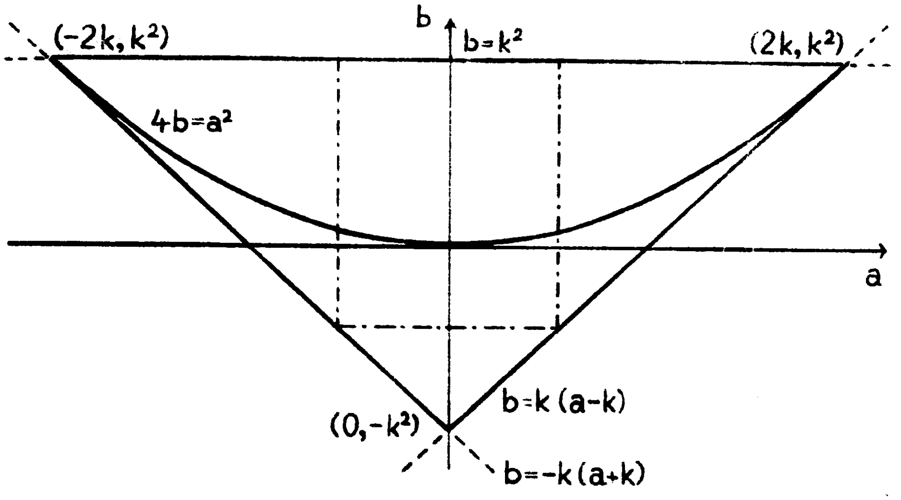

FIG. 2. Determination of optimum ω.

The possible values of $a = \omega\sigma$ and $b = \omega(1 - \tau) + 1$ lie in a rectangle which is a projection from the point $(0, -1)$ of the rectangle $-S \leqslant a \leqslant S$ and $-T \leqslant b \leqslant T$. Starting with $\omega = 0$, $\max|\lambda|^{1/2} = 1$ and decreases as ω increases until it takes its minimum value k when the rectangle is in the position shown in Fig. 2; that is, when

$$\omega(1 + T) - 1 = k^2$$

and

$$\omega(1 - T) - 1 = k(\omega S - k)$$

or

$$1 + k^2 = \omega(1 + T)$$

and

$$2(\omega - 1) = k\omega S.$$

Eliminating ω, we obtain

$$Sk^3 - 2k^2 + Sk + 2T = 0.$$

We require the least positive root of this equation. Using the fact that $S + T < 1$, we find that, if $T > 0$, the left-hand side is positive for $k = 0$ and negative for $k = -1$ and 1. Thus the roots lie in $(-1, 0)$, $(0, 1)$, and $(1, \infty)$ and we require the root satisfying $0 < k < 1$.

If $T > 0$ and $S + T = 1 - \eta$, where $\eta^2 \ll \eta$, we find the approximate values

$$k = 1 - \frac{\frac{1}{2}\eta}{T},$$

$$\omega = \frac{2 - \eta/T}{1 + T},$$

and

$$\max|\lambda| = k^2 = 1 - \frac{\eta}{T}.$$

For $T = 0$, corresponding to $E = 0$, there are two roots in the immediate neighborhood of $k = 1$ and the approximation above is invalid. We have instead the well-known result $k = 1 - (2\eta)^{1/2}$ for ordinary relaxation without delay. We see that the introduction of the delay in the relaxation scheme reduces the optimum rate of convergence from $0(h)$ to $0(h^2)$.

For the model problem,

$$S = T = \frac{2 \cos \pi h}{6 - 2 \cos \pi h} \simeq \tfrac{1}{2}(1 - \eta),$$

where $\eta = \frac{3}{4}\pi^2 h^2$. This gives $\omega \simeq \frac{4}{3} \cos \pi h$, in the range already found, $0 < \omega \leqslant \frac{4}{3}$, and $\max|\lambda| = 1 - \frac{3}{2}\pi^2 h^2$. It is worth pointing out that, if $r \leqslant \frac{1}{2}n$, the scheme we have been considering can be modified so that lines being processed at the same time are uncoupled. This can be done by treating first the odd-numbered lines of one plane and then the even-numbered lines of the same plane. For $\frac{1}{2}n < r \leqslant n$, it can be done by treating the odd-numbered lines of the first plane and the even-numbered lines of the second plane together and then the even-numbered lines of the first plane and the odd-numbered lines of the second plane, and so on through the cube. The process is then equivalent to a standard over-relaxation and the optimum rate of convergence is $0(h)$.

If $G \neq 0$, we replace (3.1) by

$$(\lambda\omega F + \omega F^* - (\lambda + \omega - 1)D + \omega(E + E^*) + \lambda\omega G + \lambda^{-1}\omega G^*)z = 0.$$

Assuming that the splitting (2.1) has a double property A, we may replace this by

$$(\lambda^{1/2}\omega(F + F^*) - (\lambda + \omega - 1)D + \omega(G + E + E^* + G^*))w = 0$$

if $\lambda \neq 0$.

4. ANOTHER THEOREM

In Theorem 2.1 let us take $\alpha = 2c$, for some $c > 0$, and $G = 0$. Let us replace A by $D^{-1/2}AD^{-1/2}$ so that $D = I$. Then, setting $D + E + E^* = M$, a sufficient condition for M to be positive definite is $\sum_{j \neq i} |m_{ij}| < c$ for all i. We shall give an independent generalization of this result based on the proof of Theorem 4.1 of [1]. In this way we shall demonstrate its independence of the order of the rows and columns of A.

We return to the notation of (1.3). Since we are still considering periodic schemes, we have $k(m, j) = r - 1$, all m and j. We are relaxing one component at a time. That is, the $(m + 1)$th iterate differs from the mth in the component $h(m)$. In ordinary relaxation this operation is equivalent to minimizing (Ax^m, x^m) by a variation of the $h(m)$ component of x. That is, α^m is chosen to minimize (Ax^{m+1}, x^{m+1}) with

$$x^{m+1} = x^m - \alpha^m e_{h(m)}, \tag{4.1}$$

where e_i is the unit vector in the ith coordinate direction. (We have taken $a_{ii} = 1$). Therefore

$$\alpha^m = e^T_{h(m)} A x^m. \tag{4.2}$$

If we are using delayed relaxation, we have instead

$$\alpha^m = e^T_{h(m)} A x^{m-r+1}. \tag{4.2'}$$

To allow for overrelaxation we replace (4.1) by

$$x^{m+1} = x^m - \beta^m e_{h(m)}, \tag{4.1'}$$

leaving the relation of β^m to α^m until later.

Now

$$\begin{aligned} x^m &= -\beta^{m-1} e_{h(m-1)} + x^{m-1} \\ &= -\beta^{m-1} e_{h(m-1)} - \beta^{m-2} e_{h(m-2)} - \cdots - \beta^{m-r+1} e_{h(m-r+1)} + x^{m-r+1}, \end{aligned}$$

so that

$$\begin{aligned} &(Ax^{m+1}, x^{m+1}) - (Ax^m, x^m) \\ &\quad = -2\beta^m e^T_{h(m)} A x^m + (\beta^m)^2 \\ &= 2\beta^m (a_{h(m)h(m-1)}\beta^{m-1} + \cdots + a_{h(m)h(m-r+1)}\beta^{m-r+1}) - 2\alpha^m\beta^m + (\beta^m)^2. \end{aligned}$$

Define

$$Q_{m+1} = (Ax^{m+1}, x^{m+1}) + \rho_1(\beta^m)^2 + \rho_2(\beta^{m-1})^2 + \cdots + \rho_{r-1}(\beta^{m-r+1})^2,$$

where $\rho_1, \rho_2, \ldots, \rho_{r-1}$ are positive constants whose values are to be assigned later. Then

$$\begin{aligned} Q_m - Q_{m+1} = {} & (-2 + \sigma_1)(\beta^m)^2 + \sigma_2(\beta^{m-1})^2 + \cdots + \sigma_r(\beta^{m-r+1})^2 + 2\alpha^m\beta^m \\ & - 2(a_{h(m)h(m-1)}\beta^m\beta^{m-1} + \cdots + a_{h(m)h(m-r+1)}\beta^m\beta^{m-r+1}), \end{aligned} \tag{4.3}$$

where $\sigma_1 = 1 - \rho_1$, $\sigma_2 = \rho_1 - \rho_2, \ldots, \sigma_r = \rho_{r-1}$. Thus

$$\begin{aligned} Q_m - Q_{m+1} = {} & \left(\sigma_1 - \frac{a^2_{h(m)h(m-1)}}{\sigma_2} - \cdots - \frac{a^2_{h(m)h(m-r+1)}}{\sigma_r} - 2\right)(\beta^m)^2 \\ & + 2\alpha^m\beta^m + \sigma_2\left(\beta^{m-1} - \frac{\beta^m a_{h(m)h(m-1)}}{\sigma_2}\right)^2 \\ & + \cdots + \sigma_r\left(\beta^{m-r+1} - \frac{\beta^m a_{h(m)h(m-r+1)}}{\sigma_r}\right)^2. \end{aligned} \tag{4.4}$$

We shall choose the σ_s so that $Q_m - Q_{m+1}$ is positive. Suppose there exists $c_s > 0$ such that

$$|a_{h(m)h(m-s+1)}| \leqslant c_s, \qquad 2 \leqslant s \leqslant n, \text{ all } m. \tag{4.5}$$

If $\beta^m \equiv \alpha^m$ we have the constraints

$$\sigma_2 + \cdots + \sigma_r = 1 - \sigma_1 \quad \text{and} \quad \frac{c_2^2}{\sigma_2} + \cdots + \frac{c_r^2}{\sigma_r} \leqslant \sigma_1. \tag{4.6}$$

Let us take $\sigma_s/c_s = (1 - \sigma_1)/c$, where $c = c_2 + \cdots + c_r$. This minimizes the left-hand side of (4.6), which becomes

$$c^2 \leqslant \sigma_1(1 - \sigma_1).$$

In the case $\beta^m \neq \alpha^m$ we require

$$\left(\sigma_1^2 - \frac{c^2}{1 - \sigma_1} - 2\right)(\beta^m)^2 + 2\alpha^m\beta^m \geqslant 0. \tag{4.7}$$

With $\beta^m = \omega\alpha^m$ this becomes

$$(K-2)\omega^2 + 2\omega \geqslant 0$$

and, looking at Fig. 3, we see that this is true for $0 \leqslant \omega \leqslant 2/(2-K)$ and that, the larger the value of K, the larger is the upper limit on ω.

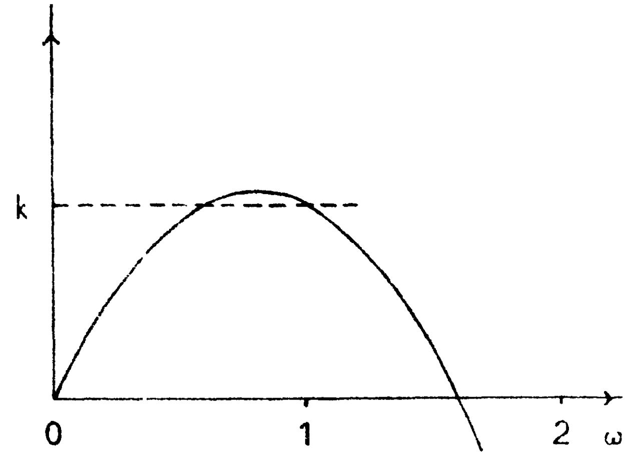

FIG. 3. Bounds for ω.

Looking at the expression for K, we can see that this maximum is attained for $\sigma_1 = 1 - c$ when $K = 1 - 2c$. The value $\omega = 1$ is allowed provided $A \geqslant 0$ or $c \leqslant \frac{1}{2}$. We have the following result, which may be compared with Theorem 2.1.

THEOREM 4.1. *Let A be a positive definite matrix with $a_{ii} = 1$. Let periodic delayed relaxation be used according to the scheme described in this section. If there exist positive constants $c_2, \ldots, c_r$ such that $|a_{h(m)h(m-s+1)}| \leqslant c_s$ for all m, $2 \leqslant s \leqslant r$, then the scheme converges with overrelaxation for $0 \leqslant \omega \leqslant 1/(c + \frac{1}{2})$, where $c = c_2 + \cdots + c_r$.*

Proof. We have just seen that $Q_m \geqslant Q_{m+1} \geqslant 0$, so that Q_m tends to a limit and $Q_m - Q_{m+1}$ tends to zero. If (4.7) is strictly positive, this implies that β^m tends to zero and, by (4.1′), x^m tends to x satisfying, by (4.2′), $Ax = 0$ and thus $x = 0$. This proof fails only when $\omega = 1/(c + \frac{1}{2})$ and $|a_{h(m)h(n-s+1)}| = c_s$, all m, $2 \leqslant s \leqslant r$. In this case (4.4) implies that

$$\gamma^i = \lim_{m\to\infty} \{\beta^m : h(m) = i\}$$

exists and that $\gamma^j = \gamma^i \operatorname{sign}(a_{ij})$ for all pairs i, j such that $h(m) = i$ and $h(m - s + 1) = j$, for some m, s, $2 \leqslant s \leqslant r$. If $\gamma^i \neq 0$, this requires the existence of a sign vector e with $|e_i| = 1$, $1 \leqslant i \leqslant n$, and

$$e_i\gamma^i = |\gamma^i| = \gamma, \qquad 1 \leqslant i \leqslant n.$$

For (4.1′) we have

$$\lim_{m=\infty} x^{m+n} = \lim_{m=\infty} x^m + \gamma e$$

and, from (4.2′),

$$\lim_{m=\infty} Ax^{m+n} = \lim_{m=\infty} Ax^m,$$

which gives $\gamma Ae = 0$ or $\gamma = 0$. Thus β^m tends to zero and the rest of the proof has been given.

Remarks. For $a_{ii} \neq 1$ the condition (4.5) is

$$|a_{h(m)h(m-s+1)}| \leqslant c_s a_{h(m)h(m)}^{1/2} a_{h(m-s+1)h(m-s+1)}^{1/2}.$$

We have proved convergence in the case $\omega = 1/(c + \frac{1}{2})$ only when $r > 1$ and $c > 0$. Ordinary relaxation with $\omega = 2$ does not converge. The conditions on the elements of A may appear somewhat restrictive but, examining the symmetric circulant matrix with $a_{ii} = 1$, $a_{ij} = c_k$, where $k = (i - j + 1)$ (modulo n) for $2 \leqslant k \leqslant r$ and a_{ij} zero otherwise, we see that $c < \frac{1}{2}$ is necessary for A to be positive definite. The case $r = 2$ with $\omega = 1$ and $c = \frac{1}{2}$ was proved as Theorem 4.2 of [1]. Finally we notice that we do not actually require $h(m)$ to take its values periodically. Provided only that $\omega \leqslant 1/(c + \frac{1}{2})$ at each step and (4.5) is satisfied, the process will converge.

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