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A new iterative method for solving linear systems ☆

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Abstract

A new iterative method for solving linear systems is derived. It can be considered as a modification of the Gauss–Seidel method. The modified method can be two times faster than the original one.

Keywords: Linear system; Iterative method; Gauss–Seidel method; Modification

1. Introduction

In recent years a number of authors have considered iterative methods for solving linear systems. In Refs. [1–7] we mentioned only few of published articles.

In this paper, we consider the linear system of equations $Ax = b$, where A is a positive definite matrix of order n and $b \in R^n$ is a given element. As we know, such a system can be solved by the well-known Gauss–Seidel method. This method can be derived such that we seek an approximate solution of the problem

$$f(x) = \frac{1}{2}(Ax, x) - (b, x) \to \inf .$$

We form the sequence

$$x_{k+1} = x_k + \alpha_{k_i} e_i, \quad k = 0, 1, 2, \ldots, \ \ i = 1, 2, \ldots, n,$$

where $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1)$, $\alpha_{k_i} \in R$ and $x_0 \in R^n$ is a given element. The parameters α_{k_i} are chosen such that the method converges. In such a way, the Gauss–Seidel method examine equations of the system $Ax = b$ one at a time in sequence and previously computed results are used as soon as they are available.

Here, we give a new iterative method for solving linear systems. It can be considered as a modification of the Gauss–Seidel method. The modified method updates two components of the approximate solution x_k at the same time. In the modification of the Gauss–Seidel method, considered in this paper, a sequence of approximate solutions is formed by the formula

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$$x_{k+1} = x_k + \alpha_{k_i} e_i + \gamma_{k_i} e_j,$$

where $\gamma_{k_i} \in R$ and j depends on i. As we know, the Gauss–Seidel method requires $\sim 2n^2$ arithmetic operations per one iteration cycle. The modified method requires only $4n$ arithmetic operations more than the Gauss–Seidel method, per one iteration cycle (it does not require $2n^2 + 2n^2$ operations, as we could expect). At the same time, it can be two times faster than the original method, since it updates two components of the approximate solution.

Besides, we consider a general method, i.e. we form the sequence

$$x_{k+1} = x_k + \alpha_{k_i} e_i + \gamma_{k_i} q_j,$$

where $q_j \in R^n$ is an arbitrary element. We also show that this general method gives a better reduction of the error $x_k - x^*$ ($Ax^* = b$) than the Gauss–Seidel method. Of course, the above mentioned modification also give a better reduction of the error.

2. Preliminary results

We consider the linear system of equations:

$$Ax = b, \tag{2.1}$$

where A is a symmetric positive definite matrix of order n and $b \in R^n$ is a given element. We also consider the problem:

$$f(x) = \frac{1}{2}(Ax, x) - (b, x) \to \inf. \tag{2.2}$$

Since $f'(x) = Ax - b$ and $f''(x) = A$, the system (2.1) and the problem (2.2) have the same solution $x^* \in R^n$.

We seek an approximate solution of the problem (2.2). For that purpose, we form a sequence (x_k) by the procedure:

$$
\begin{aligned}
&\text{for} \quad k = 0, 1, 2, \ldots \\
&\quad y_1 = x_k \\
&\text{for} \quad i = 1, 2, \ldots, n \\
&\quad y_{i+1} = y_i + \alpha_i e_i \\
&\qquad \text{end} \quad \text{for } i \\
&\quad x_{k+1} = y_{n+1} \\
&\qquad \text{stopping criteria} \\
&\qquad \text{end} \quad \text{for } k
\end{aligned}
$$

where $x_0 \in R^n$ is a given element, α_i are chosen in a given way and $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1)$.

If we choose

$$\alpha_i = -\frac{p_i}{a_{ii}}, \quad p_i = (a_i, y_i) - b_i, \; i = 1, 2, \ldots, n \tag{2.3}$$

then we get the Gauss–Seidel method.

We have

$$f(y + h) = f(y) + (f'(y), h) + \frac{1}{2}(Ah, h) \tag{2.4}$$

if the function f is given by (2.2). Substituting $y = y_i$, $h = \alpha_i e_i$ in (2.4) we get

$$f(y_i + \alpha_i e_i) = f(y_i) + \alpha_i (Ay_i - b, e_i) + \frac{1}{2}\alpha_i^2 (Ae_i, e_i) = f(y_i) + \alpha_i p_i + \frac{1}{2}\alpha_i^2 a_{ii}. \tag{2.5}$$

From (2.3) and (2.5) we get

$$f(y_{i+1}) - f(y_i) = -\frac{1}{2}\frac{p_i^2}{a_{ii}} \leqslant 0, \tag{2.6}$$

i.e. $f(y_{i+1}) \leqslant f(y_i)$.

On the other hand, we can show that the reduction of f is equivalent to the reduction of the error $g_i = y_i - x^*$. Let us do that.

If A is a symmetric positive definite matrix of order n then A defines a new scalar product in R^n by the formula

$$\langle x, y\rangle = (Ax, y), \quad x, y \in R^n.$$

The corresponding norm is

$$\|x\|_A^2 = (Ax, x), \quad x \in R^n.$$

We have

$$\begin{aligned}\|y_{i+1} - x^*\|_A^2 - \|y_i - x^*\|_A^2 &= (Ay_{i+1} - Ax^*, y_{i+1} - x^*) - (Ay_i - Ax^*, y_i - x^*),\\ &= (Ay_{i+1}, y_{i+1}) - 2(b, y_{i+1}) - [(Ay_i, y_i) - 2(b, y_i)] = 2f(y_{i+1}) - 2f(y_i).\end{aligned}$$

Hence, the reduction of the function f is equivalent to the reduction of the error in the A-norm.

We have

$$\|g_{i+1}\|_A^2 - \|g_i\|_A^2 = -\frac{p_i^2}{a_{ii}} \leqslant 0, \tag{2.7}$$

i.e. $\|g_{i+1}\|_A \leqslant \|g_i\|_A$.

A consequence is the convergence of Gauss–Seidel method.

3. A general method

Let

$$y_{i+1} = y_i + h_i, \quad h_i = -\frac{p_i}{a_{ii}} e_i, \tag{3.1}$$

where

$$p_i = (a_i, y_i) - b_i. \tag{3.2}$$

Let us consider the element

$$z_{i+1} = y_i + h_i + \gamma_i q_i, \tag{3.3}$$

where $\gamma_i \in R$, $q_i \in R^n$. From (2.4) we get

$$\begin{aligned}f(y_i + h_i + \gamma_i q_i) &= f(y_i) + (f'(y_i), h_i + \gamma_i q_i) + \frac{1}{2}(A(h_i + \gamma_i q_i), h_i + \gamma_i q_i)\\ &= f(y_i) + (Ay_i - b, h_i) + \gamma_i (Ay_i - b, q_i) + \frac{1}{2}(Ah_i, h_i) + \gamma_i (Aq_i, h_i) + \frac{1}{2}\gamma_i^2 (Aq_i, q_i)\\ &= f(y_{i+1}) + \gamma_i [(Ay_i - b, q_i) + (Aq_i, h_i)] + \frac{1}{2}\gamma_i^2 (Aq_i, q_i).\end{aligned} \tag{3.4}$$

We now define the function

$$g(\gamma) = \gamma[(Ay_i - b, q_i) + (Aq_i, h_i)] + \frac{1}{2}\gamma^2 (Aq_i, q_i) \tag{3.5}$$

such that

$$\begin{aligned} g'(\gamma) &= (Ay_i - b, q_i) + (Aq_i, h_i) + \gamma(Aq_i, q_i), \\ g''(\gamma) &= (Aq_i, q_i) > 0. \end{aligned}$$

From the equation $g'(\gamma) = 0$ we get

$$\gamma_i = -\frac{(Ay_i - b, q_i) + (Aq_i, h_i)}{(Aq_i, q_i)}. \tag{3.6}$$

From (3.6) and (3.5) we have

$$\begin{aligned} g(\gamma_i) &= -\frac{[(Ay_i - b, q_i) + (Aq_i, h_i)]^2}{(Aq_i, q_i)} + \frac{1}{2}\frac{[(Ay_i - b, q_i) + (Aq_i, h_i)]^2}{(Aq_i, q_i)^2}(Aq_i, q_i) \\ &= -\frac{1}{2(Aq_i, q_i)}[(Ay_i - b, q_i) + (Aq_i, h_i)]^2 \leqslant 0. \end{aligned} \tag{3.7}$$

From (3.3), (3.4) and (3.7) we get

$$f(z_{i+1}) - f(y_i) = f(y_{i+1}) - f(y_i) - \frac{1}{2(Aq_i, q_i)}[(Ay_i - b, q_i) + (Aq_i, h_i)]^2. \tag{3.8}$$

From (3.8) we see that the element z_{i+1} gives a better reduction of the function f than the element y_{i+1}. As we know, this means that the element z_{i+1} gives better reduction of the error than the element y_{i+1}. From (3.8) and (2.7) we get

$$\begin{aligned} &\|z_{i+1} - x^*\|_A^2 - \|y_i - x^*\|_A^2 \\ &= -\frac{p_i^2}{a_{ii}} - \frac{1}{(Aq_i, q_i)}[(Ay_i - b, q_i) + (Aq_i, h_i)]^2. \end{aligned} \tag{3.9}$$

We now describe the modification of the Gauss–Seidel method. We form a sequence (x_k) by the procedure

$$\begin{aligned} \text{for} \quad k &= 0, 1, 2, \ldots \\ z_1 &= x_k \\ \text{for} \quad i &= 1, 2, \ldots, n \\ z_{i+1} &= z_i + h_i + \gamma_i q_i \\ &\text{end} \quad \text{for } i \\ x_{k+1} &= z_{n+1} \\ &\text{end} \quad \text{for } k \end{aligned}$$

where

$$h_i = -\frac{\hat{p}_i}{a_{ii}}, \quad \hat{p}_i = (a_i, z_i) - b_i, \tag{3.10}$$

$$\gamma_i = -\frac{(Az_i - b, q_i) + (Aq_i, h_i)}{(Aq_i, q_i)}. \tag{3.11}$$

Note that q_i can be any element of R^n. However, we choose the elements q_i such that the modified method is effective in applications.

4. A particular method

We choose: $q_i = e_j$, where j depends on i, $j \neq i$. Then we have

$$z_{i+1} = z_i + h_i + \gamma_i e_j, \tag{4.1}$$

$$h_i = -\frac{\hat{p}_i}{a_{ii}} e_i, \quad \hat{p}_i = (a_i, z_i) - b_i, \tag{4.2}$$

$$\gamma_i = -\frac{(Az_i - b, e_j) + (Ae_j, h_i)}{(Ae_j, e_j)} = -\frac{\hat{p}_j - \frac{\hat{p}_i}{a_{ii}} a_{ij}}{a_{jj}} = -\frac{a_{ii}\hat{p}_j - \hat{p}_i}{a_{jj} a_{ii}},$$

i.e.

$$\gamma_i = -\frac{\hat{p}_j}{a_{jj}} - \hat{p}_i \frac{a_{ij}}{a_{ii} a_{jj}}, \tag{4.3}$$

where

$$\hat{p}_j = (a_j, z_i) - b_j. \tag{4.4}$$

We can choose the indices j in different ways. Here, we choose $j = i - 1$, if $i = 2, 3, \ldots, n$ and $j = n$ if $i = 1$. Then we have

$$z_i = z_{i-1} + h_{i-1} + \gamma_{i-1} e_{i-2}, \quad h_{i-1} = -\frac{\hat{p}_{i-1}}{a_{i-1,i-1}} e_{i-1}, \tag{4.5}$$

where $\gamma_0 = \gamma_n$, $e_0 = e_n$, $e_{-1} = e_{n-1}$ and $h_0 = h_n$, $\hat{p}_0 = \hat{p}_n$, $a_{00} = a_{nn}$.

From (4.4) and (4.5) we get

$$\begin{aligned}\hat{p}_j &= (a_j, z_i) - b_j = (a_{i-1}, z_{i-1} + h_{i-1} + \gamma_{i-1} e_{i-2}) - b_{i-1} \\ &= (a_{i-1}, z_{i-1}) - b_{i-1} - \frac{\hat{p}_{i-1}}{a_{i-1,i-1}} (a_{i-1}, e_{i-1}) + \gamma_{i-1}(a_{i-1}, e_{i-2}) = \hat{p}_{i-1} - \frac{\hat{p}_{i-1}}{a_{i-1,i-1}} a_{i-1,i-1} + \gamma_{i-1} a_{i-1,i-2} \\ &= \gamma_{i-1} a_{i-1,i-2},\end{aligned} \tag{4.6}$$

where $a_{0,-1} = a_{n,n-1}$, $a_{1,0} = a_{1,n}$. From (4.3) and (4.6) we get

$$\gamma_i = -\gamma_{i-1} \frac{a_{i-1,i-2}}{a_{i-1,i-1}} + \frac{\hat{p}_{i-1}}{a_{i-1,i-1} a_{ii}} a_{i,i-1}. \tag{4.7}$$

The above results provide the next algorithm.

Algorithm 1. Choose $x_0 = (x^1, \ldots, x^n) \in R^n$ and set $x = x_0$.

for $i = 1, 2, \ldots, n$
$j = i - 1$
$m = i - 2$
if $i = 1$ then
$m = n - 1$
$j = n$
endif
if $i = 2$ then $m = n$
$r_i = -\frac{a_{jm}}{a_{jj}}$
$t_i = \frac{a_{ij}}{a_{ii} a_{jj}}$
end for i
$p_n = (a_n, x) - b_n$
$p_1 = (a_1, x) - b_1$

$\gamma_n = -\frac{p_n}{a_{nn}} + p_1 \frac{a_{1n}}{a_{11}a_{nn}}$
for $k = 0, 1, \ldots$
for $i = 1, 2, \ldots, n$
$j = i - 1$
if $i = 1$ then $j = n$
$p_i = (a_i, x) - b_i$
$x^i = x^i - \frac{p_i}{a_{ii}}$
$\gamma_i = \gamma_j r_i + p_i t_i$
$x^j = x^j + \gamma_i$
end for i
stopping criteria
end for k

We must note that this modification of the Gauss–Seidel method requires only four arithmetic operations (additions and multiplications) per one iteration step more than the original Gauss–Seidel method. On the other hand, it can be two times faster than the Gauss–Seidel method as we can see if we solve the next example.

Example 1. Let the matrix A be given by

$$a_{ii} = 4n, \quad a_{i,i+1} = a_{i+1,i} = n, \quad a_{ij} = 0.5, \quad j \neq i, \ i+1.$$

Let $b = \sum_{k=1}^{n} a_{ik}$ such that $x = (1, 1, \ldots, 1)$ is the exact solution of the system $Ax = b$.

If we choose $n = 1000$, the initial guess $x_0 = (x^1, \ldots, x^n)$, $x^i = 0.001 * i$, $i = 1, 2, \ldots, n$, the stopping criteria $\|x_{k+1} - x_k\| < 0.000001$ and solve the above problem by the Gauss–Seidel method and by the modified method then we shall see that the modified method is two times faster than the Gauss–Seidel method. In fact, the Gauss–Seidel method stops after $k = 11$ iterations and the modified method stops after $k = 5$ iterations. The time of execution for the Gauss–Seidel method is 1.42 s and for the modified method it is 0.632 s.

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