Basic Data Structures: Dynamic Arrays and Amortized Analysis

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Data Structures
Data Structures and Algorithms

Outline

1. Dynamic Arrays
2. Amortized Analysis—Aggregate Method
3. Amortized Analysis—Banker's Method
4. Amortized Analysis—Physicist's Method

Problem: static arrays are static!

```
int my_array[100];
```

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Semi-solution: dynamically-allocated arrays:

```
int *my_array = new int[size];
```

Problem: might not know max size when allocating an array

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Solution: *dynamic arrays* (also known as *resizable arrays*)

Idea: store a pointer to a dynamically allocated array, and replace it with a newly-allocated array as needed.

Definition

Dynamic Array:

Abstract data type with the following operations (at a minimum):

---

*must be constant time

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- PushBack(val): Adds val to the end

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- `Get`(i): returns element at location i^*
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- `PushBack`(val): Adds val to the end
- `Remove`(i): Removes element at location i

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- `Get`(i): returns element at location i*
- `Set`(i, val): Sets element i to val*
- PushBack(val): Adds val to the end
- `Remove`(i): Removes element at location i
- `Size`(): the number of elements

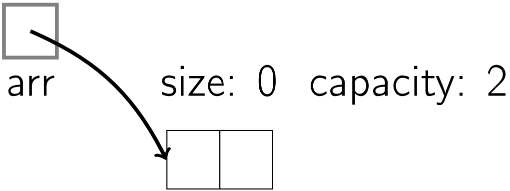
*must be constant time

Implementation

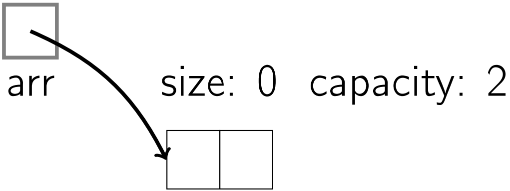
Store:

- `arr`: dynamically-allocated array
- `capacity`: size of the dynamically-allocated array
- `size`: number of elements currently in the array

Dynamic Array Resizing

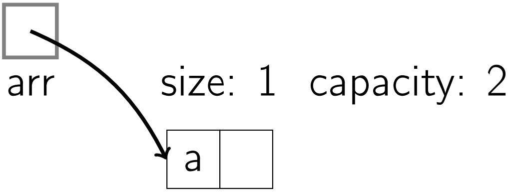


Dynamic Array Resizing



`PushBack(a)`

Dynamic Array Resizing



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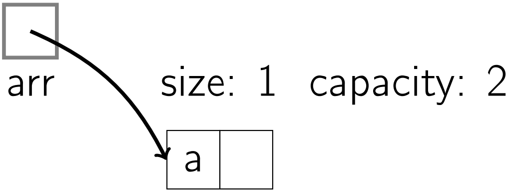

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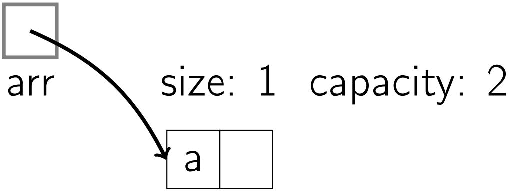
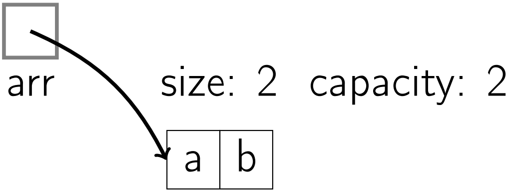


`PushBack(b)`

Dynamic Array Resizing



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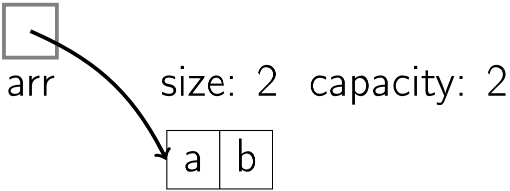

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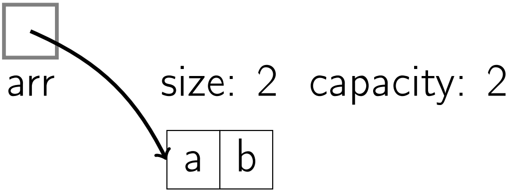
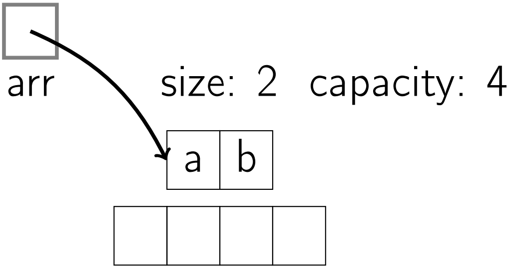


`PushBack(c)`

Dynamic Array Resizing



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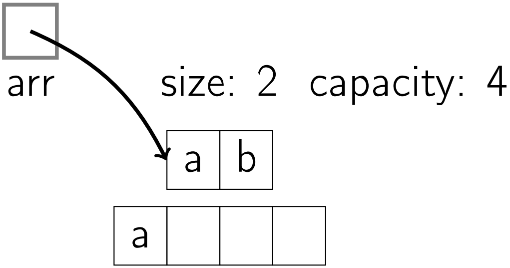

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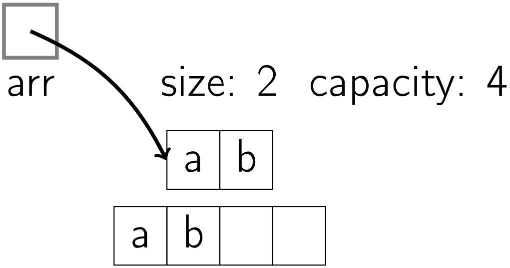

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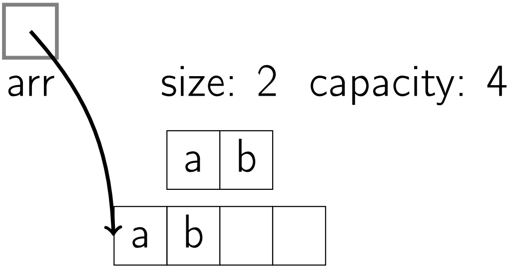

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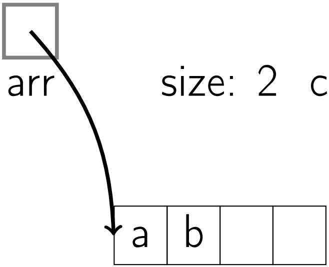

size: 2 capacity: 4

`PushBack(c)`

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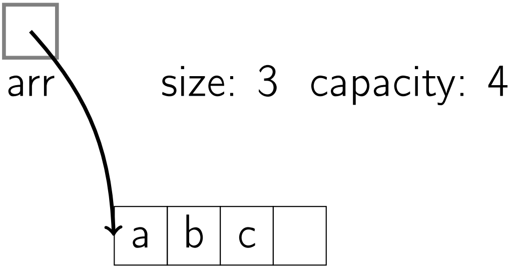

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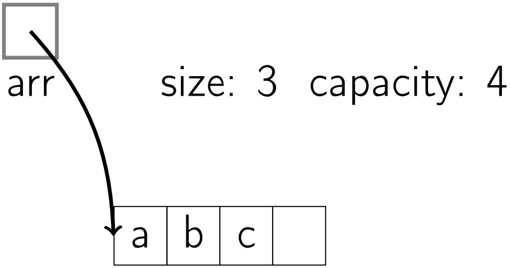

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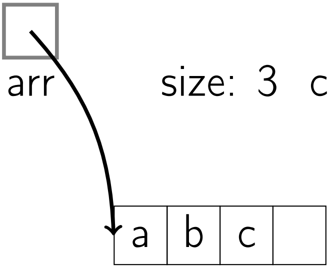
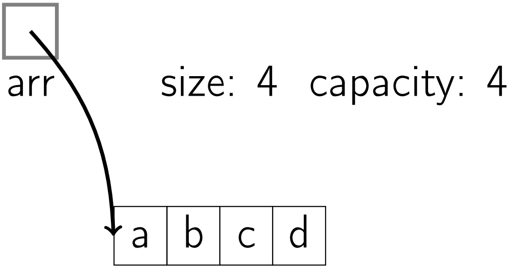


size: 3 capacity: 4

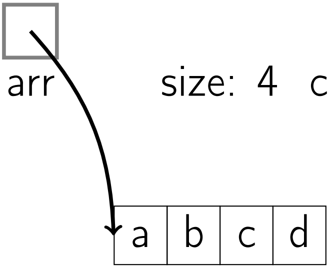
`PushBack(d)`

Dynamic Array Resizing



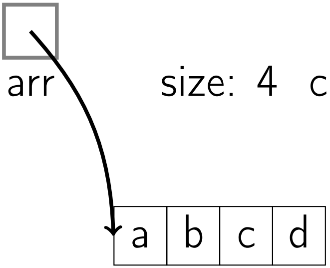
`PushBack(d)`

Dynamic Array Resizing



size: 4 capacity: 4

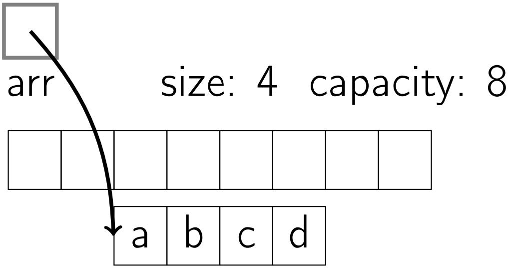
Dynamic Array Resizing



size: 4 capacity: 4

`PushBack(e)`

Dynamic Array Resizing



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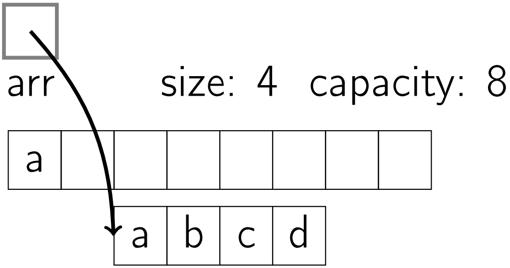

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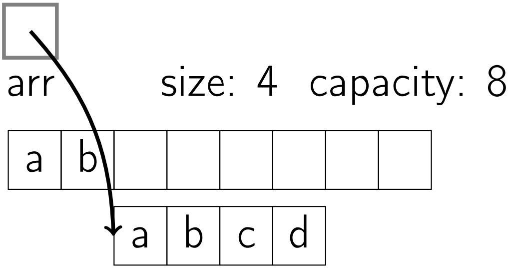

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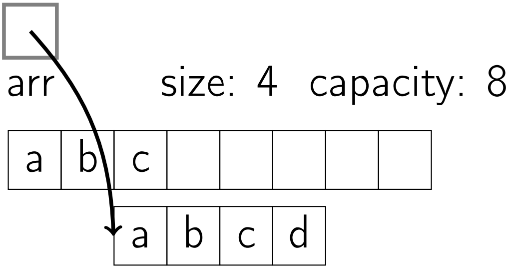

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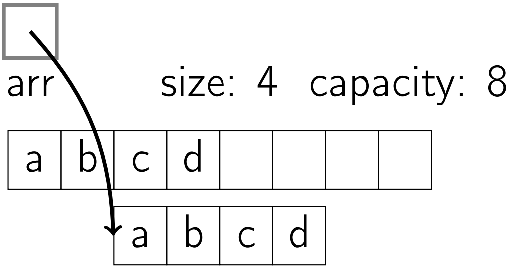

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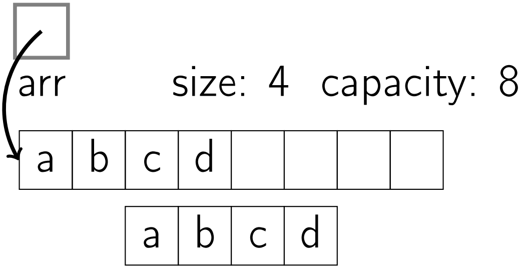

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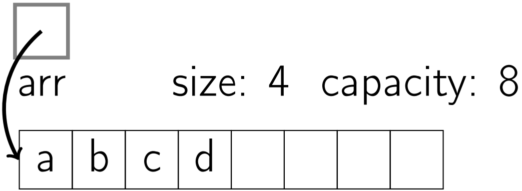

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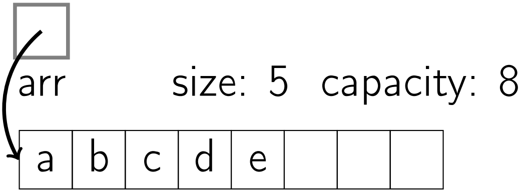


```
PushBack(e)
```

Get(i)

```
if i < 0 or i ≥ size:
  ERROR: index out of range
return arr[i]
```

$\texttt{Set}(i, val)$

```
if i < 0 or i ≥ size:
  ERROR: index out of range
arr[i] = val
```

PushBack(val)

```
if size = capacity:
    allocate new_arr[2 × capacity]
    for i from 0 to size − 1:
        new_arr[i] ← arr[i]
    free arr
    arr ← new_arr; capacity ← 2 × capacity
arr[size] ← val
size ← size + 1
```

Remove(i)

```
if i < 0 or i ≥ size:
  ERROR: index out of range
for j from i to size − 2:
  arr[j] ← arr[j + 1]
size ← size − 1
```

`Size()`

`return` *size*

Common Implementations

- **C++**: `vector`
- **Java**: `ArrayList`
- **Python**: `list` (the only kind of array)

Runtimes

$\texttt{Get}(i)$	$O(1)$

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Runtimes

$\texttt{Get}(i)$	$O(1)$
$\texttt{Set}(i, val)$	$O(1)$
$\texttt{PushBack}(val)$	$O(n)$
$\texttt{Remove}(i)$	$O(n)$
$\texttt{Size}()$	$O(1)$

Summary

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- Unlike static arrays, dynamic arrays can be resized.
- Appending a new element to a dynamic array is often constant time, but can take $O(n)$.
- Some space is wasted—at most half.

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1. Dynamic Arrays
2. Amortized Analysis—Aggregate Method
3. Amortized Analysis—Banker's Method
4. Amortized Analysis—Physicist's Method

Sometimes, looking at the individual worst-case may be too severe. We may want to know the total worst-case cost for a sequence of operations.

Dynamic Array

We only resize every so often.
Many $O(1)$ operations are followed by an $O(n)$ operations.
What is the total cost of inserting many elements?

Definition

Amortized cost: Given a sequence of n operations, the amortized cost is:

$$\frac{\text{Cost}(n \text{ operations})}{n}$$

Aggregate Method

Dynamic array: n calls to `PushBack`

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Dynamic array: n calls to PushBack

Let c_i = cost of i'th insertion.

$$c_i = 1 + \begin{cases} i-1 & \text{if } i-1 \text{ is a power of 2} \\ & \end{cases}$$

Aggregate Method

Dynamic array: n calls to PushBack

Let c_i = cost of i'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a power of 2} \\ 0 & \text{otherwise} \end{cases}$$

Aggregate Method

Dynamic array: n calls to PushBack

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$$\frac{\sum_{i=1}^{n} c_i}{n}$$

Aggregate Method

Dynamic array: n calls to PushBack

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$$c_i = 1 + \begin{cases} i-1 & \text{if } i-1 \text{ is a power of } 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\sum_{i=1}^{n} c_i}{n} = \frac{n + \sum_{j=1}^{\lfloor \log_2(n-1) \rfloor} 2^j}{n}$$

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Dynamic array: n calls to PushBack

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$$c_i = 1 + \begin{cases} i-1 & \text{if } i-1 \text{ is a power of } 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\sum_{i=1}^{n} c_i}{n} = \frac{n + \sum_{j=1}^{\lfloor \log_2(n-1) \rfloor} 2^j}{n} = \frac{O(n)}{n}$$

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$$\frac{\sum_{i=1}^{n} c_i}{n} = \frac{n + \sum_{j=1}^{\lfloor \log_2(n-1) \rfloor} 2^j}{n} = \frac{O(n)}{n} = O(1)$$

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Banker's Method

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- Use the tokens to pay for expensive operations.

Like an amortizing loan.

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- Resize needed: To pay for moving the elements, use the token that's present on each element that needs to move.

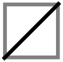
Banker's Method

Dynamic array: n calls to `PushBack`

Charge 3 for each insertion: 1 token is the raw cost for insertion.

- Resize needed: To pay for moving the elements, use the token that's present on each element that needs to move.
- Place one token on the newly-inserted element, and one token $\frac{capacity}{2}$ elements prior.

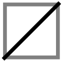
Dynamic Array Resizing



arr

size: 0 capacity: 0

Dynamic Array Resizing



arr

size: 0 capacity: 0

PushBack(a)

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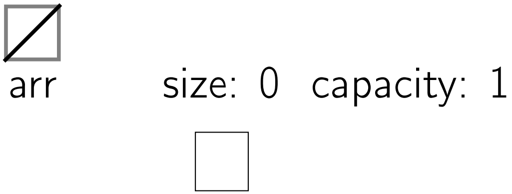

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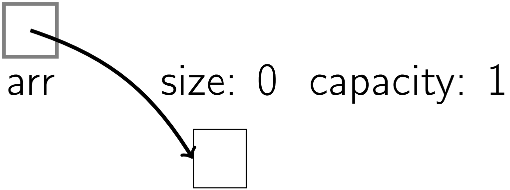

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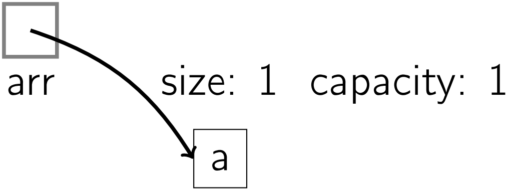

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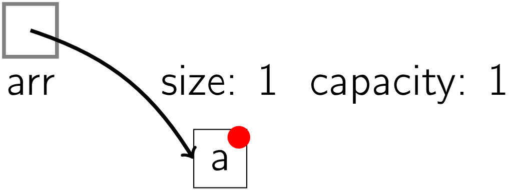

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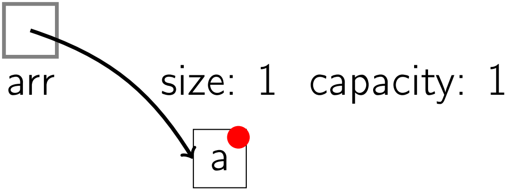

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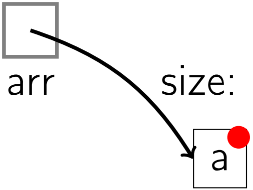
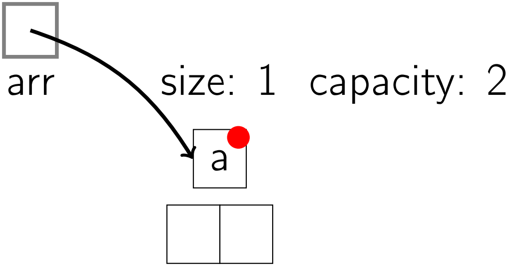


size: 1 capacity: 1

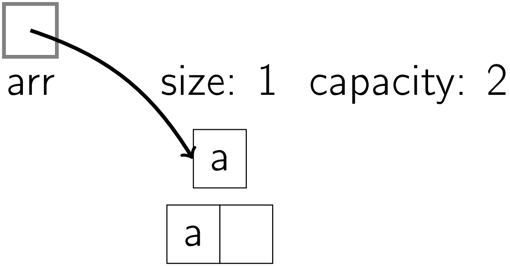
PushBack(b)

Dynamic Array Resizing



PushBack(b)

Dynamic Array Resizing



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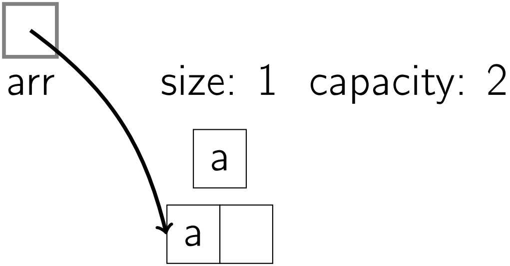

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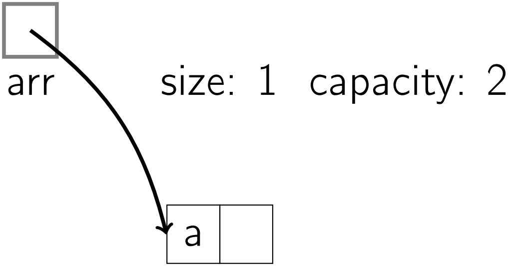

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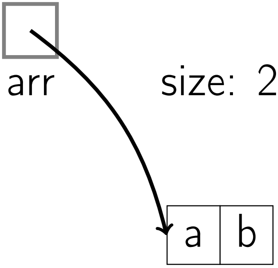

size: 2 capacity: 2

PushBack(b)

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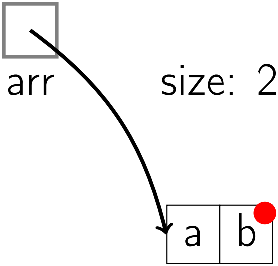

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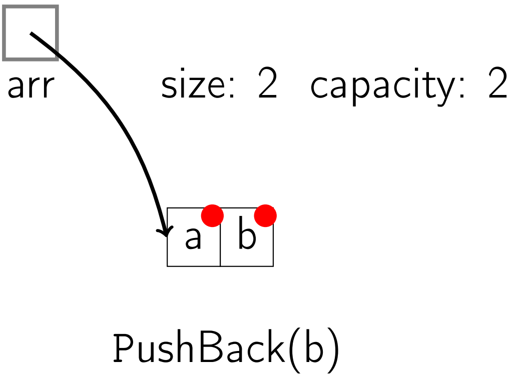

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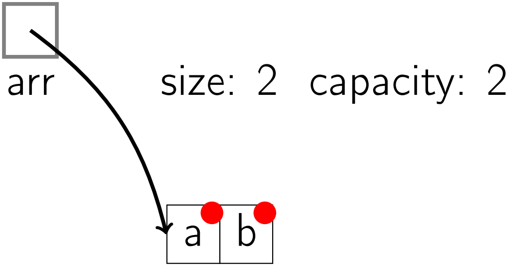

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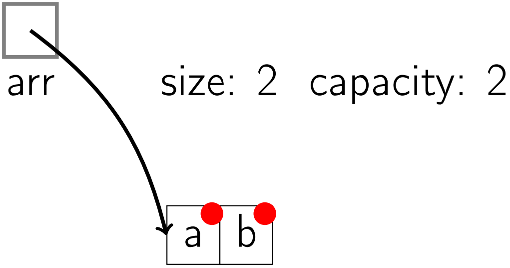
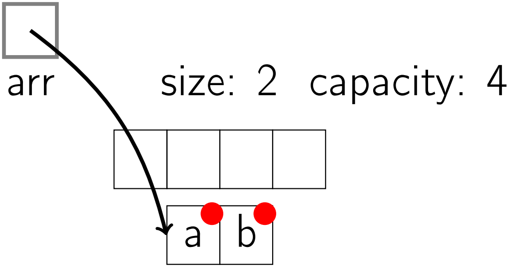


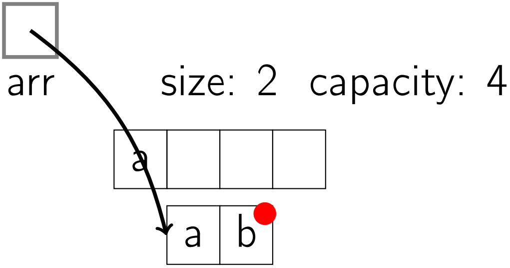
PushBack(c)

Dynamic Array Resizing



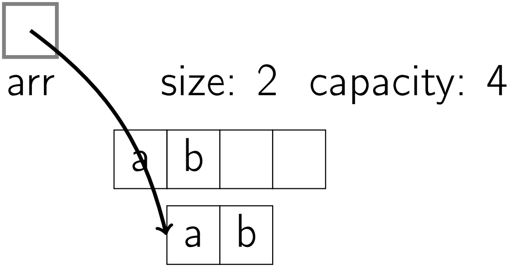
PushBack(c)

Dynamic Array Resizing



PushBack(c)

Dynamic Array Resizing



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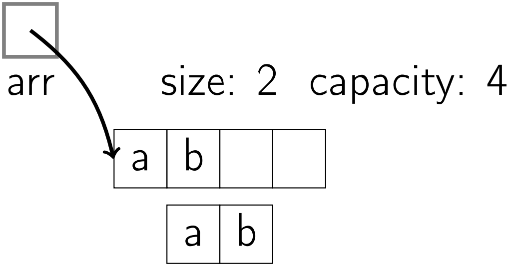
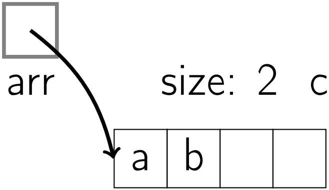
Dynamic Array Resizing



PushBack(c)

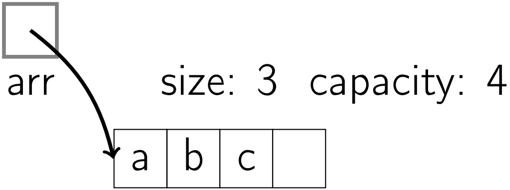
Dynamic Array Resizing



size: 2 capacity: 4

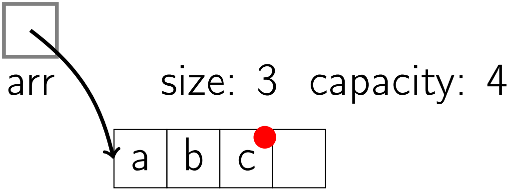
PushBack(c)

Dynamic Array Resizing



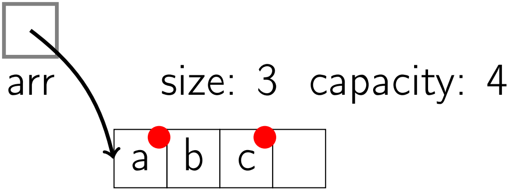
PushBack(c)

Dynamic Array Resizing



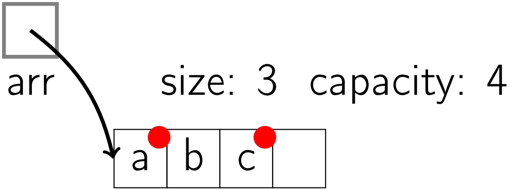
PushBack(c)

Dynamic Array Resizing

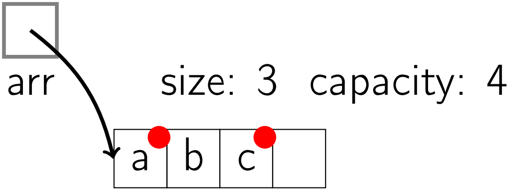


PushBack(c)

Dynamic Array Resizing

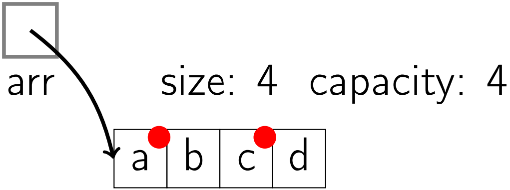


Dynamic Array Resizing



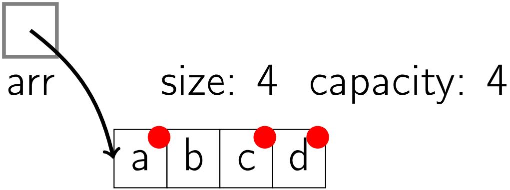
PushBack(d)

Dynamic Array Resizing



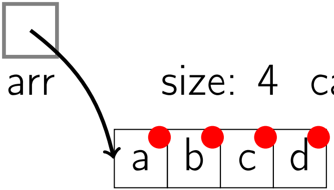
PushBack(d)

Dynamic Array Resizing



PushBack(d)

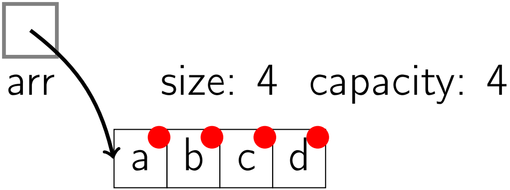
Dynamic Array Resizing



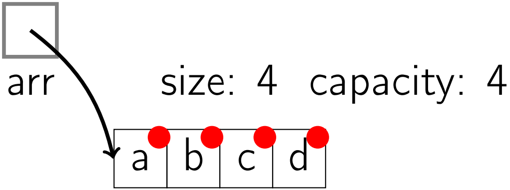
size: 4 capacity: 4

PushBack(d)

Dynamic Array Resizing

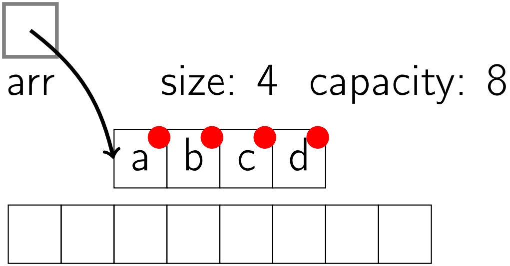


Dynamic Array Resizing



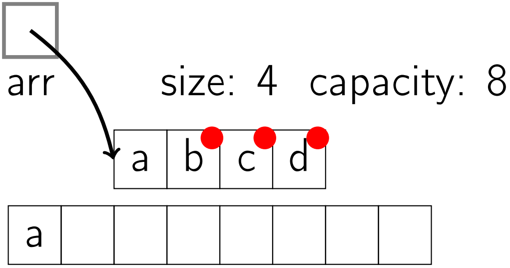
PushBack(e)

Dynamic Array Resizing



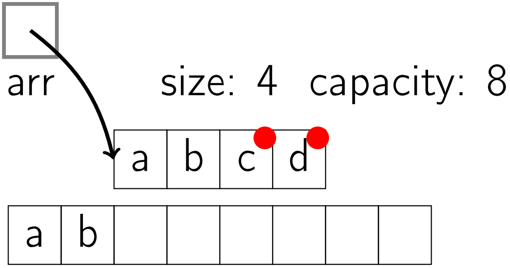
PushBack(e)

Dynamic Array Resizing



PushBack(e)

Dynamic Array Resizing



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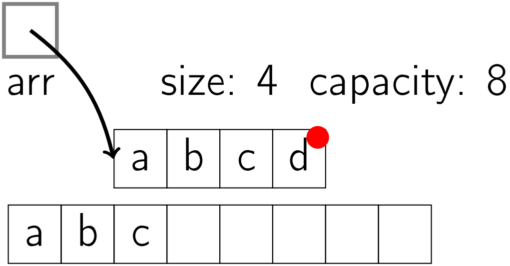

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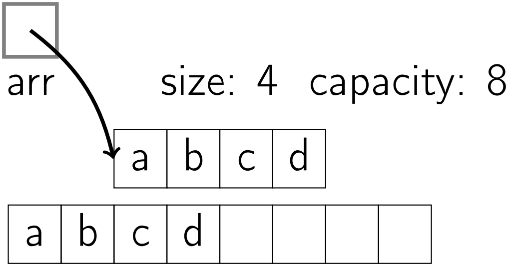

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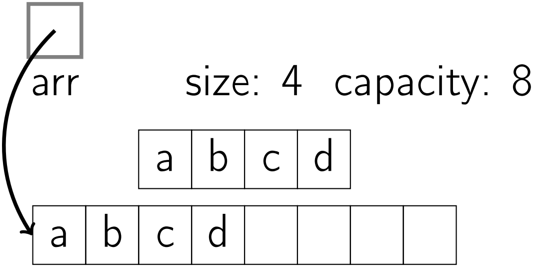

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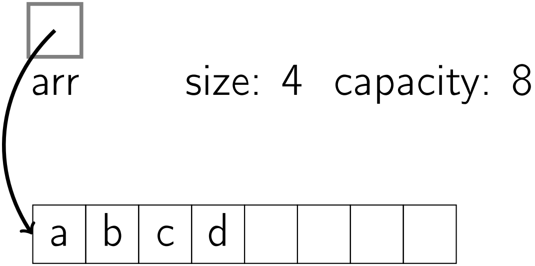
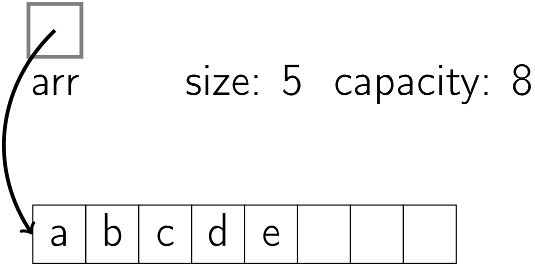


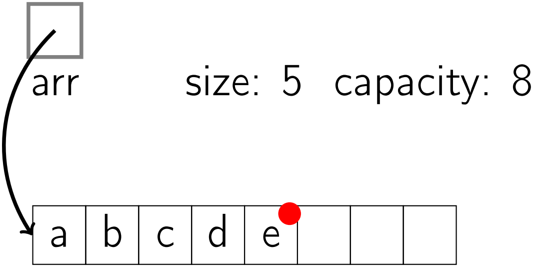
PushBack(e)

Dynamic Array Resizing



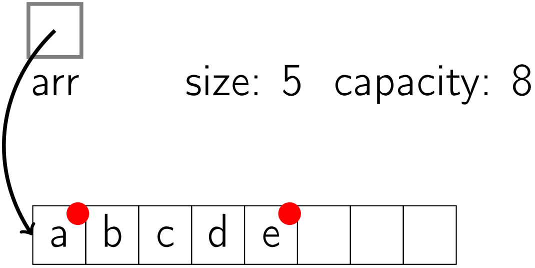
PushBack(e)

Dynamic Array Resizing



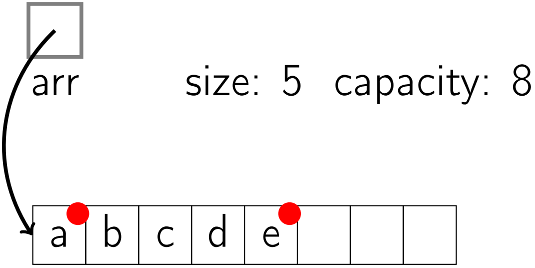
PushBack(e)

Dynamic Array Resizing



PushBack(e)

Dynamic Array Resizing



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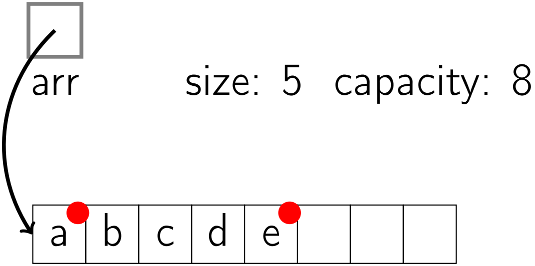

$O(1)$ amortized cost for each `PushBack`

Banker's Method

Dynamic array: n calls to `PushBack`

Charge 3 for each insertion. 1 coin is the raw cost for insertion.

- Resize needed: To pay for moving the elements, use the coin that's present on each element that needs to move.
- Place one coin on the newly-inserted element, and one coin $\frac{capacity}{2}$ elements prior.

Outline

1. Dynamic Arrays
2. Amortized Analysis—Aggregate Method
3. Amortized Analysis—Banker's Method
4. Amortized Analysis—Physicist's Method

Physicist's Method

- Define a *potential function*, Φ which maps states of the data structure to integers:

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  $c_t + \Phi(h_t) - \Phi(h_{t-1})$

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Choose Φ so that:

- if c_t is small, the potential increases
- if c_t is large, the potential decreases by the same scale

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$$\begin{aligned} =&c_1 + \Phi(h_1) - \Phi(h_0) + \\ &c_2 + \Phi(h_2) - \Phi(h_1) \cdots + \\ &c_n + \Phi(h_n) - \Phi(h_{n-1}) \end{aligned}$$

Physicist's Method

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- The sum of the amortized costs is:

$$
\begin{aligned}
&\sum_{i=1}^{n}(c_i + \Phi(h_i) - \Phi(h_{i-1})) \\
=&c_1 + \Phi(h_1) - \Phi(h_0) + \\
&c_2 + \Phi(h_2) - \Phi(h_1) \cdots + \\
&c_n + \Phi(h_n) - \Phi(h_{n-1}) \\
=&\Phi(h_n) - \Phi(h_0) + \sum_{i=1}^{n} c_i
\end{aligned}
$$

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- The sum of the amortized costs is:

$$\sum_{i=1}^{n} (c_i + \Phi(h_i) - \Phi(h_{i-1}))$$

$$= c_1 + \Phi(h_1) - \Phi(h_0) +$$

$$c_2 + \Phi(h_2) - \Phi(h_1) \cdots +$$

$$c_n + \Phi(h_n) - \Phi(h_{n-1})$$

$$= \Phi(h_n) - \Phi(h_0) + \sum_{i=1}^{n} c_i \geq \sum_{i=1}^{n} c_i$$

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Dynamic array: n calls to `PushBack`

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Let $\Phi(h) = 2 \times \mathit{size} - \mathit{capacity}$

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Let $\Phi(h) = 2 \times size - capacity$

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Physicist's Method

Dynamic array: n calls to PushBack

Let $\Phi(h) = 2 \times \mathit{size} - \mathit{capacity}$

- $\Phi(h_0) = 2 \times 0 - 0 = 0$
- $\Phi(h_i) = 2 \times \mathit{size} - \mathit{capacity} > 0$ (since $\mathit{size} > \frac{\mathit{capacity}}{2}$)

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i:

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Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i:

$$c_i + \Phi(h_i) - \Phi(h_{i-1})$$

$$= 1 + 2 \times size_i - cap_i - (2 \times size_{i-1} - cap_{i-1})$$

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i:

$$
\begin{aligned}
& c_i + \Phi(h_i) - \Phi(h_{i-1}) \\
= & 1 + 2 \times \mathit{size}_i - \mathit{cap}_i - (2 \times \mathit{size}_{i-1} - \mathit{cap}_{i-1}) \\
= & 1 + 2 \times (\mathit{size}_i - \mathit{size}_{i-1})
\end{aligned}
$$

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i:

$$\begin{aligned}
& c_i + \Phi(h_i) - \Phi(h_{i-1}) \\
=& 1 + 2 \times size_i - cap_i - (2 \times size_{i-1} - cap_{i-1}) \\
=& 1 + 2 \times (size_i - size_{i-1}) \\
=& 3
\end{aligned}$$

Dynamic Array Resizing

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Then:

$\Phi(h_{i-1}) = 2size_{i-1} - cap_{i-1} = 2k - k = k$

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Then:

$$\Phi(h_{i-1}) = 2size_{i-1} - cap_{i-1} = 2k - k = k$$

$$\Phi(h_i) = 2size_i - cap_i = 2(k+1) - 2k = 2$$

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Amortized cost of adding element i:

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Amortized cost of adding element i:

$$
\begin{aligned}
&c_i + \Phi(h_i) - \Phi(h_{i-1}) \\
=&(size_i) + 2 - k
\end{aligned}
$$

Dynamic Array Resizing

With resize when adding element i

Let $k = size_{i-1} = cap_{i-1}$

Then:

$\Phi(h_{i-1}) = 2size_{i-1} - cap_{i-1} = 2k - k = k$

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Amortized cost of adding element i:

$$\begin{aligned} & c_i + \Phi(h_i) - \Phi(h_{i-1}) \\ =& (size_i) + 2 - k \\ =& (k+1) + 2 - k \end{aligned}$$

Dynamic Array Resizing

With resize when adding element i

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Then:

$\Phi(h_{i-1}) = 2size_{i-1} - cap_{i-1} = 2k - k = k$

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Amortized cost of adding element i:

$$
\begin{aligned}
& c_i + \Phi(h_i) - \Phi(h_{i-1}) \\
=& (size_i) + 2 - k \\
=& (k+1) + 2 - k \\
=& 3
\end{aligned}
$$

Alternatives to Doubling the Array Size

We could use some different growth factor (1.5, 2.5, etc.).
Could we use a constant amount?

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i'th insertion.

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$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a multiple of } 10 \\ & \end{cases}$$

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If we expand by 10 each time, then:

Let c_i = cost of i'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a multiple of 10} \\ 0 & \text{otherwise} \end{cases}$$

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$$\frac{\sum_{i=1}^{n} c_i}{n}$$

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$$\frac{\sum_{i=1}^{n} c_i}{n} = \frac{n + \sum_{j=1}^{(n-1)/10} 10j}{n}$$

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$$= \frac{n + 10O(n^2)}{n}$$

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$$= \frac{n + 10O(n^2)}{n} = \frac{O(n^2)}{n} = O(n)$$

Summary

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Summary

- Calculate amortized cost of an operation in the context of a sequence of operations.
- Three ways to do analysis:
    - Aggregate method (brute-force sum)
    - Banker's method (tokens)
    - Physicist's method (potential function, Φ)
- Nothing changes in the code: runtime analysis only.