A Study on *Preconditioning Sparse Matrices with Alternating and Multiplicative Operator Splittings* *

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1 SUMMARY

1.1 PROBLEM SETUP Given a linear system $Ax = b$, where A is an $n \times n$ real matrix, we can derive the basic iterative method with the matrix splitting $A = M + (A - M)$, where M is an invertible matrix. Then it could lead to the iterative defect correction process (iterative methods to solve linear systes) with the initial value $x^0 := 0$ and $k = 0, ..., k_{last} - 1$ as the following:

$$\begin{aligned} x^{k+1} &= M^{-1}(b - (A - M)x^k) \\ &= x^k + M^{-1}(b - Ax^k) \\ &= Gx^k + M^{-1}b, \quad G := I - M^{-1}A \end{aligned}$$

This iteration could alternate by using different splitting such as $\overline{M}_0, ..., \overline{M}_{m-1}$ which could result in the repetition of the alternating corrections as the following:

$$\begin{aligned} x^{mk+1} &= x^{mk+0} + \overline{M}_0^{-1}(b - Ax^{mk+0}) \\ x^{mk+2} &= x^{mk+1} + \overline{M}_1^{-1}(b - Ax^{mk+1}) \\ &\vdots \\ x^{mk+m} &= x^{mk+m-1} + \overline{M}_{m-1}^{-1}(b - Ax^{mk+m-1}) \end{aligned}$$

Since we try to find a way to express this set of equations just by using one single equation. Then the preconditioner, $\overline{M}_{ALT-i}$ was introduced and it has the form as the following:

$$\overline{M}_{ALT-i} = A(I - \overline{G}_{ALT-i})^{-1}, \quad \overline{G}_{ALT-i} = \prod_{l=m-1}^{0} (I - \overline{M}_l^{-1}A)$$

Then, we have $x^{k+1} = x^k + \overline{M}_{ALT-i}^{-1}(b - Ax^k)$. However, it has the shortcoming which is that it requires $m - 1$ additional multiplication with A but we don't want to see that. We need to find a new splitting, M_{MOS}, which is defined as the following:

$$M_{MOS} = T \prod_{l=m-1}^{0} M_l$$

Then we have $x^{k+1} = x^k + M_{MOS}^{-1}(b - Ax^k)$.

2 MOTIVATION

2.1 IMPLICIT LU METHOD (ILU) The multiplicative operater splitting method in the paper is claimed to be better than ADI and ILU methods because it generalizes the characteristics of these methods and offer much more flexibility in preconditioner construction. Here we give a very brief conclusion of the ILU and ADI method:

ILU is an approximate factorization method used as a preconditioner for solving large, sparse linear systems of equations from PDE problems. As indicated in its name, it is based on the LU factorization, where a given matrix A is decomposed into a lower triangular matrix L and an upper triangular matrix U. There are certainly downsides for this method, for example, the decomposition is not exact, and the splitting only produce two matrices L and U.

2.2 ALTERNATING DIRECTION IMPLICIT METHOD (ADI) ADI is an iterative method used to solve multidimensional PDE problems, for example, heat equation and wave equation which we are really familiar. The idea of the method is solving the PDE along one coordinate direction, for example, x, at a time while keeping other directions fixed. Then the method performs an alternation between difference coordinate directions. In short, this ADI method is basically reducing a multidimensional problem into a series of one dimensional problem. However, it also has downsides. For example, the ADI framework requires banded matrices with low bandwidth. This forces a lot of multidimensional PDE problems to be out of the party since not all problems will satisfy that requirement.

The MOS method proposed in the paper aims to solve the disadvantages of these two methods.

3 IMPLEMENTATION

3.1 DIRECT CONSTRUCTION

$$E = \frac{1}{m} T^{-1}(diag(A) - T) \leq 0, \quad T(I + mE) = diag(A),$$

$$J = I + E, \quad \frac{m-1}{m} I \leq J \leq I,$$

$$M_l' = (TE + A_l'), \quad A_l' = J^{-(m-1-l)} A_l'' J^{-l},$$

$$M_l = (I + T^{-1} M_l') = (J + T^{-1} A_l')$$

Here, M_l is the factor which is defined based on some off-diagonal matrices $A_0'', ..., A_{m-1}''$ and on an invertible diagonal matrix T with $unit(T) = unit(diag(A))$ and $|T| \geq |diag(A)|$.

The MOS-d preconditioner can be expressed as:

$$\begin{aligned} M_{MOS-d} &= T \prod_{l=m-1}^{0} (I + T^{-1} M_l') \\ &= T \prod_{l=m-1}^{0} (J + T^{-1} A_l') \end{aligned}$$

3.2 ADAPTIVE CONSTRUCTION

Similar to direct construction, it's also a multiplicative setting:

$$M_{MOS-a} = T \prod_{l=m-1}^{0} M_l$$

Here, M_l is constructed adaptively with immediate pruning as following:

$$B_0 = prune(A, S_0^B), \quad M_0 = prune(B_0, S_0^M)$$

$$B_1 = prune(B_0 M_0^{-1}, S_1^B), \quad M_1 = prune(B_1, S_1^M)$$

$$\vdots$$

$$B_{m-1} = prune(B_{m-2}M_{m-2}^{-1}, S_{m-1}^{B}), \qquad M_{m-1} = prune(B_{m-1}, S_{m-1}^{M})$$

$$B_m = prune(B_{m-1}M_{m-1}^{-1}, S_m^{B}), \qquad T = diagp_1(B_m)$$

where $prune(A, S)$ and $diagp1(A)$ are helper functions defined in the reference paper.

4 EXPERIMENTS

Our eventual goal is to reproduce the results in the paper, which is to show that as m increases, the relative residue norm will decrease with a faster rate for both the MOS direct construction and adaptive construction. Figure 1 below is the result produced by the direct construction. As we can see, with increasing matrices in the multiplication process, the relative residue norm indeed reduces faster. The value of the relative residue norm is also smaller for larger m.

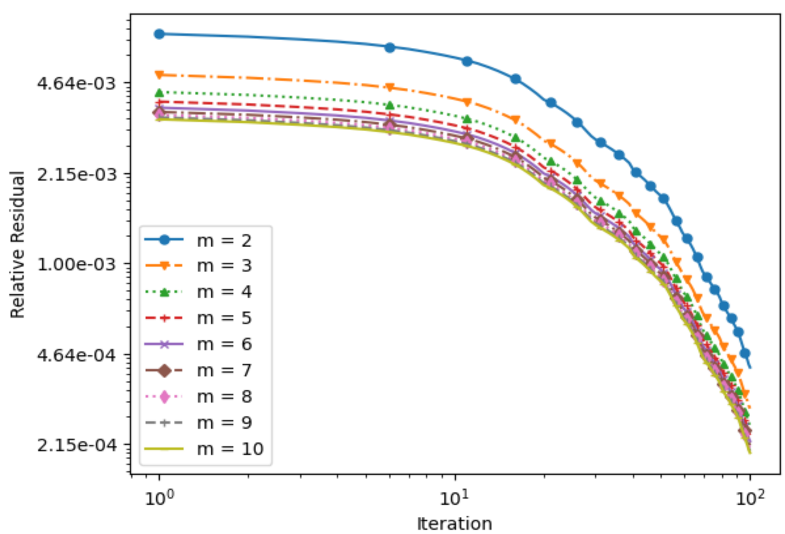

Figure 1: $M_{mos-direct}$ for $2D$ Five Point Stencil
Param.: $n = 100$

However, for the adaptive construction, we are restrained by the CPU power so that discrimination between different m is very hard to visualize even though we have $n = 75$. For larger n, the program will not be stable on the CPU. In the paper, the authors performed the experiments under a parallel computing setting using the GPU with n equals millions. In that case, a superior performance on larger n can be concluded.

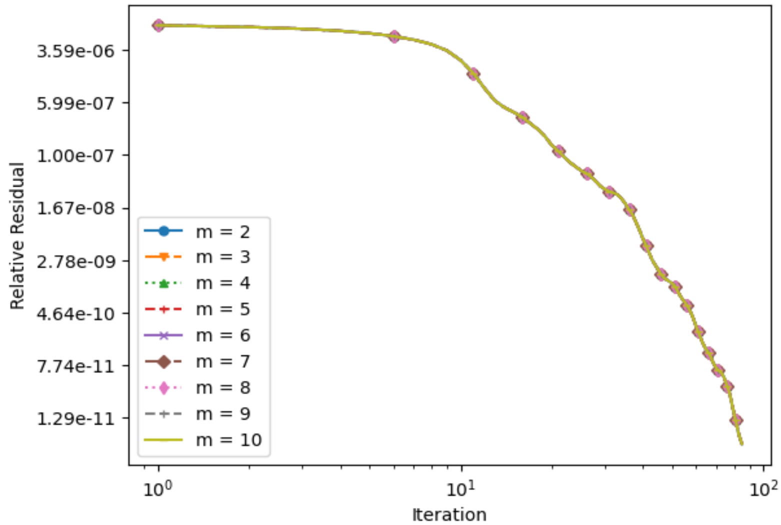

Figure 2: $M_{mos-adaptive}$ for $2D$ Five Point Stencil
Param.: $n = 75$

5 Conclusion

This paper introduced a new operator splitting method to solve problems with large, sparse matrices. The new multiplicative operator splitting method generalizes the Implicit LU method and Alternating Direction Implicit method. Out of the four implementations in the paper, we implemented and tested the direct and adaptive construction for general sparse matrices. The results were aligning with the conclusions in the paper. However, many more aspects need to be further investigated. For example, in the direct construction, the off-diagonal matrices should be problem specific. A general pattern for which off-diagonal matrices should be implement for which kinds of problems need to be derived theoretically. Moreover, the MOS method seems to be unstable for some specific problems, for example, diagonally dominant problems, which the performance is not as good as the traditional ILU method. When to implement the MOS method, and how to specify free parameters during construction are pending investigations.

REFERENCES

1. Christoph Klein, Robert Strzodka. *Preconditioning Sparse Matrices with Alternating and Multiplicative Operator Splittings*. SIAM J. Sci. Comput. Vol. 45, No. 1, pp. A25-A48. 2023.