Preconditioning Sparse Matrices with Alternating and Multiplicative Operator Splittings[1]

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Part I: Theory & Construction

Our Common Goal: Try to Solve A**x** = **b**
(A is an n x n matrix, **b** is an n x 1 vector)

- Basic Iterative Method: A = M + (A - M), where M is an invertible matrix. It leads to the iterative defect correction with $\boldsymbol{x}^0 := 0, \mathrm{k} = 0, \dots, k_{last} - 1$

$$\begin{aligned} \boldsymbol{x}^{k+1} &:= M^{-1}\left(\boldsymbol{b} - (A - M)\boldsymbol{x}^k\right) \\ &:= \boldsymbol{x}^k + M^{-1}(\boldsymbol{b} - A\boldsymbol{x}^k) \\ &:= G\boldsymbol{x}^k + M^{-1}\boldsymbol{b}, \quad G := I - M^{-1}A. \end{aligned}$$

- This iteration could alternate between different splittings $\overline{M}_0, \dots, \overline{M}_{m-1}$, resulting in the repetition (from k = 0 to $k_{last} - 1$) of the alternating corrections as shown in the following.

$$\begin{aligned} \boldsymbol{x}^{mk+1} &:= \boldsymbol{x}^{mk+0} + \overline{M}_0^{-1}\left(\boldsymbol{b} - A\boldsymbol{x}^{mk+0}\right) \\ &\vdots \\ \boldsymbol{x}^{mk+m} &:= \boldsymbol{x}^{mk+m-1} + \overline{M}_{m-1}^{-1}\left(\boldsymbol{b} - A\boldsymbol{x}^{mk+m-1}\right) \text{ with } alternating\ splittings\ \overline{M}_0, \dots, \overline{M}_{m-1}. \end{aligned}$$

Try to Reduce the Operational Cost

- By using the appropriate preconditioner $\overline{M}_{\mathrm{ALT}-i}$, we are able to express the alternating corrections as a single correction as the following:

  $$\overline{M}_{\mathrm{ALT}-i} := A\left(I - \overline{G}_{\mathrm{ALT}-i}\right)^{-1}, \quad \overline{G}_{\mathrm{ALT}-i} := \prod_{l=m-1}^{0}\left(I - \overline{M}_l^{-1}A\right),$$

  $$\boldsymbol{x}^{k+1} := \boldsymbol{x}^k + \overline{M}_{\mathrm{ALT}-i}^{-1}(\boldsymbol{b} - A\boldsymbol{x}^k)$$

  However, $\overline{M}_{\mathrm{ALT}-i}$ has a disadvantage which is that it requires m - 1 additional multiplications with A (It is expensive!). Hence, in addition to the alternating splittings as we introduced above like $\overline{M}_l$, we need to introduce a new splittings which is called *multiplicative operator splittings* (MOS). The definition as the following:

  $$M_{\mathrm{MOS}} := T \prod_{l=m-1}^{0} M_l$$

  At here, T is an invertible diagonal matrix and M_l are some invertible matrices.
- Advantage: The application of all $M_l(A)$ doesn't require additional multiplication with A (It is cheaper!).

  $$\boldsymbol{x}^{k+1} := \boldsymbol{x}^k + M_{\mathrm{MOS}}^{-1}(\boldsymbol{b} - A\boldsymbol{x}^k)$$

Outer Solver: GMRES

- We can also use the operator splitting preconditioner with GMRES as an outer solver to achieve faster convergence.

$$\boldsymbol{x}^{k+1} := \mathrm{GMRESstep}^{k}(A, M)$$

Similarly, we can also apply different splittings $\overline{M}_l$ in alternating iterations of FGMRES (A GMRES variant which could support different preconditioners).

$$\begin{aligned}\boldsymbol{x}^{km+1} &:= \mathrm{FGMRESstep}^{km+0}(A, \overline{M}_l)\\ &\vdots\\ \boldsymbol{x}^{km+m} &:= \mathrm{FGMRESstep}^{km+m-1}(A, \overline{M}_{m-1})\end{aligned}$$

- Combining those two GMRES iterations with the alternating and multiplicative preconditioners, we can get four different approaches which we will introduce later.

ADI Method

- The ADI method is about elliptic and parabolic PDEs on a 2D tensor product grid, where the elliptic operator splits into two direction:

  continuous Laplace $\Delta = \partial^2/\partial Y^2 + \partial^2/\partial X^2$

  discrete Laplace $A = L_1 + L_0$ with tridiagonal L_1, L_0, which allow fast inversion

- Then **Ax = b** can be solved by iterative method with $\mathbf{x}_0 = 0$, $k = 0, \ldots, k_{last} - 1$ as the following:

$$(L_1 + \rho_k I)\boldsymbol{x}^{k+\frac{1}{2}} := -(L_0 - \rho_k I)\boldsymbol{x}^k + \boldsymbol{b}$$

$$(L_0 + \rho_k I)\boldsymbol{x}^{k+1} := -(L_1 - \rho_k I)\boldsymbol{x}^{k+\frac{1}{2}} + \boldsymbol{b},$$

with alternating acceleration factors $\rho_k > 0$

- This is the same as the iterative defect correction process in the previous slide with:

$$\begin{aligned} G &:= (L_0 + \rho_k I)^{-1}(L_1 - \rho_k I)(L_1 + \rho_k I)^{-1}(L_0 - \rho_k I) \\ &:= (L_0 + \rho_k I)^{-1}(L_1 + \rho_k I)^{-1}(L_1 - \rho_k I)(L_0 - \rho_k I) \end{aligned}$$

$$\begin{aligned} M &:= A(I - G)^{-1} = \frac{1}{2\rho_K}(L_1 + \rho_k I)(L_0 + \rho_k I) \\ &= (T + \left(L_1 - \frac{T}{m}\right))T^{-1}(T + \left(L_0 - \frac{T}{m}\right)) \end{aligned}$$

It is a special case $(m = 2, T = 2\rho_k I)$ of our MOS

ILU Factorization

- In our MOS method, the invertible diagonal matrix T appears only on the left of MOS splitting, but we can express the matrix factor M_l in a form which could highlight the symmetry of the MOS w.r.t. T.

$$M_l \coloneqq (I + T^{-1}M_l') = T^{-1}(T + M_l')$$

$$M_{\mathrm{MOS}} \coloneqq T \prod_{l=m-1}^{0} M_l = (T + M_{m-1}')T^{-1}(T + M_{m-2}')T^{-1} \cdots T^{-1}(T + M_0')$$

- Generalization of ILU:

  Setting m = 2, M_1' strictly lower triangular and M_0' strictly upper triangular. Then, we can generalize ILU as the special case of MOS splitting as the following:

$$M_{\mathrm{MOS}} = (T + M_1')T^{-1}(T + M_0') = (I + M_1'T^{-1})(T + M_0')$$

- Therefore, the MOS preconditioners could be used to generalize ILU to multiple factors (m > 2) and more general form of factors as shown above.

Diagonal Preconditioners

$$(\mathrm{diagp}_0(A))_{i,i} := \left(\mathrm{diag}(A)\right)_{i,i} = A_{i,i},$$

$$(\mathrm{diagp}_1(A))_{i,i} := \mathrm{unit}(A_{i,i})\max(|A_{i,i}|, \textstyle\sum_{j:j\neq i} |A_{i,j}|),$$

$$(\mathrm{diagp}_2(A))_{i,i} := \mathrm{unit}(A_{i,i}) \textstyle\sum_j |A_{i,j}|,$$

$$(\mathrm{diagp}_{l1}(A))_{i,i} := A_{i,i} \textstyle\sum_{j:j\neq i} |A_{i,j}|,$$

$$\mathrm{unit}(x) = \begin{cases} 1 & \text{If x=0} \\ x/|x| & \text{else.} \end{cases}$$

Sparsity Patterns

The spasticity pattern is the index set of the nonzero coefficients in the matrix. Let's first define some sparsity patterns:

$$S^{\text{diag}} := \{(i,j) \mid i = j\}, \qquad S^{\text{tri}} := \{(i,j) \mid |i-j| \leq 1\},$$

$$S^{\text{low}} := \{(i,j) \mid i \geq j\}, \qquad S^{\text{low}-s} := \{(i,j) \mid i > j\},$$

$$S^{\text{max-n}}(A, q) := \{(i,j) \mid j \in \bar{S}_i^{\max-n}(A, q_i)\}$$

Then we want to scale coefficients in A based on our sparsity pattern:

$$(\text{prune}(\mathrm{A}, \mathrm{S}))_{i,j} = \begin{cases} A_{i,j} & \text{if } (i,j) \in S, \\ 0 & else, \end{cases}$$

$$(\text{scale}(\mathrm{A}, \mathrm{S}, \omega))_{i,j} = \begin{cases} \omega A_{i,j} & \text{if } (i,j) \in S, \\ A_{i,j} & else. \end{cases}$$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{bmatrix}$$

Direct Construction

$$E := \frac{1}{m} T^{-1}(\mathrm{diag}(A) - T) \leq 0, \qquad T(I + mE) = \mathrm{diag}(A),$$

$$J := I + E, \qquad \frac{m-1}{m} I \leq J \leq I,$$

$$M_l' := (TE + A_l'), \qquad A_l' := J^{-(m-1-l)} A_l'' J^{-L},$$

$$M_l := (I + T^{-1} M_l') = (J + T^{-1} A_l')$$

Here, M_l is factors defined based on some off-diagonal matrices $A_0'', \dots, A_{m-1}''$ and on an invertible diagonal matrix T with $\mathrm{unit}(T) = \mathrm{unit}(\mathrm{diag}(A))$ and $|T| \geq |\mathrm{diag}(A)|$

Then

$$I \leq J^{-m} \leq 4I, \qquad 0 \leq J^m - (I + mE) \leq \binom{m}{2} E^2 \leq \frac{1}{2}(mE)^2$$

Direct Construction (Continued)

Now we have the MOS-d preconditioner as a permutation of the matrix A.

$$M_{\text{MOS\_d}} := T \prod_{l=m-1}^{0} (I + T^{-1} M_l') = T \prod_{l=m-1}^{0} (J + T^{-1} A_l')$$

$$= TJ^m + \textstyle\sum_{l=0}^{m-1} J^{m-1-l} A_l' J^l + R_2$$
$$= A + R, \qquad R := R_0 + R_1 + R_2,$$

$$R_2 := \textstyle\sum_{l_1,l_2=0;l_1>l_2}^{m-1} A_{l_1}'' J^{-m} T^{-1} A_{l_2}'' + R_{\text{tail}}(||T^{-1}||^2),$$

$$R_1 = \left(\text{diag}(A) + \textstyle\sum_{l=0}^{m-1} A_l''\right) - A,$$

$$R_0 = T\left(J^m - (I + mE)\right), \; ||R_0|| \le \frac{1}{2} ||T|| \, ||mE||^2.$$

Adaptive Construction

Let's start with MOS:

$$M_{\mathrm{MOS}-a} := T \prod_{l=m-1}^{0} M_l \,,$$

Except for now, M_l are constructed adaptively with immediate pruning (scaling)

$$B_0 := \mathrm{prune}(A, S_0^B), \qquad M_0 := \mathrm{prune}(B_0, S_0^M),$$

$$B_1 := \mathrm{prune}(B_0 M_0^{-1}, S_1^B), \qquad M_1 := \mathrm{prune}(B_1, S_1^M),$$

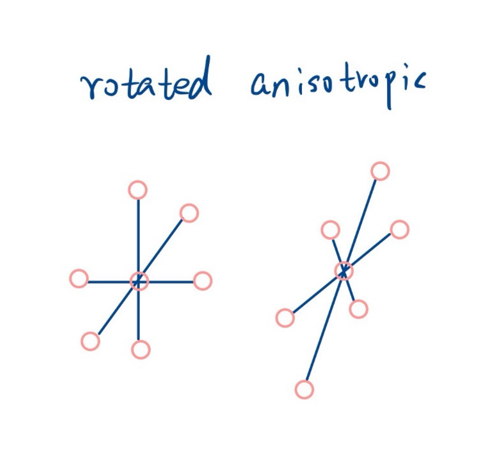
$$\vdots$$

$$B_{m-1} := \mathrm{prune}(B_{m-2} M_{m-2}^{-1}, S_{m-1}^B), \qquad M_{m-1} := \mathrm{prune}(B_{m-1}, S_{m-1}^M),$$

$$B_m := \mathrm{prune}(B_{m-1} M_{m-1}^{-1}, S_m^B), \qquad T := \mathrm{diagp}_1(B_m),$$

$$S_l^B \supseteq S_l^M .$$

Part II: Solver & Result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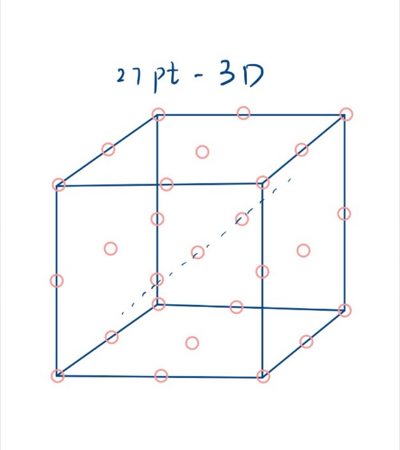
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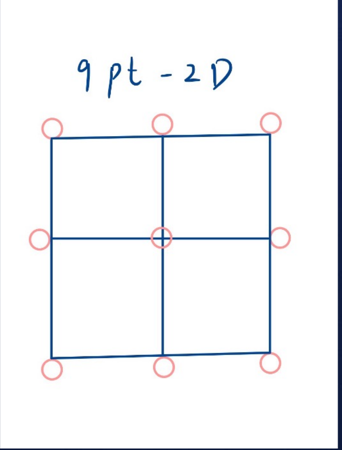
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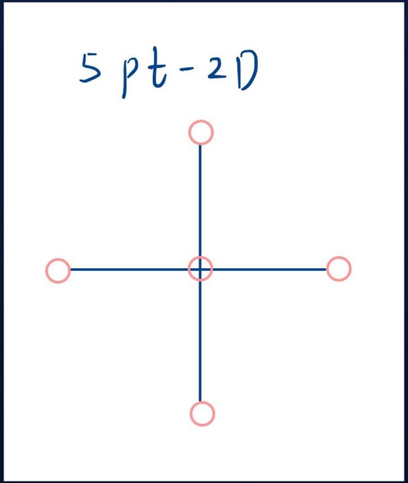
5 pt - 2D

Reference List

[1] Christoph Klein, Robert Strzodka. *Preconditioning Sparse Matrices with Alternating and Multiplicative Operator Splittings.* SIAM J. Sci. Comput. Vol. 45, No. 1, pp. A25-A48. 2023.