Understanding the Dot Product and the Cross Product

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Introduction

One of the first steps in tackling differential calculus in many dimensions is simply knowing how to abstract the idea of a *number*. The objects that we get are *vectors*. We can add two vectors, just like how we can add two numbers, but things get a little tricky when we try to multiply vectors. It turns out that there are two useful ways to do this: the dot product, and the cross product. Here, we will talk about the geometric intuition behind these products, how to use them, and why they are important.

The Dot Product

Definitions and Properties

First, we will define and discuss the dot product. Let's start out in two spatial dimensions. Given two vectors

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

we define their **dot product** to be the following:

$$\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_1 b_1 + a_2 b_2 \tag{1}$$

In words, we take the corresponding components, multiply them, and add everything together.

The first thing to notice is that the dot product of two vectors gives us a *number*. Certain basic properties follow immediately from the definition. For any vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$, and any real number λ,

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. In words, the order of multiplication doesn't matter.
2. $(\lambda \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda(\mathbf{a} \cdot \mathbf{b})$. We can move scalars in and out of each of the vectors without changing the value.
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$. The dot product distributes over addition of vectors.

I'm not going to prove all of these here, but they all follow from the definition and the properties of real numbers. Notice that the statements in $1, 2$ and 3 mimic the properties of multiplication in the real numbers — so in a way, the dot product is a very natural analogue to number multiplication!

One more thing to note: recall that the **norm** / **length** / **magnitude** of a vector is defined to be the square root of the sum of the squared components. So if we take the dot product of a vector with itself, we get the square of the length of that vector:

$$\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 = \|\mathbf{a}\|^2$$

Geometric Intuition

The algebraic properties of the dot product are important (and you should know them well!) but they're not very interesting. Here's what I would want to know if I were you: *what does the dot product mean*? In other words, how can we interpret the value of the dot product geometrically?

To tackle this question, I'm going to present an alternative (but equivalent!) way to define the dot product[1]: given vectors **a** and **b**, let θ be the angle between them, and define their **dot product** to be:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta \tag{2}$$

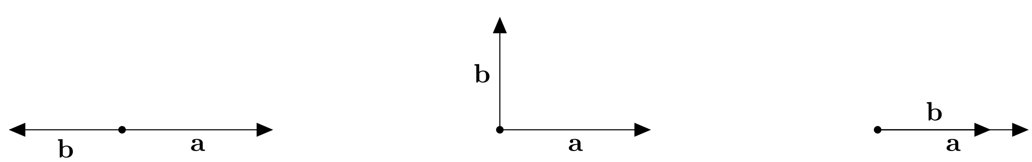
This formulation of the dot product is conceptually nice for many reasons. For one, we can immediately see that the dot product **encodes information about the angle between two vectors**. So, for example, if we're given two vectors **a** and **b** and we want to calculate the angle θ between them, we can solve for θ in (2) to get:

$$\theta = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|}\right)$$

The fact that the dot product carries information about the angle between the two vectors is the basis of our geometric intuition. Consider the formula in (2) again, and focus on the $\cos\theta$ part. We know that the cosine achieves its most positive value when $\theta = 0$, its most negative value when $\theta = \pi$, and its smallest magnitude when $\theta = \pi/2$. Explicitly,

$$\cos\pi = -1 \qquad \cos\frac{\pi}{2} = 0 \qquad \cos 0 = 1$$

Geometrically, these particular angles correspond respectively to the following pictures:



So on the far left, when **a** and **b** are going in exactly opposite directions, the dot product will be as negative as possible. In the middle case, when the vectors are perpendicular, the dot product will be 0.[2] On the far right, when the vectors are heading in the same direction, the dot product will be as positive as possible.

[1] We really should prove that these are equivalent formulations, but for the purpose of this discussion, I'll let you believe me.

[2] In fact, this is a complete characterization of perpendicularity, also called **orthogonality**. If $\mathbf{a} \cdot \mathbf{b} = 0$, then **a** and **b** are orthogonal.

Summarizing this, we see that **the dot product measures how similar two vectors are, or, how well they travel together**. In other words, if they are parallel (i.e. traveling in the same direction), the dot product will be as big as possible (either negatively big or positively big), and if the vectors are perpendicular (and so don't travel well together at all), the dot product will be zero. Also, consider the following: the argument above (which boils down to the fact that $\cos\theta$ is always between -1 and 1) combined with equation (2) tells us that for any vectors $\mathbf{a}$ and $\mathbf{b}$,

$$-\|\mathbf{a}\|\,\|\mathbf{b}\| \leq \mathbf{a}\cdot\mathbf{b} \leq \|\mathbf{a}\|\,\|\mathbf{b}\| \qquad \Rightarrow \qquad |\mathbf{a}\cdot\mathbf{b}| \leq \|\mathbf{a}\|\,\|\mathbf{b}\| \tag{3}$$

This inequality is known as the **Cauchy-Schwarz Inequality**, and gives an alternate way to summarize what we talked about above.

Projections

As stated above, the dot product gives us a way to measure how similar two vectors are. The problem with the dot product, though, is that it spits out a number. Sometimes we want a way to measure how well vectors travel together *while still preserving some information about direction.* In other words, we want a dot-product-like measurement that returns the same information as a *vector* rather than a *scalar.*

How should we do this?

Well, given vectors $\mathbf{a}$ and $\mathbf{b}$, the quantity $\mathbf{a}\cdot\mathbf{b}$ measures how well they travel together. We could rephrase this, use $\mathbf{a}$ as our "reference direction" and say that $\mathbf{a}\cdot\mathbf{b}$ measures *how well* $\mathbf{b}$ *travels in the direction of* $\mathbf{a}$. Since we want to preserve information about the direction we're travelling in, we can just multiply $\mathbf{a}\cdot\mathbf{b}$ by the vector $\mathbf{a}$!

$$(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a}$$

The only issue here is that the length of $\mathbf{a}$ is going to mess up our measurement, so to be safe, we should instead multiply the dot product by *a unit vector in the* $\mathbf{a}$ *direction*:

$$(\mathbf{a}\cdot\mathbf{b})\,\frac{\mathbf{a}}{\|\mathbf{a}\|}$$

We could improve on one more thing. Since $\mathbf{a}$ is our reference *direction*, we (again) don't want the length of $\mathbf{a}$ messing up our measurements. So we could normalize the coefficient of our vector by dividing once more by the length of $\mathbf{a}$:

$$\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\,\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}$$

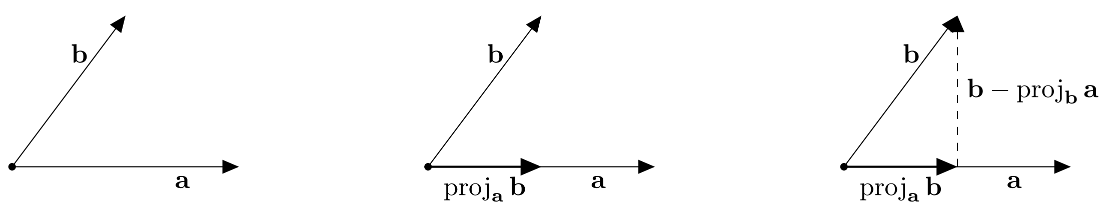
Cool! This is a normalized-vector-version of the dot product. We give this measurement a special name: **the projection of b onto a**:

$$\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\,\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a} \tag{4}$$

The reason this is called the *projection* is because it has a very nice geometric interpretation: given vectors $\mathbf{a}$ and $\mathbf{b}$, $\operatorname{proj}_{\mathbf{a}}\mathbf{b}$ gives a vector that represents the *component of* $\mathbf{b}$ *in the* $\mathbf{a}$ *direction.* In other words, if we smashed down $\mathbf{b}$ directly on top of $\mathbf{a}$, $\operatorname{proj}_{\mathbf{a}}\mathbf{b}$ is the vector that we get:

Alternatively, the vector $\text{proj}_{\mathbf{b}}\,\mathbf{a}$ smashes $\mathbf{a}$ directly onto $\mathbf{b}$ and gives us the component of $\mathbf{a}$ in the $\mathbf{b}$ direction:

It turns out that this is a very useful construction. For example, **projections give us a way to make orthogonal things**. By the nature of "projecting" vectors, if we connect the endpoints of $\mathbf{b}$ with its projection $\text{proj}_{\mathbf{b}}\,\mathbf{a}$, we get a vector orthogonal to our reference direction $\mathbf{a}$. In other words, the vector $\mathbf{b} - \text{proj}_{\mathbf{b}}\,\mathbf{a}$ is orthogonal to $\mathbf{a}$:



So projections give us one way to construct perpendicular directions. If we need a normal vector, a perpendicular bisector, the shortest distance between a point and a line, etc., we can use projections!

One last thing about the dot product. We started this discussion under the assumption that our vectors were two dimensional. But (other than the fact that vectors in two dimensions are easy to visualize) we never really used this fact! We could just as easily define the dot product for n-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

The pictures will be hard (and even impossible) to visualize if n is big, but the same properties hold. Also, if $n = 1$ (so that $\mathbf{a}$ and $\mathbf{b}$ are just numbers) we get regular multiplication! Pretty cool.

The Cross Product

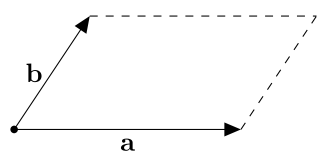
Motivation

Now it's time to talk about the second way of "multiplying" vectors: *the cross product.* Defining this method of multiplication is not quite as straightforward, and its properties are more complicated. But the cross product is an extremely powerful tool, and understanding it well will pay off.

First, I want to provide some motivation for the cross product, because at first glance the definition will seem rather ugly and arbitrary. So having developed the theory of the dot product, here are some interesting questions we might ask:

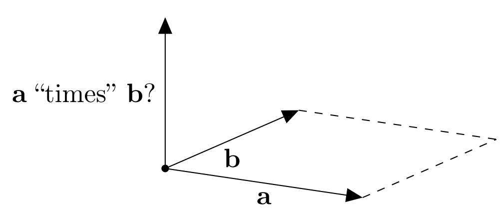
1. **Duality to the Dot Product**: We have a product that measures how **similar** two vectors are (the dot product). Can we construct a product that measures how **different** (i.e. perpendicular) two vectors are?

2. **Area**: Given two vectors $\mathbf{a}$ and $\mathbf{b}$, they define a parallelogram:



   Can we construct an "area product"? In other words, can we multiply $\mathbf{a}$ and $\mathbf{b}$ in a way that gives us the area of the parellelogram they span?

3. **Orthogonality**: We have a vector product that returns a *scalar* (the dot product). Can we define a useful product that returns a *vector*? In particular, since we know that orthogonality is a nice property, can our product return a vector that is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$?[3]



Each of these questions will be good to keep in mind as we move forward, because the cross product is the answer to all of them!

Alright, take a deep breath, because here we go:

[3]Note that this only makes sense in at least 3 dimensions.

Definition

For the cross product, we are going to restrict ourselves to three spatial dimensions (so even if we only care about vectors in two dimensions, we can view them in three dimensions by adding a 0 in the last component). So let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$. We define their **cross product** to be:

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{5}$$

As I mentioned, this is not a very easy definition to digest. Unfortunately, this is something that you'll have to memorize and learn to work with — but I promise, it *does* make sense! It just takes some time and practice to get the intuition.

Here's an alternative way to remember the definition, using the determinant of a matrix. Recall that the standard vectors in three dimensions are $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$:[4]

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

To calculate the cross product of $\mathbf{a}$ and $\mathbf{b}$, we can make a 3×3 matrix with $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ across the top row, the components of $\mathbf{a}$ across the middle row, and the components of $\mathbf{b}$ across the bottom row. Then, take its determinant:[5]

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} + (-1)(a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\ &= \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \end{aligned}$$

There are a number of ways to memorize the formula for the cross product — this is just one of them.

Before we list the algebraic properties of the cross product, take note that unlike the dot product, the cross product spits out a *vector*. This alone goes to show that, compared to the dot product, the cross product is a different kind of beast.

Before we get to the interesting part of interpreting the meaning of the cross product, we need to know how to work with it algebraically:

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[4] You might see these referred to as $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$.

[5] Here we are using what is known as *Laplace Cofactor Expansion* across the top row.

Properties

Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors in three dimensions, and let λ be a real number. Then the following holds:[6]

1. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$. In words, *the order does matter*! Calculating $\mathbf{b} \times \mathbf{a}$ gives us a vector in the opposite direction as $\mathbf{a} \times \mathbf{b}$.

2. $(\lambda \mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda \mathbf{b}) = \lambda (\mathbf{a} \times \mathbf{b})$. Just like with the dot product, we can move scalars in an out of each vector.

3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$. The cross product distributes across vector addition, just like the dot product.

Like the dot product, the cross product behaves a lot like regular number multiplication, *with the exception of property* 1. The cross product is *not commutative.*

There are a lot of other algebraic properties and identities that can be uncovered using the definition, but the ones listed above are the most important.

Geometric Intuition

Now it's time to understand what the cross product represents geometrically. We will use the three motivating questions at the beginning of the section to guide us. Let's tackle these one by one:

1. **Duality to the Dot Product**: The cross product acts as a sort of dual to the dot product in that it measures how *different* two vectors are, rather than how *similar* they are. More formally, the cross product provides a way to measure *orthogonality*: the more orthogonal $\mathbf{a}$ and $\mathbf{b}$ are, the longer the cross product $\mathbf{a} \times \mathbf{b}$ will be.

   Remember the formula $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$? The similarity of $\mathbf{a}$ and $\mathbf{b}$ was encapsulated in the $\cos \theta$, where θ represented the angle between the two vectors. So if the cross product represents the "dual" measurement to the dot product, we can use the "dual" to $\cos \theta$: $\sin \theta$! It turns out that we can calculate the length of the cross product vector $\mathbf{a} \times \mathbf{b}$ in the following way:

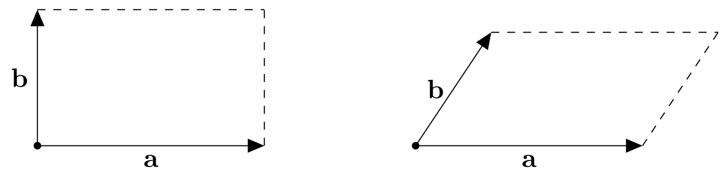
   $$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta \tag{6}$$

   The biggest (in magnitude) $\sin \theta$ gets is when $\theta = \pi/2, 3\pi/2$, where $\sin \theta = 1, -1$ respectively. In other words, the cross product achieves its largest magnitude when $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. The smallest (in magnitude) that $\sin \theta$ gets is when $\theta = 0, \pi$, where $\sin \theta = 0$. This occurs when $\mathbf{a}$ and $\mathbf{b}$ are parallel.

   Cool!

[6]Like before, I won't prove all of these here. They are a straightforward, albeit tedious, consequence of the definition.

2. **Area**: The previous discussion should make a lot of sense in the context of area. Consider the picture below:

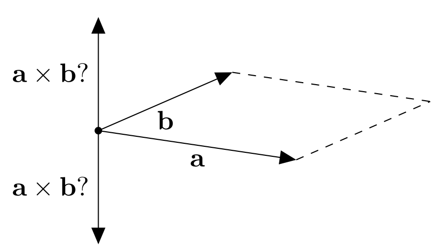


Because of the fact that the area of a parallelogram is the length of the base times the *vertical* height, it shouldn't be hard to convince yourself that the rectangle on the left has more area than the parallelogram on the right. In other words, the more orthogonal $\mathbf{a}$ and $\mathbf{b}$ are, the larger amount of area they span! This is really just a reinterpretation of the discussion in question 1. The cool part is that the cross product let's us actually calculate the exact area of the parallelogram. Not coincidentally, it is the formula in (6):

$$\text{Area of parallelogram spanned by } \mathbf{a} \text{ and } \mathbf{b} = \|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta$$

Note that it follows immediately that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for all vectors $\mathbf{a}$, since one vector cannot determine a parallelogram![7]

3. **Orthogonality**: The first two questions dealt with the *length* of the cross product, but it's important to keep in mind that $\mathbf{a} \times \mathbf{b}$ is a *vector*. As it turns out, $\mathbf{a} \times \mathbf{b}$ is a vector that is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$![8] In other words, $\mathbf{a} \times \mathbf{b}$ is a vector normal to the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. While this is pretty cool, there is one small complication: given two vectors $\mathbf{a}$ and $\mathbf{b}$, there are *two possible directions that cross product could go.* We could have a normal vector that sticks out either "above" or "below" $\mathbf{a}$ and $\mathbf{b}$:



So which one is it?

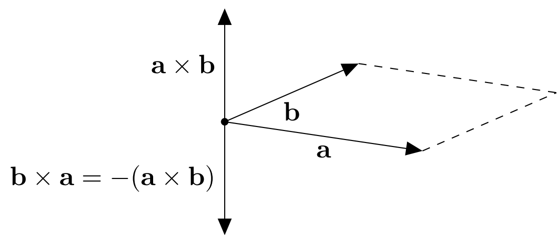
Recall the first property of the cross product:

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$$

[7] We could prove this using the definition (it isn't hard), but the geometric implications of the cross product give us a conceptually nice argument for why this is true.

[8] Explicitly: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. Again, we could prove this using the definition.

Again, this means that changing the order of multiplication gives us a vector in the opposite direction. That means that, in the picture above, one of the vectors will be $\mathbf{a} \times \mathbf{b}$ and the other will be $\mathbf{b} \times \mathbf{a}$. Which one is which? This is where the **right hand rule** comes in. In calculating $\mathbf{a} \times \mathbf{b}$, take your right hand, curl your fingers starting at $\mathbf{a}$ and in the direction of $\mathbf{b}$. The direction that your thumb points is the direction of $\mathbf{a} \times \mathbf{b}$! Next, $\mathbf{b} \times \mathbf{a}$: take your right hand, curl your fingers from $\mathbf{b}$ to $\mathbf{a}$. Notice that your thumb points downwards, in the opposite direction of $\mathbf{a} \times \mathbf{b}$. This completes our picture:



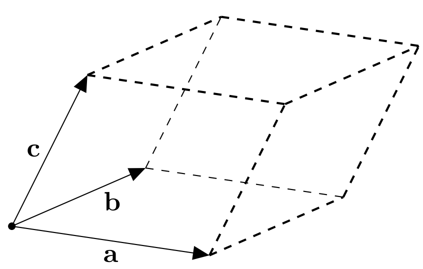
So the $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, and its direction is determined by the right hand rule.

These three questions demonstrate the usefulness of the cross product: $\mathbf{a} \times \mathbf{b}$ measures the area spanned by $\mathbf{a}$ and $\mathbf{b}$ (and thus encodes their perpendicularity), and produces a vector that is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Some Applications

The Scalar Triple Product

Here, we're going to put both the dot product and the cross product to use. Any three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in three dimensions determine a **parallelepiped** (i.e. a three dimensional parallelogram-like box), just like how any two vectors determine a parallelogram:



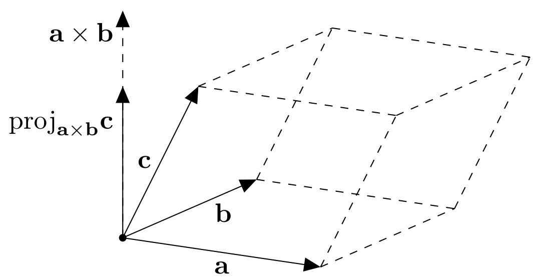
Let's calculate the volume of this parallelepiped. We know that the formula for the volume of a parallelepiped is:

$$V = (\text{area of base})\,(\text{vertical height}) \tag{7}$$

Note that the base is the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. We just learned how to calculate this area! The area of the base is exactly the length of the cross product: $\|\mathbf{a} \times \mathbf{b}\|$. So equation (7) becomes:

$$V = \|\mathbf{a} \times \mathbf{b}\| \text{ (vertical height)}$$

Next, we need to calculate the vertical height of the parallelepiped. In other words, we need to calculate the *component of* $\mathbf{c}$ *in the direction perpendicular to the base*. Explicitly, we want the component of $\mathbf{c}$ in the direction of $\mathbf{a} \times \mathbf{b}$. This should ring a bell: *projections*. In particular, we want the length of the projection of $\mathbf{c}$ onto $\mathbf{a} \times \mathbf{b}$!



So let's recall how to calculate that projection:

$$\text{proj}_{\mathbf{a}\times\mathbf{b}}\mathbf{c} = \frac{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}}{\|\mathbf{a} \times \mathbf{b}\|} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|}$$

This is just an application of projection formula in (4). What we want is the vertical height, which is just the magnitude of this vector:

$$\left\|\text{proj}_{\mathbf{a}\times\mathbf{b}}\mathbf{c}\right\| = \left\| \frac{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}}{\|\mathbf{a} \times \mathbf{b}\|} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \right\| = \frac{|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|}{\|\mathbf{a} \times \mathbf{b}\|} \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a} \times \mathbf{b}\|} = \frac{|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|}{\|\mathbf{a} \times \mathbf{b}\|}$$

Therefore, the volume of the parallelepiped spanned by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is:

$$V = \|\mathbf{a} \times \mathbf{b}\| \left\|\text{proj}_{\mathbf{a}\times\mathbf{b}}\right\| = \|\mathbf{a} \times \mathbf{b}\| \frac{|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|}{\|\mathbf{a} \times \mathbf{b}\|} = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| \tag{8}$$

Cool! We used both the cross product and the dot product to prove a nice formula for the volume of a parallelepiped: $V = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$. The product that appears in this formula is called the **scalar triple product**:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

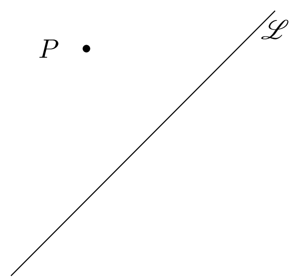
One last comment: there was nothing special about that fact that we chose the base of our parallelepiped to be determined by $\mathbf{a}$ and $\mathbf{b}$ — we could calculate this volume in any order we want!

$$V = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}| = |(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}|$$

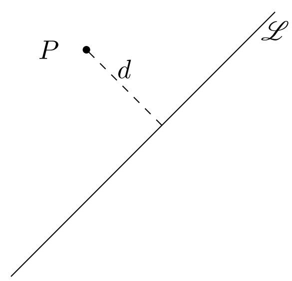
Note that this is because we're taking the absolute value of the number we get from the scalar triple product. Since it contains a cross product, switching the order *could* change the sign of the scalar triple product itself.

Calculating Shortest Distances

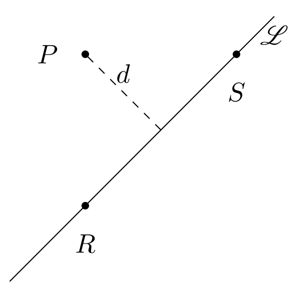
Let's do one more application of both the dot product and the cross product. Here's a motivating question: given a point P and a line $\mathscr{L}$, what is the shortest distance from P to $\mathscr{L}$?

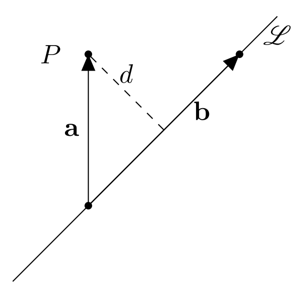


Geometrically, the distance d we want is indicated by the dashed line below. Note that the dashed line is perpendicular to $\mathscr{L}$:



So if we can find a vector that goes from P to the intersection of the dashed line with $\mathscr{L}$, we can take the length of that vector! Here's the gameplan: first, pick *any two points on* $\mathscr{L}$, R and S.[9] Then, construct the vectors $\mathbf{a} = \overrightarrow{RP} = P - R$ and $\mathbf{b} = \overrightarrow{RS} = S - R$:

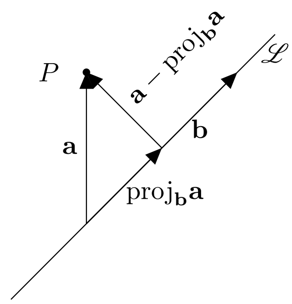




The diagram on the right looks very similar to the projection diagram on page 4. If we calculate the projection of $\mathbf{a}$ onto $\mathbf{b}$, then calculate $\mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a}$, this is exactly the vector that we want!

See the diagram below:

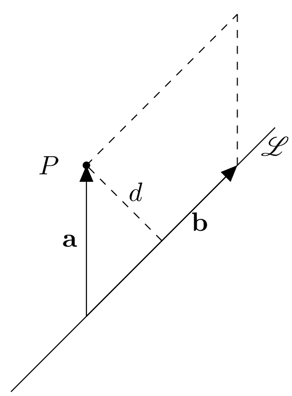
[9] If the equation for $\mathscr{L}$ is given as a parametric equation, you can do this by picking two arbitrary values of t.



Therefore, the shortest distance from P to $\mathscr{L}$ is exactly $d = \|\mathbf{a} - \mathrm{proj}_{\mathbf{b}}\mathbf{a}\|$.

We just used projections (and so indirectly, the dot product) to calculate the shortest distance from P to $\mathscr{L}$. Alternatively, we could use the cross product! Here's how:

Note that our two vectors $\mathbf{a}$ and $\mathbf{b}$ define a parallelogram:



As we now know, the area of this parallelogram is $\|\mathbf{a} \times \mathbf{b}\|$. Alternatively, the area of any parallelogram is the length of the base times the vertical height. If we make the base of the parallelogram the vector $\mathbf{b}$, the vertical height is exactly d! This tells us that:

$$\text{Area} = (\text{Length of Base})(\text{Vertical Height}) = \|\mathbf{b}\|\ d$$

But we also know that:

$$\text{Area} = \|\mathbf{a} \times \mathbf{b}\|$$

Therefore,

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{b}\|\ d \qquad \Rightarrow \qquad d = \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{b}\|}$$

The cross product gives us an alternative (and perhaps slicker) way to calculate this minimum distance.