

# Toric and their Destructction

- integrability  $\Rightarrow$  can write as action-angle form:
- $$\begin{cases} \frac{d\underline{I}}{dt} = \underline{\omega} \cdot \frac{d\underline{\theta}}{dt} = \underline{\omega}(\underline{I}) \\ \text{const } \underline{I}. \end{cases}$$

$\Rightarrow$  motion defines toric



$$\frac{d\theta}{dt} = \omega_1(I_1)t$$

$$\frac{d\phi}{dt} = \omega_2(I_2)t$$

scanning  $I_1, I_2$  (linked to  $E$ )

$\Rightarrow$

define nested toric



etc.

eg. box

$$\omega_1 = \pi^2 I_1 / m a^2$$

$$\omega_2 = \pi^2 I_2 / m b^2$$

$$E = I_1 \omega_1 + I_2 \omega_2$$

- motion on each toroidal surface will cover surface ergodically, unless  $\underline{\omega}$  rational.
- many surfaces  $\Rightarrow$   $\underline{\omega}_2$  define volume of phase space.

- motion is conditionally periodic

i.e. ergodic motion on bounded surface  
 $\Rightarrow$  Poincaré recurrence guarantees nearby return to c.p.

$\Rightarrow$  How robust are toroidal surfaces?

i.e. if  $H \rightarrow H_0(\underline{I}) + \epsilon H(\underline{I}, \underline{\phi})$

$\uparrow$   
 symmetry breaking  
 perturbation

can we integrate the perturbed system to some order in  $\epsilon$ ?

i.e. transform  $\underline{I}, \underline{\phi} \rightarrow \underline{J}, \underline{\phi}$

s.t.  $\left. \begin{array}{l} \underline{\dot{J}} = 0 \\ \underline{\dot{\phi}} = \omega(\underline{J}) \end{array} \right\}$  to specified order in P.T.?

This is equivalent to exploring "fragility of surfaces"  $\Rightarrow$  i.e. can nested structure be maintained with  $o(\epsilon)$  deformation?

n.b.  $\Rightarrow$  intro to canonical perturbation theory

→ start with 7 deg freedom:

$$J = I + o(\epsilon)$$

$$q = Q + o(\epsilon)$$

then: old:  $I, Q$

new:  $J, q$

o/t  $J = Q$   
to  $o(\epsilon)$

so have C-T. problem:

$$\begin{aligned} p &\leftrightarrow I \\ q &\leftrightarrow Q \\ (\text{old}) \end{aligned}$$

$$\begin{aligned} p &= J \\ Q &= q \\ (\text{new}) \end{aligned}$$

so

$$\begin{aligned} \text{index} \\ q &\leftrightarrow Q \\ p &\leftrightarrow J \end{aligned}$$

$$\begin{aligned} \text{def} \\ p &\leftrightarrow I \\ Q &= q \end{aligned}$$

$$p = \frac{\partial F}{\partial q} = \frac{\partial \mathcal{H}}{\partial q}$$

so

$$\begin{aligned} F &= S \\ \text{here,} \\ S &= H - J \\ \text{Fctn.} \end{aligned}$$

$$I = \partial S / \partial \phi$$

$$\phi = \partial S / \partial J$$

where:  $S = S_0 + \epsilon S_1$   $\rightarrow$  unknown

$$= J\phi + \epsilon S_1$$

now here:

$$S = S_0 + \epsilon S_1$$

$$H'(J) \equiv K(J)$$

$\downarrow$   
new, integrated  
Hamiltonian  $\rightarrow$  fctn of  $J$ , only

$\rightarrow$  re-label.

and can expand:

$$K(J) = K_0(J) + \epsilon K_1(J) + \dots$$

$\equiv$

$$K(J) = H(I, \phi)$$

$$= H_0\left(\frac{\partial S}{\partial \phi}, \phi\right) + \epsilon H_1\left(\frac{\partial S}{\partial \phi}, \phi\right) + \dots$$

n.b: 
$$\begin{aligned} S' &= S_0 + \epsilon S_1 \\ &= J_0 + \epsilon S_1 \end{aligned}$$

$$I = J + \epsilon \frac{\partial S_1}{\partial \theta} \quad \Rightarrow \quad J = I - \epsilon \frac{\partial S_1}{\partial \theta}$$

$$\phi = \theta + \epsilon \frac{\partial S_1}{\partial J} \quad \phi = \theta + \epsilon \frac{\partial S_1}{\partial J}$$

now, plugging  $J$  in to relation for  $H'$  etc

$$K_0(J) + \epsilon K_1(J) + \epsilon^2 K_2(J)$$

$$= H_0 \left( J + \epsilon \frac{\partial S_1}{\partial \theta} + \epsilon^2 \frac{\partial S_2}{\partial \theta} + \dots \right)$$

$$+ \epsilon H_1 \left( J + \epsilon \frac{\partial S_1}{\partial \theta} + \dots, \theta \right)$$

cranking expansion to  $O(\epsilon^2)$ :

$$K_0(J) + \epsilon K_1(J) + \epsilon^2 K_2(J) + \dots =$$

$$H_0(J) + \epsilon \frac{\partial S_1}{\partial \theta} \frac{\partial H_0}{\partial J} + \epsilon^2 \frac{\partial S_2}{\partial \theta} \frac{\partial H_0}{\partial J}$$

$$+ \epsilon H_1(J, \theta) + \epsilon^2 \frac{\partial S_1}{\partial \theta} \frac{\partial H_1(J)}{\partial J}$$

$$+ \frac{1}{2} \epsilon^2 \left( \frac{\partial S_1}{\partial \theta} \right)^2 \frac{\partial^2 H_0}{\partial J^2}$$

matching order-by-order:

$$H_0 = K_0$$

$$K_1(J) = \frac{\partial S_1}{\partial \theta} \frac{\partial H_0}{\partial J} + H_1(J, \theta)$$

$$K_2(J) = \frac{1}{2} \left( \frac{\partial S_1}{\partial \theta} \right)^2 \frac{\partial^2 H_0}{\partial J^2} + \frac{\partial S_2}{\partial \theta} \frac{\partial H_0}{\partial J}$$

$$+ \frac{\partial S_1}{\partial \theta} \frac{\partial H_1}{\partial J} + H_2$$

etc.

if  $\theta$  present.

For  $O(\epsilon)$ :

$$\begin{aligned}
 K_1(J) &= \frac{\partial S_1}{\partial \theta} \frac{\partial H_0}{\partial J} + H_1(J, \theta) \\
 &= \frac{\partial S_1}{\partial \theta} \omega_0(J) + H_1(J, \theta)
 \end{aligned}$$

$\downarrow$   
 winding frequency

where understand:

$$\begin{aligned}
 I &= J + O(\epsilon) \\
 \phi &= \theta + O(\epsilon)
 \end{aligned}$$

$$\begin{aligned}
 \theta &= \phi - \epsilon \frac{\partial S_1}{\partial J} \\
 I &= J + \epsilon \frac{\partial S_1}{\partial \theta}
 \end{aligned}$$

Now, if define:

$$H_1 = \underbrace{\langle H_1 \rangle}_{\text{avg.}} + \underbrace{\tilde{H}_1}_{\substack{\text{dep piece} \\ \text{(symmetry breaking)}}}$$

$$\langle H_1 \rangle = \oint_{\frac{2\pi}{2\pi}}^{\frac{2\pi}{2\pi}} H_1$$

(mean part)

then



averaging  $K_1(J)$  eqn  $\Rightarrow$

$$\boxed{K_1(J) = \langle H_1 \rangle}$$

and for  $S_1$ , from solvability:

$$\begin{aligned} \omega_0(J) \frac{\partial S_1}{\partial \theta} &= K_1(J) - H_1 \\ &= \tilde{K}_1(J) - \langle \tilde{H}_1 \rangle - \tilde{H}_1 \\ &= -\tilde{H}_1 \end{aligned}$$

$$\boxed{\omega_0(J) \frac{\partial S_1}{\partial \theta} = -\tilde{H}_1}$$

Now, from before, as motion closed and periodic:

$$\tilde{H}_1 = \sum_{n=1}^{\infty} H_n(J) e^{in\theta}$$

$$S_1 = \sum_{n=1}^{\infty} S_n e^{in\theta}$$

$$J = J_0 + \epsilon S_1$$

$\Rightarrow$



$$\mathcal{S}_1 = - \sum_n \frac{H_n(J)}{in \omega_0(J)} e^{in\theta}$$

so can finally write full solution to  $\mathcal{S}(\epsilon)$ :

$$\begin{aligned}\phi &= \theta + \epsilon \frac{\partial \mathcal{S}_1}{\partial J}(J, \theta) \\ J &= I - \epsilon \frac{\partial \mathcal{S}_1}{\partial \theta}(J, \theta) \\ \omega &= \omega_0(J) + \epsilon \frac{\partial}{\partial J} K_1(J)\end{aligned}$$

where:

$$\begin{aligned}K_1 &= \langle H_1 \rangle \\ \mathcal{S}_1 &= \sum_n \frac{i H_n(J)}{n \omega_0(J)} e^{in\theta}\end{aligned}$$

so, on 1 d.o.f; can define strategy of perturbative 'integration'.

BUT, if # d.o.f's  $> 1$ :

$\Theta \rightarrow \underline{\Theta}$  (i.e.  $\Theta, \phi$  toroidal angles)

$$n\omega_0(\underline{J}) \rightarrow \underline{n} \cdot \underline{\omega}_0(\underline{J})$$

$$\left( \begin{array}{l} \text{i.e. } \underline{n} \cdot \underline{\omega}_0 = n\omega_1(J_1) + m\omega_2(J_2) \\ \text{where } E = J_1\omega_1 + J_2\omega_2 \end{array} \right)$$

then if

$$\underline{n} \cdot \underline{\omega}_0(\underline{J}) \rightarrow 0$$

denominator  
vanishes and  
perturbation theory  
fails

$\Rightarrow$  welcome to  
the "problem of  
small divisors"

$\Rightarrow$  identifies resonant surfaces

i.e. special surfaces of nested torus

where pitch of perturbation  
 $n/m = \text{pitch of winding } \frac{\omega_2}{\omega_1}$

These seem (and are) most fragile  
surfaces

These surfaces are "resonant surfaces"

Classic example:

- tokamak  
field lines

$$m = n Z(r)$$

$$Z(r) = m/n$$

↓  
pitch  
of lines

(note shear)

↓  
pitch of  
perturbation

→ wave particle

$$v = \omega/k$$

n.b. here  
time makes  
resonance

$$\partial \phi / \partial t = H - H'$$

①

↓  
particle  
velocity

↓  
wave  
phase velocity

⇒ in vicinity of resonant surfaces,  
perturbative integration fails

② since action  $\propto \omega$  are set  $\omega \rightarrow 0$   
on whole #'s, resonant surfaces  
are in some sense special,  
of measure

→ sneak preview

distortions called "cusps" form  
(const. H surface)



+ resonant  
perturbation  
 $m = 4$   
 $n = 2$



Filamentation  
occurs.

$$W_H \sim \sqrt{\delta B}$$

upshot: - trajectory undertakes excursion  
from surface but remains near  
- phase space structure  
resembles that of pendulum.

→ caveat: secular <sup>canonical</sup> perturbation  
theory works for 1 resonance,  
only.

strategy:

- remove resonance by transformation to  
frame co-rotating with resonant variables  
Akin removal by frame change.  
n.b. really avg. over fast variable

- limitation to removal of 1  
fast variable

i.e. works as resonance  $\leftrightarrow$  slow

Now,

$$H = H_0(\underline{I}) + \epsilon H_1(\underline{I}, \underline{\theta})$$

if resonance:  $r\omega_1 - s\omega_2 \approx 0$   
 $\rightarrow$  resonance

$\Rightarrow$

$$\omega_1 = \frac{d\theta_1}{dt}$$

$$\theta = r\theta_1 - s\theta_2 \text{ "slow"}$$

$$\omega_2 = \frac{d\theta_2}{dt}$$

so

$$\begin{aligned} (\underline{\omega} \cdot \underline{\nabla}_{\underline{\theta}}) f(\underline{\theta}) &= (\omega_1 \partial_{\theta_1} + \omega_2 \partial_{\theta_2}) f \\ &= (r\omega_1 - s\omega_2) F_{\text{is}} \end{aligned}$$

$\rightarrow \theta$ , near resonance.

F dependence on  $\theta$  is h.o.  $\rightarrow$  slow.

thus, before:

$$\underline{I}, \underline{Q} \rightarrow \underline{J}, \underline{\phi}$$

now:

$$\left. \begin{array}{l} I_1, Q_1 \\ I_2, Q_2 \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \overset{\text{slow}}{\downarrow} r Q_1 - s Q_2, \hat{J}_1 \\ Q_2, \hat{J}_2 \end{array} \right.$$

2 Fast  $\rightarrow$  1 slow, 1 Fast

$$\begin{aligned} F &= S'(\text{old positions, new momenta}) \\ &= S(Q_1, Q_2; \hat{J}_1, \hat{J}_2) \end{aligned}$$

and type 2, so:

$$S = \underbrace{(r Q_1 - s Q_2)}_{S_0} \hat{J}_1 + Q_2 \hat{J}_2 + \epsilon S_1$$

S



$$I_1 = \partial S / \partial \theta_1 = r \hat{J}_1 + \epsilon \partial S_1 / \partial \theta_1$$

$$I_2 = \partial S / \partial \theta_2 = (\hat{J}_2 - s \hat{J}_1) + \epsilon \partial S_1 / \partial \theta_2$$

$$\phi_1 = \partial S / \partial \hat{J}_1 = r \theta_1 - s \theta_2 + \epsilon \partial S_1 / \partial \hat{J}_1$$

$$\phi_2 = \partial S / \partial \hat{J}_2 = \theta_2 + \epsilon \partial S_1 / \partial \hat{J}_2$$

$$H = H_0(\underline{I}) + \epsilon H_1(\underline{I}, \underline{\theta})$$

$$H_1 = \sum_{\ell, m} H_{\ell m}(\underline{I}) e^{i(\ell \theta_1 + m \theta_2)} \quad \ell, m \neq 0$$

but know:

$$\phi_1 = r \theta_1 - s \theta_2 + \cancel{\mathcal{O}(\epsilon)} \quad \text{Slow}$$

$$\phi_2 = \theta_2 + \cancel{\mathcal{O}(\epsilon)} \quad \text{Fast}$$

$\Rightarrow$

$$\theta_1 \cong (\phi_1 + s \phi_2) / r$$

$$\theta_2 \cong \phi_2$$



re-writing:

$$H_1 = \sum_{\ell, m} H_{\ell, m}(\underline{\hat{J}}) \exp \left[ i \left( \frac{\ell}{r} \phi_1 + \frac{(\ell s + m r)}{r} \phi_2 \right) \right]$$

$\phi_2 \rightarrow \text{fast}$

$\phi_1 \rightarrow \text{slow}$

} distinction only possible  
near resonance where  
 $r\omega_1 - s\omega_2 \rightarrow 0$

[Now, average out fast  $\phi_2$   
dependence, and focus on  
evolution near resonance.  $\Rightarrow$  isolates  
region  
near resonance]

Thus, will have

$$K_1 = K_1(\underline{\hat{J}}, \phi_1) = \langle H_1 \rangle_{\phi_2}$$

$$\langle H_1 \rangle_{\phi_2} = \left\langle \sum_{\ell, m} H_{\ell, m}(\underline{\hat{J}}) \exp \left[ i \left( \frac{\ell}{r} \phi_1 + \frac{(\ell s + m r)}{r} \phi_2 \right) \right] \right\rangle_{\phi_2}$$

on

Simply put:

$$\frac{\partial}{\partial m} = -\frac{n}{s}$$

 $\Rightarrow$ 

mode # pitch of  
perturbation must  
match pitch of  
resonance

so

$$\sum_{\ell, m} \rightarrow \sum_{p(-r/s)}$$

$\Rightarrow$  sum over  
all harmonics  
of, perturbation  
resonant

 $\therefore$ 

$$\sum_{\ell, m} \rightarrow \sum_p F_{-p, \frac{r}{s}} p s$$

upon  $\phi_2$  integration:  $l_s = -mr$

$$\frac{l}{m} = \frac{-r}{s} \quad \text{but} \quad r\omega_1 - s\omega_2 \sim 0$$

$$\sim \frac{\omega_2}{\omega_1} \Rightarrow \frac{l}{m} \text{ ratio set by resonance.}$$

$$\Rightarrow H_{l, m} \rightarrow H_{-m \frac{r}{s}, m} \quad l = -\frac{r}{s} m$$

$$\rightarrow H_{-mr, ms}$$

relabel

$$\rightarrow H_{-pr, ps}$$

$$\text{also} \quad \frac{l}{r} = -\frac{m}{s} \quad \text{relabel: } -\frac{m}{s} \rightarrow -m$$

$$-m \rightarrow -p$$

$$\Rightarrow \langle \rangle_{\phi_2} \text{ perturbation is}$$

just harmonics of resonant pair  $-r, s$ .

$$\langle H_1 \rangle_{\phi_2} = \sum_{p=0}^{\infty} \sum_{-r, p, s, p} H_1 e^{-i p \phi_1}$$

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$$\langle H \rangle = H_0(J) + \epsilon \sum_{p=0}^{\infty} H_{-p, p, s}^{(1)} e^{-i p \phi_1}$$

From C-T rules:

$$\frac{\partial \langle H \rangle}{\partial \phi_2} = 0 \Rightarrow \frac{d \hat{J}_2}{dt} = 0 \rightarrow \text{adiabatic invariant}$$

and from C-T rules:

$$I_1 = r \hat{J}_1$$

$$I_2 = \hat{J}_2 - s \hat{J}_1$$

$\Rightarrow$

$$\hat{J}_2 = I_2 + \frac{s}{r} I_1$$

is adiabatic inv. of  
avgd Hamiltonian

$\phi, \phi \rightarrow \text{res.}$

19.

so  $\frac{d\vec{J}_2}{dt} = 0 \Rightarrow \frac{d\phi_2}{dt} = \frac{\partial \langle H \rangle}{\partial \vec{J}_2} \equiv \omega(\vec{J}_2)$

Now,  $\langle H \rangle = \langle H(\vec{J}_1, \phi_1, \vec{J}_2) \rangle$

→ For solution, need understand motion in  $\vec{J}_1, \phi_1$

→ without loss of generality, simplify by:

$p = 0, \pm 1$  harmonics only, contribute

so 
$$\langle H \rangle = H_0(\vec{J}) + \epsilon H_{0,0}(\vec{J}) + 2\epsilon H_{0,\pm 1}(\vec{J}) \cos \phi$$

$H_{-0,5} = H_{0,-5}$

and seek motion near fixed points, as characterization.

so,  $\begin{matrix} \dot{\vec{J}}_1 = 0 \\ \dot{\phi}_1 = 0 \end{matrix} \Rightarrow \text{f.p.} \Leftrightarrow \begin{matrix} \partial \langle H \rangle / \partial \phi_1 = 0 \\ \partial \langle H \rangle / \partial \vec{J}_1 = 0 \end{matrix}$

these define:  $\begin{cases} \vec{J}_1, 0 = 0 \\ \phi_1, 0 = 0 \end{cases} \left. \vphantom{\begin{matrix} \vec{J}_1, 0 = 0 \\ \phi_1, 0 = 0 \end{matrix}} \right\} \text{fixed pts of motion}$

18

$$\frac{\partial \langle H \rangle}{\partial \phi} = 0 \Rightarrow -2\epsilon H_{0,5}^{(1)} \sin \phi_1 = 0$$

$$\phi_1 = 0, \pm \pi$$

fixed pts.

and

$$\begin{aligned} \frac{\partial \langle H \rangle}{\partial \vec{J}_1} = 0 &\Rightarrow \frac{\partial H_0(\vec{J})}{\partial \vec{J}_1} + \epsilon \frac{\partial H_{0,0}(\vec{J})}{\partial \vec{J}_1} \\ &+ 2\epsilon \frac{\partial H_{0,5}^{(1)}}{\partial \vec{J}_1} \cos \phi_1 = 0 \end{aligned}$$

Now

$$\frac{\partial}{\partial \vec{J}_1} = \frac{d\vec{I}_1}{d\vec{J}_1} \frac{\partial}{\partial \vec{I}_1} + \frac{\partial \vec{I}_2}{\partial \vec{J}_1} \frac{\partial}{\partial \vec{I}_2}$$

C-T  
rules

$$= r \frac{\partial}{\partial \vec{I}_1} - s \frac{\partial}{\partial \vec{I}_2}$$

so,  $\partial \langle H \rangle / \partial \vec{J}_1 = 0 \Rightarrow$  re-express

$$0 = \left( r \frac{\partial}{\partial I_1} - s \frac{\partial}{\partial I_2} \right) H_0(\underline{I}) + \epsilon \frac{\partial}{\partial \vec{J}_1} H_{0,0} + 2\epsilon \frac{\partial H_{1,0}^{(4)}}{\partial \vec{J}_1} \cos \phi_1$$

$$= (r\omega_1 - s\omega_2) + \epsilon \left( \frac{\partial H_{0,0}^{(4)}}{\partial \vec{J}_1} + 2 \frac{\partial H_{1,0}^{(4)}}{\partial \vec{J}_1} \cos \phi_1 \right)$$

0 on resonance!

so, to lowest order:

$$\partial \langle H \rangle / \partial \vec{J}_1 = 0 \Leftrightarrow d\phi_1/dt = 0$$

is satisfied by resonance condition.

so  $\vec{J}_{1,0}$  defined by resonance condition.



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fixed points:

$$\hat{J}_{1,0} \leftrightarrow \text{resonant position} \\ r \omega_1(I) - s \omega_2(J) = 0$$

$$\phi_{1,0} \leftrightarrow \sin \phi_1 = 0.$$

n.b.  
see 22b

$$\begin{aligned} \langle H \rangle &= \langle H(\hat{J}_1, \hat{J}_2, \phi_1) \rangle \\ &= \langle H(\underbrace{\hat{J}_{1,0}}_{\text{resonance}} + \underbrace{\delta \hat{J}_1}_{\text{excursion}}, \underbrace{\phi_1}_{\text{IOM}} | \hat{J}_2) \rangle \end{aligned}$$

118, expanding:

$$\begin{aligned} \langle H(\hat{J}_1, \phi_1) \rangle &\approx H_0(\hat{J}_{1,0}) + \epsilon(H_{0,0}^{(4)}(\hat{J}_{1,0})) \\ &\quad + \cancel{\frac{\partial H_0}{\partial \hat{J}_1}(\hat{J}_1 - \hat{J}_{1,0})} + \frac{1}{2} \frac{\partial^2 H_0}{\partial \hat{J}_1^2} (\hat{J}_1 - \hat{J}_{1,0})^2 \\ &\quad + 2 \epsilon H_{1,-5}^{(4)} \cos \phi_1 \end{aligned}$$

reson      $\hat{J}_{1,0}$

 $\Rightarrow$ 

$$\langle H(\hat{J}_1, \phi_1) \rangle \approx \cancel{\text{const.}} + \frac{1}{2} \frac{\partial^2 H_0}{\partial \hat{J}_1^2} (\hat{J}_1 - \hat{J}_{1,0})^2 + 2 \epsilon H_{1,-5}^{(4)} \cos \phi_1$$

so, have arrived at averaged Hamiltonian near resonance:

$$\langle H(\hat{J}_1, \phi_1) \rangle = \frac{1}{2} (\hat{J}_1 - \hat{J}_{1,0})^2 \left. \frac{\partial^2 H_0}{\partial \hat{J}_1^2} \right|_{\hat{J}_{1,0}} - F \cos \phi_1$$

$$= \frac{G}{2} (\hat{J}_1 - \hat{J}_{1,0})^2 - F \cos \phi_1$$

$$G = \left. \frac{\partial^2 H_0}{\partial \hat{J}_1^2} \right|_{\hat{J}_{1,0}}, \quad F = \hbar \epsilon H_{1,5}^{(4)}$$

→ isomorphic to pendulum!

Recall for pendulum:

$$L = \frac{m l^2}{2} \dot{\theta}^2 - m g l (1 - \cos \theta)$$

$$H = p \dot{\theta} - L = \frac{p^2}{2 m l^2} - m g l \cos \theta$$

$$\Rightarrow H(\hat{J}_1, \phi) = \frac{\sigma}{2} (\hat{J}_1 - J_{b,0})^2 - F \cos \phi$$

is form of Hamiltonian near resonance.

Note:

- assumes  $\frac{\partial^2 H}{\partial \hat{J}_1^2} = \frac{\partial^2 W}{\partial \hat{J}_1^2} \neq 0$  (NL/shear)

"accidental" resonance.

- for properties:

$$\langle H(\hat{J}_1, \phi) \rangle = \frac{\sigma}{2} (\hat{J}_1 - J_{b,0})^2 - F \cos \phi$$

$\downarrow$  shear/NL parameter       $\downarrow$  perturbation amplitude

and so:

$$\begin{aligned} \Delta \dot{J} &= -F \sin \phi \\ \dot{\phi} &= \sigma \Delta J \end{aligned}$$

$$\begin{aligned} \phi &= 0 + \delta \phi \\ \Delta \dot{J} + F \sigma \delta \phi &= 0 \\ \text{near } \phi &= 0 \end{aligned}$$

c.e.

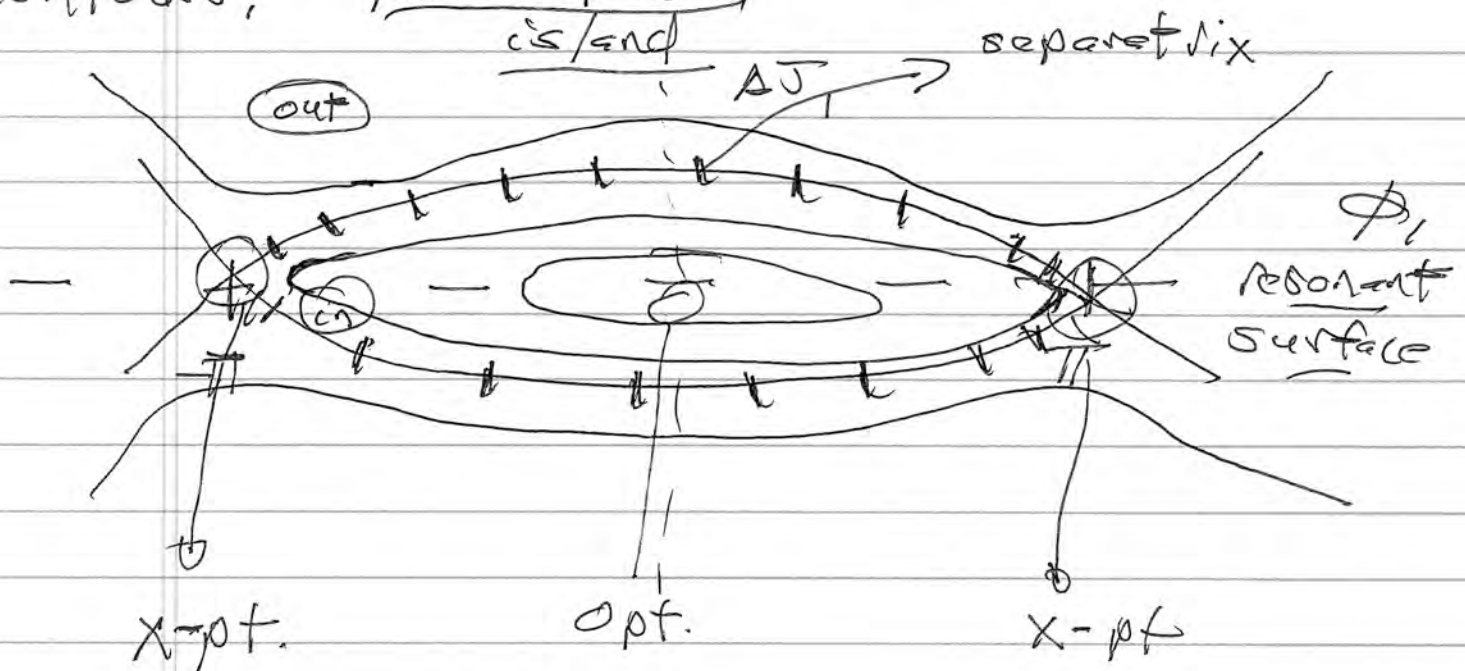
$$\Delta \ddot{J}_1 = -F \cos \phi_{1,0} G \Delta J$$

$$\Delta \ddot{J}_1 + FG \cos \phi_{1,0} \Delta J = 0$$

$FG > 0 \Rightarrow \phi_1 = 0$ , stable Fixed point  
(o-pt/elliptic point) ↗

$\phi_1 = \pm \pi \Rightarrow$  unstable Fixed pt.  
(x-pt/hyperbolic pt.)

Contours: phase space  
is/and



→ stable fixed pt.  $\Leftrightarrow$  elliptic point  $\Leftrightarrow$  O pt.  
 - island center

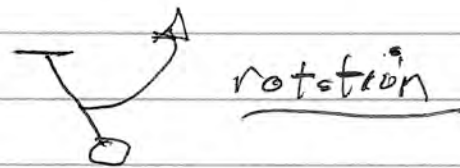
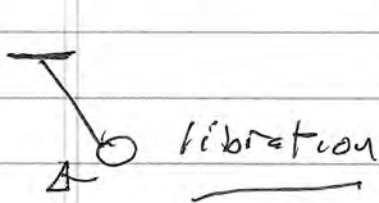
- center of trapped or libration region

→ unstable fixed point  $\Leftrightarrow$  hyperbolic point  $\Leftrightarrow$  X pt.

- island edge

- separatrix crossing point

→ separatrix ('separator') region of rotation (i.e. untrapped) from region of trapped (i.e. libration)



rotation

→ libration: elliptic orbits  
 rotation: hyperbolic orbits

- width of separatrix - "island width"

$$\Delta J)_{\max} \approx 2(F/G)^{1/2}$$

$$\approx 2 \left( -2E H_{0-5} / \left. \frac{\partial^2 H}{\partial J_1^2} \right|_{J_1=0} \right)^{1/2}$$

i.e. particle + wave:

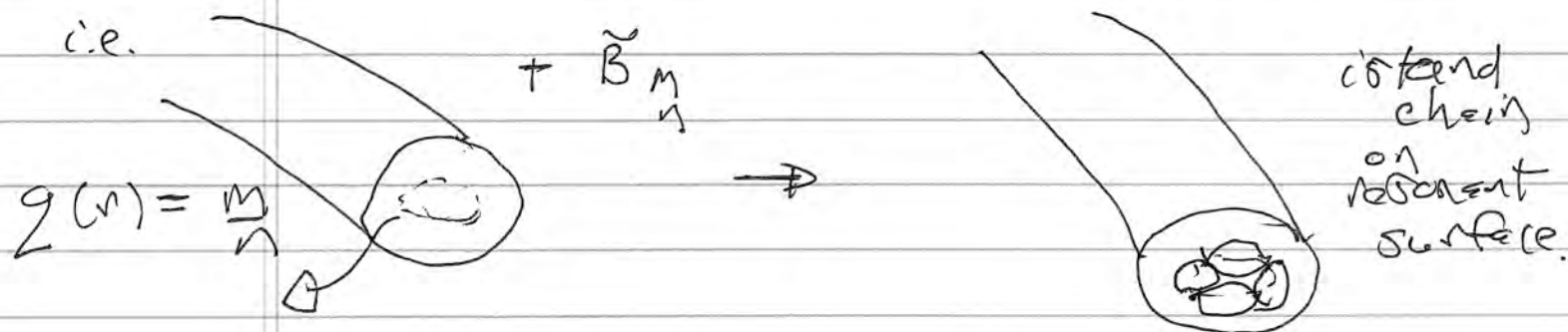
$$H = (p + m\omega/k)^2 / 2m + \sum \phi_0 \cos kx$$

$$\Delta p = (\sum \phi_0 m)^{1/2}$$

$$\Delta V \approx (\sum \phi_0 / m)^{1/2} \rightarrow \text{trapping width}$$

$\Rightarrow$  the Big Picture:

- resonant perturbation distorts and foliate resonant tori in phase space, forming island chain structures.





Note:

- structure localized to resonant surface
- $\left. \begin{array}{l} \text{trapped} \\ \text{untrapped} \end{array} \right\} \text{ orbits stay } \left\{ \begin{array}{l} \text{trapped} \\ \text{untrapped} \end{array} \right.$
- resonant surface is foliated but not destroyed.
- motion remains on surface, though surface is ruffled...