Why? — List Important MFE Problems

- Islands
- RMP
- Multi-scale
- ELM / Ped collapse
- Disruption / Thermal Quench
- SOL Width / Heat Load

Review

→ Resonances $\underline{k} \cdot \underline{B} = 0$

$q = m/n$

pitch match

overlap → $\frac{w_i + w_{i+1}}{r_{i+1} - r_i} > 1$ ⇒ chaos

→ $D_M = \sum_{\underline{k}} |b_{r_k}|^2 \delta(k_\parallel)$ QL

"test particle"

$= \sum_{\underline{k}} |b_{r_k}|^2 l_{ac}$

→ $l_{ac} < l_c < l_{mfp}$ collisionless

$l_{ac} < l_{mfp} < l_c$ collisional

From scales $l_{ac} = |\Delta k_\parallel|^{-1}$

$l_c = \left(\frac{k_0^2 D_M}{L_s^2} \right)^{-1/3}$

→ $K_u = \frac{l_{ac}}{\Delta_\perp} b_r$ $K_u \sim 1$ saturated / critical balance

Lecture II

Stochastic Fields, continued.

→ Why?

→ Review

→ Heat Transport

N. B.

$\left\{\begin{array}{ll} 2006 & 2014 \\ \text{Lect.} & 9\text{–}10 \end{array}\right.$

Can transform:

$$\langle H(\hat{J}_1, \phi_1)\rangle = \frac{1}{2}(\hat{J}_1 - \hat{J}_{1,0})^2 \frac{\partial^2 H_0}{\partial J^2}\bigg| - F\cos\phi_1$$

$$= \frac{G}{2}(\hat{J}_1 - \hat{J}_{1,0})^2 - F\cos\phi_1$$

$$G = \frac{\partial^2 H_0}{\partial J^2} = \frac{\partial \omega}{\partial J}, \qquad F = -2 G H'_{1,r}$$

etc.

→ For QL regime validity:

$\boxed{l_{ac} < l_c.}$ → $Ku < 1$ (show!) ($l_c < l_{ac}$ → ? prop.)

and another (particle) length: l_{mfp}.

$$\boxed{\begin{array}{l} \Rightarrow\ l_{ac} < l_c < l_{mfp} \rightarrow \text{so called "\underline{collisionless regime}"} \\ l_{ac} < l_{mfp} < l_c \rightarrow \underline{\text{collisional}} \end{array}}$$

which brings us to: something physical

⇒(Electron) Heat Transport

Theme: - waves. - interacting processes

N.B. - nobody cares about "line" diffusion

why do this?

- people (i.e. experimentalists) do care about:
  - → heat
  - → particle
  - → momentum

} transport.

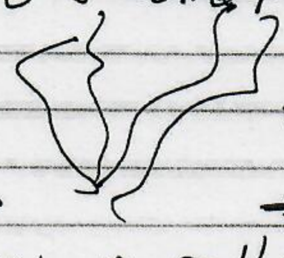
Rechester + Rosenbluth PRL '78

a MUST!

∴ lets begin with <u>heat</u> transport!

→ consider $l_{ac} < l_c < l_{mfp}$:

— lines wander



heat diffusivity

What is $\chi_\perp$?

⇒ "of course cf $\chi_\perp \sim V_{th} D_M$"

→ but is it so simple?

<u>Recall</u> - thought prblm.

— <u>but</u>, lets assume parallel collisions (only) happen. (Particle stays on line!).

∴ motion <u>along</u> line is <u>diffusive</u>

$$\delta z^2 \sim D_{\parallel} t \sim \chi_{\parallel} t$$

$D_\parallel$: V_{th}^2/ν

$\chi_\parallel$: parallel thermal diffusion

→ so! for slug heat:

$$\langle \delta r^2 \rangle \sim D_M z \sim D_M (\chi_{\parallel} t)^{1/2}$$

$\langle \delta r^2 \rangle$: radial scatter

so:

$$\chi_\perp = d\langle \delta r^2 \rangle / dt \sim D_M (\chi_\parallel)^{1/2} / \underline{t^{1/2}}$$

$\to 0$!

Point: → line may wander

but

→ particle kicked back ~~at~~ along line

→ even though $l_0 < l_{mfp}$,

no net radial wander as particle kicked back.

Lessons:

→ collisions control irreversibility

⊕ → $\boxed{\text{need get kicked off field line}}$

→ Need:

– ⊥ coarse graining:

– FLR → ρ_e

– $\chi_\perp$

– drifts.

{ minimum resolution scale

⇒ applied every l_{max}

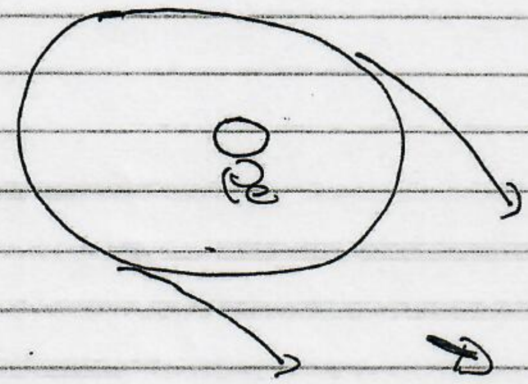
smear particle location over a resolution cell.

⇒ coarse graining resets "active volume".

so

→ consider the following argument:

① Consider disk of $r \sim \rho_e$



Disk

②

⇒

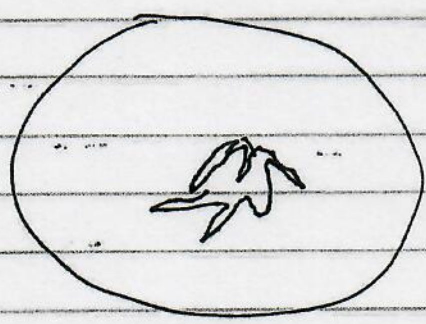
Map disk forward, noting that $\underline{\nabla} \cdot \underline{B} = 0$
⇒ map is area preserving

after ~ l_{mfp}

1 $h_L > 0$

1 $h_L < 0$

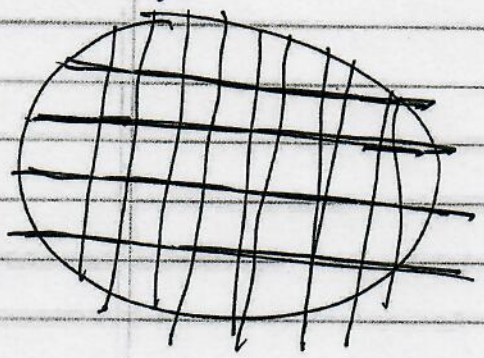
(h → Lyapunov Exp)



width ↳ $w \sim \rho_e e^{-l_{mfp}/l_c}$

$(\ell \sim \rho_e e^{+l_{mfp}/l_c})$

③ but coarse graining occurs at lmfp

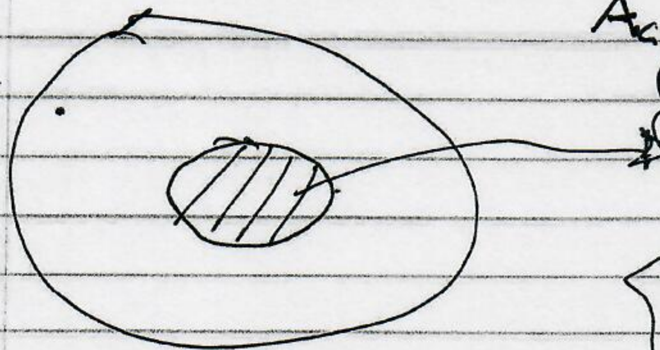


["particle/contour ~~re-gets~~ /smeared" to nearest grid site]

ans.

so

④



$A_{CG}\bar{f} = A_0 f_0$
$(\bar{f} < f_0)$

coarse graining of structure from previous V.G.

$\bar{f} = \frac{A_0 f_0}{A_{CG}}$

and on continue....

⑤ Ludwig Boltzmann assures us no memory between steps ($\underline{1 \text{ lmfp}}$ / collision time)

so initial spot expands, with random walk, as

$\boxed{\langle \delta r^2 \rangle \sim D_M\, l_{mfp}.}$ in lmfp i.e. $\boxed{\text{coarse graining interval sets } \langle \delta r^2 \rangle \text{ step !}}$

⇒

⑥ then, for $\chi_\perp$ (on l_{mfp}): $1/\tau_c \sim \nu_c$

$$\chi_\perp \sim \langle \delta r^2 \rangle / \tau_c \sim D_M \frac{l_{mfp}}{\tau_c}$$

$$\sim v_{th} D_M.$$

⇒ $\boxed{\chi_\perp \sim v_{th} D_M}$!

→ collisionless stochastic field heat diffusivity

→ manifestly independent of collisionality

→ yet clearly dependent on collisions and coarse graining !

Lesson: Coarse graining essential to irreversibility ||

~~[illegible]~~

Collisions → arrow of time.

on

Coarse graining essential to kick particle off field line, or else collisional back-scatter reverses wander.

Stoch Fields, cont'd

Exercises (suggested):

i.) Derive the magnetic diffusivity with magnetic drifts. How do these modify D_M? Explain why high energy particles (runaways) are confined longer than thermals.

ii.) Formulate the theory of diffusion due stochastic fields in toroidal geometry using ballooning mode formalism for the fluctuations.

iii.) What happens to net cross field transport in a standing spectrum of e.s. and magnetic perturbations. When might transport vanish?, Why?

→ Collisional Regime — More challenging

Here: $l_{c0} < l_{mfp} < l_c$

(short mean free path)

Point: what is going on?

→ $l_{mfp} < l_c$ ⇒ particle random walks parallel and undergoes many kicks in l_c. So parallel motion is diffusive.

→ perpendicular motion is continuous coarse graining/spreading, at

$D_\perp \sim \rho_e^2 \nu_{ee} \sim \rho_e^2 \frac{V_{The}}{l_{mfp}}$

So, can write:

$\langle \delta r^2 \rangle \sim D_M l_{c,d}$

parallel correlation length (d signifies diffusive regime)

must kick off line.

but also note that parallel motion is diffusive, so:

but time set by :

$$\chi_\parallel / l_{c,\delta}^2 \sim 1/t$$

$$\Rightarrow \frac{\langle \delta r^2 \rangle}{t} \sim \frac{\chi_\parallel}{l_{c,\delta}^2} D_M \, l_{c,\delta}$$

$$\sim D_M \frac{\chi_\parallel}{l_{c,\delta}} \sim D_M \chi_\parallel / l_{c,\delta}$$

$$\boxed{\chi_\perp = D_M \frac{\chi_\parallel}{l_{c,\delta}}} \rightarrow$$ [perpendicular heat conductivity in collisional regime.]

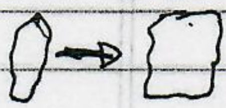
$$\chi_\perp = v_{th} D_M \frac{l_{mfp}}{l_{c,\delta}}$$

some correlation.

Now, what is $l_{c,\delta}$?

Notice $l_{c,\delta}$ is set by competition between 2 processes:

① width of δ increases due diffusion (coarse graining)



so $$(d\delta)^2 \sim (D_\perp dt)$$

$$d\delta \sim (D_\perp dt)^{1/2}$$

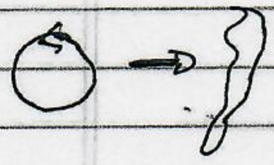
but
$$\chi_\parallel/(dL)^2 \sim 1/dt$$

$\Rightarrow$
$$d\delta \sim \left(\frac{D_\perp (dL)^2}{\chi_\parallel}\right)^{1/2}$$

$$\frac{\chi_\parallel}{L^2} \sim \frac{D_\perp}{\Delta^2}$$

$$\boxed{d\delta \sim \left(D_\perp/\chi_\parallel\right)^{1/2} dL}$$

② Width shrinks, due stochastic instability and area conservation:



$$d\delta/dL = -\delta/l_c \qquad \text{(exponential decay)}$$

then balance at:

$$d\delta \sim \underbrace{\left(D_\perp/\chi_\parallel\right)^{1/2} dL}_{\text{smearing}} \sim \underbrace{\frac{\delta}{l_c} dL}_{\text{thinning}}$$

$$\boxed{\delta \sim l_c \left(D_\perp/\chi_\parallel\right)^{1/2}}$$

N.B.: Can select δ from:

$$\partial_t T - \chi_\parallel \nabla_\parallel^2 T - D_\perp \nabla_\perp^2 T = 0$$

$$\Rightarrow \quad \frac{\chi_\parallel}{l_c^2} \sim \frac{D_\perp}{\delta^2}. \qquad \boxed{\delta \sim l_c \left(D_\perp/\chi_\parallel\right)^{1/2}}$$

Finally, need ~~[illegible]~~ correlation length $l_{c\delta}$ for chunk size δ. Assume set by k_θ

$$k_\theta^{-1} \sim \delta e^{z/l_c}\Big|_{l_{c\delta}} \sim \delta e^{l_{c\delta}/l_c}$$

19. 6.

$$\boxed{l_{c\delta} \sim l_c \ln\left(1/k_0\delta\right)}$$

$$\boxed{l_{c\delta} \sim l_c \ln\left(\chi_\parallel/D_\perp\right)^{1/2}}$$

$$\boxed{l_{c\delta} \sim l_c \ln\left(\left(\chi_\parallel/D_\perp\right)^{1/2}/k_0 l_c\right)}$$

$$\Rightarrow \boxed{\chi_\perp \simeq D_M \, \chi_\parallel / l_{c\delta}}$$

Apart from log factor:

$$\boxed{\chi_\perp \simeq v_{Th} D_M \underbrace{\left(l_{mfp}/l_c\right)}_{<1}}$$

⇒ reduced relative to collisionless values

Lesson: — collisions reduce ($l_{mfp} < l_c$) reduce χ_{eff} relative to "collisionless case"

— interplay of perp and parallel diffusion

— again, critical to knock particle off field line.

Now, the above calculation requires thought. It's much more convenient to crank ~~[illegible]~~ mindlessly. BOUT++ relevant.

⇒ Hydro approach: Kadomtsev and Pogutse (not mindless but systematic)

Consider heat flux along wiggling fields

i.e.

$$\underline{q} = -\chi_{\parallel} \nabla_{\parallel} T \,\hat{b} \;\; - \chi_{\perp} \underline{\nabla_{\perp}} T$$

parallel conduction

perp. conduction

tricky in codes

$\chi_{\parallel} \gg \chi_{\perp}$

Here: $\underline{b} = \underline{b_0} + \underline{\tilde{b}}$

$\underline{b_0}$ → unperturbed; $\underline{\tilde{b}}$ → fluctuating

$\nabla_{\parallel} = \partial_z + \underline{\tilde{b}} \cdot \nabla_{\perp}$

$\underline{\tilde{b}} \cdot \nabla_{\perp}$ → piece along wiggling line

So, seek mean radial heat flux

$$\begin{aligned} \langle q_r \rangle = & - \chi_{\parallel} \langle \tilde{b}_r^2 \rangle \partial_r \langle T \rangle \quad (1) \\ & - \chi_{\parallel} \langle \tilde{b}_r \partial_z \tilde{T} \rangle \quad (2) \\ & - \chi_{\parallel} \langle \tilde{b}_r \tilde{b}_r \partial_r \tilde{T} \rangle \quad (3) \\ & - \chi_{\perp} \nabla_r \langle T \rangle \end{aligned}$$

(1), (2) → usual quadratic

(3) → cubic

Now $\frac{(3)}{(2)} \sim \frac{\cancel{\chi_{\parallel}} \tilde{b}_r \tilde{b}_r \tilde{T} / \Delta_r}{\cancel{\chi_{\parallel}} \tilde{b}_r \tilde{T} / l_{ac}}$

$\sim \frac{\tilde{b}_r l_{ac}}{\Delta_r} \sim Ku$

so cubic nonlinearity dominates for $Ku > 1$.

$Ku < 1 \Rightarrow$ drop cubic.

To compute $\langle q_r \rangle$, need

- retain ① (usual), and ②
- iterate for $\tilde{T}$ using $\nabla \cdot \underline{q} = 0$ i.e. a la QLT.

Thinking (gasp!) first:

$$\langle q_r \rangle \cong -\chi_{\parallel} \left[\langle \tilde{b}_r^2 \rangle \partial_r T + \langle \tilde{b}_r \partial_z \tilde{T} \rangle \right] - \chi_{\perp} \nabla_r \langle T \rangle$$

$$\cong -\chi_{\parallel} \left[\langle \tilde{b}_r \, \widetilde{\underline{b} \cdot \nabla T} \rangle \right] - \chi_{\perp} \nabla_r \langle T \rangle$$

linearization:

$$\tilde{b}_r \partial_r \langle T \rangle + \partial_z \tilde{T}$$

$$\chi_{\parallel} \, \underline{b} \cdot \nabla T = \chi_{\parallel} \nabla_{\parallel} T$$

$$\langle b_r q_{\parallel} \rangle$$

$\therefore$

Point: - need non-zero $b \cdot \nabla T$ fluctuation to drive heat flux

- $\boxed{\text{i.e. temperature can't be constant along line, to drive parallel, and thus perp heat flux,}}$

- $\underline{\nabla} \cdot \underline{q} = 0 \Rightarrow$ result

$\nabla_\parallel \tilde{q}_\parallel \neq 0$
$\Rightarrow \nabla_\perp q_\perp \neq 0$

must imply $\underline{\chi_\perp}$ dependence!
$\chi_\perp$ to balance

$$\langle q_r \rangle = -\chi_\parallel \left[\langle \tilde{b}_r^2 \rangle \partial_r \langle T \rangle + \langle \tilde{b}_r \partial_z \tilde{T} \rangle \right] - \chi_\perp \nabla_\perp \langle T \rangle$$

$$\nabla \cdot q = 0$$

$$\Rightarrow \nabla_\parallel \tilde{q}_\parallel + \nabla_\perp \cdot \tilde{q}_\perp = -\chi_\parallel \partial_z \tilde{b} \, \partial \langle T \rangle / \partial r$$

i.e.

24. ~~$\frac{1}{2}$~~

$$q = -\chi_{\parallel}\left[(\partial_z + \tilde{\underline{b}}\cdot\underline{\nabla})(T_0 + \tilde{T})(\underline{b}_0 + \tilde{\underline{b}})\right] - \chi_{\perp}\nabla_{\perp} T$$

So

$$-\chi_{\parallel}\partial_z^2 \tilde{T} - \chi_{\perp}\nabla_{\perp}^2 \tilde{T} = -\chi_{\parallel}\partial_z \tilde{b}\frac{\partial\langle T\rangle}{\partial r}$$

$\Rightarrow$

$$\tilde{T}_{\underline{k}} = \frac{-\chi_{\parallel} i k_z \tilde{b}_{\underline{k}}\ \partial\langle T\rangle/\partial r}{\left(\chi_{\parallel} k_z^2 + \chi_{\perp} k_{\perp}^2\right)}$$

So

$$-\chi_{\parallel}\langle\tilde{b}^2\rangle\frac{\partial\langle T\rangle}{\partial r} - \chi_{\parallel}\langle\tilde{b}\,\partial_z\tilde{T}\rangle$$

$$= -\chi_{\parallel}\sum_{\underline{k}}\left(-\frac{\chi_{\parallel}k_{\parallel}^2|\tilde{b}_{\underline{k}}|^2}{\chi_{\parallel}k_z^2 + \chi_{\perp}k_{\perp}^2} + |\tilde{b}_{\underline{k}}|^2\right)\frac{\partial\langle T\rangle}{\partial r}$$

$$= -\chi_{\parallel}\frac{\partial\langle T\rangle}{\partial r}\sum_{\underline{k}}\left(\frac{\cancel{-\chi_{\parallel}k_{\parallel}^2}}{\chi_{\parallel}k_z^2 + \chi_{\perp}k_{\perp}^2} + \frac{\cancel{\chi_{\parallel}k_{\parallel}^2} + \chi_{\perp}k_{\perp}^2}{\chi_{\parallel}k_z^2 + \chi_{\perp}k_{\perp}^2}\right)$$

80

$$\boxed{\langle q_r \rangle_{NL} = -\chi_{\parallel} \frac{\partial \langle T \rangle}{\partial r} \sum_{\underline{k}} \frac{\chi_{\perp} k_{\perp}^2 |b_{\underline{k}}|^2}{\chi_{\parallel} k_{\parallel}^2 + \chi_{\perp} k_{\perp}^2}}$$

Note explicit dependence on $\chi_{\perp}$!

80.

$$\langle q_r \rangle_{NL} \cong -\cancel{\chi_{\parallel}} \frac{\partial \langle T \rangle}{\partial r} \int dk_{\perp} \int dk_z \frac{\chi_{\perp} k_{\perp}^2 \langle \tilde{b}_{\underline{k}}^2 \rangle}{\cancel{\chi_{\parallel}} \left(k_z^2 + \frac{\chi_{\perp} k_{\perp}^2}{\chi_{\parallel}} \right)}$$

$$= -\frac{\partial \langle T \rangle}{\partial r} \int dk_{\perp} \int dk_z \frac{\chi_{\perp} k_{\perp}^2 \langle \tilde{b}^2 \rangle_{\underline{k}}}{\left(\frac{k_z^2}{(\chi_{\perp}/\chi_{\parallel}) k_{\perp}^2} + 1 \right) \left(\frac{\chi_{\perp}}{\chi_{\parallel}} k_{\perp}^2 \right)}$$

$$= -\frac{\partial \langle T \rangle}{\partial r} \int dk_{\perp} \frac{k_{\perp}^2 (\chi_{\parallel} \chi_{\perp})^{1/2}}{\sqrt{k_{\perp}^2}} \langle \tilde{b}^2 \rangle_{\underline{k}} \text{fac}$$

~~[illegible]~~
auto correlation
bandwidth

autocorrelation l_{ac} enters via normalization

$\Rightarrow$

$$\boxed{\langle q_r \rangle_{M} \cong -\sqrt{\chi_{\parallel} \chi_{\perp}} \; \underbrace{\langle \tilde{b}^2 \rangle l_{ac}} \; \underbrace{\sqrt{\overline{k_{\perp}^2}}} \; \frac{\partial \langle T \rangle}{\partial r}}$$

Note: - need $\nabla_{\parallel} \hat{T} \neq -\tilde{b}_r \, \partial \langle T \rangle / \partial r$

$(\widetilde{B \cdot \nabla T} \neq 0)$ for $\perp$ heat flux

- $\langle \tilde{b}^2 \rangle l_{ac} \sim D_M$

$\sqrt{\overline{k_{\perp}^2}} \sim 1/\Delta_{\perp}$

so

$$\langle q_r \rangle \cong -\chi_{\perp eff} \, \partial \langle T \rangle / \partial r \; - \chi_{\perp} \, \partial \langle T \rangle / \partial r$$

$$\chi_{\perp eff} \cong \sqrt{\chi_{\parallel} \chi_{\perp}} \, \frac{D_M}{\Delta_{\perp}} .$$

$$\left(\begin{array}{c} \chi_{\parallel} \chi_{\perp} \sim \\ \frac{v_{th e}^2}{\nu} \rho_e^2 \nu \sim D_B \end{array} \right)$$

$$\boxed{\chi_{\perp\text{eff}} \simeq \frac{D_B\, D_M}{\Delta_\perp}}$$

- $\chi_{\perp\text{eff}}$ scales with Bohm, not Spitzer ($\chi_\parallel$)
- kicking off line important, again.

To compare R&R:

$$\chi_\perp \sim \sqrt{\chi_\parallel \chi_\perp}\, \frac{\langle \tilde{b}^2 \rangle}{\Delta_\perp} l_{ac}$$

what is $\Delta_\perp$?

Now $\frac{\chi_\parallel}{l_c^2} \sim \frac{\chi_\perp}{\Delta_\perp^2}$ {diffusion set $\perp$ scale.

$$\Delta_\perp \sim l_c \sqrt{\chi_\perp / \chi_\parallel}$$

enters spectrum (small layer)

Basic study → BOUT++ ?
Lesson Ku

28. ~~15.~~

⇒

$$\chi_\perp \sim \sqrt{\chi_\parallel \chi_\perp} \frac{\langle \tilde{b}^2 \rangle l_{ac}}{l_c (\chi_\perp/\chi_\parallel)^{1/2}}$$

$$\boxed{\chi_\perp \sim \frac{\chi_\parallel}{l_c} D_M}$$

∴ so – modulo Ku, $\Delta_\perp$; agrees with RT R to within log. factor

– $\chi_\perp \sim V_{th} D_M \frac{l_{mfp}}{l_c}$

⇒ covers diffusion in $Ku < 1$ stochastic fields

⇒ **Lesson: Take care re: irreversibility !**

Next → Momentum