Recall

Remind → student notes

In stochastic field,

$$\frac{\partial \langle n_e \rangle}{\partial t} = - \partial_r \Gamma_{stoch} + \dots$$

$$\Gamma_{stoch} = V \langle n \rangle$$

$$\langle V \rangle = - \frac{D_B}{B} \underbrace{\partial_r \langle b_r b_\theta \rangle} - D_M \partial_r \langle V_{\parallel} \rangle$$

$$= \frac{D_B \partial \langle V_E' \rangle}{B \partial r} \langle b_r^2 \rangle \tilde{\tau}_c] - D_M \partial_r \langle V_{\parallel} \rangle$$

likewise:

$$\frac{\partial \langle V_{\parallel} \rangle}{\partial t} = - c_s \partial_r \langle \tilde{b}_r \frac{\tilde{n}}{n_0} \rangle$$

$$= \cancel{+} c_s \partial_r D_M \frac{\partial \langle n \rangle}{n_0 \partial r}$$

etc.

Before Ku > 1, etc. revisit time scales.

1D QL

$$D_v = \sum_k \left(\frac{q}{m}\right)^2 |E_k|^2 \pi \delta(\omega - kv)$$

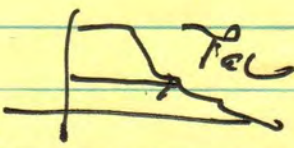
$$\sim \langle \tilde{E}^2 \rangle \tau_{ac}$$

$$= \langle (\tilde{E} \tau_{ac})^2 \rangle / \tau_{ac}$$

$$1/\tau_{ac} = \Delta(\omega - kv) \to |v_{ph} - v_{gr}| \Delta k$$

$\tau_{ac} \to \langle E(0) E(\tau) \rangle$

(self) field correlation time



sets step in Random Walk.

$\tau_c \to$ time for particle (orbit) to be scattered from its linear/unperturbed/straight orbit

$\to$ also time for exponential growth

$$\frac{dv}{dt} = \frac{q}{m} E$$

$$v = v_0 + \delta v$$

$$\frac{d}{dt} \delta v = \frac{q}{m} E$$

$$\delta v = \int^t \frac{q}{m} E$$

$$\frac{dx}{dt} = v = v_0 + \delta v$$

$$\frac{d\,\delta x}{dt} = \int \delta v$$

$$\langle \delta x^2 \rangle = \int dt \int dt \; \langle \delta v^2 \rangle$$

$$\langle \delta v^2 \rangle \sim D_v t$$

$$\langle \delta x^2 \rangle \sim D_v t^3$$

contrast Langevin

$\rightarrow$ scale of comparison.

$$\overline{k^2} \langle \delta x^2 \rangle \sim 1 \quad \Rightarrow \quad \overline{k^2} D_v \tau_c^3 = 1$$

$$1/\tau_c = \left(k^2 D_v\right)^{1/3}$$

Now, obviously:

$$\begin{matrix} l_{ac} & \longleftrightarrow & \tau_{ac} \\ l_c & & \tau_c \end{matrix}$$

$l_{ac} \rightarrow \quad \langle \tilde{B}_r(l_1)\, \tilde{B}_r(l_2) \rangle$

$$= \langle \tilde{B}_r(0)\, \tilde{B}_r(l_2 - l_1) \rangle$$

self-correlation length of field perturbation field

$l_{ac} = 1/|\Delta(k_{r\parallel})|$

For l_c

$$\frac{dz}{B_0} = \frac{r\,d\theta}{B_\theta} = \frac{dr}{B_r}$$

$$\frac{r\,d\theta}{dz} = \frac{B_\theta}{rB_0} = \frac{R\,B_\theta}{rB_0}\frac{1}{R}$$

$$= 1/R\,\Sigma(r)$$

$$\frac{dr}{dz} = \frac{\tilde{B}_r}{B_0}$$

$$\delta r = \int dz\, \tilde{B}_r / B_0$$

$$\frac{d}{dz}\delta\theta = -\frac{1}{Rq^2} q' \delta r$$

$$\frac{d(r\delta\theta)}{dz} = -\frac{\hat{s}}{R\xi} \delta r$$

$$r\delta\theta = -\frac{\hat{s}}{R\xi}\int d\cancel{z}\, \delta r$$

$$\langle r^2 \delta\theta^2 \rangle = \frac{1}{L_s^2}\int dz \int dz\, \langle \delta r^2 \rangle$$

$$= \frac{D_M}{L_s^2} z^3$$

$$\overline{k_\theta^2} \langle r^2 \delta\theta^2 \rangle \sim 1$$

$$\Rightarrow 1/l_c = \left(\frac{D_M \overline{k_\theta^2}}{L_s^2}\right)^{1/3}.$$

Physics 235

Notes 4

→ To $K_u > 1$. → How.

→ Recall, have been concerned with transport and diffusion.

Focus: $$D_M = \int_0^\infty dl \sum_k \left|\frac{\delta B_k}{B_0}\right|^2 e^{ik_\parallel l}$$

→

[$\Delta(k_M)$ cuts off integral.

$$\sim \left\langle \left(\frac{\delta B}{B_0}\right)^2 \right\rangle l_{ac}$$

Scattering: $k_\parallel = 0$ resonances.

→ $$K_u \sim \frac{l_{ac}}{\Delta}\frac{\delta B}{B_0} \sim \frac{1}{\Delta}\, \delta B/B_0 / [\Delta(k_\parallel)]$$

→ What Happens for $Ku \geq 1$?

— Recall:

~ stoch Fields

$$Ku \sim \frac{l_{ac}}{\Delta_\perp} \delta B/B_0 \sim \frac{1/\Delta_\perp \; \delta B/B_0}{\Delta |k_\parallel|}$$

$$\sim l_{ac}/l_c$$

$$\sim 1/l_{NL} / 1/l_{ac} \sim l_{ac}/l_{NL}$$

[ratio of autocorrelation to NL mixing length]

[Large Ku → NL scatt. processes control time/sp. scale.]

~ flow

$$Ku \sim \tilde{\tau}_{ac} \tilde{V}/\Delta \sim 1/\tau_{NL} / 1/\tau_{ac} \sim \tau_{ac}/\tau_{NL}$$

$\tilde{V}/\Delta$ iy

i.e. collisional DW's:

$$\tau_{ac} \sim (\Delta |x_\parallel k_\parallel^2|)^{-1}$$

$$Ku \sim 1/\tilde{\tau}_\perp (\Delta |x_\parallel k_\parallel^2|)$$

$Ku > 1$

<u>2D GC Plasma</u> – Simple / Compelling Example.

cf: Taylor + McNamara.

so Point:

$\therefore D_\perp \simeq \int_0^\infty d\tau \langle \tilde{V}(0)\tilde{V}(\tau)\rangle$ — Diffusion coeff as integral of correlation (i.e. time history ⊕ rep)

<u>Kubo</u>

<u>Formula</u> $\sim \int_0^\infty d\tau \sum_{\underline{k}} |\tilde{V}_{\underline{k}}|^2 R(\tau)$ — Diffusion coefficient as integral of correlation function

QL → γ_{ab} ... memory fctn of velocity field

$R(\tau)_\omega = e^{+i(\omega - k_\parallel v_\parallel)\tau} \; e^{-\tau/\tau_1}$ → <u>scattering</u>

from upo

$Ku >> 1$ limit corresponds to:

→ $k_\parallel \to 0$

→ $\omega \to 0$

⇒ time integral controlled by <u>non linear</u> scattering, not wave packet dispersion.

⇒ 2D GC Plasma / Fluid:

<u>i.e.</u> $\frac{\partial \rho}{\partial t} + \underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla}\rho = D_0 \nabla^2 \rho$

$\nabla^2 \phi = -4\pi \rho$ (Taylor + McNamara)

→ 2D Fluid

→ GC Plasma.

Then generally;

$$D_\perp \cong \int_0^\infty d\tau \sum_k |\tilde{V}_k|^2 e^{-ik\cdot r_0} e^{+ik\cdot r(-\tau)}$$

$$= r_0(-\tau) + \delta r(-\tau)$$

but $r(-\tau) = r_0 + \delta r(-\tau)$

↓

only evolution off of r_0 → scattering

δr → stochastic

→ need ensemble average [stochastic phase dynamics]

$$D_\perp \cong \int_0^\infty \sum_k |\tilde{V}_k|^2 \langle e^{ik\cdot\delta r(-\tau)}\rangle d\tau$$

↓

$$\langle D_\perp\rangle \cong \int_0^\infty d\tau \sum_k |\tilde{V}_k|^2 e^{-k_\perp^2 D_\perp \tau} d\tau$$

↓

turbulence/scatt. itself controls correl. time.

e.g.

$$\langle e^{ik\cdot\delta r(-\tau)}\rangle \cong \left\langle \left(1 + \cancel{ik\cdot\delta r(-\tau)} - \frac{(k\cdot\delta r)^2}{2} + \ldots\right)\right\rangle$$

$$\cong \left\langle \left(1 - \frac{(k\cdot\delta r)^2}{2} + \right)\right\rangle$$

$$\cong \left\langle \left(1 - k_\perp^2 D_\perp \tau - \right)\right\rangle$$

$$\cong e^{-k_\perp^2 D_\perp \tau}$$

scatt

$$D_\perp = \int_0^\infty d\tau \sum_k |\tilde{V}_k|^2 e^{-k_\perp^2 D_\perp \tau} d\tau$$

long coherence / large scale can immediately note:

$$= \sum_k |\tilde{V}_k|^2 \frac{1}{k_\perp^2 D_\perp}$$

— $D_\perp$ controls τ_c

— conservation $\partial_t \int \phi \, d^2x \to 0 \Rightarrow$ slow τ_c at large scale

compare to dispersion

N.B. → note recursive structure of diffusivity!

$Ku > 1$ regime

inferred divergence → τ_c in integral set by scattering $\sim 1/k_\perp^2 D$, $k_\perp^2$ from conservation ($\nabla \cdot \underline{V}$)

$\Rightarrow$ → $\underline{V} \sim \frac{c}{B} \underline{E} \times \hat{z}$ → $D_\perp \sim 1/B_0 \Rightarrow$ Bohm.

$$\boxed{(D_\perp)^2 \cong \sum_k |\tilde{V}_k|^2 / k_\perp^2}$$

→ recursive definition

→ $\sim \left(\frac{\langle \tilde{V}^2 \rangle}{k_\perp^2}\right)^{1/2} \sim \frac{c\, \phi\, T}{B_0\, T\, L}$ $\sim \rho_s c_s \left(\frac{e\phi}{T}\right)$

but 2D, assuming symmetric spectrum

$$D_\perp^2 \cong \int dk_\perp\, \cancel{k_\perp}\, |\tilde{V}_k|^2 / k_\perp^{\cancel{2}}$$

$\Rightarrow$

$$D_\perp^2 \cong \frac{c^2}{B_0^2}\int dk_\perp \frac{|E_k|^2}{k_\perp}$$

$$\boxed{D_\perp \sim \frac{c}{B_0}\left(\int dk_\perp \frac{|E_k|^2}{k_\perp}\right)^{1/2}}$$ $\Rightarrow$ Diffusivity

$\sim 1/B_0$

Now, can explore different spectra:

(cf. TPM)

~ thermal equilibrium:

$\hookrightarrow$ thermal

$$|E_k|^2 = \frac{4\pi}{\ell} k_B T \Big/ (1 + k^2 \lambda_D^2)$$ $\hookrightarrow$ Debye screening

$e/\ell \to$ charge/length

$$= \frac{4\pi}{e}\left(\frac{e}{\ell}\right) k_B T \Big/ \left(1 + k_\perp^2 \lambda_D^2\right)$$

$$D_\perp \sim \frac{c}{B_0}\left(\int_{1/L_0}^{1/\lambda_D} dk_\perp \left[\frac{4\pi}{e}\left(\frac{e}{\ell}\right) k_B T \Big/ \left(\left(1 + k_\perp^2 \lambda_D^2\right) k_\perp\right)\right]\right)^{1/2}$$

$$\sim \underbrace{\frac{c k_B T}{eB}}\left[(n\lambda^2)^{-1} \ln(L_0/\lambda)\right]^{1/2}$$

system size

so.

$$D_\perp \sim D_B \left[(k_\perp \lambda_D^2)^{-1} \ln (L_0/\lambda) \right]^{1/2}$$

- recovers basic Bohm scaling, even from thermal flctns.
- scales (weakly) with L_0 ⇒ not pure local, or "intensive".

⇒ simple example of "non-locality".

- "non-locality" appears from "slow mode" i.e. $1/\tau_c \to 0$. $\sim k_\perp^2 D_\perp$ as $k_\perp^2 \to 0$

ρ is conserved → "conserved order parameter" $\omega \sim k_\perp^2 D_\perp$

- if shear flow:

$$R(t) = \int_0^\infty d\tau \, \exp\left[i(\omega - k_\theta V_\theta)\tau - \tau/\tau_c \right]$$

Interesting to note:

- can consider diffusion due random array charges ("rods")

$\underline{\nabla}_{x}$

For spectrum:

$\underline{\nabla}\cdot\underline{E} = 4\pi\rho$

$= \frac{4\pi}{\ell}\sum_i q_i\,\delta(\underline{x}-\underline{x}_i)$

$i\underline{k}\cdot\underline{E}_k = \left(\frac{4\pi}{\ell}\right)\sum_i q_i\, e^{-i\underline{k}\cdot x_i}$

symmetric distribution $\Rightarrow$ { random array, discrete charges

$|E_k|^2 = \frac{1}{k_\perp^2}\left(\frac{4\pi}{\ell}\right)^2\left\langle\sum_{ij} q_i q_j\, e^{ik\cdot(x_j-x_i)}\right\rangle$

$= \frac{16\pi^2\, Nq^2}{k_\perp^2\ell^2}$

$\Rightarrow \quad 1/k_\perp^2$

$D_\perp^2 \sim \frac{c^2}{B_0^2}\int dk_\perp\,\frac{16\pi^2 Nq^2}{k_\perp^2\ell^2 k_\perp^2}\,k_\perp$

$\sim \frac{c^2}{B_0^2}\int_{k_{min}}^{k_{max}}\frac{16\pi^2 Nq^2}{k_\perp^3\ell^2} \sim \frac{c^2}{B_0^2}\left(\frac{16\pi^2 Nq^2}{\ell^2}\right)\frac{1}{k_{\perp min}^2}$

$k_{\perp\,min} \sim 1/L_o$

system size

$$\boxed{D_\perp \simeq \frac{c}{B_o}\, 4\pi \left(\frac{n q^2}{\ell}\right)^{1/2} L_o}$$

strong dependence on system size.

contd

Stochastic Fields — Toward High Ku, Random Conductivity

→ So far:

- reviewed theory of Hamiltonian chaos
- derived QL D_M
- derived $\chi_{\perp e}$ due stochastic fields in Ku < 1 regime — diffusion
- focused on interaction of scattering (electrons), collisions, coarse graining
- discussed transport in GC plasma, as example of $\tau_{ac} \to \infty$ regime.

→ Observations

- idea of resonance (small denominator problem) and resonance overlap fundamental to Hamiltonian chaos.
- Ku ~ $\tau_{ac}/\tau_{scatt.}$
- might ask: unified treatment that combines Ku < 1, Ku > 1 regimes ⇒ renormalized response?
- in hydro treatment of $\chi_{\perp eff}$, what of nominal 3rd order contribution vs $\chi_{\perp, coll.}$ See (K+P)?
- is diffusive treatment of high Ku regime (as in Taylor + McNamara) valid? See Bachenko, Gusakov papers.
- $\tau_{ac} \to \infty$ can recover strong $D_\perp$ at modest fluctn. level.

Here:

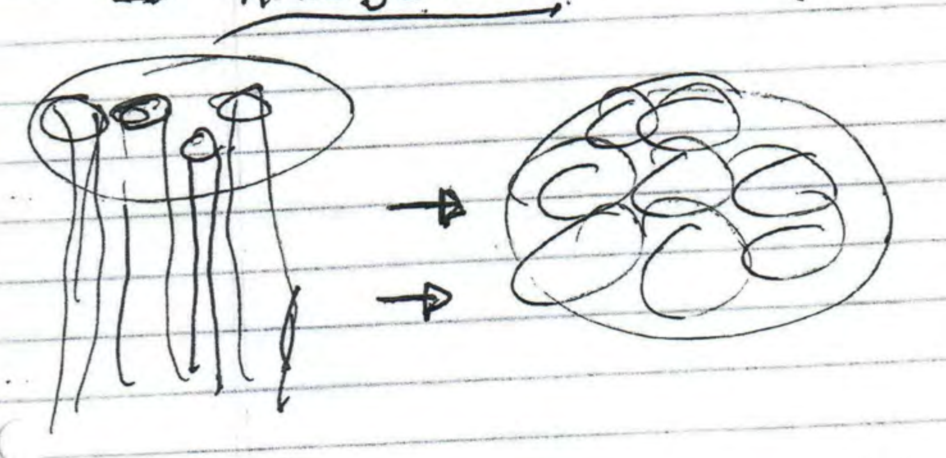
- general analysis of diffusion
- aspects of percolating, large Ku regime
- Dykhne method $\Rightarrow$ conduction in random media.

To Linear

$\Rightarrow$ Recall, $Ku \sim l_{ac}\, \delta B / B \Delta_\perp$

- have considered low Ku, with
  - finite l_{ac}
  - $\Rightarrow$ inhomogeneity in z

- now, $\boxed{\text{consider } Ku \to \infty \text{ limit, opposite}}$
  - $\Rightarrow$ random field, in x, y.
  - $\Rightarrow$ homogeneous in z.



i.e. thin rods

i.e. $\begin{cases} \dfrac{dx}{dz} = b_r = \dfrac{\partial A}{\partial y} \\ \\ \dfrac{dy}{dz} = b_\theta = -\dfrac{\partial A}{\partial x} \end{cases}$

from: $\dfrac{dr}{dz} = b_r = \dfrac{\partial A}{\partial y}$

$$r\frac{d\theta}{dz} = \cancel{\frac{\langle B_\theta \rangle}{B_0}} + b_y = -\frac{\partial A}{\partial x}$$

equivalent, of $\overset{(L_s \to \infty)}{\text{course}}$, $\overset{L_s \to \infty,\ \text{high } k_\parallel}{\text{to G.C. plasma:}}$

$$\begin{cases} \dfrac{dx}{dt} = -\dfrac{c}{B}\,\partial_y \phi \\ \\ \dfrac{dy}{dt} = \dfrac{c}{B}\,\partial_x \phi \end{cases}$$

$\Rightarrow$ motivates study of random media transport!

Formally, can extend D_M calculation to include resonance broadening

i.e. $$\frac{\partial}{\partial z}\tilde{f} + \tilde{b}\cdot\nabla\tilde{f} = -\tilde{b}_r \frac{\partial\langle f\rangle}{\partial r}$$

on beloved model

$$\Rightarrow$$

$$D_M = \sum_k |\tilde{b}_{r,k}|^2 \frac{i}{k_\parallel + i k_\perp^2 D_M}$$

where
$$k_\perp^2 D_M / k_\parallel \sim Ku^2$$

$$D_M = \sum_k \int_0^\infty e^{i k_\parallel l_\parallel} \left\langle e^{-i k_\perp\cdot\delta r(l)} \right\rangle \left|\frac{\delta B}{B}\right|^2_k$$

$$D_M = \int_0^\infty dl \left\langle \frac{\delta B_r(0)}{B_0} \frac{\delta B_r(l)}{B_0} \right\rangle$$

For $Ku < 1 \Rightarrow$

$$D_M \simeq \sum_k |\tilde{b}_{r,k}|^2 \delta(k_\parallel)$$, à la RST

For $Ku \gg 1$
$$D_M \simeq \sum_k |\tilde{b}_{r,k}|^2 / k_\perp^2 D_M$$

a la Taylor, McNamara.

$$\Rightarrow \quad D_M \cong \left(\sum_k |\tilde{A}_k|^2 \right)^{1/2}$$

$$\sim \tilde{b}\, \Delta$$

So $Ku < 1 \Rightarrow D_M \sim \tilde{b}^2 l_{ac}$

$Ku > 1 \Rightarrow D_M \sim \tilde{b}\, \Delta$

and transport $\sim \langle A^2 \rangle^{1/2}$

* But, is $Ku > 1$ regime always really diffusive ? *

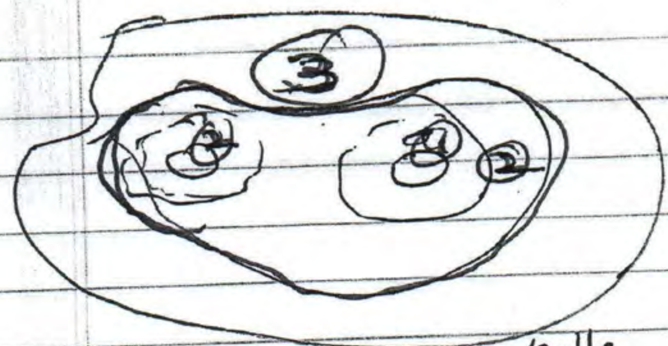
→ recall: $\frac{dx}{dz} = \nabla_\perp A \times \hat{z}$

{ akin 2D random media, for A indep z. } similar

→ Can view physically as :

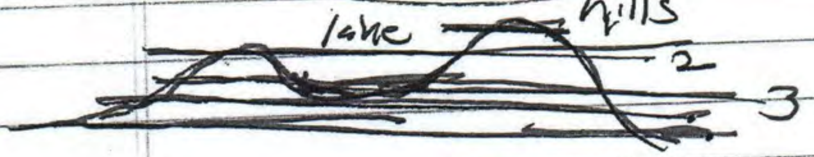
topographical map

i.e



Map

(what with ambient diff'n ?)



levels

now, as $\frac{dx}{\partial_y A} = \frac{+dy}{-\partial_x A} = \frac{dz}{1}$ $\qquad B_0 \to I$

$\Rightarrow$ $dy/dx = -\partial_x A / \partial_y A$ $\qquad$ lines

$\Rightarrow$ $\boxed{\underline{\nabla} A \cdot d\underline{x} = 0}$

$\Rightarrow$ – Lines traverse const A contours, as on map

– $\langle A \rangle = 0$, $\langle A^2 \rangle = A_0^2$

– [avg. depth/height of "lakes", "hills" set by $\underline{A_0}$

— most contours closed, isolated
⇒ little contribution to transport

— but contours along "passes"

, i.e 3, can take on long

path lengths.

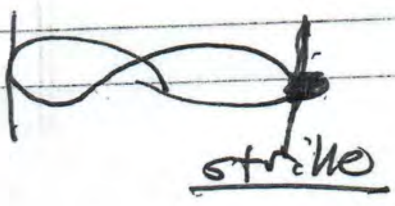
⇒ transport occurs primarily along
there.

large connections ⇒ percolation, not diffusion *

[a → b transport
isolated along
contour c.

* ~ [more like 'lightning bolt' than
diffusion. Heat "channeled" along c]

~ signature would be sharply
localized strike mark (if b → PFC)
and not periodic or smooth.



vs spread load

→ percolation → extension of mean length as $A \to 0$

$$\boxed{l_A \sim A^{-\gamma}}$$

→ replaces concept of m.f.p.

Message:

- to understand $Ku > 1$ regime, useful to examine;

→ transport in random media

→ percolation theory

$$l \sim (p - p_c)^{-\nu}$$

$$\to (A_c - A)^{-\nu}$$

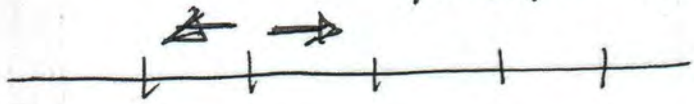
$$\to A^{-\gamma}$$

To percolation $\longrightarrow$

Percolation processes 1

Percolation vs. Diffusion $\to$ Comparison/ contrast

a) Diffusion $\Rightarrow$ 1D random walk
prob. 1/2 for each way, each step



- medium fixed
- particle motion stochastic

$$\delta x^2 \sim Dt$$

$$\sim N$$

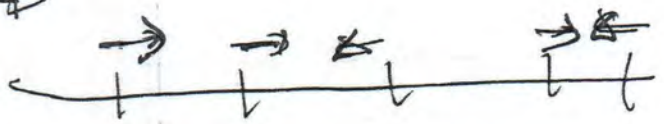
$\hookrightarrow$ # steps

and particle can return.

b.) Percolation - assign left/right orientation to each site, with probability 1/2

particle

$\Rightarrow$



etc.,

- Medium stochastic
- particle motion deterministic

i.e. random conductivity

Simply: { → diffusion: medium deterministic, motion stochastic $Ku \ll 1$

→ percolation: motion deterministic, medium stochastic $Ku > 1$ }

Examples: ~ "random" media at low, high Ku.

a.) Cascade process

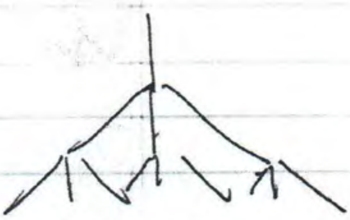
$(p+q) = 1$

descendant

0 desc. 1 desc. 2 desc.

q^2 $2pq$ p^2.

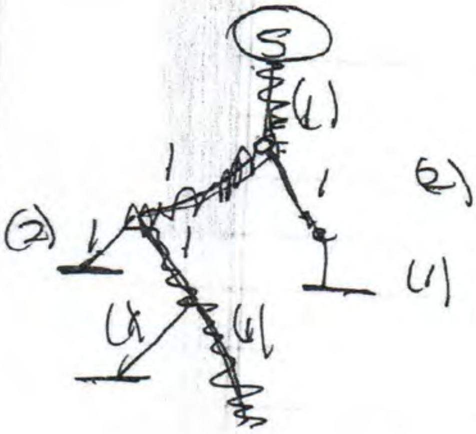
can think of as diffusion



or, does one generation "reach" N into future generations.

$\approx$

0 1 or 2



Σ probability of blocked
p of 1, 2 cont.

percolation

In percolation:

→ intrinsic and random properties of medium determine motion. i.e. cell arrangement

→ in diffn / stoch motion indeterminate. prop. particle fundamental (i.e. stochastic orbits)..

Origin of random characteristics of medium:

→ randomly dam / cut connections
→ dimensionality matters

→ result is random maze.

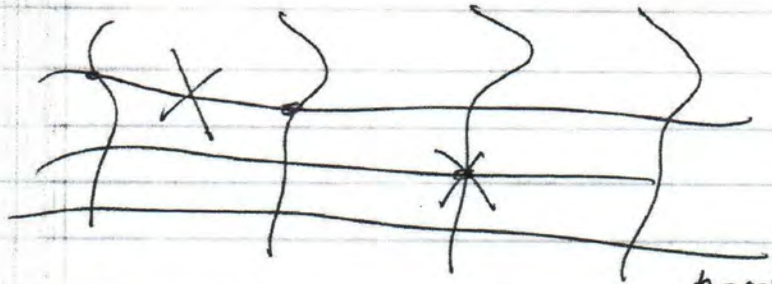
→ flow can traverse A→B only if there is an un-dammed, un-cut self-avoiding random walk connecting A,B

SAW visits intermediates at most once.

→ General aspects (mostly topological)

– can have "bond" or "site" percolation

i.e



$\lfloor N$

bond → some fraction of ~~[illegible]~~ bonds/connections missing

site → some fraction of sites missing

⇒ game played either way.

– Consider:

→ lattice of N sites, $N \gg 1$

→ concentration of allowed sites $\underline{X}$

→ can envision increasing X from below, i.e.

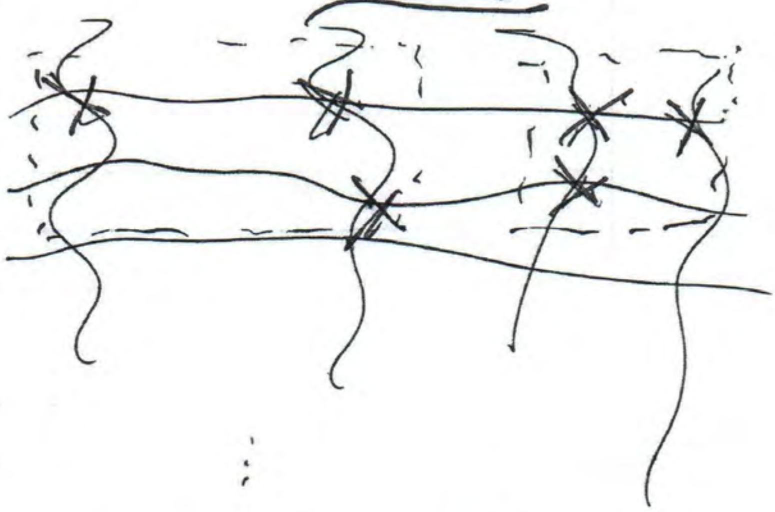
1½

i.e.

2 cluster

X_1

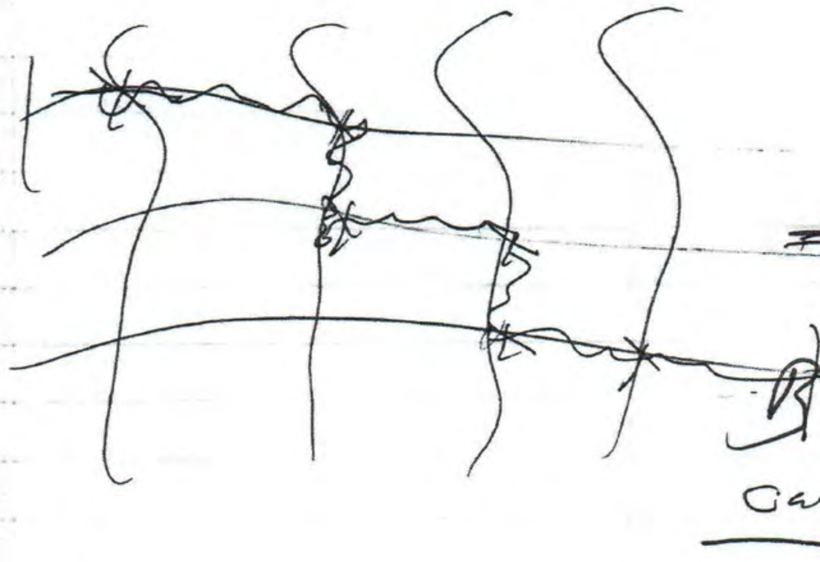
3 clusters



$X_2 > X_1$

$N \gg 1$

:



connection

⇒ cluster spans network

can traverse.

∞ $N \to \infty$.

- $x \ll x_c$ → isolated small clusters
- $x\uparrow$ → larger clusters form.
- $\ell(x)$ ↑ with x
  size
- as $x\uparrow$, few large clusters form
- $\ell(x) \to \infty$ as $N \to \infty$
- one cluster for $x > x_c$

$\boxed{\text{phase transition}}$

$P^s(x) \equiv$ site percolation probability

$\equiv$ ratio of # sites in big (infinite) cluster to # sites in lattice

$\equiv$ fraction of system in which DC conduction is possible.

And:

- there is a threshold concentration of active sites, x_c
- near threshold:

$$\rho^{s}(x) \sim \cancel{(x - x_c)}^{s}$$

→ perc. exponent ↓ percolation exponent.

↓ fraction of conduction

geometry universality

- 3D: $s \sim .3 \to .4$ corresponding lattice structure

⇒

- percolation is a type of phase transition

i.e. $s \Leftrightarrow$ critical exponent

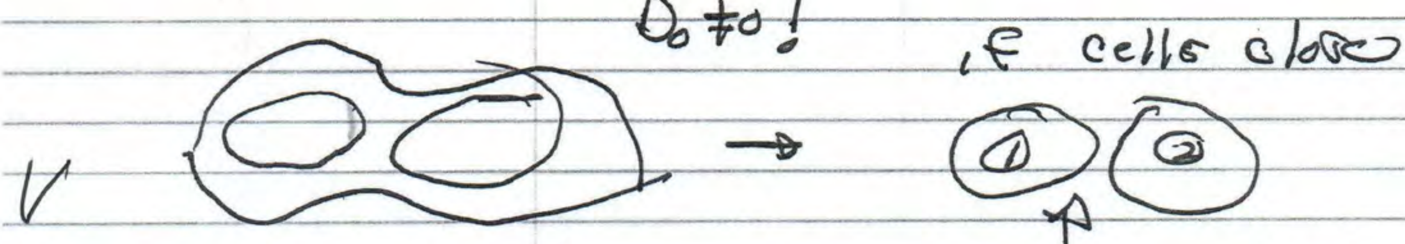
$x_c \leftrightarrow$ critical temp / pt.

$\ell_A \sim \ell_c$.

- key question is effective medium model near x_c.
- connection to turbulence: intermittency → reduced packing fraction.

Notes 5 — Collisional Diffusion + Scattering
→ Taylor Cells
→ Taylor Shear Dispersion

→ Recall 2D transport in stochastic field



theme: diffusive kick off line.

collisional diffusion can kick ① → ②

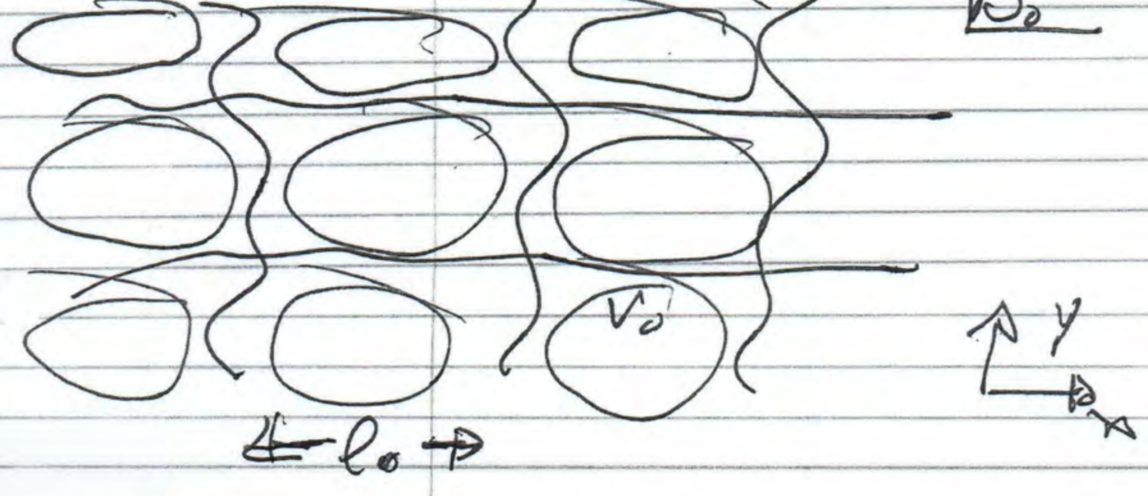
$$D_{\perp\,\text{eff}} = D_\perp(\tilde{b}, \Delta_\perp, D_0)$$

new overlap — corresponds to D_M here

→ [Many problems involve synergy between turbulent scattering and collisional diffusion]

→ Taylor Problem – the Classic

[Geometry matters]



For passive scalar problem:

$$\frac{\partial n}{\partial t} + \underline{v} \cdot \underline{\nabla} n - D_0 \nabla^2 n = 0$$

can define: $Pe = V_0 l_0 / D_0$
→ Peclet number
→ Parm of interest

Interest: – Effect transport coefficient, i.e. diffusivity, for scales $L \gg l_0$

(*) – effective medium problem with 2 transport process:

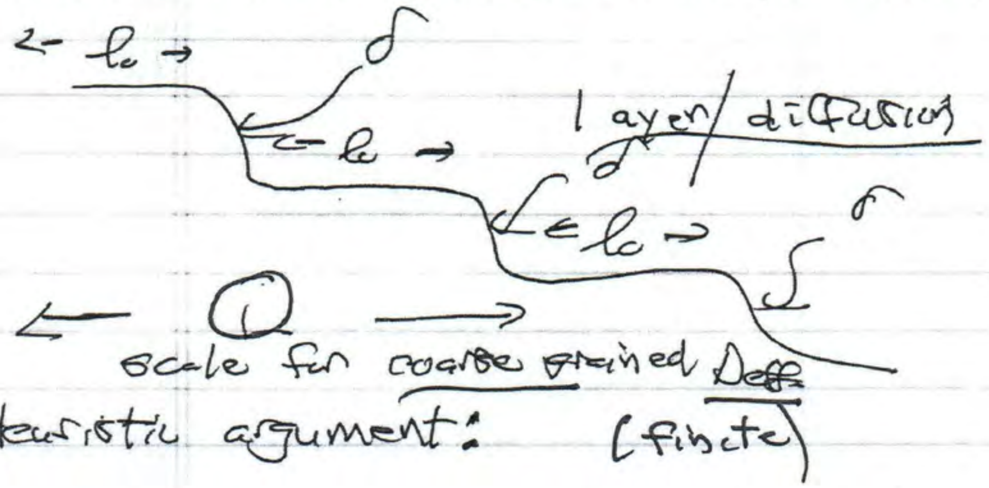
fast → convection → operating on cells
slow → diffusion → operating on boundary layers

What is D_{eff}?

Point: – transport is hybrid of ⟨ fast kicks thru cell | slow diffn thur BL |

– [diffusion is ultimate origin of irreversibility for static cells. Only BL particles transported.

Can envision concentration profile:



"staircase"?!

Heuristic argument:

→ for random walk:

$$D_{eff} \simeq \text{factor} \, \frac{(\Delta x)^2}{\Delta t}$$

↱ fraction of active region → fraction where diffusion occurs.

factor: active fraction for diffusion

$\sim \underline{\delta / \ell_0}$ small in BL thickness

Δt: cell circulation time

$\sim \underline{\ell_0 / V_0}$

then, $$\boxed{\delta^2 \sim D_0 \, \Delta t \sim D_0 \frac{\ell_0}{V_0}}$$

[Diffusion in transit time thru layer

and $\Delta x \sim l_0$ (cell scale).

so

$$D_{eff} \cong \frac{\delta}{l_0} \frac{l_0^2}{l_0/v_0}$$

$$\cong \left(D_0 \frac{l_0}{v_0}\right)^{1/2} \frac{l_0 v_0}{l_0}$$

$$\cong (D_0 v_0 l_0)^{1/2}$$

$$D_{eff} \cong (D_0 D_{cell})^{1/2} \cong \boxed{D_0 (Pe)^{1/2}}$$

→ D_{eff} is geometric mean of D_0 (slow) and D_{cell} (fast), $\sim Pe^{1/2}$, large Pe

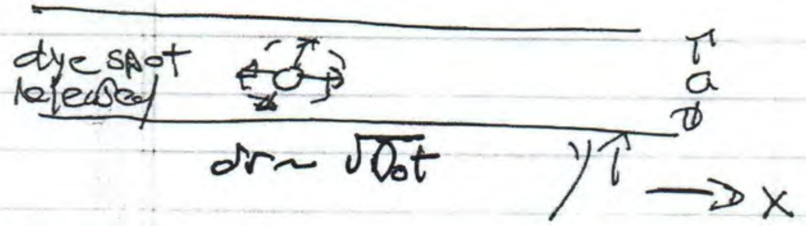
→ resembles Dykhne result, but
⇒ Dykhne → equal areas σ_1, σ_2 cells → $\delta_{active}/l_0 \ll 1$

→ see Rosenbluth, et. al. 1987 for details of calculation (tedious).

→ Result is not simple addition

Related problem : Shear dispersion

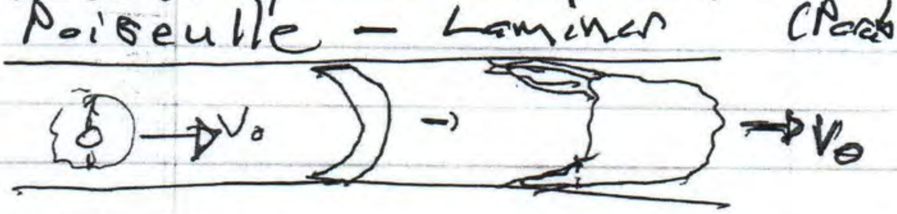
see Taylor 1953 et seq (many postings)
Young & S. Jones especially good
Problem stated by comparison of three laminar flows, into which dye with molecular diffusion D_0 is injected.



Flow (1) stationary water.

Pipe (2) Laminar Flow at V_0. Slip BC

necessarily a shear flow,
. Poiseulle — Laminar (Parabolic)



Pipe (3) No-slip BC.

Pipe (2) : – slug CM advects at V_0

– slug expands axially at $(D_0 t)^{1/2}$ → molecular diffusion

Shear Dispersion:

→ what is effective along stream diffusivity of passive scalar in a laminar shear flow.

simplest - Poiseuille Flow
- BC → shear

" the fundamental character of the result that differential unidirectional convection and transverse diffusion yield a longitudinal diffusion process far downstream"

Laminar Flow

→ Lecture Piece ③:

- experiments (cf. 1953 paper) indicate more rapid dispersal of dye in sheared flow, i.e. effective axial diffusion enhanced

$$D_{eff\ axial} > D_0$$

$$\Rightarrow D_{eff} = D_0 + D_{shear}$$

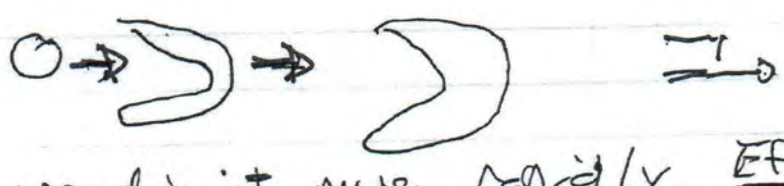
why?

$\langle \ \rangle \to \perp$ avg

$\frac{\partial c}{\partial t} + \underline{v} \cdot \nabla c = D_0 \nabla^2 c \rightarrow \partial_t \langle c \rangle + \partial_x \langle \tilde{v} \tilde{c} \rangle = D_0 \partial_x^2 \langle c \rangle$

- though cm velocity same, V_0!

- velocity shear stretches cloud,



spreading it more rapidly. Effect is shear + D_0

⊥ collisional scattering sets

How calculate ?!

Consider dye concentration field $c(x, y, t)$

Calculation - simple

Now, consider scalar equation:

$$\frac{\partial c}{\partial t} + \underline{v} \cdot \underline{\nabla}\, c = D_0 \nabla^2 c$$

$$\langle c \rangle = \frac{1}{a} \int_{-a/2}^{a/2} dy\; c(x, y, t)$$

$$\langle v \rangle = \frac{1}{a} \int_{-a/2}^{a/2} dy\; v(y, x)$$

y across section avg (2D, 3D)

$$\frac{\partial c}{\partial t} + \underline{v} \cdot \underline{\nabla}\, c = D_0 \nabla^2 c \qquad (1)$$

$$= D_0 (\partial_x^2 + \partial_y^2)\, c$$

$$\partial_t \langle c \rangle + \langle v \rangle \partial_x \langle c \rangle + \partial_x \langle \tilde{v} \tilde{c} \rangle \qquad (2)$$

$$= D_0 \partial_x^2 \langle c \rangle$$

$$\tilde{v} = v - \langle v \rangle$$

$$\tilde{c} = c - \langle c \rangle$$

Now, subtract (2) from (1):

$$\partial_t \langle C \rangle + \partial_t \tilde{C} + \langle V \rangle \partial_x \langle C \rangle + \tilde{V} \partial_x \langle C \rangle + \langle V \rangle \cdot \nabla \tilde{C} + \partial_x \langle \tilde{V} \tilde{C} \rangle + \nabla \cdot \widetilde{\tilde{V} \tilde{C}} = D_0 \left(\partial_x^2 \langle C \rangle \right) + D_0 \partial_x^2 \tilde{C} + D_0 \, \cancel{\partial_y^2 \langle C \rangle} + D_0 \partial_y^2 \tilde{C} \qquad (1')$$

$$\partial_t \langle C \rangle + \langle V \rangle \partial_x \langle C \rangle + \partial_x \widetilde{\langle \tilde{V} \tilde{C} \rangle} = D_0 \partial_x^2 \langle C \rangle \qquad (2')$$

subtracting:

$$\partial_t \tilde{C} + \langle V \rangle \cdot \nabla \tilde{C} - D_0 \nabla^2 \tilde{C} + \tilde{V} \partial_x \langle C \rangle = D_0 \partial_y^2 \tilde{C}$$

Define:

$$\left(\frac{d}{dt} \tilde{C} \right) = \partial_t \tilde{C} + \langle V \rangle \cdot \nabla \tilde{C} - D_0 \nabla^2 \tilde{C}$$

$$\left(\frac{d}{dt}\tilde{c}\right)_{slow} + \tilde{v}\,\partial_x \langle c\rangle = \underbrace{D_0\,\partial_y^2\,\tilde{c}}_{fast}$$

Now,

$\uparrow L_\perp \downarrow$

→ pipe has finite ⊥ scale

→ this defines natural time scale – in frame co-moving with mean flow – of

$$\tilde{\tau}_{diff} \sim L_\perp^2 / D_0$$

$\left[\begin{array}{l} L_\perp \ll L_\parallel \\ \to \text{time scale separation} \end{array}\right]$

For $t \gg \tau_{diff} \cong L_\perp^2/D_0$, $\tilde{c}$ diffusively damped.

So, time asymptotically, i.e. $t \gg \tilde{\tau}_{\perp\,diff}$ have dominant balance:

$$\tilde{v}_x\,\partial_x \langle c\rangle \cong D_0\,\partial_y^2\,\tilde{c}$$

Recall: $\tilde{v}_x = v - \langle v\rangle$

Fourier expand:

$\sim$ diff from channel eqs.

$$\tilde{c} = \sum_{k_y} e^{ik_y} \tilde{c}_{k_y}$$

$\tilde{v}$ similar.

$k_{y\,min} \cong 2\pi / L_\perp$

$\Rightarrow$

$$\tilde{V}_{x_{k_y}} \partial_x \langle C \rangle = -k_y^2 D_0 \tilde{C}_{k_y}$$

$$\tilde{C}_{k_y} = \frac{-1}{k_y^2 D_0} \tilde{V}_{x_{k_y}} \partial_x \langle C \rangle$$

so, what is sought:

$$\langle \tilde{V}_x \tilde{C} \rangle = \sum_{k_y} - \frac{|\tilde{V}_{x_{k_y}}|^2}{k_y^2 D_0} \frac{\partial \langle C \rangle}{\partial x}$$

so, recalling:

$$\partial_t \langle C \rangle + \partial_x \langle \tilde{V}_x \tilde{C} \rangle = D_0 \partial_x^2 \langle C \rangle$$

so

$$D_{eff} = D_0 + D_{shear}$$

$$D_{shear} = \sum_{k_y} |\tilde{V}_{k_y}|^2 / k_y^2 D_0$$

$$\rightarrow \# \frac{V_0^2 L_\perp^2}{D_0}$$

$$\boxed{\begin{aligned} D_{eff} &= D_0 + D_{shear} \\ &= D_0 + \# \frac{V_0^2 L_\perp^2}{D_0} \end{aligned}}$$

Note that for Laminar flow

$D_{shear} > D_0$ quite possible

$\Rightarrow$ enhanced along stream / in frame diffusivity.

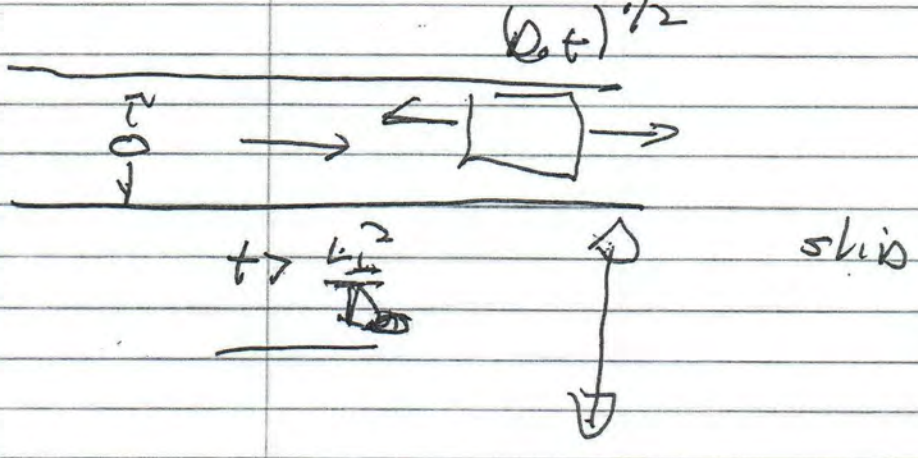
$\Rightarrow$ enhanced dispersion / "shear dispersion"

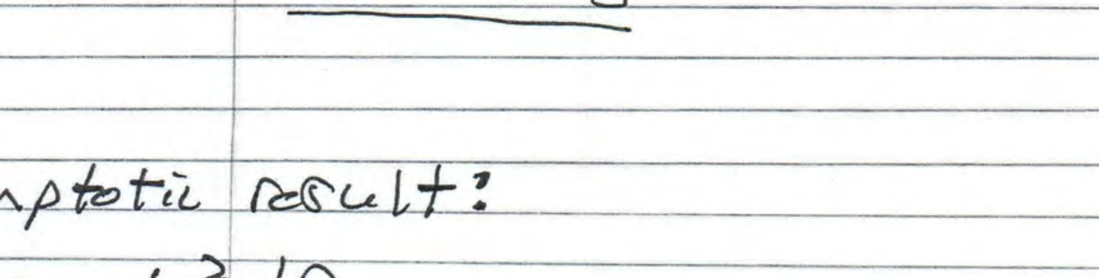
Several Points:

→ What is happening?

"the fundamental character of the result / differential unidirectional convection and transverse difusion yield a longitudinal diffusion process for downstream"

i.e.

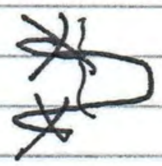




→ asymptotic result:

$t \gg L_\perp^2 / D_0$

→ uni-directional velocity essential



→ Details:

$$v(y,z) = 2(u/a^2)(a^2 - y^2 - z^2)$$

$$\# = 1/48$$

$$D_{eff} = D_0 + \frac{1}{48}\frac{V_0^2 a^2}{D_0}$$

$D_{sh} > D_{eff}$ for $P_e > \sqrt{48}$ ✓

→ For turbulent flow,

$$D_{shear} \sim \# \frac{u_*^2 a^2}{D_{shear}}$$

$$D \sim u_* a.$$

More generally:

→ Origin of irreversibility is D_0 (Laminar flow). $\tilde{\tau}_c^{-1} \sim D_0 / \ell_\perp^2$

→ Akin cell, laminar flow + molecular diffusion yields transport

→ Taylor proposed shear dispersion as mechanism for distributing nutrients in blood flow.

→ time scale ordering is crucial.

→ Excellent topic for further study.