next section) appears because of the four-wave processes and can take waves out of resonance if the set of wavevectors is discrete, owing to a finite box size (it is a mechanism of instability restriction for finite-dimensional systems like an oscillator – swing frequency decreases with amplitude, for instance). If, however, the box is large enough, the frequency spectrum is close to continuous and there are waves in resonance for any non-linearity. In this case, the saturation of instability is caused by renormalization of the damping and pumping. The renormalization (increase) of γ_k appears because of the waves of the third generation that take energy from a_1, a_2. The pumping renormalization appears because of the four-wave interaction, for example, (3.25) adds $-ia_2^* \int T_{1234} a_3 a_4 \delta(\mathbf{k}-\mathbf{k}_3-\mathbf{k}_4)\, d\mathbf{k}_3 d\mathbf{k}_4$ to $\dot{a}_1$.

3.3 Non-linear Schrödinger equation (NSE)

This section is devoted to a non-linear spectrally narrow wave packet. Consideration of the linear propagation of such a packet in Section 3.1.4 taught us the notions of phase and group velocities. In this section, the account of non-linearity brings equally fundamental notions of the Bogoliubov spectrum of condensate fluctuations, modulational instability, solitons, self-focusing, collapse and wave turbulence.

3.3.1 Derivation of NSE

Consider a quasi-monochromatic wave packet in an isotropic non-linear medium. Quasi-monochromatic means spectrally narrow, that is the wave amplitudes are non-zero in a narrow region Δk of $\mathbf{k}$-space around some $\mathbf{k}_0$. In this case the processes changing the number of waves (like $1 \to 2+3$ and $1 \to 2+3+4$) are non-resonant because the frequencies of all waves are close. Therefore, all the non-linear terms can be eliminated from the interaction Hamiltonian except $\mathcal{H}_4$ and the equation of motion has the form

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = -i \int T_{k123} a_1^* a_2 a_3 \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3. \tag{3.27}$$

Consider now $\mathbf{k} = \mathbf{k}_0 + \mathbf{q}$ with $q \ll k_0$ and expand, similar to (3.13),

$$\omega(k) = \omega_0 + (\mathbf{qv}) + \frac{1}{2} q_i q_j \left(\frac{\partial^2 \omega}{\partial k_i \partial k_j} \right)_0 ,$$

where $\mathbf{v} = \partial\omega/\partial\mathbf{k}$ at $k = k_0$. In an isotropic medium ω depends only on modulus k and

$$q_i q_j \frac{\partial^2 \omega}{\partial k_i \partial k_j} = q_i q_j \frac{\partial}{\partial k_i} \frac{k_j}{k} \frac{\partial \omega}{\partial k} = q_i q_j \left[\frac{k_i k_j \omega''}{k^2} + \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \frac{v}{k} \right]$$
$$= q_\parallel^2 \omega'' + \frac{q_\perp^2 v}{k}.$$

Let us introduce the temporal envelope $a_k(t) = \exp(-\mathrm{i}\omega_0 t)\psi(\mathbf{q},t)$ into (3.27):

$$\left[\mathrm{i}\frac{\partial}{\partial t} - (\mathbf{qv}) - \frac{q_\parallel^2 \omega''}{2} - \frac{q_\perp^2 v}{2k} \right] \psi_q = T \int \psi_1^* \psi_2 \psi_3 \delta(\mathbf{q} + \mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3)\, d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3.$$

We assumed the non-linear term to be small, $T|a_k|^2(\Delta k)^{2d} \ll \omega_k$, and took it at $k = k_0$. This result is usually represented in r-space for $\psi(\mathbf{r}) = \int \psi_q \exp(\mathrm{i}\mathbf{qr})\, d\mathbf{q}$. The non-linear term is local in r-space:

$$\int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \psi^*(\mathbf{r}_1)\psi(\mathbf{r}_2)\psi(\mathbf{r}_3) \int d\mathbf{q} d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3 \delta(\mathbf{q} + \mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3)$$
$$\times \exp\Big[\mathrm{i}(\mathbf{q}_1\mathbf{r}_1) - \mathrm{i}(\mathbf{q}_2\mathbf{r}_2) - \mathrm{i}(\mathbf{q}_3\mathbf{r}_3) + \mathrm{i}(\mathbf{qr}) \Big]$$
$$= \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \psi^*(\mathbf{r}_1)\psi(\mathbf{r}_2)\psi(\mathbf{r}_3)\delta(\mathbf{r}_1 - \mathbf{r})\delta(\mathbf{r}_2 - \mathbf{r})\delta(\mathbf{r}_3 - \mathbf{r}) = |\psi|^2\psi,$$

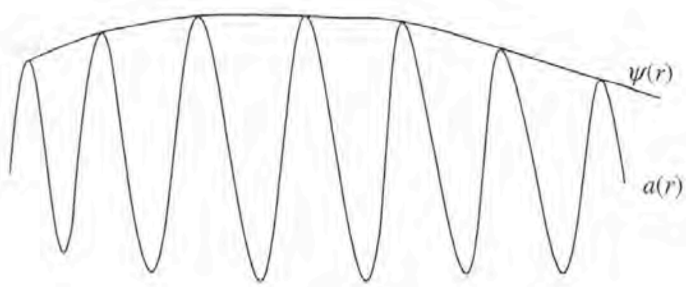
and the equation takes the form

$$\frac{\partial \psi}{\partial t} + v\frac{\partial \psi}{\partial z} - \frac{\mathrm{i}\omega''}{2}\frac{\partial^2 \psi}{\partial z^2} - \frac{\mathrm{i}v}{2k}\Delta_\perp \psi = -\mathrm{i}T|\psi|^2\psi.$$

Here the term $v\partial_z$ is responsible for propagation with the group velocity, $\omega''\partial_{zz}$ for dispersion and $(v/k)\Delta_\perp$ for diffraction. One may ask why in the expansion of $\omega_{\mathbf{k}+\mathbf{q}}$ we kept the terms both linear and quadratic in small q. This is because the linear term (which gives $\partial\psi/\partial z$ in the last equation) can be eliminated by the transition to the moving reference frame $z \to z - vt$. We also renormalize the transversal coordinate by the factor $\sqrt{k_0\omega''/v}$ and obtain the celebrated non-linear Schrödinger equation

$$\mathrm{i}\frac{\partial \psi}{\partial t} + \frac{\omega''}{2}\Delta\psi - T|\psi|^2\psi = 0. \tag{3.28}$$

Sometimes (particularly for $T < 0$) it is called the Gross–Pitaevsky equation after the scientists who derived it to describe a quantum condensate. This equation is meaningful at different dimensionalities. It may describe the evolution of a three-dimensional packet, as in a Bose–Einstein condensation of cold atoms. When $\mathbf{r}$ is two-dimensional, it may correspond either to the evolution of the packet in a 2D medium (say, for surface waves) or to steady propagation in 3D described by $iv\psi_z + (v/2k)\Delta_\perp\psi = T|\psi|^2\psi$, which turns into (3.28) upon relabelling $z \to vt$. In a steady case, one neglects ψ_{zz} since this term is much less than ψ_z. In a non-steady case, this is not necessarily so, since ∂_t and $v\partial_z$ might be about to annihilate each other and one is interested in the next terms. And, finishing with dimensionalities, the one-dimensional NSE corresponds to a stationary two-dimensional case.



Different media provide for different signs of the coefficients. Apart from hydrodynamic applications, the NSE also describes non-linear optics. Indeed, Maxwell's equation for waves takes the form $[\omega^2 - (c^2/n)\Delta]E = 0$. The refraction index depends on the wave intensity: $n = 1 + 2\alpha|E|^2$. There are different reasons for that dependence (and so different signs of α may be realized in different materials), for example: electrostriction, heating and the Kerr effect (orientation of non-isotropic molecules by the wave field). We consider waves moving mainly in one direction and pass into the reference frame moving with the velocity c, i.e. change $\omega \to \omega - ck$. Expanding

$$ck/\sqrt{n} \approx ck_z(1 - \alpha|E|^2) + ck_\perp^2/2k,$$

substituting it into

$$(\omega - ck - ck/\sqrt{n})(\omega - ck + ck/\sqrt{n})E = 0,$$

and retaining only the first non-vanishing terms in diffraction and non-linearity, we obtain the NSE after the inverse Fourier transform. In particular, the one-dimensional NSE describes light in optical fibres.

3.3.2 Modulational instability

The simplest effect of the four-wave scattering is frequency renormalization. Indeed, the NSE has a stationary solution as a plane wave with a renormalized frequency $\psi_0(t) = A_0 \exp(-\mathrm{i}TA_0^2 t)$ (in quantum physics, this state, coherent across the whole system, corresponds to a Bose–Einstein condensate). Let us describe small perturbations of the condensate. We write the perturbed solution as $\psi = A\mathrm{e}^{\mathrm{i}\varphi}$ and assume the perturbation to be one-dimensional (along the direction which we denote ξ). Then,

$$\psi_\xi = (A_\xi + \mathrm{i}A\varphi_\xi)\mathrm{e}^{\mathrm{i}\varphi}, \quad \psi_{\xi\xi} = (A_{\xi\xi} + 2\mathrm{i}A_\xi\varphi_\xi + \mathrm{i}A\varphi_{\xi\xi} - A\varphi_\xi^2)\mathrm{e}^{\mathrm{i}\varphi}.$$

Introduce the current wavenumber $K = \varphi_\xi$. The real and imaginary parts of the linearized NSE take the form

$$\tilde{A}_t + \frac{\omega''}{2}A_0 K_\xi = 0, \quad K_t = -2TA_0\tilde{A}_\xi + \frac{\omega''}{2A_0}\tilde{A}_{\xi\xi\xi}.$$

We look for the solution in the form where both the amplitude and the phase of the perturbation are modulated:

$$\tilde{A} = A - A_0 \propto \exp(\mathrm{i}k\xi - \mathrm{i}\Omega t), \quad K \propto \exp(\mathrm{i}k\xi - \mathrm{i}\Omega t).$$

The dispersion relation for the perturbations then takes the form:

$$\Omega^2 = T\omega'' A_0^2 k^2 + \omega''^2 k^4/4. \tag{3.29}$$

When $T\omega'' > 0$, it is called the Bogoliubov formula for the spectrum of condensate perturbations. We have an instability when $T\omega'' < 0$ (the Lighthill criterion). I first explain this criterion using the language of classical waves and at the end of the section I give an alternative explanation in terms of quantum (quasi)-particles. Classically, we define the frequency as minus the time derivative of the phase: $\varphi_t = -\omega$. For a non-linear wave, the frequency is generally dependent on both the amplitude and the wavenumber. The factors T and ω'' are the second derivatives of the frequency with respect to the amplitude and the wavenumber, respectively. That is, instability happens when the surface $\omega(k, A)$ has a saddle point at $k = 0 = A$. Intuitively, one can explain the modulational instability in the following way: consider, for instance, $\omega'' > 0$ and $T < 0$. If the amplitude acquires a local minimum as a result of perturbation then the frequency has a maximum there because $T < 0$. The time derivative of the current wavenumber is as follows: $K_t = \varphi_{\xi t} = -\omega_\xi$. The local maximum in ω means that K_t changes sign, that is K will grow to the right of the ω maximum and decrease to the

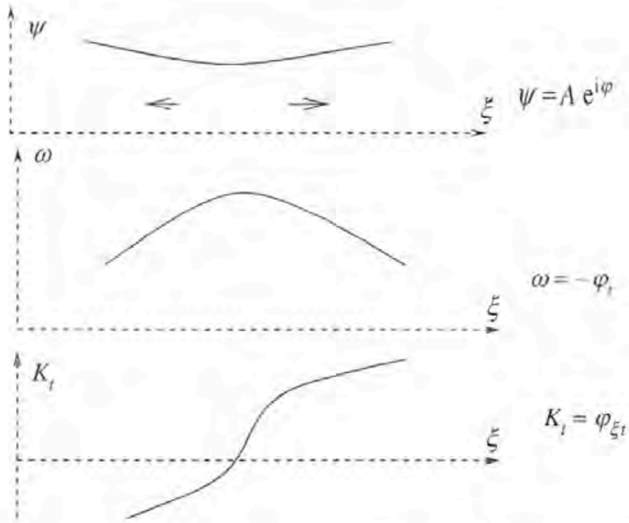


Figure 3.5 Space dependencies of the wave amplitude, frequency and time derivative of the wavenumber, which demonstrate the mechanism of the modulational instability for $\omega'' > 0$ and $T < 0$.

left of it. The group velocity ω' grows with K since $\omega'' > 0$. Then the group velocity grows to the right and decreases to the left so that the parts separate (as the arrows show) and the perturbation deepens, as shown in Figure 3.5.

The result of this instability can be seen on the beach, where waves coming to the shore are modulated. Indeed, for long water waves $\omega_k \propto \sqrt{k}$ so that $\omega'' < 0$. As opposed to a pendulum and somewhat counterintuitively, the frequency grows with the amplitude and $T > 0$; it is related to the change of wave shape from sinusoidal to that forming a sharpened crest, which reaches 120° for sufficiently high amplitudes. A long water wave is thus unstable with respect to longitudinal modulations (Benjamin–Feir instability, 1967). The growth rate is maximal for $k = A_0\sqrt{-2T/\omega''}$, which depends on the amplitude (Figure 3.6). Still, folklore has it that approximately every ninth wave is the largest.

For transverse propagation of perturbations, one has to replace ω'' by v/k, which is generally positive so the criterion of instability is $T < 0$ or $\partial\omega/\partial|a|^2 < 0$, which also means that for instability the wave velocity has to decrease with amplitude. This can be easily visualized: if the wave is transversely modulated then the parts of the front where the amplitude is larger will move slower and further increase the amplitude because of focusing from neighbouring parts, as shown in Figure 3.7.

Let us now find a quantum explanation for the modulational instability. Remember that the NSE (3.28) is a Hamiltonian system ($i\psi_t = \delta\mathcal{H}/\delta\psi^*$) with

$$\mathcal{H} = \frac{1}{2}\int\left(\omega''|\nabla\psi|^2 + T|\psi|^4\right)d\mathbf{r}. \tag{3.30}$$

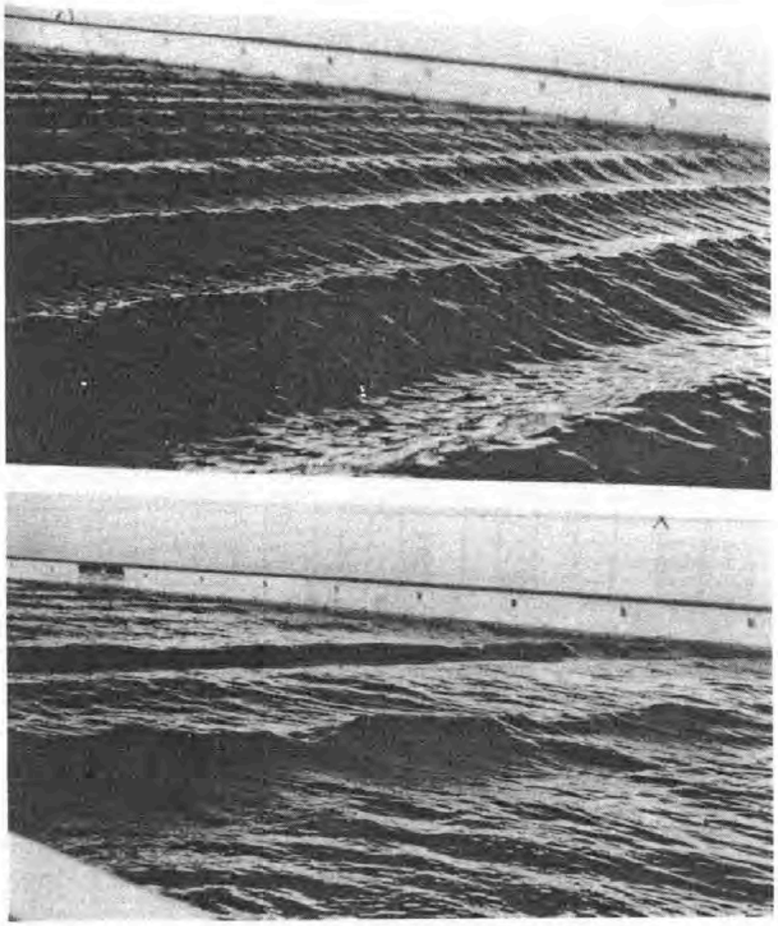


Figure 3.6 Disintegration of the periodic wave due to modulational instability as demonstrated experimentally by Benjamin and Feir (1967). The upper photograph shows a regular wave pattern close to a wavemaker. The lower photograph is made some 60 metres (28 wavelengths) away, where the wave amplitude is comparable, but spatial periodicity is lost. The instability was triggered by imposing on the periodic motion of the wavemaker a slight modulation at the unstable side-band frequency; the same disintegration occurs naturally over longer distances. Photograph by J. E. Feir, reproduced from *Proc. R. Soc. Lond. A*, **299**, 59 (1967).

The Lighthill criterion means that the modulational instability happens when the Hamiltonian is not sign-definite. The overall sign of the Hamiltonian is unimportant, as one can always change $\mathcal{H} \to -\mathcal{H}$, $t \to -t$; it is important that the Hamiltonian can have different signs for different configurations of $\psi(\mathbf{r})$. Consider $\omega'' > 0$. Using the quantum language one can interpret the first term in the Hamiltonian as the kinetic energy of (quasi)-particles and the second term as their potential energy. For $T < 0$, the interaction is attractive, which leads to the instability. For the condensate, the kinetic energy (or pressure) is balanced by the interaction; a local perturbation with more particles (higher $|\psi|^2$) will

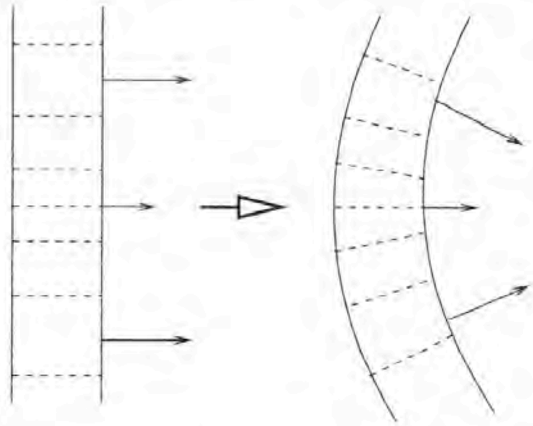


Figure 3.7 Transverse instability for the velocity decreasing with the amplitude.

make the interaction stronger, which leads to the contraction of perturbation and further growth of $|\psi|^2$.

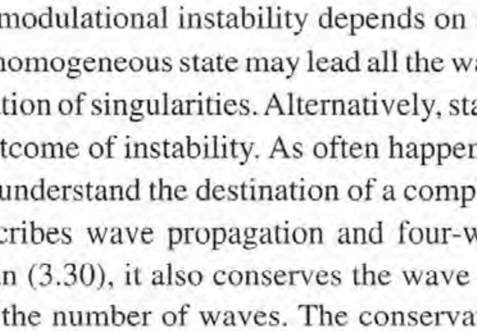
3.3.3 Soliton, collapse and turbulence

The outcome of the modulational instability depends on space dimensionality. The breakdown of a homogeneous state may lead all the way to small-scale fragmentation or the creation of singularities. Alternatively, stable finite-size objects may appear as an outcome of instability. As often happens, analysis of conservation laws helps to understand the destination of a complicated process. Since the NSE (3.28) describes wave propagation and four-wave scattering, apart from the Hamiltonian (3.30), it also conserves the wave action $N = \int |\psi|^2 \, d\mathbf{r}$, which one may call the number of waves. The conservation follows from the continuity equation

$$2i\partial_t |\psi|^2 = \omega'' \nabla(\psi^* \nabla \psi - \psi \nabla \psi^*) \equiv -2\mathrm{div}\, \mathbf{J} \,. \tag{3.31}$$

Note also the conservation of the momentum or total current, $\int \mathbf{J} \, d\mathbf{r}$, which does not play any role in this section but is important for Exercise 3.7.

Consider a wave packet characterized by the generally time-dependent size l and the constant value of N.



Since one can estimate the typical value of the envelope in the packet as $|\psi|^2 \simeq N/l^d$, then $\mathcal{H} \simeq \omega'' N l^{-2} + TN^2 l^{-d}$ – remember that the second term is negative

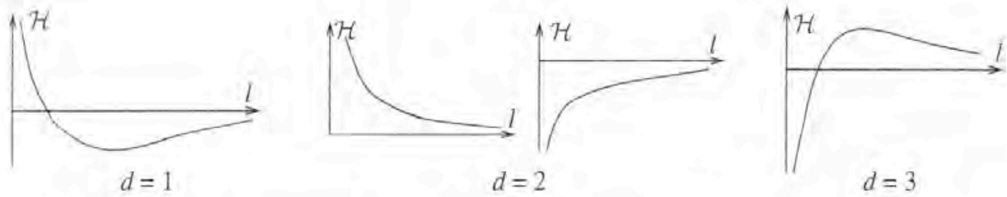


Figure 3.8 The Hamiltonian $\mathcal{H}$ as a function of the packet size l under fixed N.

here. We consider the conservative system, so the total energy is conserved yet we expect the radiation from the wave packet to bring it to the minimum of energy. In the process of weak radiation, wave action is conserved since it is an adiabatic invariant. This is particularly clear for a quantum system, like a cloud of cold atoms, where N is their number. Whether this minimum corresponds to $l \to 0$ (which is called self-focusing or collapse) is determined by the balance between $|\nabla\psi|^2$ and $|\psi|^4$. The Hamiltonian $\mathcal{H}$ as a function of l in three different dimensionalities is shown in Figure 3.8.

(i) $d=1$. At small l repulsion dominates with $\mathcal{H} \simeq \omega'' N l^{-2}$ while attraction dominates at large l with $\mathcal{H} \simeq -TN^2 l^{-1}$. It is thus clear that a stationary solution must exist with $l \sim \omega''/TN$, which minimizes the energy. Physically, the pressure of the waves balances the attraction force. Such a stationary solution is called a *soliton*, short for solitary wave. It is a travelling-wave solution of (3.28) with the amplitude function just moving, $A(x,t) = A(x-ut)$, and the phase having both a space-dependent travelling part and a uniform non-linear part linearly growing with time: $\varphi(x,t) = f(x-ut) - TA_0^2 t$. Here, A_0 and u are soliton parameters. We substitute the travel solution into (3.28) and separate the real and imaginary parts:

$$A'' = \frac{2T}{\omega''}\left(A^3 - A_0^2 A\right) + Af'\left(f' - \frac{2u}{\omega''}\right), \quad \omega''\left(A'f' + \frac{Af''}{2}\right) = uA'. \quad (3.32)$$

For the simple case of the standing wave ($u=0$) the second equation gives $f =$ const., which can be put equal to zero. The first equation can be considered as a Newtonian equation $A'' = -\mathrm{d}U/\mathrm{d}A$ for the particle with coordinate A in the potential $U(A) = -(T/2\omega'')(A^4 - 2A^2A_0^2)$ and the space coordinate x replacing the particle's time. The soliton is a separatrix, that is a solution that requires for particle an infinite time to reach zero, or in original terms where $A \to 0$ as $x \to \pm\infty$. The upper part of Figure 3.9 presumes $T/\omega'' < 0$, that is a case of modulational instability. Let me mention in passing that the separatrix also exists for $T/\omega'' > 0$ but in this case the running wave is a kink, that is a transition between two different values of the stable condensate (the lower part of the figure). The kink is seen as a dip in intensity $|\psi|^2$.

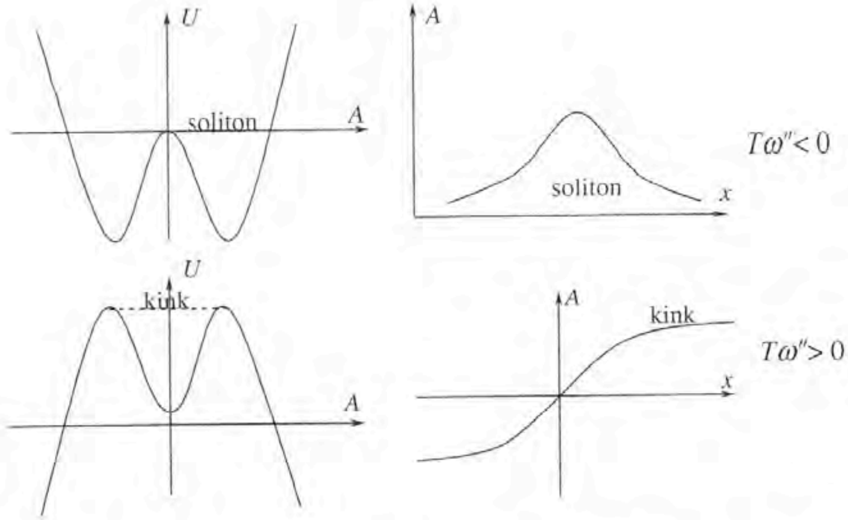


Figure 3.9 Energy as a function of the amplitude of a running wave, and the profile of the wave. The upper part corresponds to the case of an unstable condensate, where a steady solution is a soliton, the lower part to a stable condensate, where it is a kink.

Considering a general case of a travelling soliton (at $T\omega'' < 0$), one can multiply the second equation by A and then integrate: $\omega'' A^2 f' = u(A^2 - A_0^2)$, where by choosing the constant of integration we defined A_0 as A at the point where $f' = 0$. We can now substitute f' into the first equation and get the closed equation for A. The soliton solution has the form:

$$\psi(x,t) = \sqrt{2} A_0 \cosh^{-1}\left[\left(\frac{-2T}{\omega''}\right)^{1/2} A_0(x - ut)\right] e^{i(2x-ut)u/2\omega'' - iTA_0^2 t}.$$

Note that the Galilean transformation for the solutions of the NSE appears as $\psi(x,t) \to \psi(x - ut, t)\exp[iu(2x - ut)/2\omega'']$. In the original variable $a(\mathbf{r})$, our envelope solitons appear as shown in Figure 3.10.

(ii) $d = 2, 3$. When the condensate is stable, there exist stable solitons analogous to kinks, which are localized minima in the condensate intensity. In optics they can be seen as grey and dark filaments in a laser beam propagating through a non-linear medium. The wave (condensate) amplitude turns into zero in a dark filament, which means that it is a vortex, i.e. a singularity of the wave phase, see Exercise 3.6.

When the condensate is unstable, there are no stable stationary solutions for $d = 2, 3$. From the dependence $\mathcal{H}(l)$ shown in Figure 3.8 we expect that the character of evolution will be completely determined by the sign of the Hamiltonian at $d = 2$: the wave packets with positive Hamiltonian spread because the

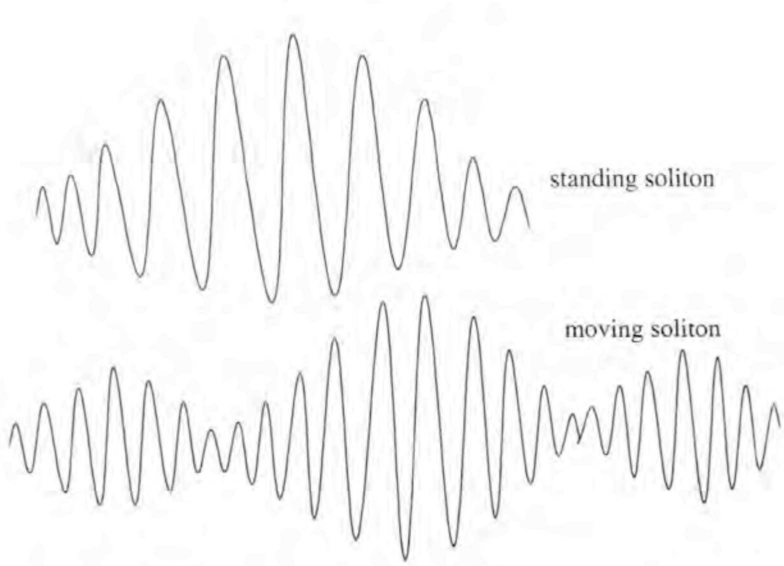


Figure 3.10 Standing and travelling solitons of the envelope of an almost monochromatic wave.

wave dispersion (kinetic energy or pressure, in other words) dominates while the wave packets with negative Hamiltonian shrink and collapse. Let me stress that this way of arguing based on the dependence $\mathcal{H}(l)$ is non-rigorous and suggestive at best. A rigorous proof of the fact that the Hamiltonian sign determines whether the wave packet spreads or collapses in 2D is called Talanov's theorem, which is the expression for the second time derivative of the packet size squared, $l^2(t) = \int |\psi|^2 r^2 \, d\mathbf{r}$. To obtain that expression, differentiate over time using (3.31), then integrate by parts, then differentiate again:

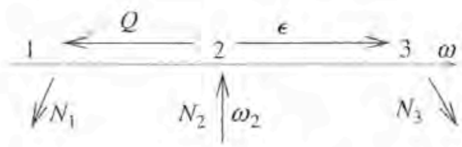
$$\begin{aligned}\frac{d^2 l^2}{dt^2} &= \frac{i\omega''}{2}\partial_t \int r^2 \nabla\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) d\mathbf{r} \\ &= i\omega'' \partial_t \int r_\alpha \left(\psi^* \nabla_\alpha \psi - \psi \nabla_\alpha \psi^*\right) d\mathbf{r} = 2\omega''^2 \int |\nabla \psi|^2 \, d\mathbf{r} \\ &\quad + d\omega'' T \int |\psi|^4 \, d\mathbf{r} = 4\mathcal{H} + 2(d-2)\omega'' T \int |\psi|^4 \, d\mathbf{r}.\end{aligned}$$

Consider an unstable case with $T\omega'' < 0$. We see that indeed for $d \geq 2$ one has an inequality $\partial_{tt} l^2 \leq 4\omega'' \mathcal{H}$ so that

$$l^2(t) \leq 2\omega'' \mathcal{H} t^2 + C_1 t + C_2$$

and for $\omega'' \mathcal{H} < 0$ the packet shrinks to singularity in a finite time (this is the singularity in the framework of NSE, which is itself valid only for the scales much larger than the wavelength of the carrier wave $2\pi/k_0$). This, in particular, describes self-focusing of light in non-linear media. For $d=2$ and $\omega''\mathcal{H}>0$, on the contrary, one has dispersive expansion and decay.

Turbulence with two cascades. As mentioned, any equation (3.27) that describes only four-wave scattering necessarily conserves two integrals of motion, the energy $\mathcal{H}$ and the number of waves (or wave action) N. For waves of small amplitude, the energy is approximately quadratic in wave amplitudes, $\mathcal{H}\approx\int\omega_k|a_k|^2\,d\mathbf{k}$, as well as $N=\int|a_k|^2\,d\mathbf{k}$. The existence of two quadratic positive integrals of motion in a closed system means that if such system is subject to external pumping and dissipation, it may develop turbulence consisting of two cascades.



Indeed, imagine that the source at some ω_2 pumps N_2 waves per unit time. It is then clear that for a steady state one needs two dissipation regions in ω-space (at some ω_1 and ω_3) to absorb the inputs of both N and E. Conservation laws allow one to determine the numbers of waves, N_1 and N_3, absorbed per unit time in the regions of low and high frequencies, respectively. Schematically, solving $N_1+N_3=N_2$ and $\omega_1N_1+\omega_3N_3=\omega_2N_2$ we get

$$N_1=N_2\frac{\omega_3-\omega_2}{\omega_3-\omega_1},\qquad N_3=N_2\frac{\omega_2-\omega_1}{\omega_3-\omega_1}\,. \tag{3.33}$$

We see that for a sufficiently large left interval (when $\omega_1\ll\omega_2<\omega_3$) most of the energy is absorbed by the right sink: $\omega_2N_2\approx\omega_3N_3$. Similarly at $\omega_1<\omega_2\ll\omega_3$ most of the wave action is absorbed at small ω: $N_2\approx N_1$. When $\omega_1\ll\omega_2\ll\omega_3$ we have two cascades with the fluxes of energy ϵ and wave action Q. The Q-cascade towards large scales is called the inverse cascade (Kraichnan, 1967; Zakharov, 1967); it corresponds (somewhat counterintuitively) to a kind of self-organization, i.e. the creation of larger and slower modes out of small-scale fast fluctuations.[7] The limit $\omega_1\to0$ is well-defined; in this case the role of the left sink can actually be played by a condensate, which absorbs an inverse cascade. Note in passing that consideration of thermal equilibrium in a finite-size system with two integrals of motion leads to the notion of negative temperature.[8]

An important hydrodynamic system with two quadratic integrals of motion is a two-dimensional ideal fluid. In two dimensions, the velocity $\mathbf{u}$ is perpendicular to the vorticity $\omega=\nabla\times\mathbf{u}$, so that the vorticity of any fluid element is conserved by virtue of the Kelvin theorem. This means that the space integral of any function of vorticity is conserved, including $\int \omega^2 \, d\mathbf{r}$, called enstrophy. We can write the densities of the two quadratic integrals of motion, energy and enstrophy, in terms of the velocity spectral density: $E = \int |\mathbf{v_k}|^2 \, d\mathbf{k}$ and $\Omega = \int |\mathbf{k} \times \mathbf{v_k}|^2 \, d\mathbf{k}$. Assume now that we excite turbulence with a force having a wavenumber k_2 while dissipation regions are at k_1, k_3. Applying the consideration similar to (3.33) we obtain

$$E_1 = E_2 \frac{k_3^2 - k_2^2}{k_3^2 - k_1^2}, \qquad E_3 = E_2 \frac{k_2^2 - k_1^2}{k_3^2 - k_1^2}. \tag{3.34}$$

We see that for $k_1 \ll k_2 \ll k_3$, most of the energy is absorbed by the left sink, $E_1 \approx E_2$, and most of the enstrophy is absorbed by the right one, $\Omega_2 = k_2^2 E_2 \approx \Omega_3 = k_3^2 E_3$. We conclude that conservation of both energy and enstrophy in two-dimensional flows requires two cascades: that of the enstrophy towards small scales and that of the energy towards large scales (opposite to the direction of the energy cascade in three dimensions). Large-scale motions of the ocean and planetary atmospheres can be considered to be approximately two-dimensional; the creation and persistence of large-scale flow patterns in these systems is probably related to inverse cascades.[9]

3.4 Korteveg–de-Vries (KdV) equation

Here we consider another universal limit: weakly non-linear long waves. More often than not the dispersion relation of such waves is close to acoustic. We derive the respective KdV equation for shallow-water waves. We then consider some remarkable properties of this equation and of such waves.

3.4.1 Waves in shallow water

Linear gravity-capillary waves have $\omega_k^2 = (gk + \alpha k^3/\rho) \tanh kh$, see (3.12). That is, for sufficiently long waves (when the wavelength is larger than both h and $\sqrt{\alpha/\rho g}$) their dispersion relation is close to linear:

$$\omega_k = \sqrt{gh}\, k - \beta k^3, \qquad \beta = \frac{\sqrt{gh}}{2} \left(\frac{h^2}{3} - \frac{\alpha}{\rho g} \right). \tag{3.35}$$

Therefore, one can expect a quasi-simple plane wave propagating in one direction, like that described in Sections 2.3.2 and 2.3.3. Let us derive the equation satisfied by such a wave. From the dispersion relation, we obtain the linear part