Translated from Russian by **Herbert Lashinsky**
University of Maryland

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Edited by Acad. M. A. Leontovich

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INTRODUCTION

In linear plasma theory an arbitrary perturbation can be expressed as a superposition of characteristic eigenmodes, each of which is independent of the others. In the nonlinear theory the eigenmodes interact with each other as a result of the nonlinearity. This interaction is, in many respects, reminiscent of the interaction between motions on different scales in hydrodynamic turbulence. In a plasma, however, the pattern of this interaction can frequently be represented in the familiar form of a superposition of linear eigenmodes if account is taken of the fact that the nonlinearity only leads to a weak interaction between the modes. This means that the coefficients in an expansion in characteristic eigenmodes become slowly varying functions of time and ultimately take on values which are very different from the values given by the linear theory.

This approach is now generally called the theory of weak turbulence. The equations of this theory can be derived from first principles by means of an expansion of original equations for the plasma in powers of a small parameter, the ratio of energy in the oscillations to the total energy in the plasma. The energy source for the perturbations in this theory are usually the various plasma instabilities.

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The theory of weak turbulence was developed at the beginning of the 1960's; at the present time, by means of this theory it has been possible to explain a number of important nonlinear effects: the interaction of a beam of charged particles with a plasma; turbulent heating of a plasma; the dissipation mechanism in collisionless shock waves; and anomalous resistivity. The methods of weak turbulence go beyond the framework of plasma physics and have been applied successfully in the analysis of nonlinear dispersive media in general, and, in particular, in the nonlinear dynamics of water waves. Thus, it has been possible to formulate a quantitative theory of water ripples, effects which were only amenable to a qualitative description for a long time.

The theory of weak turbulence has been the subject of a number of books and reviews in the last decade*; nevertheless, it is felt that a need has arisen to summarize the results of the theory of weak turbulence from a single point of view, including phenomenon that are not found in laboratory plasma physics.

It is useful to consider nonlinear plasma theory in terms of three basic interactions: nonlinear wave–wave interactions, wave–particle interactions, and finally, wave–particle–wave interactions (sometimes called the nonlinear wave–particle interactions).

The first interaction, the wave – wave interaction is frequently called resonance wave – wave scattering. The resonance conditions can be written

$$\sum_i \omega_i = 0, \ \sum_i \mathbf{k}_i = 0, \ i = 1, 2, \ldots,$$

where ω_i and $\mathbf{k}_i$ are the frequencies and wave vectors of the waves which participate in the interaction. The simplest interaction of this kind is the one in which three waves are involved. The coupling between the waves is especially strong if the resonance condition is satisfied. Since this interaction does not involve resonance particles, it can be described by means of the fluid equations (in other words, it is not necessary to use the kinetic equations). The wave–wave interaction lies at the basis of many effects in nonlinear wave dynamics: parametric wave instabilities (the case of small amplitudes corresponds to the well-known decay instability); the modulational instability of wave packets in a plasma and in nonlinear optics; and self-focusing of waves in nonlinear optics. If the quantities ω and $\mathbf{k}$ are interpreted as the energy and momentum of the photon associated with the ω, $\mathbf{k}$ wave, it will be evident that the resonance condition is simply a statement of the conservation of energy and momentum in the elementary process in which a single photon decays into two other photons or in the inverse process. Consequently, it is not surprising that the wave interaction conserves the total energy and momentum.

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The second interaction can be pictured as being almost linear (or quasilinear). The wave – particle interaction is especially strong near resonance $\omega = \mathbf{k} \cdot \mathbf{v}$ ($\mathbf{v}$ is the velocity of the particle that participates in the interaction). If this so-called Landau resonance condition is satisfied, the particle maintains a constant phase with respect to the wave and is effectively accelerated (or retarded) by the electric field associated with the wave. An analogous resonance arises in a magnetic field when the following condition is satisfied:

$$\omega - l\omega_H = \mathbf{k}\cdot\mathbf{v}, \quad l = 0, \pm 1, \ldots,$$

where ω_H is the particle gyrofrequency. Since this interaction involves resonant particles it is necessary to make use of the kinetic equations. From the quantum-mechanical point of view the resonance condition for this interaction is a statement of the conservation of energy and momentum in the elementary process involving the emission or absorption of a photon with energy $\hbar\omega$ and momentum $\hbar\mathbf{k}$ by a particle moving with velocity $\mathbf{v}$. Thus it is not surprising that the wave–particle interaction conserves the total energy and momentum of the waves and particles (rather than the energy of the waves alone). The change in the wave amplitude associated with this interaction is called Landau damping (or inverse Landau damping) while the corresponding change in the velocity distribution of the particles is called quasilinear diffusion.

The third interaction, the wave–particle–wave interaction, is frequently called nonlinear Landau damping. The resonance condition for this interaction is written in the form $\omega_1 - \omega_2 = (\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{v}$, while the basic mechanism is reminiscent of the linear wave–particle interaction. In the present case, however, the particle maintains a constant phase with respect to the beat wave produced by two waves. This interaction also involves resonant particles and must be considered within the framework of kinetic theory. The resonance condition written above taken with the plus sign corresponds to the elementary process involving simultaneous emission or absorption of two photons by a particle.

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The third interaction, the w a v e – p a r t i c l e – w a v e interaction, is frequently called n o n l i n e a r L a n d a u d a m p i n g. The resonance condition for this interaction is written in the form $\omega_1 - \omega_2 = (\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{v}$, while the basic mechanism is reminiscent of the linear wave–particle interaction. In the present case, however, the particle maintains a constant phase with respect to the beat wave produced by two waves. This interaction also involves resonant particles and must be considered within the framework of kinetic theory. The resonance condition written above taken with the plus sign corresponds to the elementary process involving simultaneous emission or absorption of two photons by a particle.

With the minus sign the resonance condition refers to the elementary process involving emission of a single photon and absorption of another (in other words, the scattering process). In addition to the conservation of the total energy and the total momentum of the waves and particles in the scattering process it also turns out that the total number of photons is conserved. In the classical case the number of photons can be defined as the energy of the wave W_k divided by the frequency [that is to say, W_k/ω_k is the action of the $(\omega, \mathbf{k})$ wave].

It will be evident that in general all three interactions described above can occur in a plasma at the same time; the behavior of the plasma is then determined by the total effect of all three interactions. The problem of anomalous resistivity in the plasma is treated in a separate chapter as an example of this interaction.

Nonlinear phenomenon in a plasma cannot always be treated by means of the theory of weak turbulence. Many plasma effects are a result of strong turbulence, which is similar to the usual hydrodynamic turbulence. At the present time no reliable quantitative methods are available in the theory of strong turbulence. As a rule, one tries to obtain reasonable estimates as to the orders of magnitude involved. Certain examples of this kind are discussed in various sections of the present review.

The original step in the writing of this review was the presentation of lectures by the authors at the International Center for Theoretical Physics in Trieste in 1966. A report containing the lectures [1] was edited by Book and O'Neil and published in 1969.

In the ensuing period so many new results have been obtained in nonlinear plasma theory that it has become necessary to subject the original version to a thorough revision. In the process we have also added a new chapter on anomalous resistivity in plasmas. It is assumed that the reader of the present review is familiar with the linear theory of plasma waves and instabilities.*

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Chapter 1

WAVE–WAVE INTERACTION

§ 1.1. Resonance Interaction between Plasma Waves

Let us consider the nonlinear wave–wave interaction between three plasma waves. An example of this kind of interaction is the process in which a wave of finite amplitude decays into two daughter waves, which was first treated by Oraevskii and Sagdeev [1]. In order for this interaction to occur, the wave vectors and frequencies must satisfy a resonance condition, that is to say, $\mathbf{k}_0 = \mathbf{k}_1 \pm \mathbf{k}_2$ and $\omega_0 = \omega_1 \pm \omega_2$. It will be evident that the frequencies and wave vectors of each of the waves are coupled by the linear dispersion equation $\omega = \omega(\mathbf{k})$. The nature of the dispersion plays an important role in determining whether or not a resonance interaction is possible for a given set of waves. In order to illustrate this point we note the difference between nonlinear resonances which dominate the situation in ordinary gas dynamics and those which are important in plasma physics. In the case of a monochromatic acoustic wave of large amplitude, gas dynamic theory predicts that the primary mechanism responsible for the nonlinear distortion of the wave is the steepening of the wave front (Fig. 1). This steepening can be understood in terms of the resonance generation of higher harmonics. If the original large-amplitude wave is characterized by a frequency ω and wave vector $\mathbf{k}$, the nonlinear interaction of the wave leads to the appearance of a second harmonic $(2\omega, 2\mathbf{k})$. Since the dispersion relation for the acoustic wave is a linear relation $\omega = kc_s$, the harmonics, like the fundamental mode, are characteristic modes of the system and are thus at all times in resonance with the fundamental mode; thus,

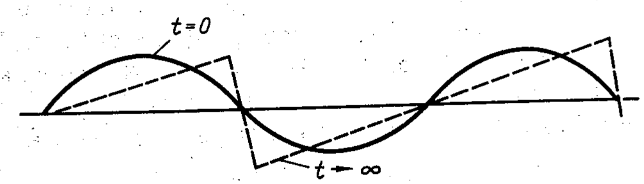

Fig. 1. Steepening of the profile of an acoustic wave of finite amplitude.

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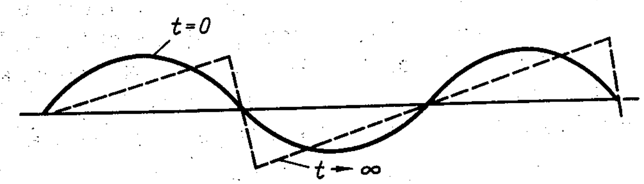

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the harmonics grow in time. Higher harmonics grow in similar fashion and the appearance of higher and higher values of **k**, that is to say, shorter wavelengths, is responsible for the steepening of the front (Fig. 1).

On the other hand, as a rule plasma oscillations exhibit a strong dispersion (i.e., the frequency ω is usually a nonlinear function of **k**) so that the harmonics of the normal modes are themselves not normal modes of the system. Consequently, harmonics of a plasma wave of large amplitude are usually limited at a very low level and only lead to an insignificant distortion of the wave shape [2]. However, this result does not mean that plasma waves always propagate without change of shape. Even if a plasma wave does not interact with its own harmonics, it can still be in resonance with two other waves.

Resonance generation of harmonics can also be suppressed by a choice of wave polarization such that the matrix element of the interaction operator between the wave and its harmonics can vanish identically. An example of this kind is furnished by transverse Alfvén waves. Nonetheless, as we have seen earlier, this does not mean that these waves propagate in a plasma without change of shape.

For purposes of illustration we now consider a wave–wave interaction between an Alfvén wave of large amplitude, an Alfvén wave of low amplitude, and an acoustic wave. The magnetic field, the electric field, the velocity of the electron fluid, and its density are written in the form

$$\left.\begin{aligned} \mathbf{H} &= \mathbf{H}_0 + \delta\mathbf{H}_\perp(z, t) + \mathbf{h}_\perp(z, t);\\ \mathbf{E} &= \delta\mathbf{E}_\perp(z, t) + \mathbf{e}_\perp(z, t);\\ \mathbf{v} &= \delta\mathbf{V}_\perp(z, t) + \mathbf{v}_\perp(z, t) + \mathbf{v}_\parallel(z, t);\\ N &= N_0 + N(z, t), \end{aligned}\right\} \qquad (1.1)$$

where $\mathbf{H}_0$ is a fixed magnetic field in the z direction; the functions $\delta\mathbf{H}_\perp(z, t)$, $\delta\mathbf{E}_\perp(z, t)$, and $\delta\mathbf{V}_\perp(z, t)$ characterize the finite-amplitude Alfvén wave; $\mathbf{h}_\perp$, $\mathbf{e}_\perp$, and $\mathbf{v}_\perp$ are the perturbations of the fields and fluid velocity of the low-amplitude Alfvén wave; N and $\mathbf{v}_\parallel$ are the perturbations in the density and fluid velocity of the acoustic wave. It is assumed that all three waves propagate along the fixed magnetic field $\mathbf{H}_0$.

We now solve the two-fluid MHD equations by perturbation theory, taking the quantities $\mathbf{H}_0$, $\mathbf{N}_0$, $\delta\mathbf{H}_\perp$, $\delta\mathbf{E}_\perp$ and $\delta\mathbf{V}_\perp$ to be the

unperturbed quantities; the quantities $\mathbf{h}_\perp$, $\mathbf{e}_\perp$, $\mathbf{v}_\perp$, N, and $\mathbf{v}_\parallel$ are small perturbations.

The Alfvén wave of finite amplitude can be represented as a circularly polarized wave which is written in the form

$$\delta V_x - i\delta V_y = \delta V \exp(-i\,\omega_0 t + i k_0 z) + \text{c.c.} \qquad (1.2)$$

The connection between the field amplitudes $\delta\mathcal{H}$ and $\delta\mathcal{E}$, which are defined in similar fashion, and the flow velocity δV is given by the equations of motion of electron fluid and Maxwell's equations

$$\left.\begin{aligned} -i(\omega_0 + \omega_H)\delta V &= -(e/m)\delta\mathcal{E};\\ k_0\delta\mathcal{H} &= -(4\pi e N_0/c)\delta V - (i\omega_0/c)\delta\mathcal{E};\\ k_0\delta\mathcal{E} &= (i\omega_0/c)\delta\mathcal{H};\\ \omega_{Hj} = e_j H_0/m_j c;\ \omega_{He} \equiv \omega_H,\ &\omega_{Hi} \equiv \Omega_H,\ m_e \equiv m,\ m_i \equiv M. \end{aligned}\right\} \qquad (1.3)$$

The solution of these equations leads to the linear dispersion relation

$$\left.\begin{aligned} &k_0^2 c^2 = \omega_0^2\,\varepsilon(\omega_0); \quad \varepsilon(\omega) \equiv 1 - [\omega_p^2/\omega(\omega + \omega_H)];\\ &\omega_{pj}^2 = 4\pi e_j^2 N_0/m_j; \quad \omega_{pe}^2 \equiv \omega_p^2; \quad \omega_{pi}^2 \equiv \Omega_p^2. \end{aligned}\right\} \qquad (1.4)$$

The first-order perturbation equations are of the form

$$\partial v_\parallel/\partial t + (c_s^2/N_0)(\partial N/\partial z) = -(\partial/\partial z)(\mathbf{h}_\perp \cdot \delta\mathbf{H}_\perp/4\pi N_0 M); \qquad (1.5)$$

$$\partial N/\partial t + N_0(\partial v_\parallel/\partial z) = 0; \qquad (1.6)$$

$$\begin{aligned} m(\partial\mathbf{v}_\perp/\partial t) + e\{\mathbf{e}_\perp + (1/c)[\mathbf{v}_\perp \times \mathbf{H}_0]\} = -m v_\parallel(\partial/\partial z)\delta\mathbf{V}_\perp -\\ -(e/c)[\mathbf{v}_\parallel \times \delta\mathbf{H}_\perp]; \end{aligned} \qquad (1.7)$$

$$\operatorname{curl}\mathbf{h}_\perp - (1/c)(\partial\mathbf{e}_\perp/\partial t) + (4\pi e/c)N_0\mathbf{v}_\perp = (-4\pi e/c)\,N\delta\mathbf{V}_\perp; \qquad (1.8)$$

$$\operatorname{curl}\mathbf{e}_\perp + (1/c)(\partial\mathbf{h}_\perp/\partial t) = 0. \qquad (1.9)$$

The left-hand sides of Eqs. (1.5) and (1.6) describe the acoustic wave while the right-hand sides of these equations represent the coupling between the acoustic wave and the two Alfvén waves. Similarly, the left-hand side of Eqs. (1.7)–(1.9) describe the low-amplitude Alfvén wave while the right-hand sides indicate the couplthe harmonics grow in time. Higher harmonics grow in similar fashion and the appearance of higher and higher values of **k**, that is to say, shorter wavelengths, is responsible for the steepening of the front (Fig. 1).

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The connection between the field amplitudes $\delta\mathcal{H}$ and $\delta\mathcal{E}$, which are defined in similar fashion, and the flow velocity δV is given by the equations of motion of electron fluid and Maxwell's equations

$$\left.\begin{aligned} -\mathrm{i}\,(\omega_0 + \omega_H)\delta V &= -(e/m)\delta\mathcal{E};\\ k_0\delta\mathcal{H} &= -(4\pi e N_0/c)\delta V - (\mathrm{i}\omega_0/c)\delta\mathcal{E};\\ k_0\delta\mathcal{E} &= (\mathrm{i}\omega_0/c)\delta\mathcal{H};\\ \omega_{Hj} = e_j H_0/m_j c;\ \ \omega_{He} \equiv \omega_H,\ \ \omega_{Hi} &\equiv \Omega_H,\ \ m_e \equiv m,\ \ m_i \equiv M. \end{aligned}\right\} \qquad (1.3)$$

The solution of these equations leads to the linear dispersion relation

$$\left.\begin{aligned} k_0^2 c^2 = \omega_0^2\,\varepsilon(\omega_0);\quad \varepsilon(\omega) \equiv 1 - \left[\omega_p^2/\omega(\omega+\omega_H)\right];\\ \omega_{pj}^2 = 4\pi e_j^2 N_0/m_j;\quad \omega_{pe}^2 \equiv \omega_p^2;\quad \omega_{pi}^2 \equiv \Omega_p^2\,. \end{aligned}\right\} \qquad (1.4)$$

The first-order perturbation equations are of the form

$$\partial v_\parallel/\partial t + (c_s^2/N_0)(\partial N/\partial z) = -(\partial/\partial z)(\mathbf{h}_\perp\cdot\delta\mathbf{H}_\perp/4\pi N_0 M); \qquad (1.5)$$

$$\partial N/\partial t + N_0\,(\partial v_\parallel/\partial z) = 0; \qquad (1.6)$$

$$\begin{aligned} m\,(\partial\mathbf{v}_\perp/\partial t) + e\{\mathbf{e}_\perp + (1/c)[\mathbf{v}_\perp \times \mathbf{H}_0]\} = -m v_\parallel\,(\partial/\partial z)\delta\mathbf{V}_\perp -\\ -(e/c)[\mathbf{v}_\parallel \times \delta\mathbf{H}_\perp]; \end{aligned} \qquad (1.7)$$

$$\operatorname{curl}\mathbf{h}_\perp - (1/c)\,(\partial\mathbf{e}_\perp/\partial t) + (4\pi e/c)N_0\mathbf{v}_\perp = (-4\pi e/c)\,N\delta\mathbf{V}_\perp; \qquad (1.8)$$

$$\operatorname{curl}\mathbf{e}_\perp + (1/c)(\partial\mathbf{h}_\perp/\partial t) = 0. \qquad (1.9)$$

The left-hand sides of Eqs. (1.5) and (1.6) describe the acoustic wave while the right-hand sides of these equations represent the coupling between the acoustic wave and the two Alfvén waves. Similarly, the left-hand side of Eqs. (1.7)–(1.9) describe the low-amplitude Alfvén wave while the right-hand sides indicate the coupling of this wave with the acoustic wave and the finite-amplitude Alfvén wave.

In order to simplify Eqs. (1.7)-(1.9) we now make an additional assumption: $\beta = c_s^2/v_A^2 \ll 1$ ($v_A^2 = H_0^2/4\pi N_0 M$). It is possible to determine the order of magnitude of the terms on the right-hand sides of these equations by comparing them with the linear terms:

$$R_1 = [v_\parallel (\partial/\partial z)\delta V_\perp](\partial v_\perp/\partial t)^{-1} \sim (k_0 v_\parallel \delta V_\perp/\omega_0 v_\perp);$$
$$R_2 = [(e/mc) v_\parallel \delta H_\perp](\partial v_\perp/\partial t)^{-1} \sim (\omega_H v_\parallel \delta H_\perp/\omega_0 H_0 v_\perp);$$
$$R_3 = [(4\pi e/c) N \delta V_\perp][(4\pi e/c) N_0 v_\perp]^{-1} \sim (N\delta V_\perp/N_0 v_\perp).$$

By using the linear relations between the parameters $\mathbf{h}_\perp$, $\mathbf{v}_\perp$ and N, $v_\parallel$ in the wave we can rewrite the ratios given above in the form

$$R_2 \sim R_1 \approx \frac{k_0 c_s}{\omega_0} R_3 \ll R_3.$$

It then follows that in Eqs. (1.7) and (1.8) we need only retain the nonlinearity which corresponds to R_3, i.e., the term $4\pi e N \delta \mathbf{V}_\perp / c$ in Eq. (1.8).

Since we are interested in the time evolution of the perturbation we now expand the amplitudes of the perturbations as functions of the coordinates in a Fourier series. Let $\mathbf{k}_A$ be the wave vector of the Alfvén perturbation and $\mathbf{k}_s$ the wave vector for the acoustic perturbation. The nonlinear term on the right side of Eq. (1.5) must then have a spatial dependence of the form $\exp(ik_s z)$. This means that the product of the two exponentials $h\delta H^* \sim \exp[i(\mathbf{k}_A - \mathbf{k}_0)z]$ has the same dependence, that is to say, $k_s = k_A - k_0$. In this case the spatial dependence of the nonlinear term in the Alfvén-wave equation, which is proportional to the product $N\delta H \sim \exp[i(k_s + k_0)z]$, coincides with $\exp(ik_A z)$.

In a uniform plasma the right-hand side in Eq. (1.5) vanishes and we obtain the acoustic dispersion relation $\omega_s^2 = k_s^2 c_s^2$ from Eqs. (1.5)-(1.6). In the presence of the unperturbed wave this term is nonvanishing and the acoustic oscillation is coupled to the Alfvén wave $h_\perp \backsim \exp(ik_A z)$. This leads to a frequency shift $\delta\omega$ for the coupled oscillations. The frequencies of the Alfvén waves are generally much larger than the frequency of the acoustic wave ($\omega_A \gg \omega_s$). Hence we assume that $\omega_A \gg \delta\omega$ and write the solution of Eqs. (1.7)-(1.8) in the form of a product of a slowly varying amplitude and a rapidly oscillating exponential:

$$h_x - ih_y = h_\perp(t) \exp[i(k_A z - \omega_A t)]. \tag{1.10}$$

We will not write the explicit time dependence of the acoustic perturbations for the moment:

$$N(z, t) = N(t) \exp(ik_s z). \tag{1.11}$$

The variables $\mathbf{e}_\perp$, $\mathbf{v}_\perp$, and $v_\parallel$ are now eliminated from Eqs. (1.5)-(1.9), providing two equations for N(t) and $h_\perp(t)$:

$$[(\partial^2/\partial t^2) + k_s^2 c_s^2] N(t) = (k_A - k_0)^2 (\delta \mathcal{H}^* h_\perp(t)/4\pi N_0 M) \times \times \exp[i(k_A - k_0 - k_s)z - i(\omega_A - \omega_0)t]; \tag{1.12}$$

$$[k_A - (\omega_A^2/k_A c^2)\,\varepsilon(\omega_A)]/h_\perp(t) = k_0 (\delta\mathcal{H} N/N_0) \times \times \exp[i(k_s + k_0 - k_A)z - i(\omega_0 - \omega_A)t]. \tag{1.13}$$

Averaging these equations over the rapid oscillations in space and time we find N, $h_\perp$ = const so long as the three wave vectors do not satisfy the resonance condition $k_A = k_0 + k_s$ and the Alfvén frequencies are not approximately the same ($\omega_A - \omega_0 \backsim N/N$).

However, if the resonance conditions on k are satisfied and if the Alfvén frequencies are close, it is possible to write a solution of the equations in the form

$$N(t) \backsim \exp[-i\Omega t], \qquad h_\perp \exp[-i\omega_A t] \backsim \exp[-i(\Omega + \omega_0)t].$$

In this case the system of Eqs. (1.12)-(1.13) reduces to the matrix equation

$$\begin{pmatrix} \Omega^2 - k_s^2 c_s^2 & k_s^2\, \delta\mathcal{H}^*/4\pi M \\ k_0\, \delta\mathcal{H}/N_0 & k_A - [(\omega_0 + \Omega)^2/k_A c^2]\, \varepsilon(\omega_0 + \Omega) \end{pmatrix} \begin{pmatrix} N \\ h_\perp \end{pmatrix} = 0. \tag{1.14}$$

The solubility condition for this equation (vanishing of the determinant) then gives the dispersion relation for Ω.

Decay Instability. If the amplitude of the unperturbed Alfvén wave is small the solution of the system in (1.14) can be found by perturbation theory. In the first approximation the solution describes the propagation of normal modes with frequencies that satisfy the resonance condition

$$\left.\begin{array}{l} \omega_s^2 = k_s^2 c_s^2; \quad k_A^2 c^2 = \omega_A^2\, \varepsilon(\omega_A); \\ \omega_A = \omega_0 + \omega_s. \end{array}\right\} \tag{1.15}$$

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By using the linear relations between the parameters $\mathbf{h}_\perp$, $\mathbf{v}_\perp$ and N, $v_\parallel$ in the wave we can rewrite the ratios given above in the form

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In a uniform plasma the right-hand side in Eq. (1.5) vanishes and we obtain the acoustic dispersion relation $\omega_s^2 = k_s^2 c_s^2$ from Eqs. (1.5)-(1.6). In the presence of the unperturbed wave this term is nonvanishing and the acoustic oscillation is coupled to the Alfvén wave $h_\perp \propto \exp(ik_A z)$. This leads to a frequency shift $\delta\omega$ for the coupled oscillations. The frequencies of the Alfvén waves are generally much larger than the frequency of the acoustic wave ($\omega_A \gg \omega_s$). Hence we assume that $\omega_A \gg \delta\omega$ and write the solution of Eqs. (1.7)-(1.8) in the form of a product of a slowly varying amplitude and a rapidly oscillating exponential:

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Averaging these equations over the rapid oscillations in space and time we find N, $h_\perp$ = const so long as the three wave vectors do not satisfy the resonance condition $k_A = k_0 + k_s$ and the Alfvén frequencies are not approximately the same ($\omega_A - \omega_0 \propto N/N$).

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In the next approximation it is necessary to take account of the off-diagonal elements in the matrix (1.14); in the diagnonal elements use is made of an expansion in terms of the small frequency shift ($\delta\Omega = \Omega - \omega_s$). The result is

$$\begin{pmatrix} 2\delta\Omega\cdot\omega_s & k_s^2\,\delta\mathscr{H}/4\pi M \\ k_0\,\delta\mathscr{H}/N_0 & -\delta\Omega\,\dfrac{\partial}{\partial\omega_A}\cdot\dfrac{\omega_A^2}{k_A c^2}\,\varepsilon(\omega_A) \end{pmatrix}\begin{pmatrix} N \\ h_\perp \end{pmatrix}=0. \tag{1.16}$$

When the solubility condition is imposed we obtain the dispersion relation [3]

$$\delta\Omega^2 = -(k_s^2 k_0 v_{gA}/8\omega_s)\cdot(|\delta\mathscr{H}|^2/2\pi N_0 M), \quad v_g \equiv \partial\omega/\partial k. \tag{1.17}$$

The characteristic frequencies and wave vectors then satisfy the resonance condition

$$k_0 = k_A - k_s; \quad \omega_0 = \omega_A - \omega_s. \tag{1.18}$$

We note that only the squares of the quantities k_0, k_A, k_s, and ω_s appear in the dispersion equations; therefore, the signs of these quantities can be chosen arbitrarily.

The signs of the Alfvén frequencies are fixed, being positive. Furthermore, if the signs of the wave vectors are the same (parallel propagation), it follows from Eq. (1.17) that $k_s = 0$, so that $\delta\Omega = 0$.

Hence, the signs of k_0 and k_A must be different. By virtue of the inequality $\omega_0 \gg \omega_s$ we immediately find that $k_A \approx -k_0$, $k_s \approx -2k_0$. It will be evident from Eq. (1.7) that small perturbations will only grow when $\omega < 0$. In other words, if the energy of the photon associated with the initial wave is larger than the energy of each of the photons of the perturbation waves (i.e., if $\omega_0 > \omega_A, |\omega_s|$) then the initial wave will be unstable against decay into two waves, and this is the *decay instability*.

Modulational Instability. A change in the thermodynamic parameters of the medium (density, pressure, etc.) due to the effect of an electromagnetic wave in the medium leads to a change in the dielectric properties of the medium, so that the frequency of the oscillations depends on the wave amplitude. When an Alfvén wave propagates through a plasma the change in the dielectric permittivity of the medium is due to a local perturbation

of the density caused by the pressure associated with the electromagnetic field. Hence, the wave frequency is proportional to the square of its magnetic field:

$$\omega_0 = \omega(\mathbf{k}_0) + \alpha\,|\delta\mathscr{H}|^2/2\pi. \tag{1.19}$$

The dependence of oscillation frequency on amplitude leads to the possibility of self-modulation of the wave and splitting into separate packets when the Lighthill criterion is satisfied [4]:

$$\alpha\,(\partial^2\omega/\partial k_0^2) < 0. \tag{1.20}$$

The instability mechanism can be understood from an examination of Fig. 2, in which we show the change of magnetic field in a weakly modulated Alfvén wave. When $\alpha > 0$, the phase velocity of the wave is larger in regions of maximum amplitude than in regions of minimum amplitude. Correspondingly, the wave number, which is proportional to the number of nodes of the function $\delta\mathscr{H}(z)$, increases in approaching a region of minimum amplitude and decreases in moving away from such a region. If the group velocity decreases with increasing wave number $\left(v_g' \equiv \frac{\partial^2\omega}{\partial k^2} < 0\right)$, the oscillations in region 1 in front of the amplitude minimum lag behind; in region 2, behind the minimum, they move ahead. Under these conditions the amplitude minimum becomes more pronounced. The first examples of the modulational instability were reported in 1957 by Sagdeev, who considered the instability of an electromagnetic field at the interface with a plasma [5], and by Volkov,

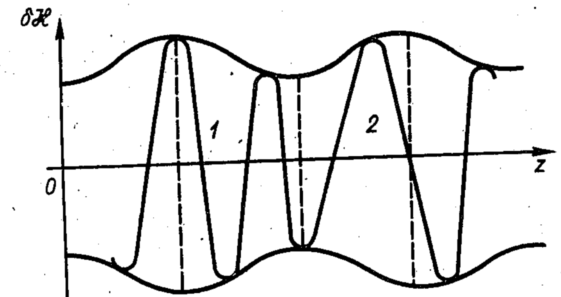

Fig. 2. Diagram to show the development of the modulation of the amplitude of an Alfvén wave.

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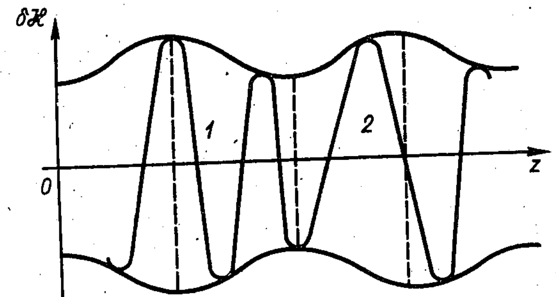

Fig. 2. Diagram to show the development of the modulation of the amplitude of an Alfvén wave.

who found that electromagnetic waves are unstable in propagation through a plasma [6] (cf. Problem 6 of this section).

The modulational instability can be described in terms of resonant wave interactions through the use of the nonlinear equations (1.5)-(1.9). However, in the present case, the perturbations consist of a pair of Alfvén waves with wave vectors $k_\pm = k_0 \pm q$ and amplitudes $h_\pm$ and oscillations (non-eigenmodes) of the density of the electron fluid characterized by frequency $\Omega = (d\omega_0/dk_0)q$. Correspondingly, on the right-hand side of the equation for the density oscillations there is a contribution due to beats between the fundamental wave and waves characterized by the upper $\omega(k_+)$ and lower $\omega(k_-)$ sideband frequencies. Furthermore, it is now necessary to add another row in the matrix in (1.14); this is the equation for the perturbation associated with the sideband frequency:

$$\begin{pmatrix} \Omega^2-q^2c_s^2 & (k_+-k_0)^2\dfrac{\delta\mathcal{H}^*}{4\pi M} & (k_0-k_-)^2\dfrac{\delta\mathcal{H}}{4\pi M} \\ k_0\,\delta\mathcal{H}/N_0 & k_+-\dfrac{(\omega_0+\Omega)^2}{k_+c^2}\,\varepsilon(\omega_0+\Omega) & 0 \\ k_0\,\delta\mathcal{H}^*/N_0 & 0 & k_--\dfrac{(\omega_0-\Omega)^2}{k_-c^2}\,\varepsilon(\omega_0-\Omega) \end{pmatrix}\begin{pmatrix} N \\ h_+ \\ h_-^* \end{pmatrix}=0. \qquad (1.21)$$

The dispersion equation for the Alfvén wave is now expanded in terms of the small difference $[\Omega - q(d\omega_0/dk_0)]$:

$$k_\pm^{-1}\left[\frac{\partial}{\partial\omega}\cdot\frac{\omega^2}{c^2}\,\varepsilon(\omega)\right]\bigg|_{\omega=\omega(k_\pm)}(\omega_0\pm\Omega-\omega(k_\pm))=$$
$$=\frac{1}{k_\pm}\left[\frac{\partial}{\partial\omega}\cdot\frac{\omega^2}{c^2}\,\varepsilon(\omega)\right]_{\omega=\omega(k_\pm)}\left\{\pm\left(\Omega-q\frac{\partial\omega_0}{\partial k_0}\right)-q^2\frac{\partial^2\omega_0}{2\partial k_0^2}\right\}. \qquad (1.22)$$

The solubility condition for Eq. (1.21) then gives the dispersion equation for Ω:

$$\Omega^2-q^2c_s^2=-\frac{k_0q^4\,\dfrac{\partial^2\omega_0}{\partial k_0^2}\cdot\dfrac{\partial\omega_0}{\partial k_0}}{\left(\Omega-q\dfrac{\partial\omega_0}{\partial k_0}\right)^2-q^4\left[\dfrac{\partial^2\omega_0}{2\partial k_0^2}\right]^2}\cdot\frac{|\delta\mathcal{H}|^2}{8\pi N_0 M}. \qquad (1.23)$$

Perturbations due to the modulation of the wave propagate with group velocity $(\Omega/q) \approx (d\omega/dk)$ and are characterized by a growth

rate

$$\operatorname{Im}\left(\Omega-\frac{\partial\omega}{\partial k}q\right)=q\sqrt{-\frac{\partial^2\omega_0}{\partial k_0^2}\left(\alpha\frac{|\delta\mathcal{H}|^2}{2\pi}+\frac{\partial^2\omega_0}{4\partial k_0^2}q^2\right)}, \qquad (1.24)$$

where $\alpha=k_0\dfrac{\partial\omega_0}{\partial k_0}\Big/4MN_0\left[c_s^2-\left(\dfrac{\partial\omega_0}{\partial k_0}\right)^2\right]$ is a coefficient of proportionality which relates the nonlinear frequency change to the quantity $|\delta\mathcal{H}|^2/2\pi$.

The relation in (1.24) demonstrates the validity of the Lighthill criterion (1.20). It should be emphasized that the instability occurs only for small values of q, corresponding to low modulation frequencies. The maximum growth rate for the modulational instability is proportional to the second power of the field amplitude, as in the decay of the second harmonic of a primary wave into a pair of sideband waves which propagate in the same direction as the primary wave (cf. § 1.3).

In the case being considered here, a low-pressure plasma ($c_s^2 \ll v_A^2$), the coefficient α is negative [$\alpha = -k_0/MN_0(D\omega_0/dk_0)$]; Hence the criterion for the modulational instability assumes the form [7]

$$(\partial^2\omega/\partial k^2)>0 \quad \text{or} \quad \omega<(1/4)\,|\omega_H|. \qquad (1.25)$$

Aperiodic Instability. In a low-pressure plasma ($\beta \ll 1$) the growth rate for the decay instability for a low-amplitude wave ($\delta\mathcal{H} \ll H_0$) can exceed the frequency of the characteristic acoustic oscillations; the growth of these oscillations is in the nature of an essentially aperiodic exponential growth from the initial amplitude. This branch appears as an extension of the decay instability to the region $\nu \gg \omega_s$ and can then be called the modified decay instability. Decay instabilities are parametric in nature, being analogous to the parametric resonance in a system of two coupled oscillators (the coupling is realized through the high-amplitude primary wave, which is called the pump wave). The structure of the corresponding region in the Mathieu diagram is considered in § 1.3.

When the pump wave is an electromagnetic wave ($k_0 \ll k_s$) a degeneracy arises and it is necessary to consider three oscillators [8, 9]. In the case of the instability being discussed here there is a parameter region for which perturbation theory is again valid who found that electromagnetic waves are unstable in propagation through a plasma [6] (cf. Problem 6 of this section).

The modulational instability can be described in terms of resonant wave interactions through the use of the nonlinear equations (1.5)-(1.9). However, in the present case, the perturbations consist of a pair of Alfvén waves with wave vectors $k_\pm = k_0 \pm q$ and amplitudes $h_\pm$ and oscillations (non-eigenmodes) of the density of the electron fluid characterized by frequency $\Omega = (d\omega_0/dk_0)q$. Correspondingly, on the right-hand side of the equation for the density oscillations there is a contribution due to beats between the fundamental wave and waves characterized by the upper $\omega(k_+)$ and lower $\omega(k_-)$ sideband frequencies. Furthermore, it is now necessary to add another row in the matrix in (1.14); this is the equation for the perturbation associated with the sideband frequency:

$$\begin{pmatrix} \Omega^2-q^2c_s^2 & (k_+-k_0)^2\dfrac{\delta\mathcal{H}^*}{4\pi M} & (k_0-k_-)^2\dfrac{\delta\mathcal{H}}{4\pi M} \\ k_0\,\delta\mathcal{H}/N_0 & k_+-\dfrac{(\omega_0+\Omega)^2}{k_+c^2}\varepsilon(\omega_0+\Omega) & 0 \\ k_0\,\delta\mathcal{H}^*/N_0 & 0 & k_--\dfrac{(\omega_0-\Omega)^2}{k_-c^2}\varepsilon(\omega_0-\Omega) \end{pmatrix}\begin{pmatrix} N \\ h_+ \\ h_-^* \end{pmatrix}=0. \tag{1.21}$$

The dispersion equation for the Alfvén wave is now expanded in terms of the small difference $[\Omega - q(d\omega_0/dk_0)]$:

$$k_\pm^{-1}\left[\frac{\partial}{\partial\omega}\cdot\frac{\omega^2}{c^2}\varepsilon(\omega)\right]\Bigg|_{\omega=\omega(k_\pm)}(\omega_0\pm\Omega-\omega(k_\pm))=$$
$$=\frac{1}{k_\pm}\left[\frac{\partial}{\partial\omega}\cdot\frac{\omega^2}{c^2}\varepsilon(\omega)\right]_{\omega=\omega(k_\pm)}\left\{\pm\left(\Omega-q\frac{\partial\omega_0}{\partial k_0}\right)-q^2\frac{\partial^2\omega_0}{2\partial k_0^2}\right\}. \tag{1.22}$$

The solubility condition for Eq. (1.21) then gives the dispersion equation for Ω:

$$\Omega^2-q^2c_s^2=-\frac{k_0q^4\dfrac{\partial^2\omega_0}{\partial k_0^2}\cdot\dfrac{\partial\omega_0}{\partial k_0}}{\left(\Omega-q\dfrac{\partial\omega_0}{\partial k_0}\right)^2-q^4\left[\dfrac{\partial^2\omega_0}{2\partial k_0^2}\right]^2}\cdot\frac{|\delta\mathcal{H}|^2}{8\pi N_0M}. \tag{1.23}$$

Perturbations due to the modulation of the wave propagate with group velocity $(\Omega/q) \approx (d\omega/dk)$ and are characterized by a growth

rate

$$\operatorname{Im}\left(\Omega-\frac{\partial\omega}{\partial k}q\right)=q\sqrt{-\frac{\partial^2\omega_0}{\partial k_0^2}\left(\alpha\frac{|\delta\mathcal{H}|^2}{2\pi}+\frac{\partial^2\omega_0}{4\partial k_0^2}q^2\right)}, \tag{1.24}$$

where $\alpha=k_0\dfrac{\partial\omega_0}{\partial k_0}\Big/4MN_0\left[c_s^2-\left(\dfrac{\partial\omega_0}{\partial k_0}\right)^2\right]$ is a coefficient of proportionality which relates the nonlinear frequency change to the quantity $|\delta\mathcal{H}|^2/2\pi$.

The relation in (1.24) demonstrates the validity of the Lighthill criterion (1.20). It should be emphasized that the instability occurs only for small values of q, corresponding to low modulation frequencies. The maximum growth rate for the modulational instability is proportional to the second power of the field amplitude, as in the decay of the second harmonic of a primary wave into a pair of sideband waves which propagate in the same direction as the primary wave (cf. § 1.3).

In the case being considered here, a low-pressure plasma ($c_s^2 \ll v_A^2$), the coefficient α is negative [$\alpha = -k_0/MN_0(D\omega_0/dk_0)$]; Hence the criterion for the modulational instability assumes the form [7]

$$(\partial^2\omega/\partial k^2) > 0 \quad \text{or} \quad \omega < (1/4)\,|\omega_H|. \tag{1.25}$$

<u>Aperiodic Instability.</u> In a low-pressure plasma ($\beta \ll 1$) the growth rate for the decay instability for a low-amplitude wave ($\delta\mathcal{H} \ll H_0$) can exceed the frequency of the characteristic acoustic oscillations; the growth of these oscillations is in the nature of an essentially aperiodic exponential growth from the initial amplitude. This branch appears as an extension of the decay instability to the region $\nu \gg \omega_s$ and can then be called the modified decay instability. Decay instabilities are parametric in nature, being analogous to the parametric resonance in a system of two coupled oscillators (the coupling is realized through the high-amplitude primary wave, which is called the pump wave). The structure of the corresponding region in the Mathieu diagram is considered in § 1.3.

When the pump wave is an electromagnetic wave ($k_0 \ll k_s$) a degeneracy arises and it is necessary to consider three oscillators [8, 9]. In the case of the instability being discussed here there is a parameter region for which perturbation theory is again valid (the small parameter $\delta\mathscr{H}/H_0 \ll 1$). Under these conditions the growth of the perturbation can be described by means of Eq. (1.14). We neglect the term $k_s^2 c_s^2$ as compared with Ω^2 in the first row of the determinant; in this way we obtain the dispersion equation

$$\Omega = \left(\frac{1 \pm i\sqrt{3}}{2}\right) [k_0 k_s^2 v_g |\delta\mathscr{H}|^2/(8\pi N_0 M)]^{1/3} \gg (\omega_0 - \omega_A). \qquad (1.26)$$

The growth rate is a maximum when an Alfvén wave is produced which propagates in the direction opposite to the original Alfvén wave; obviously, the growth rate increases less rapidly with increasing amplitude.

PROBLEMS

1. Find the growth rate for the instability associated with the decay of a plasma wave (ω_0, k_0) into another plasma wave (ω_l, k_l) and an acoustic wave (ω_s, k_s) (the process $l \to l + s$) [2], [III].

The growth rate for the instability is a maximum when the perturbation propagates in the same direction as the original wave. Hence, we can limit the analysis to the case of one-dimensional decay. Furthermore, we limit the analysis to the description of waves with wavelengths much larger than the Debye radius, in which case the plasma remains quasineutral in the presence of the acoustic waves:

$$n_i = N_0 + N, \quad n_e = N_0 + N + n^l + \delta n.$$

The nonlinear effect of the plasma waves on the "slow" motion of the medium can be described by the ponderomotive force (sometimes called the dynamic pressure force or the Miller force) which has been derived independently in [5, 10 and 11]:

$$F = -m\frac{\partial}{\partial z}(v^l \delta v^*) = -\frac{\partial}{\partial z}\frac{e^2 E^l \delta E^*}{m\omega_l \omega_0} \exp\{i(k_l - k_0)z - i(\omega_l - \omega_0)t\}.$$

In this case the equation of motion and the equation of continuity for the slow motion can be reduced to an equation like (1.26), in which the magnetic pressure is now replaced by the dynamic pres-

sure:

$$\left(\frac{\partial^2}{\partial t^2} + k_s^2 c_s^2\right) N(t) = -k_s^2 \frac{e^2 E^l \delta E^*}{mM\omega_l \omega_0} \exp[i(k_l - k_0 - k_s)z - i(\omega_l - \omega_0)t]. \qquad (1)$$

In the equations for the plasma oscillations we need only retain the nonlinear term which describes the modulation of the electron density by the acoustic waves:

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2(k_l)\right) E^l(t) = (4\pi e^2/m) N(t)\, \delta E \exp[i(k_0 + k_s - k_l)z - i\omega_0 t]. \qquad (2)$$

In matrix form Eqs. (1) and (2) become

$$\begin{pmatrix} \Omega^2 - k_s^2 c_s^2 & k_s^2 \dfrac{e^2 \delta E^* N_0}{mM\omega_0 \omega_l} \\ \omega_p^2 \dfrac{\delta E}{\omega_0 N_0} & (\omega_0 + \Omega)\, \varepsilon(k_l, \omega_0 + \Omega) \end{pmatrix} \begin{pmatrix} N \\ E^l \end{pmatrix} = 0. \qquad (3)$$

Whence we obtain the growth rate for the decay instability:

$$\nu_d^2 \equiv -\delta\Omega^2 = \frac{m}{4M}\frac{\omega_p^2}{\omega_l \omega_s} k_s^2 |\delta v|^2, \quad \delta v = -\frac{ie\delta E}{m\omega_0}. \qquad (4)$$

2. Find the growth rate for the decay of an electromagnetic wave into another electromagnetic wave and an acoustic wave (inverse parametric scattering).

The problem is essentially the same as the preceding problem. The growth rate for the instability, which corresponds to scattering of the electromagnetic wave through an angle Θ, is given by [cf. Eq. (4) of Problem 1]

$$\nu_d^2 = \frac{\omega_p^4 \omega_s}{\omega_0^2 \omega_t} \cdot \frac{|\delta E|^2}{16\pi n_0 T_e}, \quad \omega_s^2 \approx \frac{2\omega_0 \omega_t T_e}{Mc^2}(1 - \cos\Theta).$$

When $\nu > \omega_s$ the usual decay process is replaced by a modified form:

$$\nu \approx (\sqrt{3}/2)(2\omega_s \nu_d^2)^{1/3}.$$

(the small parameter $\delta\mathcal{H}/H_0 \ll 1$). Under these conditions the growth of the perturbation can be described by means of Eq. (1.14). We neglect the term $k_s^2 c_s^2$ as compared with Ω^2 in the first row of the determinant; in this way we obtain the dispersion equation

$$\Omega = \left(\frac{1 \pm i\sqrt{3}}{2}\right) [k_0 k_s^2 v_g |\delta\mathcal{H}|^2/(8\pi N_0 M)]^{1/3} \gg (\omega_0 - \omega_A). \qquad (1.26)$$

The growth rate is a maximum when an Alfvén wave is produced which propagates in the direction opposite to the original Alfvén wave; obviously, the growth rate increases less rapidly with increasing amplitude.

PROBLEMS

1. Find the growth rate for the instability associated with the decay of a plasma wave (ω_0, k_0) into another plasma wave (ω_l, k_l) and an acoustic wave (ω_s, k_s) (the process $l \to l + s$) [2], [III].

The growth rate for the instability is a maximum when the perturbation propagates in the same direction as the original wave. Hence, we can limit the analysis to the case of one-dimensional decay. Furthermore, we limit the analysis to the description of waves with wavelengths much larger than the Debye radius, in which case the plasma remains quasineutral in the presence of the acoustic waves:

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In this case the equation of motion and the equation of continuity for the slow motion can be reduced to an equation like (1.26), in which the magnetic pressure is now replaced by the dynamic pressure:

$$\left(\frac{\partial^2}{\partial t^2} + k_s^2 c_s^2\right) N(t) = -k_s^2 \frac{e^2 E^l \delta E^*}{mM\omega_l \omega_0} \exp\,[i(k_l - k_0 - k_s)z - i(\omega_l - \omega_0)t]. \qquad (1)$$

In the equations for the plasma oscillations we need only retain the nonlinear term which describes the modulation of the electron density by the acoustic waves:

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$$\begin{pmatrix} \Omega^2 - k_s^2 c_s^2 & k_s^2 \dfrac{e^2 \delta E^* N_0}{mM\omega_0 \omega_l} \\ \omega_p^2 \dfrac{\delta E}{\omega_0 N_0} & (\omega_0 + \Omega)\,\varepsilon(k_l, \omega_0 + \Omega) \end{pmatrix} \begin{pmatrix} N \\ E^l \end{pmatrix} = 0. \qquad (3)$$

Whence we obtain the growth rate for the decay instability:

$$\nu_d^2 \equiv -\delta\Omega^2 = \frac{m}{4M}\frac{\omega_p^2}{\omega_l \omega_s} k_s^2 |\delta v|^2, \quad \delta v = -\frac{i e \delta E}{m\omega_0}. \qquad (4)$$

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When $\nu > \omega_s$ the usual decay process is replaced by a modified form:

$$\nu \approx (\sqrt{3}/2)(2\omega_s \nu_d^2)^{1/3}.$$

3. Find the threshold for the decay instability considered in Problem 1 in the case in which the plasma is isothermal and the acoustic waves are damped with a damping rate $\nu_s \lessdot \omega_s$ [12].

The fact that the ion-acoustic wave is damped does not have any fundamental effect on the instability other than to retard its growth rate. Hence, in determining the threshold it is necessary to take account of the infrequent collisions between electrons, characterized by frequency ν_e. In this case the system of equations in problem 1 denoted by (3) assumes the form

$$\begin{pmatrix} -\Omega^2 - 2i\,\Omega\nu_s + k_s^2 c_s^2 & k_s^2 \dfrac{N_0 e^2\,\delta E}{mM\omega_0\omega_l} \\ \omega_p^2 \dfrac{\delta E}{\omega_0 N_0} & (\omega_0+\Omega) - \dfrac{\omega_p^2}{\omega_0+\Omega+i\nu_e} \end{pmatrix} \begin{pmatrix} N \\ E^l \end{pmatrix} = 0. \qquad (1)$$

Consider the first row of the determinant which describes the acoustic perturbation; in expansion of the diagonal element in terms of the small difference $(\Omega^2 - k_s^2 c_s^2)$ we can neglect the slowly varying change of amplitude and need only take account of the Landau damping (assuming that $\nu_s \gg \nu_d$). In the second row we need only retain the small terms which describe the dissipation of the wave due to particle collisions. Using the solubility condition for (1) we then find the required equation for the instability growth rate:

$$\nu = \frac{\nu_e}{2}\left[\frac{\omega_0\omega_s}{2\nu_e\nu_s}\,\frac{|\delta E|^2}{4\pi N_0 T_e} - 1\right]. \qquad (2)$$

Special interest attaches to the case in which the damping ν_s is associated with the resonance interaction of the waves $\omega_s/k_s = (\omega_0 - \omega_l)/(k_0 - k_l) \sim V_{T_i}$ with the ions. It will be evident that a parametric instability can also occur when $T_i \gtrsim T_l$, i.e., when it is really not possible to think in terms of ion-acoustic waves. This case is considered in detail in § 3.1, since a parametric instability of this kind can be described conveniently in terms of stimulated scattering of plasmons on ions. In this case, using the familiar expression for the dielectric permittivity of a highly non-isothermal plasma

$$\varepsilon(\Omega,\ k_l - k_0) = 1 - \frac{\Omega_p^2}{\Omega^2} + \frac{\Omega_p^2}{k_s^2 c_s^2}, \qquad (3)$$

we rewrite the determinant in Eq. (1) in terms of the dielectric permittivity

$$\begin{vmatrix} \dfrac{\Omega_p^2\,\varepsilon(\omega_l-\omega_0,\ k_l-k_0)}{|\varepsilon_e(\omega_l-\omega_0,\ k_l-k_0)|^2} & (k_l-k_0)^2\,\dfrac{e^2\,\delta E^* N_0}{mM\omega_0\omega_l} \\ \omega_p^2\,\delta E/N_0\omega_0 & \omega_l\,\varepsilon(\omega_l,\ k_l) \end{vmatrix} = 0. \qquad (4)$$

It is then an easy matter to obtain the growth rate of the instability:

$$\frac{\partial\varepsilon}{\partial\omega_l}\nu = -\operatorname{Im}\varepsilon(\omega_l,\ k_l) + \frac{|\delta E|^2}{4\pi n_0 T_l}(k\lambda_D)^{-2}\operatorname{Im}\frac{1}{\varepsilon(\omega_l-\omega_0)}. \qquad (5)$$

4. Find the growth rate for the aperiodic instability of an electromagnetic wave ($\omega_0 \gtrsim \omega_p$) in a cold plasma which leads to the production of a plasma wave and an acoustic wave (the process $t \to l + s$ [13]).

The equations which describe the decay of a transverse wave into a plasma wave and an acoustic wave are the same as the equations for the decay of a plasma wave into another plasma wave and an acoustic wave (cf. Problem 1). The reason lies in the fact that Eqs. (1) and (2) of Problem 1 only contain the electric field of the pump wave and the electron velocity in the wave. The actual nature of the electric field (whether the pump is transverse or longitudinal) does not play a role.

For a reasonably large amplitude we note an essential feature of the present problem which is associated with the smallness of the wave vector of the pump wave as compared with the wave vector of the plasma wave. Let us write $k_0 = 0$ and consider large-amplitude pump waves for which the growth rate of the decay instability becomes larger than the frequencies of the characteristic acoustic waves. It then turns out that both beats of the pump wave with the acoustic wave (both the sum and difference frequencies $\Omega \pm \omega_0$) are in resonace with the frequency of the characteristic plasma wave. Hence the determinant of the matrix equation (3) of Problem 1 now contains an additional row:

$$\begin{vmatrix} \Omega^2 - k^2 c_s^2 & N_0 k^2 \dfrac{e^2\,\delta E^*}{mM\omega_0\omega_l} & N_0 k^2 \dfrac{e^2\,\delta E}{mM\omega_0\omega_l} \\ \omega_p^2 \dfrac{\delta E}{\omega_0 N_0} & (\Omega+\omega_0)\,\varepsilon(\Omega+\omega_0,\ k) & 0 \\ -\omega_p^2 \dfrac{\delta E^*}{\omega_0 N_0} & 0 & (\Omega-\omega_0)\,\varepsilon(\Omega-\omega_0,\ k) \end{vmatrix} = 0. \qquad (1)$$

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we rewrite the determinant in Eq. (1) in terms of the dielectric permittivity

$$\begin{vmatrix} \dfrac{\Omega_p^2\, \varepsilon\,(\omega_l - \omega_0,\ k_l - k_0)}{|\,\varepsilon_e\,(\omega_l - \omega_0,\ k_l - k_0)\,|^2} & (k_l - k_0)^2\, \dfrac{e^2\, \delta E^*\, N_0}{mM\omega_0\, \omega_l} \\ \omega_p^2\, \delta E / N_0\, \omega_0 & \omega_l\, \varepsilon\,(\omega_l,\ k_l) \end{vmatrix} = 0. \qquad (4)$$

It is then an easy matter to obtain the growth rate of the instability:

$$\frac{\partial \varepsilon}{\partial \omega_l}\,\nu = -\operatorname{Im} \varepsilon\,(\omega_l,\ k_l) + \frac{|\,\delta E\,|^2}{4\pi n_0\, T_l}\,(k\,\lambda_D)^{-2}\, \operatorname{Im} \frac{1}{\varepsilon\,(\omega_l - \omega_0)}. \qquad (5)$$

4. Find the growth rate for the aperiodic instability of an electromagnetic wave ($\omega_0 \gtrless \omega_p$) in a cold plasma which leads to the production of a plasma wave and an acoustic wave (the process $t \to l + s$ [13]).

The equations which describe the decay of a transverse wave into a plasma wave and an acoustic wave are the same as the equations for the decay of a plasma wave into another plasma wave and an acoustic wave (cf. Problem 1). The reason lies in the fact that Eqs. (1) and (2) of Problem 1 only contain the electric field of the pump wave and the electron velocity in the wave. The actual nature of the electric field (whether the pump is transverse or longitudinal) does not play a role.

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$$\begin{vmatrix} \Omega^2 - k^2\, c_s^2 & N_0\, k^2\, \dfrac{e^2\, \delta E^*}{mM\omega_0\, \omega_l} & N_0\, k^2\, \dfrac{e^2 \delta E}{mM\omega_0\, \omega_l} \\ \omega_p^2\, \dfrac{\delta E}{\omega_0\, N_0} & (\Omega + \omega_0)\, \varepsilon\,(\Omega + \omega_0,\ k) & 0 \\ -\omega_p^2\, \dfrac{\delta E^*}{\omega_0\, N_0} & 0 & (\Omega - \omega_0)\, \varepsilon\,(\Omega - \omega_0,\ k) \end{vmatrix} = 0. \qquad (1)$$

In the first row we can now neglect $k^2c_s^2$ as compared with Ω^2 and expand the diagonal elements in the next two rows in terms of the small difference $\Delta = \omega_0 - \omega_p$; in this way we obtain an equation for Ω:

$$-\operatorname{Im}\Omega^2 = \frac{\Delta^2}{4} - \frac{1}{2}\left[\Delta\omega_s^2\omega_p \frac{|\delta E|^2}{4\pi n T_l}\right]^{1/2}. \qquad (2)$$

The growth rate reaches a maximum $\nu_{max} = (\sqrt{3}/2)(\omega_s^2\omega_p|\delta E|^2/8\pi n T)^{1/3}$ when $\Delta = 2\nu_{max}/\sqrt{3}$.

5. Consider the decay instability in an inhomogeneous medium.

In addition to effects associated with damping in an inhomogeneous plasma there is an additional mechanism which imposes a limitation from below on the amplitude of the pump wave for which a parametric instability can be excited. The point here is that in the presence of an inhomogeneity any wave packet will, in the propagation process, be also smeared out in wave vector space (in other words, its mean wave number will change). It then follows that the resonance condition can only be satisfied at one point which, for simplicity, we will locate at the origin. At any other point the resonance condition for the wave vectors suffers from a frequency mismatch. For example, let us assume that the wave packets of the pump wave (ω_0, k_0) and the first wave (ω_1, k_1) move in a plasma which is inhomogeneous in the x direction; consequently, the wave vectors $k_{0,1}$ change from point to point in accordance with the equations

$$\omega_i(s,\ k_{ix}(x)) = \text{const}. \qquad (1)$$

In this case the frequency of the second wave will change for two reasons: because of the change in the resonance wave vector $k_{2x} = k_{1x} - k_{0x}$ and because of the explicit dependence of the frequency ω_2 on x, that is to say,

$$\frac{d\omega_2}{dx} = \frac{\partial\omega_2}{\partial x} + \frac{\partial\omega_2}{\partial k_{2x}}\left(\frac{dk_{1x}}{dx} - \frac{dk_{0x}}{dx}\right). \qquad (2)$$

But the forcing term in the equation for the second wave is proportional to the quantity $\exp[-i(\omega_0 - \omega_1)t]$ and maintains its frequency. Hence, the resonance between $\omega_2(x)$ and $\omega_0 - \omega_1$ is gradually disturbed. When the frequency mismatch $\delta = \omega_0 + \omega_2 - \omega_1$ exceeds

the maximum allowable frequency (the width of the instability region) the growth of the pair of coupled waves is terminated. Ultimately, the wave packet can even fall into a damping region, i.e., the parametric instability in the case at hand can only occur when the gain for the propagation of the wave packet of plasmons through the resonance region $(|\delta| < 2\nu_d)$ is sufficiently large. The gain can be estimated as follows. In disintegration of the wave packet in the x direction, the time differential $dt = dx/(d\omega/dk_x)$. Then the wave intensity in the instability region will go as

$$\exp(\Gamma) \equiv \exp\left[2\int \nu(x)(\partial\omega_1/\partial k_{1x})^{-1}\,dx\right], \qquad (3)$$

where the integration is carried out only over the instability region.

This integral can be computed in several particular cases. Close to the resonance point the wave vectors can be expanded as functions of the coordinate:

$$k_{ih}(x) = k_{ix}(0) + k'_{ix}(0)\,x.$$

Then the frequency mismatch due to the change of frequency is also proportional to the displacement of the wave packet in the x direction:

$$\delta = -\frac{\partial\omega_2}{\partial k_{2x}}\left[\frac{d}{dx}(k_{0x} + k_{2x} - k_{1x})\right]x, \qquad (4)$$

where the derivative of the vector $\mathbf{k}_i$ is calculated at a fixed frequency ω_i. In the general case the growth rate for the decay instability in the presence of frequency mismatch is (cf. § 1.2)

$$\nu = [\nu_d^2 - \delta^2(x)/4]^{1/2}.$$

Integration of Eq. (3) then leads to the Piliya–Rosenbluth formula

$$\Gamma = 2\pi\nu_d^2 \Big/ \left|\frac{\partial\omega_1}{\partial k_{1x}} \cdot \frac{\partial\omega_2}{\partial k_{2x}}\,\frac{d}{dx}(k_{0x} + k_{2x} - k_{1x})\right|. \qquad (5)$$

This formula is not valid if one of the waves lies in the vicinity of a turning point $(k_{1x} = 0)$. In this case the group velocity becomes vanishingly small $(\partial\omega_1/\partial k_{1x} \approx 0)$ and the integral in (3) must be computed taking account of this circumstance. The gain is then ex-

In the first row we can now neglect $k^2c_s^2$ as compared with Ω^2 and expand the diagonal elements in the next two rows in terms of the small difference $\Delta = \omega_0 - \omega_p$; in this way we obtain an equation for Ω:

$$-\mathrm{Im}\,\Omega^2 = \frac{\Delta^2}{4} - \frac{1}{2}\left[\Delta\omega_s^2\omega_p\frac{|\delta E|^2}{4\pi n T_i}\right]^{1/2}. \qquad (2)$$

The growth rate reaches a maximum $\nu_{max} = (\sqrt{3}/2)(\omega_s^2\omega_p|\delta E|^2/8\pi n T)^{1/3}$ when $\Delta = 2\nu_{max}/\sqrt{3}$.

▶ 5. Consider the decay instability in an inhomogeneous medium.

In addition to effects associated with damping in an inhomogeneous plasma there is an additional mechanism which imposes a limitation from below on the amplitude of the pump wave for which a parametric instability can be excited. The point here is that in the presence of an inhomogeneity any wave packet will, in the propagation process, be also smeared out in wave vector space (in other words, its mean wave number will change). It then follows that the resonance condition can only be satisfied at one point which, for simplicity, we will locate at the origin. At any other point the resonance condition for the wave vectors suffers from a frequency mismatch. For example, let us assume that the wave packets of the pump wave (ω_0, k_0) and the first wave (ω_1, k_1) move in a plasma which is inhomogeneous in the x direction; consequently, the wave vectors $k_{0,1}$ change from point to point in accordance with the equations

$$\omega_i(s,\ k_{ix}(x)) = \mathrm{const}. \qquad (1)$$

In this case the frequency of the second wave will change for two reasons: because of the change in the resonance wave vector $k_{2x} = k_{1x} - k_{0x}$ and because of the explicit dependence of the frequency ω_2 on x, that is to say,

$$\frac{d\omega_2}{dx} = \frac{\partial\omega_2}{\partial x} + \frac{\partial\omega_2}{\partial k_{2x}}\left(\frac{dk_{1x}}{dx} - \frac{dk_{0x}}{dx}\right). \qquad (2)$$

But the forcing term in the equation for the second wave is proportional to the quantity $\exp[-i(\omega_0 - \omega_1)t]$ and maintains its frequency. Hence, the resonance between $\omega_2(x)$ and $\omega_0 - \omega_1$ is gradually disturbed. When the frequency mismatch $\delta = \omega_0 + \omega_2 - \omega_1$ exceeds

the maximum allowable frequency (the width of the instability region) the growth of the pair of coupled waves is terminated. Ultimately, the wave packet can even fall into a damping region, i.e., the parametric instability in the case at hand can only occur when the gain for the propagation of the wave packet of plasmons through the resonance region $(|\delta| < 2\nu_d)$ is sufficiently large. The gain can be estimated as follows. In disintegration of the wave packet in the x direction, the time differential $dt = dx/(d\omega/dk_x)$. Then the wave intensity in the instability region will go as

$$\exp(\Gamma) \equiv \exp\left[2\int\nu(x)(\partial\omega_1/\partial k_{1x})^{-1}dx\right], \qquad (3)$$

where the integration is carried out only over the instability region.

This integral can be computed in several particular cases. Close to the resonance point the wave vectors can be expanded as functions of the coordinate:

$$k_{ik}(x) = k_{ix}(0) + k'_{ix}(0)\,x.$$

Then the frequency mismatch due to the change of frequency is also proportional to the displacement of the wave packet in the x direction:

$$\delta = -\frac{\partial\omega_2}{\partial k_{2x}}\left[\frac{d}{dx}(k_{0x} + k_{2x} - k_{1x})\right]x, \qquad (4)$$

where the derivative of the vector $\mathbf{k}_i$ is calculated at a fixed frequency ω_i. In the general case the growth rate for the decay instability in the presence of frequency mismatch is (cf. § 1.2)

$$\nu = [\nu_d^2 - \delta^2(x)/4]^{1/2}.$$

Integration of Eq. (3) then leads to the Piliya–Rosenbluth formula

$$\Gamma = 2\pi\nu_d^2\Big/\left|\frac{\partial\omega_1}{\partial k_{1x}}\cdot\frac{\partial\omega_2}{\partial k_{2x}}\frac{d}{dx}(k_{0x} + k_{2x} - k_{1x})\right|. \qquad (5)$$

This formula is not valid if one of the waves lies in the vicinity of a turning point $(k_{1x} = 0)$. In this case the group velocity becomes vanishingly small $(\partial\omega_1/\partial k_{1x} \approx 0)$ and the integral in (3) must be computed taking account of this circumstance. The gain is then ex-

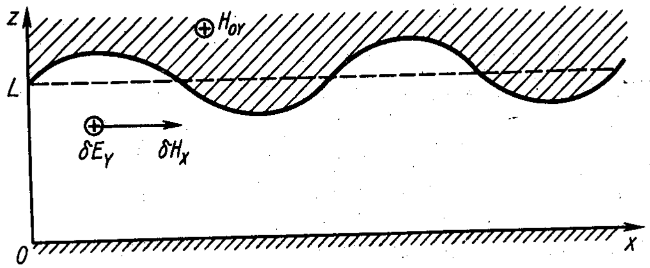

Fig. 3. Perturbation of the boundary between a plasma and a standing electromagnetic wave.

pressed in terms of total elliptic integrals of the first and second kind:

$$\Gamma = (8/3)\dot{K}(\varkappa)\left[2\varkappa^2 \frac{E(\varkappa)}{K(\varkappa)} - \left(\frac{E(\varkappa)}{K(\varkappa)} - 1 + \varkappa^2\right)\right]\times$$
$$\times v_d^{3/2}/(\partial\omega_1/\partial k_{1x}^2)\left|\frac{dk_{1x}^2}{dx}\cdot\frac{d\delta}{dx}\right|^{1/2},$$
$$0 \leqslant \varkappa^2 \equiv 1 - 0.5\,[1 + \delta(0)/2v_d - k_{1x}^2(0)\,\Lambda^2] \leqslant 1,$$
$$\Lambda^{-2} = 2v_d\,|dk_{1x}^2/dx\,|\,/\,|d\delta/dx\,|.$$

▶ 6. Obtain the dispersion equation for oscillations at the interface of a plasma and a standing electromagnetic wave [5].

We shall investigate the stability with respect to one-dimensional perturbations of the plasma boundary (cf. Fig. 3). The instability is manifested in the modulation of the electromagnetic field in the vacuum region and the accompanying excitation of magnetoacoustic waves in the plasma. Within the vacuum region and in the plasma the oscillations are described by linear equations for the vacuum electric field and the vector displacement of the plasma ξ:

$$[\Delta - \partial^2/\partial c^2 t^2]\,E_y^{(1)} = 0, \quad 0 \leqslant z \leqslant L, \tag{1}$$
$$\partial^2\xi/\partial t^2 - (c_s^2 + v_A^2)\,\nabla\,\mathrm{div}\,\xi = 0.$$

The coupling between these waves arises from the pressure balance condition and the absence of a tangential component of the electric field at the surfaces of the plasma and the metal (sic):

$$\left[p^{(1)} + \frac{H_0 H_y^{(1)}}{4\pi}\right]_{z=L} = \left.\frac{H_x^{(0)} H_x^{(1)}}{4\pi}\right|_{z=L}, \tag{2}$$
$$E_y^{(1)}(L) + \xi_z\left(\partial\delta E_y^{(0)}/\partial z\right)_{z=L} = 0, \quad E_y^{(1)}(0) = 0,$$

where

$$p^{(1)} = \gamma p_0\,\mathrm{div}\,\xi, \quad H_y^{(1)} = H_0\,\mathrm{div}\,\xi$$
$$\delta E_y^{(0)} = \delta E_0 \sin\omega z/c, \quad \partial H_x^{(1)}/\partial ct = \partial E_y^{(1)}/\partial z.$$

The solution of Eq. (1) with boundary conditions (2) leads to the following equation for the growth rate:

$$\sqrt{\omega^2/c^2 - k^2}\ \cot\sqrt{\omega^2/c^2 - k^2} = \nu^2/c_s^2\sqrt{k^2 + \nu^2/\left(c_s^2 + v_A^2\right)}\,.$$

§ 1.2. Interaction of Finite-Amplitude Waves

The stability of Alfvén waves of finite amplitude has been considered in § 1.1. In the case of perturbations in the form of a sum of an acoustic wave and Alfvén wave we have written two coupled equations for the perturbed longitudinal velocity and the transverse magnetic field. These equations describe the interaction between the acoustic wave and the Alfvén wave.

It has been shown that the frequency rule for the decay instability corresponds to a quantum-mechanical conservation relation. Consequently, we may anticipate that the equations can be simplified considerably if they are written in Hamiltonian form through the use of quantum-mechanical parameters such as the energy in the photon associated with the wave and the number of photons. In order to achieve this objective we replace N and $h_\perp$ by $[N(t), h_\perp(t)]\exp(ikz - i\omega t)$, where N(t) and $h_\perp(t)$ are the slowly varying amplitudes while ω and **k** satisfy the linear dispersion equations for the acoustic wave and the Alfvén wave respectively. We shall also use the subscript 1 to denote parameters associated with the Alfvén wave and the subscript 2 for parameters associated with the acoustic wave:

$$\left.\begin{aligned} h_\perp &\equiv h_1; \quad k_A = k_1; \quad \omega_1^2\,\varepsilon(\omega_1) = k_1^2 c^2;\\ N &\equiv N_2; \quad k_s = k_2; \quad \omega_2^2 = k_2^2 c_s^2. \end{aligned}\right\} \tag{1.27}$$

Equations (1.14) and (1.15) now assume the form

$$\mathrm{i}\,\frac{\partial N_2}{\partial t} = \frac{k_2^2}{2\omega_2}\frac{\delta\mathscr{H}^*}{4\pi M}h_1\exp[-\mathrm{i}(\omega_1 - \omega_0 - \omega_2)t]; \tag{1.28}$$

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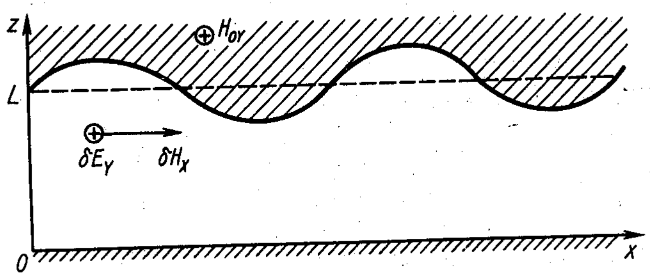

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The number of photons can be defined as the total energy in a given mode divided by the frequency of the mode. For the acoustic wave we have

$$n_{\mathbf{k}} = \omega_{\mathbf{k}}^{-1}\left(\frac{MN_0 v_{\mathbf{k}}^2}{2} + \frac{MN_{\mathbf{k}}^2 c_s^2}{2N_0}\right). \tag{1.30}$$

The energy of the Alfvén wave can be found from the familiar relation for the field energy in a dispersive medium [14]:

$$n_{\mathbf{k}} = \frac{\partial}{\partial\omega}\left[\frac{\omega^2}{c^2}\,\varepsilon(\mathbf{k},\omega)\right]\frac{|\delta\mathcal{H}|^2}{4\pi k^2}. \tag{1.31}$$

Equations (1.28) and (1.29) can be written in symmetric form by introducing the probability amplitudes ($|C_j|^2 = n_j$). In the case at hand the probability amplitudes are defined as follows:

$$C_0(t) = \frac{\delta\mathcal{H}}{(2\pi k_0 v_{g0})^{1/2}};\quad C_1(t) = \frac{h_1}{(4\pi k_1 v_{g1})^{1/2}};\quad C_2(t) = \frac{N_2}{(N_0 k_2^2/|\omega_2|)^{1/2}}. \tag{1.32}$$

When these variables are used Eqs. (1.28) and (1.29) can be rewritten in the form of a Schrödinger equation in the interaction representation [15]:

$$i\frac{\partial C_1}{\partial t} = V_{\mathbf{k}_1,\mathbf{k}_0,\mathbf{k}_2} C_0 C_2, \tag{1.33}$$

$$i\frac{\partial C_2}{\partial t} = V_{\mathbf{k}_2,-\mathbf{k}_0,\mathbf{k}_1} C_0^* C_1, \tag{1.34}$$

where

$$V_{\mathbf{k}_2,-\mathbf{k}_0,\mathbf{k}_1} = V_{\mathbf{k}_1,\mathbf{k}_0,\mathbf{k}_2}\,\mathrm{sign}\,(\omega_2 k_1 v_{g1}) \equiv -\left[\frac{|k_0 v_{g0} k_1 v_{g1}\omega_2|}{8MN_0 c_s^2}\right]^{1/2}\mathrm{sign}\,\omega_2.$$

Equations in the form of (1.33) and (1.34) for the amplitudes, C_j are valid for any undamped waves. The only difference that can arise is associated with the actual expression for the matrix element which will always satisfy the symmetry condition (because a Hamiltonian system is treated). Equations (1.33) and (1.34) can be generalized to the case in which there is frequency detuning $\delta = \omega_0 + \omega_2 - \omega_1 \neq 0$. In this case the right-hand side of Eq. (1.33) will contain the factor $\exp(i\delta t)$ while Eq. (1.34) will contain the factor $(-i\delta t)$.

In the general case with a frequency mismatch the decay instability can be described by Eqs. (1.33) and (1.34). These equations have unstable solutions characterized by the growth rate

$$\nu = \left[-|V_{\mathbf{k}_1,\mathbf{k}_0,\mathbf{k}_2}|^2\,|C_0|^2\,\mathrm{sign}\,(\omega_1\omega_2) - \frac{1}{4}\delta^2\right]^{1/2}. \tag{1.35}$$

The analogy with the parametric resonance in a system consisting of two coupled oscillators becomes especially clear if we do not separate the rapidly varying factor $\exp(i\omega t)$ in the equation for the C amplitudes [cf., for example, Eq. (1.12)]; in this case the system in Eqs. (1.33) and (1.34) is replaced by the system

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These equations not only describe the decay instability; they also describe the modified decay instability with the growth rate

$$\nu = (\sqrt{3}/2)\,(2\omega_2|V_{\mathbf{k}_1,\mathbf{k}_0,\mathbf{k}_2}|^2|C_0|^2)^{1/3}. \tag{1.36}$$

Up to this point we have only obtained equations which describe the growth of the perturbation waves; we have neglected completely the feedback effect of these waves on the primary (decaying) wave. In order to describe the relaxation of the primary wave it is necessary to take account of effects due to the finite amplitudes of the perturbations and to retain the nonlinear terms in the equation for the primary wave. When C_0, C_1, and C_2 are of the same order of magnitude we may expect that the appropriate equation for C_0 will be of the form

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Obviously this equation can be derived form the same MHD equations that are used for the derivation of Eqs. (1.33) and (1.34). In order to solve the system of coupled equations (1.33), (1.34), and (1.37), we write the probability amplitude in the form

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$$C_j(t) = a_j(t)\exp(i\Phi_j(t)), \quad \mathrm{Im}\,a_j = \mathrm{Im}\,\Phi_j = 0.$$

When $\omega_0 > \omega_1 > |\omega_2|$ and $\omega_2 < 0$, from the symmetry properties of the V's we have

$$V_{\mathbf{k}_1,\mathbf{k}_0,\mathbf{k}_2} = -H;$$
$$V_{\mathbf{k}_2,-\mathbf{k}_0,\mathbf{k}_1} = V_{\mathbf{k}_1,\mathbf{k}_0,\mathbf{k}_2}\,\mathrm{sign}\,(\omega_1\cdot\omega_2) = H;$$
$$V_{\mathbf{k}_0,\mathbf{k}_1,-\mathbf{k}_2} = V_{\mathbf{k}_1,\mathbf{k}_0,\mathbf{k}_2}\,\mathrm{sign}\,(\omega_1\cdot\omega_0) = -H.$$

Separating the real and imaginary parts of Eqs. (1.33), (1.34), and (1.37) and using the variables $a_j(t)$ and $\Theta = \Phi_1 - \Phi_0 - \Phi_2$, we obtain the following equations [16]:

$$\left.\begin{aligned}(\partial a_1/\partial t) &= Ha_2a_0 \sin\Theta;\\ \partial a_2/\partial t &= Ha_1a_0 \sin\Theta;\\ (\partial a_0/\partial t) &= -Ha_1a_2 \sin\Theta;\\ (\partial\Theta/\partial t) &= H\,[(a_0a_2/a_1) + (a_0a_1/a_2) -\\ &- (a_1a_2/a_0)]\cos\Theta = \cot\Theta\,(\partial/\partial t)\ln(a_0a_1a_2).\end{aligned}\right\} \quad (1.38)$$

By integrating the last equation we find the relation $a_0a_1a_2\cos\Theta \equiv \Gamma = \mathrm{const}$. Making use of the frequency resonance condition we obtain the first integral for the remaining equations:

$$a_1^2\omega_1 + a_2^2\,|\,\omega_2\,| - a_0^2\omega_0 = \mathrm{const}. \quad (1.39)$$

Integrating $a_0(da_0/dt) + a_1(da_1/dt)$, etc., we then obtain the following constants of the motion:

$$\left.\begin{aligned}m_1 &\equiv n_0 + n_1 = \mathrm{const};\\ m_2 &\equiv n_0 + n_2 = \mathrm{const};\\ m_3 &\equiv n_1 - n_2 = \mathrm{const}.\end{aligned}\right\} \quad (1.40)$$

These relations are very well known in the theory of parametric amplifiers [17], where they are called vector Manley–Rowe relations (taken in the direction of propagation). These relations can be easily understood by introducing the diagram for the three-wave process (Fig. 4), if it is noted that the disappearance of one photon mode $\omega_{\mathbf{k}_0}$ is accompanied by the appearance of a photon for mode $\omega_{\mathbf{k}_1}$ and a photon for mode $\omega_{\mathbf{k}_2}$ so that $\Delta n_0 = -1$, $\Delta n_1 = 1 = \Delta n_2$. By using Eqs. (1.38)-(1.40) we then obtain an equation for n_0:

$$(dn_0/dt) = -2H\,[n_0\,(m_1 - n_0)(m_2 - n_0) - \Gamma^2]^{1/2}. \quad (1.41)$$

If the three roots of the equation $n_0(m_1 - n_0)(m_2 - n_0) = \Gamma^2$ are arranged in descending order, Eq. (1.41) can be written in the

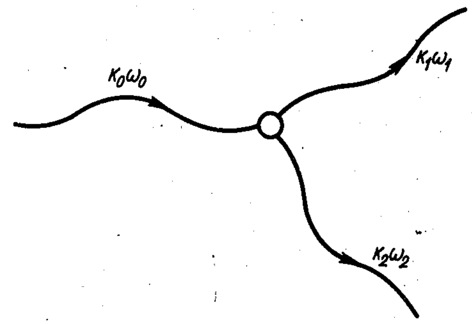


Fig. 4. Graphical representation of the three-wave interaction.

form

$$H(t - t_0) = -\frac{1}{2}\int\limits_{n_0(t_0)}^{n_0(t)} \frac{dn_0}{[(n_0 - n_c)(n_0 - n_b)(n_0 - n_a)]^{1/2}}, \quad n_c \geqslant n_b \geqslant n_a \geqslant 0.$$

The integral which appears here can be reduced to an elliptic integral by the change of variables

$$y(t) \equiv [(n_0(t) - n_a)/(n_b - n_a)]^{1/2};$$
$$\varkappa = [(n_b - n_a)/(n_c - n_a)]^{1/2}.$$

If the time t_0 is chosen so that $y(t_0) = 0$, we can write the resulting equation in the standard form:

$$H(t - t_0)\sqrt{n_c - n_a} = -\int\limits_0^{y(t)} \frac{dy}{[(1 - y^2)(1 - \varkappa^2y^2)]^{1/2}}. \quad (1.42)$$

Consequently,

$$y(t) = \mathrm{sn}\,[H\sqrt{n_c - n_a}\,(t_0 - t);\quad \varkappa],$$

and the general solution is obtained from the definition of $y(t)$:

$$n_0(t) = n_a + (n_b - n_a)\,\mathrm{sn}^2\,(H\sqrt{n_c - n_0}\,(t_0 - t);\ \varkappa). \quad (1.43)$$

Let us consider two simple examples [16].

Case A. At time $t = 0$, $n_0(0) = 0$, $n_1(0) \gg n_2(0)$. Without loss of generality we can take $\Gamma = 0$ in Eq. (1.41). The three roots $n_{a,b,c}$ are then written $m_1 \equiv n_1(0) = n_c \gg m_2 \equiv n_2(0) = n_b > 0 = n_a$.

When $\omega_0 > \omega_1 > |\omega_2|$ and $\omega_2 < 0$, from the symmetry properties of the V's we have

$$V_{\mathbf{k}_1,\mathbf{k}_0,\mathbf{k}_2} = -H;$$
$$V_{\mathbf{k}_2,-\mathbf{k}_0,\mathbf{k}_1} = V_{\mathbf{k}_1,\mathbf{k}_0,\mathbf{k}_2}\,\mathrm{sign}\,(\omega_1\cdot\omega_2) = H;$$
$$V_{\mathbf{k}_0,\mathbf{k}_1,-\mathbf{k}_2} = V_{\mathbf{k}_1,\mathbf{k}_0,\mathbf{k}_2}\,\mathrm{sign}\,(\omega_1\cdot\omega_0) = -H.$$

Separating the real and imaginary parts of Eqs. (1.33), (1.34), and (1.37) and using the variables $a_j(t)$ and $\Theta = \Phi_1 - \Phi_0 - \Phi_2$, we obtain the following equations [16]:

$$\left.\begin{aligned}(\partial a_1/\partial t) &= Ha_2a_0 \sin\Theta;\\ \partial a_2/\partial t &= Ha_1a_0 \sin\Theta;\\ (\partial a_0/\partial t) &= -Ha_1a_2 \sin\Theta;\\ (\partial\Theta/\partial t) &= H\,[(a_0a_2/a_1) + (a_0a_1/a_2) -\\ -\,(a_1a_2/a_0)]\cos\Theta &= \cot\Theta\,(\partial/\partial t)\ln(a_0a_1a_2).\end{aligned}\right\} \quad (1.38)$$

By integrating the last equation we find the relation $a_0a_1a_2\cos\Theta \equiv \Gamma = \mathrm{const}$. Making use of the frequency resonance condition we obtain the first integral for the remaining equations:

$$a_1^2\omega_1 + a_2^2|\omega_2| - a_0^2\omega_0 = \mathrm{const}. \quad (1.39)$$

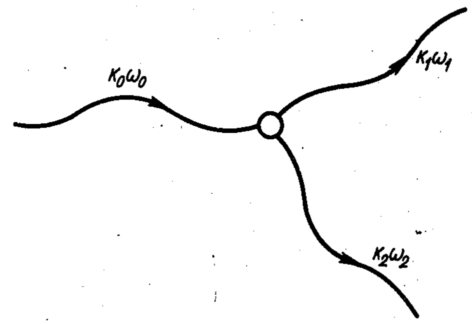
Integrating $a_0(da_0/dt) + a_1(da_1/dt)$, etc., we then obtain the following constants of the motion:

$$\left.\begin{aligned}m_1 &\equiv n_0 + n_1 = \mathrm{const};\\ m_2 &\equiv n_0 + n_2 = \mathrm{const};\\ m_3 &\equiv n_1 - n_2 = \mathrm{const}.\end{aligned}\right\} \quad (1.40)$$

These relations are very well known in the theory of parametric amplifiers [17], where they are called vector Manley–Rowe relations (taken in the direction of propagation). These relations can be easily understood by introducing the diagram for the three-wave process (Fig. 4), if it is noted that the disappearance of one photon mode $\omega_{\mathbf{k}_0}$ is accompanied by the appearance of a photon for mode $\omega_{\mathbf{k}_1}$ and a photon for mode $\omega_{\mathbf{k}_2}$ so that $\Delta n_0 = -1$, $\Delta n_1 = 1 = \Delta n_2$. By using Eqs. (1.38)-(1.40) we then obtain an equation for n_0:

$$(dn_0/dt) = -2H\,[n_0\,(m_1 - n_0)(m_2 - n_0) - \Gamma^2]^{1/2}. \quad (1.41)$$

If the three roots of the equation $n_0(m_1 - n_0)(m_2 - n_0) = \Gamma^2$ are arranged in descending order, Eq. (1.41) can be written in the



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$$n_0(t) = n_2(0)\sin^2\left(Ht\sqrt{n_1(0)}\right),$$
$$n_2(t) = n_2(0)\cos^2\left(Ht\sqrt{n_1(0)}\right),$$
$$n_1(t) = n_1(0) - n_2(0)\sin^2\left(Ht\sqrt{n_1(0)}\right).$$

The time dependence of the occupation number is shown in Fig. 5. This figure describes the case in which the frequency of the finite-amplitude wave is smaller than the frequency of the perturbation wave; consequently it is stable against decay. The small periodic variation in its amplitude is a consequence of the fact that a small fraction of the energy is stored initially in the perturbation at the low frequency; in the photon coupling process this energy can be transferred to the perturbation at the high frequency.

Case B. We now consider the decay of a finite-amplitude wave when $n_2(0) = 0$, $n_0(0) \gg n_1(0)$ at time $t = 0$. Again taking $\Gamma = 0$ we find the constants $n_{a,b,c}$: $n_c = m_1 \equiv n_0(0) + n_1(0) > n_b = m_2 \equiv n_0(0) > n_a = 0$. Making use of Eqs. (1.40) and (1.43) in this case we have

$$n_0(t) = n_0(0)\,\mathrm{sn}^2\left[H(t-t_0)\sqrt{n_c};\ \varkappa\right],$$
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$$n_2(t) = n_0(0)\left\{1 - \mathrm{sn}^2\left[H(t-t_0)\sqrt{n_c};\ \varkappa\right]\right\},$$

where $1 - \varkappa^2 = n_1(0)/[n_0(0) + n_1(0)] \ll 1$. Since $n_1 = n_1(0)$ when $t = 0$, we can write $1 - \mathrm{sn}^2[Ht_0\sqrt{n_c};\ \varkappa] = 0$. Consequently, the quantity

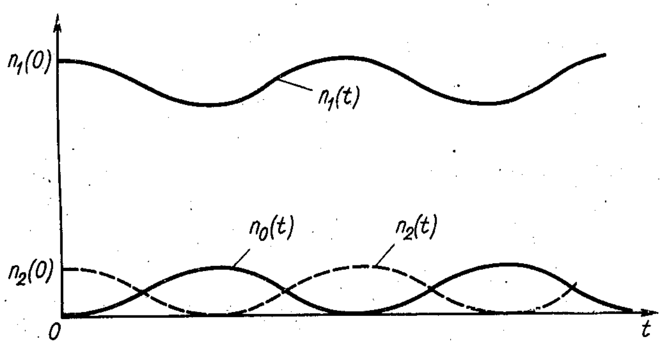


Fig. 5. Time dependence of the amplitude of a wave which is stable against the decay instability.

$Ht_0\sqrt{n_c}$ is equal to one quarter period sn : $Ht_0\sqrt{n_c} = K(\varkappa)$. Since $1 - \varkappa^2 \ll 1$, the complete elliptic integral of the first kind $K(\varkappa)$ can be expanded in terms of the small parameter $\varkappa' = (1 - \varkappa^2)^{1/2}$:

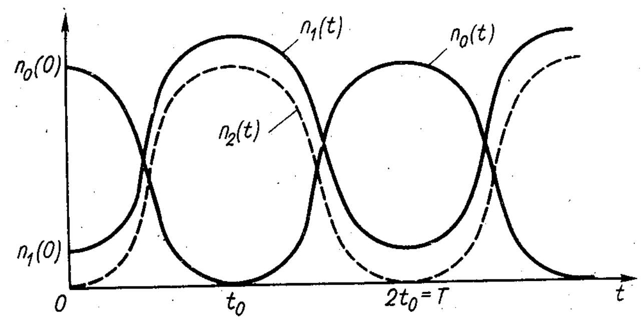
$$K(\varkappa) \approx -(\varkappa'/2)^2 + \ln(4/\varkappa')[1 + (\varkappa'/2)^2 + 0(\varkappa'^4)].$$

As a result we find

$$t_0 = (1/2H\sqrt{n_0(0)})\ln(n_0(0)/n_1(0)).$$

in this same time the amplitude of the primary wave $n_0(0)$ is reduced to zero so that we call this time the decay time. The reciprocal of t_0 only differs from the linear growth rate of the instability $H\sqrt{n_0}$ by a logarithmic factor. This factor takes account of the fact that even with exponential growth, the amplitude of the small perturbations reaches the amplitude of the primary wave in a time of order $t = (1/2\nu)\ln(n_0/n_1)$. The time behavior of the number of photons in the modes is shown in Fig. 6. It turns out that after a sufficiently long time the amplitude of the perturbations again falls to zero. Thus, the decay process in the interaction of three separate waves is reversible.

In the present section we have limited the analysis to three waves although an exact solution for the system of equations that describes a four-wave interaction has now been obtained. The reader interested in this solution and many other problems of nonlinear optics, where these interactions play an important role, is referred to well-known monographs [18, 19].



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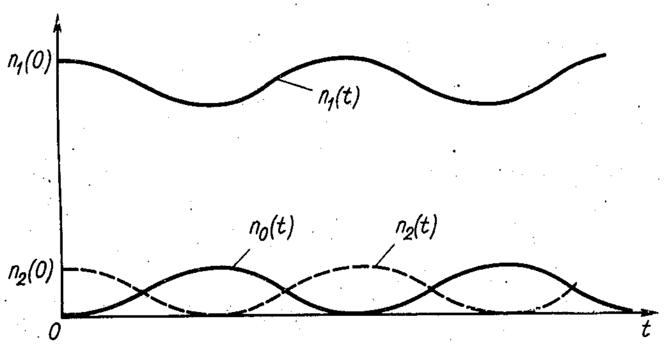

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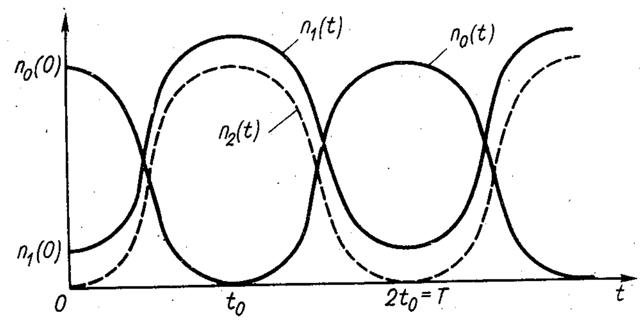

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PROBLEM

1. Find the growth rate for the decay instability in the case in which the perturbations are characterized by damping rates ν_1 and ν_2.

In Eqs. (1.33) and (1.34) $\partial C_{1,2}/\partial t$ is replaced by $\partial C_{1,2}/\partial t + \nu_{1,2}C_{1,2}$. Writing the solution in the form (νt), we then find

$$\nu = -(1/2)(\nu_1+\nu_2) \pm \sqrt{(1/4)(\nu_1-\nu_2)^2 + |V_{\mathbf{k}_1,\mathbf{k}_0,\mathbf{k}_2}|^2 |C_0|^2}.$$

It will be evident that the instability has a threshold given by [20]

$$|C_0|^2 > \frac{\nu_1\nu_2}{|V_{\mathbf{k}_1,\mathbf{k}_0,\mathbf{k}_2}|^2}.$$

In particular, it is then possible to use this relation to obtain the solution of Problem 2 in § 1.1.

§ 1.3. Higher-Order Parametric Instabilities

The necessary condition for a resonance interaction in a system of three plasma waves is that Eq. (1.18) for the frequencies and wave vectors must be satisfied. It will be obvious that these relations cannot be satisfied for an arbitrary dispersion relation. Various possible spectra and dispersion relations are shown in Fig. 7. By making use of the vector inequality $|\mathbf{k}_1 + \mathbf{k}_2| \leq |\mathbf{k}_1| + |\mathbf{k}_2|$ it is easy to show that the resonance condition can only be

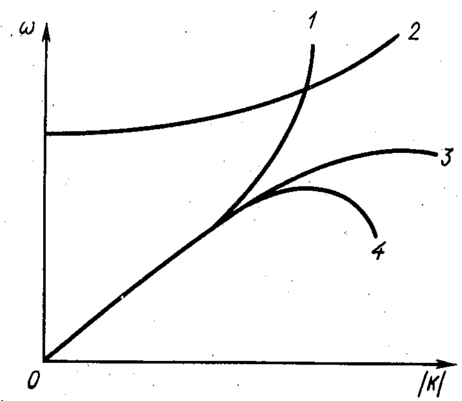

Fig. 7. Various forms for wave dispersion in a plasma.

satisfied for dispersion relations similar to curves 1 and 4 (cf. Fig. 7); conversely, for dispersion relations like 2 and 3 the resonance condition cannot be satisfied. If the dispersion relation has more than one branch the resonance condition may be satisfied by waves that correspond to different branches. In the general case the resonance condition can be satisfied when it is possible to draw a curve similar to curves 1 or 4 through the three points which correspond to the modes $(\omega_0, \mathbf{k}_0)$, $(\omega_1, \mathbf{k}_1)$, and $(\omega_2, \mathbf{k}_2)$ (these points can lie on different branches).

When the resonance conditions cannot be satisfied in a three-wave interaction it is possible to include a fourth wave. It will be evident that to obtain a finite growth rate this fourth wave must be of finite amplitude. Consequently, it is necessary to consider the stability of the second harmonic of a finite amplitude wave. The resonance conditions now assume the form

$$2\omega(\mathbf{k}_0) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2); \; 2\mathbf{k}_0 = \mathbf{k}_1 + \mathbf{k}_2 \tag{1.44}$$

By making use of the decay condition $2|\omega(\mathbf{k}_0)| > |\omega(\mathbf{k}_1)|, |\omega(\mathbf{k}_2)|$ and these resonance conditions it can be shown that in second order the resonance condition can, in fact, be satisfied for modes characterized by dispersion curve 3 (cf. Fig. 7) while it cannot be satisfied by modes characterized by curve 2. One then expects that the diagram of the unstable regions in the frequency–amplitude plane of the wave will be qualitatively like the corresponding diagram for the parametric resonance problem (Fig. 8).

The width of the unstable region close to the n-th harmonic is of the order of the growth rate of the instability and is proportional to the n-th power of the amplitude. It will be evident that

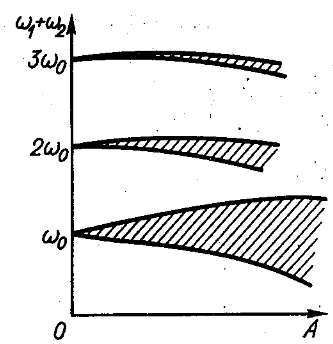

Fig. 8. Stability diagram in the frequency–amplitude plane (the unstable regions are crosshatched).

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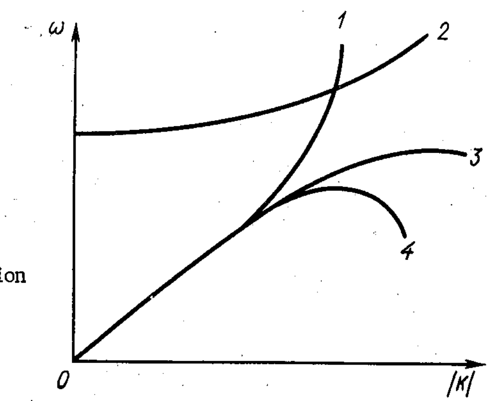

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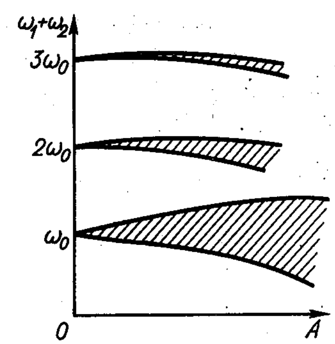

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the resonance condition must be satisfied:

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An interesting example of a high-order decay instability is the instability of a high-amplitude Alfvén wave with a sawtooth profile against decay into perturbations with frequencies ω_1 and ω_2, where $\max(\omega_1, \omega_2) \gg v_A/\lambda$, and λ is the wavelength which corresponds to the separation between the two teeth of the profile of the Alfvén wave [21].

We now consider the simpler problem of the decay of a second-harmonic oscillation at the surface of a deep fluid [22, 23].

The oscillations of the fluid surface are described by two functions: the velocity potential $\varphi(x, y, z, t)$ and the displacement of the fluid surface from the equilibrium position $\eta(x, y, t)$. The equations for these functions are well known [24]:

$$\left.\begin{aligned} &\Delta\varphi \equiv \nabla^2\varphi + \varphi_{zz} = 0;\\ &\eta_t + \nabla\eta\cdot[\nabla\varphi]_{z=\eta} - [\varphi_z]_{z=\eta} = 0;\\ &g\eta + [\varphi_t + (1/2)(\nabla\varphi^2) + (1/2)\varphi_z^2]_{z=\eta} = 0. \end{aligned}\right\} \tag{1.45}$$

In the case of interest here, small oscillations, the quantity $[\varphi]_{z=\eta}$ can be expanded in powers of $(k\eta) \ll 1$. Retaining terms to third order inclusively, we have

$$\left.\begin{aligned} \varphi_z &= \eta_t + \operatorname{div}(\eta\nabla\varphi) + \operatorname{div}[(1/2)\eta^2\nabla\varphi_z] - \varphi_{zz}\eta - (1/2)\varphi_{zzz}\eta^2,\\ -g\eta &= \varphi_t + \varphi_{zt}\eta + \varphi_{zzt}\eta^2/2 + (1/2)[(\nabla\varphi)^2 + \varphi_z^2] +\\ &\quad + (1/2)[(\nabla\varphi)^2 + \varphi_z^2]_z\cdot\eta, \end{aligned}\right\} \tag{1.46}$$

where the velocity potential is computed using the unperturbed surface. Reducing these equations to a single equation for φ, we obtain the equation of motion for the wave at a water surface:

$$\begin{aligned} \varphi_{tt} + g\varphi_z = \{&g^{-1}\varphi_t\varphi_{zt} - g^{-2}\varphi_t\varphi_{zt}^2 - 0.5g^{-2}\varphi_{zzt}\,\varphi_t^2 -\\ &-(1/2)[(\nabla\varphi^2) + \varphi_z^2] + (1/2g)\varphi_t\,[(\nabla\varphi)^2 + \varphi_z^2]_z\}_t + \operatorname{div}\{-\varphi_t\nabla\varphi +\\ &+ 0.5g^{-1}\varphi_t^2\,\nabla\varphi_z - (1/2)[(\nabla\varphi)^2 + \varphi_z^2]\nabla\varphi + g^{-1}\varphi_t\varphi_{zt}\nabla\varphi\}. \end{aligned} \tag{1.47}$$

The second-order nonlinear terms in Eq. (1.47) describe the generation of the second harmonic. By retaining these terms alone it is an easy matter to obtain two terms in the Stokes expansion for the primary wave. In this case it turns out that the second harmon-

ic vanishes in the expansion of the velocity potential in terms of the amplitude:

$$\delta\varphi_0 = a_0 \exp(|\mathbf{k}_0|z)\sin(\mathbf{k}_0\cdot\mathbf{r} - \omega_0 t).$$

As in the three-wave interaction, it is convenient to introduce a probability amplitude, C_0, defined in such a way that the surface energy density of the wave is equal to $|C_0|^2\omega(\mathbf{k}_0)$. It is then evident that

$$C_0 = (|\mathbf{k}_0|/4\omega(\mathbf{k}_0))^{1/2}\,a_0,$$

where, for simplicity, the fluid density has been set equal to unity. The third-order terms in Eq. (1.47) are responsible for the nonlinear frequency shift and also describe the interaction of the perturbations $(\omega_1, \mathbf{k}_1)$ and $(\omega_2, \mathbf{k}_2)$ with the second harmonic of the surface waves when the frequencies and wave vectors of the perturbations satisfy the resonance condition in (1.44). As first shown by Benney [25] the equations for C_0, C_1, and C_2 have the symmetric form

$$\left.\begin{aligned} i(\partial C_1/\partial t) &= \alpha_{10}|C_0|^2 C_1 + (1/2)\beta C_0^2 C_2^* \times\\ &\quad\times \exp(-i(2\omega_0 - \omega_1 - \omega_2)t),\\ i(\partial C_2/\partial t) &= \alpha_{20}|C_0|^2 C_2 + (1/2)\beta C_0^2 C_1^* \times\\ &\quad\times \exp(-i(2\omega_0 - \omega_1 - \omega_2)t),\\ i(\partial C_0/\partial t) &= \alpha_{00}|C_0|^2 C_0, \end{aligned}\right\} \tag{1.48}$$

where α_{i0} and β are real homogeneous functions of the third degree. The nonlinear frequency shift for the wave of finite amplitude is known from the Stokes expansion:

$$\delta\omega_0 \equiv \alpha_{00}|C_0|^2 = 2|\mathbf{k}_0|^3|C_0|^2.$$

In the general case, the nonlinear frequency shifts of the perturbations are given by a complicated expression which, however, can be simplified considerably for one-dimensional perturbations that propagate in the same direction as the primary wave. If the amplitude of the wave is small, the decay occurs to perturbations with close-lying wave vectors $k_{1,2} = \mathbf{k}_0 \pm \mathbf{q}$, $q \ll \mathbf{k}_0$. It is then evident that [25] $\alpha_{10} = \alpha_{20} = \beta = 2\alpha_{00}$. The solution of Eq. (1.48) is written the resonance condition must be satisfied:

$$n\omega(\mathbf{k}_0) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2);\; n\mathbf{k}_0 = \mathbf{k}_1 + \mathbf{k}_2.$$

An interesting example of a high-order decay instability is the instability of a high-amplitude Alfvén wave with a sawtooth profile against decay into perturbations with frequencies ω_1 and ω_2, where $\max(\omega_1, \omega_2) \gg v_A/\lambda$, and λ is the wavelength which corresponds to the separation between the two teeth of the profile of the Alfvén wave [21].

We now consider the simpler problem of the decay of a second-harmonic oscillation at the surface of a deep fluid [22, 23].

The oscillations of the fluid surface are described by two functions: the velocity potential $\varphi(x, y, z, t)$ and the displacement of the fluid surface from the equilibrium position $\eta(x, y, t)$. The equations for these functions are well known [24]:

$$\left.\begin{aligned} &\Delta\varphi \equiv \nabla^2\varphi + \varphi_{zz} = 0;\\ &\eta_t + \nabla\eta\cdot[\nabla\varphi]_{z=\eta} - [\varphi_z]_{z=\eta} = 0;\\ &g\eta + [\varphi_t + (1/2)(\nabla\varphi^2) + (1/2)\varphi_z^2]_{z=\eta} = 0. \end{aligned}\right\} \qquad (1.45)$$

In the case of interest here, small oscillations, the quantity $[\varphi]_{z=\eta}$ can be expanded in powers of $(k\eta) \ll 1$. Retaining terms to third order inclusively, we have

$$\left.\begin{aligned} \varphi_z &= \eta_t + \operatorname{div}(\eta\nabla\varphi) + \operatorname{div}[(1/2)\eta^2\nabla\varphi_z] - \varphi_{zz}\eta - (1/2)\varphi_{zzz}\eta^2,\\ -g\eta &= \varphi_t + \varphi_{zt}\eta + \varphi_{zzt}\eta^2/2 + (1/2)[(\nabla\varphi)^2 + \varphi_z^2] +\\ &\quad + (1/2)[(\nabla\varphi)^2 + \varphi_z^2]_z\cdot\eta, \end{aligned}\right\} \qquad (1.46)$$

where the velocity potential is computed using the unperturbed surface. Reducing these equations to a single equation for φ, we obtain the equation of motion for the wave at a water surface:

$$\begin{aligned} \varphi_{tt} + g\varphi_z = \{&g^{-1}\varphi_t\varphi_{zt} - g^{-2}\varphi_t\varphi_{zt}^2 - 0.5g^{-2}\varphi_{zzt}\,\varphi_t^2 -\\ &-(1/2)[(\nabla\varphi^2) + \varphi_z^2] + (1/2g)\varphi_t\,[(\nabla\varphi)^2 + \varphi_z^2]_z\}_t + \operatorname{div}\{-\varphi_t\nabla\varphi +\\ &+ 0.5g^{-1}\varphi_t^2\,\nabla\varphi_z - (1/2)[(\nabla\varphi)^2 + \varphi_z^2]\nabla\varphi + g^{-1}\varphi_t\varphi_{zt}\nabla\varphi\}. \end{aligned} \qquad (1.47)$$

The second-order nonlinear terms in Eq. (1.47) describe the generation of the second harmonic. By retaining these terms alone it is an easy matter to obtain two terms in the Stokes expansion for the primary wave. In this case it turns out that the second harmonic vanishes in the expansion of the velocity potential in terms of the amplitude:

$$\delta\varphi_0 = a_0 \exp(|\mathbf{k}_0|z)\sin(\mathbf{k}_0\cdot\mathbf{r} - \omega_0 t).$$

As in the three-wave interaction, it is convenient to introduce a probability amplitude, C_0, defined in such a way that the surface energy density of the wave is equal to $|C_0|^2\omega(\mathbf{k}_0)$. It is then evident that

$$C_0 = (|\mathbf{k}_0|/4\omega(\mathbf{k}_0))^{1/2} a_0,$$

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$$\mathrm{Re}\,\nu = \pm(1/2)\{\beta^2|C_0|^4 - [q^2(\partial^2\omega/\partial k^2) + (\alpha_{10}+\alpha_{20}-2\alpha_{00})|C_0|^2]^2\}^{1/2} =$$
$$= \pm q\{-(\partial^2/\omega/\partial k^2)[\alpha_{00}|C_0|^2 + (1/4)q^2(\partial^2\omega/\partial k^2)]\}^{1/2}.$$

Thus, in the one-dimensional case, the necessary condition for decay of the second harmonic, the nonlinear compensation of the frequency mismatch due to the dispersion of the group velocity is mathematically identical with the Lighthill criterion. In the case of perturbations which propagate at a large angle with respect to the primary wave the resonance condition can only be satisfied for a curve like 3 in Fig. 7. In this case, $\partial^2\omega/\partial k^2 < 0$ and $\alpha_{00} > 0$, so that it is again necessary that the Lighthill criterion be satisfied.

§ 1.4. Geometric Optics Approximation

The interaction of waves with significantly different frequencies and space scales, which results in a slow change in the amplitude and frequency of the primary wave (cf. § 1.1), can be described in the language of geometric optics. For example, let us assume that a quasimonochromatic Alfvén wave propagates in a plasma. The Fourier expansion of this wave will only contain components which are close to some mean value of the frequency and wave vector. Thus, in the equation for the amplitude component $[\omega - \omega(\mathbf{k})] \times$ $\delta\mathscr{H}(\omega, \mathbf{k}) = 0$ it is possible to expand the function $\omega(\mathbf{k})$ in terms of the small deviation of the wave vector from its mean value

$$[\omega - \omega(k_0) - \partial\omega/\partial k_0\,(k_z - k_0) - 1/2\;\partial^2\omega/\partial k_0^2\,(k_z - k_0)^2]\delta\mathscr{H}(\omega, k_z) = 0.$$

Transforming to the variables z and t and taking account of the nonlinear correction to the frequency of the Alfvén wave, we obtain a parabolic equation which has been used earlier in a number of problems in diffraction [26], the theory of superconductivity [27], "self-effects" in optics [28], and the nonlinear theory of wave propagation [29]:

$$i\left(\frac{\partial}{\partial t} + \frac{\partial\omega}{\partial k}\frac{\partial}{\partial z}\right)\delta\mathscr{H} + \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}\frac{\partial^2\delta\mathscr{H}}{\partial z^2} - \alpha|\delta\mathscr{H}|^2\delta\mathscr{H} = 0, \tag{1.49}$$

where $\alpha = k/4\pi NM(d\omega/dk)$ [cf. Eq. (1.24)].

As an example we consider the modulational instability of the Alfvén wave. The complex amplitude of the unperturbed wave is written in the form $\delta\mathscr{H}(z, t) = a\exp(i\varphi)$. The slow variation of the amplitude in space and time can be described in terms of small perturbations in amplitude and phase:

$$a = a_0 + a';\quad \varphi = \varphi_0 + \varphi', \tag{1.50}$$

where the quantities a' and φ' are assumed to be real. The equation for these quantities can be easily obtained from Eq. (1.49) by substituting the complex amplitude $a\exp(i\varphi)$ in it and separating the real and imaginary parts:

$$\left(\frac{\partial}{\partial t} + \frac{\partial\omega}{\partial k}\cdot\frac{\partial}{\partial z}\right)\varphi + \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}\left(\frac{\partial\varphi}{\partial z}\right)^2 + \alpha a^2 - \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}a^{-1}\frac{\partial^2 a}{\partial z^2} = 0; \tag{1.51}$$

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Linearizing these equations with respect to small perturbations, we have

$$a_0\left(\frac{\partial}{\partial t} + \frac{\partial\omega}{\partial k}\frac{\partial}{\partial z}\right)\varphi' - \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}\frac{\partial^2 a'}{\partial z^2} + 2\alpha a_0^2 a' = 0;$$

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Since the equation that has been obtained is linear, a solution can be sought in the form $\exp(-i\Omega t + iqz)$. As a result we obtain a dispersion relation for Ω which coincides with Eq. (1.24):

$$\Omega = q\frac{\partial\omega}{\partial k} \pm \left[\frac{\partial^2\omega}{\partial k^2}q^2\left(\alpha a^2 + \frac{1}{4}\frac{\partial^2\omega}{\partial k^2}q^2\right)\right]^{1/2}.$$

Equation (1.49) allows a stationary solution which describes a self-compressing wave packet. As in the dynamics of nonlinear periodic waves [II], the possibility of a stationary solution arises from the competition between two factors: the nonlinear self-compression of the wave packet and the expansion of the wave packet due to the dispersion of the group velocity. For this reason the demensions of the packet and its amplitude are coupled.

We write the stationary solution of Eq. (1.49) in the form

$$\delta\mathscr{H} = f(z - v_g t)\exp(-i\delta\omega t). \tag{1.53}$$

in the form $C_{1,2} = \exp(\nu t - i\alpha_{1,20}|C_0|^2 t)$. In this case the following dispersion relation is obtained for Re ν:

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The function f is governed by the equation

$$\frac{\partial^2 f}{\partial z^2}=\frac{2\alpha}{v_g'}f^3-\frac{2\delta\omega}{v_g'}f, \quad v_g'\equiv\frac{\partial^2\omega}{\partial k^2}.$$

As in problems involving the nonlinear dynamics of periodic waves [II], the equation for the envelope of the wave packet can be written in the form of an equation of motion for a nonlinear oscillation in a potential field given by $U(f)=-\alpha f^4/2v_g' + \delta\omega f^2/v_g'$. When $\alpha v_g' < 0$ the potential $U(f)$ is in the form of a well. In this case Eq. (1.53) allows solutions in the form of periodic or soliton wave envelopes. For example, the solution for the soliton case is of the form [30]

$$\delta\mathcal{H}(z, t)=\delta\mathcal{H}_0 \operatorname{sech}[\delta\mathcal{H}_0(-\alpha/v_g')^{1/2}(z-v_g t)]; \ \delta\omega=(1/2)\alpha\delta\mathcal{H}_0^2. \quad (1.54)$$

In conclusion we note that if one goes beyond the one-dimensional analysis and takes account of perturbations which propagate at a small angle with respect to the z-axis a similar approach can be used to analyze the self-focusing of light first reported by Askaryan [31].

§ 1.5. Wave Interaction in the Random-Phase Approximation

Since the resonance conditions can be satisfied for a large number of sets of three waves, the wave interaction in a plasma is usually not a highly ordered process (cf. Fig. 6). Under these conditions the reversible nature of the process is violated (Fig. 9)

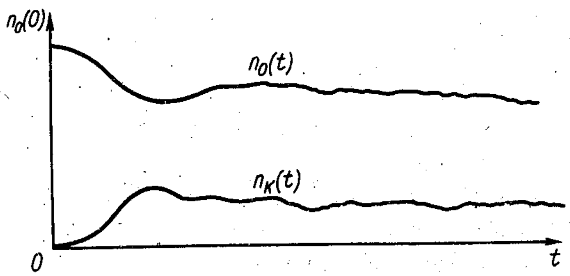

Fig. 9. Time dependence of the amplitude of a wave which decays into a wave spectrum.

so that if the frequencies of the different modes are incommensurate the phase shifts between them may be regarded as random after some time period elapses. In this case one can use the random-phase approximation to describe the evolution of perturbations (that is to say, one examines the wave amplitude in some detail but only takes a general average over phases). Under these conditions it is not meaningful to distinguish the primary wave from the other wave so that the equation for the wave amplitudes is written in the form

$$i(dC_k/dt)=\sum_{k'} V_{k,k';k-k'}C_{k'}(t)C_{k-k'}(t)\exp[-i(\omega_{k'}+\omega_{k-k'}-\omega_k)t]. \quad (1.55)$$

With an appropriate normalization the wave amplitudes, the C_k, can be interpreted as probability amplitudes so that the occupation numbers are equal to the squares of these amplitudes:

$$n_k=|C_k|^2. \quad (1.56)$$

With this normalization the matrix elements exhibit the following symmetry properties (these may be compared with the matrix elements which describe the interaction between Alfvén waves and the acoustic wave):

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$$C_k(t)=C_k^{(0)}+C_k^{(1)}+C_k^{(2)}+\ldots$$

and substituting in Eq. (1.55), we have

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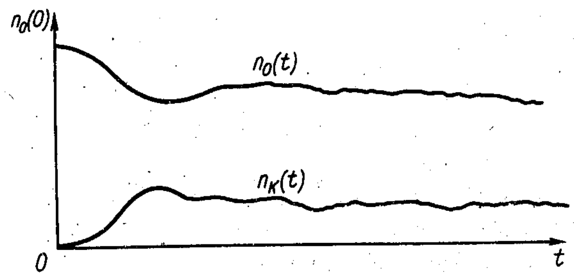

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$$V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t) \equiv V_{\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}'} \exp\left[i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''})t\right]\delta_{\mathbf{k},\mathbf{k}'+\mathbf{k}''}, \qquad (1.59)$$

$$\delta_{\mathbf{k},\mathbf{q}} = \begin{cases} 1, & \mathbf{k}=\mathbf{q} \\ 0, & \mathbf{k}\neq\mathbf{q}. \end{cases}$$

The quantities $C_{\mathbf{k}}^{(0)}$ are time independent and correspond to the solution in the absence of interactions between modes. They can be written in the form of a product of a positive amplitude and a phase factor $\exp i\Phi_{\mathbf{k}}$. Although the phases $\Phi_{\mathbf{k}}$ are given by the initial conditions in any particular experiment, it is still reasonable to take them as random quantities in view of the fact that they become smeared out in time [32] (i.e., $\langle C_{\mathbf{k}}^{(0)} C_{\mathbf{k}'}^{(0)}\rangle = |C_{\mathbf{k}}^{(0)}|^2 \delta_{\mathbf{k},-\mathbf{k}'}$). Exploiting this property, we can average over the change in the occupation numbers (i.e., the quantities $|C_{\mathbf{k}}(t)|^2 - |C_{\mathbf{k}}(t_0)|^2$). To lowest order in the amplitude we have

$$\langle |C_{\mathbf{k}}(t)|^2\rangle = |C_{\mathbf{k}}^{(0)}(t_0)|^2 + \langle |C_{\mathbf{k}}^{(1)}|^2\rangle + \langle C_{\mathbf{k}}^{(0)} C_{\mathbf{k}}^{(2)*} + C_{\mathbf{k}}^{(2)*} C_{\mathbf{k}}^{(2)}\rangle. \qquad (1.60)$$

Substituting the values found earlier in Eq. (1.58), we find

$$|C_{\mathbf{k}}(t)|^2 - |C_{\mathbf{k}}(0)|^2 =$$

$$= \sum_{k',k'',q',q''} \Bigg[C_{\mathbf{k}'}^{(0)} C_{\mathbf{k}''}^{(0)} C_{\mathbf{q}'}^{(0)*} C_{\mathbf{q}''}^{(0)*} \int_0^t V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t')\,dt' \times$$

$$\times \int_0^t V^*_{\mathbf{k},\mathbf{q}',\mathbf{q}''}(t')\,dt' - \mathrm{Re}\, 2\, C_{\mathbf{k}}^{(0)*} C_{\mathbf{k}'}^{(0)} C_{\mathbf{q}'}^{(0)} C_{\mathbf{q}''}^{(0)} \times$$

$$\times \int_0^t dt'\, V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t') \int_0^{t'} V_{\mathbf{k}'',\mathbf{q}',\mathbf{q}''}(t'')\,dt'' -$$

$$- \mathrm{Re}\, 2\, C_{\mathbf{k}}^{(0)*} C_{\mathbf{k}''}^{(0)} C_{\mathbf{q}'}^{(0)} C_{\mathbf{q}''}^{(0)} \int_0^t dt'\, V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t') \int_0^{t'} dt''\, V_{\mathbf{k}',\mathbf{q}',\mathbf{q}''}(t'') \Bigg]. \qquad (1.61)$$

When this equation is averaged over random phases the product of the four $C_{\mathbf{k}}^{(0)}$ reduces to the product of two occupation numbers. Two possible combinations of the amplitudes are shown here by the dashed and solid connecting lines. In the first term, the $C_{\mathbf{k}}^{(0)}$ are combined in a product $|C_{\mathbf{k}'}^{(0)}|^2 \cdot |C_{\mathbf{k}''}^{(0)}|^2 = n_{\mathbf{k}'}^{(0)} n_{\mathbf{k}''}^{(0)}$ in the other two they are combined as $|C_{\mathbf{k}}^{(0)}|^2 |C_{\mathbf{k}'}^{(0)}|^2 = n_{\mathbf{k}}^{(0)} n_{\mathbf{k}'}^{(0)}$ and $|C_{\mathbf{k}}^{(0)}|^2 |C_{\mathbf{k}''}^{(0)}|^2 = n_{\mathbf{k}}^{(0)} n_{\mathbf{k}''}^{(0)}$ respectively. Making use of the symmetry properties of the matrix elements we note that the product of any two matrix elements in Eq. (1.61) can be written in the form of the square of

its modulus $\left|\int_0^t V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t)\,dt\right|^2$ with sign which depends on the signs of the frequencies $\omega_{\mathbf{k}}$; $\omega_{\mathbf{k}'}$; $\omega_{\mathbf{k}-\mathbf{k}'}$. For time intervals much longer than the longest period the time integrals can be evaluated approximately as

$$\left|\int_0^t V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t)\,dt\right|^2 \simeq \frac{4\sin^2\left[(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''})t/2\right]}{(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''})} |V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}|^2 \times$$

$$\times \delta_{\mathbf{k},\mathbf{k}'+\mathbf{k}''} = 2\pi\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''})\delta_{\mathbf{k},\mathbf{k}'+\mathbf{k}''} |V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}|^2 \cdot t.$$

Consequently the time variation of the occupation numbers is given by

$$\Delta n_{\mathbf{k}} = 4\pi\Delta t \sum_{\mathbf{k}',\mathbf{k}''} |V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}|^2 \left[n_{\mathbf{k}'}^{(0)} n_{\mathbf{k}''}^{(0)} - \mathrm{sign}(\omega_{\mathbf{k}}\omega_{\mathbf{k}''})\, n_{\mathbf{k}}^{(0)} n_{\mathbf{k}'}^{(0)} -\right.$$

$$\left. - \mathrm{sign}(\omega_{\mathbf{k}}\omega_{\mathbf{k}'})\, n_{\mathbf{k}}^{(0)} n_{\mathbf{k}''}^{(0)}\right] \delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''})\, \delta_{\mathbf{k},\mathbf{k}'+\mathbf{k}''}. \qquad (1.62)$$

We now assume that no correlation of the phases occurs as the system evolves in time. The result which has been obtained can then be written in the form of a differential equation if it is assumed that the averaging procedure used above can be carried out at any arbitrary time, thus determining the change in the occupation number for the subsequent time interval $t + \Delta t$. In other words,

$$\Delta n_{\mathbf{k}}/\Delta t \approx dn_{\mathbf{k}}/dt; \quad n_{\mathbf{k}}^{(0)} \equiv n_{\mathbf{k}}(t).$$

This procedure thus leads to a wave kinetic equation [15, 33, 34]:

$$dn_{\mathbf{k}}/dt = 4\pi \sum_{\mathbf{k}',\mathbf{k}''} |V_{\mathbf{k},\mathbf{k}',\mathbf{k}'}|^2 \{n_{\mathbf{k}'} n_{\mathbf{k}''} - \mathrm{sign}(\omega_{\mathbf{k}}\omega_{\mathbf{k}''})\, n_{\mathbf{k}} n_{\mathbf{k}'} -$$

$$- \mathrm{sign}(\omega_{\mathbf{k}}\omega_{\mathbf{k}'})\, n_{\mathbf{k}} n_{\mathbf{k}''}\} \delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''})\, \delta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}. \qquad (1.63)$$

This equation, which is written for waves with positive frequencies, can be obtained directly from the dynamic equation for the amplitudes by using quantum-mechanical perturbation theory and the "golden rule." For example, we can consider the interaction of a wave $n_{\mathbf{k}}$ with other waves at lower frequencies (i.e., $\omega_{\mathbf{k}} > \omega_{\mathbf{k}'}$, $\omega_{\mathbf{k}''} > 0$). In this case the interaction process consists of a set of decay processes for the wave with frequency $\omega_{\mathbf{k}}$ and the inverse process of combination of two waves with frequencies $\omega_{\mathbf{k}'}$, $\omega_{\mathbf{k}''}$. The change in occupation number in these processes is described

$$V_{k,k',k''}(t) \equiv V_{k,k',k-k'} \exp[i(\omega_k-\omega_{k'}-\omega_{k''})t]\,\delta_{k,k'+k''}, \qquad (1.59)$$

$$\delta_{k,q} = \begin{cases}1, & k=q\\ 0, & k\neq q.\end{cases}$$

The quantities $C_k^{(0)}$ are time independent and correspond to the solution in the absence of interactions between modes. They can be written in the form of a product of a positive amplitude and a phase factor $\exp i\Phi_k$. Although the phases Φ_k are given by the initial conditions in any particular experiment, it is still reasonable to take them as random quantities in view of the fact that they become smeared out in time [32] (i.e., $\langle C_k^{(0)} C_{k'}^{(0)}\rangle = |C_k^{(0)}|^2\delta_{k,-k'}$). Exploiting this property, we can average over the change in the occupation numbers (i.e., the quantities $|C_k(t)|^2 - |C_k(t_0)|^2$). To lowest order in the amplitude we have

$$\langle|C_k(t)|^2\rangle = |C_k^{(0)}(t_0)|^2 + \langle|C_k^{(1)}|^2\rangle + \langle C_k^{(0)}C_k^{(2)*} + C_k^{(2)*}C_k^{(2)}\rangle. \qquad (1.60)$$

Substituting the values found earlier in Eq. (1.58), we find

$$|C_k(t)|^2 - |C_k(0)|^2 =$$

$$= \sum_{k',k'',q',q''}\Big[C_{k'}^{(0)} C_{k''}^{(0)} C_{q'}^{(0)*} C_{q''}^{(0)*} \int_0^t V_{k,k',k''}(t')\,dt' \times$$

$$\times \int_0^t V^*_{k,q',q''}(t')\,dt' - \mathrm{Re}\,2\, C_k^{(0)*} C_{k'}^{(0)} C_{q'}^{(0)} C_{q''}^{(0)} \times$$

$$\times \int_0^t dt'\, V_{k,k',k''}(t') \int_0^{t'} V_{k'',q',q''}(t'')\,dt'' -$$

$$- \mathrm{Re}\,2\, C_k^{(0)*} C_{k''}^{(0)} C_{q'}^{(0)} C_{q''}^{(0)} \int_0^t dt'\, V_{k,k',k''}(t') \int_0^{t'} dt''\, V_{k',q',q''}(t'')\Big]. \qquad (1.61)$$

When this equation is averaged over random phases the product of the four $C_k^{(0)}$ reduces to the product of two occupation numbers. Two possible combinations of the amplitudes are shown here by the dashed and solid connecting lines. In the first term, the $C_k^{(0)}$ are combined in a product $|C_{k'}^{(0)}|^2\cdot|C_{k''}^{(0)}|^2 = n_{k'}^{(0)}n_{k''}^{(0)}$ in the other two they are combined as $|C_k^{(0)}|^2|C_{k'}^{(0)}|^2 = n_k^{(0)}n_{k'}^{(0)}$ and $|C_k^{(0)}|^2|C_{k''}^{(0)}|^2 = n_k^{(0)}n_{k''}^{(0)}$ respectively. Making use of the symmetry properties of the matrix elements we note that the product of any two matrix elements in Eq. (1.61) can be written in the form of the square of

its modulus $\left|\int_0^t V_{k,k',k''}(t)\,dt\right|^2$ with sign which depends on the signs of the frequencies ω_k; $\omega_{k'}$; $\omega_{k-k'}$. For time intervals much longer than the longest period the time integrals can be evaluated approximately as

$$\left|\int_0^t V_{k,k',k''}(t)\,dt\right|^2 \simeq \frac{4\sin^2[(\omega_k-\omega_{k'}-\omega_{k''})t/2]}{(\omega_k-\omega_{k'}-\omega_{k''})}|V_{k,k',k''}|^2 \times$$

$$\times\,\delta_{k,k'+k''} = 2\pi\delta(\omega_k-\omega_{k'}-\omega_{k''})\delta_{k,k'+k''}|V_{k,k',k''}|^2\cdot t.$$

Consequently the time variation of the occupation numbers is given by

$$\Delta n_k = 4\pi\Delta t\sum_{k',k''}|V_{k,k',k''}|^2\,[n_{k'}^{(0)}n_{k''}^{(0)} - \mathrm{sign}(\omega_k\omega_{k''})\,n_k^{(0)}n_{k'}^{(0)} -$$

$$-\,\mathrm{sign}(\omega_k\omega_{k'})\,n_k^{(0)}n_{k''}^{(0)}]\,\delta(\omega_k-\omega_{k'}-\omega_{k''})\,\delta_{k,k'+k''}. \qquad (1.62)$$

We now assume that no correlation of the phases occurs as the system evolves in time. The result which has been obtained can then be written in the form of a differential equation if it is assumed that the averaging procedure used above can be carried out at any arbitrary time, thus determining the change in the occupation number for the subsequent time interval $t + \Delta t$. In other words,

$$\Delta n_k/\Delta t \approx dn_k/dt; \quad n_k^{(0)} \equiv n_k(t).$$

This procedure thus leads to a wave kinetic equation [15, 33, 34]:

$$dn_k/dt = 4\pi\sum_{k',k''}|V_{k,k',k'}|^2\{n_{k'}n_{k''} - \mathrm{sign}(\omega_k\omega_{k''})\,n_k n_{k'} -$$

$$-\,\mathrm{sign}(\omega_k\omega_{k'})\,n_k n_{k''}\}\,\delta(\omega_k-\omega_{k'}-\omega_{k''})\,\delta_{k,k',k''}. \qquad (1.63)$$

This equation, which is written for waves with positive frequencies, can be obtained directly from the dynamic equation for the amplitudes by using quantum-mechanical perturbation theory and the "golden rule." For example, we can consider the interaction of a wave n_k with other waves at lower frequencies (i.e., $\omega_k > \omega_{k'}$, $\omega_{k''} > 0$). In this case the interaction process consists of a set of decay processes for the wave with frequency ω_k and the inverse process of combination of two waves with frequencies $\omega_{k'}$, $\omega_{k''}$. The change in occupation number in these processes is described by the equation [34]:

$$dn_{\mathbf{k}}/dt = -4\pi \sum_{\mathbf{k}',\mathbf{k}''} |V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}|^2 \{n_{\mathbf{k}}(n_{\mathbf{k}'}+1)(n_{\mathbf{k}''}+1) - n_{\mathbf{k}'}n_{\mathbf{k}''}(n_{\mathbf{k}}+1)\}\,\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''})\,\delta_{\mathbf{k},\mathbf{k}'+\mathbf{k}''}. \tag{1.64}$$

In the classical limit, in which the number of photons is large ($n_{\mathbf{k}} \gg 1$) we recover Eq. (1.63) directly. The "collision" term for the four-wave interaction is derived in the same way. This term is proportional to the third power of the occupation number. Consequently, in the absence of a three-wave interaction (as is the case, for example, for surface waves in water [36]) the interaction between waves can only arise in the wave energy.

In the form given in Eq. (1.36) the wave kinetic equation has already been used for a long time in solid-state physics in the description of phonon interactions associated with lattice irregularities [35]. However, there is a fundamental difference between the application of this equation to phonons and to plasma turbulence. In solid-state physics one usually deals with a state which is close to thermodynamic equilibrium. Under these conditions nonlinear effects only lead to small corrections to the equilibrium occupation number. In a plasma, on the other hand, nonlinear effects frequently play a dominant role since the mean free path for a wave in a turbulent plasma can be very small and equipartition of the energy over different modes need not necessarily be realized.

PROBLEMS

▶ 1. Given the dynamic equation for waves with a nondecay spectrum

$$i(\partial C_{\mathbf{k}}/\partial t) = \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} C_{\mathbf{k}_1} C_{\mathbf{k}_2} + \sum_{\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}+\mathbf{k}_1} W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} C^*_{\mathbf{k}_1} C_{\mathbf{k}_2} C_{\mathbf{k}_3} \tag{1}$$

derive a kinetic equation for the waves.

It should first be noted that beats between two waves in a nondecay spectrum never lead to resonance with characteristic eigenmodes of the medium. Consequently, these waves do not lose their energy and can be regarded as an additional ensemble of the modes of the medium; in lowest order in an energy expansion the number of photons associated with these modes is conserved. The change in the number of photons in four-wave processes can be conveniently described by means of a dynamic equation on the right-hand side of which similar rules are used to take account of the terms which describe the resonance interaction for the four waves as well as terms which correspond to the resonance interaction for the two beat waves:

$$i(\partial C_{\mathbf{k}}/\partial t) = \sum_{\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}+\mathbf{k}_1} U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} C^*_{\mathbf{k}_1} C_{\mathbf{k}_2} C_{\mathbf{k}_3}, \tag{2}$$

where

$$U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} - 2V_{\mathbf{k},-\mathbf{k}_1,\mathbf{k}+\mathbf{k}_1} V_{\mathbf{k}+\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}/(\omega_{\mathbf{k}+\mathbf{k}_1}-\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}_3}) - \frac{2V_{\mathbf{k},\mathbf{k}_2,\mathbf{k}-\mathbf{k}_2} V_{\mathbf{k}-\mathbf{k}_2,-\mathbf{k}_1\mathbf{k}_3}}{\omega_{\mathbf{k}-\mathbf{k}_2}+\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_3}} - \frac{2V_{\mathbf{k},\mathbf{k}_3,\mathbf{k}-\mathbf{k}_3} V_{\mathbf{k}-\mathbf{k}_3,-\mathbf{k}_1,\mathbf{k}_2}}{\omega_{\mathbf{k}-\mathbf{k}_3}+\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}}.$$

The wave kinetic equation is obtained from the dynamic equation given above either by the procedure described in the text or by use of the "golden rule"; this equation then has the form

$$dn_{\mathbf{k}}/dt = 6\pi \sum_{\mathbf{k}+\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3} |U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}|^2 (n_{\mathbf{k}_1}n_{\mathbf{k}_2}n_{\mathbf{k}_3} + n_{\mathbf{k}}n_{\mathbf{k}_2}n_{\mathbf{k}_3} - n_{\mathbf{k}}n_{\mathbf{k}_1}n_{\mathbf{k}_2} - n_{\mathbf{k}}n_{\mathbf{k}_1}n_{\mathbf{k}_3})\,\delta(\omega_{\mathbf{k}}+\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}_3}). \tag{3}$$

▶ 2. Find the radiation intensity for an electromagnetic wave at frequency $\omega \approx 2\omega_p$ in the presence of a gas of plasma oscillations [37].

The equation for the electric field associated with the electromagnetic wave is

$$[k^2c^2 - \omega^2\varepsilon(\omega)]\,\mathbf{E}^t = -4\pi i\omega \mathbf{j}_{nl} \tag{1}$$

where the oscillations of the current at frequency $2\omega_p$ are due to the presence of the plasma oscillations $(\omega_1, \mathbf{k}_1)$ and $(\omega_2, \mathbf{k}_2)$

$$\mathbf{j}_{nl} = -e\{n_1\mathbf{v}_2 + n_2\mathbf{v}_1 + N_0\mathbf{v}^{(2)}\}.$$

The perturbation of the electron velocity, which is quadratic in the wave amplitude, makes no contribution in Eq. (1) since

$$\mathbf{v}^{(2)} = (1/\omega_1+\omega_2)[(\mathbf{v}_1\mathbf{k}_2)\mathbf{v}_2 + (\mathbf{v}_2\mathbf{k}_1)\mathbf{v}_1] = (\mathbf{k}_1+\mathbf{k}_2/\omega_1+\omega_2)(\mathbf{v}_1\mathbf{v}_2).$$

by the equation [34]:

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The wave kinetic equation is obtained from the dynamic equation given above either by the procedure described in the text or by use of the "golden rule"; this equation then has the form

$$dn_{\mathbf{k}}/dt = 6\pi \sum_{\mathbf{k}+\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3} |U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}|^2 (n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{k}} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_3}) \delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3}). \tag{3}$$

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where the oscillations of the current at frequency $2\omega_p$ are due to the presence of the plasma oscillations $(\omega_1, \mathbf{k}_1)$ and $(\omega_2, \mathbf{k}_2)$

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The perturbation of the electron velocity, which is quadratic in the wave amplitude, makes no contribution in Eq. (1) since

$$\mathbf{v}^{(2)} = (1/\omega_1+\omega_2)[(\mathbf{v}_1\mathbf{k}_2)\mathbf{v}_2 + (\mathbf{v}_2\mathbf{k}_1)\mathbf{v}_1] = (\mathbf{k}_1+\mathbf{k}_2/\omega_1+\omega_2)(\mathbf{v}_1\mathbf{v}_2).$$

Using the linear relations we write the expression for the current in the form

$$\mathrm{j}_{nl} = \frac{e^3}{m^2 \omega_1 \omega_2} \left(\mathrm{k}_1 \mathrm{E}_1^l\right)\left(\mathrm{k}_2 \mathrm{E}_2^l\right) \left\{ \frac{\mathrm{k}_2}{\omega_1 k_2^2} + \frac{\mathrm{k}_1}{\omega_2 k_1^2} \right\}. \tag{2}$$

Substituting Eq. (2) in Eq. (1) and multiplying by E^{t*} we obtain the dynamic equation for the amplitudes in symmetric form:

$$(-\mathrm{i}/2) \frac{\partial}{\partial t} |C_{\mathrm{k}}^t|^2 = \sum_{\mathrm{k}_1 + \mathrm{k}_2 = \mathrm{k}} V_{\mathrm{k}\mathrm{k}_1\mathrm{k}_2} C_{\mathrm{k}}^{t*} C_{\mathrm{k}_1}^l C_{\mathrm{k}_2}^l, \tag{3}$$

where

$$|C_{\mathrm{k}}^t|^2 = \frac{\partial}{\omega^2 \partial\omega} [\omega^2 \varepsilon(\omega)] \cdot \frac{|\mathrm{E}_{\mathrm{k}}^t|^2}{8\pi}; \quad |C_{\mathrm{k}}^l|^2 = \frac{\partial}{\omega\partial\omega} [\omega\varepsilon(\omega)] \frac{|\mathrm{E}_{\mathrm{k}}^l|^2}{8\pi};$$

$$V_{\mathrm{k}\mathrm{k}_1\mathrm{k}_2} = \frac{\omega_p^3}{\sqrt{N_0 m |\omega_1 \omega_2 \omega_3|}} \frac{([\mathrm{k}_1 \times \mathrm{k}_2]\,\mathrm{e}_h)}{k_1 k_2 k} \left(\frac{k_1^2}{\omega_1} - \frac{k_2^2}{\omega_2} \right); \quad \mathrm{e}_h \equiv \frac{\mathrm{H}_{\mathrm{k}}^t}{|\mathrm{H}_{\mathrm{k}}^t|}.$$

The radiation intensity is then found from the kinetic equation

$$(\partial/\partial t) n_{\mathrm{k}}^t \approx 4\pi \int \frac{d^3 \mathrm{k}_1}{(2\pi)^3} |V_{\mathrm{k},\mathrm{k}_1,\mathrm{k}-\mathrm{k}_1}|^2 n_{\mathrm{k}_1}^l n_{\mathrm{k}-\mathrm{k}_1}^l \,\delta(\omega_{\mathrm{k}} - \omega_{\mathrm{k}_1} - \omega_{\mathrm{k}_2}). \tag{4}$$

Because of the low frequency of the radiation the wave vectors for the plasma waves must be close to each other ($|\mathrm{k}_1 - \mathrm{k}_2|\lambda_D \ll v_{Te}/c$). Hence, the radiation intensity contains the factor (v_{Te}/c) to a high power:

$$\dot{n}_{\mathrm{k}}^t \approx \int \frac{d^3 \mathrm{k}}{2\pi^2} \frac{n_{\mathrm{k}}^l \omega_{\mathrm{k}}^2}{N_0 T_e} \left(\frac{v_{Te}}{c} \right)^5 n_{\mathrm{k}}^l.$$

§ 1.6. Weak Turbulence and the Wave Kinetic Equation

In § 1.5, the random-phase approximation has been used to derive a wave kinetic equation. In this section we shall investigate the properties of this equation and consider certain solutions.

Investigations of the stability of a plasma confined by a magnetic field frequently show that the plasma goes to a state of disordered motion as a result of small perturbations. In the general case this motion must be described in terms of all of the physical parameters, the velocity, temperature etc. (in the fluid description of the plasma) at each point in space and time. If the deviation from the equilibrium state is small (or if the total turbulence energy is small) it is possible to describe this turbulent motion as a superposition of linear eigenmodes:

$$\mathrm{v}(\mathrm{r}, t) = \sum_{\mathrm{k}} \mathrm{v}_{\mathrm{k}} \exp[-\mathrm{i}\omega_{\mathrm{k}} t + \mathrm{i}\mathrm{k}\cdot\mathrm{r}], \tag{1.65}$$

where the frequencies satisfy the dispersion relation $\omega_{\mathrm{k}} = \omega(\mathrm{k})$. Thus, the turbulent state is described by assigning amplitudes to these characteristic oscillations as functions of the wave vector and the time t. The distribution of energy over the different turbulent scales can then be found from the equations which describe the wave interactions. These equations can be obtained easily by using the random-phase approximation, which is obviously valid when the interaction involves a large number of waves.

This approach to the problem is called the theory of *weak plasma turbulence*. The results of this kind of solution can be given either in terms of the energy of the waves or the occupation numbers n_{k}. In the latter case the wave kinetic equation can be written in the form

$$\partial n_{\mathrm{k}}/\partial t \approx 2\gamma_{\mathrm{k}} n_{\mathrm{k}} + \mathrm{St}(n_{\mathrm{k}}). \tag{1.66}$$

The collision integral which appears in this equation has been obtained in the preceding section. In the nonlinear analysis of stability it is necessary to introduce an energy source and an energy sink; in Eq. (1.66) these are described by the first term on the right-hand side. In the steady-state description of the turbulence the term dn_{k}/dt can be omitted, in which case the right-hand side of Eq. (1.66) is set equal to zero.

In this section we shall discuss the relation between the theory of weak plasma turbulence and the Kolmogorov theory of turbulence. In the theory of hydrodynamic turbulence one finds that it is very difficult to develop a rigorous description of the energy exchange between different turbulent scales since the turbulence motion cannot be described in terms of an ensemble of characteristic modes; there is not available a statistical description which can give an equation like the wave kinetic equation obtained above. For this reason, in conventional hydrodynamics the most reliable estimates are obtained from dimensional arguments.

Using the linear relations we write the expression for the current in the form

$$j_{nl} = \frac{e^3}{m^2 \omega_1 \omega_2} (\mathbf{k}_1 \mathbf{E}_1^l)(\mathbf{k}_2 \mathbf{E}_2^l) \left\{ \frac{\mathbf{k}_2}{\omega_1 k_2^2} + \frac{\mathbf{k}_1}{\omega_2 k_1^2} \right\}. \tag{2}$$

Substituting Eq. (2) in Eq. (1) and multiplying by E^{t*} we obtain the dynamic equation for the amplitudes in symmetric form:

$$(-i/2) \frac{\partial}{\partial t} |C_k^t|^2 = \sum_{k_1 + k_2 = k} V_{kk_1 k_2} C_k^{t*} C_{k_1}^l C_{k_2}^l, \tag{3}$$

where

$$|C_k^l|^2 = \frac{\partial}{\omega^2 \partial \omega} [\omega^2 \varepsilon(\omega)] \cdot \frac{|E_k^l|^2}{8\pi}; \quad |C_k^t|^2 = \frac{\partial}{\omega \partial \omega} [\omega \varepsilon(\omega)] \frac{|E_k^t|^2}{8\pi};$$

$$V_{kk_1 k_2} = \frac{\omega_p^3}{\sqrt{N_0 m |\omega_1 \omega_2 \omega_3|}} \frac{([\mathbf{k}_1 \times \mathbf{k}_2] \mathbf{e}_h)}{k_1 k_2 k} \left(\frac{k_1^2}{\omega_1} - \frac{k_2^2}{\omega_2} \right); \quad \mathbf{e}_h \equiv \frac{\mathbf{H}_k^t}{|\mathbf{H}_k^t|}.$$

The radiation intensity is then found from the kinetic equation

$$(\partial/\partial t) n_k^t \approx 4\pi \int \frac{d^3 k_1}{(2\pi)^3} |V_{k, k_1, k-k_1}|^2 n_{k_1}^l n_{k-k_1}^l \delta(\omega_k - \omega_{k_1} - \omega_{k_2}). \tag{4}$$

Because of the low frequency of the radiation the wave vectors for the plasma waves must be close to each other ($|\mathbf{k}_1 - \mathbf{k}_2| \lambda_D \ll v_{Te}/c$). Hence, the radiation intensity contains the factor (v_{Te}/c) to a high power:

$$\dot{n}_k^t \approx \int \frac{d^3 k}{2\pi^2} \frac{n_k^l \omega_k^2}{N_0 T_e} \left(\frac{v_{Te}}{c} \right)^5 n_k^l.$$

§ 1.6. Weak Turbulence and the Wave Kinetic Equation

In § 1.5, the random-phase approximation has been used to derive a wave kinetic equation. In this section we shall investigate the properties of this equation and consider certain solutions.

Investigations of the stability of a plasma confined by a magnetic field frequently show that the plasma goes to a state of disordered motion as a result of small perturbations. In the general case this motion must be described in terms of all of the physical parameters, the velocity, temperature etc. (in the fluid description of the plasma) at each point in space and time. If the deviation from the equilibrium state is small (or if the total turbulence energy is small) it is possible to describe this turbulent motion as a superposition of linear eigenmodes:

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where the frequencies satisfy the dispersion relation $\omega_k = \omega(\mathbf{k})$. Thus, the turbulent state is described by assigning amplitudes to these characteristic oscillations as functions of the wave vector and the time t. The distribution of energy over the different turbulent scales can then be found from the equations which describe the wave interactions. These equations can be obtained easily by using the random-phase approximation, which is obviously valid when the interaction involves a large number of waves.

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The collision integral which appears in this equation has been obtained in the preceding section. In the nonlinear analysis of stability it is necessary to introduce an energy source and an energy sink; in Eq. (1.66) these are described by the first term on the right-hand side. In the steady-state description of the turbulence the term dn_k/dt can be omitted, in which case the right-hand side of Eq. (1.66) is set equal to zero.

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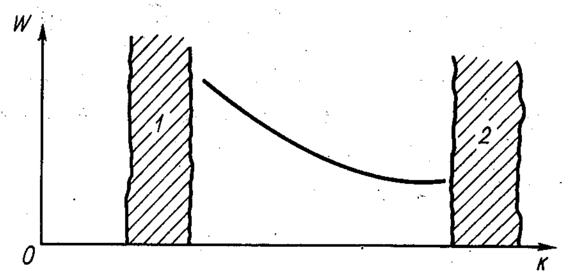

Fig. 10. Turbulence spectrum for constant energy flow along the spectrum in the equilibrium range between regions of excitation and damping.

Let us consider a situation in which the source of large-scale turbulence (small **k**) (the region marked 1 in Fig. 10) is separated from the region in which the turbulence is damped rapidly because of an increase in the viscous dissipation associated with small-scale motion (the damping region 2). In this case the energy flows continuously from the large-scale region to the small-scale region in **k**-space. Well-known dimensional arguments then lead to the following spectrum in the intermediate region (between 1 and 2): $W_k \sim k^{-5/3}$ (the Kolmogorov–Obukhov law), where W_k is the energy per unit volume per unit wave-number interval. It is assumed in the derivation of this relation that the turbulence is isotropic, that it can be described in terms of the local characteristics of the medium, and that the energy is transferred from the large-scale region to the small-scale region by resonance-mode interactions, passing successively through diminishing turbulent scales [24].

It is of interest to find an analog for the Kolmogorov spectrum for weak plasma turbulence because in this case we would have an equation for the spectral energy density derived from first principles. In a plasma, however, one usually encounters a large number of different kinds of characteristic modes so that a universal spectrum with a simple power dependence of the form $n_k \sim k^{-s}$ does not exist. For this reason, the notion of a simple spectrum is only meaningful in particular cases in which specific modes are excited.

An interesting example of a weakly turbulent medium which supports the propagation of one type of wave is furnished by the well-known wave spectrum associated with surface waves in a deep

liquid. The dispersion relation for the surface waves

$$\omega_{\mathbf{k}} = \sqrt{gk} \tag{1.67}$$

is such that the three-wave resonance interaction is not possible (cf. Fig. 7). The four-wave interaction can only occur in third order in the wave energy and is described by a kinetic equation of the form (cf. § 1.4, Problem 1)

$$\partial n_{\mathbf{k}}/\partial t = \int U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} (n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{k}} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} - \\ - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_3}) \, \delta(\omega + \omega_1 - \omega_2 - \omega_3) \, \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \, d^2\mathbf{k}_1 \, d^2\mathbf{k}_2 \, d^2\mathbf{k}_3, \tag{1.68}$$

where $U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_2}$ is a homogeneous function of sixth degree (cf. § 1.3).

A standard program for finding the turbulence spectrum, proposed by Zakharov [38], is based on the assumption that the turbulence is uniform and isotropic and that an equilibrium range exists; the analysis exploits the homogeneous nature of the kernel of the integral equation (1.68) and its symmetry properties

$$U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = U_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2\mathbf{k}_3} = U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_3\mathbf{k}_2} = U_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}\mathbf{k}}. \tag{1.69}$$

According to this program, Eq. (1.68) is averaged over angle in **k**-space; in the equation obtained in this way one then converts to the variable $\omega = \omega(\mathbf{k})$ (obviously the frequency spectrum must also be isotropic). As a result, the problem is reduced to a search for the power solutions of the equation

$$\int T_{\omega,\,\omega_1+\omega_2-\omega,\,\omega_1,\,\omega_2} \{ n_{\omega_1} n_{\omega_2} n_{\omega_1+\omega_2-\omega} + n_\omega n_{\omega_1} n_{\omega_2} - \\ - 2 n_\omega n_{\omega_1} n_{\omega_1+\omega_2-\omega} \} \, d\omega_1 \, d\omega_2 = 0. \tag{1.70}$$

The integration is carried out over the regions shown in Fig. 11, where curves 1-4 are described by the following equations respectively:

$$(\omega_1 + \omega_2 - \omega)^2 = -\omega_1^2 - \omega_2^2 + \omega^2,$$
$$(\omega_1 + \omega_2 - \omega)^2 = \omega_1^2 + \omega_2^2 + \omega^2,$$
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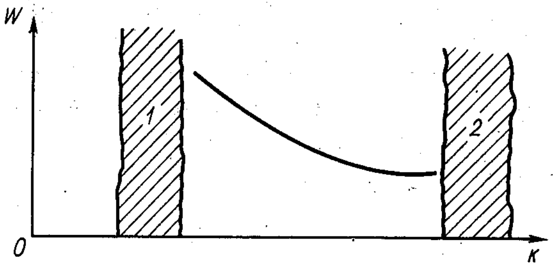
As before, the kernel of Eq. (1.70) remains homogeneous (the degree of homogeneity can be found by comparing this equation



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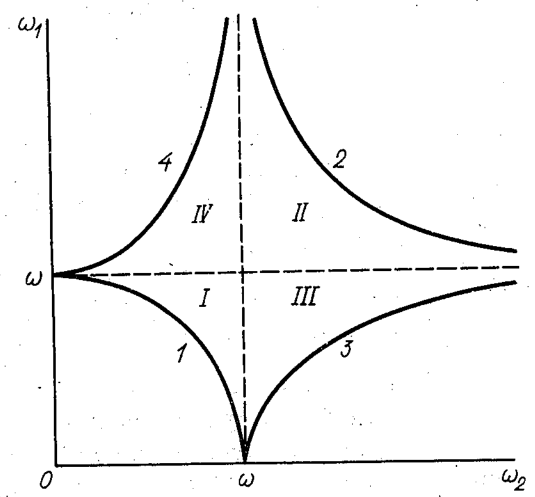


Fig. 11. Region of integration in the (ω_1, ω_2) plane.

with Eq. (1.68); it is equal to 20) and positive. The solution of the equation is written in the form

$$n_\omega = A\omega^s. \tag{1.71}$$

Each of the regions II, III, and IV is taken with respect to region I according to the following rule:

1. For region II, $\omega_1 \to \omega_2\omega/(\omega_1+\omega_2-\omega)$; $\omega_2 \to \omega_1\omega/(\omega_1+\omega_2-\omega)$.
2. For region III, $\omega_2 \to \omega^2/\omega_2$; $\omega_1 \to (\omega_1+\omega_2-\omega)\omega/\omega_2$.
3. For region IV, $\omega_1 \to \omega^2/\omega_1$; $\omega_2 \to (\omega_1+\omega_2-\omega)\omega/\omega_1$.

Using the symmetry and homogeneity of the nucleus T the integrals over regions II–IV can be expressed in terms of the integral over region I. Equation (1.70) can then be written in the form

$$\int\limits_I \frac{T_{\omega,\,\omega_1+\omega_2-\omega,\,\omega_1,\,\omega_2}}{(\omega_1+\omega_2-\omega)^{23+3s}\,\omega_1^{23+3s}\,\omega_2^{23+3s}}\,[\omega_1^s\omega_2^s(\omega_1+\omega_2-\omega)^s+$$
$$+\omega^s\omega_1^s\omega_2^s-2\omega^s\omega_1^s(\omega_1+\omega_2-\omega)^s]\,\{[\omega_1\omega_2(\omega_1+\omega_2-\omega)]^{23+3s}+$$
$$+[\omega\omega_1\omega_2]^{23+3s}-2\,[\omega\omega_2(\omega_1+\omega_2-\omega)]^{23+3s}\}\,d\omega_1\,d\omega_2. \tag{1.72}$$

The integrand vanishes when s = −1 and when s = −8. The first of the solutions which is obtained is the Rayleigh–Jeans distribution, and when the integration is carried out it is found that the integral diverges at large k. The second solution corresponds to the Kolmogorov spectrum of hydrodynamic turbulence and, for surface waves, gives the energy spectrum [36]

$$W_\omega = \omega^4 n_\omega = A'\omega^{-4}. \tag{1.73}$$

Transformation of the integral equation to the form in (1.72) can be useful in finding the level of turbulence. As far as the exponent of the spectrum is concerned, as in the Kolmogorov spectrum it can be found from considerations of the constant flow of energy along the spectrum in the equilibrium interval and by taking account of the fact that the interaction between the turbulence scales can be described within the framework of the theory of a weakly turbulent medium.

The constancy of the energy flow along the spectrum is indicated by the relation

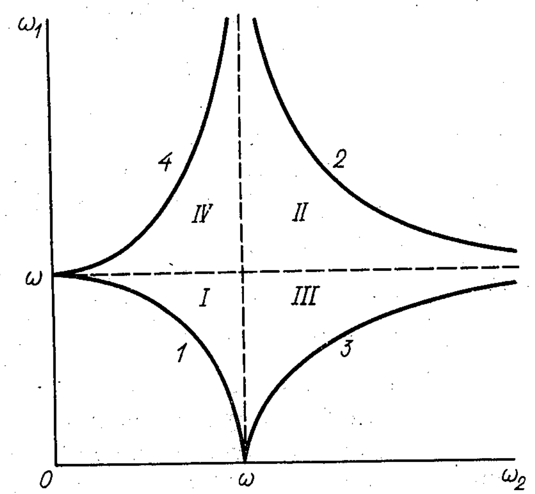
$$W_k k/\tau_k = \text{const}, \tag{1.74}$$

where $W_k dk$ is the surface energy density of the turbulent fluctuations with scale k^{-1} in the interval of wave numbers dk; τ_k is the characteristic interaction time for turbulent fluctuations with scale k^{-1}. Since the interaction only appears in third order in the energy in the case of surface waves, we have $\tau_k^{-1} = \omega U(k)(W_k k)^2$.

When the problem has no characteristic quantity with the dimensions of length, the function U(k) is specified beforehand to be homogeneous. The degree of its homogeneity is determined from dimensional arguments, and the function U(k) must have the dimensionality of the inverse square of the energy density, that it to say, $U(k) = \{MN_0(\omega/k)^2k^{-1}\}^{-2}$. Here, $N_0M\omega^2/k^2$ is a characteristic volume density of gravitational energy in the fluid with respect to the level of the unperturbed surface of the water. The factor k^{-1} takes account of the fact that the amplitude of the surface wave is damped as a function of depth in proportional to k^{-1}; hence, in the problem we must introduce a gravitational energy for the surface layer of the fluid with thickness k^{-1}.

It follows from the foregoing that τ_k is described by the equation

$$\tau_k^{-1} \sim \sqrt{gk}\left(\frac{W_k k}{MN_0\, g/k^2}\right)^2. \tag{1.75}$$



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It follows from the foregoing that τ_k is described by the equation

$$\tau_k^{-1} \sim \sqrt{gk}\left(\frac{W_k k}{MN_0\, g/k^2}\right)^2. \tag{1.75}$$

Using Eqs. (1.74) and (1.75) we find the turbulence spectrum

$$W_k \approx Ak^{-5/2}, \tag{1.73a}$$

which corresponds to $W_\omega \approx \omega^{-4}$.

We note that the derivation is not only based on dimensional arguments but also depends on the assumption that the theory of weak turbulence can be applied. Hence it is not surprising that the characteristic exponent (1.73) is different from that obtained by Phillips [39] for the case in which strong wave action leads to wave breaking so that the theory of weak coupling between modes is no longer valid.

Thus, we have shown that constancy of the energy flow along the spectrum in the equilibrium range and weak mode coupling (in terms of the mode energy) are sufficient for finding the characteristic exponent for the spectrum of uniform isotropic turbulence in cases in which the problem does not exhibit a parameter with the dimensions of length. If, on the other hand, such a parameter l does appear in the problem, the degree of homogeneity of the nucleus U(k) cannot be found exclusively from dimensional arguments since multiplication by any function (kl) does not change the dimensionality of the nucleus. In this case the degree of homogeneity must be found by direct calculation.

PROBLEM

▶ 1. Find the turbulence spectrum for capillary waves [40].

The frequency spectrum for capillary waves is described by a dispersion curve of type 1 (cf. Fig. 7)

$$\omega(k) = \sqrt{\sigma k^3/\rho}, \tag{1}$$

where σ is the coefficient of surface tension; $\rho \equiv N_0 M$. Thus, the three-wave interaction is allowed in a wave system of this kind. Correspondingly, the characteristic time for the interaction is inversely proportional to the wave energy (rather than the square of the wave energy as in the case of the four-wave interaction described in the text) [29]

$$\tau_k^{-1} \sim \omega\left[\frac{W_k k}{\rho_0(\omega/k)^2}\right]. \tag{2}$$

Combining this equation with Eq. (1.74) we obtain the following spectrum:

$$W_k \sim k^{-11/4}. \tag{3}$$

§ 1.7. Negative-Energy Instabilities

There is a much richer variety of physical phenomena to be found in plasma dynamics than in the dynamics of fluids. For example, plasmas exhibit what are known as negative-energy waves. The term "negative energy" used in this context means that the total energy (kinetic plus electromagnetic) of the medium is reduced as the wave amplitude increases. The possible existence of negative energy waves in a plasma was first noted by Kadomtsev, Mikhailovskii, and Timofeev [41]. In order to understand the origin of a change in the sign of the energy of a wave, we first consider the well-known expression for the energy of an electromagnetic field in a dispersive medium:

$$W = \frac{1}{8\pi}\left[\frac{d}{d\omega}(\omega\varepsilon)\langle \mathbf{E}^2\rangle + \frac{d}{d\omega}(\omega\mu)\langle \mathbf{H}^2\rangle\right], \tag{1.76}$$

where ε, μ are the dielectric permittivity and the magnetic permeability respectively. If we limit the analysis to electrostatic waves it follows that the sign of the energy depends only on $d\varepsilon/d\omega$ and can be negative in a medium which is not in thermodynamic equilibrium (in an equilibrium medium this possibility is ruled out by the Kramers–Kronig relations). The lack of equilibrium in a medium can arise from the fact that it is inhomogeneous or anisotropic.

As an example we consider a plasma with an anisotropic ion velocity distribution (specifically, $T_\parallel/T_\perp \to 0$). The dielectric permittivity for a wave with frequency close to the ion-cyclotron frequency $\Omega_H = eH_0/MC$ is then given by

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2}\cdot\frac{k_\parallel^2}{k^2} - \frac{\Omega_p^2\Gamma_n}{(\omega - n\Omega_H)^2}\cdot\frac{k_\parallel^2}{k^2}, \tag{1.77}$$

where $\Gamma_n = I_n(b_\perp)\exp(-b_\perp)$; $b_\perp = (k_\perp^2 T_\perp)/M\Omega_H^2$; I_n is the modified Bessel function of n-th order. Even within the framework of the linear analysis waves of this kind exhibit unusual properties. The

Using Eqs. (1.74) and (1.75) we find the turbulence spectrum

$$W_k \approx Ak^{-5/2}, \tag{1.73a}$$

which corresponds to $W_\omega \approx \omega^{-4}$.

We note that the derivation is not only based on dimensional arguments but also depends on the assumption that the theory of weak turbulence can be applied. Hence it is not surprising that the characteristic exponent (1.73) is different from that obtained by Phillips [39] for the case in which strong wave action leads to wave breaking so that the theory of weak coupling between modes is no longer valid.

Thus, we have shown that constancy of the energy flow along the spectrum in the equilibrium range and weak mode coupling (in terms of the mode energy) are sufficient for finding the characteristic exponent for the spectrum of uniform isotropic turbulence in cases in which the problem does not exhibit a parameter with the dimensions of length. If, on the other hand, such a parameter l does appear in the problem, the degree of homogeneity of the nucleus U(k) cannot be found exclusively from dimensional arguments since multiplication by any function (kl) does not change the dimensionality of the nucleus. In this case the degree of homogeneity must be found by direct calculation.

PROBLEM

▶ 1. Find the turbulence spectrum for capillary waves [40].

The frequency spectrum for capillary waves is described by a dispersion curve of type 1 (cf. Fig. 7)

$$\omega(k) = \sqrt{\sigma k^3/\rho}, \tag{1}$$

where σ is the coefficient of surface tension; $\rho \equiv N_0 M$. Thus, the three-wave interaction is allowed in a wave system of this kind. Correspondingly, the characteristic time for the interaction is inversely proportional to the wave energy (rather than the square of the wave energy as in the case of the four-wave interaction described in the text) [29]

$$\tau_k^{-1} \sim \omega \left[\frac{W_k k}{\rho_0 (\omega/k)^2} \right]. \tag{2}$$

Combining this equation with Eq. (1.74) we obtain the following spectrum:

$$W_k \sim k^{-11/4}. \tag{3}$$

§ 1.7. Negative-Energy Instabilities

There is a much richer variety of physical phenomena to be found in plasma dynamics than in the dynamics of fluids. For example, plasmas exhibit what are known as negative-energy waves. The term "negative energy" used in this context means that the total energy (kinetic plus electromagnetic) of the medium is reduced as the wave amplitude increases. The possible existence of negative energy waves in a plasma was first noted by Kadomtsev, Mikhailovskii, and Timofeev [41]. In order to understand the origin of a change in the sign of the energy of a wave, we first consider the well-known expression for the energy of an electromagnetic field in a dispersive medium:

$$W = \frac{1}{8\pi}\left[\frac{d}{d\omega}(\omega\varepsilon)\langle \mathbf{E}^2\rangle + \frac{d}{d\omega}(\omega\mu)\langle \mathbf{H}^2\rangle\right], \tag{1.76}$$

where ε, μ are the dielectric permittivity and the magnetic permeability respectively. If we limit the analysis to electrostatic waves it follows that the sign of the energy depends only on $d\varepsilon/d\omega$ and can be negative in a medium which is not in thermodynamic equilibrium (in an equilibrium medium this possibility is ruled out by the Kramers–Kronig relations). The lack of equilibrium in a medium can arise from the fact that it is inhomogeneous or anisotropic.

As an example we consider a plasma with an anisotropic ion velocity distribution (specifically, $T_\parallel/T_\perp \to 0$). The dielectric permittivity for a wave with frequency close to the ion-cyclotron frequency $\Omega_H = eH_0/MC$ is then given by

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2}\cdot\frac{k_\parallel^2}{k^2} - \frac{\Omega_p^2\Gamma_n}{(\omega - n\Omega_H)^2}\cdot\frac{k_\parallel^2}{k^2}, \tag{1.77}$$

where $\Gamma_n = I_n(b_\perp)\exp(-b_\perp)$; $b_\perp = (k_\perp^2 T_\perp)/M\Omega_H^2$; I_n is the modified Bessel function of n-th order. Even within the framework of the linear analysis waves of this kind exhibit unusual properties. The amplitude of a wave reflected from a region with negative dispersion ($d\omega/dk < 0$) is larger than the amplitude of the incident wave. In conventional hydrodynamics it is well known that the amplitude of a wave reflected from a medium moving with supersonic velocity ($v > c_s$) is increased for the same reason. When $v > 2c_s$, there exists an angle of incidence for which the transmitted wave and the reflected wave both exhibit finite amplitudes with an incident wave of infinitesimally small amplitude. (This, of course, is the well-known Cerenkov radiation.) Similarly, absorption of this wave in the medium in which it propagates [42] or loss through the ends of the system [43] lead to an increase in amplitude.

A peculiar kind of nonlinear instability can arise in a system of waves whose energies carry different signs. If a negative-energy wave loses energy to a positive-energy wave, it turns out that the amplitudes of both waves increase. A simple instability mechanism (it is called the *explosive* instability in the literature) is provided by the decay of a negative energy wave into two waves, one of each type. This instability was first illustrated by means of a kinetic equation which takes proper account of the interaction of waves with random phases and with different signs for the energies [44].

Let us consider the case of waves with fixed phases; the evolution of the perturbations in time is described analytically in the same way as for the interaction of positive-energy waves in § 1.2. The dynamic equation for the waves is obtained by the substitution:

$$\left.\begin{aligned} C_k(t) &= \left[\frac{k^2}{8\pi}\left|\frac{\partial\varepsilon(\omega)}{\partial\omega_k}\right|\right]^{1/2}\Phi_k(t); \\ \operatorname{sign}\omega_k &\to \operatorname{sign}\left[\omega_k(\partial/\partial\omega_k)\omega_k\,\varepsilon(\omega_k)\right] = \operatorname{sign}(\partial\varepsilon/\partial\omega_k). \end{aligned}\right\} \quad (1.78)$$

The matrix elements then exhibit the following symmetry properties:

$$V_{k_2,\,-k_0,\,k_1} = V_{k_1,\,k_0,\,k_2}\operatorname{sign}\left(\frac{\partial\varepsilon}{\partial\omega_1}\cdot\frac{\partial\varepsilon}{\partial\omega_2}\right). \quad (1.79)$$

As a result, Eq. (1.39) is replaced by

$$\left.\begin{aligned} \partial a_1/\partial t &= Ha_0a_2\sin\Theta, \\ \partial a_2/\partial t &= Ha_0a_1\sin\Theta, \\ \partial a_0/\partial t &= Ha_1a_2\sin\Theta, \\ \partial\Theta/\partial t &= \cot\Theta\,\frac{\partial}{\partial t}\ln a_0a_1a_2, \end{aligned}\right\} \quad (1.80)$$

where the waves denoted by (ω_0, k_0) and (ω_1, k_1) are negative-energy waves. The Manley–Rowe relations for this case are also symmetric:

$$\left.\begin{aligned} m_1 &= n_0 - n_1 = \text{const}, \\ m_2 &= n_0 - n_2 = \text{const}, \\ m_0 &= n_1 - n_2 = \text{const}. \end{aligned}\right\} \quad (1.81)$$

In order to simplify the calculations we limit the analysis to the particular case in which $\Theta = \pi/2$ (a more general case is treated in [45]). When the Manley–Rowe relations are used the first equation in (1.80) can be rewritten in the form

$$2Ht = \int_{(n_0)0}^{(n_0)t} dn_0/\sqrt{n_0(n_0 - m_1)(n_0 - m_2)}. \quad (1.82)$$

Taking $n_0(0) > m_1 > m_2$ and introducing the new variables

$$y(t) = [m_1/n_0(t)]^{1/2}, \quad \varkappa = (m_2/m_1)^{1/2}, \quad (1.83)$$

we transform the integral in (1.82) to the standard form for the elliptic integral:

$$H\sqrt{m_1}\,t = -\int_{y(0)}^{y(t)} dy/\sqrt{(1-y^2)(1-\varkappa^2y^2)}.$$

Consequently,

$$n_0(t) = m_1/[(m_1/n_0(0))^{1/2} - \operatorname{sn}(H\sqrt{m_1}\,t,\ \varkappa)]^2. \quad (1.84)$$

By hypothesis, the wave characterized by frequency ω_0 has the largest probability amplitude, that is to say, $n_0(0) > m_1$; thus, the expression becomes infinite as $(t_\infty - t)^2$ at some finite t_∞. In other words, the instability is said to be an explosive instability.

PROBLEM

▶ 1. Find the solution of the kinetic equations which describe the interaction of waves with random phases with different signs of the energy in the one-dimensional case [44].

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PROBLEM

▶ 1. Find the solution of the kinetic equations which describe the interaction of waves with random phases with different signs of the energy in the one-dimensional case [44].

Introducing the substitution in (1.78) and using Eq. (1.63), we obtain the kinetic equation for the occupation numbers $n_k = |C_k|^2$:

$$\frac{d}{dt} n_k = 2 \int dk'\, dk'' \, |V_{kk',\,k''}|^2 \, \delta(\omega_k - \omega_{k'} - \omega_{k''}) \times$$
$$\times\, \delta(k - k' - k'') \left[n_{k'} n_{k''} - \operatorname{sign}\left(\frac{\partial \varepsilon}{\partial \omega_{k'}} \frac{\partial \varepsilon}{\partial \omega_k} \right) \times \right.$$
$$\left. \times\, n_k n_{k''} - \operatorname{sign}\left(\frac{\partial \varepsilon}{\partial \omega_k} \frac{\partial \varepsilon}{\partial \omega_{k''}} \right) n_k n_{k'} \right]. \qquad (1)$$

Assume that there are three wave branches: two with positive energy and one with negative energy. In the one-dimensional case, for each fixed k there are generally only one pair of values k', k" for which the resonance conditions can be satisfied. Hence the system of interacting waves separates into triplets. We shall consider three interacting waves, one of which is a negative-energy wave. In this case the equation for the occupation numbers assumes the form

$$\varepsilon_{12} \dot{n}_3 = \varepsilon_{31} \dot{n}_2 = \varepsilon_{23} \dot{n}_1 = |V_{123}|^2 (n_1 n_2 + n_1 n_3 + n_2 n_3), \qquad (2)$$

where

$$\varepsilon_{\alpha\beta} = \left| \frac{\partial \omega(k_\alpha)}{\partial k_\alpha} - \frac{\partial \omega(k_\beta)}{\partial k_\beta} \right|, \quad V_{k_1 k_2 k_3} \equiv V_{123}.$$

By means of the obvious conservation relations

$$m_1 = \varepsilon_{23} n_1 - \varepsilon_{12} n_3 = \text{const}, \quad m_2 = \varepsilon_{31} n_2 - \varepsilon_{12} n_3 = \text{const} \qquad (3)$$

we reduce the system in (2) to a single equation for n_3:

$$\frac{d}{dt} n_3 = \frac{|V_{123}|^2}{\varepsilon_{12} \varepsilon_{23} \varepsilon_{31}} \{ n_3^2 \varepsilon_{12} (\varepsilon_{12} + \varepsilon_{23} + \varepsilon_{31}) +$$
$$+ n_3 [m_1 (\varepsilon_{23} + \varepsilon_{12}) + m_2 (\varepsilon_{31} + \varepsilon_{12})] + m_1 m_2 \}. \qquad (4)$$

The general solution of this equation is extremely complicated. We shall consider a limiting case in which one of the quantities $\varepsilon_{\alpha\beta}$ (say ε_{12}) is much smaller than the other two

$$\varepsilon_{12} \ll \varepsilon_{13} = \varepsilon_{23}. \qquad (5)$$

In this case the solution of Eq. (4) assumes the much simpler form

$$n_3 = n_3^{(0)} (A - 1)(\exp Bt)/(A - \exp Bt), \qquad (6)$$

where

$$A = \varepsilon_{23} (n_1^{(0)} + n_2^{(0)}) / 2\varepsilon_{12} n_3^{(0)}, \quad B = |V_{123}|^2 n_3^{(0)} (A - 1)/\varepsilon_{23}.$$

It will be evident that the occupation numbers become infinite for some $t_\infty = B^{-1} \ln A$ in accordance with the relation $n_3 \sim |t_\infty - t|^{-1}$, that is to say, at a rate which is much smaller than in the case of waves with fixed phases [(cf. Eq. (1.84)].

§ 1.8. Adiabatic Approximation (Interaction between High-Frequency and Low-Frequency Waves)

Even with the use of a number of simplifications (in § 1.1, the assumption that β is small) the calculation of the kernel of the wave kinetic equation is an extremely complicated problem. Hence it is desirable to simplify the calculational scheme even further. A case in which this has been done is represented by the interaction of modes with very different scales, a problem which was first investigated by Vedenov and Rudakov [46]. If the analysis is concerned with the interaction of modes with frequencies $\omega_{\mathbf{k}}$ and $\Omega_{\mathbf{q}}$ such that $\omega_k \gg \Omega_q$ and $|\mathbf{k}| \gg |\mathbf{q}|$, the adiabatic approximation can be used. By the term adiabatic here we mean that one of the modes propagates in a slowly varying weakly inhomogeneous medium. The change in the parameters in the medium is due to the presence of the second (low-frequency mode). The latter obviously experiences an effect due to the high-frequency modes, and this effect is taken into account by averaging over the fast oscillations of the high-frequency modes.

For example, let us consider the interaction between high-frequency Alfvén waves with low-frequency acoustic waves. This problem has been treated in §§ 1.1 and 1.2 for the case of three waves with fixed phases.

The starting equation is the Liouville equation for the occupation number of the Alfvén waves in the phase space consisting

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Assume that there are three wave branches: two with positive energy and one with negative energy. In the one-dimensional case, for each fixed k there are generally only one pair of values k', k" for which the resonance conditions can be satisfied. Hence the system of interacting waves separates into triplets. We shall consider three interacting waves, one of which is a negative-energy wave. In this case the equation for the occupation numbers assumes the form

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By means of the obvious conservation relations

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$$\frac{d}{dt} n_3 = \frac{|V_{123}|^2}{\varepsilon_{12}\varepsilon_{23}\varepsilon_{31}} \{ n_3^2 \varepsilon_{12} (\varepsilon_{12} + \varepsilon_{23} + \varepsilon_{31}) + + n_3 [m_1(\varepsilon_{23} + \varepsilon_{12}) + m_2(\varepsilon_{31} + \varepsilon_{12})] + m_1 m_2 \}. \qquad (4)$$

The general solution of this equation is extremely complicated. We shall consider a limiting case in which one of the quantities $\varepsilon_{\alpha\beta}$ (say ε_{12}) is much smaller than the other two

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It will be evident that the occupation numbers become infinite for some $t_\infty = B^{-1} \ln A$ in accordance with the relation $n_3 \sim |t_\infty - t|^{-1}$, that is to say, at a rate which is much smaller than in the case of waves with fixed phases [(cf. Eq. (1.84)].

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For example, let us consider the interaction between high-frequency Alfvén waves with low-frequency acoustic waves. This problem has been treated in §§ 1.1 and 1.2 for the case of three waves with fixed phases.

The starting equation is the Liouville equation for the occupation number of the Alfvén waves in the phase space consisting of coordinates and wave vectors:

$$\left.\begin{aligned} &\frac{\partial n_k}{\partial t}+\frac{\partial \omega_k}{\partial k_z}\cdot\frac{\partial n_k}{\partial z}-\frac{\partial \omega}{\partial z}\cdot\frac{\partial n_k}{\partial k_z}=0,\\ &n_k=\left[\frac{\partial}{\partial \omega}\omega^2\varepsilon(\omega)\right]\frac{|\mathcal{H}_k|^2}{8\pi k^2c^2}. \end{aligned}\right\} \tag{1.85}$$

In a uniform plasma the change in the frequency $[\omega(z)]$ as a function of coordinates is due to the presence of the acoustic waves; as before we describe these waves by fluid equations like those used in § 1.1:

$$N\left(\frac{\partial v_{\parallel}}{\partial t}+v_{\parallel}\frac{\partial}{\partial z}v_{\parallel}\right)=-c_s^2\frac{\partial N}{\partial z}-\frac{\partial}{\partial z}\cdot\frac{|\mathcal{H}_k|^2}{8\pi M}, \tag{1.86}$$

$$\partial N/\partial t+\partial N v_{\parallel}/\partial z=0. \tag{1.87}$$

Introducing the new variable ξ, which is the displacement of a volume element of a fluid, we can write Eqs. (1.85)-(1.87) in the symmetric form

$$\frac{\partial n_k}{\partial t}+\frac{\partial \omega_k}{\partial k_z}\cdot\frac{\partial n_k}{\partial z}-s\frac{\partial^2\xi}{\partial z^2}\cdot\frac{\partial n_k}{\partial k_z}=0, \tag{1.88}$$

$$N_0M\left(\frac{\partial^2\xi}{\partial t^2}-c_s^2\frac{\partial^2\xi}{\partial z^2}\right)=-s\frac{\partial}{\partial z}\sum_k n_k, \tag{1.89}$$

where use has been made of the dispersion relation $k^2-(\omega^2/c^2)\varepsilon(\omega, N)=0$ in computing the derivative of the frequency with respect to coordinate:

$$\left.\begin{aligned} &\frac{\partial \omega_k}{\partial z}=\frac{\partial\omega^2\varepsilon/\partial N}{\partial\omega^2\varepsilon/\partial\omega}\cdot\frac{\partial N}{\partial z}=s\frac{\partial^2\xi}{\partial z^2},\\ &s=[\partial\omega^2\varepsilon(\omega)/\partial\omega]^{-1}k^2c^2. \end{aligned}\right\} \tag{1.90}$$

<u>Damping of Acoustic Waves in a Gas of Alfvén Waves.</u> The evolution of small acoustic perturbations in a gas of plasmons is described by Eqs. (1.88) and (1.89), which are linearized with respect to ξ. From Eq. (1.88) we find the correction to the Alfvén wave distribution function due to the small perturbations:

$$-\delta n_k=\frac{is}{\Omega-q\partial\omega/\partial k}q^2\xi\frac{\partial n_k}{\partial k}.$$

Substituting this result in Eq. (1.89) we obtain the dispersion relation between the frequency Ω and the wave vector $\mathbf{q}$ of the ion-acoustic wave:

$$\Omega^2-q^2c_s^2=\frac{q^2s^2}{2\pi MN}\int\frac{q\frac{\partial n_k}{\partial k}dk}{\Omega-q\frac{\partial\omega}{\partial k}+i0}. \tag{1.91}$$

For small amplitudes of the Alfvén wave, the damping of the acoustic wave is small and the frequency Ω can be written in the form

$$\left.\begin{aligned} &\Omega=\pm|q|c_s+i\Gamma_q^{\pm},\\ &\Gamma_q^{\pm}=\frac{s^2q}{4MN_0c_s}\int dkq\frac{\partial n_k}{\partial k}\delta\left(\pm|q|c_s-q\frac{\partial\omega}{\partial k}\right). \end{aligned}\right\} \tag{1.92}$$

The analogy with Landau damping on particles will be evident: In the present case the acoustic waves are damped on quasiparticles characterized by velocity $d\omega/dk$ and a distribution function n_k. It will be evident that a distribution will relax under the effect of the acoustic wave (cf. problems in §§ 2 and 3).

<u>Instability of a Plasmon Gas.</u> We shall now show that a system consisting of a plasmon gas with close-lying frequencies and acoustic waves is subject to the Lighthill instability criterion (cf. § 1.1). If the plasmon spectrum is narrow it can be treated as that of a monochromatic wave:

$$n_k=2\pi n_0\delta(k-k_0). \tag{1.93}$$

Substituting Eq. (1.93) in Eq. (1.91), we have

$$\Omega^2-q^2c_s^2=-\frac{q^4s^2}{MN_0}\cdot\frac{n_0\,\partial^2\omega/\partial k^2}{(\Omega-q\,\partial\omega/\partial k)^2}. \tag{1.94}$$

For a nonresonant perturbation ($\Omega\approx q d\omega/dk+i\nu$) we can then immediately find the growth rate for the modulational instability:

$$\nu^2=\frac{q^2s^2n_0}{N_0M(\partial\omega/\partial k)^2}\cdot\frac{\partial^2\omega}{\partial k^2}>0.$$

Taking account of the definition of the occupation number (1.85) and of coordinates and wave vectors:

$$\left.\begin{aligned} &\frac{\partial n_k}{\partial t}+\frac{\partial \omega_k}{\partial k_z}\cdot\frac{\partial n_k}{\partial z}-\frac{\partial \omega}{\partial z}\cdot\frac{\partial n_k}{\partial k_z}=0,\\ &n_k=\left[\frac{\partial}{\partial\omega}\omega^2\varepsilon(\omega)\right]\frac{|\mathcal{H}_k|^2}{8\pi k^2 c^2}. \end{aligned}\right\} \tag{1.85}$$

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$$\frac{\partial n_k}{\partial t}+\frac{\partial \omega_k}{\partial k_z}\cdot\frac{\partial n_k}{\partial z}-s\frac{\partial^2\xi}{\partial z^2}\cdot\frac{\partial n_k}{\partial k_z}=0, \tag{1.88}$$

$$N_0 M\left(\frac{\partial^2\xi}{\partial t^2}-c_s^2\frac{\partial^2\xi}{\partial z^2}\right)=-s\frac{\partial}{\partial z}\sum_k n_k, \tag{1.89}$$

where use has been made of the dispersion relation $k^2 - (\omega^2/c^2)\varepsilon(\omega, N) = 0$ in computing the derivative of the frequency with respect to coordinate:

$$\left.\begin{aligned} &\frac{\partial\omega_k}{\partial z}=\frac{\partial\omega^2\varepsilon/\partial N}{\partial\omega^2\varepsilon/\partial\omega}\cdot\frac{\partial N}{\partial z}=s\frac{\partial^2\xi}{\partial z^2},\\ &s=[\partial\omega^2\varepsilon(\omega)/\partial\omega]^{-1}k^2c^2. \end{aligned}\right\} \tag{1.90}$$

D a m p i n g o f A c o u s t i c W a v e s i n a G a s o f A l f v é n W a v e s. The evolution of small acoustic perturbations in a gas of plasmons is described by Eqs. (1.88) and (1.89), which are linearized with respect to ξ. From Eq. (1.88) we find the correction to the Alfvén wave distribution function due to the small perturbations:

$$-\delta n_k=\frac{\mathrm{i}\,s}{\Omega-q\partial\omega/\partial k}\,q^2\xi\,\frac{\partial n_k}{\partial k}.$$

Substituting this result in Eq. (1.89) we obtain the dispersion relation between the frequency Ω and the wave vector q of the ion-acoustic wave:

$$\Omega^2-q^2c_s^2=\frac{q^2s^2}{2\pi MN}\int\frac{q\frac{\partial n_k}{\partial k}\,dk}{\Omega-q\frac{\partial\omega}{\partial k}+\mathrm{i}0}. \tag{1.91}$$

For small amplitudes of the Alfvén wave, the damping of the acoustic wave is small and the frequency Ω can be written in the form

$$\left.\begin{aligned} &\Omega=\pm|q|c_s+\mathrm{i}\Gamma_q^{\pm},\\ &\Gamma_q^{\pm}=\frac{s^2 q}{4MN_0c_s}\int dkq\frac{\partial n_k}{\partial k}\delta\left(\pm|q|c_s-q\frac{\partial\omega}{\partial k}\right). \end{aligned}\right\} \tag{1.92}$$

The analogy with Landau damping on particles will be evident: In the present case the acoustic waves are damped on quasiparticles characterized by velocity $d\omega/dk$ and a distribution function n_k. It will be evident that a distribution will relax under the effect of the acoustic wave (cf. problems in §§ 2 and 3).

I n s t a b i l i t y o f a P l a s m o n G a s. We shall now show that a system consisting of a plasmon gas with close-lying frequencies and acoustic waves is subject to the Lighthill instability criterion (cf. § 1.1). If the plasmon spectrum is narrow it can be treated as that of a monochromatic wave:

$$n_k=2\pi n_0\delta(k-k_0). \tag{1.93}$$

Substituting Eq. (1.93) in Eq. (1.91), we have

$$\Omega^2-q^2c_s^2=-\frac{q^4s^2}{MN_0}\cdot\frac{n_0\,\partial^2\omega/\partial k^2}{(\Omega-q\,\partial\omega/\partial k)^2}. \tag{1.94}$$

For a nonresonant perturbation ($\Omega \approx qd\omega/dk + i\nu$) we can then immediately find the growth rate for the modulational instability:

$$\nu^2=\frac{q^2s^2n_0}{N_0M(\partial\omega/\partial k)^2}\cdot\frac{\partial^2\omega}{\partial k^2}>0.$$

Taking account of the definition of the occupation number (1.85) and the nonlinear frequency correction $\delta\omega_0 = -s^2n_0/N_0M(d\omega/dk)^2$, the criterion which is obtained reduces to the Lighthill criterion. However, the dispersion equation does not contain terms which describe the stabilization of an instability in the presence of a group velocity (cf. § 1.1).

PROBLEM

▶ 1. Find the intersection conditions for the decay and modulational branches of the plasmon instability (Vedenov and Rudakov [46]).

In a system consisting of a gas of plasmons plus acoustic waves we know that Eqs. (1.85)-(1.87) are valid with the difference that the magnetic pressure is replaced by the ponderomotive force $(N_0e^2/2m)\ \nabla \sum_k |E_k|^2/\omega_k^2$. in the second of these equations. Correspondingly, $s = \omega_p/2$. For plasma waves $\partial^2\omega/\partial k^2 > 0$ so that according to the Lighthill criterion the plasma oscillations are unstable against the modulational instability if the nonlinear correction to the frequency is negative. This correction is negative when

$$\partial\omega/\partial k \ll c_s, \quad \text{i.e.,} \quad k^2\lambda_D^2 \ll c_s/v_{Te}. \tag{1}$$

The smallness of the group velocity of the plasma waves provides the possibility of intersection of the branches corresponding to the decay and modulation instabilities ($c_s \approx \partial\omega/\partial k$), when the growth rate of the instability is of the order of the fourth power of the energy of the primary wave. With large amplitudes this intersection can occur over a wide range of wave numbers. In order to treat this effect we seek the solution of Eq. (1.94) in he form

$$\Omega = (q/2)\partial\omega/\partial k_0 + i\nu \gg qc_s. \tag{2}$$

We then find that ν is described by the dispersion equation

$$\nu^2 = -\frac{1}{4}\left(q\cdot\frac{\partial\omega}{\partial k}\right)^2 \pm q^2c_s^2\left[\frac{s^2\int\frac{\partial^2\omega}{\partial k^2}n_k\frac{d^3k}{(2\pi^3)}}{N_0Mc_s^4}\right]^{1/2}$$

From this result we obtain the instability criterion

$$\int\omega_k n_k(d^3k/(2\pi)^3) > (M/m)N_0T_e(k\lambda_D)^4\left[\frac{(k\cdot q)}{(|k|\cdot|q|)}\right]^4 \tag{3}$$

and the growth rate

$$\nu = qc_s\left[\frac{M}{m}\int\omega_k n_k\frac{d^3k}{(2\pi)^3}\bigg/N_0T_e\right]^{1/4}. \tag{4}$$

The condition of applicability of the weak turbulence approximation and inequality (2) for the growth rate indicate the limitation on the range of wave vectors in which the instability can occur, $m/M > (k\lambda_D)^4 > (m/M)^2$.

Chapter 2

WAVE–PARTICLE INTERACTIONS

§ 2.1. Wave – Particle Interaction for a Monochromatic Wave

We now consider the linear (or quasilinear) wave–particle interaction associated with the resonance condition. When the particle and wave satisfy this resonance condition the particle maintains a constant phase with respect to the wave and is accelerated in the constant electric field associated with the wave. Since the interaction is characteristic of resonance particles, it cannot be described by fluid equations; rather, it is necessary to make use of the kinetic equations and Maxwell's equations in the self-consistent approximation.

We start by considering a relatively simple problem that involves the resonance interaction between electrons and a monochromatic plasma wave in the one-dimensional case. The system of equations used for solving this problem is

$$\left.\begin{aligned}\partial f/\partial t + v\,\partial f/\partial x + (e/m)(\partial\Phi/\partial x)(\partial f/\partial v) &= 0,\\ \partial^2\Phi/\partial x^2 &= 4\pi ne\,(1 - \textstyle\int f dv).\end{aligned}\right\} \tag{2.1}$$

Here, $f(x; v, t)$ is the electron distribution function and $\Phi(x, t) = (1/2)\Phi\cos(kx - \omega t)$ is the electric potential associated with the monochromatic wave. The only nonlinear term in these equations is the third term in the kinetic equation $(d\Phi/dx)(df/dv)$. This term can be linearized by replacing df/dv by df_0/dv or by setting the amplitude of the wave equal to a constant [(i.e., by replacing $\Phi\cos(kx - \omega t)$ by $\Phi_0\cos(kx - \omega t)$]; Landau [47] replaced df/dv by df_0/dv and obtained the well-known result that the amplitude of the wave is damped according to $\exp(\gamma_L t)$, where γ_L =

the nonlinear frequency correction $\delta\omega_0 = -s^2 n_0/N_0 M(d\omega/dk)^2$, the criterion which is obtained reduces to the Lighthill criterion. However, the dispersion equation does not contain terms which describe the stabilization of an instability in the presence of a group velocity (cf. § 1.1).

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The condition of applicability of the weak turbulence approximation and inequality (2) for the growth rate indicate the limitation on the range of wave vectors in which the instability can occur, $m/M > (k\lambda_D)^4 > (m/M)^2$.

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$(\pi/2)(\omega_p^2\omega/k^2)(df_0/dv)|_{v=\omega/k}$. When $(df_0/dv) > 0$, the wave exhibits growth (the so called "bump-on-tail" instability).

We now wish to examine the key difference between the two methods of linearization in this problem. If $\Phi \cos(kx - \omega t)$ is replaced by $\Phi_0 \cos(kx - \omega t)$ it is possible to follow the behavior of the distribution function in the region of resonance velocities. One expects that the characteristic time scale required for a change in the distribution function in this region will be of the order of an oscillation period of the resonance electrons which are trapped by the electric field of the wave, that is to say, $\tau_b = (2m/ek^2\Phi_0)^{1/2}$.

The linearization procedure used by Landau is valid if the wave amplitude changes much more rapidly than df/dv. This condition can be written in the form $|\gamma_L \tau_b| \gg 1$. In other words, the amplitude of the wave must be smaller than some specified value, i.e., $\Phi_0 \ll \gamma_L^2\, m/k^2e$. The other limiting case arises when the quantity df/dv varies much faster than the wave amplitude. This condition can be described by $|\gamma_L \tau_b| \ll 1$ or $\Phi_0 \gg m\gamma_L^2/k^2e$. It will be evident that when $\gamma_L \tau_b$ is of the order of unity the problem is essentially nonlinear and neither linearization procedure is applicable.

In investigating the evolution of the distribution function it is useful to consider the electron trajectories in the phase plane. In the reference system that moves with the wave $\Phi_0 \cos(kx - \omega t)$ (Fig. 12) these trajectories are described by the equation $\mathscr{E} = mv^2/2 - (1/2)\, e\Phi_0 \cos kx$. Electrons for which $\mathscr{E} < e\Phi_0/2$ are trapped by the wave, while electrons for which $\mathscr{E} > e\Phi_0/2$ are not trapped. It is now convenient to introduce the angle-energy variables $(\mathscr{E}, \vartheta)$, where $\mathscr{E}$ determines some trajectory while ϑ is a point on this trajectory. In terms of the new variables the distribution function is given by $f(\vartheta, \mathscr{E})$. This function can be time independent only when f does not depend on ϑ (in other words, when it is a constant along the particle trajectories). If this distribution function is made self-consistent with the electric field it is possible to construct a stationary solution of the system in (2.1) which will describe a steady-state nonlinear wave. This solution is called the Bernstein–Greene–Kruskal solution (BGK) [48]. In the problem at hand f depends on both ϑ, and $\mathscr{E}$, initially; however, in the course of time it tends asymptotically to one of the particular BGK solutions. In order to demonstrate this feature we consider the behavior of the trapped particles. Two particles on close-lying trajectories, i.e., two particles with somewhat different

energies $\mathscr{E}$, will, in general, have somewhat different rotational frequencies in phase space (cf. Fig. 12), $\omega_2 - \omega_1 = (d\omega/d\mathscr{E})(\mathscr{E}_2 - \mathscr{E}_1)$. If these particles start with the same phase ϑ, after a time interval $\Delta t \approx 1/\omega_2 - \omega_1)$, the phases of the two particles will differ by an amount $\Delta\vartheta \sim 1$. This leads to a smearing of the phases and f becomes constant along the trajectory if one considers averaging of the distribution function even over a small range $\mathscr{E}$. Similar arguments hold for the untrapped particles if f is periodic in space. In an actual plasma scrambling of the fine-scale fluctuations in the distribution function will occur as a result of collisions between particles. As the phases become mixed the distribution function becomes a very fine-scaled function of $\mathscr{E}$ or ϑ. After a sufficiently long time the scale of these fine-grain fluctuations becomes so small that one must take account of the d^2f/dv^2 term in the Landau collision integral so that even in a collisionless plasma collisions will ultimately smooth out the fine-scale variations of the distribution function causing it to approach some average value. The time required for this process to occur is finite and, in practice, is insensitive to the collision frequency. These observations indicate the way in which the entropy can increase in a collisionless plasma (the details of this mechanism for entropy "production" are discussed in [49, 50]).

The pattern described here is qualitatively similar to the pattern in the theory of hydrodynamic turbulence: The turbulence first develops on large scales and then on ever-decreasing smaller scales. Ultimately, at a sufficiently small scale length the real damping occurs (due to viscosity) and the turbulence energy is dissipated. The difference lies in the fact that the breakup of the scales in the present case occurs in velocity space $[f = f(v)]$.

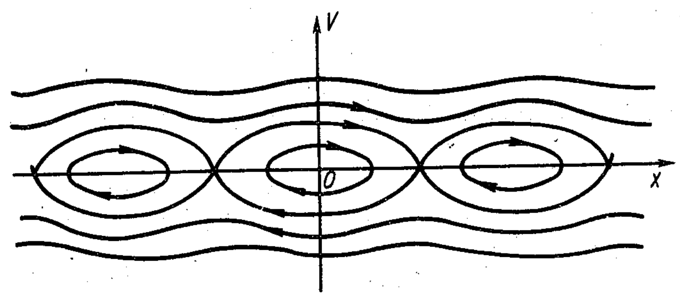

Fig. 12. Phase trajectories of particles moving in the field of a monochromatic wave.

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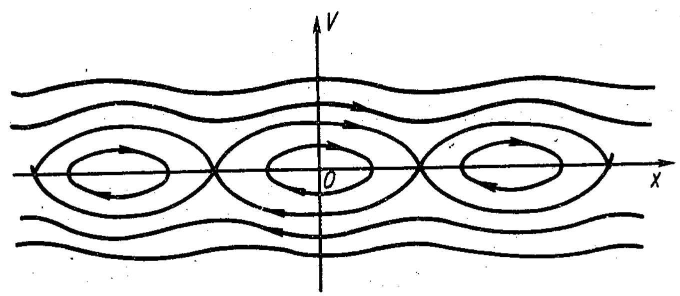

Fig. 12. Phase trajectories of particles moving in the field of a monochromatic wave.

The evolution of the distribution function in the field of a wave characterized by the amplitude

$$\Phi = \Phi_0 (1 - \cos kx)/2 \equiv \Phi_0 \sin^2 (kx/2) \tag{2.2}$$

(this normalization of the potential in the wave will be useful for the calculations below) is solved by integrating over the trajectory. If the wave amplitude is not too large the initial distribution function (at time t = 0) in the resonance region can be expanded

$$f_0 (v) = f_0 (\omega/k) + (\partial f_0/\partial v)\,|_{v=\omega/k}(v - \omega/k) + \ldots. \tag{2.3}$$

This distribution function at t > 0 can then be obtained by integrating over the trajectory. In Eq. (2.3) this means replacing the velocity as follows:

$$(v - \omega/k) \to \sigma \sqrt{(2/m)\{\mathscr{E} - e\Phi_0\,[\sin (k/2)x_0 (x,\ \mathscr{E},\ t)]^2\}},$$

where $x_0(x, \mathscr{E}, t)$ is the initial coordinate of an electron with energy $\mathscr{E}$ which, at time t, is located at point x; $\sigma = \pm 1$ determines the direction of motion. The motion of the particles in the wave field can be expressed in terms of elliptic functions by introducing the substitution $\sin\xi = \varkappa \sin kx/2$, $\varkappa^2 = e\Phi_0/\mathscr{E}$.

Integrating the equation of motion for the trapped particles

$$dx/[(2/m)(\mathscr{E} - e\Phi_0 \sin^2 kx/2)]^{1/2} = dt,$$

we have

$$F (1/\varkappa,\ kx_0/2) = F (1/\varkappa,\ kx/2) - t/\tau_b,$$

where $F(\varkappa, \varphi) = \int\limits_0^{\varphi} d\xi\,(1 - \varkappa^2 \sin^2 \xi)^{-1/2}$ is an elliptic integral of the first kind:

$$\tau_b = [2m/k^2 e\Phi_0]^{1/2}. \tag{2.4}$$

The distribution function for the trapped particles f (v, t) is then written in the form

$$f(x,\ \mathscr{E},\ t) = \left\{ f_0\left(\frac{\omega}{k}\right) + f_0'\,\sigma \sqrt{\frac{2\mathscr{E}}{m}}\,\mathrm{cn}\left[F\left(\frac{1}{\varkappa},\ \frac{kx}{2}\right) - \frac{t}{\tau_b},\ \frac{1}{\varkappa}\right]\right\} \Big/ \left[2m\left(\mathscr{E} - e\Phi_0 \sin^2 \frac{kx}{2}\right)\right]^{1/2}$$

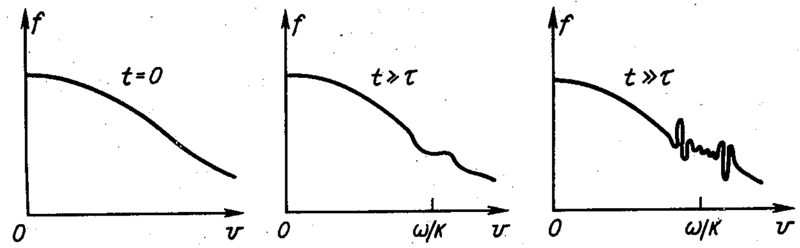

Fig. 13. Distortion of the resonance-particle distribution function in the field of a monochromatic wave.

The elliptic cosine is an oscillating function of $\varkappa$ for which the oscillation period in $\varkappa$ diminishes as t increases. As a consequence, the distribution function also oscillates. The qualitative behavior of the distribution function is shown in Fig. 13. When this distribution function is averaged the second term vanishes. Thus, the averaged distribution function $\langle f(\mathscr{E})\rangle$ is the same for all trajectories [i.e., $\langle f\rangle = f(\omega/k)$]. In other words, in the region of phase space which corresponds to the trapped particles (Fig. 14) one finds that a plateau is formed. The formation of this plateau is a very general consequence of the wave–particle interaction and

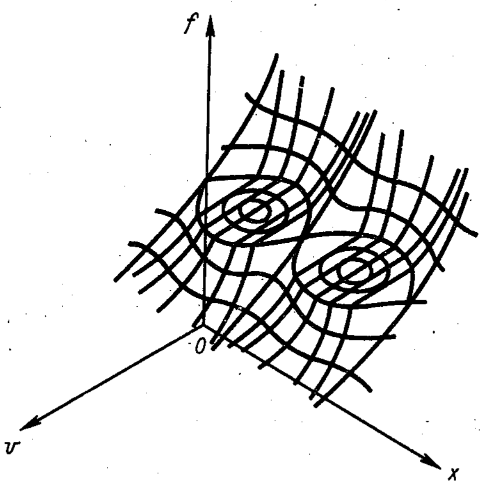

Fig. 14. Averaged distribution function formed as a result of relaxation in the field of a monochromatic wave.

The evolution of the distribution function in the field of a wave characterized by the amplitude

$$\Phi = \Phi_0 (1 - \cos kx)/2 \equiv \Phi_0 \sin^2 (kx/2) \tag{2.2}$$

(this normalization of the potential in the wave will be useful for the calculations below) is solved by integrating over the trajectory. If the wave amplitude is not too large the initial distribution function (at time t = 0) in the resonance region can be expanded

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This distribution function at t > 0 can then be obtained by integrating over the trajectory. In Eq. (2.3) this means replacing the velocity as follows:

$$(v - \omega/k) \rightarrow \sigma\sqrt{(2/m)\{\mathscr{E} - e\Phi_0 [\sin (k/2)x_0 (x, \mathscr{E}, t)]^2\}},$$

where $x_0(x, \mathscr{E}, t)$ is the initial coordinate of an electron with energy $\mathscr{E}$ which, at time t, is located at point x; $\sigma = \pm 1$ determines the direction of motion. The motion of the particles in the wave field can be expressed in terms of elliptic functions by introducing the substitution $\sin \xi = \varkappa \sin kx/2$, $\varkappa^2 = e\Phi_0/\mathscr{E}$.

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The distribution function for the trapped particles $f(v, t)$ is then written in the form

$$f(x, \mathscr{E}, t) = \left\{ f_0\left(\frac{\omega}{k}\right) + f_0'\,\sigma \sqrt{\frac{2\mathscr{E}}{m}}\,\mathrm{cn}\left[F\left(\frac{1}{\varkappa}, \frac{kx}{2}\right) - \frac{t}{\tau_b}, \frac{1}{\varkappa}\right]\right\} \Big/ \left[2m\left(\mathscr{E} - e\Phi_0 \sin^2 \frac{kx}{2}\right)\right]^{1/2}$$

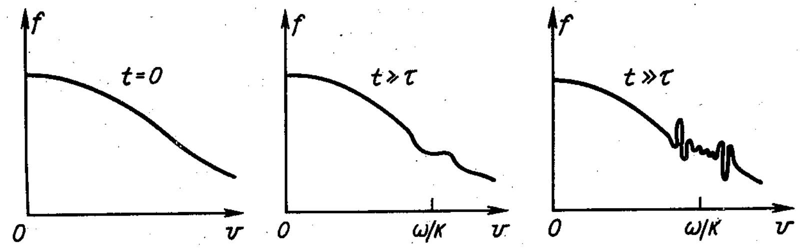

Fig. 13. Distortion of the resonance-particle distribution function in the field of a monochromatic wave.

The elliptic cosine is an oscillating function of $\varkappa$ for which the oscillation period in $\varkappa$ diminishes as t increases. As a consequence, the distribution function also oscillates. The qualitative behavior of the distribution function is shown in Fig. 13. When this distribution function is averaged the second term vanishes. Thus, the averaged distribution function $\langle f(\mathscr{E})\rangle$ is the same for all trajectories [i.e., $\langle f\rangle = f(\omega/k)$]. In other words, in the region of phase space which corresponds to the trapped particles (Fig. 14) one finds that a plateau is formed. The formation of this plateau is a very general consequence of the wave–particle interaction and

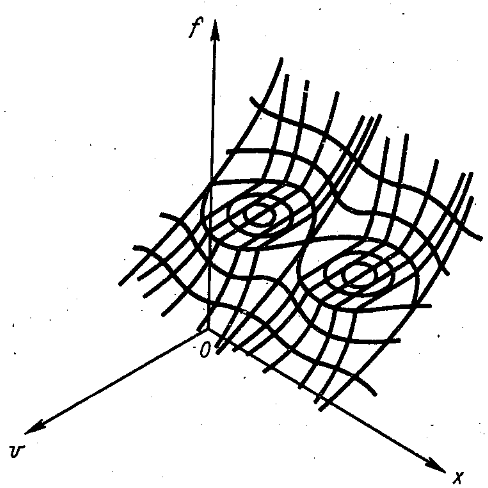

Fig. 14. Averaged distribution function formed as a result of relaxation in the field of a monochromatic wave.

we shall encounter it in the many-wave case (i.e., in the interaction of a particle with a wave packet). The distribution function can be obtained in the same way for the trajectories of untrapped electrons:

$$f(x, \mathscr{E}, t)=\left\{f_0+f_0'\sigma\sqrt{\frac{2\mathscr{E}}{m}}\, dn\left[F\left(\varkappa, \frac{kx}{2}\right)-\right.\right.$$
$$\left.\left.-\frac{t}{\varkappa\tau_b}, \varkappa\right]\right\} \Big/ \left[2m\left(\mathscr{E}-e\Phi_0\sin^2\frac{kx}{2}\right)\right]^{1/2}. \tag{2.5}$$

The mean value of the elliptic function dn [...] is different from zero. Hence, when Eq. (2.5) is averaged we find

$$\langle f\rangle \equiv f_{QL}=\left\{f_0'\left(\frac{\omega}{k}\right)+\left.\frac{df_0}{dv}\right|_{v=\omega/k}\times\right.$$
$$\left.\times\frac{\pi}{k\varkappa K(\varkappa)\tau_b}\right\} \Big/ \left[2m\left(\mathscr{E}-e\Phi_0\sin^2\frac{kx}{2}\right)\right]^{1/2}, \tag{2.6}$$

where $K(\varkappa) = F(\varkappa, \pi/2)$ is the complete elliptic integral of the first kind.

Up to this point we have assumed that the wave amplitude is constant and have computed the change in the distribution function that results from the interaction with this wave. We are now in a position to use the change in the distribution function that has been obtained in order to compute a small correction to the wave amplitude. This correction is of order $\Delta\Phi \sim \gamma_L\tau_b\Phi_0$ so that the procedure can be regarded as an expansion in the small parameter $\gamma_L\tau_b$. We now introduce a time-dependent damping rate

$$\gamma(t) = dW/2Wdt, \tag{2.7}$$

where $W(t) = \int_0^\lambda (dx/\lambda)(d\Phi/dx)^2/4\pi$ is the wave energy.

Conservation of energy is now used to find dW/dt:

$$dW/dt = ne\int_0^\lambda (dx/\lambda)(\partial\Phi/\partial x)\int_{-\infty}^{+\infty} dvf(x, v, t)v. \tag{2.8}$$

The expression for the time-dependent distribution function $f(x, v, t)$, which is needed to compute the integral in Eq. (2.8), can be found for two different profiles for the electric potential in the

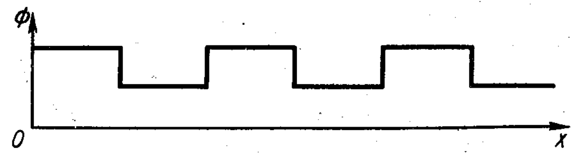

Fig. 15. Square wave potential.

wave. In the simplest case, which is treated [51], the potential is assumed to be a square wave (Fig. 15). The damping of this wave is first characterized by the quantity γ_L as determined from the linear Landau theory; the damping subsequently oscillates with a period of order τ_b, the mean oscillation period of electrons trapped by the wave. The damping vanishes when the phases of the trapped electrons are completely smeared out. In this idealized wave profile the contribution to the damping only comes from the trapped electrons.

The case of a sinusoidal wave has been treated by O'Neil [52]. In this case the untrapped electrons also contribute to the damping. The damping coefficient can be found by substituting the distribution functions from (2.4) and (2.5) in Eq. (2.8):

$$\gamma(t)=\gamma_L\sum_{n=0}^{\infty}\frac{64}{\pi}\int_0^1 d\varkappa\left\{\frac{2n\pi^2\sin\left[\frac{\pi nt}{\varkappa K\tau_b}\right]}{\varkappa^5K^2(1+q^{2n})(1+q^{-2n})}+\right.$$
$$\left.+\frac{(2n+1)\pi^2\varkappa\sin\left[\frac{(2n+1)\pi t}{\varkappa K\tau_b}\right]}{K^2(1+q^{2n+1})(1+q^{-2n-1})}\right\}, \tag{2.9}$$

where $K' = K(1-\varkappa^2)^{1/2}$ and $q = \exp[\pi K'/K]$. The dependence of the damping coefficient on time is shown in Fig. 16. As recently

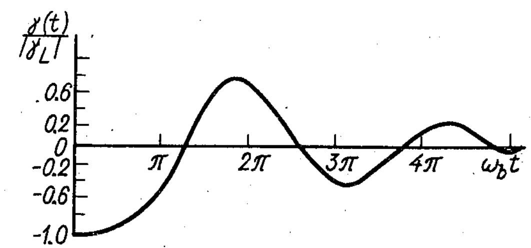

Fig. 16. Time behavior for the damping coefficient of a sinusoidal wave.

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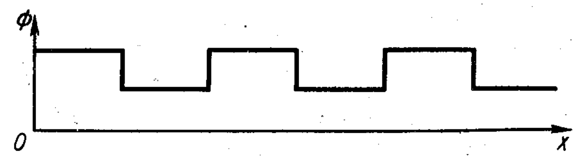

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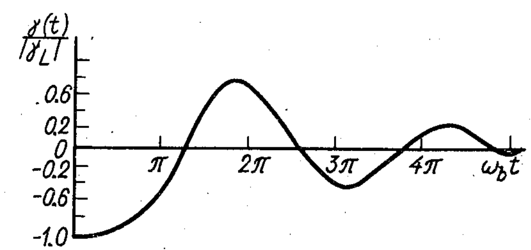

Fig. 16. Time behavior for the damping coefficient of a sinusoidal wave.

noted by Morales and O'Neil [53], in the approximation used above, in which only the change in wave amplitude due to Landau damping is taken into account, the change in electron energy due to the modification of the distribution function is not compensated by the change in the energy of the wave due to the change in wave amplitude. This is not surprising since changing only one of the variables (amplitude) means that it is not possible to satisfy two conservation conditions, namely, those for momentum and energy. It will be evident that the change in the oscillation frequency must be taken into account. In order to describe the frequency change properly it is necessary to retain higher-order terms in the expansion in (2.3):

$$f_0(v) = f_0\left(\frac{\omega}{k}\right) + \left.\frac{\partial f_0}{\partial v}\right|_{v=\omega/k}\left(v - \frac{\omega}{k}\right) + \left.\frac{\partial^2 f_0}{\partial v^2}\right|_{v=\omega/k}\left(v - \frac{\omega}{k}\right)^2 + \ldots \qquad (2.3')$$

An estimate of the nonlinear frequency shift can be obtained easily once we know the change in the well-known expression for the dielectric permittivity caused by nonlinear effects which lead to the formation of a plateau on the distribution function:

$$\delta\omega\,|_{t\to\infty}\,\frac{\partial\mathscr{E}}{\partial\omega} \approx \frac{\omega_p^2}{k^2}\int \frac{k\dfrac{\partial}{\partial v}[f_0(v) - f_{QL}(v)]}{\omega - kv}\,dv,$$

where f_{QL} is the value of the distribution functions (2.4) and (2.5) averaged over the fast motion of the particles in the wave [cf. Eq. (2.6)]. From considerations of parity, the change in frequency can only be due to the last term in Eq. (2.3'). The basic contribution in the integral comes from particles with velocities $\Delta v \approx (2e\Phi_0/m)^{1/2}$:

$$\delta\omega\,(t=\infty) \approx -\Omega_0 = -\left(\frac{2e\Phi_0}{m}\right)^{1/2}\left(\frac{\omega_p}{k}\right)^2\left(\frac{\partial^2 f_0}{\partial v^2}\right)\bigg|_{v=\omega/k}\left(\frac{\partial\varepsilon}{\partial\omega}\right)^{-1}.$$

More precise calculations lead to the following result [53]:

$$\delta\bar{\omega}\,(t=\infty) = -\Omega_0\,\frac{16}{\pi}\int_0^1 d\varkappa\left[\frac{\varkappa}{K(\varkappa)}\,(2E(\varkappa) - K(\varkappa))^2 + \frac{[2(E-K) + \varkappa^2 K]^2}{\varkappa^6 K}\right].$$

The dependence of the frequency shift on time is shown in Fig. 17.

The BGK solution, which is the asymptotic limit in the theory discussed here, can ultimately be distorted because of effects that

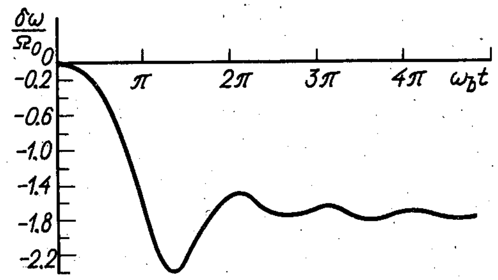

Fig. 17. Time dependence of the frequency shift of a sinusoidal wave.

have not been included in the present analysis. For example, it may be unstable against decay, caused by a nonlinear wave–wave interaction, into another plasma wave and an ion-acoustic wave (cf. § 1.1). It will be evident that the BGK solution is ultimately distorted because of ordinary collisions. The effect of collisions on the wave–particle interaction is very important in certain cases.

In the linear theory (Landau approximation) the problems of wave damping ($df_0/dv(v = \omega/k) < 0$) and growth ($df_0/dv(v = \omega/k) > 0$) are solved in the same way. But if one attempts to treat the nonlinear distortion of the distribution function in the resonance region $v \approx \omega/k$ for growing (unstable) waves ($df_0/dv(v = \omega/k) > 0$) one immediately encounters a difficulty if it is desired to use the Mazitov–O'Neil approximation in this case. Actually, the initial low-amplitude wave must grow, by definition, so that the expression for the potential of the wave is of the form

$$\Phi = (1/2)\Phi_0(t)\cos(kx - \omega t). \qquad (2.10)$$

The motion of particles in a potential well which changes in time is complicated. One of the most important effects here is the transition of particles from a region in which $\mathscr{E} > e\Phi_0/2$ through the separatrix into a region in which $\mathscr{E} < e\Phi_0/2$, i.e., the conversion of untrapped particles into trapped particles, Ultimately, a plateau type distribution arises for the trapped particles. This leads to a situation in which the growth of the unstable wave is terminated. In view of the complexity of the problem we will obtain a qualitative estimate of the order of magnitude of the saturation amplitude for the instability.

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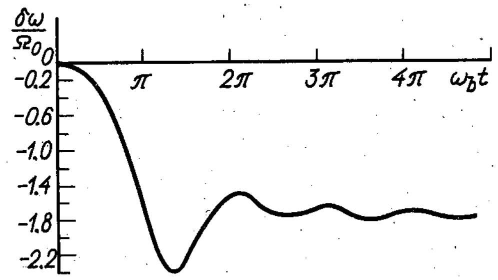

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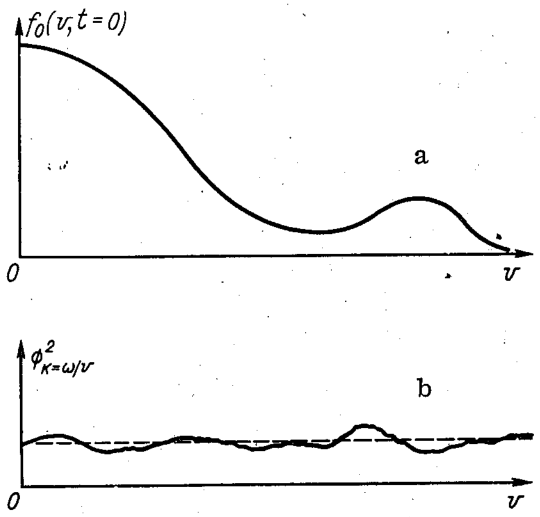


Fig. 18. Initial particle distribution (a) and noise spectrum (b) for a beam.

In Fig. 18 we show the unstable distribution function $f_0(v)$ with a smeared-out beam. Let us assume that for some reason a perturbation arises in the form of a low-amplitude monochromatic wave $\Phi = (1/2)\Phi_0 \cos(kx - \omega t)$ in the spectrum of initial fluctuations. We first expand the distribution function in the resonance region

$$f_0(v) = f_0(\omega/k) + (df_0/dv)\,|_{v=\omega/k}(v - \omega/k).$$

Ultimately, when the wave amplitude reaches its maximum value and a plateau is established in the resonance region $[f(v) = f_0(\omega/k)]$ with width $\Delta v = (2e\Phi_0/m)^{1/2}$, part of the kinetic energy of the particles

$$\Delta\mathscr{E} = \frac{mn}{2}\int\limits_{\omega/k-\sqrt{2e\Phi_0/m}}^{\omega/k+\sqrt{2e\Phi_0/m}} v^2 f_0(v)dv - \frac{mn}{2}\int\limits_{\omega/k-\sqrt{2e\Phi_0/m}}^{\omega/k+\sqrt{2e\Phi_0/m}} v^2 f_0\left(\frac{\omega}{k}\right) dv \qquad (2.11)$$

is converted into wave energy

$$W = k^2\Phi_0^2/16\pi. \qquad (2.12)$$

It then follows from energy conservation $\Delta\mathscr{E} = W$ that [$\Delta\mathscr{E}$ is com-

puted by means of the expansion for f_0 given in Eq. (2.3)]

$$\frac{2}{3}\, mn \left(\frac{2e\Phi_0}{m}\right)^{3/2}\left(\frac{\omega}{k}\right)\frac{df_0}{dv} = \frac{k^2\Phi_0^2}{16\pi}. \qquad (2.13)$$

The wave amplitude is then found to be

$$k\sqrt{\frac{e\Phi_0}{2m}} = \frac{16}{3}(\omega_p^2\,\omega/k^2)\left.\frac{df_0}{dv}\right|_{v=\omega/k}. \qquad (2.14)$$

Using the expression which relates the slope of the initial distribution function df_0/dv with the growth rate in the linear theory, γ_L, we can write this relation in the form

$$\omega_b/\gamma_L = 32/3\pi, \qquad (2.15)$$

where $\omega_b = k(e\Phi_0/2m)^{1/2}$ is the mean bounce frequency of the trapped particles in the wave field.

Thus, the final result assumes a simple universal form. It will be evident that this estimate cannot be taken to have any high degree of accuracy.

A numerical solution of this problem carried out on a high-speed computer is given in [54] (cf. also [55]). In this solution account is taken of the joint motion of all the electrons in the field of the growing wave. It turns out that the numerical coefficient on the right side of Eq. (2.15) is changed by a factor of 1.5:

$$\omega_b/\gamma_L \approx 3.2. \qquad (2.16)$$

The time dependence of the wave amplitude obtained in [54] is shown in Fig. 19. The amplitude oscillations occur with a fre-

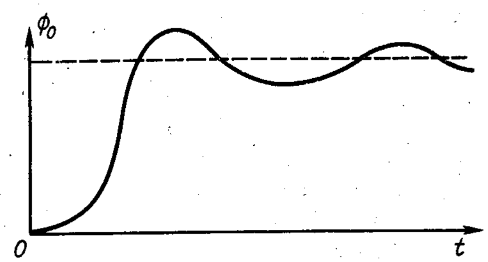
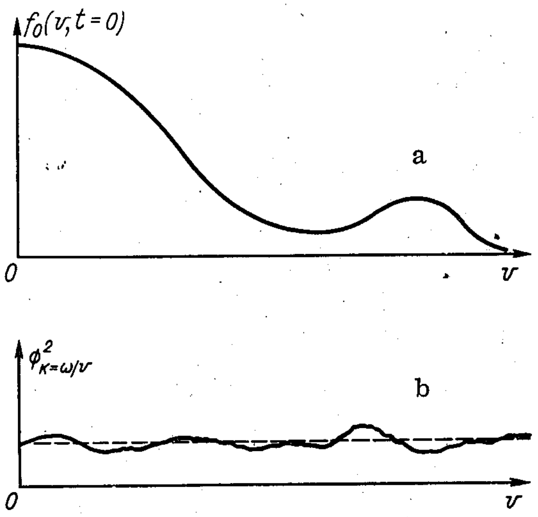


Fig. 19. Establishment of the amplitude of an unstable wave in a plasma with a beam with a weak bump.



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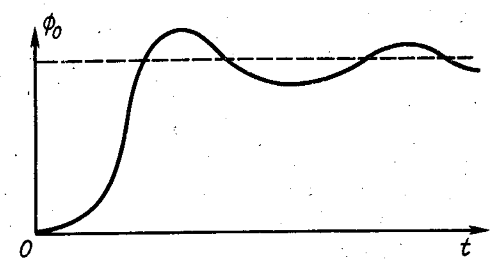

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quency of order ω_b (as in the Mazitov–O'Neil problem) and are damped as $t \to \infty$ because of the smearing of the particle phases.

§ 2.2. The Many-Wave Case (One-Dimensional Spectrum)

We shall first consider the problem of two waves of equal amplitude. If these waves have well-separated phase velocities they will not interact with each other and the overall situation that arises can be considered in terms of the superpositon of processes which occur for each wave separately. However, if the phase velocities of the waves are approximately the same $\Delta(\omega/k) \lesssim (e\Phi/m)^{1/2}$, where Φ is the potential of the wave, then the situation becomes completely different. One then expects to find overlapping or, to put it another way, collective effects associated with the trapped particles. Extrapolating to cases with three, four, or more waves (in which case a rigorous analysis becomes hopelessly complicated) one can only expect to obtain rough qualitative results. On the other hand, in the limiting case of a large number of waves it becomes possible to use a statistical approach and the random-phase approximation.

Let us assume that within some velocity range $(\omega/k)_{max} > v > (\omega/k)_{min}$, waves whose phase velocity occupy this entire range experience a collective interaction of resonance particles between two neighboring waves. If the phases of these waves are random, the velocity of any given particle executes Brownian motion. In phase space this Brownian motion along the velocity axis is superposed on the free-streaming of the particles so that resulting phase trajectories of the particles are of the form shown in Fig. 20.

In the preceding section we have reached the conclusion that the asymptotic solution for the distribution function is constant along the particle trajectories, at least when the smoothing effect due to Coulomb scattering at small angles is taken into account. Extending this conclusion to the many-wave case, we can show that the distribution function will tend asymptotically to a constant value in a segment of phase space between $v = (\omega/k)_{min}$ and $v = (\omega/k)_{max}$, since the particle trajectories will fill this region between the two velocity values in ergodic fashion (cf. Fig. 20). It should be noted that without the smoothing effect of Coulomb collisions the distribution function would be extremely complicated and

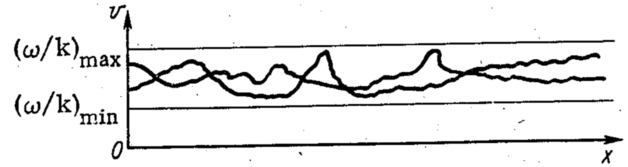

Fig. 20. Brownian motion of electrons in phase space.

exhibit fine-scale variations. The smoothing of the fine-scale variations can be achieved by averaging. Obviously the true (fine-scale) distribution function conserves entropy while the smoothed function does not. The time evolution of the smoothed averaged distribution function is governed by the so-called quasilinear diffusion equation [2, 56-58]. The most rigorous way of obtaining the quasilinear equation together with the criterion for overlap of resonance regions of neighboring monochromatic waves has been developed by Al'tshul' and Karpman [59] (cf. also [60]). We shall limit ourselves to a simpler derivation below.

We start by carrying out a Fourier transformation in coordinate space for the one-dimensional Vlasov equation

$$\frac{\partial}{\partial t} f_k + ivkf_k - \frac{ie}{m} k\Phi_k \frac{\partial f_0}{\partial v} = \frac{ie}{m} \sum{}' (k-q)\,\Phi_{k-q} \frac{\partial f_q}{\partial v}, \qquad (2.17)$$

where the primes to the right of the summation sign indicate that the term with $q = 0$ is not included in the summation. When $k = 0$, this equation assumes the form

$$\partial f_0/\partial t = -(ie/m)\Sigma' q\Phi_{-q}\,(\partial f_q/\partial v). \qquad (2.18)$$

When $k \neq 0$ the terms on the right side of Eq. (2.18) describe the nonlinear coupling between different modes. We will not take further account of mode–mode coupling since we are concerned here only with the linear (or quasilinear) wave–particle interaction. Correspondingly, when $k \neq 0$ we use the linear equation

$$\partial f_k/\partial t + ivkf_k - (ie/m)k\Phi_k\,(\partial f_0/\partial v) = 0. \qquad (2.19)$$

The solution of this equation is of the form

$$f_k(v,\ t) = (ie/m)k\int_0^t dt' \exp[ikv(t'-t)]\Phi_k(t')(\partial f_0(v,\ t')/\partial v). \qquad (2.20)$$

quency of order ω_b (as in the Mazitov–O'Neil problem) and are damped as $t \to \infty$ because of the smearing of the particle phases.

§ 2.2. The Many-Wave Case (One-Dimensional Spectrum)

We shall first consider the problem of two waves of equal amplitude. If these waves have well-separated phase velocities they will not interact with each other and the overall situation that arises can be considered in terms of the superpositon of processes which occur for each wave separately. However, if the phase velocities of the waves are approximately the same $\Delta(\omega/k) \lessdot (e\Phi/m)^{1/2}$, where Φ is the potential of the wave, then the situation becomes completely different. One then expects to find overlapping or, to put it another way, collective effects associated with the trapped particles. Extrapolating to cases with three, four, or more waves (in which case a rigorous analysis becomes hopelessly complicated) one can only expect to obtain rough qualitative results. On the other hand, in the limiting case of a large number of waves it becomes possible to use a statistical approach and the random-phase approximation.

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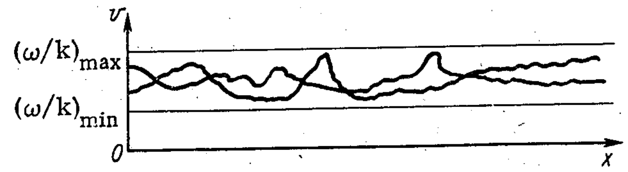

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The solution of this equation is of the form

$$f_k(v,\ t) = (ie/m)k\int_0^t dt' \exp\,[ikv\,(t'-t)]\Phi_k\,(t')(\partial f_0\,(v,\ t')/\partial v). \qquad (2.20)$$

We now substitute this solution into Poisson's equation

$$k^2\Phi_k(t) = 4\pi en \int dv f_k(v,\ t) \tag{2.21}$$

and use the WKB approximation in time, that is to say, it is assumed that $f_0(v, t)$ only changes slightly during one oscillation period ω_p^{-1}:

$$\Phi_k(t) = \Phi_k(0)\exp\left\{\int_0^t [-i\omega_k + \gamma_k(t')]dt'\right\}, \tag{2.22}$$

$$\omega_k = \omega_p(1 + 3k^2\lambda_D^2/2),\ \gamma_k(t) = (\pi/2)(\omega_k\omega_p^2/k^2)(\partial f_0/\partial v)|_{v=\omega/k}.$$

In order to find the time dependence of the distribution function $f_0(v, t)$ we substitute (2.20) and (2.22) in Eq. (2.18):

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left\{\int_0^t dt' \frac{e^2}{m^2}\sum_k k'|\Phi_k(t)|^2 \times \right.$$
$$\left. \times \exp\left[i(kv-\omega_k)(t'-t) + \int_t^{t'} \gamma_k(t'')dt''\right]\frac{\partial f_0(v,t')}{\partial v}\right\}. \tag{2.23}$$

If it assumed that the wave spectrum is wide, i.e., $\Delta(kv - \omega_k) \gg \gamma_k$, τ_R^{-1} [where τ_R is the relaxation time for $f_0(v, t)$] then the summation over k, $\sum_k k^2|\Phi_k(t)|^2 \exp\,[i(kv - \omega_k)(t' - t)]$, gives a vanishing result as a consequence of phase mixing for $(t' - t) > \omega_k^{-1}$. Hence, we can take $f_0(v, t') \approx f_0(v, t)$ and $\gamma_k(t'') \approx \gamma_k(t)$. Making use of this feature and carrying out the integration over t', we have

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\cdot\frac{e^2}{m^2}\sum_k k^2|\Phi_k|^2\,\frac{1-\exp\,[i(kv-\omega_k)t+\gamma_k t]}{i(kv-\omega_k)+\gamma_k}\cdot\frac{\partial f_0}{\partial v}. \tag{2.24}$$

Using the asymptotic relation

$$\frac{1-\exp\,[i(kv-\omega_k)t+\gamma_k t]}{i(kv-\omega_k)+\gamma_k} = P\,\frac{1}{i(kv-\omega_k)+\gamma_k} + \pi\delta(kv-\omega_k)$$

and the condition $\omega_{-k} = -\omega_k$, $\gamma_{-k} = \gamma_k$ (which follows from the fact that the electric potential and the distribution function are real) we can reduce Eq. (2.18) to the form

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}D(v)\frac{\partial f_0}{\partial v}, \tag{2.25}$$

where

$$D(v) = \frac{e^2}{m^2}\sum_k k^2|\Phi_k(t)|^2\left[P\,\frac{\gamma_k}{(kv-\omega_k)^2+\gamma_k^2} + \pi\delta(kv-\omega_k)\right].$$

It will be evident that this diffusion equation is to be suplemented by an equation for the wave amplitude:

$$\left.\begin{aligned}(\partial/\partial t)|\Phi_k|^2 &= 2\gamma_k|\Phi_k|^2,\\ \gamma_k &= (\pi/2)\omega_k(\omega_p/k)^2(\partial f_0/\partial v)(v = \omega/k).\end{aligned}\right\} \tag{2.26}$$

The two terms in the diffusion coefficient (specifically the term with the δ-function and the term containing the principal part of the integral) have very different physical origins. The term with the δ-function is positive definite and corresponds to the smoothing of the distribution function in the resonance region. This is an irreversible process. On the other hand, the term with the principal part describes a reversible process (i.e., the quantity $2\gamma_k|\Phi_k|^2 \equiv \partial|\Phi_k(t)|^2/\partial t$ changes sign when the sign of the time variable is changed). This "apparent" (or "adiabatic") diffusion describes the response of nonresonance particles to the change in wave amplitude. For example, when the wave amplitude increases the oscillatory kinetic energy of the particles associated with the wave also increases and the nonresonance particles appear to be heated (cf. § 2.6). It will be evident, however, that this "heating" does not lead to any entropy change.

The k = 0 Fourier component of the distribution function $f_0(v, t)$ plays a special role in this theory since it represents the zeroth approximation to the function $f(x, v, t)$. This result is reasonable physically since $f_0(v, t)$ is the average of $f(x, v, t)$ taken along the unperturbed particle trajectories and the particle motion actually leads to an averaging of $f(x, v, t)$ along these trajectories. When there is an external field and the unperturbed particle trajectories in the phase plane are more complicated, to obtain the zeroth approximation to $f(x, v, t)$ it is necessary to average along these complicated trajectories (in this case the average distribution function will obviously not be the same k = 0 Fourier component).

If the Doppler frequency shift of the particles is taken into account it can be shown that the condition on the width of the wave spectrum $\Delta(kv - \omega_k) \gg \gamma$, τ_R^{-1} is equivalent to the condition that

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$$f = \bar{f} + f'. \tag{2.27}$$

The rapidly varying part is described by the linearized equation

$$\frac{\partial f'}{\partial t} + v\frac{\partial f'}{\partial x} - \frac{e}{m}\frac{\partial \Phi}{\partial x}\cdot\frac{\partial \bar{f}}{\partial v} = 0, \tag{2.28}$$

where we neglect the product of two rapidly varying factors $(\partial\Phi/\partial x)\,(\partial f'/\partial v)$. Over one oscillation period the quantity $\bar{f}$ only varies insignificantly so that in finding $\bar{f}$ we can use the WKB approximation in time [cf. Eq. (2.22)]. In the equation for $\bar{f}$ we then carry out an average over the fast motion:

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Substituting for $\bar{f}$ and averaging the resulting expression over the unperturbed trajectories we can then reduce this equation to the form of Eq. (2.25).

Using the equations of the quasilinear approximation (2.25) and (2.26) we have considered the relaxation of the instability associated with a velocity distribution with a smeared-out beam (cf. Fig. 17a) [56-57]. We assume that the initial wave spectrum is a smooth function of ω/k (cf. Fig. 17b). Waves with phase velocities for which $\partial f/\partial v > 0$ should grow and after some time interval the amplitudes of the spectrum of these waves will become large enough to cause the formation of a plateau on the distribution function. After the formation of the plateau the wave growth is terminated (Fig. 21). We note that in the limit $t \to \infty$ the

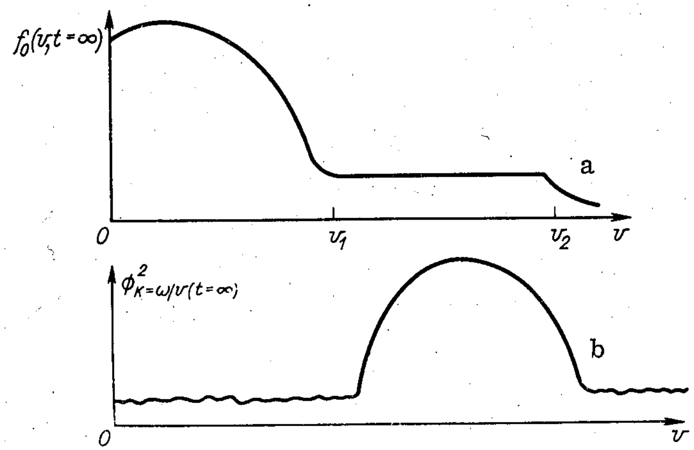

Fig. 21. Establishment of the particle distribution (a) and wave spectrum (b) for a beam with a weak bump.

nonresonant (smooth) part of the distribution function is somewhat displaced to the right in accordance with conservation of momentum. The momentum originally associated with the gentle bump has now been converted into ocillatory motion. In the resonance region the asymptotic distribution is determined uniquely by the conservation by the number of particles:

$$\left.\begin{aligned} &\int_{v_1}^{v_2} \bar{f}(v, t=0)\,dv = \bar{f}(t=\infty)(v_2 - v_1);\\ &\bar{f}(v_1, t=0) = \bar{f}(v_2, t=0) = \bar{f}(t=\infty). \end{aligned}\right\} \tag{2.30}$$

In order to find the asymptotic form of the spectrum we substitute Eq. (2.26) into Eq. (2.25) only keeping the term with the δ-function:

$$\frac{\partial}{\partial t}\left[\bar{f}(v, t) - \frac{\partial}{\partial v}\left(\frac{e}{m}\right)^2 \sum_k \frac{k^4}{\omega_p^3}\,\delta(kv - \omega_k)\,|\Phi_k|^2\right] = 0. \tag{2.31}$$

Assuming that the initial energy in the spectrum is small compared with the energy of the bump, as a result of integration we obtain the relation

$$\begin{aligned} &\frac{e^2}{m^2}\sum_k k^4\,|\Phi_k(t=\infty)|^2\,\delta(kv - \omega_k) = \\ &= \omega_p^3 \int_{v_1}^{v} dv\,\left[\bar{f}(t=\infty) - \bar{f}(v, t=0)\right]. \end{aligned} \tag{2.32}$$

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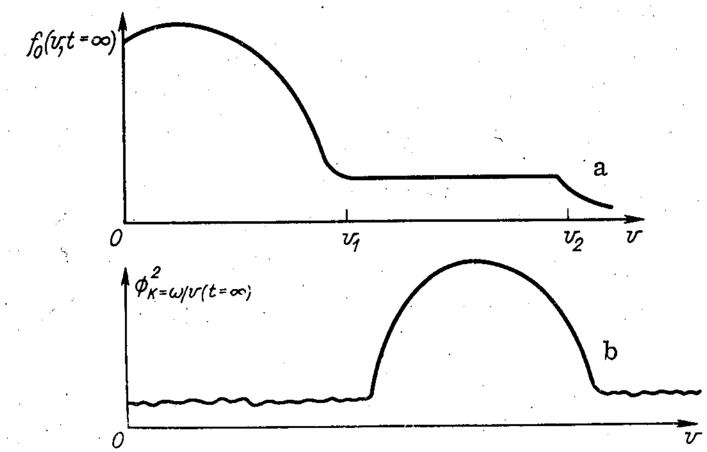

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It will be evident that this is not yet a steady-state spectrum: In the derivation of the quasilinear theory we have neglected the coupling between modes (that is to say, the nonlinear wave–wave and wave–particle interactions). After a sufficiently long time interval these effects will cause a modification of the spectrum. The time associated with the quasilinear relaxation and the time associated with the nonlinear mode interaction, in general, will be inversely proportional to the wave energy; hence it is necessary to have additional small parameters if one is to neglect mode–mode interactions in the quasilinear relaxation process.

PROBLEM

1. Find the self-similar solution of the quasilinear equations which describe the relaxation of a weak smeared-out beam of electrons moving with velocity much larger than the electron thermal velocity in a plasma [61].

We consider the establishment in time of the plateau on the distribution function and show that at any given time the distribution of electrons in the beam is in the form of a step with a sharp front moving toward lower velocities.

Introducing the dimensionless variables

$$F(V)=\frac{v_b}{n_b}f_c(v),\quad w=\frac{W_k\omega_p}{2\pi m n_b v_b^3},\quad V=\frac{v}{v_b},\quad \tau=\pi\omega_p\frac{n_b}{n_0}t, \tag{1}$$

where $W_k = k^2|\Phi_k|^2/4\pi$ is the spectral energy density of the waves per unit wave number k and $f_b(v)$ is the distribution function for the electrons in the beam normalized to the particle density in the beam, we write Eqs. (2.25) and (2.26) in the dimensionless form

$$\partial F/\partial\tau=(\partial/\partial V)(wV^2\,\partial F/\partial V), \tag{2}$$

$$\partial w/\partial\tau=wV^2\,\partial F/\partial V, \tag{3}$$

$$\int F dV=1. \tag{4}$$

From Eqs. (2) and (3) we obtain a relation between the distribution function and the noise [cf. Eq. (2.31)]:

$$F(V,\ \tau)-F_0(V)=(\partial/\partial V)[w(V,\tau)-w_0], \tag{5}$$

where F_0 and w_0 are the distribution function and the function w at

the initial time. By means of this relation we can eliminate F from Eq. (3) and obtain an equation which only contains w:

$$\frac{\partial w}{\partial\tau}=V^2 w\,\frac{\partial^2(w-w_0)}{\partial V^2}+Vw\,\frac{\partial F_0}{\partial V}. \tag{6}$$

In the region outside of the beam, where $\partial F_0/\partial V = 0$, this equation is reminiscent of the equation which describes the propagation of heat in space when the thermal conductivity is a power function of the heat content. As is well known, in this case the heat propagates in the form of a wave with a sharp front [24].

We seek the self-similar solution of Eq. (6) in the region where

$$\sqrt{T_b/mv_b^2}\ll 1-V\ll 1. \tag{7}$$

We now introduce the self-similar variable and a new function

$$\xi=1-V/\sqrt{\tau};\quad \varphi=w(V,\ \tau)-w_0(V). \tag{8}$$

Then in the region to the left of the beam Eqs. (5) and (6) assume the form

$$F=-\varphi'/\sqrt{\tau}; \tag{9}$$

$$-(\xi/2)\,\varphi'=(\varphi+w_0)\,\varphi''. \tag{10}$$

Integrating Eq. (9) with respect to V from a point in front of the leading edge of the jump where $\varphi = 0$ to a point close to the beam, where we can still neglect F_0, and using the fact that the number of particles in the jump is approximately equal to the total number of particles in the beam [when (7) is satisfied], we obtain the following boundary condition for φ:

$$\varphi(0)=1. \tag{11}$$

The boundary condition for $\xi\to\infty$ is obviously

$$\varphi(\infty)=0, \tag{12}$$

since the noise has still not increased in front of the step propagating to the left. In the general case the solution of Eq. (10) with

It will be evident that this is not yet a steady-state spectrum: In the derivation of the quasilinear theory we have neglected the coupling between modes (that is to say, the nonlinear wave–wave and wave–particle interactions). After a sufficiently long time interval these effects will cause a modification of the spectrum. The time associated with the quasilinear relaxation and the time associated with the nonlinear mode interaction, in general, will be inversely proportional to the wave energy; hence it is necessary to have additional small parameters if one is to neglect mode–mode interactions in the quasilinear relaxation process.

PROBLEM

1. Find the self-similar solution of the quasilinear equations which describe the relaxation of a weak smeared-out beam of electrons moving with velocity much larger than the electron thermal velocity in a plasma [61].

We consider the establishment in time of the plateau on the distribution function and show that at any given time the distribution of electrons in the beam is in the form of a step with a sharp front moving toward lower velocities.

Introducing the dimensionless variables

$$F(V)=\frac{v_b}{n_b}f_c(v),\quad w=\frac{W_k\omega_p}{2\pi m n_b v_b^3},\quad V=\frac{v}{v_b},\quad \tau=\pi\omega_p\frac{n_b}{n_0}t, \tag{1}$$

where $W_k = k^2|\Phi_k|^2/4\pi$ is the spectral energy density of the waves per unit wave number k and $f_b(v)$ is the distribution function for the electrons in the beam normalized to the particle density in the beam, we write Eqs. (2.25) and (2.26) in the dimensionless form

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In the region outside of the beam, where $\partial F_0/\partial V = 0$, this equation is reminiscent of the equation which describes the propagation of heat in space when the thermal conductivity is a power function of the heat content. As is well known, in this case the heat propagates in the form of a wave with a sharp front [24].

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since the noise has still not increased in front of the step propagating to the left. In the general case the solution of Eq. (10) with boundary conditions (11) and (12) can only be obtained numerically. However, in the case which arises in practice, in which $\ln \varphi/w_0 \gg 1$, an approximate solution can be found. For this purpose we rewrite Eq. (10) $d\varphi'/d\xi = -(1/2)\xi(d/d\xi)\ln(\varphi + w_0)$. Then, in front of the jump ($\xi - \xi_0 \ll \xi_0$), where there is a sharp change in φ, we obtain the following solution of this equation:

$$\varphi' = -(1/2)\xi_0 \ln(1 + \varphi/w_0). \tag{13}$$

It will be evident that the noise level decays in the region in front of the jump, this decay being exponential $\varphi \sim \exp[(-\xi_0/2w_0)\xi]$. Equation (13) is also valid for $\xi - \xi_0 \sim \xi_0$ but only with logarithmic accuracy. Integrating this equation, taking account of the boundary condition in (12), we find

$$\varphi = (1/2)\xi_0(\xi_0 - \xi)\ln(1/w_0). \tag{14}$$

The constant ξ_0 can be found from the second boundary condition $\varphi(0) = 1$

$$\xi_0 = \sqrt{2}/\sqrt{\ln(1/w_0)}. \tag{15}$$

It is then a simple matter to estimate the characteristic time in which quasilinear relaxation of the beam occurs in the velocity interval Δv:

$$t = \frac{1}{2\gamma}\ln\frac{1}{w_0}; \quad \gamma = \pi\omega_p\frac{n_b}{n_0}\left(\frac{v_b}{\Delta v}\right)^2. \tag{16}$$

§ 2.3. The Many-Wave Case (Two- and Three-Dimensional Spectra)

Up to this point we have only considered the quasilinear theory for the one-dimensional case. In the two- and three-dimensional cases the quasilinear diffusion equation is very similar to its one-dimensional analog:

$$\left.\begin{aligned}
&\frac{\partial \bar f}{\partial t} = \sum_{\alpha,\beta}\frac{\partial}{\partial v_\alpha}D_{\alpha\beta}\frac{\partial \bar f}{\partial v_\beta};\\
&D_{\alpha\beta} = \frac{e^2}{m^2}\sum_{\mathbf{k}} k_\alpha k_\beta|\Phi_{\mathbf{k}}|^2 \times\\
&\times\left[P\frac{\gamma_{\mathbf{k}}}{(\mathbf{k}\cdot\mathbf{v} - \omega_{\mathbf{k}})^2 + \gamma_{\mathbf{k}}^2} + \pi\delta(\mathbf{k}\cdot\mathbf{v} - \omega_{\mathbf{k}})\right].
\end{aligned}\right\} \tag{2.33}$$

On the other hand, the solution is much more complicated. In a certain sense the one-dimensional case can be regarded as a degenerate case since the resonance particles are bounded within a fixed range of the variable v.

In the two- and three-dimensional cases, even when the wave packet is localized in k-space the resonance region is expanded considerably.

Let us consider the two-dimensional case. The curves corresponding to equal levels of f will be assumed to be circles with center at the origin in the (v_x, v_y) plane as shown in Fig. 22. It is assumed that a narrow wave packet propagates in the x-direction. As a result of the formation of the quasilinear plateau within this narrow resonance region there arises a new system of level curves (parallel to v_x). These must obviously be connected to the circles in the velocity region outside of the resonance band. It is evident that a finite amount of energy is required in order to construct the distributions in this way, as shown in Fig. 22.

Now let us assume that there are wave packets that propagate at different angles. In Fig. 22 each packet must correspond to its own system of level curves. Since the level curves for different packets intersect, in order to make the distribution function constant along all of these level curves it is necessary that f be constant in some region of space which extends to infinity. It follows

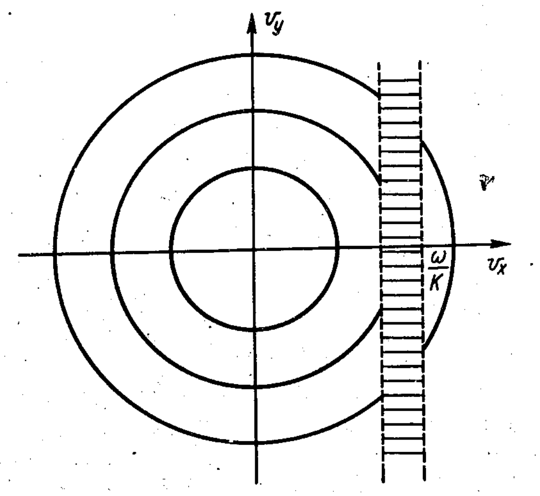

Fig. 22. Initial and final level curves for the particle distribution in a one-dimensional wave packet.

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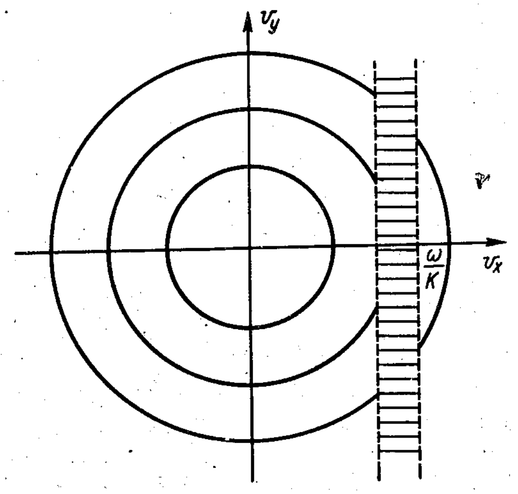

Fig. 22. Initial and final level curves for the particle distribution in a one-dimensional wave packet.

that this modification of the distribution function would require an infinite amount of energy. Thus, since the quasilinear plateau in the two-dimensional case is not a steady-state, even with narrow wave packets, we can really only think of it in the sense of a "quasi-plateau."

Let us assume that the wave packets propagate in all directions with phase velocity ω/k. The resonance region in the (v_x, v_y) plane will lie outside a circle of radius (ω/k) since any portion of this region corresponds to at least two different resonance bands. In order for the level curves to satisfy all of the intersecting resonances with the wave packets, the distribution function outside of the circle $v_x^2 + v_y^2 = (\omega/k)^2$ must be a constant. But the energy of this distribution is infinite. It then follows that the steady-state solution corresponding to a wave packet of finite energy is not physically meaningful and that the wave spectrum must ultimately vanish [III].

In order to see what actually happens to a distribution function we now consider the simple case of a two-dimensional wave packet which exhibits cylindrical symmetry in space. Then, f is isotropic: $f = f(v_x^2 + v_y^2)$. Substituting this form of the distribution function in Eq. (2.27), we have

$$\frac{\partial \bar{f}}{\partial t} = \frac{e^2}{m^2} \cdot |\Phi|^2 (\omega^2/k_0) \frac{\partial}{v \partial v} \cdot \frac{1}{\sqrt{v^2 - (\omega/k_0)^2}} \cdot \frac{\partial \bar{f}}{v \partial v}, \qquad (2.34)$$

where the summation over k has been replaced by an integration and we have limited ourselves to the case of a narrow spectrum $|\Phi_k|^2 = 2\pi |\Phi|^2 k^{-1} \delta(k - k_0)$. The quasilinear theory is still valid since the distribution function is smeared out over velocity because of the spread in angle within the wave packet.

When $\omega/k > v$, we have $\partial f/\partial t = 0$. Equation (2.34) together with Eq. (2.26) allow an exact solution if we introduce an additional simplification. If the initial energy of the wave is large enough, as a result of diffusion in the region of high velocities ultimately a state is reached in which the inequality $v \gg \omega/k$ will be satisfied over most of velocity space. With this simplification Eq. (2.34) can now be solved. An example of this case, in which ω/k can be neglected compared with v, is the interaction between electrons and ion-acoustic waves, since $\omega/k = \sqrt{T_e/M} \ll \sqrt{T_e/m} \sim v$. In the case of plasma waves this relation holds only after a sufficient time has passed, in which case the distribution function has spread out toward large values of v.

We introduce the variable

$$\tau = (25 e^2/m^2) \int_0^t (\omega^2/k) |\Phi|^2 dt' \equiv \int_0^t D(t')\, dt',$$

in which case Eq. (2.28) assumes the form

$$\frac{\partial \bar{f}}{\partial \tau} = \frac{4}{25} \cdot \frac{\partial}{\partial v^2} \left(\frac{1}{v} \cdot \frac{\partial}{\partial v^2} f \right). \qquad (2.35)$$

To solve Eq. (2.35) with a specified initial distribution $\bar{f}$ we make use of the Laplace transform and express the solution by means of a Green's function in terms of modified Bessel functions. This solution is not very convenient but can be simplified by considering asymptotic behavior as $t \to \infty$. At long times $\bar{f}$ is independent of the structure of the function $\bar{f}_0$ and exhibits the self-similar form [62]:

$$\bar{f} = A \left[\int_0^t D(t')\, dt \right]^{-2/5} \exp \left[-v^5 \Big/ \int_0^t D(t')\, dt' \right], \qquad (2.36)$$

where

$$A \approx (5/\Gamma(2/5)) \int_{\omega/k}^{\infty} f_0(v)\, v\, dv; \quad v > \omega/k.$$

In order to find γ we now use the relation

$$\gamma = \frac{\pi}{2} \omega_k \frac{\omega_p^2}{k^2} \int \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}} \delta(\omega_k - \mathbf{k} \cdot \mathbf{v})\, d^2 \mathbf{v}.$$

Substituting the asymptotic solution (2.63) we then find

$$\gamma_k = -\frac{\beta_k}{\left[\int_0^t D(t')\, dt' \right]^{3/5}}; \quad \beta_k = A \frac{\omega_k^3 \omega_p^2 \Gamma(2/5)}{10 k^3}. \qquad (2.37)$$

It will be noted that ω/k does not appear in (2.37) for γ_k since the basic contribution to γ_k comes from velocities $v \gg \omega/k$.

Thus, the original system of equations is reduced to a single equation (2.37), which can be transformed to a second-order differential equation. The solution of this equation is extrememly complicated. However, the qualitative behavior of the solution is that this modification of the distribution function would require an infinite amount of energy. Thus, since the quasilinear plateau in the two-dimensional case is not a steady-state, even with narrow wave packets, we can really only think of it in the sense of a "quasi-plateau."

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The picture that has been given here can be applied to the theory of what is known as turbulent heating. Let us consider a system in which turbulent heating is being carried out. This means that the current that flows in the plasma excites some instability as a consequence of the relative motion between the electrons and ions. If the drift velocity of the relative motion exceeds $(T_e/M)^{1/2}$, then ion-acoustic waves can be excited [VI]. As noted above, the quantity ω/k does not appear in the final result; thus, even without knowing the spectrum of waves which are excited as a result of the instability it is clear that after heating, the electron distribution will not be a Maxwellian; rather its form will be determined by Eq. (2.36). It is possible that the form of the distribution at very high velocities will differ from the quasilinear result in (2.36) as a result of effects which have not been taken into account, e.g., wave–wave interactions, etc.

It should be noted that the wave spectrum has been assumed to be isotropic in the calculations given above. In ion-acoustic turbulent heating this situation does not hold since there is a preferred direction (in the direction of the current). Let us assume that the current flow is perpendicular to the magnetic field (a situation frequently encountered in practice): j is in the x-direction and H is in the z-direction. At small but finite values of the magnetic field the situation is equivalent to the case of the isotropic spectrum. As a result of gyration of particles in the magnetic field a smearing occurs in the (v_x, v_y) plane. This situation can be regarded as a rotation of the wave packet rather than a gyration of the particles, and the electron distribution function will only depend on $v_x^2 + v_y^2$ even for a one-dimensional wave packet.

We thus reach the conclusion that there is an important difference when a magnetic field is applied, even in the one-dimensional quasilinear theory. Obviously, H cannot be too large because otherwise the simple pattern of the plasma dynamics which has been described above will be modified substantially. In the present case the magnetic field only causes a mixing of particles in the (v_x, v_y) plane. But the effect of H can be neglected if we consider the properties of longitudinal electron oscillations if $\omega_p^2 \gg \omega_H^2$, as is usually the case in turbulent heating.

PROBLEMS

1. Show that to first-order in terms of an expansion in the parameter $(\omega/kv) \ll 1$ the interaction of ion-acoustic waves with particles leads to the formation of an isotropic particle distribution [63].

We first transform the quasilinear equation (2.33) to spherical coordinates with axis along the external electric field **E** which drives the current. Let (k, θ', φ') be the spherical coordinates of the wave vector and let (v, θ, φ) denote the particle velocity. Integration with respect to the difference in the azimuthal angles $(\varphi - \varphi')$ which appear in the argument of the δ-function in the expression for the diffusion coefficient leads to the explicit form

$$\int_0^{2\pi} \pi\delta(\mathbf{k}\cdot\mathbf{v})\,d\varphi' = \frac{2\pi}{kv}\left[1-\cos^2\theta-\cos^2\theta'\right]^{-1/2}$$

The remaining angle variables appear only as $\xi = \cos\theta$, and $x = \cos\theta'$ so that the quasilinear equation becomes

$$\begin{aligned}\frac{\partial}{\partial t} f_e + \frac{e}{m} E \frac{\partial}{v\partial\xi} f_e + \frac{e}{m} E\xi \frac{\partial}{\partial v} f_e = \frac{\partial}{\partial\xi}(1-\xi^2) D_{\xi\xi}\frac{\partial f_e}{\partial\xi} + \\ + \frac{\partial}{\partial\xi}\sqrt{1-\xi^2}\, D_{v\xi}\, v\frac{\partial f_e}{\partial v} + \frac{\partial}{v^2\partial v} v^3\sqrt{1-\xi^2}\, D_{v\xi}\frac{\partial f_e}{\partial\xi} + \\ + \frac{\partial}{v^2\partial v} v^4 D_{vv}\frac{\partial f_e}{\partial v},\end{aligned} \tag{1}$$

where

$$D_{\alpha\beta} = \frac{M}{4\pi^2 mN_0 v^3}\int_0^\infty dk k^3 \int_{-1}^{+1} d\eta \frac{W(k, \eta\sqrt{1-\xi^2})}{\sqrt{1-\eta^2}} A_{\alpha\beta}\left(\frac{s_k}{v}, \eta\right);$$

$$A_{\xi\xi} = \eta^2; \quad A_{\xi v} = A_{v\xi} = \eta\,(s_k/v); \quad A_{vv} = (s_k/v)^2;$$

$$W(k, x) = \omega\frac{\partial\varepsilon}{\partial\omega}\frac{k^2|\Phi_k|^2}{8\pi} \approx \frac{\Omega_p^2}{s_k^3}\frac{|\Phi_k|^2}{4\pi}; \quad s_k \equiv \frac{\omega(k)}{k}.$$

apparent. The wave energy is damped and ultimately vanishes when $t \to \infty$. The damping coefficient at the outset is nothing more than the Landau coefficient; it then undergoes significant changes (being reduced) as a consequence of the change in the slope in the distribution function. When $t \to \infty$, $\gamma_k \to$ const because ultimately the energy required for further modification of f is exhausted. This situation is clearly different from the situation in the one-dimensional case, in which a plateau is formed.

The picture that has been given here can be applied to the theory of what is known as turbulent heating. Let us consider a system in which turbulent heating is being carried out. This means that the current that flows in the plasma excites some instability as a consequence of the relative motion between the electrons and ions. If the drift velocity of the relative motion exceeds $(T_e/M)^{1/2}$, then ion-acoustic waves can be excited [VI]. As noted above, the quantity ω/k does not appear in the final result; thus, even without knowing the spectrum of waves which are excited as a result of the instability it is clear that after heating, the electron distribution will not be a Maxwellian; rather its form will be determined by Eq. (2.36). It is possible that the form of the distribution at very high velocities will differ from the quasilinear result in (2.36) as a result of effects which have not been taken into account, e.g., wave–wave interactions, etc.

It should be noted that the wave spectrum has been assumed to be isotropic in the calculations given above. In ion-acoustic turbulent heating this situation does not hold since there is a preferred direction (in the direction of the current). Let us assume that the current flow is perpendicular to the magnetic field (a situation frequently encountered in practice): j is in the x-direction and H is in the z-direction. At small but finite values of the magnetic field the situation is equivalent to the case of the isotropic spectrum. As a result of gyration of particles in the magnetic field a smearing occurs in the (v_x, v_y) plane. This situation can be regarded as a rotation of the wave packet rather than a gyration of the particles, and the electron distribution function will only depend on $v_x^2 + v_y^2$ even for a one-dimensional wave packet.

We thus reach the conclusion that there is an important difference when a magnetic field is applied, even in the one-dimensional quasilinear theory. Obviously, H cannot be too large because otherwise the simple pattern of the plasma dynamics which has been described above will be modified substantially. In the present case the magnetic field only causes a mixing of particles in the (v_x, v_y) plane. But the effect of H can be neglected if we consider the properties of longitudinal electron oscillations if $\omega_p^2 \gg \omega_H^2$, as is usually the case in turbulent heating.

PROBLEMS

1. Show that to first-order in terms of an expansion in the parameter $(\omega/kv) \ll 1$ the interaction of ion-acoustic waves with particles leads to the formation of an isotropic particle distribution [63].

We first transform the quasilinear equation (2.33) to spherical coordinates with axis along the external electric field **E** which drives the current. Let (k, θ', φ') be the spherical coordinates of the wave vector and let (v, θ, φ) denote the particle velocity. Integration with respect to the difference in the azimuthal angles $(\varphi - \varphi')$ which appear in the argument of the δ-function in the expression for the diffusion coefficient leads to the explicit form

$$\int_0^{2\pi} \pi\delta(\mathbf{k}\cdot\mathbf{v})\, d\varphi' = \frac{2\pi}{kv}\left[1-\cos^2\theta-\cos^2\theta'\right]^{-1/2}$$

The remaining angle variables appear only as $\xi = \cos\theta$, and $x = \cos\theta'$ so that the quasilinear equation becomes

$$\frac{\partial}{\partial t} f_e + \frac{e}{m} E \frac{\partial}{v\partial\xi} f_e + \frac{e}{m} E\xi \frac{\partial}{\partial v} f_e = \frac{\partial}{\partial\xi}(1-\xi^2) D_{\xi\xi}\frac{\partial f_e}{\partial\xi} + \frac{\partial}{\partial\xi}\sqrt{1-\xi^2}\, D_{v\xi}\, v\frac{\partial f_e}{\partial v} + \frac{\partial}{v^2\partial v} v^3\sqrt{1-\xi^2}\, D_{v\xi}\frac{\partial f_e}{\partial\xi} + \frac{\partial}{v^2\partial v} v^4 D_{vv}\frac{\partial f_e}{\partial v}, \tag{1}$$

where

$$D_{\alpha\beta} = \frac{M}{4\pi^2 m N_0 v^3}\int_0^\infty dk k^3 \int_{-1}^{+1} d\eta \frac{W(k, \eta\sqrt{1-\xi^2})}{\sqrt{1-\eta^2}} A_{\alpha\beta}\left(\frac{s_k}{v}, \eta\right);$$

$$A_{\xi\xi} = \eta^2; \quad A_{\xi v} = A_{v\xi} = \eta\,(s_k/v); \quad A_{vv} = (s_k/v)^2;$$

$$W(k, x) = \omega\frac{\partial\varepsilon}{\partial\omega}\frac{k^2|\Phi_k|^2}{8\pi} \approx \frac{\Omega_p^2}{s_k^3}\frac{|\Phi_k|^2}{4\pi}; \quad s_k \equiv \frac{\omega(k)}{k}.$$

In computing the diffusion coefficients we neglect terms of order $(s/v) \ll 1$. It will be evident that the basic effect here is the quasilinear diffusion in angle, which leads to an isotropic distribution. Hence, the solution of Eq. (1) can be written in the form

$$f(v,\ \xi,\ t)=f_0(v,\ t)+f_1(v,\ \xi,\ t). \tag{2}$$

Substituting the solution in this form in Eq. (1) and only retaining first-order terms we have

$$\frac{\partial f_1}{\partial \xi}=-\frac{eE\sqrt{1-\xi^2}/m+vD_{v\xi}}{D_{\xi\xi}\sqrt{1-\xi^2}}\frac{\partial f_0}{\partial v}. \tag{3}$$

In the second-order expansion in the small parameter $(s/v) \ll 1$, using Eq. (1) we obtain a quasilinear equation for $f_0(v, t)$ which describes the particle heating:

$$\frac{\partial f_0}{\partial t}=\frac{1}{v^2}\frac{\partial}{\partial v}v^4\int_{-1}^{+1}d\xi\left[D_{vv}-\frac{D_{v\xi}^2}{D_{\xi\xi}}+\frac{e^2E^2(1-\xi^2)}{m^2D_{\xi\xi}}\right]\frac{\partial f_0}{\partial v}. \tag{4}$$

In this approximation the expression for the growth rate is

$$\gamma(k,\ x)=\frac{2\pi M}{m}ks_k^3\left\{\int_0^\infty f_0\,dv-x\int_0^\infty dv\int_{-1}^{+1}\frac{d\mu}{\sqrt{1-\mu^2}}\cdot\frac{\partial f_{1e}}{\partial \xi}\bigg|_{\xi=\mu\sqrt{1-x^2}}\right\}. \tag{5}$$

In the limit $t \to \infty$ the solution of this equation is of the same nature as in the case of an isotropic wave spectrum $[\sim \exp(-\alpha v^5)]$.

▶ 2. Derive the relativistic quasilinear equation which describes the relaxation of a relativistic electron beam injected into a plasma [64, 65].

The quasilinear equation for the electron distribution function f_e and the equation for the spectral density of the energy of the plasma waves $W_\mathbf{k}$ in the relativistic case are (the beam moves along the z axis):

$$\frac{3v_T^2k_z}{\omega_p}\cdot\frac{\partial W_\mathbf{k}}{\partial z}-\frac{\partial \omega_p}{\partial z}\cdot\frac{\partial W_\mathbf{k}}{\partial k_z}=2\gamma_\mathbf{k}W_\mathbf{k}, \tag{1}$$

$$c\,\frac{p_z}{p}\cdot\frac{\partial f}{\partial z}=\frac{\partial}{\partial p_\alpha}D_{\alpha\beta}\frac{\partial f}{\partial p_\beta}, \tag{2}$$

where

$$D_{\alpha\beta}=\frac{m\omega_p^2}{N_0}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\cdot\frac{k_\alpha k_\beta}{k^2}W_\mathbf{k},$$

$$\gamma_\mathbf{k}=\frac{m\omega_p^2}{N_0}\int d^3p\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{p}}\pi\delta(\omega_p-\mathbf{k}\cdot\mathbf{p}c/p).$$

As in the previous problem, it is convenient to use spherical coordinates (p, Θ, φ) in momentum space and (k, Θ', φ') in wave-vector space (the angles Θ and Θ' are computed from the z axis). The expressions for the diffusion coefficients in spherical coordinates are rather complicated because now the phase velocity (ω_p/k) is comparable with the particle velocity. As before, the integration with respect to φ' is carried out in explicit form

$$\pi\int_0^{2\pi}d\varphi'\,\delta(\omega_p-kc[\sin\Theta\sin\Theta'\cos(\varphi'-\varphi)+\cos\Theta\cos\Theta'])=$$
$$=2\pi/kc\,[(\cos\Theta_1'-\cos\Theta')(\cos\Theta'-\cos\Theta_2')]^{1/2}, \tag{3}$$

where

$$\cos\Theta_{1,2}'=(\omega_p/kc)\left\{\cos\Theta\pm\sin\Theta\sqrt{k^2c^2/\omega_p^2-1}\right\}.$$

When account is taken of the resonance condition in spherical coordinates the differentiation operator becomes

$$\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}}\equiv\frac{\omega_p}{c}\left\{\frac{\partial}{p\partial p}+\zeta\frac{\partial}{p\partial\Theta}\right\},\quad \zeta\equiv\frac{\cos\Theta-kc\cos\Theta'/\omega_p}{\sin\Theta}. \tag{4}$$

Using Eqs. (3) and (4) we now write Eqs. (1) and (2) in the form

$$\frac{3kv_T^2}{\omega_p}\cos\Theta'\frac{\partial W}{\partial z}-\frac{\partial\omega_p}{\partial z}\left(\cos\Theta'\frac{\partial W}{\partial k}-\frac{\sin\Theta'}{k}\cdot\frac{\partial W}{\partial\Theta'}\right)=2\gamma W, \tag{5}$$

$$c\,\frac{p_z}{p}\cdot\frac{\partial f}{\partial z}=\frac{1}{p^2}\frac{\partial}{\partial p}p^2\left(D_{pp}\frac{\partial f}{\partial p}+\frac{1}{p}D_{p\Theta}\frac{\partial f}{\partial\Theta}\right)+$$
$$+\frac{1}{p\sin\Theta}\cdot\frac{\partial}{\partial\Theta}\sin\Theta\left(D_{p\Theta}\frac{\partial f}{\partial p}+\frac{1}{p}D_{\Theta\Theta}\frac{\partial f}{\partial\Theta}\right), \tag{6}$$

where

$$\left.\begin{matrix}D_{pp}\\D_{p\Theta}\\D_{\Theta\Theta}\end{matrix}\right\}=\frac{m\omega_p^4}{(2\pi)^3N_0c^3}\int_{\omega_p/c}^{\infty}\frac{dk}{k}\int_{\Theta_1'}^{\Theta_2'}\frac{\sin\Theta'\,W(k,\Theta')\,d\Theta'}{\sqrt{(\cos\Theta_1'-\cos\Theta')(\cos\Theta'-\cos\Theta_2')}}\left\{\begin{matrix}1\\\zeta\\\zeta^2\end{matrix}\right\}$$

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$$\left.\begin{matrix}D_{pp}\\ D_{p\Theta}\\ D_{\Theta\Theta}\end{matrix}\right\}=\frac{m\omega_p^4}{(2\pi)^3N_0c^3}\int_{\omega_p/c}^{\infty}\frac{dk}{k}\int_{\Theta_1'}^{\Theta_2'}\frac{\sin\Theta' W(k,\Theta')\,d\Theta'}{\sqrt{(\cos\Theta_1'-\cos\Theta')(\cos\Theta'-\cos\Theta_2')}}\begin{Bmatrix}1\\ \zeta\\ \zeta^2\end{Bmatrix}$$

$$\gamma(k,\ \Theta') =$$

$$= -\frac{m\omega_p^4}{4\pi N_0 k^3 c^2}\int\limits_{\Theta_1}^{\Theta_2}\frac{\sin\Theta d\Theta}{\sqrt{(\cos\Theta_1-\cos\Theta)(\cos\Theta-\cos\Theta_2)}}\left(g-\frac{1}{2}\zeta\cdot\frac{\partial g}{\partial\Theta}\right);$$

$$\cos\Theta_{1,2}=\frac{\omega_p}{kc}\left(\cos\Theta'\pm\sin\Theta'\sqrt{\frac{k^2c^2}{\omega_p^2}-1}\right);\qquad g\equiv 2\pi\int\limits_0^\infty f p dp\,.$$

§ 2.4. Effect of Collisions on Wave – Particle Interactions

One of the most important consequences of the wave–particle interaction is the deformation of the particle velocity distribution function, especially in the resonance region $v = \omega/k$. In some cases the distortion of the distribution function is so large that it becomes necessary to take account of collisions. In particular, the collision integral St$\{f\}$ in the Landau form contains a term with the second derivative of the distribution function. It is specifically this term which becomes important if the wave–particle interaction distorts the distribution function in a narrow resonance region. Collisions tend to modify the slope of the distribution function df/dv, moving it toward the equilibrium value. As a result, a competition arises between the effect of the waves and the effect of collisions. In order to illustrate this process using a quantitative example, we now consider the problem of a one-dimensional wave packet, treating the quasilinear diffusion coefficient in the resonance region:

$$D(v) = (\pi e^2/m^2)\sum_{\mathbf{k}}|\mathbf{E}_{\mathbf{k}}|^2\delta(\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v}) \approx e^2\langle E^2\rangle/m^2\omega.$$

Because of the competition between the quasilinear effect of the waves on the particles and the collisions in the resonance region we find there is a quasistationary distribution ($df/dt = 0$) which is governed by the equation

$$(dD(v)/dv)(df/dv) = \mathrm{St}\{f\}. \tag{2.38}$$

The expression for St$\{f\}$ will only contain the term with the second derivative d^2f/dv^2. For simplicity we can write this in the following symbolic form: $\nu(\omega/k)^2(d^2/dv^2)(f_M - f)$, where ν is the mean collision frequency for electrons for velocity $v = \omega/k$. This simplified form of the collision integral gives a proper description of the relaxation to local equilibrium (with $f = f_M$, where f_M is a Maxwellian distribution). Integrating Eq. (2.38) once we have

$$\frac{df}{dv}=\frac{df_M}{dv}\cdot\frac{1}{1+e^2\langle E^2\rangle/m^2\omega\nu(\omega/k)^2}.$$

The slope of the distribution function given by this expression is substituted in the damping relation $\gamma = (\pi/2)(\omega^3/k^2)(df/dv)$ ($v = \omega/k$). This procedure gives the damping coefficient

$$\gamma = \gamma_L/[1 + e^2\langle E^2\rangle/m^2\omega\nu(\omega/k)^2]. \tag{2.39}$$

We now investigate this expression in various limiting cases. If the amplitudes of the waves in the packet are small enough $e^2\langle E^2\rangle \ll m^2\omega\nu(\omega/k)^2$, the damping coefficient approaches γ_L, the usual Landau damping rate. This follows from the fact that the collisions are capable of equalizing the distribution function in the resonance region, thereby maintaining the Maxwellian slope df_M/dv. At large amplitudes $e^2\langle E^2\rangle \gg m^2\omega\nu(\omega/k)^2$, the damping coefficient becomes essentially nonlinear and, as follows from Eq. (2.39), falls off with amplitude in accordance with the relation $\sim 1/\langle E^2\rangle$.

Equation (2.39) can be interpreted as follows. We first write the damping rate in the form

$$\gamma = \gamma_L/(1 + \tau_1/\tau_2), \tag{2.40}$$

where τ_1 is the characteristic time required for the establishment of a local Maxwellian distribution and τ_2 is the characteristic time for the distortion of the distribution function due to the effect of the wave packet. If $\tau_1 \ll \tau_2$, that is to say, if collisions dominate, the usual Landau damping is obtained. As the amplitude of the wave increases the distorting effect it has on the distribution function becomes so large that particle collisions are not able to cause the distribution function to become a Maxwellian, in which case the damping rate is reduced.

A more complicated problem is that of taking account of the effect of collsions on the interaction with particles in a monochromatic wave of finite amplitude.

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A more complicated problem is that of taking account of the effect of collsions on the interaction with particles in a monochromatic wave of finite amplitude.

In contrast with Eq. (2.33), which is an ordinary differential equation for the stationary case d/dt = 0, in the interaction with a monochromatic wave $\Phi = \Phi_0 \cos kx$ the kinetic equation becomes a partial differential equation $[f = f(x, v)]$. However, with a monochromatic wave the damping rate can be obtained from qualitative considerations by analogy with the case of the wave packet. In [2] it has been found that a relation like (2.40) also applies for a monochromatic wave for reasonable choices of τ_1 and τ_2. With a monochromatic wave the width of the resonance region in which the distribution function is distorted significantly is of the order of $\Delta v \approx (e\Phi/m)^{1/2}$. Small-angle Coulomb collisions restore the local equilibrium in this region in a time of order $\tau_1 = \nu^{-1} e\Phi/m(\omega/k)^2$. On the other hand, the time associated with the nonlinear distortion of the distribution function due to the wave field (cf. § 1.1) is of order $\tau_2 = \lambda/(e\Phi/m)^{1/2}$, where λ is the wavelength. Thus, from Eq. (2.40) we have

$$\gamma = \frac{\gamma_L}{1 + (e\Phi)^{3/2}/\nu\lambda m^{3/2}(\omega/k)^2}. \qquad (2.41)$$

At low amplitudes $\tau_1 \ll \tau_2$ and the damping rate approaches the linear value; at higher amplitudes, with $\tau_1 \gg \tau_2$, the damping rate falls off as $\Phi^{-3/2}$. This qualitative interpretation has been verified by direct investigation of the equation for the distribution function in the field of a monochromatic wave, with collisions included [66].

In some sense we are considering here the modification of the distribution function in a steady-state Bernstein–Greene–Kruskal wave. The damping of the wave can be found from the relation $\gamma = (1/2W)(dW/dt) = -\langle \mathbf{j}\mathbf{E}\rangle/2W$, where $\mathbf{j}$ is the current caused by the wave (in a reference system fixed in the wave); W is the energy density in the wave. The averaging is carried out over the wave period.

Thus, the problem has been reduced to finding the electron distribution function. In the reference system that moves with the wave, the kinetic equation for this function is

$$u\, \partial f/\partial y - \varphi'(y)\partial f/\partial u = \nu'(\partial/\partial u)[\partial f/\partial u + (\alpha + u)f]. \qquad (2.42)$$

We have introduced the following dimensionless variables:

$$\varphi(y) = e\Phi/T,\ y = kx,\ \Phi(x) = \Phi_0 \sin^2 kx/2;\ v_T = \sqrt{2T/m},$$
$$u = v\sqrt{2}/v_T,\ \alpha = \omega\sqrt{2}/kv_T,\ \nu' = 3/\sqrt{2}kv_T\tau_D,$$
$$\tau_D = m^2\omega^3/8\pi e^4 N_0 k^3 \,\mathrm{Ln},$$

while the collision integral is taken in the Fokker–Planck form (Ln is the Coulomb logarithm).

In order to simplify the problem we introduce the following ordering: $\nu' \ll \varphi_0 \ll 1$; this corresponds to a wave of finite but reasonably small amplitude and a low collision rate. We then introduce a new variable, the energy $\varepsilon = u^2/2 + \varphi(y)$, where ε is the dimensionless energy of a particle in the wave field. Under these conditions Eq. (2.42) assumes the form

$$\frac{\partial f}{\partial y} = \nu \frac{\partial}{\partial \varepsilon}\left\{\sigma[\varepsilon - \varphi(y)]^{1/2}\left(f + \frac{\partial f}{\partial \varepsilon}\right) + V_f f\right\}, \qquad (2.43)$$
$$V_f = \alpha/\sqrt{2} = \omega/kv_T, \qquad \nu = \nu'\sqrt{2} = 3/kv_T\tau_D,$$

where $\sigma = \pm 1$ corresponds to the two directions of motion of the electrons (the "plus" sign applies for particles which overtake wave). The quantity ν plays the role of a dimensionless parameter. We therefore seek a solution in the form

$$f(\varepsilon, y) = f_0(\varepsilon) + f_1(\varepsilon, y) + \ldots. \qquad (2.44)$$

In finding a solution of Eq. (2.43) it is necessary to consider two regions: $\varepsilon > \varphi_0$ (the outer region) and $\varepsilon < \varphi_0$ (the inner region). Substituting the expansion in (2.44) in Eq. (2.43) we have

$$\frac{\partial f_1}{\partial y} = \nu \frac{\partial}{\partial \varepsilon}\left\{\sigma[\varepsilon - \varphi(y)]^{1/2}\left(f_0 + \frac{\partial f_0}{\partial \varepsilon}\right) + V_f f_0\right\}; \qquad (2.45)$$

where $f_0(\varepsilon)$ (the zeroth approximation) for the outer region is determined from the condition of periodicity of f_1 as a function of the y-coordinate if it is assumed that when $\varepsilon \gg \varphi_0$ the function $f_0(\varepsilon)$ approaches a Maxwellian distribution asymptotically. Integrating Eq. (2.45) with respect to y, taking account of the periodicity, we obtain an equation for f_0 when $\varepsilon > \varphi_0$:

$$\frac{d}{d\varepsilon}\left\{\sqrt{\varepsilon}\, E\left(\sqrt{\frac{\varphi_0}{\varepsilon}}\right)\left[f_0 + \frac{df_0}{d\varepsilon}\right] + \frac{\pi\sigma}{2} V_f f_0\right\} = 0, \qquad (2.46)$$

where $E(\varkappa) = \int_0^{\pi/2} \sqrt{1 - \varkappa^2 \sin^2 t}\, dt$ is a complete elliptic integral of the second kind. We now write the solution of this equation, going over to a Maxwellian distribution for values far from the resonance

In contrast with Eq. (2.33), which is an ordinary differential equation for the stationary case d/dt = 0, in the interaction with a monochromatic wave $\Phi = \Phi_0 \cos kx$ the kinetic equation becomes a partial differential equation $[f = f(x, v)]$. However, with a monochromatic wave the damping rate can be obtained from qualitative considerations by analogy with the case of the wave packet. In [2] it has been found that a relation like (2.40) also applies for a monochromatic wave for reasonable choices of τ_1 and τ_2. With a monochromatic wave the width of the resonance region in which the distribution function is distorted significantly is of the order of $\Delta v \approx (e\Phi/m)^{1/2}$. Small-angle Coulomb collisions restore the local equilibrium in this region in a time of order $\tau_1 = \nu^{-1} e\Phi/m(\omega/k)^2$. On the other hand, the time associated with the nonlinear distortion of the distribution function due to the wave field (cf. § 1.1) is of order $\tau_2 = \lambda/(e\Phi/m)^{1/2}$, where λ is the wavelength. Thus, from Eq. (2.40) we have

$$\gamma = \frac{\gamma_L}{1 + (e\Phi)^{3/2}/\nu\lambda m^{3/2}(\omega/k)^2}. \tag{2.41}$$

At low amplitudes $\tau_1 \ll \tau_2$ and the damping rate approaches the linear value; at higher amplitudes, with $\tau_1 \gg \tau_2$, the damping rate falls off as $\Phi^{-3/2}$. This qualitative interpretation has been verified by direct investigation of the equation for the distribution function in the field of a monochromatic wave, with collisions included [66].

In some sense we are considering here the modification of the distribution function in a steady-state Bernstein–Greene–Kruskal wave. The damping of the wave can be found from the relation $\gamma = (1/2W)(dW/dt) = -\langle \mathbf{jE} \rangle/2W$, where $\mathbf{j}$ is the current caused by the wave (in a reference system fixed in the wave); W is the energy density in the wave. The averaging is carried out over the wave period.

Thus, the problem has been reduced to finding the electron distribution function. In the reference system that moves with the wave, the kinetic equation for this function is

$$u\,\partial f/\partial y - \varphi'(y)\partial f/\partial u = \nu'(\partial/\partial u)[\partial f/\partial u + (\alpha + u) f]. \tag{2.42}$$

We have introduced the following dimensionless variables:

$$\varphi(y) = e\Phi/T,\ y = kx,\ \Phi(x) = \Phi_0 \sin^2 kx/2;\ v_T = \sqrt{2T/m},$$
$$u = v\sqrt{2}/v_T,\ \alpha = \omega\sqrt{2}/kv_T,\ \nu' = 3/\sqrt{2}kv_T\tau_D,$$
$$\tau_D = m^2\omega^3/8\pi e^4 N_0 k^3 \,\mathrm{Ln},$$

while the collision integral is taken in the Fokker–Planck form (Ln is the Coulomb logarithm).

In order to simplify the problem we introduce the following ordering: $\nu' \ll \varphi_0 \ll 1$; this corresponds to a wave of finite but reasonably small amplitude and a low collision rate. We then introduce a new variable, the energy $\varepsilon = u^2/2 + \varphi(y)$, where ε is the dimensionless energy of a particle in the wave field. Under these conditions Eq. (2.42) assumes the form

$$\frac{\partial f}{\partial y} = \nu \frac{\partial}{\partial \varepsilon}\left\{\sigma[\varepsilon - \varphi(y)]^{1/2}\left(f + \frac{\partial f}{\partial \varepsilon}\right) + V_f f\right\}, \tag{2.43}$$
$$V_f = \alpha/\sqrt{2} = \omega/kv_T, \qquad \nu = \nu'\sqrt{2} = 3/kv_T\tau_D,$$

where $\sigma = \pm 1$ corresponds to the two directions of motion of the electrons (the "plus" sign applies for particles which overtake wave). The quantity ν plays the role of a dimensionless parameter. We therefore seek a solution in the form

$$f(\varepsilon, y) = f_0(\varepsilon) + f_1(\varepsilon, y) + \ldots. \tag{2.44}$$

In finding a solution of Eq. (2.43) it is necessary to consider two regions: $\varepsilon > \varphi_0$ (the outer region) and $\varepsilon < \varphi_0$ (the inner region). Substituting the expansion in (2.44) in Eq. (2.43) we have

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where $f_0(\varepsilon)$ (the zeroth approximation) for the outer region is determined from the condition of periodicity of f_1 as a function of the y-coordinate if it is assumed that when $\varepsilon \gg \varphi_0$ the function $f_0(\varepsilon)$ approaches a Maxwellian distribution asymptotically. Integrating Eq. (2.45) with respect to y, taking account of the periodicity, we obtain an equation for f_0 when $\varepsilon > \varphi_0$:

$$\frac{d}{d\varepsilon}\left\{\sqrt{\varepsilon}\,E\left(\sqrt{\frac{\varphi_0}{\varepsilon}}\right)\left[f_0 + \frac{df_0}{d\varepsilon}\right] + \frac{\pi\sigma}{2} V_f f_0\right\} = 0, \tag{2.46}$$

where $E(\varkappa) = \int_0^{\pi/2} \sqrt{1 - \varkappa^2 \sin^2 t}\, dt$ is a complete elliptic integral of the second kind. We now write the solution of this equation, going over to a Maxwellian distribution for values far from the resonance region:

$$f_0(\varepsilon)=\frac{1}{\sqrt{\pi}\,v_T}\exp\left[-\varepsilon-0.5\pi\sigma V_f\sqrt{\varphi_0}\int_1^{\varepsilon/\varphi_0}\frac{d\varkappa^2}{\varkappa E(1/\varkappa)}-V_f^2\right]. \tag{2.47}$$

In order to obtain the solution it is necessary to find a correction from Eq. (2.45) and to compute the work of the wave field:

$$\mathbf{j}\cdot\mathbf{E}=e\int_0^{\lambda}\frac{dx}{\lambda}\int_{-\infty}^{+\infty}\frac{\partial\Phi(x)}{\partial x}\left(\frac{\omega}{k}\right)f_1\,dv=$$
$$=\frac{1}{2}\,\omega T v_T\int_{-\pi}^{\pi}\frac{dy}{2\pi}\sum_{\sigma=\pm 1}\int_{\varphi(y)}^{\infty}\frac{\partial\varphi}{\partial y}\cdot f_1\frac{d\varepsilon}{\sqrt{\varepsilon-\varphi(y)}}.$$

However, by integrating by parts and carrying out a further analysis of Eq. (2.45) to determine df_1/dy, it is possible to express the work of the field in terms of the known function $f_0(\varepsilon)$ and thus to compute the contribution from the outer region:

$$\dot{W}_{\text{outer}}=-\nu\omega v_T T\int_{-\pi}^{\pi}\frac{dy}{2\pi}\sum_{\sigma}\int_{\varphi_0}^{\infty}d\varepsilon\sqrt{\varepsilon-\varphi(y)}\,\frac{\partial}{\partial\varepsilon}\times$$
$$\times\left\{\sigma\sqrt{\varepsilon-\varphi(y)}\left(\frac{\partial f_0}{\partial\varepsilon}+f_0\right)+V_f f_0\right\}=$$
$$=-\nu\pi^{-3/2}V_f e^{-V_f^2}\,\omega N_0 T\sqrt{\varphi_0}\left[\int_1^{\infty}\frac{d\varkappa^2}{\varkappa}\left(K\left(\frac{1}{\varkappa}\right)-\frac{\pi^2}{4E(1/\varkappa)}\right)\right].$$

A similar procedure can be carried out to find the distribution function in the inner region $\varepsilon < \varphi_0$. However, it should be noted that to a high degree of accuracy the distribution function is symmetric for particles that oscillate within the potential well $f(\sigma = \pm 1) = f(\sigma = -1)$. This means that the contribution to damping from the inner region is small and can be neglected. A more complete calculation verifies this conclusion.

Finally, it is necessary to investigate the behavior of the distribution function in the narrow transition region that lies between the inner and outer regions. After finding the distribution function in this region it is necessary to compute its contribution to the work of the wave field. However, the determination of the distribution function in this region turns out to be an extremely complicated problem. Fortunately, as usually happens in the analysis of a boundary layer, which is essentially what the present problem is,

the final result does not depend on the detailed structure of the transition region. In the problem at hand the work of the wave field associated with this region is also insensitive to the details of the distribution function. A knowledge of the value of $f_0(\varepsilon)$ on both sides of the transition region is sufficient for our purposes. It turns out that

$$\dot{W}_{\text{trans}}=-\nu\omega v_T T\int_{-\pi}^{\pi}\frac{dy}{2\pi}\sum_{\sigma}\int_{\varphi_0-\delta}^{\varphi_0+\delta}d\varepsilon\sqrt{\varepsilon-\varphi(y)}\,\frac{\partial}{\partial\varepsilon}\times$$
$$\times\left\{\sigma\sqrt{\varepsilon-\varphi(y)}\left(\frac{\partial f_0}{\partial\varepsilon}+f_0\right)+V_f f_0\right\},$$

where δ is the width of the transition region. Integrating by parts and only considering the major terms (of the order of the derivatives of the distribution function) we have

$$\dot{W}_{\text{trans}}=-\nu\omega v_T T\int_{-\pi}^{\pi}\frac{dy}{2\pi}\cos^2(y/2)\sum_{\sigma}\sigma\left(\left.\frac{\partial f_0(\varepsilon,\sigma)}{\partial\varepsilon}\right|_{\varepsilon=\varphi_0+\delta}-\left.\frac{\partial f_0(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varphi_0-\delta}\right).$$

The only remaining problem is to substitute the values of $df_0/d\varepsilon$ in the outer ($\varepsilon = \varphi_0 + \delta$) and inner ($\varepsilon = \varphi_0 - \delta$) regions using the expressions found for the distribution function in these regions (2.42). Finally we have

$$\dot{W}\approx-\frac{\sqrt{\pi}}{4}\left[1-\int_1^{\infty}\frac{d\varkappa^2}{\varkappa}\left(\frac{1}{E(1/\varkappa)}-\frac{4}{\pi^2}K(1/\varkappa)\right)\right]\nu\varphi_0^{1/2}\,\omega N_0 T V_f e^{-V_f^2}. \tag{2.48}$$

The damping coefficient for the wave is [66]

$$\gamma=\left[1-\int_1^{\infty}\frac{d\varkappa^2}{\varkappa}\left(\frac{1}{E(1/\varkappa)}-\frac{4}{\pi^2}K(1/\varkappa)\right)\right]\nu\varphi_0^{-3/2}\gamma_L, \tag{2.49}$$
$$\gamma_L\approx\pi^{1/2}\omega V_f^3 e^{-V_f^2}$$

which, to within a factor of order unity, coincides with Eq. (2.41), which is obtained on the basis of qualitative considerations of the quasilinear analysis.

Effects of this kind, which arise when account is taken of collisions in problems concerned with the interaction of a monochromatic wave with particles, are found to be important in the so-called neoclassical theory of plasma diffusion in region:

$$f_0(\varepsilon)=\frac{1}{\sqrt{\pi}\,v_T}\exp\left[-\varepsilon-0.5\pi\sigma V_f\sqrt{\varphi_0}\int_1^{\varepsilon/\varphi_0}\frac{d\varkappa^2}{\varkappa E(1/\varkappa)}-V_f^2\right]. \tag{2.47}$$

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$$=-\nu\pi^{-3/2}V_f e^{-V_f^2}\,\omega N_0 T\sqrt{\varphi_0}\left[\int_1^{\infty}\frac{d\varkappa^2}{\varkappa}\left(K\left(\frac{1}{\varkappa}\right)-\frac{\pi^2}{4E(1/\varkappa)}\right)\right].$$

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Effects of this kind, which arise when account is taken of collisions in problems concerned with the interaction of a monochromatic wave with particles, are found to be important in the so-called n e o c l a s s i c a l t h e o r y o f p l a s m a d i f f u s i o n i n magnetic confinement systems (cf. the present volume, page 257).

PROBLEM

▶ 1. It has been proposed to retard the spreading of plasma along a magnetic field by the use of a "bumpy" field with period $l \ll \lambda$ (λ is the mean free path) [67, 68]. Calculate the retarding force and estimate the loss time for a plasma in a system of length $L \gg \lambda$.

The magnetic field is written in the form $H = H_0 + \Delta H \cos^2(\pi z/l)$. The work performed by the plasma on the magnetic field as the plasma spreads with velocity U is found from Eq. (2.48), in which the field amplitude Φ_0 is replaced by $\mu \Delta H (\mu = mv_\perp^2/2H)$ and the phase velocity (ω/k) is replaced by U. The work is equal to the product of the frictional force F_{fr} and the velocity; thus from Eq. (2.48) we have

$$F_{fr} = -\frac{\sqrt{\pi}}{4}\left[1 - \int_0^\infty \frac{d\varkappa^2}{\varkappa}\left(\frac{1}{E(1/\varkappa)} - \frac{4}{\pi^2}K(1/\varkappa)\right)\right]\nu_{ii}\left(\frac{\Delta H}{H_0}\right)^{1/2} N_0 MU, \quad (1)$$

$$\nu_{ii} = N_0 e^4 \,\mathrm{Ln}/M^2 v_{Ti}^3 .$$

By balancing the pressure against the frictional force we can determine the spreading velocity and the loss time

$$U \sim (\lambda/L)(H_0/\Delta H)^{1/2} c_s, \quad \tau = (L^2/\lambda c_s)(\Delta H/H_0)^{1/2}. \quad (2)$$

§ 2.5. Quasilinear Theory of Electromagnetic Modes

In this section the quasilinear theory will be used to analyze the interaction of particles with electromagnetic modes in a plasma in a uniform magnetic field $\mathbf{H}_0$. We shall consider the simplest cases, specifically, ion and electron cyclotron waves (whistlers) which propagate along $\mathbf{H}_0$ [2, 69]. Limiting ourselves to the case of parallel propagation simplifies the algebra without introducing any fundamental changes in the physical results.

We start from the kinetic equation

$$\frac{\partial f_j}{\partial t} + \mathbf{v}\frac{\partial f_j}{\partial \mathbf{r}} + \frac{e_j}{m_j c}[\mathbf{v}\cdot\mathbf{H}_0]\frac{\partial f_j}{\partial \mathbf{v}} + \frac{e_j}{m_j}\left\{\mathbf{E}_\perp + \frac{1}{c}[\mathbf{v}\times\mathbf{H}_\perp]\right\}\frac{\partial f_j}{\partial \mathbf{v}} = 0 \quad (2.50)$$

and Maxwell's equations. If the distribution function is written in the form of a slowly varying part and a rapidly varying part, the equation for the slowly varying part is

$$\frac{\partial \bar f_j}{\partial t} = \left(\frac{e_j}{m_j}\right)^2 \sum_k \left[-\frac{kv_\perp}{\omega_{-k}}\frac{\partial}{\partial v_z} + \left(1 + \frac{kv_z}{\omega_{-k}}\right)\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp\right]\times$$
$$\times \frac{|E_k|^2}{(-i\omega_k \pm i\omega_{Hj} + ikv_z)}\left[\left(1 - \frac{kv_z}{\omega_k}\right)\frac{\partial \bar f_j}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial \bar f_j}{\partial v_z}\right]. \quad (2.51)$$

Here, we have introduced cyclindrical coordinates in velocity space. The symbol $\pm$ in the resonance denominator corresponds to right- and left-hand circularly polarized waves. In the resonance region this equation can be simplified as follows:

$$\frac{\partial \bar f_j}{\partial t} = \left(\frac{e_j}{m_j}\right)^2 \sum_k \left[\left(1 - \frac{kv_z}{\omega_k}\right)\frac{\partial}{v_\perp \partial v_\perp}v_\perp + \frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z}\right]|\mathbf{E}_k|^2\times$$
$$\times \pi\delta(\omega_k - kv_z \pm \omega_{Hj})\left[\left(1 - \frac{kv_z}{\omega_k}\right)\frac{\partial}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z}\right]\bar f_j. \quad (2.52)$$

The basic contribution comes from particles with resonance velocities $v_z = \omega \pm \omega_{Hj}/k$. As a result of the Doppler shift the frequency of a wave in a coordinate system moving with this velocity is equal to the gyrofrequency. Hence, resonance particles gyrate in the magnetic field H_0 with the same frequency as the electric vector of the wave $\mathbf{E}_\perp$ and are therefore accelerated. As a first example of the application of Eq. (2.52) we consider the problem of a one-dimensional wave packet as shown in Fig. 23. This region lies in the left-half plane since $\omega \pm \omega_{Hj}$ is negative for whistlers. The circles in this figure show the level curves for the original isotropic velocity distribution. As in the quasilinear theory for plasma oscillations, the resonance particles continue to diffuse until a steady-state "plateau" is formed. If the wave packet is narrow $[\Delta(\omega/k) \ll \omega/k]$ the steady state is described by the relation

$$\left[\left(1 - \frac{kv_z}{\omega_k}\right)\frac{\partial}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z}\right]\bar f_j = 0. \quad (2.53)$$

This equation describes the "plateau" for the case at hand and is analogous to $d\bar f/dv = 0$ (the plateau in the quasilinear theory of plasma oscillations). The level curves for the steady-state velocmagnetic confinement systems (cf. the present volume, page 257).

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$$\times \frac{|\mathbf{E}_k|^2}{(-i\omega_k \pm i\omega_{Hj} + ikv_z)}\left[\left(1-\frac{kv_z}{\omega_k}\right)\frac{\partial \bar f_j}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial \bar f_j}{\partial v_z}\right]. \quad (2.51)$$

Here, we have introduced cyclindrical coordinates in velocity space. The symbol $\pm$ in the resonance denominator corresponds to right- and left-hand circularly polarized waves. In the resonance region this equation can be simplified as follows:

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$$\times \pi\delta(\omega_k - kv_z \pm \omega_{Hj})\left[\left(1-\frac{kv_z}{\omega_k}\right)\frac{\partial}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z}\right]\bar f_j. \quad (2.52)$$

The basic contribution comes from particles with resonance velocities $v_z = \omega \pm \omega_{Hj}/k$. As a result of the Doppler shift the frequency of a wave in a coordinate system moving with this velocity is equal to the gyrofrequency. Hence, resonance particles gyrate in the magnetic field H_0 with the same frequency as the electric vector of the wave $\mathbf{E}_\perp$ and are therefore accelerated. As a first example of the application of Eq. (2.52) we consider the problem of a one-dimensional wave packet as shown in Fig. 23. This region lies in the left-half plane since $\omega \pm \omega_{Hj}$ is negative for whistlers. The circles in this figure show the level curves for the original isotropic velocity distribution. As in the quasilinear theory for plasma oscillations, the resonance particles continue to diffuse until a steady-state "plateau" is formed. If the wave packet is narrow $[\Delta(\omega/k) \ll \omega/k]$ the steady state is described by the relation

$$\left[\left(1-\frac{kv_z}{\omega_k}\right)\frac{\partial}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z}\right]\bar f_j = 0. \quad (2.53)$$

This equation describes the "plateau" for the case at hand and is analogous to $d\bar f/dv = 0$ (the plateau in the quasilinear theory of plasma oscillations). The level curves for the steady-state velocity distribution, which satisfies Eq. (2.53), are given by the equation

$$v_\perp^2/2 + v_z^2/2 - \omega v_z/k = \text{const.} \qquad (2.54)$$

These curves are also circles but with centers displaced to the right by an amount ω/k (shown by the dashed curves in Fig. 23). It is well known from the linear theory of plasma waves that the imaginary part of the whistler frequency vanishes when

$$\int dv_\perp v_\perp^2 \left[\left(1 - \frac{kv_z}{\omega_k}\right)\frac{\partial \bar{f}_j}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\cdot\frac{\partial \bar{f}_j}{\partial v_z}\right]\Bigg|_{v_z=\frac{\omega\pm\omega_{Hj}}{k}} = 0. \qquad (2.55)$$

Comparing the "plateau condition" (2.53) with Eq. (2.55), we see that the whistler damping rate in the plateau state also vanishes (as in the case of the plasma oscillations).

To extend this analogy even further, we now reduce the two-dimensional operator for quasilinear diffusion in Eq. (2.52) to one-dimensional form. Introducing the variable

$$w = v_\perp^2/2 + v_z^2/2 - \omega v_z/k, \qquad (2.56)$$

we find that the derivative d/dw vanishes and Eq. (2.52) assumes

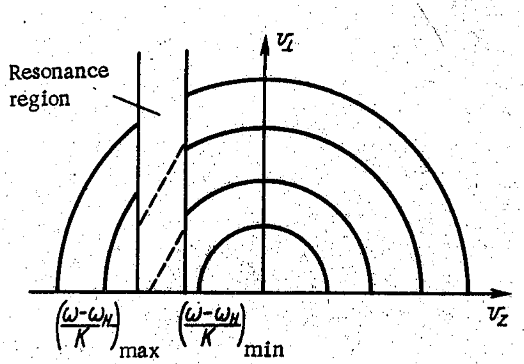

Fig. 23. Initial and final level curves for the particle for a whistler wave packet.

the following form in terms of the new variables w and $v \equiv v_z$:

$$\left.\begin{aligned}\frac{\partial \bar{f}_j}{\partial t} &= \sum_k \left[\left(1 - \frac{kv}{\omega}\right)\frac{1}{v_\perp} + \frac{kv_\perp}{\omega}\cdot\frac{\partial}{\partial v}\right]\frac{e_j^2}{m_j^2}|\mathbf{E}_k|^2 \pi\delta \times \\ &\times (\omega - kv \pm \omega_{Hj})\frac{kv_\perp}{\omega}\cdot\frac{\partial \bar{f}_j}{\partial v}; \\ \frac{\partial \bar{f}_j}{\partial t} &= \left(\frac{e_j}{m_j}\right)^2 \frac{\partial}{\partial v}\left[v_\perp(w, v)\frac{|\mathbf{H}_k|^2(v)}{\left|v - \frac{d\omega}{dk}\right|}\cdot\frac{\partial \bar{f}_j}{\partial v}\right].\end{aligned}\right\} \qquad (2.57)$$

The integral in (2.55), which determines the imaginary part of the frequency, also reduces to one-dimensional form in terms of the new variables (2.56) [1, 69]:

$$\operatorname{Im}\omega \sim \int dv_\perp v_\perp^2 \left[\left(1 - \frac{kv_z}{\omega}\right)\frac{\partial \bar{f}_j}{\partial v_\perp} + \frac{kv_\perp}{\omega}\frac{\partial \bar{f}_j}{\partial v_z}\right]_{v_z=\frac{\omega\pm\omega_{Hj}}{k}} \equiv$$

$$\equiv \int_{w_{min}(v)}^{\infty} v_\perp^2(w, v)\frac{\partial \bar{f}_j}{\partial v}\, dw. \qquad (2.58)$$

The integrals in Eqs. (2.55) and (2.58) can become positive, indicating an instability, in a plasma with an anisotropi- velocity distribution. One of the most important examples of an anisotropic velocity distribution is the so-called loss-cone distribution. A distribution of this kind arises in magnetic-mirror plasma confinement systems. The loss-cone distribution function can be written in the form

$$f = f_0(v_\perp^2 + v_z^2)\,\eta(\alpha v_\perp^2 - v_z^2), \qquad (2.59)$$

where $\alpha \equiv (H_{max} - H)/H$ and $\eta(x)$ is shown in Fig. 24. For a specified α the level curves are of the form shown in Fig. 25. Substitution of the distribution function in (2.59) in the whistler stability cri-

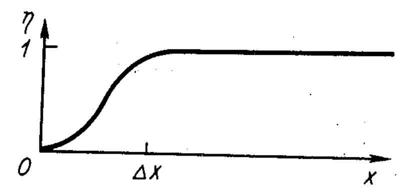

Fig. 24. The function $\eta(x)$ for a loss-cone distribution.

ity distribution, which satisfies Eq. (2.53), are given by the equation

$$v_\perp^2/2 + v_z^2/2 - \omega v_z/k = \text{const.} \tag{2.54}$$

These curves are also circles but with centers displaced to the right by an amount ω/k (shown by the dashed curves in Fig. 23). It is well known from the linear theory of plasma waves that the imaginary part of the whistler frequency vanishes when

$$\int dv_\perp v_\perp^2 \left[\left(1-\frac{kv_z}{\omega_k}\right)\frac{\partial \bar{f}_j}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\cdot\frac{\partial \bar{f}_j}{\partial v_z}\right]\Bigg|_{v_z=\frac{\omega\pm\omega_{Hj}}{k}} = 0. \tag{2.55}$$

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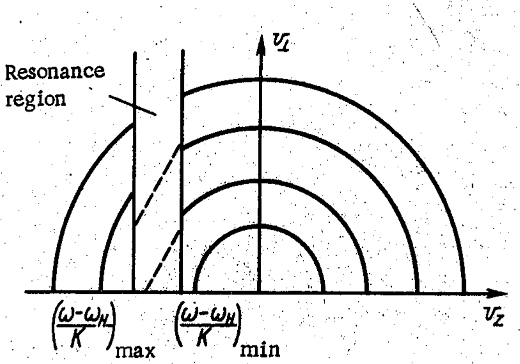

Fig. 23. Initial and final level curves for the particle for a whistler wave packet.

the following form in terms of the new variables w and $v \equiv v_z$:

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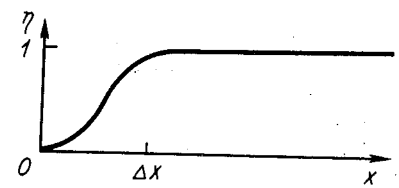

Fig. 24. The function η(x) for a loss-cone distribution.

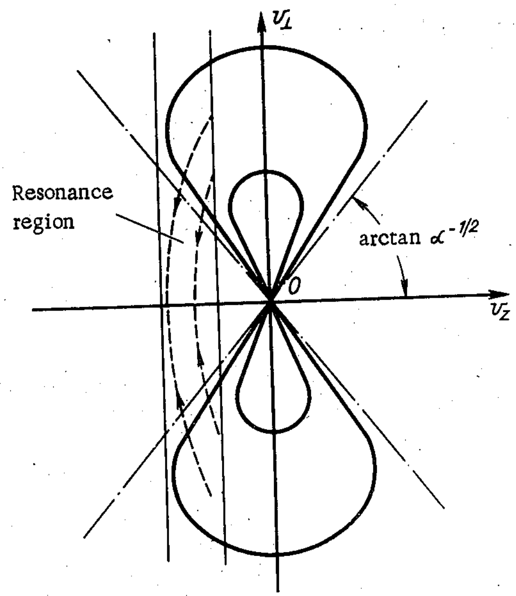

Fig. 25. Level curves for a loss-cone distribution and quasilinear diffusion in the loss cone.

terion (2.55) yields

$$\int_0^\infty \varepsilon_\perp \left[\left(1-\frac{kv_z}{\omega}\right)\frac{\partial}{\partial \varepsilon_\perp}(f_0\eta)+\frac{k}{\omega}\cdot\frac{\partial}{\partial m v_z}(f_0\eta)\right]d\varepsilon_\perp<0, \tag{2.60}$$

where $\varepsilon_\perp = mv_\perp^2/2$. Integrating the first term by parts and carrying out the indicated operations in the other two terms, we have

$$-\int_0^\infty f_0\,\eta\, d\varepsilon_\perp - \frac{2kv_z}{\omega m}\int_0^\infty \varepsilon_\perp \eta\left(\frac{\partial f_0}{\partial v_\perp^2}-\frac{\partial f_0}{\partial v_\parallel^2}\right)d\varepsilon_\perp - \frac{kv_z}{\omega}(\alpha+1)\int_0^\infty d\varepsilon_\perp\, \varepsilon_\perp\, \eta' f_0<0. \tag{2.61}$$

The second term in this expression vanishes identically. Assuming that the region in which there is an important change in the function is small as compared with v_{Tj}^2, we replace $\eta'(x)$ by $\delta(x)$ in the third term. Then,

$$-f_{0j}\, v_{Tj}^2\left[1+\frac{kv_z^3(\alpha+1)}{\omega\alpha^2 v_{Tj}^2}\right]<0, \tag{2.62}$$

where $v_z = (\omega \pm \omega_{Hj})/k$. Since v_z is negative, in accordance with the criterion in (2.62) an instability will occur for sufficiently large values of v_z or sufficiently small values of α. Near the region of maximum magnetic field in a mirror the quantity α approaches zero so that there is always an unstable region, although it is small. However, violation of the stability criterion in a small region along the line of force Δz does not necessary mean that a wave will grow. In order for the amplitude of an initially small fluctuation to grow to significant values before the corresponding wave packet moves out of the unstable region it is necessary that

$$\int_{\Delta z} dz\,\frac{\operatorname{Im}\omega}{\partial\omega/\partial k_z}\gg 1.$$

When the integration is carried out, this criterion can be written in the form of an inequality

$$\Delta z \gg cF(\beta_j)/\omega_{pj},$$

where F is a function which depends on the ratio of the pressure of the given plasma component to the pressure of the magnetic field $\beta_j = 8\pi N_0 T_j/H_0^2$; this function increases without limit as β goes to zero. In present-day experimental devices this criterion can only be satisfied by electron whistlers in the case in which β_j is reasonably large.

According to Eq. (2.61) the plasma is also unstable if the resonant velocity is much larger than the thermal velocity. In any plasma which is reasonably thermalized particles with velocities of this value will be exponentially small in number so that the corresponding instability has an exponentially small growth rate. In the magnetosphere of the earth there are trapped particles with energies much higher than the mean thermal energy. The magnetic field of the earth exhibits the properties of a magnetic-mirror confinement device as far as these particles are concerned. The physical analysis of the quasilinear interaction of particles with a whistler wave packet in the presence of an instability such as that described above has been applied to the magnetosphere of the earth. It is found that quasilinear diffusion into the loss-cone is an important mechanism which determines the particle lifetime in the radiation belts in the magnetic field of the earth [70-72].

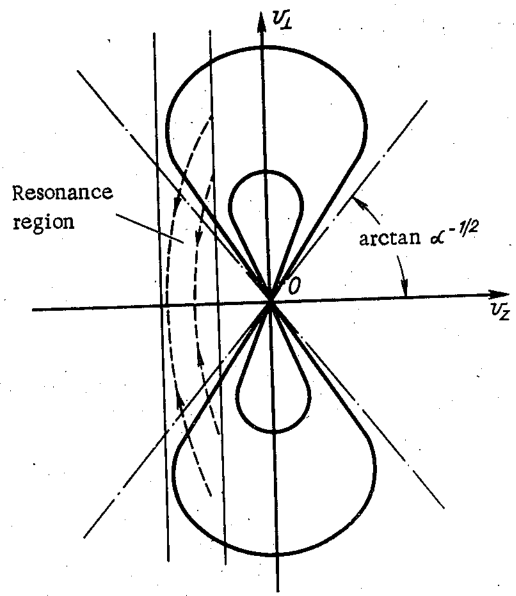

Fig. 25. Level curves for a loss-cone distribution and quasilinear diffusion in the loss cone.

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PROBLEM

1. Find the coefficient for quasilinear diffusion of electrons in pitch angle α ($\sin\alpha = v_\perp/v$) in interactions with whistle waves [71].

Introducing spherical coordinates (v, α, φ) we write Eq. (2.51) in the form

$$\frac{\partial f_e}{\partial t} = \omega_H^2 \sum_k \left[\frac{\partial}{v\partial v} v + \left(\cos^2\alpha - \frac{(\omega - |\omega_H|)}{\omega}\right)\frac{1}{v\cos\alpha\sin\alpha}\cdot\frac{\partial}{\partial\alpha}\right]\times$$

$$\times \frac{\omega_k^2 |\mathbf{H}_k|^2}{k^2 H_0^2}\pi\sin^2\alpha\,\delta(kv\cos\alpha - \omega + |\omega_H|)\times$$

$$\times\left[\frac{\partial}{\partial v} + \left(\cos^2\alpha - \frac{(\omega - |\omega_H|)}{\omega}\right)\frac{1}{v\cos\alpha\sin\alpha}\cdot\frac{\partial}{\partial\alpha}\right] f_e.$$

If it is assumed that the diffusion occurs primarily in pitch angle, this equation can be simplified considerably ($\omega \ll \omega_H$):

$$\frac{\partial f_e}{\partial t} = \frac{\partial}{\partial\alpha}(\omega_H^2/v\,|\cos\alpha|)\frac{|\mathbf{H}(k \approx \omega_H/v)|^2}{H_0^2}\cdot\frac{\partial}{\partial\alpha} f.$$

§ 2.6. Nonresonant Wave – Particle Interactions

Up to this point we have only considered resonant interactions between waves and particles. In an analysis of nonresonant (or adiabatic) interactions it is necessary to take account of the principal-part term in the quasilinear diffusion equation [cf. Eq. (2.25)]. As has already been noted in connection with Eq. (2.25) in the case of plasma waves, this term in the qusilinear diffusion equation describes the contribution of nonresonance electrons (the main part of the distribution function) with the plasma waves. For example, an increase in the oscillatory kinetic energy of the electrons due to an increase in wave amplitude leads to an apparent "heating" of the main part of the distribution function. In order to investigate this effect quantitatively, we now write Eq. (2.25) with the nonresonant part

$$\frac{\partial f}{\partial t} = \frac{e^2}{m^2}\cdot\frac{\partial}{\partial v}\sum_k \frac{\gamma_k |\mathbf{E}_k|^2}{(kv - \omega_k)^2 + \gamma_k^2}\cdot\frac{\partial f}{\partial v} \approx \frac{e^2}{m^2}\cdot\frac{\partial}{\partial v}\sum_k \frac{\gamma_k |\mathbf{E}_k|^2}{\omega_p^2}\cdot\frac{\partial f}{\partial v}, \tag{2.63}$$

where, as an approximation, $[(kv - \omega_k)^2 + \gamma_k^2]$ has been replaced by ω_p^2. When the equation for wave growth is taken into account, Eq. (2.63) assumes the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\cdot\frac{1}{N_0 m}\left(\frac{d}{dt}\sum_k \frac{|\mathbf{E}_k|^2}{8\pi}\right)\frac{\partial f}{\partial v}. \tag{2.64}$$

Multiplying both sides of this equation by $mv^2/2$ and integrating over velocity, we have

$$\frac{d}{dt}\frac{m}{2}\int_{-\infty}^{+\infty} dv\, v^2 \bar{f}(v, t) = \frac{d}{dt}\sum_k \frac{|\mathbf{E}_k|^2}{8\pi}. \tag{2.65}$$

In other words, the kinetic energy of the electrons in the main body of the distribution function increases together with the electrostatic energy of the oscillations. This is nothing more than a reflection of the well-known fact that the total energy of the plasma waves contains two equal parts: the electrosatic energy and the electron kinetic energy. In order to be convinced of the fact that the increase in the kinetic energy of the electrons oscillating under the effect of the oscillatory field can be looked upon as a kind of heating, in Eq. (2.64) we replace the variable t by the variable $\tau = \sum_k |\mathbf{E}_k|^2/4\pi N_0$:

$$\frac{\partial f}{\partial\tau} = \frac{1}{2m}\cdot\frac{\partial^2 f}{\partial v^2}. \tag{2.66}$$

We now assume the initial conditions

$$f(v, \tau = 0) = \sqrt{m/2\pi T}\exp[-mv^2/2T]. \tag{2.67}$$

This equation then has the solution

$$f(v, \tau) = \left[\frac{m}{2\pi(T + \tau)}\right]^{1/2}\exp[-mv^2/2(T + \tau)]. \tag{2.68}$$

It can be shown in similar fashion that the main body of the distribution also carries momentum associated with the waves. However, in order to demonstrate this feature, it would be necessary to retain the velocity dependence in the denominator $[(\omega_k - kv)^2 + \gamma_k^2]$ in Eq. (2.63). This dependence shifts the maximum of the distribution function in the direction of wave propaga-

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$$\times\frac{\omega_k^2\,|\mathbf{H}_k|^2}{k^2H_0^2}\pi\sin^2\alpha\,\delta\left(kv\cos\alpha-\omega+|\omega_H|\right)\times$$

$$\times\left[\frac{\partial}{\partial v}+\left(\cos^2\alpha-\frac{(\omega-|\omega_H|)}{\omega}\,\frac{1}{v\cos\alpha\sin\alpha}\cdot\frac{\partial}{\partial\alpha}\right]f_e.$$

If it is assumed that the diffusion occurs primarily in pitch angle, this equation can be simplified considerably ($\omega \ll \omega_H$):

$$\frac{\partial f_e}{\partial t}=\frac{\partial}{\partial\alpha}\left(\omega_H^2/v\,|\cos\alpha|\right)\frac{|\mathbf{H}\,(k\approx\omega_H/v)|^2}{H_0^2}\cdot\frac{\partial}{\partial\alpha}f.$$

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Multiplying both sides of this equation by $mv^2/2$ and integrating over velocity, we have

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It can be shown in similar fashion that the main body of the distribution also carries momentum associated with the waves. However, in order to demonstrate this feature, it would be necessary to retain the velocity dependence in the denominator $[(\omega_k-kv)^2+\gamma_k^2]$ in Eq. (2.63). This dependence shifts the maximum of the distribution function in the direction of wave propagation, thereby taking account of the momentum. In contrast with the instabilities which have been treated so far in this chapter, there are many plasma instabilities which are algebraic in nature, in which resonant wave–particle interactions play no rôle. In cases of this kind it is necessary to use the quasilinear theory with nonresonant diffusion in the description of the relaxation of the instabilities. For example, let us consider the "firehose" instability [73]. In this instability the imaginary part of the frequency is

$$\gamma_k = kv_{Ti}\sqrt{(T_\parallel - T_\perp)/T} \tag{2.69}$$

where $\beta \gg 1$. The real part of the frequency is negligibly small if the wavelength of the perturbation exceeds the mean ion Larmor radius ($kv_{Ti} \ll \omega_{Hi}$). If the plasma is close to marginal stability ($\Delta T/T \ll 1$), then $\gamma_k \ll kv_{Ti}$ and the quasilinear theory can be used. Since $\omega_{Hi} \gg kv_{Ti}$, resonance particles do not play a role in the firehose instability. In order to obtain the diffusion equation, in Eq. (2.63) we introduce the substitution $\omega_k \to i\gamma_k$ and make use of the condition $\gamma_{-k} = \gamma_k$. As a result we have [74]

$$\frac{\partial f_j}{\partial t} = \frac{e_j^2 \sum_k \frac{d}{dt}|\mathbf{H}_k|^2}{2m_j^2\omega_{Hj}^2c^2}\left(v_\perp^2\frac{\partial^2 f_j}{\partial v_z^2} + v_z^2\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp\frac{\partial f_j}{\partial v_\perp} - 2v_\perp\frac{\partial}{\partial v_z}v_z\frac{\partial f_j}{\partial v_\perp}\right). \tag{2.70}$$

In the derivation of this relation use has also been made of the relation

$$|\mathbf{E}_k|^2 = (\omega^2/k^2c^2)|\mathbf{H}_k|^2. \tag{2.71}$$

The following approximation procedure can be used to solve Eq. (2.70). Since the plasma is near marginal stability ($\Delta T \ll T$), the distribution function can be written in the form $f = f_M + \Delta T f_1/T$, where f_M is the Maxwellian function and f_1 is a correction which takes account of the anisotropy. If the right-hand side of the quasilinear equation (2.70) is linearized, we have

$$\frac{\partial f_1}{\partial t} = \frac{(v_\perp^2/2 - v_\parallel^2)}{v_T^2} f_M \frac{d}{dt}\sum_k \frac{|\mathbf{H}_k|^2}{H_0^2}. \tag{2.72}$$

The solution of this equation is

$$\bar{f} = f_0(v_\perp^2, v_\parallel^2) + \frac{(v_\perp^2/2 - v_\parallel^2)}{v_T^2} f_M \sum_k \frac{|\mathbf{H}_k|^2}{H_0^2}. \tag{2.73}$$

This distribution function can now be substituted in Eq. (2.69), which gives the growth rate:

$$\gamma_k = k\left[\int d^3\mathbf{v}\,\bar{f}_i(v_\parallel^2 - v_\perp^2/2)\right]^{1/2} = kv_{Ti}\left[(\Delta T/T)_0 - 3\sum_k |\mathbf{H}_k|^2/H_0^2\right]^{1/2}. \tag{2.74}$$

Thus, we finally obtain a relatively simple nonlinear differential equation for the wave amplitude ($kr_{Hi} = (\Delta T/T)^{1/2}$ [75]):

$$\frac{d}{dt}\sum_k |\mathbf{H}_k|^2 = \Omega_H \sum_k |\mathbf{H}_k|^2\left[(\Delta T/T)_0 - 3H_0^{-2}\sum_k |\mathbf{H}_k|^2\right]. \tag{2.75}$$

The solution of this equation is of the form shown in Fig. 26 [74]. Evidently, the quasilinear relaxation of the instability being considered leads to the restoration of the isotropic distribution. This process can be described by the following simple picture. Since the instability grows slowly, the integral

$$\mathcal{J} = \int v_\parallel\, dl \tag{2.76}$$

must be conserved (since it is the adiabatic invariant of the particle motion in a slowly varying field). At first the lines of force are parallel lines, but as the perturbation amplitude increases the lines of force become more and more curved and distorted. Since the length of the line of force [and with it the length of the path of integration in Eq. (1.76)] increases, $v_\parallel$ must be reduced and this leads to a reduction in $T_\parallel$.

Although nonresonant diffusion describes the interaction of a wave with all of the particles, the effectiveness of this interaction

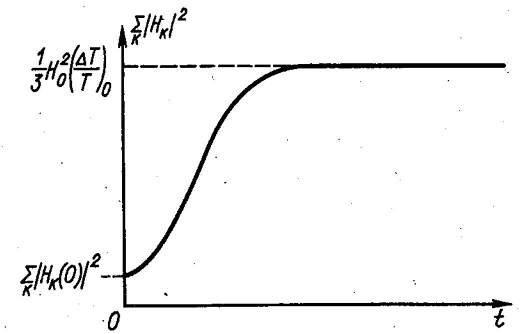

Fig. 26. Nonlinear evolution of the energy density spectrum in the firehose instability.

tion, thereby taking account of the momentum. In contrast with the instabilities which have been treated so far in this chapter, there are many plasma instabilities which are algebraic in nature, in which resonant wave–particle interactions play no rôle. In cases of this kind it is necessary to use the quasilinear theory with non-resonant diffusion in the description of the relaxation of the instabilities. For example, let us consider the "firehose" instability [73]. In this instability the imaginary part of the frequency is

$$\gamma_k = kv_{Ti}\sqrt{(T_\parallel - T_\perp)/T} \tag{2.69}$$

where $\beta \gg 1$. The real part of the frequency is negligibly small if the wavelength of the perturbation exceeds the mean ion Larmor radius ($kv_{Ti} \ll \omega_{Hi}$). If the plasma is close to marginal stability ($\Delta T/T \ll 1$), then $\gamma_k \ll kv_{Ti}$ and the quasilinear theory can be used. Since $\omega_{Hi} \gg kv_{Ti}$, resonance particles do not play a role in the firehose instability. In order to obtain the diffusion equation, in Eq. (2.63) we introduce the substitution $\omega_k \to i\gamma_k$ and make use of the condition $\gamma_{-k} = \gamma_k$. As a result we have [74]

$$\frac{\partial f_j}{\partial t} = \frac{e_j^2 \sum_k \frac{d}{dt}|\mathbf{H}_k|^2}{2m_j^2\omega_{Hj}^2 c^2}\left(v_\perp^2 \frac{\partial^2 f_j}{\partial v_z^2} + v_z^2 \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} v_\perp \frac{\partial f_j}{\partial v_\perp} - 2v_\perp \frac{\partial}{\partial v_z} v_z \frac{\partial f_j}{\partial v_\perp}\right). \tag{2.70}$$

In the derivation of this relation use has also been made of the relation

$$|\mathbf{E}_k|^2 = (\omega^2/k^2c^2)|\mathbf{H}_k|^2. \tag{2.71}$$

The following approximation procedure can be used to solve Eq. (2.70). Since the plasma is near marginal stability ($\Delta T \ll T$), the distribution function can be written in the form $f = f_M + \Delta T f_1/T$, where f_M is the Maxwellian function and f_1 is a correction which takes account of the anisotropy. If the right-hand side of the quasilinear equation (2.70) is linearized, we have

$$\frac{\partial f_1}{\partial t} = \frac{(v_\perp^2/2 - v_\parallel^2)}{v_T^2} f_M \frac{d}{dt}\sum_k \frac{|\mathbf{H}_k|^2}{H_0^2}. \tag{2.72}$$

The solution of this equation is

$$\bar f = f_0(v_\perp^2, v_\parallel^2) + \frac{(v_\perp^2/2 - v_\parallel^2)}{v_T^2} f_M \sum_k \frac{|\mathbf{H}_k|^2}{H_0^2}. \tag{2.73}$$

This distribution function can now be substituted in Eq. (2.69), which gives the growth rate:

$$\gamma_k = k\left[\int d^3\mathbf{v}\,\bar f_i(v_\parallel^2 - v_\perp^2/2)\right]^{1/2} = kv_{Ti}\left[(\Delta T/T)_0 - 3\sum_k |\mathbf{H}_k|^2/H_0^2\right]^{1/2}. \tag{2.74}$$

Thus, we finally obtain a relatively simple nonlinear differential equation for the wave amplitude ($kr_{Hi} = (\Delta T/T)^{1/2}$ [75]):

$$\frac{d}{dt}\sum_k |\mathbf{H}_k|^2 = \Omega_H \sum_k |\mathbf{H}_k|^2\left[(\Delta T/T)_0 - 3H_0^{-2}\sum_k |\mathbf{H}_k|^2\right]. \tag{2.75}$$

The solution of this equation is of the form shown in Fig. 26 [74]. Evidently, the quasilinear relaxation of the instability being considered leads to the restoration of the isotropic distribution. This process can be described by the following simple picture. Since the instability grows slowly, the integral

$$\mathcal{J} = \int v_\parallel \, dl \tag{2.76}$$

must be conserved (since it is the adiabatic invariant of the particle motion in a slowly varying field). At first the lines of force are parallel lines, but as the perturbation amplitude increases the lines of force become more and more curved and distorted. Since the length of the line of force [and with it the length of the path of integration in Eq. (1.76)] increases, $v_\parallel$ must be reduced and this leads to a reduction in $T_\parallel$.

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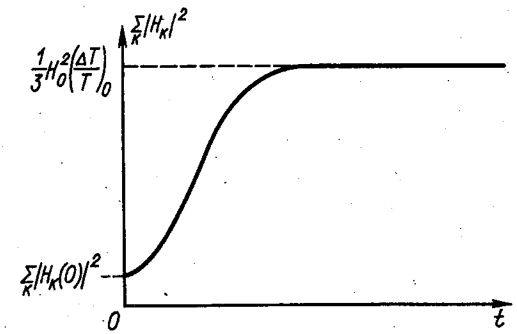

Fig. 26. Nonlinear evolution of the energy density spectrum in the firehose instability.

can vary considerably in different regions in velocity space. In these situations the quasilinear distribution function can assume some very unusual forms in the relaxation process. For example, consider the case of an electromagnetic mode in a plasma in the absence of an external magnetic field H_0. If the electron distribution function (or ion distribution function) is not isotropic, these modes can become unstable.

Let us consider an anisotropic distribution function $f(v_z^2, v_x^2)$. As is evident from Fig. 27, the effective temperature in the x-direction is greater than the temperature in the z-direction. It is not difficult to show that the pure transverse perturbation is unstable even with a small anisotropy [76-78]. In the linear theory the correction to the distribution function for a perturbation of the form $\exp[i(k_z z - \omega t)]$ is given by the equation

$$-i(\omega - k_z v_z) f_j + \frac{e_j}{m_j}\left\{E_x - \frac{1}{c} v_z H_y\right\}\frac{\partial f_{0j}}{\partial v_x} + \frac{e_j}{m_j c} v_x H_y \frac{\partial f_{0j}}{\partial v_z} = 0, \qquad (2.77)$$

where the fields E_x and H_y can be found from Maxwell's equations:

$$-ik_z H_y = (4\pi e/c)\int (f_i - f_e) v_x d^3v, \qquad (2.78)$$

$$ik_z E_x = i\omega H_y/c. \qquad (2.79)$$

Thus, there are four equations for the four quantities f_i, f_e, H_y,

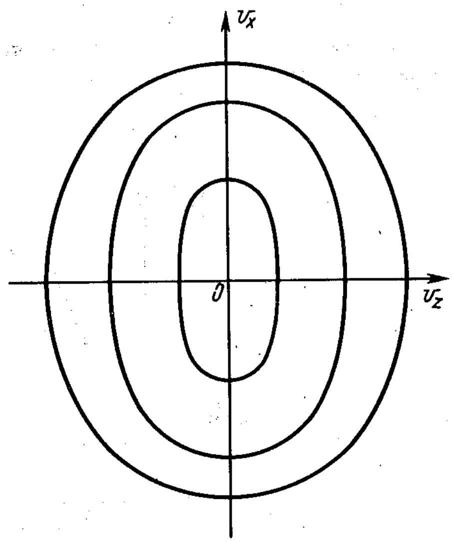

Fig. 27. Level curves for an anisotropic particle distribution.

and E_x. Expressing the fields in terms of H_y, we have

$$f_j = \frac{1}{i(\omega - k_z v_z)}\cdot\frac{e_j H_y}{m_j c}\left\{v_x \frac{\partial f_{0j}}{\partial v_z} - v_z \frac{\partial f_{0j}}{\partial v_x} + \frac{\omega}{k_z}\cdot\frac{\partial f_{0j}}{\partial v_x}\right\}. \qquad (2.80)$$

Let us assume that the ion distribution is anisotropic (if f_{0i} is isotropic, terms corresponding to the Lorentz force vanish). Substituting the current density in Eq. (2.79), and using the correction to the distribution function, we have

$$ik_z H_y = -\frac{4\pi i}{c^2}\sum_j \frac{e_j^2}{m_j} H_y \int \frac{d^3v}{\omega - k_z v_z + i0}\left(v_z f_j + v_x^2 \frac{\partial f_j}{\partial v_z} - \frac{\omega}{k_z} f_j\right). \qquad (2.81)$$

The dispersion equation now assumes the simple form

$$k_z^2 = \sum_j \frac{4\pi e_j^2}{m_j^2 c^2}\left(\int d^3v \frac{k_z v_x^2}{\omega - k_z v_z + i0}\frac{\partial f_j}{\partial v_z} - N_0\right). \qquad (2.82)$$

In an isotropic plasma this relation becomes the usual dispersion relation for electromagnetic waves in a plasma in the absence of an external electromagnetic field. If the distribution is anisotropic new roots appear for this equation, these roots representing low-frequency modes. Let us assume that $f \sim \exp[-(mv_x^2/2T_x) - (mv_z^2/2T_z)]$. It is then obvious that if $(1 - T_x/T_z) < 0$ the wave is unstable for small values of k_z^2. If the distribution function is of more general form, the instability condition becomes

$$\sum_j \frac{\omega_{pj}^2}{c^2}\int d^3v\left(\frac{v_x^2}{v_z}\cdot\frac{\partial f_j}{\partial v_z} + f_j\right) < 0. \qquad (2.83)$$

Since the frequency ω has no real part, the instability is aperiodic. In the other limit $T_x < T_z$, the instability appears for perturbations which propagate in the x-direction. Thus, we are dealing with an absolutely unstable situation, even for an arbitrarily small anisotropy. The following relation obtains for the wave vector k_z at marginal stability: $k_z^2 \leq \Delta T\omega_p^2/Tc^2$. Let us now apply the quasilinear theory to the analysis of this instability. The equation for the averaged distribution function is

$$\frac{\partial \bar{f}}{\partial t} + \left\langle \frac{e}{m}\left(\mathbf{E}' + \frac{1}{c}[\mathbf{v}\times\mathbf{H}']\right)\cdot\frac{\partial f'}{\partial \mathbf{v}}\right\rangle = 0, \qquad (2.84)$$

where the primes denote rapidly varying quantities. We now substitute the expression for the rapidly varying function f' found in can vary considerably in different regions in velocity space. In these situations the quasilinear distribution function can assume some very unusual forms in the relaxation process. For example, consider the case of an electromagnetic mode in a plasma in the absence of an external magnetic field H_0. If the electron distribution function (or ion distribution function) is not isotropic, these modes can become unstable.

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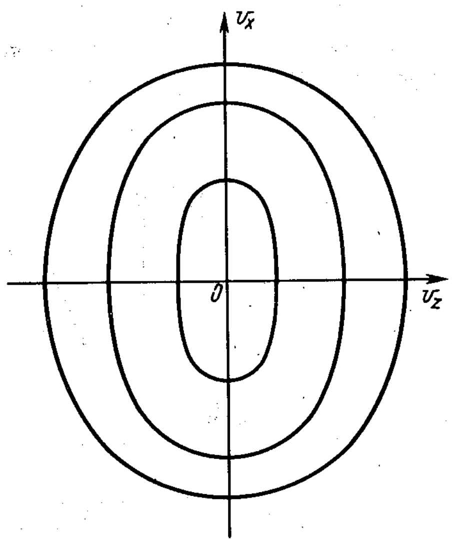

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The dispersion equation now assumes the simple form

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$$\frac{\partial \bar f}{\partial t} + \left\langle \frac{e}{m}\left(E' + \frac{1}{c}[v\times H']\right)\cdot\frac{\partial f'}{\partial v}\right\rangle = 0, \qquad (2.84)$$

where the primes denote rapidly varying quantities. We now substitute the expression for the rapidly varying function f' found in the linear theory and take averages in the usual way in order to obtain the quasilinear equation

$$\frac{\partial \bar{f}}{\partial t} = \frac{e^2}{m^2 c^2} \left\langle [\mathbf{v} \times \mathbf{H}'] \frac{\partial}{\partial \mathbf{v}} \cdot \frac{i}{\omega - \mathbf{k}\cdot\mathbf{v}} (c\mathbf{E}' + [\mathbf{v} \times \mathbf{H}']) \frac{\partial \bar{f}}{\partial \mathbf{v}} \right\rangle,$$

where the angle brackets denote the averaging operation. The quantity $\mathrm{Im}[1/(\omega - k_z v_z)]$ can be written $(d/dt)(\omega^2 + k_z^2 v_z^2)^{-1}$ and ω^2 can be neglected in the denominator since $w^2 \ll k_z^2 v_z^2$.

We now carry out calculations analogous to those that have been performed in the analysis of the firehose instability. The quasilinear equation then assumes the form

$$\frac{\partial \bar{f}}{\partial t} = \frac{e^2}{m^2 c^2} \sum_k \frac{1}{k^2} \cdot \frac{d}{dt} |\mathbf{H}_k|^2 \left(v_x \frac{\partial}{\partial v_z} - v_z \frac{\partial}{\partial v_x} \right) \frac{v_x}{v_z^2} \cdot \frac{\partial \bar{f}}{\partial v_z}. \tag{2.85}$$

The physical interpretation of quasilinear diffusion in this situation is obvious: the instability leads to the growth of fluctuating magnetic fields. These magnetic fields affect the particle motion (essentially, the particles scatter on these magnetic fluctuations).

Although Eq. (2.85) describes the adiabatic interaction of perturbations with all the particles, it will be evident that the quasilinear diffusion coefficient is especially large for particles characterized by small v_z. Hence we expect that a significant modification of the particle distribution function will first arise only for $v_z \ll v_{Ti}$. In view of these considerations, it is possible to neglect the derivative with respect to v_z in this equation. Introducing the new variable $h = e^2/m^2c^2 \Sigma k^{-2} |H_k|^2$, we then reduce Eq. (2.85) to the simple form

$$\frac{\partial \bar{f}}{\partial h} = \frac{\partial}{\partial v_z} \cdot \frac{v_x^2}{v_z^2} \cdot \frac{\partial}{\partial v_z} \bar{f}.$$

This equation allows an analytic solution in terms of the initial distribution function. Let us assume that the initial distribution function is a bi-Maxwellian with different temperatures. In this case the following solution is obtained [79]:

$$\bar{f}(h) = \frac{\Gamma(3/4)}{(2\pi)^{3/2} 2^{1/4}} \frac{m^2 |v_z|^{3/2}}{T_z^{3/2} T_x^{1/2}} \exp(-m v_x^2/2T_x) \times$$
$$\times \int_0^\infty \frac{\exp(-4 v_x^2 \lambda^2 h) \lambda^{1/4} \mathcal{J}_{-3/4}(\lambda v_z^2)\, d\lambda}{(m/2T_z)^2 + \lambda^2}. \tag{2.86}$$

It will be evident that the simplification of the distribution in the region of small v_z (not a complete equalization of the temperature) leads to the stabilization of the instability when $h \sim (\Delta T/T)^4$.

PROBLEMS

1. Estimate the level of fluctuations of the magnetic field as a consequence of an instability of the neutral sheet in the tail of the magnetosphere [79].

Assume that the magnetic field in the tail of the magnetosphere is in the z-direction and that it is inhomogeneous in the vertical direction (y-axis):

$$H_z(y) = H_0 \tanh(y/L). \tag{1}$$

The inhomogeneity of the magnetic field is due to the flow of electron current along the x-axis, so that the electron distribution is described by a Maxwellian distribution shifted by the velocity u [80]

$$f_{0e} = \frac{N_0}{\pi^{3/2} v_{Te}^3} \exp\left\{ -\frac{v^2}{v_{Te}^2} + 2u \left(v_x - \frac{e}{mc} A_{0x} \right) \Big/ v_{Te}^2 \right\}, \tag{2}$$

where A_{0x} is the vector potential of the unperturbed magnetic field. As first shown by Laval et al. [81], this state is unstable against electromagnetic perturbations which propagate along the weak magnetic field. Since the basic contribution to the growth rate comes from electrons in a narrow band close to the neutral sheet $[|y| < (r_{He}L)^{1/2}]$, where it is it is permissible to neglect the effect of the magnetic field on the electron trajectories, the equations for the perturbations as well as the quasilinear equation for the perturbed distribution function coincide with Eqs. (2.77) and (2.85); these expressions have been obtained for the case in which the excess energy of the particles in the x-direction is not due to the electron current but to a somewhat higher temperature than in the z-direction. For convenience in the linear analysis of the stability of the neutral sheet we express the fields in terms of the vector potential A_x:

$$H_y = ik_z A_x(y) \exp(i k_z z - i\omega t),$$
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In this case Maxwell's equation for A_x becomes

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In this case Maxwell's equation for A_x becomes

$$-d^2 A_x/dy^2 + k^2 A_x = 4\pi/c \sum_j e_j \int d^3 v\, v_x f'_{kj}. \qquad (4)$$

The rapidly oscillating part of the distribution function contains two terms. Outside of the region of the neutral sheet ($|y| > d_e$) we can neglect the effect of the turbulence on the magnetized electrons; it can be assumed that they are in a magnetostatic equilibrium $f_{kj} = \partial f_{0j} A_x / \partial A_{0x}$. Inside the sheet ($|y| < d_e$) we describe the distribution function by Eq. (2.81) which has been obtained earlier; in this equation we neglect the effect of the magnetic field on the particle motion. We then find that Eq. (4) is given by

$$-d^2 A_x/dy^2 + V(y) A_x = \lambda A_x, \qquad (5)$$

where

$$V(y) = \begin{matrix} V^{>} & , & |y| > d_e, \\ V^{<} + |\omega/k|^2 Q, & & |y| < d_e, \end{matrix} \qquad \lambda \equiv -k^2$$

$$V^{>} = \sum_j \frac{4\pi e_j}{c} \int d^3 \mathrm{v}\, v_x \frac{\partial f_{0j}}{\partial A_{0x}} < 0;$$

$$V^{<} = \frac{4\pi e^2}{mc^2} \int d^3 \mathrm{v} \left(\bar{f}_e + \frac{v_x^2}{v_z} \cdot \frac{\partial \bar{f}_e}{\partial v_z} \right) > 0; \quad Q = -\frac{4\pi e^2}{mc^2} \int d^3 \mathrm{v}\, \frac{v_x^2}{v_z^2} \cdot \frac{\partial^2 \bar{f}_e}{\partial v_z^2}.$$

The growth of the magnetic fluctuations and the relaxation of the particle distribution are terminated when the plasma reaches the stability threshold. In this case Eq. (5) will not have solutions with characteristic values $\lambda < 0$ and growth rate $\omega = 0$ [82]. Since $V^{<} > 0$, this obviously occurs when $V^{<} d_e \sim 1$. Making use of Eq. (2.86) for the averaged distribution function we rewrite this condition in the form

$$d_e^2 V^{<} \approx \frac{\omega_p^2}{c^2} \cdot \frac{h^{1/4}}{v_{Te}^{1/2}} d_e^2 \sim 1. \qquad (6)$$

It then follows that the amplitude of the fluctuating fields is given by

$$\sum_k |\mathbf{H}_k|^2 / H_0^2 \approx k^2 r_{He}^2 (r_{He}/L)^4. \qquad (7)$$

▶ 2. Calculate the velocity of propagation of a weak shock wave along a magnetic field in a plasma with cold electrons under the assumption that the dissipation of energy in the wave front is due to the development of the firehose instability [83].

The problem reduces to a search for the critical velocity of a finite amplitude perturbation ($-U_0$) for which the quasilinear relaxation leads to an increase in the pressure anisotropy at the leading edge of the wave. The difference from the case of quasilinear relaxation in time lies in the fact that now there is a rearrangement of the distribution of "resonance" ions, which move with the wave; this rearrangement yields the basic contribution in the change of the pressure anisotropy. Introducing the variable $h = \sum_k |\mathbf{H}_k|^2/H_0^2$ in Eq. (2.70) and retaining the term with the second derivative in the longitudinal velocity, we have

$$(v_{\parallel} - U_0)(\partial f/\partial h) = -U_0 v_{\perp}^2 (\partial^2 f/\partial v_{\parallel}^2). \qquad (1)$$

In contrast with the usual quasilinear plateau parallel to the abscissa axis, this equation describes the establishment of a plateau with a definite slope. Hence, at small wave velocities, the particles lose some part of their longitudinal energy to the wave, but at large velocities they acquire energy (the difference $p_{\parallel} - p_{\perp}$ is increased). Using the Laplace transform we can find the distortion of the distribution near the resonance velocity and compute the change in anisotropy:

$$p_{\parallel} - p_{\perp} \approx \frac{4p_0}{3\sqrt{\pi}} \Gamma\left(\frac{4}{3}\right) \left(\frac{9hMU_0^2}{2T}\right)^{2/3} \exp\left(-\frac{MU_0^2}{2T}\right) \times$$
$$\times \left[\frac{MU_0^2}{2T} - \frac{5}{6} - \frac{\pi\sqrt{3}}{\Gamma^3(1/3)}\right]. \qquad (2)$$

The wave velocity is then found to be

$$3p_0/\rho_0 > U_0^2 \geqslant \left[\frac{5}{3} + \frac{2\pi\sqrt{3}}{\Gamma^3(1/3)}\right] p_0/\rho_0 > \frac{5}{3} \cdot \frac{p_0}{\rho_0}. \qquad (3)$$

If one retains terms of the next order ($\sim h^{4/3}$) in the expansion of the pressure anisotropy in the wave amplitude it is easy to show that a further increase in the field amplitude h causes the anisotropy to disappear so that the particle distribution becomes isotropic behind the wave front.

§ 2.7. Quasilinear Theory of the Drift Instability

The drift instability of an inhomogeneous plasma, first described by Rudakov and Sagdeev [84], leads to an anomalous increase in the flux of particles and heat across the magnetic field, and thus represents an important limitation on the lifetime of a

The rapidly oscillating part of the distribution function contains two terms. Outside of the region of the neutral sheet ($|y| > d_e$) we can neglect the effect of the turbulence on the magnetized electrons; it can be assumed that they are in a magnetostatic equilibrium $f_{kj} = \partial f_{0j} A_x / \partial A_{0x}$. Inside the sheet ($|y| < d_e$) we describe the distribution function by Eq. (2.81) which has been obtained earlier; in this equation we neglect the effect of the magnetic field on the particle motion. We then find that Eq. (4) is given by

$$-d^2 A_x/dy^2 + V(y) A_x = \lambda A_x, \tag{5}$$

where

$$V(y) = \begin{cases} V^{>}, & |y| > d_e, \\ V^{<} + |\omega/k|^2 Q, & |y| < d_e, \end{cases} \qquad \lambda \equiv -k^2$$

$$V^{>} = \sum_j \frac{4\pi e_j}{c} \int d^3 \mathrm{v}\, v_x \frac{\partial f_{0j}}{\partial A_{0x}} < 0;$$

$$V^{<} = \frac{4\pi e^2}{mc^2} \int d^3 \mathrm{v} \left(\bar{f}_e + \frac{v_x^2}{v_z} \cdot \frac{\partial \bar{f}_e}{\partial v_z} \right) > 0; \quad Q = -\frac{4\pi e^2}{mc^2} \int d^3 \mathrm{v}\, \frac{v_x^2}{v_z^2} \cdot \frac{\partial^2 \bar{f}_e}{\partial v_z^2}.$$

The growth of the magnetic fluctuations and the relaxation of the particle distribution are terminated when the plasma reaches the stability threshold. In this case Eq. (5) will not have solutions with characteristic values $\lambda < 0$ and growth rate $\omega = 0$ [82]. Since $V^{<} > 0$, this obviously occurs when $V^{<} d_e \sim 1$. Making use of Eq. (2.86) for the averaged distribution function we rewrite this condition in the form

$$d_e^2 V^{<} \approx \frac{\omega_p^2}{c^2} \cdot \frac{h^{1/4}}{v_{Te}^{1/2}}\, d_e^2 \sim 1. \tag{6}$$

It then follows that the amplitude of the fluctuating fields is given by

$$\sum_k |\mathbf{H}_k|^2 / H_0^2 \approx k^2 r_{He}^2 (r_{He}/L)^4. \tag{7}$$

▶ 2. Calculate the velocity of propagation of a weak shock wave along a magnetic field in a plasma with cold electrons under the assumption that the dissipation of energy in the wave front is due to the development of the firehose instability [83].

The problem reduces to a search for the critical velocity of a finite amplitude perturbation ($-U_0$) for which the quasilinear relaxation leads to an increase in the pressure anisotropy at the leading edge of the wave. The difference from the case of quasilinear relaxation in time lies in the fact that now there is a rearrangement of the distribution of "resonance" ions, which move with the wave; this rearrangement yields the basic contribution in the change of the pressure anisotropy. Introducing the variable $h = \sum_k |\mathbf{H}_k|^2 / H_0^2$ in Eq. (2.70) and retaining the term with the second derivative in the longitudinal velocity, we have

$$(v_\parallel - U_0)(\partial f / \partial h) = -U_0 v_\perp^2 (\partial^2 f / \partial v_\parallel^2). \tag{1}$$

In contrast with the usual quasilinear plateau parallel to the abscissa axis, this equation describes the establishment of a plateau with a definite slope. Hence, at small wave velocities, the particles lose some part of their longitudinal energy to the wave, but at large velocities they acquire energy (the difference $p_\parallel - p_\perp$ is increased). Using the Laplace transform we can find the distortion of the distribution near the resonance velocity and compute the change in anisotropy:

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We shall only consider one of the many branches of the drift wave which develops in a plane, low-pressure ($\beta \ll 1$) plasma; the growth rate is much smaller than the oscillation frequency. The basic ideas and detailed derivations are given in the well-known monograph of Mikhailovskii [VI], to which the reader is referred. We first expand the potential of the electric field of the perturbation in terms of plane waves where

$$\Phi = \sum_{\mathbf{k},\,\omega} \Phi(\mathbf{k}, \omega) \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)] \tag{2.87}$$

and describe the plasma particles by means of the so-called "drift" kinetic equation:

$$\frac{\partial}{\partial t} f_j + v_z \frac{\partial}{\partial z} f_j - \frac{c}{H_0}\cdot\frac{\partial \Phi}{\partial y}\cdot\frac{\partial f_j}{\partial x} - \frac{e_j}{m_j}\cdot\frac{\partial \Phi}{\partial z}\cdot\frac{\partial f_j}{\partial v_z} = 0, \tag{2.88}$$

where the z-axis is taken in the direction of the unperturbed magnetic field $H_0 = \{0, 0, H_0\}$ while the x-axis is in the direction of the gradient of the plasma density n(x). Linearizing Eq. (2.88) with respect to small perturbations, we find the rapidly oscillating correction to the particle distribution function:

$$f_{1j} = -\frac{e_j \Phi}{m_j}\left[k_z \frac{\partial f_{0j}}{\partial v_z} + \frac{k_y}{\omega_{Hj}}\cdot\frac{\partial f_{0j}}{\partial x}\right](\omega_k - k_z v_z)^{-1}. \tag{2.89}$$

We consider oscillations with phase velocities in the range $v_{Ti} \ll \omega/k_z \lessgtr v_A \ll v_{Te}$. The electrons (except for the resonant electrons) are able to maintain a Boltzmann distribution in the electrostatic field of the wave. Thus, the perturbation in electron density can be written

$$n_{1,e} = \frac{e\Phi}{T_e} n_0 \left\{1 - \frac{i\pi T_e}{|k_z| m n_0}\left(k_z \frac{\partial}{\partial z} + \frac{k_y}{\omega_H}\cdot\frac{\partial}{\partial x}\right) f_{0e}(v_z, x)\right\}, \tag{2.90}$$

where the distribution of nonresonant electrons is assumed to be a Maxwellian at temperature T_e. We neglect terms that describe the longitudinal motion in the ion equation and only consider the electric drift. The remaining terms are contained in the equation

of continuity:

$$\partial n_{1i}/\partial t - (c/H_0)(\partial\Phi/\partial y)(\partial n_0/\partial x) = 0. \tag{2.91}$$

Using the plasma neutrality condition and Eqs. (2.90) and (2.91), we obtain a dispersion equation for the frequency and growth rate of the oscillations:

$$\omega = k_y v_*^e; \quad v_*^e \equiv -\frac{cT_e}{eH_0 n_0}\cdot\frac{dn_0}{dx}; \tag{2.92}$$

$$\gamma \approx \frac{\pi\omega T_e}{mn_0 |k_z|}\left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\omega_H}\cdot\frac{\partial}{\partial x}\right) f_{0e}(v_x, x). \tag{2.93}$$

It will be evident that the growth rate of the instability is zero in a plasma with a Maxwellian electron distribution and a uniform temperature (neutral equilibrium). A mechanism for excitation of the oscillations in this case is found in the fact that the ion Larmor radius is finite; this feature has not been taken into account here. In the limit in which the wavelength is much larger than the ion Larmor radius the correction to the frequency associated with this effect can be found by introducing higher-order drift terms in the equation of continuity (inertial drift):

$$\frac{\partial n_{1i}}{\partial t} + n_0 \operatorname{div}\left\{-\frac{c}{H_0^2}[\nabla\Phi \times \mathbf{H}_0] - \frac{c}{\Omega_H H_0}\frac{\partial}{\partial t}\left(\nabla\Phi + \frac{T_{0i}}{n_0 e}\nabla n_{1i}\right)\right\} = 0.$$

From Eq. (2.90) we then have

$$\left.\begin{aligned} \omega_k &= k_y v_*^e (1 - k^2 R_H^2), \\ \gamma_k^{(M)} &= \frac{\sqrt{\pi}\,\omega}{|k_z| v_{Te}}(\omega - k_y v_*^e), \end{aligned}\right\} \tag{2.94}$$

where $\gamma_k^{(M)}$ is the growth rate for the drift wave for a Maxwellian electron distribution. It will be evident that the growth rate is very small so that even small nonlinear distortions of the distribution function can have an effect on the stability of the plasma with respect to finite perturbations.

We shall now consider two such nonlinear effects. The first of these is associated with the broadening of the region of resonance velocities in the field of a monochromatic wave of finite amplasma in a confinement device. For this reason a great deal of importance attaches to a correct description of the nonlinear stage of the instability and the resulting turbulent state.

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We shall now consider two such nonlinear effects. The first of these is associated with the broadening of the region of resonance velocities in the field of a monochromatic wave of finite amplitude, this broadening being given approximately by $\sim(e\Phi_0/m)^{1/2}$. This broadening process is capable of limiting the growth of the drift wave. The second mechanism is the saturation of the instability due to the relaxation of the distribution of resonant electrons.

Nonlinear Stability of a Monochromatic Drift Wave. We choose the electric potential of the monochromatic wave to be of the form

$$\Phi(y, z, t) = -\Phi_0 [\cos(k_y y + k_z z - \omega t) + 0(e\Phi_0/T)],$$

where, for simplicity, we have taken $k_z \equiv 0$. In the coordinate system which moves with the wave the rate of reduction of the kinetic energy of the resonant particles, which must be equal to the growth rate of the wave, can be written in the form

$$\frac{d\mathscr{E}}{dt} = \frac{n_0 m}{2} \int_{-\lambda_z/2}^{\lambda_z/2} \frac{dz}{\lambda_z} \int_{-\infty}^{+\infty} dv_z \left(v_z + \frac{\omega}{k_z}\right)^2 \frac{\partial f_e}{\partial t}. \tag{2.95}$$

In the drift approximation, the general solution for the kinetic equation is

$$f_e(\mathbf{r}, v_z, t) = f_e[\mathbf{r}_0(\mathbf{r}, v_z, t), v_{z0}(\mathbf{r}, v_z, t), 0],$$

where $f_e(\mathbf{r}_0, v_{z0}, 0)$ is the initial distribution while $(\mathbf{r}_0, v_{z0})$ is the initial position of the particle. The distribution function can be divided into two parts:

$$f_e(\mathbf{r}_0, v_{z0}, 0) = f_0(x_0, v_{z0}) + f_1(x_0, v_{z0}, 0)\cos(k_y y_0 + k_z z_0), \tag{2.96}$$

where the first part f_0 is a local Maxwellian function while the second part describes the perturbation of the particle distribution due to the wave. The second part only makes a contribution in the generation of harmonics and can be neglected in computing the energy balance. We then have

$$\frac{\partial f_e}{\partial t} = \frac{\partial f_e(x_0, v_{z0}, 0)}{\partial x_0} \cdot \frac{\partial x_0}{\partial t} + \frac{\partial f_e(x_0, v_{z0}, 0)}{\partial v_{z0}} \cdot \frac{\partial v_{z0}}{\partial t} =$$
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Here, we have used the drift equations to describe the particle motion:

$$\left.\begin{aligned} \dot{y} &= 0; \\ \dot{x} &= c(k_y \Phi_0/H_0)\sin(k_y y + k_z z); \\ \ddot{z} &= -(ek_z \Phi_0/m)\sin(k_y y + k_z z). \end{aligned}\right\} \tag{2.98}$$

The solution of these equations can be given in terms of elliptic integrals. We first introduce the new variable $2\xi = k_y y + k_z z$ and rewrite the energy-conservation equation

$$1/2 m\dot{z}^2 - e\Phi_0 \cos(k_y y + k_z z) = \mathscr{E} \tag{2.99}$$

in the form

$$\dot{\xi}^2 = 1/\varkappa^2\tau^2 (1 - \varkappa^2 \sin^2 \xi), \tag{2.100}$$

where

$$\varkappa^2 = 2e\Phi_0/(\mathscr{E} + e\Phi_0), \quad \tau = (m/e\Phi_0 k_z^2)^{1/2}.$$

When $\varkappa^2 < 1$, the solution of Eq. (2.100) is

$$F(\varkappa, \xi_0) = F(\varkappa, \xi) - t/\varkappa\tau. \tag{2.101}$$

When $\varkappa^2 > 1$, use is made of the substitution

$$\varkappa \sin \xi = \sin \zeta, \quad \dot{\zeta}^2 = 1/\tau^2 (1 - 1/\varkappa^2 \sin^2 \zeta).$$

The solution of the equation is then

$$F(1/\varkappa, \zeta_0) = F(1/\varkappa, \zeta) - t/\tau. \tag{2.102}$$

In order to find the time dependence of the growth rate we need only substitute the solutions that have been found in Eqs. (2.95) and (2.97). This problem has been solved by O'Neil in the approximation in which the result coincides with the linear growth rate at $t = 0$ (cf. § 2.1).

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We wish to take account of nonlinear effects which operate even at $t = 0$, so that it is necessary to include more terms than have been treated in the O'Neil analysis. Substituting Eq. (2.97) into Eq. (2.95) we have

$$\frac{d\dot{\mathscr{E}}}{dt}=\frac{16n_0\,e\omega\Phi_0}{\pi k_z^2}\int_0^{\infty}d\dot{\xi}\int_0^{\pi/2}d\xi\dot{\xi}\,\Bigg[\left(k_z\,\frac{\partial f_{0e}}{\partial v_{z0}}+\frac{k_y}{\omega_H}\cdot\frac{\partial f_{0e}}{\partial x_0}\right)\Bigg|_{\substack{x_0=x\\ v_{z0}=0}}+$$
$$+\frac{\partial^2}{\partial v_z^2}\left(k_z\,\frac{\partial f_{0e}}{\partial v_{z0}}+\frac{k_y}{\omega_H}\cdot\frac{\partial f_{0e}}{\partial x_0}\right)\Bigg|_{\substack{x_0=x\\ v_{z0}=0}}\dot{\xi}^2/4k_z^2\Bigg], \tag{2.103}$$

where we have neglected even powers because of symmetry considerations. Carrying out the integration and, for simplicity, limiting ourselves to the limit $t/\tau \ll 1$ (the general case is treated in [1]), we find the instability growth rate

$$\gamma_k=-\frac{2T_e}{n_0\,e^2\,|\Phi_0|^2}\,\frac{dW}{dt}=\pi^{1/2}\,\frac{\omega^2}{|k_z|\,v_{Te}}\left(k_\perp^2\,r_{Hi}^2-\frac{4e\,\Phi_0}{T_e}\right). \tag{2.104}$$

Consequently, a wave of finite amplitude can become stabilized in the nonlinear stage if its linear growth rate is small enough. Here we are only treating the particular case in which the oscillations are excited by taking account of the finite Larmor radius, but these results also apply for the case in which excitation is due to current flow or any other mechanism. Hence, one may reasonably expect to use this analysis in discussing instabilities in a Q-machine plasma. Since the amplitude of the oscillations found from the condition $\gamma_k = 0$ is small, $e\Phi_0/T_e \approx \frac{1}{4}k^2r_{Hi}^2 \ll 1$, the amplitudes of the higher harmonics observed in certain experiments can be estimated from a simple expansion of the kinetic equation with respect to $e\Phi_0/T_e$. For example, the second harmonic is given by $e\Phi_{2k}^{(2)}/T_e \approx (1/2)(e\Phi_k^{(1)}/T_e)^2$. Consequently one expects that the amplitude of the harmonics will fall off exponentially as a function of frequency.

The pattern described here will also hold in the case of a narrow wave packet: $\Delta(\omega/k_z) \ll (e\Phi_0/m)^{1/2}$. However, in the experiments the instability can develop over a wide range of phase velocities and this condition is violated. The stabilization effect is then due to the relaxation of the electron distribution function.

Quasilinear Relaxation of the Particle Distribution and Transport Processes [85, 86]. In the usual way, the particle distribution function is written in the form of a sum of a slowly varying part and a rapidly varying part (i.e., $f_j = \bar{f}_j + \delta f_j$) and the kinetic equation is averaged over the fast

oscillations; using this procedure we obtain the quasilinear equation for the slowly varying distribution function:

$$\frac{\partial}{\partial t}\bar{f}_j=\left\langle\frac{c}{H^2}[\nabla\Phi\times H]\cdot\nabla\delta f_j+\frac{e}{m}\cdot\frac{\partial\Phi}{\partial z}\cdot\frac{\partial}{\partial v_z}\delta f_j\right\rangle.$$

Substituting the expression obtained eariler for the rapidly varying part of the distribution function [cf. Eq. (2.89)] we rewrite it in the form

$$\left.\begin{aligned}&\partial\bar{f}_e/\partial t=\mathrm{St}_{QL}(\bar{f}_e);\\&\mathrm{St}_{QL}(\bar{f}_e)=\frac{e^2}{m^2}\sum_k\left(k_z\,\frac{\partial}{\partial v_z}+\frac{k_y}{\omega_H}\cdot\frac{\partial}{\partial x}\right)|\Phi_k|^2\,\pi\delta\,\times\\&\qquad\times(\omega_k-k_z\,v_z)\left(k_z\,\frac{\partial}{\partial v_z}+\frac{k_y}{\omega_H}\cdot\frac{\partial}{\partial x}\right)\bar{f}_e.\end{aligned}\right\} \tag{2.105}$$

The principal part of the operator $(\omega_k - k_zv_z)$ has been neglected since the relaxation of the distribution of nonresonant electrons occurs at a much smaller rate than the distribution of resonant electrons. On the other hand, in the quasilinear ion equation we can neglect the exponentially small fraction of resonant particles and only take account of the adiabatic change in the distribution of nonresonant particles:

$$\frac{\partial\bar{f}_i}{\partial t}=\frac{e^2}{M^2}\sum_{\mathrm{k}}\left(k_z\,\frac{\partial}{\partial v_z}+\frac{k_y}{\Omega_H}\cdot\frac{\partial}{\partial x}\right)\frac{\gamma_{\mathrm{k}}\,|\Phi_{\mathrm{k}}|^2}{\omega_k^2}\left(k_z\,\frac{\partial}{\partial v_z}+\frac{k_y}{\Omega_H}\cdot\frac{\partial}{\partial x}\right)\bar{f}_i. \tag{2.106}$$

In the shortwave limit we then substitute the electric field averaged over the ion Larmor orbit $[|\langle\Phi_k\rangle|^2 \approx |\Phi_k|^2 I_0^2(k_\perp v_\perp/\Omega_H)]$.

We first consider the formation of a plateau. For this purpose it is necessary to simplify the diffusion equation in (x, v_z)-space, making use of the assumption that the basic contribution to the diffusion coefficient comes from waves characterized by the maximum growth rate (that is to say, account is taken only of waves with wave vector $\mathbf{k} = \bar{\mathbf{k}}$, where $\bar{k}_z = \omega_k/v_A$ while $\bar{k}_y$ is determined, as will be shown below, by the competing effects of plateau formation and the establishment of a Maxwellian distribution by collisions). If the new variables

$$\eta=v_z^2/2,\quad \xi=v_z^2/2-\omega_k\,\omega_H\,x/2\bar{k}_y, \tag{2.107}$$

are introduced, the differential operator in Eq. (2.105) is reduced into Eq. (2.95) we have

$$\frac{d\dot{\mathscr{E}}}{dt}=\frac{16n_0\,e\omega\Phi_0}{\pi k_z^2}\int\limits_0^\infty d\dot{\xi}\int\limits_0^{\pi/2} d\xi\dot{\xi}\,\left[k_z\,\frac{\partial f_{0e}}{\partial v_{z0}}+\frac{k_y}{\omega_H}\cdot\frac{\partial f_{0e}}{\partial x_0}\right)\Bigg|_{\substack{x_0=x\\ v_{z0}=0}}+$$
$$+\frac{\partial^2}{\partial v_z^2}\left(k_z\,\frac{\partial f_{0e}}{\partial v_{z0}}+\frac{k_y}{\omega_H}\cdot\frac{\partial f_{0e}}{\partial x_0}\right)\Bigg|_{\substack{x_0=x\\ v_{z0}=0}}\dot{\xi}^2/4k_z^2\Bigg], \tag{2.103}$$

where we have neglected even powers because of symmetry considerations. Carrying out the integration and, for simplicity, limiting ourselves to the limit $t/\tau \ll 1$ (the general case is treated in [1]), we find the instability growth rate

$$\gamma_k=-\frac{2T_e}{n_0\,e^2\,|\,\Phi_0\,|^2}\,\frac{dW}{dt}=\pi^{1/2}\frac{\omega^2}{|\,k_z\,|\,v_{Te}}\left(k_\perp^2\,r_{Hi}^2-\frac{4e\,\Phi_0}{T_e}\right). \tag{2.104}$$

Consequently, a wave of finite amplitude can become stabilized in the nonlinear stage if its linear growth rate is small enough. Here we are only treating the particular case in which the oscillations are excited by taking account of the finite Larmor radius, but these results also apply for the case in which excitation is due to current flow or any other mechanism. Hence, one may reasonably expect to use this analysis in discussing instabilities in a Q-machine plasma. Since the amplitude of the oscillations found from the condition $\gamma_k = 0$ is small, $e\Phi_0/T_e \approx \frac{1}{4}k^2r_{Hi}^2 \ll 1$, the amplitudes of the higher harmonics observed in certain experiments can be estimated from a simple expansion of the kinetic equation with respect to $e\Phi_0/T_e$. For example, the second harmonic is given by $e\Phi_{2k}^{(2)}/T_e \approx (1/2)(e\Phi_k^{(1)}/T_e)^2$. Consequently one expects that the amplitude of the harmonics will fall off exponentially as a function of frequency.

The pattern described here will also hold in the case of a narrow wave packet: $\Delta(\omega/k_z) \ll (e\Phi_0/m)^{1/2}$. However, in the experiments the instability can develop over a wide range of phase velocities and this condition is violated. The stabilization effect is then due to the relaxation of the electron distribution function.

Quasilinear Relaxation of the Particle Distribution and Transport Processes [85, 86]. In the usual way, the particle distribution function is written in the form of a sum of a slowly varying part and a rapidly varying part (i.e., $f_j = \bar{f}_j + \delta f_j$) and the kinetic equation is averaged over the fast

oscillations; using this procedure we obtain the quasilinear equation for the slowly varying distribution function:

$$\frac{\partial}{\partial t}\bar{f}_j=\left\langle\frac{c}{H^2}[\nabla\Phi\times H]\cdot\nabla\delta f_j+\frac{e}{m}\cdot\frac{\partial\Phi}{\partial z}\cdot\frac{\partial}{\partial v_z}\delta f_j\right\rangle.$$

Substituting the expression obtained eariler for the rapidly varying part of the distribution function [cf. Eq. (2.89)] we rewrite it in the form

$$\left.\begin{aligned}&\partial\bar{f}_e/\partial t=\mathrm{St}_{QL}(\bar{f}_e);\\ &\mathrm{St}_{QL}(\bar{f}_e)=\frac{e^2}{m^2}\sum_k\left(k_z\,\frac{\partial}{\partial v_z}+\frac{k_y}{\omega_H}\cdot\frac{\partial}{\partial x}\right)|\,\Phi_k|^2\,\pi\delta\times\\ &\qquad\times(\omega_k-k_z\,v_z)\left(k_z\,\frac{\partial}{\partial v_z}+\frac{k_y}{\omega_H}\cdot\frac{\partial}{\partial x}\right)\bar{f}_e.\end{aligned}\right\} \tag{2.105}$$

The principal part of the operator $(\omega_k - k_zv_z)$ has been neglected since the relaxation of the distribution of nonresonant electrons occurs at a much smaller rate than the distribution of resonant electrons. On the other hand, in the quasilinear ion equation we can neglect the exponentially small fraction of resonant particles and only take account of the adiabatic change in the distribution of nonresonant particles:

$$\frac{\partial\bar{f}_i}{\partial t}=\frac{e^2}{M^2}\sum_k\left(k_z\,\frac{\partial}{\partial v_z}+\frac{k_y}{\Omega_H}\cdot\frac{\partial}{\partial x}\right)\frac{\gamma_k\,|\,\Phi_k\,|^2}{\omega_k^2}\left(k_z\,\frac{\partial}{\partial v_z}+\frac{k_y}{\Omega_H}\cdot\frac{\partial}{\partial x}\right)\bar{f}_i. \tag{2.106}$$

In the shortwave limit we then substitute the electric field averaged over the ion Larmor orbit $[|\langle\Phi_k\rangle|^2 \approx |\Phi_k|^2 I_0^2(k_\perp v_\perp/\Omega_H)]$.

We first consider the formation of a plateau. For this purpose it is necessary to simplify the diffusion equation in (x, v_z)-space, making use of the assumption that the basic contribution to the diffusion coefficient comes from waves characterized by the maximum growth rate (that is to say, account is taken only of waves with wave vector $\mathbf{k} = \bar{\mathbf{k}}$, where $\bar{k}_z = \omega_k/v_A$ while $\bar{k}_y$ is determined, as will be shown below, by the competing effects of plateau formation and the establishment of a Maxwellian distribution by collisions). If the new variables

$$\eta=v_z^2/2,\quad \xi=v_z^2/2-\omega_k\,\omega_H\,x/2\bar{k}_y, \tag{2.107}$$

are introduced, the differential operator in Eq. (2.105) is reduced to the form

$$\frac{\partial}{\partial \eta} \equiv \left(\frac{1}{v_z} \cdot \frac{\partial}{\partial v_z} + \frac{\bar{k}_y}{\omega_{\mathbf{k}} \omega_H} \cdot \frac{\partial}{\partial x} \right).$$

Consequently, the level curves along which the plateau is established are described by the equation $x - \bar{k}_y v_z^2 / 2\omega_H \omega_{\mathbf{k}} \equiv \xi = \text{const}$. Since $v \ll v_{Te}$ in the resonance region, the level curves for a Maxwellian distribution can be described approximately by the relation $x - v_z^2 / 2\omega_H v_*^2 = \text{const}$. For the linear waves, $\omega_k = k_y v_*^e \times (1 - k_\perp^2 R_H^2)$ and the two families of level curves differ from each other only as a consequence of the finite Larmor radius of the ions (Fig. 28). Hence, the energy evolved as a result of the relaxation of the electron distribution is also small in this limit.

If the relaxing electron velocity distribution v_z is plotted for fixed x it will be evident that the slope of the distribution function in the resonance region becomes sharper, as a consequence of which the Landau damping is increased so that the wave growth is saturated. In the relaxation process the electrons lose energy associated with their motion along the magnetic field. It can then be said that the source of energy of the growing fluctuations is the energy associated with the thermal motion of the electrons along the field. It is possible to estimate the electron displacement in the relaxation process from the condition $\xi = \text{const}$:

$$\delta x = \frac{\bar{k}_y}{\omega_{\mathbf{k}} \omega_H} \cdot \frac{\delta v_z^2}{2}. \tag{2.108}$$

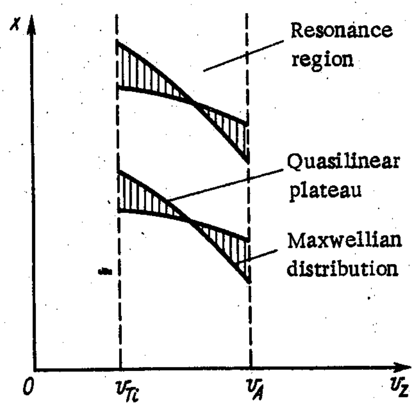


Fig. 28. Level curves for a Maxwellian distribution and a distribution with a quasilinear plateau in the resonance region.

Since $\delta v_z^2 \ll v_A^2$ the displacement of resonance electrons is much smaller than the plasma radius ($\delta x \sim n_0 v_A^2 / n_0' v_{Te}^2$). In other words, the instability is self-saturating so that no significant change of the electron density occurs. Integrating the quasilinear equations for the ions and electrons one can easily show that both plasma components diffuse across the magnetic field at the same rate. For example, taking Eq. (2.105) for the electrons we have

$$\frac{\partial n}{\partial t} = \frac{e^2}{m^2} \sum_{\mathbf{k}} \frac{k_y}{\omega_H} \cdot \frac{\partial}{\partial x} |\Phi_{\mathbf{k}}|^2 \int d^3 \mathbf{v} \pi \delta (\omega_{\mathbf{k}} - k_z v_z) \left(k_z \frac{\partial}{\partial v_z} + \right.$$
$$\left. + \frac{k_y}{\omega_H} \cdot \frac{\partial}{\partial x} \right) f_e = \frac{\partial}{\partial x} \left(\frac{c^2}{H_0^2} \sum_{\mathbf{k}} k_y^2 |\Phi_{\mathbf{k}}|^2 \frac{\gamma_{\mathbf{k}}}{\omega_{\mathbf{k}} k_y v_*^e} \cdot \frac{\partial n}{\partial x} \right), \tag{2.109}$$

where use has been made of Eq. (2.93) for the growth rate γ_k. If, furthermore, one takes account of the fact that $\omega_k \approx k_y v_*^e$, it turns out that the coefficient of resonance diffusion for the electrons coincides with the diffusion coefficient for the nonresonant ions:

$$D_\perp \approx \sum_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \cdot \frac{c^2}{H_0^2} k_y^2 |\Phi_{\mathbf{k}}|^2 .$$

Furthermore, since there is no plasma diffusion in the absence of collisions, to obtain a diffusion effect it is necessary to take account of the small collisional rate on the right side of Eq. (2.105). Collisions tend to establish a Maxwellian distribution and thus prevent the formation of a plateau. Hence when collisions are included we may properly expect to find diffusion. The electron equation assumes the form

$$\partial f_e / \partial t = \mathrm{St}_{QL} + \mathrm{St}_{\mathrm{coll}}, \tag{2.110}$$

where

$$\mathrm{St}_{\mathrm{coll}}(f_e) \equiv \nu_e v_{Te}^2 (\partial^2 (f_e - f_{Me}) / \partial v_z^2).$$

Since we are interested in the case of low collision rates (i.e., $\mathrm{St}_{\mathrm{coll}} \ll \mathrm{St}_{QL}$) Eq. (2.110) can be solved by successive approximations [86]. The distribution function is written in the form $f_e = f_e^{(0)} + f_e^{(1)}$, where $f_e^{(0)}$ is the solution of the quasilinear equation

$$\mathrm{St}_{QL} [f_e^{(0)}] = 0. \tag{2.111}$$

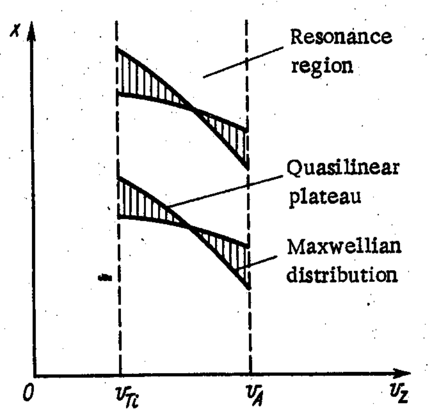
to the form

$$\frac{\partial}{\partial \eta} \equiv \left(\frac{1}{v_z} \cdot \frac{\partial}{\partial v_z} + \frac{\bar{k}_y}{\omega_k \omega_H} \cdot \frac{\partial}{\partial x}\right).$$

Consequently, the level curves along which the plateau is established are described by the equation $x - \bar{k}_y v_z^2 / 2\omega_H \omega_k \equiv \xi = \text{const}$. Since $v \ll v_{Te}$ in the resonance region, the level curves for a Maxwellian distribution can be described approximately by the relation $x - v_z^2/2\omega_H v_*^2 = \text{const}$. For the linear waves, $\omega_k = k_y v_*^e \times (1 - k_\perp^2 R_H^2)$ and the two families of level curves differ from each other only as a consequence of the finite Larmor radius of the ions (Fig. 28). Hence, the energy evolved as a result of the relaxation of the electron distribution is also small in this limit.

If the relaxing electron velocity distribution v_z is plotted for fixed x it will be evident that the slope of the distribution function in the resonance region becomes sharper, as a consequence of which the Landau damping is increased so that the wave growth is saturated. In the relaxation process the electrons lose energy associated with their motion along the magnetic field. It can then be said that the source of energy of the growing fluctuations is the energy associated with the thermal motion of the electrons along the field. It is possible to estimate the electron displacement in the relaxation process from the condition $\xi = \text{const}$:

$$\delta x = \frac{\bar{k}_y}{\omega_k \omega_H} \cdot \frac{\delta v_z^2}{2}. \tag{2.108}$$



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$$\left. + \frac{k_y}{\omega_H} \cdot \frac{\partial}{\partial x}\right) f_e = \frac{\partial}{\partial x}\left(\frac{c^2}{H_0^2} \sum_{\mathbf{k}} k_y^2 |\Phi_{\mathbf{k}}|^2 \frac{\gamma_{\mathbf{k}}}{\omega_{\mathbf{k}} k_y v_*^e} \cdot \frac{\partial h}{\partial x}\right), \tag{2.109}$$

where use has been made of Eq. (2.93) for the growth rate γ_k. If, furthermore, one takes account of the fact that $\omega_k \approx k_y v_*^e$, it turns out that the coefficient of resonance diffusion for the electrons coincides with the diffusion coefficient for the nonresonant ions:

$$D_\perp \approx \sum_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \cdot \frac{c^2}{H_0^2} k_y^2 |\Phi_{\mathbf{k}}|^2 .$$

Furthermore, since there is no plasma diffusion in the absence of collisions, to obtain a diffusion effect it is necessary to take account of the small collisional rate on the right side of Eq. (2.105). Collisions tend to establish a Maxwellian distribution and thus prevent the formation of a plateau. Hence when collisions are included we may properly expect to find diffusion. The electron equation assumes the form

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Since we are interested in the case of low collision rates (i.e., $\mathrm{St}_{\mathrm{coll}} \ll \mathrm{St}_{QL}$) Eq. (2.110) can be solved by successive approximations [86]. The distribution function is written in the form $f_e = f_e^{(0)} + f_e^{(1)}$, where $f_e^{(0)}$ is the solution of the quasilinear equation

$$\mathrm{St}_{QL}\left[f_e^{(0)}\right] = 0. \tag{2.111}$$

In this case $f_e^{(1)}$ is given by the equation

$$\mathrm{St}_{QL}\left[f_e^{(1)}\right]+\mathrm{St}_{\mathrm{coll}}\left[f_e^{(0)}\right]=0. \tag{2.112}$$

Equation (2.111) has a simple solution with a plateau so that

$$\frac{1}{v_z}\cdot\frac{\partial f_e^{(0)}}{\partial v_z}=-\frac{k_y}{\omega_{\mathbf{k}}\,\omega_H}\cdot\frac{\partial f_e^{(0)}}{\partial x}. \tag{2.113}$$

In order to solve Eq. (2.112) we integrate it with respect to v_z:

$$\int_0^{v_z} dv_z\left(\frac{e}{m}\right)^2\sum_{\mathbf{k}}\left(k_z\frac{\partial}{\partial v_z}+\frac{k_y}{\omega_H}\cdot\frac{\partial}{\partial x}\right)|\Phi_{\mathbf{k}}|^2\,\pi\delta(\omega_{\mathbf{k}}-k_z v_z)\times$$
$$\times\left(k_z\frac{\partial}{\partial v_z}+\frac{k_y}{\omega_H}\cdot\frac{\partial}{\partial x}\right)f_e^{(1)}=\nu_e v_{Te}^2\frac{\partial}{\partial v_z}\left(f_{Me}-f_e^{(0)}\right). \tag{2.114}$$

Since the dependence of $f_e^{(1)}$ on v_z is much stronger than on x, the derivative with respect to x in the first parentheses in Eq. (2.114) can be neglected. If, furthermore, use is made of the equation for the plateau (2.111) in order to estimate the second term on the right side of Eq. (2.114), we find

$$\left(\frac{e}{m}\right)^2\sum_{\mathbf{k}}|\Phi_{\mathbf{k}}|^2 k_z\,\pi\delta(\omega_{\mathbf{k}}-k_z v_z)\left(k_z\frac{\partial}{\partial v_z}+\frac{k_y}{\omega_H}\cdot\frac{\partial}{\partial x}\right)f_e^{(1)}=$$
$$=\nu_e v_{Te}^2 v_z\left(\frac{1}{v_z}\cdot\frac{\partial}{\partial v_z}+\frac{k_y}{\omega_{\mathbf{k}}\,\omega_H}\cdot\frac{\partial}{\partial x}\right)f_{Me}. \tag{2.115}$$

Integration of this expression with respect to $\mu = mv_\perp^2/2H$ then gives

$$\left(\frac{e}{m}\right)^2\sum_{\mathbf{k}}k_z^2|\Phi_{\mathbf{k}}|^2\gamma_{\mathbf{k}}^{(1)}\,\pi\delta(\omega_{\mathbf{k}}-k_z v_z)=\nu_e v_{Te}^2\,\varphi_{\mathbf{k}}^{(M)}, \tag{2.116}$$

where we have introduced the growth rate for the relaxing distribution in the presence of collisions $\gamma_{\mathbf{k}}^{(1)}$ and the growth rate for the Maxwellian plasma [cf. Eq. (2.94)]. Although this expression cannot be used to estimate the growth rate or the wave energy separately, it can still be used to estimate the diffusion coefficient, which depends specifically on the product of these quantities. Using Eqs. (2.109) and (2.116) we can then obtain the diffusion coefficient

$$D_\perp=\nu_e\left(\frac{n_0}{\partial n_0/\partial x}\right)^2\frac{\gamma_{\mathbf{k}}^{(M)}}{\bar{k}_y v_*^e}\left(\frac{\bar{k}_y v_*^e}{k_z v_{Te}}\right)^2. \tag{2.117}$$

In order to estimate the diffusion coefficient we must find the characteristic wave number $\bar{\mathbf{k}}$. In the case in which collisions can establish the Maxwellian distribution the turbulence spectrum has a peak at $kr_{Hi}\approx 1$. This feature is associated with the fact that the stimulated scattering of waves on ions quickly transfers particles from the shortwave region, where the linear growth rate is appreciably larger, into the region $kr_{Hi}\sim 1$ [87]. As a result of the reduction in the collision frequency the electron distribution relaxes and the growth rate of the instability for wavelengths of the order of the Larmor radius becomes negative (damping). Stabilization of the shortwave region of the spectrum requires a much sharper slope for the distribution function in the resonance region $T_e df_e^{(0)}/dmv_z^2\approx k_y v_*^e/\omega_{\mathbf{k}}\gg 1$. Hence, when perturbations with wavelengths of the order of the ion Larmor radius have been stabilized, the nonlinear transfer of energy from the shortwave region is reduced and the peak in the spectral energy density is displaced into this region. The coefficient of spatial diffusion can be estimated from Eq. (2.117):

$$D_\perp\approx\nu_e\left[\frac{n_0}{\partial n_0/\partial x}\right]^2\left(\frac{m}{M\beta}\right)^{3/2}\bar{k}^2 r_{Hi}^2. \tag{2.118}$$

The relations in (2.117) and (2.118) allow of a simple interpretation. We first estimate the energy which is released as a result of the relaxation of the particles distribution (this goes into an increase of wave energy):

$$\sum_k\frac{n_0 e^2|\Phi_k|^2}{2T_e}\approx\frac{1}{2}m\int v_z^2\,[f^{(0)}(v_z)-f_M(v_z)]\,dv_z\sim nT_e\left(\frac{\omega}{k_z v_{Te}}\right)^4. \tag{2.119}$$

The electric drift of the resonance electrons causes them to be displaced over a distance which coincides with the estimate given earlier in (2.108):

$$\delta x\approx\frac{c\bar{k}_y\varphi_\infty}{H_0\,\omega_{\mathbf{k}}}\approx\left[\frac{n_0}{\partial n_0/\partial x}\right]\left(\frac{\omega_{\mathbf{k}}}{k_z v_{Te}}\right)^2\frac{k_y v_*^e}{\omega_{\mathbf{k}}}. \tag{2.120}$$

The diffusion flux of electrons (2.117) can now be written in the form

$$D_\perp\frac{dn_0}{dx}\equiv\nu_{\mathrm{eff}}(\delta x)^2\frac{d\delta n}{dx}, \tag{2.121}$$

where $\nu_{eff}=\nu_e\left(\frac{v_{Te}}{\omega/k_z}\right)^2$ is an effective collision frequency for parti-

In this case $f_e^{(1)}$ is given by the equation

$$\mathrm{St}_{QL}\left[f_e^{(1)}\right] + \mathrm{St}_{\mathrm{coll}}\left[f_e^{(0)}\right] = 0. \tag{2.112}$$

Equation (2.111) has a simple solution with a plateau so that

$$\frac{1}{v_z} \cdot \frac{\partial f_e^{(0)}}{\partial v_z} = -\frac{k_y}{\omega_{\mathrm{k}}\,\omega_H} \cdot \frac{\partial f_e^{(0)}}{\partial x}. \tag{2.113}$$

In order to solve Eq. (2.112) we integrate it with respect to v_z:

$$\int_0^{v_z} dv_z \left(\frac{e}{m}\right)^2 \sum_{\mathrm{k}} \left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\omega_H} \cdot \frac{\partial}{\partial x}\right) |\Phi_{\mathrm{k}}|^2 \pi\delta(\omega_{\mathrm{k}} - k_z v_z) \times \\ \times \left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\omega_H} \cdot \frac{\partial}{\partial x}\right) f_e^{(1)} = \nu_e v_{Te}^2 \frac{\partial}{\partial v_z}(f_{Me} - f_e^{(0)}). \tag{2.114}$$

Since the dependence of $f_e^{(1)}$ on v_z is much stronger than on x, the derivative with respect to x in the first parentheses in Eq. (2.114) can be neglected. If, furthermore, use is made of the equation for the plateau (2.111) in order to estimate the second term on the right side of Eq. (2.114), we find

$$\left(\frac{e}{m}\right)^2 \sum_{\mathrm{k}} |\Phi_{\mathrm{k}}|^2 k_z \pi\delta(\omega_{\mathrm{k}} - k_z v_z)\left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\omega_H} \cdot \frac{\partial}{\partial x}\right) f_e^{(1)} = \\ = \nu_e v_{Te}^2 v_z \left(\frac{1}{v_z} \cdot \frac{\partial}{\partial v_z} + \frac{k_y}{\omega_{\mathrm{k}}\,\omega_H} \cdot \frac{\partial}{\partial x}\right) f_{Me}. \tag{2.115}$$

Integration of this expression with respect to $\mu = mv_\perp^2/2H$ then gives

$$\left(\frac{e}{m}\right)^2 \sum_{\mathrm{k}} k_z^2 |\Phi_{\mathrm{k}}|^2 \gamma_{\mathrm{k}}^{(1)} \pi\delta(\omega_{\mathrm{k}} - k_z v_z) = \nu_e v_{Te}^2 \varphi_{\mathrm{k}}^{(M)}, \tag{2.116}$$

where we have introduced the growth rate for the relaxing distribution in the presence of collisions $\gamma_{\mathrm{k}}^{(1)}$ and the growth rate for the Maxwellian plasma [cf. Eq. (2.94)]. Although this expression cannot be used to estimate the growth rate or the wave energy separately, it can still be used to estimate the diffusion coefficient, which depends specifically on the product of these quantities. Using Eqs. (2.109) and (2.116) we can then obtain the diffusion coefficient

$$D_\perp = \nu_e \left(\frac{n_0}{\partial n_0/\partial x}\right)^2 \frac{\gamma_{\mathrm{k}}^{(M)}}{\bar{k}_y v_*^e} \left(\frac{\bar{k}_y v_*^e}{k_z v_{Te}}\right)^2. \tag{2.117}$$

In order to estimate the diffusion coefficient we must find the characteristic wave number $\bar{\mathrm{k}}$. In the case in which collisions can establish the Maxwellian distribution the turbulence spectrum has a peak at $\mathrm{k}r_{\mathrm{Hi}} \approx 1$. This feature is associated with the fact that the stimulated scattering of waves on ions quickly transfers particles from the shortwave region, where the linear growth rate is appreciably larger, into the region $\mathrm{k}r_{\mathrm{Hi}} \sim 1$ [87]. As a result of the reduction in the collision frequency the electron distribution relaxes and the growth rate of the instability for wavelengths of the order of the Larmor radius becomes negative (damping). Stabilization of the shortwave region of the spectrum requires a much sharper slope for the distribution function in the resonance region $T_e df_e^{(0)}/dmv_z^2 \approx k_y v_*^e/\omega_{\mathrm{k}} \gg 1$. Hence, when perturbations with wavelengths of the order of the ion Larmor radius have been stabilized, the nonlinear transfer of energy from the shortwave region is reduced and the peak in the spectral energy density is displaced into this region. The coefficient of spatial diffusion can be estimated from Eq. (2.117):

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The relations in (2.117) and (2.118) allow of a simple interpretation. We first estimate the energy which is released as a result of the relaxation of the particles distribution (this goes into an increase of wave energy):

$$\sum_k \frac{n_0 e^2 |\Phi_k|^2}{2T_e} \approx \frac{1}{2} m \int v_z^2 [f^{(0)}(v_z) - f_M(v_z)]\, dv_z \sim nT_e \left(\frac{\omega}{k_z v_{Te}}\right)^4. \tag{2.119}$$

The electric drift of the resonance electrons causes them to be displaced over a distance which coincides with the estimate given earlier in (2.108):

$$\delta x \approx \frac{c\bar{k}_y \varphi_\infty}{H_0 \omega_{\mathrm{k}}} \approx \left[\frac{n_0}{\partial n_0/\partial x}\right] \left(\frac{\omega_{\mathrm{k}}}{k_z v_{Te}}\right)^2 \frac{k_y v_*^e}{\omega_{\mathrm{k}}}. \tag{2.120}$$

The diffusion flux of electrons (2.117) can now be written in the form

$$D_\perp \frac{dn_0}{dx} \equiv \nu_{\mathrm{eff}} (\delta x)^2 \frac{d\delta n}{dx}, \tag{2.121}$$

where $\nu_{eff} = \nu_e \left(\frac{v_{Te}}{\omega/k_z}\right)^2$ is an effective collision frequency for particles with resonant velocities $v_z \approx \omega/k_z$ while $\delta n = n_0(\omega/k_z v_{Te})$ is the density of resonance electrons [the growth rate is proportional to the number of resonance electrons $\gamma_k \approx (\delta n/n_0)k_y v_*^e$].

In concluding this section we offer several remarks on the limits of applicability of Eq. (2.118). First of all, in the derivation of this equation it has been assumed that the instability is weak, that is to say, $\gamma_k \ll \omega_k$ or $kr_{Hi} \ll (M\beta/m)^{1/2}$. We note that in a finite-pressure plasma the extremely shortwave oscillations $(k > k_c)$ will be suppressed because of the resonance between ions moving with the diamagnetic drift velocity and the drift wave [VI]. Hence, in the limit $\nu_e \to 0$ one should substitute $k_c(\beta)$ in Eq. (2.118). Secondly, it has been assumed that $St_{coll} \ll St_{QL}$. This latter condition obviously is the same as the condition

$$D_{QL}=\nu_e\left(\frac{n}{\Delta n}\right)^2\frac{\gamma_k k_y v_*^e}{k_z^2 v_{Te}^2} \ll D_{\rm turb}=\frac{\gamma_k^2}{\omega_k k_x^2}, \tag{2.122}$$

where the turbulent diffusion coefficient is obtained from Eq. (2.109) taking account of the fact that the spectral energy of the drift wave is limited to a level proportional to the linear growth rate so that it contains the small parameter γ_k/ω_k (a more detailed discussion of this point is given in the review in [88] and in [III]).

PROBLEM

1. Estimate the growth rate for the nonlinear instability of a plasma with an inhomogeneous electron temperature with respect to finite amplitude perturbations and the diffusion coefficient that arises because of the instability [89].

The analysis of quasilinear stabilization of the drift wave given in the text shows that collisions are capable of maintaining the growth of fluctuations of the drift wave. The increment for this nonlinear instability is estimated from Eq. (2.16) and found to be (cf. the results in § 1.3)

$$\gamma_k^{(1)}=\gamma_k^{(M)}/[1+\tau_e/\tau_{QL}], \tag{1}$$

where $\tau_e = \nu_{eff}^{-1} = \nu_e^{-1}(\Delta v_z/v_{Te})^2$ is the time between collisions of resonant particles while τ_{QL} is the time associated with the quasilinear distortion of the distribution function. As in § 2.3 it is assumed that this relation is also valid in the case of finite perturbations in the form of a monochromatic drift wave. In this case, however, it is necessary to make the substitution $\tau_e \to \nu_e^{-1}(e\Phi_0/T_e)$, while $\tau_{QL} \to \tau_b = \lambda_z/(e\Phi_0/m)^{1/2}$ is the time associated with the nonlinear distortion for the distribution of electrons trapped in the wave. In the limit $\tau_b \ll \tau_e$, from Eq. (1) we have

$$\gamma_k^{(1)}=-\gamma_e\left(\frac{T_e}{e\Phi}\right)^{1/2}\left[\frac{k_y v_*^e}{k_z(e\Phi/m)^{1/2}}\right]^2\eta,\qquad \nu_e\left(\frac{T_e}{e\Phi}\right)\ll k_z\sqrt{\frac{e\Phi}{m}}, \tag{2}$$

where we have used the well-known relation for the linear growth rate $\gamma_k^{(M)}=-\omega^2\eta/|k_z|v_{Te}$, $\eta \equiv d\ln T_e/d\ln n_0$. The wave energy comes from the longitudinal energy of the electrons which becomes available as a result of the release of electrons trapped in the potential well. The diffusion associated with the development of the nonlinear instability can be estimated from Eq. (2.121), where the displacement in the wave field and the number of resonant particles are given by

$$\delta x=\frac{ck_y\Phi}{H_0}\tau_b=r_{He}\left(\frac{k_y}{k_z}\right)\sqrt{\frac{e\Phi}{T_e}},\qquad \delta n=n_0\sqrt{\frac{e\Phi}{T_e}}.$$

It turns out that the diffusion is a weak function of the field amplitude:

$$D_\perp=\sqrt{\frac{e\Phi}{T_e}}\cdot\frac{k_y^2}{k_z^2}\nu_e r_{He}^2. \tag{3}$$

Chapter 3

NONLINEAR WAVE–PARTICLE INTERACTIONS

§ 3.1. Turbulence Associated with Electron Plasma Waves

We now consider the last mechanism for nonlinear interactions between waves in a plasma; for simplicity we shall consider a plasma without a magnetic field.

Since a plasma is a nonlinear medium, if two waves with frequencies ω_1 and ω_2 propagate in a plasma, then beats at the beat frequencies $(\omega_1 \pm \omega_2)$ with wave vectors $(\mathbf{k}_1 \pm \mathbf{k}_2)$ will be produced.

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It turns out that the diffusion is a weak function of the field amplitude:

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A resonance between this beat wave and a third wave $(\omega_3, \mathbf{k}_3)$ is properly described within the framework of the decay interaction of waves and has been discussed in Chapter 1. By analogy with the linear theory, it is also possible to have a particle that will be in resonance with the beat wave, this particle being characterized by a velocity $\mathbf{v}$ such that

$$(\mathbf{k}_1 \pm \mathbf{k}_2)\cdot \mathbf{v} = \omega_1 \pm \omega_2. \tag{3.1}$$

This process was first considered by Drummond and Pines [57] for the case of a one-dimensional wave packet, and by Kadomtsev and Petviashvili [34] in the general case. In the language of quantum mechanics this process can be described as stimulated scattering of waves on particles. The rate of energy exchange as a consequence of this process is proportional to the spectral energy density in the incident and scattered waves. Consequently, within the framework of classical perturbation theory, which is used for the description of stimulated scattering in [90], it is necessary to retain third-order terms in the expansion in terms of the wave amplitude. In this case the effect can be important when the number of particles that resonate with one of the waves is small but the number of particles that resonate with the beat wave is large.

As a first example we consider a wave packet of plasma waves with random phases. The frequency of the plasma waves is almost constant [i.e., $\omega^2 = \omega_p^2(1 + 3k^2\lambda_D^2/2)$, where $k\lambda_D \ll 1$]. Consequently, the interaction of waves with waves only appears in third-order in the wave energy (it is only in this order that a four-wave interaction occurs for which it is possible to satisfy the resonance condition $\omega_1 + \omega_2 = \omega_3 + \omega_4$). This interaction can be neglected as compared with the scattering of plasmons on particles which occurs in second order in the wave energy.

Let us consider scattering on particles with velocities which satisfy resonance condition $\omega_1 - \omega_2 = (\mathbf{k}_1 - \mathbf{k}_2)\cdot\mathbf{v}$. We first expand the electric potential of the waves in Fourier series in both time and space. The expansion in time assumes reasonable behavior of the potential as $t \to \infty$. Although this condition is known beforehand to be violated in the linear approximation if wave damping or excitation occurs, nonlinear effects which saturate the growth of the perturbation can justify this assumption. In the usual way, we expand the distribution function in powers of the wave amplitude,

using the iteration relation

$$\left.\begin{aligned} f_j(\mathbf{k}, \omega, \mathbf{v}) &= \sum_{n=0}^{\infty} f_j^{(n)}(\mathbf{k}, \omega, \mathbf{v}), \\ \tilde{f}_j^{(n)}(\mathbf{k}, \omega, \mathbf{v}) &= \frac{ie_j}{m_j} \sum_{\substack{\mathbf{k}'+\mathbf{k}''=\mathbf{k} \\ \omega'+\omega''=\omega}} \int_{-\infty}^{t} dt'\, \tilde{\Phi} \times \\ &\times (\mathbf{k}', \omega')\, \mathbf{k}' \frac{\partial}{\partial \mathbf{v}} \tilde{f}_j^{(n-1)}(\mathbf{k}'', \omega'', \mathbf{v}). \end{aligned}\right\} \tag{3.2}$$

Here, $\tilde{f}_j$ and $\tilde{\Phi}$ are the Fourier components multiplied by the quantity $\exp[i(\mathbf{kr} - \omega t)]$. Substituting this expression in Poisson's equation we obtain the dynamic wave equation

$$\begin{aligned} &\varepsilon_{\mathbf{k}}^{(1)}(\omega)\,\Phi(\mathbf{k}, \omega) + \sum_{\substack{\mathbf{k}'+\mathbf{k}''=\mathbf{k} \\ \omega'+\omega''=\omega}} \varepsilon_{\mathbf{k}',\mathbf{k}''}^{(2)}(\omega', \omega'')\,\Phi(\mathbf{k}', \omega')\,\Phi(\mathbf{k}'', \omega'') + \\ &+ \sum_{\substack{\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''=\mathbf{k} \\ \omega'+\omega''+\omega'''=\omega}} \varepsilon_{\mathbf{k}',\mathbf{k}'',\mathbf{k}'''}^{(3)}(\omega', \omega'', \omega''')\,\Phi(\mathbf{k}', \omega')\,\Phi \times \\ &\times (\mathbf{k}'', \omega'')\,\Phi(\mathbf{k}''', \omega''') + \dots, \end{aligned} \tag{3.3}$$

where $\Phi(\mathbf{k}, \omega)$ is the Fourier transform of the potential and

$$\left.\begin{aligned} &\varepsilon_{\mathbf{k}}^{(1)}(\omega) = 1 + \sum_j \frac{\omega_{pj}^2}{k^2} \int d^3\mathbf{v}\, \frac{\mathbf{k}\cdot(\partial f_{0j}/\partial \mathbf{v})}{\omega - \mathbf{k}\cdot\mathbf{v} + i0}; \\ &\varepsilon_{\mathbf{k}',\mathbf{k}''}^{(2)}(\omega', \omega'') = -\sum_j \frac{\omega_{pj}^2}{2(\mathbf{k}'+\mathbf{k}'')^2} \times \\ &\times \frac{e_j}{m_j} \int d^3\mathbf{v}\, \frac{1}{\omega'+\omega''-(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{v}+i0} \left(\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{v}} \frac{1}{\omega''-\mathbf{k}''\cdot\mathbf{v}+i0}\mathbf{k}'' \times \right. \\ &\left. \times \frac{\partial}{\partial \mathbf{v}} + \mathbf{k}''\cdot\frac{\partial}{\partial \mathbf{v}} \frac{1}{\omega'-\mathbf{k}'\cdot\mathbf{v}+i0}\cdot\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{v}}\right) f_{0j}. \end{aligned}\right\} \tag{3.4}$$

The infinitesimal positive quantity "+ 0" is introduced in order to provide the proper contour around the poles in the velocity integration. This quantity does not arise in a natural way as in the case of the linear theory when the Laplace transform is used for the solution of the initial-value problem, but is introduced simply to maintain causality (providing a slow switching on the interaction for $t = -\infty$). We solve the dynamic equation assuming that the quantity $|\Phi_{\mathbf{k}}|^2 \sim \gamma_k/\omega_k$ is a small parameter. It will be evident that $\Phi(\omega, \mathbf{k})$ has a narrow peak close to the characteristic frequency

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$\omega = \omega(\mathbf{k})$, this peak having a width of order $\gamma_{\mathbf{k}}$, so that

$$\Phi(\mathbf{k}, \omega) \approx \Phi_{\mathbf{k}}^{(1)} \delta(\omega - \omega(\mathbf{k})), \tag{3.5}$$

where $\omega(\mathbf{k})$ is the solution of the equation $\mathrm{Re}\, \varepsilon_{\mathbf{k}}^{(1)}(\omega) = 0$. In the next order Eq. (3.3) gives

$$\Phi^{(2)}(\mathbf{k}, \omega) = -\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \frac{\varepsilon_{\mathbf{k}'\mathbf{k}''}^{(2)}(\omega', \omega'')}{\varepsilon_{\mathbf{k}}^{(1)}(\omega)} \Phi_{\mathbf{k}'}^{(1)} \Phi_{\mathbf{k}''}^{(1)} \delta(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''} - \omega). \tag{3.6}$$

In order to derive the wave kinetic eqution, we multiply Eq. (3.3) by $\Phi^*(\mathbf{k}, \tilde{\omega}) \exp[i(\tilde{\omega} - \omega)t]$ and integrate over $d\omega d\tilde{\omega}$. The first term in the resulting relation is

$$\int d\omega \int d\tilde{\omega} \varepsilon_{\mathbf{k}}^{(1)}(\omega) \Phi(\mathbf{k}, \omega) \Phi^*(\mathbf{k}, \tilde{\omega}) \exp[i(\tilde{\omega} - \omega)t]. \tag{3.7}$$

Since $\Phi(\mathbf{k}, \omega)$ has a peak close to $\omega_{\mathbf{k}}$ we can rewrite the imaginary part of this expression in the form

$$\begin{aligned} \mathrm{Im}\left[\int d\omega \int d\tilde{\omega} \varepsilon_{\mathbf{k}}^{(1)}(\omega) \Phi(\mathbf{k}, \omega) \Phi^*(k, \tilde{\omega}) \exp[i(\tilde{\omega} - \omega)t]\right] \approx \\ \approx \frac{1}{2} \frac{\partial \varepsilon_{\mathbf{k}}^{(1)\prime}}{\partial \omega_{\mathbf{k}}} \cdot \frac{d|\Phi_{\mathbf{k}}(t)|^2}{dt} + \varepsilon_{\mathbf{k}}^{(1)\prime\prime}(\omega_{\mathbf{k}}), \end{aligned} \tag{3.8}$$

where $\varepsilon_{\mathbf{k}}^{(1)}(\omega) \equiv \varepsilon_{\mathbf{k}}^{(1)\prime}(\omega) + i\varepsilon_{\mathbf{k}}^{(1)\prime\prime}(\omega)$. Substituting Eqs. (3.5) and (3.6) in the remaining terms and carrying out an average over phase (i.e., $\langle \Phi_{\mathbf{k}}^{(1)} \Phi_{\mathbf{k}'}^{(1)*} \rangle = |\Phi_{\mathbf{k}}^{(1)}|^2 \delta_{\mathbf{k},\mathbf{k}'}$) we obtain the well-known wave kinetic equation [34, 57, 90, 91, V]

$$\begin{aligned} -\frac{1}{2} \frac{\partial \varepsilon_{\mathbf{k}}^{(1)\prime}}{\partial \omega_{\mathbf{k}}} \cdot \frac{\partial |\Phi_{\mathbf{k}}|^2}{\partial t} = -\varepsilon_{\mathbf{k}}^{(1)\prime\prime}(\omega_{\mathbf{k}}) |\Phi_{\mathbf{k}}|^2 + \\ + \mathrm{Im} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \frac{2|\varepsilon_{\mathbf{k}'\mathbf{k}''}^{(2)}(\omega_{\mathbf{k}'}, \omega_{\mathbf{k}''})|^2 |\Phi_{\mathbf{k}'}|^2 |\Phi_{\mathbf{k}''}|^2}{\varepsilon_{\mathbf{k}'+\mathbf{k}''}^{(1)*}(\omega_{\mathbf{k}} + \omega_{\mathbf{k}''})} + \\ + \mathrm{Im} \sum_{\mathbf{k}'} \left[\frac{4\varepsilon_{\mathbf{k}',\mathbf{k}-\mathbf{k}'}^{(2)}(\omega_{\mathbf{k}'}, \omega_{\mathbf{k}} - \omega_{\mathbf{k}'}) \varepsilon_{\mathbf{k},-\mathbf{k}'}^{(2)}(\omega_{\mathbf{k}}, -\omega_{\mathbf{k}'})}{\varepsilon_{\mathbf{k}-\mathbf{k}'}^{(1)}(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})} - \right. \\ \left. - 3\varepsilon_{\mathbf{k}',-\mathbf{k}',\mathbf{k}}^{(3)}\right] |\Phi_{\mathbf{k}} \Phi_{\mathbf{k}'}|^2, \end{aligned} \tag{3.9}$$

where we have retained terms of second order in the wave energy inclusively and have omitted the upper subscript on the wave amplitude $\Phi_{\mathbf{k}}^{(1)}$. The first term on the right side of this equation describes the linear damping (growth) of the waves. The contributions in the second terms come from poles which arise when the beat frequency coincides with any one of the characteristic frequencies $\mathrm{Im}[\varepsilon^{(1)}(\omega, \mathbf{k})]^{-1} \approx -\pi\delta(\varepsilon^{(1)}(\omega, \mathbf{k}))$. Obviously this term describes the combination of two waves $\Phi_{\mathbf{k}'}$ and $\Phi_{\mathbf{k}''}$ to form a third wave $\Phi_{\mathbf{k}}$. The third term corresponds to the decay process $(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} = \omega_{\mathbf{k}''})$ and the process of stimulated scattering $[\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} = (\mathbf{k} - \mathbf{k}') \cdot \mathbf{v}]$ which we have interpreted earlier as the decay of a primary wave $\Phi_{\mathbf{k}}$ into a wave with a nearby frequency $\Phi_{\mathbf{k}'}$ on a density fluctuation which decays rapidly because of the resonance with the plasma ions $(\omega - \omega')/|\mathbf{k} - \mathbf{k}'| \approx v_{Ti}$ (§ 1.1, cf. Problem 2). The fourth term obviously corresponds to Compton scattering of the plasma waves. (Details concerning the role of the last two terms in stimulated scattering on electrons are given in § 3.3.)

It is not possible to satisfy the frequency resonance condition for three plasma waves, so that the decay interaction does not occur. Furthermore, as has been shown by Drummond and Pines [57], the contribution in the induced scattering of waves by electrons described by the last two terms in Eq. (3.9) is small because they cancel each other approximately. The physical mechanism associated with the reduction of scattering is associated with the shielding of the electron charge by the ion cloud. With narrow wave packets the beats can fall into resonance with the thermal ions, in which case the basic contribution in the right side of Eq. (3.9) comes from the third term; in this term the nonlinear dielectric permittivity $\varepsilon^{(2)}$ is due to the nonlinear equation of motion of the electrons while the contribution to the imaginary part $\varepsilon^{(1)}$ comes from the ions. In order to estimate $\varepsilon_{\mathbf{k},-\mathbf{k}'}^{(2)}(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})$, we expand the integral in Eq. (3.4) in powers of the small parameter $(\omega - \omega')/|\mathbf{k} - \mathbf{k}'| v_{Te}$:

$$\begin{aligned} \varepsilon_{\mathbf{k},-\mathbf{k}'}^{(2)}(\omega_{\mathbf{k}}, -\omega_{\mathbf{k}'}) = \frac{\omega_p^2}{2(\mathbf{k} - \mathbf{k}')^2} \cdot \frac{e}{m} \int d^3\mathbf{v} \, [\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \\ - (\mathbf{k} - \mathbf{k}') \cdot \mathbf{v} + i0]^{-1} \left[\frac{\mathbf{k} \cdot \mathbf{k}'}{(\omega_{\mathbf{k}} - \mathbf{k} \cdot \mathbf{v})^2} \mathbf{k} \cdot \frac{\partial f_e}{\partial \mathbf{v}} - \frac{\mathbf{k} \cdot \mathbf{k}'}{(\omega_{\mathbf{k}'} - \mathbf{k}' \cdot \mathbf{v})^2} \mathbf{k}' \cdot \frac{\partial f_e}{\partial \mathbf{v}} + \right. \\ \left. + \frac{\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - (\mathbf{k} - \mathbf{k}') \cdot \mathbf{v}}{(\omega_{\mathbf{k}} - \mathbf{k} \cdot \mathbf{v})(\omega_{\mathbf{k}'} - \mathbf{k}' \cdot \mathbf{v})} \left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}}\right) \left(\mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}}\right) f_e \right] \approx \\ \approx \frac{e}{2m} \frac{\mathbf{k} \cdot \mathbf{k}'}{\omega_{\mathbf{k}} \omega_{\mathbf{k}'}} \cdot \frac{\omega_p^2}{(\mathbf{k} - \mathbf{k}')^2} \int d^3\mathbf{v} \frac{(\mathbf{k} - \mathbf{k}') \cdot \partial f_e / \partial \mathbf{v}}{\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - (\mathbf{k} - \mathbf{k}') \cdot \mathbf{v} + i0} = \\ = \frac{e}{2m} \cdot \frac{\mathbf{k} \cdot \mathbf{k}'}{\omega_{\mathbf{k}} \omega_{\mathbf{k}'}} \varepsilon_{\mathbf{k}-\mathbf{k}'}^{(1)e}(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'}). \end{aligned} \tag{3.10}$$

$\omega = \omega(\mathbf{k})$, this peak having a width of order $\gamma_{\mathbf{k}}$, so that

$$\Phi(\mathbf{k}, \omega) \approx \Phi_{\mathbf{k}}^{(1)} \delta(\omega - \omega(\mathbf{k})), \tag{3.5}$$

where $\omega(\mathbf{k})$ is the solution of the equation $\operatorname{Re} \varepsilon_{\mathbf{k}}^{(1)}(\omega) = 0$. In the next order Eq. (3.3) gives

$$\Phi^{(2)}(\mathbf{k}, \omega) = -\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \frac{\varepsilon^{(2)}_{\mathbf{k}',\mathbf{k}''}(\omega', \omega'')}{\varepsilon_{\mathbf{k}}^{(1)}(\omega)} \Phi_{\mathbf{k}'}^{(1)} \Phi_{\mathbf{k}''}^{(1)} \delta(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''} - \omega). \tag{3.6}$$

In order to derive the wave kinetic eqution, we multiply Eq. (3.3) by $\Phi^*(\mathbf{k}, \tilde{\omega}) \exp\{i(\tilde{\omega} - \omega)t\}$ and integrate over $d\omega d\tilde{\omega}$. The first term in the resulting relation is

$$\int d\omega \int d\tilde{\omega} \varepsilon_{\mathbf{k}}^{(1)}(\omega) \Phi(\mathbf{k}, \omega) \Phi^*(\mathbf{k}, \tilde{\omega}) \exp[i(\tilde{\omega} - \omega)t]. \tag{3.7}$$

Since $\Phi(\mathbf{k}, \omega)$ has a peak close to $\omega_{\mathbf{k}}$ we can rewrite the imaginary part of this expression in the form

$$\operatorname{Im}\left[\int d\omega \int d\tilde{\omega} \varepsilon_{\mathbf{k}}^{(1)}(\omega) \Phi(\mathbf{k}, \omega) \Phi^*(k, \tilde{\omega}) \exp[i(\tilde{\omega} - \omega)t]\right] \approx$$
$$\approx \frac{1}{2} \frac{\partial \varepsilon_{\mathbf{k}}^{(1)'}}{\partial \omega_{\mathbf{k}}} \cdot \frac{d|\Phi_{\mathbf{k}}(t)|^2}{dt} + \varepsilon_{\mathbf{k}}^{(1)''}(\omega_{\mathbf{k}}), \tag{3.8}$$

where $\varepsilon_{\mathbf{k}}^{(1)}(\omega) \equiv \varepsilon_{\mathbf{k}}^{(1)'}(\omega) + i\varepsilon_{\mathbf{k}}^{(1)''}(\omega)$. Substituting Eqs. (3.5) and (3.6) in the remaining terms and carrying out an average over phase (i.e., $\langle \Phi_{\mathbf{k}}^{(1)} \Phi_{\mathbf{k}'}^{(1)*} \rangle = |\Phi_{\mathbf{k}}^{(1)}|^2 \delta_{\mathbf{k},\mathbf{k}'}$) we obtain the well-known wave kinetic equation [34, 57, 90, 91, V]

$$-\frac{1}{2} \frac{\partial \varepsilon_{\mathbf{k}}^{(1)'}}{\partial \omega_{\mathbf{k}}} \cdot \frac{\partial |\Phi_{\mathbf{k}}|^2}{\partial t} = -\varepsilon_{\mathbf{k}}^{(1)''}(\omega_{\mathbf{k}}) |\Phi_{\mathbf{k}}|^2 +$$
$$+ \operatorname{Im} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \frac{2|\varepsilon^{(2)}_{\mathbf{k}',\mathbf{k}''}(\omega_{\mathbf{k}'}, \omega_{\mathbf{k}''})|^2 |\Phi_{\mathbf{k}'}|^2 |\Phi_{\mathbf{k}''}|^2}{\varepsilon^{(1)*}_{\mathbf{k}'+\mathbf{k}''}(\omega_{\mathbf{k}} + \omega_{\mathbf{k}''})} +$$
$$+ \operatorname{Im} \sum_{\mathbf{k}'} \left[\frac{4\varepsilon^{(2)}_{\mathbf{k}',\mathbf{k}-\mathbf{k}'}(\omega_{\mathbf{k}'}, \omega_{\mathbf{k}} - \omega_{\mathbf{k}'}) \varepsilon^{(2)}_{\mathbf{k},-\mathbf{k}'}(\omega_{\mathbf{k}}, -\omega_{\mathbf{k}'})}{\varepsilon^{(1)}_{\mathbf{k}-\mathbf{k}'}(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})} - \right.$$
$$\left. - 3\varepsilon^{(3)}_{\mathbf{k}',-\mathbf{k}',\mathbf{k}} \right] |\Phi_{\mathbf{k}} \Phi_{\mathbf{k}'}|^2, \tag{3.9}$$

where we have retained terms of second order in the wave energy inclusively and have omitted the upper subscript on the wave amplitude $\Phi_{\mathbf{k}}^{(1)}$. The first term on the right side of this equation describes the linear damping (growth) of the waves. The contributions

in the second terms come from poles which arise when the beat frequency coincides with any one of the characteristic frequencies $\operatorname{Im}[\varepsilon^{(1)}(\omega, \mathbf{k})]^{-1} \approx -\pi\delta(\varepsilon^{(1)}(\omega, \mathbf{k}))$. Obviously this term describes the combination of two waves $\Phi_{\mathbf{k}'}$ and $\Phi_{\mathbf{k}''}$ to form a third wave $\Phi_{\mathbf{k}}$. The third term corresponds to the decay process $(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} = \omega_{\mathbf{k}''})$ and the process of stimulated scattering $[\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} = (\mathbf{k} - \mathbf{k}') \cdot \mathbf{v}]$ which we have interpreted earlier as the decay of a primary wave $\Phi_{\mathbf{k}}$ into a wave with a nearby frequency $\Phi_{\mathbf{k}'}$ on a density fluctuation which decays rapidly because of the resonance with the plasma ions $(\omega - \omega')/|\mathbf{k} - \mathbf{k}'| \approx v_{Ti}$ (§ 1.1, cf. Problem 2). The fourth term obviously corresponds to Compton scattering of the plasma waves. (Details concerning the role of the last two terms in stimulated scattering on electrons are given in § 3.3.)

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$$\varepsilon^{(2)}_{\mathbf{k},-\mathbf{k}'}(\omega_{\mathbf{k}}, -\omega_{\mathbf{k}'}) = \frac{\omega_p^2}{2(\mathbf{k}-\mathbf{k}')^2} \cdot \frac{e}{m} \int d^3\mathbf{v} \, [\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} -$$
$$- (\mathbf{k} - \mathbf{k}') \cdot \mathbf{v} + i0]^{-1} \left[\frac{\mathbf{k} \cdot \mathbf{k}'}{(\omega_{\mathbf{k}} - \mathbf{k} \cdot \mathbf{v})^2} \mathbf{k} \cdot \frac{\partial f_e}{\partial \mathbf{v}} - \frac{\mathbf{k} \cdot \mathbf{k}'}{(\omega_{\mathbf{k}'} - \mathbf{k}' \cdot \mathbf{v})^2} \mathbf{k}' \cdot \frac{\partial f_e}{\partial \mathbf{v}} + \right.$$
$$\left. + \frac{\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - (\mathbf{k} - \mathbf{k}') \cdot \mathbf{v}}{(\omega_{\mathbf{k}} - \mathbf{k} \cdot \mathbf{v})(\omega_{\mathbf{k}'} - \mathbf{k}' \cdot \mathbf{v})} \left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} \right) \left(\mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} \right) f_e \right] \approx$$
$$\approx \frac{e}{2m} \frac{\mathbf{k} \cdot \mathbf{k}'}{\omega_{\mathbf{k}} \omega_{\mathbf{k}'}} \cdot \frac{\omega_p^2}{(\mathbf{k} - \mathbf{k}')^2} \int d^3\mathbf{v} \frac{(\mathbf{k} - \mathbf{k}') \cdot \partial f_e / \partial \mathbf{v}}{\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - (\mathbf{k} - \mathbf{k}') \cdot \mathbf{v} + i0} =$$
$$= \frac{e}{2m} \cdot \frac{\mathbf{k} \cdot \mathbf{k}'}{\omega_{\mathbf{k}} \omega_{\mathbf{k}'}} \varepsilon^{(1)e}_{\mathbf{k}-\mathbf{k}'}(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'}). \tag{3.10}$$

Similarly,

$$\mathbf{k}^2 \varepsilon^{(2)}_{\mathbf{k}', \mathbf{k}-\mathbf{k}'} = \frac{e}{2m} \cdot \frac{\mathbf{k}\cdot\mathbf{k}'}{\omega_{\mathbf{k}}\,\omega_{\mathbf{k}'}} (\mathbf{k}-\mathbf{k}')^2 \varepsilon^{(1)\,e^*}_{\mathbf{k}-\mathbf{k}'} (\omega_{\mathbf{k}} - \omega_{\mathbf{k}'}) \tag{3.11}$$

$$\varepsilon^{(1)}_{\mathbf{k}-\mathbf{k}'} (\omega_{\mathbf{k}} - \omega_{\mathbf{k}'}) \approx 1 + \frac{1}{(\mathbf{k}-\mathbf{k}')^2 \lambda^2_{De}} - \frac{i\sqrt{\pi}}{(\mathbf{k}-\mathbf{k}')^2 \lambda^2_{Di}} \times \\ \times \frac{\omega_{\mathbf{k}} - \omega_{\mathbf{k}'}}{|\mathbf{k}-\mathbf{k}'|\, v_{Ti}} W\left(\frac{\omega_{\mathbf{k}} - \omega_{\mathbf{k}'}}{|\mathbf{k}-\mathbf{k}'|\, v_{Ti}} \right), \tag{3.12}$$

where $W(z) = \frac{i}{\pi} \int\limits_{-\infty}^{+\infty} \frac{e^{-t^2}\,dt}{z - t + i0}$. As a result, Eq. (3.9) assumes the form

$$\frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial \varepsilon^{(1)'}_{\mathbf{k}}}{\partial \omega_{\mathbf{k}}} \mathbf{k}^2 |\Phi_{\mathbf{k}}|^2 \right) = \sum_{\mathbf{k}'} \frac{e^2 |\Phi_{\mathbf{k}}|^2 \cdot |\Phi_{\mathbf{k}'}|^2 (\mathbf{k}\cdot\mathbf{k}')^2}{m^2 \omega^2_{\mathbf{k}}\, \omega^2_{\mathbf{k}'}} \times \\ \times (\mathbf{k}-\mathbf{k}')^2 \left| \frac{\varepsilon^{(1)\,e}_{\mathbf{k}-\mathbf{k}'} (\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})}{\varepsilon^{(1)}_{\mathbf{k}-\mathbf{k}'} (\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})} \right|^2 \operatorname{Im} \varepsilon^{(1)\,i}_{\mathbf{k}-\mathbf{k}'} (\omega_{\mathbf{k}} - \omega_{\mathbf{k}'}). \tag{3.13}$$

Using this equation it is not difficult to show that the number of waves is conserved:

$$\frac{\partial}{\partial t} \sum_{\mathbf{k}} n_{\mathbf{k}} = (\partial/\partial t) \sum_{\mathbf{k}} \frac{\partial}{\omega_{\mathbf{k}}\, \partial \omega_{\mathbf{k}}} \left| \omega_{\mathbf{k}} \varepsilon^{(1)'}_{\mathbf{k}} (\omega_{\mathbf{k}}) \right| \mathbf{k}^2 |\Phi_{\mathbf{k}}|^2 / 8\pi = 0.$$

This result can be understood easily if one takes account of the fact that the resonance condition $\omega_1 + \omega_2 = (\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{v}$ cannot be satisfied for waves with phase velocity larger than the particle thermal velocity. The process which occurs when the second resonance condition $\omega_1 - \omega_2 = (\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{v}$ is satisfied represents wave scattering, in which the resonance condition guarantees the conservation of energy $\Delta\mathscr{E} = d\mathscr{E}\Lambda p/dp \equiv h(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{v} = h(\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})$. It is reasonable that the number of waves is conserved in the process, and this result can be proved in the general case without invoking quantum-mechanical arguments that make use of the symmetry properties of the coefficients $\varepsilon^{(n)}$.

Equation (3.13) allows an analytic solution in the case in which the wave distribution in k-space is isotropic and the spread in the wave phase velocities is much greater than the ion thermal velocity:

$$\Delta k / k \gg (m/M)^{1/2}. \tag{3.14}$$

Under these conditions the only strong interactions in the radiation spectrum will be for waves whose frequencies are close to each other, so that the integrand in Eq. (3.13) (we convert to a continuous spectrum) can be expanded in powers of the difference in frequency between the interacting waves. The integrals over frequency which arise in this case represent well-known integrals in the theory of dispersion relations. Carrying out the integration, we reduce Eq. (3.13) to a differential equation in **k**-space:

$$\frac{\partial N_{\mathbf{k}}}{\partial \tau} + N_{\mathbf{k}} \frac{\partial N_{\mathbf{k}}}{\partial \chi} = 0, \tag{3.15}$$

where

$$(2\pi)^3 N_{\mathbf{k}} = \frac{4\pi k^2 k_0}{3} \cdot \frac{k^2 |\Phi_{\mathbf{k}}|^2}{4\pi N_{0e} T_e}; \quad \tau = \frac{4\pi m \omega_p}{9 M k_0 \Delta k \lambda^2_{De}}; \quad \chi = \frac{k^2}{k_0^2 \Delta k}.$$

The stimulated scattering of waves in the longwave region of the spectrum leads to a reduction in the wave energy and a sharpening of the leading edge of the radiation line in **k**-space [90]; we note that Eq. (3.15) no longer applies if the leading edge of the wave front forms a vertical profile. The evolution pattern of the profile of the line can be traced simply in the case in which a weak line moves against the background of almost uniform distribution of spectral energy. In this case Eq. (3.15) must be supplemented by higher-order terms in the expansion in the parameter in (3.14). From parity considerations it is apparent that a term in the third derivative $N_{\mathbf{k}} \partial^3 N_{\mathbf{k}} / \partial \chi^3$ appears in Eq. (3.15) and the analysis of the profile of the line becomes analogous to the problem of evolution of the profile of an initial perturbation in a dispersive nonlinear medium. As is well known [II], dispersion of the velocity of a line in **k**-space establishes a curvature of the front; in addition, solitons, are emitted from the leading edge and move ahead or behind, depending on the sign of the dispersion.

If we are interested in the relaxation of an individual line of plasma waves in **k**-space, then the gradients that are formed in **k**-space are not small and there is no small parameter like that in (3.14). In this case it is necessary to solve the exact integral equation. However, the qualitative picture of relaxation remains the same as before: The leading edge of the line becomes more curved and emits solitons [92].

Similarly,

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PROBLEMS

1. Calculate the intensity of scattering of plasma waves on fluctuations in the electron density [VII, VIII].

The calculations are completely analogous to those which have been carried out in Problems 1-3 in § 1.1, in which we have considered the scattering on acoustic oscillations of the electron density. The current which excites the scattered wave is

$$\mathbf{j}_{\mathbf{k},\omega}=\frac{ie^2}{m\omega_0}\,n_{\Delta\omega,\Delta k}\,\mathbf{E}_{\mathbf{k}_0}. \tag{1}$$

When the scattered waves are longitudinal, the wave amplitudes are found from the equation

$$-i\omega\varepsilon(\omega,\mathbf{k})\,\mathbf{E}_{\mathbf{k}}=4\pi\mathbf{k}\,(\mathbf{k}\cdot\mathbf{j}_{\mathbf{k},\omega})/k^2. \tag{2}$$

Calculating the work of the electrons in the field of the scattered wave, we find the radiation intensity

$$\frac{\partial}{\partial t}\cdot\frac{\partial}{\partial\omega}[\omega\varepsilon]\frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi}=\int\frac{d^3\mathbf{k}_0}{(2\pi)^3}\cdot\frac{4\pi e^4(\delta n_e^2)_{\Delta\omega,\Delta\mathbf{k}}}{m^2\omega_0^2\omega_{\mathbf{k}}(\partial\varepsilon/\partial\omega_{\mathbf{k}})}\cdot\frac{|(\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0})|^2}{\mathbf{k}^2}. \tag{3}$$

For the case of a Maxwellian plasma, using Eq. (11) of Appendix A we rewrite Eq. (3) in the form

$$\frac{\partial}{\partial t}\cdot\frac{|\mathbf{E}_{\mathbf{k}}|^2}{4\pi}\approx\int\frac{d^3\mathbf{k}_0}{(2\pi)^3}\frac{e^2}{m^2\omega_0^2}\frac{|\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0}|^2}{\mathbf{k}^2}\cdot\frac{(\Delta\mathbf{k})^2\,\mathrm{Im}}{\Delta\omega\,|\varepsilon(\Delta\omega,\Delta\mathbf{k})|^2}\{|1+\varepsilon_i|^2T_e\varepsilon_e+$$
$$+|\varepsilon_e|^2T_i\varepsilon_i\}. \tag{4}$$

2. Estimate the noise level and the heating rate in a plasma with $T_i \approx T_e$ due to the development of the decay instability of an incident wave [93].

The electric field induced in the plasma from the external source is written in the form

$$\mathbf{E}(t)={}^1/_2\mathbf{E}_0\exp(-i\omega_0 t)+\text{c.c.} \tag{1}$$

The decay of this wave in an almost isothermal plasma into a plasma wave and a rapidly damped acoustic perturbation is essentially the process of stimulated scattering on ions and is described by Eq. (3.13). However, in order to simplify the problem, rather than using this exact expression we shall use the approximate expression for the growth rate which has been obtained in the fluid description in Problem 3 of § 1.1.

The saturation mechanism for the instability is assumed to be the stimulated scattering of plasma waves on ions. As in [93], it will be assumed that the plasma waves which are produced exhibit a small angular spread ($2\sin\Theta/2 < 1$) so that the Doppler broadening of the beat frequency is much smaller than the width of the frequency spectrum; consequently, the scattering can be described by a differential equation like that in (3.15). In addition to these two processes we also consider the scattering of waves (1) on fluctuations in the electron density (cf. Problem 1), thus obtaining the following approximate equation for the spectral density of the plasma waves:

$$\frac{\partial}{\partial\tau}W(\Omega,\xi)=W(\Omega,\xi)\left\{\frac{E^2\xi^2}{1+\Omega^2}-1-\frac{\partial}{\partial\Omega}\alpha\int_{-1}^{+1}d\xi'(1-\xi\xi')\times\right.$$
$$\left.\times\left[\xi^2\xi'^2+\frac{1}{2}(1-\xi^2)(1-\xi'^2)\right]W(\Omega,\xi')\right\}+\delta\xi^2, \tag{2}$$

where

$$W(\Omega,\xi)=|\mathbf{E}_{\mathbf{k}}|^2/4\pi;\quad \Omega=(\omega_{\mathbf{k}}+\omega_s-\omega_0)/\nu_s;\quad \xi=\cos\Theta;$$

$$E^2=(\omega_{\mathbf{k}}/16\nu_e)(\omega_s/\nu_s)(E_0^2/4\pi N_0T_e);\quad \alpha=\frac{m}{12\pi M}\cdot\frac{\omega_{\mathbf{k}}^e}{\nu_e\nu_s}\cdot\frac{k^3}{N_0T_e};$$
$$\delta=\frac{\omega_p^4}{\omega_0^2\nu_s\nu_e}\cdot\frac{E_0^2}{4\pi N_0};\quad \tau=\nu_e t.$$

We have introduced a system of polar coordinates with axis along $\mathbf{E}_0$ and have used the azimuthal symmetry of the problem. Equation (2) has the following formal solution, which is even in the variable ξ:

$$W=\delta\xi^2/[a^2(\Omega)-b^2(\Omega)\xi^2]. \tag{3}$$

The integral of this equation with respect to ξ can be taken immediately:

$$\overline{W}=\int_{-1}^{+1}W(\Omega,\xi)\,d\xi=\frac{\delta}{2ab}\ln\frac{a/b+1}{a/b-1}. \tag{4}$$

PROBLEMS

▶ 1. Calculate the intensity of scattering of plasma waves on fluctuations in the electron density [VII, VIII].

The calculations are completely analogous to those which have been carried out in Problems 1-3 in § 1.1, in which we have considered the scattering on acoustic oscillations of the electron density. The current which excites the scattered wave is

$$\mathbf{j}_{\mathbf{k},\omega}=\frac{ie^2}{m\omega_0}n_{\Delta\omega,\Delta k}\,\mathbf{E}_{\mathbf{k}_0}. \tag{1}$$

When the scattered waves are longitudinal, the wave amplitudes are found from the equation

$$-i\omega\varepsilon(\omega,\mathbf{k})\,\mathbf{E}_{\mathbf{k}}=4\pi\mathbf{k}\,(\mathbf{k}\cdot\mathbf{j}_{\mathbf{k},\omega})/k^2. \tag{2}$$

Calculating the work of the electrons in the field of the scattered wave, we find the radiation intensity

$$\frac{\partial}{\partial t}\cdot\frac{\partial}{\partial\omega}[\omega\varepsilon]\frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi}=\int\frac{d^3\mathbf{k}_0}{(2\pi)^3}\cdot\frac{4\pi e^4(\delta n_e^2)_{\Delta\omega,\Delta\mathbf{k}}}{m^2\omega_0^2\,\omega_{\mathbf{k}}(\partial\varepsilon/\partial\omega_{\mathbf{k}})}\cdot\frac{|(\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0})|^2}{\mathbf{k}^2}. \tag{3}$$

For the case of a Maxwellian plasma, using Eq. (11) of Appendix A we rewrite Eq. (3) in the form

$$\frac{\partial}{\partial t}\cdot\frac{|\mathbf{E}_{\mathbf{k}}|^2}{4\pi}\approx\int\frac{d^3\mathbf{k}_0}{(2\pi)^3}\,\frac{e^2}{m^2\omega_0^2}\,\frac{|\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}_0}|^2}{\mathbf{k}^2}\cdot\frac{(\Delta\mathbf{k})^2\,\mathrm{Im}}{\Delta\omega\,|\varepsilon(\Delta\omega,\Delta\mathbf{k})|^2}\{|1+\varepsilon_i|^2T_e\varepsilon_e+$$
$$+|\varepsilon_e|^2T_i\varepsilon_i\}. \tag{4}$$

▶ 2. Estimate the noise level and the heating rate in a plasma with $T_i\approx T_e$ due to the development of the decay instability of an incident wave [93].

The electric field induced in the plasma from the external source is written in the form

$$\mathbf{E}(t)={}^1/_2\mathbf{E}_0\exp(-i\omega_0 t)+\mathrm{c.c.} \tag{1}$$

The decay of this wave in an almost isothermal plasma into a plasma wave and a rapidly damped acoustic perturbation is essen-

tially the process of stimulated scattering on ions and is described by Eq. (3.13). However, in order to simplify the problem, rather than using this exact expression we shall use the approximate expression for the growth rate which has been obtained in the fluid description in Problem 3 of § 1.1.

The saturation mechanism for the instability is assumed to be the stimulated scattering of plasma waves on ions. As in [93], it will be assumed that the plasma waves which are produced exhibit a small angular spread ($2\sin\Theta/2<1$) so that the Doppler broadening of the beat frequency is much smaller than the width of the frequency spectrum; consequently, the scattering can be described by a differential equation like that in (3.15). In addition to these two processes we also consider the scattering of waves (1) on fluctuations in the electron density (cf. Problem 1), thus obtaining the following approximate equation for the spectral density of the plasma waves:

$$\frac{\partial}{\partial\tau}W(\Omega,\xi)=W(\Omega,\xi)\left\{\frac{E^2\xi^2}{1+\Omega^2}-1-\frac{\partial}{\partial\Omega}\alpha\int_{-1}^{+1}d\xi'(1-\xi\xi')\times\right.$$
$$\left.\times\left[\xi^2\xi'^2+\frac{1}{2}(1-\xi^2)(1-\xi'^2)\right]W(\Omega,\xi')\right\}+\delta\xi^2, \tag{2}$$

where

$$W(\Omega,\xi)=|\mathbf{E}_{\mathbf{k}}|^2/4\pi;\quad \Omega=(\omega_{\mathbf{k}}+\omega_s-\omega_0)/v_s;\quad \xi=\cos\Theta;$$

$$E^2=(\omega_{\mathbf{k}}/16v_e)(\omega_s/v_s)(E_0^2/4\pi N_0T_e);\quad \alpha=\frac{m}{12\pi M}\cdot\frac{\omega_{\mathbf{k}}^e}{v_ev_s}\cdot\frac{k^3}{N_0T_e};$$
$$\delta=\frac{\omega_p^4}{\omega_0^2v_sv_e}\cdot\frac{E_0^2}{4\pi N_0};\quad \tau=v_et.$$

We have introduced a system of polar coordinates with axis along E_0 and have used the azimuthal symmetry of the problem. Equation (2) has the following formal solution, which is even in the variable ξ:

$$W=\delta\xi^2/[a^2(\Omega)-b^2(\Omega)\xi^2]. \tag{3}$$

The integral of this equation with respect to ξ can be taken immediately:

$$\overline{W}=\int_{-1}^{+1}W(\Omega,\xi)\,d\xi=\frac{\delta}{2ab}\ln\frac{a/b+1}{a/b-1}. \tag{4}$$

Since the spontaneous emission of electrons moving in the wave field is weak ($E^2 - 1 \gg \alpha\delta$), in Eq. (4) we can obviously take $a/b - 1 \backsim \exp(-\overline{W}/\delta)$ and rewrite Eq. (3) in the form

$$W(\Omega, \xi) = \delta\xi^2 / [1 - \xi^2 + 2\exp(-\overline{W}/\delta)], \tag{5}$$

where

$$\overline{W} = (1/\alpha)\{E^2[\arctan\sqrt{E^2-1}] - \Omega - \sqrt{E^2-1}\}.$$

Making use of these relations it is not difficult to estimate the high-frequency conductivity of the plasma. The effective collision frequency is then given by

$$\nu_{\text{eff}}\frac{E_0^2}{4\pi} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\nu_e W(\Omega, \xi) \approx \frac{\pi\nu_e^2 N_0 T_e}{\omega_\mathbf{k}} E^4, \tag{6}$$

that is to say, the Coulomb collision frequency does not appear in the result.

§ 3.2. Ion-Acoustic Turbulence

One of the principal mechanisms responsible for anomalous resistivity in the plasma (this problem is discussed in detail in the next chapter) is the excitation of ion-acoustic turbulence by the flow of a current.

For ion-acoustic waves with phase velocities between the ion and electron thermal velocities the linear dispersion relation is of the form

$$\varepsilon_\mathbf{k}^{(1)}(\omega) \equiv 1 - \frac{\Omega_p^2}{\omega^2} + \frac{\omega_p^2}{k^2 c_s^2}\left[1 + i\sqrt{\frac{\pi m}{2M}}\left(\frac{\omega}{|\mathbf{k}|c_s} - \frac{V_d}{c_s}\cos\Theta\right)\right], \tag{3.16}$$

where $c_s = (T_e/M)^{1/2}$ is the acoustic velocity, V_d is the electron drift velocity, and Θ is the angle between the wave vector $\mathbf{k}$ and the directed current V_d. In the longwave limit the phase velocity of the waves coincides with the acoustic velocity $\omega/k = c_s$; in the shortwave limit it diminishes as the wavelength is reduced since the frequency remains constant ($\omega = \Omega_p$; cf. Fig. 7, curve 3). If longwave oscillations are to be excited the drift velocity must exceed the acoustic velocity. On the other hand, the excitation of shortwave perturbations in a highly nonisothermal plasma ($T_e \gg T_i$) can occur at lower velocities. Since the ion-acoustic instability is

a resonant instability with a small growth rate, the turbulent state which develops as a result of this instability can be described in terms of a gas of interacting waves which are described by the wave kinetic equation. We have shown in Chapter 1 that a three-wave resonance interaction is not possible for waves whose dispersion is given by curve 3 of Fig. 7. Consequently, the basic contribution in the nonlinear relaxation of the turbulent spectrum comes from scattering on particles as described by the last two terms of Eq. (3.9). The interaction with electrons can be neglected since the number of particles in resonance with the beats is small: ($\delta n \approx N_0(\omega \pm \omega')/|\mathbf{k} + \mathbf{k}'| v_{Te} \ll N_0$). Furthermore, since the phase velocity of the wave is much larger than the ion thermal velocity, the ions can only scatter (but not absorb) waves, since the number of waves is conserved. Taking $\omega'' = \omega_\mathbf{k} - \omega_{\mathbf{k}'} \sim kv_{Ti}$, $k'' \sim k$, in lowest order in the expansion in ω''/ω we have

$$\left.\begin{aligned} \varepsilon^{(1)}(\mathbf{k}'', \omega'') &= \frac{\Omega_p^2}{\mathbf{k}''^2}\int \frac{\mathbf{k}''(\partial f_i/\partial \mathbf{v})}{\omega'' - \mathbf{k}''\cdot\mathbf{v} + i0}\, d^3\mathbf{v}, \\ k^2 \varepsilon^{(2)}_{\mathbf{k}', \mathbf{k}-\mathbf{k}'} &= k''^2 \varepsilon^{(2)}_{\mathbf{k}, -\mathbf{k}'} = \frac{M\omega_\mathbf{k}\omega_{\mathbf{k}'}}{e(\mathbf{k}\cdot\mathbf{k}')} k^2 \varepsilon^{(3)}_{\mathbf{k}', \mathbf{k}, -\mathbf{k}'} = \\ &= \frac{e}{M}\,\frac{(\mathbf{k}\cdot\mathbf{k}')}{\omega_\mathbf{k}\omega_{\mathbf{k}'}}\mathbf{k}''^2 \varepsilon^{(1)}_{\mathbf{k}-\mathbf{k}'}(\omega''). \end{aligned}\right\} \tag{3.17}$$

However, if these expressions are substituted in the wave kinetic equation the nonlinear terms are not evident. Consequently, it is necessary to take account of higher-order terms. The quantity $\varepsilon^{(3)}$ is given by

$$\begin{aligned} \operatorname{Im}\varepsilon^{(3)}_{\mathbf{k}', \mathbf{k}-\mathbf{k}'} = \frac{\pi\Omega_p^2}{k^2}\cdot\frac{e^2}{M^2}\cdot\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{\omega^2}\int d^3\mathbf{v}\left[1 + \frac{4\mathbf{k}\cdot\mathbf{v}}{\omega} + 10\frac{(\mathbf{k}\cdot\mathbf{v})^2}{\omega^2}\right]\times \\ \times\, \mathbf{k}''\frac{\partial f_i}{\partial \mathbf{v}}\delta(\omega'' - \mathbf{k}''\cdot\mathbf{v}). \end{aligned} \tag{3.18}$$

In order to carry out the integration over velocity we divide $\mathbf{k}\cdot\mathbf{v}$ into two parts:

$$\mathbf{k}\cdot\mathbf{v} = (\mathbf{k}''\cdot\mathbf{v})(\mathbf{k}\cdot\mathbf{k}'')/\mathbf{k}''^2 + [\mathbf{k}''\times\mathbf{v}]\cdot[\mathbf{k}''\times\mathbf{k}]/k''^2. \tag{3.19}$$

The first term is expressed simply in terms of the argument of the δ-function, while the second is independent of it. Consequently, the integration over velocity in Eq. (3.18) can be carried out with-

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$$\left.\begin{aligned}
&\varepsilon^{(1)}(\mathbf{k}'', \omega'') = \frac{\Omega_p^2}{k''^2}\int \frac{\mathbf{k}''(\partial f_i/\partial \mathbf{v})}{\omega'' - \mathbf{k}''\cdot\mathbf{v} + i0}\,d^3\mathbf{v},\\
&k^2\varepsilon^{(2)}_{\mathbf{k}',\mathbf{k}-\mathbf{k}'} = k''^2\varepsilon^{(2)}_{\mathbf{k},-\mathbf{k}'} = \frac{M\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}{e(\mathbf{k}\cdot\mathbf{k}')}\,k^2\varepsilon^{(3)}_{\mathbf{k}',\mathbf{k},-\mathbf{k}'} =\\
&\qquad = -\frac{e}{M}\,\frac{(\mathbf{k}\cdot\mathbf{k}')}{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}\,\mathbf{k}''^2\varepsilon^{(1)}_{\mathbf{k}-\mathbf{k}'}(\omega'').
\end{aligned}\right\} \tag{3.17}$$

However, if these expressions are substituted in the wave kinetic equation the nonlinear terms are not evident. Consequently, it is necessary to take account of higher-order terms. The quantity $\varepsilon^{(3)}$ is given by

$$\begin{aligned}\operatorname{Im}\varepsilon^{(3)}_{\mathbf{k}',\mathbf{k}-\mathbf{k}'} = \frac{\pi\Omega_p^2}{k^2}\cdot\frac{e^2}{M^2}\cdot\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{\omega^2}\int d^3\mathbf{v}\left[1 + \frac{4\mathbf{k}\cdot\mathbf{v}}{\omega} + 10\,\frac{(\mathbf{k}\cdot\mathbf{v})^2}{\omega^2}\right]\times\\
\times\,\mathbf{k}''\,\frac{\partial f_i}{\partial \mathbf{v}}\,\delta(\omega'' - \mathbf{k}''\cdot\mathbf{v}).\end{aligned} \tag{3.18}$$

In order to carry out the integration over velocity we divide $\mathbf{k}\cdot\mathbf{v}$ into two parts:

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The first term is expressed simply in terms of the argument of the δ-function, while the second is independent of it. Consequently, the integration over velocity in Eq. (3.18) can be carried out without difficulty; we rewrite it in the form

$$\operatorname{Im}\varepsilon^{(3)}_{\mathbf{k}',\,\mathbf{k},\,-\mathbf{k}'} \approx -\frac{e^2(\mathbf{k}\cdot\mathbf{k}')^2}{M^2\omega^4}\left[1+\frac{4(\mathbf{k}\cdot\mathbf{k}'')\omega''}{k''^2\omega}+\frac{10(\mathbf{k}\cdot\mathbf{k}'')^2\omega''^2}{k''^2\omega^2}+\right.$$
$$\left.+\frac{10[\mathbf{k}\times\mathbf{k}'']^2 v_{Ti}^2}{\omega^2 k''^2}\right]\operatorname{Im}\varepsilon^{(1)}(\omega'',\mathbf{k}''). \qquad (3.20)$$

Similarly, we can compute

$$k^2\varepsilon^{(2)}_{\mathbf{k}',\,\mathbf{k}''} = k''^2\varepsilon^{(2)}_{\mathbf{k},\,-\mathbf{k}'} = \frac{e}{M}\cdot\frac{(\mathbf{k}\times\mathbf{k}')}{\omega^2}\times$$
$$\times\left[1+\frac{2(\mathbf{k}\times\mathbf{k}'')\omega''}{k''^2\omega_k}+\frac{3(\mathbf{k}\times\mathbf{k}'')^2\omega''^2}{k''^4\omega^2}+\frac{3[\mathbf{k}\times\mathbf{k}'']^2 v_{Ti}^2}{k''^2\omega^2}\right]\varepsilon^{(1)}(\omega'',\mathbf{k}''). \qquad (3.21)$$

When Eqs. (3.20) and (3.21) are used, the kinetic equation (3.8) can be written in the form [94, III]

$$\frac{\Omega_p^2}{\omega_{\mathbf{k}}^2}\cdot\frac{\partial|\Phi_{\mathbf{k}}|^2}{\partial(\omega_{\mathbf{k}}t)} = \frac{\Omega_p^2}{k^2c_s^2}\sqrt{\frac{\pi m}{2M}}\left(\frac{V_d}{c_s}\cos\Theta-\frac{\omega_{\mathbf{k}}}{kc_s}\right)|\Phi_{\mathbf{k}}|^2+$$
$$+\frac{16\pi^2e^4n}{M^3k^2}\sum_{\mathbf{k}'}\frac{(\mathbf{k}\cdot\mathbf{k}')^2[\mathbf{k}\times\mathbf{k}']^2v_{Ti}^2}{\omega_{\mathbf{k}}^3\omega_{\mathbf{k}'}^3k''^2}\int d^3\mathbf{v}\,\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\mathbf{k}''\cdot\mathbf{v})\,\mathbf{k}''\times$$
$$\times\frac{\partial f_i}{\partial\mathbf{v}}|\Phi_{\mathbf{k}}|^2|\Phi_{\mathbf{k}'}|^2. \qquad (3.22)$$

As expected, the number of waves is conserved in the nonlinear processes treated here:

$$\sum_{\mathbf{k}} n_{\mathbf{k}} \equiv \sum_{\mathbf{k}}(\Omega_p^2/\omega_{\mathbf{k}}^3)(k^2|\Phi_{\mathbf{k}}|^2/4\pi) = \text{const}.$$

Consequently, the interaction of the modes (scattering) cannot lead to increase in the total number of waves as a consequence of the linear instability and it is necessary to introduce some other additional process if the turbulence spectrum is to be stationary. As a result of scattering on ions with a Maxwellian distribution we find that energy flows along the spectrum from the shortwave region to the longwave region.

Let us assume that damping of the long waves as a consequence of ion–ion collisions plays the role of a sink in the flow of energy and waves [III]. When this assumption is used we can truncate the spectrum in the longwave region; in the remaining wave-vector space we then form a stationary solution in which the linear growth of the instability is balanced by the flow of energy into the longwave region of the spectrum.

To illustrate this kind of turbulence we now consider the case in which the wave vectors associated with the oscillations have only two allowed directions. Going from the discrete set of wave vectors to a continuous set we can write $|\Phi_{\mathbf{k}}|^2$ in the form

$$|\Phi_{\mathbf{k}}|^2 = I(k)\,\delta(\Phi)\,\delta(\cos\Theta-\cos\Theta_0),\quad k\lambda_D \ll 1, \qquad (3.23)$$

where (k, Θ, Φ) are spherical coordinates in k-space with polar axis in the direction of current flow.

We consider the turbulence spectrum in the longwave region, assuming that the drift velocity is only slightly higher than the acoustic velocity. In this case, the instability will only lead to the excitation of waves which propagate at a small angle with respect to the current (i.e., $\Theta_0 \ll 1$) so that Eq. (3.22) can be simplified considerably:

$$\left(\frac{\pi m}{2M}\right)^{1/2}\left(\frac{V_d}{c_s}\cos\Theta_0-1\right)I(k) \approx -\frac{e^2}{T_e^2}\cdot\frac{T_i}{\pi^2}I(k)\,k\times$$
$$\times\frac{\partial}{\partial k}[k^3I(k)]\cos^2(2\Theta_0)\sin^2(2\Theta_0), \qquad (3.24)$$

where we have used the relation

$$\int d^3\mathbf{v}\,\mathbf{k}''\cdot\frac{\partial f_{0i}}{\partial\mathbf{v}}\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\mathbf{k}''\cdot\mathbf{v}) = -k''\frac{\partial}{\partial\omega_{\mathbf{k}'}}\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})$$

and have converted from a summation to an integration with respect to k, making use of the following rule:

$$\sum_{\mathbf{k}} = \int_0^{2\pi}d\Phi\int_0^{\pi}\sin\Theta\,d\Theta\int_0^{\infty}\frac{k^2\,dk}{(2\pi)^3}.$$

The general solution of Eq. (3.24) is

$$\frac{e^2}{\pi^2T_e^2}I(k) \approx -\sqrt{\frac{\pi m}{2M}}\left(\frac{V_d}{c_s}\cos\Theta-1\right)\frac{T_e}{T_i\Theta_0^2k^3}\ln kD. \qquad (3.25)$$

In this equation we truncate the spectrum at some wavelength D, that is to say, I(D) = 0.

The solution that has been obtained is obviously not unique. Moreover, it is unstable since any perturbation that propagates in the direction of the current has a large growth rate and a smaller saturation effect due to the presence of other modes which propaout difficulty; we rewrite it in the form

$$\operatorname{Im} \varepsilon^{(3)}_{\mathbf{k}', \mathbf{k}, -\mathbf{k}'} \approx -\frac{e^2 (\mathbf{k}\cdot\mathbf{k}')^2}{M^2 \omega^4}\left[1 + \frac{4(\mathbf{k}\cdot\mathbf{k}'')\,\omega''}{k''^2\omega} + \frac{10(\mathbf{k}\cdot\mathbf{k}'')^2\,\omega''^2}{k''^2\omega^2} + \right.$$
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$$+\frac{16\pi^2 e^4 n}{M^3 k^2}\sum_{\mathbf{k}'}\frac{(\mathbf{k}\cdot\mathbf{k}')^2[\mathbf{k}\times\mathbf{k}']^2 v_{Ti}^2}{\omega_{\mathbf{k}}^3\,\omega_{\mathbf{k}'}^3\,k''^2}\int d^3\mathbf{v}\,\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \mathbf{k}''\cdot\mathbf{v})\,\mathbf{k}''\times$$
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In this equation we truncate the spectrum at some wavelength D, that is to say, I(D) = 0.

The solution that has been obtained is obviously not unique. Moreover, it is unstable since any perturbation that propagates in the direction of the current has a large growth rate and a smaller saturation effect due to the presence of other modes which propagate at an angle Θ_0. These considerations lead to the conclusion that a spectrum of this kind will tend to collapse.

An attempt to obtain a more realistic solution of Eq. (3.22) has been described by Akhiezer [95], who gives a self-similar solution. It turns out that angular distribution of the spectral energy density oscillates between the Cerenkov cone [i.e., $I(\Theta) \sim \delta(\cos\Theta - c_s/V_d)$] and the current, that is to say, $I(\Theta) = \delta(1-\cos\Theta)$ with a period proportional to the wave energy.

PROBLEM

1. Find the turbulence spectrum in the shortwave region [I].

In the shortwave region Eq. (3.22) can be written in the form

$$\frac{\partial I(k,\Theta,t)}{\partial(\Omega_p t)} = \frac{\Omega_p^2}{k^2 c_s^2}\left(\frac{\pi m}{2M}\right)^{1/2}\left(\frac{V_d}{c_s}\cos\Theta - \frac{\Omega_p}{k c_s}\right) I(k,\Theta,t) +$$

$$+\frac{e^2 I(k,\Theta,t)}{2\pi^2 T_e^2}\frac{T_i}{T_e}(k\lambda_D)^5\frac{\partial}{\partial k}(k^7\lambda_D^5)\int_0^{2\pi} d\Phi'\int_{-1}^{+1} d\cos\Theta' I(k,\Theta',t)\times$$

$$\times[\mathbf{l}\times\mathbf{l}']^2(\mathbf{l}\cdot\mathbf{l}')^2, \quad \mathbf{l}\equiv\frac{\mathbf{k}}{|\mathbf{k}|}. \tag{1}$$

In the stationary case this equation allows a power solution with respect to k:

$$e^2 k^3 I(k)/T_e^2 \approx \frac{T_e}{T_i}\left(\frac{V_d}{v_{Te}}\right)(k\lambda_D)^{-10}. \tag{2}$$

§ 3.3. Stimulated Scattering of Light in a Plasma (Basic Equations)

The description of a system consisting of electromagnetic radiation plus a plasma is given by the set of equations consisting of the kinetic equations for the particle distribution functions and Maxwell's equations for the electric and magnetic fields. If the oscillations in the electric and magnetic fields are small, as in the cases treated earlier, the solution of the kinetic equation can be written in the form of an expansion in the amplitude of the

electric fields [cf. Eq. (3.1)]*

$$f = f^{(0)}(v,t) + \int\frac{d^3\mathbf{k}}{(2\pi)^3} f_{\mathbf{k}}^{(1)} + \iint\frac{d^3\mathbf{k}}{(2\pi)^3}\cdot\frac{d^3\mathbf{k}'}{(2\pi)^3} f_{\mathbf{k},\mathbf{k}'}^{(2)} + \ldots \tag{3.26}$$

The distribution function in each successively higher approximation in the wave amplitude can be found from the interaction relation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) f^{j\,(n+1)}_{\mathbf{k},\mathbf{k}',\ldots,\mathbf{k}^{(n)}} =$$

$$= -\frac{1}{(n+1)!}\sum\frac{e_j}{m_j}\left\{\mathbf{E}_{\mathbf{k}}^{(1)} + \frac{1}{c}\left[\mathbf{v}\times\mathbf{H}_{\mathbf{k}}^{(1)}\right]\right\}\frac{\partial}{\partial\mathbf{v}} f^{j\,(n)}_{\mathbf{k}',\ldots,\mathbf{k}^{(n)}}. \tag{3.27}$$

The result is symmetrized with respect to all possible permutations of the subscripts $\mathbf{k}, \mathbf{k}', \ldots, \mathbf{k}^{(n)}$. The stimulated scattering of light waves with random phase is a second-order effect in terms of the wave energy; hence, a proper description of this process requires a calculation of the scattering flux with an accuracy to third order in the wave amplitude, presupposing that the correction to the distribution function $f^{(3)}$ has been determined.

In the linear approximation the description of the propagation of electromagnetic waves in a plasma reduces to a determination of the dielectric permittivity of the medium:

$$\varepsilon(\omega,\mathbf{k}) = 1 + \sum_j \varepsilon_j(\omega,\mathbf{k}); \quad \varepsilon_j = \frac{4\pi e_j^2}{m_j k^2}\int\frac{\mathbf{k}\cdot\frac{\partial f^{j\,(0)}}{\partial\mathbf{v}}}{\omega - \mathbf{k}\cdot\mathbf{v} + i0}. \tag{3.28}$$

In computing $f^{j(2)}$ it should be kept in mind that the Doppler correction to the wave frequency (but not the beats) is very small in a nonrelativistic plasma. Retaining second-order corrections inclusively in v/c, we have

$$f^{j\,(2)}_{\mathbf{k},-\mathbf{k}'} = \frac{1}{2}\left(\frac{ie_j}{m_j}\right)^2\frac{(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f^{j\,(0)}}{\partial\mathbf{v}}}{\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v} + i0}\cdot\frac{(\mathbf{E}_{\mathbf{k}}\cdot\mathbf{E}_{\mathbf{k}'}^*)}{\omega_{\mathbf{k}}\,\omega_{\mathbf{k}'}}. \tag{3.29}$$

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In computing $f^{j(2)}$ it should be kept in mind that the Doppler correction to the wave frequency (but not the beats) is very small in a nonrelativistic plasma. Retaining second-order corrections inclusively in v/c, we have

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*As we have already noted many times, the nonlinear interaction for both plasma waves and electromagnetic waves is described by the same equation. Here, we present a short derivation of more complete equations than those which have been obtained in Section 3.1.

In the expression for $f^{j(3)}$ we have focused attention only on the term which gives a contribution to the scattering flux for waves with frequency ω_k:

$$f^{j(3)}_{k', k-k'} = -\frac{ie_j}{m_j}\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}}\left\{\mathbf{E}_{k'} + \frac{1}{c}[\mathbf{v}\times\mathbf{H}_{k'}]\right\}\cdot\frac{\partial f^{j(2)}_{k,-k'}}{\partial\mathbf{v}}. \qquad (3.30)$$

The radiation intensity of the scattered wave is equal to the work of the particle in the field of the scattered wave taken with reverse sign. However, the scattering flux determined this way only determines part of the total scattering effect due to free electrons. In a plasma the electrons are surrounded by a shielding cloud of electrons and ions. It will be evident that the oscillations of the electron and the shielding cloud are in opposite phase; hence, scattering on the shielding cloud will act to neutralize scattering on the electron itself.

The oscillations of the shielding cloud are described by the electrostatic potential $\Phi^{(2)}_{\Delta k,\Delta\omega}$. The oscillation amplitude is found from Poisson's equation for the potential:

$$\Delta k^2\,\varepsilon(\Delta\omega,\Delta\mathbf{k})\,\Phi^{(2)}_{\Delta k,\Delta\omega} = -4\pi e\int f^{e(2)}_{k,-k'}\,d^3\mathbf{v}, \qquad (3.31)$$

whence, from Eq. (3.29), we have

$$\Phi^{(2)}_{\Delta k,\Delta\omega} = \frac{e}{2m_e}\cdot\frac{(\mathbf{E}_k\cdot\mathbf{E}^*_{k'})}{\omega_k\,\omega_{k'}}\cdot\frac{\varepsilon_e(\Delta\omega,\Delta\mathbf{k})}{\varepsilon(\Delta\omega,\Delta\mathbf{k})}, \qquad (3.32)$$

$$\Delta\omega = \omega_k - \omega_{k'}, \quad \Delta\mathbf{k} = \mathbf{k} - \mathbf{k}'.$$

The electron distribution in the field of the virtual electrostatic wave $\Phi^{(2)}$ is

$$f^{j(1)}_{\Delta k}(\Delta\omega) = \frac{e_j}{m_j}\,\Phi^{(2)}_{\Delta k,\Delta\omega}\,\frac{\Delta\mathbf{k}\cdot(\partial f^{j(0)}/\partial\mathbf{v})}{\Delta\omega - \Delta\mathbf{k}\cdot\mathbf{v} + i0}. \qquad (3.33)$$

The oscillations of the electrons in the field of the incident wave, with the electron density being modulated in the field of the virtual wave, give a contribution to the scattering flux which is comparable with that found earlier. This effect is described by a second-order correction to the particle distribution function; in the iteration equation the function for the first approximation is taken from (3.33):

$$f^{j(2)}_{k',\Delta k} = \frac{ie}{m}\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}}\left\{E_{k'} + \frac{1}{c}[\mathbf{v}\times H_{k'}]\right\}\frac{\partial}{\partial\mathbf{v}}f^{j(1)}_{\Delta k}. \qquad (3.34)$$

Summing the contributions in the scattered flux due to the oscillations of the electrons and the shielding cloud, and computing the radiation intensity at the frequency of the scattered wave, we obtain an equation for the oscillation amplitude [96, 97, V]:

$$\frac{\partial}{\partial t}\frac{|\mathbf{E}_k|^2}{4\pi} = \frac{e^2}{2m^2}\int\frac{d^3\mathbf{k}'}{(2\pi)^3}\cdot\frac{|\mathbf{E}_k\cdot\mathbf{E}^*_{k'}|^2}{\omega_k\,\omega^2_{k'}}\times$$
$$\times\frac{\Delta k^2}{4\pi|\varepsilon(\Delta\omega,\Delta\mathbf{k})|^2}\,\mathrm{Im}\{|1+\varepsilon_i(\Delta\omega,\Delta\mathbf{k})|^2\varepsilon_e(\Delta\omega,\Delta\mathbf{k}) +$$
$$+|\varepsilon_e(\Delta\omega,\Delta\mathbf{k})|^2\varepsilon_i(\Delta\omega,\Delta\mathbf{k})\}. \qquad (3.35)$$

The equation for the fields must be supplemented by a quasilinear equation for the averaged particle distribution function. The latter is obtained by taking an average of the nonlinear terms in the kinetic equation, averaging over the rapid oscillations in time of the electric and magnetic fields. Retaining terms to second order inclusively in the wave energy, we have

$$\frac{\partial f^{j(0)}}{\partial t} = -\frac{e_j}{m_j}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\left\{\mathbf{E}^*_k + \frac{1}{c}[\mathbf{v}\times\mathbf{H}^*_k]\right\}\frac{\partial}{\partial\mathbf{v}}\left(f^{j(3)}_{k',k,-k'} + f^{j(2)}_{k',\Delta k}\right) -$$
$$-\frac{ie_j}{m_j}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\Phi^{(2)*}_{\Delta\omega,\Delta k}\,\Delta\mathbf{k}\,\frac{\partial}{\partial\mathbf{v}}\left(f^{j(1)}_{\Delta k} + f^{j(2)}_{k,-k'}\right).$$

From Eqs. (3.29-3.34) we then have

$$\partial f^{j(0)}/\partial t = (\partial/\partial v_\alpha)D^j_{\alpha\beta}(\partial f^{j(0)}/\partial v_\beta), \qquad (3.36)$$

where

$$D^j_{\alpha\beta} = \frac{\pi e^4}{4m^2m_j^2}\iint\frac{d^3\mathbf{k}}{(2\pi)^3}\cdot\frac{d^3\mathbf{k}'}{(2\pi)^3}\cdot\frac{|\mathbf{E}_k\cdot\mathbf{E}^*_{k'}|^2}{\omega_k^2\,\omega_{k'}^2}\,\Delta k_\alpha\,\Delta k_\beta\,\delta(\Delta\omega - \Delta\mathbf{k}\cdot\mathbf{v})\times$$
$$\times\{\delta_{je}|1+\varepsilon_i(\Delta\omega,\Delta\mathbf{k})|^2 + \delta_{ji}|\varepsilon_e(\Delta\omega,\Delta\mathbf{k})|^2\}/|\varepsilon(\Delta\omega,\Delta\mathbf{k})|^2.$$

It is not difficult to show by direct calculation using Eqs. (3.35) and (3.36) that the energy and momentum are conserved in the system comprising the plasma particles plus radiation.

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The oscillations of the electrons in the field of the incident wave, with the electron density being modulated in the field of the virtual wave, give a contribution to the scattering flux which is comparable with that found earlier. This effect is described by a second-order correction to the particle distribution function; in the iteration equation the function for the first approximation is taken from (3.33):

$$f^{j(2)}_{\mathbf{k}',\Delta\mathbf{k}} = \frac{ie}{m}\,\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}}\left\{E_{\mathbf{k}'}+\frac{1}{c}[\mathbf{v}\times H_{\mathbf{k}'}]\right\}\frac{\partial}{\partial\mathbf{v}}f^{j(1)}_{\Delta\mathbf{k}}. \tag{3.34}$$

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The equation for the fields must be supplemented by a quasilinear equation for the averaged particle distribution function. The latter is obtained by taking an average of the nonlinear terms in the kinetic equation, averaging over the rapid oscillations in time of the electric and magnetic fields. Retaining terms to second order inclusively in the wave energy, we have

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$$\frac{1}{(2\pi)^6}\iint d^3\mathbf{k}\, d^3\mathbf{k}'\, \Phi_j(\mathbf{k}, \mathbf{k}') < T_j, \tag{3.37}$$

where

$$\Phi_i = \frac{e^2(\mathbf{E}_k\cdot\mathbf{E}_{k'}^*)}{2m\omega_k\,\omega_{k'}}\cdot\frac{\varepsilon_e(\Delta\omega, \Delta\mathbf{k})}{\varepsilon(\Delta\omega, \Delta\mathbf{k})},\quad \Phi_e = \frac{e^2(\mathbf{E}_k\cdot\mathbf{E}_{k'}^*)}{2m\omega_k\,\omega_{k'}}\cdot\frac{1+\varepsilon_i(\Delta\omega, \Delta\mathbf{k})}{\varepsilon(\Delta\omega, \Delta\mathbf{k})}.$$

The low-intensity limit for the region of applicability of Eq. (3.35) is imposed by spontaneous scattering of light on inhomogeneities in the electron density (cf. Problem 1).

PROBLEM

1. Compute the intensity of scattering of light on fluctuations in the electron density [VII-VIII].

In contrast with the spontaneous emission of longitudinal waves, optical emission is characterized by components of the scattering flux that are perpendicular to the wave vector. We also assume that the coupling of a given mode to all other modes is balanced by an inverse process; Eq. (3) (cf. Problem 1 of § 3.1) is then replaced by the equation

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\omega_k\,\partial\omega_k}\cdot\omega_k^2\,\varepsilon(\omega_k)\frac{|\mathbf{E}_k|^2}{8\pi}\right) = \frac{4\pi e^4}{m^2}\int\frac{d^3\mathbf{k}'}{(2\pi)^3}(\delta n_e^2)_{\omega-\omega',\,\mathbf{k}-\mathbf{k}'}\times$$
$$\times\left\{\frac{|[\mathbf{k}\times\mathbf{E}_{k'}]|^2/k^2\omega^2}{\frac{\partial}{\omega\partial\omega}\,\omega^2\,\varepsilon(\omega)} - \frac{|[\mathbf{k}'\times\mathbf{E}_k]|^2/k'^2\omega^2}{\frac{\partial}{\omega'\partial\omega'}\,\omega'^2\,\varepsilon(\omega')}\right\}. \tag{1}$$

The last term here describes the well-known Thompson scattering of light with total cross section $\sigma_T = (8\pi/3)(e^2/mc^2)^2$.

§ 3.4. Relaxation of a Radiation Line in a Plasma

The equations obtained above apply to the relaxation of a radiation line in a plasma. To simplify the problem we assume that the radiation is not polarized and carry out an average over polarization directions, making use of the relations $\langle e_\parallel e_\perp\rangle = 0$, $\langle e_\parallel e_\parallel\rangle = \langle e_\perp e_\perp\rangle = 1/2$, where $e_\parallel$, $e_\perp$ are the projections of the polarization vector in the scattering plane and normal to it. The quantity $|\mathbf{E}_k\cdot\mathbf{E}_{k'}^*|^2$ is then found to be

$$\langle|\mathbf{E}_k\cdot\mathbf{E}_{k'}^*|^2\rangle = (1/4)[1+(\mathbf{l}\cdot\mathbf{l}')^2],\quad \mathbf{l}\equiv\mathbf{k}/|\mathbf{k}|,$$

where $\mathbf{l}$ is a unit vector in the direction of propagation of the wave. In analyzing unpolarized radiation it is convenient to replace the oscillation amplitude by the occupation number

$$n(\nu, \mathbf{l}) = |\mathbf{E}_k|^2/4\pi h\nu, \tag{3.38}$$

where $\nu = \omega/2\pi$ is the frequency of the light; h is Planck's constant. The occupation number defined in this way is related simply to the spectral density of the radiation:

$$W_\nu = \frac{2h\nu^3}{c^3}\int n(\nu, \mathbf{l})\, d^2\mathbf{l}. \tag{3.39}$$

Finally, it is assumed that the radiation frequency is much larger than the plasma frequency $\nu \gg \nu_{pe} = \sqrt{N_0 e^2/\pi m}$. Then Eq. (2.35) becomes

$$\partial n(\nu, \mathbf{l}, t)/\partial t = \frac{3hN_0\sigma_T}{16\pi^2 mc\nu_{pe}^2}\, n(\nu, \mathbf{l}, t)\int \nu'^2 d\nu' d^2\mathbf{l}'[1+(\mathbf{l}\cdot\mathbf{l}')^2](1-\mathbf{l}\cdot\mathbf{l}')\times$$
$$\times n(\nu', \mathbf{l}', t)\,\mathrm{Im}\{|1+\varepsilon_i|^2\varepsilon_e(\Delta\nu, \Delta\varkappa)+|\varepsilon_e|^2\varepsilon_i(\Delta\nu, \Delta\varkappa)\}/|\varepsilon(\Delta\nu, \Delta\varkappa)|^2, \tag{3.40}$$

where $\sigma_T = (8\pi/3)(e^2/mc^2)^2$ is the Thompson cross section for scattering of the light, $\Delta\nu = \nu - \nu'$, $e\Delta\varkappa = \nu\mathbf{l} - \nu'\mathbf{l}'$. The diffusion coefficient for the particles is then found to be

$$D_{\alpha\beta} = \frac{3h^2\sigma_T}{32\pi m_j^2 c^2}\int d^2\mathbf{l}\, d^2\mathbf{l}'\int \nu d\nu\nu'\, d\nu'\,[1+(\mathbf{l}\cdot\mathbf{l}')^2]\, n(\nu, \mathbf{l}, t)\times$$
$$\times n(\nu', \mathbf{l}', t)\,\Delta\varkappa_\alpha\,\Delta\varkappa_\beta\,\delta(\nu-\nu'-\Delta\varkappa\cdot\mathbf{v})\{\delta_{je}|1+\varepsilon_i|^2+\delta_{ji}|\varepsilon_e|^2\}/|\varepsilon|^2. \tag{3.41}$$

This equation can be simplified considerably in two limiting cases: when the linewidth of the radiation is much greater than or much smaller than the Doppler broadening of the beat frequency:

$$\nu_{Dj} = 2\nu\nu'\, v_{Tj}^2/c^2. \tag{3.42}$$

tial energy of the ions in the electrostatic field of the beats, with a potential given by Eq. (3.32), and that the potential energy of the electrons in the field of the beats can be set equal to the kinetic energy of their oscillations in the high-frequency field [97], we write the criterion for the applicability of Eqs. (3.35) and (3.36) in the form

$$\frac{1}{(2\pi)^6}\iint d^3\mathbf{k}\, d^3\mathbf{k}'\, \Phi_j(\mathbf{k}, \mathbf{k}') < T_j, \tag{3.37}$$

where

$$\Phi_i = \frac{e^2(\mathbf{E}_\mathbf{k}\cdot\mathbf{E}^*_{\mathbf{k}'})}{2m\omega_\mathbf{k}\,\omega_{\mathbf{k}'}}\cdot\frac{\varepsilon_e(\Delta\omega, \Delta\mathbf{k})}{\varepsilon(\Delta\omega, \Delta\mathbf{k})}, \quad \Phi_e = \frac{e^2(\mathbf{E}_\mathbf{k}\cdot\mathbf{E}^*_{\mathbf{k}'})}{2m\omega_\mathbf{k}\,\omega_{\mathbf{k}'}}\cdot\frac{1+\varepsilon_i(\Delta\omega, \Delta\mathbf{k})}{\varepsilon(\Delta\omega, \Delta\mathbf{k})}.$$

The low-intensity limit for the region of applicability of Eq. (3.35) is imposed by spontaneous scattering of light on inhomogeneities in the electron density (cf. Problem 1).

PROBLEM

▶ 1. Compute the intensity of scattering of light on fluctuations in the electron density [VII-VIII].

In contrast with the spontaneous emission of longitudinal waves, optical emission is characterized by components of the scattering flux that are perpendicular to the wave vector. We also assume that the coupling of a given mode to all other modes is balanced by an inverse process; Eq. (3) (cf. Problem 1 of § 3.1) is then replaced by the equation

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\omega_\mathbf{k}\,\partial\omega_\mathbf{k}}\cdot\omega_\mathbf{k}^2\,\varepsilon(\omega_\mathbf{k})\frac{|\mathbf{E}_\mathbf{k}|^2}{8\pi}\right) = \frac{4\pi e^4}{m^2}\int\frac{d^3\mathbf{k}'}{(2\pi)^3}(\delta n_e^2)_{\omega-\omega',\,\mathbf{k}-\mathbf{k}'}\times$$
$$\times\left\{\frac{|[\mathbf{k}\times\mathbf{E}_{\mathbf{k}'}]|^2/k^2\omega^2}{\dfrac{\partial}{\omega\partial\omega}\,\omega^2\,\varepsilon(\omega)} - \frac{|[\mathbf{k}'\times\mathbf{E}_\mathbf{k}]|^2/k'^2\omega^2}{\dfrac{\partial}{\omega'\partial\omega'}\,\omega'^2\,\varepsilon(\omega')}\right\}. \tag{1}$$

The last term here describes the well-known Thompson scattering of light with total cross section $\sigma_T = (8\pi/3)(e^2/mc^2)^2$.

§ 3.4. Relaxation of a Radiation Line in a Plasma

The equations obtained above apply to the relaxation of a radiation line in a plasma. To simplify the problem we assume that

the radiation is not polarized and carry out an average over polarization directions, making use of the relations $\langle e_\parallel e_\perp\rangle = 0$, $\langle e_\parallel e_\parallel\rangle = \langle e_\perp e_\perp\rangle = 1/2$, where $e_\parallel$, $e_\perp$ are the projections of the polarization vector in the scattering plane and normal to it. The quantity $|\mathbf{E}_\mathbf{k}\cdot\mathbf{E}^*_{\mathbf{k}'}|^2$ is then found to be

$$\langle|\mathbf{E}_\mathbf{k}\cdot\mathbf{E}^*_{\mathbf{k}'}|^2\rangle = (1/4)[1+(\mathbf{l}\cdot\mathbf{l}')^2], \quad \mathbf{l}\equiv\mathbf{k}/|\mathbf{k}|,$$

where $\mathbf{l}$ is a unit vector in the direction of propagation of the wave. In analyzing unpolarized radiation it is convenient to replace the oscillation amplitude by the occupation number

$$n(\nu, \mathbf{l}) = |\mathbf{E}_\mathbf{k}|^2/4\pi h\nu, \tag{3.38}$$

where $\nu = \omega/2\pi$ is the frequency of the light; h is Planck's constant. The occupation number defined in this way is related simply to the spectral density of the radiation:

$$W_\nu = \frac{2h\nu^3}{c^3}\int n(\nu, \mathbf{l})\, d^2\mathbf{l}. \tag{3.39}$$

Finally, it is assumed that the radiation frequency is much larger than the plasma frequency $\nu \gg \nu_{pe} = \sqrt{N_0e^2/\pi m}$. Then Eq. (2.35) becomes

$$\partial n(\nu, \mathbf{l}, t)/\partial t = \frac{3hN_0\sigma_T}{16\pi^2 mc\nu_{pe}^2}\, n(\nu, \mathbf{l}, t)\int \nu'^2 d\nu' d^2\mathbf{l}'\,[1+(\mathbf{l}\cdot\mathbf{l}')^2](1-\mathbf{l}\cdot\mathbf{l}')\times$$
$$\times n(\nu', \mathbf{l}', t)\,\mathrm{Im}\{|1+\varepsilon_i|^2\varepsilon_e(\Delta\nu, \Delta\varkappa)+|\varepsilon_e|^2\varepsilon_i(\Delta\nu, \Delta\varkappa)\}/|\varepsilon(\Delta\nu, \Delta\varkappa)|^2, \tag{3.40}$$

where $\sigma_T = (8\pi/3)(e^2/mc^2)^2$ is the Thompson cross section for scattering of the light, $\Delta\nu = \nu - \nu'$, $e\Delta\varkappa = \nu\mathbf{l} - \nu'\mathbf{l}'$. The diffusion coefficient for the particles is then found to be

$$D_{\alpha\beta} = \frac{3h^2\sigma_T}{32\pi m_j^2 c^2}\int d^2\mathbf{l}\, d^2\mathbf{l}'\int \nu d\nu\nu'\, d\nu'\,[1+(\mathbf{l}\cdot\mathbf{l}')^2]\, n(\nu, \mathbf{l}, t)\times$$
$$\times n(\nu', \mathbf{l}', t)\,\Delta\varkappa_\alpha\Delta\varkappa_\beta\,\delta(\nu-\nu'-\Delta\varkappa\cdot\mathbf{v})\{\delta_{je}|1+\varepsilon_i|^2+\delta_{ji}|\varepsilon_e|^2\}/|\varepsilon|^2. \tag{3.41}$$

This equation can be simplified considerably in two limiting cases: when the linewidth of the radiation is much greater than or much smaller than the Doppler broadening of the beat frequency:

$$\nu_{Dj} = 2\nu\nu'\, v_{Tj}^2/c^2. \tag{3.42}$$

If the radiation is anisotropic, the spectral width must be compared with the effective Doppler shift of the photon frequency in scattering $\Delta\nu_j = \nu_{Dj}\sqrt{1-\cos\Theta}$ (Θ is the angular aperture of the radiation beam).

Wide Frequency Spectrum. The only strong interaction in the radiation spectrum occurs between components which are near to each other; hence, the integrand in Eq. (3.40) can be expanded in powers of the frequency difference between interacting components. In this case it can be reduced to a differential equation much in the way as has been done for plasmons. As a result we have [98]

$$\frac{\partial}{\partial t} n(\nu, \mathbf{l}, t) = \frac{3hN_0\sigma_T}{16\pi mc} h(\nu, \mathbf{l}, t) \sum_i \int d^2\mathbf{l}' [1+(\mathbf{l}\cdot\mathbf{l}')^2](1-\mathbf{l}\cdot\mathbf{l}') \times \times \left[\frac{\partial}{\partial \nu'} \nu'^2 w_j n(\nu', \mathbf{l}', t)\right]_{\nu'=\nu}, \tag{3.43}$$

where

$$w_e = \frac{1}{\pi\nu_{pe}^2} \operatorname{Im} \int_{-\infty}^{+\infty} \frac{\nu\, d\nu}{1+\varepsilon_e(\nu, \Delta\varkappa)},$$

$$w_i = \frac{|\varepsilon_e(0, \Delta\varkappa)|^2}{\pi\nu_{pe}^2} \operatorname{Im} \int_{-\infty}^{+\infty} \frac{\nu\, d\nu}{1+\varepsilon_e(0, \Delta\varkappa)+\varepsilon_i(\nu, \Delta\varkappa)}$$

and the integrals are computed by means of the dispersion relations [14]

$$\left.\begin{aligned} w_e^{tot} &= 1; \\ w_i^{tot} &= (m/M)\,\nu_{pe}^4/[\nu_{pe}^2+\nu_{De}^2(1-\cos\Theta)]^2. \end{aligned}\right\} \tag{3.44}$$

The quantities w_j do not characterize the scattering itself, but the effectiveness of energy transfer in the scattering. This effective cross section for scattering on electrons and ions is not the same, but differs by a factor of m/M. It follows from Eq. (3.44) that in the case of a wide spectrum the effects of shielding can be neglected. Furthermore, if $\nu_{pe} \ll \nu_{De}$, the shielding effects disappear in the scattering of light on fluctuations in the electron density. In this case we can carry out the integration over angle in Eq. (3.43) to

obtain the well-known equation given by Kompaneets [99]:

$$\frac{\partial n(\nu, t)}{\partial t} = \frac{\sigma_T N_{0e} \hbar}{mc} \cdot \frac{1}{\nu^2} \cdot \frac{\partial}{\partial \nu} \nu^4 \left[n + n^2 + \frac{\partial n}{\partial \nu} \cdot \frac{T_e}{\hbar}\right], \tag{3.45}$$

where the first and last terms are associated with the scattering of light on fluctuations in the electron density (cf. § 3.3, Problem 1). In the present case, which corresponds to an equilibrium distribution of the radiation, the right side of this equation vanishes and the Planck distribution is obtained.

Wide Radiation Line ($\nu_{pe} > \delta > \nu_{De}\sqrt{1-\cos\Theta}$). In this case, we must eliminate the contribution due to the decay of the incident photon of the electromagnetic wave into a photon of a plasma wave and a scattered photon in Eq. (3.43). As already noted in § 3.1, the latter is determined by the contributions in the integral in (3.43) from poles at points where the dielectric permittivity vanishes:

$$w^{col} = (1/\nu_{pe}^2) \int \nu\, d\nu\, \delta[\varepsilon(\nu, \Delta\varkappa)]. \tag{3.46}$$

Subtracting (3.46) from (3.44) we find the transfer of energy in scattering on single electrons (but not collective oscillations):

$$w_e^{part} = 6(\nu_{De}/\nu_{pe})^4(1-\cos\Theta)^2, \qquad \nu_{pe} > \delta > \nu_{De}\sqrt{1-\cos\Theta}. \tag{3.47}$$

Similarly, in a nonisothermal plasma with hot electrons, the scattering of a radiation line with width much smaller than the ion acoustic frequency only occurs on the ions:

$$w_i^{part} = 6\frac{\nu_{pe}^4 T_i^2}{T_e^2} \Big/ [\nu_{pe}^2 + \nu_{De}^2(1-\cos\Theta)]^2, \qquad 1 < \frac{\delta}{\nu_{Di}} < \sqrt{\frac{T_e}{T_i}}. \tag{3.48}$$

Since the equations which have been obtained exhibit the same structure as the equations which describe the relaxation of a packet of plasma waves in a plasma, the qualitative pattern of the relaxation is the same as before: The lines are shifted in the direction of lower frequencies and the leading edge of the profile becomes sharper and emits solitons [92]. The characteristic time for the relaxation of the profile of the line can be estimated from Eq. (3.46).

Relaxation of a Spectrally Narrow Line. The relaxation of a spectrally narrow line will be analyzed under the

If the radiation is anisotropic, the spectral width must be compared with the effective Doppler shift of the photon frequency in scattering $\Delta\nu_j = \nu_{Dj}\sqrt{1-\cos\Theta}$ (Θ is the angular aperture of the radiation beam).

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$$\frac{\partial}{\partial t} n(\nu, \mathbf{l}, t) = \frac{3hN_0\sigma_T}{16\pi mc} h(\nu, \mathbf{l}, t) \sum_i \int d^2\mathbf{l}' [1 + (\mathbf{l}\cdot\mathbf{l}')^2](1 - \mathbf{l}\cdot\mathbf{l}') \times \\ \times \left[\frac{\partial}{\partial \nu'} \nu'^2 w_j n(\nu', \mathbf{l}', t)\right]_{\nu'=\nu}, \tag{3.43}$$

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Relaxation of a Spectrally Narrow Line. The relaxation of a spectrally narrow line will be analyzed under the assumption that the particle distribution is isotropic. An isotropic distribution will be frequently produced for a number of reasons. For example, when the time required for binary collisions is much shorter than the time associated with the quasilinear distortion of the distribution function the collisions are capable of producing an isotropic distribution. We have shown earlier (cf. § 2.3, Problem 1) that quasilinear diffusion in the field of beat waves with small phase velocity, corresponding to the case of a narrow line, also leads to an isotropic distribution, even if the beat spectrum is anisotropic. Finally, in the field of a narrow axially symmetric radiation beam ($\Theta \ll 1$) the distribution is always isotropic in the plane of the wave vectors associated with the beat waves (it is this symmetry which provides the validity of of the formulas obtained below). In addition to exhibiting an isotropic velocity distribution, it is found that the plasma is also heated (the use of laser radiation for heating a plasma has been proposed in [102, 103]). It will be assumed that in the field of a narrow spectral line the phase velocity of the beats can be of the same order as the ion thermal velocity, so that the possibility of heating the plasma ions exists [104]. The distribution of particles established in the field of the beat waves with small phase velocity has been found in § 2.3, where it has been shown that in the limit $t \to \infty$, the final distribution is independent of the initial distribution. In this case the mean particle energy increases with the field energy: $m\langle v^2\rangle \sim [\sum_k E_k^2]^{4/5}$ Thus, in the cases listed above (obviously including the case of completely isotropic radiation), in Eq. (3.36) we can carry out an average over the directions of the velocities of the resonant particles (in the case of a narrow radiation beam these lie in the plane perpendicular to the beam). Then the imaginary part of the dielectric permittivity which appears in the wave equation is of the form

$$\operatorname{Im} \varepsilon_j = \frac{\nu_{pj}^2}{\Delta\varkappa^2}\int d^3 v \pi \delta(\Delta\nu - \Delta\varkappa\cdot v)\,\Delta\varkappa\frac{\partial f_{0j}}{\partial v} \equiv \equiv -\frac{\nu_{pj}^2}{\Delta\varkappa^2}\cdot\frac{\Delta\nu\cdot(\partial f_{0j}/v_\perp\,\partial v_\perp)}{\sqrt{(2\nu\nu'/c^2)\,v_\perp^2\,(1-\cos\Theta)-\Delta\nu^2}}. \tag{3.49}$$

We can also write the equation for the occupation number:

$$\frac{\partial n}{\partial t} = -\frac{3h\sigma_T c}{32\pi^2 m\nu}\int d^2 \mathbf{l}' \int \nu'\,d\nu'\,[1+(\mathbf{l}\cdot\mathbf{l}')^2]\,n(\nu,\,\mathbf{l},\,t)\,n(\nu',\,\mathbf{l}',\,t)\,\times \times\sum_j w_j \int \frac{\Delta\nu\,(\partial f_j/v_\perp\,\partial v_\perp)}{\sqrt{\nu_{Dj}^2 v_\perp^2 | v_{Tj}^2 - \Delta\nu^2}}\,d^2 v, \tag{3.50}$$

where

$$w_e = \begin{cases} (1-\nu_{pi}^2/\Delta\nu^2)^2/(1+\nu_{pe}^2/\nu_{De}^2\,(1-\cos\Theta)-\nu_{pi}^2/\Delta\nu^2)^2, \\ \qquad \nu_{De} > \dfrac{\Delta\nu}{\sqrt{1-\cos\Theta}} > \nu_{Di}; \\ (1-\cos\Theta+\nu_{pi}^2/\nu_{Di}^2)^2/(1-\cos\Theta+\nu_{pi}^2/\nu_{Di}^2+\nu_{pe}^2/\nu_{De}^2)^2, \\ \qquad \Delta\nu < \nu_{Di}\sqrt{1-\cos\Theta}; \end{cases}$$

$$w_i = \frac{m}{M}\left(\frac{\nu_{pe}}{\nu_{De}}\right)^4 \Big/ [1-\cos\Theta+\nu_{pi}^2/\nu_{Di}^2+\nu_{pe}^2/\nu_{De}^2]^2, \quad \Delta\nu < \nu_{Di}\sqrt{1-\cos\Theta}.$$

It will be evident that the scattering of waves in the plasma described by Eqs. (3.43) and (3.50) has a very complicated dependence on the ratio of the spectral width of the line to the Doppler width, as well as on the plasma density. The qualitative form of the dependence of the ratio of the intensity of the scattering process in the plasma to the intensity of the Thompson cross section for scattering by unshielded electrons is shown in Fig. 29. In illustrating the nature of the relaxation of a narrow line in the plasma we limit ourselves to rather dense plasmas ($\nu_{pe} \gg \nu_{De}$), i.e., $N_e \gg m\nu^2 v_{Te}^2/e^2c^2$, and very narrow lines of isotropic radiation

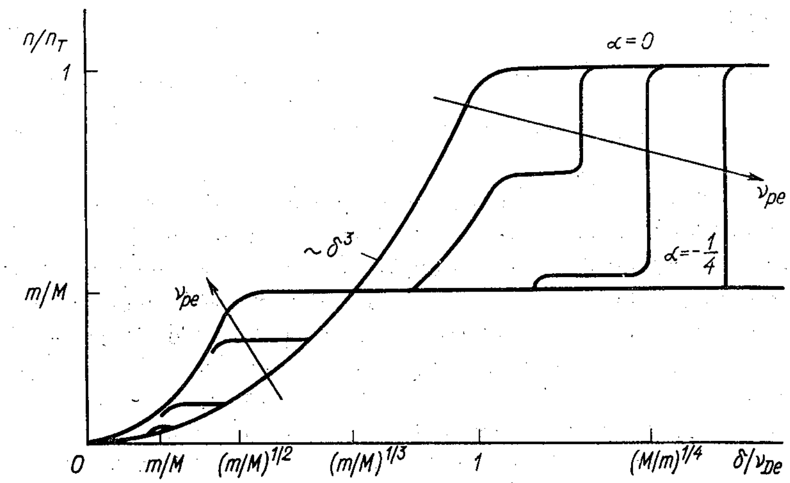

Fig. 29. The ratio of the intensity of light scattering in a plasma to the intensity of Thompson scattering as a function of the ratio of the spectral width of the emission line and the plasma frequency to the Doppler broadening of the beat frequency.

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We can also write the equation for the occupation number:

$$\frac{\partial n}{\partial t} = -\frac{3h\sigma_T c}{32\pi^2 m\nu}\int d^2\mathbf{l}'\int \nu'\,d\nu'\,[1+(\mathbf{l}\cdot\mathbf{l}')^2]\,n(\nu,\mathbf{l},t)\,n(\nu',\mathbf{l}',t)\times$$
$$\times\sum_j w_j\int\frac{\Delta\nu\,(\partial f_j/v_\perp\,\partial v_\perp)}{\sqrt{\nu_{Dj}^2\,v_\perp^2\,|\,v_{Tj}^2-\Delta\nu^2}}\,d^2\mathbf{v}, \tag{3.50}$$

where

$$w_e = \begin{cases} (1-\nu_{pi}^2/\Delta\nu^2)^2/(1+\nu_{pe}^2/\nu_{De}^2(1-\cos\Theta)-\nu_{pi}^2/\Delta\nu^2)^2, \\ \qquad \nu_{De} > \dfrac{\Delta\nu}{\sqrt{1-\cos\Theta}} > \nu_{Di}; \\ (1-\cos\Theta+\nu_{pi}^2/\nu_{Di}^2)^2/(1-\cos\Theta+\nu_{pi}^2/\nu_{Di}^2+\nu_{pe}^2/\nu_{De}^2)^2, \\ \qquad \Delta\nu < \nu_{Di}\sqrt{1-\cos\Theta}; \end{cases}$$

$$w_i = \frac{m}{M}\left(\frac{\nu_{pe}}{\nu_{De}}\right)^4 \Big/ [1-\cos\Theta+\nu_{pi}^2/\nu_{Di}^2+\nu_{pe}^2/\nu_{De}^2]^2,\quad \Delta\nu < \nu_{Di}\sqrt{1-\cos\Theta}.$$

It will be evident that the scattering of waves in the plasma described by Eqs. (3.43) and (3.50) has a very complicated dependence on the ratio of the spectral width of the line to the Doppler width, as well as on the plasma density. The qualitative form of the dependence of the ratio of the intensity of the scattering process in the plasma to the intensity of the Thompson cross section for scattering by unshielded electrons is shown in Fig. 29. In illustrating the nature of the relaxation of a narrow line in the plasma we limit ourselves to rather dense plasmas ($\nu_{pe} \gg \nu_{De}$), i.e., $N_e \gg m\nu^2 v_{Te}^2/e^2c^2$, and very narrow lines of isotropic radiation

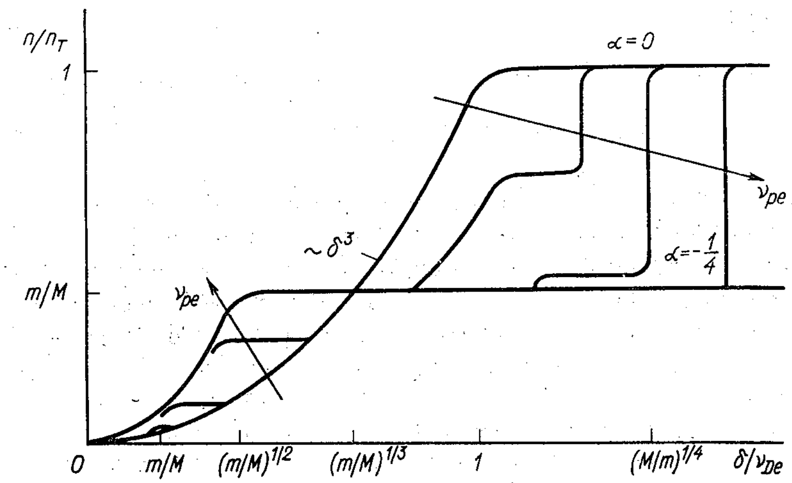

Fig. 29. The ratio of the intensity of light scattering in a plasma to the intensity of Thompson scattering as a function of the ratio of the spectral width of the emission line and the plasma frequency to the Doppler broadening of the beat frequency.

($\delta \ll \nu_{Di}$) (cf. Problems). In this limit Eq. (3.50) assumes the simple form

$$\partial n(\nu, t)/dt = -K'(0)n(\nu,t)\int \nu'^2 n(\nu',t)(\nu-\nu')\,d\nu',$$

where

$$K'(0) = 11\sqrt{\pi}\, hN_{0i}\,\sigma_T/5M v_{Ti}^3 c. \tag{3.51}$$

Following the treatment in [100], we write the solution of Eq. (3.51) in terms of the unknown photon energy W(t) which conserves the total number of photons N_γ and the initial photon distribution $n(\nu, 0)$:

$$n(\nu, t) = n(\nu, 0)\exp\left\{\frac{K'(0)}{h}\int_0^t [W(t') - h\nu N_\gamma]\,dt'\right\}, \tag{3.52}$$

where

$$W(t) = (h/c^3)\int \nu^3 n(\nu, t)\,d\nu, \quad N_\gamma = c^{-3}\int \nu^2 n(\nu, t)\,d\nu.$$

Let us assume that the initial profile of the line is Gaussian $n(\nu,0) = (1/\sqrt{2\pi}\delta)\exp[-(\nu-\nu_0)^2/2\delta^2]$. Equation (3.52) then yields the relations

$$\left.\begin{aligned} n(\nu, t) &= (1/\sqrt{2\pi}\delta)\exp\{-[\nu-\nu_0+\delta^2 K' N_\gamma t]^2/2\delta^2 + \varphi(t)\},\\ \varphi(t)/K'(0) &= -\nu_0 N_\gamma t + \delta^2 N_\gamma^2 K' t^2/2 + \int^t W(t')\,dt'/h. \end{aligned}\right\} \tag{3.53}$$

It will be evident that a Gaussian line profile is not distorted by stimulated scattering but is only shifted as a whole in the direction of lower frequencies at a constant rate. It is found that the Gaussian profile – this is the particular case of a degenerate spectrum – does not change its form in time (all other profiles undergo distortion). If the drop in the wings of the line occurs at a slower rate than in a Gaussian profile (an example being the Lorenz profile) the line is spread out because of the transfer of photons into the low-frequency wing. If, however, the wings are suppressed (for example, they vanish for a rectangular profile), then the low-frequency edge of the line remains fixed and the line itself is compressed as a consequence of the transfer of photons to the line edge [100].*

PROBLEMS

1. Estimate the limiting brightness temperature of a saturated astrophysical maser [98].

To simplify the problem we consider a uniform isotropic medium filled with the active molecules such as OH or H_2O, neutral atoms and molecules, electrons, and protons. It is assumed that a pumping mechanism exists which provides population inversion in the levels of the OH or H_2O molecule. Assuming that the Doppler width of the emission line Δ_M due to thermal motion of the heavy molecules is less than the Doppler width of the beat frequency on the plasma protons, we can write the following equation for the population number:

$$\frac{\partial}{\partial t} n(\nu, t) = \frac{B\bar{n}_H}{\sqrt{2\pi}\,\Delta_M}\exp\left[-\frac{(\nu-\nu_0)^2}{2\Delta_M^2}\right] - K'(0)\,n(\nu, t)\int(\nu-\nu')\,\nu'^2 n(\nu', t)\,d\nu', \tag{1}$$

where $\bar{n}_H$ is the number of photons at which the maser is saturated, while the coefficient B is expressed by means of the probability for a spontaneous transition A_{mn} of the population of level f_m with multiplicity g_m and the density of active molecules N_M:

$$B_{mn} = (c^3/8\pi\nu^3)A_{mn}[f_m - (g_m/g_n)f_n]N_M. \tag{2}$$

Spontaneous emission is neglected in Eq. (1). The qualitative description of the relaxation of the emission line described by Eq. (1) is rather simple. The emission intensity increases linearly in time until a critical value is reached at the center of the line, this value being $n \sim [B\bar{n}_H/\nu_0^2\Delta_M^3 K'(0)]^{1/2}$. For this value the stimulated Compton scattering on plasma protons leads to a shift of the line

*For a very narrow line the growth in the fluctuations of the field far from the line can occur much more rapidly in the region of the maximum growth rate for wave scattering. This jumplike displacement of the line in the low-frequency direction is discussed in [101].

($\delta \ll \nu_{Di}$) (cf. Problems). In this limit Eq. (3.50) assumes the simple form

$$\partial n(\nu, t)/dt = -K'(0) n(\nu, t) \int \nu'^{2} n(\nu', t)\,(\nu - \nu')\,d\nu',$$

where

$$K'(0) = 11\sqrt{\pi}\, h N_{0i}\, \sigma_T / 5 M \nu_{Di}^{3} c. \tag{3.51}$$

Following the treatment in [100], we write the solution of Eq. (3.51) in terms of the unknown photon energy W(t) which conserves the total number of photons N_γ and the initial photon distribution $n(\nu, 0)$:

$$n(\nu, t) = n(\nu, 0)\exp\left\{\frac{K'(0)}{h}\int_0^t [W(t') - h\nu N_\gamma]\,dt'\right\}, \tag{3.52}$$

where

$$W(t) = (h/c^3)\int \nu^3 n(\nu, t)\,d\nu, \quad N_\gamma = c^{-3}\int \nu^2 n(\nu, t)\,d\nu.$$

Let us assume that the initial profile of the line is Gaussian $n(\nu, 0) = (1/\sqrt{2\pi}\delta)\exp[-(\nu - \nu_0)^2/2\delta^2]$. Equation (3.52) then yields the relations

$$\left.\begin{aligned} n(\nu, t) &= (1/\sqrt{2\pi}\delta)\exp\{-[\nu - \nu_0 + \delta^2 K' N_\gamma t]^2/2\delta^2 + \varphi(t)\}, \\ \varphi(t)/K'(0) &= -\nu_0 N_\gamma t + \delta^2 N_\gamma^2 K' t^2/2 + \int^t W(t')\,dt'/h. \end{aligned}\right\} \tag{3.53}$$

It will be evident that a Gaussian line profile is not distorted by stimulated scattering but is only shifted as a whole in the direction of lower frequencies at a constant rate. It is found that the Gaussian profile – this is the particular case of a degenerate spectrum – does not change its form in time (all other profiles undergo distortion). If the drop in the wings of the line occurs at a slower rate than in a Gaussian profile (an example being the Lorenz profile) the line is spread out because of the transfer of photons into the low-frequency wing. If, however, the wings are suppressed (for example, they vanish for a rectangular profile), then the low-frequency edge of the line remains fixed and the line itself is com-

pressed as a consequence of the transfer of photons to the line edge [100].*

PROBLEMS

▶ 1. Estimate the limiting brightness temperature of a saturated astrophysical maser [98].

To simplify the problem we consider a uniform isotropic medium filled with the active molecules such as OH or H_2O, neutral atoms and molecules, electrons, and protons. It is assumed that a pumping mechanism exists which provides population inversion in the levels of the OH or H_2O molecule. Assuming that the Doppler width of the emission line Δ_M due to thermal motion of the heavy molecules is less than the Doppler width of the beat frequency on the plasma protons, we can write the following equation for the population number:

$$\frac{\partial}{\partial t} n(\nu, t) = \frac{B\bar{n}_H}{\sqrt{2\pi}\,\Delta_M}\exp\left[-\frac{(\nu - \nu_0)^2}{2\Delta_M^2}\right] - $$
$$-K'(0)\, n(\nu, t)\int (\nu - \nu')\,\nu'^2 n(\nu', t)\,d\nu', \tag{1}$$

where $\bar{n}_H$ is the number of photons at which the maser is saturated, while the coefficient B is expressed by means of the probability for a spontaneous transition A_{mn} of the population of level f_m with multiplicity g_m and the density of active molecules N_M:

$$B_{mn} = (c^3/8\pi\nu^3) A_{mn}[f_m - (g_m/g_n) f_n] N_M. \tag{2}$$

Spontaneous emission is neglected in Eq. (1). The qualitative description of the relaxation of the emission line described by Eq. (1) is rather simple. The emission intensity increases linearly in time until a critical value is reached at the center of the line, this value being $n \sim [B\bar{n}_H/\nu_0^2 \Delta_M^3 K'(0)]^{1/2}$. For this value the stimulated Compton scattering on plasma protons leads to a shift of the line

*For a very narrow line the growth in the fluctuations of the field far from the line can occur much more rapidly in the region of the maximum growth rate for wave scattering. This jumplike displacement of the line in the low-frequency direction is discussed in [101].

with a Doppler profile toward the low-frequency region. As before, part of the radiation occurs at frequency ν_0; it is connected with the continuing operation of the maser. The intensity of this component falls since the Compton effect rapidly transfers the photons which are produced into the drifting line. When the line has shifted by an amount of order ν_{Di}, the possibility arises of forming a new line close to the frequency ν_0 and the process is repeated. It is assumed that the maser operates in a highly saturated regime so that the radiation at frequency ν_0, even after being reduced, lies above the threshold: $n(\nu_0) \gg \bar{n}_H$ (the general case is treated in [98]). The number of photons in the moving line increases linearly with time until a shift by a distance of order ν_{Di} is achieved:

$$\left.\begin{aligned} n(\nu, t) &\approx \frac{B\bar{n}_H t}{\sqrt{2\pi}\Delta_M} \exp\left[-\frac{(\nu-\nu_*)^2}{2\Delta_M^2}\right], \\ \nu_* &= \nu_0 - \Delta_M^2 \nu_0^2 K'(0) B\bar{n}_H t^2/2. \end{aligned}\right\} \tag{3}$$

We now introduce special notation for the radiation flux density at frequency ν_0 for which the time of stimulated scattering on plasma protons $t_c^{-1} = \nu_0^2\Delta_M^2 K'(0) n_*$ is comparable with the time for the exponential growth of the emission t_M (obviously for an unsaturated maser)

$$I_* \equiv 8\pi h\nu_0^3 n_* \Delta_M \approx \frac{T_{0i}}{t_M \sigma_T} \cdot \frac{\nu_{Di}/\Delta_M}{N_{0i}\lambda^3}; \quad t_M = \sqrt{2\pi}\Delta_M/B; \quad \lambda = c/\nu. \tag{4}$$

Then during the time in which the line drifts,

$$t_R = \left[\frac{2\nu_{Di}}{\nu_0^2 \Delta_M^2 K'(0) B\bar{n}_H}\right]^{1/2} \approx \left(\frac{\nu_{Di}}{\Delta_M}\right)^{1/2} \left(\frac{n_*}{\bar{n}_H}\right)^{1/2} t_M, \tag{5}$$

the drifting line increases to the limiting value

$$n = n_H t_R/t_M \approx (\bar{n}_H n_*)^{1/2} (\nu_{Di}/\Delta_M)^{1/2}; \quad \bar{n}_H < n_* \left(\frac{\Delta_M}{\nu_{Di}}\right)^{3/2} \tag{6}$$

The brightness temperature is related to the limiting value of the occupation number by the relation $T_b \sim h\nu n$.

▶ 2. Compute the pressure of the stimulated light in a plasma in the case of a wide radiation spectrum [105]; cf. also [98].

This force can be computed from the quasilinear equation (3.36) or from the loss of momentum of the electromagnetic waves.

In the case of a wide radiation spectrum, we multiply Eq. (3.43) by $\nu \mathbf{l}/c$ and integrate over phase space to obtain

$$\mathbf{F}_j = \frac{3h^2 N_{0e}\sigma_T}{32\pi mc^3} \int d^2\mathbf{l}\, d^2\mathbf{l}' \int \nu^2 d\nu \Big\{\nu^2 n(\nu, \mathbf{l}, t)\, w_j(\nu, \nu; \Theta) \times$$
$$\times n(\nu, \mathbf{l}', t)\, \mathbf{l} + n(\nu, \mathbf{l}, t)\Big\} \left[\frac{\partial}{\partial \nu'} \nu'^2 w_j(\nu, \nu', \Theta)\, n(\nu', \mathbf{l}', t)\right]_{\nu'=\nu} \nu(\mathbf{l}' - \mathbf{l}).$$

Chapter 4

ANOMALOUS RESISTIVITY IN A PLASMA

§ 4.1. Formulation of the Problem. Conservation Relations

In the first three parts of this monograph we have described nonlinear interactions between waves and waves and between waves and particles; all of these processes can be realized in a plasma which, as a consequence of an instability, goes from a laminar state to a turbulent state. The macroscopic consequences of this change in the state of the plasma are represented by a change in its transport properties (the transport coefficients) such as diffusion, thermal conductivity, electrical resistance, etc. Under these conditions one speaks of anomalous transport coefficients. The basic problem of the theory then is to relate the values of these anomalous transport coefficients to the underlying cause which produces the original instability (in other words, with the source of "free energy" which drives the instability).

The anomalous electrical resistivity is the most important example of a problem of this kind. In this chapter we shall show how the methods of the theory of plasma turbulence developed in Chapters 1-3 are applied to this problem.

The anomalous resistivity of a plasma usually arises when the magnitude of the electrical current that flows in the plasma exceeds some critical value. Sometimes this critical value, above which the plasma resistivity changes abruptly, is extremely small. The density of the flowing current is expressed in terms of the so-called drift velocity $\mathbf{V}_d$. If the electron distribution function is characterized by some velocity $\mathbf{V}_d$ with respect to the ion distribution function, and if this velocity exceeds the critical value, then with a Doppler profile toward the low-frequency region. As before, part of the radiation occurs at frequency ν_0; it is connected with the continuing operation of the maser. The intensity of this com- ponent falls since the Compton effect rapidly transfers the photons which are produced into the drifting line. When the line has shifted by an amount of order ν_{Di}, the possibility arises of forming a new line close to the frequency ν_0 and the process is repeated. It is assumed that the maser operates in a highly saturated regime so that the radiation at frequency ν_0, even after being reduced, lies above the threshold: $n(\nu_0) \gg \bar{n}_H$ (the general case is treated in [98]). The number of photons in the moving line increases linearly with time until a shift by a distance of order ν_{Di} is achieved:

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$$1 - (\Omega_p^2/\omega^2) - \omega_p^2/(\omega - \mathbf{k}\mathbf{V}_d)^2 = 0. \tag{4.1}$$

The growth rate is

$$\operatorname{Im} \omega \approx \mathbf{k}\mathbf{V}_d \, (m/M)^{1/2} \leqslant \Omega_p \tag{4.2}$$

when $\mathbf{k}\mathbf{V}_d \ll \omega_p$ and reaches a maximum value

$$\operatorname{Im} \omega = \left(\frac{M}{4m}\right)^{1/6} \Omega_p \tag{4.3}$$

when $\mathbf{k}\mathbf{V}_d \approx \omega_p$.

Another example of an instability which arises in practice is the ion-acoustic instability. This instability appears when

Instability	Threshold	Frequency	Growth rate
Buneman	$V_d \geqslant v_{Te}$	$\sim \Omega_p$	$\sim \Omega_p$
Ion-acoustic	$V_d > c_s$	$\lesssim \Omega_p$	$\leqslant \Omega_p \dfrac{V_d}{v_{Te}}$
Drummond–Rosenbluth	$V_d > c_s$	$\sim \Omega_H$	$\sim \Omega_H \dfrac{V_d}{v_{Te}}$
Electric mode $k_\perp^2 \gg k_\parallel^2$	Very low, sometimes $< v_{Ti}$	$\ll \omega_H$	$\sqrt{\omega_H \Omega_H}$
Bernstein mode		$l\omega_H$	$\omega_H V_d / v_{Te}$

the electron drift velocity is smaller than the thermal velocity. The dispersion relation for the ion-acoustic instability is

$$0 = 1 - \frac{\Omega_p^2}{\omega^2} + \frac{\Omega_p^2}{k^2 c_s^2}\left\{1 + i\left(\frac{\pi m}{2M}\right)^{1/2}\left(\frac{\omega}{kc_s} - \frac{\mathbf{k}\cdot\mathbf{V}_d}{kc_s}\right)\right\}, \tag{4.4}$$

and the growth rate is

$$\gamma \approx \omega \sqrt{\frac{\pi m}{8M}}\left(\frac{\omega}{kc_s} - \frac{\mathbf{k}\mathbf{V}_d}{kc_s}\right) \bigg/ \left(1 + \frac{k^2 c_s^2}{\Omega_p^2}\right). \tag{4.5}$$

The growth rate for the ion-acoustic instability (the imaginary part of the frequency) is the ion-plasma frequency reduced by a factor equal to the ratio of the electron drift velocity to the electron thermal velocity. In the limiting case $V_d \to v_{Te}$, the ion-acoustic instability goes over into the Buneman instability.

In the presence of a magnetic field a number of new instabilities can appear. One of the instabilities which is also a consequence of the imaginary part of the electron term (the electron pole) in the ion-cyclotron mode is called the Drummond–Rosenbluth instability [108]. This instability arises when the current flows along the magnetic field, whereas the first two instabilities that have been considered are not affected by a magnetic field if the field is reasonably small ($\omega_H \ll \omega_p$). The Drummond–Rosenbluth instability is usually not discussed in connection with anomalous resistivity because it is characterized by a small growth rate and is evidently suppressed by simple quasilinear effects such as the formation of a plateau.

A more important role is played by a class of instabilities associated with electrostatic perturbations for which the wave vector along the magnetic field is much smaller than the transverse component of the wave vector and for which the frequency is much smaller than the electron gyrofrequency, but larger than the ion gyrofrequency. This mode is reminiscent of the well-known Post–Rosenbluth mode which arises in the presence of a loss cone [109]:

$$1 + \frac{\omega_p^2}{\omega_H^2} + \frac{\Omega_p^2}{k^2}\int \frac{\mathbf{k}\cdot\dfrac{\partial f_i}{\partial \mathbf{v}}\, d^3\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v} + i0} + \frac{\omega_p^2}{k^2}\int \frac{k_\parallel \dfrac{\partial f}{\partial v_\parallel}\, \partial v_\parallel}{\omega - \mathbf{k}\cdot\mathbf{V}_d - k_\parallel v_\parallel + i0} = 0. \tag{4.6}$$

In the approximation in which $\omega \gg kv_{Ti}$ and $\mathbf{k}\mathbf{V}_d \gtrless k_\parallel v_{Te}$, Eq. (4.6) becomes the dispersion equation for the so-called modified an instability can arise. When this instability does arise, in addition to losing momentum by binary collisions, electrons also lose momentum because of the interaction with oscillations and waves of various kinds. It is convenient to start with a table of relevant instabilities which arise when a critical velocity is exceeded (cf. Table 1). In this table we have listed all of the basic instabilities which bear on the problem of anomalous resistivity in a plasma. The simplest instability is the so-called Buneman instability [106, 107]. In this case the original distribution functions for the electrons and ions are two δ-functions which are shifted with respect to each other by the mean velocity $\mathbf{V}_d$. The instability is manifest in the excitation of longitudinal electrostatic plasma oscillations with a growth rate of the order of the ion-plasma frequency. The well-known dispersion equation for the Buneman instability is

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$$0=1-\frac{\Omega_p^2}{\omega^2}+\frac{\Omega_p^2}{k^2c_s^2}\left\{1+i\left(\frac{\pi m}{2M}\right)^{1/2}\left(\frac{\omega}{kc_s}-\frac{\mathbf{k}\cdot\mathbf{V}_d}{kc_s}\right)\right\}, \tag{4.4}$$

and the growth rate is

$$\gamma\approx\omega\sqrt{\frac{\pi m}{8M}}\left(\frac{\omega}{kc_s}-\frac{\mathbf{k}\mathbf{V}_d}{kc_s}\right)\Big/\left(1+\frac{k^2c_s^2}{\Omega_p^2}\right). \tag{4.5}$$

The growth rate for the ion-acoustic instability (the imaginary part of the frequency) is the ion-plasma frequency reduced by a factor equal to the ratio of the electron drift velocity to the electron thermal velocity. In the limiting case $V_d \to v_{Te}$, the ion-acoustic instability goes over into the Buneman instability.

In the presence of a magnetic field a number of new instabilities can appear. One of the instabilities which is also a consequence of the imaginary part of the electron term (the electron pole) in the ion-cyclotron mode is called the Drummond–Rosenbluth instability [108]. This instability arises when the current flows along the magnetic field, whereas the first two instabilities that have been considered are not affected by a magnetic field if the field is reasonably small ($\omega_H \ll \omega_p$). The Drummond–Rosenbluth instability is usually not discussed in connection with anomalous resistivity because it is characterized by a small growth rate and is evidently suppressed by simple quasilinear effects such as the formation of a plateau.

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In the approximation in which $\omega \gg k v_{Ti}$ and $\mathbf{k}\mathbf{V}_d \gtrsim k_\parallel v_{Te}$, Eq. (4.6) becomes the dispersion equation for the so-called modified

Buneman instability [110]

$$1+\frac{\omega_p^2}{\omega_H^2}-\frac{\Omega_p^2}{\omega^2}-\frac{\omega_p^2 k_\parallel^2/k^2}{(\omega-\mathbf{k}\cdot\mathbf{V}_d)^2-k_\parallel^2 v_{Te}^2}=0. \tag{4.7}$$

The growth rate is

$$\gamma(\mathbf{k})=\sqrt{\omega_H\Omega_H}\ll\Omega_p;\quad \bar{k}r_H\approx 1;\quad \bar{k}_\parallel=\bar{k}\,(V_d/v_{Te}). \tag{4.8}$$

The approximation in (4.7) is valid if the drift velocity is much greated than v_{Ti}. If this condition is not satisfied then we are dealing with the so-called **electron-acoustic instability** ($\omega_p \gg \omega_H$):

$$\mathrm{Re}\,(\omega-\mathbf{k}\cdot\mathbf{V}_d)\approx k_\parallel\left(\frac{T_i}{m}\right)^{1/2}\Big/\ 1+k^2r_{H*}^2,\quad \mathrm{Im}\,\omega=\pi^{1/2}\frac{\omega}{2\,|\mathbf{k}|\,v_{Ti}}(\omega-\mathbf{k}\mathbf{V}_d). \tag{4.9}$$

An instability of this kind arises when the current flows across the magnetic field. The instability in (4.7) has a very small growth rate and is important only at comparatively small currents, in which case the stronger instabilities such as the Buneman instability or the ion-acoustic instability are not excited.

Finally, recent interest has been given to an instability associated with Bernstein modes [111]. This instability is characterized by a relatively large growth rate and arises when the current flows across the magnetic field. The dispersion relation for this instability is rather complicated.

Up to the present time, primary attention has been given to two kinds of instabilities; the Buneman instability and the ion-acoustic instability. In his first paper Buneman proposed a heuristic expression for the nonlinear stage of the instability. He proposed that the effective collision frequency for the electrons should be of the order of the imaginary part of the frequency as determined from the linear theory, that is to say, the order of the ion-plasma frequency. This simple formulation, in which the ion-plasma frequency appears in place of ν_{eff} in Ohm's law, is called the Buneman conductivity. It will be clear that this formulation cannot give a very accurate description of experimental results; it can only give the appropriate order of magnitude.

A rigorous formulation of the problem of determining the conductivity σ must be carried out taking account of the exchange

of momentum between the electrons and the waves. The well-known expression for the plasma conductivity

$$\sigma = Ne^2/m\nu \tag{4.10}$$

contains ν, the frequency of collisions of electrons with scattering centers (ions, neutrals) in terms of the loss of momentum. If the plasma electrons excite some kind of oscillation or wave as a consequence of the instability, there will be an anomalous loss of momentum (transfer by the oscillations, i.e., collective ion motion). In order to find ν_{eff} we can use the conservation of momentum for the system consisting of the **electron and the wave**. The mean momentum loss of the electrons per unit time is

$$\nu_{eff} mN\mathbf{V}_d \approx -\mathbf{F}. \tag{4.11}$$

If this momentum is transferred to a wave with energy density W, then the change in the wave momentum is

$$\int \gamma_{\mathbf{k}}^e W_{\mathbf{k}}\frac{\mathbf{k}}{\omega_{\mathbf{k}}}\cdot\frac{d^3\mathbf{k}}{(2\pi)^3}, \tag{4.12}$$

where $\gamma_{\mathbf{k}}^e$ is the electron contribution in the imaginary part of the frequency. Equating (4.11) and (4.12), we have

$$\nu_{eff}\, mN\mathbf{V}_d\approx\int\gamma_{\mathbf{k}}^e W_{\mathbf{k}}\frac{\mathbf{k}}{\omega_{\mathbf{k}}}\cdot\frac{d^3\mathbf{k}}{(2\pi)^3}, \tag{4.13}$$

i.e.,

$$\nu_{eff}=\frac{1}{mNV_d}\int\gamma_{\mathbf{k}}^e W_{\mathbf{k}}\frac{\mathbf{k}}{\omega_{\mathbf{k}}}\cdot\frac{d^3\mathbf{k}}{(2\pi)^3}. \tag{4.14}$$

Thus, the problem reduces to finding $W_{\mathbf{k}}$; the quantity $\gamma_{\mathbf{k}}^e$ is to be understood in the quasilinear sense.

The validity of Eq. (4.13) can be demonstrated through the use of the quasilinear diffusion equation for the electrons. For example, with ion-acoustic oscillations,

$$\frac{\partial f_e}{\partial t}=\frac{e^2}{m^2}\int\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}|\Phi_{\mathbf{k}}'|^2\,\pi\delta(\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v})\,\mathbf{k}\cdot\frac{\partial f_e}{\partial\mathbf{v}}\cdot\frac{\partial^3\mathbf{k}}{(2\pi)^3}. \tag{4.15}$$

Buneman instability [110]

$$1+\frac{\omega_p^2}{\omega_H^2}-\frac{\Omega_p^2}{\omega^2}-\frac{\omega_p^2 k_\parallel^2/k^2}{(\omega-\mathbf{k}\cdot\mathbf{V}_d)^2-k_\parallel^2 v_{Te}^2}=0. \tag{4.7}$$

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$$\mathrm{Re}\,(\omega-\mathbf{k}\cdot\mathbf{V}_d)\approx k_\parallel\left(\frac{T_i}{m}\right)^{1/2}\Big/\,1+k^2r_{H*}^2,\quad \mathrm{Im}\,\omega=\pi^{1/2}\frac{\omega}{2\,|\mathbf{k}|\,v_{Ti}}(\omega-\mathbf{k}\mathbf{V}_d). \tag{4.9}$$

An instability of this kind arises when the current flows across the magnetic field. The instability in (4.7) has a very small growth rate and is important only at comparatively small currents, in which case the stronger instabilities such as the Buneman instability or the ion-acoustic instability are not excited.

Finally, recent interest has been given to an instability associated with Bernstein modes [111]. This instability is characterized by a relatively large growth rate and arises when the current flows across the magnetic field. The dispersion relation for this instability is rather complicated.

Up to the present time, primary attention has been given to two kinds of instabilities; the Buneman instability and the ion-acoustic instability. In his first paper Buneman proposed a heuristic expression for the nonlinear stage of the instability. He proposed that the effective collision frequency for the electrons should be of the order of the imaginary part of the frequency as determind from the linear theory, that is to say, the order of the ion-plasma frequency. This simple formulation, in which the ion-plasma frequency appears in place of ν_{eff} in Ohm's law, is called the Buneman conductivity. It will be clear that this formulation cannot give a very accurate description of experimental results; it can only give the appropriate order of magnitude.

A rigorous formulation of the problem of determining the conductivity σ must be carried out taking account of the exchange

of momentum between the electrons and the waves. The well-known expression for the plasma conductivity

$$\sigma = Ne^2/m\nu \tag{4.10}$$

contains ν, the frequency of collisions of electrons with scattering centers (ions, neutrals) in terms of the loss of momentum. If the plasma electrons excite some kind of oscillation or wave as a consequence of the instability, there will be an anomalous loss of momentum (transfer by the oscillations, i.e., collective ion motion). In order to find ν_{eff} we can use the conservation of momentum for the system consisting of the **electron and the wave**. The mean momentum loss of the electrons per unit time is

$$\nu_{eff} mN\mathbf{V}_d \approx -\mathbf{F}. \tag{4.11}$$

If this momentum is transferred to a wave with energy density W, then the change in the wave momentum is

$$\int \gamma_{\mathbf{k}}^e W_{\mathbf{k}}\frac{\mathbf{k}}{\omega_{\mathbf{k}}}\cdot\frac{d^3\mathbf{k}}{(2\pi)^3}, \tag{4.12}$$

where $\gamma_{\mathbf{k}}^e$ is the electron contribution in the imaginary part of the frequency. Equating (4.11) and (4.12), we have

$$\nu_{eff}\, mN\mathbf{V}_d\approx\int \gamma_{\mathbf{k}}^e W_{\mathbf{k}}\frac{\mathbf{k}}{\omega_{\mathbf{k}}}\cdot\frac{d^3\mathbf{k}}{(2\pi)^3}, \tag{4.13}$$

i.e.,

$$\nu_{eff}=\frac{1}{mNV_d}\int \gamma_{\mathbf{k}}^e W_{\mathbf{k}}\frac{\mathbf{k}}{\omega_{\mathbf{k}}}\cdot\frac{d^3\mathbf{k}}{(2\pi)^3}. \tag{4.14}$$

Thus, the problem reduces to finding $W_{\mathbf{k}}$; the quantity $\gamma_{\mathbf{k}}^e$ is to be understood in the quasilinear sense.

The validity of Eq. (4.13) can be demonstrated through the use of the quasilinear diffusion equation for the electrons. For example, with ion-acoustic oscillations,

$$\frac{\partial f_e}{\partial t}=\frac{e^2}{m^2}\int \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}}\,|\Phi_{\mathbf{k}}|^2\,\pi\delta(\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v})\,\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}}\cdot\frac{\partial^3\mathbf{k}}{(2\pi)^3}. \tag{4.15}$$

Multiplying this equation by mv and integrating over velocity we have

$$\nu_{\rm eff}\, mN\mathbf{V}_d = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\gamma_{\mathbf{k}}^{e}\,\frac{\partial\varepsilon(\omega)}{\partial\omega_{\mathbf{k}}}\cdot\frac{h^2|\Phi_{\mathbf{k}}|^2}{8\pi}\,\mathbf{k} \tag{4.16}$$

since $W_{\mathbf{k}} = (\partial\omega\varepsilon/\partial\omega)\,(k^2|\Phi_{\mathbf{k}}|^2/8\pi)$, and the validity of Eq. (4.13) is demonstrated.

The existence of an anomalous resistivity leads to the anomalous generation of Joule heat in a plasma j^2/σ_{an}. This kind of plasma heating is frequently called t u r b u l e n t h e a t i n g since the mechanism responsible for the anomalous resistivity of the plasma is the turbulence associated with the instability. In the absence of binary collisions the turbulent heating is different for the electron and ion components of the plasma. Furthermore, it is not really valid to talk about increasing the "temperatures" of the electrons and ions if temperature is understood in the traditional sense (a Maxwellian particle distribution). Under the plasma conditions described here the term temperature is usually taken to mean the mean random energies of the components.

As a rule the electron temperature increases more rapidly in turbulent heating of a plasma. It is possible to establish a simple criterion which relates the rate of electron heating to the rate of ion heating. The derivation of this criterion is based on the conservation of momentum and energy in the interaction of electrons and ions with the waves. As we have seen, the electrons in the plasma experience a frictional force

$$\mathbf{F} = -\nu_{\rm eff}\, Nm\mathbf{V}_d. \tag{4.17}$$

The work performed by this force goes into heating the plasma electrons:

$$\frac{d\mathcal{E}_e}{dt} \sim \nu_{\rm eff}\, mNV_d^2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\gamma_{\mathbf{k}}^{e} W_{\mathbf{k}}\frac{(\mathbf{k}\cdot\mathbf{V}_d)}{\omega_{\mathbf{k}}}. \tag{4.18}$$

In the stationary saturated state, which is reached when the growth of the instability is limited by nonlinear effects, the momentum of the waves (along with their energy) is transferred to the ions. Thus, in the saturated state the ions must absorb the oscillation energy at a rate of order $\int\gamma_{\mathbf{k}}^{e} W_{\mathbf{k}}\, d^3\mathbf{k}$. As a result, the ion heating

proceeds at a rate

$$\frac{d\mathcal{E}_i}{dt} \sim \int \gamma_{\mathbf{k}}^{e} W_{\mathbf{k}}\frac{d^3\mathbf{k}}{(2\pi)^3}. \tag{4.19}$$

Now let us divide Eq. (4.18) by Eq. (4.19):

$$\frac{d\mathcal{E}_e}{d\mathcal{E}_i} \sim \int \gamma_{\mathbf{k}}^{e} W_{\mathbf{k}}\frac{(\mathbf{k}\cdot\mathbf{V}_d)}{\omega_{\mathbf{k}}}\,d^3\mathbf{k} \Big/ \int \gamma_{\mathbf{k}}^{e} W_{\mathbf{k}}\, d^3\mathbf{k}. \tag{4.20}$$

If we write

$$\int \gamma_{\mathbf{k}}^{e} W_{\mathbf{k}}\frac{(\mathbf{k}\cdot\mathbf{V}_d)}{\omega_{\mathbf{k}}}\,d^3\mathbf{k} \approx \frac{kV_d}{\omega_{\mathbf{k}}}\int \gamma_{\mathbf{k}}^{e} W_{\mathbf{k}}\, d^3\mathbf{k}$$

in Eq. (4.20), the ratio of the rate of electron heating to the rate of ion heating is easily obtained from the following estimate [112]:

$$d\mathcal{E}_e/d\mathcal{E}_i \sim V_d/\omega/k. \tag{4.21}$$

In the form in which it has been obtained here this relation is independent of the nature of the instability, and is thus a universal relation. For most instabilities this relation leads to a more rapid heating of the electrons. Thus, in the ion-acoustic and Buneman instabilities $V_d \gg \omega/k$ so that $d\mathcal{E}_e/d\mathcal{E}_i \gg 1$. The ratio $d\mathcal{E}_e/d\mathcal{E}_i$ is especially large [of order $(M/m)^{1/2}$] for the Buneman instability.

It is desirable to write Eq. (4.14) in a more useful form in dealing with the ion-acoustic and Buneman instabilities. For this purpose, in Eq. (4.14) we substitute the well-known value for the maximum growth rate of the ion-acoustic instability (4.5). The maximum growth rate obtains when $k \sim \lambda_D^{-1}$. Making the substitution $\gamma_{\mathbf{k}}^{e} \approx \omega V_d/v_{Te}$, we obtain the following relation:

$$\nu_{\rm eff} = \omega_p W/N_0 T_e. \tag{4.22}$$

Thus, a knowledge of the energy density of the waves W in the saturated regime of the instability can be used to find $\nu_{\rm eff}$. The quantity W in nonlinear plasma theory can be obtained by the usual methods of w e a k p l a s m a t u r b u l e n c e. However, this method does not always apply. Even the simplest case of the Buneman instability must be treated from the viewpoint of strong turbulence. However, the available theories for strong turbulence can only

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§ 4.2. Anomalous Resistivity Due to the Ion-Acoustic Instability

The ion-acoustic instability furnishes a convenient example for the application of the methods of weak plasma turbulence. The imaginary part of the frequency is much smaller than the real part since the drift velocity can be much smaller than the mean thermal velocity of the electrons. The nonlinear theory of the ion-acoustic instability and the anomalous resistivity have been treated by many authors. We shall consider this question in some detail. The energy density of a mode W_k characterized by wave vector **k** grows exponentially at small amplitudes. Then, at large amplitudes, effects associated with nonlinear saturation come into play and it is possible for a steady or quasisteady state situation to arise. We can then neglect the left side and find the spectrum W_k, equating the linear growth rate to one of the effects associated with the nonlinear saturation. The nonlinear effects can be written symbolically in the following form:

$$0=\left\{2\gamma_{\mathbf{k}}-A\omega_{\mathbf{k}}\left(\frac{W}{N_0T_e}\right)-B\omega_{\mathbf{k}}\left(\frac{W}{N_0T_e}\right)^2\right\}W_{\mathbf{k}}. \tag{4.23}$$

The quadratic effects (proportional to the square of the wave amplitude) represent wave–wave interactions. Resonant three-wave interactions are forbidden in the ion-acoustic instability so that the only effect which can give a term of order W^2 is nonlinear scattering of the wave on the ions [III]. This effect, associated with the presence of a denominator like $\omega - \omega' = (\mathbf{k} - \mathbf{k}') \cdot \mathbf{v}$, represents Landau resonances for the nonlinear beats associated with a given wave pair. These beat waves fall in resonance with the ions; one part of the energy is absorbed while another part goes to the wave with the lower frequency. The quadratic term actually represents a rather complicated integral expression (cf. § 3.2) in which the integral is taken over all wave vectors. The value of

this term can be estimated as follows. Since the effect is associated with the thermal motion of the ions, the operator **A** contains a small numerical factor T_i/T_e (since we are discussing ion-acoustic waves, then by definition it is necessary that $T_i \ll T_e$).

The next effect, the cubic effect, relates to the four-wave interaction. The four-wave interaction has been solved and taking account of it leads to a rather complicated nonlinear operator which contains the energy density of the waves to the third power. In the theory of weak turbulence this effect is weaker than nonlinear ion scattering. Thus, the basic contribution, which is decisive, is the balance between the linear growth and the first (quadratic) nonlinear term. A problem of this kind has been solved approximately by Kadomtsev who has been able to convert a complicated integral operator to differential form by assuming that the nonlinear interaction only causes a small frequency change (cf. § 3.2). In solving the balance equation Kadomtsev has observed that in the region of wave numbers much smaller than the Debye wave number (wavelengths much larger than the Debye wavelength), there is a simple dependence: The energy density is proportional to k^{-3} [III]. When $k\lambda_D \sim 1$ the integral operator, i.e., the collision term for the waves, does not reduce to a simple form. But it is possible to investigate the opposite limit of large wave vectors, that is to say, wavelengths shorter than the Debye radius (where the dispersion relation for ion-acoustic waves is simple: $\omega \approx \Omega_p$). It turns out that when $k\lambda_D > 1$ the spectrum falls off rapidly ($\sim k^{-13}$), [IA]. The Kadomtsev spectrum has a logarithmic divergence: The total wave energy diverges at small wave numbers. But this logarithmic divergence is not described because Eq. (4.14) for ν_{eff} does not contain the energy density but rather the momentum lost by the electrons; that is to say, in Eq. (4.14) we deal with another integral (k and ω are approximately proportional to each other for long wavelengths). There is an additional factor, the imaginary part of γ_k, a quantity which is proportional to the frequency; i.e., the quantity k appears again. Thus, now there is no divergence at small values of k. On the contrary, the contribution to the integral comes from the region of large $k\lambda_D \approx 1$. It is reasonable to make the assumption that a cut-off must be introduced at wave vectors of the order of the Debye radius (beyond this point the Kadomtsev spectrum is not valid and sharp damping occurs). The calculation of this integral leads to the following formula for the effective hope to give estimates of orders of magnitude. In the case of the Buneman instability an estimate of this kind can be made, as an example, using the following approach. The ratio of the ion energy and electron energy is written in the form $\mathscr{E}_i/\mathscr{E}_e \sim \omega/kV_d$, where $\mathscr{E}_e \sim m\langle v^2\rangle/2$, while the energy density of the waves $W \lesssim \mathscr{E}_i$. Since $\omega/k \approx \sqrt{m/M}\,\langle v\rangle$, using Eq. (4.14) we find $\nu_{\text{eff}} \sim \Omega_p$.

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$$\nu_{\mathrm{eff}} = 10^{-2}\Omega_p\,(V_d/c_s)(T_e/T_i)\Theta^{-2} \tag{4.24}$$

The factor 10^{-2} arises in the calculations (cf. also [113]). Thus, if it is possible to propagate a current in the plasma (this current being considerably above the critical value) so that the electrons lose momentum because of coherent emission of phonons, i.e., ion-acoustic waves, ultimately a stationary spectrum will be established (more precisely, a quasistationary spectrum) and ν_{eff} will be determined by Eq. (4.24). This relation has a deeper significance than Eq. (4.22) for the Buneman conductivity because of the fact that it reflects the specific nonlinear saturation of the instability. Nonetheless, it is only approximate in nature since the stationary Kadomtsev spectrum (3.25) is only an approximate description of the steady-state ion-acoustic waves. A solution of this kind could only be rigorous in the absence of an angular dependence in the expression for the growth rate of the ion-acoustic instability. This approximation is sometimes called the isotropic growth rate approximation. In the best case, the error incurred through the use of this approximation leads to an undetermined numerical factor of the order of unity. However, the danger exists that the nonlinear steady-state solution obtained by Kadomtsev may itself be unstable with respect to the formation of an elongated cone of unstable modes in **k**-space. This would lead to the reduction of the angle Θ, whose square appears in the denominator of the expression for the effective collision frequency (4.24). At the present time this question has not yet been resolved, although it is shown in [95] that solutions exist in which the angle Θ_0 varies in time about some mean value.

In conclusion we note another feature: If a current flows under conditions corresponding to the anomalous Buneman resistivity in a plasma in which the ions and electrons are initially isothermal (in which case ion-acoustic waves cannot be excited), sooner or later these conditions will become favorable for the ion-acoustic instability. This feature follows from the fact that the electrons are heated more rapidly than the ions by a factor of kV_d/ω in the Buneman instability; ultimately, therefore, the plasma must exhibit a difference between the electron and ion temperatures. In this sense, the ion-acoustic instability appears to be self-sustaining because when $V_d > c_s$ the electrons will always acquire more heat than the ions.

The ion-acoustic wave spectrum is a nondecay spectrum; however, as has been pointed out by Tsytovich [114], in the region of small wave numbers, where the dispersion relation is almost linear, there is a small imaginary frequency component which arises because of the nonlinear broadening of the line. It is then possible that the three-wave resonance condition can be satisfied. Tsytovich includes the three-wave resonance in the wave kinetic equation as a nonlinear term which leads to saturation, that is to say, as a decay process; the kinetic equation is then solved in this form. In this case one obtains a spectrum which is like the Kadomtsev spectrum, since the nonlinearity is quadratic; however, the small parameter T_i/T_e which appears in nonlinear scattering on ions does not appear. It is obvious that a somewhat different value will be obtained for the quantity W_k. However, the Tsytovich spectrum can only obtain for sufficiently small wave numbers, in which case the small nonlinearity can cause overlapping of the resonances so that the three-wave interaction can be realized. However, the basic contribution in the momentum loss (in contrast with the wave energy) comes from the short wavelengths. At the short wavelengths (frequencies of the order of the ion-plasma frequency) the deviation from the linear dispersion relation becomes very large and a very strong nonlinearity would be needed in order to satisfy the three-wave resonance condition. Thus, the Tsytovich model has a very limited range of applicability, the more so since strong nonlinearities mean that it is necessary to treat the problem from the point of view of strong turbulence.

Finally, we note that the formal application of perturbation theory, as in § 3.1, for computing the electron distribution function leads to a somewhat paradoxical result: The linear theory for the excitation of ion-acoustic waves must be treated for a level of turbulence which is much lower than that reached in the steady-state turbulence [115]. The point here is that the basic contribution to the nonlinear correction to the work in the field of the wave, as computed by means of the distribution function $f^{e(3)}_{k',k,-k'}(\mathbf{v})$ (cf. § 3.1), gives particles with velocities that are not higher than the wave phase velocity:

$$\mathbf{j}^{(3)}\,\mathbf{E} \approx \mathrm{Im}\int d^3\mathbf{v}\,(\mathbf{v}\cdot\mathbf{E})\left(\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}+i0}\,\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{v}}\right)^3 f_e^{(0)} \approx \mathbf{J}^{(1)}\cdot\mathbf{E}\,\frac{E^2}{4\pi N_0 m\,(\omega/k)^2}.$$

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The factor 10^{-2} arises in the calculations (cf. also [113]). Thus, if it is possible to propagate a current in the plasma (this current being considerably above the critical value) so that the electrons lose momentum because of coherent emission of phonons, i.e., ion-acoustic waves, ultimately a stationary spectrum will be established (more precisely, a quasistationary spectrum) and ν_{eff} will be determined by Eq. (4.24). This relation has a deeper significance than Eq. (4.22) for the Buneman conductivity because of the fact that it reflects the specific nonlinear saturation of the instability. Nonetheless, it is only approximate in nature since the stationary Kadomtsev spectrum (3.25) is only an approximate description of the steady-state ion-acoustic waves. A solution of this kind could only be rigorous in the absence of an angular dependence in the expression for the growth rate of the ion-acoustic instability. This approximation is sometimes called the isotropic growth rate approximation. In the best case, the error incurred through the use of this approximation leads to an undetermined numerical factor of the order of unity. However, the danger exists that the nonlinear steady-state solution obtained by Kadomtsev may itself be unstable with respect to the formation of an elongated cone of unstable modes in **k**-space. This would lead to the reduction of the angle Θ, whose square appears in the denominator of the expression for the effective collision frequency (4.24). At the present time this question has not yet been resolved, although it is shown in [95] that solutions exist in which the angle Θ_0 varies in time about some mean value.

In conclusion we note another feature: If a current flows under conditions corresponding to the anomalous Buneman resistivity in a plasma in which the ions and electrons are initially isothermal (in which case ion-acoustic waves cannot be excited), sooner or later these conditions will become favorable for the ion-acoustic instability. This feature follows from the fact that the electrons are heated more rapidly than the ions by a factor of kV_d/ω in the Buneman instability; ultimately, therefore, the plasma must exhibit a difference between the electron and ion temperatures. In this sense, the ion-acoustic instability appears to be self-sustaining because when $V_d > c_s$ the electrons will always acquire more heat than the ions.

The ion-acoustic wave spectrum is a nondecay spectrum; however, as has been pointed out by Tsytovich [114], in the region of small wave numbers, where the dispersion relation is almost linear, there is a small imaginary frequency component which arises because of the nonlinear broadening of the line. It is then possible that the three-wave resonance condition can be satisfied. Tsytovich includes the three-wave resonance in the wave kinetic equation as a nonlinear term which leads to saturation, that is to say, as a decay process; the kinetic equation is then solved in this form. In this case one obtains a spectrum which is like the Kadomtsev spectrum, since the nonlinearity is quadratic; however, the small parameter T_i/T_e which appears in nonlinear scattering on ions does not appear. It is obvious that a somewhat different value will be obtained for the quantity W_k. However, the Tsytovich spectrum can only obtain for sufficiently small wave numbers, in which case the small nonlinearity can cause overlapping of the resonances so that the three-wave interaction can be realized. However, the basic contribution in the momentum loss (in contrast with the wave energy) comes from the short wavelengths. At the short wavelengths (frequencies of the order of the ion-plasma frequency) the deviation from the linear dispersion relation becomes very large and a very strong nonlinearity would be needed in order to satisfy the three-wave resonance condition. Thus, the Tsytovich model has a very limited range of applicability, the more so since strong nonlinearities mean that it is necessary to treat the problem from the point of view of strong turbulence.

Finally, we note that the formal application of perturbation theory, as in § 3.1, for computing the electron distribution function leads to a somewhat paradoxical result: The linear theory for the excitation of ion-acoustic waves must be treated for a level of turbulence which is much lower than that reached in the steady-state turbulence [115]. The point here is that the basic contribution to the nonlinear correction to the work in the field of the wave, as computed by means of the distribution function $f^{e(3)}_{\mathbf{k}',\mathbf{k},-\mathbf{k}'}(\mathbf{v})$ (cf. § 3.1), gives particles with velocities that are not higher than the wave phase velocity:

$$\mathbf{j}^{(3)}\,\mathbf{E} \approx \operatorname{Im}\int d^3\mathbf{v}\,(\mathbf{v}\cdot\mathbf{E})\left(\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}+i0}\,\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{v}}\right)^3 f_e^{(0)} \approx \mathbf{J}^{(1)}\cdot\mathbf{E}\,\frac{E^2}{4\pi N_0 m\,(\omega/k)^2}.$$

Thus, according to this estimate the nonlinear correction to the growth rate becomes larger than the linear growth rate when

$W > mN_0T_e/M$. The resolution of the paradox lies in the fact that in an earlier stage of the development of the instability it is necessary to take account of the nonlinear broadening of the resonance velocities by an amount of order $\Delta v \approx (eE/mk)^{1/2}$. The Doppler broadening of the resonance due to this effect starts to play a role when $(eE/mk)^{1/2} > \omega/k$, that is to say, when $E^2/4\pi > mN_0T_e/M$. In view of these considerations it is not difficult to estimate the nonlinear correction to the growth rate [115]:

$$\delta\gamma/\gamma^L \approx E^2/4\pi m N_0 \Delta v^2 \approx (E^2/4\pi N_0 T_e)^{1/2} \ll 1.$$

Thus, the conclusion regarding the importance of taking account of the nonlinear electron contribution in the growth rate is not justified.

§ 4.3. Quasilinear Effects in Anomalous Resistivity Due to the Ion-Acoustic Instability

Although it is not yet completely reliable, indirect experimental evidence is available which shows that Eq. (4.24) has been verified in certain limiting cases. However, these experimental data must still be treated with caution. The point here is that no one of the four quantities which appear in this equation, V_d, c_s, T_e, i, has its usual physical significance because we are dealing with a plasma in which true binary collisions are replaced by scattering on fluctuations. Let us start with the electron temperature. If binary collisions do not occur it is difficult to believe that the distribution function will be a Maxwellian. Even if it is not required that the electron distribution function be a Maxwellian with some mean thermal spread, it is necessary that the function f_e have a rather rapidly converging tail. Under these conditions one can discuss the idea of a single temperature for the electrons. The situation is more complicated for the ions because at the outset it is obvious that the ion distribution function will be rather unusual if the ions only interact with the waves (without binary collisions). Finally, there is the mean drift velocity. Usually the particle distribution in a plasma in which a current flows is like that shown in Fig. 30. Here, we have an ion distribution function and an electron distribution function which is displaced with re-

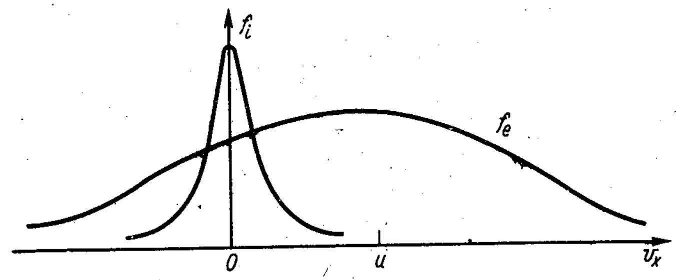

Fig. 30. Maxwellian particle distribution in a plasma with a current.

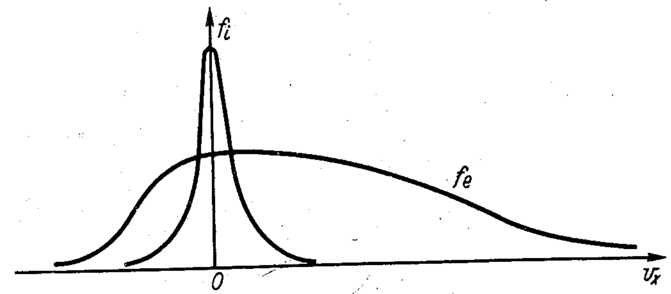

Fig. 31. Example of a stable electron distribution in a plasma with a current.

spect to the ion function under the assumption that the electron distribution is displaced as a whole. However, in principle, it is possible to have a situation (Fig. 31) in which the electron distribution has its maximum at the same point as the ion distribution, but in which the electron distribution is distorted so as to become highly asymmetric (cf. Fig. 31).

At this point it is useful to discuss the possible form of the distribution functions for the ions and electrons. It will be convenient to use the two-dimensional pattern shown in Fig. 32. Along the abcissa axis we have plotted the component of the particle velocity in the direction of current flow; the transverse component is plotted along the ordinate axis. Assume that initially we have the usual Maxwellian distribution of electrons and ions. With a Maxwellian distribution the curves corresponding to equal values of the distribution function in this plane will be circles. The interaction between waves and particles is especially strong when the Landau resonance is realized. A wave with phase velocity ω/k

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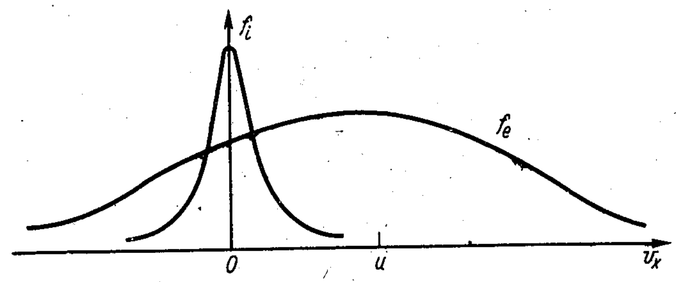

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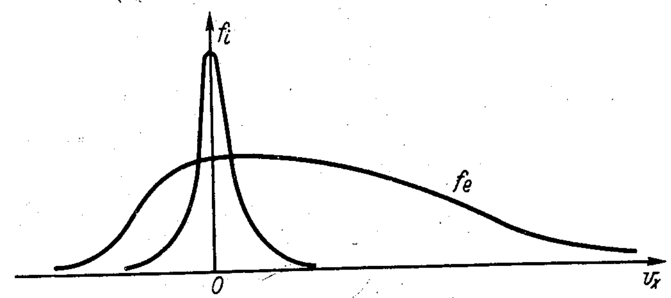

Fig. 31. Example of a stable electron distribution in a plasma with a current.

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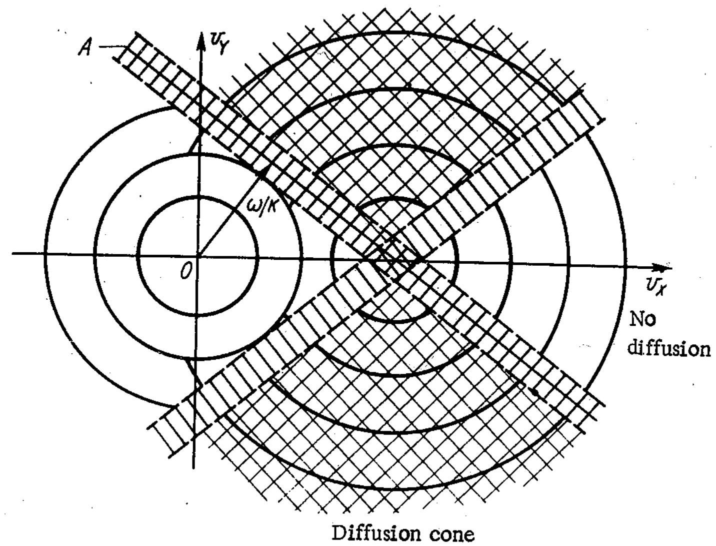

Fig. 32. Diffusion cone for ion-acoustic turbulence in a plasma with a current.

interacts with particles located close to the line A (cf. Fig. 32). These are the particles which take part in the resonance interaction. If we consider waves with all possible directions and phase velocities it is easy to show that all particles located in that part of the plane (v_x, v_y) in which it is possible to draw a line which corresponds to the Landau resonance condition will experience the effect of the random field of the wave. However, the ion-acoustic spectrum does not contain waves with velocities below some critical value ($\omega/k \lessapprox v_{Ti}$). In the quasilinear approximation the particles noted above will not interact with waves. The interaction is found to be much weaker, being associated with nonlinear effects in the next approximation. The number of ions in the wave interaction region is rather small since only a small fraction of the ions experience a strong effect from the wave. In the zeroth approximation the distribution function in the primary region is then essentially undeformed. On the other hand, a strong deformation is produced in the resonance region. In the language of quasilinear theory, this deformation is nothing more than the diffusion of particles in velocity space. There is a large number of experiments which have been carried out in the intermediate regime in which this diffusion process is relatively slow and binary particle colli-

sions are capable of producing something like a Maxwellian distribution. This is the case in which the plasma resistivity is somewhat higher than the classical value. We only discuss the most extreme case, in which binary collisions do not play a role. In this case the change in the ion distribution function is very important. It has a strong effect on the ion imaginary part (the ion pole, which is proportional to the number of resonance ions). As a rule, this is a very small number and is very sensitive to the features of the tail of the ion distribution. At the present time there is still not available a self-consistent theory which is capable of describing the change in the ion distribution function at long times (with the exception of the one-dimensional case, which is discussed below). It is possible, however, to make certain estimates which make use of the two-temperature approximation, that is to say, the division of the ions into two groups: cold ions, for which essentially nothing happens, and hot ions in the tail of the distribution function which are characterized by a high temperature.

The situation for the electrons is as follows: The region of forbidden velocities, within which there is no resonance between particles and waves, is relatively small because the acoustic velocity is $(M/m)^{1/2}$ times smaller than the mean thermal velocity of the electrons. In principle, it is possible to neglect effects which occur within this small group. However, the situation is more complicated. It is very difficult to think of a situation in which the current flowing in this direction can excite a wave which is almost perpendicular to the current. Actually, it is well known from the theory of the ion-acoustic instability that a wave with a large wave-vector component perpendicular to the direction of current flow has a small imaginary part; in practice, waves of this kind can be regarded as stable. Thus, we conclude that there is formed a small cone in velocity space in which there are no waves that can resonate with the electrons. These electrons are freely accelerated by the electric field which produces the current flow in the plasma. The contribution of these electrons can have a strong reduction effect on the plasma resistance. It is then of interest to ask what fraction of the electrons fall in this loss cone and are freely accelerated. The problem can be considered in two limiting cases. It is first useful to isolate the simpler case, in which there is a weak magnetic field H_0 in the plane perpendicular to V_d. This magnetic field causes a slow gyration of the electrons (slow as compared with the frequency of the plasma oscillations). However,

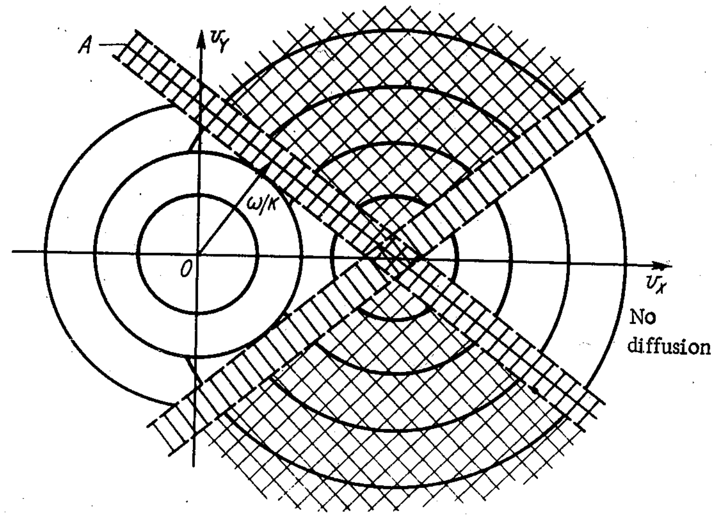

Fig. 32. Diffusion cone for ion-acoustic turbulence in a plasma with a current.

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However, there are still certain difficulties associated with the ion distribution. The point here is that ion mixing does not occur in the magnetic field, so that the ion distribution may acquire an unusual form which, ultimately, will be very different from a Maxwellian distribution. One expects that the bulk of the ions will be cold and that some fraction of the ions (starting from a velocity of the order of the acoustic velocity) will be heated. Without the use of numerical methods it will evidently not be possible in the near future to find the form of this complicated ion distribution. On the other hand, certain conclusions can be drawn from the following considerations. If there is no interaction with the walls during the process in which the current flows and if the electron and ion energies increase, that is to say, if there is no loss

of heat to an external sink, one expects some kind of self-similar forms for both the distribution functions and the spectra.

The situation should be much simpler in the case of a weak nonlinearity, in which it is possible to rely on the quasilinear approximation. In this approximation, saturation of the instability is reached as a consequence of the quasilinear deformation of the ion distribution function; as a result, even in a nonisothermal plasma there will be a group of ions with large velocities. These ions, which absorb the ion-acoustic waves in resonant fashion, then balance the excitation of the waves by electrons.

Let us consider the nonlinear saturation process in this case. Assume that the equation for the spectrum of unstable waves is given by the symbolic form [compare with Eq. (4.23)]

$$dW_k/dt = 2\gamma_k^e W_k - 2\gamma_k^i W_k - A\,(W/N_0T_e)W_k - B\,(W/N_0T_e)^2 W_k. \tag{4.25}$$

The wave growth when $V_d > V_c$ leads to an increase in the resistance, that is to say, a friction force arises which acts on the electrons. If it is assumed that the electric field which drives the current is not too large, then as a consequence of the reaction on the electrons due to the increasing resistance, the velocity V_d will continue to be reduced as long as the instability threshold is not reached. This means that the nonlinear terms in Eq. (4.25) will play a small role and that the saturation of the waves is determined by the condition $\gamma_k^e \approx \gamma_k^i$ for all originally unstable waves. In other words the following condition must be satisfied for all originally unstable waves:

$$\gamma_k = \gamma_k^e - \gamma_k^i \approx [df_e/dv + (m/M)(df_i/dv)] = 0. \tag{4.26}$$

In this form, Eq. (4.25) does not contain the amplitudes of the steady-state waves W_k so that it cannot be used to compute ν_{eff} directly. However, the effective collision frequency for this saturation regime of the instability (this is sometimes called the "threshold" or quasilinear regime) can be found from Ohm's law by substituting the expression found for $j = eNV_d \approx eNV_c$. Writing $j = eNV_c = \sigma E$, we have

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$$\nu_{eff} = eE/mV_c. \qquad (4.27)$$

Now, using Eq. (4.14) which relates ν_{eff} to the wave energy, we can find W:

$$W/N_0T_e \approx (eE\lambda_{De}/N_0T_e)(v_{Te}/V_c). \qquad (4.28)$$

As E increases the quantity W also increases, and at sufficiently large fields E, it is no longer possible to neglect the nonlinear terms in Eq. (4.25). At this point the apparently simple formula for the threshold (quasilinear) regime becomes complicated. At first sight it might appear that in finding $V_d \approx V_c$ it is sufficient to use the linear expression for the imaginary part of the frequency (4.26). However, γ_k^i is very sensitive to the form of the ion distribution for large ion velocities, that is to say, it is sensitive to the tail of the distribution function. These ions, which damp the waves, increase their energy and rapidly change (in quasilinear fashion) the form of the distribution function. As a result, γ_k^i changes rapidly, as does V_c.

If the thermal energy absorbed by the plasma particles is not transferred to an external sink, one expects that ultimately a universal self-similar ion distribution function will be established. The assumption that such a self-similar solution (corresponding to the quasilinear regime) actually exists is verified in [116]. The self-similar variables in which the equations of weak turbulence assume a simple form are found in this work; however, these equations still cannot be used to solve the general case. Nonetheless we can proceed as follows. When the current flows across the magnetic field and the wave spectrum is three-dimensional, the ion distribution can be divided into cold ions and hot ions. In this two-group approximation we can find the quantities which characterize the current flow. The group of ions which are in resonance with the ion-acoustic waves and are then accelerated is relatively small. We denote the fractional concentration of these hot ions by X. The effective temperature of these resonance ions will be called T_{hi}. Then from Eq. (4.21) we have

$$T_e/T_{hi} \approx XV_d/c_s. \qquad (4.29)$$

Estimating the ion damping coefficient as $\gamma_i \approx (\omega^3/k^3)\,(Xc_s/(T_{hi}/M^{3/2})$ and comparing it with the electron coefficient (4.5), we have

$$X = (m/M)^{1/4}\,(T_{hi}/T_e)^{1/4}, \qquad (4.30)$$

$$V_d \approx c_s\,(M/m)^{1/4}\,(T_e/T_{hi})^{5/4}. \qquad (4.31)$$

We now consider the momentum of the resonance ions. The momentum lost by the electrons in scattering is transferred to the ions, $\mathbf{P}_i = Nm\mathbf{V}_d\nu_{\rm eff}$. Since $T_{hi} \gg T_e$, the ion distribution function can be written in the form $f_i(\mathbf{v}, \vartheta) = f_{0i}(\mathbf{v}) + f_{1i}(\mathbf{v}, \vartheta)$ (cf. § 2.3), where the anisotropic part $f_{1i} \sim [c_s/(T_{hi}/M^{1/2})]\,f_{0i} \lesssim f_{0i}$. Thus, $|\mathbf{P}_i| \approx \approx |\int M\mathbf{v}_s f_{1i} d^3\mathbf{v}| \approx N_0XMc_s$. Finally we have

$$V_d \approx c_s\,(M/m)^{1/4}, \quad T_{hi} \approx T_e, \quad X \approx (m/M)^{1/4}. \qquad (4.32)$$

Thus, the mean random energy of these hot ions is approximately the same as the random energy of the bulk of the electrons. Obviously, Eq. (4.32) contains factors of the order of unity; without knowing the exact solution, these factors can only be determined by comparison with experimental data. The only exception is the idealized case of the one-dimensional spectrum, which allows an exact analytic solution. This solution is of great interest from a procedural point of view, and we shall consider it in some detail.

Let us assume that there are waves which only propagate in the direction of the current ($\parallel \chi$). The ion distribution function for the ions which interact with these waves is also one-dimensional. However, because of the magnetic field the electron distribution will be axially symmetric in the plane v_x, v_y around the point V_d, 0. The interaction of the electrons with the waves in this problem corresponds to the case that has been treated in § 2.3. The electron distribution function is of the form $f_e \sim \exp(-\alpha v_\perp^5)$ with the origin taken at the point V_d, 0. However, not all of the electrons interact with the waves. This point can be made clear from consideration of Fig. 33. For example, if only the usual ion-acoustic

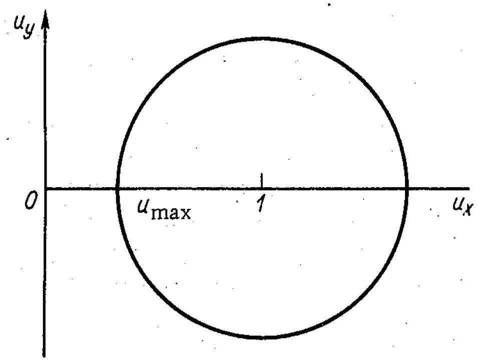

Fig. 33. Interaction of one-dimensional waves with electrons in a magnetic field.

find W:

$$W/N_0T_e \approx (eE\lambda_{De}/N_0T_e)(v_{Te}/V_c). \qquad (4.28)$$

As E increases the quantity W also increases, and at sufficiently large fields E, it is no longer possible to neglect the nonlinear terms in Eq. (4.25). At this point the apparently simple formula for the threshold (quasilinear) regime becomes complicated. At first sight it might appear that in finding $V_d \approx V_c$ it is sufficient to use the linear expression for the imaginary part of the frequency (4.26). However, γ_k^i is very sensitive to the form of the ion distribution for large ion velocities, that is to say, it is sensitive to the tail of the distribution function. These ions, which damp the waves, increase their energy and rapidly change (in quasilinear fashion) the form of the distribution function. As a result, γ_k^i changes rapidly, as does V_c.

If the thermal energy absorbed by the plasma particles is not transferred to an external sink, one expects that ultimately a universal self-similar ion distribution function will be established. The assumption that such a self-similar solution (corresponding to the quasilinear regime) actually exists is verified in [116]. The self-similar variables in which the equations of weak turbulence assume a simple form are found in this work; however, these equations still cannot be used to solve the general case. Nonetheless we can proceed as follows. When the current flows across the magnetic field and the wave spectrum is three-dimensional, the ion distribution can be divided into cold ions and hot ions. In this two-group approximation we can find the quantities which characterize the current flow. The group of ions which are in resonance with the ion-acoustic waves and are then accelerated is relatively small. We denote the fractional concentration of these hot ions by X. The effective temperature of these resonance ions will be called T_{hi}. Then from Eq. (4.21) we have

$$T_e/T_{hi} \approx XV_d/c_s. \qquad (4.29)$$

Estimating the ion damping coefficient as $\gamma_i \approx (\omega^3/k^3)\,(Xc_s/(T_{hi}/M^{3/2})$ and comparing it with the electron coefficient (4.5), we have

$$X = (m/M)^{1/4}\,(T_{hi}/T_e)^{1/4}, \qquad (4.30)$$

$$V_d \approx c_s\,(M/m)^{1/4}\,(T_e/T_{hi})^{5/4}. \qquad (4.31)$$

We now consider the momentum of the resonance ions. The momentum lost by the electrons in scattering is transferred to the ions, $\dot{\mathbf{P}}_i = Nm\mathbf{V}_d\nu_{\text{eff}}$. Since $T_{hi} \gtrsim T_e$, the ion distribution function can be written in the form $f_i(\mathbf{v}, \vartheta) = f_{0i}(\mathbf{v}) + f_{1i}(\mathbf{v}, \vartheta)$ (cf. § 2.3), where the anisotropic part $f_{1i} \sim [c_s/(T_{hi}/M^{1/2}]\,f_{0i} \lesssim f_{0i}$. Thus, $|\mathbf{P}_i| \approx$ $\approx |\int M\mathbf{v}_s t_{1i} d^3\mathbf{v}| \approx N_0XMc_s$. Finally we have

$$V_d \approx c_s\,(M/m)^{1/4}, \quad T_{hi} \approx T_e, \quad X \approx (m/M)^{1/4}. \qquad (4.32)$$

Thus, the mean random energy of these hot ions is approximately the same as the random energy of the bulk of the electrons. Obviously, Eq. (4.32) contains factors of the order of unity; without knowing the exact solution, these factors can only be determined by comparison with experimental data. The only exception is the idealized case of the one-dimensional spectrum, which allows an exact analytic solution. This solution is of great interest from a procedural point of view, and we shall consider it in some detail.

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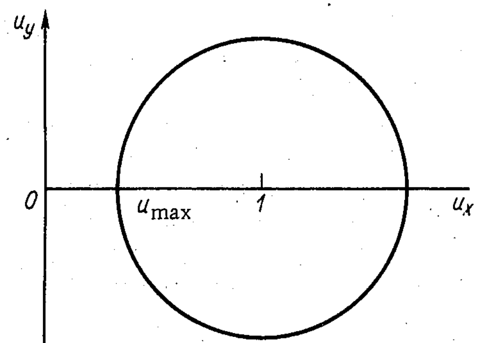

Fig. 33. Interaction of one-dimensional waves with electrons in a magnetic field.

waves are excited in the plasma the phase velocity cannot be larger than ω/k_{max}. Since the spectrum is one-dimensional, for drift velocities $V_d > \omega/k_{max}$ some of the electrons within the circle (cf. Fig. 32) will not interact with the waves. Hence, the electrons in this region remain cold and the corresponding value of the distribution function will become larger and larger as compared with the value of f_e in the resonance region. This process will continue as long as the dispersion relation for the ion-acoustic waves does not change so much that waves with phase velocities in the range from the initial value ω/k_{max} to V_d do not appear. In this case the circle (cf. Fig. 33) is elongated and the electrons in the circle can be described by a distribution in the form of a δ-function $f_e \sim X_e \delta(v_x - V_d)\delta(v_y)$. The relative fraction of these nonresonant electrons X_e will be determined below.

In terms of the dimensionless variables $u_x = v_x/V_d$, the quasi-linear kinetic equation for the ions can be used to write the distribution function $f_i(v_x, t) = (N/V_d) g_i(u_x)$ in the form

$$-(d/du_x)u_x g_i = (m/M)^2 (d/du_x)D(u_x)(dg_i/du_x), \tag{4.33}$$

where

$$D(v_x) = \frac{8\pi^2 e^2}{m^2} \int W_k \delta(\omega_k - kv_x) dk/(2\pi).$$

As in the preceding problem, it is expected that the bulk of the ions will not interact with the waves. The relative fraction of resonant ions described by Eq. (4.33) is $(1 - X_i)$. The condition $\gamma_i + \gamma_e = 0$ in the range of phase velocities between 0 and V_d $(0 < u_x < 1)$ now assumes the form

$$dg_i/du_x = (M/m)(dh_e/du_x), \tag{4.34}$$

where $h_e(u_x)$ is the electron distribution (of the form $\exp[-\alpha V_\perp^5]$) in the self-similar variables, integrated over v_y. The system of equations in (4.33) and (4.34) can be solved (cf. Problem 1 at the end of this section).

It is interesting to note that the ratio of the directed electron velocity V_d to the electron thermal velocity, and the number of resonant ions v_{Te}, can be estimated from simple considerations based on the conservation relations. Scattering of electrons on waves is associated with an electron-ion frictional force F_{fr} which transfers momentum from the electrons to the ions. The momentum of the ions is denoted by P_i and we have $dP_i/dt = F_{fr}$. This relation is equivalent to the result obtained by computing the

electron distribution function as projected along the parallel velocity is then found to be highly elongated in the direction of current flow, and the ratio of mean electron drift velocity to mean electron thermal velocity can be equal to unity [117]. It is difficult to predict the exact numerical value. So far data obtained from laboratory experiments have not been very useful. The point is that most experiments in which anomalous resistivity is investigated are carried out in discharges with so-called *open ends*, that is to say, discharges in which the electrons in the plasma can move freely and be lost along the lines of force of the magnetic field. Under these conditions it is impossible to obtain an extended self-similar regime since heat is lost continuously. Furthermore, since the particle mean free path for high-velocity particles is large, the fast particles are lost rapidly from the system. For this reason a continuous truncation of the tail of the electron distribution function occurs. The problem is extremely complicated and is very sensitive to the boundary conditions, and for this reason loses much of its general application. It can be shown that the ratio V_d/v_{Te} will remain much smaller than unity in this case. In principle it is not possible to exclude a number of other mechanisms which terminate the runaway of electrons and thus lead to a ratio $V_d/v_{Te} \ll 1$. The combined effect of ion-acoustic and cyclotron instabilities on the electrons is discussed in [118]. Along the same lines, in [119] the authors have introduced the notion of the extended existence of so-called *macroparticles* [120]. However, these ideas are essentially of phenomenological nature.

There do exist, however, certain idealized limiting cases in which the weak turbulence equations can be solved almost exactly in treating anomalous resistivity. These are one-dimensional cases. Just as there is a class of one-dimensional models amenable to solution in statistical thermodynamics, in the theory of weak turbulence the one-dimensional models are found to be very much simpler. In certain cases the one-dimensional models can have actual physical meaning. For example, if the magnetic field is so large that the electron gyrofrequency is much greater than the plasma frequency, electron motion across the lines of force of the magnetic field is essentially forbidden and one deals with what is essentially a one-dimensional motion. In these cases the one-dimensional theory can give an adequate description of the actual situation. Self-similar equations in the one-dimensional formulation for the quasilinear regime have been solved exactly [117].

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In terms of the dimensionless variables $u_x = v_x/V_d$, the quasilinear kinetic equation for the ions can be used to write the distribution function $f_i(v_x, t) = (N/V_d)g_i(u_x)$ in the form

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Let us consider a constant uniform electric field parallel to the magnetic field in a uniform plasma. After a certain time interval the mean-square velocity of the plasma can become so large that the plasma "forgets" its original state; in this case the further evolution of the system is of a universal nature, being independent of the initial conditions. Formally, this regime corresponds to the transition to the case in which quasilinear equations with self-similar variables can be used. It follows from simple dimensional arguments that the particle velocities must be measured in units of eEt/m and the wave vectors must be measured in units of $m\omega_p/eEt$.

The electron and ion distribution functions $f_{e,i}$ and the spectral density of the electrostatic wave energy W are then of the form

$$\left.\begin{aligned} f_e = mNg_e(u)/eEt;\ f_i = mNg_i(u)/eEt;\ W(k, t) = m\omega_p^4 t^2 U(q);\\ u = mv/eEt;\quad q = keEt/m\omega_p. \end{aligned}\right\} \tag{4.36}$$

Substituting these functions in the quasilinear equations written in the reference system that moves with the freely accelerating ions, we have

$$(-d/du)(u - 1 - \mu)g_e = (d/du)D(u)(dg_e/du), \tag{4.37}$$

$$-(d/du)ug_i = \mu^2 (d/du)D(u)(dg_i/du), \tag{4.38}$$

where D(u) is the quasilinear diffusion coefficient and $\mu = m/M$. Taken together with the marginal stability criterion

$$(d/du)(g_e + \mu g_i) = 0, \tag{4.39}$$

Eqs. (4.37) and (4.38) form a closed system with the following solution:

$$\left.\begin{aligned} g_e = Cu/(u+\mu^2),\ g_i = C\mu(1-u)/(u+\mu^2),\ D = u^2(1-u)/\mu^2\\ \text{for } 0 < u < 1;\\ g_e = g_i = D = 0 \quad \text{for } u < 0,\ u > 1, \end{aligned}\right\} \tag{4.40}$$

where C is an arbitrary positive constant. The functions $g_e(u)$ and $g_i(u)$ must be supplemented by a certain number of freely accelerating electrons and ions which, in the self-similar solution, correspond to δ-functions at the point u = 1 for the electrons and u = 0 for the ions. Denoting the fraction of freely accelerating particles by X_e and X_i, from the normalization condition we find that $X_e + C = 1$, $X_i + 2C\mu \ln \mu^{-1} = 1$. Knowing the functions g_e and g_i we can then easily write the dispersion relation

$$\varepsilon(q, \omega) = 1 - (1 - C)/(\omega - q)^2 - \mu/\omega^2 + C/\omega q - C/(\omega - q)q.$$

The function $\varepsilon(q, \omega)$ must satisfy the following requirements: All waves must be stable; all waves must have phase velocities in the range (0, 1). From these conditions it is possible to determine the constant C uniquely (which is found to be equal to $2\mu^{1/2}$) and, thus, the distribution function $g_{e,i}$.

Thus, ultimately one finds that a universal self-similar electron distribution is formed. In this case there is a plateau region from V = 0 extending up to the velocity of free acceleration characteristic of the bulk of the electrons.

Numerical experiments in the one-dimensional case give distribution functions of approximately the same form [121]. However, there is some difference between the distribution functions in the numerical experiments and the self-similar experiments. In particular, in the self-similar analysis the number of electrons in the plateau region is proportional to $N_0(m/M)^{1/2}$. On the other hand, in the numerical experiments a significant fraction of electrons is found in the plateau region. This difference stems from the fact that the quasilinear theory ignores nonlinear effects.

PROBLEM

▶ 1. Find the exact solution for the quasilinear equations (4.33) and (4.34) for the problem of an anomalous resistance with respect to a current which flows across H in the one-dimensional problem [117].

The electron distribution function $f_e(v_\perp, t) = (N/v)g_e(u_\perp)$ can be expressed in terms of the quasilinear diffusion coefficient $g_e = C_1 \exp(-u_\perp^5/5\bar{D})$, where $\bar{D} = \pi^{-1}\int_0^1 D(u_x)(u_x - 1)^2\,du_x$, while the constant C_1 is found from the normalization condition $2\pi\int_0^\infty u_\perp g_e\,du_\perp = 1$

$$C_1 = 1/\pi\Gamma(7/5)[5\bar{D}]^{2/5}$$

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$$C_1 = 1/\pi\Gamma(7/5)[5\bar{D}]^{2/5}$$

From the condition $\gamma = 0$ in the phase velocity range (0, 1) we have

$$\frac{dg_i}{du_x} = -\frac{M}{m} \cdot \frac{dg_e}{du_x} \approx \frac{2M}{m}(1-u_x)\int\limits_0^\infty \frac{du_\perp}{u_\perp} \cdot \frac{dg_e}{du_\perp} =$$
$$= -\frac{M}{m}(1-u_x)(5\bar{D})^{-3/5}\frac{5\Gamma(9/5)}{2\pi\Gamma(7/5)}$$

(in computing dg_e/du_x we have taken account of the fact that the electron thermal velocity is much greater than the electron drift velocity which, in the variables used here, is equal to unity). We then find the ion distribution function

$$g_i = (M/m)(1-u_x)^2(5\bar{D})^{-3/5}[5\Gamma(9/5)]/[4\pi\Gamma(7/5)]$$

and, from Eq. (4.33), the diffusion coefficient $\bar{D} = 1/40\pi(m/M)^2$. Now, using the relation

$$\bar{u}_\perp^2 = \int\limits_0^\infty u_\perp^3 g_e\, du_\perp \Bigg/ \int\limits_0^\infty u_\perp g_e\, du_\perp = (5\bar{D})^{2/5}\Gamma(9/5)/2\Gamma(7/5)$$

we can find the mean-square electron velocity $(\bar{u}_\perp^2)^{1/2}$, which is found to be $0.38(M/m)^{2/5}$. Thus, the ratio of the directed electron velocity to the thermal velocity is $2.65(m/M)^{2/5}$ in this model.

As before, the number of ions that interact with the waves is small,

$$1 - X_i = \int\limits_0^1 g_i(u_x)\, du_x = (m/M)^{1/5}\frac{10\Gamma(9.5)}{3(8\pi)^{2/5}\Gamma(7/5)} \approx 0.95(m/M)^{1/5},$$

and in determining X_e we make use of the dispersion relation

$$\varepsilon(\omega, q) = 1 - \frac{1}{q^2}\int\limits_{-\infty}^{+\infty}\frac{du_x}{u_x}\cdot\frac{dg_e}{du_x} - \frac{X_e}{-q)^2} - \frac{mg_i(0)}{M\omega q} - \frac{mX_i}{M\omega^2} = 0$$

or

$$1 + \frac{8.2}{q^2}\left(\frac{m}{M}\right)^{4/5} = \frac{X_e}{(\omega-q)^2} + \frac{m}{M\omega^2} + \frac{2.86}{\omega q}\left(\frac{m}{M}\right)^{6/5}$$

From the requirements on stability and the existence of waves with all phase velocities in the range (0, 1), which is completely

analogous to the case of propagation of the current along **H** [cf. Eqs. (4.36-4.40)], we find the number of particles in the electron root $X_e \approx 8.2(m/M)^{4/5}$.

§ 4.4. Anomalous Resistivity Caused by Other Instabilities

Let us now return to the Table 1 in §4.1 and consider other instabilities. The Drummond–Rosenbluth instability leads to a relatively small imaginary frequency component; furthermore, from the point of view of the quasilinear approximation, this is a one-dimensional instability. Hence, an electron plateau arises which is similar to the plateau that arises in the one-dimensional ion-acoustic model (4.3); this plateau causes rapid saturation of the instability. The current can increase further, but in a small region of velocity space, the electron distribution will have a plateau and the plasma will not be unstable.

The instabilities with the lowest excitations threshold are the electrostatic instabilities with $k_\| \ll k_\perp$ in a plasma in which the current flows across the magnetic field. When $\omega \gtrsim kv_{Ti}$, $V_d > k_\| v_{Te}/k$ these oscillations are described by (4.6). The nonlinear saturation of this kind of instability is not amenable to analysis within the theory of weak turbulence. For example, we may consider the case of the modified Buneman instability [cf. Eq. (4.7)]. The dispersion relation in (4.7) differs from the usual Buneman equation only in that Ω_p is replaced by $\Omega_p/(1+\omega_p^2/\omega_H^2)^{1/2}$ and ω_p by $\omega_p k_\|/k(1+\omega_p^2/\omega_H^2)^{1/2}$. It is possible to estimate the wave amplitude in the saturation regime, as is frequently done in the theory of strong turbulence. We compare the linear term $\partial v/\partial t$ and the nonlinear term $(v\cdot\nabla)v$ in the electron equation:

$$\partial \mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{v} = e\{\mathbf{E} + (1/c)[\mathbf{v}\times\mathbf{H}]\}.$$

In the nonlinear stage these terms compete, and this competition leads to a quasistationary regime in which the instability is saturated. Equating these terms (by order of magnitude) we find $kV_d \sim (kc/H_0)\sum_q q\Phi_q$. It is then possible to obtain the following estimate for the energy density of the waves:

$$\sum_k N_0 e^2|\Phi_k|^2/2T_e \approx mNV_d^2, \quad kr_{He} \approx 1. \tag{4.41}$$

From the condition $\gamma = 0$ in the phase velocity range (0, 1) we have

$$\frac{dg_i}{du_x} = -\frac{M}{m} \cdot \frac{dg_e}{du_x} \approx \frac{2M}{m}(1-u_x)\int_0^\infty \frac{du_\perp}{u_\perp} \cdot \frac{dg_e}{du_\perp} =$$

$$= -\frac{M}{m}(1-u_x)(5\bar{D})^{-3/5}\frac{5\Gamma(9/5)}{2\pi\Gamma(7/5)}$$

(in computing dg_e/du_x we have taken account of the fact that the electron thermal velocity is much greater than the electron drift velocity which, in the variables used here, is equal to unity). We then find the ion distribution function

$$g_i = (M/m)(1-u_x)^2(5\bar{D})^{-3/5}[5\Gamma(9/5)]/[4\pi\Gamma(7/5)]$$

and, from Eq. (4.33), the diffusion coefficient $\bar{D} = 1/40\pi\,(m/M)^2$. Now, using the relation

$$\bar{u}_\perp^2 = \int_0^\infty u_\perp^3 g_e\, du_\perp \Big/ \int_0^\infty u_\perp g_e\, du_\perp = (5\bar{D})^{2/5}\,\Gamma(9/5)/2\Gamma(7/5)$$

we can find the mean-square electron velocity $(\bar{u}_\perp^2)^{1/2}$, which is found to be $0.38(M/m)^{2/5}$. Thus, the ratio of the directed electron velocity to the thermal velocity is $2.65(m/M)^{2/5}$ in this model.

As before, the number of ions that interact with the waves is small,

$$1 - X_i = \int_0^1 g_i(u_x)\,du_x = (m/M)^{1/5}\frac{10\Gamma(9.5)}{3(8\pi)^{2/5}\Gamma(7/5)} \approx 0.95\,(m/M)^{1/5},$$

and in determining X_e we make use of the dispersion relation

$$\varepsilon(\omega, q) = 1 - \frac{1}{q^2}\int_{-\infty}^{+\infty}\frac{du_x}{u_x} \cdot \frac{dg_e}{du_x} - \frac{X_e}{-q)^2} - \frac{mg_i(0)}{M\omega q} - \frac{mX_i}{M\omega^2} = 0$$

or

$$1 + \frac{8.2}{q^2}\left(\frac{m}{M}\right)^{4/5} = \frac{X_e}{(\omega - q)^2} + \frac{m}{M\omega^2} + \frac{2.86}{\omega q}\left(\frac{m}{M}\right)^{6/5}$$

From the requirements on stability and the existence of waves with all phase velocities in the range (0, 1), which is completely

analogous to the case of propagation of the current along **H** [cf. Eqs. (4.36–4.40)], we find the number of particles in the electron root $X_e \approx 8.2(m/M)^{4/5}$.

§ 4.4. Anomalous Resistivity Caused by Other Instabilities

Let us now return to the Table 1 in §4.1 and consider other instabilities. The Drummond–Rosenbluth instability leads to a relatively small imaginary frequency component; furthermore, from the point of view of the quasilinear approximation, this is a one-dimensional instability. Hence, an electron plateau arises which is similar to the plateau that arises in the one-dimensional ion-acoustic model (4.3); this plateau causes rapid saturation of the instability. The current can increase further, but in a small region of velocity space, the electron distribution will have a plateau and the plasma will not be unstable.

The instabilities with the lowest excitations threshold are the electrostatic instabilities with $k_\parallel \ll k_\perp$ in a plasma in which the current flows across the magnetic field. When $\omega \gg kv_{Ti}$, $V_d > k_\parallel v_{Te}/k$ these oscillations are described by (4.6). The nonlinear saturation of this kind of instability is not amenable to analysis within the theory of weak turbulence. For example, we may consider the case of the modified Buneman instability [cf. Eq. (4.7)]. The dispersion relation in (4.7) differs from the usual Buneman equation only in that Ω_p is replaced by $\Omega_p/(1+\omega_p^2/\omega_H^2)^{1/2}$ and ω_p by $\omega_p k_\parallel/k(1+\omega_p^2/\omega_H^2)^{1/2}$. It is possible to estimate the wave amplitude in the saturation regime, as is frequently done in the theory of strong turbulence. We compare the linear term $\partial \mathbf{v}/\partial t$ and the nonlinear term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ in the electron equation:

$$\partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{v} = e\{\mathbf{E} + (1/c)[\mathbf{v}\times\mathbf{H}]\}.$$

In the nonlinear stage these terms compete, and this competition leads to a quasistationary regime in which the instability is saturated. Equating these terms (by order of magnitude) we find $kV_d \sim (kc/H_0)\sum_q q\Phi_q$. It is then possible to obtain the following estimate for the energy density of the waves:

$$\sum_k N_0 e^2|\Phi_k|^2/2T_e \approx mNV_d^2, \quad kr_{He} \approx 1. \qquad (4.41)$$

Now, using Eq. (4.14) we have [I]

$$\nu_{\rm eff} = \omega_H V_d / v_{Te}. \tag{4.42}$$

It is useful to estimate the ratio of electron heating to ion heating in this instability making use of Eq. (4.21):

$$\dot{T}_e/\dot{T}_i \approx V_d/v_{Ti}. \tag{4.43}$$

This instability, as a rule, is a slower instability than the ion-acoustic instability (for $T_e \gg T_i$); however, it can occur in a plasma with a high ion temperature ($T_i \sim T_e$) in which the ion-acoustic instability cannot be excited. When the ratio T_e/T_i is reduced below some critical value we can no longer neglect the ion thermal motion in the dispersion relation (4.6). In this limiting case one is dealing with a so-called electron acoustic mode [cf. Eqs. (4.6) and (4.9)]. It is possible to estimate $\nu_{\rm eff}$ by again carrying out a procedure similar to that used in strong turbulence.

When $\omega_p \gg \omega_H$, the Bernstein modes can be unstable in a plasma that supports a current. This kind of instability can be described as follows. The Bernstein modes are waves for which the wave vector is perpendicular, or almost perpendicular, to the magnetic field and in which the frequencies are grouped close to the harmonics $l\omega_H$. These are pure electron oscillations at rather high frequencies, and no interaction with the ions occurs. Let us assume that a current flows through a plasma and that there are no waves in a reference system that moves with the electrons. Because of the Doppler effect, in the laboratory reference system in which the ions are at rest the oscillation frequency is shifted $l\omega_H - \mathbf{k}\cdot\mathbf{V}_d$. If the drift velocity is large enough, then with sufficiently large k (and k can be chosen as large as the Debye wave vector) the frequency can be reduced significantly in the ion reference system, so that these oscillations can interact with the ions $(l\omega_H - \mathbf{k}\cdot\mathbf{V}_d) \sim kv_{Ti}$. In this case an instability characterized by a negative energy will be excited. The usual Maxwellian ion distribution can then be unstable if one takes account of the imaginary part of the ion interaction with the Bernstein modes due to the Landau resonance. It turns out that this instability has a rather large imaginary part, this value being of the order of the electron Larmor frequency reduced by a factor equal to the ratio of the electron drift velocity to the electron thermal velocity. This instability is not very sensitive to the temperature ratio. In contrast with the ion-acoustic instability, this instability does not require that the electron temperature be much higher than the ion temperature. One would expect a high anomalous resistivity in this case. It turns out, however, that a very small nonlinearity (small effective collision frequency) which arises in the development of this instability is sufficient to completely suppress it. The electron inertia is important in these waves. This means that electron collisions can make a large contribution to the imaginary part. It will be evident that the quantity $\nu_{\rm eff}$ is subtracted from the growth rate. It would appear at first sight that one could find $\nu_{\rm eff}$ by equating these two quantities. However, collisions give a still larger contribution. To understand this it should be recalled that the oscillating part of the distribution function contains a factor $\exp(i[\mathbf{k}\times\mathbf{v}]/\omega_H)$ so that $\nu_{\rm eff}$ will appear with the Pitaevskii factor $k^2 r_{He}^2$ [122]. Since we are discussing very large values of $kr_{He} \approx v_{Te}/V_d$ (short wavelengths) this factor plays an important role. As a result even a small nonlinearity is sufficient to suppress the Bernstein instability.

The effective collision frequency $\nu_{\rm eff}$ for unstable Bernstein modes can be found approximately by the following method. The well-known dispersion equation from the linear theory of the instability is modified in order to include collisions with the frequency being determined, $\nu_{\rm eff}$. This can be done by adding a Fokker–Planck collision term $\partial^2 f/\partial v_\perp^5$ in the linearized kinetic equation for the correction to the electron distribution function. In the dispersion equation which results we assume that nonlinear effects (which take account of $\nu_{\rm eff}$) lead to saturation of the instability:

$$\gamma_k - \nu_{\rm eff}\, k^2 r_{He}^2 = 0. \tag{4.44}$$

Then, substituting the growth rate $\gamma = \omega_H V_d/v_{Te}$ and the wave number $k \approx \omega_H/V_d$ for the growing waves, we have [123]

$$\nu_{\rm eff} = 10^{-1}\,\omega_H (V_d/v_{Te})^3. \tag{4.45}$$

The approach described here, in which nonlinear effects are taken into account by introducing in the linear analysis turbulent transport coefficients with values such that the system returns to the stability threshold, has been used frequently in problems on anomalous diffusion and thermal conductivity in inhomogeneous plas-

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$$\gamma_k - k^2 D = 0, \tag{4.46}$$

where

$$D \approx \frac{c^2}{H_0^2} \sum_k \gamma_k^{-1} |E_k|^2 \frac{1}{\sqrt{\pi} k r_{He}}.$$

It is then an easy matter to estimate the level of the turbulence:

$$W/N_0 T_e \approx (\omega_H^2/\omega_p^2)(V_d/v_{Te})^3. \tag{4.47}$$

The estimate for the effective collision frequency then coincides with that in Eq. (4.45), since $D_\perp \approx \nu_{eff} r_{He}^2$.

CONCLUSION

The relation between the various elements of the theory of weak plasma turbulence can be given in the form of a chart derived from the equations that are used.

Theory of Weak Plasma Turbulence

Vlasov equation (for each particle species) + Maxwell's equations

Use of statistical approach

Particle kinetic equation

Kinetic equation for each plasma wave

The general symbolic forms for these equations are

$$\boxed{\frac{df(\mathbf{v})}{dt} = \mathrm{St}\,[f(\mathbf{v})]} \qquad \boxed{\frac{dn(\mathbf{k})}{dt} = \mathrm{St}\,[n(\mathbf{k})]}$$

The collision term can be written as follows:

In the first approximation

$$\boxed{\mathrm{St} = \mathrm{St}_{QL}[\bar{f}(\mathbf{v})]} \qquad \boxed{\mathrm{St}\,[n(\mathbf{k})] = 2\,\mathrm{Im}\,\omega_k\,[f(\mathbf{v})]\cdot n_k}$$

The quantity $\mathrm{Im}\,\omega_k[f(\mathbf{v})]$ symbolizes the dependence of the growth rate on the distribution function $f(\mathbf{v})$. This approximation corresponds to the quasilinear approximation of Chapter 2 and only takes account of the linear interaction between the waves and resonant particles, in accordance with the resonance relation $\omega - \mathbf{k}\cdot\mathbf{v} = 0$.

In the second approximation

(a) Wave–wave interaction (Chapter 1)

$$\omega_1 + \omega_2 = \omega_3,$$
$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3.$$

The collision term in the particle kinetic equation describes the adiabatic interaction of waves with particles which participate in the wave motion	St(n) = n · n is a symbolic notation which reflects the quadratic nature of the three-wave interaction

(b) Nonlinear wave–particle interaction (Chapter 2)

$$\omega_1 - \omega_2 = (\mathbf{k}_1 - \mathbf{k}_2)\cdot\mathbf{v}.$$

The collision term describes the resonance interaction between particles and "beat" waves	St(n) – n · n · f is a symbolic notation which reflects the fact that the particles also participate in the interaction

In the third approximation

Again, only the adiabatic interaction between particles and waves is included	St(n) = n · n · n. These processes are important for the nondecay spectrum (cf. § 1.3)

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The approach described in this review cannot be used in the case of strong turbulence since no small parameter is available; this means higher-order interactions give the same contributions as the terms which are retained in our present analysis. Some simple examples of estimates for strong turbulence have been given in the problem on anomalous plasma resistivity (cf. Chapter 4).

In this review we have not considered problems associated with the so-called plasma echoes first reported by Gold, O'Neal, and Malmberg [127] (further details are given in the review by Kadomtsev [128]). An echo effect is a nonlinear effect related to mode mixing and its presence is due to a conserved modulation of the particle distribution function in the field of a wave described by exp ($i\mathbf{kv}t$) (in other words, the existence of a memory in the system).

In recent years, effects associated with the existence of a memory have been studied in connection with the problem of wave propagation in an inhomogeneous plasma [129], the evolution of wave packets [130], and nonlinear mode interactions [120].

APPENDIX

THERMAL FLUCTUATIONS IN WEAK PLASMA TURBULENCE

In the weak-turbulence approach a plasma is described as an ensemble of gases: particles (the charges in the plasma) and "quasiparticles" (waves). The plasma instabilities play the role of wave sources. However, low-amplitude oscillations and waves exist even in the absence of instabilities. These represent thermal fluctuations (or the equilibrium noise). The level of this noise also plays a role in quasilinear diffusion and can be easily taken into account by the general methods used in weak turbulence. Taking proper account of equilibrium oscillations requires an analysis of the spontaneous emission and absorption of waves by individual particles [VII, VIII].

We introduce the fluctuating part of the distribution function $\delta f(\mathbf{r}, \mathbf{v}, t)$, which satisfies the correlation law for an ideal gas, (assuming a plasma with no magnetic field):

$$\langle \delta f(\mathbf{r}, \mathbf{v}, t)\, \delta f(\mathbf{r}', \mathbf{v}, t')\rangle = \delta(\mathbf{v}-\mathbf{v}')\, \delta[\mathbf{r}-\mathbf{r}'-\mathbf{v}(t-t')]\, f(v). \tag{A.1}$$

In the simplest case of longitudinal oscillations

$$\left.\begin{aligned} &\varepsilon(\mathbf{k}, \omega)\, \Phi_{\mathbf{k}\omega} = \sum_j \frac{4\pi e_j}{k^2} \int \delta f^j_{\mathbf{k}\omega}\, d^3\mathbf{v}; \\ &\delta f_{\mathbf{k}\omega} = \int \delta f(\mathbf{r}, \mathbf{v}, t) \exp[i\omega t - i\mathbf{k}\cdot\mathbf{r}]\, dt\, d^3\mathbf{r}; \\ &\varepsilon(\mathbf{k}, \omega) = 1 + \sum_j \frac{4\pi e_j^2}{k^2} \int \frac{\mathbf{k}\cdot(\partial f_j/\partial \mathbf{v})\, d^3\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v} + i0}. \end{aligned}\right\} \tag{A.2}$$

The fluctuating δf is an additional source of electric field. However, we are interested in the square of the fluctuations $|\Phi_{\mathbf{k}\omega}|^2$. Multiplying Eq. (A.2) by $\Phi^*_{\mathbf{k}\omega}$ we have

$$\varepsilon(\mathbf{k}, \omega)\, |\Phi_{\mathbf{k}\omega}|^2 \approx -\frac{4\pi e}{k^2}\, \Phi^*_{\mathbf{k}\omega} \int \delta f^e_{\mathbf{k}\omega}\, d^3\mathbf{v} \tag{A.3}$$

or

$$\varepsilon(\mathbf{k}, \omega)\, |\Phi_{\mathbf{k}\omega}|^2 = \frac{(4\pi e)^2}{k^4\, \varepsilon^*(\mathbf{k}, \omega)} \int \delta f_{\mathbf{k}\omega}\, d^3\mathbf{v} \int \delta f^*_{\mathbf{k}\omega}\, d^3\mathbf{v}. \tag{A.4}$$

Using this symbolic notation

$$\varepsilon(\mathbf{k}, \omega) = \varepsilon[\mathbf{k}, \operatorname{Re}\omega + i\partial/\partial t] \approx \varepsilon(\mathbf{k}, \operatorname{Re}\omega) + i\,(\partial\varepsilon/\partial\omega)\,(\partial/\partial t)$$

and averaging $\delta f \cdot \delta f^*$, in accordance with the correlation relation (A.1) we easily find

$$\frac{\partial \varepsilon(\mathbf{k}, \omega_{\mathbf{k}})}{\partial \omega_{\mathbf{k}}} \cdot \frac{\partial |\Phi_{\mathbf{k}}|^2}{\partial t} = \operatorname{Im} \frac{16\pi^2 e^2}{k^4} \int \frac{f_{0e}(\mathbf{v})}{\varepsilon^*(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})}\, \delta(\omega - \mathbf{k}\cdot\mathbf{v})\, d^3\mathbf{v}. \tag{A.5}$$

This is an additional term which describes the spontaneous emission of waves by the plasma and which must be added to the right-hand side of the wave kinetic equation. The corresponding recoil effect (the particle reaction to spontaneous emission) gives an additional term in the quasilinear equation for the distribution function:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{fluct}} = \left\langle \frac{e\nabla\delta\Phi}{m} \cdot \frac{\partial \delta f}{\partial \mathbf{v}} \right\rangle, \tag{A.6}$$

where $\nabla\delta\Phi$ is the correction to the electric field due to the spontaneous fluctuations. Averaging $\delta\Phi\,\delta f$ and taking account of Eqs. (1)

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where $\nabla\delta\Phi$ is the correction to the electric field due to the spontaneous fluctuations. Averaging $\delta\Phi\,\delta f$ and taking account of Eqs. (1)

and (2), we find the required additional term in the kinetic equation:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{fluct}} = -\frac{\pi N e^4}{m^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 k^4}\int d^3\mathbf{v}'\,\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}}\,\frac{\delta(\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}')}{|\varepsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2}\,f_e(\mathbf{v})\,\mathbf{k}\cdot\frac{\partial f_e(\mathbf{v}')}{\partial \mathbf{v}'} \tag{A.7}$$

It will be evident that these supplementary terms [in the wave kinetic equation (5) and the quasilinear equation (7)] are negligibly small when one considers an unstable plasma. On the other hand, in a stable plasma the balance between the term in (5) and Landau damping determines the equilibrium level of the thermal fluctuations. After substitution in the quasilinear equation, this equilibrium level gives

$$\left(\frac{\partial f}{\partial t}\right)_{QLA} = \frac{\pi N e^4}{m^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 k^4}\int d^3\mathbf{v}'\,\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}}\,\frac{\delta(\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}')}{|\varepsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2}\,f_e(\mathbf{v}')\,\mathbf{k}\cdot\frac{f_e(\mathbf{v})}{\partial \mathbf{v}}. \tag{A.8}$$

In place of the terms in (A.7) and (A.8) we can use the well-known form of the collision integral given by Lenard and Balescu [125, 126]. Thus, we see how the addition of simple supplementary effects to the binary distribution function (f_2), making use of the methods of weak turbulence, yields the usual kinetic equation for a stable plasma. In addition to treating the spontaneous emission of waves by the plasma in the wave kinetic equation, one should take account of the Cerenkov emission by particles introduced into the plasma from external sources (nonequilibrium radiation). In the method of weak plasma turbulence this effect appears as scattering of the wave on thermal fluctuations of the electron density. Hence to be complete one should really compute the quantity

$$\langle \delta n_{\mathbf{k}\omega}\,\delta n_{\mathbf{k}'\omega'}\rangle = (2\pi)^4\langle \delta n_e^2\rangle_{\mathbf{k}\omega}\,\delta(\mathbf{k}-\mathbf{k}')\,\delta(\omega-\omega'). \tag{A.9}$$

The fluctuating part of the electron distribution function can be written in the form

$$f^e_{\mathbf{k}\omega}(\mathbf{v}) = \delta f^e_{\mathbf{k}\omega} + \frac{e}{m}\,\frac{\mathbf{k}\,(\partial f_{0e}/\partial \mathbf{v})}{\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v}}\,\Phi_{\mathbf{k}\omega}, \tag{A.10}$$

where the potential $\Phi_{\mathbf{k}\omega}$ is related to $\delta f_{\mathbf{k}\omega}$ by Eq. (A.2), in which we now carry out a summation over particle species. Substitution

of Eq. (A.10) in Eq. (A.9) leads to the well-known result

$$e^2\langle \delta n_e^2\rangle_{\mathbf{k}\omega} = \frac{2k^2}{\omega\,|\varepsilon(\mathbf{k},\omega)|^2}\{A_e\,|1+\varepsilon_i|^2 + A_i\,|\varepsilon_e|^2\}, \tag{A.11}$$

where

$$A_j = \frac{\pi e_j^2\omega}{k^2}\int f_{0j}(\mathbf{v})\,\delta(\omega-\mathbf{k}\cdot\mathbf{v})\,d^3\mathbf{v}.$$

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As we have already noted, the quasilinear approximation can be applied for weak nonlinearities. This approximation is valid if the electric field is small so that the frictional force due to coherent emission of ion-acoustic waves retards the electrons and prevents them from acquiring a mean velocity greater than the critical velocity needed for the instability. In other words, at all times the plasma is essentially at the threshold of the instability. In this case the nonlinear regime described in § 4.2 must correspond to the case of large electric fields. The quantity E_c, that is to say, the limiting value which separates the two regimes being considered, can be found as follows. Let V_c be the electron drift velocity which corresponds to the instability threshold ($\gamma_i + \gamma_e = 0$). The case of large electric fields (4.24) is realized if V_d, as computed from the relation $V_d = eE/m\nu_{eff}$, exceeds V_c ($V_d \gg V_c$). Using Eq. (4.24) for ν_{eff}, we have

$$E \gg 10^{-2}\,(mM^3)^{1/4}\Omega_p c_s/e. \qquad (4.35)$$

The functional dependence $j = j(E)$ can be represented qualitatively by a curve like that shown in Fig. 34. Here, there is a classical region which obtains at low electric fields, in which the plasma is far from the unstable regime; at moderate electric fields $j = eNV_c$ (quasilinear regime); $j \sim E^{1/3}$ in the nonlinear regime, in which case ν_{eff} is determined from Eq. (4.24).

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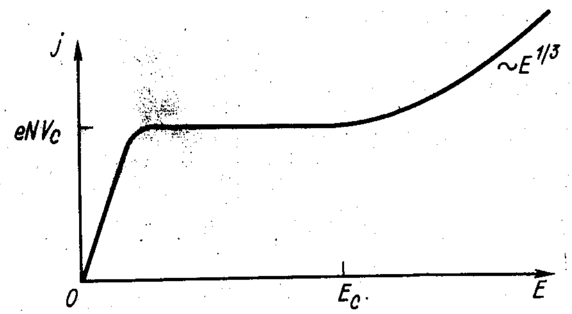

Fig. 34. Ohm's law with ion-acoustic turbulence.

All of the discussion in this section refers to the case in which the current flow is perpendicular, or almost perpendicular, to the magnetic field. Now let us consider the situation in which the magnetic field is directed in the direction of current flow or in which there is no magnetic field. In this case the mechanism which mixes the electrons no longer operates and the problem of determining the electron distribution function becomes more complicated. The question arises as how to proceed in this case. We shall assume here that after some time a self-similar distribution is established [117]. The electron distribution function assumes some universal form: Further heating of the electrons occurs and increases their mean velocity, but the form of the function remains the same. In practice, however, it appears that these self-similar variables cannot be used: It is evidently not possible to introduce a two-temperature electron distribution because there are no two clearly defined groups of electrons, as in the ion case. On the contrary, there is a smooth transition from the slow electrons to the fast electrons; in the course of time, as is shown by a qualitative analysis of the equations in the self-similar variables, a significant fraction of the electrons fall into the velocity region in which there are essentially no waves. This effect is reminiscent of electron runaway in a gas with Lorenz collisions, in which the collision frequency falls off with velocity as v^{-3}. The interaction of electrons with ion-acoustic waves exhibits precisely the same properties [63].

The question of the ultimate form that is assumed by Ohm's law in such a plasma remains open at the present time. One of the points of view that is presently held is the following: A significant fraction of the electrons fall into the runaway regime; the

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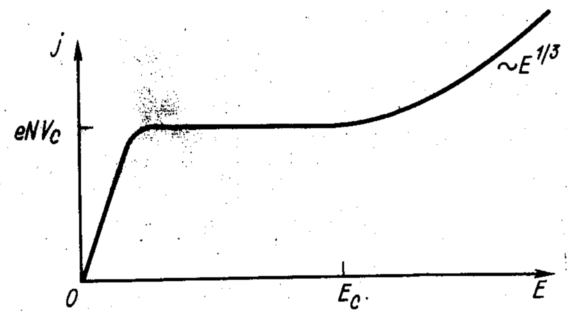

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WAVE PROCESSES IN AN INHOMOGENEOUS PLASMA

N. S. Erokhin and S. S. Moiseev

INTRODUCTION

Investigations into the effect of plasma inhomogeneities on wave interactions and wave propagation first appeared in the literature many years ago (cf. [1, 2] and the bibliographies contained therein). In recent years this branch of plasma physics has received intensive development and the results have found broad practical application [3-5]. The following trends in the development in this interesting field can be distinguished: (a) introduction of non-one-dimensional inhomogeneities in a medium; (b) the investigation of the joint effect of nonlinearity and inhomogeneity (harmonic generation, decay processes, transformation of nonlinear waves); (c) wave conversion in nonequilibrium media; (d) kinetic effects in wave conversion in an inhomogeneous plasma (linear and nonlinear nonlocal wave reflection, nonlocal wave conversion); (e) an extension of work on wave propagation in media with one-dimensional inhomogeneities in the linear MHD approximation, in which a number of questions still remain to be resolved.

In many important respects, the properties of wave motion in an inhomogeneous plasma are qualitatively different from those in a homogeneous plasma. For example, in a spatially inhomogeneous plasma longitudinal and transverse waves are coupled together and the wave vector **k** is a function of the coordinates; furthermore, the "decay" region for interacting waves is bounded in space, leading to the inhibition of decay instabilities as a consequence of the transport of energy out of the decay region and the