

Turbulence Theory

- An Introduction

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i.e.

I.) Basics of Fluid Turbulence (30)



Characteristics of Fluid Turbulence:

"turbulence" vs "noise" \rightarrow energy flux

- broad range of spatio-temporal scales center-TAM
excited

Ref.

U. Frisch

- decay of large scale energy \rightarrow need "Turbulence input/stirring" to maintain stationarity - The legacy of A.N. Kolmogorov

- energy input dissipated as heat (to maintain stationarity) \rightarrow viscosity

\rightarrow irreversibility

- irreversible mixing occurs \rightarrow i.e. passive tracer

- intermittency manifested
i.e. spatial \rightarrow coherent structures (i.e. vortices)

occur
temporal \rightarrow bursts measure probe
trace

- self-similarity / scale-similarity :

turbulence looks the same on all scales,
except the very largest (stirring) and
the very smallest (dissipation)

Caveat: Intermittency - memory of large scales on small

(c) Navier-Stokes Equation - Describes
Fluid

$$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} = -\nabla p + \nu \nabla^2 \underline{V}$$

$$\left. \begin{array}{l} \\ \end{array} \right\} p$$

advection/ pressure
strain rate viscous diffusion of
 \rightarrow nonlinearity momentum

$\rho = 1$
hereafter

$$\nabla \cdot \underline{V} = 0 \quad \text{incompressibility}$$

Note: Pressure determined from incompressibility

i.e.

$$\nabla \cdot \left[\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right] = -\nabla^2 p + \nu \nabla^2 (\nabla \cdot \underline{V})$$

$$\nabla^2 p = -\nabla \cdot \nabla \underline{V}$$

$$p = -\frac{1}{2} \nabla \cdot \nabla \underline{V}$$

$$= \frac{1}{4\pi D} \int \frac{\partial^3 \underline{V}(x') \nabla(x') \cdot \nabla_{x'} \underline{V}(x')}{|x-x'|} dx'$$

More generally, can eliminate p

$$\frac{\partial}{\partial t} V_i + (\partial_i - \partial_i \nu \nabla^{-2}) \partial_j (V_j V_i) = \nu \nabla^2 V_i$$

Key Parameters: Reynolds #

$$Re = |V \cdot \nabla V| / (\nu \delta^2 V)$$

$$\sim \frac{V(L) L}{\nu}$$

$\sim \frac{\text{nonlinearity}}{\text{collisional diffusion}}$

measure of strength
of NL.

- Re usually referenced to largest scale

$$L = L_{\max}$$

$V(L) =$ largest scale velocity

- Re always referenced to a particular scale

$$L_{\max}, \lambda = \left[\langle (\delta_i V_j)^2 \rangle / \langle V_i^2 \rangle \right]^{-1/2} \quad \lambda_{\text{dissip.}}, \\ (\text{Taylor Scale}) \quad (Re=1)$$

- $Re \gg 1$ in turbulent { pipe flow
atmosphere
etc }

$$Re \sim 10^6 - 10^8, \text{ etc.}$$

i.e. planetary boundary layer : $l_{\text{out}} \sim 1 \text{ km}$
 $\sim 10^5 \text{ cm}$
 $l_{\text{disc}} \sim 1 \text{ cm}$

$\Rightarrow 6$ decades!

- Re : measure of ratio of inertial mixing of momentum to collisional mixing

(4)

(ii) Experimental Laws' of Fully Developed Turbulence

Much / most of turbulence theory is empirically motivated. Experimental info / results preceded sophisticated theoretical analyses.

The experimental facts:

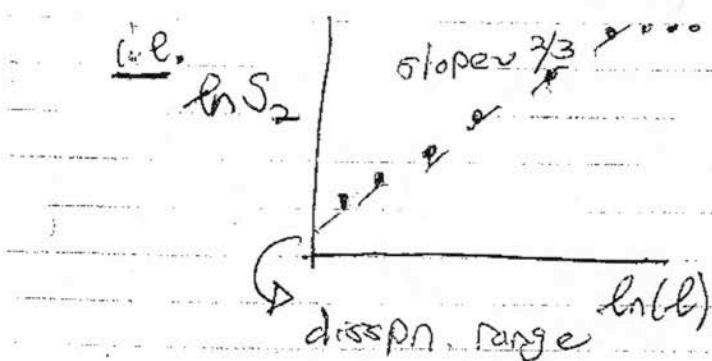
1.) 2/3 Law (Mundane)

In a turbulent flow with $Re \gg 1$,
 $\langle dV(l)^2 \rangle$ (mean square velocity increment
between two scales) separated by distance
 l) scales as $l^{2/3}$.

c.f.
 $dV(l) = |V(r+l) - V(r)| \Rightarrow$ a difference!

$$S_2(l) = \langle dV(l)^2 \rangle \sim l^{2/3} \rightarrow \text{fundamental scaling relation}$$

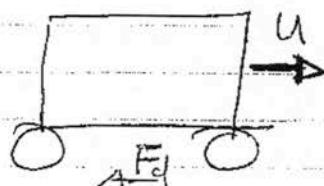
2nd order structure function



B) Law of Finite Energy Dissipation (Profound)

If in an experiment on turbulent flow all the control parameters are kept the same, except the viscosity, which is lowered as much as possible, the energy dissipation per unit mass dE/dt behaves in a way consistent with a finite limit.

What means 'Energy Dissipation Rate'?

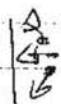


Consider a car experiencing atmospheric drag

$$\bar{F}_d = \frac{1}{2} C_D \rho S U^2$$

↑
face surface area

c.e.



$$\rho = \rho S (U^2) U \rightarrow \text{momentum in air of slug}$$

$$M = \rho S U^2 \rightarrow \text{mass}$$

If assume air momentum completely transferred to car

$$\frac{dp_{car}}{dt} = \bar{F}_d = \rho S U^2$$

$C_D(Re)$ = drag coefficient (slowly varying function of Re , depends on shape, etc.)

$$\therefore F_d = \frac{C_D}{2} \rho S U^2$$

Now, power dissipated by drag force

$$P_d = F_d U$$

$$\Rightarrow P_d = \frac{C_D}{2} \rho S U^3$$

Energy dissipation rate $\epsilon = P_d / \text{Mass}$
(per unit volume)

$$= \frac{C_D}{2} \frac{U^3}{L}$$

$$\text{also } NS \Rightarrow \partial_t \langle U^2 \rangle \sim - \langle U \partial_x U^2 \rangle \sim \frac{U^3}{L}$$

→ Why should we care?

Note energy budget:

$$\frac{\partial V_i}{\partial t} + V_j \partial_j V_i - \nu \nabla^2 V_i = - \partial_i \rho$$

$$\partial_t \frac{V_i^2}{2} + \partial_j \frac{V_i V_j}{2} - \nabla V_i \cdot \nabla^2 V_i = -V_i \partial_i \rho$$

$\langle \quad \rangle \equiv$ ensemble (Fast space-time avg.)

$$\partial_t \langle \frac{V_i^2}{2} \rangle + \langle \partial_j \frac{V_i V_j}{2} \rangle - \nabla \langle V_i \cdot \nabla V_i \rangle$$

surface terms = $\langle \partial_i \chi_i \rho \rangle$
upon IBP



$$\partial_t \langle \frac{V_i^2}{2} \rangle = -\nabla \langle |\nabla V_i|^2 \rangle$$

but $\epsilon = -\partial_t \langle \frac{V_i^2}{2} \rangle$! (- dissipation)

$\boxed{\epsilon = \nabla \langle |\nabla V_i|^2 \rangle}$!

\Rightarrow experiments suggest that $\epsilon \rightarrow$ finite
as $r \rightarrow 0$. \leftrightarrow remarkable

\Rightarrow suggests that extremely large ∇V
forms as $r \rightarrow 0$, singular vortex sheets

\Rightarrow singular velocity gradients formed in
limit of weak viscosity

{ Heart of turbulence problem is grappling with
singularity (especially its degree) of velocity gradients.

No. B.: { Dissipation Law
 Singularity formation is at the
 heart of why turbulence is
 a "hard" problem.

Re: Dissipation Law:

$$\epsilon \sim \frac{U^3}{L} \sim U^2(L/U)$$

$$\sim \frac{\text{K.E. per Mass}}{\text{Conculation Time}}$$

i.e. \rightarrow in 1 macro circulation time, a finite fraction of (macro) kinetic energy is dissipated by viscosity.

\rightarrow dissipation time scale is (L/U) .

V.) Kalmogorov's Hypotheses and their Predictions / Implications. \rightarrow K41 Theory of Turbulence

1: In the limit of $Re \rightarrow \infty$, all possible symmetries of the Navier-Stokes equation, usually broken by the mechanisms producing the turbulent flow, are restored in a statistical sense at small scales and away from boundaries.

lose memory

What means?

- "small scales": $l \ll l_0$

$\frac{l}{\text{integral scale}} \rightarrow \text{characteristic of production}$

- symmetries

First, symmetries of Navier-Stokes Eqn. I.

a.) space translations $\Gamma \rightarrow \Gamma + \Delta$
(no explicit Γ dep.)

b.) time translation $t \rightarrow t + \tau$
(no t dep.)

* c.) Galilean boosts $\begin{cases} \Gamma \rightarrow \Gamma - Ut \\ V \rightarrow V + U \end{cases} \quad V = U + \underline{V}(r - Ut, t)$
(no frame dep.)

$$\text{i.e. } \frac{\partial V}{\partial t} + V \cdot \nabla V = -\nabla P + \rho \nabla^2 V$$

insert \Rightarrow $-U \frac{\partial V}{\partial t} + U \cdot \nabla V + \frac{\partial V}{\partial t} + V \cdot \nabla V = -\nabla P + \rho \nabla^2 V$

d.) Parity left-right
(no preferred direction) $X \rightarrow -X, V \rightarrow -V$

e.) Rotation (no preferred direction) $\begin{cases} \Gamma \rightarrow R \Gamma \\ V \rightarrow R V \end{cases}$

* e.) Scaling (for $r \rightarrow 0$) \Rightarrow critical: scale elimination
and passing

$$\Gamma, V, f \rightarrow \lambda \Gamma, \lambda^h V, \lambda^{h_f} f$$

$$\text{i.e. } \frac{\partial V}{\partial t} + V \cdot \nabla V = - \nabla P$$

$$V \rightarrow \lambda^a V$$

$$f \rightarrow \lambda^b f$$

$$\frac{\lambda^a \partial V}{\lambda^b \partial t} + \frac{\lambda^{2a}}{\lambda} V \cdot \nabla V = - \nabla P$$

\hookrightarrow From $\nabla \cdot V = 0$

$$\lambda^{2a-1} = \lambda^{a-b}$$

$$b = 1-a$$

$$\Rightarrow \lambda^{-(a-1)} = \lambda^b$$

$$\therefore \text{scalings } \Gamma \rightarrow \lambda \Gamma, V \rightarrow \lambda^h V, f \rightarrow \lambda^{h_f} f$$

Now, turbulence onset \Rightarrow symmetry breaking!

i.e. ① KH : shear breaks $\begin{cases} \text{translational} \\ \text{rotational} \end{cases}$
invariance.

② rigid body boundary flow, etc.

③ Flushing toilet $\begin{cases} \text{space} \\ \text{time} \end{cases}$

etc.

- * However, fully developed turbulence tends to restore symmetry, except near boundaries, on small scale.

b.c. \Leftrightarrow boundary conditions

i.e. if $\delta V(r, \ell) = V(r+\ell) - V(r)$
 \Rightarrow

$$\delta V(r+\ell) = \delta V(r)$$

Similarly: isotropy, parity
 facilitates scaling approach

- H2 For $Re \rightarrow \infty$ turbulence, at small scales and away from boundaries, the flow is self-similar at small scales

i.e. possesses a unique scaling exponent
 h s/t

$$\delta V(r, \lambda \ell) \rightarrow \lambda^h \delta V(r, \ell)$$

(→ addresses 2/3 Law)

- H3 With assumptions similar to H1, the turbulent flow has a finite, nonvanishing mean rate of dissipation ϵ' per unit mass.

$Re \rightarrow \infty \Rightarrow r \rightarrow \infty$ with $V_0 = V_{max}$ to fixed

$$\epsilon' = V_0^3 / f_0$$

Alternative (not necessary): Kolmogorov's Second Universality Assumption: In the limit of infinite Reynolds number all the small-scale statistical properties are uniquely and universally determined by the scale ℓ and the mean energy dissipation rate ϵ .
 (First: $\frac{4}{5} \epsilon \ell = \langle u^3 \rangle$)

i.e. Implications:

$$-\langle \delta V(e)^2 \rangle = S_2 \quad ?$$

$$S_2 \sim L^2/T^2 \text{, dimensionally}$$

$$\text{Now } \epsilon \sim L^2/T^3$$

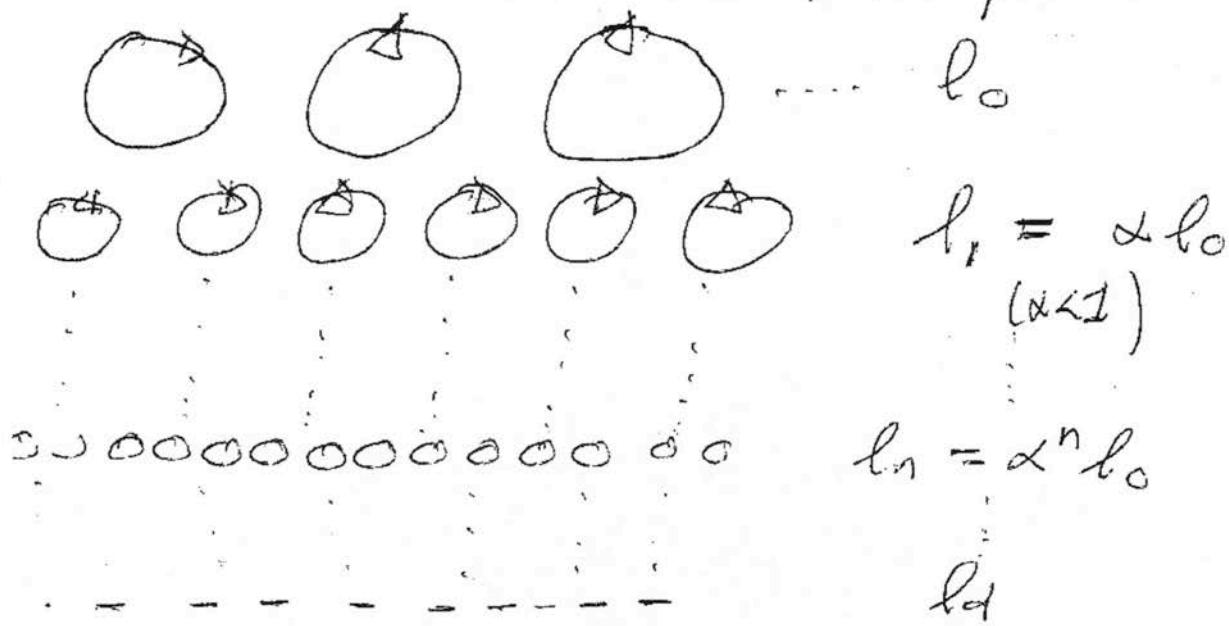
$$\Rightarrow \langle \delta V(e)^2 \rangle \sim \epsilon^{2/3} \ell^{2/3} \Rightarrow \text{recovers } 2/3 \text{ law.}$$

also, implies $h = 1/3 \Rightarrow$ scaling exponent, etc.

H1, H2, H3 (2nd Universality Assumption) \Rightarrow
 H4/phenomenology.

K41 Phenomenology

Picture: (Richardson) Cascade / Eddy Mifosis



Key Idea: ① \rightarrow Flux of energy in 'scale space',
from l_0 (integral scale) to l_d ,
(dissipation scale)

- ② \rightarrow energy flux is self-similar
- ③ symmetry restoration.

Flux \leftrightarrow ④ \rightarrow energy dissipation \rightarrow finite limit of $\eta \rightarrow 0$ (i.e. entropy re-adjustment)

⑤ If-similarity \leftrightarrow 2/3 Law $S_2 = C(l/l_0)^{2/3}$
 $l_0 \rightarrow l_0$, $C \rightarrow C \alpha^2$
etc.

Ingredients in K41 Phenomenology:

$\rightarrow \ell$: scale parameter : eddy scale

$\rightarrow V(\ell) : \tilde{V}(\ell) \sim (k \langle V_u(\ell)^2 \rangle)^{1/2}$

$$\stackrel{\text{eddy}}{\text{velocity}} \quad \delta V_{\parallel} \sim (\underline{V}(\underline{l} + \underline{\ell}) - \underline{V}(\underline{l})) \cdot \frac{\underline{\ell}}{\ell}$$

\equiv longitudinal velocity increment

$\rightarrow V_0$: rms velocity fluctuation
(large scale dominates)

$$V(\ell_0) \sim V_0$$

$\rightarrow \tau(\ell)$: eddy lifetime / turn-over rate
characteristic rates of transfer thru
scale ℓ .

self-similarity : energy thru-put rate is
 $\stackrel{\text{scale } \ell}{\text{dissipation}}$ $\stackrel{\text{scale invariant}}{\text{energy}}$

$$\stackrel{\text{dissipation}}{\epsilon} = \frac{V(\ell)^2}{\tau(\ell)} \quad \begin{matrix} \text{energy below / throu} \\ \text{scale } \ell \end{matrix}$$

, now, $\tau(\ell)$?

compare $\tau(l)$ with $\tau_{\text{av}}(l)$

1962

$\sim(l) \rightarrow$ lifetime of structure of scale l

\rightarrow i.e. time for structure to be distorted out of existence

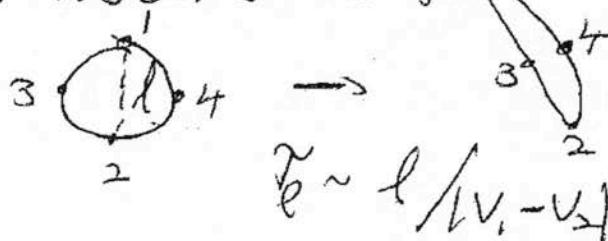
scales $l' \gg l$:

\rightarrow advection eddys \rightarrow apply Goldstein heat, but don't affect life-time.
irrelevant/ \rightarrow symmetry under random Goldstein
transformations \rightarrow would also violate symmetry restoration.

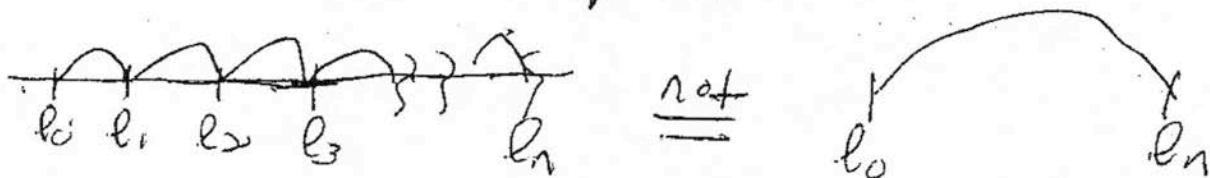
scales $l' \ll l$:

\rightarrow irrelevant as very little energy/shear in such eddys/scales

seek scales which:



$\rightarrow \tau(l) \sim \frac{l}{\partial(l)}$: cascade local in wave space.



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$$\epsilon = \frac{V(\ell)^3}{\ell}$$

$$\Rightarrow \left\{ \begin{array}{l} V(\ell) \sim (\ell \epsilon)^{1/3} \\ V(\ell)^3 \sim \epsilon \ell^{2/3} \end{array} \right. : \text{K41 scaling relation}$$

- verifies $\ell^{2/3}$ law

- for spectrum:

$$\text{if } E(k) = |\psi(k)|^2$$

$$\text{so } E = \int dk E(k) \quad \left\{ \begin{array}{l} \text{c.e. absrbs} \\ \text{density of states} \end{array} \right.$$

$$\text{then } V(\ell) = \int_{k_\ell}^{k_{\ell n}} dk E(k)$$

$$V(\ell)^3 \sim \epsilon \ell^{2/3} = \epsilon^{2/3} k_\ell^{-2/3}$$

$$\Rightarrow \left\{ E(k) = \epsilon^{2/3} k^{-5/3} \right\} : \text{Kolmogorov Spectrum}$$

at ℓ_0 :

$$V_0 \sim \epsilon^{1/3} \ell_0 \Rightarrow \frac{V_0^3}{\ell_0} = \epsilon.$$

198.

For dissipation scale:

ℓ_d occurs in k -space when cascade terminated
viscosity asserts itself $\rightarrow Re(k) \rightarrow 1$

$$1/\tau(e) \sim 1/\tau_d = \nu/\ell^2$$

$$\Rightarrow \epsilon^{1/3} \ell^{-2/3} = \frac{\nu}{\ell^2}$$

$$\ell^{4/3} = \nu/\epsilon^{1/3}$$

$$\boxed{\ell_d \equiv 1, \text{ in Frisch}}$$

$$\Rightarrow \boxed{\ell_d = \nu^{3/4}/\epsilon^{1/4}}$$

Recall: $E = \nu \langle (\nabla V)^2 \rangle$

$\Rightarrow \nu \rightarrow 0 \Rightarrow \langle (\nabla V)^2 \rangle$ divergent

$$\langle (\nabla V)^2 \rangle = \int_{k_0}^{k_d} dk k^2 \epsilon^{2/3} k^{-5/3}$$

$$= \int_{k_0}^{k_d} dk k^{4/3} \epsilon^{2/3}$$

$$= k_d^{4/3} \epsilon^{2/3}$$

$$= \frac{\epsilon^{1/3}}{\nu} \epsilon^{2/3} = \epsilon / \nu$$

$\Rightarrow \boxed{\langle (\nabla V)^2 \rangle \text{ divergent}}$

→ Counting Degrees of Freedom

How big is the inertial range?

$$\begin{aligned} \text{# of } l's &\sim \frac{l_0}{l_{\text{d}}} \sim \frac{l_0}{(\nu^3/\epsilon)^{1/4}} \\ &\sim \frac{l_0}{r^{3/4}} \left(V_0^3/l_0 \right)^{1/4} \sim \left(\frac{V_0 l_0}{r} \right)^{3/4} \\ &\sim Re^{3/4} \end{aligned}$$

∴ number of degrees of freedom for 3D turbulence is;

$$N \sim Re^{9/4} \quad] : \quad \begin{array}{l} \text{would be (minimum)} \\ \# \text{ grid points to resolve} \\ \text{range of scales in numerical} \\ \text{simulation} \end{array}$$

Now, i.e. atmospheric boundary layer:

$$\begin{aligned} l_0 &\sim 1 \text{ km} \\ l_{\text{d}} &\sim 1 \text{ mm} \end{aligned}$$

$$n \sim 10^6 \Rightarrow \begin{cases} N \sim 10^{18} \\ Re \sim 10^8 - 10^9 \end{cases}$$

⇒ Subgrid scale modelling ...

i.B.: Sometimes able to exploit reduced degrees of freedom models, i.e. when some class of scales slaved to others.

Exercises:

→ Consider passive scalar, with concentration C :

$$\frac{\partial C}{\partial t} + \nabla \cdot \nabla C - R D^3 C = \tilde{f}_C$$

Concentration rate in K41 turbulence $\equiv \zeta$

i.e. $\zeta = C_0^2 \frac{V_0}{l_0}$

⇒ a) Calculate K41 spectrum for C .

b.) What if $M \ll V$?

→ Consider incompressible turbulence with

$$M \equiv \frac{V_0}{C_0} \ll 1.$$

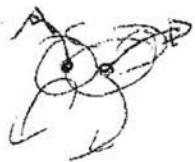
Show: $\frac{l_d}{l_{mfp}} \sim M^{-1} Re^{1/4}$

⇒ Validity of continuum hydrodynamics gets broken at high Re .

200.
q

Particle Separation/Richardson Law

Consider 2 particles (test) in K41 turbulence.
Rate of separation?



- larger eddys advect both
- smaller eddys do Ⓛ nothing

▷ divergence controlled by eddys of scale $\lambda \sim |\underline{x}_1 - \underline{x}_2|$.

$$\therefore \text{if } \lambda \equiv |\underline{x}_1 - \underline{x}_2|$$

$$\nabla \frac{d\lambda}{dt} = v(\lambda) = \epsilon^{1/3} \lambda^{1/3}$$

$$\lambda^{2/3} = \epsilon^{1/3} t$$

$$\therefore \boxed{\lambda \sim \epsilon^{1/2} t^{3/2}}$$

Richardson's 3/2 Law

N.B.: Non-diffusive!

$$x^2 \sim \epsilon t^3 \Rightarrow \boxed{T_{sep.} \sim \lambda^{2/3} / \epsilon^{1/3}}$$

N.B.:

→ process is self-accelerating \Rightarrow large eddys move faster

• non-diffusive.

\rightarrow [Momentum] Flux Driver
Turbulence

265.

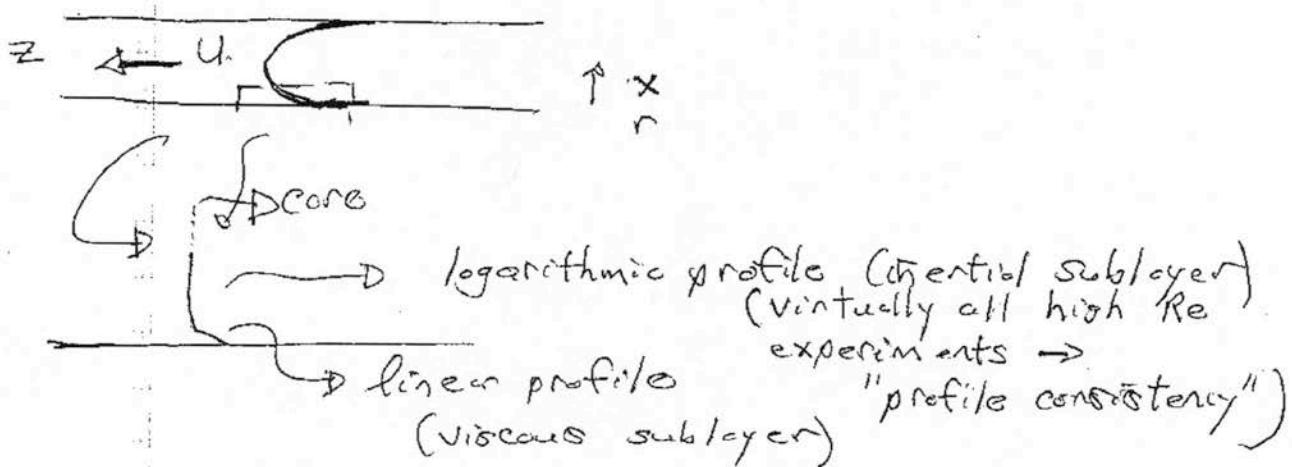
Turbulent Pipe Flow

(cf. Landau, Lifshitz "Fluid Mechanics")

Till now \rightarrow homogeneous flow in a periodic box
 \rightarrow cascade in scale space (Kolmogorov)

Now \rightarrow inhomogeneous flow in a pipe
 \rightarrow momentum transport in a turbulent boundary layer (Prandtl)

Consider turbulent pipe flow:



Common features of pipe flows:

- linear \rightarrow logarithmic $U(x)$ profile
- logarithmic profile persists over a broad range of Re

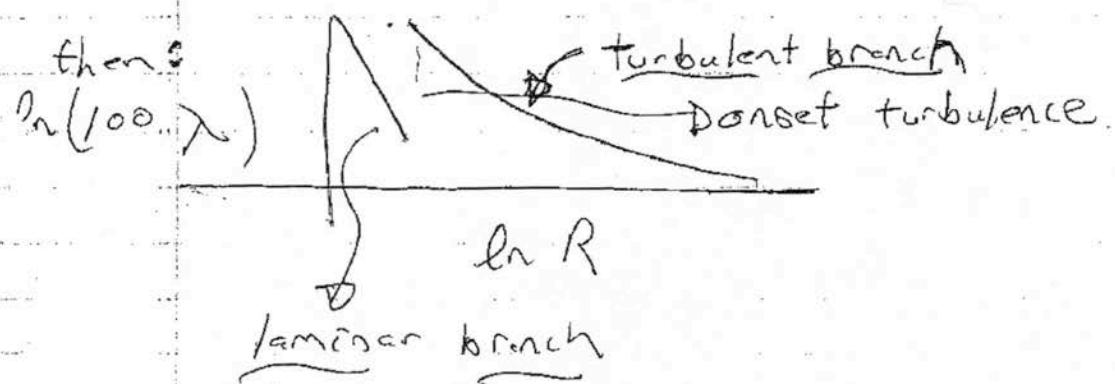
$$(Re = 2Ua/\nu)$$

• logarithmic profile "universal" (Prandtl
increases)
("Law of the Wall")

- resistance increases with increasing Re ,
discontinuously \rightarrow pressure drop/length

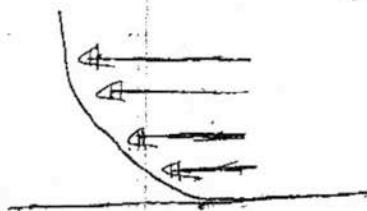
$$\lambda = \frac{2a \Delta P / l}{\frac{1}{2} \rho U^2}$$

$\frac{1}{2} \rho U^2$
 \hookrightarrow mean flow energy



- turbulent resistance curve universal.

• What is going on?



no slip boundary condition
 $U = U(x) \rightarrow 0$
 $x \rightarrow 0$

$\therefore U = U(x) \Rightarrow \left\{ \begin{array}{l} \text{momentum flux} \\ \text{to wall} \end{array} \right.$

26%

→ Momentum flux to wall \Rightarrow stress on the wall

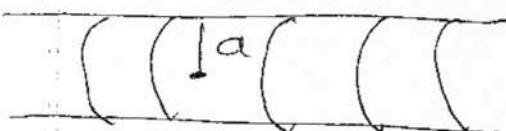
→ Wall stress must balance pressure drop, for steady flow

so wall stress: ρU_*^2
 $U_* \equiv$ friction velocity

i.e.
 $\frac{\int \Sigma V - \Sigma V}{F_{\text{wall}}^{\text{friction}}} = \Delta P \sim \frac{\Delta P}{l}$
 $2\pi U_*^2 l R \sim \Delta P \pi R^2$

$$\rho U_*^2 2\pi a l = \Delta P \pi a^2$$

$$A = 2\pi a L$$



$$\Delta P \rightarrow$$

Force on wall \approx

$$\rho U_*^2 A_{\text{wall}}$$

(pressure drop) A_{flow}

$$\Delta P \xrightarrow{\text{S}} \underline{\text{pressure drop}}$$

= Force on Fluid

$$\text{friction only} \Rightarrow \rho U_*^2 (2\pi a l) = (\Delta P) \pi a^2$$

$$U_* = \left[\frac{(\Delta P / 2\rho)}{\left(\frac{a}{2l} \right)} \right]^{1/2}$$

Friction Velocity

U_* = friction velocity

= "typical" velocity of turbulence in turbulent pipe

Deriving the inertial sublayer profile:

i) dimensional reasoning

in pipe flow inertial sublayer, have

3 dimensional parameters ρ , τ , X

Key Point: - Assumption
of scale invariance

$\left\{ \begin{array}{l} \text{density} \\ \text{wall stress} \\ U_* \end{array} \right\}$ \rightarrow distance from wall

on scale $l_{vs} = \frac{V}{U_*} < X < a$

\Rightarrow universality of logarithmic profile motivated scale invariance assumption

Now, seek velocity gradient dU/dx ,

$\frac{dU}{dx} : U_*, X, \rho$

so simplest form for dU/dx is:

$$\frac{dU}{dx} = \frac{U_*}{X}$$

$$\Rightarrow \boxed{U = \frac{U_* \ln(X/X_0)}{K}}$$

$$= \frac{U_* \ln X}{K} + \text{const.}$$

→ log arithmetic profile (consequence of scale invariance in pipe flow)

→ $K \approx 4$ universal constant → von-Karman constant

→ $X_0 \rightarrow$ width of viscous sublayer $\sim V/U_*$

i.) Physical Reasoning

stationary flow \Rightarrow

Momentum flux to wall = pressure drop

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$$\langle \tilde{v}_x \tilde{v}_z \rangle = u_*^2$$

Reynolds stress

$$\rho(v_x v_z) = \int_p \xrightarrow{\text{momentum Flux}}$$

$$T_p/\varrho = c_k^2$$

Now, to calculate

$\langle \tilde{V}_X \tilde{V}_Z \rangle :$

\Rightarrow take velocity fluctuation as generated by mixing of $U(x)$, so

$$\Rightarrow V_2 \sim l \frac{\partial u}{\partial x}$$

↓

"mixing length"

analogous to Chapman-Enskog expansion), i.e.

$\delta \leftrightarrow \text{long}$

$$J_x \Leftrightarrow \vee_{T_h}$$

hence scale invariance $\Rightarrow l \sim x$

$\left. \begin{matrix} \\ \end{matrix} \right\}$
mixing length set by
distance from wall

so

$$\langle \tilde{v}_x \tilde{v}_z \rangle = \langle v_x l \rangle \frac{\partial u}{\partial x}$$

$$\approx u_* x \frac{\partial u}{\partial x}$$

$$V_T = u_* x \rightarrow \frac{\text{"eddy viscosity"}^{\text{key}}}{\text{"turbulent viscosity"}^{\text{concept}}} \quad \boxed{\text{rate of turbulent transport}}$$

\Rightarrow rate of turbulent transport
of momentum

then momentum balance \Rightarrow

$$u_* x \frac{\partial u}{\partial x} = u_*^2$$

 \Rightarrow

$$u = \frac{u_*}{H} \ln \left(\frac{x}{x_0} \right) \rightarrow \text{Logarithmic profile}$$

\rightarrow Law of the Wall

2020

Some comments:

→ as in h41, clear phenomenology critical to guiding the approximations \rightarrow scale invariance

⇒ "Mixing length theory always works ... provided you know the mixing length..."
- P.D.

⇒ why a single value of velocity, i.e. U_{∞} ?

Consistent with mixing length hypothesis,
velocity fluctuations generated by
mixing of mean flow gradient, i.e.

$$\nabla \sim l \frac{\partial U}{\partial X} \sim X \frac{\partial U}{\partial X}$$

$\sim X U_{\infty}$ absence of preferred scale

consistent. \rightarrow Assumption consistent with:
- logarithmic profile
- scale invariance

⇒ viscous sublayer / cut-off of inertial layer?

∴ when: $\gamma_t < \gamma$

$\left\{ \begin{array}{l} \text{molecular viscosity} \\ \text{dominates mixing} \end{array} \right.$

$$\Rightarrow u_* x \lesssim \gamma$$

$$x \lesssim \gamma/u_* \equiv x_*$$

$\underbrace{\gamma}_{\text{viscous sublayer scale.}}$

In viscous sublayer, flow linear:

$$\gamma \frac{\partial u}{\partial x} = u_*^2$$

$$\therefore u = \frac{u_*^2}{\gamma} x$$

⇒ note effect of turbulence is to:

- flatten profile → higher transport at fixed wall stress
- reduce central velocity
- limit Q (quality factor)

- matching, far. const:

$$X_0 = \frac{V}{U_k} \text{ so}$$

$$U_k = \frac{U_k}{K} \ln \left(\frac{U_k Y}{V} \right)$$

Note: Flow in viscous sublayer is turbulent, but mixing there affected by dissipation length scales \Rightarrow linear profile

Now - turbulent dissipation?

Consider NSE:

$$\frac{\partial \hat{V}}{\partial t} + \hat{V} \cdot \nabla \hat{V} + \langle \hat{V} \rangle \frac{\partial}{\partial z} \hat{V} + \hat{V}_x \frac{\partial}{\partial x} \langle V_z \rangle$$

$$- \nabla \hat{p} + \nu \nabla^2 \hat{V}$$

\hat{V} and \hat{v}_z \Rightarrow

$$\frac{\partial \langle \hat{V}^2 \rangle}{\partial t} + \langle \hat{V} \cdot \hat{V} \cdot \nabla \hat{V} \rangle + \langle V_z \rangle \underbrace{\langle \hat{V} \cdot \frac{\partial \hat{V}}{\partial z} \hat{V} \rangle}_{\text{odd}}$$

$$+ \langle \hat{V}_x \hat{V}_z \rangle \frac{\partial}{\partial x} \langle V_z \rangle = \cancel{\langle \hat{V} \cdot \hat{p} \rangle} - \nu \langle (\nabla \hat{V})^2 \rangle$$

i.b.-P

input \rightarrow mean flow mixing 276.

obviously: $\nu \langle (\delta v)^2 \rangle = Y_f \left(\frac{\partial u}{\partial x} \right)^2$

↓
small scale
dissipation

and

$$\begin{aligned} E &= (U_f X) \left(\frac{U_f}{X} \right)^2 && \text{(ignoring } R \text{)} \\ &= \frac{U_f^3}{X} \end{aligned}$$

→ sets dissipation rate.

i.e. $E = \frac{V_0^3}{l}$ $V_0 \leftrightarrow U_f$
 $l \leftrightarrow X$

→ E finite as $V \rightarrow 0$ (i.e. viscous sublayer gradient diverges, then)

Additional References:

- S.B. Pope, "Turbulent Flows"
- H. Tennekes and J. Lumley, "A First Course in Turbulence"

For net energy budget:

$$\frac{d\epsilon}{dt} = - \langle \hat{U}_x \hat{V}_z \rangle \frac{\partial \langle V_z \rangle}{\partial x} - \nu \langle (\vec{v}^3)^2 \rangle$$

$\underbrace{\qquad}_{\text{input to fluctuations}} \\ \text{by relaxation of} \\ \text{mean shear flow} \\ (\text{Reynolds work})$

$\underbrace{\qquad}_{\text{dissipation}} \\ \text{of fluctuation} \\ \text{energy by viscosity}$

∴ can define:

$$\epsilon_t = \langle \hat{U}_x \hat{V}_z \rangle \frac{\partial U}{\partial x}$$

\downarrow
 turbulent
 dissipation
 rate

and using mixing length theory:

$$\langle \hat{U}_x \hat{V}_z \rangle = u_* x \frac{\partial U}{\partial x}$$

$$\Rightarrow \epsilon = (u_* x) \left(\frac{\partial U}{\partial x} \right)^2 = \gamma_t \left(\frac{\partial U}{\partial x} \right)^2$$

\downarrow
 rate of "heating" by
 turbulent relaxation
 of mean flow.



Now interesting to tabulate comparison between Pipe Flow and K41 Problem

Pipe Flow (Prandtl) K41 (Kolmogorov)

scales: $a_s, x, \sqrt{u_*}$

l_0, l_n, l_d

invariance: $x \rightarrow$ real space

$l \rightarrow$ scale space

inertial sublayer

inertial range
dissipation range

viscous sublayer

$$\text{balance: } U_*^2 = \nu_f \frac{\partial U}{\partial x}$$

$$\epsilon = \frac{U(l)^2}{\tau(l)}$$

dynamics: eddy viscosity

turn-over rate

$$\nu_f = U_* x$$

$$\tau(l) = \frac{U(l)}{l}$$

$$\text{result: } U = \frac{U_* l_n(x)}{R}$$

$$U(l) = \epsilon^{1/3} l^{1/3}$$

universal profile

universal spectral scaling

$$\text{dissipation: } \nu = \nu_f$$

$$\nu(l)/l = \nu/l^2$$

$$x_0 = \nu/U_*$$

$$l_d = \nu^{3/4}/\epsilon^{1/4}$$

→ Practical Issues

Resistance Law \Rightarrow Pipe Flows.

have: $\frac{V}{U_f} < x \leq \frac{a}{r}$
but
radius

can push to $x \approx a$, with logarithmic accuracy

$$U \approx \frac{U_f}{R} \ln \left(\frac{U+a}{V} \right)$$

but

$$V_f = U_f = \left(\frac{a}{l} \frac{\Delta P}{2\rho} \right)^{1/2}$$

⇒ re-write:

$$U = \left(\frac{a \Delta P}{2 \rho l h^2} \right)^{1/2} \ln \left(a \left(\frac{a \Delta P}{2 \rho l h^2} \right)^{1/2} / r \right)$$

Convenient to define:

$$X = \frac{2a \Delta P / l}{\frac{1}{2} \rho U^2} \rightarrow \begin{array}{c|c} \text{friction} & \text{resistance} \\ \text{factor} & \text{coefficient} \end{array}$$

↳ Flow KE

279.

$$\Rightarrow \text{taking } Re = 2aU/v$$

can re-write friction law as:

$$\frac{1/\lambda}{Re} = .88 \ln(Re/\lambda) - .85$$

↓
phenom.

$$Re = 2aU/v$$

$$\lambda = \frac{2a \Delta P / l}{\frac{1}{2} \rho U^2}$$

→ good fit to pipe flow data.

Problems :

1a) A very strong explosion, with energy released ΔE , creates a spherical blast wave in an atmosphere of pressure P_0 , density ρ . Use dimensional analysis to derive the radius of the blast front as a function of time, i.e. $r(t)$? When does this scaling fail?

b) 333 A hot surface produces thermal convection above it. Assuming the convection is turbulent, use scaling arguments to calculate the temperature profile above the plate, assuming the hot plate drives a surface heat flux Q . (See Chapter 5; Landau).

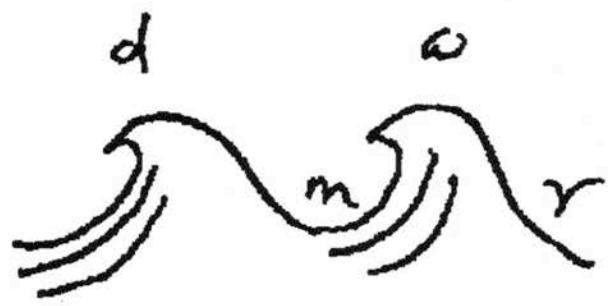
Wave kinetics

2.

"Why Wave kinetics?"

"A wave is never found alone, but is mingled with as many other waves as there are uneven places in the object where the said wave is produced. At one and the same time there will be moving over the greatest wave of a sea innumerable other waves proceeding in different directions"

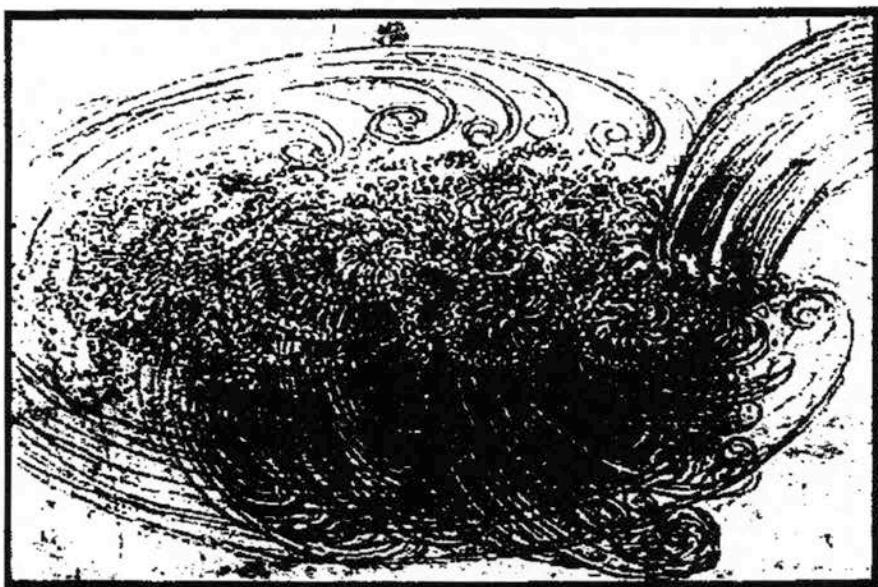
- Leonardo da Vinci
Codice Atlantico, c.1500.



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[Leonardo
on waves...]



4.

From Asian art...

The great wave at Kanagawa Hokusai

