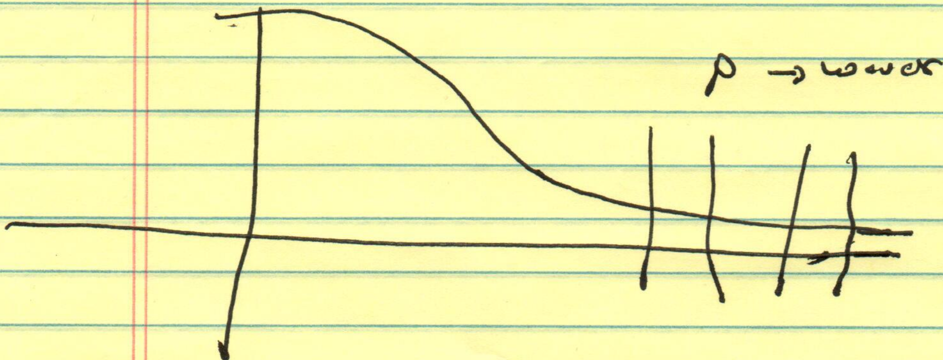
PKU – Lecture II

→ Applications of Mean Field Theory (QL)

(i) R–O–T Relaxation

compare:



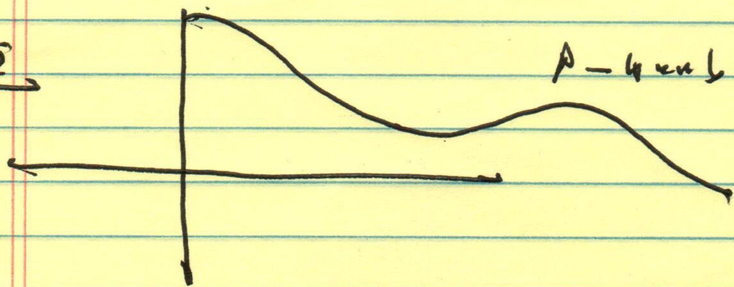
∂_t RPKED > 0 $\Rightarrow$ ∂_t WED < 0

∂_t RPMD > 0 $\Rightarrow$ ∂_t WMD < 0

as ∂_t FED < 0 $\Rightarrow$ ∂_t NRPMD < 0

∂_t NRPKED < 0

vs



∂_t RPMD < 0 $\Rightarrow$ ∂_t WMD > 0

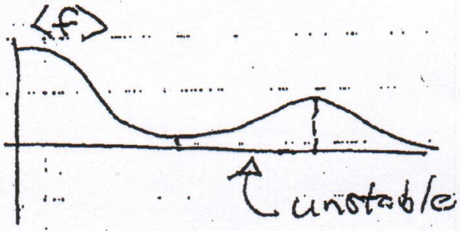
∂_t RPKED < 0

∂_t NRPMD > 0

∂_t WED > 0.

(iv.) Applications of Quasilinear Theory (I)

→ Bump on Tail



↑ unstable phase velocities. (bump on tail)

$\omega_k = \omega_{pe}(1+\frac{3}{2}k^2\lambda_D^2)^{1/2}$

Quasi-linear Equations:

$$\epsilon(k, \omega_k) = 0 \quad \Rightarrow \omega(k),\ \gamma(k) \quad \text{from } \langle f\rangle$$

$$\frac{\partial\langle f\rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial\langle f\rangle}{\partial v}$$

$$D = D^R + D^{NR}$$

$$= \sum_k \frac{q^2}{m^2}|E_k|^2 \left\{\pi\,\delta(\omega_k - kv) + \frac{|\gamma_k|}{\omega_k^2}\right\}$$

$$\frac{\partial}{\partial t}\left(|E_k|^2/8\pi\right) = 2\gamma_k |E_k|^2/8\pi$$

Observe: - resonant diffusion describes dynamics of tail particles

- non-resonant diffusion describes dynamics of bulk Maxwellian

Expect: - tail flattening

with ↕

- adjustment of core/bulk profile (i.e. effective "temperature")

Now, first consider resonant particles (i.e. on bump):

$$\frac{\partial\langle f\rangle}{\partial t} = \frac{\partial}{\partial v} D^R \frac{\partial\langle f\rangle}{\partial v} \qquad * \langle f\rangle \text{ and i.b.p.} \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial t}\int_{res} \frac{\langle f\rangle^2}{2} = -\int_{res} dv\, D^R\left(\frac{\partial\langle f\rangle}{\partial v}\right)^2$$

{ generalization ⇒ Zeldovich Thm.

stationarity ⇒

$$D^R\left(\partial\langle f\rangle/\partial v\right)^2 = 0$$

Now "res" → some finite interval of phase velocities

so

stationarity $\Rightarrow$ $D^R = 0$; i.e. fluctuations decay and damp

or

$\partial \langle f \rangle / \partial v = 0$; plateau forms, removing growth

N.B.: – In 1D $\rightarrow$ plateau
– can generalize

To resolve:

$$D^R = \frac{8\pi^2 q^2}{m^2} \sum_k \frac{|E_k|^2}{8\pi} \delta(\omega - kv)$$

$$\cong \frac{16\pi^2 q^2}{m^2} \int dk \; \mathcal{E}_F(k) \, \delta(\omega - kv)$$

$$D^R = \frac{16\pi^2 q^2}{m^2 v} \mathcal{E}_F(\omega_{pe}/v)$$

so

$$\partial_t D^R = \frac{16\pi^2 q^2}{m^2 v} \left(2 \gamma_{\omega_{pe}/v}\right) \mathcal{E}(\omega_{pe}/v)$$

Now, $\gamma_k = -\epsilon_{IM} / \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k}$

$$\gamma_k = \gamma_{\frac{\omega_{pe}}{v}} = \pi v^2 \omega_p \frac{\partial \langle f \rangle}{\partial v}$$

so
$$\frac{\partial D^R}{\partial t} = \frac{16\pi^2 \xi^2}{m^2 v} \left(2\pi v^2 \omega_p \frac{\partial \langle f \rangle}{\partial v} \right) \mathcal{E}(\omega_p / v)$$

$$= \left(\pi \omega_p v^2 \frac{\partial \langle f \rangle}{\partial v} \right) D^R \quad \text{, using } D^R \text{ defn.}$$

so

$$D^R(v,t) = D^R(v,0) \exp\left[\pi \omega_{pe} v^2 \int_0^t dt' \frac{\partial \langle f \rangle}{\partial v} \right]$$

and:

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D^R \frac{\partial \langle f \rangle}{\partial v}$$

$$= \frac{\partial}{\partial t} \frac{\partial}{\partial v} \left[\frac{D^R}{\pi \omega_{pe} v^2} \right] \quad \left\{ \text{using } \gamma_k, D \text{ definitions} \right.$$

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so

$$\langle f(v,t)\rangle - \langle f(v,0)\rangle = \frac{\partial}{\partial v}\left[\frac{D^R(v,t) - D^R(v,0)}{\pi\omega_p V^2}\right]$$

∴ have:

$$\boxed{\begin{aligned} D^R &= D^R(v,0)\exp\left[\pi\omega_p V^2\int_0^t dt'\,\frac{\partial\langle f\rangle}{\partial v}\right] \\ \langle f(v,t)\rangle &= \langle f(v,0)\rangle + \frac{\partial}{\partial v}\left[\frac{D^R(v,t) - D^R(v,0)}{\pi\omega_p V^2}\right]\end{aligned}}$$

Now, recall seeks to know if:

i) $D^R \to 0 \Rightarrow \partial\langle f\rangle/\partial v\Big|_{t\to\infty} < 0$ (fluctuations damp)

ii) $\partial\langle f\rangle/\partial v \to 0 \Rightarrow$ finite D^R, distribution plateaus.

Now, if $D^R \to 0$,

$$\langle f(v,t)\rangle = \langle f(v,0)\rangle - \frac{\partial}{\partial v}\left[\frac{D^R(v,0)}{\pi\omega_p V^2}\right]$$

$$D^R(0) = \frac{16\pi^2 q^2}{m^2 v}\,\mathcal{E}(\omega_p/v, 0)$$

fluctuation energy

but $\frac{16\pi^2 q^2}{m^2 V} \frac{\mathcal{E}_k(0)}{\pi \omega_p V^2} = 2 E_F(0) / (n m v_b^2/2)$

$\ll 1$, as $n \gg n_b$

$\therefore \quad \langle f(v,t)\rangle \cong \langle f(v,0)\rangle$, to good approx.

but, for resonant velocities,

$\Rightarrow$ linear instability $\Rightarrow \partial\langle f\rangle/\partial v > 0$

$\Rightarrow D^R \underset{t\to\infty}{\to} 0 \quad \Rightarrow \quad \partial\langle f\rangle/\partial v < 0$

but have (for $D^R \to 0$) $\quad \langle f(t)\rangle = \langle f(0)\rangle$!

$\therefore$ $\boxed{\text{contradiction follows from assumption of } D^R(v,t) \to 0}$

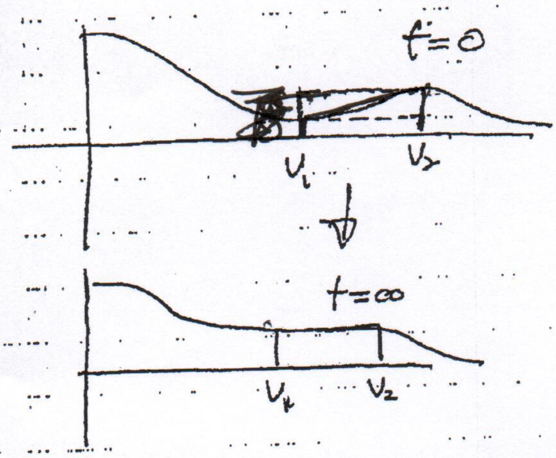
$\therefore$ have established that

$\partial\langle f\rangle/\partial v \Big\}_{res} \to 0 \quad \Rightarrow$ plateau forms!

For plateau formation, can immediately determine saturation levels from

$$\frac{\partial}{\partial t}(\mathrm{RPKED}) + \frac{\partial}{\partial t}(\mathrm{WED}) = 0$$

i.e.



t→∞

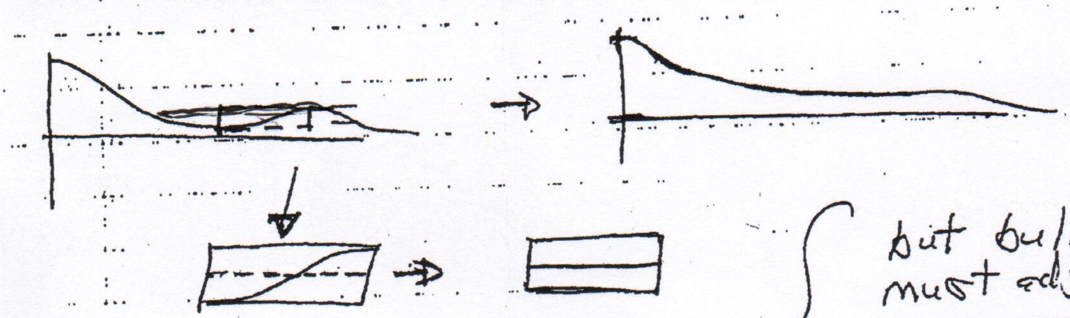
$k = \omega/v$

$$\Delta\left(\int_{v_1}^{v_2} \frac{mv^2}{2}\langle f\rangle\right) = -\Delta\int_{k_1}^{k_2} W_k \, dk$$

but $W_k = 2\,\mathcal{E}(k)$

$$\Rightarrow \quad \Delta\left(\int_{v_1}^{v_2} dv\, \frac{mv^2}{2}\langle f\rangle\right) = -2\Delta\int_{k_1}^{k_2} \mathcal{E}(k)\, dk$$

→ can estimate A (RPKED) analytically via construction



i.e. beam slows down

{ but bulk must adjust to conserve momentum!

i.e. bulk spreads outward, to conserve momentum as beam slows (bump flattened inward)

Now, for non-resonant particles:

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D^{NR} \frac{\partial \langle f \rangle}{\partial v}$$

$$= \frac{\partial}{\partial v} \frac{q^2}{m^2} \sum_k |E_k|^2 \frac{\gamma_k}{(\omega - kv)^2} \frac{\partial \langle f \rangle}{\partial v}$$

$\gamma > 0$

$$\simeq \frac{8\pi q^2}{m^2} \int dk\, \mathcal{E}(k) \frac{\gamma_k}{\omega_{pe}^2} \frac{\partial^2 \langle f \rangle}{\partial v^2}$$

so, using γ definition:

$$\frac{\partial \langle f \rangle}{\partial t} = \left(\frac{1}{nm} \frac{\partial}{\partial t} \int dk \; \mathcal{E}(k) \right) \frac{\partial^2 \langle f \rangle}{\partial v^2}$$

now define $T(t) = \frac{2}{n_e} \int dk \; \mathcal{E}(k,t)$

so $\Rightarrow$

$$\frac{\partial \langle f \rangle}{\partial T} = \frac{1}{2m} \frac{\partial^2 \langle f \rangle}{\partial v^2}$$

thus for initial Maxwellian:

$$\langle f \rangle = \left[m/2\pi \left[T + T(t) - T(0) \right] \right]^{1/2} \exp\left[\frac{-mv^2/2}{\left[T + T(t) - T(0) \right]} \right]$$

Thus, for non-resonant particles

- at saturation

$$T/2 \rightarrow T/2 + \frac{1}{n} \int dk \left[\mathcal{E}(k,\infty) - \mathcal{E}(k,0) \right]$$

i.e. electrons 'heated' by net increase in field energy

- can also note:

$$\frac{\partial}{\partial t}(RPKED) + \frac{\partial}{\partial t}(WED) = 0$$

for plasma waves,

$$\frac{\partial}{\partial t}(RPKED) = -2\frac{\partial}{\partial t}(FED)$$

so $\Delta(RPKED) = -2\Delta(FED)$

but $\Delta(PKED) = -\Delta(FED)$

so $\Delta(RPKED) = +2(\Delta(PKED))$

$$\Rightarrow 0 = \Delta(RPKED) + 2\Delta(NRPKED)$$

and,

$$\Delta(PKED) - \Delta(RPKED) = -\Delta(FED) - (-2)\Delta(FED)$$

$$\boxed{\Delta(NRPKED) = \Delta(FED)}$$

as shown above.

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→ heating is one-sided, to conserve momentum.

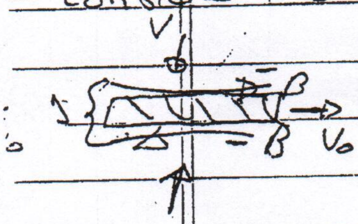
Applications II

Anomalous Resistivity → { Application of QLI } { Follows / expands on Galeev / Sagdeev Rev. Pl. Phys. 7.

→ an instructive and important example of quasilinear theory is anomalous resistivity

→ here, try approach classic current-driven ion acoustic instability (CDIA) model of anomalous resistivity via coupled micro-macro dynamics

– consider Sweet-Parker model, i.e.



$$VL = V_{out}\Delta$$

$$V_{out} = V_A$$

$\langle E\rangle = \langle \frac{v}{c} B\rangle < 0$ (into page)

$$2V\frac{B^2}{8\pi}L = \eta \bar{J}^2 L\Delta$$

cf. 218B notes

$$\frac{\Delta}{L} = \frac{V}{V_A} \Rightarrow \boxed{\Delta^2 = \frac{(L)(\eta)}{V_A}} \Rightarrow \text{layer width}$$

$$\frac{\Delta}{L} \sim 1/\sqrt{R_m}$$

What happens as η decreased?

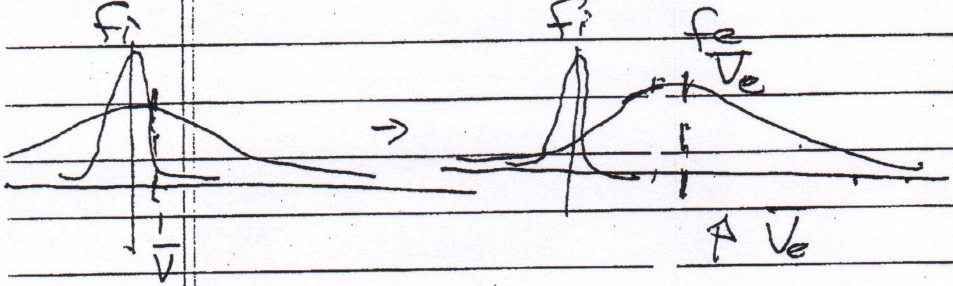
$$-\frac{c}{4\pi\Delta}B = J = nq\bar{V}_e$$

electron drift speed

so $\bar{V}_e = cB/4\pi n q\Delta = \frac{d_{skin}^2}{\Delta}\Omega_e$

Now, $\bar{v}_e \sim B/An \Rightarrow \bar{v}_e \uparrow$ as
- $\Delta_b \downarrow \Rightarrow$ narrow layer
- $n_b \downarrow \Rightarrow$ few charge carriers
- $B \uparrow \Rightarrow$ stronger field (drive)

$A \uparrow \Rightarrow$

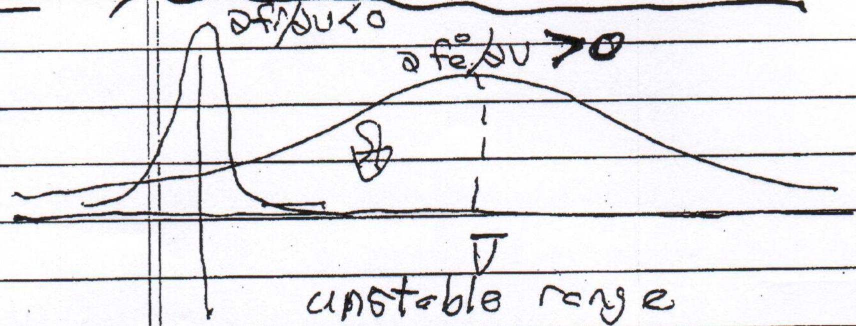


$\Rightarrow$ decreasing A raises $\bar{v}$

$\rightarrow$ up-shifts f_e centroid relative to f_i

$\Rightarrow$ destabilizes CDIA!

i.e. classic scenario of CDIA



∴ expect CDIA will:

$\rightarrow$ exchange momentum between electrons and waves

so $\rightarrow$ slow down electrons, reduce $\bar{v}_e$

$\rightarrow$ act as "anomalous" "turbulent" resistivity i.e. $\begin{cases} \Delta^2 = \frac{L}{V_A}\left(\eta + \eta_A(\bar{V})\right) \\ \quad \to \text{anomalous resistivity} \\ \bar{V} = \frac{cB}{4\pi} / nq\Delta \end{cases}$

How calculate:

① Brute Force

- confining oneself to 1D model, ignoring Layer structure, have:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m} E \frac{\partial f}{\partial V} = -C(f)$$

here $x \to$ vertical
$U \to$ vertical velocity

$\underline{\Delta\, V \| E}$

$\to$ electron velocity

vertical $\to$ $\perp$ to layer

$m_e V$ * $\to$

$$\frac{\partial \langle p_e \rangle}{\partial t} - q\langle E \int dv f \rangle = -\nu_{e,i} n_0 m_e \bar{V}_e$$

collisional loss to ions

$$\frac{\partial \langle p_e \rangle}{\partial t} - q n_0 \langle E \rangle - q\langle \tilde{E}\tilde{n} \rangle = -\nu_{e,i} n_0 m_e \bar{V}_e$$

so

$$\langle E \rangle + \frac{\langle \tilde{E}\tilde{n} \rangle}{n_0} - \frac{1}{n_0 q}\frac{\partial \langle p_e \rangle}{\partial t} = + \frac{\nu_{ei} m_e n_0 q \bar{V}_e}{n_0 q^2}$$

at ⓐ stationary state,

$$\langle E \rangle + \langle \tilde{E} \frac{\tilde{n}}{n_0} \rangle = \eta \langle J \rangle$$

$\langle E \rangle$: driving field $\sim \langle VB \rangle$

$\langle \tilde{E} \frac{\tilde{n}}{n_0} \rangle$: electron acceleration by turbulence ⇒ "anomalous resistivity"

$\eta \langle J \rangle$: collisional resistivity

to calculate:

$$\langle \tilde{E} \frac{\tilde{n}}{n_0} \rangle = \sum_k + ik \hat{\phi}_{-k} \frac{\tilde{n}_k^e}{n_0}$$

$$= \int dv \sum_k ik \hat{\phi}_{-k} \tilde{f}_k^e$$

$\tilde{f}_k^e$: electron density perturbation

$f_k^e \to f_k^{e\,L}$

- quasilinear calculation
- stationarity ⇒ resonant transport

ⓑ Conservation Argument

- as in ⓐ anticipate stationarity ⇒ resonant quasilinear evolution

- recall,

$$\frac{\partial}{\partial t}\left(\mathcal{E}^{RP} + \mathcal{E}^{wave}\right) = 0$$

$$\frac{\partial}{\partial t}\left(p^{RP} + p^{wave}\right) = 0$$

→ # quanta

$$\mathcal{E}^{w}_{k} = \omega_k \frac{\partial \epsilon}{\partial \omega}\Big|_k \frac{|E_k|^2}{8\pi} \equiv \omega_k N_k$$

$$p^{w}_{k} = \frac{k}{\omega} \mathcal{E}^{w}_{k} = k N_k$$

as waves (CDIA) electrostatic, can ignore field momentum

so, for resonant electrons:

$$\frac{\partial}{\partial t} p^{RP}_{e} = -\frac{\partial p^{w}}{\partial t} = -\sum_k (2\gamma^e_k)\frac{k}{\omega_k} \mathcal{E}^{w}_{k}$$

$\gamma^e_k \equiv$ electron (resonant) growth rate

but

$\frac{\partial}{\partial t} P^{RP}_{electron}$ → slowing down

→ macro-representation as effective collision frequency

so $\frac{\partial P^{RP}_{electron}}{\partial t} = -n m_e \underbrace{\nu_{eff}}_{\text{effective collision frequency}} \overline{V}$

slowing down by resonant scattering (resonant particle interaction)

$$\boxed{n m_e \nu_{eff} \overline{V} = \sum_k (2\gamma^e_k) \frac{k}{\omega_k} \mathcal{E}^{\omega}_k}$$

- defines $\overline{V}$
- for macro-micro link
  $\overline{V} = \frac{cB}{4\pi n q \Delta}$

* - n.b. if 2D, 3D theory, i.e. $\perp$ dynamics ⇒ non-resonant scattering ⇔ wave driven momentum flux i.e. $\Pi_{\perp,\parallel}$ → $\perp$ radiation $\parallel$ momentum. Relation to whistler interpretation of Bellan ?! There need include wave radiation in energy balance.

so now have,

$$\boxed{\begin{aligned} & n\, m\, \nu_{eff}(B_0, \Delta)\bar{V} = \sum_k (2\gamma_k^e)\frac{k}{\omega_k}\,\mathcal{E}_k^w \qquad (1) \\ & \Delta^2 = \frac{k}{V_A}\left(1 + \frac{c^3}{\omega_{pe}^2}\nu_{eff}\right) \end{aligned}}$$

⇒ need γ_k^e, $\mathcal{E}_k^w$ and $\langle f^e \rangle$ evolution

at simplest level, proceed via linear/quasilinear theory in 1D

- at more advanced level:
  - consider 1D phase space structures
    - → electron/ion clumps, momentum exchange
    - → electron scattering off ion hole
  - consider 3D $J_\parallel$ driven instability with electron viscosity

Now, proceed in usual fashion:

γ_k^e → linear theory

$\mathcal{E}_k^w$ → nonlinear saturation

$\langle f_e \rangle$ → QL equation — flattening

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For linear theory of CDIA ;

$$\nabla^2 \hat{\phi} = -4\pi n_0 |e| \left(\frac{\hat{n}_i}{n_0} - \frac{\hat{n}_e}{n_0} \right)$$

$$\hat{n}_i / n_0 = \frac{k^2 c_s^2}{\omega^2} \frac{|e|\phi}{T}$$

$$\frac{\hat{n}_e}{n_0} = \frac{|e|\hat{\phi}}{T} \left[1 - i\Gamma(k) \right]$$

$\Gamma(k)$,

$$\frac{\partial \tilde{f}}{\partial t} + v \frac{\partial \tilde{f}}{\partial x} = -\frac{|e|}{m_e} \tilde{E} \frac{\partial \langle f \rangle}{\partial v}$$

$$\tilde{f} = \frac{|e|\hat{\phi}}{T} \langle f \rangle + g$$

$$\frac{\partial \tilde{g}}{\partial t} + v \frac{\partial \tilde{g}}{\partial x} = -v \frac{\partial}{\partial x} \left(\frac{|e|\tilde{\phi}}{T} \langle f \rangle \right) + \frac{|e|}{m_e} \frac{\partial \tilde{\phi}}{\partial x} \frac{\partial \langle f \rangle}{\partial v}$$
$$- \frac{\partial}{\partial t} \left(\frac{|e|\tilde{\phi}}{T_e} \langle f \rangle \right)$$
$$= v \frac{\partial \tilde{\phi}}{\partial x} \frac{|e|}{T} \langle f \rangle + \frac{|e|}{\cancel{m_e}} \frac{\partial \phi}{\partial x} \left(-\frac{v - \bar{v}}{T/\cancel{m_e}} \right) \langle f \rangle - \frac{\partial}{\partial t} \frac{|e|\tilde{\phi}}{T_e} \langle f \rangle$$
$$= -\frac{\partial}{\partial t} \frac{|e|\tilde{\phi}}{T} \langle f \rangle + \bar{v} \frac{\partial}{\partial x} \frac{|e|\tilde{\phi}}{T} \langle f \rangle$$

$$\Rightarrow g_k = \frac{i(\omega - k\bar{v})}{-i(\omega - kv)} \frac{|e|\hat{\phi}_k}{T} \langle f \rangle$$

$$= -\left(\frac{\omega - k\bar{v}}{\omega - kv}\right) \frac{|e|\hat{\phi}_k}{T} \langle f \rangle$$

$$\omega_k^{2\,(0)} = \frac{k^2 c_s^2}{1 + k^2 \lambda_D^2}$$

$$-i\,r(k) = \int dv\, \frac{+(\omega - k\bar{v})}{(\omega - kv)} \langle f \rangle$$

$$\bar{f} = \frac{1}{\sqrt{\pi}} \exp\left[\frac{-(\omega/k - \bar{v})^2}{v_{Th}^2}\right]$$

$$= -\frac{(\omega - k\bar{v})(i\pi)}{|k| v_{Th}} \bar{f}\Big|_{\omega/kv_{Th}}$$

$$1 + k^2 \lambda_D^2 = \frac{k^2 c_s^2}{\omega^2} + \frac{(\omega - k\bar{v})}{|k| v_{Th}} (i\pi) \bar{f}\Big|_{\omega/kv_{Th}}$$

$$\omega \to \omega + \delta\omega$$

$$0 = -\frac{2\delta\omega}{\omega} + \frac{(\omega - k\bar{v})(-i\pi)}{|k| v_{Th}} \bar{f}\Big|_{\omega/kv_{Th}}$$

$$\frac{\delta\omega}{\omega} = -\frac{i\pi}{2} \frac{(\omega - k\bar{v})}{|k| v_{Th}} \bar{f}\Big|_{\omega/kv_{Th}}$$

$\delta\omega \Rightarrow i\gamma_k$ growth rate

(2) $$\boxed{\gamma_k \sim \frac{+\pi}{2} \omega_k \frac{(\omega - k\bar{v})}{|k| v_{Th}} \bar{f}\Big|_{\omega/kv_{Th}}} \Rightarrow$$

$\gamma > 0$ for $\bar{v} > c_s$

$\Rightarrow$ critical velocity

for $\langle f \rangle$ evolution,

$$\frac{\partial \langle f \rangle}{\partial t} = +\frac{\partial}{\partial v} \sum_k \frac{+|e|}{m_e} \tilde{E}_{-k} \tilde{g}_k$$

$$= +\frac{\partial}{\partial v} \sum_k \frac{|e|}{m_e} \tilde{E}_{-k} \left(\frac{-(\omega - k\bar{v})}{(\omega - kv)} \frac{|e|\tilde{\phi}_k}{T} \langle f \rangle \right)$$

$$= \frac{\partial}{\partial v} \sum_k \frac{T_e |e|}{m_e} \frac{+ik\tilde{\phi}_k}{T_e} \left(-(\omega - kv) i\pi\delta(\omega - kv) \right) \frac{|e|\tilde{\phi}_k}{T} \langle f \rangle$$

$$= \frac{\partial}{\partial v} \sum_k (-v_{th}^2) \left| \frac{|e|\tilde{\phi}_k}{T} \right|^2 k(\omega - k\bar{v}) \, \pi\delta(\omega - kv) \langle f \rangle$$

$$\boxed{\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v_k} \sum (-v_{th}^2) \left| \frac{|e|\tilde{\phi}_k}{T} \right|^2 k(\omega - k\bar{v}) \, \pi\delta(\omega - kv) \langle f \rangle} \quad (3)$$

– mean evolution

Note:

– really only assumed $\langle f \rangle = \langle f((v-\bar{v})^2/2v_{th}^2) \rangle$

so $\dfrac{\partial \langle f \rangle}{\partial v} = \left(\dfrac{v - \bar{v}}{v_{th}^2} \right) \langle f \rangle$

and

$\langle f \rangle' = -\langle f \rangle$

→ minimal assumption on structure

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- can write as $\bar{v}$ evolution

$$\bar{v} = \int dv\, v \langle f\rangle \Big/ \int dv \langle f\rangle$$

$$\Rightarrow \frac{\partial \bar{v}}{\partial t} = + \int dv \sum_k v_{Th}^2 \left|\frac{e\phi_k}{T}\right|^2 k^2 \left(\frac{\omega}{k} - \bar{v}\right) \pi\delta(\omega - kv) \langle f\rangle$$

$\omega/k < \bar{v} \rightarrow \partial\bar{v}/\partial t < 0$

$> \bar{v} \rightarrow \partial\bar{v}/\partial t > 0$

Remains to determine fluctuation intensity level

Generically, can write:

$$\frac{\partial}{\partial t}\mathcal{E}_k^w = \overset{(a)}{\gamma_k \mathcal{E}_k^w} - \left(\overset{(b)}{\sum_{k'} \omega_k C(k,k')} \frac{\mathcal{E}^w_{k'}}{nT}\right)\mathcal{E}_k^w$$

$$- \left(\sum_{k',k''} \overset{(c)}{\omega_k C_2(k,k',k'')} \frac{\mathcal{E}^w_{k'}}{nT} \frac{\mathcal{E}^w_{k''}}{nT}\right)\mathcal{E}_k^w$$

spectral equation constituents:

(a) - linear growth

(b) - quadratic nonlinearity $\rightarrow$ 3 wave coupling / NL ion-wave interaction

(c) - "cubic NL" $\rightarrow$ wave coupling

Nloong

Now, for ion-acoustic wave:

- 3 wave coupling effects negligible
  $\Rightarrow$ can't satisfy resonance
- NL wave-particle effects weak $\Rightarrow$ intrinsically

$\Rightarrow$ consider 4 wave process

$$\boxed{\frac{\partial \mathcal{E}_k^w}{\partial t} = \left[\gamma_k - \omega_k B(\omega, k) \left(\frac{\mathcal{E}_k^w}{nT} \right)^2 \right] \mathcal{E}_k^w} \quad (5)$$

- 'cartoon' NL saturation equation

Now, (1) - (5) $\Rightarrow$ $\begin{cases} \text{coupled, non-stationary} \\ \text{micro-macro system} \end{cases}$

$\Rightarrow$ describe anomalous resistivity dynamics and its effect on reconnection

$\Rightarrow$ coupled solution corresponds to solution of the problem

(1)
$$\begin{cases} n m_e \nu_{eff}(B,\Delta)\,\overline{V} = \sum_k 2\gamma_k^e \frac{k}{\omega_k} \mathcal{E}_k^w \\ \Delta^2 = \frac{L}{V_A}\left(M + \frac{c^2}{\omega_{pe}^2}\nu_{eff}\right), \quad \overline{V} = cB/4\pi n q \Delta \end{cases}$$

(2)
$$\gamma_k^e = -\frac{\pi}{2}\,\omega_k \frac{(\omega - k\overline{V})}{|k| v_{Th}} \overline{F}\Big|_{\omega/k v_{Th}}$$

(4)
$$\frac{\partial \overline{V}}{\partial t} = \int dv \left(\sum_k v_{Th}^2 \left|\frac{e\tilde{\phi}_k}{T}\right|^2 k^2 \left(\frac{\omega}{k} - \overline{V}\right) \pi \delta(\omega - kv) \langle F \rangle\right)$$

(5)
$$\frac{\partial \mathcal{E}_k^w}{\partial t} = \left[\gamma_k - \omega_k\, B(\omega,k)\left(\frac{\mathcal{E}_k^w}{nT}\right)^2\right]\mathcal{E}_k^w$$

Now, stationarity $\Rightarrow$

$$\mathcal{E}_k^w = nT\,(\gamma_k/\omega_k B)^{1/2}$$

$$\gamma_k = +\frac{\pi}{2}\left(\frac{\overline{V} - C_s}{|k v_{Th}|}\right) k\omega_k \overline{F}\Big|_{\omega/k v_{Th}}.$$

So, for scalings:

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$$\nu_{eff} = \frac{1}{nm\bar{V}} \sum_k 2\gamma_k^e \frac{k}{\omega_k} \mathcal{E}_k^{\omega}$$

$$\sim \frac{1}{nm\bar{V}} \frac{(\bar{V}-c_s)}{|k|v_{Th}} k\omega_k \bar{f}\Big|_{\frac{\omega_k}{kv_{Th}}} \frac{k(nT)}{\omega_k}\left(\frac{\gamma_k}{\omega_k B}\right)$$

$$\sim \frac{(1-c_s/\bar{V})}{nm} \frac{k^2 nT}{|k|v_{Th}} \left(\frac{\gamma_k}{\omega_k B}\right) \bar{f}\Big|_{\frac{\omega_k}{kv_{Th}}}$$

$$\sim (1-c_s/\bar{V})\, \bar{f}\Big|_{\omega/kv_{Th}} \left(\frac{k^2 v_{Th}}{|k|}\right)\left(\frac{\gamma_k}{\omega_k B}\right)$$

$$\sim (1-c_s/\bar{V}) \left(\frac{k\omega_k}{|k v_{Th}|} \frac{(\bar{V}-c_s)}{\omega_k B} \bar{f}\right)^{1/2} \bar{f}\, \frac{k^2 v_{Th}}{|k|}$$

$$\sim \left[(\bar{V}-c_s)k\right]^{3/2} \bar{f}^{3/2} \frac{k(v_{Th}/\bar{V})}{|k|\,|k v_{Th}|^{1/2}}$$

$$\boxed{\nu_{eff} \sim \frac{\left[(\bar{V}-c_s)k\right]^{3/2}}{|k v_{Th}|^{1/2}} \frac{v_{Th}}{\bar{V}} \bar{f}^{3/2} \left(\frac{k}{|k|}\right)}$$

[Turbulent collision frequency

so have

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$$\Delta^2 = \frac{L}{V_A}\left(\eta + \frac{c^2}{\omega_{pe}^2}\nu_{eff}\right)$$

$$\bar{V} = cB_0 / 4\pi n_0 q \Delta$$

$$\nu_{eff} = \frac{\left[(\bar{V}-C_S)k_\parallel\right]^{3/2}}{|k v_{Th}|^{1/2}} \frac{v_{Th}}{\bar{V}} \bar{F}^{3/2} \left(\frac{k}{|k|}\right)$$

with:

$$\frac{\gamma_k}{\omega_k} = \frac{\pi}{2}(\bar{V}-C_S)\frac{k}{|k|v_{Th}}\bar{F}$$

$$\bar{F} = \frac{1}{\sqrt{\pi}}\exp\left[-(\omega/k_\parallel - \bar{V})^2/2v_{Th}^2\right]$$

→ characterize micro-macro coupling with anomalous resistivity

→ now can envision structure

- finite current, $\Delta \sim (L\eta/V_A)^{1/2}$

  $\nu_{eff} = 0$

so if:

- decrease η ⇒ Δ decreases

- Δ decreases ⇒ $\bar{V}$ increases

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- $\bar{V}$ increases $\Rightarrow \gamma_{\parallel} > 0$

- $\gamma_{\parallel} > 0 \Rightarrow \begin{cases} \Sigma_{\parallel}^{\omega} > 0 \\ \nu_{eff} > 0 \end{cases}$

- $\nu_{eff} > 0 \Rightarrow \begin{cases} \Delta \text{ increases} \\ \bar{V} \text{ decreases} \end{cases}$

$\Rightarrow$ decreasing ν so Δ decreases triggers feedback so Δ increases $\Rightarrow$ self regulation / $\oplus$ feedback

$\therefore$ in this model, can expect:

- at low $\nu_{collisional}$, so $\bar{V}\big|_c \sim cB_0 / 4\pi n_0 e \Delta_c \sim C_s$

$\Rightarrow$ CDIA "hovers" near marginal stability

- for stronger drive (i.e. $B_0 \uparrow$)

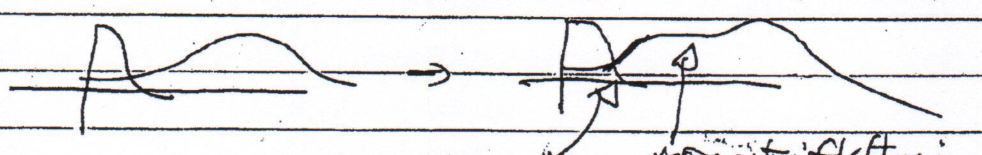
$\Rightarrow$ ion interaction important

$\Rightarrow$ strong ion distortion possibly significant

$\Rightarrow$ granulation formation important

$\Rightarrow$ distortion of electron distribution function need be considered

i.e.



58. → hyper-resistivity

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⇒ useful extensions:

- 1D avalanche model → avalanching ⇒ jitter effects on Δ, $\overline{V}$

- nonlinear noise effects via fluctuations

- 2D, 3D ⇒ $\boxed{\text{wave radiation, esp. wave momentum flux} \perp \text{layer}}$

* - granulation effects ⇒ strong distortion hole (cef. later in course)

Comment:

This simple problem is surprisingly poorly understood. Excellent example of:

→ micro-macro feedback

→ self-regulation

→ marginal stability.

Mean Field Theory - DW. → Transport

Consider DKE → simplicity

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} - \frac{c}{B_0}\nabla\phi\times\hat{z}\cdot\nabla f - \frac{|e|}{m_e}E_z\frac{\partial f}{\partial v_z} = 0$$

(i.e. electrons) → $\underline{B_0} = B_0\hat{z}$ strt.

then

$$\partial_t\langle f\rangle + \underbrace{\partial_r\langle \tilde{v}_r\,\delta f\rangle}_{\text{transport}} + \underbrace{\partial_{v_z}\langle \tilde{a}_z\,\delta f\rangle}_{\text{heating}} = 0$$

n.b. $\delta f = \frac{|e|\phi}{T}\langle f\rangle + \tilde{h}$

In linear theory, with Maxwellian:

$$\partial_t\left(\frac{|e|}{T}\phi\langle f\rangle + \tilde{h}\right) + v_z\partial_z\left(\cancel{\frac{|e|\phi}{T}}\langle f\rangle + \tilde{h}\right)$$

$$- \frac{c}{B_0}\nabla\tilde{\phi}\times\hat{z}\cdot\nabla\langle f_0\rangle - \frac{c}{B_0}\nabla\tilde{\phi}\times\hat{z}\cdot\nabla\left[\cancel{\frac{|e|\tilde{\phi}}{T}}\langle f\rangle + \tilde{h}\right]$$

$$- \frac{|e|}{m_e}E_z\partial_{v_z}\left[\cancel{\langle f\rangle} + \frac{|e|\tilde{\phi}}{T}\langle f\rangle + \tilde{h}\right] = 0$$

ignoring parallel nonlinearity

$$\partial_t \tilde{h} + v_z \partial_z \tilde{h} - \frac{c}{B} \nabla\phi \times \hat{z} \cdot \nabla \tilde{h} - \frac{|e|}{m_e} E_z \partial_{v_z} \tilde{h}$$

$$= -\partial_t \frac{|e|\phi}{T} \langle f \rangle - \frac{c}{B_0} \nabla\phi \times \hat{z} \cdot \nabla \langle f_0 \rangle$$

$$(\omega - \omega_*)$$

Scattering,

$$\delta f^{(0)} = \frac{\phi_k}{\omega - k_z v_z} L_k \langle f \rangle$$

$$L_k = -\frac{c k_\theta}{B_0} \frac{\partial}{\partial r} + \frac{|e|}{m_e} k_z \frac{\partial}{\partial v_z}$$

scattering operator

$$\partial_t \langle f \rangle = \overset{①}{\partial_r D_{r,r} \partial_r \langle f \rangle} + \overset{②}{\partial_r D_{r,v} \partial_{v_z} \langle f \rangle}$$

$$+ \overset{③}{\partial_{v_z} D_{v_z,r} \partial_r \langle f \rangle} + \overset{④}{\partial_{v_z} D_{v,v} \partial_{v_z} \langle f \rangle}$$

$$D_{r,r} = \mathrm{re} \sum_{k} \frac{c^2}{B_0^2} k_\theta^2 |\phi_k|^2 \frac{i}{\omega - k_z v_z}$$

symmetry owing to $\underbrace{\quad}$

$$D_{r,v} = D_{v,r} = \mathrm{re} \sum_{k} \frac{c}{B_0} \frac{|e|}{m_e} \overbrace{k_\theta k_z} |\phi_k|^2 \frac{i}{\omega - k_z v_z}$$

$$D_{v,v} = \mathrm{re} \sum_{k} \frac{e^2}{m_e^2} k_z^2 |\phi_k|^2 \frac{i}{\omega - k_z v_z}$$

1 + 2 $\rightarrow$ radial transport

i.e. $\sim \partial_r \Gamma_{v,r}$

4 $\rightarrow$ parallel heating

Illuminating to examine momentum flow

$$\partial_t \int \langle f \rangle v_z = \partial_r \Big[\overset{①}{\int dv_z\, v_z D_{r,r} \partial_r \langle f \rangle} \overset{②}{+ \int dv_z\, v_z D_{r,v_z} \partial_{v_z} \langle f \rangle} \Big]$$
$$\overset{③}{- \int dv_z\, D_{v_z,r} \partial_r \langle f \rangle}$$
$$- \int dv_z\, D_{v_z,v_z} \partial_{v_z} \langle f \rangle$$

① → χ_ϕ, etc. ; Π_{resid}
$\partial_r\langle f\rangle \to \partial_r\langle T\rangle$ etc.

② → Π_{resid}, also.

i.e. $\Pi_{resid} = \int dv_z \; v_z \sum_k \frac{c}{B}\frac{|e|}{m} k_\theta k_\theta |\tilde{\phi}_k|^2 \frac{\partial}{\omega - k_z v_z} \partial_{v_z}\langle f\rangle$ ✓

③, ④ → Turbulent acceleration
(aka' Li Wang, P.D.)

③ $= -\int dv_z \; D_{v_z,r} \; \partial_r\langle f\rangle$

$= -\int dv_z \sum_k \frac{c}{B}\frac{|e|}{m} \underbrace{k_\theta k_z}_{\text{symmetry}} |\tilde{\phi}_k|^2 \frac{\partial}{\omega - k_z v_z} \partial_r\langle f\rangle$

and <u>not</u> ~ $\nabla \cdot \Pi$, $\partial_r\langle f\rangle \to$ driver flow.

④ → another piece turbulent source

→ Correlation Times

→ Useful here to explore correlation times

$$D_{\perp,k} = \sum_{k} \frac{c^2}{B_0^2} |\phi_k|^2 k_\theta^2 \frac{\partial}{\omega - k_z v_z}$$

$$|\phi_k|^2 = |\phi_0|^2 \left(\frac{\Delta k_\theta}{(k_\theta - \bar{k}_\theta)^2 + (\Delta k_\theta)^2} \right) \frac{\Delta k_z}{(k_z^2 + \Delta k_z^2)}$$

$$D_{\perp,t} \sim \frac{c}{\omega_{\bar{k}_\theta} - \bar{k}_z v_z + i\left[|\Delta k_\theta| \left|\frac{d\omega}{dk_\theta}\right| + |{-v_z}| |\Delta k_z| + \frac{d\omega}{dk_z} \Delta k_z \right]}$$

$$\sim \frac{c}{\omega_{\bar{k}} - \bar{k}_z v_z + i |\Delta k_\theta| \left|\frac{d\omega}{dk_\theta}\right| - |v_z| |\Delta k_z| \, c}$$

$$\boxed{1/\tau_{ac} \sim |\Delta k_\theta| \left|\frac{d\omega}{dk_\theta}\right| + \left|\frac{\omega}{k_z}\right| |\Delta k_z|}$$

resonant

- contrast 1D
- extra dof $\rightarrow$ $\boxed{\text{reduced resn. disp.}}$
- if non res: $\Delta k_\theta \left|\frac{d\omega}{dk_\theta}\right| - |v_z| \Delta k_z$

Other cases

→ precession:

$$D = \frac{c^2}{B_0^2} \sum_n \frac{n^2}{r^2} |\overline{\tilde{\phi}_n}|^2 \frac{\hat{c}}{\omega - \omega_D}$$

as before

$$\frac{\hat{c}}{\omega - \omega_D} \sim \frac{\cancel{\hat{c}}}{\cancel{\hat{c}} \left|\frac{d\omega}{dk_\theta}\right| \Delta k_\theta - \overline{V}_D \epsilon \Delta k_\theta}$$

$$\epsilon \sim \frac{\omega}{\omega_D} \qquad \sim \frac{1}{\left|\frac{d\omega}{dk_\theta}\right| \Delta k_\theta - \cancel{V_D} |\Delta k_\theta| \frac{\omega}{k_\theta \cancel{V_D}}}$$

$$\sim \frac{1}{|\Delta k_\theta| \left[\left|\frac{d\omega}{dk_\theta}\right| - \frac{\omega}{k_\theta} \right]}$$

→ à la' 1D → dispersion sensitive

→ CTEM tend to run $k_\theta \uparrow$, as

$\gamma_{ac} \uparrow$.

→ **Relaxation**

$$\frac{k_\theta}{m} v_z \to \frac{k_\theta}{m_0} \frac{\omega}{v_z}$$

from resonance

$$L_k \langle f \rangle = \frac{k_\theta}{\Omega_0} \frac{\partial}{\partial r} \langle f \rangle + \frac{\omega_k}{v_z} \frac{\partial}{\partial v_z} \langle f \rangle$$

and

$$\partial_t \langle f \rangle = \sum_k \left[L_k \, |\phi_k|^2 \pi \delta(\omega - k_z v_z) \, L_k \right] \langle f \rangle$$

relaxn:

$$L_k \to 0$$

$$-\frac{k_\theta}{\Omega_0} \frac{\partial}{\partial r} \langle f \rangle + \frac{\omega}{v_z} \frac{\partial}{\partial v_z} \langle f \rangle \to 0$$

$$\left(\frac{-k_\theta}{\Omega_0} \frac{1}{\Delta r} + \frac{\omega / v_z}{\Delta v_z} \right) \langle f \rangle \to 0$$

$$\Delta r - \Delta \frac{v_z^2}{2} \frac{k_\theta}{\Omega_0 \omega_k} = 0$$

$$\Delta \left(r - \frac{v_z^2}{2} \frac{k_\theta}{\Omega_0 \omega_k} \right) = 0$$

$$\boxed{\begin{aligned} \Delta r &= \frac{\Delta V_z^2}{2}\,\frac{k_\theta}{\Omega_e \omega_*} \\[1em] r - \frac{V_z^2}{2}\,\frac{k_\theta}{\Omega_e \omega_*} &= 0 \end{aligned}}$$

→ i.e. $\left\{ \begin{array}{l} \text{any } \Delta r > 0 \text{ displacement} \Rightarrow \\ \text{heating } (\Delta V_z^2 > 0) \end{array} \right.$

how QL saturate without $DN \to 0$

→ DW expends energy on heating via Landau damping ⇒ route for non-trivial QL satn.

→ $\Delta V_z^2 \sim \Delta r \, \frac{\Omega_e \omega_*}{k_\theta}$ $\cancel{k_\theta}\ \frac{\rho_e V_{Te}^2}{L_n \cancel{\Omega_e}}$

$$\boxed{\frac{\Delta V_z^2}{V_{The}^2} \sim \frac{\Delta r}{L_n}\, V_{Te}^2}$$

Lesson

- DW's $\underline{\underline{\text{do}}}$ distort distribution function

- relevant to ITG near marginal.

and Energetics:

- Now DKE / QL

$$\partial_t \langle E_{kin} \rangle + \frac{\partial}{\partial r} Q_e - \langle E_z J_z \rangle = 0$$

radial transport

P-Thm

$$\partial_t W_{DW} + \partial_r S_r + \langle E_z J_z \rangle_R = 0$$

$$\Rightarrow \langle E_z J_z \rangle_R = \partial_t \langle E_{kin} \rangle_R + \partial_r Q_{e\,R}$$

$\therefore$

$$\boxed{\partial_t \langle E_{kin} \rangle_R + \partial_t W_{DW} + \partial_r \left(Q_{e,R} + \partial_r S_r \right) = 0}$$

RP + wave, as before

transport

wave transport

coupling

n.b. $S_r \longleftrightarrow Q_{e,R}$

$S_r \sim V_{gr} \, \mathcal{E}_{DW}$ $\rightarrow$ ZF

Dominant Balance:

$$\partial_t \langle E_{kin} \rangle_R + \partial_r Q_{e\,R} = 0$$

N.B. show: $\Gamma_e = \Gamma_i$

res NR.

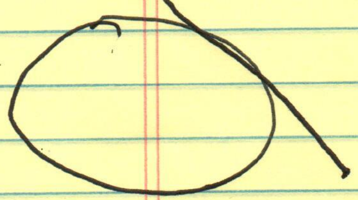
→ PV Mixing + G-D Thms

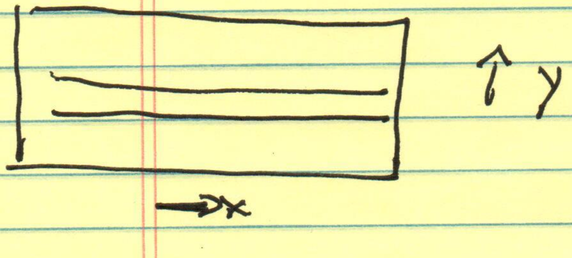
Closely related to Vlasov Dynamics
is β-Plane / QG

$$\frac{\partial}{\partial t}(\nabla^2\phi + \beta y) + \langle v\rangle\cdot\nabla(\nabla^2\phi + \beta y)$$
$$+ \tilde{v}\cdot\nabla(\nabla^2\phi + \beta y) - \nu\nabla^2(\nabla^2\phi + \beta y)$$
$$= 0$$

$$\partial_t \nabla^2\phi + \langle v\rangle_x \partial_x \nabla^2\phi + \underline{\tilde{v}}\cdot\nabla\nabla^2\phi$$
$$-\nu\nabla^2\nabla^2\phi = -\tilde{v}_y(\beta + \langle\nabla^2\phi\rangle)$$

Mean vorticity





obeys:

$$\partial_t \zeta + \underline{v}\cdot\nabla\zeta = +\nu\nabla^2\nabla^2\zeta$$

$$q = PV = \nabla^2\phi + \beta y$$

fluid, local

planetary

potential vorticity

→ 2D conservative advection, according Hamiltonian system

$$\frac{dx}{dt} = -\nabla\phi \times \hat{z}$$

→ N.B.:

$$\frac{dq}{dt} = 0 +$$

$$q = \begin{cases} \nabla^2\phi & \rightarrow 2D \\ \nabla^2\phi + \beta y & \rightarrow QG \\ \ln\frac{n}{n_0} + \phi - \nabla^2\phi & \rightarrow HM \\ + \\ \vdots \end{cases}$$

⟺

$$\frac{df}{dt} = 0 \quad , \quad \partial_t f + v\,\partial_x f + \frac{q}{m} E\,\partial_v f = 0$$

$$H = \frac{mv^2}{2} + q\phi$$

$$f = \langle f \rangle + \delta f$$

→ A key result: Taylor Identity

$$\langle v_y \nabla^2 \phi \rangle_x = \langle \partial_x \phi \, (\partial_x^2 \phi + \partial_y^2 \phi) \rangle_x$$

$$= \cancelto{0}{\left\langle \partial_x \frac{(\partial_x \phi)^2}{2} \right\rangle_x}$$

$$+ \langle \partial_x \phi \, \partial_y^2 \phi \rangle_x$$

$$= 0 + \partial_y \langle \partial_x \phi \, \partial_y \phi \rangle - \cancelto{0}{\langle \partial_{xy} \phi \, \partial_y \phi \rangle}$$

$$= \partial_y \underbrace{\langle \partial_x \phi \, \partial_y \phi \rangle_x}_{R\text{-}S} - \left\langle \partial_x \cancel{\frac{(\partial_y \phi)^2}{2}} \right\rangle_x$$

$$\to \; = - \partial_y \langle \tilde{v}_y \tilde{v}_x \rangle$$

vorticity (+ PV) flux = Reynolds Force.

N.B. Need 1 direction of symmetry.