PKU Lectures – 2019-2020

Magnetic Fluctuations and Resonances

Statics $\underline{k} \cdot \underline{B} = 0$ Physics 235 Spring 2019

→ Stochastic Fields

→ Heat Transport

→ Fields, Flows and Particle Transport

→ Marginal Cells / Percolation

Dynamics $k_{\parallel} + k_{\parallel}' \to 0$

→ Turbulence, MHD Turbulence I
Richardson

→ MHD Turbulence II
Richardson.

→ Helicity and Taylor Relaxation

Lecture I

Physics 235

Prologue

What is this course about?

- transport in disordered / random / turbulent system, different regimes
- system evolution due such transport, i.e. relaxation

Topics:

- Diffusion & beyond:
  → transport in stochastic magnetic fields
  - scattering + collisions → what is origin of irreversibility
  - $Ku < 1$ to $Ku > 1$
  → shear dispersion, cellular arrays, random media
  → Percolation and $Ku > 1$ transport
  → Intermittency: (beyond Fokker-Planck)
  → fractals, multi-scaling ↔ scaling, power laws
  → Hurst exponent (correlations)
  → Fat tails, Levy Flights

→ CTRW and Fractional Kinetics

→ Relaxation → Avalanching.

→ Self-Organized Criticality (SOC)

→ Traffic Flow, Jams

→ Models of SOC

→ Turbulence spreading and avalanching

→ Selected Topics, TBD.

why?
⇒ RMP
⇒ islands
⇒ divertors
→ :

1.

Turbulent Transport

"How many magnetic field lines in the universe?" → 1

I) Case Study: Transport in Stochastic Fields (Buried bodies in QLT).

A) Review – Basics of Hamiltonian Chaos (cf. Ott, and other supplementary material)

Ott, Chapt. 7

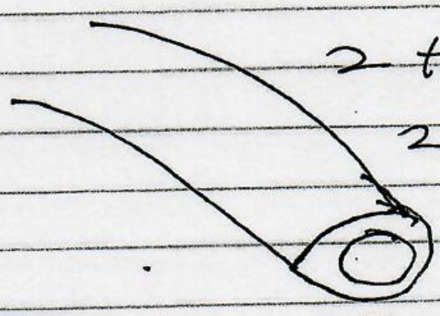
If integrable system, can write:

$H = H_0(\underline{J})$

$\underline{J} \equiv$ action variable
$\underline{\theta} \equiv$ angle variable

so $\frac{d\underline{\theta}}{dt} = \frac{\partial H}{\partial \underline{J}} = \underline{\omega}(\underline{J})$

$\frac{d\underline{J}}{dt} = 0$



trajectories lie on toroidal surfaces.

For 2-torus, have:

$\omega_1/\omega_2 = P/Q$ → rational number
closed trajectory

$\omega_1/\omega_2 =$ irrational → ergodic trajectory, fills surface

recall: Poincaré's recurrence....

Surfaces where $\omega_1 / \omega_2 = p/q$ are 'rational' surfaces, and define natural resonances of system

Now if perturb:

$$H = H_0(\underline{J}) + \epsilon H_1(\underline{J}, \underline{\theta})$$

then must implement perturbation theory such that canonical structure maintained, so ΔS (correction to action) needed:
→ perturbation in Liouville eqn.

$\underline{\Delta S \text{ of}}$

and $\Delta S \sim \epsilon H_1(\underline{J})_{\underline{m}} / \underline{\omega} \cdot \underline{m}$

$\underline{m} \cdot \underline{\omega} = 0$ → "small denominator problem" → resonance.

$\omega - kv$

what happens? → central issue in chaos theory

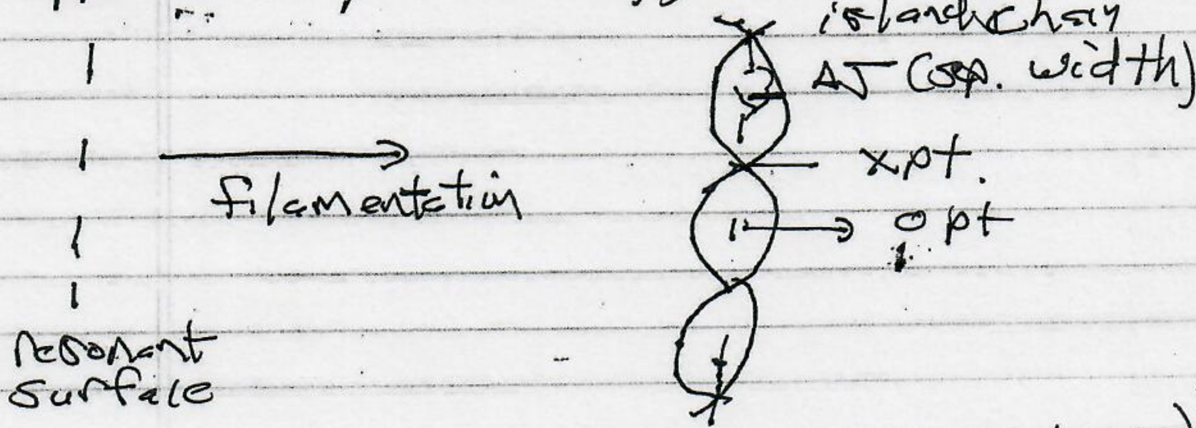
Small denominator problem ⟷ resonance phenomena (n.b. akin Landau resonance)

i.e. $m \omega_1 + n \omega_2 = 0$

$$m/n = -\omega_2/\omega_1 = -q/p$$

pitch of perturbation ... pitch of trajectory

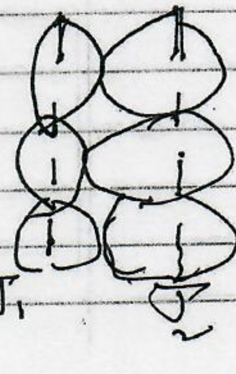
Now, can (for single resonance) resolve small denominator problem by secular perturbation theory (see supplementary notes), so



$$\Delta J \sim \left(\underbrace{\epsilon H_1}_{\text{perturbation strength}} \Big/ \underbrace{\partial\omega/\partial J}_{\text{shear (diff'nt'l rotation in phase space)}} \right)^{1/2}$$

lines on perturbed surfaces

Now, this fix-up works in the region of a single resonance. But if resonances overlap, i.e.



trajectories:
- wander in radius
- fill volume, not surface
- ⇒ chaos results

Chaos:

- trajectory separation exhibits linear instability, exponentially growing

i.e. $J_1 - J_2 = \Delta J_0 \, e^{\gamma t}$

$\Rightarrow$ 1 (at least) Lyapunov exponent > 0

- chaotic motion $\Rightarrow$ statistical approach for prediction / characterization

$\Rightarrow$ Fokker-Planck Eqn.

or

$\Rightarrow$ Hamiltonian dynamics (Liouville Thm) + chaos

$\therefore$ Quasilinear eqn. $(f \rightarrow \text{pdf})$

(F-P and QLT equiv. for Hamiltonian $\nabla \cdot V_n = 0$)

N.B.: Approaches limited to $Ku \ll 1$

- criterion (working) for chaos:

Chirikov overlap!

~~spacing~~ island width

$$\frac{\Delta J_1 + \Delta J_2}{|J_1 - J_2|} > 1$$

spacing

(good working criterion)

i.e. islands $\frac{\Delta W_1 + \Delta W_2}{|\Omega_2 - \Omega_1|} > 1$

- KAM theorem is concerned with ruggedness of irrational surfaces but chaos onset concerned with rational surfaces.

Prime example: Field Lines in Torus

- magnetic field lines + perturbation

$$\hat{B}_r = \sum_{m,n} B_{m,n}\, e^{i(m\theta - n\phi)}$$

- seek D_M → diffusivity of field lines in chaotic regime

but who cares about "lines"?! → seek impact on

- heat, particle, momentum transport

and

- is chaotic dynamics always diffusive ?!

i.e. $Ku \equiv l_{ac}\, \delta B/B \,/\, \Delta r$ <1 / >1

$\Downarrow$

Kubo #. What of $Ku > 1$?

<u>Line Wandering / Diffusion</u>

if $f = f(r, \theta, z)$ → line density
i.e. magnetic flux

then, $\underline{B} \cdot \underline{\nabla} f = 0$ i.e. $\frac{dr}{B_r} = \frac{r\,d\theta}{B_\theta} = \frac{B_z\, dz}{B_0}$
z as time

so if $\underline{B} = B_0 \hat{z} + B_\theta(r)\hat{\theta} + \tilde{B}_r \hat{r} + \tilde{B}_\theta \hat{\theta}$

B_0: toroidal, strong; $B_\theta(r)$: poloidal

then
- Hamiltonian system
- f → phase space density.

$\frac{dr}{dz} = \frac{\tilde{B}_r}{B_0}$

$r\,d\theta/dz = \frac{\langle B_\theta \rangle + \tilde{B}_\theta}{B_0}$

$$B_0 \partial_z f + \frac{B_\theta(r)}{r} \partial_\theta f + \underline{\tilde{B}} \cdot \nabla \tilde{f} = 0$$

$$\partial_z f + \frac{B_\theta(r)}{B_0 r} \partial_\theta f + \frac{\underline{\tilde{B}}_\perp}{B_0} \cdot \underline{\nabla} \tilde{f} = 0$$

$$\Rightarrow \partial_z f + \frac{1}{R q(r)} \partial_\theta f + \frac{\underline{\tilde{B}}}{B_0} \cdot \underline{\nabla} f = 0$$

N.B.: $z \rightarrow$ plays role of time
- periodicity of fast scale perturbations
- irreversibility of $\langle f \rangle$ evolution

$\theta \rightarrow$ periodic

<u>2 scale</u>

so, for $\langle f \rangle$,

$$\partial_z \langle f \rangle + \frac{\partial}{\partial r} \left\langle \frac{\tilde{B}_r}{B_0} \tilde{f} \right\rangle = 0$$

$$\Gamma_{r,B} = \left\langle \frac{\tilde{B}_r}{B_0} \tilde{f} \right\rangle \quad \text{so Fick's Law.}$$

$\Gamma_{r,B}$ flux of line density

How close?

Now, characteristics of Liouville Eqn. $\Rightarrow$ equations of lines

$$\frac{dr}{B_r} = \frac{r\,d\theta}{\langle B_\theta(r) \rangle + \tilde{B}_\theta} = \frac{dz}{B_{z0}}$$

so radial excursion given by:

$$\langle \delta r \delta r \rangle = \int_0^{l^+} dl_+ \int_0^{l_-} dl_- \langle b_r(l_+ - l_-) b_r(l_+ + l_-) \rangle \frac{\delta}{}$$

$$\sim \int^{l^+} |b_r|^2 l_{ac} \, dl_+$$

$$dr/dz = \tilde{B}_r / B_0$$

so $\delta r \cong \int_0^{l} (\tilde{B}_r / B_0) \, dz$

$$\delta r = \int_0^{l'} \tilde{b}_r \, dz$$

$$\delta r = \int_0^{l''} \tilde{b}_r \, dz$$

Now, line trajectory de-coheres from perturbation for $l > l_{ac}$

→ autocorrelation length

sum

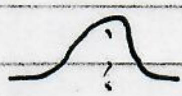
$- |\Delta k_\parallel| z$

e

$l_{ac} \cong 1 / |\Delta (k_\parallel)|$ i.e. inverse spatial bandwidth

$\therefore \boxed{\delta r \cong l_{ac} \tilde{B}_r / B_0} \Rightarrow$ {line excursion in 1 l_{ac}

Can identify $\Delta_r \equiv$ scatterer radial correlation length (i.e. spatial spectral width) — radial coherence length.



then.

$$K_u \cong \delta r / \Delta_r \cong \frac{l_{ac}}{\Delta_r} \tilde{B}_r / B_0 \Rightarrow \text{Kubo \#}$$

and can then posit:

→ $K_u < 1 \Rightarrow$ many kicks in coherence length

⇒ diffusion process

$\left\{ \begin{array}{l} \boxed{Ku \sim 1 \rightarrow \text{B.B.K. "natural state of EM turbulence"}} \\ Ku \sim 1 \rightarrow \text{critical balance} \end{array} \right.$

$\rightarrow Ku > 1$ $\rightarrow$ more than one Δ_r in $k_\theta \tilde{b}_r$
$\rightarrow$ strong scattering $\Leftrightarrow$ percolation.

QLT

Here $Ku \lesssim 1$, at first. So, proceed via Quasilinear theory.

$$\Gamma_M = \left\langle \frac{\tilde{B}_r}{B_0} \tilde{f} \right\rangle$$

$$= \sum_{\underline{k}} \frac{\tilde{B}_{r-\underline{k}}}{B_0} \tilde{f}_{\underline{k}}$$

$$-i\left(k_z - \frac{k_\theta B_\theta}{B_0}\right)\tilde{f}_{\underline{k}} = -\tilde{B}_{r_{\underline{k}}} \frac{\partial\langle f\rangle}{\partial r}$$

so

$$\Gamma_M = -D_M\, \partial\langle f\rangle/\partial r$$

$$D_M = \sum_{\underline{k}} \left|\frac{\tilde{B}_{r\underline{k}}}{B_0}\right|^2 \pi\delta(k_z - k_\theta B_\theta/B_0)$$

magnetic diffusivity

$$= \sum_{\underline{k}} \left|\frac{\tilde{B}_{r\underline{k}}}{B_0}\right|^2 \pi\delta(k_\parallel)$$

(RSTZ '66)

$$\sim \left\langle \left(\frac{\delta B_r}{B_0}\right)^2 \right\rangle l_{ac}$$

What is l_{ac}?

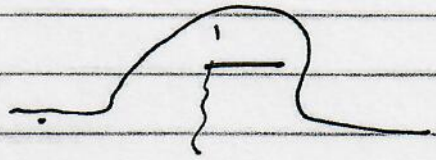
l_{ac}

N.B.: $\sum_{\underline{n}} = \sum_{m,n}$ spatial spread scatterers $\Rightarrow l_{ac}$

$n = \frac{m}{q}$, $dn = \frac{m}{q^2} q'/dx$

$\Rightarrow$ spatial scale of spectral width (Δr) sets $|k_{ll}| \sim \left|\frac{k_\theta \Delta r}{L_s}\right|$

$$\boxed{l_{ac} \sim L_s / |k_\theta \Delta r|}$$



Lines then diffuse as:

$\langle \delta r^2 \rangle \sim D_M Z$

Broader Perspective $\rightarrow$ Orbit averaging

N.B. Line Liouville eqn can be obtained by reducing / simplify in DKE
(*) $\rightarrow v_D$ large

$$\frac{\partial f}{\partial t} + v_{||} \hat{n}_0 \cdot \underline{\nabla} f + \underline{v}_D \cdot \underline{\nabla} f - \frac{c}{B_0} \underline{\nabla} \phi \times \hat{z} \cdot \underline{\nabla} f$$
$$+ v_{||} \frac{\delta B_\perp}{B_0} \cdot \underline{\nabla} f - \frac{|e|}{m_e} E_{||} \frac{\partial f}{\partial v_{||}} = C(f)$$

$$\Rightarrow \quad \hat{n}_0 \cdot \nabla f + \frac{\delta B_\perp}{B_0} \cdot \nabla f = 0 \quad \checkmark$$

Scales

Now, scales:

l_{ac} → (scatters)
→ field line memory length
self-coherence of scattering field.

l_c → line decorrelation length
(length over which line scattered from its upos)

i.e. $$r \frac{d\theta}{dz} = \frac{B_\theta(r)}{B_0}$$

but r scattered, ⇒

$$\frac{dy}{dz} = B_\theta(r_0) + \frac{B_\theta'(r_0)}{B_0} \delta r$$

so

$$\frac{d}{dz} \delta y \cong \frac{B_\theta''(r_0)}{B_0} \delta r$$

$$\langle \delta y^2 \rangle = \left\langle \left(\int^z \left(\frac{B_\theta'}{B_0} \right) \delta r \, dz' \right)^2 \right\rangle$$

$\Rightarrow$

$$\langle \delta y^2 \rangle \sim \frac{B_\theta'^2}{B_0^2} z^2 \langle (\delta r)^2 \rangle$$

$$\sim \frac{B_\theta'^2}{3B_0^2} D_M z^3$$

akin

$$\langle \delta x^2 \rangle \sim D_v \frac{\tau^3}{3} \quad \text{in 1D}$$

For orbit decorrelation length:

$$k_\theta^2 \langle \delta y^2 \rangle \sim k_\theta^2 \frac{B_\theta'^2}{3B_0^2} D_M z^3$$

$\Rightarrow$

$$l_{c_K} \sim \left(k_\theta^2 \frac{B_\theta'^2}{3B_0^2} D_M \right)^{-1/3}$$

$$\sim \left(\frac{k_\theta^2}{L_s^2} \frac{D_M}{3} \right)^{-1/3}$$

analogous to shear dispersion $\left(\frac{k_\theta^2 V'^2 D}{3} \right)^{1/3}$

Also: orbit exponentiation length (separation) → $\boxed{\text{stretching}}$

show via 2pt. $\langle \delta f(1)\, \delta f(2) \rangle$

chaotic → stretching