Date

PKU Lectures – Lecture VII

Lecture VII :

From Violent Relaxation to

Phase Space Density Holes

→ 'Coherent Structures' in Phase Space.

Recall:

→ classes of galaxies emit light fitting isothermal sphere, but $\tau_{evoln.} \ll \tau_{coll.}$!!

→ "Violent relaxation" ⟷ phase mixing, chaos

→ dynamical time scale not collisional

i.e $\tau_A = \left(n^2 \left(\frac{\partial \Omega}{\partial J}\right)^2 \nu\right)^{-1/3}$

In galaxies, violent relaxation driven by gravity ⟷ i.e. Jean's modes. Time scale?

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Plan

→ Review → : violent relaxation - gravitational collapse time scale

→ What of Vlasov time scale, structure.
  - gravitating blob
  - plasma

→ Phase space density hole as coherent structure

Aside :

- what makes coherent structure coherent ?

- difference of coherent structure, statistical entity ?

Friday

→ Hole as most probable BGK .

→ Hole aggregation .

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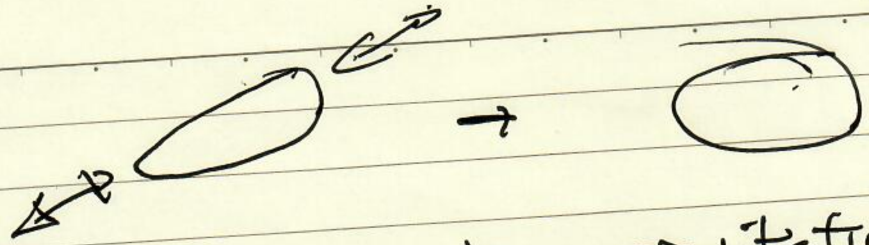
i.e. 

~ large scale gravitational field

$\Rightarrow$ potential $\phi(x,t) \rightarrow \psi(x,t)$

if $\epsilon = m\left(\frac{1}{2}v^2 - \psi\right)$

$$\frac{d\epsilon}{dt} = -m\,\partial_t \psi$$

$$\tau_{Relax} \approx \left[\left\langle\left(\frac{d\epsilon}{dt}\right)^2\right\rangle / \epsilon^2\right]^{-1/2}$$

$$\approx \left[(\partial\psi/\partial t)^2/\epsilon^2\right]^{-1/2}$$

How does ψ change?

aside

$\Rightarrow$ Virial Theorem: (see notes)

$$I_{ij} = \int d^3x\, \rho\, x_i x_j$$

$\Rightarrow$ substitution, using fluid equations:

akin moment of inertia

this brings us to ....

→ Virial Theorems in MHD

- what is a virial theorem
- why yet another theorem?

→ Virial Theorems are:

- space/time averaged energy theorems
- "lumped parameter" relations for energies in complex, multi-element interacting systems
- useful for 'back-of-envelope' estimates, etc.
- logically extend the moment program:

$$\underbrace{f(\underline{x}, \underline{v}, t)}_{\text{phase space fluid}} \xrightarrow[\underline{v}\text{ moments}]{} \underbrace{n(\underline{x}, t), \underline{v}, T}_{\text{position space fluid}} \xrightarrow[\text{virial integrals}]{} \underbrace{E_v,\ E_B,\ \text{etc.}}_{\#}$$

virial integrals → space integrals

Before proceeding:

¿ Can an isolated blob of MHD plasma confine itself without self gravity ?

Easily answered by Virial Theorem....

Recall, for system of particles, Virial theorem derived by considering:

$$\frac{d}{dt}\left(\sum_i \underline{p}_i \cdot \underline{x}_i\right) = \sum_i \underline{p}_i \cdot \dot{\underline{x}}_i + \sum_i \dot{\underline{p}}_i \cdot \underline{x}_i$$

$$= \underset{\substack{\text{kinetic}\\ \text{energy}}}{2T} + \underset{\text{via Newton's Law}}{\sum_i \left(-\frac{\partial U}{\partial \underline{x}_i}\right) \cdot \underline{x}_i}$$

Now, if $\sum_i \underline{p}_i \cdot \underline{x}_i$ bounded,

$$\left\langle \frac{d}{dt}\sum \underline{p}_i \cdot \underline{x}_i \right\rangle = \frac{1}{T}\int_0^T dt \frac{d}{dt}\left(\sum_i \underline{p}_i \cdot \underline{x}_i\right)$$

$$\underset{T \to \infty}{\longrightarrow} 0$$

so ...

→ (First) Virial of system

$$2\langle T\rangle = \left\langle \sum_i \frac{\partial U}{\partial \underline{x}_i}\cdot \underline{x}_i \right\rangle$$

Further, if $U = U(\underline{x}_1, \underline{x}_2, \dots \underline{x}_n)$

where $U(\alpha \underline{x}_1, \alpha \underline{x}_2, \dots \alpha \underline{x}_n) = \alpha^k U(\underline{x}_1, \underline{x}_2, \dots \underline{x}_n)$

(scaling ⟺ structure of power-law potentials → i.e. h.o. → $k=2$, Coulomb → $k=-1$)

↓

homogeneous function

$$\Rightarrow \boxed{2\langle T\rangle = k\langle U\rangle}$$

but of course:

$$T + U = \langle T\rangle + \langle U\rangle = E$$

then $\left(\frac{k}{2}+1\right)\langle U\rangle = E$

$$\boxed{\langle U\rangle = \frac{2}{k+2}E}, \quad \langle T\rangle = \frac{kE}{k+2}$$

check: $k=2$, $\langle U\rangle = \frac{1}{2}E$, $\langle T\rangle = \frac{1}{2}E$

$k=-1$, $\langle T\rangle = -E$ ✓ ⇒ (bounded motion only if total energy negative (i.e. bound state))

($\Rightarrow E<0$)

Aside: Simplest realization of negative specific heat 'paradox', i.e.

R → consider 'blob' of self gravitating matter

$$E \sim -1/R$$

if radiation  → E decreases ⟷ R decreases

∴ $(-E)$ increases ⟶ kinetic energy ⟹ $\langle T \rangle$ increases

but $\langle T \rangle \sim$ temperature, so have cycle of: radiative cooling ⟹ temperature increase!

$$\Rightarrow \; c < 0 \;!?$$

c ⟶ specific heat

In the days before the discovery of nuclear fusion this was thought to be what heated stars. Kelvin, in particular, was a proponent.

Now, proceeding to full Virial theorem ...

→ Consider equation of motion

$$\underbrace{\frac{\partial}{\partial t}(\rho v_i)}_{\text{momentum}} = -\frac{\partial}{\partial x_j}\underbrace{T_{ij}}_{\underline{\text{full}}\text{ stress tensor}}$$

$$T_{ij} = \rho v_i v_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \cancel{\rho \phi \delta_{ij}}$$

Now, recalling relation of virial to $\frac{d}{dt}(\underline{p}\cdot\underline{x})$
⇒ consider:

$$I_{ij} = \int d^3x\, \rho\, x_i x_j \qquad (\sim \text{moment of inertia})$$

↦ Virial theorem is for tensor...

and
$$\frac{d}{dt} I_{ij} = \int d^3x\, \frac{\partial \rho}{\partial t} x_i x_j$$

$$= -\int d^3x\, \frac{\partial}{\partial x_t}(\rho v_t)\, x_i x_j$$

integrating by parts, assuming ρ compact (i.e. 'blob' of interest)

$$= \int d^3x\, [\rho x_i v_j + \rho x_j v_i]$$

so

$$\frac{d^2 I_{ij}}{dt^2} = \int d^3x \left[x_i\left(\frac{\partial}{\partial t}\rho v_j\right) + x_j \frac{\partial}{\partial t}(\rho v_i)\right]$$

but $\frac{\partial}{\partial t}(\rho V_i) = -\frac{\partial}{\partial x_k} T_{ik}$

$\Rightarrow$

$$\frac{d^2 I_{ij}}{dt^2} = -\int d^3x \left[x_i \frac{\partial T_{j,t}}{\partial x_t} + x_j \frac{\partial T_{i,t}}{\partial x_t} \right]$$

and integrating by parts, assuming $\left\{ \begin{array}{l} \text{compact blob,} \\ \text{no external linkage} \end{array} \right.$

$\Rightarrow$

$$\frac{d^2 I_{i,j}}{dt^2} = +\int d^3x \left[\delta_{i,t} T_{j,t} + \delta_{j,t} T_{i,t} \right]$$

$\partial x_i / \partial x_t = 0$ unless $i = t$

$$= +\int d^3x \left[T_{j,i} + T_{i,j} \right]$$

and as T_{ij} manifestly symmetric $\Rightarrow$

$$\left\{ \begin{array}{l} \frac{1}{2} \frac{d^2 I_{ij}}{dt^2} = +\int d^3x \, T_{ij} \\ T_{ij} = \rho V_i V_j + \left(P + \frac{B^2}{8\pi} \right) \delta_{ij} - \frac{B_i B_j}{4\pi} + \rho \phi \delta_{ij} \end{array} \right.$$

note unlike simple pt particle example, time dependence remains.

— tensor virial theorem.

Now, to make contact with notions of energy, etc., useful to contract the tensor

$$I = I_{ii} = \mathrm{tr}\, I_{ij}$$

repeated indexes summed

$\mathrm{tr}\,(\mathrm{V.T.}) \Rightarrow$

$$\mathrm{tr}\, \frac{1}{2} \frac{d^2 I_{ij}}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x\, \frac{\rho x^2}{2}\right)$$

$$= \mathrm{tr} \int d^3x \left[\rho v_i v_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho\phi\, \delta_{ij} \right]$$

$$= \int d^3x \left[\rho v^2 + 3\left(p + \frac{B^2}{8\pi}\right) - \frac{B^2}{4\pi} + 3\rho\phi \right]$$

$\therefore \quad I \equiv \int d^3x\, \rho x^2/2 \Rightarrow$

$$\boxed{\frac{d^2 I}{dt^2} = \int d^3x \left[\rho v^2 + 3p + \frac{B^2}{8\pi} + 3\rho\phi \right]}$$

$\rightarrow$ Scalar Virial Theorem.

Now, first neglect self-gravitation ⇒

$$\frac{d^2 I}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x \, \frac{\rho x^2}{2}\right)$$

$$= \int d^3x \left[\rho v^2 + 3p + B^2/8\pi\right]$$

Now → can an isolated blob of MHD fluid confine itself ??

If 'self-confined' ⇒ $\frac{dI}{dt} \leq 0$

$\frac{d^2 I}{dt^2} \leq 0$

i.e. quiescent ⇒ $\dot{I}, \ddot{I} = 0$

stable pulsation ⇒ $\ddot{I} = -\Omega^2 I < 0$

but have $\ddot{I} = \int d^3x \left[\rho v^2 + 3p + B^2/8\pi\right]$

so even if $v^2 = 0$ (no fluid motion in blob) ⇒

$p > 0$, $B^2/8\pi > 0$ ⇒ $\ddot{I} > 0$!

∴ No → isolated blob can't confine itself.

More generally, noting that

$$E_V = \int d^3x\, \rho V^2/2$$

$$E_P = \int d^3x\, \frac{P}{\gamma - 1} = \frac{3}{2}\int d^3x\, P \qquad (\text{gas})$$

$$E_B = \int d^3x\, \frac{B^2}{8\pi}$$

can write scalar Virial theorem in form:

$$\boxed{\frac{d^2 I}{dt^2} = 2E_V + 2E_P + E_B}$$

simple relation in terms energies.

Aside: $\Rightarrow$ So, isolated blob can't confine itself

$\Rightarrow$ how is
- tokamak $\rightarrow B_T$ for stability; not transport
- or - better macro-confinement
- RFP $\rightarrow$ weak external B_T guide (negligible)

confined ?? Confinement by wall is unacceptable ...

Answer: → toroidal plasma tends to expand toroidally

→ held in place by $\left\{ \begin{array}{l} \text{conducting shell (often undesireable) or} \\ \text{" vertical field"} \end{array} \right.$

i.e.

→ additional external P_{Mag} to oppose toroidal expansion — vertical field

→ image currents in close-in conducting shell can do likewise.

JET anecdote re: vertical field failure ...

Now, retaining self-gravitation:

$$T_{ij}\Big|_{\text{gravity}} = \rho\phi\,\delta_{ij} = 2\underbrace{\left(\frac{\rho\phi}{2}\right)}_{\mathcal{E}_{\text{gravity}}}\delta_{ij}$$

to calculate: $\nabla^2\phi = 4\pi G\rho$

$$\Rightarrow \quad \phi = -G\int d^3x\,\frac{\rho(x')}{|x-x'|}$$

so

$$T_{ij}\Big|_{\text{gravity}} = T\Big|_{\text{gravity}} \delta_{ij}$$

$\Rightarrow$

$$T = -\frac{G}{2}\int d^3x \int d^3x' \frac{\rho(\underline{x})\,\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

$$= + E_{\text{gravitation}} \equiv \quad E_g < 0$$

so scalar Virial theorem becomes, with gravity $\Rightarrow$

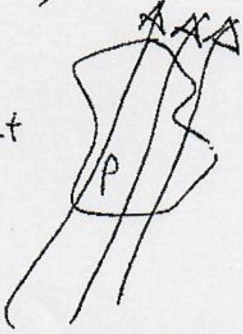
$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2E_V + 2E_P - |E_g| \quad + E_B}$$

so $\underbrace{\text{with gravity}}$, can have self-confining blob
(no surprise ...)

This brings us to another application of virial theorems, namely proto-stellar cloud collapse ....

– now, consider a plasma cloud/blob

P_{ext}



- mass M, radius R
- threaded by B
- pressure p, external pressure P_0
- no bulk motion
- frozen flux

now, easy to show for $\ddot{I} = 0$, $\underline{V} = 0$, must have:

$$2E_p - |E_g| + E_B = \underbrace{\int dA\, P_{ext}\, \hat{x}\cdot\hat{n}}_{\text{external pressure}} - \underbrace{\int dA\, \underline{x}\cdot\underline{\underline{T}}_B\cdot\hat{n}}_{\text{magnetic stress thru surface (threading fields)}}$$

surface terms

Now, can estimate:

$$M = \int \rho\, dV \rightarrow \text{total mass}$$

$$E_p \cong c_s^2 M$$

$$|E_g| \cong \underbrace{\alpha}_{\text{form factor}} \frac{GM^2}{R}$$

For frozen flux, $\Phi \sim \pi R^2 B$

so $$E_B + \int dA\, \underline{x} \cdot \underline{T}_B \cdot \hat{n} \sim \beta \Phi^2 / R$$

$\Rightarrow$ have: (eliminating extraneous factors):

$$\boxed{R^2 P_{ext} \sim \left(\frac{\beta \Phi^2}{R} - \alpha \frac{GM^2}{R} + \frac{3}{2} c_s^2 M \right)}$$

$\rightarrow$ scalar virial theorem for cloud....

Now! $$P_{ext} \sim \left(\frac{\beta \Phi^2}{R^3} - \alpha \frac{GM^2}{R^3} + \frac{3}{2} \frac{c_s^2 M}{R^2} \right)$$

$\rightarrow$ if Φ, $G \rightarrow 0$ $\rightarrow$ need $P_{int} = P_{ext}$ for confinement.....

$\rightarrow$ if $\Phi = 0$

$$P_{ext} = -\alpha \frac{GM^2}{R^3} + \frac{3}{2} \frac{c_s^2 M}{R^2}$$

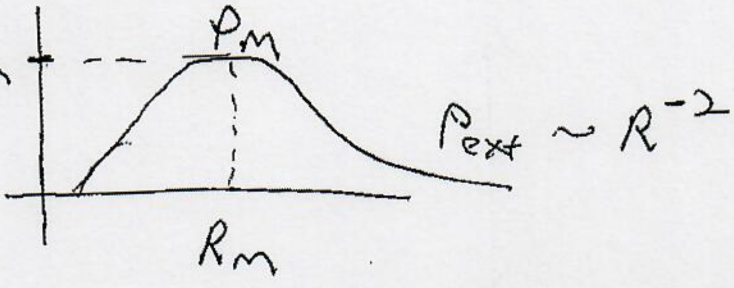
$$dP/dR = 0 \Rightarrow 3\alpha \frac{GM^2}{R^4} = 3 \frac{c_s^2 M}{R^3}$$

$$R_{max} = GM\alpha / c_s^2$$

[Note: $\Rightarrow R_m^2 = (G\rho / c_s^2)^{-1/2} \Rightarrow L^2_{Jeans}$]

54.

so P_{ext} needed



- $P > P_{max}$ $\Rightarrow$ no equilibrium
- $R < R_{max}$ $\Rightarrow$ P_{ext} must decrease to maintain equilibrium $\Rightarrow$ instability to gravitational collapse!

- $\Phi \neq 0$ (magnetic field on ...) $\rightarrow$ note immediately that magnetic support scales similarly to gravitational attraction

$\Rightarrow$

$$P_{ext} \sim \left[(\beta \Phi^2 - \alpha G M^2)/R^3 + \frac{3}{2} \frac{c_s^2 M}{R^2} \right]$$

so key point is: $(\beta \Phi^2 - \alpha G M^2) \lessgtr 0$?!

$$\Rightarrow M \gtrless M_\Phi = \sqrt{\beta/\alpha}\, \Phi / G^{1/2}$$

- $M < M_\Phi$ $\rightarrow$ magnetically subcritical mass for gravitational collapse

$M > M_\Phi$ → magnetically super-critical mass for collapse.

i.e, $M < M_\Phi$ → repulsive effects $\left\{ \begin{matrix} \text{field} \\ \text{thermal} \end{matrix} \right\}$ pressure always win
$(M_\Phi^2 - M^2 > 0)$ → no ammount of external compression can induce indefinite contraction, IF flux remains frozen in

$M > M_\Phi$ → sufficient external pressure/compression can induce gravitational collapse, even if flux frozen in.
$(M_\Phi^2 - M^2 < 0)$

[Note: IF kinetic energy contribution, NL Alfven waves can support cloud.]

For perspective, recall:

- (famous) Chandrasekhar Mass
  - $M > M_{Chandra}$ → collapse
  - $M < M_{Chandra}$ → no collapse.

$M_{Chandra}$ derived for degenerate Fermi gas equation of state → $\gamma = 4/3$, instead of $\gamma = 5/3$.

- if flux-freezing $\Rightarrow$ $\Phi \sim B R^2$
$\rho R^3 \sim M$

$\Rightarrow B \sim R^{-2} \Rightarrow B \sim \rho^{2/3}$

$\therefore B^2 \sim P_{Mag} \sim \rho^{4/3}$

$\Rightarrow$ <u>if</u> flux frozen, field obeys equation of state like Fermi gas

(i.e. flux freezing is akin to exclusion, albeit on field-lines-per-fluid-element)

$\therefore$ an analogue to Chandrasekhar mass seems quite plausible ...

Aside: Chandrasekhar Limit — Simple Derivation
(c.f.: Shapiro, Teukolsky)

→ suppose: N Fermions in star of radius R

$\therefore$ $n_{\text{fermion}} \sim N/R^3$

$\therefore$ Vol./Fermion $\sim 1/n$ (Pauli exclusion)

$p \sim \hbar/\Delta x \sim \hbar n^{1/3}$ (Heisenberg Uncertainty)

p → Fermion Momentum

⇒ Fermion energy (per Fermion) : $E_F = pc \sim \hbar c \dfrac{N^{1/3}}{R}$

replaces: (i.e. Thermal energy)

Gravitational Energy (per Fermion) : $E_{Grav} \sim -\dfrac{GM}{R} m_b$

m_b → Baryon mass

$M \sim N\ m_B$

Pressure → electron
Mass → Baryon

$\therefore$ $E = E_F + E_G$

$= \dfrac{\hbar c N^{1/3}}{R} - \dfrac{G N m_B^2}{R}$

note: $E = E_F + E_G$

$$= \frac{\hbar c N^{1/3}}{R} - \frac{G N m_B^2}{R}$$

$E > 0 \Rightarrow$ decrease E, E_F by increasing R.

but as $E_F \downarrow$, electrons non-relativistic,
$\therefore E_F \sim 1/R^2 \rightarrow$ eqbrm.

$E < 0 \Rightarrow$ decrease E without bound by decreasing R $\Rightarrow$ collapse.

$\therefore$ eqbrm: $\hbar c N^{1/3} = G N m_B^2$

$$N_{Max} = \left(\hbar c / G m_B^2\right)^{3/2} \sim 2 \times 10^{57} \quad \text{(proton)}$$

$$\therefore \quad M_{chandra} = N_{max} m_B \sim 1.5 M_\odot$$

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$$\frac{1}{2}\frac{d^2}{dt^2} I_{ij} = \int d^3x \, T_{ij}$$

$$T_{ij} = \rho v_i v_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho\psi\delta_{ij}$$

shows: isolated plasma blob can't confine itself

gravity

so, neglecting MHD and thermal pressure (stars ↓) :

$$\tfrac{1}{2}\ddot{I} = 2T + V$$

$$I = \sum m r^2$$

KE

$$E = T + V$$

eqbrm $\ddot{I} = 0$

$E = T + V$

$0 = 2T + V$

$$\Rightarrow \boxed{\begin{aligned} T &= - [redacted] E \\ V &= 2E \end{aligned}}$$

$$\left[T = -\frac{V}{2} \right.$$

away eqbrm $T \sim V/2$ oscillation

but $V \sim \tfrac{1}{2}\sum(PE)$

mutual

$$\tfrac{1}{2} m v^2 \sim \tfrac{1}{4} m \psi$$

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$$\tau_{Relax} \sim \left\langle \frac{(\partial_t \psi)^2}{\sigma^2} \right\rangle^{-1/2}$$

$$1/2\, m v^2 \sim \frac{3}{4} m \psi$$ mutual interaction

$$\sigma \sim -3/4\, \psi$$

$$\Rightarrow \tau_{relax} \sim \frac{3}{4} \left\langle \dot{\psi}^2 / \psi^2 \right\rangle^{-1/2}.$$

$$\sim \tau_{\underset{\underline{(global)}}{vibration}}.$$

What would $\tau_{vibr.}$ be ?

$\Rightarrow \tau_{Jeans}$!

i.e. if gas:

$$\partial_t \rho + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

$$\rho(\partial_t \underline{v} + \underline{v} \cdot \underline{\nabla}\, \underline{v}) = -\underline{\nabla} P + \rho \nabla \phi$$

$$\nabla^2 \psi = 4\pi G \rho$$

$$\Rightarrow \omega^2 = k^2 C_s^2 - 4\pi G \rho_0$$

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if ignore pressure:

$$\tau_{relax} \sim (4\pi G\rho)^{-1/2} / 2\pi$$

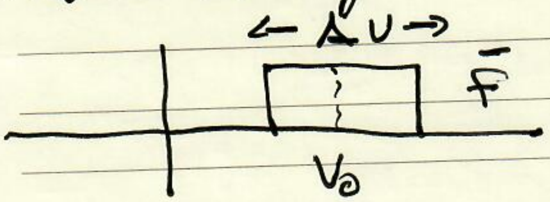
can derive from Vlasov Eqn → show.

Suggests: → relaxed state is self-bound by
- gravity - matter
- electrostatic field - plasma.

→ time scale - collapse.

→ what would localized state look like? in phase space

i.) Gravity blob



Marginal state?

$$\nabla^2 \phi = -4\pi G \int dv$$

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \nabla\phi \frac{\partial f}{\partial v} = 0$$

so:

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$$f = \bar{f} + \tilde{f}$$

$$\partial_t \tilde{f} + v \frac{\partial \tilde{f}}{\partial x} = - \partial_x \psi \, \frac{\partial f_0}{\partial v}$$

$$\tilde{f} = \frac{i k \psi_n}{-i(\omega - k v)} \left[f_0 \left(\delta\left(v_0 - \frac{\Delta v}{2}\right) - \delta\left(v_0 + \frac{\Delta v}{2}\right) \right) \right]$$

$$-k^2 \cancel{\psi_n} = 4\pi G \int \left[\frac{k \psi_n}{\omega - k v} f_0 \left(\delta\left(v_0 - \frac{\Delta v}{2}\right) - \delta\left(v_0 + \frac{\Delta v}{2}\right) \right) \right]$$

$$+k^{\cancel{2}} = +4\pi G f_0 k \left[\frac{1}{\omega - k\left(v_0 - \frac{\Delta v}{2}\right)} - \frac{1}{\omega - k\left(v_0 + \frac{\Delta v}{2}\right)} \right]$$

$$1 = -\frac{4\pi G f_0}{\cancel{k}} \left[\frac{\Delta v \cancel{k}}{(\omega - k v_0)^2 - \left(\frac{\Delta v}{2} k\right)^2} \right]$$

$\Rightarrow$

gravitational attraction

$$(\omega - k v_0)^2 = -4\pi G \left[f_0 \Delta v \right] + \left(\frac{k \Delta v}{2} \right)^2$$

self-dispersion

$k \Delta v \rightarrow$ streaming apart
dispersion

$G f_0 \Delta v \rightarrow$ self-gravity

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A blob of matter (phase space chunk), size Δv, will self-bind at:

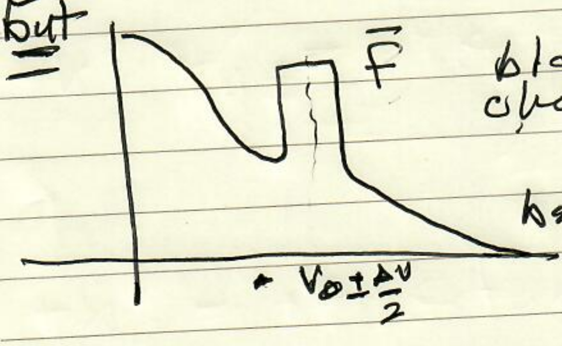
$$\boxed{\bar{f}_0 \sim \Delta v}$$

Now, explore for similar case for plasma, with background Maxwellian:

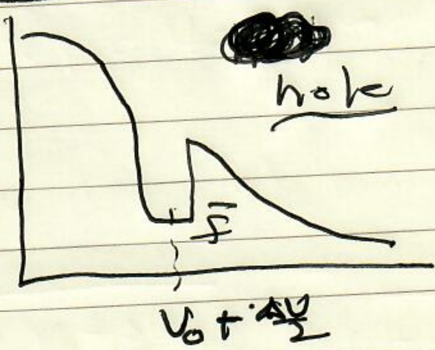
stable

Obviously here blob [illegible]

but



blob/clump in vs background



$$\left| \frac{\langle f_{max} \rangle - \bar{f}}{\langle f_{max} \rangle} \right| < 1. \qquad \frac{\delta f}{\langle f_M \rangle} < 1$$

(modest hole blob)

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int dv \, \frac{\partial/\partial v \langle f \rangle}{\omega - kv}$$

$$= 1 + \frac{\omega_p^2}{k} \int dv \, \frac{\partial/\partial v \left[\langle f_{max} \rangle + \bar{f} \right]}{\omega - kv}$$

$$= \epsilon_0(k,\omega) + \frac{\omega_p^2}{k} \int dv \, \frac{\frac{\partial}{\partial v} \bar{f}}{\omega - kv}$$

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Bare Blob Plasma

$$1 + \frac{\omega_p^2}{k}\int dv \frac{\bar{F}\left[\delta\left(v_0 - \frac{\Delta v}{2} - v\right) - f_0\,\delta\left(v - \left(v_0 + \frac{\Delta v}{2}\right)\right)\right]}{\omega - kv}$$

$= 0$

$$1 + \frac{\omega_p^2 \bar{F}}{k}\left\{\frac{1}{\omega - k\left(v_0 - \frac{\Delta v}{2}\right)} - \frac{1}{\omega - k\left(v_0 + \frac{\Delta v}{2}\right)}\right\}$$

$$1 = -\frac{\omega_p^2 \bar{F}}{\cancel{k}} \frac{-\cancel{k}\Delta v}{(\omega - kv_0)^2 - k^2\frac{\Delta v^2}{2}}$$

$$\bar{\omega}^2 = k^2\left(\frac{\Delta v}{2}\right)^2 + \underbrace{\omega_p^2 \Delta v \bar{F}}_{\sim \Omega^2}$$

stable

i.e. compress $\Rightarrow$ @ plasma oscillation

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for hole:

$$0 = \epsilon_0 (k, kv_0) + \frac{\omega_p^2}{k} \int_{v_0-\Delta v/2}^{v_0+\Delta v/2} \frac{dv}{\omega - kv} \Big[\overbrace{-\delta \bar{f}\, \delta(v-(v_0-\tfrac{\Delta v}{2}))}^{\partial f/\partial v < 0}$$

$$\qquad + \overbrace{\delta \bar{f}\, \delta(v-(v_0+\tfrac{\Delta v}{2}))}^{\partial f/\partial v > 0} \Big]$$

$$-\epsilon (k, kv_0) = -\frac{\omega_p^2 \delta \bar{f}}{\cancel{k}} \frac{\cancel{k}\Delta v}{(\omega - kv_0)^2 - \left(\frac{k\Delta v}{2}\right)^2}$$

$$(\omega - kv_0)^2 = \left(\frac{k\Delta v}{2}\right)^2 + \frac{\omega_p^2 \delta \bar{f} \Delta v}{\epsilon(k, kv_0)}$$

need $\delta \bar{f} < 0$ for $\epsilon(k, kv_0) > 0$

but in bulk of distribution,

$$\epsilon (k, kv_0) \cong 1 + \frac{1}{k^2 \lambda_D^2}$$

$v_0 < v_{th}$

$\epsilon > 0$

$\Rightarrow$ **hole in phase space is self-bound state.**

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i.e.

is Jeans marginal, with;

$$\Delta V = -\frac{\omega_p^2 \, \delta \bar{f}}{k^2} \Big/ \epsilon(k, kV_0)$$

$$\boxed{\Delta V = \frac{-\omega_p^2 \, \delta \bar{f}}{\left(\frac{1}{\lambda_D^2} + k^2\right)}}$$

blob $\rightarrow \delta \bar{f} > 0$, need $\epsilon < 0$.

variable sign
Screening $\Rightarrow$ new element beyond Gravity

$\Rightarrow$ ~ Simple calculation suggests that phase space density hole is self-bound coherent structure.

n.b.: $\delta \bar{f} > 0$ self-binding requires $\epsilon < 0$.

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Aside:

- "Coherent structure"?

- what? $\tau_{Life} > \tau_c$

identifiable ⇔ output of filter.

hole : $\langle \delta f^3 \rangle = S < S_{crit}$

Should have idea of what makes a coherent structure coherent?, i.e. what explains long τ_{Life}

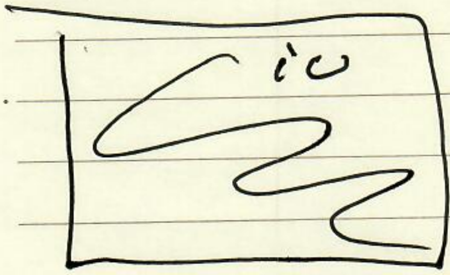
phase space

- hole : Jeans mechanism

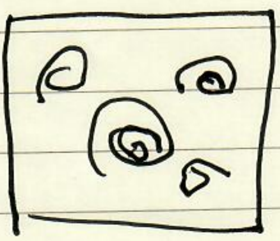
$$\Delta U = -\frac{\omega_p^2 \, \delta f}{\epsilon(k, kv)}$$

- 2D fluid — see J.C. McWilliams JFM '84

J. Weiss, Physica D '91



run down

→ 

isolated coherent vortices

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then, can derive evolution equation for $\rho = \nabla^2\phi$

Gaussian curvature ϕ

$$\partial_t \nabla\rho = \left(\underbrace{\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2}_{\text{local shear}} - \underbrace{\rho^2}_{\text{local vorticity}}\right)^{1/2}$$

- high local vorticity regions dynamically stable

- resist "cascading" in 2D turbulence.

$\Rightarrow$ Gaussian curvature emerges as "coherency criterion".

- How do coherent structures differ from concepts of structure based on statistical Theory??

$\rightarrow$ Statistical Theory Basics

'structure' $\leftrightarrow$ $\langle \delta f(1)\delta f(2)\rangle$

correlation function

$\leftrightarrow$ scale

how deriv., evolve?

$$\partial_t \delta f + v\partial_x \delta f + \frac{q}{m}E\delta f = -\frac{q}{m}E\frac{\partial\langle f\rangle}{\partial v}$$

$\delta f(2) \quad \partial_t \delta f(1)$

$\delta f(1) \quad \partial_t \delta f(2)$

$\Rightarrow$

$$\partial_t\langle\delta f(1)\delta f(2)\rangle + \left(v_1\partial_{x_1} + v_2\partial_{x_2}\right)\langle\delta f(1)\delta f(2)\rangle$$

$$+ \frac{\partial}{\partial v_1}\left\langle\frac{q}{m}E_1\,\delta f(2)\,\delta f(1)\right\rangle + \frac{\partial}{\partial v_2}\left\langle\frac{q}{m}E_2\,\delta f(1)\,\delta f(2)\right\rangle$$

$$= -\frac{q}{m}\left[\langle E(1)\,\delta f(2)\rangle\frac{\partial\langle f_1\rangle}{\partial v} + \langle E(2)\,\delta f(1)\rangle\frac{\partial\langle f_2\rangle}{\partial v}\right]$$

Recall $\qquad u \qquad \Delta$

$$\partial_t\langle f\rangle = -\frac{\partial}{\partial v}\frac{q}{m}\langle E\delta f\rangle$$

Now, recall

$$\frac{df}{dt} = 0 \quad \Rightarrow \quad \int dv\,\frac{df^2}{dt} = 0$$

average $\left\langle\int dv\,\frac{d}{dt}f^2\right\rangle = 0$

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so

$$\int dv \left[\frac{d}{dt} \langle \delta f^2 \rangle + \frac{\partial \langle f \rangle^2}{\partial t} \right] = 0$$

$$\int \left\langle \frac{d}{dt} \delta f^2 \right\rangle dv = \int dv \, -2 \frac{\varepsilon}{m} \langle E \delta f \rangle \frac{\partial \langle f \rangle}{\partial v}$$

as above.

statistical closure :
(TBD)

relative streaming

$$\partial_t \langle \delta f^2 \rangle + v_- \partial_{x_-} \langle \delta f^2 \rangle$$

$$- \partial_{v_-} D_- \frac{\partial}{\partial v_-} \langle \delta f^2 \rangle = -2 \frac{\varepsilon}{m} \langle E \delta f \rangle \frac{\partial \langle f \rangle}{\partial v}$$

relative diffusion

driven by relaxation

$$D_- \sim \langle E^2 \rangle$$

but $\to 0$ as $- \to 0$

so immediately see one key difference $\Rightarrow$ statistical theory recognizes only $\langle \delta f^2 \rangle$ i.e. does not recognize the difference between $\delta f \gtrless 0$ $\Rightarrow$ can't distinguish excess from hole.

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"Clump" is buzzword for correlation evolved / described by $\langle \delta f(1)\, \delta f(2) \rangle$, i.e. like eddy.

What is "in" $\langle \delta f^2 \rangle$?

Can observe, in relative coordinates:

$$\partial_t \langle \delta f(1)\, \delta f(2) \rangle + v_- \partial_{x_-} \langle \delta f(1)\, \delta f(2) \rangle$$

$$+ \left\langle \frac{q}{m} \big(E(1) - E(2)\big) \frac{\partial}{\partial v_-} \delta f(1)\, \delta f(2) \right\rangle = RHS$$

exact structure.

then if write:

$$\big(\partial_t + T_{1,2}(-)\big) \langle \delta f(1)\, \delta f(2) \rangle = RHS$$

then $\lim_{1 \to 2} T_{1,2}(-) = 0$

so $\langle \delta f(1)\, \delta f(2) \rangle$ diverges as $1 \to 2$, up to coarse graining

so

$$\langle \delta f(1)\, \delta f(2) \rangle = T^{-1}(-)\, RHS$$

$(-)$ singular.

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but, examine linear response

i.e. in QLT $\delta f = f^c$

$$f^c = \frac{-\frac{\varepsilon}{m} E_k \frac{\partial \langle f \rangle}{\partial v}}{-i\left(\omega - kv + \frac{i}{\tau_c}\right)}$$

$\Downarrow$

resonance broadening

$$1/\tau_c = (k^2 D)^{1/3}$$

then

$$\langle f^c(1) f^c(2) \rangle = \sum_k \frac{\varepsilon^2}{m^2} |E_k|^2 \left(\frac{\partial \langle f \rangle}{\partial v}\right)^2 e^{ik(x_1 - x_2)} \, \tau_c \frac{1/\tau_c}{(\omega - kv)^2 + 1/\tau_c^2}$$

$$= \tau_c \, D_{QL} \left(\frac{\partial \langle f \rangle}{\partial v}\right)^2$$

finite

$\rightarrow$ granulation

so

$$\delta f = f^c + \tilde{f}$$

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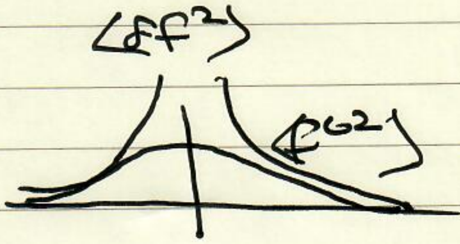
so observe:

$\langle \delta f^2 \rangle \longrightarrow$ ☹ singular

$\Downarrow$

full correlation function



$\langle \tilde{f}^2 \rangle \longrightarrow$ ☺ finite

$$\delta f = f^c + \underbrace{\tilde{f}}_{\to \text{ something else}}$$

$f^c \Downarrow$ usual

$\tilde{f} \Downarrow$ granulation

- is hole a realization of more abstract ~~xxxx~~ concept of granulation?

- effect of granulation?

observe

$$\partial_t \langle f \rangle = -\partial_v \left\langle \frac{q}{m} E \, \delta f \right\rangle$$

$$= -\partial_v \left\langle \frac{q}{m} E \left[(f^c) + \tilde{f} \right] \right\rangle$$

$$= \partial_v D_{QL} \, \partial_v \langle f \rangle - \partial_v \left\langle \frac{q}{m} E \tilde{f} \right\rangle$$

$\Downarrow$ diffusion $\qquad\qquad \Downarrow$ dynamical friction!

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$$= -\partial_v \left[\left\langle \frac{q}{m} E \tilde{f} \right\rangle - D_{QL} \partial_v \langle f \rangle \right]$$

$\longleftrightarrow$

$$= -\partial_v \left[\frac{\langle \Delta v \rangle}{\Delta t} \langle f \rangle - \frac{\partial}{\partial v} \left[D_v \langle f \rangle \right] \right]$$

dynamical friction

$\rightarrow$ drag

F-P

How calculate $\tilde{f}$? - TBD

Note analogy:

in test particle model

$$\delta f = f^c + \tilde{f} \qquad \text{where}$$

$$\langle \tilde{f}(1) \tilde{f}(2) \rangle = \frac{\langle f \rangle}{n} \delta(x_-) \delta(v_-)$$

discreteness

and

$$\frac{q}{m} \langle E \delta f \rangle = -D \frac{\partial \langle f \rangle}{\partial v} + \frac{q}{m} \langle E \tilde{f} \rangle$$

dynamical friction

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i.e. $\frac{q}{m}\langle E\tilde{f}\rangle$ corresponds to F-P drag.

Also, $\boxed{\text{statistical theory neglects self-field effects.}}$

$$\tilde{\tau}_{cl} = \tilde{\tau}_0 \ln\left[F(k x_-, v_-/\Delta v_{tr})\right]$$

set by exponential divergence of orbits.

Note:

$$1/\tilde{\tau}_0 = k\,\Delta v_{tr}$$

resonance width — trapping width.

Now, how does self-field of hole compare with $\Delta v/\Delta x \sim 1/\tau_c$?

To estimate self-potential:

hole size

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{1}{\lambda^2}\right)\phi = 4\pi n q \tilde{f}\,\Delta v$$

$$1/\lambda_D^2 = \omega_p^2\, P\int \frac{dv}{u-v}\,\partial f_0/\partial v$$

but $\tilde{f} \sim \frac{\Delta V}{V_{Th}^2}$, crudely

⇒

$$\boxed{\phi \sim 4\pi n q \frac{\Delta V^2}{V_{Th}^2}\lambda^2}$$

but

$$\Delta V_{Tr} \sim \sqrt{\frac{2q\phi}{m}} \sim \left(\frac{\lambda}{\lambda_D}\right)\Delta V$$

so $\lambda > \lambda_D$ $\Delta V_{Tr} > \Delta V$

⇒ hole must/can trap itself.

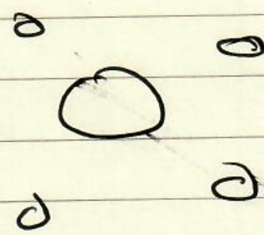
⇒ cant neglect hole self-field

- statistical theory neglects self-field.

- competition:

D_- in 2 pt eqn represents scattering/straining of clumps by other structures

$$D_- = \sum_k k^2 \chi^2 \frac{q^2|E_k|^2}{m^2}\pi\delta(\omega - kv)$$



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but

$$D_- \sim \sum_{k} x^2 \frac{q^2}{m^2} \left| \frac{\partial E_k}{\partial x} \right|^2 \pi \, \delta(\omega - kv)$$

$$\rightarrow \frac{q^2}{m^2} \left(\frac{\partial E}{\partial x} \right)^2 (\Delta x)^2 \left(\frac{\Delta x}{\Delta v} \right)$$

so competition is:

$$\left[\left(\frac{\partial E}{\partial x} \right)^2 \right]^{1/2} \quad \text{vs} \quad \left(\frac{\partial E}{\partial x} \right)_{\text{self}}$$

rms tidal force — self

competition.

For self:

$$\frac{\partial E}{\partial x} \sim 4\pi n q \, \Delta v \tilde{f}$$

$$\sim 4\pi n q \left[\Delta v \, \tilde{f} \right]_{\text{self}}$$