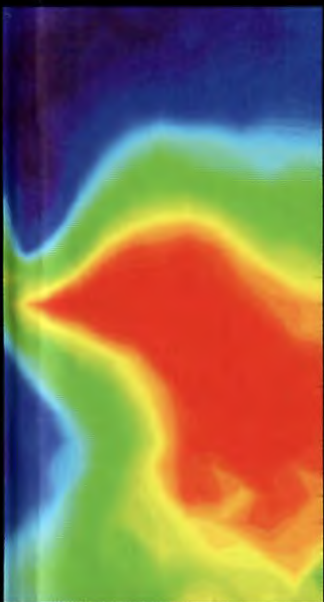

Modern Plasma Physics

Volume 1: Physical Kinetics of Turbulent Plasmas

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and Kimitaka Itoh

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Volume 1: Physical Kinetics of Turbulent Plasmas

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8

Cascades, structures and transport in phase space turbulence

It is that science does not try to explain, nor searches for interpretations but primarily constructs models. A model is a mathematical construction, which supplemented with some verbal explanation, describes the observed phenomena. Such mathematical construction is proved if and only if it works, that it describes precisely a wide range of phenomena. Furthermore, it has to satisfy certain aesthetic criteria, i.e., it has to be more or less simple compared to the described phenomena.

(J. Von Neumann)

8.1 Motivation: basic concepts of phase space turbulence

8.1.1 Issues in phase space turbulence

Up to now, our discussion of plasma turbulence has developed by following the two parallel roads shown in Figure 8.1. Following the first, well trodden, path, we have developed the theory of nonlinear mode interaction and turbulence as applied to fundamentally *fluid* dynamical systems, such as the Navier–Stokes (NS) equation or the quasi-geostrophic (QG) Hasegawa–Mima equation. Along the way, we have developed basic models such as the scaling theory of eddy cascades as in the Kolmogorov theory, the theory of coherent and stochastic wave interactions, renormalized theories of fluid and wave turbulence, the Mori–Zwanzig memory function formalism for elimination of irrelevant variables, and the theory of structure formation in Langmuir turbulence by disparate scale interaction. Following the second, less familiar trail, we have described the theory of kinetic Vlasov turbulence – i.e. turbulence where the fundamental dynamical field is the phase space density $f(x, v, t)$ and the basic equation is the Vlasov equation or one of its gyrokinetic variants. Along the way, we have discussed quasi-linear theory of mean field ($\langle f \rangle$) evolution and the theory of resonance broadening and nonlinear response $\delta f/\delta E$, and of nonlinear wave–particle interaction. We note, though, that our discussion of Vlasov turbulence has been cast *entirely* within the framework

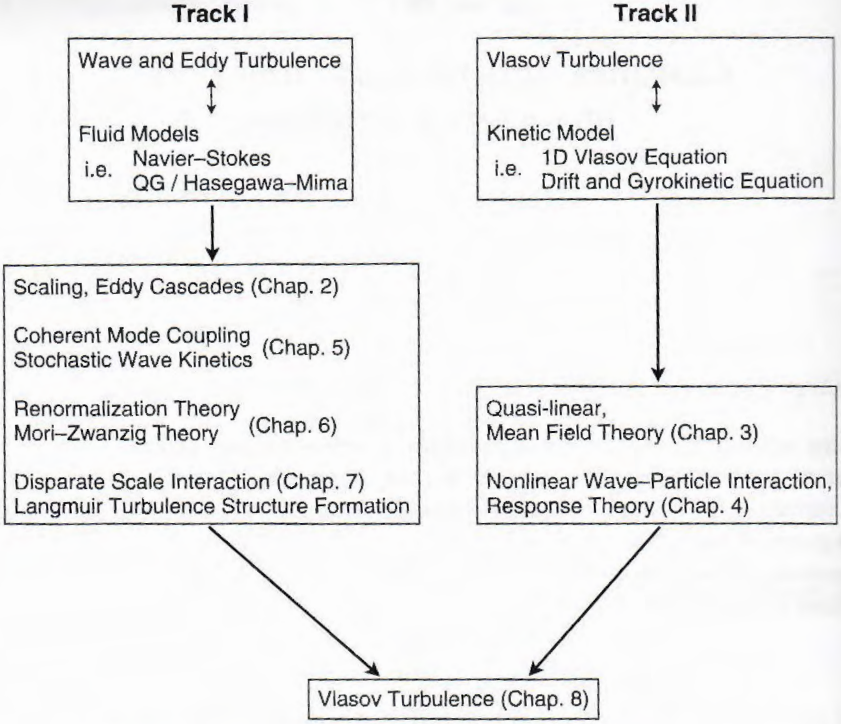

Fig. 8.1. Two paths in the discussion of plasma turbulence.

of linear and nonlinear response theory – i.e. we have focused exclusively upon the impact of Vlasov nonlinearity on the distribution function response δf to a fluctuating electric field δE, and on the consequent construction of macroscopic evolution equations. As a result of this focus, the approaches developed so far, such as the (coarse grained) quasi-linear equation or the wave kinetic equation for $\langle E^2\rangle_{\mathbf{k}}$, have *not* really described the dynamics of Vlasov turbulence at the level of its governing nonlinear phase space equation, which is the Vlasov equation. In particular, we have not adapted familiar turbulence concepts such as eddies, coherent vortices, cascades, mixing, etc. to the description of Vlasov turbulence. Furthermore, we have so far largely separated the phenomena of wave–particle resonance – which we have treated primarily using quasi-linear theory, from the process of mode–mode coupling, which we have treated using fluid equations or modal amplitude equations.

8.1.1.1 Vlasov–Poisson system

In this chapter, we present an extensive discussion of phase space turbulence, as governed by the Vlasov–Poisson system. Recall the Vlasov equation is simply the Boltzmann equation,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{qE}{m}\frac{\partial f}{\partial v} = C(f) \tag{8.1a}$$

for vanishing $C(f)$. As with the well-known distinction between the Navier–Stokes and Euler equations, one must take care to distinguish the limit of $C(f)$ small but finite, from the case where $C = 0$. The Poisson equation,

$$\frac{\partial E}{\partial x} = 4\pi n_0 q \int \mathrm{d}v\, f + 4\pi q_{\mathrm{os}} \hat{n}_{\mathrm{os}} \tag{8.1b}$$

allows a *linear* feedback channel between the electric field which evolves f and the charge density which f itself produces. For completeness, we include the possibility of additional charge perturbations induced by other species (denoted here by $q_{\mathrm{os}}\hat{n}_{\mathrm{os}}$), which couple to f evolution via Poisson's equation. The Vlasov equation is a statement of the local conservation (up to collisional coarse-graining) of a (scalar) phase space density $f(x, v, t)$ along the particle trajectories set by its Hamiltonian characteristic equations,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{q}{m}E(x,t). \tag{8.2}$$

Thus, Vlasov–Poisson turbulence may be thought of as "active scalar" turbulence, as is 2D fluid or quasi-geostrophic (QG) turbulence.

8.1.1.2 Analogy between Vlasov system and quasi-geostrophic system

Indeed, there is a close and instructive analogy between the quasi-geostrophic or Hasegawa–Mima system and the Vlasov–Poisson system. In the QG system, potential vorticity (PV) $Q(\mathbf{x}, t)$ is conservatively advected (modulo a small viscous cut-off) along the streamlines of incompressible flow $\mathbf{v}$, itself determined by the PV field $Q(\mathbf{x}, t)$ (Vallis, 2006), so the QG equation is just,

$$\partial_t Q + \{Q, \phi\} - \nu\nabla^2 Q = 0, \tag{8.2a}$$

where the velocity is expressed in terms of a stream function,

$$\mathbf{v} = \nabla\phi \times \hat{z}, \tag{8.2b}$$

ν is the viscosity, and the advected PV is related to the stream functions ϕ via,

$$Q = Q_0 + \beta x + \nabla^2\phi. \tag{8.2c}$$

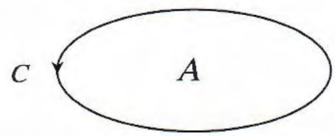

Fig. 8.2. Contour and area of integration.

(See the description of coordinates in Appendix 1.)[1] Note that the stream function is related to potential vorticity via a Poisson equation. In turn, the Vlasov–Poisson system may be rewritten in the form,

$$\partial_t f + \{f, H\} = C(f) \tag{8.3a}$$

$$f = \langle f(v, t) \rangle + \delta f(\mathbf{x}, \mathbf{v}, t) \tag{8.3b}$$

$$H = \frac{p^2}{2m} + q\phi(\mathbf{x}), \tag{8.3c}$$

where q is a charge, ϕ is an electrostatic potential, p is a momentum, and f and ϕ are related through Poisson's equation, given by Eq. (8.1b).

8.1.1.3 Circulation in QG system revisited

Circulation and its conservation are central to PV dynamics and, as we shall see, to Vlasov plasma dynamics. It is well known that an inviscid, barotropic fluid obeys Kelvin's circulation theorem,

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0, \tag{8.4a}$$

where,

$$\Gamma = \oint_C \mathbf{v} \cdot \mathbf{dl} = \int_A \boldsymbol{\omega} \cdot \mathbf{da}. \tag{8.4b}$$

Here C is a closed contour which bounds the area (Fig. 8.2). The element of thus fluid may be thought of as carrying a circulation $\Gamma \sim Vl \sim \omega A$. Conservation of circulation is fundamental to vortex dynamics and enstrophy prediction, discussed in Chapter 2, and to the very notion of an "eddy", which is little more than a conceptual cartoon of an element of circulation on a scale l. *So, Kelvin's theorem is indeed a central element of turbulence theory* (P. A. Davidson, 2004). For the QG system, a new twist is that the conserved potential vorticity is the sum of

[1] We note here that the x-direction is taken in the direction of inhomogeneity of the mean profile and y- is an ignorable coordinate after the convention of plasma physics. In geofluid dynamics, the x-direction is in the longitudinal direction (ignorable coordinate) and y- is in the latitudinal direction (in the direction of mean inhomogeneity). Thus, our notation is **not** the standard one used in geofluid dynamics.

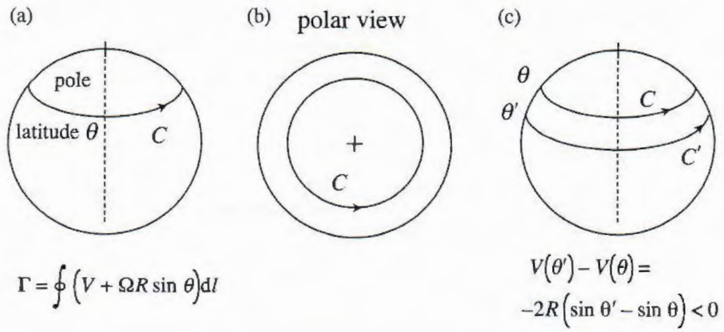

Fig. 8.3. Contour C at latitude θ and the circulation Γ (a). Polar view is illustrated in (b). When the contour is moved to lower latitude, $\theta \to \theta'$, conservation of circulation induces a westward circulation (c).

planetary (i.e. mean, i.e. $Q = Q_0 + \beta x$) and relative (i.e. fluctuating ω) pieces, so the integrated PV is,

$$\int_A Q \, da = \int_A (\omega + 2\Omega \sin\theta) \, da, \tag{8.5a}$$

and that the conserved circulation of, say, an eddy encircling the pole of latitude θ_0 (on a rotating sphere) is then,

$$\Gamma = \oint_C (V + 2\Omega R \sin\theta) \, dl, \tag{8.5b}$$

(where Ω and R are rotation frequency and radius of the sphere, respectively), as shown in Figure 8.3. The novel implication here is that since total Γ is conserved, there must be trade-offs between planetary and relative contributions when C moves. This follows because advecting a PV patch or element to higher latitude necessarily implies an increase in planetary vorticity $Q_{ot} = \beta x$, so conservation of *total* PV and circulation consequently imply that *relative* vorticity $\omega = \nabla^2\phi$ and relative circulation $\int \omega \, da = \oint \mathbf{V} \cdot d\mathbf{l}$ must *decrease*, thus generating a westward flow, i.e. see Figure 8.3. Thus, in QG fluids, Kelvin's theorem links eddy dynamics to the effective mean vorticity profile, and so governs how localized eddies interact with the mean PV gradient.

8.1.1.4 Circulation theorem for Vlasov system and granulations

Perhaps not surprisingly, then, the Vlasov equation also satisfies a circulation theorem, which we now present for the general case of electrostatic dynamics in 3D. Consider C_Γ, a closed phase space trajectory in the 6-dimensional phase space, and $C_{\mathbf{r}}$, its projection into 3-dimensional configuration space, i.e. see Figure 8.4.

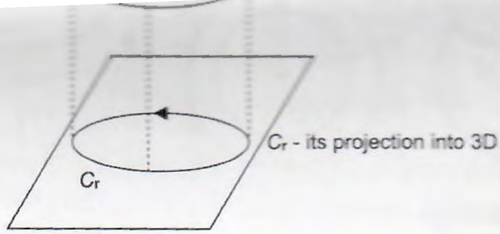

Fig. 8.4. Closed path in a phase space and its projection into real space.

Let s specify the location along the path C_Γ, so that $\mathbf{r}(s)$ corresponds to the trajectory C_r and $(\mathbf{r}(s), \mathbf{v}(s))$ corresponds to the trajectory C_Γ. Then the circulation is simply,

$$\Gamma = \oint \mathbf{v}(s) \cdot \mathrm{d}\mathbf{r} \tag{8.6a}$$

and,

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \oint \left[\frac{\mathrm{d}\mathbf{v}(s)}{\mathrm{d}t} \cdot \mathrm{d}\mathbf{r} + \mathbf{v}(s) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}\mathbf{r}(s) \right]. \tag{8.6b}$$

Since

$$\frac{\mathrm{d}\mathbf{v}(s)}{\mathrm{d}t} = -\frac{q}{m} \nabla \phi, \tag{8.6c}$$

and

$$\mathbf{v}(s) \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}\mathbf{r}(s) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} v^2(s) \right), \tag{8.6d}$$

we easily see that,

$$\mathrm{d}\Gamma / \mathrm{d}t = 0, \tag{8.6e}$$

so the *collisionless electrostatic Vlasov–Poisson system indeed conserves phase space circulation*. The existence of a Kelvin's theorem for the Vlasov–Poisson system then suggests that an "eddy" is a viable and useful concept for phase space turbulence, as well as for fluids. For the 2D phase space of (x, v), the circulation is simply the familiar integral $\Gamma = \oint v \, \mathrm{d}x$, so a Vlasov eddy or *granulation* may be thought of as a chunk of phase space fluid with effective circulation equal to its

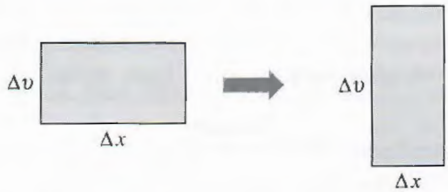

Fig. 8.5. Conservation of a volume of an element in the phase space.

conserved phase volume $\Delta x \Delta v$, a circulation time equal to the orbit bounce time and an associated conserved phase space density,

$$[f(x, v, t)]_{\Delta x, \Delta v} = \int_{x-\Delta x/2}^{x+\Delta x/2} \frac{\mathrm{d}x'}{\Delta x} \int_{v-\Delta v/2}^{v+\Delta v/2} \frac{\mathrm{d}v'}{\Delta v} f(x', v', t) . \tag{8.7}$$

Usually, the velocity scale of a phase space eddy will correspond to its trapping width Δv_{T}, so the eddy circulation time scale $\tau = \Delta x / \Delta v$ will correspond to the wave–particle correlation time τ_{c}. Just as $(\Delta v_{\mathrm{T}}) \tau_{\mathrm{c}} = k^{-1}$, $(\Delta v) \tau = \Delta x$. Note that Kelvin's theorem requires that if, say Δv increases on account of acceleration, Δx must then decrease concomitantly, to conserve circulation. This process of stretching while conserving phase volume is depicted in Figure 8.5. It suggests that a turbulent Vlasov fluid may be thought of as a tangle of thin stretched strands, each with its own value of locally conserved f, and that fine scale structure and sharp phase space gradients must develop and ultimately be limited by collisions. The correspondence between the conservative QG system and the Vlasov–Poisson system is summarized in Table 8.1.

8.1.2 Granulation – what and why

8.1.2.1 Dynamics of granulations

Like a fluid eddy, a granulation need not be related to a wave or collective mode perturbation, for which the dielectric function,

$$\epsilon(k, \omega) \to 0,$$

but may be nonlinearly driven instead, as in a fluid turbulence cascade. To see this, consider the behaviour of the two-point correlation function, discussed in detail later in this chapter. If one takes the phase space density fluctuation $\delta f = f^{\mathrm{c}}$, i.e. just the coherent response – as in the case of a wave, one has,

Table 8.1. *Comparison/contrast of quasi-geostrophic and Vlasov turbulence*

Quasi-geostrophic turbulence	Vlasov turbulence
Structure	
Conserved field	
potential vorticity – Q	pbase space density – f
Evolution	
$\{Q, \phi\}$	$\{f, H\}$
Evolver	
stream function – ϕ	electrostatic potential – ϕ, $H = p^2/2m + \phi$
Dissipation	
$-\nu\nabla^2$	$C(f)$
Feedback	
$Q = Q_0 + \beta x + \nabla^2\phi$	Poisson equation
Circulation	
$\oint_C (V + 2\Omega R \sin\theta_0)\,\mathrm{d}l$	$\oint \mathbf{v}(s)\cdot \mathbf{dr}$
Planetary + Relative vorticity	$f = \langle f\rangle + \delta f$
Element	
Vortex Patch	Granulation
$\int Q\,\mathrm{d}a$ conserved	Phase volume conserved

$$\langle \delta f(1)\,\delta f(2)\rangle^{\mathrm{c}} = \left\langle f^{\mathrm{c}}(1)\, f^{\mathrm{c}}(2)\right\rangle$$
$$= \mathrm{Re}\sum_{\mathbf{k},\omega} \frac{q^2}{m^2}\frac{\left|E_{k,\omega}\right|^2 (\partial\langle f\rangle/\partial v)^2 \exp\{ik(x_2 - x_1)\}}{(\omega - kv_1 + i/\tau_{ck})^* (\omega - kv_2 + i/\tau_{ck})} \tag{8.8a}$$

so,

$$\lim_{1\to 2}\left\langle f^{\mathrm{c}}(1)\, f^{\mathrm{c}}(2)\right\rangle = 2\tau_{\mathrm{c}} D\,(\partial\langle f\rangle/\partial v)^2, \tag{8.8b}$$

where D is just the familiar quasi-linear diffusion coefficient. Note that $\lim_{1\to 2}\langle f^{\mathrm{c}}(1)\, f^{\mathrm{c}}(2)\rangle$ is manifestly *finite* at small separations. In contrast, by using phase space density conservation to write the *exact* evolution equation for $\langle \delta f(1)\,\delta f(2)\rangle$ in the relative coordinates x_-, v_-, we see that,

$$\frac{\partial}{\partial t}\langle \delta f(1)\,\delta f(2)\rangle + v_- \frac{\partial}{\partial x_-}\langle \delta f(1)\,\delta f(2)\rangle$$
$$+ \frac{q}{m}\left\langle (E(1) - E(2))\frac{\partial}{\partial v_-}\delta f(1)\,\delta f(2)\right\rangle + (\delta f(2)\, C(\delta f(1)) + (1 \leftrightarrow 2))$$
$$= -\frac{q}{m}\langle E(1)\,\delta f(2)\rangle \frac{\partial\langle f(1)\rangle}{\partial v_1} + (1 \leftrightarrow 2). \tag{8.9a}$$

This may be condensed to the schematic form,

$$\left(\partial_t + T_{1,2}(x_-, v_-) + \nu\right)\left\langle \delta f^2 \right\rangle = P, \tag{8.9b}$$

so we see that correlation evolves via a balance between *production* by mean distribution function relaxation (i.e., $P(1,2)$, where $\lim_{x_-,v_-\to 0} P(1,2)$ is finite) and relative dispersion (i.e., $T_{1,2}(x_-, v_-)$, where $\lim_{x_-,v_-\to 0} T_{1,2}(x_-, v_-) \to 0$), cut off at small scale by collisions only. Observe that the stationary $\left\langle \delta f^2 \right\rangle$ tends to diverge as $1 \to 2$, so finiteness requires a collisional cut-off. *The small-scale divergence of* $\langle \delta f(1)\,\delta f(2)\rangle$, *as contrasted to the finiteness of* $\langle \delta f(1)\,\delta f(2)\rangle^{\mathrm{c}}$, *establishes that there must indeed be a constituent or element of the total phase space density fluctuation in addition to the familiar coherent response*, f^{c} (i.e. the piece linearly proportional to E). That piece is the *granulation* or *incoherent* fluctuation $\tilde{f}$, which is produced by nonlinear phase space mixing and which is associated with the element of conserved circulation $\Delta x \Delta v$ (Lynden-Bell, 1967; Kadomtsev and Pogutse, 1970; Dupree, 1970, 1972; Boutros-Ghali and Dupree, 1981; Diamond *et al.*, 1982; Balescu and Misguish, 1984; Suzuki, 1984; Terry and Diamond, 1984; McComb, 1990; Berk *et al.*, 1999). A large portion of this chapter is concerned with determining the macroscopic consequences of granulations.

8.1.2.2 Evolution correlation in QG turbulence

The analogy between 2D, quasi-geostrophic and Vlasov turbulence may be extended further, in order to illustrate the fundamental cascade and balance relations in Vlasov turbulence. Both the QG and Vlasov equations describe the conservative advection of a field by an incompressible flow. Thus, the key balance in QG turbulence is focused on that of the correlation of fluctuating potential enstrophy $\left\langle \delta Q^2 \right\rangle$, as a direct consequence of PV evolution,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} - \nu\nabla^2 Q = +\tilde{f}. \tag{8.10a}$$

This may be condensed to the schematic form,

$$(\partial_t + \mathbf{V}\cdot\nabla)\,\delta Q = -\tilde{V}_x \frac{\partial\langle Q\rangle}{\partial x} + \tilde{f} + \nu\nabla^2 Q, \tag{8.10b}$$

so,

$$\begin{aligned}
&(\partial_t + \mathbf{V}(1)\cdot\nabla_1 + \mathbf{V}(2)\cdot\nabla_2)\,\langle \delta Q(1)\,\delta Q(2)\rangle \\
&\quad = -\left\langle \tilde{V}_x(1)\,\delta Q(2)\right\rangle \frac{\partial\langle Q(1)\rangle}{\partial x} + \left\langle \tilde{f}(1)\,\delta Q(2)\right\rangle \\
&\quad\quad - \nu\,\langle \nabla_1 \delta Q(1)\cdot\nabla_2 \delta Q(2)\rangle + (1 \leftrightarrow 2)\,.
\end{aligned} \tag{8.10c}$$

Here the evolution of the correlation function $\langle \delta Q(1)\, \delta Q(2) \rangle$ – which necessarily determines the potential enstrophy spectrum, etc. – results from:

(i) production by forcing $\tilde{f}$ and interaction with the mean PV gradient $\partial \langle Q \rangle / \partial x$;
(ii) nonlinear transfer due to relative advection;
(iii) viscous damping at small scales.

8.1.2.3 'Phasestrophy' in Vlasov turbulence

The nonlinear transfer mechanism in Eq.(8.10) is just that of the forward enstrophy cascade in 2D turbulence, namely, the scale-invariant self-similarity of $\langle \delta f^2 \rangle$. Then, it is apparent that for Vlasov turbulence, the quadratic quantity of interest, must be the "phasestrophy", i.e. the mean-square phase space density $\langle \delta f^2 \rangle$, the spectrum of which is set by the two-point correlation function $\langle \delta f(1)\, \delta f(2) \rangle$. When $\delta f \ll \langle f \rangle$, the integrated phasestrophy equals the fluctuation entropy. Note that while all powers of f are conserved in the absence of collisions, we especially focus on the quadratic quantity phasestrophy, since,

(i) it alone is conserved on a finite mesh or interval in k,
(ii) it is directly related to field energy, etc., via the linear Poisson's equation.

Also note that the Vlasov equation states that total f, i.e. $\langle f \rangle + \delta f$, is conserved, so that scattering in v necessarily entails trade-offs between the amplitude of the mean $\langle f \rangle$ and the fluctuations δf. As in the case of enstrophy, phasestrophy evolution is determined by the balance between production by mean relaxation $\sim -(q/m) \langle E \delta f \rangle \partial \langle f \rangle / \partial v$ and collisional dissipation $\sim \nu \langle \delta f^2 \rangle$, mediated by stretching and dissipation of phase space fluid elements. As we shall see, the phasestrophy cascade closely resembles the forward cascade of enstrophy in 2D fluid turbulence, since both correspond to the increase of mean-square gradients in space and phase space, due to stretching of iso-vorticity and iso-phase space density contours, respectively. Recall that in the enstrophy cascade, a scale-independent transfer of fluctuation enstrophy to small scale occurs, with,

$$\left\langle \tilde{\omega}^2(k) \right\rangle / \tau(k) \sim \eta.$$

Since $\langle \tilde{\omega}^2(k) \rangle \sim k^3 \left\langle \tilde{V}^2(k) \right\rangle$ and $1/\tau(k) \sim k \left[k \left\langle \tilde{V}^2(k) \right\rangle \right]^{1/2}$, we have

$$\left\langle \tilde{\omega}^2(k) \right\rangle \sim \eta^{2/3}/k,$$

and $l_{\mathrm{d}} \sim \left(\nu^2/\eta\right)^{1/4}$. For a driven, self-similar phasestrophy cascade, we thus expect,

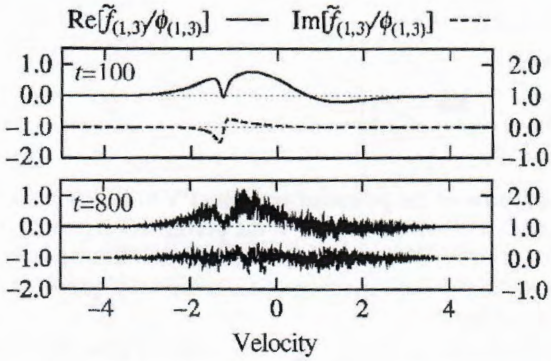

Fig. 8.6. Evolution of the distribution function from the linear phase (top) to the turbulent phase (lower) in direct nonlinear simulation. In the turbulent state, small-scale and sharp corrugations are driven until smeared by (small-but-finite) collisions. [Quoted from (Watanabe *et al.*, 2002), which explains details.]

$$\frac{\langle \delta f^2 (l) \rangle}{\tau (l)} \sim \alpha, \tag{8.11a}$$

where α is the phasestrophy flux in scale l and $\tau(l)$ is the lifetime of a phase space eddy. From dimensional considerations and Poisson's equation, we can *estimate* the lifetime $\tau(l)$ to be,

$$\frac{1}{\tau (l)} \sim \frac{q}{m} E \frac{\partial}{\partial v} \sim l\omega_p^2 \frac{\delta f \Delta v}{\Delta v} \sim l\omega_p^2 \delta f. \tag{8.11b}$$

Thus, the cascade balance for phasestrophy is,

$$\left(l\omega_p^2\right) \delta f^3 \sim \alpha, \tag{8.11c}$$

so $\delta f(l) \sim \left(\alpha / l\omega_p^2\right)^{1/3}$. Straightforward manipulation then gives $\langle \delta f^2 \rangle_k \sim \alpha^{2/3} k^{-1/3}$, so we see that considerable fine-scale structure is generated by stochastic acceleration in phase space. The phasestrophy cascade is terminated when the phase element decay rate $\sim l\omega_p^2 \delta f$ becomes comparable to the collisional diffusion rate $D_{v,\mathrm{col}}/\Delta v^2$. This gives $l_{\mathrm{d}} \sim \left(D_{v,\mathrm{col}}/\omega_p^2 \Delta v^2\right)$ as the spatial dissipation scale. Equivalently, the velocity coarse-graining scale Δv_{c} corresponding to a spatial scale l is $\Delta v_{\mathrm{c}} \sim \left(D_{v,\mathrm{col}}/l\omega_p^2\right)^{1/2}$, so that scales with $\Delta v < \Delta v_{\mathrm{c}}$ are smoothed out by collisions. The function Δv_{c} thus sets a lower bound on the thickness of phase space elements produced by stretching.

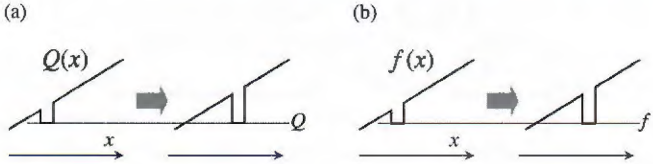

Fig. 8.7. Conservation of the potential vorticity PV in the QG system (a). When a dip moves to a region of higher mean PV, the perturbation grows. Associated with this, the gradient of mean PV relaxes. Conservation of the total number density f in the Vlasov system (b). In this illustration, a dip of f grows at the expense of the relaxation of the mean.

8.1.2.4 Generation of eddy in QG system

An essential and instructive element of the theory of phase space structures is the link via dynamics which it establishes between production and the dynamics of a localized structure, as opposed to a wave or eigenmode. Here, "production" refers to growth of fluctuation energy via its extraction from mean gradients. A prototypical example of this type of reasoning, originally due to G.I. Taylor, appears in the theory of mean flow–fluctuation interaction in 2D QG fluids, and deals with shear perturbation growth (Taylor, 1915). Periodicity of zonal flows and the relation between the mean Magnus force and PV flux tell us at the outset that the zonal flow $\langle V_y \rangle$ satisfies,

$$\frac{\mathrm{d}\langle V_y \rangle}{\mathrm{d}t} = \left\langle \tilde{V}_x \tilde{\omega} \right\rangle, \tag{8.12a}$$

while conservation of PV implies that the sum of planetary and relative vorticity is conserved, since $Q = \langle Q \rangle + \delta Q = Q_0 + \beta x + \tilde{\omega}$. Then displacing a vortex patch from x_0 to x means that its relative vorticity must change, since necessarily Q is constant at $Q = \langle Q(x_0) \rangle$, yet $\langle Q(x) \rangle \simeq \langle Q(x_0) \rangle + (x - x_0)\, \mathrm{d}\langle Q \rangle / \mathrm{d}x$. Thus the fluctuation δQ must compensate for the change in mean PV, i.e.

$$\delta Q = -(x - x_0) \frac{\mathrm{d}\langle Q \rangle}{\mathrm{d}x}, \tag{8.12b}$$

to conserve total PV, as shown in Figure 8.7. Since $\delta Q = \tilde{\omega}$, we have,

$$\begin{aligned} \frac{\mathrm{d}\langle V_y \rangle}{\mathrm{d}t} &= -\frac{\mathrm{d}\langle Q \rangle}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\langle \delta x^2 \rangle}{2} \\ &= -\frac{\mathrm{d}\langle Q \rangle}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\langle \xi^2 \rangle}{2}, \end{aligned} \tag{8.12c}$$

where ξ is the x-displacement of a fluid element. Note that $\mathrm{d}\left\langle \xi^2/2 \right\rangle / \mathrm{d}t$ is just the Lagrangian fluid element diffusivity D_Q, so,

$$\frac{\mathrm{d}\left\langle V_y \right\rangle}{\mathrm{d}t} = -D_Q \frac{\mathrm{d}\left\langle Q \right\rangle}{\mathrm{d}x}. \tag{8.12d}$$

Equation (8.12d) tells us that:

(i) diffusive relaxation of mean PV gradients will drive mean zonal flows;
(ii) for the geophysically relevant case where the latitudinal derivative of $\langle Q \rangle$ ($\cong \beta$) is positive, any scattering process which increases the latitudinal variance of fluid particles must necessarily produce a net westward zonal flow.
(iii) since, in general, shear flow instability requires that $\left\langle \xi^2 \right\rangle$ increases everywhere, while *total* flow x-momentum must be conserved (i.e. shear instabilities displace stream lines in both directions) so that $\int \mathrm{d}x \left\langle V_y \right\rangle$ is constant, $\mathrm{d}\left\langle Q \right\rangle / \mathrm{d}x$ *must change sign for some* x, in order for perturbations to grow.

The result that $\mathrm{d}\left\langle Q \right\rangle / \mathrm{d}x \to 0$ at some x is a necessary condition for instability is equivalent to Rayleigh's famous inflection point theorem. It is derived here in a short, physically transparent way, without the cumbersome methods of eigenmode theory. This, in turn, suggests that the inflection point condition is more fundamental than linear theory, as was suggested by C. C. Lin, on the basis of considering the restoring force for vortex interchange, and subsequently proven via *nonlinear* stability arguments by V. I. Arnold. We note in passing here that the result of Eq.(8.12c) is related to the well-known Charney–Drazin theorem, which constrains zonal flow momentum. The Charney–Drazin theorem will be discussed extensively in Volume 2 of this series.

8.1.2.5 Growth of granulations

A similar approach can be used to determine the condition for growth of *localized* phase space density perturbations. We can expect such an approach to bear fruit here, since the Vlasov system has the property that total f is conserved, so that δf must adjust when a localized fluctuation is scattered up or down the profile of $\langle f \rangle$, as indicated in Figure 8.7

To see this clearly, we note that since f and $f^2 = (\langle f \rangle + \delta f)^2$ are conserved along phase space trajectories, we can write directly,

$$\int \mathrm{d}\Gamma \frac{\mathrm{d}}{\mathrm{d}t} \left(\langle f \rangle + \delta f\right)^2 = 0, \tag{8.13a}$$

so,

$$\frac{\partial}{\partial t} \int \mathrm{d}\Gamma \left\langle \delta f^2 \right\rangle = -\frac{\partial}{\partial t} \int \mathrm{d}\Gamma \left(2\delta f \left\langle f \right\rangle + \left\langle f \right\rangle^2 \right). \tag{8.13b}$$

Here $\int \mathrm{d}\Gamma$ is an integral over the fluctuation's phase volume $\mathrm{d}x\mathrm{d}v$. As we are interested in localized fluctuations δf – i.e. 'blobs' or holes in phase space – we can expand $\langle f \rangle$ near the fluctuation centroid v_0 and thereafter treat it as static, giving,

$$\begin{aligned}\frac{\partial}{\partial t}\int \mathrm{d}\Gamma \left\langle \delta f^2 \right\rangle &\cong -2\frac{\partial}{\partial t}\int \mathrm{d}\Gamma \ (v - v_0)\,\delta f \left.\frac{\partial \langle f \rangle}{\partial v}\right|_{v_0} \\ &= -2\left.\frac{\partial \langle f \rangle}{\partial v}\right|_{v_0} \frac{1}{m}\frac{\mathrm{d}p_f}{\mathrm{d}t}.\end{aligned} \tag{8.13c}$$

Here $p_f = m\int \mathrm{d}\Gamma \ (v - v_0)\,\delta f$ is the net momentum associated with the phase space density fluctuation δf. For a single species plasma, momentum conservation requires that $\mathrm{d}p/\mathrm{d}t = 0$, so fluctuations cannot grow by accelerating up the mean phase space gradient since they have no place to deposit their momentum, and $\partial_t \left\langle \delta f^2 \right\rangle = 0$ is forced. However, for a two-species, electron–ion plasma, the relevant momentum conservation constraint becomes,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_{f_e} + p_{f_i}\right) = 0, \tag{8.13d}$$

enabling momentum exchange between species. Using Eq. (8.13c) to re-express the relation between momentum evolution and fluctuation growth then gives,

$$\frac{m_e}{\partial \langle f_e \rangle / \partial v|_{v_0}}\partial_t \int \mathrm{d}\Gamma \left\langle \delta f_e^2 \right\rangle = -\frac{m_i}{\partial \langle f_i \rangle / \partial v|_{v_0}}\partial_t \int \mathrm{d}\Gamma \left\langle \delta f_i^2 \right\rangle. \tag{8.13e}$$

Thus, we learn growth of a localized structure at v_0 is possible if $(\partial \langle f_e \rangle / \partial v) \times (\partial \langle f_i \rangle / \partial v)|_{v_0} < 0$ – i.e. the slopes of the electron and ion distribution functions are opposite at v_0. This condition is usually encountered in situations where the electrons carry a net current, so that $\langle f_e \rangle$ is shifted relative to $\langle f_i \rangle$, and is shown in Figure 3.2(b). While this is superficially reminiscent of the familiar textbook example of current-driven ion-acoustic instability (CDIA), it is important to keep in mind that:

(a) *linear instability – mediated by waves* – requires minimal overlap of $\langle f_e \rangle$ and $\langle f_i \rangle$, so that electron growth (due to inverse dissipation) exceeds ion Landau damping;

(b) *granulation growth, which is nonlinear, is larger* for significant overlap of the electron and ion distribution functions, and even requires significant $\partial \langle f_i \rangle / \partial v$, in order to optimize the collisionless exchange of momentum between species, as indicated by Eq. (8.13e). In this limit, the linear CDIA is strongly stabilized.

This contrast is readily apparent from consideration of the evolution of $\left\langle \delta f^2 \right\rangle$ for localized fluctuations. For an ion granulation at v_0, phase space density conservation for homogeneous turbulence implies,

$$\frac{\partial}{\partial t}\left\langle \delta f_i^2 \right\rangle = -\frac{q}{m}\left\langle E\delta f_i \right\rangle \frac{\partial \left\langle f_i \right\rangle}{\partial v}, \tag{8.14a}$$

so, since δf is localized in velocity,

$$\frac{1}{\partial \left\langle f_i \right\rangle / \partial v|_{v_0}} \partial_t \int \mathrm{d}v \left\langle \delta f_i^2 \right\rangle = -\frac{q}{m_i}\int \left\langle E\delta f_i \right\rangle = -\frac{q}{m_i}\left\langle E\tilde{n}_i \right\rangle . \tag{8.14b}$$

However, momentum balance requires that,

$$\frac{\mathrm{d}\left\langle p_{f_i} \right\rangle}{\mathrm{d}t} = +q\left\langle E\tilde{n}_i \right\rangle , \tag{8.14c}$$

and,

$$\frac{\mathrm{d}\left\langle p_{f_i} \right\rangle}{\mathrm{d}t} + \frac{\mathrm{d}\left\langle p_{f_e} \right\rangle}{\mathrm{d}t} = 0, \tag{8.14d}$$

(in the stationary state) giving,

$$\frac{m_i}{\partial \left\langle f_i \right\rangle / \partial v|_{v_0}} \partial_t \int \mathrm{d}v \left\langle \delta f_i^2 \right\rangle = +\frac{\mathrm{d}\left\langle p_{f_e} \right\rangle}{\mathrm{d}t}. \tag{8.14e}$$

Now, since electrons are not trapped, and so are weakly scattered, a simple quasi-linear estimate of $\mathrm{d}\left\langle p_{f_e} \right\rangle / \mathrm{d}t$ gives,

$$\frac{\mathrm{d}\left\langle p_{f_e} \right\rangle}{\mathrm{d}t} = -m_e \int \mathrm{d}v\, D \frac{\partial \left\langle f_e \right\rangle}{\partial v}. \tag{8.14f}$$

Here D is the velocity space quasi-linear diffusion coefficient, which is a function of velocity. Since fluctuations are localized in phase space, we can assume $D(v)$ is peaked at v_0. Thus, combining Eqs. (8.14e) and (8.14f) finally gives,

$$\partial_t \int \mathrm{d}v_i \left\langle \delta f_i^2 \right\rangle = -\frac{m_e}{m_i} \frac{\partial \left\langle f_i \right\rangle}{\partial v}\bigg|_{v_0} \left(\int \mathrm{d}v_e\, D(v) \frac{\partial \left\langle f_e \right\rangle}{\partial v} \right). \tag{8.14g}$$

Since D is maximal for $v \leq v_0$, we see again that,

$$\frac{\partial \left\langle f_i \right\rangle}{\partial v} \frac{\partial \left\langle f_e \right\rangle}{\partial v}\bigg|_{v_0} < 0$$

is necessary for growth of ion granulations. Also, it is clear that growth is nonlinear (i.e. amplitude dependent).

8.2 Statistical theory of phase space turbulence

8.2.1 Structure of the theory

We now present the *statistical* theory of Vlasov turbulence. We construct the theory with the aim of calculating the structure and evolution of the two-point phase space density correlation function $\langle \delta f(1)\, \delta f(2) \rangle$, from which we may extract all other quantities, spectra, fluxes, etc. (Here the argument (1) indicates the position and velocity of the particle 1.) After a discussion of the general structure of the theory, we proceed to in-depth studies of production, relative dispersion and the various nonlinear states which may be realized.

The basic equation is the Vlasov equation, retaining a weak residual level of collisionality. As we are interested in the fluctuation phasestrophy for the fluctuation in the distribution function, δf, we write,

$$\begin{aligned}
&\frac{\partial}{\partial t}\delta f(1) + v_1 \frac{\partial}{\partial x_1}\delta f(1) + \frac{q}{m}E(1)\frac{\partial}{\partial v_1}\delta f(1) \\
&\quad = -\frac{q}{m}E(1)\frac{\partial}{\partial v_1}\langle f(1) \rangle + C(\delta f(1)) .
\end{aligned} \tag{8.15}$$

The equation for two-point phase space density correlation is then obtained by multiplying Eq.(8.15) by $\delta f(2)$ and symmetrizing, which gives,

$$\begin{aligned}
&\frac{\partial}{\partial t}\langle \delta f(1)\, \delta f(2) \rangle + \left(v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} \right) \langle \delta f(1)\, \delta f(2) \rangle \\
&\quad + \frac{q}{m}\frac{\partial}{\partial v_1}\langle E(1)\, \delta f(1)\, \delta f(2) \rangle + \frac{q}{m}\frac{\partial}{\partial v_2}\langle E(2)\, \delta f(1)\, \delta f(2) \rangle \\
&\quad = -\frac{q}{m}\langle E(1)\, \delta f(2) \rangle \frac{\partial}{\partial v_1}\langle f \rangle - \frac{q}{m}\langle E(2)\, \delta f(1) \rangle \frac{\partial}{\partial v_2}\langle f \rangle \\
&\quad + \langle \delta f(2)\, C(\delta f(1)) \rangle + \langle \delta f(1)\, C(\delta f(2)) \rangle .
\end{aligned} \tag{8.16}$$

Equation (8.16) tells us that $\langle \delta f(1) \delta f(2) \rangle$ evolves via:

(i) linear dispersion, due to relative particle streaming (the 2nd term on the left-hand side);
(ii) mode–mode coupling, via triplets associated with particle scattering by fluctuating electric fields (the 3rd and 4th terms on the left-hand side);
(iii) production, due to the relaxation of $\langle f \rangle$ (the 1st and 2nd terms on the right-hand side);
(iv) collisional dissipation (the 3rd and 4th terms on the right-hand side). Hereafter, we take C to be a Krook operator, unless otherwise noted.

8.2.1.1 Relative evolution operator $T_{1,2}$

It is instructive to group the linear dispersion, mode–mode coupling and collision terms together into a relative evolution operator $T_{1,2}$, which is defined as,

$$\begin{aligned} T_{1,2}\langle\delta f(1)\,\delta f(2)\rangle &= \left(v_1\frac{\partial}{\partial x_1}+v_2\frac{\partial}{\partial x_2}\right)\langle\delta f(1)\,\delta f(2)\rangle \\ &+\frac{q}{m}\frac{\partial}{\partial v_1}\langle E(1)\,\delta f(1)\,\delta f(2)\rangle \\ &+\frac{q}{m}\frac{\partial}{\partial v_2}\langle E(2)\,\delta f(1)\,\delta f(2)\rangle+\nu\left\langle\delta f^2\right\rangle, \end{aligned} \tag{8.17a}$$

where a simple Krook collision model is introduced as,

$$\langle\delta f(2)\,C(\delta f(1))\rangle+\langle\delta f(1)\,C(\delta f(2))\rangle=\nu\left\langle\delta f^2\right\rangle. \tag{8.17b}$$

As mentioned in Section 8.1, its also useful to replace $x_{1,2}$ and $v_{1,2}$ with the centroid and relative coordinates,

$$x_\pm=\frac{1}{2}(x_1\pm x_2), \tag{8.18a}$$

$$v_\pm=\frac{1}{2}(v_1\pm v_2). \tag{8.18b}$$

The correlation function $\langle\delta f(1)\delta f(2)\rangle$ is far more sensitive to the relative coordinate dependencies, because the variable v_- describes the scale (in velocity space) of particle resonance or trapping, while the variable v_+ denotes the variation of the order of thermal velocity v_{Th}, so the ordering $|\partial/\partial v_-|\gg|\partial/\partial v_+|$ holds. Spatial homogeneity ensures that $|\partial/\partial x_-|\gg|\partial/\partial x_+|$. Thus, we can discard the centroid (x_+,v_+) dependency of $T_{1,2}$ to obtain,

$$\begin{aligned} T_{1,2}\left\langle\delta f^2(x_-,v_-)\right\rangle &= v_-\frac{\partial}{\partial x_-}\langle\delta f(1)\,\delta f(2)\rangle \\ &+\frac{q}{m}\left\langle(E(1)-E(2))\frac{\partial}{\partial v_-}\delta f(1)\,\delta f(2)\right\rangle+\nu\left\langle\delta f^2(x_-,v_-)\right\rangle, \end{aligned} \tag{8.19}$$

where the centroid dependency of correlation is also neglected, so $\langle\delta f(1)\,\delta f(2)\rangle \to \left\langle\delta f^2(x_-,v_-)\right\rangle$.

8.2.1.2 Limiting behaviours and necessity of granulations

Interesting limiting behaviours may be observed from Eq.(8.19). From this we see, in the limit of $1\to 2$,

$$\lim_{1\to 2}T_{1,2}\left\langle\delta f^2(x_-,v_-)\right\rangle=\nu\left\langle\delta f^2(x_-,v_-)\right\rangle, \tag{8.20}$$

so relative evolution vanishes, apart from collisions (i.e., $T_{1,2}$ becomes very small in the limit of $1 \to 2$). In contrast, the production term (the 1st and 2nd terms on the right-hand side of Eq.(8.16)),

$$P(1,2) = -\frac{q}{m}\langle E(1)\,\delta f(2)\rangle \frac{\partial}{\partial v_1}\langle f\rangle - \frac{q}{m}\langle E(2)\,\delta f(1)\rangle \frac{\partial}{\partial v_2}\langle f\rangle \tag{8.21}$$

is well behaved as $1 \to 2$, so we hereafter neglect its dependence on relative separation (x_-, v_-). Thus, $\langle \delta f(1)\,\delta f(2)\rangle$ is sharply peaked at small separation, and the singularity of $\langle \delta f(1)\,\delta f(2)\rangle$ as $x_-, v- \to 0$ is regulated *only* by collisional dissipation. This behaviour is precisely analogous to the behaviour of the two-point velocity fluctuation correlation $\left\langle \tilde{V}(1)\,\tilde{V}(2)\right\rangle$ in Navier–Stokes turbulence. We remark in passing that collisional dissipation is well known to be necessary in order to regulate small-scale entropy fluctuations in turbulent Vlasov systems.

The physical origin of $P(1,2)$ is relaxation of the mean distribution $\langle f\rangle$, as may be seen from the following argument, which neglects collisions. Consideration of phase space density ensures that,

$$\frac{\mathrm{d}}{\mathrm{d}t}f^2 = \frac{\mathrm{d}}{\mathrm{d}t}(\langle f\rangle + \delta f)^2 = 0, \tag{8.22a}$$

so its average over phase space follows,

$$\int \mathrm{d}\Gamma \frac{\mathrm{d}}{\mathrm{d}t}(\delta f)^2 = -\int \mathrm{d}\Gamma \frac{\partial}{\partial t}\langle f\rangle^2, \tag{8.22b}$$

where the surface term on the right-hand side vanishes. Since evolution of the mean $\langle f\rangle$ obeys the relation,

$$\frac{\partial}{\partial t}\langle f\rangle = -\frac{q}{m}\frac{\partial}{\partial v}\langle E\delta f\rangle, \tag{8.22c}$$

an integration by parts on the right-hand side of Eq.(8.22b) gives,

$$\int \mathrm{d}\Gamma \frac{\partial}{\partial t}(\delta f)^2 = -2\frac{q}{m}\int \mathrm{d}\Gamma \langle E\delta f\rangle \frac{\partial}{\partial v}\langle f\rangle, \tag{8.22d}$$

which is equivalent to Eq.(8.21). Equation (8.22c) manifestly links production of perturbation phasestrophy to relaxation of $\langle f\rangle$, consistent with intuition.

We see that, absent collisions, $\left\langle \delta f^2\right\rangle$ diverges as $1 \to 2$, while $P(1,2)$ remains finite. We now address, the origin of the divergence in $\left\langle \delta f^2\right\rangle$ in the limit of small collisionality. The fluctuation δf has, in general, two components, the coherent and the incoherent part. The coherent correlation function $\langle f^c(1)\, f^c(2)\rangle$ satisfies the relation (as is explained in Chapter 3),

$$\left\langle f^c(1)\, f^c(2)\right\rangle = 2\tau_{\mathrm{cor}} D_{\mathrm{QL}} \left(\frac{\partial}{\partial v}\langle f\rangle\right)^2, \tag{8.23}$$

where τ_{cor} is the wave–particle correlation time $\sim (k^2 D_{\mathrm{QL}})^{-1/3}$ and D_{QL} is the quasi-linear diffusion coefficient. The right-hand side is finite as $1 \to 2$, even in the absence of collisions. Thus, the coherent correlation function does not contribute to the divergent behaviour of the correlation function $\langle \delta f(1)\, \delta f(2)\rangle$. Hence, we confirm that δf must contain an additional constituent beyond f^c, so,

$$\delta f = f^c + \tilde{f}. \tag{8.24}$$

This incoherent fluctuation $\tilde{f}$ is the 'granulation' or 'phase space eddy' piece. We are shown later in this chapter that it drives the dynamical friction contribution to the evolution of $\langle f\rangle$, which enters in addition to the diffusive relaxation (driven by the coherent part of δf).

8.2.1.3 Impact of granulations on the evolution of the mean

To illustrate dynamical friction self-consistently, we employ the approach of the Lenard–Balescu theory (discussed in Chapter 2) to construct an evolution equation for $\langle f\rangle$ which incorporates the effect of phase space density granulations. The novel contribution from granulations is a drag term, which enters in addition to the usual quasi-linear diffusion (presented in Chapter 3).

The physics of the granulation drag is momentum loss via radiation of waves (ultimately damped), much like the way a ship loses momentum by propagation of a wave wake. Recall that emission of waves by discrete particles is explained in Chapter 2. There, fluctuations associated with the discreteness of particles are retained, in parallel to the (smooth) fluctuations due to collective modes that satisfy the dispersion relation. By analogy with this, if there are 'granulations' in the phase space, in addition to the (smooth) fluctuations that are coherent with eigenmodes, these granulations can emit waves because of their effective discreteness. We explained in the previous subsections of this chapter that 'granulations' must exist in phase space turbulence, and they are 'produced' in conjunction with the relaxation of the mean distribution function. Thus, naturally we are motivated to study the influence of granulations. (A noticeable difference between the argument here and that in Chapter 2 is that, while the 'discreteness' of particles is prescribed for thermal fluctuations, the magnitude and distribution of granulations must be determined self-consistently here, via a turbulence theory.)

Damping of the wave wake emitted by granulations opens a channel for collisionless momentum exchange between species, either with or without linear instability. Indeed, such momentum change processes induce the novel, nonlinear instability mechanisms mentioned earlier in this chapter. As in the case of forward enstrophy cascade in 2D turbulence, conservation of total phase space density links phasestrophy production to the relaxation of the mean $\langle f \rangle$, including both mean diffusion and drag. Since stationarity requires a balance (akin to the spectral balance discussed in Chapters 4 and 5) between phasestrophy production P and phasestrophy transfer, we have from Eq.(8.17),

$$\frac{\partial}{\partial t}\left\langle \delta f^2 \right\rangle + \frac{1}{\tau\left(\Delta x, \Delta v\right)}\left\langle \delta f^2 \right\rangle = P, \tag{8.25a}$$

where nonlinear interactions in the phase space are physically written by use of $\tau\left(\Delta x, \Delta v\right)$. Of course, the phase space element lifetime $\tau\left(\Delta x, \Delta v\right)$ is directly analogous to $\tau\left(l\right)$, the eddy lifetime for scale l. This relation gives,

$$\left\langle \delta f^2 \right\rangle = \tau\left(\Delta x, \Delta v\right) P, \tag{8.25b}$$

in a stationary state. The similarity to the dynamics of production in the Prandtl mixing theory discussed in Chapter 2 should be obvious. Hence, we see that the phase space eddy lifetime $\tau\left(\Delta x, \Delta v\right)$, along with production, sets $\left\langle \delta f^2 \right\rangle$. As in the case of two-particle dispersion in the 2D enstrophy-cascade range (as in Richardson, see Chapter 2), $\tau\left(\Delta x, \Delta v\right)$ can be related to the exponentiation time for relative separation of orbits stochasticized by the turbulence. Since production depends on the fluctuations,

$$P = P\left[\left\langle \delta f^2 \right\rangle\right],$$

we can thus 'close the loop' of theoretical construction and obtain a phasestrophy balance condition. The loop between phasestrophy, radiated electric field fluctuations, production and lifetime is illustrated in Figure 8.8. In the two following subsections, 8.2.2 and 8.2.3, the production P and the phase space eddy lifetime $\tau\left(\Delta x, \Delta v\right)$ are analyzed.

This development of the theory of Vlasov turbulence is illustrated in Table 8.3. The detailed comparison and contrast of quasi-geostrophic Hasegawa–Mima (QG H–M) turbulence and Vlasov turbulence is summarized in Table 8.4.

8.2.2 Physics of production and relaxation

We now turn to the detailed physics of the phasestrophy production term $P(1, 2)$. Since incoherent fluctuations (i.e., granulations) are present, $\delta f = f^c + \tilde{f}$, we have (absorbing the factor of 2),

Table 8.2. *Theoretical development*

Basic concepts	Vlasov turbulence
Eddy scale: l intensity: enstrophy $\langle \lvert\nabla \times v\rvert^2\rangle$	Phase space density granulation scale: Δx, Δv intensity: phasestrophy $\langle \delta f^2 \rangle$
Enstrophy cascade	Phasestrophy cascade
Lenard–Balescu operator $\rightarrow$ drag due to discreteness	Lenard–Balescu operator $\rightarrow$ drag due to granulations
Production versus cascade	Production versus straining

Table 8.3. *Comparison and contrast of quasi-geostrophic Hasegawa–Mima (QG–HM) turbulence and Vlasov turbulence*

Notion	QG–HM turbulence	Vlasov turbulence
Correlation	potential enstrophy $\langle \delta Q(1) \delta Q(2) \rangle$	fluctuation phasestrophy $\langle \delta f(1) \delta f(2) \rangle$
Production	$-\langle V_x \delta Q \rangle \dfrac{\partial}{\partial x} \langle Q \rangle$	$-\dfrac{q}{m} \langle E \delta f \rangle \dfrac{\partial}{\partial v} \langle f \rangle$
Transfer/ *dispersion*	$\delta \mathbf{V} \cdot \nabla_-$ $\tau^{-1} \sim \alpha^{1/3}$	$T_{1,2}$ $\tau \sim \tau_c \sim (k^2 D)^{-1/3}$
Cascade	enstrophy $\langle \delta Q^2 \rangle \sim \alpha^{2/3} k^{-1}$	phasestrophy $\langle \delta f^2 \rangle \sim \alpha^{2/3} k^{-1/3}$
Dissipation scale	l_d	l_d, $\Delta v(l_d)$

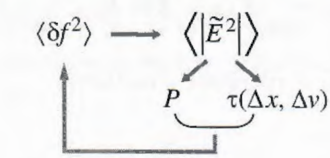

Fig. 8.8. A self-consistent loop between the phasestrophy, radiated electric field fluctuations, production and lifetime of granulations.

$$\begin{aligned} P(1,2) &= -\frac{q}{m} \langle E \delta f \rangle \frac{\partial}{\partial v} \langle f \rangle \\ &= -\frac{q}{m} \left\langle E f^c \right\rangle \frac{\partial}{\partial v} \langle f \rangle - \frac{q}{m} \left\langle E \tilde{f} \right\rangle \frac{\partial}{\partial v} \langle f \rangle . \end{aligned} \tag{8.26}$$

Here, $\langle E f^c \rangle$ yields the familiar coherent relaxation term, related to quasi-linear diffusion discussed in Chapter 3, while $\left\langle E \tilde{f} \right\rangle$ gives the new dynamical friction term (see also Adam *et al.* (1979) and Laval and Pesme (1983)). This is closely

Table 8.4. *Comparison and contrast of test particle model and theory of phase space granulations*

Notion	Test particle model	Phase space granulation
Regime	Near equilibrium Thermal fluctuations Linearly stable modes	Non-equilibrium Turbulent fluctuations, granulations mode–mode coupling Linearly stable *or* nonlinearly saturated modes
Content	Emission balances absorption Incoherent $\tilde{f} \leftrightarrow$ discreteness Dressed-test particle	Phasestrophy cascade driven production Incoherent $\tilde{f} \leftrightarrow$ granulations Clump – phase space eddy
Structure	$\left\langle \tilde{f} \right\rangle = \frac{1}{n} \langle f \rangle \, \delta(x_-)\delta(v_-)$ $J(v) \rightarrow$ diffusion + drag $D \leftrightarrow$ stochastic acceleration drag from discreteness	$\left\langle \tilde{f}^2 \right\rangle$ from closure theory $J(v) \rightarrow$ diffusion + drag $D \leftrightarrow$ stochastic acceleration drag from granulations

analogous to the outcome of the Lenard–Balescu theory of Chapter 2, where $\tilde{f}$ is due to discreteness and f^c is a linear coherent response.

8.2.2.1 *Property of coherent terms*

For a (k, ω)–Fourier component, we write the coherent part f^c as,

$$f^c_{k,\omega} = -\frac{q}{m} R\,(\omega - kv)\, E_{k,\omega} \frac{\partial}{\partial v} \langle f \rangle\,, \tag{8.27a}$$

where $R(\omega - kv)$ is the particle response function,

$$R\,(\omega - kv) = \frac{i}{\omega - kv + i\tau^{-1}_{c,k,\omega}}, \tag{8.27b}$$

and $\tau_{c,k,\omega}$ is the wave–particle coherence time for the (k, ω)–Fourier component. The coherent production term P_c is then just,

$$P_c = D_{\mathrm{QL}} \left(\frac{\partial}{\partial v} \langle f \rangle \right)^2 . \tag{8.27c}$$

Note that P_c has the classic form of production as given by mixing length theory (see the discussion of pipe flow in Chapter 2), in that it says in essence that the rearrangement of $\langle f \rangle$ produces secular growth of $\left\langle \delta f^2 \right\rangle$. This process is highlighted as follows. The perturbation due to rearrangement of $\langle f \rangle$ takes the form,

$$\delta f = \langle f(v) \rangle - \langle f(v - \Delta v) \rangle \sim \Delta v \frac{\partial}{\partial v} \langle f \rangle , \tag{8.28a}$$

that is, the mean of the statistically averaged square δf^2 is,

$$\left\langle \delta f^2 \right\rangle \sim (\Delta v)^2 \left(\frac{\partial}{\partial v} \langle f \rangle \right)^2 . \tag{8.28b}$$

In a diffusion process, the mean deviation evolves as,

$$(\Delta v)^2 \sim D_{\mathrm{QL}} t , \tag{8.28c}$$

so the time derivative of Eq.(8.28b) is given by,

$$\frac{\partial}{\partial t} \left\langle \delta f^2 \right\rangle \sim D_{\mathrm{QL}} \left(\frac{\partial}{\partial v} \langle f \rangle \right)^2 . \tag{8.28d}$$

Note that Eq.(8.28d) states that fluctuation phasestrophy must grow secularly on a transport time-scale, given the presence of a turbulence spectrum $\left\langle E^2 \right\rangle_{k,\omega}$, and phase space gradients $\partial \langle f \rangle / \partial v$. It is useful to remark here that the growth on transport time-scales discussed here is also the origin of the 'growing weight' problem in long time runs of δf PIC (particle in cell) simulation codes (Nevins *et al.*, 2005). Here the term 'weight' refers to a parameter associated with a particle that tracks its effective δf. On long, transport timescales, this unavoidable growth of δf without concomitant evolution of $\langle f \rangle$ and without dissipation of δf fluctuations via collisions, will lead to unphysical weight growth and thus to unacceptably high noise levels in the simulation.

8.2.2.2 *A note on productions*

The dynamical friction term has the appearance of a Fokker–Planck drag, since a Fokker–Planck equation for $\langle f \rangle$ has the generic structure,

$$\frac{\partial}{\partial t} \langle f \rangle = - \frac{\partial}{\partial v} J(v) , \tag{8.29a}$$

where the flux in the phase space takes the form,

$$J(v) = -D \frac{\partial}{\partial v} \langle f \rangle + F \langle f \rangle . \tag{8.29b}$$

Field emitted by granulations

To actually calculate the dynamical friction term, we must relate $\tilde{f}$ (and f^c) to E via Poisson's equation. The explicit relations are discussed below. In this explanation, we address the phase space dynamics of *ions*, and introduce the electron response in terms of a "response function" $\hat{\chi}_e$ as,

$$\nabla^2 \phi = -4\pi n_0 q \int \mathrm{d}v \delta f - 4\pi \hat{\chi}_e \frac{n_0 q^2 \phi}{T_e}, \tag{8.30a}$$

where ϕ is the (fluctuating) electrostatic potential, n_0 is the mean number density, q is a unit charge, T_e is electron temperature, and $\hat{\chi}_e$ is a linear susceptibility (response function) of electrons. For simplicity, here we consider ion phase space turbulence. Taking $\delta f = f^c + \tilde{f}$, we rewrite Eq.(8.30a) as,

$$\epsilon\,(k, \omega)\, \phi_{k,\omega} = -\frac{4\pi n_0 q}{k^2} \int \mathrm{d}v\, \tilde{f}_{k,\omega}, \tag{8.30b}$$

where the contributions of f^c and $\hat{\chi}_e$ are included in the dielectric function $\epsilon(k, \omega)$, which is,

$$\epsilon\,(k, \omega) = 1 - \frac{\omega_{p,i}^2}{k} \int \mathrm{d}v \frac{1}{\omega - kv} \frac{\partial}{\partial v} \langle f \rangle - \frac{\hat{\chi}_e}{k^2 \lambda_{\mathrm{De}}^2}. \tag{8.30c}$$

Here, λ_{De} is the Debye length and $\omega_{p,i}$ is the ion plasma oscillation frequency. Note that, just as in the test particle model in Chapter 2, the incoherent part plays the role of a source in Eq.(8.30b), and the coherent fluctuation f^c forms part of the screening response to the incoherent fluctuation $\hat{f}$, (due to granulations). For stable or over-saturated modes (i.e., waves which are nonlinearly over-damped, beyond marginal saturation), we can then write the potential in terms of the screened incoherent fluctuation,

$$\phi_{k,\omega} = -\frac{4\pi n_0 q}{\epsilon\,(k, \omega)\, k^2} \int \mathrm{d}v\, \tilde{f}_{k,\omega}. \tag{8.31a}$$

Note that Eq.(8.31a) is rigorously valid *only* in the long-time asymptotic limit, where "long" is set by the time required for the damped or over-saturated collective mode response to decay. This is a consequence of the structure of the full solution to Eq.(8.30b), which is,

$$\begin{aligned} \phi_{k,\omega} &= -\frac{4\pi n_0 q}{\epsilon\,(k, \omega)\, k^2} \int \mathrm{d}v\, \tilde{f}_{k,\omega} + \sum_j \phi_{j,0} \exp i \left(k_j x - \omega_j t\right) \\ &\rightarrow -\frac{4\pi n_0 q}{\epsilon\,(k, \omega)\, k^2} \int \mathrm{d}v\, \tilde{f}_{k,\omega}, \end{aligned} \tag{8.31b}$$

for Im $\omega_j < 0$. Obviously, then, the notion of a "screened granulation" requires reconsideration as we approach marginally from below or in transient states with growing modes.

Further progress follows by relating both $\langle E f^c \rangle$ and $\left\langle E \tilde{f} \right\rangle$ to the incoherent, or granulation, correlation function $\left\langle \tilde{f} \tilde{f} \right\rangle$, which is the effective source in the theory, again analogous to the discreteness correlation in the test particle model in Chapter 2. Taking the spectrum to be sufficiently broad so the auto-correlation time is short and renormalization unnecessary, we have,

$$\left|\phi_{k,\omega}\right|^2 = \left(\frac{4\pi n_0 q}{k^2}\right)^2 \iint \mathrm{d}v_1 \mathrm{d}v_2 \frac{\left\langle \tilde{f}(v_1) \tilde{f}(v_2) \right\rangle_{k,\omega}}{\left|\epsilon(k,\omega)\right|^2}, \tag{8.32}$$

so that the production term (8.26),

$$P = -\frac{q}{m}\left\langle E f^c \right\rangle \frac{\partial}{\partial v}\langle f \rangle - \frac{q}{m}\left\langle E \tilde{f} \right\rangle \frac{\partial}{\partial v}\langle f \rangle \equiv P_c + P_G, \tag{8.33a}$$

is given by the use of Eq.(8.32). Substituting Eq.(8.31b) into Eq.(8.27a), the coherent piece P_c is calculated as,

$$\frac{q}{m}\left\langle E f^c \right\rangle = -\frac{\partial}{\partial v}\langle f \rangle \sum_{k,\omega} \frac{q^2 k^2 \pi}{m^2} \delta(\omega - kv) \left(\frac{4\pi n_0 q}{k^2}\right)^2 \iint \mathrm{d}v_1 \mathrm{d}v_2 \frac{\left\langle \tilde{f} \tilde{f} \right\rangle_{k,\omega}}{\left|\epsilon(k,\omega)\right|^2}. \tag{8.33b}$$

The incoherent or granulation-induced correlation contribution to P_G (dynamical friction term) is given by,

$$\frac{q}{m}\left\langle E \tilde{f} \right\rangle = \frac{\partial}{\partial v}\langle f \rangle \sum_{k,\omega} \frac{qk}{m} \frac{4\pi n_0 q}{k^2} \iint \mathrm{d}v_1 \mathrm{d}v_2 \frac{\mathrm{Im}\epsilon(k,\omega)}{\left|\epsilon(k,\omega)\right|^2} \left\langle \tilde{f} \tilde{f} \right\rangle_{k,\omega}. \tag{8.33c}$$

Note that, in contrast to the usual practice in quasi-linear theory, here in Eq.(8.33), both k and ω are summed over, since the latter is not tied or restricted to wave resonances (i.e., $\omega \neq \omega_k$), and frequency broadening occurs. Indeed, we shall see that ballistic mode Doppler emission is a significant constituent in the spectrum, along with collective mode lines.

8.2.2.3 Introduction of modeling for the structure of granulations

To progress from here, we must simplify the correlation function of granulations $\left\langle \tilde{f}(v_1) \tilde{f}(v_2) \right\rangle$. To do so, keep in mind that:

(i) The correlation $\left\langle \tilde{f}(v_1)\tilde{f}(v_2)\right\rangle$ is sharply peaked at small relative velocity, i.e.,

$$\left\langle \tilde{f}\tilde{f}\right\rangle \sim F\left(\frac{v_-}{\Delta v}\right),$$

where Δv belongs to the class of fine-scale widths (of the order of wave–particle resonance or trapping width, etc.). Thus the dependence on the centroid x_+, v_+, is neglected.

(ii) For weak turbulence, we can derive $\left\langle \tilde{f}\tilde{f}\right\rangle_{k,\omega}$ from $\left\langle \tilde{f}\tilde{f}\right\rangle_k$ via the linear particle propagator, as,

$$\begin{aligned}\left\langle \tilde{f}(v_1)\tilde{f}(v_2)\right\rangle_{k,\omega} &= \mathrm{Re}\left\{\int_0^{\infty} \mathrm{d}\tau e^{i(\omega-kv)\tau} + \int_{-\infty}^{0} \mathrm{d}\tau e^{-i(\omega-kv)\tau}\right\} \\ &\quad \times \left\langle \tilde{f}(v_1)\tilde{f}(v_2)\right\rangle_k \\ &\cong 2\pi\delta(\omega-kv)\left\langle \tilde{f}(v_1)\tilde{f}(v_2)\right\rangle_k,\end{aligned} \tag{8.34}$$

so that,

$$\int \mathrm{d}v\left\langle \tilde{f}(v_1)\tilde{f}(v_2)\right\rangle_{k,\omega} = \frac{2\pi}{|k|}\left\langle \tilde{f}(u)\tilde{f}(v)\right\rangle_k, \tag{8.35}$$

where $u = \omega/k$ is the fluctuation phase velocity.

Equation (8.34) should be thought of as the resonant particle limit of the more general result,

$$\left\langle \tilde{f}(v_1)\tilde{f}(v_2)\right\rangle_{k,\omega} \cong \frac{2\tau_{c,k}^{-1}}{(\omega-kv)^2+\tau_{c,k}^{-2}}\left\langle \tilde{f}(v_1)\tilde{f}(v_2)\right\rangle_k,$$

where $\tau_{c,k}$ is a correlation time. *A distinguishing property of resonant particle, phase space turbulence is the linear proportionality of* $\left\langle \tilde{f}\tilde{f}\right\rangle_{k,\omega}$ *to* $\left\langle \tilde{f}\tilde{f}\right\rangle_k$ *via the amplitude-independent, resonant particle propagator* $\pi\delta(\omega-kv)$. This is the case of interest to our discussion of phase space density granulations. In the non-resonant limit,

$$\left\langle \tilde{f}\tilde{f}\right\rangle_{k,\omega} \cong \frac{2\tau_{c,k}^{-1}}{(\omega-kv)^2+\tau_{c,k}^{-2}}\left\langle \tilde{f}\tilde{f}\right\rangle_k \cong \frac{2\tau_{c,k}^{-1}}{\omega^2}\left\langle \tilde{f}\tilde{f}\right\rangle_k,$$

as is usually encountered. Finally, then,

$$\iint \mathrm{d}v_1\mathrm{d}v_2\left\langle \tilde{f}(v_1)\tilde{f}(v_2)\right\rangle_{k,\omega} = \frac{2\pi}{|k|}\left\langle \tilde{n}\tilde{f}(u)\right\rangle_k, \tag{8.36}$$

where $\tilde{n}$ is the density fluctuation associated with the granulations. Equation (8.36) is a particularly simple and attractive result, tying $\langle \tilde{n}\tilde{n}\rangle_{k,\omega}$ to $2\pi|k|^{-1}\left\langle \tilde{n}\tilde{f}(u)\right\rangle_k$, namely the phase space density correlation function on resonance (i.e., at $v = u$), with characteristic frequency $\sim |k|\, v_{\mathrm{T}}$.

We can simplify the production correlations, i.e. Eqs.(8.33b) and (8.33c), by use of Eq.(8.36), to obtain,

$$\frac{q}{m}\left\langle Ef^{c}\right\rangle = -\frac{\partial}{\partial v}\langle f\rangle \sum_{k,\omega}\frac{q^2k^2\pi}{m^2}\delta(\omega - kv)\left(\frac{4\pi n_0 q}{k^2}\right)^2 \frac{2\pi}{|k|}\frac{\left\langle \tilde{n}\tilde{f}(u)\right\rangle_k}{|\epsilon(k,\omega)|^2}, \quad (8.37a)$$

and,

$$\frac{q}{m}\left\langle E\tilde{f}\right\rangle = -\sum_{k,\omega}\frac{qk}{m}\frac{4\pi n_0 q}{k^2}\frac{\mathrm{Im}\varepsilon(k,\omega)}{|\epsilon(k,\omega)|^2}\frac{2\pi}{|k|}\left\langle \tilde{f}(v)\,\tilde{f}(u)\right\rangle_k. \quad (8.37b)$$

Equations (8.37a) (8.37b) may then be combined to yield the net velocity current $J(v)$,

$$J(v) = \frac{q}{m}\langle E\delta f\rangle = \sum_{k,\omega}\frac{\omega_{\mathrm{pi}}^2}{k}\frac{2\pi^2}{|k|^2}\frac{G}{|\epsilon(k,\omega)|^2}, \quad (8.38a)$$

where,

$$\begin{aligned} G(v) = {} & \frac{\omega_{\mathrm{pi}}^2}{k}\delta(v-u)\left\{\left\langle \tilde{n}\tilde{f}(u)\right\rangle_k \frac{\partial}{\partial v}\langle f\rangle - \left\langle \tilde{f}(v)\,\tilde{f}(u)\right\rangle_k \frac{\partial}{\partial v}\langle f\rangle\Big|_u\right\} \\ & - \left\langle \tilde{f}(v)\,\tilde{f}(u)\right\rangle_k \mathrm{Im}\epsilon_e(k,\omega), \end{aligned} \quad (8.38b)$$

and ϵ_e is the electrons contribution to the dielectric E.

8.2.2.4 Like-particle and inter-particle interactions

Equation (8.38) merits some detailed discussion. First, note that Eq.(8.38b) implies that ion production P_i, Eq.(8.38a), may be decomposed into like-particle (i.e., ion–ion) and interspecies (i.e., ion–electron) contributions, i.e.,

$$P_i = P_{i,i} + P_{i,e}, \quad (8.39a)$$

where,

$$P_{i,i} = -\frac{\partial}{\partial v}\langle f_i\rangle \sum_{k,\omega}\frac{\omega_{\mathrm{pi}}^2}{k}\frac{2\pi^2}{|k|^2}\frac{G_{i,i}}{|\epsilon(k,\omega)|^2}, \quad (8.39b)$$

and,

$$P_{i,e} = \frac{\partial}{\partial v} \langle f_i \rangle \sum_{k,\omega} \frac{\omega_{\mathrm{pi}}^2}{k} \frac{2\pi^2}{|k|^2} \frac{G_{i,e}}{|\epsilon(k,\omega)|^2}, \tag{8.39c}$$

with,

$$G_{i,i} = \frac{\omega_{\mathrm{pi}}^2}{k} \delta(v-u) \left\{ \left\langle \tilde{n} \tilde{f}(u) \right\rangle_k \frac{\partial}{\partial v} \langle f_i \rangle - \left\langle \tilde{f}(v) \tilde{f}(u) \right\rangle_k \frac{\partial}{\partial v} \langle f_i \rangle \bigg|_u \right\} \tag{8.39d}$$

$$G_{i,e} = \left\langle \tilde{f}(v) \tilde{f}(u) \right\rangle_k \mathrm{Im}\epsilon_e(k,\omega). \tag{8.39e}$$

Second, since the granulation correlation function is sharply localized,

$$\left\langle \tilde{f}(u) \tilde{f}(v) \right\rangle_k \simeq \delta(u-v) \left\langle \tilde{n} \tilde{f}(u) \right\rangle_k, \tag{8.40}$$

so the second term on the right-hand side of Eq.(8.39d) vanishes, to good approximation, as,

$$G_{i,i} \cong \frac{\omega_{\mathrm{pi}}^2}{k} \delta(v-u) \left\{ \left\langle \tilde{n} \tilde{f}(u) \right\rangle_k \frac{\partial}{\partial v} \langle f_i \rangle - \left\langle \tilde{n} \tilde{f}(u) \right\rangle_k \frac{\partial}{\partial v} \langle f_i \rangle \bigg|_u \right\} \to 0. \tag{8.41}$$

From this consideration we see that the like-particle contribution to relaxation and production *vanishes*! This is precisely analogous to the vanishing of like-particle contributions to relaxation in one-dimension for the Lenard–Balescu theory, discussed in Chapter 2. The underlying physics is the same as well – in 1D, interactions that conserve energy and momentum leave the final state identical to the initial state, so no relaxation can occur. Here, rather than a physical "collision", the interaction in question is the scattering of a particle with velocity v by a fluctuation with phase velocity u.

In the event that $P_{i,i} \to 0$, we have,

$$P_{i,e} = \frac{\partial}{\partial v} \langle f_i \rangle \sum_{k,\omega} \frac{\omega_{\mathrm{pi}}^2}{k} \frac{2\pi^2}{|k|^2} \frac{\mathrm{Im}\varepsilon_e(k,\omega)}{|\epsilon(k,\omega)|^2} \left\langle \tilde{f}(v) \tilde{f}(u) \right\rangle_k. \tag{8.42}$$

Once again, we see that,

$$\frac{\partial \langle f_i \rangle}{\partial v} \frac{\partial \langle f_e \rangle}{\partial v} < 0$$

is required for $P_{i,e} > 0$, and net production.

8.2.2.5 Momentum transfer channel

Two other features of these results merit special discussion. First, proximity to collective resonance (i.e., small $\epsilon(k, ku)$) can strongly enhance relaxation and transport, since in such cases, the granulations will radiate rather weakly damped waves, thus leaving a significant "wake". Second, $P_{i,e} \neq 0$ presents an interesting alternative to the quasi-linear momentum transfer "channel", discussed in Chapter 3, and so may have interesting implications for anomalous resistivity.

Recall from Chapter 3 that in quasi-linear theory (for *electrons* in 1D), resonant particle momentum is exchanged with wave momentum, while conserving the sum, so that,

$$\frac{\partial}{\partial t}\left\{\langle P_{\mathrm{res}}\rangle + \sum_k kN_k\right\} = 0. \tag{8.43a}$$

Here,

$$\frac{\partial}{\partial t}\langle P_{\mathrm{res}}\rangle = q\int \mathrm{d}v\left\langle Ef^c\right\rangle, \tag{8.43b}$$

with f^c given by the resonant, linear response, as in Eq.(8.27), and $\sum_k kN_k$ is the total momentum of waves. As a consequence, the evolution of resonant particle momentum is tied directly (and exclusively) to the wave growth, so it is difficult to simultaneously reconcile stationary turbulence with exchange of momentum by resonant particles. Indeed in 1D, Eq.(8.43b) has only the trivial solution of local plateau formulation (i.e., $\partial\langle f\rangle/\partial v \to 0$) for a stationary state, $\partial\left(\sum kN_k\right)/\partial t = 0$. In contrast, proper accounting for electron granulations opens a new channel for collisionless electron–ion momentum exchange, which does *not* rely on the presence of growing collective modes. To see this, note that for electrons,

$$\begin{aligned}\frac{\partial}{\partial t}\langle P_{\mathrm{res}}\rangle_e &= -\left|q\right|\int \mathrm{d}v\left\langle E\delta f_{e,\mathrm{res}}\right\rangle \\ &= -\left|q\right|\int \mathrm{d}v\Big[\left\langle Ef_e^c\right\rangle + \left\langle E\tilde{f}_e\right\rangle\Big]_{\mathrm{res}}.\end{aligned} \tag{8.44a}$$

Since by analogy with Eq.(8.39a), electron phasestrophy production $P_e = P_{e,i} + P_{e,e}$, and since $P_e = \left(\partial\langle f_e\rangle/\partial v\right)J_e(v)$ (here $J_e(v)$ is the electron velocity space current), then in the absence of growing waves we have,

$$\begin{aligned}\frac{\partial}{\partial t}\langle P_{\mathrm{res}}\rangle_e &= m_e\int \mathrm{d}v\left[J_{e,e}(v) + J_{e,i}(v)\right] \\ &= m_e\int dv J_{e,i}(v),\end{aligned} \tag{8.44b}$$

because $J_{e,e}(v) \to 0$ in 1D. Thus, the evolution of resonant electron momentum is finally just,

$$\frac{\partial}{\partial t}\langle P_{\mathrm{res}}\rangle_e = m_e \sum_{k,\omega} \frac{\omega_{\mathrm{pi}}^2}{k} \frac{2\pi^2}{|k|^2} \frac{\mathrm{Im}\epsilon_i(k,\omega)}{|\epsilon(k,\omega)|^2} \left\langle \tilde{n}\tilde{f}(u)\right\rangle_k . \tag{8.44c}$$

An interesting feature of Eq.(8.44c) is that it reconciles conceptually finite momentum loss by resonant electrons with stationary turbulence by replacing the dependence on wave growth in the (non-stationary) quasi-linear theory by proportionality to collisionless ion dissipation, $\mathrm{Im}\varepsilon_i(k,\omega)$.

It should be noted that there is no "automatic" momentum transfer because an interesting value of the product $\mathrm{Im}\epsilon_i(k,\omega)\left\langle \tilde{n}\tilde{f}(u)\right\rangle_k$ requires that:

(i) electron granulations be excited, so $\partial\langle f_e\rangle/\partial v|_u > 0$, assuming $\partial\langle f_i\rangle/\partial v|_u < 0$;
(ii) electron granulations resonate with ions – i.e., $u \sim v_{Th,i}$, for $\mathrm{Im}\epsilon_i(k,\omega) \neq 0$.

These two conditions require significant overlap of the $\partial\langle f_e\rangle/\partial v|_u > 0$ and $\partial\langle f_i\rangle/\partial v|_u < 0$ regions. Finally, we note that the possibility of collisionless inter-species momentum exchange in the absence of unstable waves also offers a novel, alternative mechanism for anomalous resistivity, which is related, but also complementary, to the classical paradigm involving current-driven ion-acoustic instability.

We conclude this section with Table 8.4, which compares and contrasts the physics and treatment of production and transport in the test particle model (TPM) and Lenard–Balescu theory with their counterparts in the theory of phase space density granulation.

We emphasize that the 1D cancellation is rather special. Thus, should additional degrees of freedom be present in the resonance dynamics, like-particle interchanges, which conserve total particle Doppler frequency, will become possible. In this case, the cancellation no longer need occur. This is seen in the following example of drift wave turbulence. For particles in drift wave turbulence in the presence of a sheared mean flow in the $\hat{y}$-direction, $\mathbf{V}_E = V_E' x\hat{y}$ (where $Vn_0(x)$ is the mean density gradient, and the magnetic field is in the $\hat{z}$-direction), the effective Doppler frequency becomes,

$$\omega_{\mathrm{Doppler}} = k_{\|}v_{\|} + k_y V_E' x,$$

so for that case, a class of scatterings or interchanges of $v_{\|}$ and x exists which leaves total $\omega_{\mathrm{Doppler}}$ invariant. That is, the net transport in velocity and radius can occur via like-particle interactions which scatter both $v_{\|}$ and x but leave $\omega_{\mathrm{Doppler}}$

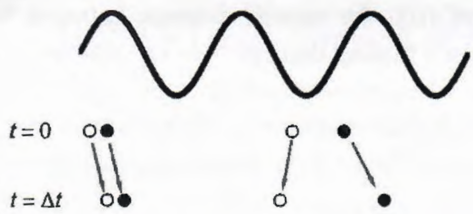

Fig. 8.9. Relative motion of two particles, being scattered by fluctuations. The growth of separation depends on the initial separation.

unchanged. Nevertheless, since a strong magnetic field always severely constrains possible wave–particle resonance, the 1D problem is an important and instructive limit which must always be kept in mind.

8.2.3 Physics of relative dispersion in Vlasov turbulence

Equation (8.16) neatly states the fundamental balance that governs Vlasov turbulence, namely that the structure of two-point correlation in phase space is set by a competition between production (discussed in detail in the previous section) and *relative dispersion*. By 'relative dispersion' we mean the tendency for two trajectories or particles to separate from one another, on account of relative streaming or relative scattering by fluctuating electric fields (see Figure 8.9.) In effect, the rate of relative dispersion assigns a characteristic lifetime $\tau_c(\Delta x, \Delta v)$ to a phase space element of scale, Δx, Δv, so that the stationary two-point correlation is simply $\langle \delta f(1)\, \delta f(2)\rangle \sim \tau_c(\Delta x, \Delta v)\, P(1,2)$. As discussed in the previous section, this picture is essentially a generalization of the Prandtl mixing length theory for turbulent pipe flow (see Chapter 2) to the case of phase space. We should keep in mind that:

(i) Production is not simply diffusive mixing by an effective eddy viscosity, but rather due to a relaxation process involving both diffusion and dynamical friction.
(ii) The calculation of dispersion must account for the structure of the governing Vlasov equation, the statistical property of phase space orbits in Vlasov turbulence, and the effect of collisions at small scale.

In this section, we turn to the calculation of dispersion and the effective phase space element lifetime, $\tau_{\text{sep}}(\Delta x, \Delta v)$. Along the way, we will further elucidate the relationship between the phasestrophy cascade and the evolution of $\langle \delta f(1) \delta f(2)\rangle$.

8.2.3.1 Richardson's theory revisited

The concept of relative dispersion has a long history in turbulence theory, starting with the seminal ideas of L. F. Richardson (discussed in Chapter 2) who considered the growth in time of $l(t)$, the natural distance between two particles in K41 turbulence. Richardson's finding that $l(t)^2 \sim \epsilon t^3$ was the first instance of super-diffusive kinetics (i.e., l increases faster than $l(t)^2 \sim D_0 t$, as for diffusion) in turbulence. A straightforward extension of Richardson's approach to dispersion in scales falling within the forward-enstrophy-cascade range of 2D turbulence gives,

$$\frac{\partial}{\partial t} l(t) \sim \eta^{1/3} l(t),$$

indicating *exponential* growth set by the enstrophy dissipation rate $\eta^{1/3}$. We shall again encounter exponentially increasing relative separation in our study of Vlasov turbulence.

8.2.3.2 *Case of Vlasov turbulence*

In considering dispersion in Vlasov turbulence, two comments are necessary at the outset. First, here we aim to develop a *statistical* weak turbulence theory for the correlation $\langle \delta f^2 \rangle$. Possible local trapping could manifest itself by a net skewness $\langle \delta f^3 \rangle$, indicative of a preferred sign in the phase space fluctuation density, and/or by violation of the weak turbulence ansatz that the spectral auto-correlation time τ_{ac} be short in comparison with the local bounce time, i.e., $\tau_{ac} < \tau_b$. Note that in a dielectric medium, trapping is related to the sign of δf, since only one sign of δf (i.e., $\delta f > 0$ for the BGK mode, or $\delta f < 0$ for a phase space density hole, but not both) is consistent with the existence of a self-trapped, stable state on a scale k^{-1} and phase velocity u_0. The sign for δf which is selected is determined by the sign of the dielectric constant $\epsilon(k, ku_0)$. We will discuss the physics of self-trapping, hole formation, etc. in Volume 2. Obviously, local trapping can surely punctuate, restrict or eliminate relative particle dispersion in a globally fluctuating plasma. However, in this section, which deals exclusively with statistical theory, we hereafter ignore trapping. Thus, the sign of δf is not determined or addressed.

Second, absent trapping in localized structures, we can expect particle orbits to be stochastic, since phase space islands will surely overlap for a broad, multi-mode spectrum. Rigorously speaking, a state of stochasticity implies at least one positive Lyapunov exponent, so neighbouring (test) particle trajectories *must* separate, with divergence increasing exponentially in time. Similarly, then, we can expect particle dispersion to grow exponentially, with the dispersion rate related to the dynamics of the underlying phase space chaos.

It is useful now to recall explicitly that,

$$\frac{\partial}{\partial t} \langle \delta f(1) \delta f(2) \rangle + T_{1,2}\left[\langle \delta f(1) \delta f(2) \rangle\right] = P(1,2), \tag{8.45a}$$

where the two-point evolution operator $T_{1,2}$ is,

$$T_{1,2}\left[\langle\delta f\,(1)\,\delta f\,(2)\rangle\right]=\left(v_1\frac{\partial}{\partial x_1}+v_2\frac{\partial}{\partial x_2}\right)\langle\delta f\,(1)\,\delta f\,(2)\rangle$$
$$+\frac{q}{m}\frac{\partial}{\partial v_1}\langle E\,(1)\,\delta f\,(1)\,\delta f\,(2)\rangle$$
$$+\frac{q}{m}\frac{\partial}{\partial v_2}\langle E\,(2)\,\delta f\,(1)\,\delta f\,(2)\rangle$$
$$+\,\nu\,\langle\delta f\,(1)\,\delta f\,(2)\rangle\,. \tag{8.45b}$$

Hence, we see immediately that any calculation of dispersion in phase space requires some closure or renormalization of the triplet terms $\sim\langle E(1)\delta f(1)\delta f(2)\rangle$ in Eq.(8.45b). Perhaps the simplest, most direct and most transparent closure is via a mean field or quasi-linear approach, as with the closure for $\langle f\rangle$ evolution. Note that here, as is quasi-linear theory, we are concerned primarily with *resonant* scattering processes.

8.2.3.3 Closure modeling for triplet correlations

To construct a quasi-linear theory for the triplet correlation $\langle E(1)\delta f(1)\delta f(2)\rangle$, we proceed by:

(i) first, calculating an effective two-point coherent response, $\langle\delta f(1)\delta f(2)\rangle^c_{k,\omega}$, which is *phase coherent* with the electric field component $E_{k,\omega}$ at *both* phase space points (1) and (2). This is simply the two-point analogue of the familiar one-point coherent response (8.27a), $f^c_{k,\omega}=-(q/m)\times R\,(\omega-kv)\,E_{k,\omega}\partial\,\langle f\rangle\,/\partial v$;
(ii) then iterating to derive a closed equation for $\langle\delta f(1)\delta f(2)\rangle$ evolution in terms of the field spectrum $\langle E^2\rangle_{k,\omega}$. This equation has the form of a bivariate diffusion equation in velocity space.

Let us progress along these lines. To obtain the coherent response $\langle\delta f\,(1)\,\delta f\,(2)\rangle^c_{k,\omega}$, we simply linearize Eq.(8.45a) in $E_{k,\omega}$, neglecting collisions (i.e., take $\tau_{\rm ac}\ll\tau_{\rm coll}$). This gives,

$$\langle\delta f\,(1)\,\delta f\,(2)\rangle^c_{k,\omega}=\mathrm{Re}\frac{q}{m}E_{k,\omega}$$
$$\times\left\{\mathrm{e}^{ikx_1}R\,(\omega-kv_1)\frac{\partial}{\partial v_1}+\mathrm{e}^{ikx_2}R\,(\omega-kv_2)\frac{\partial}{\partial v_2}\right\}\langle\delta f\,(1)\,\delta f\,(2)\rangle\,, \tag{8.46}$$

where $R(\omega-kv_1)$ is the wave–particle resonance functiou, discussed in Eq.(8.27). In practice, we may take $R(\omega-kv)\simeq\pi\delta(\omega-kv)$. Note that $\langle\delta f(1)\delta f(2)\rangle^c_{k,\omega}$

is simply the *sum* of the independent responses of particle 1 at x_1, v_1 plus that for particle 2 at x_2, v_2. These responses are dynamically independent, (i.e., correlated *only* via the driving mean correlator $\langle \delta f(1)\delta(2)\rangle$), as required by the factorizability of the Vlasov hierarchy. (In deriving the Vlasov equation, the joint probability $f(1,2)$ is approximated by multiplication of one-particle distribution functions.) Note also that $(\delta f(1)\delta f(2))^c_{k,\omega}$ is phase coherent with $\exp\left(i\theta_{k,\omega}\right)$, where $\theta_{k,\omega}$ is the phase of the k, ω field, $E_{k,\omega}$, i.e., $E_{k,\omega} = A_{k,\omega}\exp\left(i\theta_{k,\omega}\right)$. In this approach, phase coherency is fundamental, since it links $f(1)f(2)$ to $E_{k,\omega}$ at *both* points 1 and 2.

Then the quasi-linear equation for $\langle \delta f(1)\delta(2)\rangle$ follows from simply substituting Eq.(8.46) into the triplet terms of Eq.(8.45b), e.g.,

$$\frac{\partial}{\partial v_1}\langle E(1)\,\delta f(1)\,\delta f(2)\rangle \simeq \frac{\partial}{\partial v_1}\left\langle E(1)\,(\delta f(1)\,\delta f(2))^c\right\rangle.$$

With this approximation, Eq.(8.45) becomes,

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} + \nu\right)\langle \delta f(1)\,\delta f(2)\rangle \\
&-\left(\frac{\partial}{\partial v_1}D_{11}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}D_{22}\frac{\partial}{\partial v_2}\right)\langle \delta f(1)\,\delta f(2)\rangle \\
&-\left(\frac{\partial}{\partial v_2}D_{21}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_1}D_{12}\frac{\partial}{\partial v_2}\right)\langle \delta f(1)\,\delta f(2)\rangle = P(1,2),
\end{aligned} \tag{8.47a}$$

where,

$$D_{jj} = \sum_{k,\omega}\frac{q^2}{m^2}\left|E_{k,\omega}\right|^2 R\left(\omega - kv_j\right) \quad (j = 1 \text{ or } 2), \tag{8.47b}$$

and,

$$D_{12} = \sum_{k,\omega}\frac{q^2}{m^2}\mathrm{e}^{ik(x_1-x_2)}\left|E_{k,\omega}\right|^2 R\left(\omega - kv_2\right), \tag{8.47c}$$

with $1 \leftrightarrow 2$ for $D_{1,2}$. Equation (8.47a) is a bivariate diffusion equation for $\langle \delta f(1)\delta(2)\rangle$. Observe that D_{11} and D_{22} are usual quasi-linear diffusion coefficients, while D_{12} and D_{21} represent *correlated* diffusion, in that they approach unity for $|kx_-| < 1$, and tend to oscillate and so cancel for $|kx_-| > 1$. Note also that correlated scattering will occur only if v_1 and v_2 resonate with the same portion of the electric field spectrum, so $|v_-| < \Delta v_{\mathrm{T}}$ is required as well.

8.2.3.4 Alternative derivation

It is worthwhile to elaborate on the derivation of Eq.(8.47), prior to embarking on a discussion of its physics. There are at least three ways to derive Eq.(8.47). These are:

(i) the two-point quasi-linear approach, as implemented above;
(ii) a bi-variate Fokker–Planck calculation. In this approach, the validity of which is rooted in particle stochasticity, $\langle \delta f(1)\delta f(2)\rangle$ is assumed to evolve via independent random walks of particle 1 and particle 2;
(iii) a DIA-type closure of the two-point correlation equation, as presented in (Boutros-Ghali and Dupree, 1981).

It is instructive to discuss the bivariate Fokker–Planck calculation in some detail. The essence of Fokker–Planck theory is evolution via small, uncorrelated random scattering events, which add in coherently to produce a diffusive evolution. Thus $T_{1,2}\left\langle \delta f^2\right\rangle$ becomes,

$$T_{1,2}\left\langle \delta f\left(1\right)\delta f\left(2\right)\right\rangle = \left(v_1\frac{\partial}{\partial x_1}+v_2\frac{\partial}{\partial x_2}\right)\left\langle \delta f\left(1\right)\delta f\left(2\right)\right\rangle + \frac{\Delta\left\langle \delta f\left(1\right)\delta f\left(2\right)\right\rangle}{\Delta t}. \tag{8.48}$$

Here $\Delta\left\langle \delta f(1)\delta f(2)\right\rangle/\Delta t$ symbolically represents non-deterministic evolution of $\left\langle \delta f(1)\delta f(2)\right\rangle$ due to stochastic scattering events Δv_1 and Δv_2 in a scattering time step of Δt. (We expect $\Delta t\sim\tau_{\mathrm{ac}}$.) Thus, $\Delta\left\langle \delta f(1)\delta f(2)\right\rangle/\Delta t$ is represented in terms of the transition probability T as,

$$\begin{aligned}
&\frac{\Delta\left\langle \delta f\left(1\right)\delta f\left(2\right)\right\rangle}{\Delta t}\\
&=\left[\iint \mathrm{d}\left(\Delta v_1\right)\mathrm{d}\left(\Delta v_2\right)T\left(\Delta v_1,v_1;\Delta v_2,v_2;\Delta t\right)\right.\\
&\qquad\left.\times\left\langle \delta f\left(v_1-\Delta v_1\right)\delta f\left(v_2-\Delta v_2\right)\right\rangle-\left\langle \delta f\left(1\right)\delta f\left(2\right)\right\rangle\right]\Big/\Delta t.
\end{aligned} \tag{8.49}$$

Here $T\left(\Delta v_1,v_1;\Delta v_2,v_2;\Delta t\right)$ is the two-particle step probability distribution function or transition probability. Since particle scattering is statistically uncorrelated, the transition probability may be factorized, so,

$$T\left(\Delta v_1,v_1;\Delta v_2,v_2;\Delta t\right)=T\left(\Delta v_1,v_1;\Delta t\right)T\left(\Delta v_2,v_2;\Delta t\right). \tag{8.50}$$

With this factorization, the usual Fokker–Planck expansion of the right-hand side of Eq.(8.49) in small step size gives,

$$\frac{\Delta\langle\delta f(1)\,\delta f(2)\rangle}{\Delta t} =$$
$$-\frac{\partial}{\partial v_1}\left[\left(\frac{\langle\Delta v_1\rangle}{\Delta t}+\frac{\langle\Delta v_2\rangle}{\Delta t}\right)\langle\delta f(1)\,\delta f(2)\rangle\right.$$
$$\left.-\frac{\partial}{\partial v_1}\left\{\left(\frac{\langle\Delta v_1\Delta v_1\rangle}{2\Delta t}+\frac{\langle\Delta v_1\Delta v_2\rangle}{2\Delta t}\right)\langle\delta f(1)\,\delta f(2)\rangle\right\}\right]$$
$$-\frac{\partial}{\partial v_2}\left[\left(\frac{\langle\Delta v_2\rangle}{\Delta t}+\frac{\langle\Delta v_1\rangle}{\Delta t}\right)\langle\delta f(1)\,\delta f(2)\rangle\right.$$
$$\left.-\frac{\partial}{\partial v_2}\left\{\left(\frac{\langle\Delta v_2\Delta v_2\rangle}{2\Delta t}+\frac{\langle\Delta v_2\Delta v_1\rangle}{2\Delta t}\right)\langle\delta f(1)\,\delta f(2)\rangle\right\}\right], \tag{8.51a}$$

which has a form,

$$\frac{\Delta\langle\delta f(1)\,\delta f(2)\rangle}{\Delta t} =$$
$$-\frac{\partial}{\partial v_1}\left[(F_{11}+F_{12})\langle\delta f(1)\,\delta f(2)\rangle-\frac{\partial}{\partial v_1}\{(D_{11}+D_{12})\langle\delta f(1)\,\delta f(2)\rangle\}\right]$$
$$-\frac{\partial}{\partial v_2}\left[(F_{22}+F_{21})\langle\delta f(1)\,\delta f(2)\rangle-\frac{\partial}{\partial v_2}\{(D_{22}+D_{21})\langle\delta f(1)\,\delta f(2)\rangle\}\right]. \tag{8.51b}$$

Now clearly D_{11} and D_{12} (along with their counterparts with $1 \leftrightarrow 2$) correspond to single particle and correlated or cross-diffusions, respectively. Likewise, F_{11} and F_{12} (and their counterparts with $1 \leftrightarrow 2$) correspond to drag and cross-drag. It is well known that for a 1D Hamiltonian system, Liouville's theorem requires that,

$$\frac{\partial}{\partial v}\frac{\langle\Delta v\Delta v\rangle}{2\Delta t}=\frac{\langle\Delta v\rangle}{\Delta t}, \tag{8.52}$$

so *dynamical friction cancels the gradient of diffusion*. Similar cancellations occur between the cross-diffusions and cross-drags. Thus, Eq.(8.51) reduces to,

$$T_{1,2}\langle\delta f(1)\,\delta f(2)\rangle=\left(v_1\frac{\partial}{\partial x_1}+v_2\frac{\partial}{\partial x_2}\right)\langle\delta f(1)\,\delta f(2)\rangle$$
$$-\left(\frac{\partial}{\partial v_1}D_{11}\frac{\partial}{\partial v_1}+\frac{\partial}{\partial v_1}D_{12}\frac{\partial}{\partial v_2}+\frac{\partial}{\partial v_2}D_{22}\frac{\partial}{\partial v_2}+\frac{\partial}{\partial v_2}D_{21}\frac{\partial}{\partial v_1}\right)\langle\delta f(1)\,\delta f(2)\rangle, \tag{8.53}$$

where $D_{ij}=\langle\Delta v_i\Delta v_j\rangle/2\Delta t$ (i, j are 1 or 2) and is identical to Eq.(8.47). This rather formal discussion is useful since it establishes that the fundamental physics in Eq.(8.47) is just resonant diffusive scattering dynamics, which results from particle stochasticity. As we shall see, when coupled to free streaming, this results in exponential divergence of orbits, which then determines the particle dispersion rate.

We remark in passing here that the full closure theory for $\langle \delta f(1)\delta f(2)\rangle$ (Krommes, 1984) offers little of immediate utility beyond what is presented here. It is extremely tedious, conceptually unclear (at this moment) and calculationally intractable. Thus we do not discuss it in detail here.

8.2.3.5 Physics of two-particle dispersion

We now turn to a discussion of the physics of the two-particle dispersion process, as given by Eq.(8.47). First, it is clear that the essential physics is resonant scattering in velocity space, due to random acceleration by the electric field spectrum. This scattering can be uncorrelated (giving D_{11}, D_{22}) or correlated (giving D_{12}, D_{21}).

Second, given that the aim here is to calculate a two-point correlation function, and since the two-point correlation function is simply the Fourier transform of the associated field, it is useful to relate Eq.(8.53) to our earlier discussion of spectral evolution, in Chapters 5 and 6. To this end, note that Fourier transformation of Eq.(8.53) (which is in real space) gives,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left\langle \delta f(1)\,\delta f(2)\right\rangle_k + ik\left(v_1 - v_2\right)\left\langle \delta f(1)\,\delta f(2)\right\rangle_k \\
&\quad - \frac{\partial}{\partial v_1} D_{11} \frac{\partial}{\partial v_1}\left\langle \delta f(1)\,\delta f(2)\right\rangle_k - \frac{\partial}{\partial v_2} D_{22} \frac{\partial}{\partial v_2}\left\langle \delta f(1)\,\delta f(2)\right\rangle_k \\
&\quad - \sum_{\substack{\pm p, q \\ p \pm q = k}} \left(\frac{\partial}{\partial v_1}\left\{ \left\langle E^2 \right\rangle_p R\left(\omega - pv\right) \frac{\partial}{\partial v_2}\left\langle \delta f(1)\,\delta f(2)\right\rangle_q \right\} \right. \\
&\qquad\qquad \left. + \frac{\partial}{\partial v_2}\left\{ \left\langle E^2 \right\rangle_p R\left(\omega + pv\right) \frac{\partial}{\partial v_1}\left\langle \delta f(1)\,\delta f(2)\right\rangle_q \right\} \right) \\
&\quad = P_k\left(1, 2\right).
\end{aligned}
\tag{8.54}
$$

Thus, we see immediately that:

- single-particle scattering corresponds to coherent mode coupling and a Markovian 'eddy viscosity' in velocity space;
- correlated diffusion corresponds to incoherent mode coupling and thus to nonlinear noise.

Stated equivalently, the *interplay* of single particle and correlated diffusion corresponds to 'cascading', i.e., the process whereby small scales are generated from larger ones. Note here that since free streaming couples to velocity scattering and diffusion, that too enters the rate at which small scales are generated.

Third, the net stationary value of $\langle \delta f(1) \delta f(2) \rangle$ is set by the *balance* of production with the lifetime of the scale of size x_- and v_-.

8.2.3.6 Calculation of dispersion time

The actual calculation of the dispersion time $\tau_c(x_-, v_-)$ is most expeditiously pursued by working in relative coordinates. To this end, as for Eq.(8.18), we write,

$$x_\pm = \frac{1}{2}\left(x_1 \pm x_2\right),$$
$$v_\pm = \frac{1}{2}\left(v_1 \pm v_2\right),$$

and discard the slow x_+, v_+ dependence in the T_{12} operator. Thus, Equation (8.47) can be simplified to the form,

$$\left(\frac{\partial}{\partial t} + v_- \frac{\partial}{\partial x_-} - \frac{\partial}{\partial v_-} D_{rel} \frac{\partial}{\partial v_-}\right) \langle \delta f\,(1)\, \delta f\,(2) \rangle = P\,(1, 2)\,, \tag{8.55}$$

where,

$$\begin{aligned} D_{\mathrm{rel}} &= D_{11} + D_{22} - D_{12} - D_{21} \\ &= \sum_{k,\,\omega} \frac{q^2}{m^2} \left(1 - \cos\left(k x_-\right)\right) \left\langle E^2 \right\rangle_{k,\omega} R\left(\omega - k v\right). \end{aligned} \tag{8.56a}$$

Here, $D_{\mathrm{rel}}(x_-)$ is the relative diffusion function, which gives a measure of how rapidly particles (separated by x_- in phase space) diffuse apart. Note that for $\left(k^2 x_-^2\right) > 1$ (Balescu, 2005),

$$D_{rel} \simeq D_{11} + D_{22}, \tag{8.56b}$$

so that diffusion then asymptotes to the value for two uncorrelated particles. For $\left\langle k^2 {x_-}^2 \right\rangle < 1$,

$$D_{\mathrm{rel}} \simeq \frac{k_0^2 x_-^2}{2} D, \tag{8.56c}$$

so $D_{\mathrm{rel}} \to 0$ as $x_-^2 \to 0$. Here k_0^2 is a spectral average, i.e., $k_0^2 = \langle k^2 \rangle$ and $D = \left(D_{11} + D_{22}\right)/2$.

While exact calculation of the evolution of relative dispersion is lengthy and intricate, the essential behaviour can be determined by working in the $\left(k^2x_-^2\right) < 1$ limit, which captures key features, like the peaking of $\langle \delta f(1)\delta f(2)\rangle$ on small scales. Indeed, for $\left(k^2x_-^2\right) \ll 1$, $\langle \delta f(1)\delta f(2)\rangle \simeq \left\langle \tilde{f}(1)\tilde{f}(2)\right\rangle$, so no detailed calculations are needed. Of course, as we shall see, the existence of an individual small-scale peak in $\langle \delta f(1)\delta f(2)\rangle$ *requires* collisionality to be weak, i.e., $1/\tau_c \gg \nu$, to ensure that the small-scale structure of the correlation function is not smeared out. Note that the condition $1/\tau_c \gg \nu$ (which defines an effective 'Reynolds number' $Re_{\text{eff}} \sim 1/\tau_c\nu$) is equivalent to $\Delta v_{\text{T}} > 1/k\nu$, i.e., the requirement that the turbulently broadened resonance width exceeds the width set by collisional broadening. In this case we can rewrite Eq.(8.55) as an evolution equation for F, the probability density function (pdf) of relative separations x_-, v_-. As here we are concerned only with relative dispersion, and have tacitly assumed that $1/\tau_c \gg \nu$, we can now drop ν and $P(1,2)$. Thus, F satisfies the simple kinetic equation,

$$\frac{\partial F}{\partial t} + v_-\frac{\partial F}{\partial x_-} - \frac{\partial}{\partial v_-}Dk_0^2x_-^2\frac{\partial F}{\partial v_-} = 0. \tag{8.57}$$

It is now straightforward to derive a coupled set of moment equations from Eq.(8.57). Defining the moments by,

$$\langle A\left(x_-, v_-\right)\rangle \equiv \frac{\iint dx_-dv_-A\left(x_-, v_-\right)F\left(x_-, v_-; t\right)}{\iint dx_-dv_-F\left(x_-, v_-; t\right)}, \tag{8.58a}$$

we have:

$$\frac{\partial}{\partial t}\left\langle x_-^2\right\rangle = 2\left\langle x_-v_-\right\rangle, \tag{8.58b}$$

$$\frac{\partial}{\partial t}\left\langle x_-v_-\right\rangle = \left\langle v_-^2\right\rangle, \tag{8.58c}$$

$$\frac{\partial}{\partial t}\left\langle v_-^2\right\rangle = 2\left\langle D_{\text{rel}}\right\rangle. \tag{8.58d}$$

Equations (8.58b–d) combine to give,

$$\frac{\partial^3}{\partial t^3}\left\langle x_-^2\right\rangle = 4\left\langle D_{\text{rel}}\right\rangle = 4Dk_0^2\left\langle x_-^2\right\rangle, \tag{8.58e}$$

which tells us that mean square trajectory separations increase exponentially in time, at the rate $\left(Dk_0^2\right)^{1/3}$, as long as the condition $\left(k^2x_-^2\right) < 1$ is satisfied. More precisely, Eq.(8.58e), when solved for the initial conditions $\partial\langle x_-\rangle/\partial t = \langle v_-\rangle$ and $\partial\langle v_-\rangle/\partial t = 0$, gives,

$$\left\langle x_-(t)^2\right\rangle = \frac{1}{3}\left(x_-^2 + 2x_-v_-\tau_c + 2v_-^2\tau_c^2\right)\exp\left(\frac{t}{\tau_c}\right), \tag{8.59a}$$

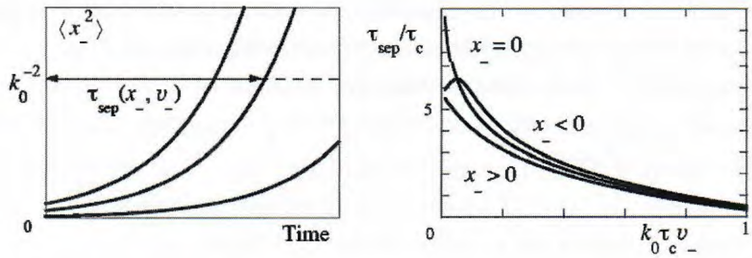

Fig. 8.10. Evolution of the statistical average of separation for various initial separations (left). The separation time τ_{sep} is defined for each initial condition. The separation time is shown as a function of initial separation in velocity space v_-.

where,

$$\tau_c^{-1} = \left(4Dk_0^2\right)^{1/3}. \tag{8.59b}$$

This gives the characteristic decorrelation rate for granulations in phase space turbulence. Equation (8.58e) has three eigensolutions, two of which are damped. To obtain the result of Eq.(8.59a), we neglect the damped solutions, which are time asymptotically subdominant.

Now finally, as we are most interested in the relative dispersion time as a function of given (initial) phase space separation x_-, v_- (i.e., corresponding to a given scale), it is appropriate to define a scale-dependent separation time $\tau_{\text{sep}}(x_-, v_-)$ by the condition,

$$k_0^2 \left\langle x_- \left(\tau_{\text{sep}}\right)^2 \right\rangle = 1, \tag{8.59c}$$

i.e., as the time needed for the pair to disperse k_0^{-1} (Fig. 8.10(a)). Hence,

$$\tau_{\text{sep}}\left(x_-, v_-\right) = \tau_c \ln \left\{ 3k_0^{-2} \left(x_-^2 + 2x_- v_- \tau_c + 2v_-^2 \tau_c^2 \right)^{-1} \right\}. \tag{8.59d}$$

We note this expression applies only when the argument of the logarithmic function is positive. An example is illustrated in Figure 8.10(b).

The expressions for τ_{sep} and τ_c in Eqs.(8.59d) and (8.59b) are the principal results of this section, and so merit some further discussion. First, we note that the basic time scale for relative dispersion is τ_c (Eq.(8.59b)), the wave–particle turbulent decorrelation time, which is also the Lyapunov time for separation of stochastic particle orbits. Thus, the calculated dispersion time is consistent with expectations from dynamical systems theory.

Second, note that $\tau_{\text{sep}} > \tau_c$ for small separations (where $k_0^2 x_-^2 \ll 1$) and/or $k_0 v_- \tau_c = v_- / \Delta v_{\text{T}} < 1$. Thus τ_{sep} is sharply peaked on scales small compared

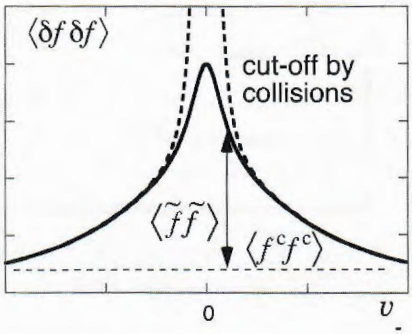

Fig. 8.11. Correlation function $\langle \delta f(1)\delta f(2)\rangle$ as a function of the separation of particles. The collisionless limit (dotted line) and the case where cut-off by collision works at $v_- = 0$. Contributions from the coherent component and granulations are also noted.

with the basic turbulence correlation scales. This peaking is consistent with the collisionless singularity of $\langle \delta f(1)\delta f(2)\rangle$, which is required for the limit $1 \to 2$, as discussed in Eq.(8.25b). Figure 8.11 illustrates schematically the correlation function $\langle \delta f(1)\delta f(2)\rangle$ and contributions from the coherent component and granulations. Retaining finite collisionality removes the singularity, and truncates the peaking of $\langle \delta f(1)\delta f(2)\rangle$ on scales $v_- < \nu/k_0$. As noted above, for this truncation to be observable, $\nu/k_0 < \Delta v_T$ is necessary. As we shall see, even in the absence of collisions, $\int \mathrm{d}v_- \tau_{\mathrm{sep}}(x_-, v_-)$ is finite, so all physical observables are well behaved.

Third, this entire calculation is predicted on the existence of k_0^2, i.e., we assume that,

$$k_0^2 = D^{-1} \sum_{k,\omega} k^2 \frac{q^2}{m^2} \left\langle E^2 \right\rangle_{k,\omega} R(\omega - kv) \tag{8.60c}$$

is finite. The requisite spectral convergence must indeed be demonstrated a posteriori. Absence of such convergence necessitates a different approach to the calculation of the phase space density correlation function.

Fourth, we remark that τ_c is also the effective 'turn-over time' or scale lifetime which determines the phasestrophy cascade in a turbulent Vlasov plasma. Here, small scales are generated by the coupled processes of relative streaming and relative scattering, rather than by eddy shearing, as in a turbulent fluid.

The structure of the theory is summarized in Figure 8.12, where the explanations in this chapter are revisited. Tbe granulations radiate the (non-modal) electric field. The intensity and spectrum of the electric field, which is excited by granulations, is given in Eq.(8.32). By use of the excited electric field, the rate of production is evaluated in Eq.(8.39), as a functional of the granulation correlation function.

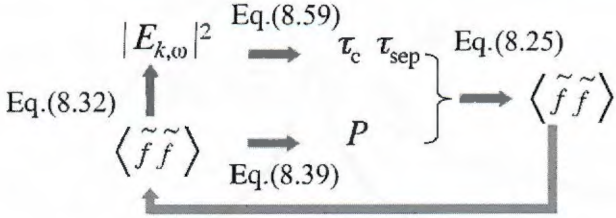

Fig. 8.12. The loop of consistency between the correlation of granulation, radiated electric field, production and separation time.

The triplet correlation is estimated by using the electric field spectrum, so that the separation time is estimated by Eq.(8.59). The self-consistency loop closes at Eq.(8.25), in which the granulation intensity is determined by the balance between the production and lifetime. This set of equations constitutes the theory that determines phase space density granulations.

8.3 Physics of relaxation and turbulent states with granulation

In the preceding sections of Chapter 8, we have discussed the physics of phase space density granulation at length, and have derived the equation of evolution for the correlation function $\langle \delta f\,(1)\,\delta f\,(2)\rangle$. We now turn to the 'bottom line' – we face the question of *what actually happens when granulations are present*. In particular, we focus on two issues, which are:

(1) what types of saturated states are possible, and what is the role of granulations in the dynamics of these states?
(2) what types of novel, nonlinear instability mechanisms may occur via granulations?

The calculations required to answer these questions quantitatively are extremely lengthy and detailed. Hence, in this section we take a 'back-of-an-envelope' approach, and only sketch the essence and key elements of the calculations, along with their physical motivations and implications. Our aim is to help the reader understand the landscape of this 'terra nova'. Once motivated and oriented, a serious reader can then consult the original research literature for details.

As we have seen, the equation of evolution for the two-point correlation function takes the generic form,

$$\left(\frac{\partial}{\partial t} + \frac{1}{\tau_{c_1}}\right) \langle \delta f\,(1)\,\delta f\,(2)\rangle = P\,(1,2)\,, \tag{8.61}$$

so that, as in a turbulent shear flow, the correlation function is set by a balance between drive by gradient relaxation (in this case, the gradients of $\langle f\rangle$) and relative shearing and dispersion, as parametrized by $\tau_{c_1}\,(x_-, v_-)$. In a stationary state, we then have simply,

$$\langle \delta f(1)\, \delta f(1) \rangle = \tau_{c_1} P(1,2). \tag{8.62}$$

The production term $P(1,2)$, as given by Eq. (8.42), is a function of the integrated granulation correlation function $\left\langle \tilde{n} \tilde{f}(u) \right\rangle_k$ and is, in essence, set by the:

(i) free energy stored in driving gradients, which makes $P(1,2) > 0$;
(ii) dielectric screening – which is particularly important near resonance between collective modes with $\omega = \omega_k$ and ballistic modes (i.e. granulations here), for which $\omega = ku$;
(iii) the spectral profile of Doppler emission by test granulations.

It is useful, then, to convert Eq.(8.62) to an integral equation for $\left\langle \tilde{n} \tilde{f}(u) \right\rangle_k$ by subtracting the coherent correlation function and $\left\langle \tilde{f} f^{c} \right\rangle$ cross-term from $\langle \delta f(1)\, \delta f(2) \rangle$, and then integrating over the relative velocity v_-. Now, since,

$$\delta f = f^{c} + \tilde{f}, \tag{8.63a}$$

we have,

$$\left\langle \tilde{f} \tilde{f} \right\rangle = \langle \delta f\, \delta f \rangle - \left\langle f^{c} f^{c} \right\rangle - 2\left\langle f^{c} \tilde{f} \right\rangle. \tag{8.63b}$$

The first subtraction ($\langle f^{c} f^{c} \rangle$) is proportional to the diffusive mixing term in $P(1,2)$, while the second ($2\left\langle f^{c} \tilde{f} \right\rangle$) is proportional to the drag term. Hence, we ultimately have just,

$$\left\langle \tilde{f}(1)\, \tilde{f}(2) \right\rangle = \left(\tau_{\mathrm{cl}}(x_-, v_-) - \tau_{\mathrm{c}}\right) P(1,2), \tag{8.63c}$$

which may be re-written in terms of a $\tau_{\mathrm{cl,eff}}$, peaked sharply as $x_-, v_- \to 0$. Hereafter, we assume this sample subtraction to be in force, and do not distinguish between τ_{cl} and $\tau_{\mathrm{cl,eff}}$. Observe that $1/\tau_{\mathrm{c}} \gg \nu$ is necessary for $\tau_{\mathrm{cl}} - \tau_{\mathrm{c}}$ to exhibit a non-trivial range of scales between the integral scale and the collisional cut-off.

Since generically, $\tau_{\mathrm{cl}} = \tau_{\mathrm{cl}}(v_-/\Delta v_{\mathrm{T}}, k_0 x_-)$, (recall Δv_{T} is a trapping width and k_0 is an integral scale) we have,

$$\begin{aligned} \left\langle \tilde{n} \tilde{f} \right\rangle &= \int \mathrm{d}v_-\, \tau_{\mathrm{cl}} P(1,2) \\ &\cong \Delta v_{Tr} \tau_{\mathrm{cl}} P(1,2). \end{aligned} \tag{8.64a}$$

Recall, though, that,

$$\tau_{\mathrm{cl}} = \tau_{\mathrm{c}} F(v_-/\Delta v_{\mathrm{T}}, k_0 x_-), \tag{8.64b}$$

so the integration over v_- necessarily yields the product $\tau_c \Delta v_T$. Since the resonance width Δv_{Tr} and correlation time τ_c are related by the definition $\Delta v_T = 1/k\tau_c$, we then have,

$$\left\langle \tilde{n} \tilde{f} \right\rangle \cong (1/k_0)\ G\left(k_0 x_-\right) P\left(1,2\right). \tag{8.64c}$$

Here $G\left(k_0 x_-\right)$ is the spatial structure function of the granulation defined by the shape of τ_{cl}. The key point here is that due to the reciprocal relation between τ_c and Δv_T, the integral equation for $\left\langle \tilde{n} \tilde{f} \right\rangle$ defined by Eq.(8.64c) is at least *formally homogeneous*. This homogeneity is ultimately a consequence of the *resonant*, linear propagator appearing in the relation $\langle \tilde{g}\tilde{g} \rangle_{k,\omega} = 2\pi\delta\left(\omega - kv\right) \langle \tilde{g}\tilde{g} \rangle_k$. Fourier transformation in space then gives,

$$\left\langle \tilde{n} \tilde{f} \right\rangle_k = A\left(k, k_0\right) P\left(1,2\right), \tag{8.64d}$$

where $A\left(k, k_0\right)$ is the spatial form factor for the granulation correlation.

Recall from Eq.(8.42), that for our generic example case of ion granulations in CDIA (current driven ion acoustic) turbulence, production is given by,

$$P\left(1,2\right) = \frac{\partial \langle f_i \rangle}{\partial v} \sum_{k,\omega} \frac{\omega_{pi}^2}{k} \frac{2\pi^2}{|k|^2} \frac{\operatorname{Im} \epsilon_e\left(k,\omega\right) \left\langle \tilde{f}\left(v\right) \tilde{f}\left(u\right) \right\rangle_k}{|\epsilon\left(k,\omega\right)|^2}. \tag{8.65}$$

Observe that only two elements in Eq.(8.59) can adjust to yield a stationary balance. These are:

- the electron distribution function, which defines $\operatorname{Im} \epsilon_e\left(k,\omega\right)$;
- the net damping of the wave resonance (i.e. mode) at $\omega = \omega_k$, which is set by $\operatorname{Im} \epsilon\left(k, \omega_k\right)$.

Thus, if we interpret Eq.(8.64d) as a stationarity condition for $\left\langle \tilde{n} \tilde{f} \right\rangle$, we see that a steady state, where production balances dissipation, requires either relaxation of $\langle f \rangle$ *or* a particular value of wave damping i.e. $\operatorname{Im} \epsilon\left(k, \omega_k\right)$. Relaxation of $\langle f \rangle$ (i.e. electron slowing down) may be calculated using the granulation-driven Lenard–Balescu equation. To determine the wave damping, note that the familiar pole approximation,

$$\begin{aligned} \frac{1}{|\epsilon\left(k,\omega\right)|^2} &\cong \frac{1}{\left(\omega - \omega_k\right)^2 \left(\partial\epsilon/\partial\omega|_{\omega_{\mathbf{k}}}\right)^2 + |\operatorname{Im}\epsilon|^2} \\ &\cong \left\{ \frac{1}{\left|\partial\epsilon/\partial\omega|_{\omega_{\mathbf{k}}}\right| |\operatorname{Im}\epsilon\left(k, \omega_{\mathbf{k}}\right)|} \right\} \delta\left(\omega - \omega_{\mathbf{k}}\right), \end{aligned} \tag{8.66}$$

allows us to perform the frequency summation in Eq.(8.65), and so to obtain,

$$P(1,2) \cong \frac{\partial \langle f_i \rangle}{\partial v} \sum_k \frac{\omega_{\mathrm{pi}}^2}{k} \frac{2\pi}{|k|^2} \frac{\operatorname{Im} \epsilon_e (k, \omega_k) \left\langle \tilde{f}(v) \tilde{f}(u_k) \right\rangle_k}{|\operatorname{Im} \epsilon (k, \omega_k)| \left| \partial \epsilon / \partial \omega|_{\omega_{\mathbf{k}}} \right|}, \tag{8.67}$$

where $u_k = \omega_k / k$. Hence, Eq.(8.65) then gives,

$$\left\langle \tilde{n} \tilde{f} \right\rangle_k = A(k, k_0) \sum_k \frac{\omega_{\mathrm{pi}}^2}{k} \frac{2\pi}{|k|^2} \frac{\operatorname{Im} \epsilon_e (k, \omega_k) \int \mathrm{d}v \, (\partial \langle f_i \rangle / \partial v) \left\langle \tilde{f}(v) \tilde{f}(u_k) \right\rangle_k}{|\operatorname{Im} \epsilon (k, \omega_k)| \left| \partial \epsilon / \partial \omega|_{\omega_{\mathbf{k}}} \right|}. \tag{8.68}$$

Though Eq.(8.68) is an integral relation, it transparently reveals the basic scaling of $\operatorname{Im}\epsilon\,(k, \omega_{\mathbf{k}})$ enforced by the stationarity condition of Eq.(8.58), which is,

$$|\operatorname{Im} \epsilon (k, \omega_k)| \sim \frac{-A(k, k_c) \operatorname{Im} \epsilon_e (k, \omega_k)}{\left| \partial \epsilon / \partial \omega|_{\omega_k} \right|}. \tag{8.69a}$$

We see that Eq.(8.69), which is a sort of "eigenvalue condition", links mode dissipation to:

(i) the granulation structure form factor $A(k, k_0)$, which is a measure of the strength and scale of granulation emissivity;
(ii) $\operatorname{Im} \epsilon_e (k, \omega_k)$, which is a measure of net free energy (i.e. electron current) available to drive relaxation. Of course, $\operatorname{Im} \epsilon_e (k, \omega_k) > 0$ is required.

Note that schematically,

$$|\operatorname{Im} \epsilon (k, \omega_k)| \sim -\operatorname{Im} \epsilon_i (k, \omega_k) \operatorname{Im} \epsilon_e (k, \omega_k) A(k, k_c). \tag{8.69b}$$

Stationarity in the presence of noise emission requires that the modes be *over-saturated*, so as to ensure a fluctuation–dissipation type balance. Then,

$$\operatorname{Im} \epsilon (k, \omega_k) \sim \operatorname{Im} \epsilon_i (k, \omega_k) \operatorname{Im} \epsilon_e (k, \omega_k) A(k, k_0), \tag{8.69c}$$

so $\operatorname{Im} \epsilon < 0$, since $A > 0$ and $\operatorname{Im}\epsilon_i \operatorname{Im}\epsilon_e < 0$.

Equation (8.69c) is, in some sense, "the answer" for the granulation problem, since several key results follow directly from it. First, since the line width (at fixed k) for mode k is just $\Delta\omega_k = |\operatorname{Im} \epsilon (k, \omega_k)| \,/\, (\partial \epsilon / \partial \omega_k)$, we see that the frequency line width at fixed k for ion acoustic modes scales as,

$$\Delta\omega_k \sim |\operatorname{Im} \epsilon_i (k, \omega_k)| \, |\operatorname{Im} \epsilon_e (k, \omega_k)| \, A(k, k_0) \,/\, \left(\partial \epsilon / \partial \omega|_{\omega_k} \right). \tag{8.70a}$$

Now, since $\text{Im}\,\epsilon\,(k,\omega_k) = \text{Im}\,\epsilon_e\,(k,\omega_k) + \text{Im}\,\epsilon_i\,(k,\omega_k)$ and $\text{Im}\,\epsilon_e \text{Im}\,\epsilon_i\, < 0$ here, the effective growth or drive in the stationary state is,

$$\gamma_k^{\text{eff}} = \left(\text{Im}\,\epsilon_e \,/ [1 - A\,(k,k_0)\,\text{Im}\,\epsilon_e\,(k,\omega_{\mathbf{k}})]\right) \Big/ \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k}. \tag{8.70b}$$

This result may be interpreted as an enhancement of effective wave growth due to the presence of granulation noise, and so is a step beyond the quasi-linear theory of Chapter 3. The result of Eq.(8.70b) is, of course, directly related to Eq.(8.70a). The mechanism of growth enhancement is granulation noise emission. Here we also add the cautionary comment that the pole approximation fails for $A\,(k,k_0)\,\text{Im}\,\epsilon_e\,(k,\omega_k)\ \rightarrow\ 1$. To go further, note that Eq.(8.68) is, in principle, an integral equation for both the k and ω spectra, but the pole approximation and Eq.(8.70a) determine only the frequency spectrum. The saturated model k-spectrum can be determined by solving the balance condition,

$$\text{Im}\,\epsilon_i = \gamma_k^{\text{eff}} (\partial\epsilon/\partial\omega|_{\omega_k}), \tag{8.70c}$$

or equivalently,

$$\text{Im}\,\epsilon_{i,\text{NL}}\,(k,\omega_k) = \gamma_k^{\text{eff}} \left(\left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k} \right) - \text{Im}\,\epsilon_{i,\,\text{L}}\,(k,\omega_k)\,. \tag{8.70d}$$

Here $\text{Im}\,\epsilon_{i,\,\text{L}}$ and $\text{Im}\,\epsilon_{i,\,\text{NL}}$ are the linear (i.e. Landau damping) and nonlinear (i.e. nonlinear wave–wave and wave–particle scattering) pieces of the ion susceptibility. Equation (8.70d) has the now familiar structure of "nonlinear damping = (granulation enhanced) growth − linear damping". To actually calculate the saturated k-spectrum requires consideration of nonlinear wave–particle and wave–wave interaction processes, as discussed in Chapters 4 and 5. Finally, it is interesting to also notice that Eq.(8.70) defines a new dynamical stability condition, due to the effects of granulation enhancement. To see this, recall that the purely *linear* instability criterion is just,

$$\text{Im}\,\epsilon\,(k,\omega_k) = \text{Im}\,\epsilon_e\,(k,\omega_k) + \text{Im}\,\epsilon_i\,(k,\omega_k) > 0. \tag{8.71a}$$

However, upon including -f granulations, the condition for non-trivial saturation at a finite amplitude becomes,

$$\gamma_k^{\text{eff}} \left(\left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k} \right) + \text{Im}\,\epsilon_{i,\,\text{L}}\,(k,\omega_k) > 0. \tag{8.71b}$$

Here ϵ_{L} indicates the *linear* dielectric. Eq.(8.71b) states that

$$\frac{\mathrm{Im}\,\epsilon_e\,(k,\omega_k)}{1 - A\,(k,k_0)\,\mathrm{Im}\,\epsilon_e\,(k,\omega_k)} + \mathrm{Im}\,\epsilon_{i\,\mathrm{L}(k,\omega_k)} > 0 \tag{8.71c}$$

is required for net relaxation and a non-trivial saturated state. We see that the effect of granulations is to augment or boost the drive for relaxation of the free energy source – in this case the current. Equation (8.71c) suggests that the formation of granulations induces an element of subcriticality or hysteresis into the instability process. To see this, recall that linear instability requires a current sufficient to make $\mathrm{Im}\,\epsilon_{\mathrm{L}} > 0$. However, for $A\,(k,k_0)\,\mathrm{Im}\,\epsilon_e\,(k,\omega_k) > 0$, the current to satisfy Eq.(8.71c) is surely *smaller* than that required to satisfy Eq.(8.71a)! Thus, the theory suggests that a viable scenario in which:

(i) the driving current is induced, so linear instability is initiated, according to Eq.(8.71a);
(ii) particle orbits go stochastic, resulting in phase space turbulence;
(iii) granulations form;
(iv) the driving current is then lowered, so Eq.(8.71c) is still satisfied.

Such a scenario predicts *sustained, sub-critical turbulence* and so may be considered as a model of nonlinear instability or relaxation due to phase space density granulations.

The possibility of a self-sustaining state evolving out of free energy levels (i.e. currents) *below* what is required for linear instability naturally motivates us to consider the broader possibility of nonlinear growth of phase space density granulations. By "granulation growth", we mean an increase in fluctuation phasestrophy $\langle \delta f^2 \rangle$ at the expense of available free energy stored in $\langle f \rangle$. Note that here "fluctuation" is not limited to eigenmodes (i.e. waves that obey a dispersion relation $\omega = \omega\,(k)$ with line width $\Delta\omega_k < \omega_k$), but also includes structures localized in phase space, which are akin to an eddy in a turbulent fluid. Thus, from a statistical perspective, the structure growth problem must be formulated at the level of the two-point correlation equation for $\langle \delta f^2 \rangle$, and "growth" should thus be interpreted as the amplification of correlation on a certain scale, i.e. $\gamma = \left(1/\left\langle \delta f\,(1)\,\delta f\,(2)\right\rangle\right)\left(\partial \left\langle \delta f\,(1)\,\delta f\,(2)\right\rangle/\partial t\right)$, rather than as eigenmode growth.

Consideration of the physics of nonlinear granulation growth again takes us to the $\langle \delta f\,(1)\,\delta f\,(2)\rangle$ equation,

$$\frac{\partial}{\partial t}\left\langle \delta f\,(1)\,\delta f\,(2)\right\rangle + \frac{1}{\tau_{c_1}}\left\langle \delta f\,(1)\,\delta f\,(2)\right\rangle = P\,(1,2)\,. \tag{8.72}$$

Multiplying through by τ_{c_1} and integrating over relative velocity (v_-) then gives,

$$(\gamma\tau_c + 1)\left\langle\tilde{n}\tilde{f}\right\rangle \cong \tau_c \Delta v_T P(1,2) \cong \frac{1}{k_0} P(1,2). \tag{8.73a}$$

Equation (8.73a) can be recognized as the balance condition of Eq.(8.65), now generalized to the case of a non-stationary state. Using the structure of $P(1,2)$, we can then write,

$$\begin{aligned}(\gamma\tau_c + 1)\left\langle\tilde{n}\tilde{f}\right\rangle_k &\cong A(k,k_0)\sum_{k,\omega}\frac{\omega_{\mathrm{Pi}}^2}{k}\frac{2\pi^2}{|k|^2}\frac{\mathrm{Im}\,\epsilon_e(k,\omega)}{|\epsilon(k,\omega)|^2}\\ &\times\left(\frac{\partial\langle f\rangle}{\partial v}\left\langle\tilde{f}(v)\tilde{f}(u)\right\rangle_k\right).\end{aligned} \tag{8.73b}$$

Equation (8.73b) constitutes a (non-stationary) spectral balance equation, formulated at the level of the Vlasov equation. Once again, using the pole approximation gives,

$$\begin{aligned}\left(\gamma_{g,k}\tau_c + 1\right)\left\langle\tilde{n}\tilde{f}\right\rangle_k &= A(k,k_0)\sum_{k}\frac{\omega_{\mathrm{Pi}}^2}{k}\frac{2\pi}{|k|^2}\\ &\times\frac{\mathrm{Im}\,\epsilon_e(k,\omega_k)\int \mathrm{d}v\,(\partial\langle f\rangle/\partial v)\left\langle\tilde{f}(v)\tilde{f}(u_k)\right\rangle_k}{|\mathrm{Im}\,\epsilon(k_0,\omega_k)|\left|\partial\epsilon/\partial\omega|_{\omega_{\mathbf{k}}}\right|},\end{aligned} \tag{8.73c}$$

which is effectively, the 'nonlinear eigenvalue' equation for growth of fluctuations on scale k. Proceeding more schematically, Eq.(8.73c) may re-written as,

$$\left(\gamma_{g,k}\tau_c + 1\right)\left\langle\tilde{n}\tilde{f}\right\rangle_k \sim \frac{A(k,k_0)(-\mathrm{Im}\,\epsilon_i(k,\omega_k))(\mathrm{Im}\,\epsilon_e(k,\omega_k))\left\langle\tilde{n}\tilde{f}\right\rangle}{|\mathrm{Im}\,\epsilon(k,\omega_k)|\left|\partial\epsilon/\partial\omega|_{\omega_k}\right|}, \tag{8.73d}$$

so finally we see that the growth rate is just,

$$\gamma_{g,k} = \frac{1}{\tau_c}\left[\frac{A(k,k_0)(-\mathrm{Im}\,\epsilon_i(k,\omega_k))(\mathrm{Im}\,\epsilon_e(k,\omega_k))}{|\mathrm{Im}\,\epsilon(k,\omega_k)|\left|\partial\epsilon/\partial\omega|_{\omega_k}\right|} - 1\right]. \tag{8.73e}$$

At long last, Eq.(8.73e) gives the nonlinear granulation growth rate! $\gamma_{g,k} = 0$ gives the marginality condition. Several features of the granulation growth rate $\gamma_{g,k}$ are apparent from inspection of Eq.(8.73e). First, we note that growth is *nonlinear* (i.e. amplitude dependent), with basic scaling $\gamma_g \sim 1/\tau_c$. Thus, fluctuation growth and granulation instability are fundamentally *explosive*. Second, we see that the stationarity condition given by Eq. (8.68) sets the effective marginality condition for instability, since putting the quantity in brackets equal to zero recovers the stationarity condition of Eq. (8.68). Third, we note that exceeding marginality or criticality requires $(-\mathrm{Im}\ \epsilon_i\ (k, \omega_k))\ (\mathrm{Im}\ \epsilon_e\ (k, \omega_k)) > 0$, so free energy must be stored in the electrons – i.e. a net current must be carried. We also see that in effect, marginality requires electron free energy (i.e. current) to exceed a critical level, which is set by the condition that the right-hand side of Eq. (8.73e) > 0. This condition is different from that required for *linear* marginality. For linearly subcritical instability, $\mathrm{Im}\epsilon_e$ and $\mathrm{Im}\epsilon_i$ should be evaluated using linear susceptibilities. Speaking pragmatically, subcritical instability requires both a sufficiently large current and also that ion Landau damping be not too strong – i.e. $\mathrm{Im}\epsilon < 0$, but not too strongly negative. In principle, subcritical nonlinear granulation growth *can* occur in linearly stable plasmas. Such an instability mechanism has already been observed in numerical simulations. Marginality can also be achieved by a state of over-saturated modes, where $\mathrm{Im}\ \epsilon$ is negative but amplitude dependent. In this case, marginality is assumed when Eq. (8.69b) is satisfied. This can occur either via $\langle f_e \rangle$ profile adjustment or by an increase in the magnitude of collective mode dissipation (i.e. adjustment of $\mathrm{Im}\ \epsilon\ (k, \omega_k)$). The physics mechanism allowing this subcritical instability mechanism is inter-species momentum transfer mediated by electron scattering of ion granulations. In this mechanism, the waves that support the 'wakes' of the granulation are damped, so they do not carry significant momentum nor do they play a significant role in the scattering process.

8.4 Phase space structures – a look ahead

Although lengthy, this chapter has only scratched the surface of the fascinating subject of phase space turbulence and phase space structures. We are especially cognizant of our omission of any discussion of intrinsic, dynamic phase space structures, such as solitons, collisionless shocks, BGK modes, double layers, phase space holes, etc. Indeed, our discussion here in Chapter 8, although lengthy is limited only to the *statistical* theory of phase space turbulence, as a logical extension of our treatment of quasi-linear theory (Chapter 3), nonlinear wave–particle scattering (Chapter 4) and nonlinear wave–wave interaction (Chapters 5, 6, 7). We defer discussion of dynamical phase space structures to Volume 2. We defer detailed discussion of the applications of phase space structures to Volume 2, as well. In particular, the subjects of anomalous resistivity and particle acceleration by shocks will be addressed there.