PKU Lectures 2018-2019

→ Nonlinear Wave-Particle Interaction

- Broadly interpreted { Vlasov, Drift wave, PV, Gravitation

- cf:

https://courses.physics.ucsd.edu/2018/Fall/physics218a/

https://courses.physics.ucsd.edu/2016/Spring/physics235/

+

PKU site

Lectures:

T 2pm
F 10 am
Sun 2pm ?

This week

T 2pm
W 5pm

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Lecture 1 Mean Field Theory ($\rightarrow$ QL)

(1 → 3)

- Mean Field theory (QL+)
  - → ubiquitous in physics (TDGL, Magnetism)
  - → much derided
  - → remains of great importance

- Point:
  - Mean/avg. field evolution in presence fluctuations

N.B. "Mean" → $\langle \ \rangle$, meaning?

- Mean Field Problems in Plasma Physics
  - → $\langle f \rangle$ - QL → $\langle \frac{q}{m} E \partial f \rangle$ et seq.
  - → $\langle V_\perp \rangle$, $\langle V_\parallel \rangle$ → flows

$\langle q \rangle$ → $\langle \tilde{V}_r \tilde{V}_\perp \rangle$, $\langle \tilde{V}_r \tilde{V}_\parallel \rangle$

Pinch → off diag

$\langle \tilde{V}_r \tilde{q} \rangle \rightarrow \langle \tilde{V}_r \nabla \tilde{\phi} \rangle$

R.S. → symm. breaking

Pinch, residuals, <u>NL sources</u>

→ $\langle n \rangle$, $\langle T \rangle$ → transport

i.e. $\langle \tilde{v}_r \tilde{T} \rangle$ $\left\{ \begin{array}{l} \text{Pinch} \\ \text{Coupling} \end{array} \right.$

* → $\langle \underline{B} \rangle$ – mean field dynamo problem

i.e. $\frac{\partial \langle B \rangle}{\partial t} = \underline{\nabla} \times \langle \tilde{\underline{V}} \times \tilde{\underline{B}} \rangle + \eta \nabla^2 \langle B \rangle$

$$\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle = \underbrace{\alpha}_{\sim \langle \tilde{\underline{v}} \cdot \tilde{\underline{\omega}} \rangle \tau_c \sim \alpha} \langle \underline{B} \rangle + \underbrace{\beta}_{\eta_T \sim \langle \tilde{v}^2 \rangle \tau_c} \langle \underline{J} \rangle + \ldots$$

c.f.: H.K. Moffatt; "Mean Field Electrodynamics"

– The Issues:

– Is the "Mean" meaningful?

i.e. $|\langle \underline{B} \rangle| \ll \langle \tilde{B}^2 \rangle^{1/2}$? a la' Ginzburg

– often relevant in dynamo
– MFT cannot predict own failure

⇒ large, power-law distributed flctns.

another: avalanche

⇒ failure of F-P expansion.
divergence $\langle \Delta x \Delta x \rangle$.

- Validity of closure?

→ $\partial_t \langle f \rangle \sim \langle f f \rangle$

etc.

→ use linear response??

- Origin Irreversibility? ↔ What is in the cross phase?

i.e.

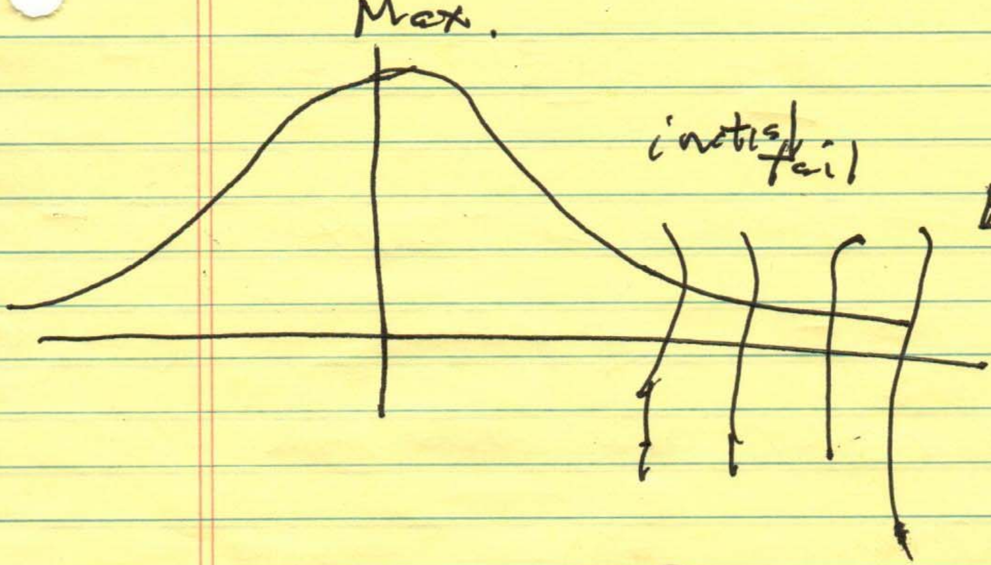
$$\langle \tilde{v}_r \tilde{n} \rangle = \left[\langle \tilde{v}_r^2 \rangle \langle \tilde{n}^2 \rangle \right]^{1/2} \cos \theta_{v,n}$$

$$\langle q \frac{E}{m} \delta f \rangle = \sum_{\underline{k}} \left(\frac{P}{\omega - kv} - i\pi \delta(\omega - kv) \right) \langle E^2 \rangle_{\underline{k}}$$

- What are the fluctuations?
  - eigenmodes – QL
  - other, + – granulations

→ Conservation Laws

i.e. QL



E_k, ~ same

plasma waves

→ What happens?

→ a.) Indiv. trapping plateau (till τ_{coll}),
  ⇝ "staircase"
  b.) Plateau from overlap.

subtlety: $-\tau_{coll}$
  $(-\nabla\langle\phi\rangle.)$

if Plateau: → ∂_t RPKED > 0
  (∂_t WKED < 0)
  → ∂_t (RPMD) > 0
  (∂_t WMD < 0)

n.b. ∂_t NRPED < 0 → cool
  ∂_t NRPMD < 0 → shift cm

∂_t PMD $= 0$, ∂_t PKED > 0 WMD = NRPED

→ Structure, Meaning
(especially symmetry breaking)

i.e. residual stress.

⋮

Issues pervade questions throughout plasma physics.

→ Quasilinear Theory for Vlasov Plasma (1D).

⇝ the Archetype of Mean Field Theory in Plasma Physics

– c.f. Oral Exam Question

– Some Review

the quasilinear system:

i.e. for

$$\partial_t f + v\,\partial_x f + \frac{q}{m} E\,\partial_v f = 0$$

$$f = \underbrace{\langle f(v,\tau)\rangle}_{\text{slow}} + \underbrace{\tilde{f}(x,v,t)}_{\text{fast}}$$

then:

- $\partial_t \langle f \rangle = \partial_v D(v)\, \partial_v \langle f \rangle$

$$D(v) = \sum_{\substack{k \\ (\text{modes})}} \frac{q^2}{m^2} |E_k|^2 \frac{|\gamma_k|}{(\omega - kv)^2 + |\gamma_k|^2}$$

from $\dfrac{i}{(\omega - kv) + i|\gamma|}$ ↳ causality

$$= \sum_k \frac{q^2}{m^2} |E_k|^2 \Big\{ \underbrace{\frac{|\gamma_k|}{\omega_k^2}}_{\text{non-resonant}} + \underbrace{\pi\,\delta(\omega - kv)}_{\text{resonant}} \Big\}.$$

- $\epsilon(k,\omega) = 0 \quad \Rightarrow \omega_k,\ \gamma_k$

$$\epsilon = 1 + \frac{\omega_p^2}{k} \int dv \frac{\partial_v \langle f \rangle}{\omega - kv}$$

$$- \partial_t |E_k|^2 = 2\gamma_k |E_k|^2$$

"$\langle f \rangle_0$, $|E_k|^2_0$" $\rightarrow$ $\epsilon(k,\omega) = 0$

$$\Big\downarrow \; \omega_k, \; |\gamma_k|$$

$$\partial_t |E_k|^2 = 2\gamma_k |E_k|^2$$

$$\Big\downarrow \; \text{up date } |E_k|^2$$

$$\partial_t \langle f \rangle = \partial_v D(\omega) \partial_v \langle f \rangle$$

up-date $\langle f \rangle$

- System analogous Landau Eqn.

$$\partial_t A^2 = 2\gamma_0 A^2 - \sigma A^4$$

i.e. if $\gamma_0 \sim V'$ shear flow

$$\begin{cases} \gamma \sim V_0' + \Delta V' \\ \Delta V' \sim -\sigma A^2 \end{cases}$$

$$\partial_t A^2 = 2\gamma_0^2 A^2 \rightarrow \text{above}$$

Thought: Evolve $\partial\langle f \rangle/\partial v$

$$\partial_t \, \partial\langle f \rangle/\partial v = \partial_v^2 D(v) \left(\frac{\partial \langle f \rangle}{\partial v} \right)$$

$$\partial_t A = \partial_v^2 [D(v) A]$$

- Key Questions

i.) Why diffusion? - origin irreversibility

ii.) When applicable, when fail,
where locate in parameter space?

iii.) relation to Fokker-Planck Theory?

iv.) how deal with energy - momentum budgets?

V.) dynamics of relaxation?

i.) → island overlap

ii.) → $\tau_{ac} < \tau_{Tr}$

$S_c > 1$

$Ku < 1$.

iii.) Resonant diffusion

iv.) $\partial_t (RPKED) + \partial_t (WED) = 0$
$\partial_t (PED) + \partial_t (EED) = 0$
$\partial_t (RPMD) + \partial_t (WMD) = 0$

→ Irreversibility

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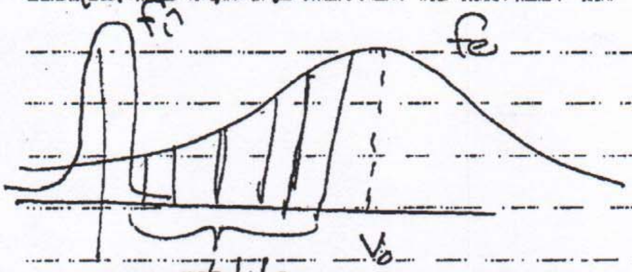
iii) Basic Scales / Regime Definition

① → Generally, Q.L.T. concerned with

i) 'broad' spectrum of:
↳ how broad?

ii) unstable waves

ie for current-driven ion-acoustic (C.D.I-A.) turbulence:



②
→ In finite system, k quantized, i.e.

$k_m = m\pi/L$, etc.

- so, have spectrum of phase velocities

$\omega_m/k_m = \omega(k_m)/k_m = V_{ph,m}$

- wave-particle resonance occurs when

$V = V_{ph,m}$

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then Sir Isaac ⇒

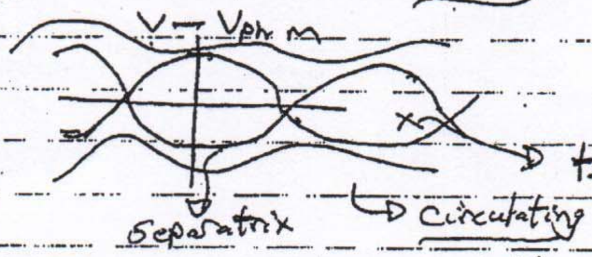
$$m\ddot{x} = \sum_m qE_m \cos(k_m x - \omega_m t)$$ n.b. $\left\{ \begin{array}{l} \text{deterministic,} \\ \text{no RPA} \end{array} \right.$

and 1 resonance dominant ⇒

$$m\ddot{x} \cong qE_{m_0} \cos(k_m x_0 + (k_m v - \omega_m) t)$$

⇒ each resonant velocity defines a phase space island

i.e.



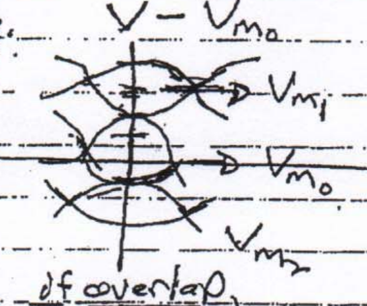
$$\Delta v \sim \left(\frac{q}{m}\phi_0\right)^{1/2}$$

∴

QLT is concerned with the case of:

⇒ multiple, $\boxed{\text{overlapping resonances}}$ ⇒ $\left\{ \begin{array}{l} \text{separatrix proximity} \Rightarrow \\ \text{destruction} \end{array} \right.$

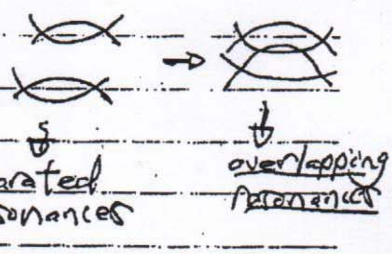
⇒ i.e. $v - v_{m_0}$



X

trapped wandering

i.e.



if overlap, hop

∴ particle can wander stochastically from resonance-to-resonance, i.e. hopping

⇒ diffusion in v / $D_v \sim \frac{(\Delta v)^2}{\tau_{ac}}$ Δv – resonance width; τ_{ac} ⇒ pattern time ↳ what is it?

ergodicity → mixing

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See Phys. 200B Notes
2014, 2015 – Google directly.

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overlap $\Leftrightarrow$ $\exists$ 1 $h_k > 0$. (Positive Lyapunov exp.) 184.

Overlap condition (B.V. Chirikov) :

$$\boxed{\frac{1}{2}\left(\Delta V_m + \Delta V_{m+1}\right) \gtrsim V_{ph_{m+1}} - V_{ph_m}}$$

$\Delta v_r \sim \sqrt{e\phi_0 / m}$

underlies diffusion eq'n. — QL

∴ → particle motion stochastic $\Leftrightarrow$ irreversibility

→ fundamental irreversibility $\Rightarrow$ orbit stochasticity (not dissipation, Landau damping .... $\Leftrightarrow$ contrast critical phenomena).

→ underpinning of diffusion equation.

③ → But, a swindle !? → [use of un-perturbed orbit in estimate !]

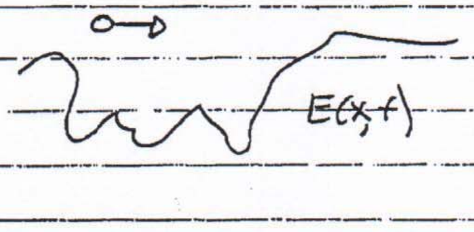
i.e. is $x \to x_0 + vt$ valid ?

Consider : linear, un-perturbed orbit ?!]

have : $E(x,t) = \sum_k E_k \exp[i(kx - \omega_k t)]$

∴ particle "sees" instantaneous pattern of electric field, from modal superposition

i.e.



→ Validity, Time scales

∴ relevant comparison is:

$\tau_L \rightarrow$ life time of 'instantaneous' pattern

$\tau_b \rightarrow$ 'bounce time' of particle in pattern

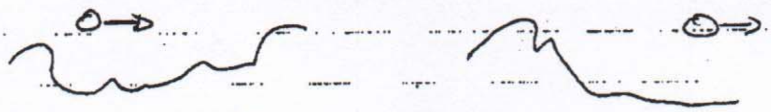
obviously, ① $\tau_L \ll \tau_b \rightarrow$ unperturbed orbit is satisfactory approximation

(pattern changes prior ⟷ bouncing)

② $\tau_L \gg \tau_b \rightarrow$ particle bounces prior pattern changes

so must consider orbit perturbation . . .

i.e.



$\tau = 0$ $\qquad \tau_b$

vs.

∴ quasilinear theory relevant to evolution when:

① → orbits stochastic (Chirikov condition satisfied)

② → $\tau_{Life} < \tau_{bounce}$ → unperturbed orbits ~ valid.

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3)

But, how relate $T_{lifetime}$, T_{bounce} to physical quantities?

Key point: Superposition patterns disperse!

$$E(x,t) \Leftrightarrow \sum_k E_k\, e^{i(kx-\omega_k t)}$$

$$= \sum_k E_k \exp\left[i\left(k\left[x - \underbrace{\omega_k/k}_{v_{ph}(k)}\, t\right]\right)\right]$$

$\therefore$ $\Delta(\omega_k/k) \equiv$ spread in phase velocities.

sets dispersal ~~rate~~ speed.

so dispersal rate is $(\text{time})^{-1}$ to disperse by one wavelength $\rightarrow$ FoM

$$1/T_{life} = k\Delta(\omega_k/k)$$

$$= k\left(\frac{d\omega_k}{dk}\frac{\Delta k}{k} - \frac{\omega_k}{k^2}\Delta k\right)$$

$$(v_g - v_{ph})$$

$$\boxed{1/T_{life} = \left(\frac{d\omega_k}{dk} - \frac{\omega_k}{k}\right)\Delta k} = (v_g(k) - v_{ph}(k))\,\Delta k$$

n.b. $T_{life} \to \infty$ for non-dispersive waves!

Generally; QLT / weak turbulence encounters trouble for non dispersive / weakly dispersive waves.

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How systematize? — E-field correlation Fctn

Consider: $\langle E(x_1,t_1)\, E(x_2,t_2)\rangle_{x,t} = C$

electric field correlation function

$C = C(x_-, \tau)$, for $\left\{\begin{matrix}\text{homogeneous}\\ \text{stationary}\end{matrix}\right\}$ fluctuations

relative coords, space/time

$x_1 = x_+ + x_- \qquad t_1 = t_+ + t_-$

$x_2 = x_+ - x_- \qquad t_2 = t_+ - t_-$

$\langle\ \rangle_{x,t} = \underline{\langle\ \rangle_{x_+,t_+}}$ what the bracket means

so

$$C(x_-,\tau) = \Big\langle \sum_{k,k'} E_k E_{k'} e^{i(k+k')x_+} e^{-i(\omega_k+\omega_{k'})t_+} \, e^{i(k-k')x_-} e^{-i(\omega_k-\omega_{k'})t_-} \Big\rangle_{x_+,t_+}$$

x_+, t_+ average $\Rightarrow k = -k' \quad \omega_k = -\omega_{k'}$

so

$$C(x_-,\tau) = \sum_k |E_k|^2 e^{ikx} e^{-i\omega_k t_-}$$

Now:

→ assume continuous spectrum – i.e. post-overlap

→ for simplicity, take model

$$|E_k|^2 = E_0^2 \Big/ \left[\left(\frac{k-k_0}{\underbrace{\Delta k}_{\text{width}}}\right)^2 + 1\right]$$

→ evaluate on u.p.o.

$$x_- = x_{0-} + V\tau$$

$$\therefore \langle E^2 \rangle = \int dk \frac{E_0^2}{\left[\frac{(k-k_0)^2}{\Delta k^2} + 1\right]} e^{ikx_0} e^{i(kV - \omega_k)\tau}$$

integrating:

$$\sim E_0^2 \underbrace{e^{ik_0 x_-} e^{-|\Delta k\, x_{0-}|}}_{\text{phase info – irrelevant}} *$$

$$\underbrace{e^{i(k_0 V - \omega_{k_0})\tau}}_{\substack{\text{oscillation} \\ (\Rightarrow \text{ on resonance})}} \underbrace{e^{-|\Delta(kV - \omega_k)|\tau}}_{\text{correlation decay}}$$

due dispersal and its interplay with resonance.

n.b.: note that spread in doppler-shifted ω is critical

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now $\Delta(kV - \omega_k) = v\,\Delta k - V_{gr}\,\Delta k$

$= |(V - V_{gr})\,\Delta k|$

$V_{gr} = \frac{d\omega}{dk}$

so

$$\langle E^2 \rangle = C(x_-, \tau)$$

$$= E_0^2\, e^{ik_0 x_-}\, e^{i(k_0 v - \omega_{k_0})\tau}\, e^{-|\Delta k\, x_0|}$$

$$* \exp-\left[|(V - V_{gr})\,\Delta k|\,\tau\right]$$

sets lifetime

$1/\tau_L = |(V - V_{gr}(k))\,\Delta k| \equiv$ (Autocorrelation Time)$^{-1}$

Note: $\equiv 1/\tau_{ac}$.

- for resonant particles, $V = \omega_k/k$

$1/\tau_L = |(V_{ph} - V_{gr})\,\Delta k|$ → recovers earlier!

- can think: $|V\Delta k| \to 1/\tau_{ac}$ wave-particle

$|V_{gr}\,\Delta k| \to 1/\tilde{\tau}_{ac}$ wave packet dispersal

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generally, shorter time dominates, except for non-dispersive waves.

So, can enumerate key time scales

→ $\tau_{ac} = |\Delta k(v_{ph} - v_{gr})|^{-1}$

≡ persistance of E pattern ($\langle E^2 \rangle$ auto correlation) for resonant particles

→ γ^{-1} = growth/damping time

→ $\tau_{Tr} = \left(k\sqrt{q\phi/m}\right)^{-1}$ ≡ trapping time

→ $\tau_{relax} = \left(\frac{1}{\langle f \rangle}\frac{\partial \langle f \rangle}{\partial t}\right)^{-1}$ ≡ avg. distribution relaxation time

so

$\tau_{ac} < \tau_{Tr}$ → u.p.o. valid

$\tau_{ac}, \gamma^{-1} < \tau_{relax}$ → $\langle f \rangle$ closure meaningful.

$\tau_{ac} < \gamma^{-1} < \tau_{relax}$ → Q.L.T. valid.

Energy, Momentum Budget

iii) Energy – Momentum Budgets

→ Key Point: There are two ways of implementing the book-keeping and accounting

i.e. $\begin{cases} \text{resonant particles} & \text{vs. 'waves'} \\ \text{or} & \\ \text{particles} & \text{vs. fields} \end{cases}$

keep in mind: Wave = Field + Non-resonant Particles

i.e. for plasma oscillation, $\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

$$\text{Wave Energy} = W = \frac{\partial}{\partial\omega}(\omega\epsilon)\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= \omega\frac{\partial\epsilon}{\partial\omega}\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= 2 \cdot \frac{|E|^2}{8\pi}$$

field non-resonant particle

(show!)

→ Resonant Particles vs. Waves ?

$$\frac{\partial \langle f \rangle}{\partial t} = -\frac{\partial}{\partial v}\frac{q}{m}\langle \tilde{E}\tilde{f}\rangle$$

$$\frac{\partial}{\partial t}\int dv \frac{mv^2}{2}\langle f \rangle = -\int dv \frac{mv^2}{2}\frac{\partial}{\partial v}\frac{q}{m}\langle \tilde{E}\tilde{f}\rangle$$

$$= \int dv\, \cancel{m} v \frac{q}{\cancel{m}}\langle \tilde{E}\tilde{f}\rangle$$

plugging in $\tilde{f}_k^{\text{linear}}$ for $\tilde{f}$:

$$\frac{\partial}{\partial t}\mathcal{E}_{kin} = -i\int dv \frac{vq^2}{m}\sum_k |E_k|^2\left(\frac{P}{\omega - kv} - i\pi\delta(\omega - kv)\right)\frac{\partial\langle f \rangle}{\partial v}$$

resonant

$$\frac{\partial}{\partial t}\mathcal{E}_{kin}^{res} = -\int dv \frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\delta(\omega/k - v)\frac{\partial\langle f \rangle}{\partial v}|E_k|^2$$

resonant only

$$= -\frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\left.\frac{\partial\langle f \rangle}{\partial v}\right|_{\omega/k}|E_k|^2$$

[As resonant particles stabilize/destabilize wave, expect resonant particles conserve energy against waves.

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For wave energy evolution:

Recall: $\epsilon = 1 + \frac{\omega_p^2}{k}\int dv \frac{\partial\langle f\rangle/\partial v}{\omega - kv}$

$\epsilon^r(\omega_k + i\gamma_k) + i\epsilon^{IM} = 0$

$i\gamma_k = \frac{-i\epsilon^{IM}}{\partial\epsilon^r/\partial\omega}$; $\cancel{i}\gamma_k = -\cancel{i}\frac{\epsilon_{IM}}{\partial\epsilon^r/\partial\omega}$

$= -\epsilon_{IM}/\partial\epsilon^r/\partial\omega$

Now, $W \equiv$ Wave Energy Density   action density

$W = \sum_k \frac{\partial}{\partial\omega}(\omega\epsilon)\frac{|E_k|^2}{8\pi} = \sum_k \omega_{\underline{k}} N_{\underline{k}}$

$= \sum_k \omega_k \frac{\partial\epsilon^r}{\partial\omega}\Big|_{\omega_k}\frac{|E_k|^2}{8\pi}$

$\frac{\partial W}{\partial t} = \sum_{\underline{k}} 2\gamma_{\underline{k}}\,\omega_k \frac{\partial\epsilon^r}{\partial\omega}\Big|_{\omega_{\underline{k}}}\frac{|E_k|^2}{8\pi}$ $|E_k|^2 = |E_k^0|^2 e^{2\gamma_k t}$

$= \sum_{\underline{k}} 2\left(\frac{-\epsilon_{IM}}{\partial\epsilon^r/\partial\omega}\right)\omega_k \frac{\partial\epsilon^r}{\partial\omega}\frac{|E_k|^2}{8\pi}$

$= \sum_{\underline{k}} -\epsilon_{IM}(k,\omega_k)\,\omega_k\left(\frac{|E_k|^2}{4\pi}\right)$

$$i\,\epsilon_{IM} = \frac{\omega_p^2}{k}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}\frac{(-i\pi)}{|k|}$$

$(n_0 = 1)$

$$\therefore\ \frac{\partial W}{\partial t} = \sum_k \pi\frac{q^2}{m}\frac{\omega_k \cancel{4\pi}}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial \cancel{k}}\right|_{\omega/k}\frac{|E_k|^2}{\cancel{4\pi}}$$

$$= +\frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}|E_k|^2$$

$$\Rightarrow\ \boxed{\partial_t E^{\text{resonant}}_{\text{kinetic}} + \partial_t W = 0}$$

$$\cancel{\partial_t \int dv \tfrac{1}{2}mv^2 f = -q\langle E\,\tilde{f}\rangle}$$

Note:

— <u>this is essentially a re-write of the Poynting theorem for plasma waves</u>, i.e.

$$\frac{\partial}{\partial t}W + \nabla\cdot\underline{S} + Q = 0$$

$Q \to \langle \underline{\tilde{E}}\cdot\underline{\tilde{J}}\rangle$ coupling

$\frac{\partial}{\partial t}W$ ↓ wave energy

$\nabla\cdot\underline{S}$ ↓ divergence of wave energy density flux

$\langle \underline{\tilde{E}}\cdot\underline{\tilde{J}}\rangle$ ↓ resonant particle heating

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For homogeneous system: $\nabla \cdot \underline{S} = 0$

so $\frac{\partial W}{\partial t} + Q = 0$

Q ~ $\langle E \cdot J \rangle$ mediated by resonant particles (DC field)

$$\Leftrightarrow \boxed{\frac{\partial W}{\partial t} + \frac{\partial}{\partial t}(\text{RPKED}) = 0}$$

RPKED ~ resonant particle kinetic energy density

Energy Thm I

Waves and Resonant particles conserve energy!

? What is the fate of RPKED for saturated waves. What must happen ??

→ Now, can observe:

$W = \text{NRPKED} + \text{FED}$

NRPKED ~ non-resonant particle kinetic energy density

FED ~ field energy density

so, simply re-grouping terms:

$$\frac{\partial}{\partial t}(\text{FED}) + \frac{\partial}{\partial t}(\text{RPKED} + \text{NRPKED}) = 0$$

$(\text{RPKED} + \text{NRPKED})$ ~ $\underline{\text{PKED}}$ → total so $\boxed{\frac{\partial}{\partial t} FED + \frac{\partial}{\partial t}(PKED) = 0}$ Energy Thm. II

i.e. fields and particles conserve energy.

What is the physics of all this ? !

$$D = \mathrm{Re} \sum_k \frac{q^2}{m^2} |E_k|^2 \left(i/\omega - kv \right)$$

QL diffusion — for general, weakly non-stationary state ---

$$= \sum_k \frac{q^2}{m^2} |E_k|^2 \left(|\gamma_k| / \left((\omega - kv)^2 + |\gamma_k|^2 \right) \right)$$

n.b. causality ⇒ no negative diffusion for damped waves

$$\cong \sum_k \frac{q^2}{m^2} |E_k|^2 \Big\{ \underbrace{\pi \delta(\omega - kv)}_{\text{resonant diffusion}} + \underbrace{\frac{|\gamma_k|}{\omega_k^2}}_{\text{non-resonant diffusion}} \Big\}$$

Resonant Diffusion → irreversible - resonance overlap is underpinning

→ rooted in particle stochasticity

→ resonant diffusion can be obtained from Fokker-Planck calculation (show this)!

→ in principle, can persist in steady state (but how balance energy ... ?!)

Non-Resonant Diffusion:

$$D^{NR} = \sum_k \frac{q^2}{m^2} |E_k|^2 \frac{|\gamma_k|}{\omega_k^2}$$

take $\gamma_k > 0$.

pondermotive energy!

$$= \frac{1}{2} \partial_t \sum_k |V_k|^2 \quad \text{where } |V_k|^2 = \sum \frac{q^2}{m^2} \frac{|E_k|^2}{\omega_k^2}$$

~ ponderomotive energy.

→ corresponds to "sloshing" motion energy of particles in wave

i.e. $D^{NR} \sim \partial_t \mathcal{E}_{quiver}$

→ thus reversible, can't be obtained from Fokker-Planck theory → aka' "fake diffusion"

→ vanishes in stationary state ($\partial_t = 0$)

Point is that can count non-resonant diffusion as:

$D_{\text{non-resonant}}$ < particle sloshing energy ~~part of wave energy density~~
or
part of total particle kinetic energy density

so two forms of energy conservation.

Note: Physically, the picture of plasma as gas { – resonant particles – waves } or equivalently

resonant particles + quasi-particles

↓ waves < $N(k, x, t)$, WKE, etc.

is appealing and will pervade this course.

N.B.: For Direct Proof of $\partial_t(PKED + FED) = 0$

see below.

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From QL equation:

$$\frac{\partial}{\partial t}(PKED) = -\sum_k \int dv \frac{\omega_p^2}{k} kv \frac{|E_k|^2}{4\pi} \frac{i}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int \frac{dv}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

so

$$\partial_t (PKED) = -i \sum_k \frac{|E_k|^2}{4\pi} \int dv \frac{\omega_p^2}{k} (\underbrace{kv - \omega}_{} + \omega) \cdot \frac{1}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

↳ { cancels denom, residual odd in k }

$$= -i \sum_k \frac{|E_k|^2}{4\pi} \int dv \frac{\omega_p^2}{k} \frac{\omega}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

using $\epsilon(k,\omega) = 0$

$$= i \sum_k \frac{|E_k|^2}{4\pi} \omega_k$$

$\omega_k = \omega_k^r + i\gamma_k$

$$= -\sum_k \frac{|E_k|^2}{8\pi} (2\gamma_k)$$

$$= -\partial_t (FED) \quad \checkmark$$

Numbers and Parameter Space

24. Further:

⇒ "Numbers" and "Ratios" ......

Numbers – as in Re ?!

– what is assumed in QLT ?

→ linear response adequate – no NL distortion

→ resonant diffusion – Markov process, a la' FPE
→ stochasticity / irreversibility
→ RPA (?)

Exercise:

a.) Derive QL (resonant diffusion) equation from Fokker-Planck theory

b.) use Hamiltonian structure of dynamics to eliminate dynamical friction term (c.f. Lichtenberg and Lieberman)

⇒ 2 dimensionless numbers:

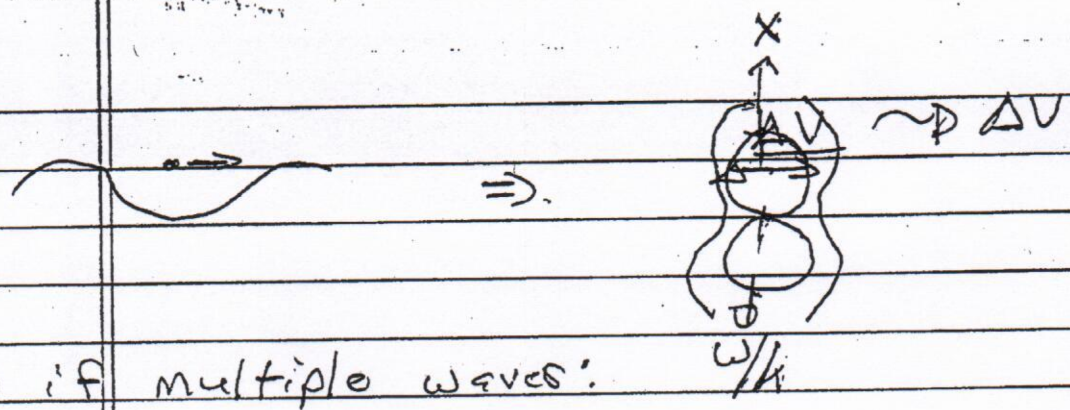
I)

$$\boxed{S_c = \Delta V / |v_{ph\,i} - v_{ph\,i+1}|}$$

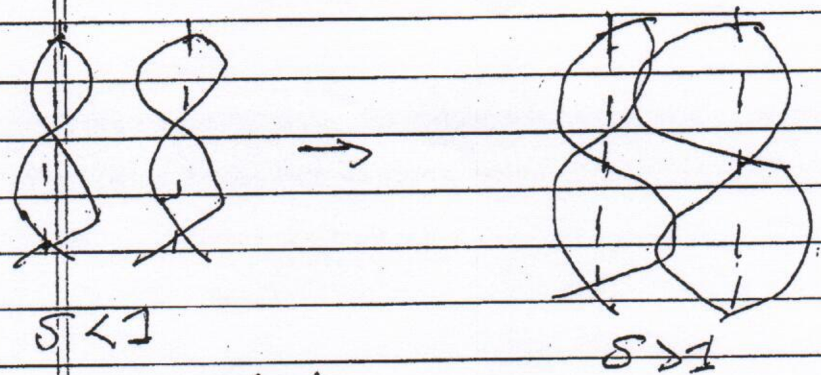
S_c ↓ Chirikov #

→ measures stochasticity of particle orbits

i.e. resonances overlap $S_c > 1$



so if multiple waves:



⇒ ~ independent resonant behavior

⇒ resonance overlap
⇒ diffusive "kicking" from 1 resonance to another.

② $\boxed{\text{Kubo \#}}$ / Strouhal # → $\boxed{\text{measures memory of flow}}$

→ field correlation time

i.e.

$$K = \frac{(qE/m)\tau_c}{(\Delta V)_c}$$

∂_t vs $\frac{qE}{m}\partial v$

→ velocity correlation length

Kubo number of prime importance.

for spatial scattering:

$$S = \frac{\tilde{v}\,\tau_c}{l_c}$$

→ scatterer correlation time

→ correlation length

Note: For $\Delta v_c \sim \Delta v_T \sim \sqrt{T/m}$

$K \sim k(\Delta v_T)\,\tau_c$

$\sim \frac{\omega_b \tau_c}{J}$

bounce freq.

$K < 1 \Rightarrow$

$\tau_c\,\omega_b < 1$

→ [pattern changes prior bounce

⇒

$k_0 S > 1$ → { ordered flow with memory; persistent scatterer pattern

$k_0 S < 1$ → { random short lived scatterers; random flow – suggests RPA

can treat $\tilde{E}$ as random variable

usually $k_0 S > 1$ ⇒ unperturbed (linear) trajectory poor approximation

⇒ need incorporate scattering field.

so, usual wisdom is that QLT valid if:

→ $\sigma_c > 1$ → need stochastic orbits

→ $K < 1$ → need avoid trapping, strong distortion, etc.

but: ?

①
→ is field / scatter correlation time the relevant time scale ?

②
→ what of $\sigma_c > 1$, $K \sim 1$? ⇒ often realistic

[③
→ what of non-resonant piece ?]

Regarding ①,

$$D = \sum_k \frac{q^2}{m^2} |E_k|^2 \int_0^\infty e^{i(\omega - kv)\tau} d\tau$$

$$|E_k|^2 = E_0^2 \, \Delta k \Big/ \left[(k - k_0)^2 + (\Delta k)^2 \right]$$

⇒

$$D = \sum_k \frac{q^2}{m^2} |E_k|^2 \int_0^\infty d\tau\, e^{i(\omega - kv)\tau}$$

$$= \int dk \frac{q^2}{m^2} E_0^2 \frac{\Delta k}{(k-k_0)^2 + (\Delta k)^2} \int_0^\infty d\tau\, e^{i(\omega - kv)\tau}$$

flip order

$$\approx \frac{q^2}{m^2} E_0^2 (2\pi) \int_0^\infty d\tau\, e^{i(\omega_{k_0} - k_0 v)\tau} \;*$$

$$\exp\left[-\left|\frac{d\omega}{dk} - v\right| \Delta k\, \tau\right]$$

→ sets correlation decay

$\Rightarrow$ if $D \sim \frac{q^2}{m^2} \langle E^2 \rangle \tau_c$

τ_c → wave-particle correlation time.

then:

$1/\tau_{ac} \sim \left|\frac{d\omega}{dk} - v\right| \Delta k$ → wave-particle decorrelation rate

if $v \sim \omega/k$ - resonance

$1/\tau_{ac\,pkt} \sim \left|\frac{d\omega}{dk} - \frac{\omega}{k}\right| \Delta k$ → packet dispersal rate

Generally: $\tau_{ac\,w\text{-}p} \neq \tau_{ac\,pkt}$ → differences more pronounced c'n 3D.

So, really need:

→ specify velocity for wave-particle decorrelation being considered.

→ need: $1/\tau_{ac} > \omega_b,\ 1/\tau_{scatt}$

δ

$(k^2 D)^{1/3}$

→ $\tilde{\tau}_{ac} \sim \tilde{\tau}_c$ only for resonant particles in 1D.

→ broad spectrum alone is not sufficient to justify QLT → effective dispersion significant!

$$1/\tau_{ac}\Big|_{res} \sim \left|\frac{d\omega}{dk} - \frac{\omega}{k}\right| \Delta k \rightarrow 0;$$ for non-dispersive waves

→ $K < 1$ criterion most accurate if one takes $\tilde{\tau}_c \sim \tilde{\tau}_{ac,pkt}$

$$K = \frac{q}{m} E\, \tau_{ac,pkt} / \Delta v_\phi = (k \Delta v_\phi)\, \tau_{ac} < 1$$

agrees with intuition.

30.

→ "phase randomization" irrelevant ⇒

- can have $\sigma > 1$, $K < 1$ with coherent phases

- QL known to well describe stochastic trajectory divergence in standard map/ magnetic fieldlines, even for static fields/ fixed phases, cf: Rechester, Rosenbluth, White PRL '80.

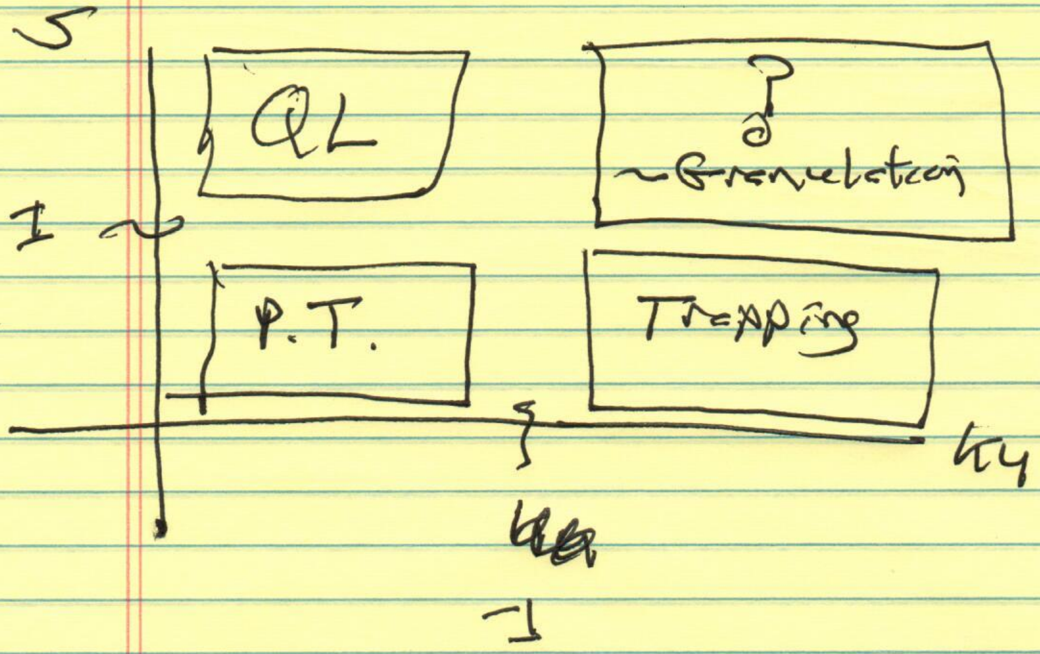
- phases fixed in Tsunoda / Malmberg experiments

→ often QLT seems to work reasonably well in limit of $\sigma > 1$, $K \sim 1$
  - unclear why.....

  - corrections due granulations needed [?]
    i.e. $K \sim 1 \Rightarrow \;\sim\; \Rightarrow$ phase space eddys formed....

→ strong non-stationarity can [boost] applicability of QLT.

→ Parameter: S, Ku



Sad reality:

→ Many relevant problems sit at:

$$Ku \sim 1 \Rightarrow \frac{\tilde{v}\,\tau_{ac}}{\Delta} \sim 1$$

$$\tilde{v} \sim \frac{\Delta}{\tau_{ac}}$$

$$\frac{\tilde{n}}{\tau_{ac}} \sim \tilde{v}\,\nabla n_0$$

$$\sim \frac{\Delta}{\tau_{ac}}\,\nabla n_0$$

$$\tilde{n}/n \sim \frac{\Delta}{L_n} \rightarrow \text{MLT}$$

and

→ $S \sim 1$ ⇒ treshold for stochastic scattering

c.f. – stiff profiles?

– $\bar{S}_0$ ↑, $S < 1 \to S > 1 \to S \sim 1$

~marg.

(c.f. Heidbrink) → stiff ~~[illegible]~~ DF in TAE is not linear threshold

but $S_c \sim 1$.

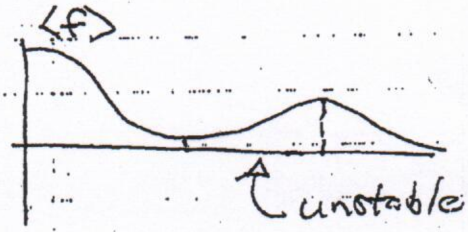
– What of hysteresis loop in S_c?

– Is $S_c \sim 1 \leftrightarrow$ stiffness point in profile.

— QL Relaxation

iv.) Applications of Quasilinear Theory (I)

→ Bump on Tail



↑ unstable phase velocities. (bump on tail)

$\omega_k = \omega_{pe}\left(1+\frac{3}{2}k^2\lambda_D^2\right)^{1/2}$

Quasi-linear Equations:

$\epsilon(k, \omega_k) = 0 \quad \Rightarrow \omega(k), \gamma(k) \quad$ from $\langle f \rangle$

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v}$$

$$D \equiv D^R + D^{NR}$$

$$= \sum_k \frac{q^2}{m^2} |E_k|^2 \left\{ \pi \delta(\omega_k - kv) + \frac{|\gamma_k|}{\omega_k^2} \right\}$$

$$\frac{\partial}{\partial t}\left(|E_k|^2/8\pi\right) = 2\gamma_k |E_k|^2/8\pi$$

Observe: - resonant diffusion describes dynamics of tail particles

- non-resonant diffusion describes dynamics of bulk Maxwellian

Expect: - tail flattening

with ⇓ ⇑

- adjustment of core/bulk profile (i.e. effective "temperature")

Now first consider resonant particles (i.e. on bump):

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D^R \frac{\partial \langle f \rangle}{\partial v} \qquad * \langle f \rangle \text{ and i.b.p.} \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial t} \int_{res} \frac{\langle f \rangle^2}{2} = - \int_{res} dv \, D^R \left(\frac{\partial \langle f \rangle}{\partial v} \right)^2$$

{ generalization ⇒ Zeldovich Thm.

stationarity ⇒

$$D^R \left(\partial \langle f \rangle / \partial v \right)^2 = 0$$

Now "res" → some finite interval of phase velocities

so

stationarity $\Rightarrow$ $D^R = 0$; i.e. fluctuations decay and damp

or

$\partial \langle f \rangle / \partial v = 0$; plateau forms, removing growth

N.B.: – In 1D $\rightarrow$ plateau
– can generalize

To resolve:

$$D^R = \frac{8\pi^2 q^2}{m^2} \sum_k \frac{|E_k|^2}{8\pi} \delta(\omega - kv)$$

$$\cong \frac{16\pi^2 q^2}{m^2} \int dk \, \mathcal{E}_F(k) \, \delta(\omega - kv)$$

$$D^R = \frac{16\pi^2 q^2}{m^2 v} \mathcal{E}_F(\omega_{pe}/v)$$

so

$$\partial_t D^R = \frac{16\pi^2 q^2}{m^2 v} (2\gamma_{\omega_{pe}/v}) \mathcal{E}(\omega_{pe}/v)$$

Now, $\gamma_k = -\epsilon_{IM} / \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k}$

$$\gamma_k = \gamma_{\frac{\omega_{pe}}{v}} = \pi v^2 \omega_p \frac{\partial \langle f \rangle}{\partial v}$$

so $$\frac{\partial D^R}{\partial t} = \frac{16\pi^3 \xi^2}{m^2 v}\left(2\pi v^2 \omega_p \frac{\partial \langle f \rangle}{\partial v}\right)\mathcal{E}(\omega_p/v)$$

$$= \left(\pi \omega_p v^2 \frac{\partial \langle f \rangle}{\partial v}\right) D^R$$, using D^R defn.

so

$$D^R(v,t) = D^R(v,0)\exp\left[\pi \omega_{pe} v^2 \int_0^t dt' \frac{\partial \langle f \rangle}{\partial v}\right]$$

and:

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D^R \frac{\partial \langle f \rangle}{\partial v}$$

$$= \frac{\partial}{\partial t}\frac{\partial}{\partial v}\left[\frac{D^R}{\pi \omega_{pe} v^2}\right]$$ $\left\{ \begin{array}{l} \text{using } \gamma_k, D \\ \text{definitions} \end{array} \right.$

35.

so

$$\langle f(v,t)\rangle - \langle f(v,0)\rangle = \frac{\partial}{\partial v}\left[\frac{D^R(v,t) - D^R(v,0)}{\pi\omega_p v^2}\right]$$

∴ have:

$$\boxed{\begin{aligned} D^R &= D^R(v,0)\exp\left[\pi\omega_p v^2\int_0^t dt'\,\frac{\partial\langle f\rangle}{\partial v}\right] \\ \langle f(v,t)\rangle &= \langle f(v,0)\rangle + \frac{\partial}{\partial v}\left[\frac{D^R(v,t) - D^R(v,0)}{\pi\omega_p v^2}\right]\end{aligned}}$$

Now, recall seeks to know if:

i.) $D^R \to 0 \Rightarrow \partial\langle f\rangle/\partial v\big|_{t\to\infty} < 0$ (fluctuations damp)

ii.) $\partial\langle f\rangle/\partial v \to 0 \Rightarrow$ finite D^R, distribution plateaus.

Now, if $D^R \to 0$,

$$\langle f(v,t)\rangle = \langle f(v,0)\rangle - \frac{\partial}{\partial v}\left[\frac{D^R(v,0)}{\pi\omega_p v^2}\right]$$

$$D^R(0) = \frac{16\pi^2 q^2}{m^2 v}\,\mathcal{E}(\omega_p/v, 0)$$

fluctuation energy

but $\frac{16\pi^2 q^2}{m^2 V} \frac{\mathcal{E}_k(0)}{\pi \omega_p V^2} = 2 E_F(0) / (n m v_b^2/2)$

$\ll 1$, as $n \gg n_b$

$\therefore \langle f(v,t) \rangle \cong \langle f(v,0) \rangle$, to good approx.

but, for resonant velocities,

→ linear instability ⇒ $\partial\langle f \rangle/\partial v > 0$

→ $D^R \underset{t\to\infty}{\to} 0$ ⇒ $\partial\langle f \rangle/\partial v < 0$

but have (for $D^R \to 0$) $\langle f(t) \rangle = \langle f(0) \rangle$!

$\therefore$ contradiction follows from assumption of $D^R(v,t) \to 0$

$\therefore$ have established that

$\partial\langle f \rangle/\partial v \big|_{res} \to 0$ ⇒ plateau forms!

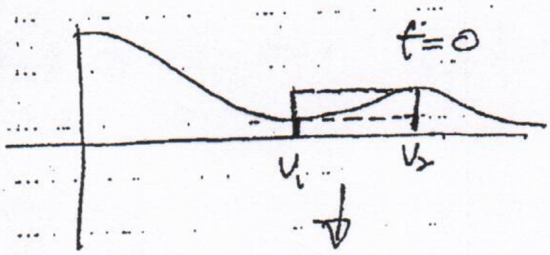
— — — — —

37.

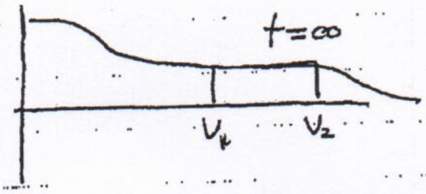
For plateau formation, can immediately determine saturation levels from

$$\frac{\partial}{\partial t}(\text{RPKED}) + \frac{\partial}{\partial t}(\text{WED}) = 0$$

i.e.



$t \to \infty$



$k_1 = \omega_p / v$

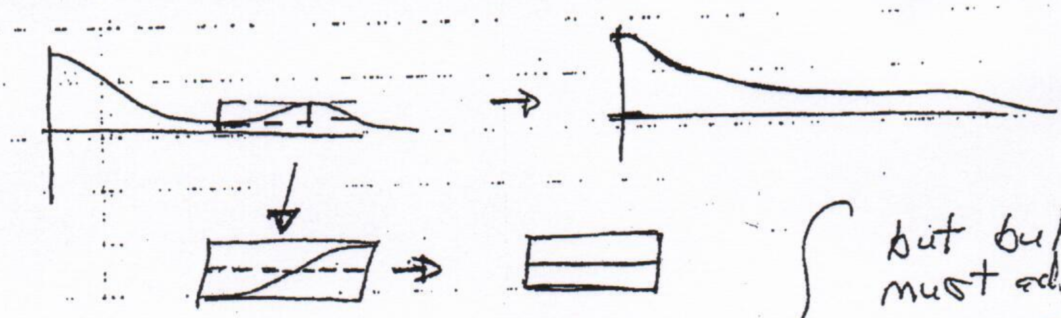
$$\therefore \quad \Delta\left(\int_{v_1}^{v_2} \frac{mv^2}{2}\langle f\rangle\right) = -\Delta\int_{k_1}^{k_2} W_k \, dk$$

but $W_k = 2\,\mathcal{E}(k)$

$$\Rightarrow \quad \Delta\left(\int_{v_1}^{v_2} dv\, \frac{mv^2}{2}\langle f\rangle\right) = -2\Delta\int_{k_1}^{k_2} \mathcal{E}(k)\, dk$$

38.

→ can estimate A (RPKED) analytically, via construction



i.e. beam slows down

but bulk must adjust to conserve momentum!

i.e. bulk spreads outward, to conserve momentum as beam slows (bump flattened inward)

Now, for non-resonant particles:

$$\frac{\partial\langle f\rangle}{\partial t} = \frac{\partial}{\partial v} D^{NR} \frac{\partial\langle f\rangle}{\partial v}$$

$$= \frac{\partial}{\partial v} \frac{q^2}{m^2} \sum_k |E_k|^2 \frac{\gamma_k}{(\omega - kv)^2} \frac{\partial\langle f\rangle}{\partial v}$$

$\gamma > 0$

$$\cong \frac{8\pi q^2}{m^2} \int dk\, \mathcal{E}(k) \frac{\gamma_k}{\omega_{p}^2} \frac{\partial^2\langle f\rangle}{\partial v^2}$$

so, using γ definition:

$$\frac{\partial \langle f \rangle}{\partial t} = \left(\frac{1}{nm} \frac{\partial}{\partial t} \int dk \, \mathcal{E}(k) \right) \frac{\partial^2 \langle f \rangle}{\partial v^2}$$

now define $T(t) = \frac{2}{n_e} \int dk \, \mathcal{E}(k,t)$

so $\Rightarrow$

$$\frac{\partial \langle f \rangle}{\partial T} = \frac{1}{2m} \frac{\partial^2 \langle f \rangle}{\partial v^2}$$

thus, for initial Maxwellian:

$$\langle f \rangle = \left[m/2\pi \left[T + T(t) - T(0) \right] \right]^{1/2} \exp\left[\frac{-mv^2/2}{\left[T + T(t) - T(0) \right]} \right]$$

Thus, for non-resonant particles

- at saturation

$$T/2 \rightarrow T/2 + \frac{1}{n} \int dk \left[\mathcal{E}(k,\infty) - \mathcal{E}(k,0) \right]$$

i.e. electrons 'heated' by net increase in field energy

- can also note:

$$\frac{\partial}{\partial t}(RPKED) + \frac{\partial}{\partial t}(WED) = 0$$

for plasma waves,

$$\frac{\partial}{\partial t}(RPKED) = -2\frac{\partial}{\partial t}(FED)$$

so $\Delta\,(RPKED) = -2\,\Delta(FED)$

but

$$\Delta(PKED) = -\Delta(FED)$$

so $\Delta\,(RPKED) = +2\,(\Delta\,(PKED))$

$$\Rightarrow\ 0 = \Delta(RPKED) + 2\Delta(NRPKED)$$

and

$$\Delta(PKED) - \Delta(RPKED) = -\Delta(FED) - (2)\Delta(FED)$$

$$\boxed{\Delta(NRPKED) = \Delta(FED)}$$

as shown above

48.

210.

→ heating is one-sided, to conserve momentum.

— — — — — — —

Q?

→ Other routes to irreversibility?

– beats, NLLD

– ~~[illegible]~~ Ray Chaos

H.O.

→ Corrections?

→ what of ~~∂_t RPKED + TWED~~?

∂_t (RPKED + TWED) = 0 in stationary state?

~ plateau?

~ other?

→ Dynamical Friction?