THE ASTROPHYSICAL JOURNAL, 471:385–399, 1996 November 1

STATISTICAL MECHANICS OF TWO-DIMENSIONAL VORTICES AND COLLISIONLESS STELLAR SYSTEMS

P. H. CHAVANIS,[1] J. SOMMERIA,[1] AND R. ROBERT[2]

Received 1996 January 4; accepted 1996 May 13

ABSTRACT

In this article, we stress the analogy between two-dimensional vortices and collisionless stellar systems. This analogy is based on the similar morphology of the Euler and Vlasov equations. These equations develop finer and finer filaments, and a statistical description is appropriate to smooth out the fluctuations and describe the macroscopic evolution of the system. We show here that the two descriptions are similar and apply the methods obtained in two-dimensional turbulence to the case of stellar systems. In particular, we propose a new evolution equation for the coarse grained distribution function $\bar{f}$ based on a general maximum entropy production principle. This equation (of a generalized Fokker-Planck type) takes into account the "incompleteness" and the "statistical degeneracy" of the violent relaxation and should be able to model the evolution of collisionless stellar systems.

Subject headings: celestial mechanics, stellar dynamics — turbulence

1. FROM JUPITER'S GREAT RED SPOT TO THE STRUCTURE OF GALAXIES

It is well known in fluid mechanics that two-dimensional turbulence governed by the *Euler equation* organizes into coherent structures (e.g., Hasegawa 1985). The robustness of Jupiter's Great Red Spot, a huge vortex persisting for more than three centuries in the turbulent shear between two zonal jets, is probably related to this general property (Sommeria et al. 1991). Some other coherent structures like dipoles (a pair of cyclone/anticyclone) and sometimes tripoles have been found in atmospheric or oceanic systems (e.g., Flierl 1987) and can persist during several days of weeks (responsible for atmospheric blocking). Some astrophysicists invoke the existence of organized vortices in the gaseous component of disk galaxies (Nezlin & Snezhkin 1993) in relation with the emission of spiral density waves. It has been proposed also that planetary formation began inside persistent gaseous vortices born out of the protoplanetary nebula (Barge & Sommeria 1995). As a result, hydrodynamical vortices occur in a wide variety of geophysical or astrophysical phenomena, and their robustness demands a general understanding.

Similarly, it is striking to observe that the galaxies themselves follow a kind of organization revealed in the Hubble classification (e.g., Binney & Tremaine 1987). Now, the dynamics of galaxies is dominated by stars under collective gravitational interaction rather than gas or hydrodynamical processes. In particular, for most stellar systems the collisions (i.e., encounters) between stars are quite negligible (the corresponding relaxation time $t_{\rm coll}$ exceeds the age of the universe), and the galaxy dynamics is well modeled by the *Vlasov equation.*

In this article, we stress the deep analogy between these two systems and try to explain their self-organization with the same theoretical tools. This analogy was mentioned by several authors (Miller 1990; Montgomery & Lee 1991; Michel & Robert 1994) but has never been studied in detail. In § 2, we show that this analogy resides in the similar morphology of the Euler and Vlasov equations that describe, respectively, two-dimensional ideal fluids and collisionless stellar systems. In § 3, we show that an equilibrium statistical mechanics of these systems is relevant if we introduce a macroscopic level of description (coarse grain). We discuss also why the complete equilibrium is not reached in many cases. As a result, the statistical theory must be extended to nonequilbrium, as stated in § 4. By analogy with two-dimensional turbulence, we derive a set of diffusion equations in phase space that should model the evolution of collisionless stellar systems at a given resolution. These equations should be able to reproduce the self-confinement of the galaxies (incomplete relaxation) via a space-dependent diffusion coefficient, as well as the "gravothermal catastrophe" (Lynden-Bell & Wood 1968). Moreover, they can set a limitation to this gravitational collapse by taking into account the *degeneracy* occurring in the statistical treatment of a collisionless stellar system. Finally, in § 5, we derive a set of simplified evolution equations in real space that should capture the basic features of the problem and facilitate a numerical implementation.

2. THE EULER AND VLASOV EQUATIONS

Let us consider first a two-dimensional incompressible and inviscid fluid. Because of incompressibility, the velocity field satisfies $\nabla \boldsymbol{u} = 0$, and this implies the existence of a stream function such that $\boldsymbol{u} = -\hat{z} \wedge \nabla\psi$ (where $\hat{z}$ is a unit vector normal to the flow). The vorticity is defined by $\nabla \wedge \boldsymbol{u} = \omega\hat{z}$ and related to the stream function by the Poisson equation:

$$\Delta\psi = -\omega \ . \tag{2.1}$$

For inviscid fluids, the vorticity is purely advected by the flow so that

$$\frac{d\omega}{dt} \equiv \frac{\partial\omega}{\partial t} + (\boldsymbol{u}\nabla)\omega = 0 \ . \tag{2.2}$$

This equation is called the Euler equation (in two dimensions) and conserves the following quantities:

1. The energy $E = \frac{1}{2}\int \omega\psi d^2\boldsymbol{r}$,[3]
2. The angular momentum $L = \int \omega r^2 d^2\boldsymbol{r}$,

[1] Laboratoire de Physique, Ecole Normale Supérieure de Lyon, 69364 Lyon cedex 07, France.

[2] Laboratoire d' Analyse Numérique, Université Lyon 1, 69622 Villeurbanne cedex, France.

[3] This quantity is just the kinetic energy of the flow $E = \frac{1}{2}\int v^2 d^2\boldsymbol{r}$, but it can be interpreted also as the energy of interaction for a system of vortices ω in a potential ψ.

3. The linear momentum $\boldsymbol{P} = \int \boldsymbol{r} \wedge \omega \hat{\boldsymbol{z}} d^2\boldsymbol{r}$,
4. The integrals $I_h = \int h(\omega) d^2\boldsymbol{r}$, where h is any continuous function of the vorticity (in particular, the total circulation $\Gamma = \int \omega d^2\boldsymbol{r}$ is conserved).

Let us turn now to stellar systems. For a wide variety of galaxies, it is possible to neglect the encounters between stars. In that case, each star can be idealized as a Hamiltonian system $H = (v^2/2) + \Phi$, where $\boldsymbol{v}$ is its velocity and $\Phi(\boldsymbol{r}, t)$ is the gravitational potential at location $\boldsymbol{r}$ and time t. A direct use of Hamilton equations shows that their flow in the individual phase space $(\boldsymbol{r}, \boldsymbol{v})$ is incompressible, i.e., $\nabla_6 \boldsymbol{U}_6 = 0$, where $\boldsymbol{U}_6 = (\boldsymbol{v}, \boldsymbol{a}) = (\boldsymbol{v}, -\nabla\Phi)$ is the generalized velocity and $\nabla_6 = (\partial/\partial\boldsymbol{r}, \partial/\partial\boldsymbol{v})$ is the generalized nabla operator (this is the Liouville theorem in the individual μ-space). The distribution function f is defined such that $f(\boldsymbol{r}, \boldsymbol{v}, t) d^3\boldsymbol{r} d^3\boldsymbol{v}$ gives the total mass of stars at location $\boldsymbol{r}$ with velocity $\boldsymbol{v}$ at time t. It is related to the gravitational field via the Newton-Poisson equation:

$$\Delta\Phi = 4\pi G \int f d^3\boldsymbol{v} , \tag{2.3}$$

where $n = \int f d^3\boldsymbol{v}$ is the local mass density. For collisionless systems, the distribution function is purely advected by the flow in phase space so that

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + (\boldsymbol{U}_6 \nabla_6) f = 0 . \tag{2.4}$$

This equation is called the Vlasov equation and conserves the following functionals:

1. The total energy (kinetic + potential) $E = \int \frac{1}{2} f v^2 d^3\boldsymbol{r} d^3\boldsymbol{v} + \frac{1}{2} \int f \Phi d^3\boldsymbol{r} d^3\boldsymbol{v}$,
2. The angular momentum $\boldsymbol{L} = \int f(\boldsymbol{r} \wedge \boldsymbol{v}) d^3\boldsymbol{r} d^3\boldsymbol{v}$,
3. The linear momentum $\boldsymbol{P} = \int f\boldsymbol{v} d^3\boldsymbol{r} d^3\boldsymbol{v}$,
4. The integrals $I_h = \int h(f) d^3\boldsymbol{r} d^3\boldsymbol{v}$ for any continuous function $h(f)$ (in particular, the total mass $M = \int f d^3\boldsymbol{r} d^3\boldsymbol{v}$ is conserved).

The basic analogy between two-dimensional vortices and stellar systems is due to the morphological similarity of the Euler and Vlasov equations if we make the correspondance between the vorticity and the distribution function $(\omega \leftrightarrow f)$ and between the stream function and the gravitational potential $(\psi \leftrightarrow \Phi)$: these two equations describe the advection of a density by an incompressible flow with which they interact via a Poisson equation. Then the density is not advected passively by the flow but is coupled to its motion. This coupling will be responsible for the strong fluctuations of the stream function or gravitational potential. These fluctuations will mix the vorticity or the phase elements at *small scale* and induce a self-organization and the appearance of structures at *larger scales*.

Note, however, some differences between the Euler and Vlasov equations: first, the Euler system evolves in the two-dimensional real space, whereas the Vlasov equation has to be solved in a phase space with six dimensions. Second, the mass of the stars is always positive (so is the distribution function f), while the vorticity ω can be either positive or negative, yielding a wider variety of structures. Third, stellar systems have an internal energy (the kinetic part), whereas two-dimensional vortices have only the potential part allowing *negative* temperatures. At last, the center of vorticity for two-dimensional vortices $\boldsymbol{X} = (1/\Gamma) \int \omega \boldsymbol{r} d^3\boldsymbol{r}$ is conserved (far from the boundaries), whereas the center of mass for stellar systems $\boldsymbol{X} = (1/M) \int f\boldsymbol{r} d^3\boldsymbol{r} d^3\boldsymbol{v}$ has a translating motion with velocity $\boldsymbol{P}/M$.

3. THE EQUILIBRIUM STATISTICAL DESCRIPTION

The Euler equation and the Vlasov equation are known to develop very complex filaments due to stirring effects, and a deterministic description of the flow (in space or phase space) would require a rapidly increasing amount of information as time goes on. Instead, it is possible to undertake a statistical description of the flow in order to smooth out the microscopic fluctuations (fine-grained) and concentrate over the locally averaged (coarse-grained) quantities. Such an analysis has been developed long ago by Lynden-Bell (1967) in the case of collisionless stellar systems and rediscovered independently by Kuzmin (1982), Miller (1990), and Robert (1990) in the context of two-dimensional turbulence. The statistical mechanics of continuous systems is not as firmly established as in the usual case of N-body systems. However, Robert (1990, 1991) has developed a mathematical justification of the procedure for two-dimensional turbulence, and Robert & Sommeria (1991) have shown its physical relevance. Applications and tests of the theory have been discussed in further works (see Chavanis & Sommeria 1996a, 1996b), and the mathematical justification has been generalized to the Vlasov equation (among other systems) by Michel & Robert (1994).

3.1. *Two-dimensional Turbulence*

In this section, we reproduce the equilibrium statistical mechanics of two-dimensional turbulence following Robert & Sommeria (1991). In the absence of viscosity, the flow developing from an initial condition $\omega_0(\boldsymbol{r})$ becomes more and more intricate, and a complete description of the fine-grained vorticity ω is impossible and useless. We define then the probability $\rho(\boldsymbol{r}, \sigma)$ of finding the vorticity level σ in a small neighbourhood of the position $\boldsymbol{r}$. The normalization condition yields at each point

$$\int \rho(\boldsymbol{r}, \sigma) d\sigma = 1 . \tag{3.1}$$

The probability field $\rho(\boldsymbol{r}, \sigma)$ represents our macrostate. In the presence of a very small viscosity, we can expect that the inertial filamentation process is preserved, but the resulting fine-scale vorticity structures are then smoothed out, leading to a vorticity field equal to the local average:

$$\bar{\omega}(\boldsymbol{r}) = \int \rho(\boldsymbol{r}, \sigma) \sigma \, d\sigma . \tag{3.2}$$

This local averaged field is called the macroscopic, or coarse-grained, vorticity field. The associated (macroscopic) stream function satisfies

$$\bar{\omega} = -\Delta\bar{\psi} . \tag{3.3}$$

It is then possible to express the conserved quantities as integrals of the macroscopic fields. These conserved quantities are the energy,

$$E = \frac{1}{2} \int \bar{\omega}\bar{\psi} d^2\boldsymbol{r} , \tag{3.4}$$

the angular momentum,

$$L = \int \bar{\omega} r^2 d^2\boldsymbol{r} , \tag{3.5}$$

the linear momentum,

$$P = \int r \wedge \bar{\omega}\hat{z} d^2 r \,, \qquad (3.6)$$

and the global probability distribution of vorticity $\gamma(\sigma)$ (i.e., the total area of each vorticity level),

$$\gamma(\sigma) = \int \rho(r, \sigma) d^2 r \,. \qquad (3.7)$$

In writing equation (3.4), we have neglected the "internal" energy $\overline{\tilde{\omega}\tilde{\psi}}$ in comparison with the explicit energy $\bar{\omega}\bar{\psi}$. Indeed, the vorticity fluctuations $\tilde{\omega}$ are of the same order as the local average $\bar{\omega}$, but the fluctuations of the stream function are $\tilde{\psi} \sim \epsilon^2 \tilde{\omega}$, where ϵ is the typical length scale of the fluctuations much smaller than the scale of motion L. As a result, $\overline{\tilde{\omega}\tilde{\psi}}$ of order $\epsilon^2 \bar{\omega}^2$ is much smaller than $\bar{\omega}\bar{\psi}$ of order $L^2 \bar{\omega}^2$.

The equilibrium distribution corresponds to the most mixed state satisfying the constraints brought by the Euler equation. Technically, it is obtained by maximizing a mixing entropy:

$$S = -\int \rho(r, \sigma) \ln \rho(r, \sigma) d^2 r d\sigma \,, \qquad (3.8)$$

with the constraints (3.4)–(3.7) and the normalization condition (3.1). This variational problem is treated by introducing Lagrange multipliers so that the first variations satisfy

$$\delta S - \beta \delta E - \int \alpha(\sigma) \delta \gamma(\sigma) d\sigma - \int \zeta(r) \delta \left(\int \rho \, d\sigma \right) d^2 r - \beta \frac{\Omega}{2} \delta L + \beta U \delta P = 0 \,, \qquad (3.9)$$

where β is the inverse temperature, $\alpha(\sigma)$ is the "chemical potential" of species σ, Ω is the rotation velocity, and U is the translation velocity. The resulting optimal probability density $\rho(r, \sigma)$ is a Gibbs state of the form

$$\rho(r, \sigma) = \frac{e^{-\alpha(\sigma) - \beta\sigma[\bar{\psi} + (\Omega/2) r^2 - (U \wedge r) z]}}{\int e^{-\alpha(\sigma) - \beta\sigma[\bar{\psi} + (\Omega/2) r^2 - (U \wedge r) z]} d\sigma} \,. \qquad (3.10)$$

In the case of a single vorticity level σ_0 (obtained if the initial condition is made of patches of uniform vorticity σ_0 embedded in an irrotational flow), we have $\bar{\omega} = \rho\sigma_0$, and the integral constraints are simply E, Γ, L, and P. Moreover, the Gibbs distribution (3.10) reduces to

$$\bar{\omega} = \frac{\sigma_0 e^{-\beta\sigma_0(\bar{\psi}' - \mu)}}{1 + e^{-\beta\sigma_0(\bar{\psi}' - \mu)}} \,, \qquad (3.11)$$

where $\bar{\psi}' = \bar{\psi} + (\Omega/2) r^2 - (U \wedge r)\hat{z}$ is the relative stream function and $\mu = -\alpha(\sigma_0)/\beta\sigma_0$ is a chemical potential.

Note first that these equilibrium states are not steady in general but translate or rotate uniformly with velocities U and Ω. Note also that these distributions belong to the Fermi-Dirac statistics: indeed, the incompressibility constraint (3.1) plays the role of an "exclusion principle" such as for fermions in quantum mechanics. However, in the "dilute limit" (or "nondegenerate" limit) at which the nonzero vorticity level occupies a small proportion of the flow ($\bar{\omega} \ll \sigma_0$), The denominator in equation (3.11) is approximately equal to one, and the equilibrium distribution becomes

$$\bar{\omega} = A e^{-\beta\sigma_0\bar{\psi}'} \,, \qquad (3.12)$$

corresponding to the Maxwell-Boltzmann statistics. The same relation is also obtained from Onsager's (1949) statistical theory of point vortices when the vortex density is locally averaged, using the mean field approach of Montgomery & Joyce (1974).

The previous analysis gives a well-defined procedure to compute the statistical equilibrium state resulting from any initial condition $\omega_0(r)$. The macroscopic stream function is obtained by solving the Poisson equation (2.1) with the vorticity (3.2) and the Gibbs distribution (3.10). The solution will depend on the Lagrange multipliers β, $\alpha(\sigma)$, Ω, and U, and these Lagrange multipliers must be related to the conserved quantities E, $\gamma(\sigma)$, L, and P by the constraints (3.4)–(3.7). Finally, we have to make sure that this solution is a true entropy *maximum* by investigating the second variations of the entropy (let us recall that the Gibbs distribution [3.10] cancels only the first variations of the entropy).

This problem can be solved numerically in the general case with the algorithm developed by Turkington & Whitaker (1996). Several computations have been performed in the case of rectangular or circular domains for a restricted number of vorticity levels and for particular values of the integral contraints. These studies exhibit a large variety of structures (monopoles, dipoles, tripoles, etc.), and it is difficult to have a clear general picture of the bifurcation diagram in the parameter space. For this reason, Chavanis & Sommeria (1996a) have investigated analytically a limit of strong mixing and obtained a classification of coherent structures in terms of a single control parameter. The opposite situation occurs when the initial state ω_0 is a stable solution of the Euler equation that is steady in the sense of Arnold (1969). In that case, it can be shown that the constraints due to the conservation laws are so strong that any mixing is forbidden: the equilibrium probability distribution at point r is $\rho(r, \sigma) = \delta[\sigma - \omega_0(r)]$. Notice that the steady flow ω^* obtained by smoothing an equilibrium state $[\omega^*(r) = \bar{\omega}(r)]$ has this stability property: it cannot mix anymore (this property can be proved when the equilibrium state is unique; see Robert & Sommeria 1991).

On the other hand, the relevance of the statistical theory has been tested by experiments and direct Navier Stokes simulations at high Reynolds number. In all these tests, the relationship $\bar{\omega} = f(\bar{\psi}')$ obtained at equilibrium was compared to the theoretical prediction (3.10). This study led to the following remarks: first, the system must converge quickly to the equilibrium, otherwise the (inherent) viscosity can alter the constants of motion and cause changes in the final state. Second, we can observe in many situations that the flow converges toward some stationary state that corresponds *locally* to the Gibbs distribution (3.10) but is not a maximum entropy in the whole domain.

A physical explanation of this self confinement has been given by Robert & Rosier (1996). The idea is to assume that the system tends to reach its global equilibrium but is restricted by some kinetic constraints: it is well known in thermodynamics that the relaxation is driven by some fluctuations (here these fluctuations are due to the fine-scale vorticity structures); however, it may happen that these fluctuations vanish before the system has achieved its global equilibrium, and in that case, the system appears to be frozen in a confined region of space. In other words, the mixing occurs only in a restricted area, and the statistical theory can be applied only in this subdomain. This mechanism justifies the existence of isolated structures like dipoles and tripoles when $\Gamma = 0$, although there is no global maximum entropy state in the infinite domain in that case (Sommeria 1994; Chavanis & Sommeria 1996b). Robert & Sommeria (1992) have proposed a set of relaxation equations to model the evolution of the system out of equilibrium. This work was pursued by Robert & Rosier (1996), who took into account the "incomplete relaxation" described previously, yielding a set of relaxation equations which converge locally toward a Gibbs state and give the shape and the size of the domain of confinement. This nonequilibrium study will be developed in detail in § 4.

3.2. *Collisionless Stellar Systems*

The statistical mechanics of collisionless stellar systems was investigated by Lynden-Bell (1967). Because of the violent fluctuations of the gravitational field in the early stage of a galaxy, the stars follow complicated paths along which the individual energies are not conserved. Accordingly, the stars will relax very rapidly toward a kind of equilibrium on a typical timescale $\sim t_D$ (dynamical time). This process is known as "violent relaxation" and was considered first by Hénon (1964) and King (1966). In this process, the (fine-grained) distribution function f never achieves equilibrium but builds up a very finely striated structure in phase space (phase mixing). However, if we average over these striations, we obtain a coarse-grained distribution function $\bar{f}$ that is smooth in phase space and likely to converge toward an equilibrium state.

In the following, we reproduce Lynden-Bell statistical mechanics of equilibrium and use notations similar to that of the previous section. Starting from some initial condition, the distribution function is stirred in phase space but conserves its values η (levels of phase density) and the corresponding hypervolumes $\gamma(\eta)$ as a property of the Vlasov equation (2.4) and the incompressibility condition $\nabla_6 \boldsymbol{U}_6 = 0$. Then we introduce the probability $\rho(\boldsymbol{r}, \boldsymbol{v}, \eta)$ of finding the level of phase density η in a small neighborhood of the position $\boldsymbol{r}$, $\boldsymbol{v}$ in phase space. The normalization condition yields at each point

$$\int \rho(\boldsymbol{r}, \boldsymbol{v}, \eta) d\eta = 1 \ . \tag{3.13}$$

The locally averaged distribution function is expressed in terms of the probability density under the form

$$\bar{f}(\boldsymbol{r}, \boldsymbol{v}) = \int \rho(\boldsymbol{r}, \boldsymbol{v}, \eta) \eta \, d\eta \ , \tag{3.14}$$

and the associated (macroscopic) gravitational potential satisfies

$$\Delta \bar{\Phi} = 4\pi G \int \bar{f} d^3 v \ . \tag{3.15}$$

As before, the conserved quantities can be expressed as integrals of the macroscopic fields. These conserved quantities are the total energy,

$$E = \int \tfrac{1}{2} \bar{f} v^2 d^3 \boldsymbol{r} d^3 \boldsymbol{v} + \tfrac{1}{2} \int \bar{f} \bar{\Phi} d^3 \boldsymbol{r} d^3 \boldsymbol{v} \ , \tag{3.16}$$

the angular momentum,

$$\boldsymbol{L} = \int \bar{f}(\boldsymbol{r} \wedge \boldsymbol{v}) d^3 \boldsymbol{r} d^3 \boldsymbol{v} \ , \tag{3.17}$$

the linear momentum,

$$\boldsymbol{P} = \int \bar{f} \boldsymbol{v} d^3 \boldsymbol{r} d^3 \boldsymbol{v} \ , \tag{3.18}$$

and the global probability distributions of phase density $\gamma(\eta)$ (i.e., the total hypervolume occupied by each level η of phase density),

$$\gamma(\eta) = \int \rho(\boldsymbol{r}, \boldsymbol{v}, \eta) d^3 \boldsymbol{r} d^3 \boldsymbol{v} \ . \tag{3.19}$$

The most likely distribution to be reached at equilibrium is obtained by maximizing a mixing entropy,

$$S = -\int \rho(\boldsymbol{r}, \boldsymbol{v}, \eta) \ln \rho(\boldsymbol{r}, \boldsymbol{v}, \eta) d\eta d^3 \boldsymbol{r} d^3 \boldsymbol{v} \ , \tag{3.20}$$

under the constraints (3.16)–(3.19) and the normalization condition (3.13). To that purpose, we introduce Lagrange multipliers and write the variational problem under the form

$$\delta S - \beta \delta E - \int \alpha(\eta) \delta \gamma(\eta) d\eta - \int \zeta(\boldsymbol{r}, \boldsymbol{v}) \delta\left(\int \rho \, d\eta\right) d^3 \boldsymbol{r} d^3 \boldsymbol{v} + \beta \boldsymbol{\Omega} \, \delta \boldsymbol{L} + \beta \boldsymbol{U} \delta \boldsymbol{P} = 0 \ , \tag{3.21}$$

where β is the inverse temperature, $\alpha(\eta)$ is the "chemical potential" of species η, $\boldsymbol{\Omega}$ is the rotation velocity, and $\boldsymbol{U}$ is the translation velocity of the system. The resulting optimal probability density is a Gibbs state which has the form

$$\rho(\boldsymbol{r}, \boldsymbol{v}, \eta) = \frac{e^{-\alpha(\eta) - \beta\eta[(v^2/2) + \bar{\Phi} - (\boldsymbol{\Omega} \wedge \boldsymbol{r} + \boldsymbol{U})\bar{\psi}]}}{\int e^{-\alpha(\eta) - \beta\eta[(v^2/2) + \bar{\Phi} - (\boldsymbol{\Omega} \wedge \boldsymbol{r} + \boldsymbol{U})\bar{\psi}]} d\eta} \ , \tag{3.22}$$

similar to equation (3.10). Once again, these equilibrium states are not steady in general but translate or rotate rigidly with velocities $\boldsymbol{U}$ and Ω. If we work in the barycentric frame of reference, the translation velocity vanishes ($\boldsymbol{U} = 0$) and the motion reduces to a pure rotation directly connected to the angular momentum of the galaxy.

The previous analysis gives a well-defined procedure to compute the statistical equilibrium states. The gravitational field is obtained by solving the Poisson equation (2.3) with the distribution function (3.14) and the Gibbs distribution (3.22). The solution depends on the Lagrange multipliers β, $\alpha(\eta)$, $\boldsymbol{\Omega}$, and $\boldsymbol{U}$ that must be related to the conserved quantities E, $\gamma(\eta)$, $\boldsymbol{L}$, and $\boldsymbol{P}$ by equations (3.16)–(3.19). Finally, we have to make sure that this solution is a true entropy *maximum* by investigating the second variations of the entropy. This problem is similar to that of two-dimensional turbulence but, to our knowledge, it has never been explored in the general case. As in the case of the Euler equation, the equilibrium states have some stability properties. Reversely, some steady solutions $f_0(\boldsymbol{r}, \boldsymbol{v})$ of the Vlasov equation cannot mix, so that $\rho(\boldsymbol{r}, \boldsymbol{v}, \eta) = \delta[\eta - f_0(\boldsymbol{r}, \boldsymbol{v})]$. The link between the stability problem and the equilibrium states of the statistical theory will be discussed in a forthcoming paper.

Following Lynden-Bell (1967), we consider a particular situation that presents interesting features. In the case of galaxies, the distribution function is always positive (by contrast with the vorticity in two-dimensional turbulence), and the assumption of a single level of phase density may be relevant; this situation occurs for an initial condition with a uniform density η_0 in part of the phase space. In that case,

$\bar{f} = \rho\eta_0$, and the Gibbs distribution (3.22) reduces to

$$\bar{f} = \frac{\eta_0 e^{-\beta\eta_0(\epsilon-\mu)}}{1 + e^{-\beta\eta_0(\epsilon-\mu)}}, \quad (3.23)$$

where $\epsilon = (v^2/2) + \bar{\Phi} - (\mathbf{\Omega} \wedge \mathbf{r})\mathbf{v}$ is the energy (per unit of mass) of the stars in the rotating frame and μ is a chemical potential. This is, apart from a reinterpretation of the constants, the distribution function for the self-gravitating Fermi-Dirac gas (here the exclusion principle is due to the incompressibility constraint [3.13]). In the near fully degenerate case, this equilibrium has been studied in connection with white dwarf stars. The form of the solution may depend crucially on the degree of degeneracy, but Lynden-Bell (1967) gives arguments according to which stellar systems would be nondegenerate. In that limit, $\bar{f} \ll \eta_0$ and equation (3.23) reduces to the Maxwell-Boltzmann statistics:

$$\bar{f} = Ae^{-\beta\eta_0\epsilon}. \quad (3.24)$$

This was in fact the initial goal of Lynden-Bell (1967): his theory of "violent relaxation" was able to justify a Maxwell-Boltzmann equilibrium distribution without recourse to collisions, on a more relevant timescale $\sim t_D \ll t_{\text{coll}}$. In addition, the individual mass of the stars never appears in his theory (based on the Vlasov equation) so that the equilibrium state in equation (3.24) does not lead to a segregation by mass. This is consistent with the observed light distributions in elliptical galaxies and contrasts with the ordinary collisional Maxwell-Boltzmann distribution, according to which heavy stars sink at the center of the galaxy whereas lighter stars orbit at the periphery.

As a matter of fact, the evolution of a galaxy may proceed in two steps: there is first a violent collisionless relaxation due to the strong fluctuations of the gravitational field, then a much slower relaxation due to collisions. However, these two successive equilibria are very similar, and it is often difficult to characterize them properly in N-body simulations (because of the limited number of particles and the associated lack of resolution). The situation is radically different in two-dimensional turbulence for which the final equilibrium is a complete state of rest because of inherent viscosity. Now, it has been revealed by a large number of numerical simulations and laboratory experiments that a first equilibrium state takes place on a very short timescale $\sim t_D$ leading to the formation of coherent structures. This equilibrium is due to the vorticity mixing and to the rapid fluctuations of the stream function and is well described by a statistical theory. This violent relaxation is completely similar to that of Lynden-Bell, and this supports the idea that this process must be at work in stellar systems also. This anology thus provides a clear justification of Lynden-Bell's statistical arguments.

3.3. *Existence of Equilibrium States in Stellar Systems*

The maximum entropy problem in the case of the Maxwell-Boltzmann distribution (3.24) is common to both collisional and collisionless systems (in the nondegenerate limit) and leads to isothermal galaxies. In the case of spherically symmetric systems with no net rotation, the solution is well known to have an infinite mass and energy (see, for example, Binney & Tremaine 1987). This means that no equilibrium can exist in an infinite domain: the density can spread indefinitely while conserving its energy and increasing the entropy. It is then necessary to confine artificially the system in a box of radius R and seek a *restricted* maximum entropy state. This problem has been investigated analytically by Antonov (1962), Lynden-Bell & Wood (1968), and Padmanabhan (1990). It is possible to prove the following results: (i) there is no global maximum for entropy; (ii) there is not even a local extremum for entropy if $-RE/GM^2 > 0.335$; (iii) a local maximum exists if $-RE/GM^2 < 0.335$. Conclusion (ii) has been called "gravothermal catastrophe" or "Antonov instability." In this case, the galaxy takes a "core-halo" configuration whose entropy can always be increased by making the center denser and denser. This famous instability can initiate the formation of black holes in the center of galaxies and has been the subject of detailed studies (see, for example, Lynden-Bell & Eggleton 1980; Heggie & Stevenson 1988).

However, we must keep in mind that these results are obtained within certain hypotheses: first of all, they apply only to nonrotating systems, and it is not clear how they are modified in the presence of a solid rotation. Second, in the context of collisionless systems, they assume that galaxies are nondegenerate (in the sense of Lynden-Bell); of course, this assumption does not hold anymore if the central nucleus becomes denser and denser. In that case, we have to come back to the general problem of equations (3.22) or (3.23) and, when degeneracy is taken into account, we know that a maximum entropy state always exists (this maximum entropy problem will be addressed specifically in a forthcoming paper). Besides, our experience in two-dimensional turbulence tells us that this form of degeneracy is not an exotic phenomenon; on the contrary, it is vindicated by a majority of laboratory experiments or direct Navier-Stokes simulations. As a result, we believe that degeneracy is relevant in collisionless stellar systems also and can put an end to the gravothermal catastrophe. Then the computation and the classification of degenerate equilibrium states is an important issue that does not seem to have received enough attention for the moment.

Moreover, there is a general weakness in the whole theory that limits its power of prediction: indeed, we must assume that the galaxy is confined arbitrarily in a box in order to avoid the "infinite mass" problem. This problem is similar to the one encountered in two-dimensional turbulence when $\Gamma = 0$: in that case, there is no equilibrium state in the whole space, and we have to seek a restricted entropy maximum. This self-confinement was justified physically in § 3.1 in terms of kinetic processes. The same explanation can be given in the case of stellar systems: here the relaxation is driven by the fluctuations of the gravitational field. Now, these fluctuations can vanish before the global equilibrium is reached, and the structure will be frozen in a subdomain of phase space (incomplete relaxation); this mechanism should be able to solve the "infinite mass problem".

The self-confinement process was described correctly in the case of two-dimensional turbulence by Robert & Rosier (1996). In the next section, we recall the main arguments of their description and extend their methods to the case of collisionless stellar systems.

4. THE NONEQUILIBRIUM STUDY: RELAXATION EQUATIONS

In this section, we consider a variational principle that appears to be efficient to describe the evolution of a system without recourse to its microscopic details. This principle, called the maximum entropy production principle (MEPP), makes capital of our ignorance and assumes that "during its evolution, the system tends to maximize its rate of entropy production while satisfying all the constraints imposed by the dynamics." This principle is reminiscent of Jaynes's (1989) ideas and is a clear extension of the well-known principle of statistical mechanics according to which "at equilibrium, the system is in a maximum entropy state consistent with all the constraints." This MEPP was applied for the first time by Robert & Sommeria (1992) and discussed further in Robert (1994) and Robert & Rosier (1996) in the context of two-dimensional turbulence. It can be seen as a substitute to the classical recipe of linear non-equilibrium thermodynamics, but easier to apply, as the notion of a local thermodynamic equilibrium is not defined clearly in this context. Accordingly, we believe that this principle is able to give interesting results in many situations including two-dimensional turbulence and collisionless stellar systems.

4.1. *Two-dimensional Turbulence*

We recall first the derivation of the relaxation equations given by Robert & Sommeria (1992) in the case of two-dimensional turbulence. We shall assume that during its evolution, the system can already be described in terms of local probabilities $\rho(\mathbf{r}, \sigma, t)$. In other words, the system has already undergone fine-scale oscillations, and the locally averaged velocity field $\bar{\mathbf{u}}(\mathbf{r}, t)$ is obtained by the integration of $\nabla \wedge \bar{\mathbf{u}} = \bar{\omega}\hat{\mathbf{z}}$, where $\bar{\omega}(\mathbf{r}, t) = \int \rho(\mathbf{r}, \sigma, t)\sigma\, d\sigma$. The vorticity patches are transported by this averaged velocity field, and we suppose that in addition they undergo a diffusion process, so that the conservation equation for each vorticity probability can be written

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\bar{\mathbf{u}}) = -\nabla \mathbf{J}\,, \tag{4.1}$$

where $\mathbf{J}(\mathbf{r}, \sigma, t)$ is the diffusion current of the patch σ. This equation conserves the total area occupied by each vorticity level. Moreover, the normalization condition (3.1) implies that $\int \mathbf{J}(\mathbf{r}, \sigma, t)d\sigma = 0$. Introducing the current $\mathbf{J}_{\omega} = \int \mathbf{J}(\mathbf{r}, \sigma, t)\sigma\, d\sigma$ of the coarse grained vorticity, we deduce from equation (4.1) an equation of the form

$$\frac{\partial \bar{\omega}}{\partial t} + \nabla(\bar{\omega}\,\bar{\mathbf{u}}) = -\nabla \mathbf{J}_{\omega}\,. \tag{4.2}$$

This evolution equation can be obtained directly from the Euler equation if we decompose the vorticity and velocity under a mean and fluctuating part (namely $\omega = \bar{\omega} + \tilde{\omega}$ and $\mathbf{u} = \bar{\mathbf{u}} + \tilde{\mathbf{u}}$) and take the average. The diffusion current is then expressed as the correlation $\overline{\tilde{\omega}\tilde{\mathbf{u}}}$ of the fine-grained fluctuations and must be related to the explicit fields by a closure model. In our case, the closure depends on the local probability distribution ρ, and we must keep track of this distribution by the detailed conservation laws in equation (4.1).

Let us finally compute the rate of change of the energy, angular momentum, linear momentum, and entropy during the convection-diffusion process. Using equations (3.4)–(3.6), (3.8) and (4.1)–(4.2), straightforward computations give

$$\dot{E} = \int \mathbf{J}_{\omega}\nabla\bar{\psi}\, d^2\mathbf{r}, \tag{4.3}$$

$$\dot{L} = -\int r^2 \nabla \mathbf{J}_{\omega}\, d^2\mathbf{r}\,, \tag{4.4}$$

$$\dot{\mathbf{P}} = -\int (\mathbf{r} \wedge \hat{\mathbf{z}})\nabla \mathbf{J}_{\omega} d^2\mathbf{r}\,, \tag{4.5}$$

$$\dot{S} = -\int \mathbf{J}\nabla(\ln \rho) d^2\mathbf{r}\, d\sigma\,. \tag{4.6}$$

We can now apply the MEPP: the system distributes its currents in order to maximize its rate of entropy production $\dot{S}$ while satisfying the following constraints:

$$\int \mathbf{J}(\mathbf{r}, \sigma, t)d\sigma = 0\,,$$

$$\dot{E} = 0, \quad \dot{L} = 0, \quad \dot{\mathbf{P}} = 0,$$

$$\int \frac{J^2}{2\rho}\, d\sigma \le C(\mathbf{r})\,.$$

The last constraint expresses the fact that the diffusion currents cannot be arbitrarily large. This variational problem can be solved by introducing Lagrange multipliers,

$$\delta\dot{S} - \beta\delta\dot{E} - \int D^{-1}\delta\left(\int \frac{J^2}{2\rho}\, d\sigma\right)d^2\mathbf{r} - \int \zeta\delta\left(\int \mathbf{J} d\sigma\right)d^2\mathbf{r} - \beta\,\frac{\Omega}{2}\,\delta\dot{L} + \beta \mathbf{U}\delta\dot{\mathbf{P}} = 0\,, \tag{4.7}$$

and it yields an optimal current of the form

$$\mathbf{J} = -D(\mathbf{r}, t)\left\{\nabla\rho + \beta(\sigma - \bar{\omega})\rho\nabla\left[\bar{\psi} + \frac{\Omega}{2}\, r^2 - (\mathbf{U} \wedge \mathbf{r})\hat{\mathbf{z}}\right]\right\}. \tag{4.8}$$

When combined with equation (4.1), we obtain a closed system for the evolution of the density probabilities. The Lagrange multipliers $\beta(t)$, $\Omega(t)$, and $\mathbf{U}(t)$ still depend on time and are determined by the conservation of energy, angular momentum and linear impulse using the expressions (4.3)–(4.5) or (3.4)–(3.6).

The MEPP cannot predict by itself the expression of the diffusion coefficient $D(\mathbf{r}, t)$ but states only that $D \ge 0$ in order to have the increase of entropy (see Appendix A). However, we can use a simple stochastic model to obtain its general behavior following Robert & Rosier (1996). Qualitatively, the diffusion coefficient is proportional to the fluctuations of the velocity. Now, these fluctuations are induced mostly by the local fine-grained fluctuations of the vorticity (because of the coupling in eq. [2.1] between velocity and vorticity). Consequently, we expect for D an expression of the form

$$D(\mathbf{r}, t) = K\epsilon^2(\overline{\omega^2} - \bar{\omega}^2)^{1/2}\,, \tag{4.9}$$

where ϵ is the spatial resolution determining the scale of the unresolved fluctuations (see Appendix B for a more detailed explanation). This diffusion coefficient will vanish where $\overline{\omega^2} - \bar{\omega}^2 = 0$, i.e., where there is no mixing of the vorticity at small scales. As a result, the diffusion will stop, and the flow will be frozen in a *subdomain* of space according to the heuristical picture given above.

In summary, the previous equations take into account all the constraints imposed by the dynamics and satisfy the increase of entropy (with an optimal rate) as well as the self-confinement. Moreover, if the system evolves toward a stationary state, it is shown in Appendix A that this state is described by the Gibbs distribution (3.10) in any region in which $D \neq 0$ (i.e., where the mixing is effective).

The relevance of these equations has been investigated numerically by Robert & Rosier (1996) in the case of dipoles or tripoles. In particular, they are able to compete very well with direct Navier-Stokes simulations at high Reynolds number and demand of course a much lower resolution, since the small scales are smoothed out.

4.2. *Collisionless Stellar Systems in Phase Space*

This approach can be applied as well to the Vlasov equations and may appear to be very useful to model these equations at large times. Here again, the nonequilibrium evolution is described by a set of local probabilities $\rho(\boldsymbol{r}, \boldsymbol{v}, \eta, t)$, and the locally averaged gravitational potential $\bar{\Phi}(\boldsymbol{r}, t)$ is obtained by the integration of equation (2.3), where $f(\boldsymbol{r}, \boldsymbol{v}, t)$ is replaced by $\bar{f}(\boldsymbol{r}, \boldsymbol{v}, t) = \int \rho(\boldsymbol{r}, \boldsymbol{v}, \eta, t)\eta\, d\eta$. The phase elements are thus transported in phase space by the corresponding averaged velocity field $\bar{\boldsymbol{U}}_6 = (\boldsymbol{v}, -\nabla\bar{\Phi})$, and we suppose that in addition they undergo a diffusion process. This diffusion occurs only in the velocity space, due to the fluctuations of the gravitational field Φ. There is no diffusion in the position space, since the velocity $\boldsymbol{v}$ is a pure coordinate and does not fluctuate. As a result, the density probabilities will satisfy a convection-diffusion equation of the form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{v}\frac{\partial \rho}{\partial \boldsymbol{r}} - \nabla\bar{\Phi}\frac{\partial \rho}{\partial \boldsymbol{v}} = -\frac{\partial \boldsymbol{J}}{\partial \boldsymbol{v}}, \tag{4.10}$$

where $\boldsymbol{J}(\boldsymbol{r}, \boldsymbol{v}, \eta, t)$ is the diffusion current of the phase element η. This equation conserves the total volume (in phase space) occupied by each phase element. Moreover, the normalization condition (3.13) implies $\int \boldsymbol{J}(\boldsymbol{r}, \boldsymbol{v}, \eta, t)d\eta = 0$. Introducing the current $\boldsymbol{J}_f = \int \boldsymbol{J}(\boldsymbol{r}, \boldsymbol{v}, \eta, t)\eta\, d\eta$ of the coarse-grained distribution function, we deduce from equation (4.10) an equation of the form

$$\frac{\partial \bar{f}}{\partial t} + \boldsymbol{v}\frac{\partial \bar{f}}{\partial \boldsymbol{r}} - \nabla\bar{\Phi}\frac{\partial \bar{f}}{\partial \boldsymbol{v}} = -\frac{\partial \boldsymbol{J}_f}{\partial \boldsymbol{v}}. \tag{4.11}$$

This evolution equation can be obtained directly from the Vlasov equation if we decompose the distribution function and the gravitational potential under a mean and fluctuating part (namely, $f = \bar{f} + \tilde{f}$ and $\Phi = \bar{\Phi} + \tilde{\Phi}$) and take the average. The diffusion current is then expressed as the correlation $-\overline{\tilde{f}\nabla\tilde{\Phi}}$ of the fine-grained fluctuations.

As before, we can compute the rate of change of the energy, angular momentum, linear momentum, and entropy during the convection-diffusion process. Using equations (3.16)–(3.18), (3.20), and (4.10)–(4.11), we obtain

$$\dot{E} = \int \boldsymbol{J}_f \boldsymbol{v} d^3r d^3v, \tag{4.12}$$

$$\dot{\boldsymbol{L}} = \int \boldsymbol{r} \wedge \boldsymbol{J}_f d^3r d^3v, \tag{4.13}$$

$$\dot{\boldsymbol{P}} = \int \boldsymbol{J}_f d^3r d^3v, \tag{4.14}$$

$$\dot{S} = -\int \boldsymbol{J}\frac{\partial \ln \rho}{\partial \boldsymbol{v}} d^3r d^3v\, d\eta. \tag{4.15}$$

We can now apply the MEPP: the system distributes its currents in order to maximize its rate of entropy production $\dot{S}$ while satisfying the following constraints:

$$\int \boldsymbol{J}(\boldsymbol{r}, \boldsymbol{v}, \eta, t)d\eta = 0,$$

$$\dot{E} = 0, \quad \dot{\boldsymbol{L}} = 0, \quad \dot{\boldsymbol{P}} = 0,$$

$$\int \frac{J^2}{2\rho} d\eta \leq C(\boldsymbol{r}, \boldsymbol{v}).$$

The last constraint expresses the fact that the diffusion currents cannot be arbitrarily large. This variational problem can be solved by introducing Lagrange multipliers,

$$\delta\dot{S} - \beta\delta\dot{E} - \int D^{-1}\delta\left(\int \frac{J^2}{2\rho} d\eta\right) d^3r d^3v - \int \zeta\delta\left(\int \boldsymbol{J}\, d\eta\right) d^3r d^3v + \beta\boldsymbol{\Omega}\,\delta\dot{\boldsymbol{L}} + \beta\boldsymbol{U}\delta\dot{\boldsymbol{P}} = 0, \tag{4.16}$$

and it yields an optimal current of the form

$$\boldsymbol{J} = -D(\boldsymbol{r}, \boldsymbol{v}, t)\left[\frac{\partial \rho}{\partial \boldsymbol{v}} + \beta(\eta - \bar{f})\rho(\boldsymbol{v} - \boldsymbol{\Omega} \wedge \boldsymbol{r} - \boldsymbol{U})\right]. \tag{4.17}$$

When combined with equation (4.10), we obtain a closed system for the evolution of the density probabilities. The Lagrange multipliers $\beta(t)$, $\boldsymbol{\Omega}(t)$, and $\boldsymbol{U}(t)$ still depend on time and are determined by the conservation of energy, angular momentum and linear impulse using equations (4.12)–(4.14) or (3.16)–(3.18).

As stated previously, the MEPP cannot predict by itself the expression of the diffusion coefficient $D(\boldsymbol{r}, \boldsymbol{v}, t)$ but states only that $D \geq 0$ in order to satisfy the increase of entropy (see Appendix A). In his article, Lynden-Bell (1967) assumes $D = K/v^3$ (where K is a constant) and obtains a stationary state of the form (King's model)

$$\bar{f} = A(e^{-\beta\epsilon} - e^{-\beta\epsilon_m}) \quad \epsilon \leq \epsilon_m, \tag{4.18}$$

$$\bar{f} = 0 \quad \epsilon \geq \epsilon_m, \tag{4.19}$$

where $\epsilon = (v^2/2) + \bar{\Phi}$ is the individual energy (per mass) of the stars and ϵ_m is the escape energy. This cutoff provides a method for confining the galaxies and solves the "infinite mass" problem. However, the escape energy ϵ_m is a free parameter, and the structure of the galaxy may depend strongly on its choice. Moreover, the underlying assumption $D \sim 1/v^3$ is correct when the fluctuations are due to two-body encounters, but in the case of stochastic processes, Kandrup (1980) argues that the diffusion coefficient must be independent of the velocity $\boldsymbol{v}$.

Here we justify the self confinment by the kinetic approach described in § 3. Qualitatively, the diffusion is due to the rapid variations of the gravitational field. These variations are induced by the fluctuations of the phase density, and if we extend (see Appendix B) the stochastic model devoloped in Robert & Rosier (1996), we obtain an expression of the form

$$D(\boldsymbol{r}, t) = KG\epsilon_r\epsilon_v^{5/2}\left[\int (\overline{f^2} - \bar{f}^2)d^3v\right]^{1/2}, \tag{4.20}$$

where ϵ_r and ϵ_v are the typical scale of the resolution in position and velocity, respectively. This diffusion coefficient vanishes where $\overline{f^2} - \bar{f}^2 = 0$, i.e., where there are no fine-scale fluctuations. As a result, the relaxation will stop and the system will be frozen in a *subdomain* of phase space. This provides a more physical mechanism for confining the galaxies and justifying truncated models. This mechanism appears to be very efficient in the case of two-dimensional turbulence and should work as well for stellar systems.

4.3. *The Case of a Single Density Level*

The evolution equations (4.10) and (4.17) can be simplified in the case of a single density level η_0 and provide a new nonlinear equation for the coarse-grained distribution function $\bar{f} = \rho\eta_0$:

$$\frac{\partial \bar{f}}{\partial t} + \boldsymbol{v}\frac{\partial \bar{f}}{\partial \boldsymbol{r}} - \nabla\bar{\Phi}\frac{\partial \bar{f}}{\partial \boldsymbol{v}} = \frac{\partial}{\partial \boldsymbol{v}}\left\{D\left[\frac{\partial \bar{f}}{\partial \boldsymbol{v}} + \beta\bar{f}(\eta_0 - \bar{f})(\boldsymbol{v} - \boldsymbol{\Omega}\wedge\boldsymbol{r} - \boldsymbol{U})\right]\right\}, \quad (4.21)$$

with the constraints (4.12)–(4.14) or (3.16)–(3.18) determining the time evolution of $\beta(t)$, $\boldsymbol{\Omega}(t)$, and $\boldsymbol{U}(t)$.

In the nondegenerate limit ($\bar{f} \ll \eta_0$), equation (4.21) takes the form of a Fokker-Planck equation:

$$\frac{\partial \bar{f}}{\partial t} + \boldsymbol{v}\frac{\partial \bar{f}}{\partial \boldsymbol{r}} - \nabla\bar{\Phi}\frac{\partial \bar{f}}{\partial \boldsymbol{v}} = \frac{\partial}{\partial \boldsymbol{v}}\left\{D\left[\frac{\partial \bar{f}}{\partial \boldsymbol{v}} + \beta\eta_0\bar{f}(\boldsymbol{v} - \boldsymbol{\Omega}\wedge\boldsymbol{r} - \boldsymbol{U})\right]\right\}. \quad (4.22)$$

This Fokker-Planck equation is well known in the case of collisional stellar systems (without the bar on f) and is usually derived from a Markov hypothesis and a stochastic Langevin equation (see Chandrasekhar 1943). It can be obtained also from the following argument: because of close encounters, the stars undergo Brownian motion and diffuse in velocity space (responsible for the first term on the right-hand side). However, under the influence of this diffusion the kinetic energy per star will diverge as $\langle v^2/2\rangle \sim 3Dt$, and one is forced to introduce an ad hoc dynamical friction $\xi = \beta D\eta_0$ (Einstein relation) in order to compensate for this divergence and recover the Maxwellian distribution of velocities at equilibrium. From these two considerations (that express the fluctuation-dissipation theorem) results the ordinary Fokker-Planck equation (4.22).

Lynden-Bell (1967) has proposed to apply the same equation for collisionless systems with the heuristic argument that the rapid fluctuations of the gravitational field during the stage of "violent relaxation" play the same role as collisions. However, the dynamical friction is difficult to justify with a stochastic model in the case of collisionless systems. Furthermore, the coefficients β and $\boldsymbol{\Omega} = \boldsymbol{U} = 0$ were assumed constant, which is incompatible with the conservation laws for an isolated system. A similar evolution equation is obtained here and justified by an argument of a very wide scope that does not refer directly to the microscopic details of the system: the Fokker-Planck equation appears to maximize the rate of entropy at each time with appropriate constraints. In this description, the "dynamical friction" (proportional to the inverse "temperature" β) appears naturally as a Lagrange multiplier associated with the conservation of energy, and the Einstein relation is satisfied automatically. Moreover, in our point of view, $\beta(t)$, $\boldsymbol{\Omega}(t)$, and $\boldsymbol{U}(t)$ are time-dependent functions that evolve in order to conserve the energy, angular momentum, and linear impulse. Finally, our procedure can take into account the degeneracy, keeping equation (4.21) instead of the nondegenerate limit in equation (4.22). This generalization is important because degeneracy is specific to collisionless systems and may be crucial for the existence of an equilibrium state (see § 3). Notice that $\beta(t)$ (the same remark holds for $\boldsymbol{\Omega}$ and $\boldsymbol{U}$) is a *global* quantity (different from the local kinetic temperature) introduced to conserve the energy globally. An alternative Fokker-Planck equation involving a *local* temperature has been proposed in Clemmow & Dougherty (1969) in the case of collisional systems. The energy is assumed to be conserved locally by the collisions, which is valid when the mean free path is much smaller than the size of the system. By contrast, this hypothesis does not seem to be justified for the violent relaxation of a collisionless system, which is rather a global process.

4.4. *Summary*

Let us review briefly the major characteristics of our evolution equations. First of all, they satisfy rigorously the conservation of energy, angular momentum, linear impulse, and phase-space hypervolumes like the Vlasov equations. Moreover, they guarantee the increase of entropy at each time ($\dot{S} \geq 0$) with an optimal rate. Of course, this H-theorem is true for the coarse-grained entropy $S = -\int \rho \ln \rho d^3\boldsymbol{r} d^3\boldsymbol{v}\, d\eta$ and not for the fine-grained entropy $S_{fg} = -\int f \ln f d^3\boldsymbol{r} d^3\boldsymbol{v}$, which is constant (as the integral of any function of f). The source of irreversibility is due to the coarse graining that smooths out the fluctuations and erases the microscopic details of the evolution. Accordingly, the coarse-grained evolution equations (4.10) and (4.17) are likely to drive the system toward an equilibrium state contrary to the Vlasov equation that develops finer and finer scales. It is shown in Appendix A that these relaxation equations converge toward the Gibbs distribution (3.22) in any region in which the diffusion coefficient is nonzero (that is to say, where the mixing is effective). This result is compatible with the equilibrium study of § 3.2, and the self-confinement of the galaxies in phase space results from kinetic processes and the special form of the diffusion coefficient (4.20).

The general evolution equations (4.10) and (4.17) can be simplified in two special limits: first of all, the distribution function is always positive, and the restriction to a single level of phase density may be relevant. Second, it is natural to believe that the galaxies are nondegenerate (in Lynden-Bell's sense) at the early stage of their evolution. Accordingly, the Fokker-Planck equation (4.22) may be quite relevant during this period. If the typical size of the galaxy is small, this equation will drive the system toward a self-confined Maxwell-Boltzmann equilibrium state (3.24). However, if the radius of the galaxy exceeds the Antonov critical value, the Fokker-Planck equation does not reach any equilibrium state anymore, and the system can always increase its entropy (while satisfying the conservation laws) by developing a singularity in its core. In fact, this "gravothermal catastrophe" will stop when the center of the galaxy will become degenerate. In that case, the Fokker-

Planck equation (4.22) is not valid anymore and must be replaced by the degenerate evolution equation (4.21) that will achieve a Fermi-Dirac equilibrium state (3.23).

The relaxation equations (4.1), (4.8), and (4.9) have been proved useful in two-dimensional turbulence to model the Euler equation. Because of the identical morphology of the Vlasov equation, we believe that equations (4.10), (4.17), (4.20), or the simplified form of equation (4.21) will work as well to describe the collisionless evolution of stellar systems.

5. COLLISIONLESS STELLAR SYSTEMS IN REAL SPACE

The previous evolution equations are written in a six-dimensional phase space, and numerical implementation is difficult (at least in the general case). Instead, it would be useful to have a set of approximate equations for the evolution of the mass density in real space (for the sake of simplicity, we restrict the discussion to the case of a single level of phase density).

5.1. *Moment Equations*

This can be obtained by taking the successive moments of equation (4.21). To that purpose, we define the local density, local velocity, and internal energy by

$$\bar{n} = \int \bar{f} d^3\boldsymbol{v} , \tag{5.1}$$

$$\bar{\boldsymbol{u}} = \frac{1}{\bar{n}} \int \bar{f}\boldsymbol{v} d^3\boldsymbol{v} , \tag{5.2}$$

$$\bar{e} = \frac{1}{\bar{n}} \int \bar{f} \frac{w^2}{2} d^3\boldsymbol{w} , \tag{5.3}$$

where $\boldsymbol{w} = \boldsymbol{v} - \bar{\boldsymbol{u}}$ is the deviation to the local velocity.

The *first moment equation* is obtained by integrating equation (4.21) over the velocities. This yields

$$\frac{\partial \bar{n}}{\partial t} + \nabla(\bar{n}\, \bar{\boldsymbol{u}}) = 0 . \tag{5.4}$$

The *second moment equation* is obtained by multiplying equation (4.21) by v_i and integrating over the velocities. This yields (the linear velocity $\boldsymbol{U}$ vanishes if we take the origin at the center of mass)

$$\frac{\partial}{\partial t}(\bar{n}\, \bar{u}_i) + \frac{\partial}{\partial x_i}(\bar{n}\, \bar{u}_i \bar{u}_j) + \frac{\partial}{\partial x_j} P_{ij} + \bar{n} \frac{\partial \bar{\Phi}}{\partial x_i} = -D\beta \int \bar{f}(\eta_0 - \bar{f})(\boldsymbol{v} - \boldsymbol{\Omega} \wedge \boldsymbol{r})_i d^3\boldsymbol{v} , \tag{5.5}$$

where $P_{ij} = \bar{n}\langle w_i w_j \rangle$ is the stress tensor.

The *third moment equation* is obtained by multiplying equation (4.21) by $\boldsymbol{v}^2/2$ and integrating over the velocities. This yields a local evolution equation for $(\bar{\boldsymbol{u}}^2/2) + \bar{e}$. From equation (5.5), we can obtain an evolution equation for the kinetic part $\bar{\boldsymbol{u}}^2/2$ by simply multiplying by $\bar{u}_i$ and rearranging terms. Then we obtain an equation for the internal energy,

$$\frac{\partial}{\partial t}(\bar{n}\, \bar{e}) + \nabla(\bar{n}\, \bar{e}\, \bar{\boldsymbol{u}}) + \nabla \boldsymbol{J}_q + P_{ij} \frac{\partial u_i}{\partial x_j} = 3D\bar{n} - D\beta \int \boldsymbol{w}\bar{f}(\eta_0 - \bar{f})(\boldsymbol{v} - \boldsymbol{\Omega} \wedge \boldsymbol{r}) d^3\boldsymbol{v} , \tag{5.6}$$

where $\boldsymbol{J}_q = \bar{n}/2\langle w^2 \bar{\boldsymbol{w}} \rangle$ is the current of heat.

This hierarchy of equations can be closed by a maximum entropy procedure: we seek the distribution function that maximizes locally the entropy $s = -1/\bar{n} \int [(\bar{f}/\eta_0)\ln(\bar{f}/\eta_0) + (1 - \bar{f}/\eta_0)\ln(1 - \bar{f}/\eta_0)] d^3\boldsymbol{v}$ at fixed density, velocity, and energy. Since the gravitational potential is determined by the density (through the Newton-Poisson equation), this is equivalent to maximizing the entropy at fixed density, velocity, and internal energy. In the case of degenerate systems, this *local thermodynamic equilibrium* will be described by the distribution function

$$\bar{f}(\boldsymbol{r}, \boldsymbol{v}, t) = \frac{\eta_0}{1 + \lambda(\boldsymbol{r}, t) e^{w^2/[2T(\boldsymbol{r},t)]}} , \tag{5.7}$$

which is a local form of equation (3.23). In this expression, the temperature $T(\boldsymbol{r}, t)$ is the Lagrange multiplier associated with the internal energy $\bar{e}$, and the fugacity $\lambda(\boldsymbol{r}, t)$ is the Lagrange multiplier associated with the density $\bar{n}$. These Lagrange multipliers are determined implicitly in term of $\bar{n}$ and $\bar{e}$ by the equations of state (obtained by the integration of $\bar{f}$ and $\bar{f}(w^2/2)$ in eq. [5.7]):

$$\bar{n} = 4\sqrt{2}\pi\eta_0 T^{3/2} I_{1/2}(\lambda) , \tag{5.8}$$

$$\bar{n}\, \bar{e} = 4\sqrt{2}\pi\eta_0 T^{5/2} I_{3/2}(\lambda) , \tag{5.9}$$

where $I_{1/2}$ and $I_{3/2}$ are the Fermi integrals:

$$I_{1/2}(\lambda) = \int_0^{+\infty} \frac{x^{1/2}}{1 + \lambda e^x} dx$$

and

$$I_{3/2}(\lambda) = \int_0^{+\infty} \frac{x^{3/2}}{1 + \lambda e^x} dx .$$

With the closure relationship of equation (5.7), the current of heat $\boldsymbol{J}_q$ is zero and the stress tensor is purely diagonal $P_{ij} = \delta_{ij}\bar{p}$, where $\bar{p} = \frac{1}{3}\int \bar{f} w^2 d^3\boldsymbol{w} = \frac{2}{3}\bar{n}\, \bar{e}$ is the pressure. Moreover, the integrals occurring in equations (5.5) and (5.6) can be explicited, and we obtain

$$\frac{\partial \bar{n}}{\partial t} + \nabla(\bar{n}\, \bar{\boldsymbol{u}}) = 0 , \tag{5.10}$$

$$\bar{n} \frac{d\bar{\boldsymbol{u}}}{dt} = -\nabla\bar{p} - \bar{n}\nabla\bar{\Phi} - D\beta\eta_0 \bar{\nu}(\bar{\boldsymbol{u}} - \boldsymbol{\Omega} \wedge \boldsymbol{r}) , \tag{5.11}$$

$$\bar{n} \frac{d\bar{e}}{dt} + \bar{p}\nabla\bar{\boldsymbol{u}} = 3D\bar{n}(1 - \beta\eta_0 T) , \tag{5.12}$$

where $\bar{\nu} = -\bar{n}\lambda(\ln I_{1/2})'(\lambda)$ is a modified density and $d/dt = \partial/\partial t + \bar{\boldsymbol{u}}\nabla$ is the convective derivative. At last, the temporal evolution of the Lagrange multipliers $\beta(t)$ and $\boldsymbol{\Omega}(t)$ is given by the conservation of the energy (eq. [3.16]) and the angular momentum (eq. [3.17]), which can be expressed in term of the local variables under the form

$$E = \int \bar{n} \frac{\bar{\boldsymbol{u}}^2}{2} d^3\boldsymbol{r} + \int \bar{n}\, \bar{e} d^3\boldsymbol{r} + \frac{1}{2}\int \bar{n}\bar{\Phi} d^3\boldsymbol{r} , \tag{5.13}$$

$$\boldsymbol{L} = \int n\boldsymbol{r} \wedge \boldsymbol{u} d^3\boldsymbol{r} . \tag{5.14}$$

Equations (5.10)–(5.12) are similar to the Navier-Stokes equations of fluid mechanics with the important difference that the dissipation is due here to a friction $\xi\bar{\boldsymbol{u}}$ or ξT, instead of a diffusion $\eta\Delta\bar{\boldsymbol{u}}$ or $\chi\Delta T$. Moreover, it is shown in Appendix A that they increase the entropy (with an optimal rate) and drive the system toward the Fermi-Dirac equilibrium state in equation (3.23).

The case of nondegenerate systems can be recovered from the previous one in the limit $\lambda \to \infty$, in which $I_{1/2}(\lambda) \sim \sqrt{\pi}/2\lambda$ and $I_{3/2}(\lambda) \sim (3\sqrt{\pi})/4\lambda$. The local thermodynamic equilibrium (5.7) becomes Maxwellian:

$$\bar{f}(\boldsymbol{r}, \boldsymbol{v}, t) = \left[\frac{1}{2\pi T(\boldsymbol{r}, t)}\right]^{3/2} \bar{n}(\boldsymbol{r}, t) e^{-w^2/[2T(\boldsymbol{r},t)]} , \quad (5.15)$$

and we recover the ordinary laws $\bar{p} = \bar{n}T$, $\bar{e} = \frac{3}{2}T$ of a classical perfect gas (we have also $\bar{v} = \bar{n}$ in that limit).

5.2. *The Limit of High Friction* (*or Large Times* $t \gg \xi^{-1}$)

The dynamical equations (5.10)–(5.12) governing the evolution of $\bar{n}$, $\bar{\boldsymbol{u}}$, and $\bar{e}$ can be simplified in the limit of high frictions $\xi = D\beta\eta_0 \to \infty$ (or equivalently for large times $t \gg \xi^{-1}$). To first order, the quantity in parentheses on the right-hand side of equation (4.21) must vanish, and we recover the local thermodynamic equilibrium in equation (5.7) with $T = 1/\eta_0 \beta(t) + O(\xi^{-1})$ and $\bar{\boldsymbol{u}} = \boldsymbol{\Omega}(t) \wedge \boldsymbol{r} + O(\xi^{-1})$. The difference between the local velocity $\bar{\boldsymbol{u}}$ and the solid-body rotation $\boldsymbol{\Omega}(t) \wedge \boldsymbol{r}$ is responsible for a diffusion current $\boldsymbol{J}_n$ of order ξ^{-1} that will drive the system toward the Fermi-Dirac equilibrium state in equation (3.23). This diffusion current can be obtained from the second moment equation (5.11): to the lowest order we can neglect the inertial term (in the rotating frame) and obtain a diffusion current of the form

$$\boldsymbol{J}_n \equiv \bar{\boldsymbol{u}} - \boldsymbol{\Omega} \wedge \boldsymbol{r} = -\frac{1}{\xi\bar{v}}(\nabla\bar{p} + \bar{n}\nabla\bar{\Phi}') + O(\xi^{-2}) , \quad (5.16)$$

where

$$\bar{\Phi}' = \bar{\Phi} - \frac{(\boldsymbol{\Omega} \wedge \boldsymbol{r})^2}{2} , \quad (5.17)$$

is the relative potential. Substituting in equation (5.10) yields an evolution equation for the density,

$$\frac{\partial\bar{n}}{\partial t} + \nabla(\bar{n}\boldsymbol{\Omega} \wedge \boldsymbol{r}) = \nabla\left[\frac{D_*}{T}(\nabla\bar{p} + \bar{n}\nabla\bar{\Phi}')\right] , \quad (5.18)$$

where $D_* = \bar{n}T/\xi\bar{v}$ is the diffusion coefficient in real space. This equation is coupled to the gravity via the Newton-Poisson equation (2.3); moreover, the temperature $T(t)$ and the angular velocity $\boldsymbol{\Omega}(t)$ must evolve in order to conserve the energy and the angular momentum. To first order, the constraints (5.13) and (5.14) can be rewritten as

$$E = \int \bar{n}\frac{(\boldsymbol{\Omega} \wedge \boldsymbol{r})^2}{2} d^3\boldsymbol{r} + \int \bar{n}\,\bar{e} d^3\boldsymbol{r} + \frac{1}{2}\int \bar{n}\bar{\Phi} d^3\boldsymbol{r} , \quad (5.19)$$

$$L_i = I_{ij}\Omega_j , \quad (5.20)$$

where $I_{ij} = \int \bar{n}(r^2\delta_{ij} - r_i r_j) d^3\boldsymbol{r}$ is the moment of inertia tensor.

These equations are able to take into account the statistical degeneracy of a collisionless stellar system and conserve the right constraints associated with the Vlasov equation. Moreover, they increase the entropy with an optimal rate and drive the system toward a Fermi-Dirac equilibrium state (which need not be spherically symmetric). As a result, they should be able to model the evolution of collisionless stellar systems and provide a simple algorithm to compute and classify their statistical equilibrium states.

In the nondegenerate limit, the diffusion equation (5.18) reduces to

$$\frac{\partial\bar{n}}{\partial t} + \nabla(\bar{n}\boldsymbol{\Omega} \wedge \boldsymbol{r}) = \nabla\left[D_*\left(\nabla\bar{n} + \frac{\bar{n}}{T}\nabla\bar{\Phi}'\right)\right] , \quad (5.21)$$

with $D_* = T/\xi$ (Einstein formula). This is a Smoluchowski equation coupled to gravity (via the Newton-Poisson equation) and with a variable temperature and rotation rate whose evolution are given by the conservation laws (5.19)–(5.20). The ordinary Smoluchowski equation describes the sedimentation of Brownian particles in a thermal bath and an *external* gravitational potential and is extended here to the case of stellar systems. This equation should drive the system toward the Maxwell-Boltzmann distribution of equation (3.24) when such an equilibrium state exists. However, this is not always the case: when the Antonov criterion is not satisfied, the system takes a "core-halo" structure and can increase its entropy with no bound by making its center denser and denser (gravothermal catastrophe). The previous equation should exhibit the onset of this instability but will fail to describe the heat processes between the core and the halo (since the temperature is strongly nonuniform in that case). This means that our previous approximations fail in that limit, and we must come back to the more general equations (5.10)–(5.12) if we want to describe the dynamical evolution of the gravitational collapse.

It is noteworthy that equation (5.18), or its nondegenerate limit (5.21), can be obtained from the MEPP also. To that purpose, we write equation (5.18) under the form

$$\frac{\partial\bar{n}}{\partial t} + \nabla(\bar{n}\boldsymbol{\Omega} \wedge \boldsymbol{r}) = -\nabla\boldsymbol{J}_n , \quad (5.22)$$

where the diffusion current $\boldsymbol{J}_n$ has to be determined by a maximum entropy prodedure. The energy and angular momentum are given by equations (5.19)–(5.20), and the entropy can be written

$$S = \int \bar{n}\left(\ln\lambda + \frac{5}{3}\frac{\bar{e}}{T}\right) d^3\boldsymbol{r} , \quad (5.23)$$

with the aid of the local thermodynamic equilibrium (5.7).

The conservation of the angular momentum $\dot{\boldsymbol{L}} = 0$ implies

$$I_{ij}\dot{\Omega}_j + (\boldsymbol{\Omega} \wedge \boldsymbol{L})_i + \Omega_j \int [2(\boldsymbol{J}_n \boldsymbol{r})\delta_{ij} - (J_{ni}r_j + J_{nj}r_i)] d^3\boldsymbol{r} = 0 , \quad (5.24)$$

and the conservation of the energy $\dot{E} = 0$ yields

$$\int \frac{\partial}{\partial t}(\bar{n}\,\bar{e}) d^3\boldsymbol{r} + \int \boldsymbol{J}_n \nabla\bar{\Phi}' d^3\boldsymbol{r} = 0 . \quad (5.25)$$

Finally, the rate of entropy production $\dot{S}$ can be determined with the aid of the Gibbs formula (which results directly from eqs. [5.7] and [5.23]):

$$T ds = d\bar{e} + \bar{p} d\left(\frac{1}{\bar{n}}\right), \tag{5.26}$$

and the use of equations (5.24)–(5.25). The final result is

$$T\dot{S} = -\int J_n\left(\frac{\nabla \bar{p}}{\bar{n}} + \nabla \bar{\Phi}'\right) d^3 r . \tag{5.27}$$

The MEPP states that the optimal current J_n maximizes the rate of entropy in equation (5.27) with the constraint $J_n^2/2n \le C_n(r, t)$ (the other constraints have been already taken into acount in the calculation of $\dot{S}$). This variational problem is solved as usual by introducing a Lagrange multiplier:

$$\delta\dot{S} - \int \frac{1}{D'}\, \delta\left(\frac{J_n^2}{2\bar{n}}\right) d^3 r = 0 , \tag{5.28}$$

and it leads to an optimal current of the form

$$J_n = -\frac{D_*}{T}\left(\frac{\nabla \bar{p}}{\bar{n}} + \nabla \bar{\Phi}'\right), \tag{5.29}$$

identical to equation (5.18). it is clear that $\dot{S} \ge 0$ and that the equilibrium state returns the Fermi-Dirac distribution (3.23).

Note finally that the previous results should be obtained by a systematic expansion of equation (4.21) in the limit of high frictions. This rigorous treatment (based on a multiple timescale analysis) would justify our main assumption (5.7) regarding the relevance of a local thermodynamic equilibrium. This expansion is classical (in the context of Brownian motion) from the Fokker-Planck equation and leads to the Smoluchowski equation (e.g., Van Kampen 1990). However, there is no result, to our knowledge, for the nonlinear equation (4.21) associated with a Fermi-Dirac statistics.

6. CONCLUSION

We have explored in this article the analogy between two-dimensional vortices and collisionless stellar systems and explained their self-organization by a similar statistical theory. In this description, the vorticity and the stream function in two-dimensional turbulence play the same role as the distribution function and the gravitational field in galaxies. As proposed by Lynden-Bell (1967), the structure of the galaxies can result from a law of chaos: there is a total lack of information at small scales, and the exciting phenomenon is that *microscopic* disorder leads to *macroscopic* order. However, the development of his statistical approach faced some difficulties, in particular the absence of equilibrium in the whole space and the need for self-confinement (incomplete relaxation). This problem is encountered also in two-dimensional turbulence to explain the local organization into a dipole or a tripole. A solution has been proposed by Robert & Sommeria (1992) in terms of a relaxation equation derived from a maximum entropy production principle and with a space-dependent diffusion coefficient (Robert & Rosier 1996). This principle can be applied to stellar systems as well and provides an evolution equation that can take into account the self-confinement process. In the case of nondegenerate systems, it reduces to the Fokker-Planck equation (4.22) with time-varying coefficients, which ensures the conservation of energy and angular momentum. In many cases, this equation never achieves equilibrium but exhibits a "gravothermal catastrophe." In the case of collisionless systems, the statistical degeneracy will set a limit to that gravitational collapse, and the system will evolve according to the nonlinear equation (4.21) until its reaches a Fermi-Dirac equilibrium state (3.23). This dynamical evolution should be well described in real space by the moments equations (5.10)–(5.12) or the more simple diffusion equations (5.18) and (5.21) for the density. These statistical arguments are being tested very carefully in the case of two-dimensional turbulence: they appear to work very well (Robert & Rosier 1996) and have allowed a synthetic classification of coherent structures such as monopoles, dipoles, etc. (Chavanis & Sommeria 1996a, 1996b). Similarly, we hope to obtain good results in the case of stellar systems and perhaps be able to recover the Hubble classification of galaxies.

We thank G. Nazarenko for first pointing out the analogy between the Euler and the Vlasov equations.

APPENDIX A

STATIONARY STATES OF THE RELAXATION EQUATIONS

In this Appendix, we make the link between the relaxation equations and the equilibrium theory. More precisely, we show that if the system converges toward a stationary state when t goes to infinity, this state is a Gibbs state.

A1. TWO-DIMENSIONAL TURBULENCE

To prove this proposition, we write equation (4.6) under the form

$$\dot{S} = -\int \frac{J}{\rho}\left\{\nabla\rho + \beta(\sigma - \bar{\omega})\rho\nabla\left[\bar{\psi} + \frac{\Omega}{2} r^2 - (U \wedge r)\hat{z}\right]\right\} d^2 r\, d\sigma + \int \frac{J}{\rho}\, \beta(\sigma - \bar{\omega})\rho\nabla\left[\bar{\psi} + \frac{\Omega}{2} r^2 - (U \wedge r)\hat{z}\right] d^2 r\, d\sigma . \tag{A1}$$

Now, the second integral is zero due to $\int J d\sigma = 0$ and the conservation laws (4.3)–(4.5). It follows that

$$\dot{S} = \int \frac{D}{\rho}\left\{\nabla\rho + \beta(\sigma - \bar{\omega})\rho\nabla\left[\bar{\psi} + \frac{\Omega}{2} r^2 - (U \wedge r)\hat{z}\right]\right\}^2 d^2 r\, d\sigma \ge 0 . \tag{A2}$$

The entropy increases at each time, and an equilibrium state is achieved if $\dot{S} = 0$. In that case, the diffusion currents J given by equation (4.8) vanish in any region in which $D > 0$. This yields

$$\nabla(\ln \rho) + \beta(\sigma - \bar{\omega})\nabla\left[\bar{\psi} + \frac{\Omega}{2} r^2 - (U \wedge r)\hat{z}\right] = 0 . \tag{A3}$$

By substracting two terms corresponding to the vorticity levels σ and σ_0, we obtain after integration

$$\ln\left(\frac{\rho}{\rho_0}\right) + \beta(\sigma - \sigma_0)\left[\bar{\psi} + \frac{\Omega}{2} r^2 - (U \wedge r)\hat{z}\right] = \ln g(\sigma) . \tag{A4}$$

The normalization condition (3.1) then returns the Gibbs state (3.10).

A2. COLLISIONLESS STELLAR SYSTEMS IN PHASE SPACE

The method is similar in the case of stellar systems. We can use the constraints (4.12)–(4.14) and the diffusion current (4.17) to write the rate of entropy (4.15) under the form

$$\dot{S} = \int \frac{D}{\rho}\left[\frac{\partial \rho}{\partial \boldsymbol{v}} + \beta(\eta - \bar{f})\rho(\boldsymbol{v} - \boldsymbol{\Omega} \wedge \boldsymbol{r} - \boldsymbol{U})\right]^2 d^3\boldsymbol{r} d^3\boldsymbol{v}\, d\eta \geq 0 . \tag{A5}$$

As a result, our system of equations guarantees the increase of entropy at each time. If an equilibrium state is achieved, we have $\dot{S} = 0$ and the diffusion currents in equation (4.17) vanish in any region in which $D > 0$:

$$\frac{\partial \rho}{\partial v} + \beta(\eta - \bar{f})\rho(\boldsymbol{v} - \boldsymbol{\Omega} \wedge \boldsymbol{r} - \boldsymbol{U}) = 0 . \tag{A6}$$

It will be convenient in the following to work in the frame of reference in which the system is stationary (that is, the frame rotating around the center of gravity with angular velocity $\boldsymbol{\Omega}$). The classical transformations of velocities and accelerations (or gravitational field) between the inertial frame and the rotating frame are given by

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\Omega} \wedge \boldsymbol{r}' , \tag{A7}$$

$$\bar{\Phi}' = \bar{\Phi} - \tfrac{1}{2}(\boldsymbol{\Omega} \wedge \boldsymbol{r}')^2 . \tag{A8}$$

In the rotating frame, the derivative of ρ with respect to the velocity is given by

$$\frac{\partial \ln \rho}{\partial \boldsymbol{v}'} + \beta(\eta - \bar{f})\boldsymbol{v}' = 0 , \tag{A9}$$

resulting from equation (A6). On the other hand, the derivative of ρ with respect to the position is given by the convection-diffusion equation (4.10), which becomes at equilibrium

$$\boldsymbol{v}' \frac{\partial \rho}{\partial \boldsymbol{r}'} - \nabla'\bar{\Phi}' \frac{\partial \rho}{\partial \boldsymbol{v}'} = 0 . \tag{A10}$$

Combining equations (A9) and (A10), we obtain

$$\left[\frac{\partial \ln \rho}{\partial \boldsymbol{r}'} + \beta(\eta - \bar{f})\nabla'\bar{\Phi}'\right]\boldsymbol{v}' = 0 . \tag{A11}$$

By subtracting two terms of the form of equation (A9) corresponding to the phase density levels η and η_0, we obtain after integration

$$\ln\left(\frac{\rho}{\rho_0}\right) + \beta(\eta - \eta_0)\frac{v'^2}{2} = A(\boldsymbol{r}', \eta) . \tag{A12}$$

Similarly, by subtracting two terms of the form of equation (A11) and using equation (A12), we obtain

$$[\nabla' A + \beta(\eta - \eta_0)\nabla'\bar{\Phi}']\boldsymbol{v}' = 0 \tag{A13}$$

This relationship must be true for any $\boldsymbol{v}'$. Since the quantity in parentheses is independent of $\boldsymbol{v}'$, it is necessary zero. This yields

$$A + \beta(\eta - \eta_0)\bar{\Phi}' = \ln g(\eta) , \tag{A14}$$

where $\ln g(\eta)$ is a constant of integration. Substituting in equation (A12) gives

$$\ln\left(\frac{\rho}{\rho_0}\right) + \beta(\eta - \eta_0)\left(\frac{v'^2}{2} + \bar{\Phi}'\right) = \ln g(\eta) . \tag{A15}$$

Then, the normalization condition (3.13) and the transformations (A7)–(A8) return finally the Gibbs distribution (3.22).

A3. COLLISIONLESS STELLAR SYSTEMS IN REAL SPACE

Let us consider the system (5.10)–(5.12) and let us compute the time derivative of the entropy functional (5.23). Using the Gibbs formula (5.26), we obtain

$$\dot{S} = \int 3D\bar{n}\left(\frac{1}{T} - \beta\eta_0\right)d^3\boldsymbol{r} \tag{A16}$$

The Lagrange multipliers $\beta(t)$ and $\boldsymbol{\Omega}(t)$ are given by the conservation of the energy and the angular momentum. Using equations (5.13) or (4.12), the conservation law $\dot{E} = 0$ implies

$$\int 3D\bar{n}d^3\boldsymbol{r} - \beta\eta_0\left(\int 3D\bar{n}Td^3\boldsymbol{r} + \int D\bar{v}\bar{\boldsymbol{u}}^2d^3\boldsymbol{r} - \Omega\mathscr{M}\right) = 0\ , \tag{A17}$$

where

$$\mathscr{M} \equiv \int D\bar{v}\boldsymbol{r} \wedge \bar{\boldsymbol{u}}d^3\boldsymbol{r}\ . \tag{A18}$$

Using equations (5.14) or (4.13), the conservation law $\dot{\boldsymbol{L}} = 0$ implies

$$\int D\bar{v}\boldsymbol{r} \wedge (\boldsymbol{\Omega} \wedge \boldsymbol{r})d^3\boldsymbol{r} = \mathscr{M}\ . \tag{A19}$$

Now, using $\boldsymbol{\Omega}[\boldsymbol{r} \wedge (\boldsymbol{\Omega} \wedge \boldsymbol{r})] = (\boldsymbol{\Omega} \wedge \boldsymbol{r})^2$ and the fact that for a fixed angular momentum the velocity field of the solid-body rotation minimizes the kinetic energy (i.e., $\int D\bar{v}(\boldsymbol{\Omega} \wedge \boldsymbol{r})^2d^3\boldsymbol{r} \leq \int D\bar{v}\bar{\boldsymbol{u}}^2d^3\boldsymbol{r}$), we obtain

$$\int D\bar{v}\bar{\boldsymbol{u}}^2d^3\boldsymbol{r} - \Omega\mathscr{M} = \frac{1}{2}\left[\int D\bar{v}\bar{\boldsymbol{u}}^2d^3\boldsymbol{r} - \int D\bar{v}(\boldsymbol{\Omega} \wedge \boldsymbol{r})^2d^3\boldsymbol{r} + \int D\bar{v}(\bar{\boldsymbol{u}} - \boldsymbol{\Omega} \wedge \boldsymbol{r})^2d^3\boldsymbol{r}\right] \geq \frac{1}{2}\int D\bar{v}(\bar{\boldsymbol{u}} - \boldsymbol{\Omega} \wedge \boldsymbol{r})^2d^3\boldsymbol{r}\ , \tag{A20}$$

from where it comes with equation (A17)

$$\beta\eta_0 \leq \frac{\int 3D\bar{n}d^3\boldsymbol{r}}{\int 3D\bar{n}Td^3\boldsymbol{r} + (1/2)\int D\bar{v}(\bar{\boldsymbol{u}} - \boldsymbol{\Omega} \wedge \boldsymbol{r})^2d^3\boldsymbol{r}}\ . \tag{A21}$$

But the Cauchy-Schwarz inequality gives

$$\left(\int 3D\bar{n}d^3\boldsymbol{r}\right)^2 \leq \int \frac{3D\bar{n}}{T}d^3\boldsymbol{r}\int 3D\bar{n}Td^3\boldsymbol{r}\ , \tag{A22}$$

so that

$$\dot{S} \geq \int \frac{3D\bar{n}}{T}d^3\boldsymbol{r}\,\frac{(1/2)\int D\bar{v}(\bar{\boldsymbol{u}} - \boldsymbol{\Omega} \wedge \boldsymbol{r})^2d^3\boldsymbol{r}}{\int 3D\bar{n}Td^3\boldsymbol{r} + (1/2)\int D\bar{v}(\bar{\boldsymbol{u}} - \boldsymbol{\Omega} \wedge \boldsymbol{r})^2d^3\boldsymbol{r}} \geq 0\ . \tag{A23}$$

If a stationary state is achieved, we have $\dot{S} = 0$, so that $\bar{\boldsymbol{u}} = \boldsymbol{\Omega} \wedge \boldsymbol{r}$. Moreover, $\dot{S} = 0$ also implies $(\int 3D\bar{n}d^3\boldsymbol{r})^2 = \int (3D\bar{n}/T)d^3\boldsymbol{r} \int 3D\bar{n}Td^3\boldsymbol{r}$, so that T is uniform in the domain in which $D > 0$ (we also have $\beta\eta_0 = 1/T$ in that case).

Coming back to equation (5.11), we obtain

$$\frac{\nabla\bar{p}}{\bar{n}} + \nabla\bar{\Phi}' = 0\ , \tag{A24}$$

which expresses the balance between the pressure forces and the gravity (modified by the centrifugal term [5.17]). With the equations of state (5.8)–(5.9) and the identity,

$$I'_{3/2}(\lambda) = -\frac{3}{2\lambda}I_{1/2}(\lambda)\ , \tag{A25}$$

(obtained by part integration), the equilibrium balance (A24) can be integrated into

$$\lambda = Ae^{\Phi'/T}\ , \tag{A26}$$

and, when substituted in equation (5.7), returns the Fermi-Dirac distribution (3.23).

APPENDIX B

DIFFUSION COEFFICIENTS

B1. TWO-DIMENSIONAL TURBULENCE

We justify here the form (4.9) of the diffusion coefficient in the case of two-dimensional turbulence. Following Robert & Rosier (1996), we consider the case $\beta = 0$ for which the energy constraint is not active. In that case, we can compute the current

J_ω by using the analogy with the diffusion of a passive scalar. Let us consider that some scalar density ω is advected by an incompressible velocity field $\boldsymbol{u}$:

$$\frac{\partial \omega}{\partial t} + (\boldsymbol{u}\nabla)\omega = 0 . \tag{B1}$$

If the field $\boldsymbol{u}$ is affected by small random fluctuations, we have $\boldsymbol{u} = \bar{\boldsymbol{u}} + \tilde{\boldsymbol{u}}$, $\omega = \bar{\omega} + \tilde{\omega}$, and it is well known that the mean value $\bar{\omega}$ will satisfy a convection-diffusion equation of the form

$$\frac{\partial \bar{\omega}}{\partial t} + (\bar{\boldsymbol{u}}\nabla)\bar{\omega} = \nabla(D\nabla\bar{\omega}) , \tag{B2}$$

where

$$D = \tfrac{1}{4}\tau\overline{\tilde{u}^2}(\boldsymbol{r}, t) , \tag{B3}$$

is the diffusion coefficient and τ is the decorrelation time of the system (i.e., the typical time during which the fluctuations $\tilde{\omega}$ are convected by the mean flow $\bar{\boldsymbol{u}}$).

In two-dimensional turbulence, the vorticity ω is not advected passively by the flow but is coupled to its motion via the relation $\omega\hat{z} = \nabla \wedge \boldsymbol{u}$. This last relation can be reversed to give

$$\boldsymbol{u}(\boldsymbol{r}, t) = \int \omega(\boldsymbol{r}', t)\boldsymbol{K}(\boldsymbol{r} - \boldsymbol{r}')d^2\boldsymbol{r}' , \tag{B4}$$

where $\boldsymbol{K}(\boldsymbol{r} - \boldsymbol{r}') = (1/2\pi)\hat{z} \wedge (\boldsymbol{r} - \boldsymbol{r}')/|\boldsymbol{r} - \boldsymbol{r}'|^2$ is the kernel (in an infinite domain). In that case, the fluctuations of the velocity are induced by the fluctuations of the vorticity, and the diffusion coefficient (B3) becomes

$$D = \frac{\tau}{4}\int \overline{\tilde{\omega}(\boldsymbol{r}', t)\tilde{\omega}(\boldsymbol{r}'', t)}\boldsymbol{K}(\boldsymbol{r} - \boldsymbol{r}')\boldsymbol{K}(\boldsymbol{r} - \boldsymbol{r}'')d^2\boldsymbol{r}'d^2\boldsymbol{r}'' . \tag{B5}$$

This integral extends from the length scale ϵ at which the oscillations of the vorticity occur (resolution scale) to the length scale a at which their influence on the value of the diffusion coefficient D in $\boldsymbol{r}$ is unimportant.

If we neglect the spatial correlations between the fluctuations of the vorticity, we have

$$\overline{\tilde{\omega}(\boldsymbol{r}', t)\tilde{\omega}(\boldsymbol{r}'', t)} = \epsilon^2\overline{\tilde{\omega}^2}(\boldsymbol{r}', t)\delta(\boldsymbol{r}' - \boldsymbol{r}'') , \tag{B6}$$

and the last expression reduces to

$$D = \frac{\tau}{4}\epsilon^2\int \overline{\tilde{\omega}^2}(\boldsymbol{r}', t)\boldsymbol{K}^2(\boldsymbol{r} - \boldsymbol{r}')d^2\boldsymbol{r}' \tag{B7}$$

If we assume in addition that $\overline{\tilde{\omega}^2}(\boldsymbol{r}', t) \simeq \overline{\tilde{\omega}^2}(\boldsymbol{r}, t)$ on the disk of radius a, we obtain

$$D = \frac{\tau}{4}\epsilon^2\overline{\tilde{\omega}^2}(\boldsymbol{r}, t)\int_\epsilon^a K^2(r')2\pi r'dr' , \tag{B8}$$

with $\overline{\tilde{\omega}^2} = \overline{\omega^2} - \bar{\omega}^2$ and $K(r') = 1/2\pi r'$. The last integral is calculated easily and yields finally

$$D(\boldsymbol{r}, t) = (\overline{\omega^2} - \bar{\omega}^2)\frac{\tau\epsilon^2}{8\pi}\ln\left(\frac{a}{\epsilon}\right) . \tag{B9}$$

In this expression, the decorrelation time τ depends on the position and is roughly estimated to be of the same order as ϵ^2/D. This yields a diffusion coefficient of the form of equation (4.9) with $K^2 = (1/8\pi)\ln(a/\epsilon)$.

B2. COLLISIONLESS STELLAR SYSTEMS

We shall use a similar method to estimate the diffusion coefficient D in the case of collisionless stellar systems. If the distribution function f was a passive scalar advected in phase space by the generalized velocity $\boldsymbol{U}_6$, the fluctuations of the force $\boldsymbol{F} = -\nabla\Phi$ (per unit of mass) would generate a diffusion current $\boldsymbol{J}_f = -D(\partial\bar{f}/\partial\boldsymbol{v})$ with

$$D = \frac{1}{6}\tau\overline{\tilde{\boldsymbol{F}}^2}(\boldsymbol{r}, t) . \tag{B10}$$

In the case of stellar systems, the gravitational force $\boldsymbol{F}$ is related to the stellar density $\int fd^3\boldsymbol{v}$ by Newton's law:

$$\boldsymbol{F}(\boldsymbol{r}, t) = \int f(\boldsymbol{r}', \boldsymbol{v}, t)\boldsymbol{K}(\boldsymbol{r} - \boldsymbol{r}')d^3\boldsymbol{r}'d^3\boldsymbol{v} , \tag{B11}$$

where $\boldsymbol{K}(\boldsymbol{r} - \boldsymbol{r}') = -G(\boldsymbol{r} - \boldsymbol{r}')/|\boldsymbol{r} - \boldsymbol{r}'|^3$ is the appropriate kernel. Accordingly, the fluctuations of the gravitational force are induced by the fluctuations of the stellar density and the diffusion coefficient in equation (B10) becomes

$$D = \frac{\tau}{6}\int \overline{\tilde{f}(\boldsymbol{r}', \boldsymbol{v}, t)\tilde{f}(\boldsymbol{r}'', \boldsymbol{V}, t)}\boldsymbol{K}(\boldsymbol{r} - \boldsymbol{r}')\boldsymbol{K}(\boldsymbol{r} - \boldsymbol{r}'')d^3\boldsymbol{r}'d^3\boldsymbol{v}d^3\boldsymbol{r}''d^3\boldsymbol{V} , \tag{B12}$$

This integral extends from the resolution scales ϵ_r, ϵ_v in position and velocity to the spatial length scale a at which the influence of the fluctuations of density is negligible for the calculation of the diffusion coefficient D in $\boldsymbol{r}$ (by contrast, there is no upper limit for the velocity because every star sufficiently close to $\boldsymbol{r}$ participates, whatever its velocity).

If we neglect the correlations between the fluctuations of the distribution function, we have

$$\overline{\tilde{f}(\boldsymbol{r}', \boldsymbol{v}, t)\tilde{f}(\boldsymbol{r}'', \boldsymbol{V}, t)} = \epsilon_r^3 \epsilon_v^3 \overline{\tilde{f}^2}(\boldsymbol{r}', \boldsymbol{v}, t)\delta(\boldsymbol{r}' - \boldsymbol{r}'')\delta(\boldsymbol{v} - \boldsymbol{V}) , \tag{B13}$$

and the last expression reduces to

$$D = \frac{\tau}{6} \epsilon_r^3 \epsilon_v^3 \int \overline{\tilde{f}^2}(\boldsymbol{r}', \boldsymbol{v}, t)\boldsymbol{K}^2(\boldsymbol{r} - \boldsymbol{r}')d^3\boldsymbol{r}'d^3\boldsymbol{v} . \tag{B14}$$

If we assume in addition that $\overline{\tilde{f}^2}(\boldsymbol{r}', \boldsymbol{v}, t) \simeq \overline{\tilde{f}^2}(\boldsymbol{r}, \boldsymbol{v}, t)$ on the sphere of radius a, we obtain:

$$D = \frac{\tau}{6} \epsilon_r^3 \epsilon_v^3 \int \overline{\tilde{f}^2}(\boldsymbol{r}, \boldsymbol{v}, t)d^3\boldsymbol{v} \int_{\epsilon_r}^{a} K^2(r')4\pi r'^2 dr' , \tag{B15}$$

with $K(r') = G/r'^2$. The last integral is calculated easily and leads to

$$D(\boldsymbol{r}, t) = \frac{2\pi G^2}{3} \tau\epsilon_r^2 \epsilon_v^3 \int (\overline{f^2} - \bar{f}^2)d^3\boldsymbol{v} . \tag{B16}$$

In this expression, the decorrelation time τ depends also on the position and is roughly estimated to be of the same order as ϵ_v^2/D. This yields a diffusion coefficient of the form of equation (4.20) with $K^2 = 2\pi/3$.

REFERENCES

Antonov, V. A. 1962, in IAU Symp. 113, Dynamics of Star Clusters, ed. P. Hut (Dordrecht: Reidel), 525
Arnold, V. I. 1966, Ann. Inst, Fourier, 16,319
Barge, P., & Sommeria, J. 1995, A&A, 295, L1
Binney, J. J., & Tremaine, S. D. 1987, Galactic Dynamics (Princeton: Princeton Univ. Press)
Chandrasekhar, S. 1943, Rev. Mod. Phys., 15, 1
Chavanis, P. H., & Sommeria, J. 1996a, J. Fluid Mech., 314, 267
———. 1996b, in preparation
Clemmow, P., & Dougherty, J. 1969, Electrodynamics of Particles and Plasmas (New York: Addison-Wesley)
Flierl, G. R. 1987, Ann. Rev. Fluid Mech., 19, 493
Hasegawa, A. 1985, Adv. Phys., 34 (1), 1
Heggie, D. C., & Stevenson, D. 1988, MNRAS, 230, 223
Hénon, M. 1964, Ann. d'Astrophys., 27, 83
Jaynes, E. T. 1989, in The Minimum Entropy Production Principle. ed. R. Rosenkrantz (Dordrecht: Kluwer)
Kandrup, E. 1980, Phys. Rep., 63, 1
King, I. R. AJ, 71, 64
Kuzmin, G. A. 1982, in Structural Turbulence, ed. M. A. Goldshtik (Acad. Naouk CCCP Novosibirsk, Inst. of Thermophysics), 103
Lynden-Bell, D. 1967, MNRAS, 136, 101
Lynden-Bell, D., & Eggleton, P. P. 1980, MNRAS, 191, 483
Lynden-Bell, D., & Wood, R. 1968, MNRAS, 138, 495
Michel, J., & Robert, R. 1994, Commun. Math. Phys., 159, 195
Miller, J. 1990, Phys. Rev. Lett., 65, 17
Montgomery, D., & Joyce, G. 1974, Phys. Fluids, 17, 6
Montgomery, D., & Lee, Y. C. 1991, ApJ, 368, 380
Nezlin, M. V., & Snezhkin, E. N. 1993, Rossby Vortices, Spiral Structures, Solitons (Berlin: Springer-Verlag)
Onsager, L. 1949, Nuovo Cimento Supp., 6, 279
Padmanabhan, T. 1990, Phys. Rep., 188, 285
Robert, R. 1990, CR Acad. Sci. Paris, 311, 1459
———. 1991, J. Stat. Phys., 65, 531
———. 1994, in Proc. XIth Int. Congress of Mathematical Physics, ed. D. Iagolnitzer (Paris: International Press)
Robert, R. & Rosier, R. 1996, J. Fluid Mech., submitted
Robert, R., & Sommeria, J. 1991, J. Fluid Mech., 229, 291
———. 1992, Phys. Rev. Lett., 69, 2776
Sommeria, J. 1994, in Modelling of Oceanic Vortices, ed. G. J. F. van Heijst (Amsterdam: North-Holland), 37
Sommeria, J., Nore, C., Dumont, T., & Robert, R. 1991, CR Acad. Sci. Paris, 312, 999
Turkington, B., & Whitaker, N. 1996, SIAM, J. Sci. Stat. Comput., in press
Van Kampen, N. G. 1990, Stochastic Processes in Physics and Chemistry (6th ed.; Amsterdam: North-Holland)