Date

# Lecture V — PKU Lectures

(Quasiparticle)

→ Modulation and Predator–Prey Model for Drift–Zonal Flow System i.e. {Energetics + Momentum}

via Langmuir

→ Homogenization and Single Wave Trapping. → PV Vlasov

Homogenization ⟺ Phase Mixing

Coarse-Graining

{ – final state
{ – time scales

N.B.

BASH → Friday PM

UCSD Balcony Astrophysics Social Hour

PKU Beida (Adjunct) Science Hour

Additional

→ Whole we're here:

Drift Wave - Zonal Flow Interaction:

Modulational Instability and the

Predator - Prey Model

- Have established connection between zonal flow and pseudo-momentum / WMD.

- Of course, coupling of WMD + flow must extract energy from turbulence ⇒ feedback loop for flow on turbulence.

- Structure closely analogous to theory of weak Langmuir turbulence

Plasma Wave ↔ Drift Wave

↕ ↕

Ion Acoustic Mode (finite frequency) — Zonal Flow (→ 0 frequency)

If know one → can derive other.

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Recall: Langmuir wave envelope

$$i\omega_0 \partial_t \mathcal{E} = \frac{\omega_{p0}^2}{2}\frac{\delta n}{n_0}\mathcal{E} - \alpha v_{The}^2 \nabla^2 \mathcal{E}$$

refraction in density perturbation — diffraction

$$\partial_t^2 \frac{\delta n}{n_0} - c_s^2 \nabla^2 \frac{\delta n}{n_0} = \nabla^2 \left( \frac{|\mathcal{E}|^2}{8\pi n m_i} \right)$$

radiation pressure

treating plasma waves at level of eikonal theory ⇒ rays, can write:

$$\frac{\partial N}{\partial t} + \underline{v}_{g} \cdot \underline{\nabla} N - \frac{\partial}{\partial \underline{x}}(\omega) \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

i.e.

$$\frac{dN}{dt} = 0$$

→ statement of conservation of wave action density

quasi-particle kinetic eqn.

Conservation Action ~~↔~~ Phase Symmetry (cf 218A notes)

$N = \frac{\mathcal{E}}{\omega_k}$ ⇒ population density in $\underline{x}, \underline{k}$ → phase space density.

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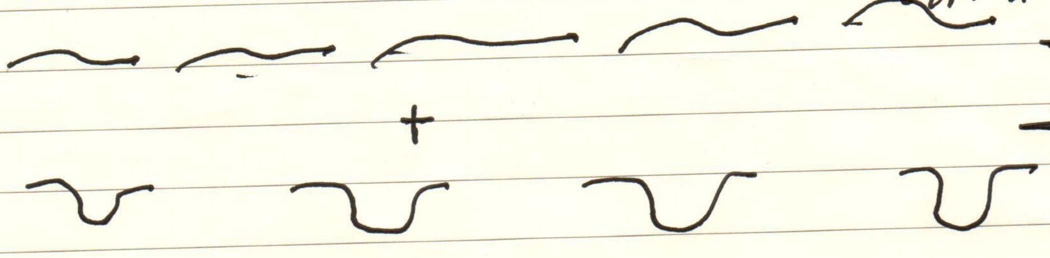
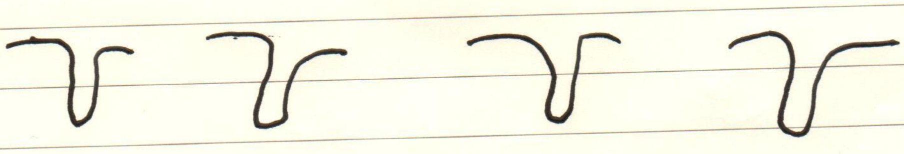
Modulation: → negative for sub-sonic

$$\delta n/n = \frac{-|E|^2}{8\pi M_i}$$



→ NLS

→ general, for meanfield

⇒ 

i.e. plasma wave gas unstable to [illegible] refracting density perturbations.

→ perturbations grow

→ plasma wave ensemble depleted.

To calculate:

– I A W growth rate

→ field energy density

$$\frac{|E|^2}{8\pi} = \left(\frac{1}{2}\right) W_k$$

$$= \int \left(\frac{1}{2}\right) \omega_k N_k \, d\underline{k}$$

modulated WAD

then:

$$\frac{\partial^2}{\partial t^2}\frac{\delta n}{n_0} - C_s^2 \nabla^2 \frac{\delta n}{n} = \nabla^2\left(\frac{1}{M_i}\int d\underline{k}\, \omega_k \tilde{N}_k\right)$$

→ couples quasi-particle to mean field

Where:

$\tilde{N}_{\underline{q},\Omega}$ ⇓ modulated WAD

$\underline{k}, \omega_k \rightarrow$ Plasma wave

$\underline{q}, \Omega \rightarrow$ Modulation field

Straightforward linearization:

$$\tilde{N}_{\underline{q},\Omega} = \frac{-\underline{q}\;\omega_{pe_0}}{\Omega - \underline{q}\cdot \underline{v}_{gr}} \frac{\tilde{n}_{\underline{q},\Omega}}{2n_0} \cdot \frac{\partial \langle N \rangle}{\partial \underline{k}}$$

where

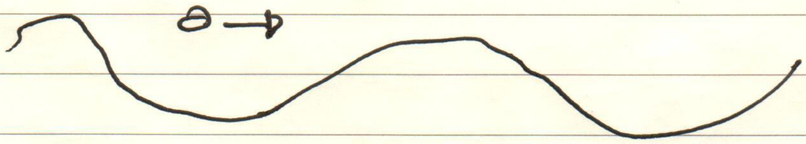
$$\frac{1}{\Omega - \underline{q}\cdot \underline{v}_{gr}} = \frac{P}{\Omega - \underline{q}\cdot \underline{v}_{gr}} - i\pi\,\delta(\Omega - \underline{q}\cdot \underline{v}_{gr})$$

i.e.

particle

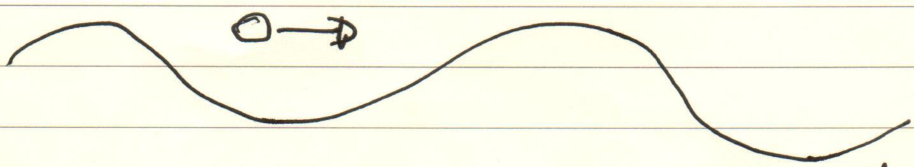
wave

V/osov



Langmuir wave packet

IAW



can have trapping, though (KAW)

Caveat re: breaking.

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so

$$\partial_t^2 \frac{\tilde{n}}{n_0} - c_s^2 \nabla^2 \frac{\tilde{n}}{n_0} = \nabla^2 \left( \frac{1}{m_i} \int dk\, \omega_{n_0}\, k \cdot \frac{-q\, \omega_{p_0}}{\Omega - q \cdot v_{gr}\, 2n_0} \frac{\tilde{n}_q}{} \cdot \frac{\partial \langle N \rangle}{\partial k} \right)$$

$\Rightarrow$

$$\underbrace{\Omega^2}_{\substack{\Omega_r + i\gamma \\ \Omega_r^2 + 2i\Omega_r\gamma}} = q^2 c_s^2 + \underbrace{q^2 \frac{\omega_p^2}{2m_i} \int dk\, i\pi\delta(\Omega - q \cdot v_{gr})\, q \cdot \frac{\partial \langle N \rangle}{\partial k}}$$

instability of IAW if:

$$\frac{\partial \langle N \rangle}{\partial k} \cdot q > 0 \quad \text{at:} \quad \Omega = q \cdot v_{gr}$$

i.e. population inversion!

$\rightarrow$ energy decays from plasma waves to IAW.

$\rightarrow$ $\Omega_{real} \approx q c_s$; as usual. $|E|^2/8\pi n T \ll 1$.

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What of Plasma Waves? → how popln respond to interaction

$\langle N \rangle$ evolves!

$$\frac{\partial \langle N \rangle}{\partial t} = \frac{\partial}{\partial \underline{k}} \cdot \left\langle \frac{\partial \tilde{\omega}}{\partial \underline{x}} \tilde{N} \right\rangle$$

can approximate, as in QL → use linear response

⇒

$$\frac{\partial \langle N \rangle}{\partial t} = \frac{\partial}{\partial \underline{k}} \cdot \underline{\underline{D}}_{\underline{k}} \cdot \frac{\partial \langle N \rangle}{\partial \underline{k}}$$

- Quasi-particle diffusion equation

- "Induced Diffusion" → random refraction etc.

$$\delta k \sim -\int \frac{\partial \tilde{\omega}}{\partial \underline{x}} \, dt$$

- Can have random advection if inhomogeneous $N$.

For energy

$$E = \int d\underline{k} \; \omega \langle N \rangle$$

$$\frac{dE}{dt} = -\int \frac{\partial \omega}{\partial \underline{k}} \cdot \underline{\underline{D}}_{k} \cdot \frac{\partial \langle N \rangle}{\partial \underline{k}}$$

sign $v_{gr}$ enters!

$$\frac{dP}{dt} = -\int d\underline{k} \; \underline{\underline{D}}_{\underline{k}} \cdot \frac{\partial \langle N \rangle}{\partial \underline{k}}$$

$\frac{\partial \omega}{\partial n} > 0$ → population inversion needed

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sign $\left[ \frac{\partial \omega}{\partial n} \frac{\partial \langle N \rangle}{\partial n} \right]$ sets net

gain / loss energy from plasmons.

$$\boxed{D_n = \pi \left( \frac{\omega_p^2}{m_g} \right) \sum_{q,\Omega} |\tilde{n}_{q,\Omega}|^2 \frac{c}{\Omega - q \cdot v_{gr}} q q}$$

- physics is stochastic refraction (induced diffusion)
- requires: ray chaos

i.e. [overlap of phase speed ↔ group speed resonances. (Q-P).

- energetics Plasmon Q-P energy vs. IAW energy.
- structure very impt.

For plasma waves $\frac{\partial \omega}{\partial n} > 0$, so

$\frac{\partial \langle N \rangle}{\partial n} >$ for decay.

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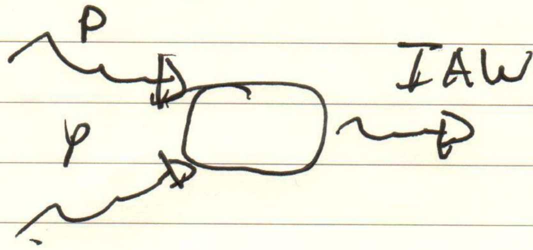
→ Wave Interaction Process




how relate?

→ consider wave interactions

$$\frac{i}{\omega_{\underline{k}+\underline{k}'} - \omega_{\underline{k}'} - \omega_{\underline{k}}}$$

$$= \frac{i}{\cancel{\omega_{\underline{k}'}} + \underline{k}\cdot\frac{\partial\omega}{\partial\underline{k}'} - \cancel{\omega_{\underline{k}'}} - \omega_{\underline{k}}}$$

$$= \frac{i}{\underline{k}\cdot\frac{\partial\omega}{\partial\underline{k}'} - \omega_{\underline{k}}}$$

$$= \frac{i}{k\cdot v_{gr}(k') - \omega_{\underline{k}}}$$ ✓

thin isoceles limit of 3 wave resonance

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If drive Plasmon Population

i.e., electron B-O-T ⇒

$$\partial_t E_{PL} = -\int dk\, \underset{\checkmark}{V_{gr}}\, \underline{\underline{D}}_k \cdot \frac{\partial \langle N \rangle}{\partial k}$$

$\uparrow \sim \langle (\frac{\delta n}{n})^2 \rangle$

$$-\beta E_{PL}^{\nu} + 2\gamma_{BOT}\left[\langle f_e \rangle\right] E_{PL}$$

$-\beta E_{PL}^{\nu}$ ↑ nstr, about IAW

$\gamma_{BOT}$ ↑ drive for plasmon population

→ should calculate B-O-T in presence Langmuir wave ensemble.

$$\partial_t \left\langle \left(\frac{\delta n}{n}\right)^2 \right\rangle = 2\gamma_{NL\,IAW}\left[\frac{\partial \langle N \rangle}{\partial k}\right] \left\langle \left(\frac{\delta n}{n_0}\right)^2 \right\rangle$$

$$- 2\gamma_{ILD} \left\langle \left(\frac{\delta n}{n}\right)^2 \right\rangle$$

↑ damping for IAW

schematically:

$\nu = 3$

$$\partial_t E_{PL} \simeq \gamma_{BOT} E_{PL} - \alpha E_{PL} E_{IAW} - \beta E_{PL}^{\nu}$$

$$+ \alpha E_{PL} E_{IAW} - \gamma_{ILD} E_{IAW}$$

→ beginnings of a predator-prey system

cf. R. May - Stability and Complexity in Model Ecosystems.

Now

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For DW-ZF:

Mauro

$$\partial_t \langle V_y \rangle = -\partial_x \langle \tilde{V}_x \tilde{V}_y \rangle - \mu \langle V_y \rangle$$

$$\partial_t N + (\underline{v}_g + \underline{V}) \cdot \nabla N - \frac{\partial (\omega + \underline{k} \cdot \underline{V})}{\partial x} \cdot \frac{\partial N}{\partial \underline{k}}$$

$$= \gamma_{\underline{k}} N + C(N)$$

growth — guaranteed sat'n absent ZF

$$\sim -\gamma N^2$$

so, $Z.F. = \langle V_y(r) \rangle$,

$$\partial_t \tilde{N} + \underline{v}_g \cdot \nabla \tilde{N} + \gamma_{\underline{k}} \tilde{N} = -\frac{\partial}{\partial x}(k_\theta V_E) \frac{\partial \langle N \rangle}{\partial k_r}.$$

of course

$$N_{\underline{k}} = \mathcal{E}_k / \omega_k = \frac{(1 + k_\perp^2 \rho_s^2)^2 |\Phi_k|^2}{2 V_* k_\theta}$$

$k_\theta$ const

$$= \Omega_{\underline{k}} \rightarrow \text{potential enstrophy} \sim \langle \delta q^2 \rangle$$

$$\tilde{V}_x \tilde{V}_y = k_x k_y |\Phi_k|^2 = \frac{\sum (k_x k_y)\, k_\theta V_* k_\theta |\Phi_k|^2}{(1 + k_\perp^2 \rho_s^2)^2}$$

Flow effect disp't

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and $$\frac{d\underline{k}}{dt} = -\frac{\partial}{\partial \underline{x}}\left(\underline{k}\cdot\underline{V} + \omega\right)$$

$$= -\frac{\partial}{\partial r}\left(k_\theta V_\theta(r)\right)$$

$$\boxed{\frac{d}{dt}k_r = -\frac{\partial}{\partial r}\left(k_\theta V_{E\theta}(r)\right) \quad k_\theta \text{ const}}$$

→ Action conservation ⟷ Potential Enstrophy Conservation in $V_y(x)$ shearing field

and straightforward to crank-out:

→ modulational growth

→ Pred - Prey

+HW

→ show energy conservation

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## HW → Hints

Key Points:

① $\frac{\partial w}{\partial \underline{k}} < 0$ so $\frac{\partial N}{\partial \underline{k}} < 0$ for

depletion DW

② Non-resonant

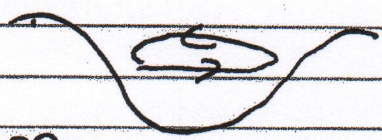
Single Wave — Important for Intuition



## Homogenization and Single Wave Nonlinear Evolution

→ Recall from Landau Damping discussion, one limitation on linear theory is:

$t < \tau_b$ → evolution time < bounce time



i.e. resonant particles suffer strong orbit distortion → linear trajectories invalid

$1/\tau_b \sim k\Delta v$

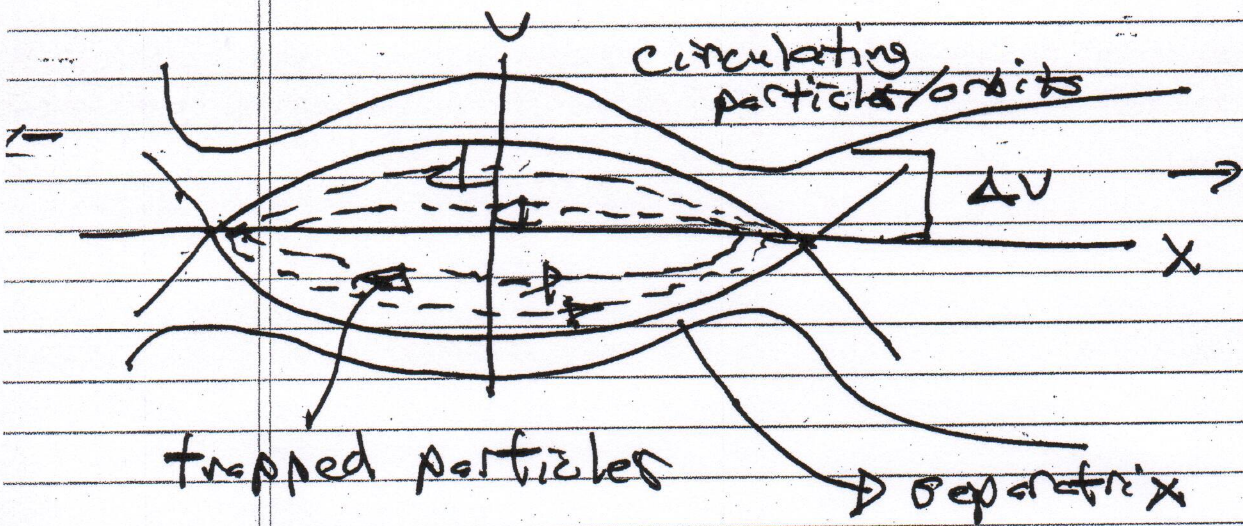
$\Delta v \sim (q\phi/m)^{1/2}$

⇒ need $\gamma_{Landau} > 1/\tau_b$ for linear Landau damping calculation to be relevant.

→ What happens for $t > \tau_b$, or $\gamma_{Landau} < 1/\tau_b$ ?

- basic orbits ~~for~~ resonant particles, strongly perturbed.
- need integrate evolution of wave using these

i.e. for trapped orbits




Trapped response $\Leftrightarrow$ calculate in
$\gamma(t)\,\tau_b < 1$ ordering $\Rightarrow$ basic orbits are those of bounce, i.e. closed.
$\downarrow$
instantaneous growthrate

Many questions emerge:

i.) What is end state?

ii.) What is mixing, decay mechanism?

iii.) What is $\gamma_k(t)$?

For insight, consider simpler closely related problem → that of PV homogenization (Prandtl - Batchelor Theorem). (also relevant to flux expulsion)

Consider 2D fluid. Then potential vorticity evolves according to:

$$\partial_t \zeta + \underline{v} \cdot \underline{\nabla} \zeta - \underline{\nabla} \cdot \nu \cdot \nabla \zeta = 0$$

here $\zeta = \nabla^2 \phi$ (PV = vorticity)

$\underline{v} = \underline{\nabla} \phi \times \hat{z}$ , ($\underline{\nabla} \cdot \underline{v} = 0$ , 2D)

$\nu \equiv$ usual viscosity → important!

so can re-write as:

$$\partial_t \nabla^2 \phi + \underline{\nabla} \phi \times \hat{z} \cdot \underline{\nabla} \nabla^2 \phi - \nu \nabla^2 \nabla^2 \phi = 0$$

– i.e. viscous 2D fluid.

Can extend to more general PV, i.e.

$$\partial_t \zeta + \underline{\nabla} \phi \times \hat{z} \cdot \underline{\nabla} \zeta - \nu \nabla^2 \zeta = 0$$

obviously $\zeta \Leftrightarrow$ charge density

System is obviously relevant to Vlasov, i.e.:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{M}E\frac{\partial f}{\partial v} = C(f) \to 0$$

$$\frac{df}{dt} = 0$$

$$\frac{\partial \zeta}{\partial t} + \underline{\nabla}\phi \times \hat{z}\cdot\underline{\nabla}\zeta = \nu\nabla^2\zeta$$

$$\frac{d\zeta}{dt} = \nu\nabla^2\zeta$$

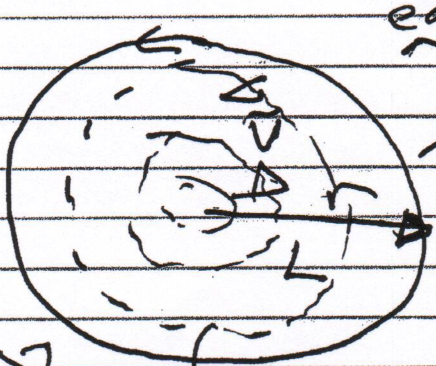
- common: Hamiltonian structure
  conservation along trajectories

  up to: dissipation, coarse graining

- different: viscosity vs $C(f)$
  $C(f) \to 0 \Rightarrow$ coarse graining.

and consider set up of
eddy with closed stream line as boundary

i.e.




→ closed boundary

$Pe = \frac{\tilde{V} L}{\nu} \gg 1$

→ eddy flow — differential rotation

Now, as for single wave:

→ What is ultimate distribution $\zeta(r)$? (N.B. Assume circle for simplicity)

→ time scales?

Observation re: viscosity,

Consider stationary state:

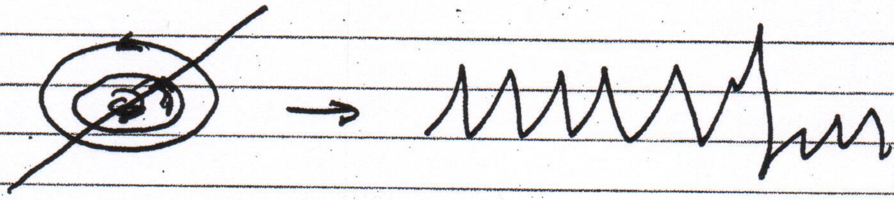
$$\cancel{\partial_t \zeta} + \nabla\phi \times \hat{z} \cdot \nabla \zeta - \nu \nabla^2 \zeta = 0$$

if $\nu \to 0$

$$\nabla\phi \times \hat{z} \cdot \nabla \zeta = 0$$

so $\zeta(\phi) = \zeta$ is solution

i.e.



- can tag each streamline arbitrarily, generate non-differentiable "crinkly" solution.
- no smoothing of sharp gradients.
- unphysical!

"Not all solutions of the Navier-Stokes [N.B. really Euler] equations are realized in nature."
— Landau, Lifshitz (Fluid Mechanics)

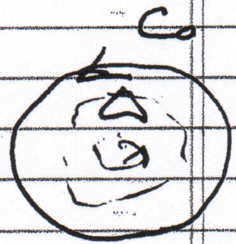
But: - with $\nu \neq 0$ will show that $\Omega(r) \rightarrow$ const is end state
- PV homogenized! i.e. $\nabla \Omega$ flattened.

Note: - $\nu = 0$ each streamline decouples
- $\nu \neq 0$, global solution.

- small dissipation makes a big difference in global state.
- see Taylor Relaxation, as well.

# Homogenization ⇒ Prandtl – Batchelor Theorem.

Theorem: Consider a region of 2D incompressible flow (i.e. vorticity advection), enclosed by a closed streamline $C_0$. Then if diffusive dissipation,



$$\text{i.e. } \partial_t q + \nabla\phi \times \hat{z} \cdot \nabla q = \nabla \cdot (\nu \nabla q)$$

then $q \to$ uniform (homogenization!) as $t \to \infty$, within $C_0$.

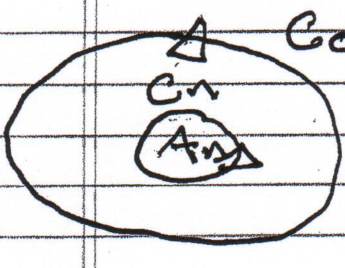
N.B. – finite $\nu$ crucial
– no comment on how long?

To show:

– $t \to \infty$, with $\nu$ finite:

$$\nabla\phi \times \hat{z} \cdot \nabla q = \nabla \cdot \nu \cdot \nabla q$$

– choose arbitrary closed $C_n$ within $C_0$. $C_n$ a streamline.



– simply connected flow.
– stationary ⇒ $\omega$ const along streamlines
– $C_0$ specified on ⊥ bndry. $C_0 \to$ B.C.

$\therefore \quad \omega \to \omega_0$ on $C_0$

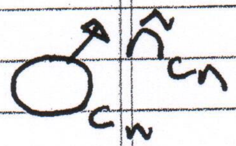
$\omega \to \omega_n$ on $C_n$

Now, for $A_n$ enclosed by $C_n$:

$$\int_{A_n} d^2x\, \underline{v}\cdot\underline{\nabla}\, q = \int_{A_n} d^2x\, \underline{\nabla}\cdot(\underline{v}\,\underline{\nabla} q)$$

but

$$\int_{A_n} d^2x\, \underline{v}\cdot\underline{\nabla}\, q = \int_{A_n} d^2x\, \nabla\cdot[\underline{v}\, q]$$




$$= \int_{C_n} dl\, \hat{n}_{C_n}\cdot(\underline{v}\, q)$$

normal to $C_n$

but $\underline{v}$ is streamline, along $C_n$, so

$$\int_{C_n} dl\, (\hat{n}_{C_n}\cdot\underline{v})\, q = 0.$$

Thus have shown:

so
$$0 = \int_{A_n} d^2x \; \underline{\nabla} \cdot (\nu \, \underline{\nabla} \, q)$$

$$= \nu \int_{C_n} d\ell \; \hat{n}_{C_n} \cdot \nabla q$$

Now, stationary state must have
$q \to$ const along streamline.

so
$$q = q(\phi)$$
$$q_{C_n} = q(\phi_n)$$

and
$$0 = \nu \int_{C_n} d\ell \; \hat{n}_{C_n} \cdot \nabla \phi_n \frac{\partial q}{\partial \phi_n}$$

$$= \nu \frac{\partial q}{\partial \phi_n} \int_{C_n} d\ell \; (\hat{n} \cdot \nabla \phi_n)$$

And: $\Gamma = \int d\underline{\ell} \cdot \underline{v} \Rightarrow$ circulation

$$= \int d\underline{\ell} \cdot (\underline{\nabla} \phi \times \hat{z})$$

$$= \int (\hat{z} \times \hat{n}) \cdot \underline{\nabla} \phi \; \hat{z} = -\int d\ell \; (\nabla \phi \cdot \hat{n})$$

$\underline{\text{so}}$ $\quad 0 = r \dfrac{\delta q}{\delta \phi_n} \Gamma_n$

$\Gamma_n \neq 0 \quad \Rightarrow \quad \boxed{\delta q / \delta \phi_n = 0}$

as $C_n$ arbitrary, $\delta q/\delta\phi_n = 0$ for all $\phi_n$ so:

$$\boxed{\delta q/\delta\phi = 0, \quad \text{all } \phi}$$

- no line-to-line variation
- $\boxed{q \text{ homogenized}}$ $\quad \nabla q \to 0$

Now,

- note <u>order</u> <u>limits</u>

<u>first</u> $t \to \infty$, <u>then</u> $\begin{cases} q \doteq q(\phi) \Rightarrow \text{concentric lines} \\ r \to \text{small} \end{cases}$

- expect $\nabla q$ large at boundary $C_0$

<u>i.e.</u> PV gradient steepening at bndry
$\Rightarrow$ barrier !?

Further:

- key assumption

  separatrix – closed, bounding streamline

  viscous dissipation ⇒ form matters.

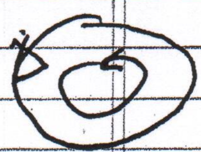
- large Re

  i.e. $\frac{\tau_{circulation}}{\tau_{diffusion}} \ll 1$

  → establish concentric circulation

  → then diffuse across to homogenize

Now, time scales:



- sheared, concentric flow
- viscous diffusion

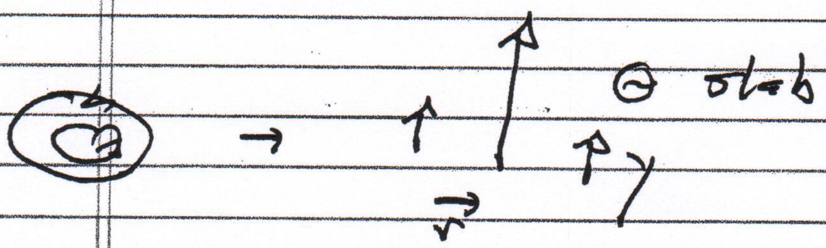
Need <u>interact</u> to homogenize on non-trivial time scale.

Look for synergism ⇒ shear dispersion

(a) pure diffusion :

$$1/\tau_d \sim \nu/L^2$$

(b) diffusion ⊕ shear :




$$r\theta = y$$

$$\frac{dy}{dt} = V_y(r)$$

so

$$\frac{d\,\delta y}{dt} = \frac{\partial V_y}{\partial r}\,\delta r$$

$$\delta y = \int dt \left(\frac{\partial V_y}{\partial r}\right) \delta r$$

$$\langle \delta y^2 \rangle = \left(\frac{\partial V_y}{\partial r}\right)^2 \langle \delta r^2 \rangle\, t^2$$

$\langle \delta r^2 \rangle = \nu t$ → molecular diffusion.

So

$$\langle \delta y^2 \rangle \sim \left(\frac{\partial V_{0y}}{\partial r}\right)^2 \nu t^3$$

Mixing occurs when mean square excursion $\sim L_y^2 \sim L^2$ $(L \sim 2\pi R)$

$$\langle \delta y^2 \rangle / L^2 \sim \left(\frac{\partial V_{0y}}{\partial r}\right)^2 \frac{\nu}{L^2} t^3 \sim 1$$

$$\boxed{\tau_c^{-1} = 1/\tau_{mix} \sim \left[\left(\frac{\partial V_{0y}}{\partial r}\right)^2 \nu / L^2\right]^{1/3}}$$

hybrid time scale, of shear and $\nu$.

$$\tau_{mix}^{-1} \sim \frac{V_0}{L} / Re^{1/3}$$

$$\boxed{\tau_{mix} \sim \tau_{circ} Re^{1/3}}$$

$\Rightarrow$ Longer time scale smoothing (on $\tau \sim L^2/\nu$) completes homogenization

For more on homogenization, see W2018 Phys. 218b Notes, Lecture 7 and references.

→ Return to Single Wave Problem.

N.B.

→ Can view:
  → exact streamlines
  → molecular diffusivity

or

  → coarse grained streamline
  → turbulent diffusivity

→ If $D_T \sim \tilde{v}\, l_{mix}$
  and $l_{mix}$ bi-stable, can
  get non-uniform / inhomogeneous
  mean field.

→ SC

$$\nabla q(\psi) = \frac{dq}{d\psi}\nabla\psi$$

(with M. Malkov)

but

$$\int_\Gamma \nabla\psi \cdot \boldsymbol{n} dl \neq 0$$

circulation in closed streamline.

$$\frac{dq}{d\psi} = 0$$

\ arbitrary $\Gamma$

$$\frac{dq}{d\psi} \equiv 0$$

in closed streamlines region

## Particles

collision operator

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = St(f) \sim \nu\frac{\partial^2 f}{\partial v^2}, \quad \nu \to 0$$

Here $f$ (charge density) plays a role of vorticity, $q = \Delta\psi$, ($\psi$ -stream function) while the particle Hamiltonian

$$H = \frac{p^2}{2m} + e\phi = \epsilon$$

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = \frac{\partial f}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial f}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial f}{\partial p}$$

$\epsilon$ is a "stream function". If $\phi$ is independent of $t$, $H = \epsilon = const.$ Any stationary solution $f(x,p) = f[H(x,p)]$ → BGK . Use $\epsilon = H$ as a new variable labeling each individual orbit (stream line). Better than $\epsilon$, is the action variable

$$J = \oint p(\epsilon, x)\, dx, \qquad S(x) = \int^x p(\epsilon, x)\, dx$$ → action

$$p = \sqrt{2m(\epsilon - e\phi)}$$

angle: $\alpha = \frac{\partial S}{\partial J}$, can be obtained through differentiation wrt $\epsilon$. It is convinient to scale $\alpha$ mod$2\pi$. $f(\alpha + 2\pi) = f(\alpha)$. The transformation from $x, p$ to $\alpha, J$ is canonical.

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial f}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial f}{\partial p} = St\,(f) \sim \nu \frac{\partial^2 f}{\partial p^2}, \qquad \nu \to 0$$

becomes (Poisson bracket is an invariant of canonical transforms)

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial J}\frac{\partial f}{\partial \alpha} - \frac{\partial H}{\partial \alpha}\frac{\partial f}{\partial J} = St\,(f) \sim \nu \frac{\partial^2 f}{\partial J^2}, \qquad \nu \to 0$$

but $H = H\,(J)$, and $\partial H/\partial J = \Omega\,(J)$ $\qquad$ $\partial H/\partial\alpha = 0$ symmetry

$$\boxed{\frac{\partial f}{\partial t} + \Omega \frac{\partial f}{\partial \alpha} \sim \nu \frac{\partial^2 f}{\partial J^2},} \qquad \nu \to 0$$

For $\nu = 0$ (following the logic of PB theorem except time-dependent $\nu \to 0$ solution), then diffuses to uniformity

$$f\,(\alpha, J, t) = f\,(\alpha - \Omega t, J, 0) = F_0\,(J) + \sum_n F_n\,(J)\, e^{in(\alpha - \Omega t)}$$

perturbation, asymmetric

$$F_0 = \langle f\,(t = 0)\rangle_\alpha$$

Phase mixing: $F_n \to 0$, $t \to \infty$, small $\nu$, but finite

$$\frac{\partial F_n}{\partial t} \sim -\nu \left(\frac{d\Omega}{dJ}\right)^2 n^2 t^2 F_n$$

All modes with $n \neq 0$, decay as

$$F_n \approx F_n^0 \exp\left\{-\frac{\nu}{3}\left(\frac{d\Omega}{dJ}\right)^2 n^2 t^3\right\}$$

as before

Critical: $\Omega' \neq 0$.

The longest survivor $F_1$ spreads $\alpha$ at the rate

$$\delta\alpha^2 \sim \nu \left(\frac{d\Omega}{dJ}\right)^2 t^3$$

To show this consider an initial distribution as a "blimp" of size $\delta\alpha \ll 1$ it dissolves as follows:

All of $F_n^0$ are of $\sim 1$. E.g. for for an initial $f = \delta(\alpha)$, $F_n^0 = 1/2\pi$, recall Poisson formula:

$$\sum \delta(\alpha - 2\pi n) = \frac{1}{2\pi} \sum e^{in\alpha}$$

The slolution for 

$$f(\alpha, J, t) - F_0(J) = \frac{1}{2\pi}\sum_n e^{in(\alpha-\Omega t)-\frac{\nu}{3}\left(\frac{d\Omega}{dJ}\right)^2 n^2 t^3}$$

many $n's$ may contribute, so

$$\sum_n \to \int dn$$

BUT: the phase function

$$\Phi = in(\alpha - \Omega t) - \frac{\nu}{3}\left(\frac{d\Omega}{dJ}\right)^2 n^2 t^3$$

has a critical point on a complex $n-$ plane. We may deform the integration path (analytic function under the integral) and pass it through the maximum of $\Phi$ on the new path (saddle point). This will be at

$$n = \frac{3}{2}\frac{i(\alpha - \Omega t)}{\nu\left(\frac{d\Omega}{dJ}\right)^2 t^3}$$

$$\int dn e^{in(\alpha-\Omega t)-\frac{\nu}{3}\left(\frac{d\Omega}{dJ}\right)^2 n^2 t^3} \sim \frac{1}{t^{3/2}} \exp\left[-\frac{3}{4}\frac{(\alpha-\Omega t)^2}{\nu\left(\frac{d\Omega}{dJ}\right)^2 t^3}\right]$$

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Note that the solution conserves $\int d\alpha$ as long as it is narrow at $\alpha-\Omega t$, gives $\delta(\alpha)$ for $t\to 0$.
Recall now the diffusion

$$\frac{\partial f}{\partial t} = D\Delta f$$

and its solution for a point source $\delta(\boldsymbol{r})$ at t=0:

$$f = \frac{1}{\sqrt{4\pi D t^3}} \exp\left[-\frac{r^2}{4Dt}\right]$$

Here we have $\alpha-\Omega t$ instead of $r$ (moving point at a speed $\Omega$) but also spreading as

$$(\alpha-\Omega t)^2 \propto \nu\left(\frac{d\Omega}{dJ}\right)^2 t^3$$

See LN 11a, simpler derivation of a similar result:

stream line along $y$ diffusion across, in $r$

$$\left\langle \delta y^2 \right\rangle \sim \nu \left( \frac{\partial V_y}{\partial r} \right)^2 t^3$$

Characteristic mixing time for trapped particles $\delta\alpha \sim 2\pi$

$$\boxed{\tau_{mix}^{-1} \sim \left\{ \nu \left( \frac{\partial V_y}{\partial r} \right)^2 \right\}^{1/3}}$$

εδ hefe[illegible]

Let us assume that the beam is monoenergetic initially, $V_b = V = V_0$. If the wave is growing slowly, one can write $m\,(V - \omega/k)^2 - 2e\phi = const$, so that $\delta V \approx (e\phi/m)\,(V_0 - \omega/k)^{-1}$. In course of time, while $\phi$ grows, particles with $V > V_0$ overtake the wave and get bunched in decelerating phase of the wave, which thus requires that initially $V_0 > \omega/k$. As a result, the beam is broken down into clusters near the decelerating phases of the wave which further increases the wave amplitude due to the increased beam modulation.

When the wave amplitude is high enough to trap the beam clusters into potential troughs, the wave ceases to grow. Indeed, as it follows from the analysis of Landau damping, when bunched particles bounce off the decelerating phase of the wave, more particles will move slower than the wave and the latter will even decay for a while until the situation is reversed again. The particle phase mixing will suppress the bunching effect and the wave amplitude oscillations decay.

The condition for the beam particles to be trapped is that in the time when they cross the trapping area (wave length) the wave amplitude should grow significantly

$$k\left|V-\frac{\omega}{k}\right|\sim\gamma,$$

or

$$e\phi/m\sim\gamma^2/k^2.$$

$k \Delta v \sim \gamma$

The growth rate

$$\gamma\sim\omega_p\left(\frac{n_b}{n_0}\right)^{1/3}$$

The saturation wave energy thus amounts to (square and use the beam resonance condition) $kV_b = \omega_p$

$$\frac{E^2}{4\pi} \sim mV_b^2 n_0 \left(\frac{n_b}{n_0}\right)^{4/3} \sim mV_b^2 n_b \left(\frac{n_b}{n_0}\right)^{1/3} .$$

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Date

→ Turbulent Homogenization

→ Fluctns Flow - stirs

Bistable Diffn.

→ Time scales

i.e. merger vs diffn.