PKU-HUST Lectures (2020-2021)

OV

→ 'Modeling Dynamical Systems', for MFE ^(Applications) (c.f. after Biophysics)

→ a useful (I hope) methods

mini-course (minimal practical knowledge)

→ Emphasis on { nonlinear dynamics, space-time scales } ⇒ MFE

→ Topics:

i.) Feedback loops, (time) predator-prey

↓

ii.) transition evolution and propagation (space-time) Fisher, FN

{ iii.) ^(traffic flow,) avalanches, bursty flux (Burgers; → kdv/solitons)

iv.) nonlinear diffusion and spreading.

v.) Negative viscosity and Cahn-Hilliard eqn

vii) Model reduction and renormalization — a look at the theory

+

History of Nonlinear Plasma Theory (1 → 2 lectures)

EPOCH

P.D., Frisch, Y. Pomeau

Sources :

- J. D. Murray, "Mathematical Biology"
- R. May, "Stability and Complexity in Model Ecosystems"
- F. Morrison, "The Art of Modelling Dynamic Systems"

+
more coming
⋮
- selected papers.

PKU-HUST Lectures I

i) Feedback Loops and Predator - Prey Systems

Plan:

- OV and Motivation $\left\{\begin{array}{l}\text{Drift Wave} \\ + \\ \text{Shear Flow}\end{array}\right.$ system
- $\left\{\begin{array}{l}\text{Constructing} \\ \text{Solving}\end{array}\right\}$ the System - $\left\{\begin{array}{l}\text{Very Simple} \\ \text{Examples}\end{array}\right.$
- Lessons from Ecology (c.f. May)
  → General structure ; constraint
  → Aside: Bigger Models, Stability + Complexity

General structure &

→ Kolmogorov Theorem ; → Sensible structure

$\left\{\begin{array}{l}\text{Fixed pts. vs. Cycles} \\ \text{Feedback}\end{array}\right.$

→ Time Delays → Cycles

- Fluctuating Environments and statistical Approaches.
  - → Physics Motivation
  - → Fluctuating Logistics - a simple example.
  - → Applications

0D
↓
Fronts

- Looking Ahead.

I) QV and Basics

→ Fundamental Problem(s)

- Drift waves + zonal shear flow
- L → H transition

- Natural candidate for Predator - Prey Model

2 animal

- what?:

Hares

$H \equiv$ prey

$P \equiv$ predator

$$\frac{dH}{dt} = H\,F(H,P)$$

rates

$$\frac{dP}{dt} = P\,G(H,P)$$

$P(t)$, $H(t)$

(OD)

can extend to $2M \times 2M$.

Lotke - Volterra

- What does it mean?:

$H \to \boxed{\mathcal{E}}$ - fluctuation intensity, energy

$P \to U = V'^2$ - mean square $V_E'^2$

OD?

$\mathcal{E} \equiv \langle \mathcal{E} \rangle$

$V'^2 = \langle V'^2 \rangle$

avg. over this layer [ignores spatial evolution]

$\langle V' \rangle$

$\mathcal{E}$

- why?

2 populations

→ self-regulation (feedback)

→ symmetry constraints ← constructed

Timo

1) self-regulation

$$\begin{array}{c} \text{Shear Flow} \\ a) \uparrow \quad \downarrow b) \\ \text{Turbulence} \end{array}$$

a) shear stabilizes, controls turbulence
→ everyone knows ...

b) turbulence drives flow, via
Reynolds stress

$$\boxed{\partial_t \langle V_\perp \rangle = -\nabla_\perp \langle \tilde{v}_r \tilde{V}_\theta \rangle} \rightarrow \nabla \cdot J = 0$$

$J = J_{pol} + J_{ps} + J_{\parallel}$

n.b. more precisely:
from vorticity eqn. $(\nabla \cdot J = 0)$

Design $\Rightarrow \partial_t \left(\boxed{\rho_s^2} \partial_r^2 \langle \phi \rangle \right) = -\partial_r \langle \tilde{v}_r \rho_s^2 \nabla_\perp^2 \tilde{\phi} \rangle + \cdots$

net polarization charge

inhomogeneity

flux of polarization charge

$\langle \tilde{v}_r \tilde{n}_{i,gc} \rangle - \langle \tilde{v}_r \tilde{n}_e \rangle$

(ρ_s generalizes
— Rosenbluth
Hinton 97)

↓

E×B shear flow

NOTE

$\boxed{\text{Reynolds force}}$

/ symm.

but: $\langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle = \partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle$

HW: show this!

(G.I.) Taylor identity (100+ yrs old)

Upshot:

- Reynolds stress, force $\leftrightarrow$ polarization charge flux

R. Wood

* - Thm: McIntyre + Wood:

"PV mixing" [i.e. polarization charge flux] + 1 direction of symmetry

$\Rightarrow$ $\boxed{\text{Z. F. generation.}}$

$\therefore$ How reconcile? $\Rightarrow$ feedback loop

'05
O.G. 2014
2019 Hahm

2) $\boxed{\text{Symmetry}^*}$ (already encountered)

Why is ZF 'special'?

(P.D., I², H '05)

ZF is mode $(\boxed{k_\theta = 0},\ \boxed{n, m \to 0})$ of:

- [redacted] minimal inertia - easily excited

$k_\parallel \to 0$ $\quad (1 + k_\perp^2 \rho^2) \Rightarrow k_\perp^2 \rho^2$ $\quad *$

HW $\qquad \Downarrow \qquad\qquad \Downarrow$

H-M $\qquad\qquad$ 2D fluid

- $\boxed{\text{minimal transport (zero)}}$ (none!) $\quad *$

- $n = 0 \Rightarrow \tilde{V}_r = 0$
$m = 0$

can't relax ∇n, ∇T, etc.

- minimal damping $\quad *$

(no Landau damping - R H, '97)

Symmetry → P.-P.

⇒ Symmetry constraints + consideration of self-regulation

⇒ Z. F. can grow/be excited only via 'feeding off' of turbulence, yet shears turbulence, thus regulating it.

⇒ $\boxed{\text{natural problem for predator-prey formulation!}}$

→ What does it look like? (cf. PD, et. al. '94)

Mixing Length process

Fluctn

$$\frac{1}{2}\frac{dE}{dt} = \gamma_0 E - \alpha_1 E^2 - \alpha_2 U E$$

growth turb. self satn. coupling key

$$\frac{1}{2}\frac{dU}{dt} = -\mu_1 U + \alpha_2 U E$$

Flow damping

→ [see table in paper for coeffs

$\langle \tilde{v} \rangle^2$

→ N.B.: – simplest

→ An infinity of extensions:

most imp't.

~ $\gamma_0 \to \bar{\gamma}_0(\nabla P) \to \bar{\gamma}_0\left(a/L_p - a/L_{p\,crit}\right)$

coupling to profile evolution

$$\partial_t \langle p \rangle + \partial_r \langle \tilde{v}_r \tilde{p} \rangle = nQ = n(\bar{Q} + \tilde{Q})$$

noise

$$\partial_t \langle p \rangle + \partial_r \chi_0 \mathcal{E} \partial_r \langle p \rangle = n(\bar{Q} + \tilde{Q})$$

0D $\qquad \partial_t \langle p \rangle + D_0 \mathcal{E} \langle p \rangle = n(\bar{Q} + \tilde{Q})$

Malkov, P.D. 2009 → $\boxed{\text{Kim, P.D. 03}}$

~ $V_E = \dfrac{\nabla P}{n} + V_0$ (Carreras, Miki)

~ 1D, toroidal flow, particles, stoch. field .....

Users guide to Aref-Aref extension

$U = \langle V_\perp \rangle'^2$

B.) $\boxed{\text{How is it Constructed ?}}$

Key term: $\boxed{\alpha_2 U E}$ → Reynolds Coupling

Point: Reynolds work transfers energy flow ⟷ fluctuations

How: 2 ways

Now, $\partial_t \langle V_\perp \rangle = - \partial_r \langle \tilde{v}_r \tilde{v}_\perp \rangle$

$$\underbrace{\partial_t \int dr \frac{\langle V_\perp \rangle^2}{2}}_{} = \underbrace{\int \langle \tilde{v}_r \tilde{v}_\perp \rangle \partial_r \langle V_\perp \rangle}_{}$$

but

$$\partial_t \underset{\text{flctn}}{\mathcal{E}_K} + \partial_t \int dr \frac{\langle V_\perp \rangle^2}{2} = 0$$

layer

so $\boxed{\partial_t \mathcal{E}_K = - \int dr \, \underline{\langle \tilde{v}_r \tilde{v}_\perp \rangle} \, \underline{\partial_r \langle V_\perp \rangle}}$

Formally $\partial_t \langle \mathcal{E}_K \rangle = - \langle \tilde{v}_r \tilde{v}_\perp \rangle \langle V_\perp \rangle'$

Layer average

What of stress?:

→ $\boxed{\text{'Shear – Eddy Tilting Feedback Loop'}}$

$$\langle \tilde{V}_r \tilde{V}_\perp \rangle = -\frac{c^2}{B^2} \sum_{k} k_x k_y |\phi_k|^2$$

Zip

$\sim \langle k_x k_y \rangle$ → eddy orientation!

i.e. $|\phi_k|^2 = f(k_x/\bar{k}_x)\, g(k_y/\bar{k}_y)$

f, g even functions

$\langle k_x k_y \rangle = 0$ ← no stress

mode structure

but

→ $\boxed{\text{Shear flow can induce correlation!}}$

i.e. $$\frac{d\underline{k}}{dt} = -\frac{\partial}{\partial \underline{x}} (\omega + \underline{k} \cdot \underline{V})$$ (Snell's Law)

$$\frac{dk_x}{dt} = -\frac{\partial}{\partial x}(k_\theta V_E) = -k_\theta \langle V_E \rangle'$$

$$\boxed{k_x = k_x^{(0)} - k_\theta \langle V_E \rangle' t}$$

$\sim$ shearing coordinate

Goldreich
L-B '65

$t \sim \tau_c$

$$k_x = k_x^{(0)} - k_\theta \langle V_E \rangle' \tau_c$$

$$\langle k_x k_y \rangle = \langle k_x^{(0)} k_y \rangle - \boxed{k_\theta^2 \langle V_E \rangle' \tau_c}$$

intrinsic correlation $\leftrightarrow$ mode structure (esp. radial propagation!)

tilting induced correlation

Pearlstein – Berk

HW: Calculate for slab DW.

so

$$\langle \tilde{v}_r \tilde{v}_\perp \rangle = + \frac{c^2}{B^2} \sum_k k_\theta^2 V_E' \tau_{c_k} |\phi_k|^2$$

$$\cong \underbrace{\langle \tilde{v}_r^2 \rangle V_E' \tau_{c_k}}_{D}$$

and

$$\partial_t \langle \mathcal{E}_k \rangle \cong -\underbrace{\langle \tilde{V}^2 \rangle \tau_{ck}}_{} \boxed{\langle V_\perp \rangle'^2}$$

note $\sim \langle V_\perp \rangle'^2$! dependence

$$\sim -D \langle V_\perp \rangle'^2$$

- Necessarily, flow eqn. must have same form, opposite sign to conserve energy
- $\langle V_\perp \rangle'^2$ is natural U variable.

⇒ basic predator-prey structure.

⇒ flow - fluctuation kinetic energy exchange

Alternative Perspective:

→ Spectral Energy Evolution

- Recall fluctuations respond adiabatically to Z.F.

i.e $\omega \sim \omega_*$ vs. $\omega \sim \Omega$

then, have for wave action density

$N = \frac{\mathcal{E}_k}{\omega_k}$ → energy density

$\frac{E}{\omega}$ → $\ell(t)$

$\partial I^2 H$, [illegible]

$$\boxed{\partial_t N + (\underline{v}_{gr} + \underline{V}) \cdot \nabla N - \frac{\partial}{\partial x}(\omega + \underline{k} \cdot \underline{V}) \cdot \frac{\partial N}{\partial \underline{k}} = C(N)}$$

relaxation

$$\partial_t \langle N \rangle + \frac{\partial}{\partial k_r} \left\langle -\frac{\partial}{\partial x}(k_\theta \tilde{V}_E)\, \tilde{N} \right\rangle = C(N)$$

HW → show

and usual quasilinear crank ⇒ see (D I² H '05)

$$\partial_t \langle N \rangle - \boxed{\partial_{k_r} D_k \partial_{k_r} \langle N \rangle} = \langle C(N) \rangle$$

$$\boxed{D_k = \sum_q k_\theta^2 \underbrace{|\tilde{V}_{E,q}|^2 \tau}_{\ell\, 2} / \varepsilon_{q,k}}$$ → [induced diffusion

random shearing

Ray chaos

Note: v_{gr} wave model $\to \Omega/q$

$$\frac{i}{\Omega - I\,v_{gr}\,q + |\gamma|} \longleftrightarrow \frac{i}{\omega - kv}$$

$$\partial_{k_r} D_k \partial_{k_r} \sim \frac{k_\theta^3}{k_r^2} \sum_q |\tilde{v}_{Eq}|^2 q^2 \, \tilde{\tau}_c$$

$\langle \tilde{v}^2 \rangle$

as before.

HW

work out correspondence

then:

$$\langle \varepsilon \rangle = \omega \langle N \rangle$$

$$\partial_t \langle \varepsilon \rangle = -\int dk \, D_k \frac{\partial}{\partial k_r} \langle N \rangle \frac{\partial \omega_k}{\partial k_r}$$

$$= -\int dk \, \frac{\partial \omega}{\partial k_r} D_k \frac{\partial}{\partial k_r} \langle N \rangle$$

< 0 $\sim \langle \tilde{v}^2 \rangle$

$\frac{\omega}{1 + k_\perp^2 \rho_s^2}$

backward wave

$\frac{\partial \langle N \rangle}{\partial k_r} < 0$

for damping

$\to$ Same structure $\checkmark$ $\partial_t \varepsilon \approx -\langle v \rangle^2 \varepsilon_k$

Aside

c.) → Lessons From Ecology (c.f. May)

→ What can be said about this type of system?

↓

1 predator - 1 prey

Prop.? Solns

General:

H → prey

→ $\frac{dH}{dt} = H \; F(H,P)$

$\frac{dP}{dt} = P \, G(H,P)$

P → predator

→ Lotka-Volterra (Linear)

breeding

Methes

$\frac{dH}{dt} = \gamma H - \alpha HP$

conservation

$\frac{dP}{dt} = -d\,P + \alpha HP$

↓

depletion

Linear osc

LCO

→ Linear oscillations / growth - damping → show

→ UO → DW − ZF

↓ departure, L-V.

$$\frac{1}{2}\frac{dE}{dt} = \gamma_0 E - \alpha_1 E^2 - \alpha_2 U E$$

DW → prey

$$\frac{1}{2}\frac{dU}{dt} = -\mu U + \alpha_2 U E$$

Shear Flow → predator.

→ For 1 pred, 1 prey:

$$\left.\begin{aligned} \frac{dH}{dt} &= H\, F(H,P) \\ \frac{dP}{dt} &= P\, G(H,P) \end{aligned}\right\} \quad **$$

⇒ Kolmogorov Thm. (1936), based on Poincaré-Bendixson Theorem.

→ Predator - Prey systems of form (**) have either:

- a stable equilibrium point

or

- a stable limit cycle

if F, G continuous, with continuous first derivatives

characterize sensible syst.

and

(i) $\partial F/\partial P < 0$ → rate of prey ~~increase~~ growth decrease as pred ↑

CS — or renewal

(ii) $H(\partial F/\partial H) + P(\partial F/\partial P) < 0$

CS ↓ resource depletion

→ prey growth rate decreases with population size

(iii) $\partial G/\partial P < 0$ → rate of increase of predators decreases with population size of pred.

CS ↓ or collapse

(iv.) $\boxed{H(\partial G/\partial H) + P(\partial G/\partial P) > 0}$

CS maintain → # predators is increasing function of population size

v.) $F(0,0) > 0$

$\frac{dH}{dt} = FH$

CS Prey have positive growth rate (~ linear), for small population

$\gamma + f$

and

$\exists$ A, B, C s/t

$\frac{dH}{dt} = HF$

$= H[\gamma - \alpha P - \cancel{\beta H}]$

vi.) $F(0, A) = 0$, with $A > 0$

$\boxed{P = \gamma/\alpha}$

→ can have predator population to stop further prey increase for low levels prey

(*) → predation induced death!

vii.) $F(B, 0) = 0$, with $B > 0$

(*) → critical prey population beyond which prey cannot increase, even absent predators!

$$\frac{\partial H}{\partial t} = H F$$
$$= H\left[\gamma - \alpha_1 P - \beta H\right]$$
$$H = \gamma/\beta$$

(*) → state prey must be able to self-saturate, as for logistic competition, absent predators $[B = \gamma_0/\alpha_1]$

viii) $\sigma(C, 0) = 0$, with $C > 0$,

There exists a critical prey size C that stops further increase in predators. $\frac{\partial P}{\partial t} = PG = P[-\mu + \alpha_2 H]$

[redacted] $[C \to \mu/\alpha_2]$

ix) $B > C$, or system collapses.

[need $\gamma_0 > \alpha_1 \mu / \alpha_2$] or flows collapse.

i) - ix) satisfied ⇒

system will have fixed point or LCO solution structure!

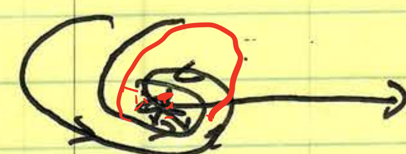
N.B.:

→ i.) → ix.) are criteria for sustainable, 'sensible', system

→ some can be $\geq$, $\leq$

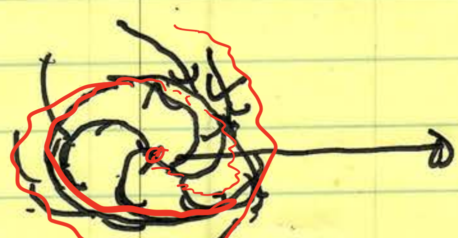
→ Criteria for (stationary state) attractor

i.e. Fixed pt



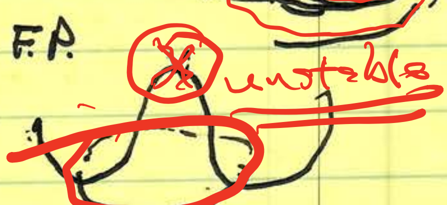
→ fixed pt stable

LCO



→ fixed pt is unstable. but LCO encircles it

but



i.e. 'Mexican hat' potential

so, if system satisfies K-Thm and fixed pt. unstable, then $\boxed{\text{LCO}}$

LCO → time delay

→ Sense of effect — Model variations
starting
From Lotka - Volterra :

$$\frac{dH}{dt} = H\,[a - \alpha P]$$

- all > 0
- often $\alpha = \beta$

$$\frac{dP}{dt} = P\,[-b + \beta H]$$

1) $a \to r(1 - H/K) + \dots$ Logistic

carrying capacity (self-satn.)

stabilizing

2) $-\alpha H P$ → prey removal by predator

→ $kP(1 - e^{cH})$ destabilizing (saturating)

→ $\frac{kHP}{H+D}$ saturation of predation at D

→ $\boxed{\text{Transitions}}$ → K Thm. Conditions

- back to PD '94 model:

Fixed pts:

- trivial: $E = U = 0$

K Thm.

- No flow: $E = \gamma_0 / \alpha_1$, $U = 0$ → self-soln, no flow

'L mode'

- Flow: $E = \mu / \alpha_2$ → turbulence set by flow damping

self!

$$U = \left(\gamma_0 - \frac{\alpha_1 \mu}{\alpha_2} \right) / \alpha_2$$

→ flow set by turb growth

$B > 0$

$$\frac{\gamma_0}{\alpha_1} > \frac{\mu}{\alpha_2}$$

Superficially counterintuitive

⇒ signature of pred-prey system

⇒ observed in GK simulations Lin, et. al. '98

$\mathcal{E} \sim \gamma$

Fluctn → FP + Flow

Clearly:

K Thm ⇒ cond. mult. F.P. transitions

- transition for:

$\gamma_0 > \frac{\alpha_1 \mathcal{U}}{\alpha_2}$ (CB > C criterion)

Power Threshold

~ P_{thresh}, as $\gamma \sim |\nabla P| \sim Q/D$

⇒ sets critical power, flux

- Can linearize about 2 fixed pts (non-trivial) $\gamma, 0$

HW

No flow:

$(\gamma_0/\alpha_1, 0)$

"mode" → eigenmode of P-P syst.

slow

$\gamma = -(\mathcal{U} - \alpha_2\gamma_0/\alpha_1)$ → U

$\gamma = -\gamma_0$ ⇒ E

B > C

Observe:

- E mode always heavily damped.

- U mode 'soft' near transition

$\gamma \to 0$ slower

excit. flow

- $\gamma_U \to 0$ at threshold

$\Rightarrow \tau_{transition} \to \infty$ (its a bifurcation).

~ 2nd order transition, soft mode is signature.

Now, since $|\gamma_E| \gg |\gamma_U|$,

- E will relax to equilibria/fixed point much faster than U

$\Rightarrow$

- 'slave' flctns to flow, i.e.

$$\frac{1}{2}\frac{dE}{dt} \to 0 = (\gamma_0 - \alpha_1 E - \alpha_2 U)\, E$$

$$E = (\gamma_0 - \alpha_2 U)/\alpha_1$$

$$\frac{1}{2}\frac{dU}{dt} = -\mu U + \alpha_2 U E$$

$$= -\mu U + \alpha_2 U (\gamma_0 - \alpha_2 U)/\alpha_1$$

⊛, system described by flow:

$$\frac{1}{2}\frac{dU}{dt} = \left(\frac{\alpha_2}{\alpha_1}\gamma_0 - 1\right)U - \frac{\alpha_2^3}{\alpha_1}U^2$$

U_1^2

$$\boxed{\frac{1}{2}\frac{dU}{dt} = \left(\frac{\alpha_2}{\alpha_1}\gamma_0 - 1\right)U - \frac{\alpha_2^3}{\alpha_1}U^2}$$

Logistic Eqn.

→ Logistic eqn, with growth threshold

+ Diffn → Fisher Eqn. → propagating transition

→ if write for $\langle V_1\rangle'$,

$$\boxed{\frac{d\langle V_1\rangle'}{dt} = \left(\frac{\alpha_2}{\alpha_1}\gamma_0 - 1\right)\langle V_1\rangle' - \frac{\alpha_2^3}{\alpha_1}\langle V_1'\rangle^3}$$

→ TDGL → 2nd order transition

and can add external torque/drive:

$$\frac{d}{dt}\langle v_1 \rangle = \left(\frac{\alpha_2 \gamma_0}{\alpha_1} - \nu\right)\langle v_1' \rangle - \frac{\alpha_2^2}{\alpha_1}\langle v_1' \rangle^3$$

$+ T_{ext}$

- Bias? (ala J-TEXT experiments)

Hext

- ala' response of ferromagnet near criticality / away from criticality

L+L Stat. Mech

⇒ Topic for serious further work....

Ki Thm → transitions

B > G

→ Furthur:

- LCO's - $\boxed{\text{time delays}}$

  next class.

- $\boxed{\text{model developments}}$

→ Near criticality → $\boxed{\text{noise}}$

$$\frac{1}{2}\frac{du}{dt} = \left(\frac{\alpha_2}{\alpha_1}\gamma_0 - \nu\right)u - \frac{\alpha_2^2}{\alpha_1}u^2$$

$Q = \bar{Q} + \tilde{Q}$

$$\gamma_0 \sim \gamma_0(\nabla p) \sim \gamma_0(Q/D)$$

$$\sim \bar{\gamma} + \tilde{\gamma}$$

as $\boxed{Q = \bar{Q} + \tilde{Q}}$

mean heat flux

↳ bursts, noise

i.e. pdf edge heat flux

pdf [v2]

viscosity 28

"const" or not const.

→ Begs the question of how deal with $\boxed{\text{noise}}$?

⇒ stochastic model → pdf

Fokker Planck Eqn.

$$D_V = \frac{\beta}{M} V_{Th}^2 \sim \frac{|\tilde{f}_0|^2}{M^2} \tau_{ac}$$

- Multiplicative Noise (Simple)

Consider Logistic Eqn → Populations

$$\frac{dN}{dt} = \underbrace{N}_{\substack{\text{Malthusian} \\ \text{growth} \\ \text{(exponential)}}}\,(k - N)$$

$(k-N)$ ⟶ saturation by competition $\sim N^2$

$N \equiv$ # of population

$$x_{n+1} = \alpha x_n (1 - x_n)$$

Logistic Map

$N = 0$, $N = k$ are fixed pts

Now, could consider variability in k, and treat as stochastic variable

$$\frac{dN}{dt} = N\left(k_0 + \tilde{\gamma}(t) - N \right) + \tilde{\alpha}(t)$$

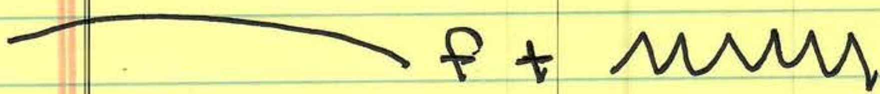
$\tilde{\gamma}(t)$ → variability in resources

⇒ multiplicative noise → rate

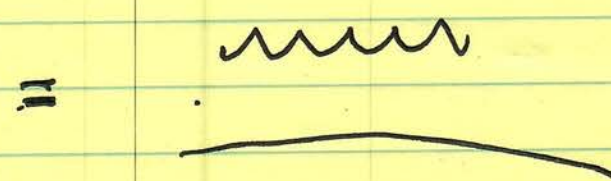
$\tilde{\alpha}(t)$ → external input variability

⇒ additive noise.

i.e. additive:

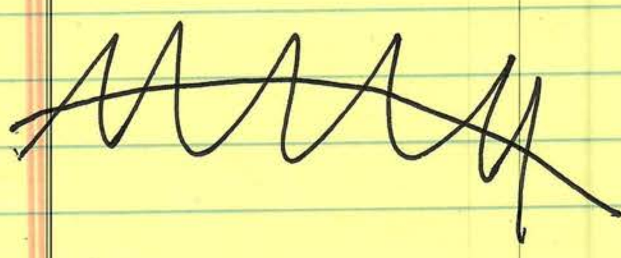


f +



noise on top deterministic base =

Multiplicative:



multiplies by fast, random quantity

$f(N,t) \to$ population pdf

How treat?

$\to$ Fokker-Planck Equation for $f(N)$

$\to$ here $\langle \tilde{\gamma}(t)\,\gamma(t')\rangle = |\tilde{\gamma}_0|^2 \tau_{ac}\, \delta(t-t')$

Delta correlated for simplicity.

N.B. This is a "textbook model".

$\to$ additive, as usual

$\langle \tilde{\alpha} \tilde{\gamma} \rangle \doteq 0$

Then : $\frac{dN}{dt} = N\left(k_0 + \tilde{\gamma}(t) - N\right) + \tilde{\alpha}$

$$\frac{\partial}{\partial t} f(N,t) = -\frac{\partial}{\partial N}\left[\left(k_0 N - N^2\right) f(N,t) - \frac{\partial}{\partial N}\left(D\, f(N)\right) \right]$$

For D;

$$\langle \Delta N\, \Delta N \rangle = \int^t dt' \int^t dt'' \, \langle \tilde{\gamma}(t') \tilde{\gamma}(t'') \rangle N^2$$

!

$$+ \int^t dt' \int^t dt'' \, \langle \alpha(t') \alpha(t'') \rangle$$

$$= |\gamma_0|^2 \tilde{\tau}_{ac} N^2 t + |\alpha_0|^2 \tilde{\tau}_{ac}$$

Nonlinearity in D

$\rightarrow$ one trademark feature of multiplicative noise

→ Note: $N \to \infty \Rightarrow D \to 0$

Rate variation ⇒ pdf spread in proportion to population.

→ Additive correction significant at low N.

Now, ignoring additive correction,

$$\boxed{\partial_t f(N) = -\frac{\partial}{\partial N}\left\{(k_0 N - N^2) f(N,t) - \frac{\partial}{\partial N}\left(\frac{|\tilde{\gamma}_0|^2}{2}\tau_{ac} N^2 f(N,t)\right)\right\}}$$

is Fokker-Planck Equation

and stationarity:

$$N(k_0 - N) f(N) = \frac{\partial}{\partial N}\left(\frac{|\tilde{\gamma}_0|^2}{2}\tau_{ac} N^2 f(N)\right)$$

so

Norm

$$\boxed{f(N) = c \cdot n^{[2(k_0/\sigma^2) - 2]} \, e^{-2N/\sigma^2}}$$

$\sigma^2 = |\tilde{\gamma}_0|^2 \tilde{\tau}_{ac}$ power

exponential tail

Need $k_0^2 > (\sigma^2/2)^2 \Leftrightarrow f > 1/n$ $n \to 0$

i.e. $\boxed{k_0 > \frac{|\tilde{\gamma}_0|^2 \tilde{\tau}_{ac}}{2}}$

to avoid log. singularity

Physics of $k_0 > \frac{|\tilde{\gamma}_0|^2 \tilde{\tau}_{ac}}{2}$?

Convenient to linearize around fixed point:

$$\frac{dN}{dt} = (k + \tilde{\gamma} - N) N$$

Validity ?

$$N = k_0 + \tilde{n}$$

$$\frac{d\tilde{n}}{dt} = (k_0 + \tilde{n})(\cancel{k_0} + \tilde{\gamma} - \cancel{k_0} - \tilde{n})$$

$$\cong k_0 \tilde{\gamma} - k_0 n + O(n^2)$$

so

$$\partial_t f(n) = -\frac{\partial}{\partial n}\left[-k_0 n f(n) - \frac{\partial}{\partial n}\left(\frac{k_0^2 \gamma_0^2 \tau_{ac}}{2} f(n) \right) \right]$$

to linearize abt fixed pt.

$$= -\frac{\partial}{\partial n}\left[-k_0 n f(n) - \frac{\partial}{\partial n}\left(\frac{k_0^2}{2} \Gamma^2 f(n) \right) \right]$$

$\Rightarrow$ zero flux / stationarity:

$$\boxed{f(n) = \underbrace{c}_{\text{const}} \exp\left[-n^2 / k_0 \Gamma^2 \right]}$$

Valid for: $\langle (\tilde{n}/N_0)^2 \rangle = \langle (\tilde{n}/k_0)^2 \rangle < 1$

Now $\langle \tilde{n}^2 \rangle = \frac{\Gamma^2 k_0}{2}$

so $\langle (\tilde{n}/k_0)^2 \rangle < 1 \Rightarrow \boxed{\frac{\Gamma^2}{2k_0} < 1}$

→ again ; $\underbrace{\sigma^2 < 2k_0}$

i.e. fluctuations small compared logistic growth.

N.B. :

- can determine time evolution

- can get moments

- spatio - temporal dynamics.