Taylor Relaxation and its Dynamics

→ Taylor Relaxation

→ Buried bodies on Taylor Hypothesis

— Relation to stochastic fields, turbulence

→ RFP

→ Mean Field Theory

→ Selective Decay.

so, ... $K_1 = \phi_1 \phi_2$ → product of fluxes

similarly $K_2 = \phi_2 \phi_1$

$\therefore$ $K = 2\phi_1 \phi_2$

if n windings $K = K_1 + K_2 = \pm 2n\phi_1\phi_2$!

⇒ helicity is measure of self-linkage of magnetic configuration.

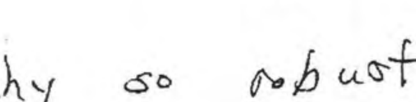
Topological complexity.

Scale?? or

Why care → Taylor Conjecture (1974) (J.B. Taylor)

- in magnetic confinement, of great interest to determine how fields, currents self-organize

$\underline{J} \times \underline{B} = 0$

- RFP  ~ toroid ~ toroidal current

well fit by $B_z = B_0 J_0(\alpha r)$
$B_\theta = B_0 J_1(\alpha r)$

$\underline{J} \times \underline{B} = 0$ ⇔ force free

⇒ why so robust?, especially since RFP so turbulent

- Taylor conjectured conservation of magnetic helicity <u>constrains</u> relaxation to force-free state.

Key Point - helicity conserved in flux tubes, to η
 - toroidal plasma $\rightarrow$ many small tubes

etc.

 - recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite η, helicity of small tubes dissipated but global helicity conserved.

i.e. $\int \underline{A} \cdot \underline{B} \, d^3x = K_0 \rightarrow$ conserved.

plasma volume

$\therefore$ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

- Taylor conjectured conservation of magnetic helicity <u>constrains</u> relaxation to force-free state.

Key Point - helicity conserved in flux tubes, to η
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$$\tau_R \sim L^{3/2}$$

 etc. $\frac{V}{L} \sim \frac{V_A}{L}/\sqrt{R_m} \sim 1/L^{3/2}$

- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite η, helicity of small tubes dissipated <u>but</u> global helicity conserved.

i.e.

$$\int_V \underline{A}\cdot\underline{B}\, d^3x \equiv K_0 \rightarrow (0) \text{ conserved.}$$

V: plasma volume

$\therefore$ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

$$\delta\left[\int_V d^3x\,\frac{B^2}{8\pi} + \lambda\int_V d^3x\,\underline{A}\cdot\underline{B}\right] = 0$$

i.e., minimize magnetic energy subject to constraint of conserved global helicity.

Comments:

→ it works! – indeed amazingly well – for RFPs, spheromaks, etc. Departures only recently being discovered.

→ inspired idea of helicity injection as way to maintain configurations

→ it is a conjecture → no proof. ||

Hypothesis: Selective Decay

energy cascades → small scale

helicity cascades → large scale (less dissipation)

Dual Cascade

– relevance to driven system?
i.e. in real RFP, transformer on.

$\tau_R \sim L^{3/2}$

→ dynamics? – how does relaxation occur
→ more in discussion of kinks, tearing..

$$\delta\left[\int d^3x\left[\frac{B^2}{8\pi}+\lambda\underline{A}\cdot\underline{B}\right]\right]=$$

$$\frac{\underline{B}}{4\pi}\cdot\delta\underline{B}+\lambda\underline{A}\cdot\delta\underline{B}=0$$

$$\frac{\nabla\times\underline{A}}{4\pi}+\lambda\underline{A}=0$$

$\underline{\nabla}\times$ $$\underline{J}=\mu\underline{B}\qquad\qquad\underline{\nabla}\times\underline{B}=\mu\underline{B}$$

force free

and

$$\frac{\underline{J}\cdot\underline{B}}{B^2}=\mu$$

const

$$\boxed{\nabla_\perp J_\parallel=0\rightarrow\text{parallel current homogenized}}$$

No. 10.

see RMP: $\left\{ \text{J.B. Taylor} \atop \text{1986} \right.$

Date

→ Taylor Relaxation

[First example of Self-organization
⇒ Improved confinement. ⇒

→ transition to "quiescent period"
"relaxation" → $\underline{\text{turbulent}}$ $\underline{\text{resistive}}$

→ magnetic energy minimization
(P_{OH} only, and $\beta << 1$)

⇒ what constraint?

→ e in ideal plasma,

$\int d^3x\, \underline{A} \cdot \underline{B}$ conserved for all

$\int d^3x$

i.e any tube, around line

$\int_{\text{tube}} d^3x\, A \cdot B = \text{const.}$

line α, β s/t $\underline{B} = \underline{\nabla}\alpha \times \underline{\nabla}\beta$

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$\Rightarrow$ if $\delta \int_{tube} d^3x \left[\frac{B^2}{8\pi} + \lambda \tilde{A} \cdot \tilde{B} \right] = 0$

$$\boxed{\nabla \times \underline{B} = \lambda(\alpha, \beta)\, \underline{B}} \;;\; \underline{B} \cdot \underline{\nabla} \lambda = 0$$

force free on micro-tubes along line

but $\lambda(\alpha, \beta) \neq \lambda(\alpha', \beta')$

i.e $\rightarrow$ each tube/line defines conserved helicity

$\rightarrow \infty$ of invariants, due freezing in.

(b) But relaxation occurs in resistive, turbulent plasma. $\tau_R \sim \frac{L}{V} \sim \tau_A \sqrt{R_m}$

$\Rightarrow$ small tubes are destroyed by reconnection $\tau_R \sim \ell^{3/2}$

$\Rightarrow$ as $t \to \infty$, only very largest tube survives $\Rightarrow$ global helicity is asymptotic survivor

could ~~also~~ view from stoch. lines

$\rightarrow$ 1 line

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motion → turbulence
resistivity → reconnection

i.e. recall, S-P:

$$V \equiv v_A/\sqrt{R_m} \sim \sqrt{\frac{v_A \eta}{L}}$$

$$1/\tau_{R_L} \sim 1/L^{3/2}$$ ⇒ smaller scales reconnect faster.

⇒ smaller tubes destroyed first.

∴ 3 arguments for conjecture of global helicity as rugged invariant:

→ enhanced dissipation (above) → largest scales reconnect most slowly

→ stochasticity → if field lines stochastic, then (cf. Fermi - MNR)
1 field line → 1 tube of conserved helicity → global helicity is only inv.

⇒ RFP has only 1 field line.

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→ selective decay → magnetic helicity (inverse cascades) in 3D (MHD)

∴ global (large scale) helicity accumulates. → magnetic energy forward cascades.

N.b. compare: energy

heuristic $\dot{W} \sim -2\eta \langle (\nabla B)^2 \rangle$ (if $\nu \to 0$)

$$K = \int d^3x\, A \cdot B \Rightarrow \dot{K} = -2\eta \langle J \cdot B \rangle$$

$$\dot{W} \sim -2\eta \frac{\langle B^2 \rangle}{L_{eff}^2}$$

$$\dot{K} \sim -\eta \frac{\langle B^2 \rangle}{L_{eff}}$$

if $\boxed{L_{eff} \sim \Delta \sim L/\sqrt{R_M} \sim \eta^{1/2}}$

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$\therefore$ $\omega \sim \eta^{0} \to$ finite $\Rightarrow$ $\boxed{\underline{\text{indep dissp'n a/a } \epsilon \text{ in turb}}}$

$K \sim \eta^{1/2} \to 0$

∞ ω diss, $K \sim$ cons!

$\Rightarrow$

Routine calc. variation:

$\nabla \times B = \mu B$

$\overline{J \cdot B / B^2} \to \text{const} = \mu$

$$\boxed{J_{\parallel}/B \text{ homogenized}}$$

n.b. $\int d^3x\, A \cdot B$ related to volt-seconds in plasma, via transformer.

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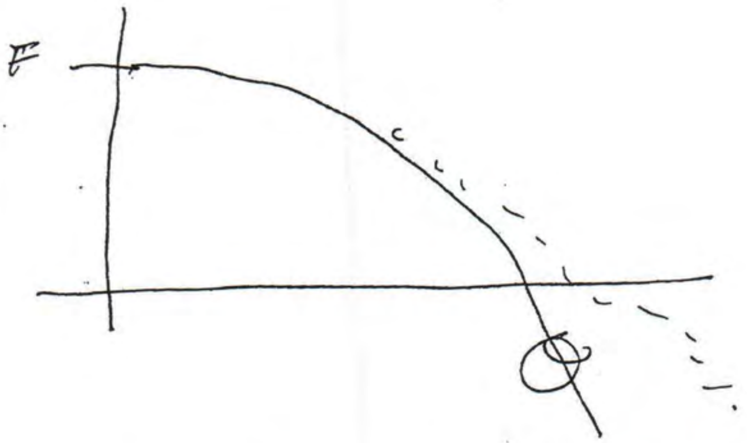
Taylor Theory predicts F – Θ
curve well



$\Theta = \mu a/2 = 2I/a B_0$

created externally.

need $\mu a > 2.4$

$\Theta > 1.2$

$F = B_{z\,wall} / \langle B \rangle$

pretty good . . . –

N.B. An unpleasant reality:

- relaxation ⟺ stoch/turb.
- stoch/turb → losses.

i.e. $\int_{V<n} d^3x\, \eta J^2 = 2\pi r R Q$

$Q = ?$ i.e. $\sim v_T \tilde{b}^2 l_{ac}$?

$\sim \frac{\chi_{\parallel}}{L} D_M$?

$\tilde{b}^2 <$ relaxation / losses

∴ heat flux driven dynamo ...

Confinement bad.

II.) Dynamics of Taylor Relaxation.

ⓐ → How represent dynamics of relaxation?
How does system evolve to Taylor state?
(general)

ⓑ → How does RFP drive poloidal currents which produce reversed toroidal field
(specific)

ⓒ → How relate to more general concepts of relaxation, dynamo? - Self-organized criticality...

ⓐ-ⓒ ⇒ Mean Field Electrodynamics

i.e. how calculate $\langle \tilde{v} \times \tilde{B} \rangle$

⇒ goal is turbulence driven EMF

→ akin $\langle E \delta f \rangle$ in QLT

→ issues: structure, symmetry
→ origin of irreversibility
conservation properties

→ topic is fundamental to subject of dynamo theory

→ flow counterpart: ~~Zonal flow generation~~
~~(Monday Lecture)~~

Good resource:

$\boxed{\text{www.igf.edu.pl/KB/HKM}}$.

items 28, 46

Keith Moffatt pubs.

(a) Structural / symmetry Argument

Approach I (Boozer '86)

Write Ohm's Law in form:
(Mean Field)

$$\langle \underline{E} \rangle + \underbrace{\cancel{\langle \underline{v} \rangle \times \langle \underline{B} \rangle}}_{\text{hereafter ignore}} = \underbrace{\langle \underline{S} \rangle}_{\substack{\text{un-resolved} \\ \text{EMF} \to \\ \text{"something"}}} + \eta \langle \underline{J} \rangle$$

some unspecified operator.

What is $\langle S \rangle$?

Taylor $\to$ i.) $\underline{S}$ must $\underline{\underline{\text{not}}}$ dissipate H_M (conserve H_M)

ii.) $\underline{S}$ must dissipate E_M.

now,

$$\partial_t \int d^3x \, \langle \underline{A} \rangle \cdot \langle \underline{B} \rangle = \partial_t \int d^3x \left[\langle \underline{A} \rangle \cdot \underline{\nabla} \times \langle \underline{A} \rangle \right]$$

$$= -2c \int d^3x \left[\left(\langle \underline{E} \rangle + \langle \underline{\nabla} \phi \rangle \right) \cdot \langle \underline{B} \rangle \right]$$

$$= -2c \int d^3x \left[\langle \underline{E} \rangle \cdot \langle \underline{B} \rangle \right] \qquad \int \underline{B} \cdot \underline{\nabla} \phi \neq 0 \text{ to ST.}$$

now

$$= -2c \int d^3x \left[\langle S \rangle \cdot \langle \underline{B} \rangle + \eta \langle \underline{J} \rangle \cdot \langle \underline{B} \rangle \right]$$

$$\partial_t \int d^3x \, \langle A \rangle \cdot \langle \underline{B} \rangle = -2c\eta \int d^3x \, \langle \underline{J} \rangle \cdot \langle \underline{B} \rangle$$
$$- 2c \int d^3x \, \langle \underline{B} \rangle \cdot \langle \underline{\mathcal{E}} \rangle$$

Now, to conserve H_M, 2^{nd} term must integrate to S.T., so:

drop $\langle \; \rangle$

$$\langle \underline{\mathcal{E}} \rangle = \frac{\underline{B}}{B^2} \, \underline{\nabla} \cdot \underline{\Gamma}_H$$

→ flux, driving helicity evolution

For form $\underline{\Gamma}_H$, consider energy:

$$\partial_t \int d^3x \, \frac{B^2}{8\pi} = \int d^3x \, \frac{\underline{B}}{4\pi} \cdot \partial_t \underline{B}$$

$$= - \int d^3x \, \frac{\underline{B}}{4\pi} \cdot c \, \underline{\nabla} \times \underline{E}$$

$$= - \int d^3x \, \underline{E} \cdot \underline{J}$$

$$= - \int d^3x \left[\eta J^2 + \left(\frac{\underline{J} \cdot \underline{B}}{B^2} \right) \underline{\nabla} \cdot \underline{\Gamma}_H \right]$$

$$= - \int d^3x \left[\eta J^2 - \underbrace{\underline{\Gamma}_H}_{\text{flux}} \cdot \underbrace{\underline{\nabla} \left(\frac{\underline{J} \cdot \underline{B}}{B^2} \right)}_{\text{force}} \right]$$

i.e. $\underset{\text{(entropy)}}{\dfrac{dS}{dt}} = \alpha \left(- \underline{\nabla} T \cdot \underline{\Gamma}_T \right) = \alpha D_T (\underline{\nabla} T)^2$, general form.

apart η,

$$\partial_t E_M = \int d^3x \, \underline{\Gamma}_H \cdot \nabla(J_\parallel / B)$$

so $\Gamma_H = -\lambda \nabla(J_\parallel/B)$ assures

$$\partial_t E_M = -\int d^3x \, \lambda \left[\nabla(J_\parallel/B)\right]^2$$

and:

$$\boxed{\langle \underline{E} \rangle = \eta \langle \underline{J} \rangle - \frac{\underline{B}}{B^2} \underline{\nabla} \cdot \left[+ \lambda \underline{\nabla} \left(\frac{\underline{J} \cdot \underline{B}}{B^2} \right) \right]}$$

. simplified form:

$$\boxed{\langle E_\parallel \rangle = \eta J_\parallel - \underline{\nabla}_\perp \cdot \lambda \underline{\nabla}_\perp J_\parallel}$$

[diffusion of current.]

$\lambda \equiv$ 'hyper-resistivity', 'electron viscosity'

structurally:

cmpl. Recon.

$$\lambda = \underbrace{\frac{c^2}{\omega_{pe}^2} D_J}_{\text{diffusivity}} , \quad \text{as} \quad \eta = \frac{c^2}{\omega_{pe}^2} \nu_{ei}$$

$\lambda \equiv \mu$.

$D_J \to$ MHD
$\to$ multi-fluid
$\to$ extended stochastic field argument

→ Exercises:

→ S-P reconnection, with $E_{\parallel} = -\mu \nabla_\perp^2 J_{\parallel}$!

$V_R/V_A = 1/(S_\mu)^{1/4}$ $\quad S_\mu = \dfrac{V_A L^3}{\mu}$ $\quad$ (J!)

$1/S \rightarrow \mu / V_A L^3$

→ derive structure of D_J for ensemble stochastic fields

(i.e. shifted electron Maxwellian → $J_{\parallel}(\underline{x})$ ---).

→ Compare D_J to χ_e for various turbulence models.

In MHD:

– as seek $\langle E_{\parallel} \rangle$, and concerned with locally strong field

Strauss

$$\left(\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \mu \underline{J} \right) \cdot \underline{B}/|B|$$

$$\Rightarrow \boxed{-\frac{1}{c} \partial_t A_{\parallel} - \underline{n} \cdot \underline{\nabla} \phi - \underline{\nabla} A_{\parallel} \times \hat{n} \cdot \nabla \phi = \mu J_{\parallel}}$$

here $\hat{n} = \underline{B}/|B|$ $\quad$ $B_0 \nabla_{\parallel} \phi$

then for mean field:

$$-\frac{1}{c}\partial_t \langle A \rangle + \partial_r \Big[\langle \nabla_\perp \tilde{\phi} \tilde{A}_\parallel \rangle \Big] = \eta \langle J_\parallel \rangle$$

fluctn. induced EMF

- note naturally in flux form.

- $\langle \nabla_\perp \tilde{\phi} \tilde{A}_\parallel \rangle \cong \langle \nabla_\perp \tilde{\phi} \, \delta A_\parallel \rangle + \langle \tilde{A}_\parallel \nabla_\perp \delta\phi \rangle$

iterate Ohm's Law ①

iterate vorticity eqn. ②

i.e.

$$\partial_t \delta A_{\parallel_{\underline{k}}} + \underbrace{\Delta\omega_{\underline{k}} \delta A_{\parallel_{\underline{k}}}}_{\text{turbulent mixing}} = \underbrace{i k_\parallel \delta\phi_{\underline{k}}}_{\text{bending}} \underbrace{- \eta k_\perp^2 \delta A_{\parallel_{\underline{k}}}}_{\text{resistive dissipn.}}$$

①

$$\langle \nabla_\perp \tilde{\phi} \, \delta A_\parallel \rangle = \sum_{\underline{k}} k_\perp k_\parallel \frac{|\tilde{\phi}_{\underline{k}}|^2 (\Delta\omega_{\underline{k}} + \eta k_\perp^2)}{\omega^2 + (\Delta\omega_{\underline{k}} + \eta k_\perp^2)^2}$$

→ in pure QLT, irreversibility from resistive diffusion, only. → can be slow unless $k_\perp^2$ large

→ if undid normalizations,

$$\langle \nabla_\perp \phi \, \delta A_\parallel \rangle = \alpha \langle B \rangle \rightarrow \text{alpha effect}$$

α = above formula.

i.e. $k_\perp k_\parallel$ → motion has handed-ness i.e. $\underline{x} \to -\underline{x} \Rightarrow \alpha \to -\alpha$

$$k_\perp k_\parallel = \frac{k_\perp^2 x}{L_s} \quad \checkmark$$

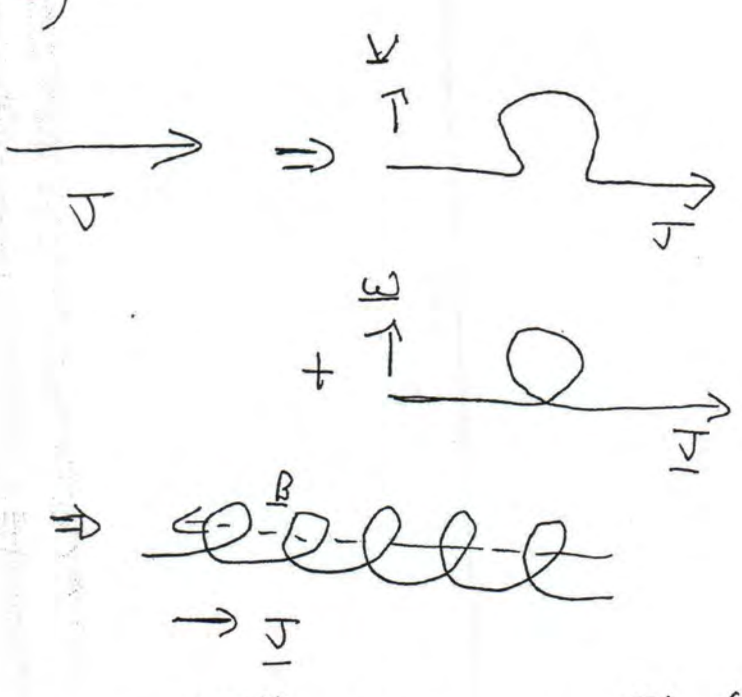
$$\Rightarrow \frac{\partial \langle A_\parallel \rangle}{\partial t} = \alpha \langle B \rangle$$

$$\frac{\partial}{\partial t} \langle B \rangle = \alpha \langle J \rangle$$

i.e. how generate a field parallel/anti-parallel to a current?

aside

(Parker)



Make B_ϕ from B_θ.

need $\langle \tilde{\underline{V}} \cdot \tilde{\underline{\omega}} \rangle \neq 0 \to$ fluctuations have net helicity.

Here $\langle \nabla_\perp \tilde{\phi} \, \nabla_\parallel \tilde{\phi} \rangle$ is magnetized analogue of handedness.

but also ---

(3) $= - \langle \nabla \tilde{A}_{\parallel} \, \delta\phi \rangle$

vorticity eqn:

$$\partial_t \nabla^2 \phi + \nabla\phi \times \hat{z} \cdot \nabla \nabla^2 \tilde{\phi}$$

$$= \frac{\tilde{B}_r}{B_0} \frac{\partial \langle J_{\parallel} \rangle}{\partial r} + \nabla_{\parallel} \tilde{J}_{\parallel} + \underline{\tilde{B}} \cdot \underline{\nabla} \tilde{J} + \mu \nabla^2 \nabla^2 \phi$$

$$\partial_t \left(-k_\perp^2 \tilde{\phi}_{\underline{k}} \right) + \Delta\omega_{\underline{k}} \left(-k_\perp^2 \tilde{\phi}_{\underline{k}} \right)$$

$$= \frac{\tilde{B}_{r\underline{k}}}{B_0} \underset{d}{\underset{\downarrow}{\frac{\partial \langle J_{\parallel} \rangle}{\partial r}}} + \underset{a}{\underset{\downarrow}{i k_{\parallel} \hat{A}_{\parallel \underline{k}} (-k_\perp^2)}} + \mu (k_\perp^2)^2 \tilde{\phi}_{\underline{k}}$$

$$\tilde{\phi}_{\underline{k}} = \frac{ - \frac{\tilde{B}_{r\underline{k}}}{B_0 k_\perp^2} \frac{\partial \langle J_{\parallel} \rangle}{\partial r} + i k_{\parallel} \hat{A}_{\parallel \underline{k}} }{ \left(-i\omega + \Delta\omega_{\underline{k}} + \mu k_\perp^2 \right) }$$

$$(2)_{(a)} = - \sum_{\underline{k}} \frac{k_\perp k_{\parallel} |\tilde{A}_{\underline{k}}|^2 \left(\Delta\omega_{\underline{k}} + \mu k_\perp^2 \right)}{\omega^2 + \left(\Delta\omega_{\underline{k}} + \mu k_\perp^2 \right)^2}$$

- magnetic α effect
- opposite in ~~[illegible]~~ sign to (1)

$$(2)_{b} = \sum_{\underline{k}} \frac{|\nabla_\perp \tilde{A}_{\parallel \underline{k}}|^2}{B_0^2 k_\perp^2} \frac{(\Delta\omega_{\underline{k}} + \eta k_\perp^2)}{\omega^2 + (\Delta\omega_{\underline{k}} + \eta k_\perp^2)^2} \frac{\partial \langle J_\parallel \rangle}{\partial r}$$

→ clearly corresponds to hyper-η.

i.e.

$$-\frac{1}{c}\frac{\partial \langle A_\parallel \rangle}{\partial t} + \partial_r \langle (\nabla_\perp \tilde{\phi}) \tilde{A}_\parallel \rangle = \eta \langle J_\parallel \rangle$$

$$\boxed{\begin{array}{l} \langle (\nabla_\perp \tilde{\phi}) \tilde{A}_\parallel \rangle = \sum_{\underline{k}} k_\perp k_\parallel \Big\{ \overbrace{|\phi_{\underline{k}}|^2 L^{\eta}_{\underline{k}}}^{\alpha_K} - \overbrace{|A_{\parallel \underline{k}}|^2 L^{\eta}_{\underline{k}}}^{\alpha_M} \Big\} \\ \left[L^{\eta} = (\Delta\omega_{\underline{k}} + \eta k_\perp^2) / \omega^2 + (\Delta\omega_{\underline{k}} + \eta k_\perp^2)^2 \right] \\ \qquad + \underbrace{\sum_{\underline{k}} \left| \frac{\tilde{B}_{r\underline{k}}}{B_0} \right|^2 \frac{L^{\eta}_{\underline{k}}}{k_\perp^2} \frac{\partial \langle J_\parallel \rangle}{\partial r}}_{\text{hyper-resistivity}} \end{array}}$$

N.B. – α's both come from bending

– α_K, α_M opposite sign.

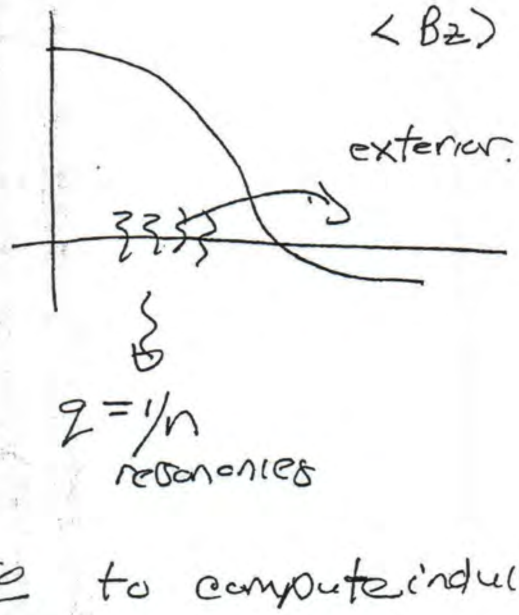
– α's from MHD exterior,

$$\tilde{A}_\parallel \to \frac{k_\parallel \tilde{\phi}_{\underline{k}}}{\omega + i\sigma}$$

- hyper-η from ~ resonance
  i.e. where vorticity driven.
  $\Rightarrow$ reconnection process site.

- hyper-η tied to basic tearing drive
- α_M + hyper η cancel in exterior
  V_K survive in exterior, vanish near Res. surf
- note total EMF encompasses more than hyper-η .....

(b) RFP



$\left. \begin{array}{l} q < 1 \\ q' < 0 \end{array} \right\} \Rightarrow$ K-S unstable

$m = 1$ paradise
(global tearing turbulence)

so to compute induced EMF, seek
$\langle \tilde{\underline{V}} \times \tilde{\underline{B}} \rangle \hat{\theta}$ in exterior.

$\underline{V} = \partial_t \tilde{\underline{\xi}}$
$\hookrightarrow$ displacement

$$\tilde{\underline{B}} = \nabla \times \underline{\xi} \times \langle \underline{B} \rangle$$

$$= -\hat{\xi} \cdot \nabla \langle B \rangle + \langle B \rangle \cdot \nabla \hat{\underline{\xi}} - \langle B \rangle \cancel{\nabla \cdot \underline{\xi}}$$

field advection irrelevant

kink incompressible

i.e. bending is key.

$$\tilde{\underline{B}} \cong \langle B \rangle \cdot \nabla \tilde{\underline{\xi}}$$

$$\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle = \sum_{\underline{k}} \gamma_{\underline{k}} \langle \tilde{\xi}_{-\underline{k}} \times \tilde{B}_{\underline{k}} \rangle$$

$$= \sum_{\underline{k}} \gamma_{\underline{k}} \, \tilde{\xi}_{-\underline{k}} \times i k_{\parallel} \langle B \rangle_{\theta} \, \tilde{\xi}_{\underline{k}}$$

→ field primarily poloidal near B_z reversal region.

$$\underline{\nabla} \cdot \underline{\xi} = 0 \Rightarrow \frac{\partial_r \tilde{\xi}_r + i k_\theta \tilde{\xi}_\theta}{i k_z} = \tilde{\xi}_z$$

then

$$\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle_{\theta} = \sum_{\underline{k}} \gamma_{\underline{k}} \, i k_{\parallel} \langle B_\theta \rangle \left[\tilde{\xi}_z^{*} \tilde{\xi}_x - \xi_x^{*} \xi_z \right]$$

$$= \sum_{\underline{k}} \frac{\gamma_{\underline{k}} \, \cancel{i} k_{\parallel} \langle B_\theta \rangle}{-\cancel{i} k_z} \, (M)$$

$$M = +\left(\partial_r \tilde{\xi}_r^* - i k_\theta \tilde{\xi}_\theta^*\right) \tilde{\xi}_r \quad -$$

$$+ \, \xi_r^* \left(\partial_r \xi_r + i k_\theta \tilde{\xi}_\theta\right)$$

$$M = + \partial_r |\tilde{\xi}_r|^2 + \cancel{\left(i k_\theta \left(\tilde{\xi}_\theta^* \xi_r - \xi_r^* \xi_\theta\right)\right)}$$

but $\tilde{\xi}_r \Big|_{\text{wall}} = 0$

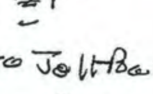
$r_{res} \sim a \Rightarrow \partial_r \gg k_\theta$

$$\boxed{\langle \tilde{v} \times \tilde{B} \rangle_\theta = +\sum_{k} \gamma_k \frac{k_\parallel}{k_z} \langle B_\theta \rangle \, \partial_r |\tilde{\xi}_k|^2}$$

$$\Rightarrow k_\parallel / k_z = \left(\frac{m}{r} B_\theta - \frac{n}{R} B_z\right) \Big/ B_\theta$$

$$\cong \frac{m}{r} - \frac{n}{r} q(r)$$

$$= 1/r \left(m - n q(r)\right)$$



so $J_\theta \parallel B_\theta$ at r_{res}

$$k_z = n/R$$

$$k_\parallel / k_z = (R/r)(m/n) - \frac{R}{r} q(r)$$

$$= (R/r)\left(q_{res} - q(r)\right)$$

$$\langle \tilde{\underline{V}} \times \tilde{\underline{B}} \rangle = -\sum_{\underline{n}} |\gamma_{\underline{n}}| \frac{R}{r} \left(q_{res} - q(r) \right) \langle B_\theta \rangle \partial_r |\tilde{\xi}_{\underline{n}}|^2$$

$\rightarrow \partial_r |\tilde{\xi}_n|^2 < 0$

$\rightarrow \gamma_{\underline{n}} \rightarrow$ irreversibity (?)

$\rightarrow q_{res} - q(r) \Rightarrow$ < 0 on axis

> 0 at r_{rev}.

so $\Gamma > 0$

$\langle E \rangle + \langle \tilde{V} \times \tilde{B} \rangle = \eta \langle J_\theta \rangle$

$\therefore \langle J_\theta \rangle \cong \frac{1}{\eta} \langle \tilde{\underline{V}} \times \tilde{\underline{B}} \rangle_\theta$

$\Rightarrow \langle B_z \rangle < 0 \rightarrow$ kinko drive reversal.

But what about irreversibility and/or locking in ?

S–T–F–R

" ⌒ =) | 17ʒ~~ʒ

$1/n \quad 1/n+1 \rightarrow \frac{2}{2n+1}$

$\searrow 0, 1$

$1/n+1, 0/1 \rightarrow 2/n+2$

m=0 drives ⇒ reconnection

→ lock in.

ii.) 4/5 Law – See Lecture I.

iii.) Cascades and Relaxation

$\Rightarrow$ Selective Decay

Recall: Taylor Relaxation { 3D; 2D – "Taylor in Flatland" }

Argued: $\int d^3x \, B^2/8\pi$ minimized subject to constraint of $\int d^3x \, A \cdot B$ conserved.

$\Rightarrow \quad J_{\parallel} = J \cdot B / B^2 \rightarrow$ const.

(2D $J/A \rightarrow$ const.).

Arguements heuristic. { Power counting (k); stoch. fields; ⋮ }

Now, – dissipation at small scale η, ν

– expect energy transfer to small scale!

∴

- Inverse {cascade / transfer} of magnetic helicity would set up "Selective Decay" scenario

i.e. magnetic energy scattered to small scale and dissipated
⇒ relaxation

magnetic helicity inverse cascades
⇒ avoids dissipation. Constraint; as survives.

c.f. {Frisch (75), Pouquet, et. al. (76) (posted)
see also: Montgomery}

- Why, where from?
→ Primarily: Statistical Mechanics
→ c.f.: Frisch '75, though not transparent.

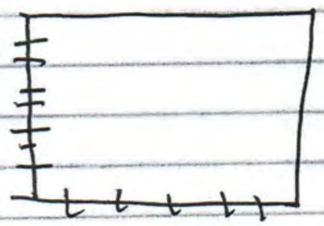
easier → "Taylor in Flatland" Problem.

Recall: Relaxation $\begin{cases} \text{minimizes } \langle B^2 \rangle \\ \text{conserving } \langle A^2 \rangle \end{cases}$.

Does this follow from Selective Decay?

$\Rightarrow$ Explore Absolute Equilibrium.

i.e.



finite box

k_{min} k_{max}

- remove forcing, dissipation etc.
- input excitation.

For 2D MHD (ignoring cross helicity):

have $A \Rightarrow X_i$

$\hookrightarrow$ modal amplitude

so

$$E_M = \sum_{i=1}^{N} k_i^2 X_i^2$$

$$H = \sum_{i=1}^{N} X_i^2 \qquad - \langle A^2 \rangle$$

$$\phi \to Y_i$$

$$E_k = \sum_{i=1}^{N} k_i^2 Y_i^2 .$$

Now,

$$\begin{array}{c} H \to \alpha \\ E = E_M + E_K \to \beta \end{array} \Big\rangle \text{ conserved}$$

conserved, so pdf of this closed system is given by micro-canonical ensemble/distribution:

$$P(X, Y) = \underset{\substack{\downarrow \\ \text{norm}}}{c} \exp\left[-\sum_{i=1}^{N}\left((\alpha + \beta k_i^2) X_i^2 + \beta k_i^2 Y_i^2\right)\right]$$

and can integrate out Y_i (KE) part, so:

$$P(X) = c \exp\left[-\sum_{i=1}^{N}(\alpha + \beta k_i^2) X_i^2\right]$$

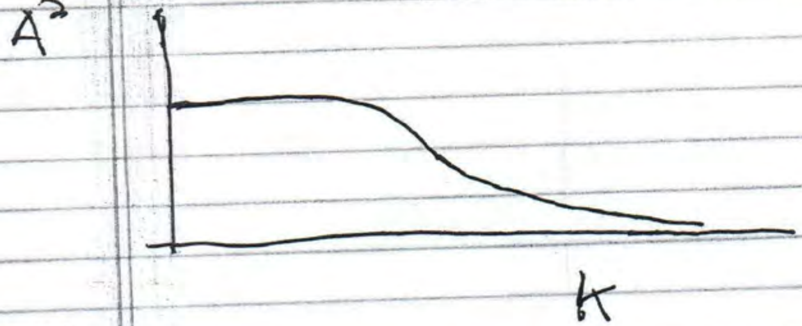
then:

$$\langle A^2(k)\rangle = \int dX_i\, X_i^2\, P(X_i)$$

$$= 1/(\alpha + \beta k^2)$$

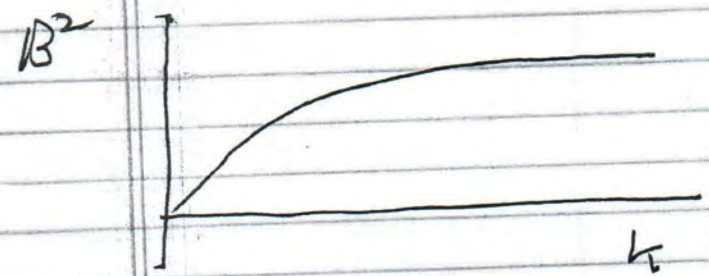
$$\langle B^2(k)\rangle = \left[k^2/(\alpha + \beta k^2)\right]$$

⊗ observe immediately:



"A^2 wants remain at large scale"

$k_{min} \to 0$



"B^2 approaches equipartition"

$\Rightarrow$ A^2 distribution most populated at large scales. Decays at small

$\Rightarrow$ B^2 distribution most populated at smaller. Approaches equipartition at small scale.

$\therefore$ - suggests A^2 populates large scales, B^2 approaches equipartition.

- suggestive of inverse cascade of A^2, along with forward cascade of energy.

- supports Selective Decay Hypothesis as foundation for "Taylor in Flatland".

- similar story for Magnetic Helicity, though more laborious.

N.B. For 2D Fluid:

$$E = \int d^2x \, (\underline{\nabla}\phi)^2 \qquad \text{— energy}$$

$$\Omega = \int d^2y \, (\nabla^2\phi)^2 \qquad \text{— enstrophy}$$

$$\Omega_i = k_i^2 E_i$$

$$\underline{V} \rightarrow \underline{X}_i$$

$\therefore$

$$P(X) = c \exp\left[-\sum_{i=1}^{N} (\alpha + \beta k_i^2) X_i^2 \right]$$

$\underline{\underline{\text{so}}}$

$$E(k) = \langle V^2(k) \rangle = 1/(\alpha + \beta k^2)$$

$$\Omega(k) = k^2/(\alpha + \beta k^2)$$

similar suggestion of dual cascade and minimum enstrophy state.

→ Is this story true ?

⇒ What does dynamics tell us ?
∴ Consider interactions in 2D MHD.

Observe:

- Reduced MHD

$$\frac{\partial \psi}{\partial t} + \underline{\nabla}_\perp \phi \times \hat{z} \cdot \underline{\nabla}_\perp \psi = B_0 \partial_z \phi + \eta \nabla_\perp^2 \psi$$

- 2D MHD

$$\frac{\partial \psi}{\partial t} + \underline{\nabla}_\perp \phi \times \hat{z} \cdot \underline{\nabla}_\perp \psi = \eta \nabla_\perp^2 \psi$$

so; with strong $\underline{B_0}$:

$\langle A \cdot B \rangle \rightarrow \langle \psi \rangle \underline{B_0}$

so mean $\langle \psi \rangle$ in 2D captures magnetic helicity dynamics in strongly magnetized system.

For $\langle A^2 \rangle_k$ transfer, consider closure of $\partial_t \langle A^2 \rangle$ equation, much akin to wave kinetics, though closure required.

See: Diamond, Hughes, Kim (posted).

Can write (see DHK):

$$\frac{1}{2}\Big[\partial_t \langle A^2\rangle_{\underline{k}} + T(k)\Big] = -\Gamma_A(k)\frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle_{\underline{k}}$$

triplet

$$\langle \nabla\cdot\langle \underline{V} A^2\rangle\rangle_{\underline{k}}$$

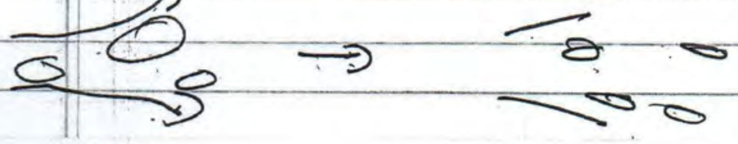
flux

$$\Gamma_A = \Big[\tilde{\tau}_c^{\phi}(\underline{k})\langle V^2\rangle_{\underline{k}} - \tilde{\tau}_c^{A}(\underline{k})\langle B^2\rangle_{\underline{k}}\Big]$$

$$T_{\underline{k}} = \sum_{\underline{k}'} (\underline{k}\cdot\underline{k}'\times\vec{z})^2 \, \Theta_{\substack{\underline{k},\underline{k}'\\ \underline{k}''}} \Big\{ \langle\phi^2\rangle_{\underline{k}'} - \frac{(k'^2-k^2)}{|\underline{k}+\underline{k}'|^2}\langle A^2\rangle_{\underline{k}'}\Big\}\{A^2_{\underline{k}}\}$$

$$- \sum_{\substack{\underline{k}=\underline{p}+\underline{q}\\ \underline{p},\underline{q}}} (\underline{p}\cdot\underline{q}\times\vec{z})^2 \, \Theta_{\underline{k},\underline{p},\underline{q}} \langle A^2\rangle_{\underline{p}}\langle\phi^2\rangle_{\underline{q}}$$

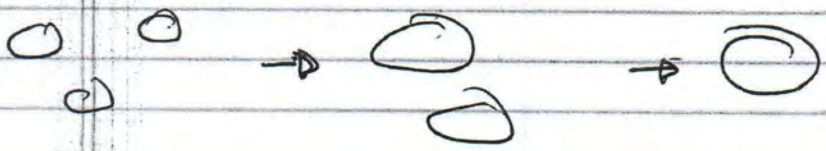
1, 3 → coherent damping, incoherent emission
→ akin to scattering of passive scalar, → small scale / chop-up.
→ conserve $\langle\psi^2\rangle$ upon $\sum_k$ together.



② → coherent damping/growth – from back reaction $\langle J \times B \rangle$ into Ohm's Law:

→ reshuffle $\langle A^2 \rangle$ to larger scale. Sign k'^2 vs k^2!

→ $\sum_{k} \langle A^2 \rangle$ conserved independently.



→ correspondence to condensation and wires (currents) attracting.

① + ② → net effective resistivity sign.

→ see Γ_A, too. – 'negative resistivity'

– Alfvenized state

⇒ $E_K > E_M$ ⇒ $\langle A^2 \rangle_k$ shuffled to smaller scale.

$E_M < E_K$ ⇒ $\langle A^2 \rangle_k$ transfered to larger scale.

and

transfer need not be local.

⇒ In dynamics ~~■~~ $\langle A^2 \rangle$; $\langle A \cdot B \rangle$ evolution is complex.

⇒ N.B. Recall flux expulsion:

$\frac{V_{A0}^2}{\tilde{V}^2} R_m < 1 \rightarrow$ A passive, B expelled

$> 1 \rightarrow$ $J \times B$ disrupts vortex, expulsion stops

$$\Rightarrow B_0^2 \lessgtr \rho_0 \langle \tilde{v}^2 \rangle / R_m$$

but $\langle \tilde{B}^2 \rangle \gg B_0^2$, upon stretching, weak B_0 is sufficient!

Zeldovich:

$$\frac{\partial A}{\partial t} + \underline{v} \cdot \underline{\nabla} A = - v_r \frac{\partial \langle A \rangle}{\partial x} + \eta \nabla^2 A$$

$\times A$ and avg. ⇒

$$\eta \langle \tilde{B}^2 \rangle = \langle \tilde{v}_r \tilde{A} \rangle \frac{\partial \langle A \rangle}{\partial x}$$

$$\langle \tilde{B}^2 \rangle = \frac{\eta_T}{\eta} B_0^2$$

$$= \frac{\tilde{v}\, \tilde{v} \tau_c}{\eta} B_0^2 \cong R_m B_0^2 \;\checkmark.$$

so, crudely:

$$\langle \tilde{B}^2 \rangle / R_m \lesssim \langle \rho \tilde{V}^2 \rangle / R_m \quad \checkmark$$

$\Rightarrow$ Questions still open!

$\therefore$ Taylor conjecture remains a conjecture!

Outline

i.) Preamble: → From Reconnection to Relaxation and Self-Organization

- → What 'Self-Organization' means
- → Why Principles are important
- → Examples of turbulent self-organization
- → Preview

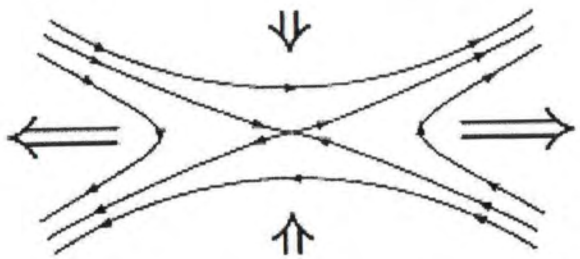
ii.) Focus I: Relaxation in R.F.P. (J.B. Taylor)

- → RFP relaxation, pre-Taylor
- → Taylor Theory
  - Summary
  - Physics of helicity constraint + hypothesis
  - Outcome and Shortcomings
- → Dynamics → Mean Field Theory
  - Theoretical Perspective
  - Pinch's Perspective
  - Some open issues
- → Lessons Learned and Unanswered Questions

I.) Preamble

→ From Reconnection to Relaxation

- Usually envision as localized event involving irreversibility, dissipation etc. at a singularity



S.-P.

$$V = V_A / Rm^{1/2}$$

- ??? - how describe global dynamics of relaxation and self-organization

→

- multiple, interacting/overlapping reconnection events

→ turbulence, stochastic lines, etc

Examples of Self-Organization Principles

→ Turbulent Pipe Flow: (Prandtl → She)

$$\sigma = -\nu_T \frac{\partial \langle v_y \rangle}{\partial x} \qquad \nu_T \sim v_* x \qquad \Rightarrow \langle v_y \rangle \sim v_* \ln x$$

Streamwise Momentum undergoes scale invariant mixing

→ Magnetic Relaxation: (Woltjer-Taylor)

(RFP, etc)
(Focus 1)   Minimize E_M at conserved global H_M $\Rightarrow$ Force-Free RFP profiles

→ PV Homogenization/Minimum Enstrophy: (Taylor, Prandtl, Batchelor, Bretherton, ...)

(Focus 2)
→ PV tends to mix and homogenize
→ Flow structures emergent from selective decay of potential enstrophy relative energy

→ Shakura-Sunyaev Accretion

→ disk accretion enabled by outward viscous angular momentum flux

II.) Focus I - Magnetic Relaxation

(Derek C Robinson)

→ Prototype of RFP's: Zeta (UK: late 50's - early 60's)

- toroidal pinch = vessel + gas + transformer
- initial results → violent macro-instability, short life time
- weak B_T → stabilized pinch ↔ sausage instability eliminated
- $I_p > Ip, crit$ ($\theta > 1+$) → access to "Quiescent Period"

→ Properties of Quiescent Period:

- macrostability - reduced fluctuations
- $\tau_E \sim 1\ msec \qquad T_e \sim 150eV$
- $B_T(a) < 0$ → reversal

→ Quiescent Period is origin of RFP

Further Developments

- Fluctuation studies:

turbulence = $\nearrow$ $m = 1$ kink-tearing → tend toward force-free state
$\searrow$ resistive interchange, ...

- Force-Free Bessel Function Model

$$B_\theta = B_0 J_1(\mu r) \qquad B_z = B_0 J_0(\mu r)$$

$$\mathbf{J} = \alpha \mathbf{B}$$

observed to correlate well with observed B structure

- L. Woltjer (1958) : Force-Free Fields at constant α

→ follows from minimized E_M at conserved $\int d^3x \mathbf{A} \cdot \mathbf{B}$

- steady, albeit modest, improvement in RFP performance, operational space

→ Needed: Unifying Principle

Theory of Turbulent Relaxation (J.B. Taylor, 1974)

→ hypothesize that relaxed state minimizes magnetic energy subject to constant global magnetic helicity

i.e. profiles follow from: $$\delta\left[\int d^3x \frac{B^2}{8\pi} + \lambda \int d^3x \mathbf{A}\cdot\mathbf{B}\right] = 0$$

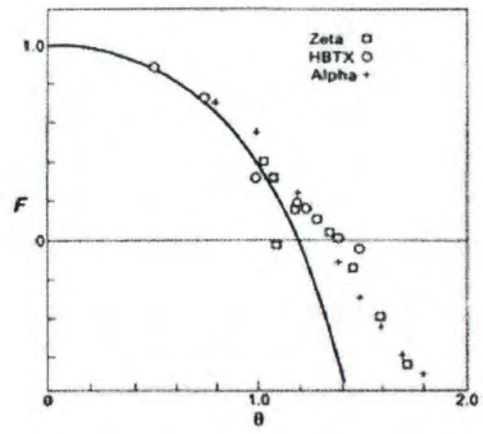
$$\Rightarrow \nabla\times\mathbf{B} = \mu\mathbf{B} \;\; ; \;\; J_{\|}/B = \frac{\mathbf{J}\cdot\mathbf{B}}{B^2} = const$$

Taylor state is:

- force free
- flat/homogenized $J_{\|}/B$
- recovers BFM, with reversal for $\theta = \frac{2I_p}{aB_0} > 1.2$

- Works amazingly well

Result:



$$\theta = \mu a/2 = \frac{2I_p}{aB_0}$$

$$F = B_{z,wall}/\langle B\rangle$$

and numerous other success stories

→ Questions:

- what is magnetic helicity and what does it mean?
- why only global magnetic helicity as constraint?
- Theory predicts end state → what can be said about dynamics?
- What does the pinch say about dynamics?

→ Central Issue: Origin of Irreversibility

Why Global helicity, Only?

- in ideal plasma, helicity conserved for each line, tube

  i.e. $\mathbf{J} = \mu(\alpha, \beta)\mathbf{B} \qquad \mu(\alpha', \beta') \neq \mu(\alpha, \beta)$

- Turbulent mixing eradicates identity of individual flux tubes, lines!

  i.e.

  - if turbulence s/t field lines stochastic, then '1 field line' fills pinch.

    1 line $\leftrightarrow$ 1 tube $\rightarrow$ only global helicity meaningful.

  - in turbulent resistive plasma, reconnection occurs on all scales, but: $\tau_R \sim l^{\alpha} \quad \alpha > 0$

    ($\alpha = 3/2$ for S-P reconnection)

    Thus larger tubes persist longer. Global flux tube most robust

  - selective decay: absolute equilibrium stat. mech. suggests possibility of inverse cascade of magnetic helicity (Frisch '75) $\rightarrow$ large scale helicity most rugged.

Comments and Caveats

→ Taylor's conjecture that global helicity is most rugged invariant remains a conjecture

→ unproven in any rigorous sense

→ many attempts to expand/supplement the Taylor conjecture have had little lasting impact (apologies to some present....)

→ Most plausible argument for global H_M is stochastization of field lines → forces confinement penalty. No free lunch!

→ Bottom Line:

- Taylor theory, simple and successful

- but, no dynamical insight!

Dynamics I:

- The question of Dynamics brings us to mean field theory (c.f. Moffat '78 and an infinity of others - see D. Hughes, Thursday Lecture)

- Mean Field Theory → how represent $\langle \tilde{v} \times \tilde{B} \rangle$?

  → how relate to relaxation?

- Caveat: - MFT assumes fluctuations are small and quasi-Gaussian. They are often NOT

  - MFT is often very useful, but often fails miserably

- Structural Approach (Boozer): (plasma frame)

$$\langle \mathbf{E} \rangle = \eta \langle \mathbf{J} \rangle + \langle \mathbf{S} \rangle$$

→ something → related to $\langle \tilde{v} \times \tilde{B} \rangle$

$\langle \mathbf{S} \rangle$ conserves H_M

$\langle \mathbf{S} \rangle$ dissipates E_M

Note this is ad-hoc, forcing $\langle \mathbf{S} \rangle$ to fit the conjecture. Not systematic, in sense of perturbation theory

Now

$$\partial_t H_M = -2c\eta \int d^3x \langle \mathbf{J} \cdot \mathbf{B} \rangle - 2c \int d^3x \langle \mathbf{S} \cdot \mathbf{B} \rangle$$

$$\therefore \langle \mathbf{S} \rangle = \frac{\mathbf{B}}{B^2} \nabla \cdot \mathbf{\Gamma}_H$$

Conservation H_M → $\langle S \rangle \sim \nabla\cdot$ (Helicity flux)

$$\partial_t \int d^3x \frac{B^2}{8\pi} = -\int d^3x \left[\eta J^2 - \mathbf{\Gamma}_H \cdot \nabla \frac{\langle \mathbf{J} \rangle \cdot \mathbf{B}}{B^2} \right]$$

so

$$\mathbf{\Gamma}_H = -\lambda \nabla (J_\parallel / B)$$, to dissipate E_M

→ simplest form consistent with Taylor hypothesis

→ turbulent hyper-resistivity $\lambda = \lambda[\langle \tilde{B}^2 \rangle]$ - can derive from QLT

→ Relaxed state: $\nabla(J_\parallel / B) \to 0$ homogenized current → flux vanishes

Dynamics II: The Pinch's Perspective

- Boozer model not based on fluctuation structure, dynamics

- Aspects of hyper-resistivity do enter, but so do other effects

  → Point: Dominant fluctuations controlling relaxation are m=1 tearing modes resonant in core → global structure

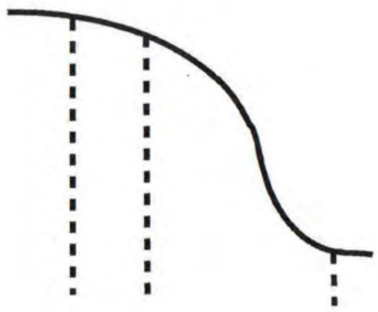
  → Issue: What drives reversal B_z near boundary?

Approach: QL $\langle \tilde{v} \times \tilde{B} \rangle$ in MHD exterior - exercise: derive!

$$\langle \tilde{\mathbf{v}} \times \tilde{\mathbf{B}} \rangle \cong \sum_k |\gamma_k| \frac{R}{r} (q_{res} - q(r)) \langle B_\theta \rangle \partial_r (|\tilde{\xi}_r|_k^2)$$

i.e. $\langle J_\theta \rangle$ driven opposite $\langle B_\theta \rangle$ → drives/sustains reversal

→ What of irreversibility - i.e. how is kink-driven reversal 'locked-in'?



→ drive $J_{\|}/B$ flattening, so higher n's destabilized by relaxation front

→ global scattering → propagating reconnection front

m=1, n $\quad$ m=1, n+1 $\quad$ → $\quad$ m=0, n=1 $\quad$ → $\quad$ driven current sheet, at r_{rev}

(difference beat)

sum beat $\left\{ \begin{array}{l} \text{m=2,} \\ \text{2n+1} \end{array} \right.$

but then $\quad$ m=1, n+2 $\quad$ driven → $\quad$ tearing activity, and relaxation region, broadens

→ Bottom Line: How Pinch 'Taylors itself' remains unclear, in detail

Summary of Magnetic Relaxation

concept: topology

process: stochastization of fields, turbulent reconnection

constraint released: local helicity

players: tearing modes

Mean Field: EMF = $\langle \tilde{v} \times \tilde{B} \rangle$

Global Constraint: $\int d^3x \mathbf{A} \cdot \mathbf{B}$

NL: Helicity Density Flux

Outcome: B-Profile

Shortcoming: Rates, confinement → turbulent transport