Lecture VI

- MHD Turbulence II
  - review of Kraichnan - Iroshnikov Theory
  - Goldreich - Sridhar and Critical Balance
  - Spectra / Triads
- Taylor Relaxation I
  - Relaxation (Dynamo) ⇒ Impact of Turbulence on Mean Field
  - Magnetic Helicity
  - Taylor Relaxation - an Intro.

Review — MHD Turbulence

→ K41

singularity, ϵ and ν

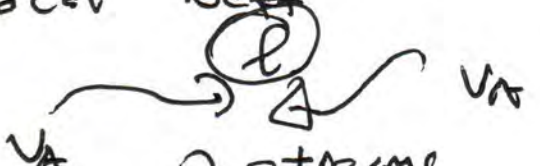
$l_d < l < l_0$, $\epsilon \sim \frac{v(l)^3}{l}$

$$v(l) \sim \epsilon^{1/3} l^{1/3}$$

$$l_d \sim (\nu^3/\epsilon)^{1/4}$$

→ MHD

Local B_{eff}



wave packet streams

2 streams

$$\delta z_l \sim \tau_A \, \partial_t z_l$$

$$\sim \tau_A \frac{z_l^2}{l}$$

random ϕ

or $\partial_t z_l \sim \frac{z_l^2}{l}$ $\Longleftrightarrow$ $\partial_t x \sim \tilde{V}$

kick time

$\rightsquigarrow \tau_{ac}$

t

$$\langle \delta z_l^2 \rangle \sim \tau_{ac} \left(\frac{z_l^2}{l}\right)^2 t$$

τ_{Tr} $\langle \delta z_l^2 \rangle \sim \langle z_l^2 \rangle$

2

→ $P_m = 1$ → effect on l_d ?

→ Result → K-I spectrum.

→ + anisotropy → G-S.

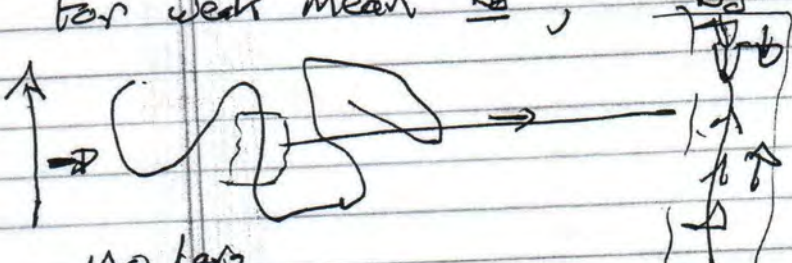
Review

Kraichnan-Iroshnikov — classic

so, some Phenomenology: (Iroshnikov, Kraichnan) ('64, '65)

→ nonlinear transfer — ⓘ isotropic turbulence

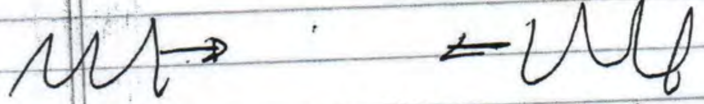
For weak mean $\underline{B_0}$, $b_0 \sim b_{rms} > |B_0|$



locally ∃ a $\underline{b_0}$

no lag

→ so, nonlinear scattering from "collisions" of counter-propagating packets



[universally true, ≡ counter Els. populations locality?]

so (modulo 4/5 !) — TBD.

$$\rightarrow \boxed{\epsilon \sim \frac{\underline{Z}_\ell^2}{\tilde{\tau}_{T\ell}}}$$

$\ell \equiv$ scale

→ Alfven wave cascade

→ const. energy thru-put rate

Now Alfvenic transit time on scale ℓ, in b_0 is:

$$\boxed{\tau_A = \ell / b_0}$$

For transfer:

- transfer by wave scattering $\Leftrightarrow$ { random walk of amplitude/intens
- define τ_{Tr} by:

$$\langle \delta Z^2(\tau_{Tr}) \rangle \sim \langle Z^2 \rangle$$

τ_{Tr} sufficient # kicks to make

i.e. randomization of scattering in amplitude

$$\underline{\delta Z_{rms} \sim Z}$$

random walk.

so

$$Z_\ell(t+\tau_A) = Z_\ell(t) + \tau_A \, \partial_t Z_\ell$$

kick in 1 Alfven transit time

diffusion process → $k_\perp$... < 1

short

$$\delta Z_\ell \simeq \tau_A \frac{Z_\ell^2}{\ell} \rightarrow \underline{\text{kick}} \text{ in } \tau_A$$

(duis)

so $\frac{Z_\ell^2}{\ell}$ → kick.

$$\Rightarrow \langle \delta Z_\ell^2 \rangle = \left(\tau_A \frac{Z_\ell^2}{\ell}\right)^2 \frac{t}{\tau_A} \rightarrow \boxed{\text{accumulated kicks}}$$

$$= \underbrace{\left(\frac{Z_\ell^2}{\ell}\right)^2 \tau_A}\, t \Rightarrow D\, t$$

thus:

$$\langle \delta Z_\ell^2 \rangle \sim \langle Z_\ell^2 \rangle \rightarrow \text{defines } \tau_{Tr}.$$

$$\Rightarrow Z_\ell^2 \simeq \tau_A^2 \frac{(Z_\ell^2)^2}{\ell^2} \frac{\tau_{Tr}}{\tau_A}$$

$$\therefore \boxed{\frac{1}{\tau_{Tr}} \simeq \frac{Z_\ell^2}{\ell^2} \tau_A} \rightarrow \text{determines transfer rate.}$$

$$\langle \delta Z_\ell^2 \rangle \sim \langle Z_\ell^2 \rangle \quad 1.$$

$$\tau_A \tau_{Tr} \frac{(Z_\ell^2)^2}{\ell^2} \sim (Z_\ell)^2$$

so $\frac{1}{\tau_{Tr}} \sim \frac{Z_\ell^2}{\ell^2} \frac{\ell}{b_0}$

$\sim \frac{Z_\ell^2}{\ell \, b_0}$

Thus:

$\epsilon \simeq \frac{Z_\ell^2}{\tau_{Tr}} \simeq \frac{Z_\ell^2 Z_\ell^2}{\ell \, b_0}$ const. rate balance

$\boxed{Z_\ell^2 \sim \sqrt{b_0 \epsilon} \, \ell^{1/2}}$ $Z_\ell \sim (b_0 \epsilon)^{1/4} \ell^{1/4}$

$\boxed{E(k) \simeq \sqrt{\epsilon b_0} \, k^{-3/2}}$ Kraichnan spectrum!

N.B. – as work with Z,
$E_M \sim E_K$.
(equal Elsasser populations)

– $1/\tau_{Tr} \simeq \frac{Z_\ell^2}{\ell^2} \frac{\ell}{b_0} \cong \underbrace{\frac{\tau_A}{\tau_{eddy}}}\frac{1}{\tau_{eddy}}$

202

$\hbar\omega_f - \hbar\omega_i$

$$\partial_t P_\alpha = \text{in} - \text{out} = \sum_{\alpha,\beta,\gamma} P_\beta P_\gamma - \sum_\gamma \Sigma_{\alpha,\gamma,\delta} P_\delta P_\alpha \qquad \frac{1}{T} \sim$$

$$T_{gr} = \sum_f |\langle f | H' | i \rangle|^2 \delta(E_f - E_i - \hbar\omega)$$

compare K41: $1/T_{Tr} \sim \frac{1}{T_{eddy}}$

Transfer rate reduced by factor $\underline{\tau_A / \tau_{eddy}}$!

Also, recall for weak wave turbulence:

$$C\{N\} = \sum_{k'} |V|^2 N_{k'} N_k \tau_{Tc}$$

→ general structure of collision integral

$|V|^2$: c.c.

τ_{Tc}: (triad) coherence time → interaction

so $$\boxed{\frac{1}{\tau_{Tr}} \simeq \sum_{k'} |V|^2 N_{k'} \tau_{Tc}}$$

but $N_k \to z_\ell^2$ (intensity) (assumes local transfer)

$$|V|^2 \sim |\nabla^2| \sim \frac{1}{\ell^2} \to \underline{\frac{1}{\ell_\perp^2}}$$ $\underline{c.c.}$

$\tau_{Tc} \sim \tau_A$ → triad coherence

τ_A: Alfvenic packet transit time

$$1/\tau_{Tr} \sim \frac{z_\ell^2}{\ell^2} \tau_A \Leftrightarrow \text{W.T.T.}$$

fundamentally wave scattering process!

$\underline{q} = \underline{k} + \underline{k}'$ (cargo)

- 'triad' → 2 AWs + cell.

  → akin NLLD.

$\omega_q = \omega_{\underline{k}} + \omega_{\underline{k}'}$

$\rightarrow 0 = k_{\parallel} V_A - k'_{\parallel} V_A$

- Alfven waves non-dispersive, but coherence time controlled by packet propagation

→ prevents negative spectra. etc.
(Coherence time not so large)

- Strong B_0. (explicit)

Some simple observations:

- turbulence clearly anisotropic
- nonlinear transfer in $k_\perp$.

So, consider weak wave turbulence!

Consider: $k_\perp \rightarrow 1/l_\perp$

$k_\parallel$

$k_\perp > k_\parallel$ so

so $$\epsilon \sim \frac{(Z(l_\perp)^2)(Z(l_\perp))^2}{l_\perp^2\ |k_\parallel V_A|}$$

$\Delta\, k_\parallel V_A$ → coh. time

i.e. tacitly $\Delta |k_\parallel V_A| \sim |k_\parallel V_A|$

as for the stoch. field

$$Z(\ell_\perp)^2 \simeq (\epsilon |k_\parallel V_A|)^{1/2} \ell_\perp$$

$\Rightarrow$

$$E(k_\perp) \simeq (\epsilon\, |k_\parallel V_A|)^{1/2} / k_\perp^2$$ sharp

and accounting for DOS in $k_\parallel$: cnt. restricted

$$\boxed{E(k_\perp, k_\parallel) \sim [\epsilon\, V_A]^{1/2} / k_\perp^2\, k_\parallel^{1/2}}$$

weak wave turbulence spectrum

- strongly anisotropic
- $k_\parallel$ frozen.

Now, $Z(\ell_\perp) \sim \delta B(\ell_\perp) \sim (\epsilon |k_\parallel V_A|)^{1/4} \ell_\perp^{1/2}$

$\omega \sim k_\parallel V_A$

recall: $\nabla_\parallel = \partial_z + \frac{\delta B_\perp}{B_0} \cdot \nabla_\perp$

Kubo #

$$\boxed{k_\parallel \sim \frac{\delta B_\perp}{B_0} \nabla_\perp \Big/ \partial_z \sim \frac{l_{ac} \delta B_\perp / B_0}{\Delta_\perp}}$$

(see 235)

kicks in coherence length

Now, $\frac{\delta B_\perp}{\ell_\perp} \sim (\epsilon |k_\parallel V_A|)^{1/4} / \ell_\perp^{1/2}$

$$k_\parallel \sim \frac{1}{|\Delta k_\parallel|} \frac{(\epsilon\, |k_\parallel V_A|)^{1/2}}{\ell_\perp^{3/2}}$$

⇒ For W.T.T., expect:

$k_u < 1$ (diffusive picture)

but,

k_u rises as $l_\perp$ drops

i.e. $k_u \uparrow$ as progress thru $l_\perp$ cascade ∴ — ?? what happens

⇒ Begs the question:

How high can k_u go and still retain physics of Alfven wave cascade? | diffusive scattering

BBK comment '78

need Pac. Tr. enter the:

∴ Critical Balance Conjecture

— Goldreich – Sridhar (1995) (cf also Kadomtsev – Pogutse, 1978)

⇒ MHD inertial range in strong field will sit at $k_u \sim 1$

i.e. $\delta B_\perp \cdot V_\perp \sim B_0 V_\parallel$

$\frac{Z(l_\perp)}{l_\perp} \simeq k_\parallel V_A$

⋮

$k_\parallel$ vs $k_\perp$ reln.

i.e. transit time sets bound on cnt. strength

$\frac{\tau_A}{\tau_{eddy}} \to 1$, $\frac{\tau_{tr}}{\tau_{eddy}}$

but: $$\epsilon \sim \frac{(Z(\ell_\perp))^2 (Z(\ell_\perp))^2}{\ell_\perp^2} \frac{1}{k_\parallel V_A}$$

$$\rightarrow \frac{(Z(\ell_\perp))^2 (Z(\ell_\perp))^2}{\ell_\perp^2 \frac{Z(\ell_\perp)}{\ell_\perp}}$$

$k_\parallel \sim 1$

$$= \frac{Z(\ell_\perp)^3}{\ell_\perp}$$

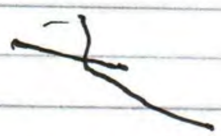
$$\Rightarrow Z(\ell_\perp) \sim (\epsilon \ell_\perp)^{1/3}$$

$$\boxed{E(k_\perp, k_\parallel) \sim \epsilon^{2/3} k_\perp^{-5/3}}$$

G-S spectrum

→ back to K41 but diff'nt physics

→ fits data



and anisotropy:

$k_\parallel \sim 1$, $B_0 \cdot \nabla \sim \tilde{B}_\perp \cdot \nabla_\perp$

$$B_0 k_\parallel \sim \epsilon^{1/3} \frac{\ell_\perp^{1/3}}{\ell_\perp}$$

$k_\parallel$ vs $k_\perp$ relation

$$\boxed{k_\parallel \sim \frac{\epsilon^{1/3}}{B_0} k_\perp^{+2/3}}$$

{ $k_\parallel$ does "cascade", though more slowly; specifies "GS cone" in $\underline{k}$ space

i.e. $k_\parallel \ll k_\perp$ → cascade develops preferentially in perp.

$$k_\parallel \sim k_\perp^{2/3}$$

– Why Believe

Is there
→ analogue of 4/5 Law (Pouquet, Politano)

$$-\frac{4}{3}\epsilon^T \ell = \langle(\delta v\cdot\delta v + \delta b\cdot\delta b)\delta v_\ell\rangle$$

↳ mom bal → form $\dot{E}_K$

$$-2\langle(\delta v\cdot\delta b)b_\ell\rangle$$

total energy cascade

induction → form $\dot{E}_M$

~ reflects flip-flop in energy between channels

or

$$-\frac{4}{3}\epsilon^{\pm}\ell = \langle(\delta z^{\pm}\cdot\delta z^{\pm})\delta z_\ell^{\mp}\rangle$$

no dissipn for 1 stream only !

→ Why? → relate induced E-field

MHD Turbulence II

→ Anisotropic Cascades and Critical Balance → A Closer Look.

→ Extending the 4/5 Law.

→ Selective Decay and Relaxation.

→ 2D MHD – A Study in Turbulent Relaxation.

i) Anisotropic Cascades and Critical Balance – A Closer Look.

Recall: I-K Phenomenology:

$$\epsilon \cong \frac{Z(\ell)^2}{\tau_{Tr}(\ell)} \qquad (\underline{B}_0 \text{ weak, but } B_{rms} \text{ generated})$$

$$1/\tau_{Tr}(\ell) \cong \frac{Z(\ell)^2\, \tau_A}{\ell^2} \qquad \tau_A \sim \frac{\ell}{b_0}$$

b_{rms}

$$\Rightarrow Z_\ell \sim (b_0 \epsilon)^{1/4}\, \ell^{1/4}$$

$$E(k) \sim \sqrt{\epsilon b_0}\; k^{-3/2}$$

and with strong $\underline{B}_0$:

$$\epsilon \sim \frac{Z(\ell_\perp)^2\, Z(\ell_\perp^2)}{\ell_\perp^2\, |k_\parallel V_A|}$$

so W.T.T. Alfvenic cascade:

$$E(k_\perp, k_\parallel) \sim (\epsilon V_A)^{1/2} / k_\parallel^{1/2} k_\perp^2$$ ~ "hard" in $k_\perp$!

However, note:

$$z(\ell_\perp) \sim \delta B(\ell_\perp) \sim (\epsilon / k_\parallel V_A)^{1/4} \ell_\perp^{1/2}$$

so

$$\delta B \cdot \nabla_\perp \sim \frac{1}{\ell_\perp^{1/2}} (\epsilon / k_\parallel V_A)^{1/4}$$

But recall:

— Alfven wave:

$$\omega \simeq k_\parallel V_A$$

derived from:

$$\partial_t A = B_0 \nabla_\parallel \phi + \dots$$

$$\partial_t \nabla^2 \phi = B_0 \nabla_\parallel J_\parallel + \dots$$

$$\nabla_\parallel = \underset{\text{Linear}}{\partial_z} + \underset{\text{Nonlinear}}{\frac{\delta B_\perp}{B_0} \cdot \nabla_\perp}$$

Ratio $\frac{\text{Nonlinear}}{\text{Linear}} = Ku = \frac{\frac{\delta B_\perp}{B_0} \cdot \nabla_\perp}{\partial z}$

$Ku \to$ Kubo #

$\Rightarrow$

$Ku \sim \frac{\delta B \, l_{ac}}{B_0 \, \Delta_\perp}$

$l_{ac} \to$ parallel autocorrelation length

$\Delta_\perp \to \perp$ correlation length.

see stochastic fields discussion of Phys 235 2016.

Point: $B_0 \partial_z C + \delta B_\perp \cdot \nabla_\perp C = 0$

$$\partial_z C + \frac{\delta B_\perp}{B_0} \cdot \nabla_\perp C = 0$$

$Ku < 1$ $\to$ C evolves by many kicks in $\Delta_\perp$

$\to$ diffusion

$\to$ in WTT wave interactions are diffusive in character.

$Ku > 1$ $\to$ C scattered $> \Delta_\perp$ in one step

$\to$ for transport in random media $\to$ percolation

Analogy $\partial_t C + \underline{v} \cdot \nabla C = 0$

$$Ku = \frac{\tilde{v}\,\tau_{ac}}{\Delta_\perp}$$

So we have a concern:

→ physics of [redacted] MHD turbulence understood in terms of Alfven wave interactions.

→ but scalings of WTT spectrum suggest that wave character lost as cascade progresses

i.e.
$$Ku \sim \frac{k_\parallel^{-1}}{\ell_\perp^{1/2}} \left[\epsilon \, |k_\parallel V_A| \right]^{1/4}$$

$$\uparrow \text{ as } \ell_\perp \downarrow$$

i.e. How high can Ku # go and still be consistent with physics of Alfven Wave Cascade

⇒ $\boxed{\text{Critical Balance Conjecture}}$

(GS '95, K+P '78)

$\Rightarrow$ $\boxed{\text{MHD inertial range in strong field will set at } ku \sim 1.}$

i.e. $\rightarrow \delta B_\perp \cdot \nabla_\perp \sim \dfrac{Z(\ell_\perp)}{\ell_\perp} \sim B_0 \nabla_\parallel$

so $\dfrac{Z(\ell_\perp)}{\ell_\perp} \simeq k_\parallel V_A$

$\rightarrow \dfrac{\tau_A}{\tau_{eddy}} \rightarrow 1 \quad \tau_{Tr} \rightarrow \tau_{eddy} \sim \tau_A.$

i.e. all timescales equalize

$\rightarrow ku \sim 1$ is maximum ku and still retain Alfvenic character.

$\rightarrow$ why?

Recall:

– WTT $\quad \tau_{T_c} \sim \tau_{ac\,wave}$ $\quad$ {Triad coherence set by wave dispersion

$$\rightarrow \pi\, \delta(\omega_{\underline{k}} - \omega_{\underline{k}''} - \omega_{\underline{k}'})$$

– STT – Renormalized Theory

$\tau_{Tc} \sim \tilde{\tau}_c$ $\quad$ {Triad coherence set by nonlinear scattering, itself.

$$\rightarrow 1 / \Delta\omega_{\underline{k}} + \Delta\omega_{\underline{k}'} + \Delta\omega_{\underline{k}''}$$

$$= \Theta_{\underline{k}, \underline{k}', \underline{k}''}$$

$\ast$ 6.

So, renormalized wave interaction theory ⇒

$$\Theta_{\underline{k},\underline{k}',\underline{k}''} = \frac{\Delta\omega_{\underline{k}} + \Delta\omega_{\underline{k}'} + \Delta\omega_{\underline{k}''}}{(\omega_{\underline{k}} - \omega_{\underline{k}''} - \omega_{\underline{k}'})^2 + (\Delta\omega_{\underline{k}} + \Delta\omega_{\underline{k}'} + \Delta\omega_{\underline{k}''})^2}$$

→ recovers both limits ✓

Now, $\Theta_{\underline{k},\underline{k}',\underline{k}''}$ clearly sets T_{tr}.

so, can re-write phenomenological transfer balance as:

$$\epsilon \sim \frac{1}{\ell_\perp^2}\frac{Z(\ell_\perp)^2 Z(\ell_\perp)^2}{T_c(\ell_\perp)}$$

$$1/T_c(\ell_\perp) = 1\Big/\left[\underbrace{(k_\parallel V_A)^2}_{①} + \underbrace{\left(\frac{Z(\ell_\perp)}{\ell_\perp}\right)^2}_{②}\right]^{1/2}$$

comparable at $k_\parallel \sim 1$

by analogy with $\Theta_{\underline{k},\underline{k}',\underline{k}''}$.

$(1) > (2) \rightarrow$ W.T.T.

$(1) \leq (2) \rightarrow$ S.T.T.

$$\epsilon \sim \frac{1}{l_\perp} \frac{Z(l_\perp)^2 Z(l_\perp)^2}{Z(l_\perp)/l_\perp}$$

$$\sim Z(l_\perp)^3 / l_\perp$$

and $\boxed{Z(l_\perp) \sim (\epsilon l_\perp)^{1/3}}$ — Back to K 41 !

Point: $\langle Z(k)^2 \rangle \sim \epsilon^{2/3} k_\perp^{-5/3}$ but different physics!

— G-S spectrum.

— softer than WTT.

— Great Power Law in sky "

— $\frac{Z(l_\perp)}{l_\perp}$ vs. $k_\parallel V_A$

$$\frac{(\epsilon l_\perp)^{1/3}}{l_\perp} \sim \frac{\epsilon^{1/3}}{l_\perp^{2/3}} \rightarrow \text{rate increases as } l_\perp \downarrow$$

$\rightarrow$ contrast constant $|k_\parallel V_A|$

— $\frac{Z(l_\perp)}{l_\perp} \sim \frac{\epsilon^{1/3}}{l_\perp^{2/3}} \rightarrow \frac{\delta B}{B_0} \cdot \nabla_\perp$

then $k_{ll} \sim 1$ $\Rightarrow$

$$\frac{\epsilon^{1/3}}{l_\perp^{2/3}} \sim k_{\parallel}$$

$$\Rightarrow \boxed{k_{\parallel} \sim \epsilon^{1/3} k_\perp^{2/3}}$$ — GS cone.

→ - Critical Balance is a hypothesis.

- Plausible answer to question of "how maintain Alfvenic cascade in state of strong (i.e. non-weak) turbulence?"

- anisotropy of spectrum supported by simulations (cf. Galtier).

BUT

- hypothesis, only

→ Computational support semi-quantitative.

→ 5/3 vs 3/2 etc. still ongoing.

→ A word about triads.

In wave turbulence cascade, must satisfy:

$$\underline{k} = \underline{p} + \underline{q}$$

$$\omega_{\underline{k}} = \omega_{\underline{p}} \pm \omega_{\underline{q}} \qquad \text{(WTT)}$$

→ resonance criterion

Conditions satisfied by:

$q_{\parallel} = 0$ (i.e. $\underline{q}$ is a cell, driven by beats)

so $k_{\parallel} = p_{\parallel}$

$$\underline{k}_{\perp} = \underline{p}_{\perp} + \underline{q}_{\perp}$$

and $\omega_k = \omega_p \pm \cancel{\emptyset}$

⇒ – deformation of Alfvenic wave packet directly related to its interaction with 2D part of wave packet travelling in opposite direction.

– interaction passive w/r $k_{\parallel}$
⇒ $\perp$ transfer in long time limit.

A.

Basics of Helicity

→ Magnetic Helicity → Constraint in Relaxation

– another conserved quantity in ideal MHD is magnetic helicity K

$$K = \int_V d^3x \; \underline{A} \cdot \underline{B}$$

{ V is taken to be the volume of a 'flux tube'.

– what, yet another invariant!?!

→ K is different ⇒ has topological interpretation

$$K = \int_V d^3x \; A \cdot \underline{\nabla} \times \underline{A}$$

→ $\underline{x} \to -\underline{x}$ flips sign of K

→ K is a pseudo-scalar

∴ has orientation or "handedness"...

Proceed via:

- show K conservation
- discuss interpretation of K
- comment on utility ⇒ Taylor Relaxation

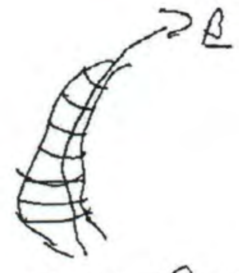
N.B.: Important → K is gauge invariant

i.e. if $\underline{A} \to \underline{A} + \underline{\nabla}\chi$

$$K \to K + \int_V d^3x \; \underline{\nabla}\chi \cdot \underline{B}$$

$$= K + \int_V d^3x \; \underline{\nabla} \cdot (\underline{B}\,\chi)$$

$= 0$, to surface term. $\left\{ \underline{B}\cdot\hat{n} = 0 \right.$ on surface of tube



Now, consider a blob of MHD fluid in motion

can show $\dfrac{dK}{dt} =$

$$\underline{E} + \frac{\underline{V}\times\underline{B}}{c} = \eta \underline{J}$$

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi$$

$\left\{ \right.$ Field in blob

$\Rightarrow$

$$\frac{\partial \underline{A}}{\partial t} = \underline{V}\times\underline{\nabla}\times\underline{A} - c\nabla\phi - c\eta\underline{J}$$

$$\frac{\partial \underline{B}}{\partial t} = -\underline{V}\cdot\underline{\nabla}\,\underline{B} + \underline{B}\cdot\underline{\nabla}\,\underline{V} - \underline{B}\,\underline{\nabla}\cdot\underline{V} + \eta\,\nabla^2\underline{B}$$

$$\frac{dK}{dt} = \frac{d}{dt}\int_V d^3x \; (\underline{A}\cdot\underline{B})$$

$$= \int d^3x \left(\frac{d\underline{A}}{dt}\cdot\underline{B} + \underline{A}\cdot\frac{d\underline{B}}{dt} \right) + \int \underline{A}\cdot\underline{B} \; \frac{d}{dt}\, d^3x$$

$$\frac{dK}{dt} = \int d^3x \left(\frac{\partial \underline{A}}{\partial t}\cdot\underline{B} + (\underline{V}\cdot\nabla\underline{A})\cdot\underline{B} + \underline{A}\cdot\frac{\partial\underline{B}}{\partial t} + \underline{A}\cdot(\underline{V}\cdot\nabla\underline{B}) \right) + \underline{A}\cdot\underline{B}\;\nabla\cdot\underline{V} \Big)$$

where $\frac{d}{dt}\, d^3x = \nabla\cdot\underline{V}$

i.e. $\frac{d}{dt}\, dV = \frac{d}{dt}\, d\underline{\sigma}\cdot d\underline{\ell} + d\sigma\cdot\frac{d}{dt}\, d\underline{\ell}$

$$= -d\underline{\ell}\cdot\nabla\underline{V}\cdot d\underline{\sigma} + (\nabla\cdot\underline{V})(d\underline{\sigma}\cdot d\underline{\ell}) + d\underline{\ell}\cdot\nabla\underline{V}\cdot d\underline{\sigma}$$

$$= \nabla\cdot\underline{V}\; d^3x$$

s.t. and $\underline{B}\cdot\vec{n}$ on surface of tube.

$$\frac{dK}{dt} = \int d^3x \Big[\big(\underline{B}\cdot\cancel{\underline{V}}\times\underline{B} - c\,\cancel{\underline{B}}\cdot\nabla\phi - c\eta\,\underline{J}\cdot\underline{B} \big) + \underline{A}\cdot\big(\nabla\times(\underline{V}\times\underline{B})\big) + \nabla\cdot\big((\underline{A}\cdot\underline{B})\underline{V}\big) + \underline{A}\cdot\eta\nabla^2\underline{B} \Big]$$

div flux

where $\underline{A}\cdot(\underline{V}\cdot\nabla\underline{B}) + \underline{B}\cdot(\underline{V}\cdot\nabla\underline{A}) + \underline{A}\cdot\underline{B}\;\nabla\cdot\underline{V} = \nabla\cdot(\underline{V}\,\underline{A}\cdot\underline{B})$

$$\therefore \frac{dK}{dt} = \int d^3x \Big[\nabla\cdot\big((\underline{A}\cdot\underline{B})\underline{V}\big) + \nabla\cdot\big((\underline{V}\times\underline{B})\times\underline{A}\big) + (\underline{V}\times\cancel{\underline{B}})\cdot(\nabla\times\underline{A}) - c\eta\,\underline{J}\cdot\underline{B} - \eta(\underline{A}\cdot\nabla\times\underline{J})\,c \Big]$$

$$\Rightarrow \frac{dK}{dt} = \int d^3x \left\{ \nabla\cdot\left[(A\cdot B)V + (V\times B)\times A + c\eta\,(A\times J)\right] - c\eta\, J\cdot B - c\eta\, J\cdot B \right\}$$

$$= \int dS\cdot\left[(A\cdot B)V + (V\times B)\times A + c\eta\, A\times J\right]$$

$$- 2\int d^3x\left[c\eta\, J\cdot B\right]$$

$$= \int dS\cdot\left[\cancel{(A\cdot B)V} - \cancel{(A\cdot B)V} + \cancel{(A\cdot V)B}\right] - c\eta\int dS\cdot J\times A$$

$B\cdot n = 0$, on tube

$$- 2c\eta\int d^3x\,(J\cdot B)$$

$$= -\int c\eta\, dS\cdot\left[\nabla B\cdot A - \cancel{A\cdot\nabla B}\right] - 2c\eta\int d^3x\, J\cdot B$$

$$= -2c\eta\int d^3x\,(J\cdot B)$$

$\Rightarrow$ have shown:

$$\boxed{\frac{dK}{dt} = -2c\eta\int d^3x\,(J\cdot B)}$$

*

and clearly! $\frac{dK}{dt} \to 0$ as $\eta \to 0$ (non-singular for $\underline{J}$)

∴ Magnetic Helicity is conserved in ideal MHD. Can assign helicity to each flux tube.

→ Magnetic Helicity conserved, but what does it mean?

- helicity is non-trivial ⇒ more than just helical field lines.

interesting to note: $q(r) = \frac{rB_z}{RB_\theta(r)} = \frac{1}{R\mu(r)}$

$\mu(r) = \frac{B_\theta(r)}{rB_z}$ → field line pitch (length scale ~~over which winding varies~~)

cylindrical plasma ⇒ $\underline{B} = \underline{B}(r)$ $B_r = 0$.

Now, $A_\theta = \frac{1}{r}\int_0^r r B_z \, dr$

$A_z = -\int_0^r B_\theta \, dr$

so $\underline{A}\cdot\underline{B} = \frac{B_\theta}{r}\int_0^r r B_z\,dr - B_z\int_0^r B_\theta\,dr$

$$= \mu B_z \int_0^r \frac{B_\theta}{\mu}\,dr - B_z\int_0^r B_\theta\,dr$$

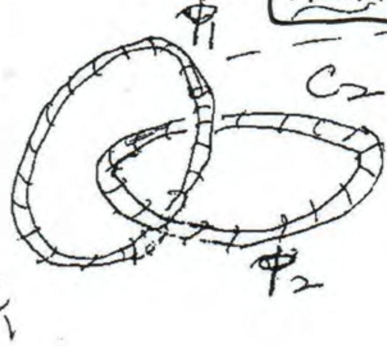
$$\underline{A}\cdot\underline{B} = B_z\left[\mu\int_0^r \frac{B_\theta}{\mu}\,dr - \int_0^r B_\theta\,dr\right]$$

$= 0$ for constant μ

∴ non-zero helicity requires $\mu = \mu(r)$
i.e. – pitch varies with radius
⇒ $\boxed{\text{magnetic shear}}$ / twist

– physically → helicity means self-linkage of 2 flux tubes

i.e.



tube 1: flux
$\Phi = \int d\underline{A}\cdot\underline{B} = \phi_1$
x-section area; const

tube 2: $\Phi = \phi_2$

field in loops, only

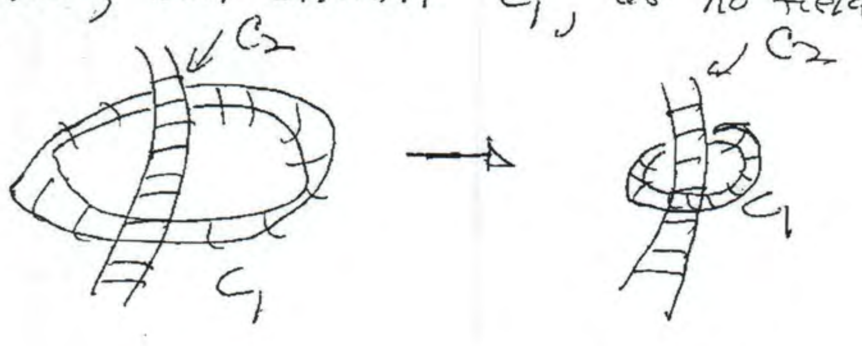
Now, for volume V_1 of tube 1

$$K = \int_{V_1} \underline{A}\cdot\underline{B}\, d^3x = \underbrace{\oint_{C_1} d\ell}_{\text{along loop}} \underbrace{\int_{A_1} \underline{dS}}_{\text{x-sect area}}\, A\cdot\underline{B}$$

$$= \oint_{C_1} \underline{A}\cdot d\underline{\ell} \int_{S_1} \underline{B}\cdot\hat{n}\, dA$$

$$= \phi_1 \oint_{C_1} \underline{A}\cdot d\underline{\ell}$$

Now, can shrink C_1, as no field outside loops



re-oriented

$\rightarrow$ in x section: 

but $\int_{C_1} A\cdot d\ell = \int_{A\ \text{enclosed}} B\cdot dS = \phi_2$

→ dynamics? – <u>how</u> does relaxation occur

→ more in discussion of kinks, tearing.

so, ... $K_1 = \Phi_1 \Phi_2$ → product of fluxes

similarly $K_2 = \Phi_2 \Phi_1$

$\therefore \; K = 2\Phi_1 \Phi_2$

if n windings $K = K_1 + K_2 = \pm 2n\Phi_1 \Phi_2$!

⟹ helicity is measure of self-linkage of magnetic configuration.

Topological complexity.

Scale?

Why care → Taylor Conjecture (1974) (J. B. Taylor)

- in magnetic confinement, of great interest to determine how fields, currents self-organize

- RFP

⇝ toroid
⇝ toroidal current

well fit by
$$B_z = B_0 J_0(\alpha r)$$
$$B_\theta = B_0 J_1(\alpha r)$$

$\underline{J} \times \underline{B} = 0$ ⟹ force free

⟹ why so robust?, especially since RFP so turbulent

- Taylor conjectured conservation of magnetic helicity <u>constrains</u> relaxation to force-free state.

Key Point – helicity conserved in flux tubes, to η
– toroidal plasma ⟹ <u>many</u> small tubes

etc.

– recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

⟹ Taylor Conjecture: At finite η, helicity of small tubes dissipated <u>but</u>, <u>global</u> helicity conserved.

<u>i.e.</u>

$$\int_{\text{plasma volume}} \underline{A} \cdot \underline{B} \, d^3x = K_0 \rightarrow \text{conserved}.$$

∴ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

$$\delta\left[\int_V d^3x\, \frac{B^3}{8\pi} + \lambda \int_V d^3x\, \underline{A}\cdot\underline{B}\right] = 0$$

i.e. minimize magnetic energy subject to constraint of conserved global helicity.

Comments:

→ it works - indeed amazingly well - for RFPs, spheromaks, etc. Departures only recently being discovered.

→ inspired idea of helicity injection as way to maintain configurations

→ it is a conjecture → no proof.

Hypothesis: Selective Decay → energy cascades → small scale; helicity cascades → large scale (less dissipation)

Dual Cascade

- relevance to driven system? i.e. in real RFP, transformer on. $\tau_R \sim L^{3/2}$

- Taylor conjectured conservation of magnetic helicity constrains relaxation to force-free state.

Key Point - helicity conserved in flux tubes, to η
- toroidal plasma $\leftrightarrow$ many small tubes

$\underline{\tau_R \sim L^{3/2}}$

etc. $\frac{V}{L} \sim \frac{V_A}{L}/\sqrt{R_M} \sim 1/L^{3/2}$

- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite η, helicity of small tubes dissipated but, global helicity conserved.

i.e.

$$\underbrace{\int}_{\text{plasma volume}} \underline{A} \cdot \underline{B} \, d^3x = K_0 \rightarrow \text{(const.) conserved.}$$

$\therefore$ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

$$\delta\left[\int_V d^3x\,\frac{B^2}{8\pi} + \lambda\int_V d^3x\,\underline{A}\cdot\underline{B}\right] = 0$$

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→ more in discussion of kinks, tearing..

$$\delta\left[\int d^3x \left[\frac{B^2}{8\pi} + \lambda \underline{A}\cdot\underline{B}\right]\right] =$$

$$\frac{\underline{B}}{4\pi}\cdot\delta\underline{B} + \lambda \underline{A}\cdot\delta\underline{B} = 0$$

$$\frac{\nabla\times\underline{A}}{4\pi} + \lambda\underline{A} = 0$$

$\nabla\times$

$$\underline{J} = \mu\underline{B} \qquad \nabla\times\underline{B} = \mu\underline{B}$$

force free

and

$$\frac{\underline{J}\cdot\underline{B}}{B^2} = \mu$$

const

$\nabla J_{\parallel} = 0$ → parallel current homogenized