Intro:

→ So far:

~ static, stochastic field prescribed

~ resonance at $\cancel{\omega} - \left(\frac{k \cdot B_0}{|B_0|}\right) v_{\parallel} = 0$

$k_{\parallel} = 0$

$k_{\parallel} v_{\parallel} = 0$

→ [illegible] l_{ac}

l_c etc.

$\frac{1}{l_c} \sim \begin{cases} k_r^2 D_M \\ \left(\frac{k_0^2 D_M}{L_s^2}\right)^{1/3} \end{cases}$

$Ku = \frac{l_{ac}}{l_c} \longleftrightarrow \underline{B \cdot \nabla}$

$\tilde{B} \cdot \nabla_\perp$ vs $B_0 \nabla_\parallel$

$V_A \frac{\tilde{B}}{B_0} \cdot \nabla_\perp \sim \tilde{V} \cdot \nabla$

in MHDT – next

→ Scaling $B \cdot \nabla$, Ku → Critical Balance

→ ideas l_{ac} → resonances.

→ Diffusive scattering/decorrelation.

all relevant.

see:
Galtier
(Kulsrud)
Frisch

MHD Turbulence I

i.) Key Features of Hydro. Turbulence

ii.) Basic Facts re: MHD Turbulence

iii.) Phenomenology a la' Kraichnan / Iroshnikov

iv.) Critical Balance and Phenomenology a la' Goldreich-[redacted] Sridhar

v.) 4/5 Law Analogue + Further.

i.) Key Features of Hydro. Turbulence (3D)
(Heavily based on experiment — C_D vs Re)

- chaotic state at high Re
  * characterized by broad self-similar range and nonlinear transfer (flux ink).

- dissipates energy; dissipation rate indep. Re, as $Re \to \infty$ (but $\nu \neq 0$).

  i.e. $\langle \tilde{f}\,\tilde{v} \rangle = \epsilon = \nu \langle (\nabla v)^2 \rangle$
  $= \nu \langle \omega^2 \rangle$

  $\Rightarrow$ singularity formation $l_\nu \sim 1/\sqrt{\nu}$

- $l_d < l < l_0$

  $\epsilon = \dfrac{v(l)}{\tau(l)} = \dfrac{v(l)^3}{l}$, etc.

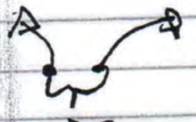
  $\dfrac{1}{\tau(l)} \to$ scale similarity constrained by Galilean invariance

  $l_d = \nu^{3/4} / \epsilon^{1/4}$

  $\Rightarrow E(k) \cong \epsilon^{2/3} k^{-5/3}$

  "turbulent flow is rough"

- alternatively



$\delta \ell$

$$\frac{d\,\delta \ell}{dt} = v(\delta \ell)$$

$$\delta \ell^2 \sim \epsilon^2 t^3 \quad \Rightarrow \text{Super-diffusive separation}$$

- Some Things You Can Trust:

a.) Karman - Howarth Eqn. (Modulo usual)

$$\partial_t \left\langle \frac{u_i u_i'}{2} \right\rangle = \frac{1}{4} \nabla_{\ell} \cdot \langle |\underbrace{\delta u}_{u(\underline{x}+\underline{\ell}) - u(\underline{x})} \cdot \delta u|\, \delta u \rangle$$

$$+ 2\nu\, \partial^2_{\ell_n, \ell_n} \left\langle \frac{u_i u_i'}{2} \right\rangle$$

$u_i = \underline{u}(\underline{x})$

$u_i' = \underline{u}(\underline{x}')$

$\underline{x}' = \underline{x} + \underline{\ell}$

with external Forcing:

$$\partial_t \left\langle \frac{u_i u_i'}{2} \right\rangle = \frac{1}{4} \nabla_{\ell} \cdot \langle (\delta u_i)^2 \delta u \rangle$$

$$+ 2\nu\, \partial^2_{\ell_n, \ell_n} \left\langle \frac{u_i u_i'}{2} \right\rangle + \mathcal{F}$$

$\Rightarrow$ scale energetics balance

b.) 4/5 Law

Balancing external stirring with energy dissipation (must for steady state):

$$\partial_t \left\langle \cancel{\frac{u_i u_i'}{2}} \right\rangle = \frac{1}{4} \nabla_\ell \cdot \langle (\delta u_i)^2 \delta \underline{u} \rangle$$

O (SS)

sole rigorous result.

$$+ 2\nu \, \partial_{\ell_k} \partial_{\ell_k} \left\langle \frac{u_i u_i'}{2} \right\rangle + \epsilon$$

i.r.

in inertial range:

$$\langle (\delta u_\ell)^3 \rangle = -\frac{4}{5} \epsilon \ell \qquad - \Rightarrow \text{decay}$$

i.e. energy transfer at scale ℓ (arbitrary in inertial range)

$$\boxed{\frac{\langle (\delta u_\ell)^3 \rangle}{\ell} = -\frac{4}{5} \epsilon}$$

proportional (with 4/5 factor) to dissipation rate.

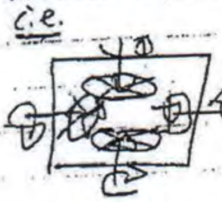
Background - turbulence

192.

Turbulence Theory

— An Introduction

P. Diamond

I.) Basics of Fluid Turbulence (3D) 

.) Characteristics of Fluid Turbulence: "turbulence" vs "noise" → $\boxed{\text{energy flux}}$

- broad range of spatio-temporal scales excited — contrast TAM

Ref. U. Frisch "Turbulence - The legacy of A. N. Kolmogorov"

- decay of large scale energy → need input/stirring to maintain stationarity

- energy input dissipated as heat (to maintain stationarity) → viscosity → irreversibility

- irreversible mixing occurs → i.e. passive tracer

- intermittancy manifested
  - i.e. spatial → coherent structures (i.e. vortices) occur
  - temporal → bursts — probe trace

- self-similarity/scale-similarity:

  turbulence "looks the same" on all scales, except the very largest (stirring) and the very smallest (dissipation)

  Caveat: Intermittancy - memory of large scales on small

iii) Navier Stokes Equation – {Describes Fluid

$$\frac{\partial \underline{V}}{\partial t} + \underbrace{\underline{V}\cdot\underline{\nabla}\,\underline{V}}_{\substack{\text{advection/}\\ \text{straining}\\ \rightarrow \text{nonlinearity}}} = \underbrace{-\frac{\underline{\nabla} P}{\rho}}_{\text{pressure}} + \underbrace{\nu \nabla^2 \underline{V}}_{\substack{\text{viscous}\\ \text{diffusion of}\\ \text{momentum}}}$$

$\rho = 1$ hereafter

$\underline{\nabla}\cdot\underline{V} = 0$ incompressibility

Note: Pressure determined from incompressibility

i.e.

$$\nabla\cdot\left[\cancel{\frac{\partial \underline{V}}{\partial t}} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right] = -\nabla^2 P + \nu\nabla^2\cancel{(\underline{\nabla}\cdot\underline{V})}$$

$$\nabla^2 P = -\nabla[\underline{V}\cdot\underline{\nabla}\,\underline{V}]$$

$$P = -\nabla^{-2}[\nabla\cdot\underline{V}\cdot\underline{\nabla}\,\underline{V}]$$

$$= \frac{1}{4\pi}\int \frac{d^3x'\,\underline{\nabla}'_{x'}[\underline{V}(x')\cdot\nabla_{x'}\underline{V}(x')]}{|\underline{x}-\underline{x}'|}$$

More generally, can eliminate P

$$\partial_t V_i + (\delta_{i\ell} - \partial_{i\ell}\nabla^{-2})\,\partial_j(V_j V_\ell) = \nu\nabla^2 V_i$$

Key Parameter: Reynolds #

$$\boxed{Re = |\underline{V}\cdot\underline{\nabla}\underline{V}| / |\nu\nabla^2 \underline{V}| \sim \frac{V(L)L}{\nu}}$$

$\sim \frac{\text{nonlinearity}}{\text{collisional diffusion}}$

measure of strength of NL.

- Re usually referenced to largest scale

$L = L_{max}$
$V(L)$ = large scale velocity

- Re always referenced to a particular scale

L_{max}, $\lambda = \left[\langle(\partial_i V_j)^2\rangle / \langle V_i^2\rangle\right]^{-1/2}$, $\lambda_{dissipn.}$
(Taylor Scale) $(Re=1)$

- $Re \gg 1$ in turbulent $\begin{cases}\text{pipe flow} \\ \text{atmosphere} \\ \text{etc.}\end{cases}$

$Re \sim 10^6 - 10^8$, etc.

i.e. planetary boundary layer: $l_{out} \sim 1\,km \sim 10^5\,cm$
$l_{diss} \sim .1\,cm$

$\Rightarrow$ 6 decades!

- Re: measure of ratio of inertial mixing of momentum to collisional mixing...

④

iii) Experimental 'Laws' of Fully Developed Turbulence

⇒ Much / most of turbulence theory is empirically motivated. Experimental info / results preceeded sophisticated theoretical analyses....

The experimental facts:

1.) 2/3 Law (Mundane)

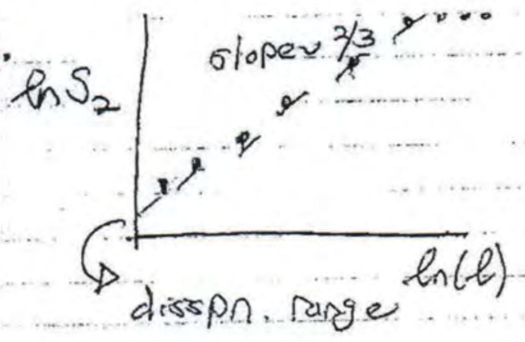
In a turbulent flow with $Re \gg 1$, $\langle \delta v(\ell)^2 \rangle$ (mean square velocity increment between two scales separated by distance ℓ) scales as $\ell^{2/3}$.

i.e. $\delta v(\ell) = |\underline{v}(\underline{r}+\underline{\ell}) - \underline{v}(\underline{r})|$ ⇒ {a difference!

$S_2(\ell) = \langle \delta v(\ell)^2 \rangle \sim \ell^{2/3}$ → fundamental scaling relation

2nd order structure function

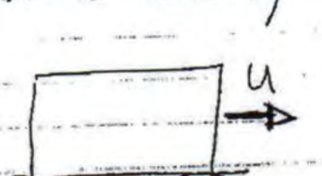
i.e.



B.) Law of Finite Energy Dissipation (Profound)

If, in an experiment on turbulent flow, all the control parameters are kept the same, except the viscosity, which is lowered as much a possible, the energy dissipation per unit mass dE/dt behaves in a way consistent with a finite limit.

- What means 'Energy Dissipation Rate'?



Consider a car, experiencing atmospheric drag

$F_d = \frac{1}{2} C_D \rho \underset{\text{face surface area}}{S} U^2$

i.e.

$$P = \rho S(U\tau)U \quad \rightarrow \text{momentum in air of slug:}$$

$M = \rho S U \tau \rightarrow$ mass

$V = U$

if assume air momentum completely transferred to car

$$\frac{dP_{car}}{dt} = F_d = \rho S U^2$$

$\frac{C_D}{2}(Re) \equiv$ drag coefficient (slowly varying function of Re, depends on shape, etc.)

$\therefore \; F_d = \frac{C_D}{2} \rho S U^2$

Now, power dissipated by drag force

$$P_d = F_d U$$

$$\Rightarrow P_d = \frac{C_D}{2} \rho S U^3$$

Energy dissipation rate (per volume) $\epsilon = P_d / \text{Mass}$

$$= \frac{C_D}{2} \frac{U^3}{L}$$

$$\sim U^3/L$$

also $NS \Rightarrow \partial_t \langle v^2 \rangle \sim -\langle \underline{v} \cdot \underline{\nabla} \, \underline{v}^2 \rangle$

$\Rightarrow$ Why should we care?

Note, energy budget:

$$\frac{\partial}{\partial t} V_i + V_j \partial_j V_i - \nu \nabla^2 V_i = -\partial_i P$$

$$\partial_t \frac{V_i^2}{2} + \partial_j \frac{V_j V_i^2}{2} - \nu V_i \nabla^2 V_i = -V_i \partial_i p$$

$\langle\ \rangle \equiv$ ensemble (fast space-time avg.)

$$\partial_t \left\langle \frac{V_i^2}{2} \right\rangle + \left\langle \partial_j \frac{V_j V_i^2}{2} \right\rangle - \nu \langle V_i \nabla^2 V_i \rangle$$

surface terms $= \langle \partial_i V_i p \rangle$ upon IBP

$$\Rightarrow \quad \partial_t \left\langle \frac{V_i^2}{2} \right\rangle = -\nu \langle |\nabla V_i|^2 \rangle$$

but $\epsilon \equiv -\partial_t \left\langle \frac{V_i^2}{2} \right\rangle$! (- dissp'n rate)

$$\therefore \quad \boxed{\epsilon = \nu \langle |\nabla V_i|^2 \rangle} \ !$$

→ experiments suggest that $\epsilon \to$ finite as $\nu \to 0$!!! ↔ re-markable

↔ suggests that extremely large ∇V forms as $\nu \to 0$, ∴ singular vortex sheets

⇒ singular velocity gradients formed in limit of weak viscosity !!

{ Heart of turbulence problem is grappling with singularity (especially its degree) of velocity gradients.

N.B. : $\left\{\begin{array}{l}\text{Dissipation Law} \\ \text{Singularity formation is at the heart of why turbulence is a "hard" problem.}\end{array}\right.$

Re: Dissipation Law:

$$\epsilon \approx \frac{U^3}{L} \sim U^2/(L/U)$$

$$\sim \frac{\text{K.E. per Mass}}{\text{Circulation Time}}$$

i.e. → in 1 macro circulation time, a finite fraction of (macro) kinetic energy is dissipated by viscosity.

→ dissipation time scale is (L/U).

V.) Kolmogorov's Hypotheses and their Predictions / Implications. → K41 Theory of Turbulence

i1 : $\left\{\begin{array}{l}\text{In the limit of } Re \to \infty \text{, all possible symmetries of the Navier-Stokes} \\ \text{equation, usually broken by the mechanisms producing the turbulent} \\ \text{flow, are restored in a statistical sense at small scales and away from} \\ \text{boundarys.}\end{array}\right.$

lose memory

What means?

- "small scales": $\ell \ll \ell_0$

ℓ_0: integral scale $\rightarrow$ characteristic of production.

- symmetries

First, symmetries of Navier-Stokes Eqn.!

a.) space translations $\quad \underline{r} \to \underline{r} + \underline{\rho}$
(no explicit $\underline{r}$ dep.)

b.) time translation $\quad t \to t + \tau$
(no t dep.)

* c.) Galilean boosts $\quad \begin{cases} \underline{r} \to \underline{r} - \underline{u}t \\ \underline{v} \to \underline{v} + \underline{u} \end{cases} \quad \underline{v} = \underline{u} + \underline{v}(\underline{r} - \underline{u}t, t)$
(no frame dep.)

i.e. $\quad \dfrac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\nabla p + \nu \nabla^2 \underline{v}$

insert $\Rightarrow$

$$\cancel{-\underline{u} \cdot \nabla \underline{v}} + \cancel{\underline{u} \cdot \nabla \underline{v}} + \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\nabla p + \nu \nabla^2 \underline{v}$$

d.) Parity $\quad$ left-right $\quad \underline{x} \to -\underline{x}, \ \underline{v} \to -\underline{v}$
(no preferred direction)

e.) Rotation $\quad \begin{cases} \underline{r} \to R\,\underline{r} \\ \underline{v} \to R\,\underline{v} \end{cases}$
(No preferred direction)

Ingredients in K41 Phenomenology:

→ ℓ : scale paramer : eddy scale

→ $V(\ell)$: $\quad \tilde{V}(\ell) \sim \left(\langle \delta V_{\parallel}(\ell)^2 \rangle \right)^{1/2}$

↓
eddy velocity

$$\delta V_{\parallel} \sim \left(\underline{V}(\underline{r}+\underline{\ell}) - V(\underline{r}) \right) \cdot \frac{\underline{\ell}}{\ell}$$

≡ longitudinal velocity increment

→ V_0 : rms velocity fluctuation (large scale dominates)

$$V(\ell_0) \sim V_0$$

→ $\tau(\ell)$: eddy lifetime / turn-over rate
↓
characteristic rate of transfer thru scale ℓ.

self-similarity : energy thru-put rate is scale invariant

⇒

$$\underset{\substack{\downarrow \\ \text{dissipation} \\ \text{rate}}}{\epsilon} = \underset{\substack{\text{scale } \ell \\ \text{energy}}}{V(\ell)^2} / \underset{\substack{\text{scale } \ell \\ \text{life-time}}}{\tau(\ell)}$$

energy balance / thruput eqn.

, ow, $\tau(\ell)$?

Compare $\tau(\ell)$ with $T_{eu}(\ell)$

$\tau(\ell)$ → 'lifetime' of structure of scale ℓ

→ i.e. time for structure to be 'distorted out of existence'

scales $\ell' \gg \ell$:

→ advect eddy ↔ apply Galilean boost, but don't affect life-time.

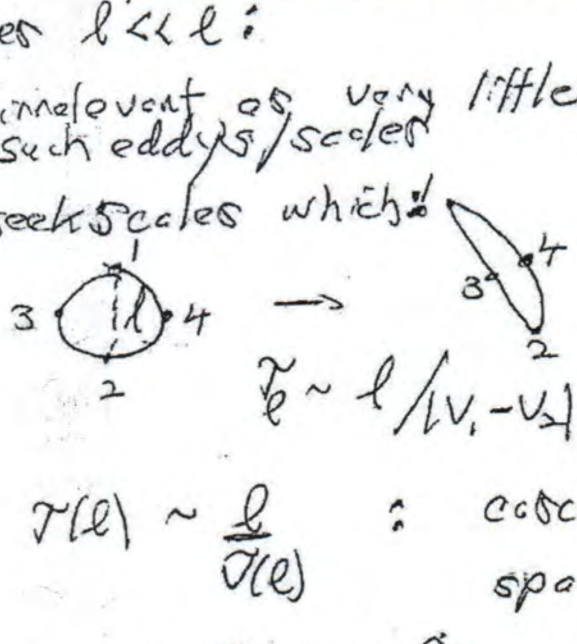
irrelevant/i → symmetry under random Galilean transformations

→ would also violate symmetry restoration.

scales $\ell' \ll \ell$:

→ irrelevant as very little energy/shear in such eddys/scales

∴ seek scales which:



$\tilde{\tau}_\ell \sim \ell / |v_1 - v_2|$

→ $\tau(\ell) \sim \dfrac{\ell}{v(\ell)}$: cascade local in scale space.

not

$$\epsilon = \frac{v(\ell)^3}{\ell}$$

$$\Rightarrow v(\ell) \sim (\epsilon \ell)^{1/3}$$: K41 scaling relation

$$\boxed{v(\ell)^2 \sim \epsilon^{2/3} \ell^{2/3}}$$

– "verifies" 2/3 Law

– for spectrum:

if $E(k) = |v(k)|^2$

s/t $E = \int dk\, E(k)$ {i.e. absorbs density of states

then $v(\ell) = \int_{k_{\ell_{n-1}}}^{k_{\ell_n}} dk\, E(k)$

$$v(\ell)^2 \sim \epsilon^{2/3} \ell^{2/3} = \epsilon^{2/3} k_\ell^{-2/3}$$

$$\Rightarrow \boxed{E(k) = \epsilon^{2/3} k^{-5/3}}$$: Kolmogorov Spectrum

at ℓ_0:

$$v_0 \sim \epsilon^{1/3} \ell_0 \Rightarrow \frac{v_0^3}{\ell_0} = \epsilon.$$

For dissipation scale:

ℓ_d occurs in ℓ-space when cascade terminates
i.e. viscosity asserts itself $\rightarrow Re(\ell) \rightarrow 1$

$$1/\tau(\ell) \sim 1/\tau_d = \nu/\ell^2$$

$$\Rightarrow \quad \epsilon^{1/3}\ell^{-2/3} = \frac{\nu}{\ell^2}$$

$$\ell^{4/3} = \nu/\epsilon^{1/3} \quad \Rightarrow \boxed{\ell_d = \nu^{3/4}/\epsilon^{1/4}}$$

$\boxed{\ell_d \equiv \eta \text{, in Frisch}}$

Recall: $\quad \epsilon = \nu\langle(\nabla\tilde{v})^2\rangle$

$\Rightarrow \nu \rightarrow 0 \Rightarrow \langle(\nabla\tilde{v})^2\rangle$ divergent

$$\langle(\nabla\tilde{v})^2\rangle = \int_{k_0}^{k_d} dk\, k^2 \epsilon^{2/3} k^{-5/3}$$

$$= \int_{k_0}^{k_d} dk\, k^{1/3} \epsilon^{2/3}$$

$$= k_d^{4/3} \epsilon^{2/3}$$

$$= \frac{\epsilon}{\nu}^{1/3} \epsilon^{2/3} = \epsilon/\nu$$

$\rightarrow \langle(\nabla\tilde{v})^2\rangle$ divergent as $\nu \rightarrow 0$.

› Counting Degrees of Freedom

How big is the inertial range:

$$\underset{\#\,l's}{n} \sim \frac{l_0}{l_d} \sim \frac{l_0}{(\nu^3/\epsilon)^{1/4}}$$

$$\sim \frac{l_0}{\nu^{3/4}}\left(V_0^3/l_0\right)^{1/4} \sim \left(\frac{V_0 l_0}{\nu}\right)^{3/4}$$

$$\sim Re^{3/4}$$

∴ number of degrees of freedom for 3D turbulence is;

$\boxed{N \sim Re^{9/4}}$: would be (minimum) # grid points to resolve range of scales in numerical simulation

Now, i.e. atmospheric boundary layer:

$l_0 \sim 1\,Km$

$l_d \sim 1\,mm$

$$n \sim 10^6 \Rightarrow \begin{cases} N \sim 10^{18} \\ Re \sim 10^8 - 10^9 \end{cases} !$$

⇒ subgrid scale modelling ...

N.B.: Sometimes able to exploit reduced degrees of freedom models, i.e. when some class of scales slaved to others.

b.) MHD (3D, Incompressible) $B_0 = 0$

B_0 strong.

① — often have strong, slowly varying external magnetic field

so $\rho(\partial_t \underline{V} + \underline{V}\cdot\underline{\nabla}\underline{V}) = -\underline{\nabla} p^* + \frac{\underline{B}\cdot\underline{\nabla}\underline{B}}{4\pi} + \nu\nabla^2\underline{V} + \tilde{f}$

$\partial_t \underline{B} + \underline{V}\cdot\underline{\nabla}\underline{B} = \underline{B}\cdot\underline{\nabla}\underline{V} + \eta\nabla^2\underline{B}$ → general → need not $\underline{B_0}$

$\underline{\nabla}\cdot\underline{B} = 0$

Now, here can (for $\underline{B_0}$) use reduced MHD:

$$\rho(\partial_t \nabla_\perp^2\phi + \underline{V}_\perp\cdot\underline{\nabla}_\perp\nabla^2\phi) = B_0\partial_z\nabla^2 A + \underline{\nabla}_\perp A\times\hat{z}\cdot\underline{\nabla}_\perp\nabla^2 A + \tilde{f} + \nu\nabla^2\nabla^2\phi$$

⊥ → emphasize

$$\partial_t A + \underline{V}_\perp\cdot\underline{\nabla}_\perp A = B_0\partial_z\phi + \eta\nabla^2 A$$

→ anisotropic turbulence, if B_0 strong. (2D + SAW in z)

→ wave component i.e. MHD turbulence is gas of waves (Alfvenic)

⇒ expect use methodology of wave turbulence.

$\underline{B_0}$ strong → wave turbulence.

* (A) → nonlinear transfer in $\perp$, i.e. $k_\perp$ (3D turbulence has coupling and 2D)

→ what controls triad coherence?

⇒ Alfvenic transit (over scale)

From HW:

(2) Nonlinear interaction exclusively via counter-propagating Elsasser populations

i.e. $\underline{Z}^{\pm} = \underline{v} \pm \underline{b}$ $\quad$ $\underline{b} \equiv$ pert. $\tilde{B}$ (as previous)

2 MHD eqns ⇒

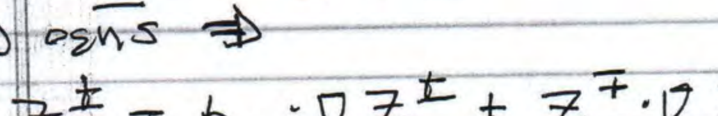
$$\partial_t \underline{Z}^{\pm} \mp \underline{b}_0 \cdot \underline{\nabla} \underline{Z}^{\pm} + \underline{Z}^{\mp} \cdot \nabla \underline{Z}^{\pm} = -\nabla p^* + \nu_+ \nabla^2 \underline{Z}^{\pm} + \nu_- \nabla^2 \underline{Z}^{\mp}$$

$$\nu_\pm = \frac{\nu \pm \eta}{2}$$

so NL coupling exclusively via $\underline{Z}^{\mp} \cdot \underline{\nabla} \underline{Z}^{\pm}$

interaction diff'nt populations

so NL ~~transfer~~



$\tau_{int} \sim l/v_A$ ⇒ coherence time.

i.e. counter-propagating packets' interaction time limited to l/v_A.

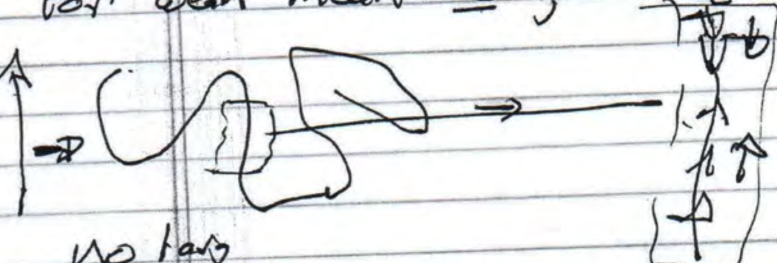
i.e. saves us from renormalization in absence of dispersion.

classic

so, some Phenomenology: (Iroshnikov, Kraichnan) ('64, '65)

→ nonlinear transfer — ≈ isotropic turbulence

For weak mean $\underline{B_0}$, $b_0 \sim b_{rms} > |B_0|$



locally, ∃ a $\underline{b_0}$

no B_0

→ so, nonlinear scattering from "collisions" of counter-propagating packets

[universally here, = counter Els. populations locality?]

so (modulo 4/5 !) — TBD.

→ $\boxed{\epsilon \sim \frac{Z_\ell^2}{\tilde{\tau}_{tr\,\ell}}}$ $\ell \equiv$ scale

→ Alfven wave cascade

⇒ const. energy thru-put rate

Now Alfvenic transit time on scale ℓ, in b_0 is:

$$\boxed{\tau_A = \ell / b_0}$$

For transfer:

- transfer by wave scattering $\Leftrightarrow$ { random walk of amplitude / intens
- define τ_{Tr} by:

$$\langle \delta Z^2(\tau_{Tr}) \rangle \sim \langle Z^2 \rangle$$

$\tilde{\tau}_{Tr}$ sufficient # kicks to make

i.e. randomization of scattering in amplitude

$$\underline{\delta Z_{rms} \sim Z}$$

random walk.

so

$$Z_\ell(t+\tau_A) = Z_\ell(t) + \tau_A \partial_t Z_\ell$$

kick in 1 Alfven transit time

diffusion process → $\underline{k_\parallel \ell < 1}$

$$\delta Z_\ell \simeq \tau_A \frac{Z_\ell^2}{\ell} \rightarrow \underline{\text{kick}} \text{ in } \tilde{\tau}_A$$

dims

so $\frac{Z_\ell^2}{\ell} \rightarrow$ kick.

$$\Rightarrow \langle \delta Z_\ell^2 \rangle = \left(\tau_A \frac{Z_\ell^2}{\ell}\right)^2 \frac{t}{\tilde{\tau}_A} \rightarrow \boxed{\text{accumulated kicks}}$$

$$= \underbrace{\left(\frac{Z_\ell^2}{\ell}\right)^2 \tau_A}\, t = D\, t$$

thus:

$$\langle \delta Z_\ell^2 \rangle \sim \langle Z_\ell^2 \rangle \rightarrow \text{defines } \tau_{Tr}.$$

$$\Rightarrow Z_\ell^2 \simeq \tau_A^2 \frac{(Z_\ell^2)^2}{\ell^2} \frac{\tau_{Tr}}{\tilde{\tau}_A}$$

$$\therefore \boxed{\frac{1}{\tau_{Tr}} \simeq \frac{Z_\ell^2}{\ell^2} \tau_A} \rightarrow \underline{\text{determines transfer rate.}}$$

so $\frac{1}{\tau_{Tr}} \sim \frac{Z_\ell^2}{\ell^2} \frac{\ell}{b_0}$

$\sim \frac{Z_\ell^2}{\ell b_0}$

Thus:

$$\epsilon \simeq \frac{Z_\ell^2}{\tau_{Tr}} \simeq \frac{Z_\ell^2 Z_\ell^2}{\ell b_0}$$ const. rate balance

$$\boxed{Z_\ell^2 \sim \sqrt{b_0 \epsilon}\, \ell^{1/2}}$$ $Z_\ell \sim (b_0 \epsilon)^{1/4} \ell^{1/4}$

$$\boxed{E(k) \simeq \sqrt{\epsilon b_0}\, k^{-3/2}}$$ Kraichnan spectrum!

N.B. – as work with Z,
$E_M \sim E_K$.
(equal Elsasser populations)

– $1/\tau_{Tr} \simeq \frac{Z_\ell^2}{\ell^2} \frac{\ell}{b_0} \simeq \underbrace{\frac{\tau_A}{\tau_{eddy}}}\, \frac{1}{\tau_{eddy}}$

2002

$$\partial_t P_\alpha = \text{in} - \text{out} = \sum_{\alpha,\beta,\gamma} c_i P_\beta P_\gamma - \sum_\gamma c_{\alpha,\gamma} P_\gamma P_\alpha \qquad \frac{1}{\tau} \sim$$

$$\hbar\omega_f - \hbar\omega_i$$

$$T_{gr} = \sum_f |\langle f|H'|i\rangle|^2 \delta(E_f - E_i - \hbar\omega)$$

compare K41: $1/\tau_{Tr} \sim \frac{1}{\tau_{eddy}}$

Transfer rate reduced by factor $\underline{\tau_A/\tau_{eddy}}$!

Also, recall for weak wave turbulence:

$$C\{N\} = \sum_{k'} |V|^2 N_{k'} N_k \tau_{Tc}$$

$\to$ general structure of collision integral

$\sum_{k'}$ ↓ c.c.

τ_{Tc} ↓ (triad) coherence time → interaction

so $$\boxed{\frac{1}{\tau_{Tr}} \simeq \sum_{k'} |V|^2 N_{k'} \tau_{Tc}}$$

but $N_k \to Z_\ell^2$ (intensity) (assumes local transfer)

$$|V|^2 \sim |\nabla^2| \sim \frac{1}{\ell^2} \to \underline{\frac{1}{\ell_\perp^2}} \qquad \underline{c.c.}$$

$\tau_{Tc} \sim \tau_A \to$ triad coherence

τ_A ↓ Alfvenic packet transit time

$$1/\tau_{Tr} \sim \frac{Z_\ell^2}{\ell^2} \tau_A \Longleftrightarrow \text{W.T.T.}$$

fundamentally wave scattering process!

- 'triad' → 2AWs + cell. $\underline{q} = \underline{k} + \underline{k}'$ (carps)

→ akin NLLD. $\omega_q = \omega_{\underline{k}} + \omega_{\underline{k}'}$

$\sim 0 = k_{\parallel} V_A - k_{\parallel}'' V_A$

- Alfven waves non-dispersive, but coherence time controlled by packet propagation

→ prevents negative spectra. etc. (Coherence time not so large)

- Strong B_0. (explicit)

Some simple observations:

- turbulence clearly anisotropic
- nonlinear transfer in $k_\perp$.

So, consider weak wave turbulence!

Consider: $k_\perp \to 1/\ell_\perp$

$k_\parallel$

$k_\perp > k_\parallel$ so

so $$\epsilon \sim \frac{\left(Z(\ell_\perp)^2\right)\left(Z(\ell_\perp)^2\right)}{\ell_\perp^2\,|k_\parallel V_A|}$$

$\Delta k_\parallel V_A$ → coh. time

i.e. tacitly $\Delta |k_\parallel V_A| \sim |k_\parallel V_A|$

$$Z(\ell_\perp)^2 \simeq (\epsilon |k_\parallel V_A|)^{1/2} \ell_\perp$$

$\Rightarrow$

$$E(k_\perp) \cong (\epsilon\ |k_\parallel V_A|)^{1/2} / k_\perp^2$$

sharp

crit. restricted

and accounting for PDS in $k_\parallel$:

$$\boxed{E(k_\perp, k_\parallel) \sim [\epsilon V_A]^{1/2} / k_\parallel^{1/2} k_\perp^2}$$

weak wave turbulence spectrum

- strongly anisotropic
- $k_\parallel$ frozen.

Now, $Z(\ell_\perp) \sim \delta B(\ell_\perp) \sim (\epsilon |k_\parallel V_A|)^{1/4} \ell_\perp^{1/2}$

$$\omega \sim k_\parallel V_A$$

recall: $\nabla_\parallel = \partial_z + \frac{\delta B_\perp}{B_0} \cdot \nabla_\perp$

Kubo #

(see 235) $\boxed{k_\parallel \sim \frac{\delta B_\perp}{B_0} \nabla_\perp / \partial_z \sim \frac{\ell_{ac} \delta B_\perp / B_0}{\Delta_\perp}}$ $\Big\}$ # kicks in coherence length

Now, $\frac{\delta B_\perp}{\ell_\perp} \sim (\epsilon k_\parallel V_A)^{1/4} / \ell_\perp^{1/2}$

⇒ For W.T.T., expect:

$k_u < 1$ (diffusive picture)

but,

k_u rises as $l_\perp$ drops

i.e. $k_u \uparrow$ as progress thru $l_\perp$ cascade ⇒ ?? what happens

⇒ Begs the question:

How high can k_u go and still retain physics of Alfven wave cascade? | diffusive scattering

BBK comment '78

enter the:

$\boxed{\text{Critical Balance Conjecture}}$

– Goldreich – Sridhar (1995) (cf also Kadomtsev – Pogutse, 1978)

⇒ $\boxed{\text{MHD inertial range in strong field will sit at } k_u \sim 1}$

i.e. $\delta B_\perp \cdot v_\perp \sim B_0 v_\parallel$

$$\frac{Z(l_\perp)}{l_\perp} \simeq k_\parallel V_A$$

i.e. transit time sets bound on int. strength

$\frac{\tau_A}{\tau_{eddy}} \to 1$, $\frac{\tau_{tr}}{\tau_{eddy}} \to$

⋮

$k_\parallel$ vs $k_\perp$ rel.

but: $\epsilon \sim \frac{(z(l_\perp))^2 (z(l_\perp))^2}{l_\perp^2} \frac{1}{k_\parallel V_A}$

$\rightarrow \frac{(z(l_\perp))^2 (z(l_\perp))^2}{l_\perp^2 \frac{z(l_\perp)}{l_\perp}}$

$= \frac{z(l_\perp)^3}{l_\perp}$

$\Rightarrow z(l_\perp) \sim (\epsilon\, l_\perp)^{1/3}$

$$\boxed{E(k_\perp, k_\parallel) \sim \epsilon^{2/3} k_\perp^{-5/3}}$$

G-S spectrum

→ back to K41 but different physics

→ fits data

and anisotropy:

$k_\parallel \sim 1$, $B_0 \cdot \nabla \sim \tilde{B}_\perp \cdot \nabla_\perp$

$B_0 k_\parallel \sim \epsilon^{1/3} \frac{l_\perp^{1/3}}{l_\perp}$

$k_\parallel$ vs $k_\perp$ relation

$$\boxed{k_\parallel \sim \frac{\epsilon^{1/3}}{B_0} k_\perp^{+2/3}}$$

specifies "GS cone" in $\underline{k}$ space

i.e. $k_\parallel << k_\perp$ → cascade develops preferentially in perp.

$k_\parallel \sim k_\perp^{2/3}$

– Why Believe

→ analogue of 4/5 Law (Pouquet, Politano)

$$-\frac{4}{3}\epsilon^T \ell = \langle (\delta \underline{v} \cdot \delta \underline{v} + \delta \underline{b} \cdot \delta \underline{b}) \delta v_\ell \rangle$$

↳ mom bal → form $\dot{E}_K$

$$-2\langle (\delta \underline{v} \cdot \delta \underline{b}) b_\ell \rangle$$

total energy cascade

induction → form $\dot{E}_M$

~ reflects flip-flop in energy between channels

or

$$-\frac{4}{3}\epsilon^{\pm} \ell = \langle (\delta Z^{\pm} \cdot \delta Z^{\pm}) \delta Z_\ell^{\mp} \rangle$$

no dissipn for 1 stream only.

→ Why ? → relate induced E-field.