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Theory of phase-space density holes

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A Bernstein–Green–Kruskal mode consisting of a depression or "hole" in the phase-space density is shown to be a state of maximum entropy subject to constant mass, momentum, and energy. The parameter space of such holes is studied. The maximum entropy property is used to develop a simplified approximate analytic method as well as to infer the results of hole collisions including coalescing and decay. The maximum entropy property suggests that random, turbulent fluctuations tend to form into such self-trapped structures and this nonperturbative concept is related to the physics of "clumps" which occur in a renormalized perturbative theory of turbulence.

I. INTRODUCTION

It has been proposed[1] that random or turbulent fluctuations of the phase space density of velocity space dimensions Δv for which the potential energy fluctuations $q\phi$ are negative and of the order of $m(\Delta v)^2/2$ tend to form into self-trapped structures (q and m are particle charge and mass, respectively). These structures, when isolated from each other are Bernstein–Green–Kruskal equilibria.[2] For fluctuation speeds of the order of, or less than, the thermal speed v_{th}, the self-trapped structures take the form of depressions or "holes" in the local phase space density. In a weakly turbulent plasma (which is the case we treat here), holes would be "small" i.e., would have $\Delta v \ll v_{\text{th}}$ and a depth $-\tilde{f}$ much less than the average phase space density f_0. We are not primarily concerned here with how these fluctuations are initially created. However, one possible mechanism and the principal motivation for this work is the turbulent conversion of the average phase space density gradient into phase space density granulations as discussed in Ref. 3. The relationship of that work to the present work is discussed in the next section.

Generally speaking, turbulent fluctuations cannot be exact Bernstein–Green–Kruskal modes since they are continually interacting or colliding with each other. However, we take the view in this paper that it is a useful approximation. In fact, we show that in a certain sense, a Bernstein–Green–Kruskal mode is the most probable state for a fluctuation of given mass M, momentum P, and energy T. In the following we compute the entropy of a Bernstein–Green–Kruskal mode as a function of M, P, T. We then investigate the interaction of Bernstein–Green–Kruskal modes by computing the entropy of the initial and final states. The present calculation is limited to a one-species, one-dimensional plasma, but the basic concepts can be extended to a three-dimensional plasma with a magnetic field, which we intend to do in future publications.

The underlying model, which is obviously highly approximate and idealized, pictures turbulent fluctuations in a plasma as consisting of a collection of isolated, noninteracting holes. This enables us to calculate many properties of the system including the entropy which we then use to infer time evolution and probable final states for the system. One justification for using such an idealized model is that it permits an analytic calculation of fluctuation self-trapping, an apparently important effect which seems very difficult to recover from a renormalized perturbation theory of plasma turbulence. On the other hand, renormalized perturbation theory treats many aspects of plasma turbulence far better than the hole model, so that the latter should be regarded as a complementary and not a competing point of view. Reference 4 contains a much improved version of renormalized perturbation theory which discusses and corrects many of the deficiencies in the original renormalized perturbation theory of clumps.

Of course, the idea of a hole in the phase-space density such as a Bernstein–Green–Kruskal mode is not new. In particular, Berk, Nielson, and Roberts[5] have investigated such holes analytically and carried out computer simulations which show the stability of such holes and their tendency to coalesce. In addition, they discussed a useful gravitational analogy in which the holes may be regarded as gravitating masses. A computer simulation of a large number of randomly interacting holes is currently in progress.[6]

The novel aspects of the present work are:

(1) We show that, subject to certain constraints, holes are states of maximum entropy for a given hole mass, momentum, and energy.

(2) We explore a two-parameter space (e.g., mass and energy or Δx and Δv, the spatial and velocity dimensions of the hole) for such holes and calculate a variety of hole properties.

(3) We use the maximum entropy property to develop a greatly simplified approximate method of calculating hole properties (rectangular hole approximation).

(4) The maximum entropy property leads to the idea that a wide variety of turbulent fluctuations tend to form themselves into such holes.

To be more specific, the principal results of this paper are:

(a) For a "local" fluctuation ($\Delta v \ll v_{\text{th}}$) we calculate a fluctuation mass M ($M<0$), charge Q, energy T, momentum P, entropy $-\sigma$, and distribution function $\tilde{f}$, all with respect to an average unperturbed state whose distribution function is f_0. If u is the hole velocity, then $P = Mu$ and $T = T_0 + \frac{1}{2}Mu^2$, where T_0 is the hole energy in its rest frame, the self-energy. We show that the

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distribution function $f = f_0 + \tilde{f}$ that makes σ stationary (minimum) subject to constant M, T, P, is a Maxwell–Boltzmann distribution with a negative temperature, and that this state is a Bernstein–Green–Kruskal mode (a Maxwell–Boltzmann hole). We interpret the maximum entropy (minimum σ) property to mean that fluctuations (with $M < 0$) tend to organize themselves into holes. The fact that $\tilde{f}$ satisfies a variational principle is also very useful for approximate analytic calculation and suggests that the important properties of a hole may not be sensitive to the details of $\tilde{f}$.

(b) The total charge and energy in a Maxwell–Boltzmann hole must satisfy

$$1 < \pi Q^2 \lambda / T_0 < 5.8 . \tag{1}$$

Less probable holes (i.e., non-Maxwell–Boltzmann) can have arbitrarily large $\pi Q^2 \lambda / T_0$ [see item (g)]. The length λ is the shielding length (22) and is of the order of the Debye length, λ_D. If two or more holes interact such that the total Q and T satisfy (1), then the most probable final state (smallest σ) is a single hole, i.e., the holes coalesce.

(c) For finite Q and $T \to 0$, one can construct an infinite collection of Maxwell–Boltzmann holes or a single non-Maxwell–Boltzmann hole. In either case $\sigma \to 0$, $\tilde{f} \to 0$, and $\Delta x \Delta v \to \infty$. The point is that charge or mass can be removed from holes and put into the interstitial background with no change in background entropy density.

(d) Therefore, for a collection of interacting holes with $\pi Q^2 \lambda / T > 5.8$, the most probable final state is one with all the energy T and an amount of charge $Q = (5.8 T / \pi \lambda)^{1/2}$ in a single hole, and the remaining charge in the interstitial background as discussed in (c).

(e) We show that the Maxwell–Boltzmann hole length Δx (not a precisely defined quantity) is a monotonically increasing function of Q^2/T (see Fig. 4) and the inequality 1 is equivalent to $0 < \Delta x / \lambda < r$ where r is of order 4. We also show that for constant T_0, σ for a Maxwell–Boltzmann hole is a monotonically decreasing function of Q^2/T_0 and therefore of $\Delta x / \lambda$. Holes with $\Delta x / \lambda \ll r$ have very large σ (see Fig. 4) and tend to coalesce to form holes with larger Δx and smaller σ. This prediction is consistent with the observation of hole coalescence reported by Berk, Nielson, and Roberts.[5]

(f) The electric field energy is proportional to the self-energy T_0. When two holes moving at velocities u_1 and u_2 coalesce, there is a loss of self-energy given by $-\frac{1}{2} M_1 u_1^2 | u_1 - u_2 / u_1 |$. This is (apparently) the principal means of fluctuation decay for a collection of interacting holes (for one dimension, one species).

(g) We also compute the entropy for holes with arbitrary Δx, i.e., $0 < \Delta x < \infty$ [these are not Maxwell–Boltzmann holes and so do not have the restriction (1) on Q^2/T]. The results for these holes are consistent with the Maxwell–Boltzmann holes. We find that $\Delta x / \lambda$ is a monotonically increasing function of Q^2/T_0 for arbitrarily large $\Delta x / \lambda$. Furthermore, at constant T_0, σ decreases with increasing $\Delta x / \lambda$ until a minimum is achieved at $\pi Q^2 / T_0 \approx 5.8$ and increases thereafter (see Fig. 4). A hole with $\Delta x / \lambda > r$ is unstable to decay into a hole with $\Delta x / \lambda = r$ (the most probable Maxwell–Boltzmann hole) and the remaining mass going into a $\sigma = 0$ hole (i.e., mixing into the background). We also derive a useful and simple relationship between the depth, $-\tilde{f}$, and the phase-space dimensions, Δv and Δx, of a hole. When $\Delta x / \lambda \ll 1$, $-\tilde{f} = \Delta v (\Delta x \omega_p)^{-2}$, and when $\Delta x / \lambda \gg 1$, $-\tilde{f} = \Delta v (\lambda \omega_p)^{-2}/6$, where $\omega_p^2 = 4\pi n q^2 / m$ and n is the average number density. Berk, Nielson, and Roberts dealt with very deep holes, i.e., $-\tilde{f} = f_0$.

(h) The Maxwell–Boltzmann entropy is used. Since this measure of entropy does not conserve phase-space density predictions (b) and (d) of the outcome of hole, interactions which rely on minimizing σ will be in error if the predicted final state does not conserve phase-space density with the initial state.

(i) The calculation of hole properties is based on isolated holes. In an actual plasma with many holes interacting at once, the properties of the resulting fluctuations will be different. The theory of the two-point correlation function in which self-energy effects are not included[3] shows that fluctuations tend to be torn apart so that their Δx and Δv dimensions are continuously reduced. This tendency is counterbalanced by the self-binding of hole material. Therefore, one would expect that in a turbulent plasma, Δx and Δv would be less than predicted by the isolated hole calculation.

II. RELATIONSHIP TO THE KINETIC THEORY OF PHASE-SPACE DENSITY GRANULATION: CLUMPS

In a turbulent plasma, particle orbits become random and stochastic. If this occurs for velocities in the range for which the velocity gradient of the average phase space density is significant, a mixing occurs, i.e., a rearrangement of elements of the phase-space density so that local values of f differ from the average. We have previously referred to these granulations or fluctuations as clumps.[3]

Mixing is an intrinsic aspect of the relaxation of a continuous system such as the Vlasov equation. Mixing results from (or is defined as) the juxtaposition of small elements of phase-space volume containing f which come from random and distant phase-space locations. This mixing is a direct result of so-called "orbit instability" and is related to the Kolmogoroff entropy. In fact, it is the occurrence of this mixing that allows us to define a "coarse-grained" entropy which is a functional of f averaged over the fine-scale.

In Ref. 3 we showed that in a "stochastic acceleration" approximation (i.e., Poisson's equation not consistently included as we shall discuss shortly), the turbulence will mix down to grain sizes of order $\Delta x \Delta v$ in a "clump lifetime"

$$\tau_{cl}(\Delta x, \Delta v) = -\tau_0 \ln\{\tfrac{1}{3}[k_0^2(\Delta x^2 - 2\Delta x \Delta v \tau_0 + 2\Delta v^2 \tau_0^2)]\}, \tag{2}$$

where $\tau_0 = (4k_0^2 D)^{-1/3}$, k_0 is an average wavenumber of the turbulence and D is the velocity diffusion coefficient. τ_0 is a mixing time and is sometimes referred to as the

Kolmogoroff entropy. Since τ_0 is finite, during any process in which the average distribution function f_0 is relaxing due to mixing (which is the only way it can relax in a collisionless system), the actual f will contain fluctuations or clumps of unmixed phase-space density which have not had time to mix down to small grain sizes. This occurs even though f_0 itself remains smooth (as it would, for example, in quasilinear theory). The mixing is due to the fact that neighboring particles feel different forces and therefore their orbits diverge. In the original clump theory, this orbit divergence was described by a relative velocity diffusion coefficient D_-.

In the theory of clumps, fluctuations are produced when the phase-space density in a small region of phase space is moved randomly to a new location in phase space where its f differs from the local average, $\langle f\rangle = f_0$. If f is preserved (i. e., not mixed) for a time τ_{c1} and the clump moves diffusively, then the mean-square fluctuation produced would be $\langle \tilde{f}^2\rangle \approx 2\tau_{c1}D(\partial f_0/\partial v)^2$. Clearly, any process that increases τ_{c1}, such as the tendency of negative $\tilde{f}$ to form into self-bound structures (holes), can enhance the fluctuations. This possibility is the main reason for our study of hole dynamics. Even a single hole can have its energy increased by this mechanism. The trapped distribution function of a hole will remain constant as it moves in phase space. Therefore, when a hole moves to a velocity region of larger $f_0(v)$, its depth, $-\tilde{f}$, will increase. This means the energy, T_0, will increase since it is proportional to $\tilde{f}^4$. The idea of modeling turbulence as a collection of randomly located coherent structures (holes) is obviously likely to be more valid if they are relatively far apart, i. e., if the turbulent fluctuations are not uniformly distributed over space but are concentrated in isolated local regions. In fluid turbulence, this situation is referred to as intermittency. The hole model introduces the concept of intermittency to plasma clump turbulence.

Because of the complexity of the phenomena, the theory in Ref. 3 is very approximate. For example, the correlation function $\langle \delta f\delta f\rangle$ was calculated with a two-point stochastic acceleration (bivariate diffusion) equation of the form

$$\left(\frac{\partial}{\partial t}+L(1,2)\right)\langle\delta f(1)\delta f(2)\rangle = S(1,2)\,, \tag{3a}$$

where

$$L(1,2) = v_-\frac{\partial}{\partial x_-} - D_-\frac{\partial^2}{\partial v_-^2}\,, \tag{3b}$$

$$S(1,1) = 2D\left(\frac{\partial f_0}{\partial v}\right)^2. \tag{3c}$$

The angular brackets $\langle\ \rangle$, denote an ensemble average. The operator L conserves phase-space density, whereas the source term $S(1,2)$ does not, i. e.,

$$\left(\frac{\partial}{\partial t}\right)\int dv_1\langle\delta f(1)^2\rangle = \int dv_1 S(1,1)\,.$$

The fluctuation δf can be written as a coherent and an incoherent (clump or hole) part denoted by $f^{(c)}$ and $\tilde{f}$, respectively. In Ref. 3, Poisson's equation was not incorporated into (3a, b, c), i. e., the potential fluctuation correlation function $\langle\phi\phi\rangle$ was assumed given and not made consistent with δf within (3a). At a second stage of the calculations, after (3a) was solved, Poisson's equation was taken into account by requiring that the assumed $\langle\phi\phi\rangle$ be equal to that produced by $\langle\delta f\delta f\rangle$. This requirement produced a so-called self-sustaining criterion. This procedure is, of course, approximate and recently a more rigorous treatment has been developed.[4] We shall not discuss the problems of constructing an analytic theory except as they relate to the hole model.

The inconsistent treatment of Poisson's equation in Ref. 3 is related to the neglect of $\langle\tilde{\phi}\tilde{f}\rangle$ in computing $\langle\delta f\delta f\rangle$ (although it was not neglected in computing $\partial\langle f\rangle/\partial t$). This omission of $\langle\tilde{\phi}\tilde{f}\rangle$ leads to (at least) two different types of physical errors. The first is that the operator L in (3b) neglects the self-field of the fluctuation acting back on itself, the self-trapping effect, which is the main subject of this paper. The other nonphysical aspect is that the right-hand side of (3a) which is a source of $\langle(\delta f)^2\rangle$ due to the rearrangement of the average phase-space density gradient is not consistent with momentum conservation and conservation of $\int f^2\,dx\,dv$ required by a Vlasov theory. These conservation requirements play an important role in hole dynamics when $\partial f_0/\partial v \neq 0$.

The self-interaction was dismissed as negligible in Ref. 3. This is not correct as can be seen by the following simple calculation. Consider a clump or fluctuation of phase-space dimensions $\Delta x, \Delta v$. The clump theory predicts that the perturbed distribution function is approximately $\tilde{f}\approx \Delta v/v_{th}^2$. This produces a self-potential given by

$$\left(-\frac{\partial^2}{\partial x^2}+\lambda^{-2}\right)\phi = 4\pi nq\tilde{f}\Delta v\,, \tag{4}$$

where λ is the shielding distance given by (22). For $\Delta x\approx\lambda$, this gives $\phi\approx 4\pi nq\Delta v^2 v_{th}^{-2}\lambda^2$. The velocity trapping width of this potential is

$$\Delta v_T = (2q\phi/m)^{1/2} = (\lambda/\lambda_D)\Delta v\,. \tag{5}$$

For $\lambda/\lambda_D > 1$, the self-field of the fluctuation is strong enought to trap itself (provided $q\phi < 1$) and, in fact, the trapping time $\Delta x/\Delta v$ is the same magnitude as the "clump lifetime" τ_{c1} from the stochastic acceleration theory. Thus, the self-energy effects are of the same order as those retained in the approximate theory.

The left-hand side of (4) is equivalent to $k^2\epsilon\phi$, where ϵ is the dielectric function and k is a wavenumber. It follows that the sign of $q\phi$ is the same as $\tilde{f}/\epsilon$. Therefore, if $\epsilon > 0$, a bound state requires $\tilde{f} < 0$, and $\epsilon < 0$ requires $\tilde{f} > 0$. In this paper we concentrate on the case $\epsilon > 0$ which occurs for velocities $0 < v < v_{th}$. In this case, the bound fluctuations $\tilde{f} < 0$ appear as depressions or holes in the local phase-space density. The negative fluctuations tend to self-trap, and positive fluctuations tend to blow apart and fill the interstitial region between the holes. Since the qualitative nature of the physics seems to depend on the sign of δf, it is difficult to see how a kinetic theory depending only on $\langle \delta f \delta f \rangle$ can describe the self-trapping effect, although it might conceivably be described as a kind of microinstability as proposed in Appendix C.

It is interesting to compare the self-binding effect of the self-field of a hole with the disruptive effect of the electric fields of other fluctuations (holes) as described by D_-. $D_- \approx \langle (\Delta v_-)^2 \rangle / 2\Delta t$ is the diffusion coefficient of the relative velocity v_- of two particles separated by Δx. One can easily show that the contribution of a single hole (denoted by a subscript 2) to D_- is

$$D_- \approx \frac{q^2}{m^2}\left(\frac{\partial E_2}{\partial x}\right)^2 (\Delta x)^3 (\Delta v)^{-1} .$$

A different hole (a "test" hole denoted by the subscript 1) is torn apart (diffused by D_-) by other holes (2) because the field of 2 at two points separated by Δx differs by $(\partial E_2/\partial x)\Delta x$ and is in a random direction with respect to 1. On the other hand, $(\partial E_1/\partial x)\Delta x$ due to the self-field of hole 1, is always in a direction to bind the hole together and counteract D_-. Therefore, the self-binding effect of a hole will dominate D_- if the $\partial E/\partial x$ is greater than that of other holes in its neighborhood. For $\Delta x < \lambda$, this criterion takes a simple form. Poisson's equation states that $\partial E/\partial x \sim 4\pi n q \bar{f} \Delta v$, and we presently show that for $\Delta x < \lambda$, $\bar{f} \sim \Delta v/(\Delta x)^2$. Therefore, $\partial E/\partial x \sim (\Delta v/\Delta x)^2$ and $D_- \sim (\Delta v/\Delta x)^3 \Delta x^2$. Thus, we can regard $\Delta v/\Delta x$ as a measure of the strength of D_- and of the self-binding effect. There are other parameters which depend only on $\Delta v/\Delta x$ and are therefore measures of self binding versus the effects of D_-. For example, the hole phase-space energy density defined as $T_0/\Delta x \Delta v \sim (\Delta v/\Delta x)^3$ and the negative hole entropy $\bar{f}^2 \Delta x \Delta v \sim (\Delta v/\Delta x)^3$ are such measures.

We now discuss the source term in (3a). The Vlasov equation requires that

$$\frac{\partial}{\partial t} \int\int dx\, dv f^2 = 0 . \tag{6}$$

For a spatially homogeneous system, an ensemble average $\langle\ \rangle$ is equal to a spatial average

$$\langle\ \rangle = \lim_{L\to\infty} (2L)^{-1} \int_{-L}^{L} dx . \tag{7}$$

If we put $f = \langle f \rangle + \delta f$, we find, from (6) and (7), that

$$\frac{\partial}{\partial t} \int dv \langle (\delta f)^2 \rangle + \frac{\partial}{\partial t} \int dv \langle f \rangle^2 = 0 . \tag{8}$$

Thus, the source term in (3a, c) should have the property

$$\int S(1,1) dv_1 + 2 \int dv \langle f \rangle \frac{\partial}{\partial t} \langle f \rangle = 0 . \tag{9}$$

Equation (9) relates the source term for fluctuations $\langle (\delta f)^2 \rangle$ to the rate of change of $\langle f \rangle$. From this relationship it is clear that if one retains $\langle \phi \tilde{f} \rangle$ in the equation for $\langle f \rangle$ (as one must to conserve energy and momentum), it must also be retained in the source term.

In fact, if momentum is conserved in a local velocity space region $\Delta v \ll v_{\mathrm{th}}$, the change in $\langle (\delta f)^2 \rangle$ is even more inhibited than (9) might indicate. We set $f = \langle f \rangle_0 + \mathfrak{F}$, where $\langle f \rangle_0$ is independent of time and is smooth over the range Δv. Conservation of particles and momentum requires

$$\frac{\partial}{\partial t} n \int dx \int dv [1, mv] \mathfrak{F} = \left[0, \frac{\partial p}{\partial t}\right] . \tag{10}$$

Equation (6) can be written

$$\frac{\partial}{\partial t} \int dv \int dx (\langle f \rangle_0^2 + 2\langle f \rangle_0 \mathfrak{F} + \mathfrak{F}^2) = 0 . \tag{11}$$

The phase-space integrals in (10) and (11) are over the region in which $\partial f/\partial t \neq 0$, typically a few Δv in velocity. The first term in (11) is zero since $\langle f \rangle_0$ is independent of time. In the second term we can expand as $\langle f(v) \rangle_0 = \langle f(u) \rangle_0 + \langle f'(u) \rangle_0 (v-u)$ and use (10) to write (11) as

$$\frac{\partial}{\partial t} nm \int dv \int dx\, \mathfrak{F}^2 = -2 \langle f'(u) \rangle_0 \frac{\partial p}{\partial t} . \tag{12}$$

For a one-species plasma, momentum conservation requires $\partial p/\partial t = 0$ and so the integral of $\mathfrak{F}^2$ must remain constant. Setting $\mathfrak{F} = \langle f(t) \rangle - \langle f(0) \rangle + \delta f$, substituting this into (12), averaging, and integrating over time we find

$$\int dv \langle [\delta f(t)]^2 \rangle = \int dv \{ \langle [\delta f(0)]^2 \rangle - [\langle f(t) \rangle - \langle f(0) \rangle]^2 \} . \tag{13}$$

Equation (13) shows that in a one-dimensional one-species plasma with local momentum conservation, $\langle (\delta f)^2 \rangle$ can only decrease, never increase with time. In this case the source term S in (9) is negative and the rearrangement of the average phase-space density does not appear to play a major role. The relevance of this point for the current work is that it provides some justification for our model in which a collection of interacting holes can only lead to another state consisting only of holes with the same total energy, momentum, and mass. This is obvious when $\partial f_0/\partial v = 0$ due to conservation of phase-space density, but for $\partial f_0/\partial v \neq 0$, it would not be the case if the source term could create new holes (and new regions with $f > f_0$) by rearranging the gradient of f_0.

The situation is quite different in a two-species plasma since electrons and ions (e, i) can exchange momentum locally, i. e., $\partial p_e/\partial t = -\partial p_i/\partial t$, and (12) is replaced by

$$\frac{n_e m_e}{\langle f'_e \rangle_0} \frac{d}{dt} \int\int dx\, dv \mathfrak{F}_e^2 = -\frac{n_i m_i}{\langle f'_i \rangle_0} \frac{d}{dt} \int\int dx\, dv\, \mathfrak{F}_i^2 . \tag{14}$$

If f'_{0e} and f'_{0i} have opposite signs, then (for example) an ion hole can move to larger f_{0i} and increase its depth, $-\bar{f}$ and its energy (as explained earlier) with the electrons providing the momentum when they are reflected from the ion holes with the consequent flattening of f'_{0e}. The growth rate γ for an ion hole can be estimated for small f'_{0e} and f'_{0i} as follows. Since $T_0 \sim \bar{f}^4$, γ is equal to $T_0^{-1} \partial T_0/\partial t \sim 4\bar{f}^{-1} \partial \bar{f}/\partial t$. The increase in hole depth is given by $\partial \bar{f}/\partial t \sim -f'_{0i} \partial u/\partial t$. The hole acceleration $\partial u/\partial t$ is determined by conserving momentum between the reflected electrons, $\partial p_e/\partial t \sim -(\Delta v_e)^4 f'_{0e} n_e m_e$, and the accelerating ion hole, $\partial p_i/\partial t \sim M \partial u/\partial t$. Using $M \sim \Delta v_i^2$ and $\bar{f} \sim \Delta v_i \omega_{pi}^{-2} \lambda^{-2}$, we find $\gamma \sim -(\Delta v_i/\lambda) f'_{0e} f'_{0i} \lambda^4 \omega_{pe}^2 \omega_{pi}^2$.

A constraint analogous to the one for momentum conservation just discussed occurs for the $\mathbf{E}\times\mathbf{B}$ relaxation of a density gradient $\partial n(x)/\partial x$ across a magnetic field. In this case (10) is replaced by the requirement that $\langle x\rangle \equiv \int dr\, xqn(r)$ summed over species be constant. Again, two species are required for a nonzero source term. Also in this case, a simple hole will grow if $\partial n_i/\partial x$ and $\partial n_e/\partial x$ have the same sign. For example, the $\mathbf{E}\times\mathbf{B}$ drift of electrons due to an ion hole will cause $\langle x\rangle$ for electrons to increase if $\partial n_e/\partial x < 0$. Therefore, $\langle x\rangle$ for the ion hole must decrease, moving the hole to larger values of n_i and causing $-\tilde{f}$ to increase.

These nonlinear hole instabilities occur even when linear theory predicts stability. They are limiting cases of the clump instability and will be discussed in a subsequent publication.

III. ENTROPY FOR A VLASOV SYSTEM

The question of an appropriate measure of entropy for a Vlasov system is difficult to resolve. The entropy is supposed to measure the number of microstates that correspond to a given macrostate. It is then hoped that the probability of observing such a macrostate is proportional to this number and that the system is likely to evolve to the most probable state. We do not even pretend to prove these assertions. To keep matters simple, we shall deal only with a one-dimensional plasma consisting of infinite mass ions (whose only role is to provide a uniform neutralizing background) and electrons whose distribution function is $f(x, v)$.

We wish to write the entropy as an integral over phase-space of some function of an appropriately averaged distribution function. For a Vlasov system it is well-known, because of the conservation of phase-space density, that if one uses the exact (unaveraged) distribution function to calculate the entropy for a Vlasov system, then the entropy would be constant in time and provide no information about time evolution. Therefore, it is essential to average the exact distribution function over some small microscales in v and x in order to calculate the entropy of larger or macroscale phenomena. Of course, this procedure will not provide information about the detailed structure of the system at the smallest scales. We shall discuss this point further.

Unfortunately, the Vlasov equation imposes a number of constraints on the allowable microstates that are not easy to incorporate into the statistics. For a system consisting of discrete particles, one could use the Maxwell–Boltzmann entropy

$$-\sigma = -n \int dx \int dv\, f \ln f, \tag{15}$$

where n is the average number density and f contains an average over microscales. However, the proper choice for the Vlasov system is not a well-understood problem. Lynden-Bell[7] has proposed an entropy similar to the Maxwell–Boltzmann entropy which conserves phase-space density. The Lynden-Bell entropy is computed by counting the number of microstates corresponding to a macrostate f. The microstates are obtained by arranging small grains of phase-space density. In each arrangement, the grain size and the phase-space density of the grains are held constant and grains cannot overlap. The Maxwell–Boltzmann entropy can be obtained as a limiting case in which the grains have phase-space density of either zero or infinity. The two statistics give similar answers when the microstates consist of disparate values of the phase-space density in neighboring grains (cells), i.e., when the macrostate f is composed of a fairly random collection of phase-space density values in adjacent grains (e.g., the fraction of unoccupied cells is not too small). However, from a physics point of view, it is hard to understand why the most probable physical state should depend on the details of microscopic grains, particularly since the details are lost in the average over microscales. Furthermore, the Lynden-Bell statistics do not take into account the fact that collisionless (Vlasov) mixing or rearrangement preserves nearest neighbors, i.e., the microstates should not consist of arrangements of unconnected grains but the winding up of long ribbons of phase-space density like the tangling of a ball of string. The Lynden-Bell statistics assume, in effect, that the tangled string can be rearranged by first cutting the string into short sections, i.e., Lynden-Bell does not take into account the irreversibility of the tangling process.

For these and other reasons neither the Maxwell–Boltzmann nor the Lynden-Bell statistics are proper for describing the relaxation of the Vlasov system. Nevertheless, in the absence of a proper understanding of this problem, we shall employ Maxwell–Boltzmann statistics because it is simple and should give some indication of the relative probabilities of various physical states.

IV. ENTROPY OF A SINGLE ISOLATED HOLE

We wish to calculate the entropy of a single isolated hole moving at velocity u. To understand the calculation it is helpful to imagine the creation of a hole. We start with an unperturbed distribution function $f_0(v)$ and very slowly remove mass M and momentum P, and put in energy T by modifying f_0 in a small region, $m(u-v)^2/2 + q\phi(x) < 0$, in such a way that outside this region f is changed nonresonantly (i.e., reversibly) so that no entropy change occurs. In the outside region f is determined by perturbation theory in terms of $\phi(x)$, $f = f_0(v) + q\phi m^{-1}(v-u)^{-1}\partial f_0/\partial u$. In the outside region, f does not contribute to the entropy change since it does not contain independent degrees-of-freedom. It is completely determined by $\phi(x)$ which, in turn, is determined by f in the inside region. Inside the mixing region, a fine-grain irreversible mixing of phase-space density occurs which causes a coarse grain or macroscale change to occur. The total entropy $(-\sigma)$ of the system is the sum of integrals over the nonmixing and mixing regions denoted p and t (for passing and trapped), respectively. From (15)

$$\sigma = n \int_p \int dx\, dv\, f_p \ln f_p + n \int_t \int dx\, dv\, f_t \ln f_t. \tag{16}$$

During the creation of the holes, as the mixing region in phase space is slowly increased in area and the unmixed region decreased in area, the unmixed region must lose phase-space density of magnitude $f_0(u)$ in an amount sufficient to fill the area of the mixing region. This follows from the fact that the phase-space volume elements are preserved by the Vlasov equation, and $f_0(u)$ is the only value of f in the outside (nonmixing) region to come in contact with the inside region during the slow creation of the hole. Since f in the outside region is changed reversibly during hole creation, the only change in entropy in the outside region is due to its reduction in area and the corresponding loss of $f_0(u)$:

$$n \int_p \int dx\, dv f_p \ln f_p = \sigma_i - n \int_p \int dx\, dv f_0(u) \ln f_0(u)\,, \qquad (17)$$

where

$$\sigma_i = n \int\int dx\, dv f_0(v) \ln f_0(v) \qquad (18)$$

is a constant equal to the initial entropy of the system. Combining (16) and (17) we obtain

$$\sigma = n \int_t \int dx\, dv [f_t(v) \ln f_t(v) - f_0(u) \ln f_0(u)] + \sigma_i\,. \qquad (19)$$

Henceforth, we shall ignore the additive constant σ_i.

We now find the function f_t in the trapped region that minimizes σ subject to the constraints that the total energy, momentum, and mass are constant. In Appendix B we show that provided one can treat the outside region as a linear dielectric medium, the mass M, the momentum P, the energy T, and the potential $\phi(x)$ of a hole are given by

$$[M, P, T] = n \int_t \int dx\, dv [f_t(x, v) - f_0(u)]\{m, mv, [mv^2 + q\phi(x)]/2\} \qquad (20)$$

$$\left(-\frac{\partial^2}{\partial x^2} + \lambda^{-2}\right)\phi(x) = 4\pi nq \int_t dv [f_t(v) - f_0(u)]\,, \qquad (21)$$

where λ^{-2} is k^2 times the real part of the linear susceptibility, i.e.,

$$\lambda^{-2} = \omega_p^2 P \int dv (u - v)^{-1} \frac{\partial f_0}{\partial v}\,, \qquad (22)$$

and P means principal value. This is, of course, the familiar linear expression for the untrapped charge density. For the steady-state case treated in this paper, the essential nonlinear behavior involves the trapped particles. The hole charge Q is equal to Mq/m.

V. THE MAXWELL-BOLTZMANN HOLE

Using Lagrange multipliers α, γ, and $-\tau^{-1}$ for mass, momentum, and energy, we wish to find the function f_t and hole boundary, $v(x)$, that make σ stationary. Setting $\delta\sigma = 0$ gives

$$0 = \int_t \int dx\, dv \left(\ln f_t + 1 + \alpha + \gamma v - \frac{E}{\tau}\right) \delta f_t\,, \qquad (23)$$

$$0 = \int_t dx \left\{ f_t(v) \ln f_t(v) - f_0(u) \ln f_0(u) + \left(\alpha + \gamma v - \frac{E}{\tau}\right)[f_t(v) - f_0(u)] \right\} \delta v(x)\,, \qquad (24)$$

where $E = mv^2/2 + q\phi(x)$ and in (23) $v = v(x)$. [Note that including the variation of f_t contained in ϕ in (20) results in twice the explicit variation.] Since δf_t is arbitrary, (23) gives, in the rest frame of the hole where we can put $\gamma = 0$,

$$f_t(v) = \exp(E/\tau - \alpha - 1)\,. \qquad (25)$$

Since $\delta v(x)$ is arbitrary, (24) requires that $f_t(v) = f_0(u)$ on the boundary. From (25), we see that this means that the boundary must be given by $E = \text{const} = q\phi_m$, where ϕ_m is the potential along the hole boundary. Putting these results together, we find

$$f_t = \begin{cases} f_0(u) \exp[(E - q\phi_m)\tau^{-1}]\,, & \text{for } q\phi(x) < E < q\phi_m\,, \\ f_0(u)\,, & \text{for } q\phi_m < E < 0\,. \end{cases} \qquad (26)$$

Equations (21) and (26) describe a Bernstein–Green–Kruskal mode, i.e., a time-independent solution of the Vlasov and Poisson equations. Thus, the most probable state is a Bernstein–Green–Kruskal mode.

Since a hole is a negative-temperature system ($-\tau < 0$), one would expect it to be unstable. Although it is indeed unstable, it is not easy for it to convert its energy into random thermal motion since it can only resonantly interact strongly with a small local velocity range Δv_T of the plasma.

The trapped particle charge density, denoted by $\rho(x)$, is given by the right-hand side of (21) divided by 4π. For f_t given by (26) we have

$$\rho(x) = 2nq \int_{q\phi(x)}^{q\phi_m} dE \frac{f_0 \exp[(E - q\phi_m)\tau^{-1}] - f_0}{\{2m[E - q\phi(x)]\}^{1/2}} m\,. \qquad (27)$$

For $(E - q\phi_m)\tau^{-1} \ll 1$, we can expand the exponential and integrate to obtain

$$\rho(x) = \frac{8 f_0}{3(2^{1/2})} \tau^{-1} nqm^{-1/2} [q\phi_m - q\phi(x)]^{3/2}\,. \qquad (28)$$

The quantity raised to the 3/2 power is to be interpreted as zero when it is negative. This expansion is not the usual weak turbulence expansion in powers of ϕ. It only implies that the hole is not very deep, i.e., $-\tilde{f} < f_0$.

Poisson's equation (21) can be written

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + V(\phi)\right] = 0\,, \qquad (29)$$

where

$$V(\phi) = -\lambda^{-2}\phi^2 + 8\pi \int^{\phi} d\phi\, \rho(\phi)\,. \qquad (30)$$

Equation (29) can be integrated once to obtain

$$\frac{\partial \phi}{\partial x} = \pm[-V(\phi)]^{1/2}\,, \qquad (31)$$

where

$$-V(\phi) = \lambda^{-2}\phi^2 - \frac{128\pi}{15(2^{1/2})} f_0 \tau^{-1} nm^{1/2} (q\phi_m - q\phi)^{5/2}\,. \qquad (32)$$

Defining a dimensionless potential energy w,

$$w = c^2 q\phi\,, \qquad (33)$$

where

$$c = 32\lambda^2\omega_p^2 m^{1/2} f_0 (15\tau 2^{1/2})^{-1} \tag{34}$$

and using (33) and (34), Eq. (31) can be written

$$dx = \lambda[w^2 - (w_m - w)^{5/2}]^{-1/2} dw . \tag{35}$$

This equation can be integrated to obtain x as a function of $\phi = w/qc^2$.

$$x = \lambda \int_{w_0}^{w_0(\phi(x)/\phi_0)} dw[w^2 - (w_m - w)^{5/2}]^{-1/2} . \tag{36}$$

The "potential" $[w^2 - (w_m - w)^{5/2}]^{1/2}$ is plotted (curve a) in Fig. 1.

At the reflection point w_0, the potential vanishes.

$$w_0^2 - (w_m - w_0)^{5/2} = 0 , \quad w_m = |w_0|^{4/5} + w_0 . \tag{37}$$

When $w_m - w < 0$, $(w_m - w)^{5/2}$ is interpreted as zero. The spatial center of the hole, $x = 0$, has the greatest negative potential w_0. Equation (36), therefore, gives the distance from the center of the hole out to a potential value of ϕ. In order for $\phi \to 0$ as $|x| \to \infty$, w_m must be negative and, therefore, w_0 be equal to or less than -1. The total charge and energy of a hole are

$$Q = 2 \int_0^{x_m} dx \rho(x) , \tag{38}$$

$$T_0 = 4n \int_0^{x_m} dx \int_{q\phi(x)}^{q\phi_m} dE[E - \tfrac{1}{2}q\phi(x)] \times \frac{f_0 \exp[(E - q\phi_m)\tau^{-1}] - f_0}{\{2m[E - q\phi(x)]\}^{1/2}} . \tag{39}$$

For $|x| = x_m$, $\phi = \phi_m$, and $f_t = f_0$ for $|x| > x_m$. The E integral in (39) is readily accomplished if we expand the exponential and use (38) and (27)

$$T_0 = \frac{8nf_0}{5(2m)^{1/2}\tau} \int_0^{x_m} dx[q\phi(x) - q\phi_m]^{5/2} + \frac{1}{2}\phi_m Q . \tag{40}$$

The x integrals can be converted to w integrals using (33), (34), and (35). We obtain

$$x_m = \lambda I_x m , \tag{41}$$

$$Q = -AqI_Q , \tag{42}$$

$$T_0 = Ac^{-2}(3I_T/10 - w_m I_Q/2) , \tag{43}$$

where

$$A = 5(8\pi q^2\lambda^2 c^2)^{-1} , \tag{44}$$

$$I_x = \int_{w_0}^{w_m} dw[w^2 - (w_m - w)^{5/2}]^{-1/2} , \tag{45}$$

$$I_Q = \int_{w_0}^{w_m} dw(w_m - w)^{3/2}[w^2 - (w_m - w)^{5/2}]^{-1/2} , \tag{46}$$

$$I_T = \int_{w_0}^{w_m} dw(w_m - w)^{5/2}[w^2 - (w_m - w)^{5/2}]^{-1/2} . \tag{47}$$

Since w_m is a function only of w_0, the I's are functions only of w_0. Equation (41) shows that w_0 is completely determined by x_m. The quantity x_m is a measure of the spatial width of a hole, but for $w_0 \to -1$, $x_m \to \infty$ because a long tail $[\tilde{f} \sim \exp(-x/\lambda)]$ develops and x_m is not a good measure of the half-width. A better measure of the width is the total charge Q divided by the maximum charge density $\rho(0)$. Using (28), (33), (34), and (42), we find

$$\Delta x = Q/\rho(0) = 2\lambda I_Q w_0^{-6/5} . \tag{48}$$

According to (48), w_0 is also completely determined by Δx. The quantity $\pi Q^2\lambda/T_0$ can be calculated using (42) and (43):

$$\pi Q^2\lambda/T_0 = (5/4)I_Q^2(3I_T/5 - w_m I_Q)^{-1} . \tag{49}$$

According to (49), Q^2/T_0 depends only on w_0 and, therefore, x_m or Δx. In other words the spatial length of the hole depends only on Q^2/T_0. The velocity width Δv of the hole at $x = 0$ (see Fig. 2)

$$\Delta v = 2\left(\frac{2q(\phi_m - \phi_0)}{m}\right)^{1/2} = 2\left(\frac{2}{m}\right)^{1/2} c^{-1}(w_m - w_0)^{1/2} = 2\left(\frac{2}{m}\right)^{1/2} c^{-1}|w_0|^{2/5} . \tag{50}$$

The maximum trapped velocity width of the hole (see Fig. 2) is

$$\Delta v_T = (2/m)^{1/2} c^{-1} |w_0|^{1/2} . \tag{51}$$

The ratio $\Delta v/\Delta v_T$ is, therefore, a function only of Δx or Q^2/T:

$$\Delta v/\Delta v_T = [(w_m - w_0)/-w_0]^{1/2} = |w_0|^{-1/10} . \tag{52}$$

By squaring (50), and dividing (43) by (42) to obtain c^{-2} we can evaluate $(\Delta v)^2$. Using (49) to simplify, we find

$$(\Delta v)^2 = (\tfrac{16}{5})(T_0/M)|w_0|^{4/5}(\pi Q^2\lambda/T_0)/I_Q . \tag{53}$$

This expression relates the velocity width of a hole to

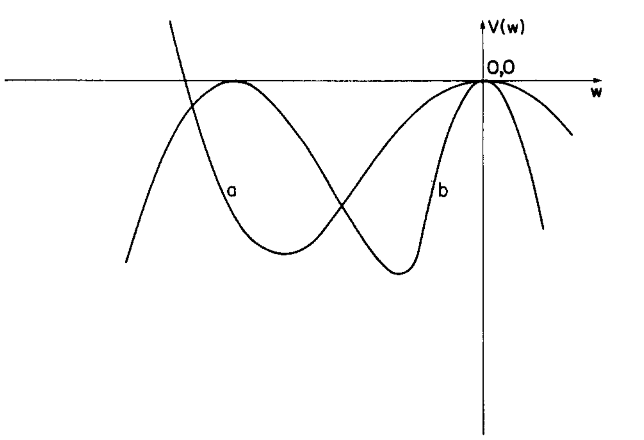

FIG. 1. The potential function $V(w) = (w_m - w)^n - w^2$ vs w with arbitrary V and w scales. Curve (a), $n = \frac{5}{2}$; curve (b), $n = \frac{3}{2}$.

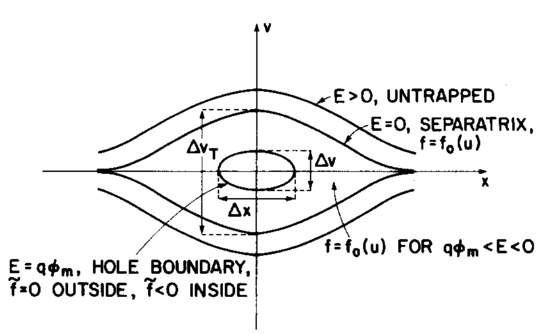

FIG. 2. Contours of constant phase-space density for a hole and background plasma.

its energy, mass, and spatial width. The maximum hole depth is

$$\tilde{f}_m = f_0\{\exp[(q\phi_0 - q\phi_m)/\tau] - 1\} = -f_0 |w_0|^{4/5}(c^2\tau)^{-1} . \tag{54}$$

Using (50) for Δv and (34) for τ, this can be written

$$\tilde{f}_m = -\Delta v \lambda^{-2}\omega_p^{-2}(\tfrac{15}{64}) |w_0|^{2/5} . \tag{55}$$

This is a useful relation between maximum hole depth, $-\tilde{f}_m$, velocity width Δv, and spatial width Δx (w_0 is a function only of Δx).

To calculate the entropy of a Maxwell–Boltzmann hole, we expand the integrand of (19) in powers of

$$\tilde{f} = f_t(x, v) - f_0(u) . \tag{56}$$

We obtain

$$\sigma = n \int_t \int dx\, dv \left(\tilde{f}[1 + \ln f_0(u)] + \frac{\frac{1}{2}\tilde{f}^2}{f_0}\right) . \tag{57}$$

According to (20), the first term in (57) is $Mm^{-1}[1 + \ln f_0(u)]$. By expanding the exponential in (26), it is not difficult to show that the second term in (57) is $2AI_T(5c^2\tau)^{-1}$. If we use a coordinate system in which u is not much larger than Δv, we can expand $f_0(u)$ and use $P = uM$ [which follows from (20)] to obtain

$$\sigma = (M/m)[1 + \ln f_0(0)] + (P/m)f_0'/f_0 + \sigma_0 , \tag{58}$$

where

$$\sigma_0 = n \int_t \int dx\, dv \frac{\tilde{f}^2}{2f_0} = 2AI_T(5c^2\tau)^{-1} . \tag{59}$$

The important piece of the entropy is σ_0 since the first two terms of (58) are always constant during any variation of $\tilde{f}$ or hole interaction (since the total M and P are constant). Using (34), (43), (44), and (49), σ_0 may be written as

$$\sigma_0 = \frac{T_0^{3/4}(10.8)^{1/2} I_T(\pi Q^2\lambda/T_0)^{3/4}}{f_0\lambda^2\omega_p^2(m^2q^2\lambda\pi 36)^{1/4} I_Q^{3/2}} . \tag{60}$$

In general, the I integrals (45)–(47) have to be evaluated numerically as a function of w_0. This has been done and various quantities are plotted in Figs. 3 and 4.

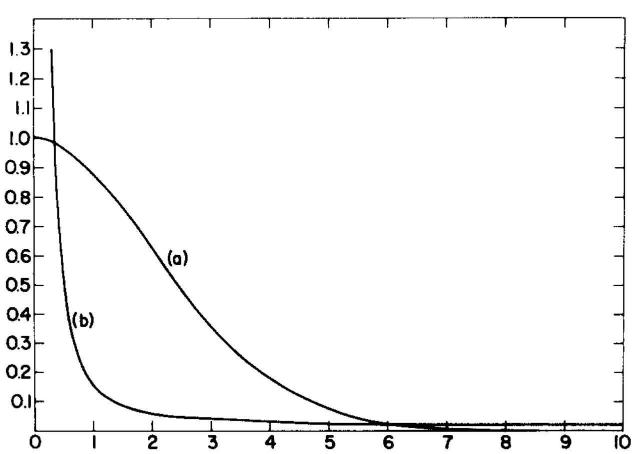

FIG. 3. Curve (a), $\phi(x)/\phi(0)$ vs x/λ from (36) for a Maxwell–Boltzmann hole with $\phi_m = 0$; curve (b), $w_p^2\lambda^2\tilde{f}/(10\Delta v)$ vs $\Delta x/\lambda$ from (78) for a rectangular hole.

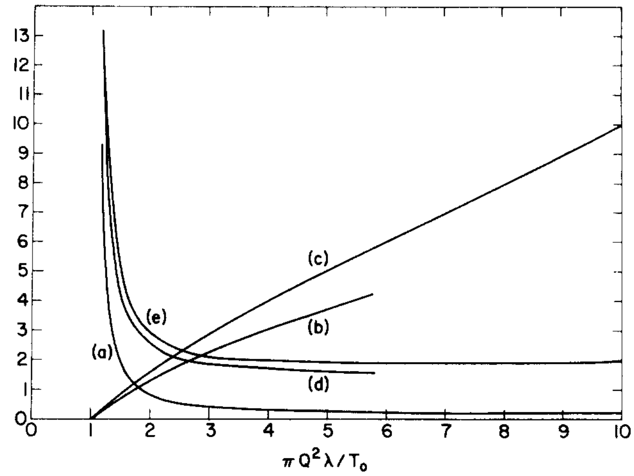

FIG. 4. Various quantities vs $\pi Q^2\lambda/T_0$. Curve (a), $w_p^2\lambda^2\tilde{f}/\Delta v$ from (55) for a Maxwell–Boltzmann hole and from (78) for rectangular hole (the two curves are virtually the same); curve (b), $\Delta x/\lambda$ from (48) for a Maxwell–Boltzmann hole; curve (c), $\Delta x/\lambda$ from (72) for a rectangular hole; curve (d), $\sigma_0 T_0^{-3/4} z$, where $z = f_0\lambda^2 u_p^2(36m^2q^2\pi\lambda)^{1/4}$ from (60) for a Maxwell–Boltzmann hole; curve (e), $\sigma_0 T^{-3/4} z$ from (80) for a rectangular hole.

For the case $w_0 = -1$, $w_m = 0$, I_Q and I_T can be evaluated analytically giving $32/15$ and $2^9/315$, respectively. The maximum value of $\pi Q^2\lambda/T$ is obtained for $w_m = 0$ and $w_0 = -1$ and is approximately 5.8 and corresponds to the Maxwell–Boltzmann hole with the largest possible value of Δx. Figure 3 shows the potential $\phi(x)/\phi(0)$ plotted as a function of x from (36) with $w_m = 0$.

VI. STEP FUNCTION HOLE

A. Exact solution

It is enlightening to consider a hole whose $f_t(v)$ has a simpler analytic form than (26), namely, a hole with constant depth. Such a $f_t(v)$ will not give as small a σ as (60), but it is useful to consider a greater variety of holes. We take f_t to be a constant equal to $f_0 + \tilde{f}$ in an inner region and equal to $f_0(u)$ in an outer region. Again, we can use (24) to determine the f_t of this form that makes σ stationary. If we use (24) to apply to the boundary between the inner and outer regions, it is easy to see that the integrand can be made to vanish all along the boundary only if f_t is a function of E only. Therefore, the boundary between the two regions must be a constant E contour, i.e.,

$$f_t = \begin{cases} f_0(u) + \tilde{f}, & \text{for } q\phi < E < q\phi_m , \\ f_0(u), & \text{for } q\phi_m < E < 0 . \end{cases} \tag{61}$$

Equations (21) and (61) can be solved together to obtain a Bernstein–Green–Kruskal mode. The quantities ϕ_m and $\tilde{f}$ are the free parameters that determine the hole instead of ϕ_m and τ as in the Maxwell–Boltzmann case. The solution for this type of hole is discussed in Appendix A.

B. Rectangular hole

One can obtain simple approximate solutions of hole equilibria by using the step function model but taking the hole boundary (i.e., the boundary between $f=f_0+\tilde{f}$ and $f=f_0$) in the shape of a rectangle instead of $E=q\phi_m$. Thus, for a hole moving with a velocity u and located at $x=0$ we get

$$f_t=\begin{cases} f_0+\tilde{f}, & \text{for } 2|v-u|<\Delta v,\quad 2|x|<\Delta x, \\ f_0(u), & \text{otherwise}. \end{cases} \tag{62}$$

We can now use the extremum property of the hole to obtain $\tilde{f}$, Δx, Δv by making σ stationary subject to constant M, P, and T. As we shall see this gives results similar to the exact solution for the step function hole obtained in Appendix A. We will also show that the results closely approximate the exact Maxwell–Boltzmann hole solution for $\Delta x/\lambda \leq r$.

The advantage in the approximate procedure is that the integrals contained in M, P, T, and σ are easy to evaluate when $\tilde{f}$ is constant and the boundaries are rectangular. Thus, the hole mass is given by (20)

$$M=nm\int_{-\Delta x/2}^{\Delta x/2} dx \int_{-\Delta v/2}^{\Delta v/2} dv\tilde{f}=nm\Delta x\Delta v\tilde{f}. \tag{63}$$

The potential is obtained by integrating (21):

$$\phi(x)=2\pi nq\tilde{f}\int_{-\Delta x/2}^{\Delta x/2} dx'\,\lambda\exp(-|x-x'|/\lambda)\int_{-\Delta v/2}^{\Delta v/2} dv'. \tag{64}$$

The potential energy is obtained from (20)

$$(nq/2)\int_{-\Delta x/2}^{\Delta x/2} dx\,\phi(x)\int_{-\Delta v/2}^{\Delta v/2} dv\tilde{f}. \tag{65}$$

Using (64), this integral may be readily evaluated to give

$$[2\pi Q^2\lambda/(\Delta x)^2][\Delta x\lambda-\lambda^2+\lambda^2\exp(-\Delta x/\lambda)]. \tag{66}$$

The trapped particle kinetic energy is

$$\frac{1}{2}nm\int_{u-\Delta v/2}^{u+\Delta v/2} dv\int_{-\Delta x/2}^{\Delta x/2} dx\,v^2\tilde{f}=\frac{M(\Delta v)^2}{24}+\frac{1}{2}Mu^2. \tag{67}$$

The momentum is

$$P=Mu. \tag{68}$$

According to (20), the total energy T is the sum of (66) and (67). It is convenient to use the rest frame ($u=0$) energy T_0

$$T_0=T-\tfrac{1}{2}Mu^2, \tag{69}$$

and write

$$T_0=\frac{M(\Delta v)^2}{24}+\frac{2\pi Q^2\lambda}{(\Delta x)^2}\left[\Delta x\lambda-\lambda^2+\lambda^2\exp\left(\frac{-\Delta x}{\lambda}\right)\right]. \tag{70}$$

The entropy is given by (57). We need consider only the σ_0 portion. Since $\tilde{f}$ is constant in our present approximation, it follows that

$$\sigma_0=M\tilde{f}[2mf_0(u)]^{-1}. \tag{71}$$

We now minimize σ_0 keeping T, P, and M constant. We solve (63) for Δv, put this into (70), multiply both sides by $(\Delta x)^2$, differentiate with respect to Δx, and set $\partial\tilde{f}/\partial(\Delta x)=0$. The result is

$$\Delta x/\lambda=(\pi Q^2\lambda/T_0)[1-\exp(-\Delta x/\lambda)]. \tag{72}$$

This may be solved for Δx in terms of Q^2/T_0. The solution is shown in Fig. 4. Two regions of interest are

$$\pi Q\lambda/T_0\gg 1,\quad \Delta x/\lambda\approx\pi Q^2\lambda/T_0, \tag{73a}$$

$$\pi Q\lambda/T_0\geq 1,\quad \Delta x/\lambda=2[1-(T_0/Q^2\lambda\pi)]. \tag{73b}$$

Using (72) in (70) we find

$$T_0=M(\Delta v)^2[1-\exp(\Delta x/\lambda)][24g(\Delta x/\lambda)]^{-1} \tag{74}$$

where

$$g(\Delta x/\lambda)=[1+2(\lambda/\Delta x)][1-\exp(-\Delta x/\lambda)]-2. \tag{75}$$

As $\Delta x/\lambda$ approaches 0 and ∞, g has the limits $-(\Delta x/\lambda)^2/6$ and -1, respectively.

Equation (74) can be solved for $(\Delta v)^2$

$$(\Delta v)^2=24(T_0/M)g(\Delta x/\lambda)[1-\exp(-\Delta x/\lambda)]^{-1}. \tag{76}$$

Using (76) for Δv in (63), solving for $\tilde{f}^2$, and using (72) we find

$$\tilde{f}^2=\tfrac{2}{3}(T_0/M)\omega_p^{-4}\lambda^{-4}g(\Delta x/\lambda)^{-1}[1-\exp(-\Delta x/\lambda)]^{-1}. \tag{77}$$

Eliminating T_0/M with (74) we obtain a relationship between $\tilde{f}$, Δv, and Δx:

$$\tilde{f}=\Delta v(6\omega_p^2\lambda^2)^{-1}g(\Delta x/\lambda)^{-1}. \tag{78}$$

Using (63) and (71), σ_0 may be written

$$\sigma_0=M^2(2nm^2f_0\Delta x\Delta v)^{-1}. \tag{79}$$

Using (76) to express Δv in terms of M and T_0, and then using (72) to eliminate M, we obtain

$$\sigma_0=T_0^{3/4}H(\Delta x/\lambda)/f_0\lambda^2\omega_p^2(36m^2q^2\pi\lambda)^{1/4}, \tag{80}$$

where

$$H(\Delta x/\lambda)=(\Delta x/\lambda)^{1/4}(-g)^{-1/2}[1-\exp(-\Delta x/\lambda)]^{-3/4}. \tag{81}$$

Note that H is also a function of Q^2/T_0.

The minimum potential, ϕ_0, is obtained by evaluating (64) at the spatial center of the hole, $x=0$:

$$\phi_0=\phi(0)=4\pi nq\lambda^2[1-\exp(-\Delta x/\lambda)]\tilde{f}\Delta v. \tag{82}$$

The maximum velocity at $x=0$ that can be trapped by this potential, $\Delta v_T/2$ (the v coordinate of the separatrix at $x=0$), is

$$(\tfrac{1}{2}\Delta v_T)^2=-2q\phi(0)/m. \tag{83}$$

Combining (78), (82), and (83) we find

$$(\Delta v/\Delta v_T)^2=-\tfrac{3}{4}g(\Delta x/\lambda)[1-\exp(-\Delta x/\lambda)]^{-1}. \tag{84}$$

In Fig. 4 we have plotted $\omega_p^2\lambda^2\tilde{f}/\Delta v$, $\Delta x/\lambda$, $H\sim\sigma_0T_0^{-3/4}$ vs $\pi Q^2\lambda/T_0$ and in Fig. 3 we have plotted $\omega_p^2\lambda^2\tilde{f}/\Delta v$ vs $\Delta x/\lambda$; all as predicted by the rectangular hole approximation. As is evident from the graphs, the rectangular hole approximation is a good approximation to the Maxwell–Boltzmann hole in the region where the two models both apply, i.e., $\pi Q^2\lambda/T_0\leq 5.8$.

VII. MULTIPLE HOLES

A. Decay of fluctuation energy

In order to relate the results of previous sections and to use entropy arguments to infer the final states of hole-hole collisions, it is useful to consider various N hole states. Consider the interaction of holes going at different velocities which changes one set of holes into another set. The total energy

$$\sum_i \left(T_{0i} + \frac{1}{2} M_i u_i^2\right)$$

must remain constant, but the total self-energy, $\sum_i T_{0i}$, need not. Since $M_i \leq 0$ and $T_{0i} \geq 0$, decreasing the u_i^2 decreases the total self-energy. Decreasing T_{0i} also leads to a smaller σ, and therefore, in a collision of interacting holes, entropy arguments would predict coalescing of holes moving at different speeds and a consequent reduction in total self-energy and electric field energy.

Consider a two hole example. Whenever two holes moving at different velocities coalesce to form a third hole $(1+2 \rightarrow 3)$, there is a loss of total self-energy since conservation of total energy in the rest frame of the coalesced hole $(u_3 = 0)$ requires

$$T_{03} = T_{01} + T_{02} + \tfrac{1}{2} M_1 u_1^2 + \tfrac{1}{2} M_2 u_2^2 .$$

Using momentum conservation, $M_1 u_1 + M_2 u_2 = 0$, the self-energy loss can be written

$$T_{01} + T_{02} - T_{03} = -\tfrac{1}{2} M_1 u_1^2 \,|\,(u_1 - u_2)/u_1\,| .$$

In fact, the only way electric field energy can be lost is the coalescing or mutual deceleration of holes moving at different speeds. If the electric energy decreases, the kinetic energy must increase. In the zero momentum frame $(u_3 = 0)$ this means that plasma must be accelerated to higher speeds which is the result of decelerating or coalescing holes.

This mechanism of hole decay is quite different from the Landau damping of waves where it is much "easier" for the wave to decay since in the wave frame the wave energy is zero and the wave only has to lose its momentum, which it can readily do by flattening the average distribution function which produces a lower σ state. An equivalent explanation is that the energy-momentum ratio for resonant particles and for waves is the same and, therefore, a wave can decay by giving its energy and momentum to the resonant particles.

B. Mass and charge loss, holes for $Q^2/T_0 \rightarrow \infty$

As we show in the next section, there is a tendency for a collection of holes to get rid of "unwanted" charge (or mass). For this reason, we examine holes that have finite Q (or M), but very small or zero T_0. In the rectangular approximation in the limit $T_0 \rightarrow 0$, Eqs. (72), (80), (63), and (78) give

$$\Delta x \rightarrow \pi Q^2 \lambda^2 / T_0 \rightarrow \infty , \quad \sigma \rightarrow (6^{1/2} \omega_p^2 \lambda^2 f_0)^{-1} (-T_0 M)^{1/2} \rightarrow 0 ,$$

$$\Delta v \Delta x \rightarrow (-Mg(\Delta x/\lambda)\Delta x)^{1/2} \rightarrow \infty .$$

We see that a hole with zero energy and finite Q has $\sigma = 0$ and occupies an infinite phase-space area. In fact, further reflection shows that such a hole has ceased to be a bound system at all, and the mass has simply become part of the background plasma or the interstitial material between other holes in phase-space. In other words, one can always get rid of charge or mass into the background at no cost in energy or entropy.

Similar results can be obtained by considering Maxwell–Boltzmann holes. For $\pi Q^2 \lambda / T_0 \gg 5.8$, we divide T_0 and Q equally among N Maxwell–Boltzmann holes where $N = Q^2 \pi / (5.8 T_0)$. Each hole will have maximum $\Delta x = r\lambda$ and $\sigma_0 \sim (-T_0 M)^{1/2}/N$. In the limit $T_0 \rightarrow 0$, the total entropy $N\sigma_0$ vanishes as $T_0^{1/2}$. One can also show that as $T_0 \rightarrow 0$, the total hole area $N \Delta x \Delta v$ diverges as $T_0^{-1/2}$ and the temperature $\tau \sim (-T_0/M)^{1/2}$.

The subject of holes with $T_0 \rightarrow 0$ is an appropriate point to consider the details of the fine grain mixing process that produces the coarse grain average f. Since small grains could have large $\tilde{f}$, that is, differ considerably from the coarse grain average f, one could wonder why the small grains do not themselves self-bind into holes and the holes coalesce into bigger holes and thwart the fine grain mixing process. The answer is that this would occur except that the fine grain holes have a very small binding energy density $T_0/\Delta x \Delta v$, compared with the macroscopic holes and are, therefore, readily torn apart by the large scale turbulence and reduced to even smaller grains. To see this we note that the binding energy density $T_0/\Delta x \Delta v$ is proportional to $\tilde{f}^2 \Delta x \Delta v$, and therefore, goes to zero as the grain (hole) area $\Delta x \Delta v$ goes to zero.

C. Most probable states for arbitrary Q^2/T

In Sec. V, we showed that the most probable $\tilde{f}$ for a given M and T is a Maxwell–Boltzmann hole provided that $1 < \pi Q^2 \lambda / T_0 < 5.8$. In Sec. VI, we considered a more general two-parameter hole for which $1 < \pi Q^2 \lambda / T_0 < \infty$. For $1 < \pi Q^2 \lambda / T_0 < 5.8$, the properties of the rectangular hole are similar to the Maxwell–Boltzmann hole. For $\pi Q^2 \lambda / T_0 > 5.8$, the properties of the rectangular hole are consistent with the Maxwell–Boltzmann hole; however, since one can show that if $\pi Q^2 \lambda / T_0 > 5.8$, a single hole is not the most probable state. In this case the most probable state is obtained by putting into one hole all the energy T_0 and a portion of the charge equal to Q_t such that $\pi Q_t^2 \lambda / T_0 = 5.8$. The remaining charge is put into a hole or holes of zero energy and, therefore, $\sigma = 0$. To prove these assertions we make an arbitrary distribution of the total energy and mass among an arbitrary number of holes and show that the σ_0 given by (80) for such a state is always greater than that of the asserted most probable state (assuming it is not in such a state to begin with). That is, we must show that

$$\sum_i T_{0i}^{3/4} H(Q_i^2/T_{0i}) > T^{3/4} H(Q_l^2/T) , \tag{85}$$

where Q_l is the lesser of Q_t or Q. T_{0i} is the energy of the ith hole in its zero momentum frame and T is the total energy of all holes in the zero momentum frame of the system

$$T = \sum_i \left(T_{0i} + \frac{1}{2} M_i u_i^2\right),$$

$$0 = \sum_i M_i u_i ,$$

$$Q=\sum_i Q_i. \tag{86}$$

For the case $\pi Q^2\lambda/T_0>5.8$, it is easy to see that (85) is satisfied. For this case, $Q_i=Q_t$ and the function H (see Fig. 4) is at a minimum and, therefore, not greater than any of the $H(Q_i^2/T_{0i})$ on the left-hand side of (85). All the H's on the left-hand side can now be replaced with the minimum values which can be divided out on both sides and the inequality is proved since $\sum_i T_{0i}^{3/4}>T^{3/4}$.

For $\pi Q^2\lambda/T_0<5.8$, the proof of (85) is more complicated and we only sketch the essential features here. First, the left-hand side of (85) is replaced with an N hole system in which each hole has the same mass M/N and energy T/N. This configuration has the minimum σ ($\delta\sigma=0$, $\delta^2\sigma>0$) for an N hole system containing T and M. The proof of (85) is now equivalent to proving

$$N(T/N)^{3/4}H(Q^2/NT)>T^{3/4}H(Q^2/T),$$

which is obviously true since H is a monotonically decreasing function in the range $1<\pi Q^2\lambda/T_0<5.8$.

D. Collisional interaction of holes

For want of a better theory, it is tempting to use entropy arguments to predict the results of hole collisions. We should regard such predictions as rough indications only. A computer simulation is currently underway to shed more light on this subject.[6]

Let us assume that by calculating the entropy of various initial and final states, one can infer what is likely to happen when holes interact or decay. Of course, entropy arguments cannot tell us which processes are dynamically possible or the growth or decay rates involved. However, one can hope to deal with these problems by various intuitive considerations. For a given total energy T and mass M, we have already calculated the most probable state in Sec. VIIC. Whether the system can ever reach this state is, of course, another question which we will discuss presently. If we neglect this important point for the moment and assume that a set of interacting holes proceeds toward the most probable hole state consistent with energy, mass, and momentum conservation, we can summarize the entropy predictions as follows: Holes with $\Delta x\ll\lambda$ tend to coalesce to form holes with larger Δx. This occurs because holes with $\Delta x\ll\lambda$ have large $\tilde{f}$ and consequently, large σ. Coalescing increases Q^2/T_0 and, therefore, increases Δx and decreases $\tilde{f}$ and σ. When Δx becomes of order λ, σ and $\tilde{f}$ become less sensitive to Δx. As we have seen, for constant T_0, σ has a weak minimum at $\Delta x/\lambda=r$. According to entropy arguments, when holes of this (large Δx) size interact, the most probable final state has all the energy in one hole along with an amount of charge equal to $(5.8T/\pi\lambda)^{1/2}$. The remaining charge goes into holes with $T_0\to 0$ and is ultimately mixed into the background. Therefore, for a large number of interacting large Δx holes the entropy arguments would predict that the energy tends to concentrate in high-energy, low-mass holes ($M\sim T_0^{1/2}$) and the mass tends to flow to low-energy holes. The high-energy holes have large $\tilde{f}$ and large negative temperature τ and the low-energy holes have small $\tilde{f}$, τ and occupy a large phase-space area. There is a kind of dual cascade with energy flowing one way and mass the other.

However, the requirements of phase-space density conservation cast considerable doubt on this dual cascade process since holes would have to be produced which have a greater $\tilde{f}$ than the initially interacting holes, even though the total σ is less. This is not necessarily a violation of phase-space density conservation since $\tilde{f}$, which is a coarse grain average, could be composed of small grains of very large $\tilde{f}$, which could be unmixed and reassembled to form a deeper hole. However, this seems a bit far-fetched and the prediction probably results from using an entropy measure that does not properly account for mixing. It should be noted, however, that the prediction of coalescing holes with $\Delta x\ll\lambda$ is not affected by this criticism since $\tilde{f}$ decreases in this process.

As we have explained earlier, to the extent that holes moving at different speeds coalesce there will be a reduction in total hole self-energy $\sum_i T_{0i}$. Of course, holes can only interact irreversibly if their relative velocity is not too great. Consider the three-hole interaction $(1+2\to 3)$ or $(3\to 1+2)$, i.e., the coalescing of two holes to form a third or the decay of a hole into two others. Such irreversible interactions that decrease σ can occur only if the trapping widths Δv_T of the three holes overlap, i.e., $|u_1-u_2|\le\Delta v_{T1}+\Delta v_{T2}$. If $\Delta v\ll\Delta v_T$, an even smaller $|u_1-u_2|$ would be required. This is not a precise criterion, since irreversible tidal effects can occur even at somewhat greater separations in velocity. Also the hole interaction modifies u and Δv, i.e., as two holes approach each other they accelerate each other since the holes attract. For example, two holes of equal Δv_T which have a relative velocity $\Delta u\simeq\Delta v_T$ when they are far apart (i.e., greater than λ) will have an increased relative velocity of approximately $\Delta u+\Delta v_T$ as they pass. Also, as they pass, Δv will be reduced (and Δx increased) due to the mutual acceleration.

In applying our entropy arguments we have focused on the most probable final state, whereas it seems more likely that actual hole-hole collisions produce a variety of fluctuations which only tend to reflect the most probable state. In a sense we have ignored the relative diffusion operator D_- from (3b). Thus, D_- will tend to disrupt the self-binding and coalescing tendency of the holes and the actual physical state should apparently be some balance between these competing effects.

There are obviously a number of assumptions implicit in our entropy arguments. One especially important one is that the fluctuations can always be described as a collection of holes, i.e., holes beget only holes. If the background distribution function f_0 in the phase-space region around the colliding holes is constant ($\partial f_0/\partial v=0$), then the product of hole collisions would have to be more holes since conservation of f dictates that no greater f than f_0 can be produced (unless f_0 or $\tilde{f}$ are themselves the average of a mixture of fine grains whose f values are larger than f_0). When $\partial f_0/\partial v \neq 0$, the result of a hole collision appears to be more complex since it is possible to produce fluctuations with f greater than (as well as less than) the local average. This is the process of clump formation. However, for a single species plasma it appears from the arguments leading to (12) that rearrangements of f that conserve M, P, and phase-space density produce mainly holes. In any case, we have assumed that the initial and final states consist only of holes with the same total mass, momentum, and energy. We shall leave unanswered the questions of whether the initial and final states are actually dynamically connected to each other and whether there exist non-hole fluctuations which are more probable than holes which are dynamically connected to the initial states.

Entropy arguments alone cannot determine the time dependence of an evolving system of colliding holes. For holes of velocity u and size $\Delta x, \Delta v$, one can define a hole distribution function $F(\Delta x, \Delta v, u)$ equal to the number of holes per unit of phase-space area. Since holes collide (interact irreversibly) if their velocities are within approximately Δv_T of each other, the hole collision frequency is of order $F\Delta v_T^2$. However, we do not know how to compute $\partial F/\partial t$ since it is not clear what happens when holes collide even if we know the most probable final state.

The electric energy density due to holes of a given size is approximately $\int du\, F T_0$. If the packing fraction (the fractional phase-space area occupied by holes) $\Delta x \Delta v F$ is of the order of one-half as it would be in the clump theory, F would be proportional to Δv^{-1} and the energy density would be proportional to $T_0/\Delta v \sim \Delta v^3$ as in the clump theory.

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APPENDIX A: SOLUTION FOR A STEP-FUNCTION HOLE

The charge density on the right-hand side of (21) is

$$\rho(x) = 2nq \int_{q\phi(x)}^{q\phi_m} dE \tilde{f}\{2m[E - q\phi(x)]\}^{-1/2}$$
$$= 2\tilde{f}(2/m)^{1/2} nq (q\phi_m - q\phi)^{1/2}, \tag{A1}$$

where the square root is to be interpreted as zero when its argument is negative. Using this in (21) and (30), the "potential" is

$$V(\phi) = -\lambda^{-2}\phi^2 - (2/m)^{1/2}(32\pi n/3)\tilde{f}(q\phi_m - q\phi)^{3/2}. \tag{A2}$$

By adjusting ϕ_m one can obtain various potential shapes and the maximum to the left of the origin can be made to occur above or below the $V=0$ axis. A particularly interesting case is $V(\phi_0)=0$ and $\partial V/\partial\phi = 0$ at $\phi=\phi_0$ which occurs for

$$\phi_m = \phi_0/4, \tag{A3}$$

and

$$\tilde{f} = -\lambda^{-2}\omega_p^{-2}(12)^{-1/2}(-2q\phi_0/6m)^{1/2}. \tag{A4}$$

This special case is shown by curve (b) in Fig. 1 [where the dimensionless potential $w = (32\pi n\tilde{f}/3)^{-2} m\lambda^{-4} \times q^{-3}2^{-1}\phi$ is used]. For this case the potential is in the neighborhood of ϕ_0 for an infinite distance before it returns to 0, i.e., $\int dx \approx \int^{\phi} d\phi(-V)^{1/2}$ diverges logarithmically at ϕ_0. This produces an infinitely long (Δx infinite) hole and for ϕ_m slightly different than (A3), a finite but long ($\Delta x \gg \lambda$) hole. The total mass of the hole is equal to m/q times the integral of (A1) over x. As we have seen for a hole with $\Delta x \gg \lambda$, the potential is equal to ϕ_0 over most of the hole except for a small region of order λ at each end. Therefore, the integral is approximately equal to Δx times the integrand evaluated at $\phi=\phi_0$:

$$M = 2\Delta x \tilde{f} n (2m)^{1/2}(q\phi_m - q\phi_0)^{1/2}.$$

Using (A3) and (A4), we obtain

$$M = m\Delta x \lambda^{-2}\phi_0/(4\pi q). \tag{A5}$$

The total energy of the hole can be evaluated in the same manner.

$$T_0 = 2n\int_{-\Delta x/2}^{\Delta x/2} dx \int_{q\phi(x)}^{q\phi_m} dE\tilde{f}[2m(E - q\phi)]^{-1/2}\left(E - \frac{q\phi}{2}\right)$$
$$= -\tfrac{4}{3}\Delta x n\tilde{f}(-3q\phi_0/4)^{3/2}(2m)^{-1/2} = \tfrac{1}{4}\Delta x\phi_0^2 4\pi\lambda^2. \tag{A6}$$

If we set $m(\Delta v)^2/8 = q(\phi_m - \phi_0)$ and use (A3) we find

$$\tfrac{1}{2}\Delta v = [-\tfrac{3}{2}(q\phi_0/m)]^{1/2}, \tag{A7}$$

where Δv is the maximum velocity width of the hole [which is less than the trapping width $(-2q\phi_0/m)^{1/2}$]. Equation (A4) can now be written

$$\tilde{f} = -\lambda^{-2}\omega_p^{-2}\Delta v/6, \tag{A8}$$

which is equal to (78) for $\Delta x \gg \lambda$. Using (A5), (A6), and (A7), we can express ϕ_0, Δv, and Δx in terms of M and T_0

$$\phi_0 = 4T_0/Q, \tag{A9}$$

$$\Delta v = (-24T_0/M)^{1/2}, \tag{A10}$$

$$\Delta x = \pi Q^2\lambda^2/T_0. \tag{A11}$$

Using (A8) and (A10) in (71) we find

$$\sigma_0 = (-T_0 M)^{1/2}/(m f_0 \lambda^2\omega_p^2 6^{1/2}). \tag{A12}$$

Expressions (A9)–(A12) are identical to (82), (76), (72), and (80) derived in the rectangular hole approximation when $\Delta x \gg \lambda$.

APPENDIX B: DIELECTRIC PROPERTIES

The potential due to a hole in its rest frame is given by

$$-\frac{\partial^2}{\partial x^2}\phi(x) = 4\pi nq\left(\int_p dv f_p + \int_t dv f_t\right) + 4\pi\rho_i, \tag{B1}$$

where ρ_i is the ion charge density. We use a linear theory expression for $f_p(v)$ which is valid in the untrapped region $|v| \geq \Delta v_T$

$$f_u(v) = f_0[(v^2 + 2q\phi/m)^{1/2}]$$
$$= f_0(v) + \frac{2q\phi}{m}\frac{\partial}{\partial v^2}f_0(v) + \left(\frac{2q\phi}{m}\right)^2\frac{\partial^2}{\partial(v^2)^2}f_0(v) + \cdots.$$

Keeping only the first two terms of this expressions, we find

$$-\frac{\partial^2\phi(x)}{\partial x^2}=4\pi nq\left(\int_{-\infty}^{-\Delta v_T}dv+\int_{\Delta v_T}^{\infty}dv\right)\frac{q\phi}{m}\frac{1}{v}\frac{\partial}{\partial v}f_0$$
$$+4\pi nq\int_{-\Delta v_T}^{\Delta v_T}(f_t-f_0)dv+4\pi\left(nq\int_{-\infty}^{\infty}du\,f_0+\rho_i\right).$$

The last term on the right-hand side vanishes because of average charge neutrality. For small Δv_T, the first term on the right-hand side can be written as $-\lambda^{-2}\phi$, where $-\lambda^{-2}$ is equal to ω_p^2 times the principal value of $\int_{-\infty}^{\infty}dv\,v^{-1}\,\partial f_0/\partial v$. For a hole moving at velocity u, λ^{-2} can be written

$$\lambda^{-2}=k^2[\mathrm{Re}\,\epsilon(k,ku)-1],\tag{B2}$$

where ϵ is the usual linear theory dielectric function for frequency ω and wavenumber k:

$$\epsilon(k,\omega)=1+\omega_p^2k^{-1}\int dv(\omega-kv)^{-1}\frac{\partial f_0}{\partial v}.\tag{B3}$$

Equation (B1) can now be written

$$-\frac{\partial^2\phi}{\partial x^2}+\lambda^{-2}\phi=4\pi nq\int_t(f_t-f_0)dv.\tag{B4}$$

We use the standard expression for the electric plus nonresonant (nontrapped) kinetic energy:

$$(16\pi^2)^{-1}\int dk\,k^2|\phi(k)|^2\left(\omega\frac{\partial\epsilon^{(r)}}{\partial\omega}+\epsilon^{(r)}\right),\tag{B5}$$

where $\omega=ku$, u is the hole velocity and the superscript (r) means real part, and $\phi(k)$ is the Fourier transform of the potential $\phi(x)$. We use a coordinate system in which the largest values of u are of order Δv or less. This is always possible since interacting holes must have velocities u whose difference cannot be much greater than Δv_T. Since u is small, we neglect the first term in parentheses in (B5) compared with the second. Note that for a wave, $\epsilon=0$ and only the first term contributes and then only for $u\neq 0$. If we now use the Fourier transform of (B4), i.e., $k^2\phi(k)\mathrm{Re}\,\epsilon(k,ku)=4\pi\tilde{\rho}(k)$, the energy (B5) becomes $\frac{1}{2}\int dx\,\phi(x)\tilde{\rho}(x)$, which is the form occurring in (20). The quantity $\tilde{\rho}(x)$ is the hole charge density, $\tilde{\rho}(x)=nq\int dv\tilde{f}$, where $\tilde{f}=f_t(v)-f_0(u)$. The nonresonant kinetic momentum is given by an expression of the form (B5) except that the quantity in parentheses is replaced by $k\,\partial\epsilon/\partial\omega$. The nonresonant momentum is of order $\phi^2\sim(\Delta v)^4$, whereas the trapped momentum [see Eq. (68)] is of order $(\Delta v)^2u\sim(\Delta v)^3$. Therefore, we shall ignore the nonresonant portion in comparison to the resonant portion in (20b).

APPENDIX C. RELATIONSHIP TO LINEAR INSTABILITY THEORY

The hole depth $\tilde{f}$ given by (78) can also be obtained (approximately) from linear stability theory. Suppose we add a long channel or hole of width Δv and depth $-\tilde{f}$ ($\tilde{f}<0$) to an otherwise smooth and stable distribution function f_0, i.e., $f=f_0+\tilde{f}$ for $|v|\leq\Delta v/2$ and $f=f_0$, otherwise. If this f is substituted into the linear dispersion relation, $\epsilon(k,\omega)=0$, [see (B3)], we obtain

$$\omega^2=k^2(\Delta v)^2[\tfrac{1}{4}+\tilde{f}\omega_p^2\lambda^2/\Delta v(1+k^2\lambda^2)],\tag{C1}$$

where $k^2\lambda^2$ is the contribution to $\epsilon(k,\omega)$ from f_0.

Equation (C1) shows that a wave of zero real frequency grows if

$$-\omega_p^2\lambda^2\tilde{f}/\Delta v>\tfrac{1}{4}(1+k^2\lambda^2).\tag{C2}$$

If we put $k\approx\Delta x^{-1}$, then (C2) is approximately (78). The interpretation of (78) is that a negative fluctuation $\tilde{f}$ will tend to stream apart, Δv will decrease, and the electric field energy will decrease until (C2) is satisfied at which point an instability develops and the electric field energy begins to increase. Thus (C2), written with an equal sign, will determine the steady state ratio of kinetic and electric field energy. The relationship between hole self-binding and linear instability was discussed by Berk *et al.*,[5] who also discussed a useful analogy with gravitating masses.

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