PKU Lectures - Lecture VIII

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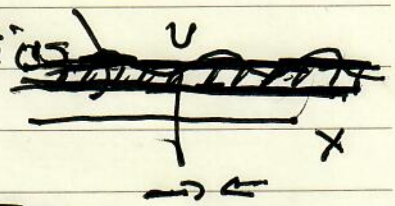
Lecture VIII

Granulations and Holes

So far: Violent relaxation $\rightarrow$ Vlasov
Jeans Eqbrm $\rightarrow$ Holes

$\Rightarrow$

- Phase space density hole as self-bound (via electrostatics) structure



$$(\omega - k v_0)^2 = \frac{k^2 (\Delta v)^2}{4} + \frac{\omega_p^2 \delta \bar{f} \Delta v}{\epsilon(k, k v_0)}$$

$\delta \bar{f} < 0 \rightarrow$ marginal state for ($\epsilon > 0$)

$$\Delta v = -\omega_p^2 \delta \bar{f} / k^2 \epsilon$$

- Hole (or any) coherent structure should be contrasted with statistical model of Vlasov turbulence (quasi-Gaussian bckgnd)

i.e. $\langle \delta f(1) \delta f(2) \rangle \rightarrow$ correlation function

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closure ⇒ mixing at rate $1/\tau_c \sim (k^2 D)^{1/3}$

evolution of correlation

$$\left(\partial_t + v_- \partial_{x_-} - \partial_{v_-} D_- \partial_{v_-}\right) \langle \delta f(1)\delta f(2)\rangle$$

$$= -\frac{2q}{m}\langle \tilde{E}\delta f\rangle \frac{\partial\langle f\rangle}{\partial v}$$

HW: Derive $\langle \delta f^2\rangle$ eqn

relaxation of mean

$$-\int \frac{\partial\langle f\rangle^2}{\partial t} dv$$

$\Delta v \Delta x \Leftrightarrow \ell$

Relaxation drives "clumps" ⇔ phase space eddies.

Theory does not address fluctuation sign.

$\langle \delta f(1)\delta f(2)\rangle$ → spectrum, etc.

– As $\tau_{cL} \gg \tau_c$ but response correlation

$$\langle \delta f^c \delta f^c\rangle = 2\tau_c D_{QL}\left(\frac{\partial\langle f\rangle}{\partial v}\right)^2$$

finite.

$$\lim_{\substack{x_- \\ v_-}\to 0} \langle \delta f(1)\delta f(2)\rangle \gg \lim_{\substack{x_- \\ v_-}\to 0} \langle \delta f^c \delta f^c\rangle$$

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$\therefore \quad \delta f = \delta f^c + \tilde{f}$

response — 'something else' granulation

Implication $\tilde{f}$

$\partial_t \langle f \rangle = -\partial_v J(v)$

Dynamical Friction

$J(v) = -D \frac{\partial \langle f \rangle}{\partial v} + F\langle f \rangle$

→ new/different channel for relaxation, as in Balescu-Lenard Eqn.

$F\langle f \rangle \leftrightarrow \frac{q}{m} \langle E \tilde{f} \rangle$

Prompts limited analogy:

$\langle \tilde{f}^2 \rangle \leftrightarrow$ TPM correlation.

$\langle \tilde{f}^2 \rangle = \frac{1}{n} \langle f \rangle \delta(x-) \delta(v-)$

H W → Re-derive $\partial \langle f \rangle / \partial t$ in 3D for TPM.

Many Questions:

i) Is (pure) violent relaxation ↔ finite time singularity (i.e. rate of decay depends on coarse graining ?!). strength of dependence ?

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ii.) Why / How do holes form?

→ Will show hole is a most probable finite state" ↔ "most probable BGK Model"
↔ stirred plasma will naturally leave residue of holes.
→ statistical theory.

iii.) What is absent in statistical theory?

→ Hole self-field!
↔ can compare self, rms field?

iv.) How do holes interact?
→ do holes merge?
→ end state of multi-hole environment.

v.) How do holes/clumps interact with free energy source.

i.) Violent Relaxation – How violent is violent?

Lynden-Bell: $1/\tau_{Relax} \sim (\delta\rho)^{1/2}$

ideal, Jeans timescale

Phase mixing: single mode $1/\tau_{relax} \sim \left(\frac{v\Omega'^2}{3}\, n^2\right)^{1/3}$

weak but explicit dependence on coarse graining

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does "Fairly violent", not purely violent ?
?

Chaotic : $1/\tau_{relax} \sim h_L \sim 1/\tau_{exp}$

$\ell = \ell_0 e^{t/\tau_{exp}}$ $\rightarrow$ [illegible] $1/\tau_{exp} \sim \ln[L/\ell_0]$

turbulent: $\delta \ell^2 \sim \epsilon t^3$

(super-diffusive)

$\boxed{\text{cascade in 1 eddy time.}}$

$$t_{relax} \sim \left(\frac{L^2}{\epsilon}\right)^{1/3}$$

$\Rightarrow$ Does 'strong' violent mixing/relaxation imply $\begin{cases} \text{chaos} \\ \text{finite time singularity} \end{cases}$

Does $\langle \delta f^2 \rangle$ cascade to small scales in finite time in Vlasov Turbulence?

Unknown.

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→ Hole as Most probable State!
  (How Form)

- Now connect hole to constrained entropy maximization.

- show hole is "most likely" BGK solution, [particle mode i.e. $V_0 < V_{Th}$]

- ignore degeneracy → use Maxwell - Boltzmann statistics.

Seek calculate entropy of single isolated hole moving at u.

Recall:

- BGK mode

$$\cancel{\frac{\partial f}{\partial t}} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$$

$$f = f\left(\frac{mv^2}{2} + \phi,\ P\right)$$

$$\nabla^2\phi = -4\pi n q \int dv\, f\left(\frac{mv^2}{2} + \phi, P\right)$$

$$u = P/M < v_{th}$$

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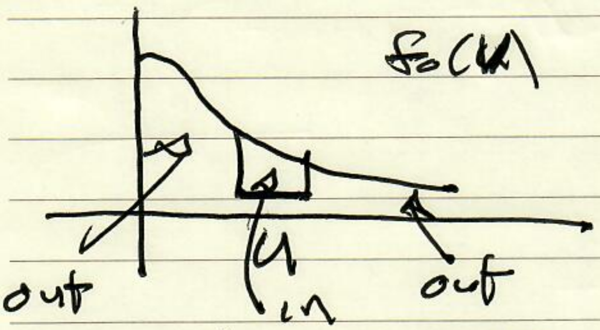
How specify f?

→ Entropy maximization!
.. but in localized region.

constrant L-B.

initial dist



- remove M, P
- add T

by modifying $f_0(u)$

in: $\frac{m}{2}(u-v)^2 + q\phi < 0$

(a la δW)

→ out → "non-resonant"
→ in → resonant

akin

so outside → no entropy change
(reversible!)
→ crank in P.T.
→ all determined by ϕ, from inside

inside → mixing
→ change in coarse grained distr.

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so:

$$S = -n \iint_{\substack{\text{trapped} \\ \text{inside}}} dx\,dv\; f_p \ln f_p \;-\; n \iint_{\substack{\text{circ.} \\ \text{outside}}} dv\,dx\; f_t \ln f_t$$

$\sigma = -S$, Dupree notation (1982)

f conserved ⇒

Outside region must change entropy

~~but~~ outside reversible

⇒

outside region changes entropy by reducing area and loses $f_0(u)$
↑
initial distr.

i.e.

$$n \iint_p dx\,dv\; f_p \ln f_p = \sigma_i - n \int_p dx \int dv\, f_0 \ln f_0$$

$$\sigma_i = \iint dx\,dv\; f_0 \ln f_0 \quad , \text{all.}$$

~~so~~

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So need maximize:

entropy

$$\mathcal{T} = n \iint_{+} dx\,dv \Big[\underset{\text{hole}}{f_t(x,v) \ln f_t(x,v)} - f_0(u) \ln f_0(u) \Big] + \mathcal{T}_i$$

Now, maximize $\mathcal{T}$ s/t

M, P, T conserved:

$$\begin{pmatrix} M \\ P \\ T \end{pmatrix} = n \int_{+} dx \int dv \left(f_t - f_0 \right) \begin{bmatrix} m \\ mv \\ \frac{m}{2}v^2 + \varepsilon\phi \end{bmatrix}$$

where

$f_t = f_t(x, v)$

$f_0 = f_0(u)$

note: M-B. not fussed re: phase volume.

$$\nabla^2 \phi = -4\pi n_0 q \int dv\, \delta f$$

$$= -4\pi n_0 q \int dv \Big[\underbrace{f_t - f_0}_{\substack{\text{hole} \\ \text{resonant}}} + \underbrace{\delta f_p}_{\substack{\text{passive} \\ \text{(NR)}}} \Big]$$

δf_p = treat as linear

$\delta f_p \sim \phi$

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so

$$-\frac{\partial^2}{\partial x^2} - 4\pi n_0 q \int dv\, \delta f_p = 4\pi n_0 q \int dv \left[f_t - f_0 \right]$$

trapping in self-field

$$\frac{+1}{\lambda^2}\phi$$

Ultimately should resemble nonlinear wave eqn.

$$1/\lambda^2 = \omega_p^2\, \phi \int \frac{dv}{u-v} \left(\frac{\partial f_0}{\partial v} \right) \rightarrow \text{screening length}$$

then: Lagrange multipliers

$\alpha \rightarrow$ mass

$\gamma \rightarrow$ momentum

$\frac{-1}{T} \rightarrow$ temperature

HW: Lynden-Bell Hole

$\rightarrow$ Maxwell-Boltzmann Hole (ignores degeneracy)

$$\boxed{S = n \int dx \int dv \left\{ \left[f_t(v) \ln f_t(v) - f_0 \ln f_0(u) \right] + \left[f_t - f_0(u) \right] \left[\alpha + \gamma V - \frac{E}{T} \right] \right\}}$$

Now, have localized state, so

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$$0 = \delta \mathcal{F}$$

$$= \iint dx\, dv \left(\ln f_t + 1 + \alpha + \gamma v - \frac{E}{T} \right) \delta f_t$$

$+$

<u>and</u> also must minimize with respect to hole bndry (localized state):

$$0 = \int_{t_r} dx \left\{ \left(f_t(v) \ln f_t(v) - f_0(u) \ln f_0 \right) + \left(\alpha + \gamma v - \frac{E}{T} \right) \left[f_t(v) - f_0(u) \right] \right\}_{\text{bndry.}} \delta v$$

$$\delta \mathcal{F} / \delta f_t = 0 \Rightarrow$$

$$\ln f_t = -1 - \alpha - \gamma v + E/T$$

in rest frame, $p = 0$

$$f_t = \exp\left[\frac{E}{T} - \alpha - 1 \right] \Rightarrow \text{hole distr.}$$

and

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$\delta T/\delta N(x) = 0$

$f_f(v) = f_0(u)$ | bndry $\left\{ \text{i.e. solves } \delta N = 0 \text{ eqn.} \right.$

so

$E = 2\phi_M$ = const on bndry.

[ϕ_M to match separatrix]

so

$$\boxed{f_f = \begin{cases} f_0(u) \exp\left[\left(\dfrac{E - 2\phi_M}{T}\right)\right] & 2\phi < E < 2\phi_M \\ f_0(u) & 2\phi_M < E < 0 \end{cases}}$$

"RGK mode", most likely.

- hole is negative temperature state, so unstable, but difficult interact resonantly.

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Now, need calculate potential:

$$\left[-\frac{\partial^2}{\partial x^2}+\frac{1}{\lambda^2}\right]\phi = 4\pi n q\int dv\,\left[f_t - f_0\right]$$

$$= 4\pi\rho$$

$$f_t = f_0(u)\exp\left[\frac{E-q\phi_M}{T}\right]$$

and,

$$dE = mv\,dv$$

$$= m\left[\frac{2(E-q\phi)}{m}\right]^{1/2} dv$$

so

$$\boxed{\rho(x) = 2nq\int_{q\phi(x)}^{q\phi_M} dE\,\frac{f_0\exp(E-q\phi_M)/T - f_0}{\left\{2m\left[E-q\phi\right]\right\}^{1/2}}}$$

and $(E-q\phi_M)/T \ll 1 \Rightarrow$ expansion

$$\boxed{\rho(x) = \frac{8f_0}{3\sqrt{2}}\,\frac{nq}{\sqrt{m}\,T}\left[q\phi_M - q\phi(x)\right]^{3/2}}$$

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so Poisson Eqn → Hole Dist. Most Likely BGK.

$$\boxed{\left(-\frac{\partial^2}{\partial x^2} + \frac{1}{\lambda^2}\right)\phi = \frac{8 f_0}{3(2)^{1/2}} \frac{1}{T} \frac{\Sigma}{m} \left[2\phi_M - 2\phi\right]^{3/2}}$$

$(\;)^{3/2} \to 0$ if <0.

- not usual $\phi^{3/2}$
- hole not deep

$\left.f_0\right] \; \underline{|-\tilde{f}| < f_0}$

note structural similarity to (ion-acoustic) soliton:

i.e. solving fluid eqns with form $\phi(x-ut)$ etc.

$$\partial_x^2 \phi = -\frac{1}{\lambda_D^2}\left(\frac{1}{\left(1-2\phi\frac{c_s^2}{u^2}\right)^{1/2}} - e^{\phi}\right)$$

similar form, though:

soliton → fluid $\quad \frac{\omega}{k} \to u > v_{Th_i}$

hole → resonant $\quad u < v_{Th}$.

deal with both by Sagdeev potential

i.e. Sagdeev potential $\not\phi'$ and integrate

so PE ⇒

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi)\right] = 0$$

$$V(\phi) = -\frac{\phi^2}{\lambda^2} + 8\pi\int^{\phi} d\phi\, \rho(\phi)$$ potential

so integrate again:

$$\frac{\partial\phi}{\partial x} = \pm\left[-V(\phi)\right]^{1/2} \qquad \int dx = \int \frac{d\phi}{\left[-V(\phi)\right]^{1/2}}$$

$$-V(\phi) = \frac{\phi^2}{\lambda^2} - \frac{128\pi}{15(\sqrt{2})}\frac{f_0}{T}\, n m^{1/2}(q\phi_m - q\phi)^{5/2}$$

Now, can further simplify:

$W = c^2 Z\phi$ (rescale ϕ)

$c = 32\lambda^2\omega_p^2 m^{1/2} f_0 (15T\sqrt{2})^{-1}$

so

$$x = \lambda\left[W^2 - (W_m - W)^{5/2}\right]^{-1/2} dW$$

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and integrate:

$$\boxed{x = \lambda \int_{w_0}^{w_0 \phi(x)/\phi_0} dw \left[w^2 - (w_m - w)^{5/2} \right]^{-1/2}}$$

solution ⇒ ① trajectory of trapped particle.

↓ ⟺ given potential contours

implicit form $\phi(x)$.

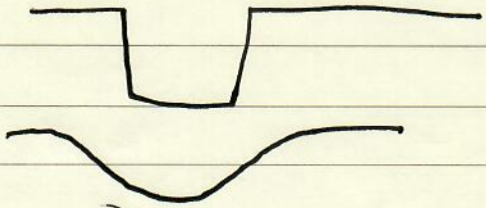
①

→ [ ] = 0 → reflection pts.

→ width

②

→ ϕ largest at center (deepest)



appears as trapping potential, particle mode.

③

→ $w_m < 0$ for $\phi \to 0$ at ∞.

and $w_0 < -1$

④

Now, can calculate:

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$$Q = 2\int_{0}^{x_M} dx\, \rho$$

charge

$$T_0 = 4n \int_{0}^{x_M} dx \int_{q\phi(x)}^{q\phi_M} dE\, [E - q\phi(x)] \ast$$

$$\left\{ \frac{f_0 \left[e^{(E - q\phi_M)/T} - 1 \right]}{2m\,[E - q\phi(x)]^{1/2}} \right\}$$

and lots crunch;

→ velocity width ($\sim \Delta V_{tr}$)

$$\Delta V = 2\left(\frac{2q(\phi_M - \phi_0)}{M}\right)^{1/2} = 2\left(\frac{2}{M}\right)^{1/2} C^{-1}\, |W_0|^{3/5}$$

for max:

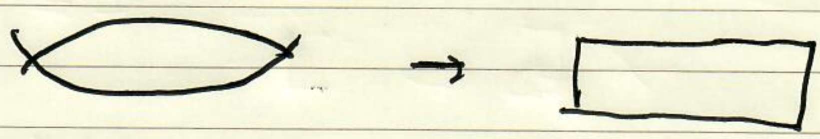
$$\Delta V_{tr} = (2/M)^{1/2}\, C^{-1}\, |W_0|^{1/2}$$

and show forms for ΔV^2, $\tilde{f}$, ∇ etc. in terms T_0, Q..

Now, to better understand holes, a convenient approx is box hole, i.e.

$$f_t = \begin{cases} f_0(u) + \overset{\text{const}}{\tilde{f}} & 2\phi < E < q\phi_M \\ f_0(u) & 2\phi_M < E < 0 \end{cases}$$

w.r.t



$$\underset{\text{bndry}}{E = 2\phi_M} \rightarrow \text{box}$$

and expanding σ for small $\tilde{f}$,
(non-degenerate)

$$\sigma = n \int_{tr} dx \int dv \left(\tilde{f}\left[1 + \ln f_0\right] + \frac{1}{2}\frac{\tilde{f}^2}{f_0} \right)$$

$$\bar{\sigma} \Leftrightarrow \left\langle \frac{\delta f^2}{f_0} \right\rangle \Leftrightarrow \Omega$$

[entropy, enstrophy, phase space intensity

and simpler:

$$f_t = \begin{cases} f_0 + \tilde{f}, & 2|v-u| < \Delta v \\ & 2|x| < \Delta x \\ f_0(u), & \text{otherwise} \end{cases}$$

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and can simplify:

$$\boxed{M = mn\,\Delta x \Delta v\,\tilde{f}} \qquad (M < 0)$$

$$\left(-\partial_x^2 + \frac{1}{\lambda^2}\right)\phi = 4\pi n_0 q \int \tilde{f}\,dv$$

$$\phi(x) = 2\pi n q \tilde{f} \int_{-\Delta x/2}^{\Delta x/2} dx' \,\lambda\, e^{-|x-x'|/\lambda} \int_{-\Delta v/2}^{\Delta v/2} dv$$

and

$$PE = \frac{nq}{2} \int_{-\Delta x/2}^{\Delta x/2} dx\,\phi(x) \int_{-\Delta v/2}^{\Delta v/2} dv\,\tilde{f}$$

so assembling:

charge

$$\boxed{T = \frac{1}{2}Mu^2 + M\frac{(\Delta v)^2}{24} + 2\pi \frac{Q^2}{(\Delta x)^2}\lambda\left[\Delta x\,\lambda - \lambda^2 + \lambda^2 \exp\left(-\frac{\Delta x}{\lambda}\right)\right]}$$

$$\boxed{\nabla = M\tilde{f}\left[2m f_0(u)\right]^{-1}}$$

by constrained minimization,
can crank out:

$$\frac{\Delta x}{\lambda} = \left(\pi \frac{Q^2 \lambda}{T_0}\right)\left[1 - \exp(-\Delta x/\lambda)\right]$$

$$\tilde{f} = \Delta V\,(6\omega_p^2 \lambda^2)^{-1} g(\Delta x/\lambda)^{-1}$$

$$\phi_0 = \phi(0) = 4\pi n q \lambda^2 \left[1 - \exp(-\Delta x/\lambda)\right] \tilde{f}\,\Delta V$$

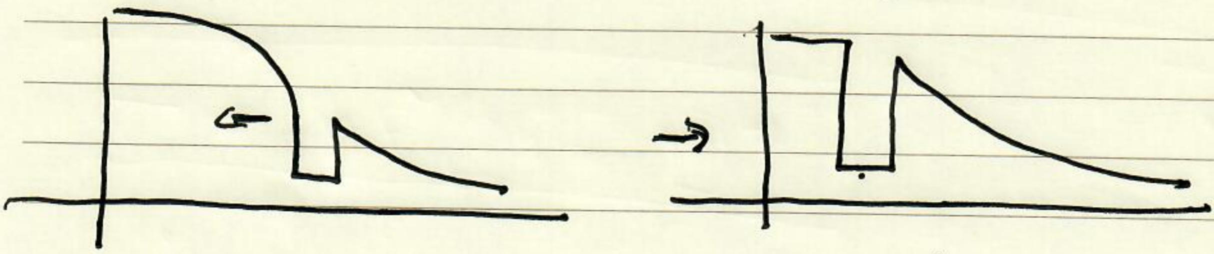
$\rightarrow$ Box Hole Works.

iii.) Self-Trapping / Field

{ Pg. 18-2
{ Lecture VII

iv.) Hole Dynamics

Do holes grow?
What does growth mean?



f conserved $\rightarrow |\tilde{f}| \uparrow$

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Holes grow by moving ↑ ϕ gradient.
But how??

⇝ Momentum Exchange

Consider the following argument:

$$f = \langle f \rangle_{g} + \tilde{f}$$

$\langle f \rangle_{g}$: indep t, smooth

$\tilde{f}$: localized part i.e. hole

$$\frac{\partial}{\partial t} n \int dx \int dv \left[m v \right] = \begin{pmatrix} 0 \\ \frac{\partial P}{\partial t} \end{pmatrix}$$

$\frac{\partial P}{\partial t}$: change in momentum.

Now:

$$df/dt = 0 \quad , \quad \frac{d}{dt} f^2 = 0$$

$$\frac{\partial}{\partial t} \int dv \int dx \left(\underbrace{\cancel{f_0^2}}_{\text{static}} + 2 f_0 \tilde{f} + \tilde{f}^2 \right) = 0$$

note $\partial_t \int \tilde{f}^2 \rightarrow$ akin $\langle \delta f^2 \rangle$ eqn.

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So

$$\partial_t \int dv \int dx\, \tilde{f}^2 = \frac{\partial}{\partial t} \int dv \int dx \left[f_0\, \tilde{f} \right]$$

$$= -\frac{\partial}{\partial t}\, 2 \int dv \int dx \left(f_0(u) + (v-u)\frac{\partial f_0}{\partial u} \right) \tilde{f}$$

$$= 0 - 2 f_0' \frac{\partial P}{\partial t}$$

Thus

$$\partial_t \int dx \int dv\, \tilde{f}^2 = -2 f_0' \frac{\partial P}{\partial t}$$

momentum balance of structure control result.

Point: Momentum budget!

1 species $\quad \frac{\partial P_h}{\partial t} = 0 \quad \rightarrow$ no growth

1 species + wave $\quad \frac{\partial P_h}{\partial t} + \frac{\partial}{\partial t} P_W = 0$

$$\frac{\partial P_h}{\partial t} = -\frac{\partial}{\partial t} P_W \quad \text{(screening)}$$

hole $\leftrightarrow$ wave coupling $\leftrightarrow$ correspondence to Linear stability!

2 species: i.e. electrons and ions active,

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$$\partial_t P_{h_e} = -\partial_t P_i$$

$$\partial_t \int dx \int dv \, \tilde{f}^2 = -2 f_0' \partial_t P_i$$

$\rightarrow$ growth possible.

Now furthur for electron-ion, can write:

$$\partial_t n m_e \int dv \int dx \, \tilde{f}_e^2 = -2 \langle f_e \rangle_0' \frac{\partial P_e}{\partial t}$$

$$\partial_t n m_i \int dx \int dv \, \tilde{f}_i^2 = -2 \langle f_i \rangle_0' \frac{\partial P_i}{\partial t}$$

Now,

$$\frac{\partial P_e}{\partial t} = -\frac{\partial P_i}{\partial t}$$ { imposed by momentum balance

$$\boxed{\frac{n m_e}{\langle f_e \rangle_0'} \frac{d}{dt} \int dx \int dv \, \tilde{f}_e^2 = -\frac{n m_i}{\langle f_i \rangle_0'} \frac{d}{dt} \int dx \int dv \, \tilde{f}_i^2}$$

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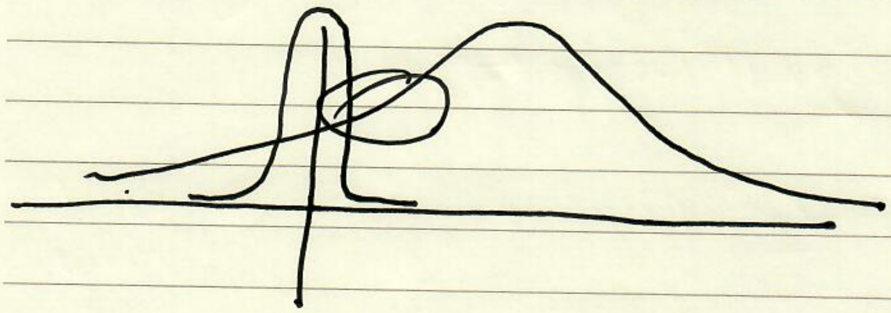
if $f_{0_e}' f_{0_i}' < 0$

→ growth possible

i.e. finite current



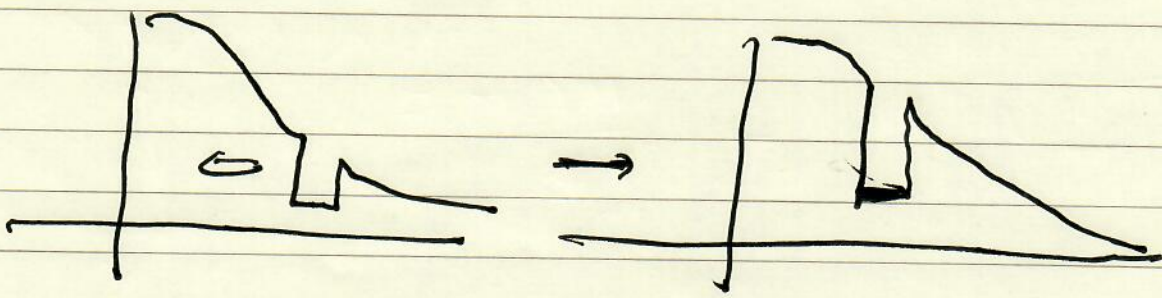
but threshold ↛ linear threshold.

indeed ⟺ strong overlap of electron and ion distribution functions!

→ bad for waves → ion Landau damping

→ good for scattering off structures.

momentum scattering can push hole up gradient



To be continued.