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Growth of phase-space density holes

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Phase-space density holes are shown to grow in a plasma for any nonzero electron–ion drift velocity. As a hole grows, its depth, velocity width, and electrostatic potential increase. For a hole with velocity u, the growth rate is of order $-v_e^2 f'_{0e}(u) v_i^2 f'_{0i}(u)$ times the bounce frequency of a particle trapped in the hole. The theoretical predictions agree reasonably well with a recent computer simulation. The results call into question the role of linear stability theory. Energy and momentum conservation are analyzed in detail, and the relationship to the clump instability is discussed.

I. INTRODUCTION

A. Phase-space holes

It has been argued[1] that fluctuations that move at characteristic particle speeds in a turbulent plasma can be modeled as a collection of phase-space density holes. In this paper we enlarge on that theme by showing that in the simple case considered, even slight departures from plasma equilibrium make such isolated holes unstable to growth. Whereas the threshold for linear instability occurs at a finite and usually significant departure from equilibrium, holes appear to require only an infinitesimal departure. This fact suggests that the prevailing view of stability theory as essentially linear stability theory should be modified. Furthermore, the fact that holes may provide the basic instability mechanism that causes the turbulence calls into question the validity of perturbative nonlinear models in which the linear theory provides the lowest-order approximation. The hole instability is related to the clump instability,[2] which also has a threshold below the linear value. We discuss this relationship which clarifies the physics and limitations of both the hole and clump models.

In this paper, we compute the hole growth rate for the simple case of a one-dimensional electron ion plasma with a relative drift velocity. The calculated growth rate is in good agreement with that observed in a recent computer simulation.[3] The simulation also contained clear evidence of holes and clump-like structures in phase space which gives further support to the theory.

This paper deals only with a one-dimensional system since it is more tractable analytically and more amenable to computer simulation. However, one can carry out a straightforward (although complicated) generalization to a three-dimensional plasma with a magnetic field—a case we will describe in a subsequent publication. The properties of holes for a one-dimensional plamsa have been discussed at length in Ref. 1. For present purposes, a hole is a BGK mode in the form of a localized depression in the phase-space density of magnitude $-\tilde{f}$, spatial width Δx, and velocity width Δv, all moving at the hole velocity u. In Sec. VIB of Ref. 1, relatively simple approximate relationships between these quantities and the hole energy $T = T_0 + \frac{1}{2}Mu^2$, self-energy T_0, momentum $P = Mu$, mass M, and charge $Q = qM/m$ are described. In a two species plasma (electrons and ions) one can have an electron hole or an ion hole, depending on whether it is the electrons or the ions which have a local depression in their phase-space density and are trapped in the resulting potential energy well.

Although in principle a hole or BGK mode can have an infinitely complex structure, we showed in Ref. 1 that the most probable (maximum entropy) holes are completely determined by three parameters, e.g., P, T_0, and M or u, Δx, and Δv. This is analogous to characterizing linear waves by the three parameters, amplitude, frequency, and wavenumber. Although the frequency and wavenumber are related by a dispersion relation, the three parameters determining hole structure are independent and arbitrary, within certain limits. For example, unlike the phase velocity of linear waves, the hole velocity u can have any value as long as the shielding distance λ given by (165) is real. This means hole velocities are of the order of, or less than, the average particle velocities. Thus, holes and waves tend to occupy different regions of velocity space. We argued in Ref. 1 that fluctuations for which $\lambda^2(u) > 0$ tend to organize themselves into holes, which can be regarded as a fundamental constituent or building block of a turbulent plasma. In this model, holes play a role analogous to that traditionally assigned to waves. We argue for this picture not only on an analytical level but on an intuitive level where historically the vocabulary of plasma physics has been heavily weighted by terminology and concepts derived from linear waves and instabilities.

The concept of a hole as a separate and identifiable entity is further enhanced by the fact that if it grows or accelerates slowly, it behaves like a macroscopic "rigid" body and obeys Newton's second law. "Slowly" means that $\gamma\tau \ll 1$ and $\dot{u}\tau/\Delta v \ll 1$ where γ and $\dot{u}$ are the hole growth rate and acceleration and τ is the hole trapping time which is approximately equal to $\Delta x/\Delta v$. This restriction is analogous to the criterion, $\gamma/\omega \ll 1$, for the existence of a wave packet with a definite energy and momentum.

We list here several useful approximate formulas from Sec. VIB of Ref. 1. The hole velocity width Δv is related to the minimum trapped-particle potential energy $q\phi_0$ by

$$\tfrac{1}{8}m\Delta v^2 = g(\Delta x/\lambda)q\phi_0, \tag{1}$$

where

$$g(z) = (1 + 2/z)[1 - \exp(-z)] - 2, \tag{2}$$

and λ is the shielding distance given by (165) and is of order of the Debye length λ_D. The quantities q, m, n, and v_α are the particle charge, mass, average number density, and thermal speed, respectively. The BGK equilibrium requires that

$$\tilde{f} = \Delta v[6\omega_p^2\lambda^2 g(\Delta x/\lambda)]^{-1}. \quad (3)$$

The hole mass is given by

$$M = nm\Delta x\Delta v\tilde{f}. \quad (4)$$

Equations (1)–(4) apply to either ion or electron holes provided the ion or electron values of q, m, n, etc., are used.

B. Hole growth rate

Conceptually, the physics of hole growth is quite simple and a rather accurate formula for γ for small u can be obtained from a simple calculation which we shall now describe. However, it should be borne in mind that the details and the justification for our simple calculations are quite complicated and occupy the bulk of this paper.

In a two-species plasma, a hole of one species will experience a Fokker–Planck drag force due to the reflection or scattering of particles of the opposite species. For example, for an isolated ion hole, the drag force due to reflected electrons is $\omega_{pe}^2\,\phi_0^2 f'_{0e}/\pi$, where $\omega_{p\alpha}^2 = 4\pi n_\alpha q_\alpha^2/m_\alpha$, $f_{0\alpha}$ is the average distribution of the αth species, and $f'_{0\alpha} = (\partial/\partial u)f_{0\alpha}(u)$. The hole acceleration, $\dot{u} = du/dt$ can be obtained by equating this force to the rate of change of the hole momentum $M\dot{u}$,

$$M\dot{u} = \omega_{pe}^2\,\phi_0^2 f'_{0e}/\pi. \quad (5)$$

Since the trapped-particle distribution function remains constant as the hole accelerates, the depth of the hole will change at the rate

$$\frac{\partial\tilde{f}}{\partial t} = -\dot{u}f'_{0i}. \quad (6)$$

Using (1) and (3)–(6), it is a simple matter to show that the growth rate of the potential ϕ_0 (which is proportional to $\tilde{f}^2$) is given by

$$\gamma \equiv \frac{1}{\phi_0}\frac{d\phi_0}{dt} = -\left(\frac{\Delta v}{\Delta x}\right) 8\lambda^4\omega_{pi}^2\omega_{pe}^2 f'_{0e}(u)f'_{0i}(u). \quad (7)$$

Thus, if the velocity gradients of $f_{0\alpha}$ for electrons and ions have opposite signs at the hole velocity u, the hole depth will grow. The characteristic frequency of the growth rate is the particle bounce or trapping frequency $\Delta v/\Delta x$. The instability is illustrated schematically in Fig. 1, showing electrons moving through the ions with a drift velocity v_D. The ion hole (dashed line) has a negative charge which reflects electrons, causing a loss in electron momentum (dashed line). To conserve momentum, the negative mass ion hole decelerates to smaller velocity u, where $f_{0i}(u)$ is also larger, and consequently the hole depth increases. Clearly, the isolated hole instability occurs for any finite value of v_D, whereas the linear ion-acoustic instability requires relatively large values of v_D and T_e/T_i. The growth rate (7) applies to either electron or ion holes when the appropriate Δv is used.

Actually (5) is not quite correct since it omits the contribution from a thin velocity layer of width Δv of passing ions which gain momentum at a rate $\gamma n_i m_i \Delta v^3 \Delta x f'_{0i}(u)$. In effect, this is the ion Landau damping rate in the limit $t \gg \tau$ and $\gamma\tau \ll 1$. When this correction is added to the left-hand side of (5), it causes λ^4 in (7) to be replaced by the smaller (approximate) value $\{\lambda^{-4} + 10\omega_{pi}^2[f'_{0i}(u)]^2\}^{-1}$. Except for numerical factors, the resulting growth rate agrees with the value (110) obtained by a more detailed and rigorous calculation. However, a detailed consideration is necessary to justify the simple procedure since the actual physics is quite complex. Furthermore, it is important to understand in detail the conservation of mass, momentum, and energy.

An instability well below the linear threshold was observed in a recent computer simulation.[3] The simulation modeled a one-dimensional electron–ion plasma with a relative drift velocity v_D, $m_i/m_e = 4$, and equal ion and electron temperatures as shown in Fig. 1. In Fig. 2 we have plotted the observed simulation growth rates for the mean square

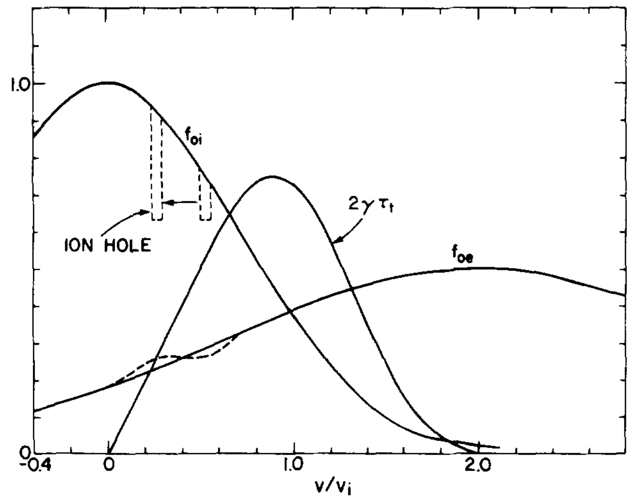

FIG. 1. Average distribution functions (Maxwell–Boltzmann) for electrons and ions with a relative drift of $v_D = 2v_i$. The growth rate curve $2\gamma\tau_t$ is for an electron hole. The dashed lines indicate the deceleration and growth of an ion hole and the relaxation of the electron distribution function.

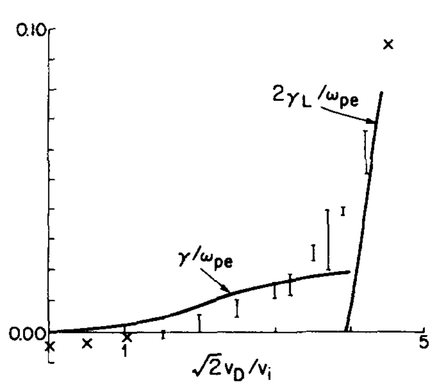

FIG. 2. Normalized growth rates, γ/ω_{pe}, for $m_i/m_e = 4$. The ×'s and bars are simulation values from Ref. 3. The curves are for electron holes and linear theory.

fluctuation (times $1/\omega_{pe}$) and the corresponding theoretical growth rate, γ/ω_{pe}, for an electron hole from (117) and (118). The linear growth rate is also shown. We computed γ for electron holes since, for the same ϕ_0, they have a Δv, and hence a γ, that is $(m_i/m_e)^{1/2}$ times that for an ion hole. We used $\Delta v/v_e = 0.4/\sqrt{2}$ $(\Delta v/\Delta x = \omega_{pe}/20)$, which is about twice as large as that cited in Ref. 3, but is consistent with more recent diagnostics on equivalent simulations.[4] These simulations also show phase-space holes with a packing fraction p less than $\frac{1}{2}$. The packing fraction is the fraction of local phase space occupied by holes. We have also calculated the hole growth rate for $m_i/m_e = 1836$. In Fig. 3, we have plotted $\gamma\tau_t \approx \gamma\Delta x/\Delta v$ given by (119) for electron and ion holes for $T_e/T_i = 1$ and 2.

C. Hole–hole collisions

So far, we have considered only the properties of single, isolated holes. In fact, it is likely that many holes will exist and interact or collide with each other. This process is difficult to analyze in detail. In Ref. 1, we discussed the coalescence and decay of colliding holes, and estimated the collision frequency to be of order $2p\Delta v/\Delta x$.

Hole collisions can be expected to change the distribution of $\tilde{f}$, Δx, and Δv of the hole fluctuations. Although there exists no real theory of this process, a computer simulation has shed considerable light on the details of hole–hole interactions.[5] For the purpose of the present calculation we shall simply assume that hole–hole collisions cause a reduction in the mean square fluctuation at the rate

$$\gamma_c = -2rp\Delta v/\Delta x, \tag{8}$$

where the factor r accounts for the fluctuation loss per collision. The validity of this formula is discussed in Ref. 5, where $r \approx \frac{1}{3}$ led to a reasonable fit with the simulation results. We assume all holes of each species have the same Δx and Δv, which we take to be an appropriate average.

When hole–hole collisions are included, the actual average fluctuation growth rate is

$$\gamma_a = \gamma + \gamma_c = (\Delta v/\Delta x)(-\tfrac{1}{2}F_eF_i - 2pr), \tag{9}$$

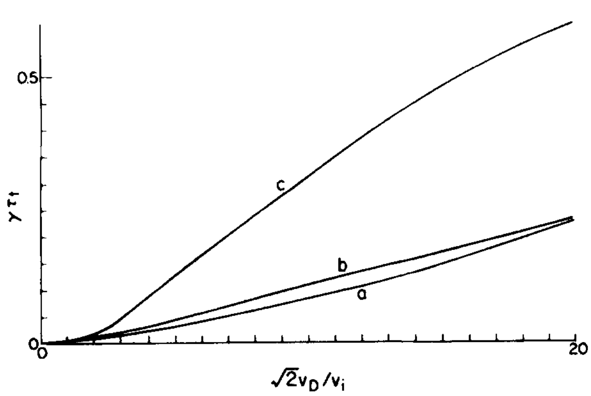

FIG. 3. Normalized growth rates, $\gamma\tau_t$, for $m_i/m_e = 1836$. Curve a, ion hole with $T_e/T_i = 1$; curve b, electron hole with $T_e/T_i = 1$; and curve c, electron hole with $T_e/T_i = 2$.

where

$$F_\alpha = 4\lambda^2\omega_{p\alpha}^2 f'_{0\alpha}(u). \tag{10}$$

The threshold ($\gamma = 0$) occurs at

$$F_eF_i = -4pr. \tag{11}$$

If we use $r = \frac{1}{3}$, $p = \frac{1}{4}$, and $\Delta v/\Delta x = \omega_{pe}/20$, which is not inconsistent with Ref. 4, we find $\gamma_c/\omega_{pe} \approx -0.008$. As one can see from Fig. 2, subtracting this value from γ, as indicated in (9), gives a value of γ_a/ω_{pe} that agrees reasonably well with the simulation growth rate in the region of the threshold. As explained later, the growth rate (9) would not be expected to hold as v_D approaches the linear instability threshold. Of course, given the approximate nature of (8) and the error in measuring p and $\Delta v/\Delta x$, good agreement between (9) and the simulation may be fortuitous. Also the clump theory shows that near $\gamma_a = 0$, for $p = \frac{1}{2}$, an additional term occurs in the growth rate [see (33)]. Furthermore, the simulations had a substantial discrete particle noise level, which means that in addition to γ_c, a discrete particle damping rate, γ_d, should also be added to γ. The effect of γ_d will be discussed later but for the simulations of Refs. 3 and 4 it is of the order of γ_c. Therefore, it would appear that the threshold observed in the simulation is determined, in part, by discrete particle noise.

Although the simulation instabilities cited had packing fractions of the order of one-half, one can imagine fluctuations with arbitrarily small p created by thermal fluctuations or the coalescing of holes.[1,5] Indeed, there would seem to be a tendency for a few large holes to dominate a turbulent system, i.e., intermittent turbulence. Statistically, some holes will always be bigger than others; but the bigger the hole, the more immune it is to destruction due to collisions with other holes (and vice versa). This effect would cause p to approach zero. Thus, it would seem that, in the absence of discrete particle effects, the ultimate stability threshold is zero.

D. Brief review of clump theory

The hole and clump instability are closely related and it is enlightening to understand this relationship. Moreover, a recent calculation of the clump growth rate[6] also agrees well with the simulation data. We shall briefly review the most recent versions of clump theory. For more details, the reader should consult Refs. 2, 6, and 8.

In the clump theory, one writes the fluctuation portion, $\delta f = f - \langle f \rangle$, of the distribution function as $\delta f = \delta f^{(c)} + \delta\tilde{f}$. The quantity $\delta f^{(c)}$ is the coherent response and $\delta\tilde{f}$ is the clump portion. Analogous quantities appear in the hole theory. The hole analogy to $\delta f^{(c)}$ is the linear solution which enters through the dielectric function and the hole analogy to $\delta\tilde{f}$ is the hole depth $-\tilde{f} = -(f_0 - f_t)$, where f_t is the distribution function for trapped particles. The equation for the two-point correlation function for a two-species plasma is (approximately)

$$\left(\frac{\partial}{\partial t} + v_-\frac{\partial}{\partial x_-} - D_-^{(\alpha)}(x_-)\frac{\partial^2}{\partial v_-^2}\right)\langle\delta f_\alpha(1)\delta f_\alpha(2)\rangle = S_\alpha. \tag{12}$$

The "source term" is given by

$$S_\alpha = -\frac{q_\alpha}{m_\alpha}\langle \delta E(1)\delta f_\alpha(2)\rangle \frac{\partial}{\partial v_1} f_{0\alpha}(1) + (1\leftrightarrow 2). \quad (13)$$

For small x_- and v_-, this becomes

$$S_\alpha = 2D_\beta^{(\alpha)}\left(\frac{\partial f_{0\alpha}}{\partial v}\right)^2 - 2\langle \dot{u}\delta\tilde{f}_\alpha\rangle_\beta \frac{\partial}{\partial v} f_{0\alpha}. \quad (14)$$

We have used "1" and "2" for the phase-space points x_1, v_1, and x_2, v_2, and $x_- = x_1 - x_2$ and $v_- = v_1 - v_2$. The quantity D_- is the diffusion coefficient for the relative coordinate v_-,

$$D_-(x_-) = \langle (\Delta v_-)^2/2\Delta t\rangle. \quad (15)$$

The quantity $D_\beta^{(\alpha)}$ is the single-particle diffusion coefficient for the αth species due to the shielded electric fields of clumps of the βth species. The quantity $\dot{u}$ is the clump acceleration and $\langle \dot{u}\delta\tilde{f}_\alpha\rangle_\beta$ describes the "drag" force exerted by the β species on clumps of the α species. As explained in Refs. 2 and 8, the like–like terms in the source term cancel due to local momentum conservation. The equation for the shielding portion of the correlation function

$$\bar{g}(1,2) \equiv \langle \delta f\delta f\rangle - \langle \delta\tilde{f}\delta\tilde{f}\rangle \quad (16)$$
$$= \langle \delta f^{(c)}\delta f^{(c)}\rangle + \langle \delta\tilde{f}(1)\delta f^{(c)}(2)\rangle + \langle \delta f^{(c)}(1)\delta\tilde{f}(2)\rangle \quad (17)$$

is given by an equation identical to (12) except that the two-particle relative diffusion coefficient $D_-^{(\alpha)}(x_-)$ is replaced with $D_-^{(\alpha)}(\infty) = 2(D_e^{(\alpha)} + D_i^{(\alpha)})$.

The equation for the hole growth rate with hole–hole collisions included can be put in a form similar to (12). When the packing fraction is approximately one-half, the destruction of holes due to hole–hole collisions can be described with the relative diffusion coefficient D_- used in the clump theory. Following the arguments of Ref. 1, we can estimate the value of D_- due to holes of size Δv, Δx as $D_- \approx 2p(\Delta v/\Delta x)^3 x_-^2$. The ratio $\Delta v/\Delta x$ is a measure of the electric field gradient $\partial E/\partial x$. Generally speaking, hole A can be torn apart by hole B only if $\partial E/\partial x$ of hole B is greater than that of hole A.

In terms of D_-, the rate of destruction of $\tilde{f}^2$ of holes of size Δx and Δv denoted γ_c is

$$-\gamma_c = rD_-(\Delta x)/\Delta v^2, \quad (18)$$

where

$$D_-(\Delta x) \approx 2p(\Delta v/\Delta x)^3 \Delta x^2. \quad (19)$$

If all holes are the same size, then (18) is equivalent to (8). Using (6) and (18), the hole growth rate can be written

$$\left(\frac{\partial}{\partial t} + \frac{r}{2}\frac{D_-(\Delta x)}{\Delta v^2}\right)\tilde{f} = -\dot{u} f_0'. \quad (20)$$

If we multiply both sides of (20) from the right by $\tilde{f}$ and average, the resulting equation,

$$\left(\frac{\partial}{\partial t} - \frac{rD_-(\Delta x)}{\Delta v^2}\right)\langle \tilde{f}\tilde{f}\rangle = -2\langle \dot{u}\tilde{f}\rangle f_0', \quad (21)$$

is very similar to the clump equations (12) and (14). The major differences between (12) and (21), the clump and hole equations, are as follows:

(a) The hole being a coherent structure, is described in terms of $\tilde{f}$, whereas the clump is a random structure and is described by a correlation function $\langle \delta\tilde{f}\delta\tilde{f}\rangle$.

(b) Only the second part of the clump source term contributes to (21).

(c) In the hole model the velocity width Δv is determined for each species separately by the structure of an individual hole [see (1)], whereas in the clump theory Δv is determined by the solution of (12) and is approximately $\Delta v_\alpha \approx [\Delta x(D_e^{(\alpha)} + D_i^{(\alpha)})]$ (Δx is the clump length). For $p \approx \frac{1}{2}$ the two models give similar values. For $p \ll \frac{1}{2}$, the plasma contains isolated holes and the clump equation (12) is not applicable.

(d) In the hole model (21) fluctuations destroy each other at the rate $rD_-/(\Delta v)^2$ or more generally $2pr\Delta v/\Delta x$, whereas for the clump theory the analogous rate is $v_-\partial/\partial x_- + D_-\partial^2/\partial v_-^2$, which can be shown to be of order $\Delta v/\Delta x \approx [D/(\Delta x)^2]^{1/3}$. The first version of clump theory[7] omitted both the drag portion of the source term and the self-binding effect, both intrinsic to hole growth. In Ref. 8, the drag source term was included in a renormalized kinetic theory but including fluctuation self-binding in such a theory proved considerably more difficult.

The hole and clump instabilities can be regarded as different regimes of the same basic physical process. If all holes have the same Δx and Δv then the hole–hole collision frequency ν is approximately $\nu = 2p\Delta v/\Delta x = 2p\tau^{-1}$. For $p \ll \frac{1}{2}$, $\nu \ll \tau^{-1}$, particles can follow well-defined trapped-particle orbits between hole–hole collisions and the hole model is valid. As $p \to \frac{1}{2}$, $\nu \to \tau^{-1}$ and holes collide before a trapped particle can execute a complete orbit. In this case, the clump picture is more realistic. In terms of the correlation time τ_c the hole model is appropriate when $\tau_c/\tau > 1$, and the clump model when $\tau_c/\tau < 1$.

Of course, the detailed effect of hole–hole collisions on the aggregate growth rate is not understood. In reality there will be a distribution of hole sizes. Holes can coalesce causing p to decrease. The larger coalesced holes will be less affected by the smaller holes and $-\gamma_c$ will be reduced. Perhaps a few large holes will dominate. This possibility is suggested by the results of a recent computer simulation[5] in which hole coalescing and the reduction in p were observed. The distribution of fluctuation amplitudes was not Gaussian and had negative skewness consistent with holes with $p < \frac{1}{2}$.

When $(\partial/\partial t)\langle \delta f\delta f\rangle = 0$, the solution to the clump equation (12) may be written $\langle \delta f\delta f\rangle \approx \tau_{cl}(x_-, v_-)S_\alpha$. The time τ_{cl} is the so-called clump lifetime and is the inversion of the operator on the left-hand side of (12) with $\partial/\partial t = 0$. The magnitude of the clump lifetime τ_{cl} is of the order of, but somewhat larger than, the trapping time $\tau \approx (D/\Delta x^2)^{-1/3}$. The solution to the $\bar{g}$ equation is $\bar{g} \approx \tau S_\alpha$. Subtracting the two solutions gives the clump correlation function

$$\langle \delta\tilde{f}_\alpha(1)\delta\tilde{f}_\alpha(2)\rangle = [\tau_{cl}(x_-, v_-) - \tau]S_\alpha. \quad (22)$$

Momentum conservation imposes a useful relationship between Fokker–Planck drag and diffusion coefficients. The drag force on clumps of species α due to species $\beta(\neq\alpha)$ is equal to minus the rate of change of momentum of species β due to the diffusion caused by the shielded electric fields of clumps of species α. Thus we can write

$$n_\alpha m_\alpha\langle \dot{u}\delta\tilde{f}_\alpha\rangle_\beta = n_\beta m_\beta D_\alpha^{(\beta)}\frac{\partial f_{0\beta}}{\partial v}. \quad (23)$$

The diffusion coefficients can be written in terms of the clump correlation function

$$D_\alpha^{(\beta)} = (q_\beta m_\beta)^2 d_\alpha, \tag{24}$$

$$d_\alpha = (4\pi n_\alpha q_\alpha)^2 \int \frac{dk}{2} \frac{|k|}{|k^2\epsilon(k,ku)|^2} \int dv_- \langle \delta\tilde{f}_\alpha^2 \rangle_k, \tag{25}$$

where k is a wavenumber, $\epsilon(k,\omega)$ is the usual dielectric function whose linear value is

$$\epsilon(k,\omega) = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{k} \int dv(\omega - kv + i\delta)^{-1} \frac{\partial f_{0\alpha}}{\partial v}, \tag{26}$$

and

$$\langle \delta\tilde{f}^2 \rangle_k = \int_{-\infty}^{\infty} dx_-\, e^{-ikx_-} \langle \delta\tilde{f}(1)\delta\tilde{f}(2) \rangle. \tag{27}$$

If we put (23) and (24) in (14) and then use the Fourier transform (27) of (22) in (25), we obtain an equation for d_α. We define the quantities

$$B(k) \equiv \int dx_- e^{-ikx_-} \int dv_- (\tau_{cl} - \tau), \tag{28}$$

$$\lambda_c^4 = \int \frac{dk}{8} \frac{B(k)|k|}{|k^2\epsilon(k,ku)|^2}, \tag{29}$$

$$F_{c\alpha} = 4\lambda_c^2 \omega_{p\alpha}^2 \frac{\partial f_{0\alpha}}{\partial v} = \left(\frac{\lambda_c}{\lambda}\right)^2 F_\alpha. \tag{30}$$

For a clump of length Δx, $B(k) \approx \Delta x^2$ when $k\Delta x < 1$. If Δx is larger than λ or λ_i [$\lambda_i \equiv \mathrm{Im}\, k^2\epsilon(k,ku)$], then (for $\lambda^2 > 0$) the integral in (29) is approximately $\lambda_c^4 \approx (\Delta x/4)^2(\lambda^{-2} + \lambda_i^{-2})^{-1}$. Using (23), (24), and (30), the source term (14) becomes

$$S_\alpha = \tfrac{1}{2}(4\pi n_\alpha q_\alpha \lambda_c^2)^{-2}(d_\beta F_{c\alpha}^2 - d_\alpha F_{c\alpha} F_{c\beta}). \tag{31}$$

The equation for d_α for a steady-state ($\partial \langle \delta f \delta f \rangle / \partial t = 0$) solution is (for $\alpha = i,e$)

$$2d_\alpha = d_\beta (F_{c\alpha})^2 - d_\alpha F_{c\alpha} F_{c\beta}. \tag{32}$$

A solution exists when

$$2F_{ce} F_{ci} = -2. \tag{33}$$

Equation (33) gives the threshold for a self-sustaining state, i.e., the clump instability When (33) is satisfied, (32) gives $-d_i F_{ce} = d_e F_{ci}$, which can be used to rewrite the first term of the source term (31) as $-d_\alpha F_{c\alpha} F_{c\beta}$. Thus, the first and second terms of the source term are equal.

It is enlightening to compare the clump and hole instability threshold conditions (33) and (11). One difference is that $F_{c\alpha}$ contains λ_c while F_α contains λ. Except near linear instability the two are of similar size. The factor of 2 on the left-hand side of (33) occurs because the clump equations contain both source terms. The right-hand sides are proportional to the hole or clump destruction rate. The smaller this rate, the lower the threshold, i.e., the criterion can be satisfied with smaller values of relative drift v_D. The factor $2pr$ in (11), which came from (8), contains the (empirical) physics of hole–hole collisions, and lowers the threshold since $2pr < 1$. Of course, such self-binding effects might also be incorporated in the clump theory. For example, a numerical solution of the clump equations shows that the threshold (33) occurs at $\sqrt{2}v_d/v_i \approx 2.5$, whereas the simulation threshold is at $\sqrt{2}v_d/v_i \approx 1.5$. However, if the right-hand side of (33) is replaced with $-2r = -\frac{2}{3}$, the observed threshold is predicted.[5,6] The decay rate of (8) is apparently more accurate than that given by the operator $v_-\partial/\partial x_- - D_-\partial^2/\partial v_-^2$ of the renormalized clump equation (12). It is interesting to note that in (12), only the $D_-\partial^2/\partial v_-^2$ term arises from the renormalization. The rest of the terms in the equation, including the source term S_α are lower order and occur in the standard two-point hierarchy equation. Therefore, the source term S_α rests on a much firmer analytical footing than does the D_- term.

The source term (14) contains two terms. The first term describes the random rearrangement of the average phase-space density gradient and was included in the early versions of the clump theory. The second term describes a coherent rearrangement due to the drag force on an existing fluctuation. Both terms rely on the fact that if the phase-space density f of a fluctuation stays constant for a time τ_{cl}, then as the fluctuation moves in phase space to a region of different $\langle f \rangle$, the fluctuation $\delta f = f - \langle f \rangle$ will change. The growth of a single isolated hole is due to the second term. For an isolated hole, $\tau_c/\tau \gg 1$, the drag force is due to reflected particles which follow coherent orbits. In the clump case ($p \approx \frac{1}{2}$) the orbits are stochastic and since $\tau_c/\tau < 1$, one can compute the drag using the standard test particle methods and orbit perturbation theory. For a given ϕ_0, the drag force is of similar magnitude for the two cases. For example, for a hole potential $\phi = \phi_0 \exp(-k_0^2 x^2)$, the perturbation theory drag is $\pi/4$ times the right-hand side of (5).

The diffusive source term does not occur in (21) or (9) because we considered only one hole of one species. However, it is easy to see its origin in a multi-hole problem. For example, a hole of species β will reflect particles of species α, creating a perturbation $\tilde{f}_\alpha \approx \Delta v_\alpha f'_{0\alpha}$ in a time $\tau_\alpha \approx \Delta x/\Delta v_\alpha$. The fluctuations will tend to organize into holes. Therefore, (21) will acquire an additional term on the right-hand side equal to $(\partial/\partial t)\langle \tilde{f}_\alpha^2 \rangle \approx (\Delta v_\alpha f'_{0\alpha})^2/\tau_\alpha \approx 2D_\beta^{(\alpha)}(f'_{0\alpha})^2$, which is the first term in (14).

In calculating the growth rate of isolated holes ($p \lessdot 0$) one can ignore the fact that holes (or clumps) are produced in the reflected species since they move off to other regions of phase space and do not interact with the original hole. However, as $p \to \frac{1}{2}$, one cannot, in general, ignore the interaction between the two types of holes, i.e., one must include the first source term as well as collisions between different types of holes. These features are, of course, included in the clump theory which is the appropriate picture when $p \to \frac{1}{2}$. It is interesting to see how the growth rate (9) approaches the clump theory result. One can compute $D_\beta^{(\alpha)}$ from (23) using the drag force from (5). The result is

$$D_\beta^{(\alpha)} = 4\omega_{p\alpha}^4 \lambda^4 \langle \tilde{f}_\beta^2 \Delta v_\beta \rangle / \Delta x, \tag{34}$$

where $\langle \tilde{f}_\beta^2 \Delta v_\beta \rangle = 2p_\beta \tilde{f}_\beta^2 \Delta v_\beta$. We may use (34) to generalize (9) to include the diffusive source term by multiplying both sides of (9) by $\tilde{f}_\alpha^2$, adding $2D_\beta^{(\alpha)}(f'_{0\alpha})^2$ to the right-hand side, multiplying both sides by $\Delta v_\alpha = \Delta x/\tau_\alpha$, and then averaging over x. We obtain

$$(\gamma\tau_\alpha + \tfrac{1}{2}F_\alpha F_\beta + 2p_\alpha r)\langle \tilde{f}_\alpha^2 \Delta v_\alpha \rangle = \tfrac{1}{2}F_\alpha^2 \langle \tilde{f}_\beta^2 \Delta v_\beta \rangle. \tag{35}$$

When $2pr \to 1$, this equation is the generalization of (32) to include finite $\gamma\tau$, since $\langle \tilde{f}_\alpha^2 \Delta v_\alpha^2 \rangle$ may be replaced with d_α. This replacement follows from (25) which shows the two are proportional as far as the α and β subscripts are concerned. In the clump theory ($2p_\alpha r = 1$), τ_α^{-1} is the damping rate due to collisions with *both* species of clumps since $\tau_\alpha^{-1} = (D_\alpha^{(\alpha)} + D_\beta^{(\alpha)})/\Delta v_\alpha^2$. However, in the hole model an additional term,

$$(D_\beta^{(\alpha)}/\Delta v_\alpha^2)\langle \tilde{f}_\alpha^2 \Delta v_\alpha \rangle = -2p_\alpha r \langle \tilde{f}_\beta^2 \Delta v_\beta \rangle, \qquad (36)$$

due to collisions between electron and ion holes, would occur on the right-hand side of (35). An equation of the form (35) (with $p_\alpha = \frac{1}{2}$) was first obtained and discussed by Tetreault,[6] who solved the clump equations to first order in $\gamma\tau$. In essence he found that the right-hand side of (22) acquires a factor $(1 + \gamma\tau_\alpha)^{-1}$, which causes the left-hand side of (32) to become $2(1 + \gamma\tau_\alpha)d_\alpha$. Equation (32) is then identical to (35) if $2p_\alpha r \rightarrow 1$, $F_\alpha \rightarrow F_{c\alpha}$, and $\langle \tilde{f}_\alpha^2 \Delta v_\alpha \rangle \rightarrow d_\alpha$.

Equation (35) can be readily solved for γ. Generally speaking, since $\tau_e/\tau_i \approx (m_e/m_i)^{1/2} \ll 1$, one finds that if the threshold (11) is exceeded, then $\gamma \sim \tau_e^{-1}$ and the growth is too rapid for the ions to respond. One can then neglect the diffusive source term and γ is given by (9) using F or F_c. If the threshold (11) is not exceeded but (33) is, then clearly ion clumps and the diffusive source term play an important role, and $\gamma \ll \tau_e^{-1}$.

For $p \ll \frac{1}{2}$ the clump picture is replaced with one of isolated holes and the additional term (36) appears on the right-hand side of (35). In this case the solution of (35) shows that γ is approximately given by (9). No additional instability region (for $\langle \tilde{f}_\alpha^2 \Delta v_\alpha \rangle > 0$), involving electron–ion hole coupling analogous to (33), appears.

E. Particle discreteness—thermal fluctuations

The role of particle discreteness, thermal fluctuations, and collisions is difficult to include in the hole model in a rigorous way, but it appears possible to estimate some of the effects. Collisional damping of holes (or clumps) will occur because the electric fields of discrete particles will tear the holes apart just as do hole–hole collisions. One can estimate this effect by computing the D_- due to particle discreteness. One can calculate D_- using the standard methods for obtaining the Lenard–Balescu collision integral. Assuming that $\lambda_c^2 \approx \lambda^2$ and $q_e = -q_i$, we find

$$D_-^{(\alpha)}(\Delta x) = 2\omega_{p\alpha}^4 \lambda^2 n^{-1} h(\Delta x/\lambda)[f_{0e}(u) + f_{0i}(u)], \qquad (37)$$

where $h(z) \approx z^2(1.57 - \ln|z|)$ for $z \ll 1$ and $h(a) \approx 1$ for $z \gg 1$.

The damping rate of ϕ_0, $-\gamma_d$, due to discrete particle (thermal) fluctuations for a hole of species α and dimensions Δx and Δv is

$$-\gamma_d = r_c D_-^{(\alpha)}(\Delta x)/\Delta v^2, \qquad (38)$$

where r_c is an empirical factor of order unity, analogous to that in (8). Using (37) for $\Delta x > \lambda$, (38) can be written

$$-\gamma_d = (2\lambda^2 r_c \omega_{p\alpha}^4/n\Delta v^2)[f_{0e}(u) + f_{0i}(u)]. \qquad (39)$$

Obviously, a hole will grow only if $\gamma + \gamma_d > 0$. This criterion can be made more illuminating by expressing it in terms of the amplitudes of the hole fluctuation and the mean square thermal fluctuation. If particles are randomly located in phase space, then the probability of finding a fluctuation $\delta N = N - \langle N \rangle$ of the number of particles in a small ($\Delta v \ll v_\alpha$) phase-space area $\Delta v \Delta x$ is

$$P(\delta N) = (2\pi\langle N \rangle)^{-1/2} \exp[-\tfrac{1}{2}(\delta N)^2/\langle N \rangle], \qquad (40)$$

where $\langle N \rangle = n\Delta x \Delta v f_{0\alpha}(v)$. If $\overline{\delta f}$ is the fluctuation of the distribution function averaged over a phase-space cell $\Delta x, \Delta v$, then the mean square thermal fluctuation is

$$\langle \overline{\delta f}^2 \rangle = \frac{\langle \delta N^2 \rangle}{(n\Delta x \Delta v)^2} = \frac{f_{0\alpha}(u)}{n\Delta x \Delta v}. \qquad (41)$$

Following (7), we set

$$a(u) = \gamma \Delta x/\Delta v, \qquad (42)$$

and use (39), (3), and (41) to write the criterion $\gamma + \gamma_d > 0$ as

$$\tilde{f}^2 > (\langle \overline{\delta f}_e^2 \rangle + \langle \overline{\delta f}_i^2 \rangle) \frac{2r_c}{a} \left(\frac{\Delta x/\lambda}{6g(\Delta x/\lambda)}\right)^2. \qquad (43)$$

The last factor has a minimum at $\Delta x/\lambda \approx 2.8$, where it is equal to 1.43. Equation (43) states that a hole will grow only if its amplitude $-\tilde{f}$ exceeds $(2.86 r_c/a)^{1/2}$ times the root mean square thermal fluctuation.

The criterion has important implications regarding the spontaneous creation of growing holes from thermal fluctuations. If $(2.86 r_c/a)^{1/2} > 1$, only a fluctuation $-\tilde{f}$ greater than the root mean square thermal fluctuation can satisfy (43). We can use (40) to estimate the probability of such a fluctuation. Consider a plasma of length L, whose phase space is divided into cells Δx by Δv. Further assume that the fluctuation in each cell is independent of the other cells and that particles are randomly redistributed among cells during every time interval $\Delta x/\Delta v$. Then, in a time t, the number of independent opportunities for a hole to be created in the velocity interval du is, approximately,

$$\frac{L}{\Delta x}\frac{du}{\Delta v}\frac{t\Delta v}{\Delta x} = \frac{du}{v_\alpha}\frac{\omega_p t(L/\lambda_D)(\lambda_D/\lambda)^2}{\sqrt{2}(\lambda/\Delta x)^2}. \qquad (44)$$

If the $\tilde{f}$'s of spontaneously created holes have a Gaussian distribution about a mean square value $\langle \overline{\delta f}^2 \rangle$, then the probability of creating a hole satisfying (43) during each "opportunity" is

$$P_1(y) = \left(\frac{2}{\pi}\right)^{1/2} \int_y^\infty dx\, e^{-x^2}, \qquad (45)$$

where

$$y^2 = \frac{f_{0e}(u) + f_{0i}(u)}{f_{0\alpha}(u)} \frac{1.43 r_c}{a(u)}. \qquad (46)$$

Of course, using (40) for the probability of creating a hole is an approximation since particle orbits are not random and uncorrelated in the vicinity of a hole. Combining (44) and (45), the probability for the spontaneous creation of a growing hole in a time t is

$$\int \frac{du}{v_\alpha} \omega_p t \left(\frac{L}{\lambda_D}\right)\left(\frac{\lambda_D}{\lambda}\right)^2 P_1[y(u)]\sqrt{2}\left(\frac{\lambda}{\Delta x}\right)^2. \qquad (47)$$

We emphasize that, at best, (47) can be considered a very rough estimate only. However, it is consistent with the simulation of Ref. 2, where turbulent fluctuations were observed to grow from discrete particle noise. For the parameters of the simulation of Ref. 3, (47) is of order one or greater for $v_D/v_i > 1.5$.

F. Organization of the paper

Although in this introduction we have discussed holes in the broader context of turbulence theory, the remaining portion of the paper is concerned with the calculation of the growth of a single isolated hole. The essential features of the calculation are contained in Secs. II–V, while most of the lengthy and detailed orbit calculations are deferred to Secs. VI–IX. In principle, the paper can be understood on three levels by reading only Sec. I, or Secs. I–V, or the entire paper.

II. MODEL OF A GROWING HOLE

The two equations that describe a growing hole are the time-dependent Vlasov equation and Poisson's equation. We consider first the solution of the Vlasov equation in a given model potential without regard to whether the solution satisfies Poisson's equation which is discussed in the next section. We consider the following model potential, which we later justify, for a slowly growing and accelerating hole:

$$\Phi(x,t) = (1+\gamma t)\phi(x - \dot{u}t^2/2). \quad (48)$$

The hole growth rate γ, the hole acceleration $\dot{u}$, and the velocity derivative of the average distribution function of the untrapped species, $f'_{0\alpha}$, are all considered to be small quantities. The "untrapped species" are the electrons for an ion hole and the ions for an electron hole. We characterize the smallness by the parameter ϵ. For an ion hole,

$$\epsilon \sim \gamma\tau \sim (\dot{u}/\Delta v_i)\tau \sim \omega_{pe}^2 \lambda^2 f'_{0e}. \quad (49)$$

Note that γ and $\dot{u}$ are the same order in ϵ although $\dot{u}$ is smaller in magnitude by a factor $\Delta v \sim (\phi_0)^{1/2}$. We shall obtain the solution to first order in ϵ. We consider the case of an ion hole, but the arguments for an electron hole are completely analogous. Since we only require a solution accurate to first order in ϵ, we can understand the result from a physical point of view by considering separately holes with $\gamma \neq 0$, $\dot{u} = 0$; and with $\gamma = 0$, $\dot{u} \neq 0$. The $\gamma = \dot{u} = 0$ case is the steady-state BGK problem discussed in Ref. 1 with the ion orbits shown in Fig. 2 of Ref. 1.

For the cases of $\gamma \neq 0$ or $\dot{u} \neq 0$, we show in Secs. VI–VIII that the orbit problem for untrapped particles of both species naturally divides into two velocity regions. For $u = 0$, the region $|v| \lesssim \Delta v$, is a "resonant layer" where the orbits cannot be expanded in integral powers of the potential ϕ and the region $|v| \gg \Delta v$ is a "nonresonant" region where the orbits can be expanded and the standard perturbative results are obtained. The ϕ^n expansion is not to be confused with the ϵ^n expansion.

Contributions to velocity integrals, such as charge density, mass, momentum, and energy, from the resonant layer involve the local value $f'_{0\alpha}(0)$; whereas contributions from the nonresonant region enter through the dielectric function and involve velocity integrals of $f_{0\alpha}(v)$. Since $\omega_{p\alpha}^2 \lambda^2 f'_{0\alpha}(u) \sim \epsilon$ for the untrapped species, we need consider only the $\dot{u} = \gamma = 0$ for this species in the resonant layer. This case is shown in Fig. 4 where electron phase-space streams of width Δv_e moving in the $\pm v$ directions are reflected by the negatively charged ion hole.

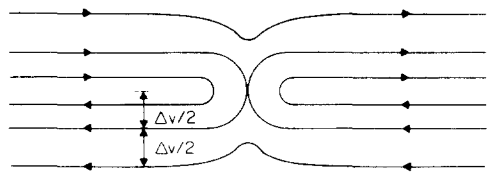

FIG. 4. Electron orbits in phase space for an ion hole with $\gamma = 0$ and $\dot{u} = 0$. Velocity (v) is plotted vertically and distance (x) is plotted horizontally.

For the trapped species the resonant layer is separated from the trapped particles by thin boundary layers of width $\delta v \approx \gamma \Delta x$ and $\dot{u}\Delta x/\Delta v$ in which the ϵ expansion is not valid. Fortunately, this layer makes a contribution to velocity integrals of order ϵ^2 and can be neglected.

Ion orbits for the case $\gamma > 0$, $\dot{u} = 0$ are shown in Fig. 5. Phase-space density streams of width $\delta v \approx \gamma \Delta x$ are trapped by the growing potential as they flow in from positive and negative v. As a consequence, streams of passing ions of width Δv, moving in both directions, are slowed down by δv to fill the void left by the trapped ions which are, in turn, filling the growing hole area.

Ion orbits for the case $\dot{u} < 0, \gamma = 0$ are shown in Fig. 6. For this case there is a plus-minus asymmetry. The stream of width Δv passing in the positive direction is accelerated by an amount $\delta v \approx \dot{u}\Delta x/\Delta v$ while the negative flowing stream is slowed down by the same amount, i.e., both streams are accelerated in the positive direction. No trapping or untrapping occurs. Since the potential in the accelerated frame differs from that in an inertial frame only by the small amount $m_i \dot{u} x/q_i$ the trapped-ion distribution function f_t in the frame of the hole is not significantly changed by $\dot{u}$. However, since f_t remains constant as u decreases, the difference between f_t and the local value of $f_{0i}(u)$, the hole depth—$\tilde{f}$, will increase if $f'_{0i}(u) < 0$.

Now let us combine the two pictures. If Δv increases as u decreases, then, for $u > 0$, the value of the phase-space density, $f_{0i}(u)$, being trapped at each instant of time will increase with time so that f_t wil be built up of concentric rings as each new layer of $f_{0i}(u)$ is added. To zero order in γ and $\dot{u}$, each ring has the shape of a trapped-particle orbit, i.e., a line of constant ion energy. Each ring corresponds to a particular value of f_t. As the hole grows and $\phi(x,t)$ increases, the rings will change, but if $\gamma\tau \ll 1$ the orbits will not cross and there will be no mixing. Therefore, since phase-space density is incompressible, the area inside a ring corresponding to a particular f_t will not change with time. This means that f_t is a

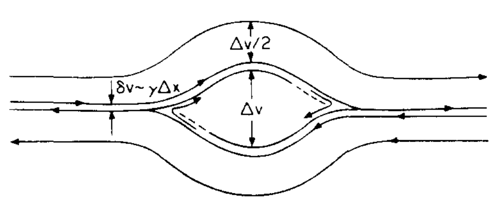

FIG. 5. Ion orbits ion phase space for an ion hole with $\gamma > 0$ and $\dot{u} = 0$. Velocity (v) is plotted vertically and distance (x) is plotted horizontally.

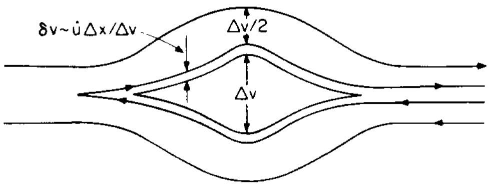

FIG. 6. Ion orbits in phase space for an ion hole with $\gamma = 0$ and $\dot{u} < 0$. Velocity (v) is plotted vertically and distance (x) is plotted hoizontally.

function of the area a only! Of course, a, in turn, is a function of E and the spatial structure of $\phi(x,t)$ so that we can write $f_t = f[a(E,t)]$.

The trapped-ion distribution function develops slowly on a long time scale γ^{-1} and must be determined self-consistently with Poisson's equation. On the other hand the distribution function for passing ions (and for passing and reflected electrons) in the region of the hole ($x \ll \Delta x/\gamma t$) follows $\phi(x,t)$ adiabatically since its response time is $\tau \ll \gamma^{-1}$.

In Secs. VI–VIII we calculate exactly to first order in ϵ, the charge density, the rate of change of hole mass M, momentum $P = Mu$, and energy, $T = Mu^2/2 + T_0$. We list here some of the principal results which are necessary to calculate and understand the hole growth rate. Although the actual calculations are complicated, it is not difficult to motivate the results by simple intuitive arguments. We use the same definitions of M, P, and T as in Ref. 1.

$$(M,P,T) = n \int dx \int dv\, \tilde{f}(x,v) \left(m, mv, \frac{mv^2 + q\phi(x)}{2}\right). \tag{50}$$

The quantities n, m, and q refer to the trapped species, and

$$\tilde{f} = f(x,v) - f_{0\alpha}(u) \tag{51}$$

inside the trapped region of phase space, $m(v-u)^2/2 + q\phi(x) < 0$, and $\tilde{f} = 0$ outside. In the frame of the hole, the rates of change of hole mass, momentum, and energy are

$$\frac{dM}{dt} = -2\dot{u}f'_{0i}(u)n_i m_i \int_{-\infty}^{\infty} dx \left(\frac{-2q_i\phi(x)}{m_i}\right)^{1/2}, \tag{52}$$

$$M\dot{u} = n_e m_e f'_{0e}(u)\left(\frac{2q_e\phi_0}{m_e}\right)^2 + \gamma n_i m_i f'_{0i}(u) \times \int_{-\infty}^{\infty} dx \left(\frac{-2q_i\phi(x)}{m_i}\right)^{3/2}, \tag{53}$$

$$\frac{dT_0}{dt} = \left(\frac{2}{3}\right)\dot{u}n_i m_i f'_{0i}(u)\int_{-\infty}^{\infty} dx \left(\frac{-2q_i\phi(x)}{m_i}\right)^{3/2}. \tag{54}$$

These formulas can be derived qualitatively as follows. The right-hand side of (52) follows from the fact that the hole depth $-\tilde{f}$ increases at the rate $\dot{u}f'_{0i}$ and that the hole mass is $M \approx \tilde{f}\Delta x \Delta v_i n_i m_i$, where $\Delta v_\alpha^2 \approx -2q_\alpha\phi/m_\alpha$. The first term on the right-hand side of (53) is the rate of momentum loss from the reflected electrons. When electrons moving in a stream of width Δv_e are reflected, they lose momentum at the rate $\frac{1}{2}n_e m_e \Delta v_e^3 f_{0e}(u + \Delta v_e/2)$. When the electrons at $-\Delta v_e/2$ are added, we obtain $n_e m_e v_e^4 f'_{0e}$, which is the first term in (53). The second term is the momentum loss of the passing ions. The calculation is similar to the electron case except that each ion in a stream of wdith Δv_i suffers a net velocity change of $-\gamma\Delta x v/|v|$ which, when summed over the streams at $u \pm \Delta v_i/2$, gives a total rate of momentum change of $-\gamma n_i m_i f'_{0i}\Delta v_i^3 \Delta x$, which is the second term in (53). The standard linear ion Landau damping calculation, valid for $t \ll \tau$, gives a similar result except that γ is replaced by τ^{-1}. Our result is valid for $t \gg \tau$ and $\gamma\tau \ll 1$. For $t \gg \tau$ and $\gamma\tau > 1$, one should use the Landau result. The right-hand side of (54) is the rate of kinetic energy loss of passing ions. The calculation is similar to the one for momentum except that each passing ion in a stream of width Δv_i at $v = u \pm \Delta v_i/2$ undergoes a kinetic energy change of $\delta v \Delta v = \dot{u}\Delta x$. The corresponding total rate of change is $\dot{u}n_i m_i f'_{0i}\Delta v_i^3 \Delta x$, which is essentially the right-hand side of (54). According to (54), in the rest frame of the hole the increased self-energy of a hole, including the electric field energy, comes from the loss in kinetic energy of the passing ions.

III. SOLUTION OF THE VLASOV AND POISSON EQUATIONS FOR A TIME-DEPENDENT HOLE

The objective is to obtain the hole solution to first order in ϵ. To this end, the potential $\phi(x)$ in (48) can be expanded as

$$\phi(x) = \phi^{(0)}(x) + \phi^{(1)}(x), \tag{55}$$

where $\phi^{(p)}$ is of order ϵ^p. Given the potential (48), one can expand in powers of ϵ the trapped and untrapped distribution functions which solve the Vlasov equation for each species:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{q}{m}\frac{\partial\Phi(x,t)}{\partial x}\frac{\partial}{\partial v}\right)f(x,v,t) = 0. \tag{56}$$

Through first order in ϵ, we obtain

$$f(x,v,t) = f^{(0)}[\Phi(x,t)] + f^{(1)}(x,v,t), \tag{57}$$

where $f^{(0)}$ is the usual BGK function of the particle energy, $mv^2/2 + q\Phi(x,t)$, but contains the instantaneous potential. More precisely

$$\left(v\frac{\partial}{\partial x} - \frac{q}{m}\frac{\partial\Phi(x,t)}{\partial x}\frac{\partial}{\partial v}\right)f^{(0)}[\Phi(x,t)] = 0, \tag{58}$$

and

$$\frac{\partial}{\partial t}f^{(0)}[\Phi(x,t)] + \left(v\frac{\partial}{\partial x} - \frac{q}{m}\frac{\partial\phi^{(0)}(x)}{\partial x}\frac{\partial}{\partial v}\right)f^{(1)}(x,v,t) = 0. \tag{59}$$

The charge density

$$\rho(x) = \sum_{\alpha=e,i} nq \int dv\, f(x,v) \tag{60}$$

can likewise be expanded in powers of ϵ:

$$\rho(x) = \rho^{(0)}(x) + \rho^{(1)}(x), \tag{61}$$

$$\rho^{(0)}(x) + \rho_a^{(1)}(x) = \sum_\alpha nq \int dv\, f^{(0)}[\Phi(x,t)], \tag{62}$$

$$\rho_b^{(1)}(x) = \sum_\alpha nq \int dv\, f^{(1)}(x,v,t), \tag{63}$$

$$\rho^{(1)} = \rho_a^{(1)} + \rho_b^{(1)}. \tag{64}$$

We show in Secs. VII and VIII that for a time-independent potential, $\phi(x) = \phi^{(0)} + \phi^{(1)}$, the solution of the Vlasov equation for an ion hole leads to a charge density

$$\sum_\alpha nq \int dv f^{(0)}[\phi(x)] = (4\pi\lambda^2)^{-1}\phi(x) + \tilde{\rho}[\phi(x)] + (|x|/x) f'_{0e}(u) F[\phi(x)] n_e q_e . \quad (65)$$

The first term is the usual Debye shielding effect. The second term describes the trapped ions and is a function of $\phi(x)$. The third term is due to reflected electrons and comes from the second term of (198), where the velocity integral has been denoted by $F[\phi(x)]$. The third term is not a function of $\phi(x)$ because of the factor $|x|/x$. Using (62) and (65), we can set, at $t=0$,

$$\rho^{(0)} = (4\pi\lambda^2)^{-1}\phi^{(0)}(x) + \tilde{\rho}[\phi^{(0)}(x)], \quad (66)$$

and

$$\rho_a^{(1)} = \phi^{(1)}\left((4\pi\lambda^2)^{-1} + \frac{\partial\tilde{\rho}[\phi^0(x)]}{\partial\phi^0(x)}\right) + n_e q_e (|x|/x) f'_{0e}(u) F[\phi^{(0)}(x)]. \quad (67)$$

If the Vlasov equation is solved with the time-dependent potential (48) (i.e., γt and $\dot{u}t^2/2$ now included), the additional terms $f^{(1)}$ and $\rho_b^{(1)}(x)$ given by (59) and (63) must now be included. In general, $\rho_b^{(1)}(x)$ will not be a function of $\phi(x)$, although it is, of course, a functional of $\phi(x)$.

The functions $\phi(x)$ and $\rho(x)$ are determined by Poisson's equation which requires that

$$-\frac{\partial^2}{\partial x^2}\phi^{(0)} = 4\pi\sum_\alpha \rho^{(0)}, \quad (68)$$

$$-\frac{\partial^2}{\partial x^2}\phi^{(1)} = 4\pi\sum_\alpha \rho^{(1)} = 4\pi\sum_\alpha (\rho_a^{(1)} + \rho_b^{(1)}). \quad (69)$$

The quantities $\phi^{(0)}$ and $\rho^{(0)}$ are the potential and charge density for a (steady-state, unaccelerated) BGK mode, and are, therefore, relatively easy to obtain. Given γ, $\dot{u}$, and $\phi^{(1)}$, the quantity $\rho_b^{(1)}$ can also be readily obtained. However, $\phi^{(1)}$ and $\rho_a^{(1)}$ must be obtained by solving (69) and the Vlasov equation simultaneously. This is difficult to do since $\rho^{(1)}$ is not a function of $\phi(x)$. Fortunately, however, one does not need $\phi^{(1)}$ to calculate γ and $\dot{u}$ to lowest order in ϵ. The reason for this is that γ and $\dot{u}$ are determined by the rate of change of particle momentum. We show in Sec. IX that $\phi^{(1)}$ makes a contribution of order ϵ^2 to the total rate of momentum change, whereas in Secs. VII and VIII we show that the γt and $\dot{u}t^2/2$ terms in $\phi^{(0)}$ each separately make a contribution of order ϵ. Therefore, we can neglect $\phi^{(1)}(x)$ in the calculation of γ and $\dot{u}$.

Consider, however, the solution with $\phi^{(1)}$ retained. Once the ion and electron distribution functions have been obtained for the model potential (48), the resulting charge densities can be substituted into Poisson's equation to obtain

$$-\frac{\partial^2\Phi(x,t)}{\partial x^2} = 4\pi\rho(x). \quad (70)$$

Both $\phi(x)$ and $\rho(x)$ are functions of x. To solve (70), it is conventional to invert $\phi(x)$ as $x(\phi)$ and write the charge density as a function of ϕ, $\rho(\phi)$. However, for a time-dependent hole $\rho(\phi)$ will, in general, be a double-valued function of ϕ since $x(\phi)$ is double valued. Following the usual procedure we define the "potential"

$$V(\phi) = 8\pi\int_{\phi_-}^{\phi} d\phi'\rho(\phi'), \quad (71)$$

so that (70) becomes

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi) = 0, \quad (72)$$

whose solution is

$$x = \pm\int_\phi^\phi \frac{d\phi'}{[-V(\phi')]^{1/2}}. \quad (73)$$

We denote the potential $\phi(x)$ at $x=+\infty$ and $x=-\infty$ by ϕ_+ and ϕ_-, respectively. Since $\rho(\phi)$ is, in general, double valued, so is $V(\phi)$. This feature plays an important role in obtaining a solution. Consider first the conventional BGK case where the hole is not time-dependent and $\rho^{(1)}=0$. Then $V(\phi)$ is single valued [and has the form of curve (b) in Fig. 1 of Ref. 1]. As shown in Ref. 1, taking $\phi_-=0$ leads to a localized hole solution, i.e., $\partial\phi/\partial x \to 0$ as $|x|\to\infty$. As $x\to\pm\infty$, $\phi\to\phi_\pm$ and in this $\gamma=0$ case, $\phi_+=\phi_-=0$ and $\phi(x)=\phi(-x)$. The criteria for such localized solution is that the maxima, $\partial V(\phi)/\partial\phi = 8\pi(\rho_e+\rho_i)=0$, occur at the same value of ϕ (0 in this case) for which $V(\phi)=0$.

Now consider the case in which the potential changes slowly in time, i.e., γ and $\dot{u}$ are nonzero. In this case $\rho^{(1)}$ is nonzero and a double-valued function of ϕ with the branches $\rho_1(\phi)$ and $\rho_2(\phi)$. As ϕ goes from ϕ_- to ϕ_0, $V(\phi)$ is given by

$$V_1(\phi) = 8\pi\int_\phi^\phi d\phi'\rho_1(\phi'). \quad (74)$$

As ϕ returns from ϕ_0 to ϕ_+, $V(\phi)$ is given by

$$V_2(\phi) = 8\pi\int_\phi^{\phi_0} d\phi'\rho_1(\phi') + 8\pi\int_{\phi_0}^{\phi} d\phi'\rho_2(\phi'). \quad (75)$$

According to (71), (70), and (65)

$$\frac{1}{2}\frac{\partial V}{\partial\phi} = -\frac{\partial^2\phi}{\partial x^2} = 4\pi\rho = -\lambda^{-2}\phi + 4\pi(\tilde{\rho} + \rho_e^{(1)}), \quad (76)$$

where $\rho_e^{(1)}$ denotes the last term in (65). If $\partial\phi/\partial x\to 0$ and $\tilde{\rho}\to 0$ as $x/\lambda\to\pm\infty$, then ϕ and $\rho_e^{(1)}$ approach constants in x, which we denote $\phi_\pm$ and $\rho_e^{(1)}(\pm\infty)$ and according to (76), $\phi_\pm = 4\pi\rho_e^{(1)}(\pm\infty)\lambda^2$. From (74) it is clear the the local solution condition is satisfied at $\phi=\phi_-$, i.e., $V_1(\phi_-)=\partial V_1(\phi_-)/\partial\phi_- = 0$. At $\phi=\phi_+$, it is again obvious that $\partial V_2(\phi_+)/\partial\phi_+ = 0$, however, the requirement that $V_2(\phi_+)=0$ is not automatically satisfied. It is

$$0 = V_2(\phi_+) = 8\pi\int_\phi^{\phi_0} d\phi'\rho_1(\phi') + 8\pi\int_{\phi_0}^{\phi_\cdot} d\phi'\rho_2(\phi'). \quad (77)$$

Integrating over x rather than ϕ', this can be written

$$\int_{-\infty}^{\infty} dx\frac{\partial\phi(x)}{\partial x}[\rho_e(x)+\rho_i(x)] = 0. \quad (78)$$

This equation simply states that the force on the electrons plus the force on the ions is zero, i.e., momentum is conserved. Since $\phi_0(x)$ is the BGK solution it is even in x and may be assumed to vanish at $|x|=\infty$. If $\phi(x)$ is odd, it is due to $\phi^{(1)}(x)$ and $\phi(x)$ at $|x|=\infty$ is given by $\phi_+^{(1)}(\pm\infty)$.

As mentioned earlier, we show in Sec. IX, that $\phi^{(1)}$ makes a contribution to the rate of momentum change that is of order ϵ^2 whereas $(1+\gamma t)\phi^{(0)}(x-\dot{u}t^2/2)$ makes a contribution of order ϵ. Therefore, in the next section, where we compute the structure of a growing and accelerating hole, we neglect $\rho^{(1)}$ and $\phi^{(1)}$. In other words, for $\gamma\tau \ll 1$, the fact that the hole is accelerating and growing has only a negligible effect on the solution so long as a solution exists, which is guaranteed by (78).

An accelerating hole also has the features of a so-called double layer. The potential change across the hole (or layer) is $\delta\phi = \phi_+ - \phi_- = 4\pi\lambda^2[\rho_e^{(1)}(+\infty) - \rho_e^{(1)}(-\infty)]$. Using the third term on the right-hand side of (198) for $\rho_e^{(1)}$, it is easy to show that $\delta\phi/\phi_0 = 4\omega_{pe}^2\lambda^2 f'_{0e}(u)$ for an ion hole.

IV. THE TRAPPED-PARTICLE DISTRIBUTION FUNCTION

In Sec. II we showed that the trapped-particle distribution $f_t(a)$ is a function only of the area a inside a trapped-particle orbit of energy E. The area is a function of E and the minimum hole potential ϕ_0. For an ion hole,

$$a(E, \phi_0) = 4\int_0^{x_E} dx' \left(\frac{2}{m_i}[E - q\phi(x')]\right)^{1/2}, \tag{79}$$

where

$$E = m_i v^2/2 + q_i\phi(x), \tag{80}$$

$$q_i\phi(x_E) = E. \tag{81}$$

We introduce a dimensionless distance, potential, and energy

$$z = x/\lambda, \tag{82}$$

$$c(z) = \phi(z)/\phi_0, \tag{83}$$

$$w = E(q_i\phi_0)^{-1}. \tag{84}$$

Note that $0 < c < 1$, $q_i\phi_0 < 1$ for an ion hole and $0 < w < 1$ for trapped ions. The area (79) can now be written

$$a(E, \phi_0) = 4\lambda\left(\frac{-2q_i\phi_0}{m_i}\right)^{1/2}\int_0^{z_w} dz[c(z) - w]^{1/2}, \tag{85}$$

where

$$c(z_w) = w. \tag{86}$$

The hole depth $-\tilde{f}$ is given by

$$\tilde{f} = f_t(a) - f_{0i}(u), \tag{87}$$

where u is the hole velocity. At the hole boundary, $E = 0$ and $\tilde{f} = 0$. For a given $\phi_0, f_t \neq 0$ for $0 < a < a(0, \phi_0)$ and $f_t = 0$ for $a > a(0, \phi_0)$. For two different values of ϕ_0, the corresponding functions $f_t(a)$ must be equal in the common region in which both are nonzero, i.e., for $0 < a < a(0, \phi'_0)$, where the ϕ'_0 is the smaller of the two potentials in absolute value. This self-similar property determines the functional form $f_t(a)$. This form can be anticipated by the following simple argument. Consider an ion hole with $u > 0, f'_{0i}(u) < 0$, and $f'_{0e}(u) > 0$. As the hole velocity decreases, the depth $-\tilde{f}$ increases. We assume, to be confirmed by the solution, that the spatial length of the hole, Δx, stays fixed near its most probable value,[1] a few times λ. According to (3), a change in the hole depth $-\delta\tilde{f}$ and a change in the hole velocity width $\delta\Delta v$ are related by $\delta\tilde{f} \sim \delta\Delta v\omega_p^{-2}\lambda^{-2}$. Since the change in hole area is $\delta a \sim \Delta v\lambda$, we obtain

$$\left(\frac{\partial\tilde{f}}{\partial a}\right)_{\Delta x} = k_0\omega_{pi}^{-2}\lambda^{-3}, \tag{88}$$

where k_0 is a positive eigenvalue to be determined by the actual solution. The solution of (88) which vanishes at the hole boundary $E = 0$ is

$$\tilde{f}(E) = -k_0\omega_{pi}^{-2}\lambda^{-3}[a(0) - a(E)]. \tag{89}$$

We will show that this form of $\tilde{f}(a)$ has the proper self-similar property.

The function $a(E)$ and the constant k_0 are determined by requiring that Poisson's equation be satisfied. As discussed in Sec. III, we use the lowest-order ($\gamma = \dot{u} = f'_{0e} = 0$) charge densities. According to (65) or (239), to zero order in ϵ, the charge density is given by a hole portion and a shielding portion just as in Ref. 1;

$$-\frac{\partial^2\phi}{\partial x^2} = -\lambda^{-2}\phi(x) + 4\pi n_i q_i\int dv\,\tilde{f}. \tag{90}$$

Using

$$dv = \{2m_i[E - q_i\phi(x)]\}^{-1/2}dE, \tag{91}$$

(90) becomes

$$-\frac{\partial^2\phi(x)}{\partial x^2} = -\lambda^{-2}\phi(x) + 8\pi n_i q_i \times\int_{q_i\phi(x)}^0 dE\frac{\tilde{f}(E)}{\{2m_i[E - q_i\phi(x)]\}^{1/2}}. \tag{92}$$

Using (89) and the dimensionless variables (82)–(84), this can be written

$$\frac{\partial^2 c(z)}{\partial z^2} - c(z) + 8k_0\int_0^{c(z)}\frac{dwA(w)}{[c(z) - w]^{1/2}} = 0, \tag{93}$$

where

$$A(w) = \int_0^\infty dz\sqrt{c(z)} - \int_0^{z_w} dz'[c(z') - w]^{1/2}. \tag{94}$$

The potential amplitude ϕ_0 and the spatial scale λ do not occur in (93) and (94). Therefore, our choice (89) for $\tilde{f}(a)$ has the required self-similar properties discussed earlier.

Following the usual procedure, we define a "potential"

$$V(c) = c^2 - 16k_0\int_0^c dc'\int_0^{c'}\frac{dw}{(c' - w)^{1/2}}A(w). \tag{95}$$

Using $V(c)$, we can write (93) as

$$\frac{\partial}{\partial z}\left[\left(\frac{\partial c(z)}{\partial z}\right)^2 - V[c(z)]\right] = 0, \tag{96}$$

which can be solved immediately to give

$$z(c) = \pm\int_0^c dc'[V(c')]^{-1/2}. \tag{97}$$

By reversing the order of integration, i.e.,

$$\int_0^c dc'\int_0^{c'} dw = \int_0^c dw\int_w^c dc', \tag{98}$$

Eq. (95) for $V(c)$ can be written

$$V(c) = c^2 - 32k_0\int_0^c dw(c - w)^{1/2}A(w). \tag{99}$$

Using the differential form of (97) for $z > 0, dz = -dc/[V(c)]^{1/2}$, we can replace z integrals with c integrals,

$$V(c) = c^2 - 32k_0 \int_0^c dw \sqrt{c-w} \left[\int_0^1 dc' \left(\frac{c'}{V(c')} \right)^{1/2} - \int_w^1 dc' \left(\frac{c'-w}{V(c')} \right)^{1/2} \right]. \tag{100}$$

Using

$$\int_0^c dw \int_w^1 dc' = \int_0^c dc' \int_0^{c'} dw + \int_c^1 dc' \int_0^c dw, \tag{101}$$

we can reverse the order of integration in the second integral of (100). The two w integrals can then be readily evaluated to give

$$V(c) = c^2 + k_0 \int_0^1 dc' \, K(c,c')[V(c')]^{-1/2}, \tag{102}$$

where

$$K(c,c') = -\tfrac{64}{3} c(cc')^{1/2} + 8(c+c')\sqrt{cc'} + 2(c-c')^2 \ln \left[(\sqrt{c} - \sqrt{c'})^4/(c-c')^2 \right]. \tag{103}$$

Equation (102) determines $V(c)$ and the eigenvalue k_0. We have solved it numerically by successive iteration. We found that $k_0 = 0.0216$. We have used the solution $V(c)$ in (97) to obtain $z(\phi/\phi_0)$, which is plotted in Fig. 7. We have also evaluated numerically the following quantities:

$$\int_{-\infty}^{\infty} dz \, c(z) = 7.00, \tag{104}$$

$$\int_{-\infty}^{\infty} dz \, c(z)^{1/2} = 9.50, \tag{105}$$

$$\int_{-\infty}^{\infty} dz \, c(z)^{3/2} = 5.76. \tag{106}$$

These integrals occur in the formula (110) for the hole growth rate. The first integral is related to the total hole charge Q (or mass M) which can be obtained by integrating (90) over x. Since the last term in (90) is 4π times the hole charge density, we obtain

$$Q = \frac{1}{4\pi\lambda} \int_{-\infty}^{\infty} dz \, \phi(z) = \frac{\phi_0}{4\pi\lambda} \int_{-\infty}^{\infty} dz \, c(z), \tag{107}$$

which holds for any shape hole.

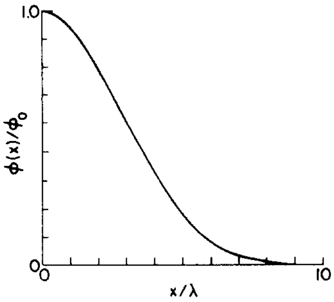

FIG. 7. Normalized potential ϕ/ϕ_0 for a self-similar hole as a function of x/λ.

By writing the integral (104) as $\lambda^{-1} \int dx \, \phi(x)/\phi(0)$, it is apparent that it is approximately equal to $\Delta x/\lambda$, where Δx is the length of the hole. From (104) we find that $\Delta x/\lambda \approx 7$. The hole parameters computed here are consistent with the prediction of the rectangle approximation of Ref. 1. If we compute the maximum hole depth from (89) using $a(0) = \Delta x \Delta v = 7\lambda\Delta v$, we find $-\tilde{f} \approx \Delta v/(7\omega_p^2\lambda^2)$, whereas (3) gives $-\tilde{f} \approx \Delta v/(6\omega_p^2\lambda^2)$.

V. HOLE GROWTH RATE

The hole growth rate γ and acceleration $\dot{u}$ are determined by (52) and (53) together with (107) for the hole mass. Since $M \sim \phi_0$, we have $\gamma \equiv (d\phi_0/dt)\phi_0^{-1} = (dM/dt)M^{-1}$. In terms of z and $c(z)$ given by (82) and (83), (52) can be written

$$\frac{\partial}{\partial t} M = \gamma M = -2\dot{u} f'_{0i} \, n_i m_i \lambda \times \left(\frac{-2q_i \phi_0}{m_i} \right)^{1/2} \int_{-\infty}^{\infty} dz \, c(z)^{1/2}. \tag{108}$$

The hole acceleration $\dot{u}$ is obtained from momentum conservation determined from (53),

$$M\dot{u} - \gamma n_i m_i f'_{0i} \lambda \left(\frac{-2q_i \phi_0}{m_i} \right)^{3/2} \int_{-\infty}^{\infty} dz \, c(z)^{3/2} - f'_{0e} \omega_{pe}^2 \, \phi_0^2 \pi^{-1} = 0. \tag{109}$$

Equations (108) and (109) can be solved together with Eq. (107) for the hole mass to obtain the ion hole growth rate γ. After some algebra we obtain

$$\gamma\lambda \left(\frac{-2q_i \phi_0}{m_i} \right)^{-1/2} = \frac{f'_{0e}(u) f'_{0i}(u) \omega_{pe}^2 \omega_{pi}^2 \lambda^4 8 \int_{-\infty}^{\infty} dz \, c(z)^{1/2}}{\left[\int_{-\infty}^{\infty} dz \, c(z) \right]^2 + (f'_{0i})^2 \omega_{pi}^4 \lambda^4 8 \int_{-\infty}^{\infty} dz \, c(z)^{1/2} \int_{-\infty}^{\infty} dz \, c(z)^{3/2}}. \tag{110}$$

The shielding $\lambda(u)$ is given by (164) and (165). If $f_{0\alpha}$ is a shifted Maxwellian,

$$f_{0\alpha}(v) = \pi^{-1/2} v_\alpha^{-1} \exp[-(v - v_{D\alpha})^2/v_\alpha^2], \tag{111}$$

then one can show that

$$\lambda^{-2} = (\lambda_{De}\lambda_{Di})^{-1} G, \tag{112}$$

$$G = b^{-1} G(x_e) + b G(x_i), \tag{113}$$

where $\lambda_{D\alpha}^{-2} = 2\omega_{p\alpha}^2/v_\alpha^2$, $b^2 = \lambda_{De}^2/\lambda_{Di}^2 = ZT_e/T_i$, $Z = -q_i/q_e$, $T_\alpha = m_\alpha v_\alpha^2/2$, and

$$G(x_\alpha) = 1 - 2x_\alpha \int_0^{x_\alpha} dy \exp(y^2 - x_\alpha^2), \tag{114}$$

$$x_\alpha = (u - v_{D\alpha})/v_\alpha. \tag{115}$$

Using (111)–(113) and (104)–(106) in (110), and the ion hole trapping width

$$\Delta v_{ti} = (-2q_i \phi_0/m_i)^{1/2}, \tag{116}$$

we obtain for an ion hole

$$\frac{\gamma}{\omega_{pi}} = \frac{\Delta v_{ti}}{v_i} (2G)^{1/2} \left(\frac{T_i}{T_e} \right)^{1/4} \Gamma_i, \tag{117}$$

where

$$\Gamma_i = \frac{(19) v_e^2 \, f'_{0e}(u) v_i^2 f'_{oi}(u)}{49G^2 + v_i^4 (f'_{0i})^2 Z (T_e/T_i)(5.76)(19)}. \tag{118}$$

Written in terms of the ion trapping time $\tau_{ti} = \Delta x/(2\Delta v_{ti})$, the growth rate is

$$\gamma\tau_{ti} = \Delta x \Gamma_i Z^{1/4}/(2\lambda) \approx 3.5 Z^{1/4}\Gamma_i. \tag{119}$$

These growth rate formulas are for an ion hole, but they can be readily converted to apply to electron holes if the subscripts i and e are switched.

We have evaluated (117) for electron holes for the simulation case of $m_i/m_e = 4$ and equal electron and ion temperature, $T_e = T_i$. For each v_D one can compute γ as a function u. A typical result is shown in Fig. 1 for $v_D = 2v_i$. For each v_D there is a maximum γ. This value as a function of v_D is plotted in Fig. 2. For evaluating (117) we have used the value $\sqrt{2}\Delta v_{ti}/v_e = 0.2$ which is consistent with the widths of holes observed in the simulation.[4] We used γ for electron holes since (116) and (117) predict that they grow approximately $(m_i/m_e)^{1/2}$ times faster than ion holes. The agreement between the theoretical and simulation growth rates is discussed in Sec. I.

According to the discussion following (107), the hole length Δx is approximately 7λ. Relative to the Debye length, the predicted half-width is then $\Delta x/(2\lambda_{De}) = 3.5(\lambda/\lambda_{De})$. The quantity λ/λ_{De} can be obtained from (112) and for values of u corresponding to the maximum γ, we find that $3.1 < \Delta x/(2\lambda_{De}) < 5.6$ for $1.5 < v_D/v_i < 3.5$, which is consistent with the value of $\Delta x/(2\lambda_{De}) \approx 4$ observed in the simulation.

For $\sqrt{2}v_D/v_i > 2.5$, the assumptions underlying (117) begin to break down. For one thing, $\gamma\tau_e$ is no longer small. When $\gamma\tau > 1$, the second term in the denominator of (110) overestimates the momentum gain of the passing ions. In (109), γ should be replaced with τ, which gives the Landau damping value. This change will cause γ to increase more rapidly with v_D as observed in the simulations. Furthermore, as v_D approaches the linear instability value, one should also include the nonresonant (wave) momentum in (109). The relationship between the hole, clump, and linear instability will be discussed in a future publication.

VI. PARTICLE ORBITS IN THE MODEL POTENTIAL

In this section and the following two, we derive in detail the expressions (65) for charge density and (52)–(54) for the rate of change of hole mass, momentum, and energy. These formulas can be obtained by solving the Vlasov equation (56) in the model potential (48).

As an alternative to solving the Vlasov equation, one can obtain the particle orbits. There are various classes of orbits. For example, for an ion hole there are trapped ions, passing ions, reflected electrons, and passing electrons. For an electron hole the roles of electrons and ions are switched, so it suffices to calculate one case. We shall arbitrarily choose an ion hole. Since the origin in time is arbitrary, we shall calculate the distribution function at $t = 0$. It is given by $f(x,v,0) = f[x(t),v(t),t]$, where $x(t)$ and $v(t)$ are the orbits whose values are x and v at $t = 0$. Also at $t = 0$, the center of the hole is assumed to be located at $x = 0$ and to have zero velocity ($u = 0$). As discussed earlier, we wish to compute the orbits to first order in γ and $\dot{u}$. Thus, we solve two separate orbit problems using the potential (48)—one with $\gamma = 0$, $\dot{u} \neq 0$, and one with $\gamma \neq 0$, $\dot{u} = 0$. To obtain $f(x,v,0)$ for passing and reflected particles, we proceed as follows. For given initial values x and v, we integrate back in time to obtain $x(t)$ and $v(t)$ for $t < 0$. We assume that the hole potential is localized in space between x_1 and x_2, i.e., $\phi(x) \neq 0$ only for $x_1 < x < x_2$ and $\Delta x = x_2 - x_1$. At $t = -\infty$ the particles have not been scattered by the hole potential and they have a spatially homogeneous distribution function $f_0(v)$. As time evolves from $t = -\infty$, a particle whose coordinates are x and v at $t = 0$, first encounters the potential at $x \equiv x_0 = x_1$ or x_2 at $t = -\tau_0$. Therefore, we can set

$$f(x,v,0) = f_0[v(-\tau_0)]. \tag{120}$$

Integrating further back in time, i.e., taking a larger value of τ_0, will not change (120) since $v(-\tau_0) = v(-\tau_0 - \Delta t)$ for $\Delta t > 0$. If we put

$$v(-\tau_0) = v + \delta v(x,v), \tag{121}$$

then (120) can be written

$$f(x,v,0) = f_0(v) + \delta v \frac{\partial f_0(v)}{\partial v}, \tag{122}$$

provided $\delta v/v_\alpha \ll 1$.

We consider first the case $\dot{u} \neq 0$. This problem can be solved exactly (to all orders in $\dot{u}$) by transforming to the accelerated frame of the hole. In the inertial frame the orbit equation is

$$m \frac{d^2}{dt^2} x(t) = F[x(t) - x_h(t)], \tag{123}$$

where $F(x) = -q(\partial/\partial x)\phi(x)$ is the force and $x_h(t) = \dot{u}t^2/2$ is the orbit of the hole in the inertial frame. The particle orbit $x_a(t)$ in the accelerated (hole) frame is

$$x_a(t) = x(t) - x_h(t), \tag{124}$$

and the equation of motion is

$$\begin{aligned} m \frac{d^2}{dt^2} x_a(t) &= F[x_a(t)] - m\dot{u} \\ &= -\frac{\partial}{\partial x_a(t)} \{q\phi[x_a(t)] + m\dot{u}x_a(t)\}. \end{aligned} \tag{125}$$

In the accelerated frame, we have a simple constant (in time) potential problem with the new potential $\phi_a(x) = \phi(x) + m\dot{u}x/q$. In the accelerated frame the "energy"

$$E_a = \frac{1}{2} m \left(\frac{dx_a(t)}{dt}\right)^2 + q\phi_a[x_a(t)] \tag{126}$$

is a constant of the motion. In the accelerated frame, the trapped ions have a steady-state distribution function which can be written as some function (say $f_t^{(a)}$) of E_a. Since $\dot{x}_h(t) = (d/dt)x_h(t) = 0$ at $t = 0$, the trapped-ion distribution function in the inertial frame at $t = 0$ is

$$f_t(x,v,0) = f_t^{(a)}[mv^2/2 + q\phi(x) + m\dot{u}x]. \tag{127}$$

For passing or reflected particles, $v(-\tau_0)$ can be calculated by using conservation of energy (126) in the accelerated frame to obtain the velocity in the accelerated frame and then using (124) to convert to the inertial frame. We find

$$v_a(t) = \frac{dx_a(t)}{dt} = \pm \left(v^2 + \frac{2q}{m}\{\phi_a(x) - \phi_a[x_a(t)]\}\right)^{1/2}, \tag{128}$$

$$v(-\tau_0) = v_a(-\tau_0) + \dot{u}\tau_0, \tag{129}$$

$$\tau_0 = -\left|\int_x^{x_0} dx' \left(v^2 + \frac{2q}{m}[\phi(x) - \phi(x')]\right)^{-1/2}\right|, \tag{130}$$

where $x_0 = x_a(-\tau_0)$ and is equal to either x_1 or x_2. If $x = x_1$ or x_2 (as required in Secs. VII and VIII) then $\phi_a(x) - \phi_a(x_0) = m\dot{u}(x - x_0)/q$. We can expand (128) and (130) to first order in $\dot{u}$ to obtain

$$\delta v(x_1,v) = v(-\tau_0) - v = \pm v - v - \dot{u}\int_{x_1}^{x_0} dx\,\{[v^2 - 2q\phi(x)/m]^{-1/2} - |v|^{-1}\}. \tag{131}$$

For passing ions and electrons the $+v$ is to be used and $x_0 = x_2$. For reflected electrons, $-v$ is to be used, $x_0 = x_1$, and the x integral is to be taken from x_1 to the reflection point (where the argument of the square root vanishes) and then back to x_1.

Unlike the accelerating potential, the orbit problem in a time-changing potential cannot, in general, be solved exactly analytically. One exception, however, is a potential square well, i.e., $\phi = (1 + \gamma t)\phi_0$ for $x_1 < x < x_2$ and $\phi = 0$ otherwise. In this case $(m/2q)[v(-\tau_0)^2 - v^2]$ is just the change in well potential $-\gamma\tau_0\phi_0$ that occurs during the transit time $\tau_0 = \Delta x(v^2 - 2q\phi_0/m)^{-1/2}$;

$$v(-\tau_0) = \{v^2 - [\gamma\Delta x 2q\phi_0/m(v^2 - 2q\phi_0/m)^{1/2}]\}^{1/2}. \tag{132}$$

Expanding to first order in γ, we obtain for passing particles $v < 0$ at x_1

$$\delta v(x_1,v) = -\gamma\Delta x q\phi_0/mv(v^2 - 2q\phi_0/m)^{1/2}. \tag{133}$$

We now obtain the orbits to first order in γ for arbitrary potential shape $\phi(x)$ by a direct expansion of (48). This procedure will also produce the result (131) for the $\dot{u}$ component of the orbit and (133) for the γ component when the potential is a square well. We expand the potential as

$$\Phi(x,t) = \Phi(x,0) + \dot{\Phi}(x,0)t + \ddot{\Phi}(x,0)t^2/2, \tag{134}$$

where $\dot{\Phi}(x,0) = \partial\Phi(x,t)/\partial t$ at $t = 0$ and $\ddot{\Phi}(x,0) = \partial^2\Phi \times (x,t)/\partial t^2$ at $t = 0$. For the model potential (48)

$$\dot{\Phi}(x,0) = \gamma\phi(x), \tag{135}$$

$$\ddot{\Phi}(x,0) = \dot{u}\frac{\partial\phi(x)}{\partial x}. \tag{136}$$

The orbit equation is

$$m\frac{d^2x(t)}{dt^2} = -q\frac{\partial}{\partial x(t)}\Phi[x(t),t]. \tag{137}$$

Using (134) and multiplying by $v(t) = dx(t)/dt$, (137) becomes

$$\frac{1}{2}m\frac{d}{dt}v(t)^2 = -q\frac{d}{dt}\phi[x(t),0] - qt\frac{d}{dt}\dot{\Phi}[x(t),0] - \frac{1}{2}qt^2\frac{d}{dt}\ddot{\Phi}[x(t),0]. \tag{138}$$

This equation can be integrated from 0 to τ_0 with $x(0) = x$ and $v(0) = v$. The last two terms on the right-hand side can subsequently be partially integrated bearing in mind that

$$\dot{\Phi}(x_0,0) = \ddot{\Phi}(x_0,0) = 0. \tag{139}$$

The result is

$$\frac{m}{2q}[v(-\tau_0)^2 - v^2] + \Phi(x_0,0) - \Phi(x,0) = \int_0^{-\tau_0} dt\,\dot{\Phi}[x(t),0] + \int_0^{-\tau_0} dt\,t\ddot{\Phi}[x(t),0]. \tag{140}$$

This equation can be readily solved to lowest order in γ and $\dot{u}$ by neglecting the $\dot{\Phi}$ and $\ddot{\Phi}$ terms. The solution, written in terms of t and x' instead of $-\tau_0$ and x_0, is

$$dt = \pm\{v^2 + 2q[\phi(x) - \phi(x')]/m\}^{-1/2}dx'. \tag{141}$$

This result can be used in (140) to convert the t integrals into x' integrals. Denoting the right-hand side of (140) by Σ, and using (135) and (136) we have

$$\Sigma = \gamma\int_x^{x_0} dx'\frac{\phi(x')}{\{v^2 + 2q[\phi(x) - \phi(x')]/m\}^{1/2}} + \dot{u}\int_x^{x_0} dx'\frac{\partial\phi(x')/\partial x'}{\{v^2 + 2q[\phi(x) - \phi(x')]/m\}^{1/2}} \times \int_x^{x'}\frac{dx''}{\{v^2 + 2q[\phi(x) - \phi(x'')]/m\}^{1/2}}. \tag{142}$$

The second term can be partially integrated to give

$$\frac{m\dot{u}}{q}\int_x^{x_0} dx'\left[1 - \left(v^2 + \frac{2q\phi(x)}{m}\right)^{1/2} \times \left(v^2 - 2q\frac{\phi(x) - \phi(x')}{m}\right)^{-1/2}\right]. \tag{143}$$

Now we solve (140) for $v(-\tau_0)^2$, take the square root, and expand to first order in Σ. We find

$$\delta v(x,v) = v(-\tau_0) - v = \pm[v^2 + 2q\phi(x)/m]^{1/2} - v \pm (q\Sigma/m)[v^2 + 2q\phi(x)/m]^{-1/2}. \tag{144}$$

If we set $x = x_1$, then $\phi(x_1) = 0$, and we have for $v < 0$

$$\delta v(x_1,v) = \pm v - v - \frac{\gamma q}{mv}\int_{x_1}^{x_0} dx\frac{\phi(x)}{[v^2 - 2q\phi(x)/m]^{1/2}} - \dot{u}\int_{x_1}^{x_0} dx\left[\left(v^2 - \frac{2q\phi(x)}{m}\right)^{-1/2} - |v|^{-1}\right]. \tag{145}$$

Note that x_0 is the exit point from the nonzero potential region as one integrates back along the orbit. If reversing the orbit causes the particle to move away from the potential, then $x_0 = x_1$ and the integrals are zero. Thus, $\delta v(x_1,v) = 0$ for $v > 0$ and $\delta v(x_2,v) = 0$ for $v < 0$. The first term $(\pm v - v)$ is 0 for passing particles and $-2v$ for reflected particles. For passing particles, $\delta v(x_1,v)$ for $v < 0$ and $\delta v(x_2,v)$ for $v > 0$ is given by (145) with $x_0 = x_2$. For reflected particles, i.e., those for which $mv^2/2 < q\phi_0$ (electrons for an ion hole, ions for an electron hole), the x integrals in (145) are path integrals to the reflection point x_r and back (see Sec. VIII). As expected, the third term of (145) agrees with (133) when the potential is a square well, and the last term of (145) agrees with (131).

From (128) it is clear that the $\dot{u}$ expansion does not converge if $v^2 < 2\dot{u}\Delta x$, and from (132) it follows that the γ expansion will not converge if $v^2 < 2\gamma\Delta x q\phi_0 m^{-1}(v^2 - 2q\phi_0/m)^{-1/2}$. Using (116) for the trapping width Δv_t and (49) for $\dot{u}$, both of these expansions can be shown to diverge in a small-velocity boundary layer of width $\delta^{(1)}v \sim \Delta v_t(\gamma\tau_t)^{1/2}$. Clearly, our calculations will not be very accurate unless $\gamma\tau_t \ll 1$.

From an examination of the formula (144) for $\delta v(x,v)$ or by intuitive consideration of the orbits as depicted in Figs. 5 and 6 one can determine the following parity of $\delta v(x,v)$. For the portion proportional to γ, $\delta v(x,v) = -\delta v(-x,-v)$, and for the portion proportional to $\dot u$, $\delta v(x,v) = \delta v(-x,-v)$. The distribution function is given by (122). Except for a boundary layer $|v| \lesssim (-2q\phi(x)/m)^{1/2}$, δv can be expanded in powers of ϕ. Since $f_0'(v)$ is essentially constant in this layer, the contribution of the boundary layer to the velocity moments of $\delta v f_0'(v)$ will have the following parity under $x \to -x$:

	γ	$\dot u$
$\langle v^0 \rangle$	odd	even
$\langle v^1 \rangle$	even	odd
$\langle v^2 \rangle$	odd	even

As we shall see in Secs. VII and VIII, except for the boundary layer contribution, the rate of change of mass, momentum, and energy can be expanded in powers of ϕ giving the usual results involving the dielectric function. Because these quantities are integrals over x, the parity properties mean that for the boundary layer $|v| < \Delta v_t$, only the $\dot u$ portion of δv contributes to mass and energy and only the γ portion to momentum.

VII. MASS, MOMENTUM, AND ENERGY FOR THE TRAPPED SPECIES

In this section we compute charge density, and the rate of change of mass, momentum, and energy for the trapped species. For an ion hole, the term "trapped species" refers to all the ions, both trapped and passing, and the "untrapped species" refers to the electrons, both reflected and passing. For an electron hole, the roles are reversed. In this section, we delete the species subscript on q, m, f_0, etc., with the understanding that the quantities for the trapped species are to be used.

We are interested in the low-order velocity moments of $f(x,v)$. For this purpose, we multiply the Vlasov equation (56) by v^l and integrate over v for $-\infty < v < \infty$ and over x for $x_1 \leqslant x \leqslant x_2$. We obtain

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} dx \int_{-\infty}^{\infty} dv\, v^l f(x,v) + \int_{-\infty}^{\infty} dv\, v^{l+1}[f(x_2,v) - f(x_1,v)] = -\frac{ql}{m}\int_{x_1}^{x_2} dx \frac{\partial \Phi}{\partial x}\int_{-\infty}^{\infty} dv\, v^{l-1} f(x,v). \quad (146)$$

The right-hand side, and therefore, the left-hand side, is independent of x_1 and x_2 as long as the hole lies between x_1 and x_2. For $l = 0,1,2$, the right-hand side is proportional to the rate of change of total mass, momentum, and kinetic energy, respectively, whereas the first term on the left-hand side is the rate of change for the same quantities inside x_1 and x_2. Therefore, the second term equals the rate of change of these quantities in the outside region, i.e.,

$$\frac{d}{dt}\left(\int_{-\infty}^{x_1} dx + \int_{x_2}^{\infty} dx\right) dv\, v^l f(x,v) = \int_{-\infty}^{\infty} dv\, v^{l+1}[f(x_2,v) - f(x_1,v)]. \quad (147)$$

To compute the rates of change to first order in γ and $\dot u$, we may use $f(x,v)$ to zero order in the first term on the left-hand side of (146) (prior to taking the time derivative) and to first order in the other two terms.

A. Inside region

We now compute

$$\langle v^l \rangle \equiv \int dv\, v^l f(x,v,t) \quad (148)$$

in the inside region (i.e., $x_1 < x < x_2$) to zero order in γ and $\dot u$. As explained earlier, a subsequent time derivative of this expression will produce a result correct to first order in γ and $\dot u$. [Later in this section, we compute (148) for $l = 1$ to first order in γ and $\dot u$ in order to obtain the ion momentum change from $\int dx\, \partial\Phi/\partial x \int dv\, f(x,v,t)$.]

In general $(\partial/\partial t)\langle v^l \rangle$ consists of a term proportional to $\dot u$ and a term proportional to γ. In computing the $\dot u$ term we set $\gamma = 0$ and vice versa. It is convenient to express $(\partial/\partial t)\langle v^l \rangle$ in the instantaneous hole frame, i.e., $u = 0$. However, in computing the $\dot u$ term we must keep u finite until after the time derivative is taken.

We begin by computing the portion of $(\partial/\partial t)\langle v^l \rangle$ proportional to $\dot u$. To zero order in γ and $\dot u$, the distribution function in the rest frame of the hole ($u = 0$) can be written as a function of the particle energy, $mv^2/2 + q\phi(x)$. In a coordinate system in which the hole has a velocity u, we have

$$\langle v^l \rangle = \left(\int_{-\infty}^{u - s(x)} dv + \int_{u+s(x)}^{\infty} dv\right) v^l \times f_0\left(\frac{v-u}{|v-u|}[(v-u)^2 - s(x)^2]^{1/2} + u\right) + \int_{u-s(x)}^{u+s(x)} dv\, v^l f_t[(v-u)^2 - s(x)^2], \quad (149)$$

where $s(x) = [-2q\phi(x)/m]^{1/2}$ is the separatrix. Setting $v - u \to v$, (149) can be written

$$\langle v^l \rangle = \left(\int_{-\infty}^{-s(x)} dv + \int_{s(x)}^{\infty} dv\right)(v+u)^l \times f_0\left(\frac{v}{|v|}[v^2 - s(x)^2]^{1/2} + u\right) + \int_{-s(x)}^{s(x)} dv (v+u)^l f_t[v^2 - s(x)^2]. \quad (150)$$

We expand $(v+u)^l = v^l + ulv^{l-1}$, set $f_0\{(v/|v|)[v^2 - s(x)^2]^{1/2} + u\} = f_0(v) + f_0'(v)\{(v/|v|)[v^2 - s(x)]^{1/2} - v + u\}$, compute $(\partial/\partial t)\langle v^l \rangle$ with $\partial s/\partial t = 0$, and set $u = 0$. We obtain

$$\frac{\partial}{\partial t}\langle v^l \rangle = \dot u l(\langle v^{l-1} \rangle - \langle v^{l-1} \rangle_0) - \dot u \int_{-s}^{s} dv\, v^l f_0'(0), \quad (151)$$

where $\langle v^l \rangle_0 \equiv \int dv\, v^l f_0(v)$ is the lth moment in the absence of a hole.

As the hole accelerates, $\partial f_t/\partial t = 0$, but its depth changes at the rate

$$\frac{\partial}{\partial t}\tilde{f}=\frac{\partial}{\partial t}[f_t-f_0(u)]=-\dot{u}f_0'(u), \tag{152}$$

which is uniform over the area of the hole. Therefore, the last term of (151) just accounts for the change of hole depth,

$$\frac{\partial}{\partial t}\int_{-s(x)}^{s(x)}dv\,v^l\tilde{f}(x,v)=-\dot{u}\int_{-s(x)}^{s(x)}dv\,v^lf_0'(v)$$
$$=\frac{-2}{l+1}\dot{u}s(x)^{l+1}f_0'(0),\quad \text{for } l=0,2,$$
$$\approx 0,\quad \text{for } l=1. \tag{153}$$

Note that the time derivative of the limit $s(x)$ does not contribute since $\tilde{f}$ vanishes on the hole boundary. Therefore, (153) has no portion proportional to γ. According to (50), the x integral of (153), with $l=0$, is proportional to the rate of changes of hole mass M;

$$\frac{\partial}{\partial t}M=-2\dot{u}f_0'(0)nm\int_{x_1}^{x_2}dx\left(\frac{-2q\phi(x)}{m}\right)^{1/2}. \tag{154}$$

For $l=2$, the x integral of (153) is proportional to the rate of change of the trapped-ion portion of the hole energy [see (50)],

$$\frac{\partial}{\partial t}nm\int_{x_1}^{x_2}dx\int_{-s(x)}^{s(x)}dv\,v^2\tilde{f}(x,v)$$
$$=-\frac{1}{3}nm\dot{u}f_0'(0)\int_{x_1}^{x_2}dx\left(\frac{-2q\phi(x)}{m}\right)^{3/2}. \tag{155}$$

We now compute $\langle v^l\rangle$ to zero order in γ and $\dot{u}$ in the instantaneous $u=0$ frame. The resulting expressions can be used in the second term of (151) and can also be differentiated in time, with u constant, to obtain the γ contribution to the inside portion of $(\partial/\partial t)\langle v^l\rangle$. Equation (149) with $u=0$ is

$$\langle v^l\rangle=\left(\int_{-\infty}^{-s(x)}dv+\int_{s(x)}^{\infty}dv\right)v^l$$
$$\times f_0\left[\frac{v}{|v|}\left(v^2+\frac{2q\phi(x)}{m}\right)^{1/2}\right]$$
$$+\int_{-s(x)}^{s(x)}dv\,v^lf_t[v^2-s(x)^2]. \tag{156}$$

First we shall evaluate (156) for $l=0$ and $l=2$. Expanding f_0 about v and using (51), Eq. (156) becomes

$$\langle v^l\rangle=\int_{-\infty}^{\infty}dv\,v^lf_0(v)+\int_{-s(x)}^{s(x)}dv\,v^l[f_0(0)-f_0(v)]$$
$$+\left(\int_{-\infty}^{-s(x)}dv+\int_{s(x)}^{\infty}dv\right)v^lf_0'(v)$$
$$\times\left(\frac{v}{|v|}[v^2-s(x)^2]^{1/2}-v\right)+\int_{-s(x)}^{s(x)}dv\,v^l\tilde{f}(v). \tag{157}$$

We can expand $f_0(v)$ about 0 in the second term and show that it is of order $f_0''\phi^{3/2}$ for $l=0$ and $f_0''\phi^{5/2}$ for $l=2$. In each case this is smaller than the terms retained and we shall neglect the second term.

The integrand of the third term can be expanded to give

$$\frac{v}{|v|}[v^2-s(x)^2]^{1/2}-v\approx\frac{q\phi(x)}{mv}-\frac{q^2\phi(x)^2}{2m^2v^3}. \tag{158}$$

The third term then involves integrals of the form

$$\left(\int_{-\infty}^{-s}dv+\int_{s}^{\infty}dv\right)\frac{f_0'(v)}{v}. \tag{159}$$

As $s\to 0$, this approaches the principal value integral

$$P\int_{-\infty}^{\infty}\frac{dv\,f_0'(v)}{v}. \tag{160}$$

One can show that the use of the expansion (158) and approximation of (159) by the principal value (160) leads to an error of order $s\sim\sqrt{\phi}$ times the principal value term. With these approximations, the third term becomes

$$-\phi(x)(4\pi nq\lambda_\alpha^2)^{-1},\quad \text{for } l=0, \tag{161}$$

$$-(q/m)\phi(x)+\phi(x)^2(8\pi nm\lambda_\alpha^2)^{-1},\quad \text{for } l=2, \tag{162}$$

where

$$\int_{-\infty}^{\infty}f_{0\alpha}(v)dv=1, \tag{163}$$

$$\lambda_\alpha^{-2}=-\omega_{p\alpha}^2\,P\int_{-\infty}^{\infty}dv\frac{f_{0\alpha}'(v)}{v}, \tag{164}$$

and $\omega_{p\alpha}^2=4\pi n_\alpha q_\alpha^2/m_\alpha$. The total shielding distance λ is given by

$$\lambda^{-2}\equiv\lambda_i^{-2}+\lambda_e^{-2}. \tag{165}$$

We now evaluate (156) with $l=1$. In this case the second integral in (156) is zero since $f_t(v^2-s^2)$ is even in v. After the transformation of integration variable, $v^2+2q\phi(x)/m\to v^2$, the remaining integral in (156) can be written

$$\langle v\rangle=\int_{-\infty}^{\infty}dv\,vf_0(v)=\langle v\rangle_0. \tag{166}$$

This result shows that in the hole frame, the creation of an ion hole does not change the total ion momentum. To obtain the terms of $(\partial/\partial t)\langle v^l\rangle$ proportional to γ, we apply $(\partial/\partial t)$ to (157) for $l=0,2$ and (166) for $l=1$. The first term in (157) gives zero. We have neglected the second. The third gives terms containing γ from (161) and (162). The last term gives only the term, proportional to $\dot{u}$ as explained following (153). For $l=1$, there is no γ term since $(\partial/\partial t)$ on (166) gives zero. The terms proportional to $\dot{u}$ and γ can now be combined to produce the first term of (146), the "inside" term, for $l=0$, 1, and 2. For $l=0$, we use (151), (153), and (161). For $l=1$, we have only the first term of (151) with $\langle 1\rangle-\langle 1\rangle_0$ given by (150) and (161). For $l=2$, we use (153) and (162). The second term of (151) is zero because of (166). After integrating over x, we find

$$\frac{d}{dt}\int_{x_1}^{x_2}dx\langle v^0\rangle=-\frac{\gamma}{4\pi nq\lambda_\alpha^2}\int_{x_1}^{x_2}dx\,\phi(x)$$
$$-2\dot{u}f_0(0)\int_{x_1}^{x_2}dx\left(\frac{-2q\phi(x)}{m}\right)^{1/2}, \tag{167}$$

$$\frac{d}{dt}\int_{x_1}^{x_2}dx\langle v\rangle=-\frac{\dot{u}}{4\pi nq\lambda_\alpha^2}\int_{x_1}^{x_2}dx\,\phi(x)+\frac{\dot{u}M}{nm}, \tag{168}$$

$$\frac{d}{dt}\int_{x_1}^{x_2}dx\langle v^2\rangle$$
$$=-\frac{q\gamma}{m}\int_{x_1}^{x_2}dx\,\phi(x)+\frac{\gamma}{4\pi nm\lambda_\alpha^2}\int_{x_1}^{x_2}dx\,\phi(x)^2$$
$$-\frac{2}{3}\dot{u}f_0'(0)\int_{x_1}^{x_2}dx\left(\frac{-2q\phi(x)}{m}\right)^{3/2}. \tag{169}$$

B. Outside region

We now calculate the second term on the left-hand side of (146), the "outside" term. According to (147), this term gives $(\partial/\partial t)\int dx\langle v^l\rangle$ in the outside region. For passing particles $f(x,v)$ is given by (145). For $x=x_1$, $v>0$ and $x=x_2$, $v<0$; $\delta v(x,v)=0$. For $x=x_1$, $v<0$ and $x=x_2$, $v>0$; δv is given by (145),

$$\delta v = -\dot{u}\int_{x_1}^{x_2} dx\left[\left(v^2-\frac{2q\phi(x)}{m}\right)^{-1/2}-|v|^{-1}\right] \pm \frac{\gamma q}{mv}\int_{x_1}^{x_2} dx\,\frac{\phi(x)}{[v^2-2q\phi(x)/m]^{1/2}}, \tag{170}$$

where the $+$ gives $\delta v(x_1,v)$ and the $-$ gives $\delta v(x_2,v)$. The square root symbols mean, in each case, the positive root. Using (122) and (170) in (147) we obtain

$$\begin{aligned}\frac{d}{dt}\left(\int_{-\infty}^{x_1} dx+\int_{x_2}^{\infty} dx\right)\langle v^l\rangle &= \int_{-\infty}^{\infty} dv\, v^{l+1}[f(x_2,v)-f(x_1,v)]\\ &= -\int_{x_1}^{x_2} dx\int_{-\infty}^{\infty} dv\, v^{l+1}f_0'(v)\\ &\quad\times\left\{\dot{u}\left[\left(v^2-\frac{2q\phi(x)}{m}\right)^{-1/2}-|v|^{-1}\right]\right.\\ &\quad\left.+\frac{q\gamma\phi(x)}{mv[v^2-2q\phi(x)/m]^{1/2}}\right\}\frac{v}{|v|}.\end{aligned} \tag{171}$$

The first term (the $\dot{u}$ term) on the right-hand side of (171) can be partially integrated to give

$$\begin{aligned}&-\int_{x_1}^{x_2} dx\,[v^l f_0'(v)]\dot{u}\left[\frac{v}{|v|}\left(v^2-\frac{2q\phi(x)}{m}\right)^{1/2}-v\right]\Bigg|_{0_-}^{0_+}\\ &+\int_{x_1}^{x_2} dx\int_{-\infty}^{\infty} dv\,[v^l f_0'(v)]'\dot{u}\\ &\times\left[\frac{v}{|v|}\left(v^2-\frac{2q\phi(x)}{m}\right)^{1/2}-v\right].\end{aligned} \tag{172}$$

The prime denotes a velocity derivative. For $l>1$, it is useful to perform another partial integration so that the second term in (172) becomes

$$\begin{aligned}&\int_{x_1}^{x_2} dx\,\frac{\dot{u}}{3v}[v^l f_0'(v)]'\left[\frac{v}{|v|}\left(v^2-\frac{2q\phi(x)}{m}\right)^{3/2}-v^3\right]\Bigg|_{0_-}^{0_+}\\ &-\int_{x_1}^{x_2} dx\int_{-\infty}^{\infty} dv\,\frac{\dot{u}}{3}\left(\frac{1}{v}[v^l f_0'(v)]'\right)'\\ &\times\left[\frac{v}{|v|}\left(v^2-\frac{2q\phi(x)}{m}\right)^{3/2}-v^3\right].\end{aligned} \tag{173}$$

The second term (the γ term) on the right-hand side of (171) can be partially integrated to give

$$\begin{aligned}&-\int_{x_1}^{x_2} dx\,\frac{q}{m}\gamma\phi(x)v^{l-1}f_0'(v)\left(v^2-\frac{2q\phi(x)}{m}\right)^{1/2}\frac{v}{|v|}\Bigg|_{0_-}^{0_+}\\ &+\int_{x_1}^{x_2} dx\,\frac{q}{m}\gamma\phi(x)\int_{-\infty}^{\infty} dv[v^{l-1}f_0'(v)]'\\ &\times\frac{v}{|v|}\left(v^2-\frac{2q\phi(x)}{m}\right)^{1/2}.\end{aligned} \tag{174}$$

To evaluate (171) for $l=0$, we use (172) for the first time in the integrand. We find

$$\begin{aligned}\frac{d}{dt}\left(\int_{-\infty}^{x_1} dx+\int_{x_2}^{\infty} dx\right)\langle 1\rangle &= \int_{x_1}^{x_2} dx\left[2f_0'(0)\dot{u}\left(\frac{-2q\phi(x)}{m}\right)^{1/2}\right.\\ &\quad\left.+(4\pi nq\lambda_\alpha^2)^{-1}\gamma\phi(x)\right].\end{aligned} \tag{175}$$

The first term in the integrand comes from the first term of (172). We have neglected the second term of (172) since it is of order $\dot{u}\phi$. The last term in (175) comes from setting $\phi=0$ in the square root of the γ term of (171).

To evaluate (171) for $l=1$, we use (172) and (174). We find

$$\begin{aligned}\frac{d}{dt}\left(\int_{-\infty}^{x_1} dx+\int_{x_2}^{\infty} dx\right)\langle v\rangle &= \int_{x_1}^{x_2} dx[(4\pi nq\lambda_\alpha^2)^{-1}\dot{u}\phi(x)\\ &\quad-\gamma f_0'(0)(-2q\phi(x)/m)^{3/2}].\end{aligned} \tag{176}$$

The first term in the integrand comes from expanding the square root in the second term of (172), and neglecting the f_0'' term which is of order $\sqrt{\phi}$ times the f_0' term. The second term of (176) comes from the first term of (174). We have neglected the second term of (174) which is of order $\sqrt{\phi}$ times the first term.

To evaluate (171) for $l=2$ we use (173) and (174). We find

$$\begin{aligned}\frac{d}{dt}\left(\int_{-\infty}^{x_1} dx+\int_{x_2}^{\infty} dx\right)\langle v^2\rangle &= \int_{x_1}^{x_2} dx\left[-\frac{4}{3}\dot{u}f_0'(0)\left(\frac{-2q\phi(x)}{m}\right)^{3/2}\right.\\ &\quad\left.+\frac{q\gamma\phi(x)}{m}+(4\pi nm\lambda_\alpha^2)^{-1}\gamma\phi(x)^2\right].\end{aligned} \tag{177}$$

The first term in the integrand comes from the first term of (173). We have neglected the second term of (173) since it is of order $\sqrt{\phi}$ times the first. The last two terms in (177) come from expanding the square root in the second term of (174) and using (164).

The rate of change for total momentum and energy for $-\infty<x<\infty$ is obtained by adding the inside and outside terms. For momentum we add (168) and (176). Two of the terms proportional to $\dot{u}$ cancel and we obtain

$$\frac{\partial}{\partial t}n\int_{-\infty}^{\infty} dx\, m\langle v\rangle = M\dot{u}-\gamma nmf_0'(0)\int_{-\infty}^{\infty} dx\left(\frac{-2q\phi(x)}{m}\right)^{3/2}. \tag{178}$$

For energy we add (169) and (177), the $\gamma\phi$ terms cancel, and we obtain

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx\,\frac{nm}{2}\langle v^2\rangle = \frac{\partial}{\partial t}(8\pi\lambda_\alpha^2)^{-1}\int_{x_1}^{x_2} dx\,\phi(x)^2$$

$$-\frac{2}{3}\dot{u}f_0'(0)nm\int_{x_1}^{x_2}dx\left(\frac{-2q\phi(x)}{m}\right)^{3/2}. \tag{179}$$

We now evaluate the third term in (146), the first term on the right-hand side. This term must, of course, equal the sum of the first two terms, so we already know its value. We evaluate it only to elucidate the process of energy and momentum conservation. For $l=0$, this term is obviously zero because of particle conservation.

For $l=0$, this term multiplied by nm is the momentum input to the trapped species by the electric field

$$-nq\int_{x_1}^{x_2}dx\int_{-\infty}^{\infty}dv\,v\frac{\partial\phi(x)}{\partial x}\frac{\partial f(x,v)}{\partial v}$$
$$=nq\int_{x_1}^{x_2}dx\frac{\partial\phi(x)}{\partial x}\int_{-\infty}^{\infty}dv\,f(x,v). \tag{180}$$

The distribution function consists of two parts—the trapped portion $f_0(0)+\tilde{f}$ and the passing portion $f_0(v)+\delta v f_0'(v)$, where δv is given by (142) and (144). The unperturbed piece f_0 will give zero when substituted into (180). Next we substitute $\delta v f_0'$ into (180), and integrate over the untrapped region. The contribution from $v>0$ and $v<0$ is the same so we integrate over $+v$ only and multiply by 2. We obtain

$$2\gamma\int_{x_1}^{x_2}dx\frac{\partial\phi(x)}{\partial x}\int_{(-q\phi(x)/m)^{1/2}}^{\infty}dv$$
$$\times\frac{nq^2f_0'(v)}{m[v^2+2q\phi(x)/m]^{1/2}}$$
$$\times\int_{x_1}^{x_2}dx'\frac{\phi(x')}{[v^2+2q\phi(x)/m-2q\phi(x')/m]^{1/2}}. \tag{181}$$

Setting $v^2+2q\phi(x)/m\rightarrow v^2$, (181) can be written

$$\frac{2nq^2\gamma}{m}\int_{x_1}^{x_2}dx\frac{\partial\phi(x)}{\partial x}$$
$$\times\int_0^{\infty}dv\frac{f_0'\{[v^2-2q\phi(x)/m]^{1/2}\}}{[v^2-2q\phi(x)/m]^{1/2}}$$
$$\times\int_{x_1}^{x}dx'\frac{\phi(x')}{[v^2-2q\phi(x')/m]^{1/2}}. \tag{182}$$

After partially integrating on x, we obtain

$$nq\gamma\int_0^{\infty}dv$$
$$\times\int_{x_1}^{x_2}dx\frac{f_0\{[v^2-2q\phi(x)/m]^{1/2}\}-f_0(v)}{[v^2-2q\phi(x)/m]^{1/2}}\phi(x). \tag{183}$$

Expanding the numerator of the integrand as in (157), we obtain

$$-n\gamma\int_{x_1}^{x_2}dx\int_0^{\infty}dv\,v\frac{f_0'(v)\phi(x)}{[v^2-2q\phi(x)/m]^{1/2}}. \tag{184}$$

After partially integrating on v, we find

$$-2nq\gamma\int_{x_1}^{x_2}dx\left[\left(v^2-\frac{2q\phi(x)}{m}\right)^{1/2}f_0'(v)\Big|_0^{\infty}\right.$$
$$\left.-\int_0^{\infty}dv\,[v^2-2q\phi(x)/m]^{1/2}f_0''(v)\right]\phi(x). \tag{185}$$

The second term in the integrand is smaller than the first by a factor of $q\phi/mv_\alpha^2$ so we neglect it. The final result is

$$2nqf_0'(0)\gamma\int_{x_1}^{x_2}dx\,\phi(x)\left(\frac{-2q\phi(x)}{m}\right)^{1/2}. \tag{186}$$

This result, for passing particles, equals the second term on the right-hand side of (178).

The momentum input to the trapped particles from the electric field is obtained by substituting the trapped-particle distribution function $f_0+\tilde{f}$ into (180). Since the trapped species and untrapped species f_0 contribution to (180) cancel, we consider only $\tilde{f}$. Next we note that in a frame accelerating at $\dot{u}$, any function of $mv^2/2+q\phi(x)+m\dot{u}x$ is a stationary solution of the Vlasov equation. Therefore, the charge density associated with $\tilde{f}$ can be written

$$nq\int\tilde{f}\,dv=\tilde{\rho}\left(\phi+\frac{n\dot{u}x}{q}\right). \tag{187}$$

Using this result, (180) becomes

$$\int_{x_1}^{x_2}dx\frac{\partial\phi(x)}{\partial x}\tilde{\rho}\left(\phi(x)+\frac{m\dot{u}x}{q}\right)$$
$$=\int_{x_1}^{x_2}dx\frac{\partial\phi(x)}{\partial x}\left(\tilde{\rho}[\phi(x)]+\frac{m\dot{u}x}{q}\frac{\partial\tilde{\rho}[\phi(x)]}{\partial\phi(x)}\right). \tag{188}$$

The first term in the integrand is zero since $\phi(x_2)=\phi(x_1)$. The second term may be partially integrated to give

$$\frac{m\dot{u}}{q}\int_{x_1}^{x_2}dx\,\tilde{\rho}[\phi(x)]=M\dot{u}. \tag{189}$$

This result gives the first term on the right-hand side of (178). When $\epsilon\ll 1$, a hole behaves like a rigid macroscopic body and obeys Newton's equation.

The total energy lost from the electric field to the trapped species is $nm/2$ times the right-hand side of (146) with $l=2$;

$$-nq\int_{x_1}^{x_2}dx\frac{\partial\phi(x)}{\partial x}\langle v\rangle=nq\int_{x_1}^{x_2}dx\,\phi(x)\frac{\partial}{\partial x}\langle v\rangle. \tag{190}$$

Using the continuity equation for ions

$$\frac{\partial}{\partial t}\langle 1\rangle+\frac{\partial}{\partial x}\langle v\rangle=0, \tag{191}$$

(190) becomes

$$-nq\int_{x_1}^{x_2}dx\,\phi(x)\frac{\partial}{\partial x}\langle 1\rangle. \tag{192}$$

If we use (167) for $\partial\langle 1\rangle/\partial t$, (192) produces the two terms on the right-hand side of (179).

VIII. MASS, MOMENTUM, AND ENERGY FOR THE UNTRAPPED SPECIES

Except for the different orbit topology, the computation of $(\partial/\partial t)\langle v^l\rangle$ for the untrapped species is similar to that for the trapped species in Sec. VII. As in the previous section we delete the species subscript on q, m, f_0, etc., with the understanding that the quantities for the untrapped species are to be used.

A. Inside region

For the untrapped species, the expression for $\langle v^l\rangle$ to zero order in γ and $\dot{u}$, analogous to (149), is

$$\langle v^l \rangle = \left(\int_{-\infty}^{-s_0(x)} dv + \int_{s_0(x)}^{\infty} dv \right) v^l f_0\left(\frac{v}{|v|} v(x) \right) + \int_{-s_0(x)}^{s_0(x)} dv\, v^l f_0\left(\frac{x}{|x|} v(x) \right), \tag{193}$$

where

$$s_0(x) = \{ [\phi_0 - \phi(x)] 2q/m \}^{1/2} \tag{194}$$

is the separatrix between passing and reflected particles, and

$$v(x) = [v^2 + 2q\phi(x)/m]^{1/2}.$$

We now expand f_0 in (193) around v

$$\langle v^l \rangle = \left(\int_{-\infty}^{-s_0(x)} dv + \int_{s_0(x)}^{\infty} dv \right) v^l \times \left[f_0(v) + f_0'(v) \left(\frac{v}{|v|} v(x) - v \right) \right] + \int_{-s_0(x)}^{s_0(x)} dv\, v^l \left[f_0(v) + f_0'(v) \left(\frac{x}{|x|} v(x) - v \right) \right]. \tag{195}$$

Using the expansion

$$\frac{v}{|v|} v(x) - v = \frac{q\phi(x)}{mv} - \frac{q^2\phi(x)^2}{2m^2v^3} + \cdots \tag{196}$$

in the first integrand in (195) and setting $f_0'(v) \simeq f_0'(0)$ in the second, we find for $l = 0,2$,

$$\langle v^l \rangle = \int_{-\infty}^{\infty} dv\, v^l f_0(v) + \left(\int_{-\infty}^{-s_0(x)} dv + \int_{s_0(x)}^{\infty} dv \right) \times f_0'(v) v^l \left(\frac{q\phi(x)}{mv} - \frac{q^2\phi(x)^2}{2m^2v^3} \right) + \frac{x}{|x|} f_0'(0) \int_{-s_0(x)}^{s_0(x)} dv\, v^l v(x). \tag{197}$$

The last term is odd in x and so will not contribute to M or T_0, but will contribute to the charge density and create an electric field which is even in x and will accelerate the hole. We now evaluate (197) for $l = 0$ and $l = 2$. Using (164) and the definition of principal value as $s_0 \to 0$ we find for $l = 0$

$$\langle 1 \rangle = 1 - \frac{\lambda_\alpha^{-2}}{4\pi nq} \phi(x) + \frac{x}{|x|} f_0'(0) \int_{-s_0(x)}^{s_0(x)} dv\, v(x). \tag{198}$$

For $l = 2$, we obtain

$$\langle v^2 \rangle = \langle v^2 \rangle_0 - \frac{q\phi(x)}{m} + \frac{\lambda_\alpha^{-2}}{8\pi nm} \phi(x)^2 + \frac{x}{|x|} f_0'(0) \int_{-s_0(x)}^{s_0(x)} dv\, v^2 v(x). \tag{199}$$

For $l = 1$, (193) may be readily evaluated by making the transformation of integration variable $v^2 + 2q\phi(x)/m \to v^2$. We find

$$\langle v \rangle = \left(\int_{-\infty}^{-(2q\phi_0/m)^{1/2}} dv + \int_{(2q\phi_0/m)^{1/2}}^{\infty} dv \right) v f_0(v) + \int_{-(2q\phi_0/m)^{1/2}}^{(2q\phi_0/m)^{1/2}} dv\, v f_0\left(\frac{x}{|x|} |v| \right). \tag{200}$$

The last term is zero, and the first two can be written

$$\langle v \rangle = \int_{-\infty}^{\infty} dv\, v f_0(v) - \int_{-(2q\phi_0/m)^{1/2}}^{(2q\phi_0/m)^{1/2}} dv\, v f_0(v). \tag{201}$$

After expanding $f_0(v)$ in the integrand, (201) becomes

$$\langle v \rangle = \int_{-\infty}^{\infty} dv\, v f_0(v) - \frac{2}{3} f_0'(0) \left(\frac{2q\phi_0}{m} \right)^{3/2}. \tag{202}$$

The portion of $(\partial/\partial t)\langle v^l \rangle$ proportional to γ can now be obtained by differentiating (198), (199), and (202) with respect to time.

Next we consider the portion of $(\partial/\partial t)\langle v^l \rangle$ proportional to $\dot{u}$. The calculation is analogous to that leading from (150) to (151). The analogy to (150) is (193) transformed to a coordinate system in which the hole has a velocity u, which gives in place of (150)

$$\langle v^l \rangle = \left(\int_{-\infty}^{-s_0(x)} dv + \int_{s_0(x)}^{\infty} dv \right) (v+u)^l f_0\left(\frac{v}{|v|} v(x) + u \right) + \int_{-s_0(x)}^{s_0(x)} dv (v+u)^l f_0\left(\frac{x}{|x|} v(x) + u \right). \tag{203}$$

We now carry out the procedure following (150) that led to (151). The calculation is virtually the same except that the presence of u in the argument of f_0 in the second term on the right-hand side of (203) (which is not present in f_t) produces an additional term which exactly cancels the last term of (151). The result for the untrapped species is

$$\frac{\partial}{\partial t} \langle v^l \rangle = \dot{u} l (\langle v^{l-1} \rangle - \langle v^{l-1} \rangle_0). \tag{204}$$

To obtain the total $(\partial/\partial t)\langle v^l \rangle$ for the inside region we add the γ portion, obtained from applying $\partial/\partial t$ to (198), (199), and (202), to the $\dot{u}$ portion obtained from (204). Again, we emphasize that the result applies only in the $u = 0$ frame. After integrating over x, the results are

$$\frac{d}{dt} \int_{x_1}^{x_2} dx \langle 1 \rangle = -\frac{\gamma}{4\pi nq\lambda_\alpha^2} \int_{x_1}^{x_2} dx\, \phi(x), \tag{205}$$

$$\frac{d}{dt} \int_{x_1}^{x_2} dx \langle v \rangle = \frac{-\dot{u}}{4\pi nq\lambda_\alpha^2} \int_{x_1}^{x_2} dx\, \phi(x) - f_0'(0)\gamma \left(\frac{2q\phi_0}{m} \right)^{3/2} \Delta x, \tag{206}$$

$$\frac{d}{dt} \int_{x_1}^{x_2} dx \langle v^2 \rangle = -\frac{q\gamma}{m} \int_{x_1}^{x_2} dx\, \phi(x) + \frac{\gamma}{4\pi nm\lambda_\alpha^2} \int_{x_1}^{x_2} dx\, \phi(x)^2 - \tfrac{4}{3} \dot{u} f_0'(0) (2q\phi_0/m)^{3/2} \Delta x. \tag{207}$$

B. Outside region

To compute $(\partial/\partial t)\langle v^l \rangle$ in the outside region, we consider separately the passing particles and the reflected particles. The distribution function is given by (122) and (145). For passing particles the "untrapped species" counterpart to (171) is

$$\int_{\text{passing}} dv\, v^{l+1} [f(x_2,v) - f(x_1,v)] = -\int_{x_1}^{x_2} dx \left(\int_{(2q\phi_0/m)^{1/2}}^{\infty} - \int_{-\infty}^{-(2q\phi_0/m)^{1/2}} dv \right) \times \left(\frac{\dot{u}}{[v^2 - 2q\phi(x)/m]^{1/2}} - \frac{\dot{u}}{|v|} + \frac{\gamma q\phi(x)}{mv} \frac{1}{[v^2 - 2q\phi(x)/m]^{1/2}} \right) v^{l+1} f_0'(v). \tag{208}$$

The portion due to reflected particles is

$$\int_{\text{reflected}} dv\, v^{l+1}\,[f(x_2,v)-f(x_1,v)]$$
$$= \int_0^{(2q\phi_0/m)^{1/2}} dv\, v^{l+1} f_0'(v)\left(-2v-2\dot{u}\int_{r_2}^{x_2}\frac{dx}{[v^2-2q\phi(x)/m]^{1/2}}-\frac{2q\gamma}{mv}\int_{r_2}^{x_2}\frac{dx\phi(x)}{[v^2-2q\phi(x)/m]^{1/2}}\right)$$
$$-\int_{-(2q\phi_0/m)^{1/2}}^{0} dv\, v^{l+1} f_0'(v)\left(-2v-2\dot{u}\int_{x_1}^{r_1}\frac{dx}{[v^2-2q\phi(x)/m]^{1/2}}-\frac{2q\gamma}{mv}\int_{x_1}^{r_1}\frac{dx\phi(x)}{[v^2-2q\phi(x)/m]^{1/2}}\right). \tag{209}$$

The x integration limits r_1 and r_2 are the reflection points and are determined by the two roots, negative and positive, respectively, of

$$v^2-2q\phi(r)/m=0. \tag{210}$$

We can reverse the order of integration of the last two terms in each integrand in (209) by using

$$\int_0^{(2q\phi_0/m)^{1/2}} dv\int_{r_2}^{x_2} dx=\frac{1}{2}\int_{x_1}^{x_2} dx\int_{(2q\phi(x)/m)^{1/2}}^{(2q\phi_0/m)^{1/2}} dv, \tag{211}$$

and

$$\int_{-(2q\phi_0/m)^{1/2}}^{0} dv\int_{x_1}^{r_1} dx=\frac{1}{2}\int_{x_1}^{x_2} dx\int_{-(2q\phi_0/m)^{1/2}}^{-(2q\phi(x)/m)^{1/2}} dv. \tag{212}$$

In obtaining these relations we have assumed $\phi(x)$ is even. Using (211) and (212) in (209), we can combine the passing and reflected terms (208) to obtain

$$\int dv\, v^{l+1} f(x,v)\Big|_{x_1}^{x_2}=A+B+C+D, \tag{213}$$

where

$$A=-\int_{x_1}^{x_2} dx\left(\int_{s(x)}^{\infty} dv-\int_{-\infty}^{-s(x)} dv\right)\times\left(\frac{\dot{u}}{[v^2-2q\phi(x)/m]^{1/2}}-\frac{\dot{u}}{|v|}\right)v^{l+1}f_0'(v), \tag{214}$$

$$B=-\int_{x_1}^{x_2} dx\left(\int_{s(x)}^{\infty} dv-\int_{-\infty}^{-s(x)} dv\right)\times\frac{\gamma q\phi(x)v^{l+1}f_0'(v)}{mv[v^2-2q\phi(x)/m]^{1/2}}, \tag{215}$$

$$C=-\int_{x_1}^{x_2} dx\left(\int_{s(x)}^{(2q\phi_0/m)^{1/2}} dv-\int_{-(2q\phi_0/m)^{1/2}}^{-s(x)} dv\right)\times\frac{\dot{u}}{|v|}v^{l+1}f_0'(v), \tag{216}$$

$$D=-2\int_0^{s_2} dv\, v^{l+2}f_0'(v)+2\int_{s_1}^{0} dv\, v^{l+2}f_0'(v). \tag{217}$$

We have used the notation $s(x)=[2q\phi(x)/m]^{1/2}$.

The last term, D, comes from the first terms in brackets in the integrands of (209). We have changed the limits of the v integration from $\pm(2q\phi_0/m)^{1/2}$ to s_1 and s_2 which contain corrections of order γ and $\dot{u}$. The rationale behind this correction is that we wish to compute $d\langle v^l\rangle/dt$ correct to first order in $\dot{u}$ and γ. Without this correction this term would be zero order. The γ correction in s_1 and s_2 takes into account the fact that a particle on the separatrix just exiting the inner region at x_1 or x_2 at time t was reflected at an earlier time $t-\Delta t$, where

$$\Delta t\approx(x_2-x_1)/[2(2q\phi_0/m)^{1/2}]. \tag{218}$$

Of course, this expression is not accurate to within $\Delta x/(2q\phi_0/m)^{1/2}$ and is, therefore, only meaningful for $x_2-x_1 \gg \Delta x$, which is not our convention since we put x_1 and x_2 at the hole boundaries, i.e., $x_2-x_1=\Delta x$. However, it is instructive to retain this term. This correction requires that the separatrix be determined by the potential evaluated at the earlier time,

$$\phi_0\to\phi_0(1-\gamma\Delta t). \tag{219}$$

The $\dot{u}$ correction in s_1 and s_2 can be understood by going to the accelerated frame. In this frame the acceleration produces an equivalent potential $m\dot{u}x/q$. Therefore, the maximum value of $q\phi(x)$ relative to the hole edges at $\mp\Delta x/2$ is $q\phi_0\pm\dot{u}\Delta x/2$. The velocity coordinate of the separatrix at x_1 and x_2 then becomes

$$\mp(2q\phi_0/m\pm\dot{u}\Delta x)^{1/2}. \tag{220}$$

Using (219) and (220), the separatrices s_1 and s_2, for reflected particles at $x=x_1$ and x_2 to first order in γ and $\dot{u}$, are

$$(s_1,s_2)=(-,+)(2q\phi_0/m)^{1/2}(1-\gamma\Delta t/2)-\dot{u}\Delta x[2(2q\phi_0/m)^{1/2}]^{-1}. \tag{221}$$

In order to evaluate (213) for $l=0,1,2$, the various terms must be dealt with in different ways for different l. The first integral, A, involving $\dot{u}$, may be partially integrated over v to obtain

$$A=\int_{x_1}^{x_2} dx\, a(x), \tag{222}$$

where

$$a(x)=-\dot{u}v^l f_0'(v)\{[v^2-2q\phi(x)/m]^{1/2}-v\}\Big|_{s(x)}^{-s(x)}+\left(\int_{s(x)}^{\infty} dv-\int_{-\infty}^{-s(x)} dv\right)\dot{u}[v^l f_0'(v)]'\times\left[\left(v^2-\frac{2q\phi(x)}{m}\right)^{1/2}-|v|\right]. \tag{223}$$

The second term may be further integrated to give

$$\frac{\dot{u}}{3}[v^l f_0'(v)]'\frac{1}{v}\left[\left(v^2-\frac{2q\phi(x)}{m}\right)^{3/2}-|v|^3\right]\Bigg|_{+s(x)}^{-s(x)}-\frac{\dot{u}}{3}\left(\int_{s(x)}^{\infty} dv-\int_{-\infty}^{-s(x)} dv\right)\left([v^l f_0'(v)]'\frac{1}{v}\right)'\times\left[\left(v^2-\frac{2q\phi(x)}{m}\right)^{3/2}-|v|^3\right]. \tag{224}$$

The integral C, containing $\dot{u}$, can be readily evaluated. It is zero when $l=1$. For $l=0,2$ it is

$$C = \int_{x_1}^{x_2} dx \frac{2}{l+1} f_0'(0)\dot{u} \times \{ [2q\phi(x)/m]^{(l+1)/2} - (2q\phi_0/m)^{(l+1)/2} \}. \quad (225)$$

The last integral, D, can be evaluated by expanding to first order in γ and $\dot{u}$. We obtain

$$D = -f_0'(0)(2q\phi_0/m)^2(1 - 2\gamma\Delta t), \quad \text{for } l = 1, \quad (226)$$

$$D = 2\dot{u}\Delta x f_0'(0)(2q\phi_0/m)^{(l+1)/2}, \quad \text{for } l = 0,2. \quad (227)$$

We shall now evaluate the terms in (213) for $l = 0,1,2$. For $l = 0$, A is given by the first term in (223), $-2f_0'(0) \times [2q\phi(x)/m]^{1/2}$, the second term is of order f_0'' and we neglect it. Here C and D are given by (225) and (227). The first term in C cancels A and the second term in C cancels D. Thus for $l = 0$ the entire contribution to (213) comes from B, which to lowest order is obtained by approximating $[v^2 - 2q\phi(x)/m]^{1/2}$ with $|v|$. When $\phi \to 0$, the integral approaches the principal value (164)

$$\int dv\, vf \Big|_{x_1}^{x_2} = B = (4\pi nq\lambda_\alpha^2)^{-1}\gamma \int_{x_1}^{x_2} dx\, \phi(x). \quad (228)$$

For $l = 1$, the first term in (223) is zero. The second term may be integrated by writing

$$[v^2 - 2q\phi(x)/m]^{1/2} - |v| \approx -q\phi(x)/(mv), \quad (229)$$

neglecting f_0'' and using (164)

$$A = \dot{u}(4\pi nq\lambda_\alpha^2)^{-1} \int_{x_1}^{x_2} dx\, \phi(x). \quad (230)$$

To evaluate B we use (215), which can be evaluated by expanding the square root. The first term will give a term of order $\phi\gamma f_0'$ and the second gives a term of order $\gamma\phi^2$ which we neglect. For $l = 1$, $C = 0$, and D is given by (226). Therefore we have

$$\int dv\, v^2 f(x,v) \Big|_{x_1}^{x_2} = \dot{u}(4\pi nq\lambda_\alpha^2)^{-1} \int_{x_1}^{x_2} dx\, \phi(x) - f_0'(0)(2q\phi_0/m)^2(1 - 2\gamma\Delta t). \quad (231)$$

For $l = 2$, we use (223) and (224) to evaluate A. The first terms of (223) and (224), the boundary terms, give

$$A = \left(-2 + \frac{4}{3}\right) \dot{u} f_0'(0) \int_{x_1}^{x_2} dx \left(\frac{2q\phi(x)}{m}\right)^{3/2}. \quad (232)$$

We neglect the second term of (224) which can be evaluated by putting $[v^2 - 2q\phi(x)/m]^{3/2} - |v|^3 = -3q\phi(x)v/m + 3[q\phi(x)/m]^2/(2v)$ in the integrand. It is easy to show that the integral is of order $\dot{u}\phi^2$, which is small compared to (232). We obtain B from (215) by expanding the square root in the integrand $[v^2 - 2q\phi(x)/m]^{-1/2} = |v|^{-1} + |v|^{-3}[q\phi(x)/m]$. We find

$$B = \gamma \int_{x_1}^{x_2} dx \left(\frac{q}{m}\phi(x) + \phi(x)^2(4\pi nm\lambda_\alpha^2)^{-1}\right). \quad (233)$$

Using (225) for $l = 2$

$$C = \frac{2}{3}\dot{u} f_0'(0) \left[\int_{x_1}^{x_2} dx \left(\frac{2q\phi(x)}{m}\right)^{3/2} - \left(\frac{2q\phi_0(x)}{m}\right)^{3/2} \Delta x\right], \quad (234)$$

and from (227)

$$D = 2\dot{u} f_0'(0)(2q\phi_0/m)^{3/2}\Delta x. \quad (235)$$

The $\phi(x)^{3/2}$ terms in A and C cancel. The $\phi_0^{3/2}$ terms in (234) and (235) may be added. We obtain finally

$$\int dv\, v^3 f \Big|_{x_1}^{x_2} = \gamma \int_{x_1}^{x_2} dx \left(\frac{q\phi(x)}{m} + \phi(x)^2(4\pi nm\lambda_\alpha^2)^{-1}\right) + \frac{4}{3}\dot{u} f_0'(0) \left(\frac{2q\phi_0(x)}{m}\right)^{3/2} \Delta x. \quad (236)$$

The total rates of change of mass, momentum, and energy are obtained by adding the inside terms (205)–(207) and outside terms (228), (231), and (236). For $l = 0$, (207) and (228) exactly cancel since total mass is conserved. For $l = 1$, terms proportional to $\dot{u}$ and γ in (206) and (231) cancel when (218) is used so that

$$\frac{\partial}{\partial t} nm \int_{-\infty}^{\infty} dx \langle v \rangle = -nmf_0'(0)\left(\frac{2q\phi_0}{m}\right)^2. \quad (237)$$

For $l = 2$, the first and third terms in (207) and (236) cancel. The second terms, proportional to λ_α^2, are equal and can be added to give

$$\frac{\partial}{\partial t}\frac{nm}{2} \int_{-\infty}^{\infty} dx \langle v^2 \rangle = \frac{1}{8\pi\lambda_\alpha^2}\frac{d}{dt} \int_{x_1}^{x_2} dx\, \phi(x)^2. \quad (238)$$

This is just the usual expression for the nonresonant kinetic energy. We note that for a localized fluctuation, half the kinetic energy comes from the hole region ($x_1 < x < x_2$) and half from outside.

C. Sum of electron and ion contributions

To zero order in ϵ, the total charge density in the inside region is obtained from (157) and (198). The first terms of (157) and (198), the $nq\int f_0(v)dv$ terms, cancel. The second term of (157) is neglected as explained. The third term of (157) is given by (161) and can be added to the second term of (198) using (165). The last term of (198) is of order ϵ. We obtain

$$\sum_\alpha n_\alpha q_\alpha \langle 1 \rangle_\alpha = -\frac{\phi(x)}{4\pi\lambda^2} + nq \int \tilde{f}\, dv, \quad (239)$$

which is the same form as that used in Ref. 1. Poisson's equation becomes

$$-\frac{\partial^2}{\partial x^2}\phi(x) = 4\pi \sum_\alpha n_\alpha q_\alpha \langle 1 \rangle_\alpha = -\lambda^{-2}\phi(x) + 4\pi nq \int dv\, \tilde{f}. \quad (240)$$

The last terms of (239) and (240) refer to the trapped species. Since we have shown that the charge in the inside and outside regions cancel, there must also be charge in the outside region. However, it is spread over a large region so that its density is very small. In fact, for the outside region $\langle 1 \rangle \to 0$ as $\epsilon \to 0$ even though $\int dx \langle 1 \rangle$ is finite.

Momentum conservation follows from the first moments of the Vlasov equation (56) for electrons and ions and from Poisson's equation (70):

$$\sum_\alpha \frac{d}{dt} \int_{-\infty}^{\infty} dx\, n_\alpha m_\alpha \langle v \rangle_\alpha = 0. \quad (241)$$

When (178) and (237) are substituted into (241), Eq. (53) results.

Similarly, (56) can be used to show energy conservation

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx\left[\frac{1}{8\pi}\left(\frac{\partial\phi(x)}{\partial x}\right)^2+\sum_\alpha\frac{n_\alpha m_\alpha}{2}\langle v^2\rangle_\alpha\right]=0. \tag{242}$$

Using (240), one can show that

$$\frac{1}{8\pi}\int_{-\infty}^{\infty} dx\left[\left(\frac{\partial\phi(x)}{\partial x}\right)^2+\frac{\phi(x)^2}{\lambda^2}\right]=\frac{nq}{2}\int dx\int dv\,\tilde{f}\phi(x). \tag{243}$$

Substituting (179), (238), (165), and (243) into (242) and using the definition of hole energy (50), we obtain (54).

IX. RATE OF MOMENTUM CHANGE DUE TO $\phi^{(1)}$

In this section we compute the rate of change of electron and ion momentum due to the first-order potential $\phi^{(1)}(x)$ as discussed in Sec. III. In this calculation the potential is considered to be stationary, i.e., we do not include the γt or $\dot{u}t^2/2$ corrections contained in (48). To do so would produce a correction of order ϵ^2. To obtain the rate of momentum change we use (146).

The inside terms have the form

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} dx\int dv\,vf(x,v), \tag{244}$$

where $f(x,v)$ is to be computed to first order in $\phi^{(1)}\sim\epsilon$. Since $\partial/\partial t$ will provide an additional factor of γ, the inside term due to $\phi^{(1)}$ is of order $\gamma\epsilon\sim\epsilon^2$ for both the trapped and the untrapped species.

Next we consider the outside region. Since the potential is stationary only two features of $\phi^{(1)}(x)$ affect the outside region. The first, which affects passing particles, is the potential difference across the hole $\phi^{(1)}(x_2)-\phi^{(1)}(x_1)\equiv\delta\phi$. [Remember that $\phi^{(0)}(x_2)=\phi^{(0)}(x_1)=0$.] Without loss of generality we can assume $\delta\phi>0$ and $\phi(x_1)=0$. The second important feature that affects the reflected particles is a change in the maximum value of $|\phi(x)|$, i.e., due to $\phi^{(1)}$, ϕ_0 is replaced by ϕ_0', where $|\phi_0-\phi_0'|\sim\delta\phi$. The rate of momentum change for the outside region is given by (147) with $l=1$, We consider the trapped species first. For this case the x_1 portion of (147) is

$$\int_{-\infty}^{\infty} dv\,v^2f(x_1,v)=\int_0^{\infty} dv\,v^2f_0(v)+\int_{-\delta s}^{0} dv\,v^2f_0(-v)+\int_{-\infty}^{-\delta s} dv\,v^2f_0\left[-\left(v^2-\frac{2q\phi(x)}{m}\right)^{1/2}\right], \tag{245}$$

where $\delta s=(2q\delta\phi/m)^{1/2}$.

The x_2 portion of (147) is

$$\int_{-\infty}^{\infty} dv\,v^2f(x_2,v)=\int_0^{\infty} dv\,vf_0\left[\left(v^2+\frac{2q\delta\phi(x)}{m}\right)^{1/2}\right]+\int_{-\infty}^{0} dv\,v^2f_0(v). \tag{246}$$

Subtracting (245) from (246) and expanding the arguments of f_0, we obtain

$$\int_0^{\infty} dv\,v^2f_0'(v)\left[\left(v^2+\frac{2q\delta\phi}{m}\right)^{1/2}-v\right]+\int_{-\delta s}^{0} dv\,v^2[f_0(v)-f_0(-v)]-\int_{-\infty}^{-\delta s} dv\,v^2f_0'(v)\left[|v|-\left(v^2-\frac{2q\phi\delta}{m}\right)^{1/2}\right]. \tag{247}$$

The first integral in (247) can be split into two pieces—a v integration from 0 to δs and an integration from δs to ∞. We combine the δs to ∞ piece with the last term in (247) and denote this quantity a. The remaining piece, 0 to δs, we will call b. The square roots in the integrands can be expanded as follows:

$$\left(v^2+\frac{2q\delta\phi}{m}\right)^{1/2}-v=\frac{q\delta\phi}{mv}-\frac{1}{2v^3}\left(\frac{q\delta\phi}{m}\right)^2+\cdots,\quad v>0, \tag{248}$$

$$|v|-\left(v^2-\frac{2q\delta\phi}{m}\right)^{1/2}=\frac{q\delta\phi}{mv}-\frac{1}{2v^3}\left(\frac{q\delta\phi}{m}\right)^2+\cdots,\quad v<0. \tag{249}$$

The first terms in the preceding expansion contribute to quantity a;

$$-\frac{q\delta\phi}{m}-\frac{q\delta\phi}{m}\int_{-\delta s}^{\delta s} dv\,vf_0'(v). \tag{250}$$

The second term above is of order $\delta\phi^2$. When the second terms, the $\delta\phi^2$ terms, from the expansion (248) and (249) are used in quantity a, the result is

$$\frac{1}{2}\left(\frac{q\delta\phi}{m}\right)^2 P\int_{-\infty}^{\infty} dv\frac{f_0'(v)}{v}. \tag{251}$$

The quantity labeled a is

$$\int_0^{\delta s} dv\,v^2f_0'(v)\left[\left(v^2+\frac{2q\delta\phi}{m}\right)^{1/2}-v\right],$$

which can be readily evaluated if we put $f_0'(v)\approx f_0'(0)$. The result is of order $\delta\phi^2$. Finally, the second integral in (247) can be easily evaluated by expanding $f_0(v)$. We obtain

$$\tfrac{1}{2}(2q\delta\phi/m)^2f_0'(0). \tag{252}$$

Therefore, the total rate of outside momentum loss is of order $\delta\phi^2$ except for the term $-nq\delta\phi$ obtained by multiplying the first term of (250) by nm. However, $-nq\delta\phi$ for the trapped species will just cancel the analogous quantity for the untrapped species. This term could, of course, have been obtained from the linear response, i.e., $nm\int dv\,v^2f^{(1)}$ where $f^{(1)}=(q\delta\phi\gamma/m)[f_0'(v)/v]$. We now consider the ocntribution to (147) due to the untrapped species. The x_1 portion is

$$\int_{-\infty}^{\infty} f(x_1,v)v^2\,dv=\int_0^{\infty} dv\,v^2f_0(v)+\int_{-(2q\phi_0'/m)^{1/2}}^{0} dv\,v^2f_0(-v)+\int_{-\infty}^{-(2q\phi_0'/m)^{1/2}} dv\,v^2f_0\left[-\left(v^2-\frac{2q\delta\phi}{m}\right)^{1/2}\right]. \tag{253}$$

The x_2 portion is

$$\int_{-\infty}^{\infty} v^2\,dv\,f(x_2,v)$$

$$= \int_{-\infty}^{0} v^2\,dv\,f_0(v) + \int_0^{[2q(\phi_0' - \delta\phi)/m]^{1/2}} dv\,v^2 f_0(-v) + \int_{[2q(\phi_0' - \delta\phi)/m]^{1/2}}^{\infty} dv\,v^2 f_0\left[\left(v^2 + \frac{2q\delta\phi}{m}\right)^{1/2}\right]. \tag{254}$$

Subtracting (253) from (254), we obtain

$$\int_{-\infty}^{\infty} dv\,v^2[f(x_2,v) - f(x_1,v)]$$

$$= \int_0^{\infty} dv\,v^2 \left\{f_0\left[\left(v^2 + \frac{2q\delta\phi}{m}\right)^{1/2}\right] - f_0(v)\right\} - \int_0^{[2q(\phi_0' - \delta\phi)/m]^{1/2}} dv\,v^2 \left\{f_0\left[\left(v^2 + \frac{2q\delta\phi}{m}\right)^{1/2}\right] - f_0(-v)\right\}$$
$$+ \int_{-\infty}^{0} dv\,v^2 \left\{f_0(v) - f_0\left[-\left(v^2 - \frac{2q\delta\phi}{m}\right)^{1/2}\right]\right\} + \int_{-(2q\phi_0'/m)^{1/2}}^{0} dv\,v^2 \left\{f_0\left[-\left(v^2 - \frac{2q\delta\phi}{m}\right)^{1/2}\right] - f_0(-v)\right\}. \tag{255}$$

The integrands of the first and third terms in (255) can be expanded in powers of $\delta\phi$ and combined to give

$$-\frac{q\delta\phi}{m} - \frac{1}{2}\left(\frac{q\delta\phi}{m}\right)^2 P\int_{-\infty}^{\infty} dv\frac{f_0'(v)}{v}. \tag{256}$$

When multiplied by nm, the first term of (256) will cancel the corresponding term in (250) for the trapped species since $n_e q_e + n_i q_i = 0$. The sum of the second and fourth terms in (255) can also be expanded in powers of $\delta\phi$. The leading terms are

$$f_0'(0)(2q\phi_0'/m)^2 + f_0'(2q\phi_0/m)^2(\delta\phi/\phi_0). \tag{257}$$

Since $|\phi_0' - \phi_0| \sim \delta\phi$, this result differs from the zero-order result (237) only by a term of order $\delta\phi f_0' \sim \epsilon^2$.

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