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PKU Lectures : Lecture VI

"Violent Relaxation" in Galaxies, Plasmas and Fluids

Generic problem:

→ in lieu of crank, extract some general principle(s) to describe how systems evolve when mixed, stirred, producing $\left\{ \begin{array}{l} \text{turbulence,} \\ \text{relaxation} \end{array} \right.$
⇒ constrained min/max.

→ some examples:

– Selective decay / Minimum Enstrophy / Taylor Relaxation

i.e. system exhibiting dual cascade

i.e.

large scales avoiding ~~dissipation~~ global magnetic helicity robust

$$E \longleftarrow \; \Omega \longrightarrow \nu$$

potential enstrophy minimized → PV homogenization

$$\langle A\cdot B\rangle \longleftarrow \; E_M \longrightarrow \eta$$

magnetic energy

$$\delta(\Omega + \lambda E) = 0$$

$$\delta\left(E_M + \lambda \int A\cdot B\right) = 0$$

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— "Violent Relaxation" → D. Lynden-Bell '67 et seq.
(see posting)

→ collisionless

→ most probable state, subject conservation constraints? which?

key questions: — entropy, for collisionless system?
— relaxation dynamics?

→ relevant { Galaxies — first
Plasmas
GFD }

→ conceptually interesting and instructive.

→ Leads to paradigm for coherent structure (self-bound)
~ Dupree '82.

—

→ Violent Relaxation in { Galaxies
Vlasov system }

the problem:

Galactic dynamics, evolution is ~ classical, governed by Vlasov Equation

c.f. Binney & Tremaine "Galactic Dynamics" so

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f + \underline{\nabla}\Psi\cdot\frac{\partial f}{\partial \underline{v}} = \cancel{C(f)}$$

with

$$\nabla^2\Psi = 4\pi G\int d^3v\, f$$

(a)
the point: $C(f) \to 0$

i.e. $\tau_{coll} \sim \tau_{universe}$.

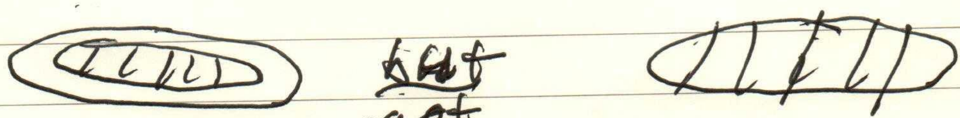
⇒ any relaxation is collisionless.

(b) light distribution in elliptical galaxies quite regular ⇒ reached equilibrium, but how?

i.e. → τ_{coll} too long

→ equipartition of energy would kick smaller stars outside

i.e.



~~hot~~ get

Mass segregation → lighter / smaller } stars collect

Kinetic energy → rim

– Physics of this 'violent relaxation'
i.e. how so fast?

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i.e. – How reach an "equilibrium"
without collisions?
⟺ n.b. is it an equilibrium?
How characterize?

recall: usually:

$$\frac{\partial}{\partial t}\delta f + \underline{v}\cdot\nabla\delta f + \frac{q}{m}E\frac{\partial f}{\partial v}$$

$$= -\frac{q}{m}E\frac{\partial\langle f\rangle}{\partial v}$$

but $\langle f\rangle \rightarrow \begin{cases}\text{Maxwellian} \\ \text{slowing down}\end{cases}$, "$C(f)=0$"

N.B. problem can be viewed as related to BGK mode, i.e.

$$\cancel{\frac{\partial f}{\partial t}} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$$

any

$$f = f\left(\frac{mv^2}{2} + q\phi\right)$$

is equilibrium solution

if:

$$\nabla^2\phi = -4\pi n_0 q\int f\,dV$$

how specify BGK equilibria? – meaning – stability

– streams, etc.
– central to Galactic dynamics

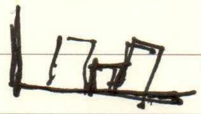
→ Interested, in a sense, in "most probable BGK" state. ⇒ likelihood – entropy

Then begs the two questions:

① — relaxation → entropy maximization

entropy → how define?

n.b. phase space density conservation — { exclusion? / Fermi-Dirac?

 →

→ constraint?

② — relaxation? → how??

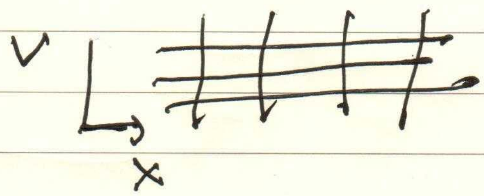
Routes to collisionless relaxation:

ⓐ → chaos + coarsegraining

i.e. Q.L. $\langle \, \rangle \equiv$ coarse grained

$$\frac{\partial}{\partial t}\langle f\rangle = \frac{\partial}{\partial v} D \frac{\partial \langle f\rangle}{\partial t} + \cancel{C(f)}$$

coarse graining → partition of phase space



then, for relaxed state:

$$\frac{\partial}{\partial t}\int dv\,\langle f\rangle^2 \to 0 \quad \Rightarrow$$

$$-\int D_R \left(\frac{\partial\langle f\rangle}{\partial v}\right)^2 \Rightarrow 0$$

i.e. resonant diffusion, only!

$\Rightarrow$ irreversibility $\Leftrightarrow$ chaos

but chaos, $\exists$ at least one $h > 0$

positive Lyapunov exponent.

Kol.-Sinai entropy

$$\mathcal{S}_{\substack{\text{Kolmogorov}\\ \text{K-S}}} = \sum_{\substack{i\\ h_i>0}} h_i$$

t + dt

t

number "active" cells increases.

$\delta(\text{info}) > 0$

$\Rightarrow$ entropy production!

Chaos $\Leftrightarrow$ Relaxation connection based on phase space partition!

and also observe:

$f \to$ exact distribution function

$\langle f\rangle \to$ coarse grained distribution (i.e. $\to$ evaluated on partition)

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then:

$$\frac{df}{dt} = 0 \quad , \quad \frac{df^2}{dt} = 0 \quad , \quad \frac{d}{dt}\int dv\, f^2 = 0$$

but:

$$\partial_t \int dv \frac{\langle f \rangle^2}{2} < 0$$

coarse graining $\Rightarrow$ {decay of coarse grained distribution. → relaxation

on

(b) "phase mixing" + coarse graining

→ What is "phase mixing"?

Phase mixing $\Rightarrow$ as for homogenization,
= phase element stretching, winding
+
coarse graining ↔ [partition finite
(scattering in action required)

i.e. consider:
— Ensemble pendulums

→ initial distribution:
Θ_0
$\Delta\Theta < \Theta_0$ distribution
random p_0

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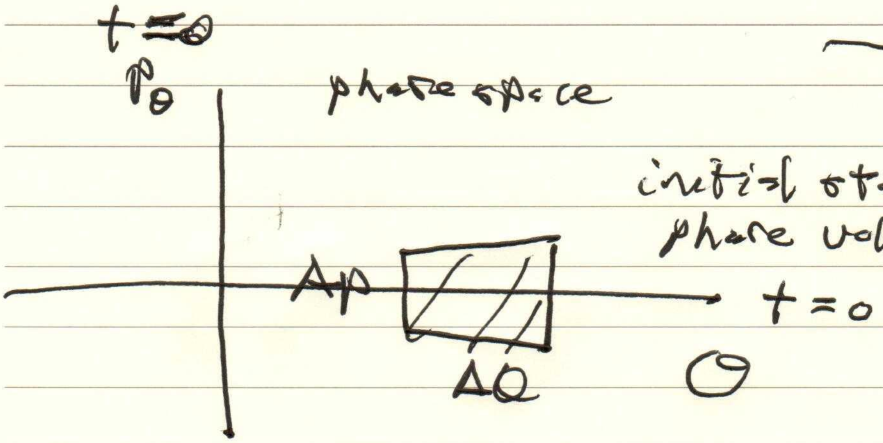
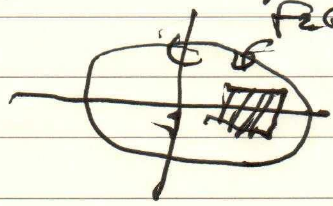
equivalently

so



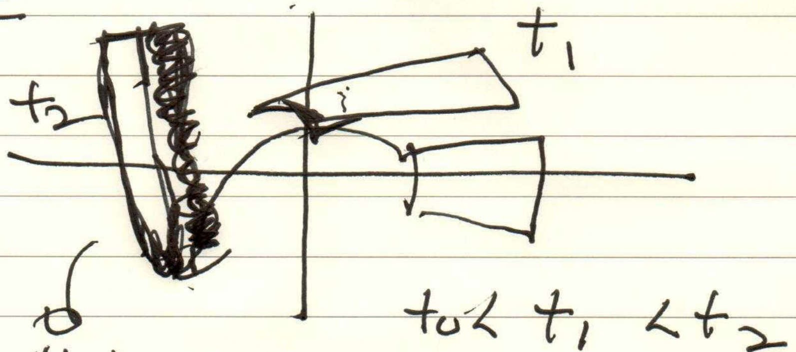


cell on eddy in differential rotation

- more energetic are slower. (larger radius)

- $\Delta p \, \Delta\theta$ = const.

so



→ thinner, longer

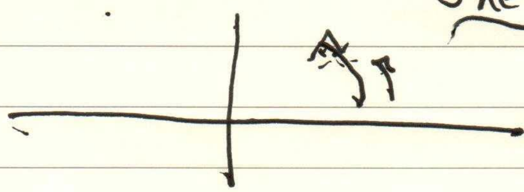
i.e.

differential phase space rotation

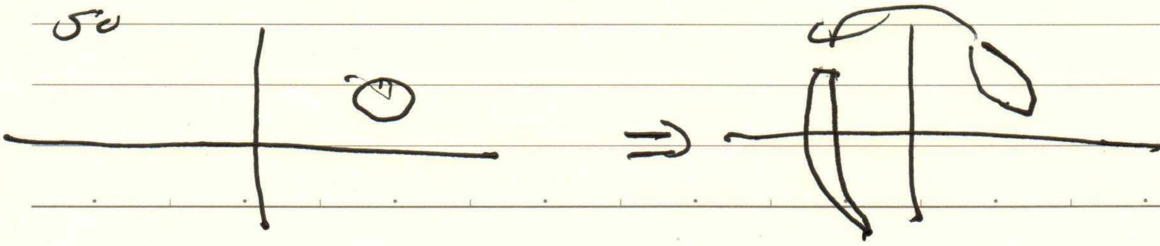
→ shearing

i.e. akin!

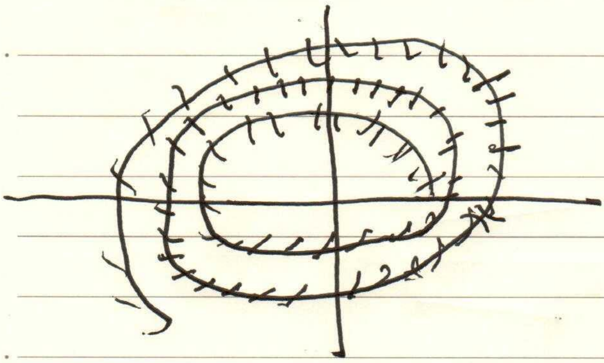
sheared eddy



so



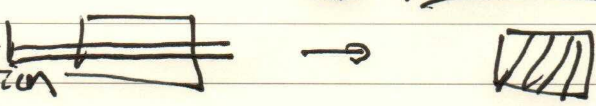
then, after several windings:



"wound noodle of finite thickness"

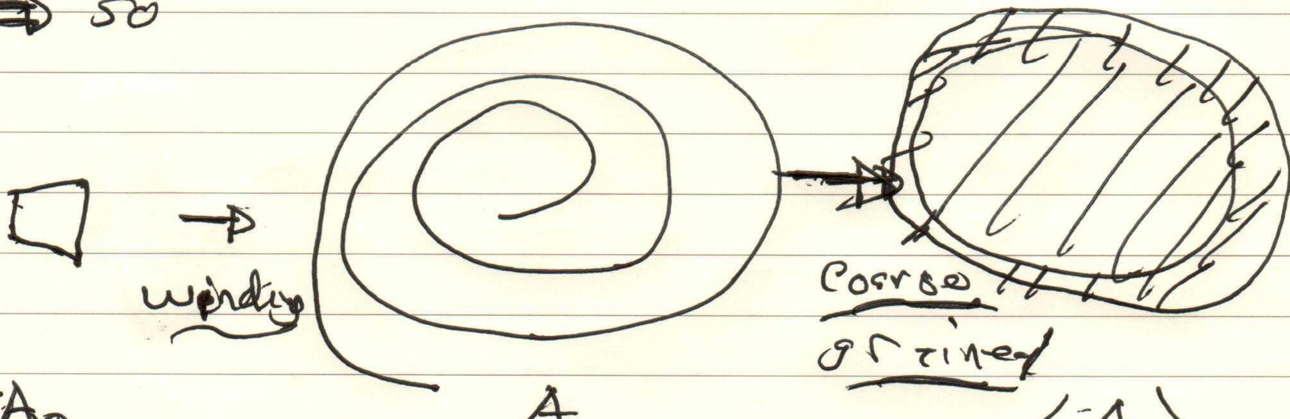
- filament wound many times
  = "thickly wound noodle," "thickness" begins to "hit" partition scale.

cell as minimal resolution



If hits any cell, that cell is "occupied"

$\Rightarrow$ so



A_0 A $\langle A \rangle$

obviously   - exact area conserved but

$A_0 \ll \langle A \rangle$ - i.e. coarse grained long time area $\uparrow$

but

$f A_0 = \langle f \rangle \langle A \rangle$, necessarily

$\Leftrightarrow$ phase space density conservation

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so necessarily:

$\langle f \rangle < f \Rightarrow$ phase mixing (differential rotation) coarse graining

$\Downarrow$

coarse grained distribution decays $\Rightarrow$ decay, relaxation of $\langle f \rangle$!

$\Downarrow$

$\Rightarrow$ collisionless relaxation..

N.B. chaos, phase mixing } fundamentally same game of:

stretching + partition finite:

- chaos $\Rightarrow$ orbit divergence (exponential)
- phasemixing $\Rightarrow$ differential rotation (algebraic)

N.B. To show phase mixing analytically (explicit coarse grained):

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{q}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = C(f) \sim \nu \frac{\partial^2 f}{\partial v^2}$$

$\nu \to 0$

i.e. as with NS vs Euler, coarse graining $\Leftrightarrow$ tiny but finite diffusion.

of course, can write as:

$$\partial_t f + [f, H] = \nu \frac{\partial^2 f}{\partial v^2}$$

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and, for:
action

$$J = \oint p(\epsilon, x) \quad , \quad S = \int p\, dx$$

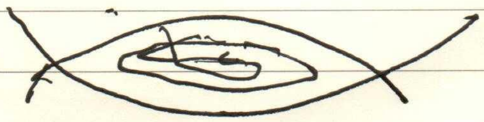
where

$$\epsilon = H = \frac{p^2}{2m} + Z\phi$$

$$p = \left[2m(H - Z\phi)\right]^{1/2}$$

$\Rightarrow$ action - angle variables

$$J, \alpha$$



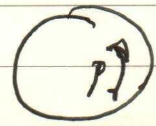
$$\frac{d\alpha}{dt} = \omega(J)$$

$\Rightarrow$ can write, for single wave:

$$\frac{\partial f}{\partial t} + \Omega(J)\frac{\partial f}{\partial \alpha} = \nu \frac{\partial^2 f}{\partial J^2}$$

n.b. writing as divergence of flux is essential

Now, $\nu \to 0$;



all particles rotate on track

$$f(\alpha, J, t) = f(\alpha - \Omega(J)t, J, 0)$$

$$= F_0(J) + \sum_n \underbrace{F_n(J)}_{\text{fine structure} \to \text{streamers.}} e^{in(\alpha - \Omega t)}$$

where: $F_0 = \langle f(t=0) \rangle_\alpha$

now, small $\underline{\nu}$ but finite:

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so

$$\left(\frac{\partial}{\partial t} + \Omega(J)\frac{\partial}{\partial \alpha} - \nu\frac{\partial^2}{\partial J^2}\right)\left[F_0(J) + \sum_n F_n(J)\, e^{in(\alpha-\Omega t)}\right] = 0$$

$$\partial_t F_0 = \nu\frac{\partial^2 F_0}{\partial J^2}$$ $\Rightarrow$ mean evolves slowly, diffusively

$$\sum_n\Big[\cancel{-in\Omega F_n} + \frac{\partial}{\partial t}F_n + \cancel{in\Omega F_n}$$

$$+\nu \text{ [redacted] } n^2\left(\frac{\partial\Omega}{\partial J}\right)^2 t^2 F_n + \dots\Big] e^{in(\alpha-\Omega t)} = 0$$

n large

$$\partial_t F_n = -\nu n^2\left(\frac{\partial\Omega}{\partial J}\right)^2 t^2 F_n$$

$$F_n = e^{-\nu\frac{n^2}{3}\left(\frac{\partial\Omega}{\partial J}\right)^2 t^3} F_n(0)$$

decay rate, due "phase mixing" (= differential rotation + coarse graining)

$$1/\tau_{decay} = \left(\nu\frac{n^2}{3}\left(\frac{\partial\Omega}{\partial J}\right)^2\right)^{1/3} \sim \nu^{1/3}\,\Omega'^{2/3}$$

tiny $\nu \Rightarrow$

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so $t \to \infty$

$F \Rightarrow F_0$, which relaxes purely diffusively.

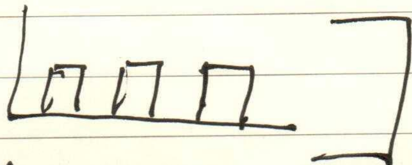
ⓐ <u>Relaxation</u> → Statistical Mechanics

Consider:

→ system stirred in phase space, starting from i.c. some finite set η's.

→ $\frac{df}{dt} = 0$,

[later: so, if for simplicity, all f's $= \eta \to \eta_0 \to$ all levels phase space density same



and since Hamiltonian,

$\underline{\nabla} \cdot \underline{V}_{\Pi} = 0 \Rightarrow$ phase volume of each element conserved.

i.e $v(\eta)$ conserved

volume of element with $f = \eta$.

then can introduce probability:

$$\boxed{\rho(\underline{x}, \underline{v}, \eta) \equiv \text{probability of finding } f=\eta \text{ at } \underline{x}, \underline{v} \text{ (small neighborhood)}}$$

so $-\int d\eta\, \rho(\underline{x},\underline{v},\eta) = 1$

and ⟶ local coarse grained distribution

$$\bar{f}(\underline{x},\underline{v}) = \int d\eta\, \eta\, \rho(\underline{x},\underline{v},\eta)$$

$$\nabla^2 \bar{\phi} = 4\pi G \int d^3v\, \bar{f}$$

$$= 4\pi G \int d^3v \int d\eta\, \eta\, \rho(\underline{x},\underline{v},\eta)$$

Now, argue that relevant entropy is:

$$S = -\int d\eta\, d^3x\, d^3v\, \rho \ln \rho$$

subject to constraints of constant:

① total phase volume occupied by level η
(i.e. element)

$$\gamma(\eta) = \int \rho(\underline{x},\underline{v},\eta)\, d^3x\, d^3v$$

→ i.e. effectively total mass of phase fluid with $f = \eta$

→ $\eta = 0, \eta_0$ only, equivalent total mass $(\eta = \eta_0)$

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② energy:

$$E = \frac{1}{2}\int f\, v^2\, d^3\cancel{x}\, d^3v + \frac{1}{2}\int d^3x\, d^3v\, \overline{f}\, \overline{\Phi}$$

③ angular momentum

④ linear momentum

Most probable state from:

$$\delta\Big[S - \underbrace{\beta}_{1/T} E - \int d\eta\, \underbrace{\alpha(\eta)}_{\text{"chemical potential" of } \eta}\, \gamma(\eta)\Big] = 0$$

$$\gamma(\eta) = \int \rho(x, v, \eta)\, d^3x\, d^3v$$

crank:

$$\boxed{\rho(\underline{x}, \underline{v}, \eta) = C \exp\left[-\alpha(\eta) - \beta\eta\left(\frac{v^2}{2} + \overline{\Phi}\right)\right]}$$

where: normalization

$$C = \int d\eta\, \exp\left[-\alpha(\eta) - \beta\eta\left(\frac{v^2}{2} + \overline{\Phi}\right)\right]$$

– Gibbs distribution

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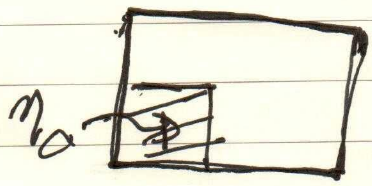
Further simplify:

- take $\eta = 0$ or $\eta = \eta_0$

i.e. single level phase space density.



i.e. $\eta = \eta_0$ in some part phase space, initially

- so $\int \eta = \sum_{\eta=0}^{\eta=\eta_0}$

and

$$\underline{\bar{f} = \rho\, \eta_0}$$

linear proportionality $\bar{f}$, ρ

$\rho = \bar{f}/\eta_0$

$\Rightarrow$

$$\boxed{\bar{f} = \frac{\eta_0 \exp[-\beta \eta_0 (\epsilon - \mu)]}{1 + \exp[-\beta \eta_0 (\epsilon - \mu)]}}$$

$\epsilon = \frac{v^2}{2} + \bar{\Phi}$, $\mu = -\alpha/\beta\eta_0$ chem pot.

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N.B. To determine $\bar{\phi}$;

$$\nabla^2\bar{\phi} = 4\pi G\int d^3v\ \bar{F}$$

$$= 4\pi G\int d^3v\ \frac{m_0 \exp\left[-\beta m_0\left(\frac{v^2}{2}+\bar{\phi}-\mu\right)\right]}{1+\exp\left[-\beta m_0\left(\frac{v^2}{2}+\bar{\phi}-\mu\right)\right]}$$

etc.

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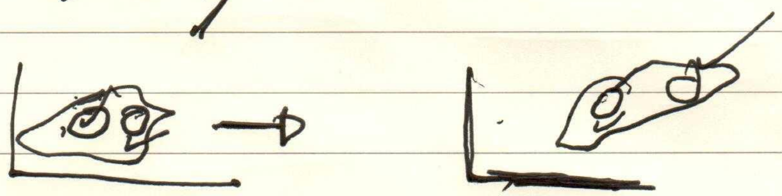
⇒ Recovers expected Fermi – Dirac statistics

→ same as F–D distribution for self-gravitating gas. (a good guess)

→ "exclusion" ↔ phase space density conservation



i.e. can't pile blobs phase fluid on each other

→ $f = \text{const}$, along trajectories

though can deform.

↳ Dupree "slicing the spaghetti" critique.

and also:

→ in non-degenerate limit:

$\bar{f} \ll \eta_0$ i.e. $\eta \to 0$

(small occupied phase volume)

$$\boxed{\bar{f} = C \cdot e^{-\beta \eta_0 G}}$$

⇒ recovers Maxwell – Boltzmann

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$\Rightarrow$ recover L-B state of Maxwell - Boltzman dist without collisions.

Further, 2D Fluid:
(β-plane $\underbrace{q}_{\text{ok}} = \beta y + \nabla^2\phi$)

Euler eqn:

$$\frac{d\omega}{dt} = 0 = \frac{\partial \omega}{\partial t} + \underline{v} \cdot \nabla \omega = 0$$

$(\nu \to 0)$

and

$$\nabla^2 \psi = -\omega$$

ψ: stream function

obviously $f \leftrightarrow \omega$

obvious analogy to Vlasov, but:

- 2D not 6D, in principle
- $f \geq 0$ while $\omega \gtrless 0 \rightarrow M$?
  $M > 0$
- in Vlasov: $E = \underbrace{\frac{1}{2} v^2}_{\text{kinetic}} + \underbrace{\int \frac{\bar{\phi}}{2} f \, d^3x \, d^3v}_{\text{potential}}$

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but fluid:

$$E = \frac{1}{2}\int \omega \psi \, d^2\underline{x} \quad (\omega \phi)$$

potential only

$\Rightarrow$ negative temperature states possible! $\left(\frac{\partial S}{\partial E} < 0\right)$

- $\underline{X}_{cm} = \frac{1}{\Gamma}\int \omega \, \underline{r} \, d^2x$ conserved.

in Vlasov, structure can have linear momentum.

- of course, phase space density conservation / Kelvin's Thm!

$$\int \omega \, d^2r = \Gamma$$ conserved (~ Mass)

indeed $\int F(\omega) \, d^2r$ conserved, any non-pathological F.

Thus can play similar game as for Vlasov,

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$\rho(\underline{x}, \sigma) \equiv$ probability of finding $\omega = \sigma$ at $\underline{x}$-neighborhood.

of course:

$$\int d\sigma\, \rho(\underline{x}, \sigma) = 1$$

and coarse grained vorticity:

$$\overline{\omega} = \int d\sigma\, \sigma\, \rho(\underline{x}, \sigma)$$

so

$$\nabla^2 \overline{\psi} = -\overline{\omega}$$ gives coarse-grained stream function.

then usual conserved quantities:

$$E = \frac{1}{2}\int d^2x\, \overline{\omega}\,\overline{\psi}$$ → energy (neglects $\overline{\tilde{\omega}\tilde{\psi}}$!) (*)

$$L = \int \overline{\omega}\, r^2 d^2r$$ → angular momentum

$$\underline{P} = \int \hat{r} \times \overline{\omega}\, \hat{z}\, d^2x$$ → linear momentum

and

$$\tilde{\omega} \sim \overline{\omega}$$

but $|\tilde{\psi}| << |\overline{\psi}|$ (scale)

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Conserve probability:

$$\gamma(\sigma) = \int \rho(\underline{x}, \sigma)\, d^2x$$

Now ignoring angular and linear momentum for simplicity: relaxed state from, with Lagrange multipliers.

$$\delta\left[S - \beta E - \int \alpha(\sigma)\,\gamma(\sigma)\right] = 0$$

where $S = -\int d^2x \int d\sigma\, \rho \ln \rho$

and, as before:

$$\rho(\underline{x}, \sigma) = C \exp\left[-\alpha(\sigma) - \beta\sigma\bar{\Psi}\right]$$

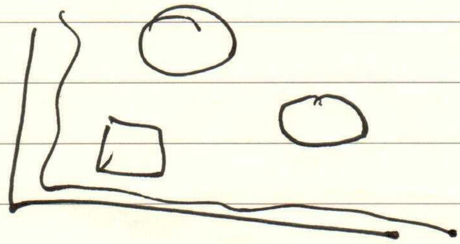
$$C = \int d\sigma \exp\left[-\alpha(\sigma) - \beta\sigma\bar{\Psi}\right]$$

$\beta = 1/T$, $\alpha \equiv$ chem potential,

and can go further:

for $\sigma = \sigma_0$, single vorticity level.

i.e.



vortices with $\sigma = \sigma_0$ in irrotational flow

$$\sigma_0 \langle \rho \rangle = \overline{\omega}$$

, defines coarse grained dist.

$\Rightarrow$

$$\boxed{\overline{\omega} = \frac{\sigma_0 \exp\left[-\beta\sigma_0(\overline{\psi} - \mu)\right]}{1 + e^{-\beta\sigma_0(\overline{\psi} - \mu)}}}$$

i.e. Fermi-Dirac, as before.

$$\mu = -\alpha / \beta\sigma_0$$

$\rightarrow$ chemical potential.

and for dilute limit:

$$\overline{\omega} = A e^{-\beta\sigma_0\overline{\psi}}$$

$\rightarrow$ Maxwell Boltzmann.