Nonlinear Dyn. for Magn. Conf.

1

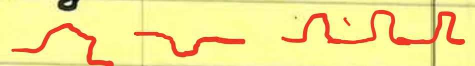
Lecture IV — $\boxed{\text{Nonlinear Waves I}}$

Pred-Prey
LCOs Events
Transitions
→ Fronts
→ Fisher
→ FN

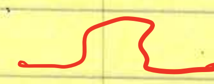
Next goal { avalanches
turbulence spreading

→ What's an avalanche?

pulse train



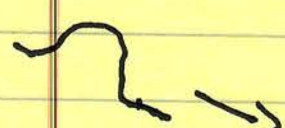
~ a pulse of a thermodynamic variable
T, n, etc., or/and its flux which
traverses a distance L at a
speed c s/t



$$\frac{c}{L} > \frac{\langle D\rangle}{L^2} \qquad \langle D\rangle \sim D_{GB}$$

avg.

Can be a pulse train



→ Pulse can be fluctuation intensity
→ spreading

"Spreading" in MFE
=
"Entrainment" in fluids

Structure of Turbulent Shear Flow
A.A. Townsend

→ Why Nonlinear waves ⟺ avalanches?

"The basic idea of wave propagation is that some recognizable feature of the disturbance moves with a finite velocity."

Landau & Lifshitz
Whitham

— G. B. Whitham
"Linear and Nonlinear Waves"

highly recommended!

cf: Front!

⇒

Nonlinear waves are underpinning for concepts re: avalanches and spreading. → Fundamental.

Burgers – avalanches

So:

①

– basics: gas dynamics ("simple waves") and plasma solitons

→ collisionless shocks

⇒ shock, pulse formation

⇒ jump conditions

⇒ steepening vs. dispersion ⇒
Ion Acoustic shock

Crucial Point:

⇒ Entropy production

then:

(II) – kinematic waves, traffic flow,
Flood waves, bores.

(III) – Avalanche physics: Micro – Macro.

(IV) – Turbulence spreading and
entrainment physics.

Gas Dynamic Shocks / "Simple Waves"

1-D Gas Dynamics — 'Simple Waves'

cf $\left\{\begin{array}{l}\text{Landau/Lifshitz Fluids}\\ \text{Whitham}\end{array}\right.$ → steepening, shocks etc.

cont.

$$\partial_t \rho + \partial_x (\rho v) = 0$$ (continuity)

ideal

1D

$$\partial_t v + v\,\partial_x v + \frac{1}{\rho}\,\partial_x P = 0$$

mom.

(momentum balance)

$$P = P(\rho)$$

Entropy: – S initially homogeneous

→ S = const, till shock forms

pulse

1D

$P = P(\rho)$

speed vs ρ rel'n of flow

so pulse follow rho.

$\frac{d\rho}{dt} = 0$ along $\left(\frac{\partial x}{\partial t}\right)_\rho = \frac{d(\rho v)}{d\rho}$

$$\partial_t \rho + \frac{d(\rho v)}{d\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\partial_t v + \left(v + \frac{1}{\rho}\frac{dP}{dv}\right)\frac{\partial v}{\partial x} = 0$$

$P = P(\rho)$
$v = v(\rho)$

$\left(\frac{\partial}{\partial t}x\right)_\rho = v + \rho\, dv/d\rho$

so

$$\left(\frac{\partial x}{\partial t}\right)_\rho = \frac{d}{d\rho}(\rho v) = \boxed{v + \rho\frac{dv}{d\rho}}$$

n.b. get on $\frac{d\rho}{dt} = 0$, $\frac{dv}{dt} = 0$ conservation law $\left\{\begin{array}{l}\text{characteristic}\\ \text{eqn.} \to \text{info}\end{array}\right.$

likewise:

$$\boxed{\left.\frac{\partial x}{\partial t}\right)_v = v + \frac{1}{\rho}\frac{dP}{dv}}$$

but

$v = v(\rho)$ viz

$P = P(\rho)$

$\Rightarrow$

$$\boxed{\left(\frac{\partial x}{\partial t}\right)_\rho = \left(\frac{\partial x}{\partial t}\right)_v}$$

pulse

$$\rho \frac{dv}{d\rho} + v = v + \frac{1}{\rho}\frac{dP}{dv}$$

$$dP = c_s^2\, d\rho$$

$$\rho \frac{dv}{d\rho} = \frac{c_s^2}{\rho}\frac{d\rho}{dv}$$

so

$$(dv/d\rho)^2 = c_s^2/\rho^2$$

$$\boxed{V = \pm\int \frac{c_s}{\rho}\, d\rho = \pm\int \frac{dP}{\rho\, c_s}}$$

← velocity

solves for flow speed in terms density

Note:

- $\boxed{V = \pm \int \frac{c_s}{\rho}\, d\rho}$

- $\boxed{P\rho^{-\gamma} = \text{const}}$ $\quad \gamma = 5/3$

$$dP = c_s^2\, d\rho = \underset{\text{const.}}{\alpha}\, \gamma\, \rho^{\gamma-1}\, d\rho$$

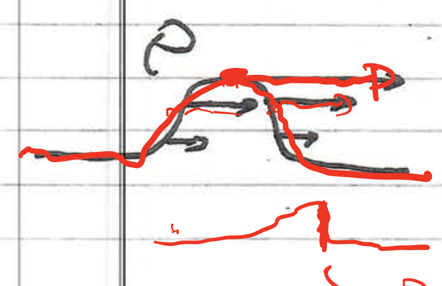
$$\boxed{c_s^2 = \alpha\gamma\rho^{\gamma-1}}$$

$\alpha = \text{const}$

$$\boxed{V = \pm \int \frac{c_s^2}{\rho c_s}\, d\rho} = \pm\, \alpha\gamma \int \frac{\rho^{1/3}}{\rho}\, d\rho$$

$$\boxed{V = \pm 3\sqrt{\alpha\gamma}\, \rho^{1/3}}$$

- flow speed increases with density



→ over-taking

- high density elements go faster

- then:

$$\left.\frac{\partial x}{\partial t}\right)_v = v + \frac{1}{\rho}\frac{dP}{dv}$$

$$v = \pm \int \frac{dP}{\rho c_s} \qquad dv = \frac{dP}{\rho c_s}$$

so $\frac{dP}{dv} = \rho c_s$

$\rho = \rho(v)$

$$\boxed{\left.\frac{\partial x}{\partial t}\right)_v = v \pm c_s(v)}$$

all cases - 1D - cont., EoM

c.c

$$\boxed{x = t\left[v \pm c_s(v)\right] + f(v)}$$

"simple wave" solution { general nonlinear

Check: linearized limit

Nonlinear wave → no char. ideal scale

$$x = t\left[\cancel{v} \pm c_s(0)\right] + \cancel{f(v)} + x_0$$

$$x = x_0 \pm c_s(0)\,t$$

→ why "simple"? — No characteristic scale!

⇒ alternate approach.

so if no characteristic scale all quantities depend only on

$$\boxed{\xi} = \boxed{x/t}$$

Similarity soln.

$f(x,t)$

$F(x/t^\alpha)$

$\begin{matrix}\rho \\ v\end{matrix} = f(\xi)$ $\quad \xi = x/t$

$\hookrightarrow$ velocity formed by x, t in absence/scale

so $\quad \boxed{\partial_x = \frac{1}{t}\,\partial/\partial\xi}$

$\boxed{\partial_t = -\frac{\xi}{t}\,\frac{\partial}{\partial\xi}}$

$\xi = \dfrac{x}{t}$

$\Rightarrow \quad \partial_t\rho + \rho\,\partial_x v + v\,\partial_x\rho = 0$

$\partial_t v + v\,\partial_x v = -\frac{1}{\rho}\,\partial_x P$

becomes:

$-\frac{\xi}{t}\rho' + \frac{\rho}{t}v' + \frac{v}{t}\rho' = 0$

$' = \partial/\partial\xi$

$-\frac{\xi}{t}v' + \frac{v}{t}v' = -\frac{c_s^2}{t}\frac{\rho'}{\rho}$

so

$$\boxed{\begin{aligned}(v-\xi)\rho' + \rho v' &= 0\\ (v-\xi)v' &= -c_s^2\frac{\rho'}{\rho}\end{aligned}}$$

like linear dispersion eq'n, treat ξ as eigenvalue.

$(v-\xi)\rho' + \rho v' = 0$ $\quad \frac{d\rho}{dt}$

$-\frac{c_s^2}{\rho}\rho' + (v-\xi)v' = 0$ $\quad \frac{dv}{dt}$

$$(v-\xi)^2 = c_s^2$$

$$\xi = v \pm c_s$$

$$\frac{x}{t} = v \pm c_s$$

$$\boxed{x = (v \pm c_s)\, t}$$

From "eigenvector", relate v, ρ etc.

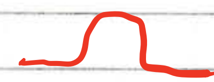
$$v - \xi = -c_s$$

$$(v-\xi)\rho' + \rho v' = 0$$

$$-c_s \rho' + \rho v'$$

$$\rho\, dv = c_s\, d\rho$$

$$dv/d\rho = c_s/\rho$$



$x = \text{i.c.} + t(v \pm c_s)$

Simple wave

$$v = \int c_s\, d\rho/\rho = \int dP/c_s\rho$$

- equivalent to previous, with $f(v) = 0$.

- corresponds to Kulsrud's version of "simple wave".

– can also write:

$$v = \int + (-dp\,dV)^{1/2}$$

$$d(1/\rho) = dV = -\frac{1}{\rho^2}\,d\rho \quad , \quad dp = c_s^2\,d\rho$$

$$\left(v = \int \left(\frac{1}{\rho^2} d\rho\, c_s^2\, d\rho \right)^{1/2} = \int \frac{d\rho}{\rho}\, c_s \right) \checkmark$$

⇒ Physics of Simple Waves:

– have established simple wave in gas dynamics, with key element of scale invariance

no scale in film.

– now elucidate simple wave physics
⇒ shocks

Similarity flow → simple wave with $f(v) = 0$

$$\boxed{x = t\,[v \pm c_s(v)]}$$

wave
front
path

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Can write general solution, for adiabatic process:

$$p\rho^{-\gamma} = \text{const}$$

$$T\,\rho^{-(\gamma-1)} = \text{const}$$

$$T = c_s^2 = \rho^{\gamma-1}$$

$$\frac{dp}{d\rho} = c_s^2 = T$$

$$\underline{p = \rho T}$$

$$\rho^{-1}\,T^{1/\gamma-1} = \text{const}$$

as $c_s^2 \sim T$

$\gamma - 1 = 2/3$

$$\boxed{\rho = \rho_0\,(c/c_0)^{2/(\gamma-1)}}$$

(elim s → clarity ↓ notation)

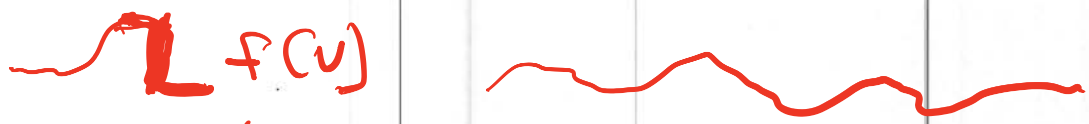
but $\quad \underbrace{V = \pm \int c \,\frac{d\rho}{\rho}}$

$c = c_s$

so

$$\boxed{V = \pm\frac{2}{\gamma-1}\,\underline{(c - c_0)}}$$

so finally can write:



$$\boxed{\begin{aligned} c &= c_0 \pm \tfrac{1}{2}(\gamma-1)v \\ \rho &= \rho_0 \left(1 \pm \tfrac{(\gamma-1)}{2} v/c_0\right)^{2/\gamma-1} \\ P &= P_0 \left(1 \pm \tfrac{1}{2}(\gamma-1) v/c_0\right)^{2\gamma/\gamma-1} \end{aligned}}$$

so then:

$$\boxed{x = t\left[v \pm c_s(v)\right] + f(v)}$$

$$\Rightarrow \quad x = t\left[v \pm \left(c_0 \pm \tfrac{1}{2}(\gamma-1)v\right)\right] + f(v)$$

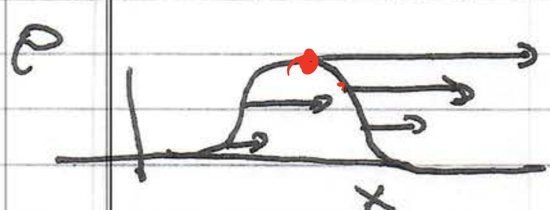
Now:

- point on wave profile moves at

$$u = \frac{\partial x}{\partial t} = v \pm c_s$$

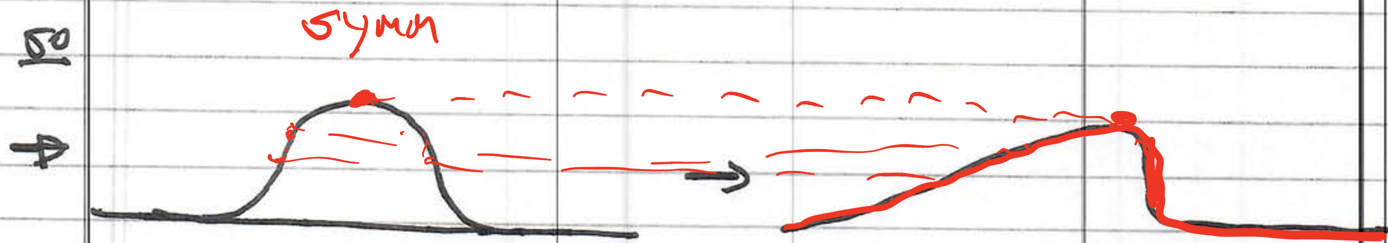
$dv/d\rho > 0$

and $dv/d\rho > 0$



→ [speed increases with density!]

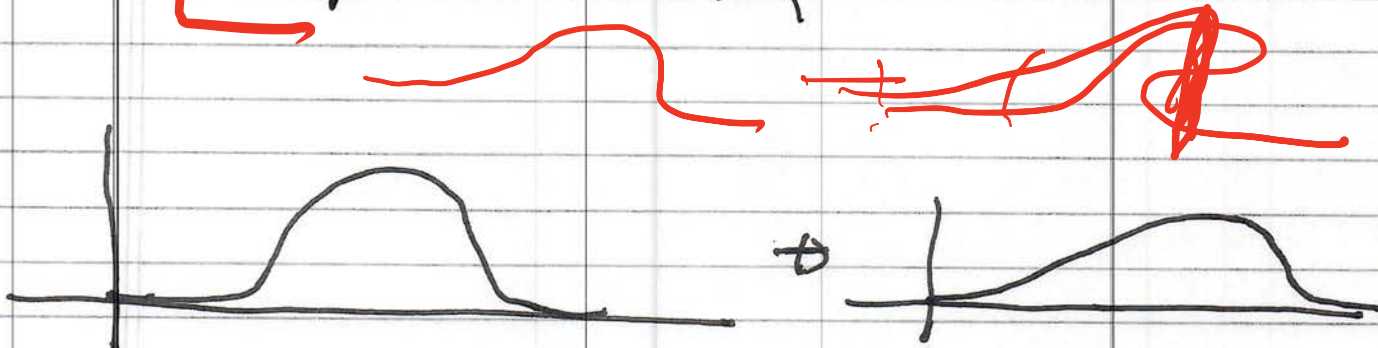
so

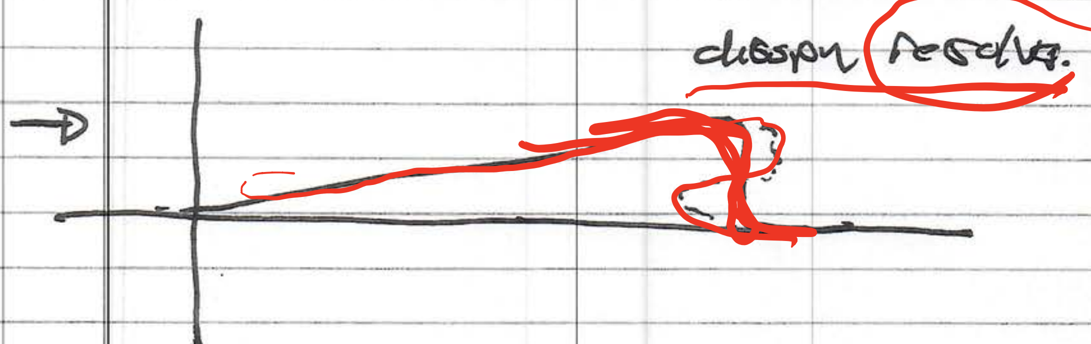


if $d\rho/dx < 0$ anywhere in i.v.d

⇒ overtaking and shock formation
discontinuity forms
dissipation no longer negligible
(prevents disc value)

i.e.





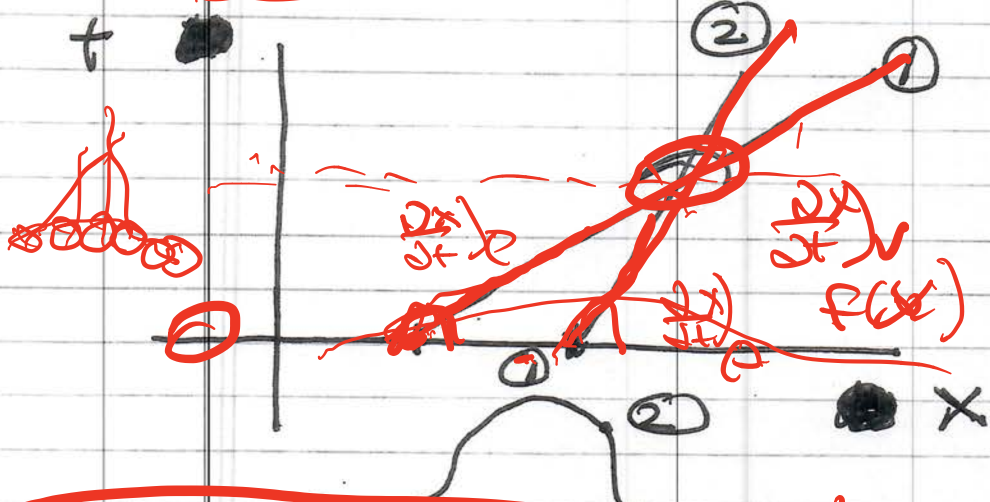
When does breaking / shock formation occur?

$v \pm c_s \leftarrow \left.\frac{\partial x}{\partial t}\right]_e \rightarrow$ characteristic eqns

$$\frac{dp}{dt} = 0 \qquad + \frac{d}{t}p + \left(\frac{\partial x}{\partial t}\right)_e \frac{\partial p}{\partial x} = 0$$

$\boxed{\text{Full nonlinear characteristic eqns.}}$

→ Breaking occurs when 2 initial points arrive at same position at same time.



overtaking

↕

[characteristics crossing

I.C.

$$\boxed{x = t\left[c_0 + \tfrac{1}{2}(\gamma+1)v\right] + f(v)}$$

$$\boxed{x = t\left[c_0 + \tfrac{1}{2}(\gamma+1)(v+\delta v)\right] + f(v+\delta v)}$$

intersect:

$$\cancel{c_0 t} + \tfrac{1}{2}\cancel{(\gamma+1)vt} + f(v)$$

$$= \cancel{c_0 t} + \tfrac{1}{2}\cancel{(\gamma+1)tv} + \delta v\,\tfrac{1}{2}(\gamma+1)t + f(v) + \delta v\, f'(v)$$

$$0 = \delta v\left[\tfrac{1}{2}(\gamma+1)\,t_{shock} + f(v)\right]$$

time for shock formation

Difference $\Rightarrow 0$ so $x = t(v\,\mathrm{ICs}) + f(v)$

$$\boxed{\boxed{t_{shock}} = \frac{-2f'(v)}{(\gamma+1)}} \rightarrow \text{time to form shock}$$

shock formation

– need $f'(v) < 0$ for overtaking

all set by i.c.

– Generally:

need:

HW → work out these. Gas Dyn. → t_{shock}, x_{shock}

~~high canf~~

$$\left(\frac{\partial x}{\partial v}\right)_t = 0$$

$$\left(\frac{\partial^2 x}{\partial v^2}\right)_t = 0 \quad \text{inflection}$$

$$\left(\frac{\partial x}{\partial v}\right)_t = \frac{1}{2}(\gamma+1)t + f' = 0 \quad ✓$$

~ formation.

→ Shocks — Flows with Discontinuity

- once wave steepen and break
  ⇒ discontinuity appears

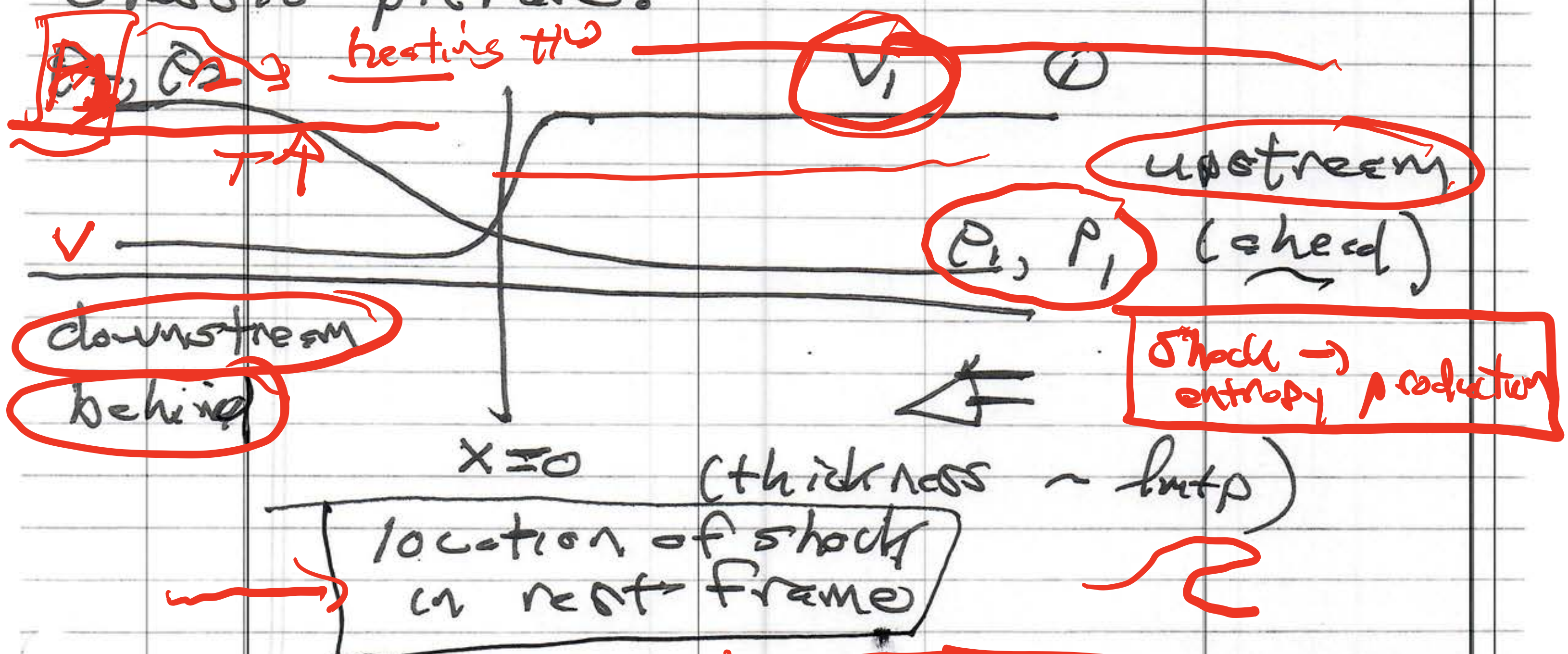
rapid jump

- "shock" — localized region of rapid change / discontinuity

- dissipation essential in shock
  (thickness ~ l_{mfp})

fluid
$\lambda > l_{mfp}$

Classic picture:



→ shock converts kinetic energy into thermal energy, compression

(engineer)

→ machines
→ gen elec → heater

Entropy Production → Insert

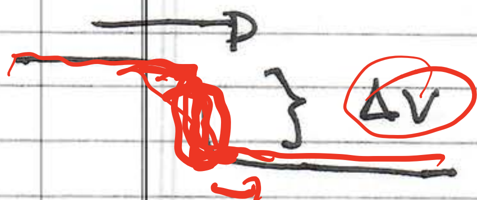
Supplement: Gas Dynamic Shocks
Lecture [redacted]

a.) Scale $\quad P = 0$

HW → Solve (gen) for Burgers (Whithams)

Consider Burgers Eqn.;

$$\partial_t V + v\, \partial_x V - \nu\, \partial_x^2 V = 0$$



$$\frac{v\,U}{w} \sim \frac{\nu}{w^2}\, V$$

$$w \sim \nu / \Delta V$$

for shock width

$$\frac{\Delta V\, \Delta V}{W} \sim \nu \frac{\Delta V}{W^2}$$

$$\boxed{W \sim \nu / \Delta V}$$

dissipation sets shock thickness

for $\nu \sim l_{mfp}\, c_s$

$$\Delta V \sim c_s$$

$$\boxed{W \sim l_{mfp}}$$

characteristic thickness of shock layer.

b.) Entropy Production

In gas-dynamics

- initially ideal dynamics
- entropy constant

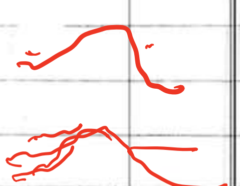
but 

pulse steepens and shocks

- sharp gradients produced in shock
  - ⇒ couple to diffusive dissipation
  - ⇒ drive collisional transport
  - ⇒ produce entropy.

N.B.: - Entropy production required in shock

- sets arrow of time.



- n.b. expansion unstable.

Flux - force

- to calculate:

factor #

$$\frac{dS}{dt} = (\ )\left(-\Gamma_x \nabla X\right) \rightarrow$$

k_B

$\nabla X \rightarrow$ thermodynamic force

$\Gamma_x \rightarrow$ flux

Ficks Law

i.e. $\Gamma_x = -D_x \nabla X$

$$dS/dt \sim (\ )\, D_x (\nabla X)^2$$

entropy production rate density

then, for Burgers shock:

$\partial_x V$ $\quad \frac{1}{W} \sim \frac{\Delta V}{\nu}$

$$dS/dt \sim \nu(\partial_x V)^2$$

$$\sim \nu \frac{(\Delta V)^2}{W^2} \sim \nu (\Delta V)^2 \frac{(\Delta V)^2}{\nu^2}$$

$$\frac{dS}{dt} \sim (\Delta V)^4/\nu$$

$\cancel{\Delta}$ ΔV

local entropy prod

$D_x (\nabla X)^2$ d(shock)

but, total entropy production is integrated over shock thickness

$$\int dx \frac{dS}{dt} \sim \frac{dS}{dt} \sim W \frac{dS}{dt}$$

$\frac{1}{W} \sim \left(\frac{\nu}{\Delta V}\right)^{-1}$

$W \sim \left[\frac{\Delta V}{\nu}\right]^{-1}$

$W \sim \frac{\nu}{\Delta V}$

$$\sim \cancel{\frac{\nu}{\Delta V}} \cancel{\frac{\nu}{\nu^2}} (\Delta V)^2 (\Delta V)^2$$

$$\frac{d}{dt} S_{tot.} \sim \boxed{(\Delta V)^3}$$

total entropy prod rate indep. of ν

so total entropy production:

$$dS/dt \sim \boxed{(\Delta V)^3}$$

entropy production independent of ν though ν required for heating.

- independent ν !
- entropy / heating produced by collisions but total dS/dt independent of ν.

Shocks $\leftrightarrow$ Turbulence $\rightarrow$ couple to small scales in finite time

finite time sing. $\quad$ t shock

$f(v) \rightarrow \frac{\partial x}{\partial t} =$ $x = t(v \pm S(v)) + f(0)$ i.c.

Conservation relations $\Rightarrow$ jump conditions

$$\partial_t \rho + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$ continuity, mass

$$\partial_t \left(\frac{1}{2}\rho v^2 + \rho \epsilon\right) + \underline{\nabla} \cdot \left(\rho \underline{v}\left(\frac{v^2}{2} + \frac{\gamma P}{(\gamma - 1)\rho}\right)\right)$$

$\Rightarrow$

energy

$$\frac{\gamma P}{\gamma - 1} = \frac{P}{\gamma - 1} + P$$

P $\hookrightarrow$ PV work on surroundings

$\frac{P}{\gamma-1}$ $\Downarrow$ energy density

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial}{\partial x_k} \Pi_{ik}$$

Π_{ik} $\hookrightarrow$ stress tensor

$$\Pi_{ik} = P \delta_{ik} + \rho v_i v_k$$

$\rightarrow$ tangential components $\underline{V}$ continuous

– so integrating.

Work in shock frame — $U = 0$

$$\partial_t \rho = -U \partial_x \rho$$

$$-\int U \partial_x \rho = (\rho_2 - \rho_1) U$$

but $U = 0$ frame

Mass

$$\boxed{\rho V_x \Big|_2 = \rho V_x \Big|_1} \quad (1)$$

continuity
mass flux

energy

$V_n \equiv V_\perp$ plane

$$\rho V_n \left(\frac{1}{2} V^2 + \frac{\gamma \, P/\rho}{\gamma - 1} \right) \Big|_2 = \rho V_n \left(\frac{V^2}{2} + \frac{\gamma \, P/\rho}{\gamma - 1} \right) \Big|_1$$

but

$$\rho V \Big|_2 = \rho V \Big|_1$$

and continuity parallel ⇒

$$V_{y_z}^2 \Big|_{②} = V_{y_z}^2 \Big|_{①}$$

so

$$\boxed{\left(\frac{V_x^2}{2} + \frac{\gamma P/\rho}{\gamma-1}\right)_{②} = \left(\frac{V_x^2}{2} + \frac{\gamma P/\rho}{\gamma-1}\right)_{①}}$$

②

and momentum conservation:

$$\boxed{\left(\rho V_x^2 + P\right)\Big|_{②} = \left(\rho V_x^2 + P\right)\Big|_{①}}$$

normal component only varies.

so, have 3 Rankine-Hugoniot jump/continuity conditions

$$[\ ] = (\ )_{②} - (\ )_{①}$$

$$\boxed{\begin{aligned} \left[\rho v_x\right] &= 0 \\ \left[\frac{v_x^2}{2} + \frac{\gamma P/\rho}{\gamma - 1}\right] &= 0 \\ \left[\rho v_x^2 + P\right] &= 0 \end{aligned}}$$

in shock frame
→ relate upstream/downstream quantities

in fixed coordinate frame:

$$V_x = V_n - U$$

V_n → normal V in fixed coords

U → shock velocity

then on to:

- shock adiabat. ↔ entropy production
- v_2/v_1, ρ_2/ρ_1, P_2/P_1

ρ_2/ρ_1 → compression

P_2/P_1 → heating

Shock – illustration – 'student' or wave.

Shocks – scale invariant

Small → non-dispersive → scale   no scale in problem

Nonlinear Waves II

gas dyn shock

- initial study : gas-dynamic 'simple waves'
  - 'scale' free
  - non-dispersive

- now : dispersive, scale, solitons → ion acoustic (simplest, not first)

→ solitons, collisionless shock

Point: $$\omega^2 = c_s^2 k^2 / 1+k^2\lambda_D^2$$

$$\left(\frac{\omega}{k}\right)^2 = c_s^2 / 1+k^2\lambda_D^2$$

dispersion

$\lambda_D \to 0$: gas dynamic limit, non-dispersive

⇒ all harmonics have same phase velocity

$$(\omega/k)^2 = c_s^2$$

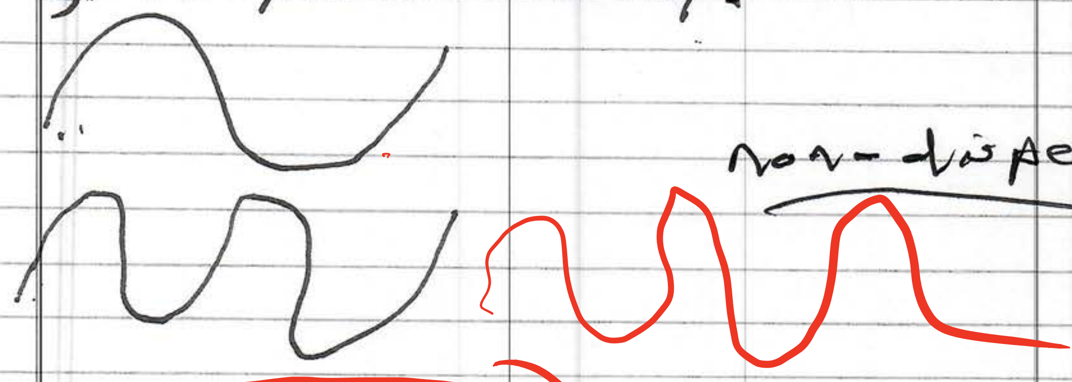
On dispersion

self-beat:

$$(2\omega/2k)^2 = c_s^2$$

etc.

As wave steepens by nonlinear interaction, <u>all</u> harmonics move at C_s as phase velocity.



non-dispersive

but <u>with</u> dispersion:

$$(\omega/k)^2 = C_s^2 / 1 + k^2\lambda_D^2$$

$$\left(\frac{2\omega}{2k}\right)^2 = C_s^2 / 1 + 4k^2\lambda_D^2$$

n^{th}

$$\omega_n^2 = C_s^2 / (1 + 2n(k\lambda_D)^2)$$

then fundamental and self-beats disperse → spread in speeds.

∴ 2 routes to balance of steepened fronts:

n.l. steepening vs. (dissipation) $\sim \partial_x^2 v$

$v \partial_x v$ → shock

dissipative structure

(conventional)

n.l. steepening vs. dispersion

→ soliton (1D)

i.e. Schematic: → "so called collisionless shock"!

recall 1D compressible hydro – Burgers Eqn.

$$\partial_t V + V\partial_x V = \nu \partial_x^2 V$$

ΔV

$$\Delta V \frac{\cancel{\Delta V}}{\cancel{\Delta x}} \sim \nu \frac{\cancel{\Delta V}}{(\Delta x)^{\cancel{2}}}$$

$\boxed{\Delta x \sim \nu / \Delta V}$ → shock layer thickness

↓

dissipn sets → ν ~ collisions.

Now, can generalize:

$$\omega^2 = k^2 C_s^2 / 1 + k^2 \lambda_D^2$$

$$\omega \cong k C_s \left(1 - \frac{k^2 \lambda_D^2}{2}\right)$$

so

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + C_s \partial_x \mathcal{E} + \mathcal{E}\partial_x \mathcal{E} = \nu \partial_x^2 \mathcal{E} + \frac{C_s \lambda_D^2}{2} \partial_x^3 \mathcal{E}}$$

dissipative

dispersión

steepening vs dispersion

steepening vs. dispersión

$c_s,\ \lambda_D \to \infty \Rightarrow$ Burgers

$\nu \to 0 \Rightarrow$ KdV   Korteweg - de - Vries

KdV

$$\boxed{\partial_t \mathcal{E} + c_s \partial_x \mathcal{E} + \mathcal{E}\partial_x \mathcal{E} = \frac{c_s \lambda_D^2}{2} \partial_x^3 \mathcal{E}}$$

dispersion controls steepening

$$\frac{\mathcal{E}}{\Delta}\mathcal{E} \sim \frac{c_s \lambda_D^2}{2}\frac{\mathcal{E}}{\Delta^3}$$

$$\Delta^2 \sim \frac{c_s \lambda_D^2}{2\mathcal{E}} \qquad \boxed{\Delta \sim \left(\frac{c_s}{2\mathcal{E}}\right)^{1/2} \lambda_D}$$

Collisionless shock

steepening vs dispersion

scale.

n.b. Where does KdV come from in hydro?

— surface wave: $\omega^2 = gk$

— surface wave with finite depth:

$$\omega^2 = gk \tanh(kd)$$

kd

then expanding:

$$\omega^2 = k^2 gd + gkd \left(-\frac{k^2 d^2}{3}\right)$$

shallow H_2O $\omega^2 = k^2 gd$

$$\omega^2 = k^2 g d\left(1 - \frac{k^2 d^2}{3}\right)$$

$$\Updownarrow$$

$$\omega^2 = \frac{k^2 c_s^2}{1 + k^2\lambda_D^2} \rightarrow k^2 c_s^2\left(1 - \frac{k^2 \lambda_D^2}{2}\right)$$

a

$$\boxed{\partial_t \varepsilon + v_0 \partial_x \varepsilon + \varepsilon\, \partial_x \varepsilon = -\frac{v_0 d^2}{3} \partial_x^3 \varepsilon}$$

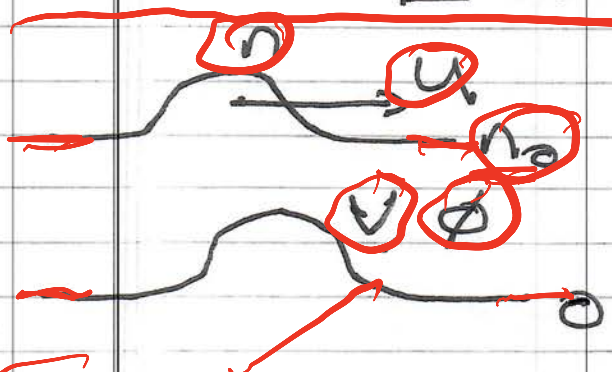
$V_0 = (gd)^{1/2}$.

KdV

classic for solitons

Now, for $\boxed{\text{ion-acoustic:}}$ $\longrightarrow$ NL ion-acoustic soliton

$n_e = n_0 \exp\left[|e|\phi / T_e\right]$ Boltzmann electrons



$$\left[\begin{array}{l} \partial_t n_i + \partial_x (n_i V) = 0 \\ \partial_t V_i + V_i \partial_x V_i = -\dfrac{|e|}{M_i} \dfrac{\partial \phi}{\partial x} \end{array}\right.$$

fluid ions.

form

$$\left\{\begin{matrix} n_i \\ v_i \\ \phi \end{matrix}\right\} = \boxed{F(x-ut)} \rightsquigarrow \text{standard form of NL pulse}$$

speed

$$-u n_i' + (n_i v)' = 0$$

$$(v-u)v' = -\frac{q}{m_i}\phi'$$

integrate;

$$\left.\begin{matrix} \phi \to 0 \\ v \to 0 \\ n \to n_0 = 1 \end{matrix}\right\} \quad |x| \to \infty$$

$$-u n_i + n_0 v = -u \qquad \text{(b.c.)}$$

$$(u-v)n_i = u$$

$$n_i = u/(u-v)$$

likewise:

$$-\frac{q\phi}{m_i} = \frac{v^2}{2} - \frac{2uv}{2} + \frac{u^2}{2} - \frac{u^2}{2}$$

$$\frac{q\phi}{m_i} = -\frac{1}{2}(u-v)^2 + \frac{u^2}{2}$$

b.c. ✓.

$n_i(\phi)$

$n_e(\phi)$

so $(u-v) = \left(u^2 - \frac{2q\phi}{M_0}\right)^{1/2}$

$\Rightarrow$

$$\partial_x^2 \phi = -4\pi n_0 q \left(n_i - n_e \right)$$

$$\partial_x^2 \left(\frac{q\phi}{T}\right) = -\frac{1}{\lambda_D^2} \left(\frac{1}{\left(1 - 2q\phi/T\right) c_s^2/u^2)^{1/2}} - \exp\left(\frac{q\phi}{T}\right) \right)$$

$\frac{q\phi}{T} \to \phi$

ϕ $\quad$ E $\quad$ u

$$\boxed{\partial_x^2 \phi = -\frac{1}{\lambda_D^2} \left(\frac{1}{\left(1 - 2\phi \frac{c_s^2}{u^2}\right)^{1/2}} - e^{\phi} \right)}$$

NL Wave eqn.

$\mathcal{M}^2 = u^2/c_s^2$

Mach #

$\Leftrightarrow$ Motion in Potential (Sagdeev)

$$\phi' \partial_x^2 \phi = -\frac{1}{\lambda_D^2} \frac{\phi'}{\left(1 - \frac{2\phi}{\mathcal{M}^2}\right)^{1/2}} - e^{\phi} \phi'$$

$$= -\left(dV(\phi)/d\phi\right) \phi'$$

and integrate: Sagdeev potential

$$\boxed{V(\phi) = \frac{-1}{\lambda_D^2}\left\{ \frac{M^2}{2}\left(1-\frac{2\phi}{M^2}\right)^{1/2} - e^{\phi}\right\} + C}$$

integration const.

→ sets possibility

⇒

$$\frac{\phi'^2}{2} + V(\phi) = 0$$

$$\phi'' = dV/d\phi$$

⇒ ion acoustic problem reduced to particle orbit.

$$\phi'' = dV/d\phi$$

$$\ddot{x} = -dU/dx$$

→ $M^2 > 2\frac{|e|\hat{\phi}}{T}$ critical velocity for soliton.

→ small ϕ

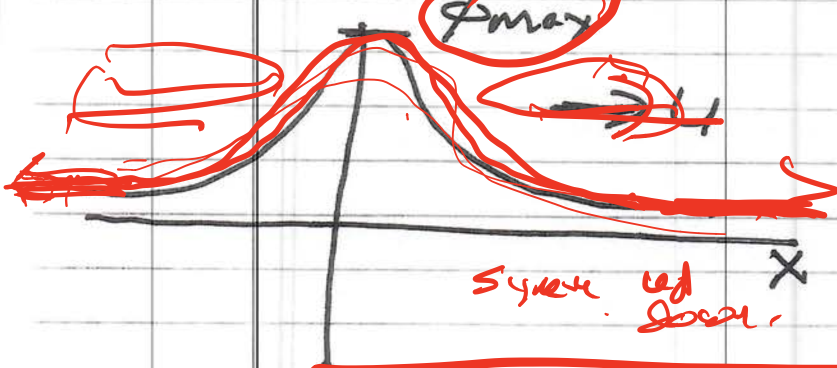
$$U(\phi) \simeq \frac{-1}{\lambda_D^2}\left\{ (1+M^2) + \phi - \phi + \frac{1}{2}\phi^2\left(\frac{-1}{M^2}+1\right)\right\}$$

so need $M^2 > 1$

Now, choose:

$\phi' \to 0$

as $\phi \to 0$



NL pulse → dispersion vs steepening

localized pulse:

$\boxed{\text{ion-acoustic soliton}}$

$$\boxed{\mathcal{M}^2 = \frac{1}{2}\,\frac{\left[\exp\phi_{max} - 1\right]^2}{\exp\phi_{max} - 1 - \phi_{max}}}$$

entropy production?

collisionless shock

entropy production

low amplitude

collisional entropy production

$$\phi \approx \frac{3}{2}\left(1 - \frac{1}{\mathcal{M}^2}\right)\operatorname{sech}^2\left\{\sqrt{\frac{\pi n_0 e^2}{T}}\sqrt{1 - \frac{1}{\mathcal{M}^2}}\;x\right\}$$

need $\mathcal{M}^2 \geq 1.6$.

Ion Acoustic collisionless shock

can consider cases.

→ Need discuss wave-particle interaction to deal with heating, entropy production etc

Sagdeev → Rev. Pl. Phys Vol. 4

KdV ✓

→ More general discussion (KdV):

Recall, had general:

Apply to: Flood wave (kinematics)

→ type?

$$\partial_t \mathcal{E} + (C_s + \mathcal{E})\,\partial_x \mathcal{E} + \frac{C_s}{2}\lambda_D^2\,\partial_x^3 \mathcal{E} = 0$$

$$a = \mathcal{E}$$

$$y = x - C_s t$$

$$\boxed{\partial_t a + a\,\partial_y a + \beta\,\frac{\partial^3 a}{\partial y^3} = 0}$$

simple KdV.

$$\underbrace{l_{\text{soliton}} \sim (\beta/a)^{1/2}}_{\text{scale amplitude reln.}}$$

Solving (reduced) KdV (integrating)

$$a = a(y - ct)$$

$$\Rightarrow$$

$$\beta a''' - c a' + a a' = 0$$

invariant
$a \to a + V$
$c \to c + V$

$$\beta a'' - c a + \frac{a^2}{2} = \frac{1}{2} C_1$$

const

$h \sim 1/\sqrt{t}$

$t \gg t_{\text{shock}}$

(Whitham) (Burgers)

$$(2a')\ *\ \left(\beta a'' - c a + \frac{a^2}{2} = \frac{C_1}{2}\right)$$

$$2\beta a' a'' - 2c a a' + a' a^2 = C_1 a'$$

intr.

$$\beta a'^2 = -\frac{1}{3} a^3 + C a^2 + C_1 a + C_2$$

and can now reduce to quadrature.

→ Convenient to factorize:

$$C,\ C_1,\ C_2 \Rightarrow a_1,\ a_2,\ a_3$$

$$\boxed{\beta a'^2 = -\frac{1}{3}(a-a_1)(a-a_2)(a-a_3)}$$

$$C = \frac{1}{3}(a_1 + a_2 + a_3)$$

3 roots again!

For: ⇒ bounded $|a(y)|$

need a_1, a_2, a_3 real

$$\Rightarrow \left[\begin{array}{l} \text{if: } a_1 > a_2 > a_3 \\ a_1 \geq a \geq a_2 \end{array}\right.$$

$a_3 = 0$ no loss generality

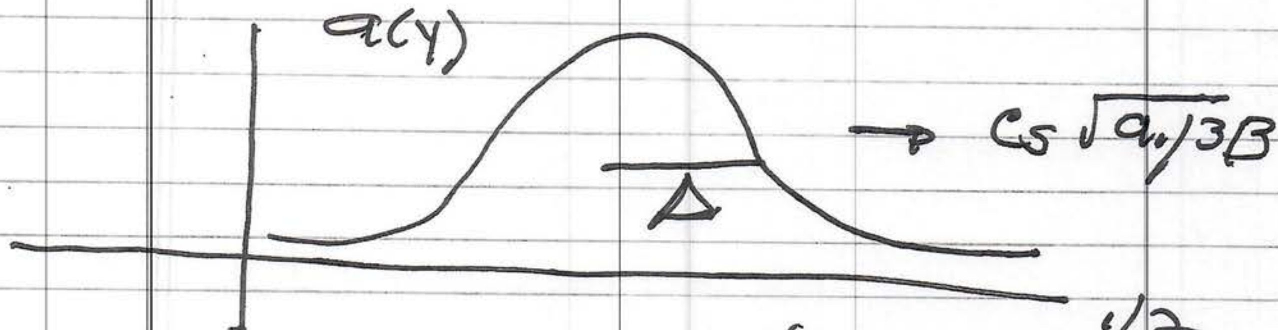
$$\Rightarrow \quad \beta a'^2 = 1/3\,(a_1 - a)(a - a_2)\,a$$

if $a_2 = 0$

$$a(y) = a_1 \cosh^{-2}\left(\tfrac{1}{2}\, y \sqrt{a_1/3\beta}\right)$$

$$\rightarrow a_1 \cosh^{-2}\left(\tfrac{1}{2}(x - c_s t)\sqrt{a_1/3\beta}\right)$$

so, have (as before)



$$\Delta \sim (3\beta/a_1)^{1/2}$$

note:

$\rightarrow\ u \sim c_s\sqrt{a_1/3\beta}$ speed – amplitude

$\rightarrow\ \Delta \sim (3\beta/a_1)^{1/2}$ width – amplitude

Note:

→ Soliton has finite width

$$\Delta \sim (3B/a_1)^{1/2} \sim \lambda_{De}$$

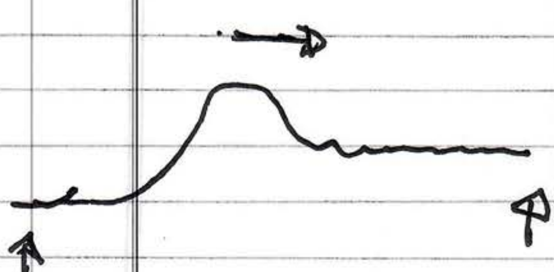
contrast shock

→ bigger solitons go faster

$$V \sim U_0 (a_1/3B)^{1/2}$$

Soliton → symmetric pulse
up/down stream
symmetric.

Collisionless shock → collisionless, asymmetry



entropy production?
TBC