Lecture III – Aspects of Nonlinear Waves, I

Recall:

- dispersionology of basic plasma waves
- most interesting → ion acoustic wave

$$\omega^2 = k^2 c_s^2 / 1 + k^2 \lambda_D^2$$

⇒ – nonlinear waves? → evolution past inf. to finite amplitude

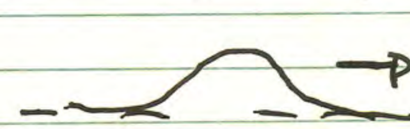
– $\left\{ \begin{array}{l} \text{shocks} \\ \text{solitons} \end{array} \right.$

⋮

⇒ Gasdynamic Waves / Shocks

Observe, for quasi-neutral ion acoustic wave: $(k^2 \lambda_D^2 \ll 1)$

$$n_e = n_0 \exp\left[\frac{|e|\phi}{T}\right]$$



$$\frac{\partial n_i}{\partial t} + \partial_x (v n_i) = 0$$

or

$$\frac{\partial n_i}{\partial t} + v \frac{\partial n_i}{\partial x} = -n_i \frac{\partial v}{\partial x}$$

and

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -|e|\frac{\partial \phi}{\partial x}$$

$$= \frac{\partial}{\partial x}\left(\cancel{-|e|}\,\frac{T}{\cancel{|e|}}\,\ln\left(\frac{n_e}{n_0}\right)\right)$$

$$= -\frac{T_e}{n_e}\frac{\partial n_e}{\partial x}$$

$$= \frac{T_e}{n_i}\frac{\partial n_i}{\partial x} \qquad \text{(QN)}$$

so

$$\begin{cases} \dfrac{\partial n_i}{\partial t} + \partial_x (n_i v) = 0 \\[2ex] \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = -\dfrac{T_e}{n_i}\dfrac{\partial n_i}{\partial x} \end{cases}$$

→ isomorphic to 1D gas-dynamics with isothermal EOS

→ so, study gas dynamics!

1D Gas Dynamics — 'Simple Waves'

cf $\left\{\begin{array}{l}\text{Landau/Lifshitz Fluids} \\ \text{Whitham}\end{array}\right.$ → steepening, shocks etc.

$$\partial_t \rho + \partial_x (\rho v) = 0$$

$$\partial_t v + v\,\partial_x v + \frac{1}{\rho}\partial_x P = 0$$

ideal

$$P = P(\rho)$$

Entropy: – S initially homogeneous

– S = const, till shock forms

1D

$P = P(\rho)$

$\left\{\begin{array}{l}\text{speed vs } \rho \text{ rel}^n \\ \text{gas flow}\end{array}\right.$ $\frac{d\rho}{dt} = 0$ along $\left(\frac{\partial x}{\partial t}\right)_\rho = \frac{d(\rho v)}{d\rho}$

so

$$\partial_t \rho + \frac{d(\rho v)}{d\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\partial_t v + \left(v + \frac{1}{\rho}\frac{dP}{dv}\right)\frac{\partial v}{\partial x} = 0$$

$P = P(\rho)$
$v = v(\rho)$

so

$$\left(\frac{\partial x}{\partial t}\right)_\rho = \frac{d}{d\rho}(\rho v) = v + \rho\frac{dv}{d\rho}$$

n.b. get on $\frac{d\rho}{dt} = 0$, $\frac{dv}{dt} = 0$ conservation for $\left\{\begin{array}{l}\text{characteristic} \\ \text{eqn.} \rightarrow \\ \text{all info}\end{array}\right.$

likewise:

$$\left.\frac{\partial x}{\partial t}\right)_V = V + \frac{1}{\rho}\frac{dP}{dV}$$

$$V = V(\rho)$$

but

$$P = P(\rho)$$

$v \leftrightarrow \rho$

$\Rightarrow$

$$\boxed{\left(\frac{\partial x}{\partial t}\right)_\rho = \left(\frac{\partial x}{\partial t}\right)_V}$$

$$\rho\frac{dV}{d\rho} + V = V + \frac{1}{\rho}\frac{dP}{dV}$$

$$dP = c_s^2\, d\rho$$

$$\rho\frac{dV}{d\rho} = \frac{c_s^2}{\rho}\frac{d\rho}{dV}$$

so

$$(dV/d\rho)^2 = c_s^2/\rho^2$$

$$\boxed{V = \pm\int\frac{c_s}{\rho}\,d\rho = \pm\int\frac{dP}{\rho c_s}}$$

solves for flow speed in terms density

Note:

- $v = \pm \int \frac{c_s}{\rho} d\rho$

- $P\rho^{-\gamma} = \text{const.} \qquad \gamma = 5/3$

$$dP = c_s^2 d\rho = \underset{\text{const}}{\alpha} \gamma \rho^{\gamma-1} d\rho$$

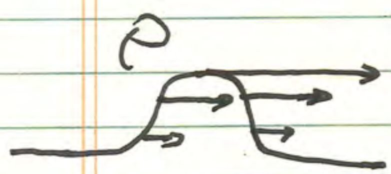
$$\boxed{c_s^2 = \alpha\gamma\rho^{\gamma-1}}$$

$\alpha = \text{const}$

$$v = \pm \int \frac{c_s^2}{\rho c_s} d\rho = \pm \alpha\gamma \int \frac{\rho^{1/3}}{\rho} d\rho$$

$$\boxed{v = \pm 3\sqrt{\alpha\gamma}\, \rho^{1/3}}$$

- flow speed increases with density



- high density elements go faster

- then:

$$\left.\frac{\partial x}{\partial t}\right)_v = v + \frac{1}{\rho}\frac{dP}{dv}$$

$$v = \pm \int \frac{dP}{\rho c_s} \qquad dv = \frac{dP}{\rho c_s}$$

so $\frac{dP}{dv} = \rho c_s$

$\rho = \rho(v)$

$$\boxed{\left.\frac{\partial x}{\partial t}\right)_v = v \pm c_s(v)}$$

$$\boxed{x = t\left[v \pm c_s(v)\right] + f(v)}$$

"simple wave" solution $\left\{\begin{array}{l}\text{general}\\ \text{nonlinear.}\end{array}\right.$

Check: linearized limit

$$x = t\left[\cancel{v} \pm c_s(0)\right] + \cancel{f(v)} + x_0$$

$$x = x_0 \pm c_s(0)t$$

→ Why "simple"? — No characteristic scale!

⇒ alternate approach

so if no characteristic scale all quantities depend only on

$$\xi = x/t$$

$$\begin{matrix}\rho \\ v\end{matrix} = f(\xi) \qquad \xi = x/t$$

→ velocity formed by x, t in absence of scales

so $\quad \partial_x = 1/t \; \partial/\partial\xi$

$$\partial_t = \frac{-\xi}{t}\frac{\partial}{\partial\xi}$$

$$\Rightarrow \partial_t \rho + \rho \partial_x v + v \partial_x \rho = 0$$

$$\partial_t v + v \partial_x v = -1/\rho \; \partial_x P$$

becomes:

$$-\frac{\xi}{t}\rho' + \frac{\rho}{t}v' + \frac{v}{t}\rho' = 0 \qquad ' = \partial/\partial\xi$$

$$-\frac{\xi}{t}v' + \frac{v}{t}v' = -\frac{c_s^2}{t}\frac{\rho'}{\rho}$$

so

$$(v-\xi)\rho' + \rho v' = 0$$

$$(v-\xi)v' = -c_s^2\frac{\rho'}{\rho}$$

like linear dispersion eq'n, treat ξ as eigenvalue.

$$(v-\xi)\rho' + \rho v' = 0$$

$$-\frac{c_s^2}{\rho}\rho' + (v-\xi)v' = 0$$

$$(v-\xi)^2 = c_s^2$$

$$\xi = v \pm c_s$$

$$\frac{x}{t} = v \pm c_s$$

$$x = (v \pm c_s)\, t.$$

From "eigenvector", relate v, ρ etc.

$$v - \xi = -c_s$$

$$(v-\xi)\rho' + \rho v' = 0$$

$$-c_s \rho' + \rho v'$$

$$\rho\, dv = c_s\, d\rho$$

$$dv/d\rho = c_s/\rho$$

$$v = \int c_s\, d\rho/\rho = \int dP/c_s\rho$$

- equivalent to previous, with $f(v) = 0$

- corresponds to Kulsrud's version of "simple wave"

- can also write:

$$v = \int + \left(-dp\, dV\right)^{1/2}$$

$$d(1/\rho) = dV = -\frac{1}{\rho^2} d\rho \quad , \quad dp = c_s^2 d\rho$$

$$\left(v = \int \left(\frac{1}{\rho^2} d\rho\, c_s^2 d\rho\right)^{1/2} = \int \frac{d\rho}{\rho} c_s \right)$$

→ Physics of simple wave

- have established simple wave in gas dynamics, with key element of scale invariance

- now elucidate simple wave physics ⇒ shocks.

similarity flow → simple wave with $f(v) = 0$

$$x = t\left[v \pm c_s(v)\right]$$

wave front rel'n

can write general solution,
for adiabatic process:

$$P \rho^{-\gamma} = \text{const}$$

$$T \rho^{-(\gamma-1)} = \text{const}$$

$$T = c_s^2 = \rho^{\gamma-1}$$

$$\frac{dP}{d\rho} = c_s^2 = T$$

$$P = \rho T$$

$$\rho \, T^{1/\gamma-1} = \text{const}$$

as $c_s^2 \sim T$

$$\boxed{\rho = \rho_0 \left(c / c_0 \right)^{2/\gamma-1}}$$

(elimis → clarity)

but $V = \pm \int c \frac{d\rho}{\rho}$

so

$$\boxed{V = \frac{\pm 2}{\gamma - 1} (c - c_0)}$$

so finally can write:

$$\boxed{\begin{aligned} c &= c_0 \pm \tfrac{1}{2}(\gamma-1)v \\ \rho &= \rho_0 \left(1 \pm \tfrac{(\gamma-1)}{2} v/c_0\right)^{2/\gamma-1} \\ p &= p_0 \left(1 \pm \tfrac{1}{2}(\gamma-1) v/c_0\right)^{2\gamma/\gamma-1} \end{aligned}}$$

so then:

$$x = t\left[v \pm c_s(v)\right] + f(v)$$

$$x = t\left[v \pm \left(c_0 \pm \tfrac{1}{2}(\gamma-1)v\right)\right] + f(v)$$

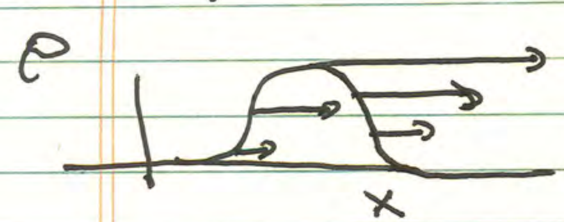
Now:

- point on wave profile moves at

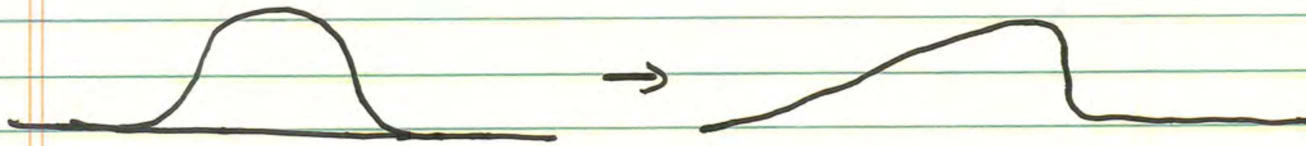
$$u = \frac{\partial x}{\partial t} = v \pm c_s$$

and $du/d\rho > 0$

speed increases with density!



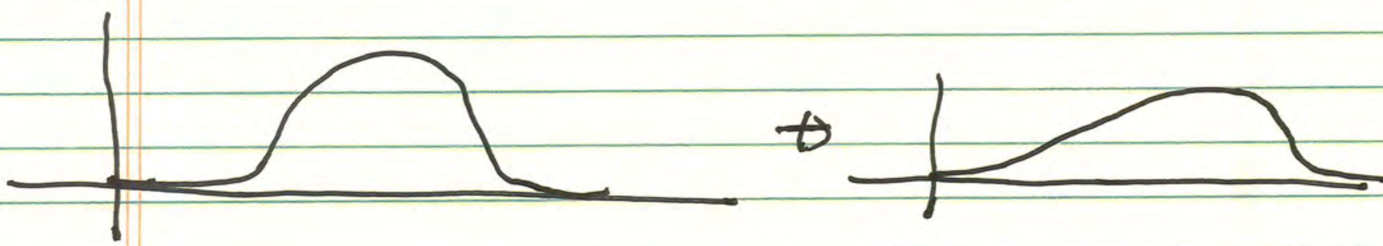
so

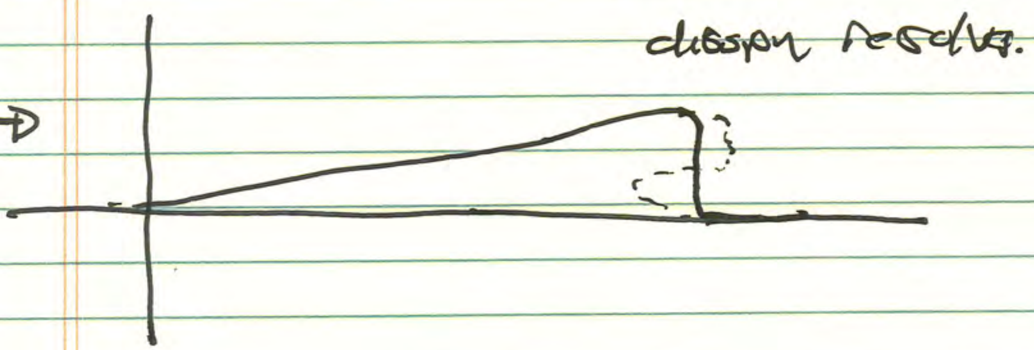
$\Rightarrow$ 

if $dc/dx < 0$ anywhere in i.v.d.

$\Rightarrow$
- overtaking and shock formation
- discontinuity forms
- dissipation no longer negligible

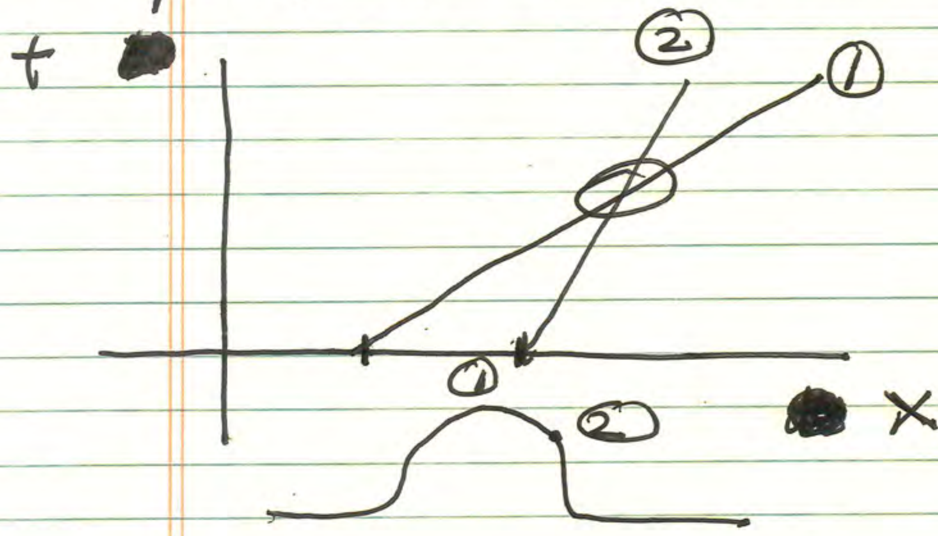
i.e.



$\rightarrow$ 

When does breaking / shock formation occur?

→ Breaking occurs when 2 initial points arrive at same position at same time.



overtaking

↕

characteristics crossing

$$x = t\left[c_0 + \tfrac{1}{2}(\gamma+1)v\right] + f(v)$$

$$x_1 = t\left[c_0 + \tfrac{1}{2}(\gamma+1)(v+\delta v)\right] + f(v+\delta v)$$

$$\cancel{c_0 t + \tfrac{1}{2}(\gamma+1)vt} + f(v)$$

$$= \cancel{c_0 t} + \cancel{\tfrac{1}{2}(\gamma+1)tv} + \delta v\,\tfrac{1}{2}(\gamma+1)t$$

$$+ f(v) + \delta v\, f'(v)$$

$$= \delta v\left[\tfrac{1}{2}(\gamma+1)t_{shock} + f'(v)\right]$$

Difference $\Rightarrow 0$ so

$$\boxed{t_{shock} = \frac{-2f'(v)}{(\gamma+1)}}$$

- need $f'(v) < 0$ for overtaking

- Generally:

need:

$$\left(\frac{\partial x}{\partial v}\right)_t = 0$$

$$\left(\frac{\partial^2 x}{\partial v^2}\right)_t = 0 \qquad \text{inflection}$$

$$\left(\frac{\partial x}{\partial v}\right)_t = \frac{1}{2}(\gamma+1)t + f' = 0 \quad ✓$$

~ formation

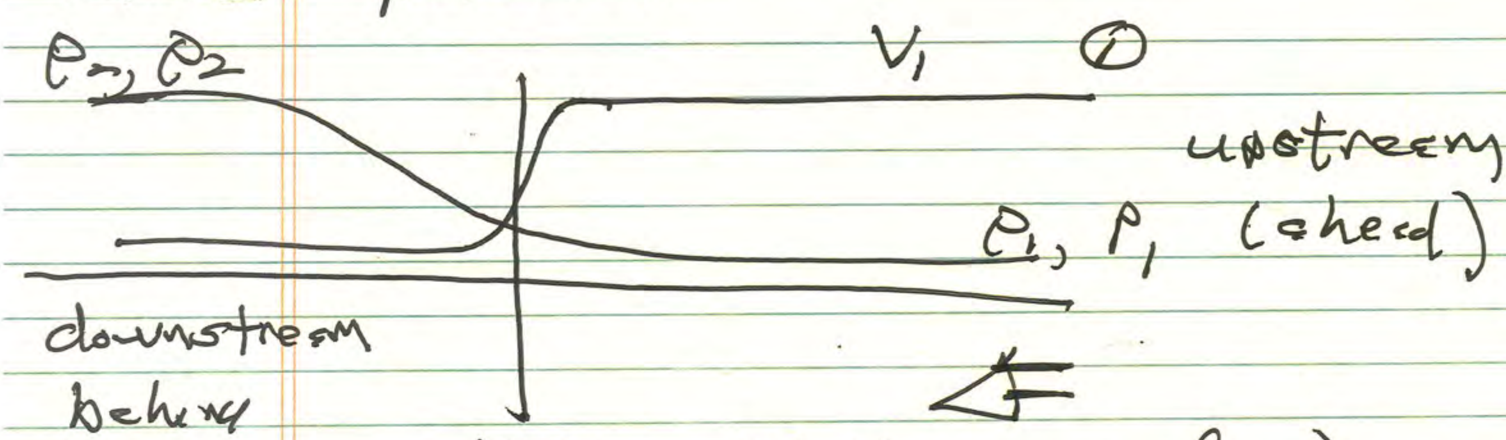
→ Shocks — Flows with Discontinuity

- once wave steepen and break
  ⇒ discontinuity appears

- "shock" — localized region of rapid change / discontinuity

- dissipation essential in shock (thickness ~ l_{mfp})

Classic picture:



$x=0$ (thickness ~ l_{mfp})
location of shock in rest frame

→ shock converts kinetic energy into thermal energy, compression

Conservation relations ⇒ jump conditions

$$\partial_t \rho + \underline{\nabla} \cdot (\rho \underline{V}) = 0$$

continuity, mass

$$\partial_t \left(\frac{1}{2} \rho V^2 + \rho \epsilon \right) + \underline{\nabla} \cdot \left(\rho \underline{V} \left(\frac{V^2}{2} + \frac{\gamma P}{(\gamma-1)\rho} \right) \right)$$

$\Rightarrow$

energy

$$\frac{\gamma P}{\gamma - 1} = \frac{P}{\gamma - 1} + P$$

P ↳ PV work on surroundings

$\frac{P}{\gamma-1}$ ↓ energy density

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial}{\partial x_k} \Pi_{ik}$$

Π_{ik} ↳ stress tensor

$$\Pi_{ik} = P \delta_{ik} + \rho v_i v_k$$

⇝ tangential components $\underline{V}$ continuous

- so integrating.

Work in shock frame — $u=0$

$$\partial_t \rho = -u\,\partial_x \rho$$

$$-\int u\,\partial_x \rho = (\rho_2-\rho_1)\,u$$

but $u=0$.

mass

$$\boxed{\rho V_x\Big|_{2} = \rho V_x\Big|_{1}} \quad (1)$$

continuity
mass flux

energy $\qquad V_n \equiv v_{\perp}$ plane

$$\rho V_n\left(\frac{1}{2}V^2+\frac{\gamma\, p/\rho}{\gamma-1}\right)\Bigg|_{2} = \rho V_n\left(\frac{V^2}{2}+\frac{\gamma\, p/\rho}{\gamma-1}\right)\Bigg|_{1}$$

but

$$\rho V\Big|_{2} = \rho V\Big|_{1}$$

and continuity parallel $\Rightarrow$

$$V_{y_z}^2 \Big|_{②} = V_{y_z}^2 \Big|_{①}$$

so

$$\boxed{\left(\frac{V_x^2}{2} + \frac{\gamma P/\rho}{\gamma - 1}\right)_{②} = \left(\frac{V_x^2}{2} + \frac{\gamma P/\rho}{\gamma - 1}\right)_{①}}$$

②

and momentum conservation:

$$\boxed{\left(\rho V_x^2 + P\right)\Big|_{②} = \left(\rho V_x^2 + P\right)\Big|_{①}}$$

normal component, only, varies.

So, have 3 Rankine-Hugoniot jump/continuity conditions

$$[\;] = (\;)_{②} - (\;)_{①}$$

$$\boxed{\begin{aligned} &[\rho v_x] = 0 \\ &\left[\frac{v_x^2}{2} + \frac{\gamma P/\rho}{\gamma - 1}\right] = 0 \\ &[\rho v_x^2 + P] = 0 \end{aligned}}$$

in shock frame

in fixed coordinate frame:

$$\underbrace{V_x}_{} = \underbrace{V_n}_{\substack{\text{normal } V \\ \text{in fixed coords}}} - \underbrace{U}_{\text{shock velocity}}$$

then on to:

- shock adiabat.
- v_2/v_1, $\underbrace{\rho_2/\rho_1}_{\text{compression}}$, $\underbrace{P_2/P_1}_{\text{heating}}$