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A THEORETICAL DISCUSSION OF EDDY-DRIVEN MEAN FLOWS *

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1. INTRODUCTION AND DISCUSSION OF THE GENERAL CIRCULATION

This paper gives a fairly detailed account of our recent results on the induction of mean circulation by eddy processes (whether wave-like or turbulent). The relevance of the work to climatology is through the role of oceans in climate, as well as the need to understand and parameterize eddies in the atmospheric circulation itself.

In Section 1 we discuss the vorticity constraints that are likely to dominate the wind-driven mean circulation, in the absence of strong eddy effects.

In Section 2 we describe the principle formulation for Eulerian-mean flow, rewriting the eddy-induction effects in the spirit of early work by Taylor (1915). Both transient and statistically steady problems, wavelike and turbulent flows, and "oceanic" and "atmospheric" geometry can be treated, although the details are distinctly different in each case. The key quantities that express the eddy effects are the Lagrangian diffusivity of marked fluid particles (κ) and the large-scale geometry of Q, where Q is the large-scale potential vorticity field. The non-dissipative theory stems from work of Taylor, but most problems are sensitive to the nature of potential vorticity dissipation by the enstrophy cascade. These new effects are summarized. The mean-flow equation is solved by integrating along characteristics. The principal restriction is that the lateral scale of the Q-field is supposed to far exceed the displacement of fluid particles, over a few eddy periods.

The important role played by the eddy flux of potential vorticity, $\langle q'\mathbf{u}'\rangle$, is

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argued in detail, and reasons are given for favoring this formulation over one involving momentum Austausch coefficients. In particular, this eddy flux will frequently have a component down the mean gradient of Q in statistically steady flows. Up-gradient fluxes of potential vorticity will occur in special regions where external forcing is directly applied or where the eddy field is a decaying transient, temporally or spatially.

The strength of the eddy-induced Eulerian circulation is estimated. This allows us to divide the fluid into turbulent and laminar regions, although there is considerable overlap (for instance, the mean circulation at a point will often be dominated by eddies, but after spatial averaging, the laminar mean flow will tend to emerge).

The force-like nature of the eddy-flux of potential vorticity is described.

Section 3 gives four examples to which this vorticity analysis is relevant: (i), the x-averaged zonal flow of a homogeneous fluid on a β-plane, induced by turbulence or waves; (ii), the related two-dimensional problem of a second-order mean motion induced on a homogeneous β-plane, by a Rossby-wave packet; (iii) the study of instability of stratified zonal flows and their back effect on the mean flow; (iv), the prediction of deep circulation in a turbulent ocean, where the mean upper-level wind-driven flow is known.

These several examples involve exact calculations of the entire field — in (i), (ii) and (iii) — and calculations of the mean field, given crude knowledge of the eddy intensity — in (i) and (iv). They stress the vastly different strengths of induced flow, according to whether the geostrophic contours extend indefinitely, are blocked by walls, or close upon themselves. Example (iv) gives an opportunity to assess the errors in the method due to the two-scale approximation and to examine weaknesses in its application, in the case of intermittent eddy fields.

The nature of the eddy stresses is discussed, particularly the inviscid pressure drag exerted from one level to the next, causing a vertical transport of horizontal momentum. Examples (iii) and (iv) involve this vertical effect, while (i) and (ii) exhibit lateral stresses only.

The formulation of Eulerian mean flow generation in terms of Lagrangian particle diffusivity follows the spirit of Taylor's (1915) and Bretherton's (1966) instability theory, Dickinson's (1969) and Green's (1970) zonal-wind generation theory, and Welander's (1973) model of lateral stresses in the ocean. The principal derivations and conclusions of this and the related paper (Rhines, 1979) are believed to be new. Of the examples: (i) appeared in Rhines' 1977 paper; (ii) is the same problem treated by McWilliams (1976) yet the essential results (eqn. (20) et seq.) are new; (iii) is a restatement of well-known instability theorems in terms of Lagrangian diffusivity (yet with remarks on topographic effects that are not encompassed by these theorems); and (iv) is from a new calculation of the eddy-induced abyssal circulation given in detail by Rhines (1979).

The classical circulation

Benjamin Franklin (1786) made an interesting observation of the North Atlantic ocean circulation, in a note to accompany his and Capt. Folger's chart of the Gulf Stream: "By observing these directions and keeping between the Stream and the shoals, the passage from the Banks of Newfoundland to New York, Delaware, or Virginia may be considerably shortened; for so you will have the advantage of an eddy current, which moves contrary to the Gulph Stream ... Note, the Nantucket captains who are acquainted with this stream, make their voyages from England to Boston in ... 20 to 30 days." *

Franklin was aware of not only the Gulf Stream, but also the more subtle southwestward "eddy current" just to the north. Such countercurrents are often the tangible result of growing instabilities upon an intense jet. In addition, gentle drifts can be induced in the quieter parts of the ocean, by eddy fields that are distant from these regions (here, unlike Franklin, we take "eddies" to mean mesoscale turbulence or waves, with time scales greater than a day, and length scales of tens to hundreds of kilometers). In the former case, the time-averaged potential vorticity equation has a strong turbulent contribution in the vicinity of the induced flow; in the latter, it is locally laminar, yet is causally related to distant eddy fluxes of potential vorticity.

This suggests a crude division of the ocean (Fig. 1) into turbulent and laminar regions. A quantitative estimate, allowing the placement of the dividing surface, and discussion of the overlap of the two regions, will be given below (eqn. 13). The interaction between the "noisy" and quiet ocean is a complex part of this story, and so we begin with notes on the dynamics of a purely steady wind-driven circulation.

If the problem is steady and non-diffusive, with large space- and time-scale, the flow beneath the surface layer responds to the wind according to

$$\hat{\mathbf{u}} \cdot \nabla Q = 0, \qquad \hat{\mathbf{u}} \cdot \nabla \rho = 0 \tag{1}$$

This is the steady conservation of potential vorticity, $Q \equiv f\partial(\ln \rho)/\partial z$, and density, ρ, along streamlines of the flow; $\hat{\mathbf{u}}$ is the total velocity and f the Coriolis frequency. For simplicity we use the Boussinesq approximation, ignoring effects of compressibility. Approximating the density field as a series of layers of thickness h_i, the (vertically averaged) potential vorticity in each layer can be written $Q = f/h_i$. The relative vorticity of the mean flow does not appear, so long as the flow is sufficiently broad. Equations (1) define the direction of a steady flow to lie along the intersection of surfaces Q = constant and ρ = constant. These paths are the geostrophic contours.

Equations (1) are consistent with the familiar relative-vorticity equation

$$\beta v = f \frac{\partial w}{\partial z} \tag{2}$$

* A recent discovery of Franklin's earlier (1768) chart of the Stream is reported by Richardson (1979).

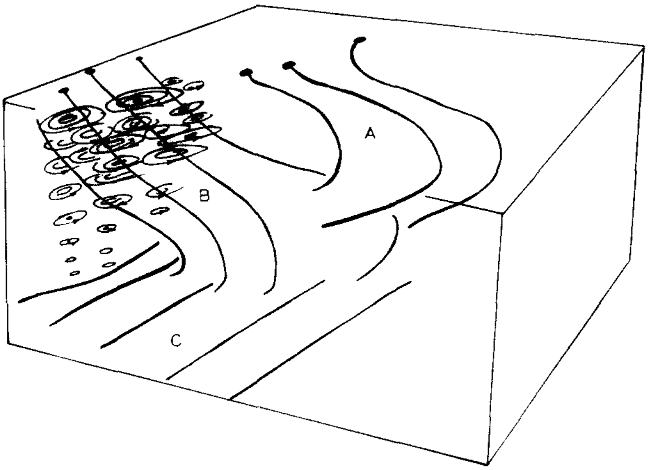

Fig. 1. Geostrophic/isopycnal contours in a mythical ocean. At the shallowest level, A, the flow originates where the contours intersect the surface mixed layer (heavy dots). At deeper levels, B, the contours encounter mesoscale eddies before surfacing. The circulation along these contours is driven by the eddies as well as surface layer convergence. At the deepest levels, C, the contours lie nearly east—west, and circulation must be driven across the geostrophic contours by eddies, if it is to occur.

where (u, v, w) are (east, north, vertical) velocities and (x, y, z) the corresponding β-plane coordinates ($\beta = \mathrm{d}f/\mathrm{d}y$). (1) has been used, for example, in non-diffusive thermocline models (e.g., Welander, 1959). Equation (2) is far more general than (1), however. For example, if eddy motion is present, eqns. (1) describe the time-averaged horizontal velocity, $\langle \mathbf{u} \rangle$, only when the turbulent flux of potential vorticity is negligible. However, when vertical stresses occur (but lateral eddy stresses are negligible), (2) is correct for the time-mean $\mathbf{u}$ and w fields. * This is because the time-average density equation can be written

$$\langle w \rangle = -\left[\nabla \cdot \langle \rho' \mathbf{u}' \rangle + \frac{\partial}{\partial z} \langle \rho' w' \rangle + \langle \mathbf{u} \rangle \cdot \nabla \langle \rho \rangle \right] \Big/ \frac{\partial \langle \rho \rangle}{\partial z}$$

where ∇ is now the horizontal gradient. Hence, $\langle w \rangle$ expresses both inclined steady flow along isopycnals and convergence of eddy density flux (eddy quantities being marked with primes). We will return later to the role of $\langle \rho' \mathbf{u}' \rangle$ as a vertical transmission of horizontal momentum by the eddies.

The depth-integrated flow has been of central interest in classical studies.

* This remark shows that Stommel and Schott's (1977) method of calculating mean velocity $\langle u, v, w \rangle$ from mean density fields alone, which uses (2), applies correctly even in the presence of strong "vertical" eddy stress effects (but not of strong lateral eddy stresses). They must require only that $\partial/\partial z \langle \rho' w' \rangle << \langle w \rangle \partial \langle \rho \rangle / \partial z$, for the "spiral" relation between $\langle w \rangle$ and current veering (with respect to depth) to hold.

It is this quantity that is the nearest to being locally and linearly related to the wind-stress curl exerted on the ocean surface. The vertical integral of (2) is

$$\beta h \bar{V} = f\mathbf{u}(z = -h) \cdot \nabla h + f\nabla \times (\tau/f)|_z \tag{3a}$$

or

$$h^2 \bar{\mathbf{U}} \cdot \nabla(f/h) = f(\mathbf{u}(z = -h) - \bar{\mathbf{U}}) \cdot \nabla h + f\nabla \times (\tau/f)|_z \tag{3b}$$

Here $\bar{\mathbf{U}} \equiv (\bar{U}, \bar{V}) = \int_{-h}^{-\delta} \mathbf{u}\, dz/h$, h is the ocean depth, δ is the Ekman surface-layer thickness, and τ the wind-stress. The bottom boundary layer has been neglected. If bottom velocities or bottom slopes are vanishingly small, (3a) gives the familiar Sverdrup balance for the total interior transport. Alternatively, if the flow is barotropic, $\mathbf{u} \equiv \bar{\mathbf{U}}$, and (3b) gives the modified Sverdrup equation relative to f/h contours. In any case except these two extremes, there is no simple relation between wind stress and total transport because of the dependence on the internal mode of flow, $\mathbf{u} - \bar{\mathbf{U}}$.

The problem is to determine that part of the circulation, $\mathbf{u} - \bar{\mathbf{U}}$, which is not so simply related to local wind-stress curl. In fact, it is a nonlocal problem in the horizontal sense. In absence of diffusion, this requires solving (1) with appropriate boundary conditions on the flow velocity and density. As we shall suggest below, the relevant boundary conditions occur where the geostrophic contours intersect the surface-layer, or eddy-containing regions. These intersection points are likely to be far away, in the horizontal, from the observation point. Figure 1 suggests the way in which the internal geostrophic contours reach from the quiescent interior to the surface or to intense eddy regions. We envisage the flow at depth as being traceable to distant points along the contours, where the higher-order regions are encountered. This dynamical picture is entirely consistent with the attempts (e.g. Montgomery, 1938) to trace kinematically the origins of intermediate-depth waters to points at the sea surface.

The above procedure is likely to describe the manner in which slight changes in boundary conditions (i.e., the imposed wind) affect the circulation. It cannot describe major changes, such as the spin-up of the circulation from rest, because the geostrophic contours are themselves determined by the flow. Thus, the nonlinear nature of this wind-driven circulation, even in absence of eddies and topography, arises from the dependence of the flow paths, as well as the domain of dependence of the solution, on the strength of the flow itself. As a result, for example, the depth of penetration of the wind-driven circulation (in addition to its strength) varies with the strength of the wind. It is instructive to show in detail that this is so.

Simply stated, unless it is highly concentrated in the vertical, a weak flow will tend to have its mean potential vorticity gradient, ∇Q, dominated by βy rather than $-f\nabla h_i/h_i$. This is because the slope of the density surfaces, which produces ∇h_i, is directly proportional to the current shear. If the geostrophic contours thus coincide with latitude circles, and if continental barriers are present, no laminar flow at all will be possible. For such flow would require both eastern and western boundary currents, and no mechanism is known which would allow eastern boundary currents to append to this kind of weak circulation.

The vertical structure of the circulation must, therefore, evolve with locally strong density tilts to allow the f/h_i (geostrophic) contours to run significantly north and south. It does so by confining the circulation to a small depth range. Specifically,

$$\frac{f\nabla\eta}{D} \geqslant \beta$$

where η is the height of an isopycnal surface and D the thickness (in z) of the region occupied by the circulation. Geostrophic thermal-wind balance is

$$N^2\nabla\eta \sim fu/D$$

where N is the buoyancy frequency. Combining these two statements, we require

$$D \leqslant \frac{f}{N}\sqrt{\frac{u}{\beta}}$$

This ensures the confinement of a weak wind-driven flow to the upper ocean (for $u = 1$ cm s^{-1}, $f/N = 0.02$, we find $D \sim 450$ m). In terms of the driving wind-stress this estimate is $D \leqslant ((f/N\beta)^2\nabla \times \tau|_z)^{1/3}$. The limit of vanishingly weak winds has the flow restricted to an infinitesimal layer beneath the surface. Vertical density and momentum diffusion may instead be invoked, to resolve the vertical structure.

2. DERIVATION

We given a compact derivation of the mean-field equation for Eulerian-induced flow. The full equation for the quasi-geostrophic potential vorticity is

$$\frac{\mathrm{D}q}{\mathrm{D}t} = F - \Delta \tag{4}$$

where

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\epsilon\frac{\partial\psi}{\partial z}\right) + f_0 + \beta y - \delta(z)\epsilon\frac{\partial\psi}{\partial z} + \delta(z+H)\left(\epsilon\frac{\partial\psi}{\partial z} - f_0h^*\right)$$

Here q is the potential vorticity appropriate to a continuously stratified fluid (which differs in dimensions from the form used in Section 1); ψ is the quasi-geostrophic stream function for the horizontal velocity, $\mathbf{u}$ (or, $(f\rho)^{-1}$ times the perturbation pressure); $\epsilon = f_0^2/N^2$, $f = f_0 + \beta y$, $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \partial(\psi,\cdot)/\partial(x,y)$; F is external forcing, confined to the upper surface layer; Δ is the dissipation; and, h^* is the depth perturbation ($h^*/H \leqslant R_0$). The boundary conditions are expressed by the delta function terms, including the effect of intersections of the isopycnal surfaces with the boundary either due to their flow-related tilt (Bretherton, 1966) or due to the tilt of the bottom. (4) is accurate to order Rossby number, $R_0 \equiv U/f_0L$, where U is a typical velocity, and L a typical lateral length scale.

Dividing (4) into eddy and mean quantities,

$$\frac{\partial Q}{\partial t} + \langle \mathbf{u} \rangle \cdot \nabla Q = -\nabla \cdot \langle q'\mathbf{u}' \rangle + \langle F \rangle - \langle \Delta \rangle \qquad \text{(mean)} \qquad (5a)$$

$$\frac{\partial q'}{\partial t} + \nabla \cdot (\mathbf{u}'q') - \nabla \cdot \langle \mathbf{u}'q' \rangle = -\mathbf{u}' \cdot \nabla Q - \langle \mathbf{u} \rangle \cdot \nabla q' + F' - \Delta' \qquad \text{(eddy)} \qquad (5b)$$

Here $q = Q(\mathbf{x}, z, t) + q'(\mathbf{x}, z, t)$, $u = \langle u(\mathbf{x}, z, t) \rangle + u'(\mathbf{x}, z, t)$ subject to some averaging procedure, $\langle \mathbf{u}' \rangle = \langle q' \rangle = 0$. The ∇ operator is horizontal and the flux forms $(\nabla \cdot q'\mathbf{u}')$ occur because $\mathbf{u}'$ is nearly horizontally non-divergent (to order R_0).

In energetic flow regions, (5a) describes how Eulerian-mean flow is driven across Q-contours by the turbulent transport of potential vorticity. An unsteady part of Q may arise to describe the adjustment of the mean flow to changes in average eddy intensity. Flow may also cross geostrophic contours readily in Ekman boundary layers or in intense lateral boundary currents. The eddy field is determined by (5b). The terms on the left-hand side, by themselves, represent unconstrained geostrophic turbulence. The right-hand terms add in the influence of the large-scale mean fields, either in the sense of Rossby-wave restoring effects or mean-flow instability.

Vorticity transport

The above notes suggest an examination of the average vorticity equation in order to diagnose the mean circulation. The near-conservation of q following the fluid motion, expressed in (4), suggests the introduction of some Lagrangian quantities. Using Taylor's (1915) ideas, we rewrite the turbulent stresses. The vorticity equation

$$\frac{\mathrm{D}(Q + q')}{\mathrm{D}t} = F - \Delta \qquad (6)$$

can be integrated formally, following the fluid, to give

$$\begin{aligned} q'(\mathbf{x}(t), t) &= q'(\mathbf{x}(0), 0) + Q(\mathbf{x}(0), 0) - Q(\mathbf{x}(t), t) + \int_0^t (F - \Delta)\,\mathrm{d}t' \\ &= q'(\mathbf{x}(0), 0) - \frac{\partial Q}{\partial x_j}(x_j(t) - x_j(0)) - t\frac{\partial Q}{\partial t} + \int_0^t (F - \Delta)\,\mathrm{d}t' + \mathrm{O}(\gamma) \\ &= -\frac{\partial Q}{\partial x_j}(x_j(t) - x_j^0) - t\frac{\partial Q}{\partial t} + \int_0^t (F - \Delta)\,\mathrm{d}t' + \mathrm{O}(\gamma) \end{aligned} \qquad (7)$$

Here, $\mathbf{x}(t)$ is the Lagrangian position of a fluid particle at time t. In the second line we make the approximation that Q is slowly varying in space and time; the typical error is $O(\gamma)$, where $\gamma = |\mathbf{x} - \mathbf{x}(0)|/L_m$, the ratio of particle displacement to the scale, L_m, of Q. (L_m is the lesser of the lateral scales of variation of ∇Q and κ.) In the third line we introduce the "rest latitude", $\mathbf{x}^0$, defined by

$$q'(\mathbf{x}(0), 0) = (x_j^0 - x_j(0)) \frac{\partial Q}{\partial x_j}$$

Thus, we reference each particle to a geostrophic contour where the perturbation potential vorticity would vanish if the particle were to move there conservatively. This artifice removes the explicit dependence on initial conditions. The rest latitude coincides with the point of origin, $\mathbf{x}(0)$, if the fluid is initially at rest and not subject to external stress curl or dissipation $(F - \Delta)$.

In transient problems, the initial conditions are important to the sense of the induced flow. When fluid is driven away from its rest latitudes, say, by the arrival of a wave-like pulse of energy, we shall see below that the eddy stresses on the mean have a pseudo-westward sense. Yet, with declining energy levels due to the departure of eddy energy from the region, fluid tends to fall back to its rest latitudes. Then, the eddy stresses have a pseudo-eastward sense. (Here we use "pseudo-westward" to mean leftward, facing large values of Q).

For continually forced problems long after initiation, however, the dissipation of q will erase all memory of initial conditions, and this referencing problem will disappear. The appropriate derivation for a Rayleigh-damping model of the dissipation is given by Rhines (1979). In that case the fluid exhibits a dependence on its prior trajectory, only for one "spin-up" time.

The object is to rewrite the eddy flux of q. This becomes

$$\langle q'u_i'\rangle = -\langle u_i'(\mathbf{x}, t)(x_j(t) - x_j^0)\rangle \frac{\partial Q}{\partial x_j} + \langle u_i'(\mathbf{x}, t) \int_0^t (F - \Delta)\, dt'\rangle + O(\gamma) \tag{8}$$

$$= -\kappa_{ij} \frac{\partial Q}{\partial x_j} + \langle u_i' \int_0^t (F - \Delta)\, dt'\rangle$$

where $\kappa_{ij} = \langle u_i'(\mathbf{x}, t)(x_j(t) - x_j^0)\rangle$, $\mathbf{x} = (x, y) = (x_1, x_2)$. κ_{ij} is essentially the Lagrangian diffusivity of fluid parcels, for $\gamma << 1$; discussion of its properties is given by Rhines (1979).

The mean-circulation equation

We wish to have a mean-field equation in which eddy effects are as transparent as possible. The form of (8), and its dissipative version (Rhines, 1979), depends on the turbulent dispersion in the eddy field (via κ_{ij}), which is a tangible kinematic property. If we now direct attention below the surface layers (taking $F = 0$) and suppose that dissipation is moderately weak (the dissipation time for q exceeding L/U, the eddy time scale), eqn. (5a) becomes, to order γ,

$$\frac{\partial Q}{\partial t} + \langle u_j \rangle \frac{\partial Q}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\kappa_{ij}\frac{\partial Q}{\partial x_j}\right) \tag{9a}$$

This equation has two useful special cases. First, for an x-averaged atmosphere (a zonally homogeneous flow in which $Q = Q(y, z)$), (9a) becomes

$$\frac{\partial}{\partial t}\hat{\Delta}\langle\psi\rangle = \frac{\partial}{\partial y}\left(\kappa_{yy}\frac{\partial Q}{\partial y}\right) \tag{9b}$$

where $\hat{\Delta} = \partial^2/\partial y^2 + \partial/\partial z(\epsilon\partial/\partial z)$ + boundary conditions, $\hat{j} \times \nabla\psi = \mathbf{u}$.

For a statistically stationary ocean, (9a) is

$$\langle u_j \rangle \frac{\partial Q}{\partial x_j} \equiv \frac{\partial(\langle\psi\rangle, Q)}{\partial(x, y)} = \frac{\partial}{\partial x_i}\left(\kappa_{ij}\frac{\partial Q}{\partial x_j}\right) \tag{9c}$$

These are exact consequences of the quasi-geostrophic equations, subject to the two-scale approximation. The restriction that γ be small is not very important if the "eddies" are waves of small steepness; the scale of Q may then be as small as that of the waves themselves.

Equation (9) is a nonlinear relation determining the Eulerian mean motion and density fields (through $Q(\mathbf{x}, t)$), given knowledge of the Lagrangian diffusivity, κ, for a particular wave or turbulence field. Many interesting problems are much simpler than this, however: in particular those in which the eddy-induced flow is so weak that Q is only slightly changed from its initial form, say $Q^{(0)}(x)$. This may occur when β, bottom topography, or the pre-existing flow dominates Q. Let

$$q = Q(\mathbf{x}, t) + q'(\mathbf{x}, t), \qquad \langle q' \rangle = 0\,.$$

$$Q(\mathbf{x}, t) = Q^{(0)}(\mathbf{x}) + Q^{(1)}(\mathbf{x}, t)\,, \qquad \langle Q^{(1)} \rangle \neq 0\,.$$

where $Q^{(1)} << Q^{(0)}$. In terms of the known field $Q^{(0)}$, eqn. (9a) becomes

$$\frac{\partial Q^{(1)}}{\partial t} + \langle u_j^{(1)} \rangle \frac{\partial Q^{(0)}}{\partial x_j} + \langle u_j^{(0)} \rangle \frac{\partial Q^{(1)}}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\kappa_{ij}\frac{\partial Q^{(0)}}{\partial x_j}\right) \tag{9d}$$

which is a linear equation for the induced circulation. For flows that eventually become statistically steady and have the lateral scale of $Q^{(0)}$ large compared to the Rossby deformation radius, we find

$$\langle u_j^{(1)} \rangle \frac{\partial Q^{(0)}}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\kappa_{ij}\frac{\partial Q^{(0)}}{\partial x_j}\right), \tag{10}$$

which might be called the turbulent Sverdrup balance. Henceforth we consider only those flows in which (9d) or (10) apply (and drop superscripts where there is no chance of confusion).

Equations (9) determine the mean Eulerian motion for a given eddy or wave field. The eddy field must be found analytically, numerically, measured in the field, or otherwise estimated. Incorrect eddy fields, which are not solutions of eqn. (5b), will of course lead to incorrect mean flows if calculated with (9). But, the simple dependence of (9) on the particle-migration behavior encourages its use as an intuitive and diagnostic tool. The role of dissipation (for instance by the enstrophy cascade), external forcing, and the Lagrangian description of both mean and eddy circulations are treated elsewhere (Rhines, 1977, 1979).

This approach is a useful alternative to analyses of the momentum or energy budgets in an eddy-containing circulation. There is much uncertainty, for example, about both the sign and magnitude of the eddy coefficient for momentum (which is the ratio of momentum flux to mean lateral gradient of velocity); Harrison (1978), for example, finds the eddy coefficients for momentum, heat and relative vorticity all to be statistically noisy in his general circulation model. The very feature that controls the present analysis, the large-scale gradient of Q, also makes wave propagation pervasive. This argues particularly against the existence of simple momentum-mixing coefficients, for there is no causal relation between local eddies and local shear. Any closure theory which is to supply the κ_{ij} field for eqns. (9) must account for the production of eddies (say, by baroclinic and barotropic instability) and for their redistribution by waves and concentration at a western boundary. Examples (i) and (ii) in Section 3 are relevant to this propagation of the eddies and hence of Lagrangian diffusivity.

Perhaps the strongest virtue of eqns. (9), however, is that they incorporate more explicitly the vertical transport of horizontal momentum. In direct analyses of the momentum equations, these vertical inviscid eddy transports are implicit in the "Coriolis torque", $f\hat{\mathbf{j}} \times \langle \mathbf{u} \rangle$, but their nature is not easily anticipated. The same eddy effects appear in the average of eqn. (2) through $f\partial\langle w \rangle/\partial z$.

The solution of an "ocean" problem, (9c), involves integration along characteristics of the equation, which are the geostrophic contours, Q = constant. Equation (9c) becomes

$$\frac{\partial \psi}{\partial p} = \nabla \cdot (\boldsymbol{\kappa} \nabla Q) \tag{10}$$

where p is arc length along a characteristic, rescaled according to

$$\frac{\partial |\mathbf{x}|}{\partial p} = |\nabla Q| \tag{11}$$

The sense of integration is pseudo-westward, beginning pseudo-eastward of any significant eddy forcing. This particular direction of influence is most clearly established as the limiting Rossby-wave field of an oscillatory forcing effect, as the frequency vanishes (e.g., Lighthill, 1967). Various vertical modes of the waves are involved; small vertical-scale disturbances involve slow baroclinic modes, the depth-averaged disturbance produces the barotropic mode, and deep disturbances produce fast baroclinic modes if bottom slopes are present. The group velocity of the former remains finite in this limit, is of order $\beta N^2 \hat{H}^2/f \sim 1$–$3$ cm s^{-1} (where $\hat{H}$ is the height scale of the motion), and is directed along the geostrophic contour (to the left, facing along the gradient of Q). More precisely, the path of the developing signal lies along the vertically averaged Q-contour, with an averaging scale $\hat{H}$. The barotropic mode, excited by the depth-averaged forcing, is far more efficient, propagating at higher group velocities $\sim\beta L^2$ (often hundreds of cm s^{-1}) along f/h contours.

The ocean circulation is thus multiple nonlinear (even in a region where the Sverdrup relation holds): turbulent fluxes excite a part of the flow; the development of the flow occurs along Q-contours which may themselves be flow-dependent, and the currents with small height-scale penetrate so slowly along those contours ($\leqslant 3$ cm s^{-1} at mid-latitudes) that the mean currents themselves can greatly alter their propagation speed. The mathematical problem for the circulation above the thermocline thus involves the propagation from boundary points which are horizontally distant as suggested by Fig. 1. The curves with Q = constant, ρ = constant, may originate at a surface boundary layer, a coastal boundary current, or indeed may intersect a region of intense turbulent vorticity flux. The relative strengths of the wind-driving and eddy effects will be discussed below. The degree of control exerted by geostrophic contours, suggested here to be strong, remains to be verified. To the extent that it is realized, in spite of dissipation, it severely limits the "mixing along isopycnal surfaces" so often invoked in studies of geochemical tracers.

Both the deep ocean flows and the barotropic mode are far more strongly controlled than the baroclinic modes relevant to the circulation in and above the main thermocline. For weak currents the appropriate Q-contours are then h = constant for deep flows and f/h = constant for the barotropic mode (h is the total depth). The relevant low-frequency wave modes that establish the currents have much faster propagation. Most significantly, when the Q-geometry is dominated by the topography and β, the wave-guide provided by the geostrophic contours is known a priori.

The nature of κ

We now want to establish the key to the predictive ability of this formulation; that if $\gamma << 1$ *the eddy transport of potential vorticity must have a component down the mean gradient, provided that friction acts to dissipate the potential enstrophy*. Here we are describing the statistically steady case, below the surface layers, where direct external forcing occurs. In transient problems, gradients of Q can cause κ to be negative in regions where the energy level is decreasing with time. In this case, the Rossby-wave restoring effects bring particles back toward their "rest latitudes" (recall that the diffusivity is here referenced to those geostrophic contours upon which the perturbation q' would vanish, rather than to the point of particle release or observation). In such regions of outward-wave radiation, where κ is negative, Q evolves into a concentrated jet, for the vorticity diffusion process is running backward.

For the statistically steady case of the ocean interior, however, the initial reference points are forgotten. The enstrophy balance equation obtained by multiplying 5(b) by q'

$$\frac{1}{2}\frac{\partial}{\partial t}\langle q'^2\rangle + \tfrac{1}{2}\nabla\cdot\langle q'^2\mathbf{u}\rangle = -\langle q'\mathbf{u}'\rangle\cdot\nabla Q - \langle\Delta' q'\rangle + \langle F' q'\rangle$$

becomes

$$\langle q'\mathbf{u}'\rangle\cdot\nabla Q = -\langle\Delta' q'\rangle \tag{12}$$

We have invoked the two-scale approximation, $\gamma << 1$, and this means that the influx of perturbation enstrophy by the turbulent velocity is small. If our dissipation mechanism is a Rayleigh damping of the full potential vorticity (that is, linear damping in both velocity- and perturbation-density equations) $\Delta' = +\lambda q'$, and the assertion is proved; both sides of (12) are then negative. If instead, the form of the damping is $\Delta' = -\nabla^2 q'$ or $\Delta' = +\nabla^4 q'$, then $\langle\Delta' q'\rangle > 0$ after spatial averaging, although it may be locally negative. It is worth noting that counter-examples can be found, for instance in models where the friction acts to dissipate relative vorticity with no attendant dissipation of density perturbations. In the case of turbulence in a state of minimum potential enstrophy (Bretherton and Haidvogel, 1976), friction can increase the potential enstrophy. Other counter-examples involve the viscous flux from energetic regions into nearby quiet fluid, but these are unlikely to occur in a problem with slowly varying spatial statistics. * In balance, it appears that exceptions will be rare, and that the effective diffusivity in stationary interior regions will usually be positive definite (in the sense that the trace $S_{ii}(\partial Q/\partial x_i)^2$ is positive definite, where S_{ij} is the diagonalized symmetric part of κ_{ij}).

The corollary of this result is that *the average stress exerted by eddies on the Eulerian mean fluid has a pseudo-westward component* (i.e., to the left, looking along the gradient of Q) wherever friction acts to dissipate the potential enstrophy, in a stationary, interior eddy field.

The magnitude of κ_{ij} is needed to complete the mean-field problem. Closure theories based on supercriticality of the local mean flow will provide a beginning, but non-local propagation and western-boundary intensification are essential elements. In Holland and Rhines (1979), for example, the quiet far-field of a particular numerical ocean simulation is very stable in the mean, yet incoming radiation of Rossby waves provides eddy transport of q' suffi-

* The restriction $\gamma << 1$ is based on the particle displacement (over one dissipation interval for the potential vorticity), divided by the scale of the mean field. The strength of the mean flow comes into this, if the mean particle trajectories are curved. The rapid transit of particles about these paths may force γ to be large, even though the eddy displacements are small. Such a case is the upper level ocean circulation in the model described by Holland and Rhines (1979). Clearly, the present formulation is best suited to strong eddies with weak mean flow, unless the mean is predominantly a parallel flow.

cient to upset the integrated Sverdrup balance.

Since eddy-energy levels are frequently known from observation, the mapping from energy density to Lagrangian diffusivity is of great interest. A crude estimate for eddies of length-scale L_ρ, the Rossby radius, is $\kappa \sim \frac{1}{2} u' L_\rho$. The numerical factor is based on rough calibration against simulations and ocean observations.

The strength of the induced circulation. Observations are also available to describe the Lagrangian diffusivity directly from both drifters at the sea surface and neutrally buoyant SOFAR floats at depth (e.g., Freeland et al., 1975). The diagonal components of κ_{ij} typically take on values of 5×10^6 cm^2 s^{-1} or less in the great, quiet regions of the deep sea. At intermediate energy levels, for example at 28°N, 70°W where MODE-73 was carried out, κ rises to about 1.0×10^7 cm^2 s^{-1} in the deep water and perhaps 3.0×10^7 cm^2 s^{-1} above the thermocline (at 500 m depth). In the near-field of the Gulf Stream, numerical models suggest values rising to 5×10^7 in the deep water and occasionally greater than 10^8 cm^2 s^{-1} above the thermocline. In such regions of rapidly varying mean properties, however, the notion of a quasi-homogeneous Taylor diffusivity must be critically examined. We shall report on the relevant experiments shortly.

A value of 1.0×10^7 cm^2 s^{-1}, appropriate to a rather quiet part of the ocean, is nevertheless equivalent to a wind-stress of 1.0 dyne cm^{-2} acting over a depth of 2.5 km (in a region where $|\nabla Q| = 2\beta$). The stronger diffusivities found in the upper ocean and near the major currents will often overwhelm the direct effect of the wind.

Assuming the contours to be open (i.e., they extend indefinitely in the horizontal) or "blocked", rather than closed, we scale analyze (9c) and find typical Eulerian-mean circulation values, U_e,

$$U_e \sim \kappa / L_m \tag{13}$$

where L_m is the lesser of the scales over which κ and ∇Q vary in the horizontal. The mean circulation is thus independent, in strength, of the Q field, except as it affects L_m. The ratio, δ, of this induced circulation to that of the classical laminar circulation, U_0, is

$$\delta \equiv \frac{U_e}{U_0} \sim \frac{\kappa}{U_0 L_m} \sim \frac{1}{2} \frac{U'}{U_0} \frac{L_\rho}{L_m} \sim \frac{1}{2} \left(\frac{\eta'}{|\nabla \overline{\eta}| L_m} \right) \frac{H_e}{H_m} \tag{14}$$

We have used here the estimate $\kappa \sim U' L_\rho / 2$, and the thermal-wind equation. The final estimate is tailored to thermocline eddies and their effect on the upper-level circulation. η' and $\overline{\eta}$ are eddy and mean thermocline displacements, and H_e and H_m are the respective height scales of the eddies and mean flow. The typically large values of U'/U_0 may be reduced in effect by small values of L_ρ / L_m.

Equation (13) shows that the eddy-induced circulation can be strong, but it cannot be both strong and have large lateral scale for moderate values of the Taylor diffusivity. For example, with $\kappa = 1.0 \times 10^7$ cm^2 s^{-1} (as in the moderately quiet MODE region), $L_m = 100$ km and $U_0 = 1$ cm s^{-1}, we find $\delta = 1$; the classical and eddy-components are comparable. The form of estimate (13) suggests that, because of the smallness of any strong eddy-induced gyre, the classical mean, U_0, should re-emerge after spatial averaging. An appropriate smoothing scale which keeps δ below 0.2, is $5\kappa/U_0$, or equivalently, about $2.5\eta' H_e/(L_m|\nabla\bar{\eta}|H_m)$. This recommended averaging distance, over which laminar dynamics may apply, is 500 km for the above values of κ and U_0.

Oceans vs. atmospheres

The geometry of the mean potential vorticity field, Q, is central to the problem of the mean circulation, whether or not eddies are important. There are some regions (like the westward-flowing subtropical North Atlantic, above the main thermocline) where density tilts nearly negate β, leaving $\nabla Q \simeq 0$. The fluid would seem to be free to wander about such a region with neither the restraint nor the propulsion afforded by $\boldsymbol{\kappa} \cdot \nabla Q$. More likely, the uniform potential-vorticity regions may represent a natural limit-point, the approach to which is governed by Q-gradients in the usual fashion.

Flows with significant variations in Q may be categorized according to whether the Q-contours are closed, open, or blocked. The strength of the induced circulations varies so greatly between these cases that a detailed review is useful here. Let τ be the driving stress, either that of the wind or that due to the average potential-vorticity flux. The response of a barotropic fluid with contours which close on themselves, by (9b), has a mean velocity along contours, $hU \sim \tau t$, or $hU \sim \tau/\lambda$ if the flow is limited by linear bottom drag with coefficient λ. In terms of the Lagrangian diffusivity, $hU \sim \kappa\nabla Q/\lambda$. When contours are blocked by "meridional" coasts, the fluid can respond with far less energy; with weak friction (9c) applies, and $hU \sim \tau/\beta L_m$ (supposing $\nabla Q \sim \beta$). The ratio of these two circulation velocities, $\lambda/\beta L_m$, is typically small. This suggests already that weak rectification mechanisms, where the perturbations are nearly linear waves, will yield mean circulations in blocked, "oceanic" geometry which are so small that they are of little practical interest (whereas, in closed "atmospheric" geometry, their effects may be considerable). The distinction is even greater for Lagrangian-mean flow: details are given in Section 4, below. These simple considerations lead us to center attention on those parts of the ocean where an active, turbulent enstrophy cascade occurs.

"Open" geostrophic contours, which extend indefinitely in space without closing on themselves or being blocked, are the transitional case. Imagine a localized stress field, with a response that has not reached far enough pseudo-westward to sense whether the contours terminate at a wall, or reconnect. Typically, within this westward extending wake, the mean velocity obeys $hU \sim \tau/\beta L$, just the Sverdrup value. If the contours are closed, with circumference D, the wake or "plume" of flow reaches back to its source with group speed $c_g \sim \beta L^2$. The ever-accelerating solution for closed contours corresponds to the successive arrivals of many such "Sverdrup" signals, each carrying currents of speed $\tau/(\beta L h)$. Hence $hU \sim n\tau/\beta L$ where n is the number of circuits about the re-entrant contour. Now $n \sim c_g t/D$ and hence $hU \sim \langle\tau\rangle t$, where $\langle\tau\rangle = \tau L/D$; hence this verbal summation agrees with the direct solution for U. One effect of a distant blockage of the contours is immediately felt, however; the net westward momentum of the circulation vanishes in any fully closed geometry, while it grows linearly with time in open geometry. This is why pseudomomentum and impulse turn out to be more general second-order mean quantities in wave-fields (e.g., Andrews and McIntyre, 1977).

Forces

The eddy transport of potential vorticity has a clear physical significance, as a stress-like quantity. Taking, for clarity, a multi-layer density field, we write the inviscid, quasi-geostrophic momentum equation as

$$\frac{\partial (h_i \mathbf{u})}{\partial t} = -\hat{\mathbf{j}} \times h_i q \mathbf{u} - H_i \nabla_{\mathrm{H}} (p + \tfrac{1}{2} |\mathbf{u}|^2) + \mathrm{O}(R_0) \tag{15}$$

$q = H_i(\zeta + f)/h_i$, $\zeta \equiv \nabla \times \mathbf{u}|_z$, $h_i \equiv H_i + h_i'$, $h_i'/H_i \sim R_0$. $\hat{\mathbf{j}}$ is a vertical unit vector, ∇_{H} the horizontal gradient, and q the potential vorticity in an isopycnal layer of varying thickness, $h_i(\mathbf{x}, t)$. Note that $h_i\mathbf{u} = H_i\mathbf{u}$ to the order of accuracy of the equation.

In eqn. (15), the Coriolis and "vortex" forces have been combined. After Eulerian averaging of (15), the net lateral transport of potential vorticity within the layer, $\langle h_i q \mathbf{u}\rangle$, gives the sole rotational stress exerted by the eddies (acting at right angles). It is Eulerian in the horizontal sense (i.e., acting on the "mean" fluid occupying a fixed observation point), yet is Lagrangian in the vertical sense (since the formulation treats fluid contained between moving density surfaces). The Bernoulli term (right-most term of (15)) has no effect upon the vorticity balance (curl of (15)), nor upon the zonal average of (15) in a channel or an "atmosphere". The horizontal velocity, $\mathbf{u}$, includes an ageostrophic, $\mathrm{O}(R_0)$, part to account correctly for the Coriolis force, $\hat{\mathbf{j}} f \times \mathbf{u}$.

Let us write down, for illustration, the component parts associated with the potential vorticity flux in a zonally homogeneous flow. Take the x-average for (15), and divide into mean and perturbation fluxes using:

$$q = H_i \frac{\zeta + f}{h_i} = f + \zeta - \frac{f h_i'}{H_i} + \mathrm{O}(R_0)$$

$$q = Q + q'; \overline{q'} \equiv 0,$$

$$h_i = H_i + h_i'; \overline{h'} \equiv 0.$$

This yields

$$\frac{\partial \overline{h_i u}}{\partial t} = \overline{h_i q v} \tag{16a}$$

$$= f\overline{h_i v} + H_i \overline{q'v'} \tag{16b}$$

$$= f\overline{h_i v} + H_i \overline{\zeta'v'} - f\overline{h_i'v'} \tag{16c}$$

$$= f\overline{h_i v} - H_i \frac{\partial}{\partial y} \overline{u'v'} + \overline{\frac{1}{\rho} p \frac{\partial h_i'}{\partial x}} \tag{16d}$$

and, using (8),

$$= f\overline{h_i v} - H_i \kappa_{yy} \frac{\partial Q}{\partial y} \tag{16e}$$

The first RHS term in (16b)–(16e) is the Coriolis force on the net meridional mass flux within the isopycnal layer. The remaining terms are direct eddy contributions given, respectively, as the sum of fluxes of relative vorticity and the "vortex-length" contribution to q', in (16c); as the sum of lateral Reynolds stress divergence and the inviscid pressure drag, $p\partial h_i/\partial x$, exerted from layers above and below, in (16d), and as the down-gradient flux $\kappa_{yy}\partial Q/\partial y$ due to the mean gradient of Q, (16e).

The important pressure drag effect appears, for example, in Green's (1970) paper as the vertical gradient of the eddy density flux. Its dynamical nature was clarified by Bretherton's (1971) general Lagrangian reformulation of wave stresses, which appear as inviscid pressure forces exerted across convoluted, marked fluid surfaces. Measurements of eddy heat flux thus give us dynamical information about the downward transmission of horizontal momentum. If we consider the deepest layer of this fluid, the pressure drag includes an important contribution from the bottom topography * (see Bretherton and Karweit, 1975).

It is more usual in meteorology to write (16) as

$$\frac{\partial \overline{h_i u}}{\partial t} = H_i f\bar{v} - H_i \frac{\partial}{\partial y} \overline{u'v'} \tag{16f}$$

(combining the first and third RHS terms of (16c)). The value of the present form is that the eddy and mean effects are more clearly separated. The corresponding vorticity equations (9) show that $f\overline{h_i v}$ is calculable as a part of the mean response to eddy forcing (whereas $f\bar{v}$ contains both eddy and mean quantities). This mean response occurs in the vortex length, $\langle f/h_i \rangle$; recall the discussion of the analogous effects occurring in $f\partial\langle w\rangle/\partial z$, eqn. (2).

We have emphasized the vertical transport by inviscid pressure-drag, because it is a strong effect in both oceans and atmospheres [$u'w'$-Reynolds stresses being negligible, $O(R_0)$]. Comparison of (16d) and (16e) suggests its simple physical characterization as a generalized Rossby wave drag (Fig. 2). Only in the presence of a mean potential vorticity gradient can the moving hills and valleys of the isopycnal surfaces exert a wave drag on the fluid above or below. This effect dominates example (iv) in Section 3.

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* Rigid topography at the base of the lowest layer acts in the same manner as the delta-function topographic contributions to the continuous form of q, in the defining eqn. (4).

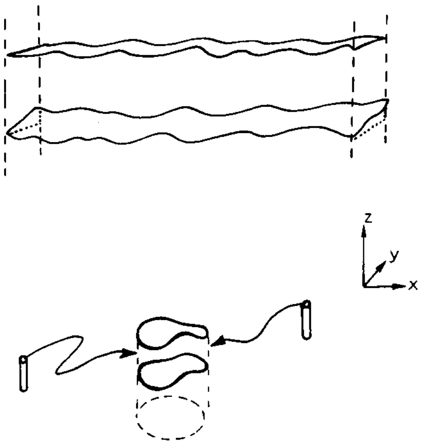


Fig. 2. Control volumes for calculations of average atmospheric (above) and oceanic (below) circulation. In the zonally oriented atmospheric region, the average x-momentum contained between fixed latitudes and moving isopycnal surfaces (solid curves) is altered by both lateral and vertical momentum transport. The vertical transport by inviscid pressure drag requires a mean potential vorticity gradient (dotted triangles) to be effective. In the 'oceanic' control volume, if the field is statistically steady, the potential vorticity transported into the region by passing fluid parcels (mean plus turbulent contributions) must vanish.

The vertically integrated momentum balance, of course, only feels these internal, vertically transmitted stresses if a net stress is exerted by the seafloor or surface. Otherwise, the eddy flux of relative vorticity is dominant, corresponding to dominance of the lateral Reynolds stresses in the momentum balance, as in Green's (1970) and Welander's (1973) papers. Such studies of the vertically integrated circulation, however, are not dependent only on the vertically averaged eddy fields; the vertical structure of κ_{ij} and Q must also be known.

3. FOUR EXAMPLES

The analysis of eddy potential vorticity transport can clarify a flow problem whose solution is already known, or it can give predictions of flows entirely unsuspected in nature. We give here, in some detail, examples of both kinds. They illustrate the use of the formulation in terms of Lagrangian diffusivity even for wave problems. The first example, forced motion in a homogeneous-fluid, β-plane channel has a linear regime, in which both the waves and rectified flow are predictable in detail, and a turbulent regime which is qualitatively predictable. Three different kinds of "mean circulation" are calculated. The second example, also on a barotropic β-plane, is a Rossby wave packet. This is an explicit calculation of induced mean motion which demonstrates the diffusivity of a wave field, and the pseudo-westward propagation of the induced field. It also demonstrates the great restrictions placed on the Lagrangian mean flow in a non-cascading eddy field. The third example, a familiar one, shows the value of the diffusivity formulation for the case of zonal-flow instability and its subsequent rectification. The fourth example sketches the vorticity-transport in a numerically simulated deep ocean beneath a shallow, intensely variable wind-driven circulation. The four-gyre abyssal circulation is predicted from the (known) upper-level mean flow and known pattern of κ.

(i): Zonal-flow induction in a barotropic β-plane channel

This "atmospheric" example is by now well known but is so straightforward that it is worth recounting. It is discussed by Kuo (1951) and Dickinson (1969) for the case of linearized perturbations, and by Rhines (1977) for turbulence. For a single homogeneous layer of fluid with depth-independent velocity, (9b) can be integrated in y, giving

$$\frac{\partial\langle u\rangle}{\partial t} = -\beta\kappa_{yy} \text{ or } \langle u\rangle - \langle u^0\rangle = -\tfrac{1}{2}\,\beta\langle(y-y^0)^2\rangle\,, \qquad (17)$$

$$\kappa_{yy} \equiv \langle v(y-y^0)\rangle = \mathrm{d}/\mathrm{d}t\,\langle\tfrac{1}{2}\,(y-y^0)^2\rangle$$

(The second relation uses the large-scale nature assumed for κ.) It holds, providing that no external forces have acted on the particles of the ensemble and that dissipation is slight. The formula guarantees that, if fluid particles are at rest at $t = 0$ and are set into motion by a disturbance arriving from a distance, they systematically transport potential vorticity down its mean gradient. This must be so because any movement of fluid across a control latitude, y^0, replaces high-latitude water with low-latitude water, reducing the potential vorticity north of y^0. The planetary vorticity there is unchanged ($= f_0 + \beta y$), so the relative vorticity decreases. Integrating the relative vorticity over the entire fluid north of y^0 leads to a westward Eulerian circulation along $y = y^0$ (given vanishing of the disturbance at indefinitely large y).

Eddy energy does not come from nowhere and, in its source region, the induced flow is different. Imagine that the eddies are generated in a narrow band of latitude by a wave maker. If the external control is contrived to exert no time-average force on the fluid, the general nature of the mean circulation in this region is determined as well. For, $\int_{-\infty}^{\infty}\langle u\rangle\,\mathrm{d}y = 0$ always; an ever strengthening eastward jet must occupy the region of the forcing. Its total momentum is just $\tfrac{1}{2}\,\beta\int\langle(y-y^0)^2\rangle\,\mathrm{d}y$, with the integral taken over all "free" latitudes, for which (17) applies. The simplicity of this circulation is slightly deceptive, for the strong westward influence causes the circulation to be set up unevenly, just as described above. The formulas apply whether or not the wavemaker and flow are homogeneous in x.

Solutions in the special case of linearized motion (e.g., Uryu, 1975; Rhines, 1977) show the exact patterns of q-flux (Fig. 3), in that case, associated with barotropic Rossby waves. The rectification is consistent with the "herring-bone" pattern of wave crests (with $u'v' < 0$ to the north of the source, $u'v' > 0$ to the south), directing eastward momentum at the source, while westward momentum is carried away from it. The wavemaker may absorb that eastward momentum by exerting a new westward force; if not, the eastward jet will appear.

The particle motion in this channel has a mean westward component, $\langle u_l \rangle$, identical to the Eulerian mean, $\langle u \rangle$, to order γ (based on estimates given below, of the turbulent Stokes drift). To emphasize the care needed in defining "mean circulation", we point out the apparently paradoxical behavior of a third such quantity, the familiar "Kelvin" circulation, $\Gamma = \oint \mathbf{u} \cdot \mathrm{d}\mathbf{l}$. $\mathcal{C}$ is a contour that moves with the fluid which initially lay about a latitude circle. To calculate Γ, we integrate the barotropic potential vorticity equation. If $\mathcal{C}$ is not distorted enough to fold over and intersect a north—south meridian twice, then the answer is simple:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \iint\limits_{S} \nabla \times \mathbf{u}|_z \, \mathrm{d}x \, \mathrm{d}y = -\frac{\mathrm{d}}{\mathrm{d}t} \iint\limits_{S} \beta y \, \mathrm{d}x \, \mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t} \beta \int \tfrac{1}{2} y^2 \, \mathrm{d}x$$

or

$$\Gamma - \Gamma|_{t=0} = \beta \int \tfrac{1}{2} y^2 \, \mathrm{d}x \tag{18}$$

where S is the fluid region north of $\mathcal{C}$, dl is a line element of $\mathcal{C}$, and y is evaluated on $\mathcal{C}$, in the last two expressions. This kind of circulation is thus equal and opposite to $\langle \mathrm{u} \rangle$ or $\langle \mathrm{u}_l \rangle$, so long as $\mathcal{C}$ does not become multi-valued in y. In words, (18) says simply that any movement of $\mathcal{C}$, off its original latitude circle, will cause less planetary vorticity to lie north of $\mathcal{C}$ (since f increases with y). Hence, the relative vorticity north of $\mathcal{C}$ must increase. There is no contradiction with (17) in spite of the opposing signs, because Γ does not

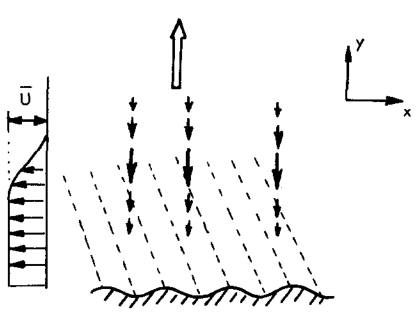


Fig. 3. Mean-flow induction by Rossby waves, generated by an infinite (in x) corrugated wall moving westward. Wave crests (dashed lines), potential vorticity flux (heavy arrows), group velocity (double arrow). The zonal mean Eulerian flow is shown; its amplitude is $\langle \beta(y - y^0)^2/2 \rangle$. If the fields are slowly varying, this is equal to the Lagrangian mean flow.

represent either the time-average motion of any fluid particles or the time-average motion at a fixed point.

(ii): Mean motions induced by a Rossby-wave packet

In any but the truly homogeneous (in x) problems, the notions of tilted troughs acting to separate the zonal momentum is incomplete. To see this, we consider the exact theory for rectification due to a barotropic wave packet. An amplitude expansion gives

$$\frac{\partial}{\partial t}\nabla^2\psi^{(1)} + \beta\frac{\partial\psi^{(1)}}{\partial x} = 0 \tag{19a}$$

$$\frac{\partial}{\partial t}\nabla^2\psi^{(2)} + \beta\frac{\partial\psi^{(2)}}{\partial x} = -J(\psi^{(1)}, \nabla^2\psi^{(1)}) = -\nabla\cdot q'\mathbf{u}' \tag{19b}$$

$$\psi = R_0\Psi^{(1)} + R_0^2\Psi^{(2)} + \ldots.$$

We are interested only in the slowly varying part of $\psi^{(2)}$. If $\psi^{(1)} = A(x, t)$-$e^{i\mathbf{k}\cdot x - \omega t}$ where A is a slowly varying function, the magnitude of the average vorticity flux is simply, from (19b),

$$|\langle q'\mathbf{u}'\rangle| = |\nabla\epsilon| \tag{20}$$

where ϵ is the kinetic energy density, $\frac{1}{2}\langle|\nabla\psi^{(1)}|^2\rangle$. The manipulations are given in the appendix to this paper.

The direction of $\langle q'\mathbf{u}'\rangle$ is such that the average eddy stress, $\hat{\mathbf{j}}\times\langle q'\mathbf{u}'\rangle$ is found by reflecting $\nabla\epsilon$ about the wavevector $\mathbf{k}$. This result depends only on the form of the nonlinear terms and not on β-plane dynamics. The vorticity flux, plotted in Fig. 4, always has a component down the mean gradient (i.e., southward), at the leading face of the packet. At the trailing edge, where the perturbation amplitude is falling with time, $\langle q'\mathbf{u}'\rangle$ has an up-gradient component. This is an example of the negative diffusivity of potential vorticity, and hence up-gradient flux, that occurs in transient problems, wherever the motions are decaying with time. It is the expression of the restoring force exerted by large-scale potential vorticity gradients, causing fluid to fall back to its rest latitude unless it is continuously being energized. To show most succinctly that negative κ is necessary in unforced systems, we can write the global conservation of zonal momentum in a completely unforced, inviscid fluid, with $Q = Q(y, z)$: $\iiint \kappa_{yy}\,\partial Q/\partial y\,\mathrm{d}V = 0$, or for this example, $\beta\iint \kappa_{yy}\,\mathrm{d}S = 0$. ($\mathrm{d}V$ and $\mathrm{d}S$ are volume and area elements, respectively).

The envelope of the packet moves steadily with the group velocity,

$$\frac{\partial\omega}{\partial\mathbf{k}} = -\frac{\beta}{|\mathbf{k}|^2}(\sin 2\theta, \cos 2\theta),\ \omega = -\frac{\beta}{|\mathbf{k}|}\cos\theta$$

where θ is the angle between $\mathbf{k}$ and east. The induced mean motions satisfy a rewritten version of (19b), driven by the divergence of $\langle q'\mathbf{u}'\rangle$. This stress-curl

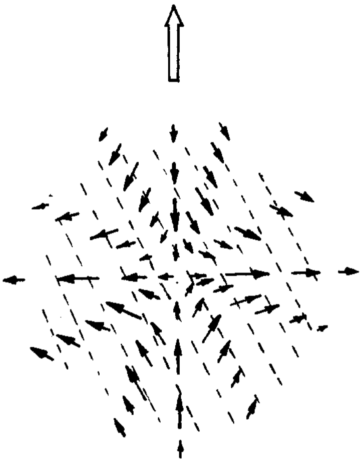


Fig. 4. Mean flow induction by an isolated northward propagating Rossby wave packet. The mean stress exerted on the fluid is the potential vorticity flux (solid arrows) rotated by −90°. This flux has a down-gradient component (southward, that is) wherever particles are being driven away from their rest latitudes, and conversely. Wave crests (dashed), group velocity (double arrow).

may be simplified by rewriting it in new coordinates, (x', y'), which lie along and perpendicular to $\mathbf{k}$, respectively. The result is

$$\nabla \cdot \langle q'\mathbf{u}'\rangle = 2\,\frac{\partial^2 \epsilon}{\partial x' \partial y'}$$

An equivalent problem was solved by McWilliams (1976), although without giving this particular form of the wave stresses. The average potential vorticity flux for packets of baroclinic-mode Rossby waves is given by exactly the same expression.

Because the wave-packet moves steadily, the induction equation becomes a classical moving-source problem,

$$-(\mathbf{c}_g \cdot \nabla)\nabla^2\langle\psi\rangle^{(2)} + \beta\,\frac{\partial\langle\psi\rangle^{(2)}}{\partial x} = -2\,\frac{\partial^2 \epsilon}{\partial x' \partial y'} \qquad (21)$$

Thus, the "mean motions" are also Rossby waves, of very low frequency. In the vicinity of the packet the response is nearly Sverdrup (first term small) as long as $(kL_m)^2$ is large, where L_m is the north–south scale of the packet. A plume of alternating zonal flow reaches along Q-contours, far to the west of the packet, trailing very slightly south of west. Lighthill's (1967) method readily produces the far-field pattern. See also Ichiye (1976).

Let us make a specific calculation for the choice

$$A(\mathbf{x}, t) = e^{-(x^2 + (y - c_y t)^2)}$$

where $\mathbf{c}_g \equiv (0, c_y)$. The parameter γ is here the inverse number of waves in the packet (divided by 2π). It is presumed small, and this allows (21) to be written, in the far-field of the packet,

$$-c_y \frac{\partial^3\langle\psi\rangle}{\partial y^3} + \beta \frac{\partial\langle\psi\rangle}{\partial x} = 0$$

The corresponding group velocities are directed nearly westward. In Fig. 5 we show the time development of such a second-order flow, radiating westward from the wave-packet. kL_m is not much larger than unity for this figure, and so the flow is not so tightly confined in latitude as it must be for large kL_m. Note the similarity with the pseudo-westward influence invoked earlier,

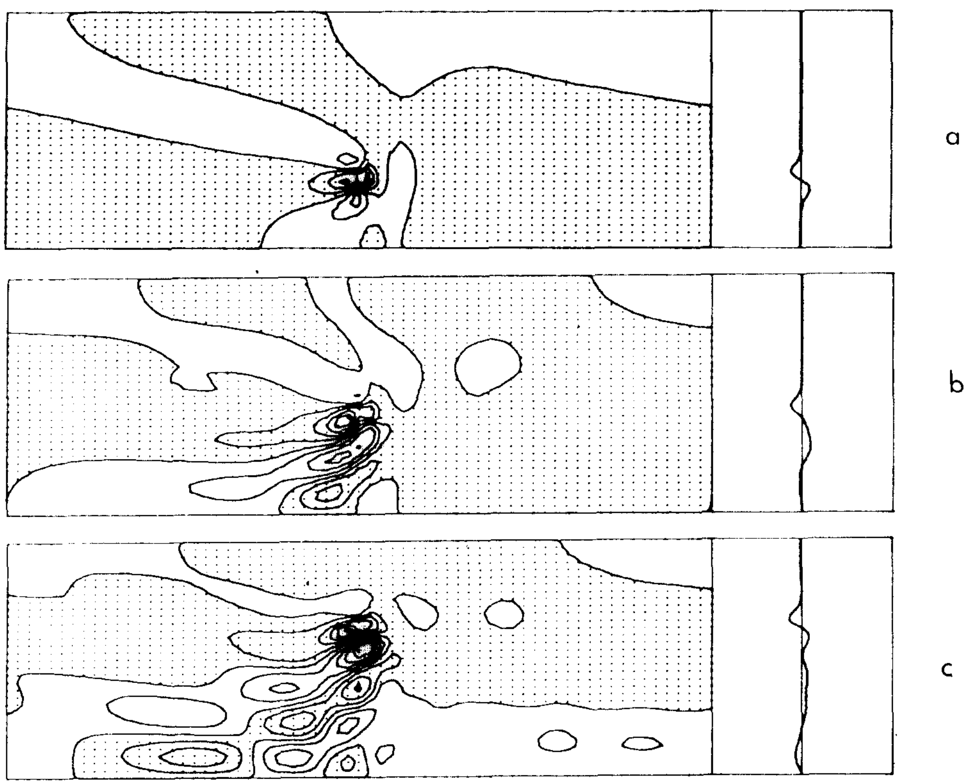


Fig. 5. Induced mean circulation due to the wave packet in Fig. 4, calculated numerically, using the initial-value form of eqn. (21). As the packet moves northward, the mean flow radiates westward. For values of γ smaller than used here, the radiation pattern is more purely westward. The zonal mean flow is plotted at the right. Friction in the numerical model causes the eastward mean flow region to expand behind the westward-flow region at the head of the packet. Simulation courtesy of Dr. G. Williams. (a), t = 17 days; (b), t = 52 days; (c), t = 109 days. The wave field itself is not shown. The domain occupies 30° of latitude and 90° of longitude on a sphere.

eqn. (10). In that statistically stationary undamped problem, such "plumes" of induced flow reached infinitely far along Q-contours. They are consistent with the limit $kL_m \to \infty$ of the present "wake".

The wave-induced large-scale motion above makes no use of the Taylor diffusivity. The "diffusive" form of the wave stresses is entirely equivalent, however. Manipulation of the wave packet solution yields, in (x', y') coordinates,

$$\kappa_{ij} = (\omega k)^{-1} \begin{bmatrix} 0 & \dfrac{\partial \epsilon}{\partial y'} \\ -\dfrac{\partial \epsilon}{\partial y'} & \dfrac{k}{\omega}\dfrac{\partial \epsilon}{\partial t} \end{bmatrix} \qquad (k \equiv |\mathbf{k}|) . \tag{22}$$

where $\kappa_{ij} \equiv \langle u_i(x_j' - x_j'^0)\rangle$. The envelope gradients determine the rate at which fluid particles depart from their rest latitudes as the wave arrives.

The principle diffusion, κ_{22}, is simply the growing oscillatory motion along wave-crests (followed by negative values of κ_{22}, as the wave packet departs). The off-diagonal terms, $\kappa_{12} = -\kappa_{21}$, determine a rotary motion of the particles (more exactly, the angular momentum $\langle(\mathbf{x} - \mathbf{x}^0) \times \mathbf{u}\rangle|_z$ of the particles is $\kappa_{21} - \kappa_{12}$).

Some further manipulations verify that the three expressions for the eddy-stress curl are equivalent,

$$\begin{aligned} -\frac{\partial}{\partial x_i}\langle q' u_i'\rangle &= -\frac{\partial}{\partial x_i}\left(\frac{\partial \epsilon}{\partial x_i}\ \mathrm{rot}\ \phi\right) \qquad (\phi \text{ is twice the angle between } \mathbf{k} \text{ and east, minus } \pi/2) \\ &= -2\,\frac{\partial^2 \epsilon}{\partial x' \partial y'} \\ &= \frac{\partial}{\partial x_i}\left(\kappa_{ij}\frac{\partial Q}{\partial x_j}\right) \equiv \beta\,\frac{\partial \kappa_{i2}}{\partial x_i'} \end{aligned}$$

where rot ϕ means rotation through an angle ϕ.

The Lagrangian mean velocity is of great interest. We will demonstrate here that, without dissipation, the north—south displacement must be severely limited, far more so than the Eulerian pattern might suggest. The scaling of the problem dictates that, in the neighborhood of the wave packet, the mean Eulerian response be dominated by the Sverdrup term in (22),

$$\beta\langle v_e\rangle = \beta \partial \kappa_{i2}/\partial x_i ,$$

where, for clarity, $\langle v_e\rangle$ is the Eulerian mean, $\partial\langle\psi\rangle/\partial x$. To the Eulerian mean velocity we add the Stokes drift, $\partial \kappa_{ij}/\partial x_j$. * This gives for the Lagrangian

* For a wave field, the difference between Eulerian and Lagrangian mean flow, the Stokes drift, is conveniently written $\langle \mathbf{x} \cdot \nabla \mathbf{u}\rangle = \partial \kappa_{ij}/\partial x_j$ to order amplitude squared. It is also the correct expression for a diffusion-equation approximation to turbulence (Freeland et al., 1975).

mean northward flow,

$$\langle v_l \rangle = \frac{\partial}{\partial x_i} (\kappa_{i2} + \kappa_{2i})$$

Using expression (22) for κ_{ij}, we find that the righthand side vanishes; $\langle v_l \rangle = 0$ at this level of approximation. Physically, this must be so. If the Eulerian-mean flow were free to carry fluid particles a distance Y across geostrophic contours, it would lead to a mean relative vorticity $\beta Y \sim \beta \langle v_e \rangle T$, during the transit of the packet, $T \sim L_m/\beta L^2$. This would cause a relative vorticity of strength $(\langle v_e \rangle / L_m)(L_m/L)^2$, that exceeds the Eulerian mean vorticity by the large factor $(L_m/L)^2$, which is impossible. $\langle v_l \rangle / \langle v_e \rangle \leqslant (L/L_m)^2$, as our above calculation indicated.

Important Lagrangian circulation does occur, however. In unbounded geometry, the total westward momentum associated with the packet is just $\int \epsilon / c_x \, dS$, the total energy divided by the westward phase speed. This equals $\frac{1}{2} \beta \int (y - y^0)^2 \, dS$, from example (i). A similar amount of momentum must occur in the Lagrangian description. In fact, outside of the wave packet, the Stokes correction vanishes because κ itself does: $\langle \mathbf{u}_l \rangle \simeq \langle \mathbf{u}_e \rangle$ there. The elongated wake indeed leads to particle migration, with a cross-contour velocity $\langle v_l \rangle = \langle v_e \rangle \sim \langle u_e \rangle (L/L_m)^2$. Now we have the average relative vorticity $\beta Y \sim u_e / L_m$ which is just correct to sustain the mean motion. Andrews and McIntyre (1977) and Beardsley (1975) emphasize the importance of dissipation to wave-induced mean Lagrangian flow.

These two examples reiterate the difference in mean response, between closed and open geostrophic contours. The oceanic case of open or blocked contours allows only a weak Eulerian response ($\langle u_e \rangle \sim \kappa / L_m \sim u' \times$ (wave steepness) $\times$ (L/L_m)), one that is far weaker for waves than for turbulence, and far weaker (across Q-contours) for non-dissipative Lagrangian flow than for Eulerian mean motion. During the transit of the wave packet, the total east–west displacement of fluid drift is merely (wave steepness) $\times$ L, which is no larger than the wave amplitude itself.

This linearized case is of pedagogic interest, but it also serves to emphasize that, unless the freedom of closed contours is available, rectified flows of practical interest will only occur in response to fully developed geostrophic turbulence. Furthermore, Lagrangian flow across Q-contours cannot occur extensively without rapid enstrophy dissipation, which attends these same turbulence fields.

(iii): Zonal acceleration induced by instability of a zonal current

This familiar meteorological problem shows the close relation between the stability properties of a flow, and induced mean-flow changes. The generalized Rayleigh criterion (or Bretherton's (1966) generalization of Taylor's (1915) version of the Rayleigh theorem) is an expression of zonal momentum balance.

In the simple case of level, isopycnal boundaries, the complete problem of determining the zonal-mean adjustment to an instability (Phillips, 1954) is given by eqn. (9b)

$$\frac{\partial}{\partial t}\hat{\Delta}\bar{\psi} = \frac{\partial}{\partial y}\left(\kappa_{yy}\frac{\partial Q}{\partial y}\right) \qquad \hat{\Delta}\psi \equiv \frac{\partial^2\psi}{\partial y^2} + \frac{\partial}{\partial z}\left(\epsilon\frac{\partial\psi}{\partial z}\right).$$

The RHS is a known function of the basic growing wave. The LHS incorporates both zonal acceleration, $-\partial^2\psi/\partial y\partial t$, and mass-field adjustment, $\partial^2\psi/\partial z\partial t$, i.e., the first and second terms of (16e).

The simplicity of this Poisson-type equation means that the nature of these mean-field changes is readily visualized: κ_{yy} is positive (by definition, for an instability), and $\partial Q/\partial y$, known for the initial flow, must take on both positive and negative values. $\partial\psi/\partial t$ is then analogous to the membrane deflection produced by a load (and lift) distribution $\partial/\partial y(\kappa_{yy}\partial Q/\partial y)$. The boundary conditions are "free" $\partial\psi/\partial z = 0$.

Thus, for example, it is easy to demonstrate that the zonal momentum of a westerly (i.e., eastward) upper-level flow must be transmitted downward by the instability, because $\kappa_{yy}\partial Q/\partial y$ is positive above and negative below. The amplitude restriction on this formulation is rather slight, requiring only that $\gamma << 1$.

Vertical integration of (9b) eliminates the mass-field adjustment term (since $\int_{-h}^{0}\bar{v}\ dz = 0$ in a channel), and yields simple information about the vertically averaged zonal momentum:

$$\frac{\partial}{\partial t}\int_{-n}^{0}\bar{u}\ dz = \int_{-h}^{0}\overline{q'v'}\ dz \tag{22a}$$

$$= \int\overline{\zeta'v'}\ dz + f_0\overline{\psi\frac{\partial h^*}{\partial x}}\bigg|_{z=-H} \tag{22b}$$

$$= -\int\overline{\kappa_{yj}\frac{\partial Q}{\partial x_j}}\ dz \qquad \text{if} \qquad Q = Q(x, y, z) \tag{22c}$$

$$= -\int\overline{\kappa_{yy}}\frac{\partial Q}{\partial y}\ dz \qquad \text{if} \qquad Q = Q(y, z) \tag{22d}$$

$$= -\beta\int\overline{\kappa_{yy}}\ dz \qquad \text{if} \qquad Q = \beta y + f_0 \tag{22e}$$

$h^*(x, y)$ is the topography of the lower boundary $z = -H + h^*$. There is neither friction nor external forcing, and rigid plane boundaries lie at $z = 0$, $y = \pm a$. Integrating now over the whole domain,

$$\frac{\partial}{\partial t}\int_{-a}^{a}\int_{-h}^{0}\bar{u}\ dz\ dy = \iint\overline{q'v'}\ dz\ dy \tag{23a}$$

$$= f_0\int\overline{\psi\frac{\partial h^*}{\partial x}}\bigg|_{z=-H}\ dy \tag{23b}$$

$$= -\iint \overline{\kappa_{yj} \frac{\partial Q}{\partial x_j}} \,dz\, dy \tag{23c}$$

$$= -\iint \overline{\kappa_{yy}} \frac{\partial Q}{\partial y} \,dz\, dy \qquad \text{if} \qquad Q = Q(y, z) \text{ only.} \tag{23d}$$

$$= 0 \qquad \text{if} \qquad h^* = 0 \tag{23e}$$

These conclusions follow immediately for flows with no bottom topography:

(a), if we define instability to be the occurrence of positive definite values for κ, everywhere in the domain, then $\partial Q/\partial y$ must take on both signs in the domain for instability to occur, (23d),

(b), if the fluid is stable, with $\partial Q/\partial y$ of one sign, then κ must take on both signs in the domain. In regions of negative diffusivity, (referenced to the rest latitudes, y^0) particles are falling back toward their y^0. This compensates for regions in which particles are being driven away from their y^0, owing to the arrival of increasing energy, by propagation or advection. There is an up-gradient flux of potential vorticity where κ is negative. In a corresponding steady problem with external forcing, the directly forced regions assume this role, and contain up-gradient flux.

(c), if a narrow jet lies surrounded by quiet fluid on a β-plane, and it is unstable ($\kappa > 0$), then the outer region accelerates to the west, by (22e), exactly as in example (i). The inner region, near the jet must accelerate to the east (regardless of whether the jet is westerly or easterly), by (22d). This is accommodated by κ, and hence the north—south particle migration, becoming particularly strong at depths where $\partial Q/\partial y$ is negative, by (22e). This is, of course, the point made by Kuo (1951), Green (1970), Held (1975), and others, but often with logic based on linearized instability theory, which has not been used here.

To a lesser extent than on a β-plane, jet sharpening is a property of f-plane baroclinic instability, as in the model of McIntyre (1968), and in Starr's heuristic models of negative viscosity.

(d), if topography is present, $h^* \neq 0$, and $\partial Q/\partial y$ is of one sign (say positive) then κ may now be positive everywhere (23e does not hold). If this is so, (23b), (23d) yield a westward topographic wave drag, and thus a systematic westward acceleration whose amplitude is given by observable quantities, κ and Q (Bretherton and Karweit, 1975). In the small-amplitude limit, fewer Rossby waves are involved, but the effect is more general.

(e), when topography is present, the Rayleigh theorem no longer holds. Rough topography can greatly alter baroclinic instability; indeed, instability is then hard to define, for the flow cascades to large wavenumber by topographic scattering, even in total absence of advection (i.e., for vanishingly small mean flow speed). Experiments with this kind of behavior are reported by Manabe and Terpstra (1974) and Rhines (1977).

(iv): Abyssal circulation of a wind-driven ocean

The above examples emphasize the role of eddy potential vorticity flux in theoretical models. There is value in similar analyses of both numerical simulations and direct oceanic and atmospheric observations. For example, Bryden (1978, 1979) has demonstrated the ability, even in quiet parts of the circulation, to use moored current meters and thermistors to measure the lateral eddy heat flux and Eulerian-mean horizontal velocity, and to infer the mean vertical velocity. Freeland et al. (1975) give explicit measurements of the Lagrangian mean flow and diffusivity, κ_{ij}, from the Rossby—Webb neutrally buoyant floats. All these data are central to the diagnosis of the mean circulation. When coupled with knowledge of the mean field, Q, they suggest detailed tests of the potential vorticity dynamics used here. One of the least known quantities, the dissipation time for potential vorticity (via the enstrophy cascade), should be investigated experimentally, for this tells us how much control the Q-geometry really exerts upon the Lagrangian mean flow.

Experiments to diagnose eddy effects on the mean ocean circulation have often assumed something analogous to a zonally homogeneous "channel" dynamics (e.g., Thompson, 1971; Webster, 1965) and have considered primarily the lateral Reynolds stresses. Such attempts at deciding the mean momentum balance are imperiled by the unmeasurable Coriolis torques (including the form-drag effect) and mean absolute pressure gradients. The fully three-dimensional nature of the circulation suggests that point-wise time- or ensemble-averaged analysis of eqn. (9c) be applied. Diagnosis or prediction of such a circulation is a stringent test of the method, for it cannot take advantage of the double averaging (for example, time- and longitude-averaging) possible in a re-entrant channel.

A compelling example of much recent interest is the deep circulation of the western North Atlantic. Both observations (Worthington, 1976; Schmitz, 1977) and numerical models (Holland and Lin, 1975; Semtner and Mintz, 1977) agree upon the existence of one or more abyssal gyres. These intense flows lie near the Gulf Stream. In the models they owe their existence entirely to eddy induction. (The classical thermohaline forcing was not a feature of the simulations.) In the ocean, we suspect that they are a superposition of thermohaline- and eddy-driven circulation.

Using eqn. (9c) it is easy to predict the eddy-driven component of the abyssal mean motion, given the time-averaged upper-level flow (which sets $Q^{(0)}$ through its control of the thermocline geometry) and given the distribution of κ (Rhines, 1979; Holland and Rhines, 1979). As shown earlier, the analysis is simplest when the eddy-induced velocity field is weaker than the pre-existing upper-level velocity, i.e., $Q^{(1)} << Q^{(0)}$. This describes well the regime found in these numerical models. In the North Atlantic near the Gulf Stream, however, corrections to $Q^{(0)}$ may well be needed to account for the stronger, more barotropic inertial circulation found there (Schmitz, 1977).

The 1800-day time-average flow fields from a two-layer numerical simula-

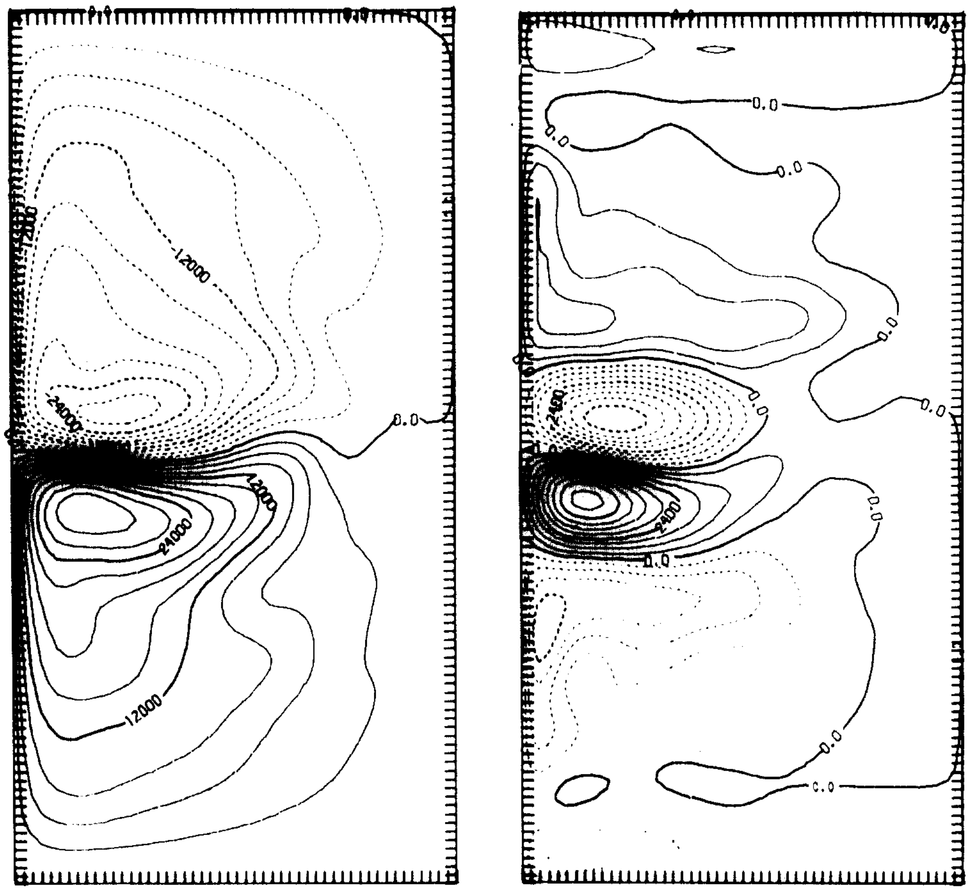
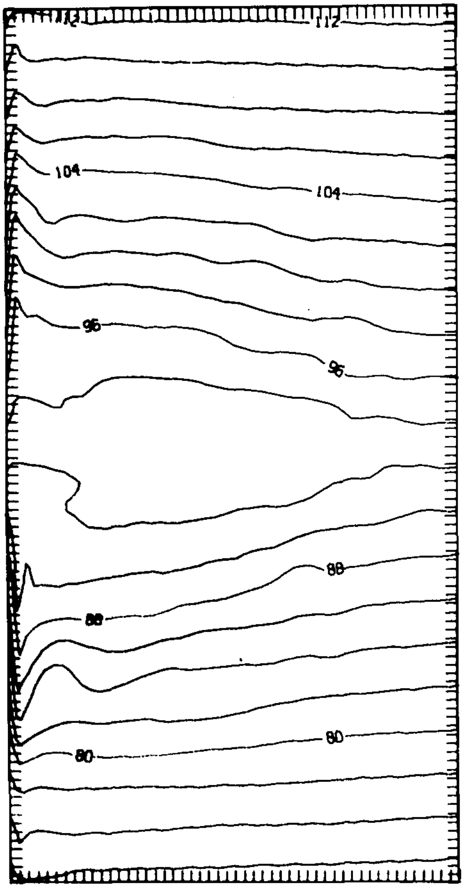


Fig. 6. 1800-day Eulerian mean flow pattern in upper (left) and lower (right) layer of a two-gyre ocean circulation, respectively, from Holland (1978). The domain is 2000 × 1000 km. Driving is by steady zonal winds. In absence of eddies, there would be no lower layer mean flow.

tion (Holland, 1978) are shown in Fig. 6. This 1000 × 2000 km basin is driven by steady zonal wind-stress, symmetric about the middle latitude. An intense "Gulf Stream" runs along the western boundary and separates to form an inertial jet. The jet meanders and produces detached eddies, in addition to innate instabilities of the broader westward flow.

The abyssal Eulerian mean circulation, Fig. 6b, contains two intense gyres rotating with the upper-level flow, and in addition, two "outer" gyres which rotate in the opposite sense. The associated deep western boundary currents flow alternately northward and southward.

The mean potential vorticity field for the lower layer is shown in Fig. 7. It is dominated by βy and the thermocline tilt provided by the upper-level flow. The vorticity-transport model consists simply of solving eqn. (9d) sub-



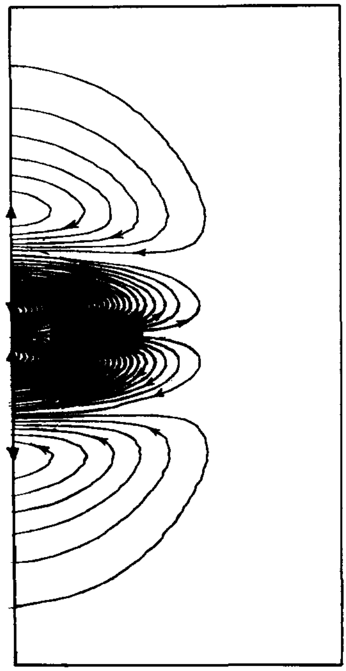


Fig. 7. Mean potential vorticity contours for the lower layer of Fig. 6. The beta effect dominates where the flow is weak.

Fig. 8. Lower layer mean Eulerian circulation according to the vorticity transport model. The 4-gyre pattern results from the flat region in the potential vorticity surface of Fig. 7.

ject to this observed $Q^{(0)}$ pattern, and subject to the measured distribution of κ. In fact, for simplicity, we have taken analytical expressions for $Q^{(0)}$ and κ, which approximate those of the full simulation. The result of integrating (9c) pseudo-westward along characteristics (i.e., along Q-contours) shown in Fig. 8 is the deep Eulerian mean circulation given by the vorticity-transport calculations. It has not been "tuned" for close agreement, but nevertheless exhibits correctly the deep Gulf Stream and four gyres of Fig. 6b. It is significant that a derivation based upon the two-scale approximation has succeeded in this case. In the free-jet region, the scale of $Q^{(0)}$ is not really much greater than the fluid-particle excursions, and thus we could not have been certain that the model would work.

The strength predicted for the deep flow varies directly with the magnitude of κ, $\langle u \rangle \sim \kappa/L_m$. For values found in the experiment, the transport in the deep, separated Gulf Stream jet is about 86×10^6 m^3 s^{-1}, while the western boundary current in the southernmost gyre transports about 11×10^6 m^3 s^{-1} southward. These compare favorably with the actual flows found in the full simulation.

This kind of analysis is both "predictive" and "diagnostic"; the large-scale shape of the thermocline and bottom topography are readily measured at sea, and some idea of the distribution of κ may be had from observations or from estimates of the eddy-generating potential of the large-scale flows. The induced circulation may then be predicted. Alternatively, the nature of the eddy stresses driving a perfectly known circulation can be diagnosed.

Comparison of fluxes

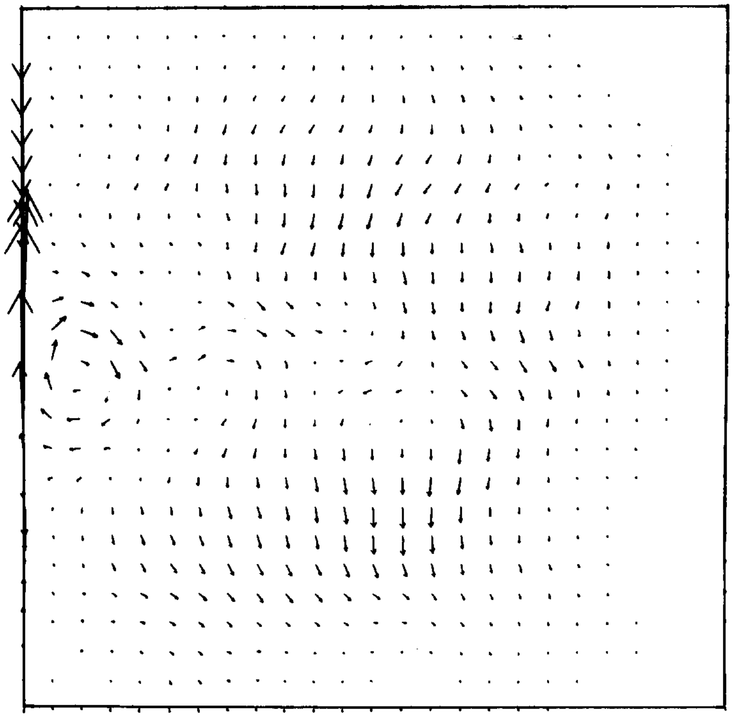
The result of the theory, that the eddy potential vorticity flux, $\langle q'\mathbf{u}' \rangle$, should have a down-gradient component, may be tested directly. Figure 9 shows the 900-day average of $\langle q'\mathbf{u}' \rangle$ in the deeper layer, as well as its component parts. Over most of the basin the flux vectors indeed have a down-gradient (i.e., southward) component *. The reduction of $\partial Q/\partial y$ at mid-basin, visible in Fig. 8, yields a reduced eddy transport, $\sim \kappa \cdot \nabla Q$, there. The divergence of these flux vectors drives the Eulerian-mean north—south velocity, $\langle v \rangle$, by (9c); that divergence pattern yields four abyssal gyres because of the mid-basin minimum in the flux itself.

The three-dimensional study of average vector-potential vorticity and heat fluxes is useful with atmospheric circulations as well. Lau and Wallace (private communication) report an interesting study in which the east—west structure of the eddy transports is important. It leaves one with a very different impression of the maintenance of the circulation than the classical, zonally averaged diagnostics.

There is great qualitative difference between the two component parts of $\langle q'\mathbf{u}' \rangle$. The flux of relative vorticity, $\langle \zeta'\mathbf{u}' \rangle$, is a weak, noisy small-scale field. Even after 1800 days of averaging, it fails to settle down to the pattern (symmetric $\langle \zeta' v' \rangle$, antisymmetric $\langle \zeta' u' \rangle$) that it must eventually take on. The flux of "vortex length", $(f/H)\langle h'\mathbf{u}' \rangle$, on the other hand, is a strong smooth field everywhere but at the separation point of the inertial jet. Its pattern is well established after a few hundred days.

These differences have both kinematic and dynamic origins. $\langle \zeta'\mathbf{u}' \rangle$ is twice more differentiated, in space, than $\langle h'\mathbf{u}' \rangle$, so that it is inevitably fine-grained. It is conceivable that $\langle \zeta'\mathbf{u}' \rangle$ may be dominated by intermittent events (e.g.,

* There is no expectation that the potential vorticity flux should point directly down the mean Q-gradient, unless κ happened to be isotropic. In particular, flows with non-vanishing mean vorticity may cause particle clouds to "swirl", and the off-diagonal diffusivity components to be significant.



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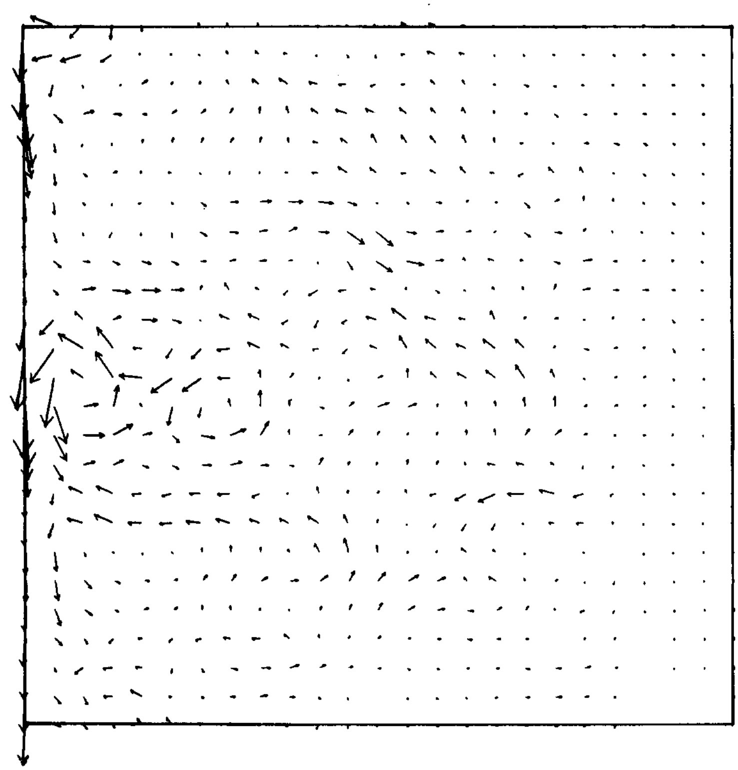

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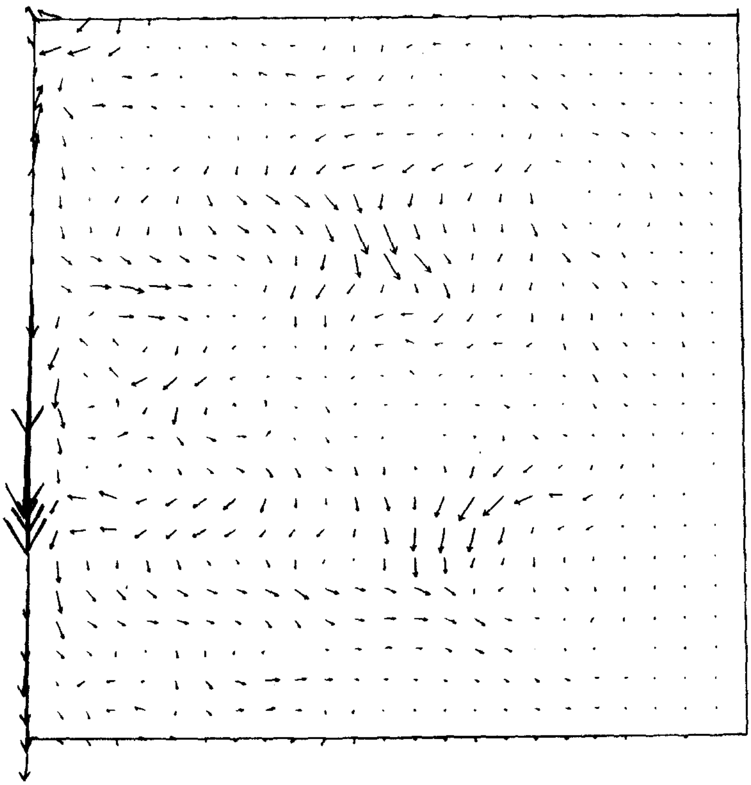

Fig. 9. 900-day average potential vorticity flux vectors in the central 1000 km region of the full numerical simulation. (a), the contribution of the flux of vortex length; (b), the contribution of the flux of relative vorticity; (c), the total flux. The contribution (a), corresponding to downward momentum flux from overhead thermocline eddies, dominates the large-scale circulation. The divergence of this vector field drives the turbulent Sverdrup equation (9c).

by Gulf Stream rings) and hence be poorly approximated by $\kappa \cdot \nabla Q$. The dynamical nature of $\langle h'\mathbf{u}'\rangle$ is such that it is necessarily the stronger of the two terms when the deep currents are weak, for it represents the wave-drag exerted by the more active upper ocean. An estimate of the ratio of the two terms is $H_2 u_2' \zeta_2 / u_2' \eta_2' \sim \gamma U_2'/U_1' << 1$, where the subscripts refer to the upper and deep ocean, respectively.

This calculation emphasizes the value of studying the abyssal ocean in isolation, rather than restricting attention to the depth-averaged transport. The lateral stresses corresponding to the flux of relative vorticity will dominate the depth-averaged diagnosis, unless strong bottom topography is present. A more complete diagnosis of the numerical simulation is given by Holland and Rhines (1979).

The Lagrangian mean flow in this numerical simulation has also been investigated, and will be reported in the literature. Fluid particles crudely tend to follow the Eulerian four-gyre pattern for moderate times (a few weeks); thereafter, turbulent dispersion becomes intense, forcing the Lagrangian and Eulerian means to diverge. The relative stability of the western boundary currents in this simulation assures us that, in some stochastic sense, the gyre pattern will be visible in the particle trajectories.

4. CONCLUSION

Mean-field equations occur in a number of turbulent flow problems; for example, in studies of the generation of large-scale mean magnetic fields by magneto-hydrodynamic turbulence (e.g., Moffatt, 1978). A central question in any such study is just how small γ need be for validity of the theory. Example (iv), above, suggests that even when the spatial scales of the particle motion and the Q-field are not well separated, the vorticity-transport theory gives results with the correct sense. The particularly questionable regions are the vicinity of the separated Gulf Stream jet, which is instantaneously thin, and the near field of the intense, detached rings. Whether the analysis will successfully predict the flow when lateral stresses dominate, is not yet known (although examples (i) and (ii) give some cause for optimism).

Further attention is needed particularly in the following areas: analysis of the wavenumber spectra of the potential vorticity fluxes; behaviour of the fluxes due to intense, intermittent eddies; the nature of fluxes in externally forced regions; and the effect of topographic roughness. Topography provides an efficient agent by which eddies can exert pseudo-westward forces on the entire vertical fluid column (see particularly the experiments of Bretherton and Karweit, 1975). Although its effects are included here, the formalism requires a clean scale separation between small-scale roughness and broader slopes.

We have repeatedly invoked the strong control over the flow (particularly the Lagrangian mean flow) by the large-scale potential vorticity field, Q. The enstrophy cascade rate becomes a quantity of practical interest, for it decides just how strong this control may be. Crudely, we wish to divide the ocean into cascading and non-cascading regions. In the latter, the fluid particles are closely bound to follow geostrophic contours, Q = constant. This constraint puts limits on processes like "mixing along potential density surfaces" so often imagined in the deep sea.

Regions where Q is nearly uniform, over hundreds of kilometers, seem to occur in the upper ocean, on the equatorial side of the subtropical gyres. The potential vorticity equation exerts little control there. It is intriguing to wonder, however, whether these are limiting states, the approach to which is governed by the usual Q-dependent dynamics.

It would be desirable to divide the mean circulation smartly into two regions, the one governed by turbulent dynamics and the other by laminar dynamics. This is not quite such a simple separation as that of the eddy dynamics, given above; the two kinds of mean flow regime overlap in many ways. The "turbulent Sverdrup balance" may apply locally, while the laminar Sverdrup balance applies to a spatially averaged flow (particularly where small stationary gyres are induced over topographic features). Eddy-induced Eulerian-mean motion may at times be strong, while the Lagrangian mean motion is weak. The local circulation may be entirely laminar (obeying eqn. (1)), while in the distance, mesoscale eddies set the strength of that circulation. The upper ocean may exhibit a near perfect Sverdrup balance (integrating (2) down to the thermocline from the surface) while, in the same locale, the deep ocean is driven in the mean by radiated eddies.

The reason for adopting a diffusivity for the large-scale potential vorticity field, rather than for momentum, is that it is well-defined in terms of particle histories, and that eqn. (12) gives us strong suspicions about its sign. The potential vorticity flux sums up the entire eddy forcing of the mean, whereas the momentum transport by the Reynolds stresses is only a part. The sign of the momentum diffusivity is, in addition, controversial and not likely to be simply related to the local velocity gradient. Thus, when Rossby waves propagate into still fluid, they induce a systematic zonal acceleration (given by $\kappa_{yy}\beta$); yet, little predictive value is found in expressing this acceleration in terms of the lateral diffusion of the mean flow itself. The method will fail, however, if γ is not sufficiently small.

With or without the diffusivity formulation, the analysis of eddy potential vorticity flux is of value experimentally. The amount of averaging necessary for stable mean fields depends on the relative role of the two contributions to the flux. The relative vorticity flux (related to the lateral transmission of momentum) is far slower to settle down to meaningful values, than is the flux of vortex length (which is related to the vertical transmission of horizontal momentum). This difference reflects the intermittent, high-pass filtered nature of the flux of relative vorticity. To the extent that the large-scale circulation is of greatest interest, it is the spatially smoothed pattern of vorticity flux that is most relevant. The "circulation integral" analysis of example (iv), suggested by these remarks, is given by Holland and Rhines (1979). In appropriate problems, double averaging is helpful to convergence of the statistics (say, zonal- and time-averaging in a zonal channel).

Although discussion of the Lagrangian aspects of this problem has been very brief here, a large body of recent literature emphasizes the direct derivation of Lagrangian mean flow (e.g., Andrews and McIntyre, 1978, Bretherton, 1978). The key difficulty which prevents direct application of this work to turbulent flows is that it assumes particle trajectories to be "wavelike"; that is, particles are referenced to their mean trajectory, and their displacements from it are strictly bounded. There is a well-defined averaging time (a wave period). Turbulent flows, by contrast, cause ensembles of particles to disperse widely. The Lagrangian mean flow changes continually with time-since-release of the ensemble. In addition, the enstrophy cascade caused by the turbulence provides a mechanism by which fluid forgets its reference potential vorticity.

The particle origins are effectively lost, being continally replaced by new ones. An explicit model of this eddy damping, and some further discussion of Lagrangian aspects, is given by Rhines (1979).

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APPENDIX: DERIVATION OF THE EDDY POTENTIAL VORTICITY FLUX IN A ROSSBY WAVE PACKET

Consider a slowly varying wavetrain of the form

$$\psi = A \cos \theta \qquad A = A(\mathbf{x}, \mathbf{t}),\ \theta = \mathbf{k} \cdot \mathbf{x} - \omega \mathbf{t} + \Delta$$

The eddy stress (the q-flux rotated by $\pi/2$) is

$$\begin{aligned}\nabla^2\psi\nabla\psi = {} & \nabla^2 A\nabla A \cos^2\theta + \nabla^2 A(-\mathbf{k}A) \sin\theta \cos\theta - 2(\mathbf{k}\cdot\nabla A)\nabla A \sin\theta\cos\theta \\ & + 2(\mathbf{k}\cdot\nabla A)\mathbf{k}A \sin^2\theta - |\mathbf{k}|^2 A\nabla A \cos^2\theta + |\mathbf{k}|^2 A^2 \mathbf{k} \sin\theta\cos\theta \\ & -(\nabla\cdot\mathbf{k})A\nabla A \sin\theta\cos\theta + \mathbf{k}\nabla\cdot\mathbf{k}A^2\sin^2\theta\end{aligned}$$

Averaging over a wave period, and neglecting all but the dominant terms,

$$\begin{aligned}\langle\nabla^2\psi\nabla\psi\rangle &= (\mathbf{k}\cdot\nabla A)\mathbf{k}A - \tfrac{1}{2}|\mathbf{k}|^2 A\nabla A + \tfrac{1}{2}\mathbf{k}(\nabla\cdot\mathbf{k})A^2 \\ &= \tfrac{1}{2}[\nabla\cdot(\mathbf{k}A^2)]\mathbf{k} - \tfrac{1}{4}|\mathbf{k}|^2\nabla(A^2)\end{aligned}$$

Now specialize to a wave-packet for which $\mathbf{k}$ changes negligibly, relative to the gradients of A:

$$\langle\nabla^2\psi\nabla\psi\rangle = -\nabla\epsilon + 2(\nabla\epsilon\cdot\mathbf{k})\mathbf{k}/|\mathbf{k}|^2$$

where ϵ is the kinetic energy density. If α is the angle between $\nabla\epsilon$ and $\mathbf{k}$, this reduces to

$$-\hat{\mathbf{j}} \times \langle q\mathbf{u}\rangle = \langle\nabla^2\psi\nabla\psi\rangle = |\nabla\epsilon|\ (\cos\alpha, -\sin\alpha)$$

which is the gradient of the kinetic-energy density, reflected about $\mathbf{k}$ and $\hat{\mathbf{j}}$ is a vertical unit vector. The same expression yields the potential vorticity flux of a baroclinic Rossby-wave packet, which, however, depends upon the vertical coordinate.

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