PKU - HUST Lectures 2020

Lecture III : Space - Time Evolution of Transitions

- Comment on 'Time Delay' → end

- From time → space - time

→ Transition Fronts (c.f. Murray) J.D. Murray

→ Fisher Equation (unstable) epidemics

→ Fitzhugh - Nagumo Equation (bistable)

→ 2 basic prototypes

→ Overview of Fronts

(a)

– Recall Predator-Prey, with fast fluctuations slaved

→ reduced model reduced to:

[Logistic]

$(T - T_c)$ $\partial[T]$

$$\boxed{\partial_t u = (\gamma - \mu c_1) u - c_2 u^2}$$

(a la' TDGL)

2 non-trivial fixed-pts:

[transition as soft mode]

$\gamma < \mu c_1$ → $u = 0$

$\gamma > \mu c_1$ → $u = (\gamma - \mu c_1)/c_2$

$O[1]$

But model is layer-averaged:

What if relax → $\boxed{\text{spatial evolution?}}$

⇒ i.e. viscous diffusion — neoclassical / turbulent

viscosity

$\rightarrow_j$ NL

$u = v'^2$

then: $\rightarrow$

$$\partial_t u - \partial_x D \partial_x u = (\gamma - u c_1) u - c_2 u^2$$

transition evolves in space, time

$\Rightarrow$ 'bare-bones' prototype: $D \equiv$ const

Pearl-Ahy Logistics + space

$$\boxed{\partial_t f - D \partial_x^2 f = \gamma f - c f^2}$$

Fisher Eqn.

KPP (1937)

$\rightarrow$ Fisher Eqn.

= Logistic + Diffusion

(N.B. Also can have Fisher – Burgers)

$\rightarrow$ useful for epidemics, 'slow' combustion, populations as well as plasmas

see W. Van Saarloos: review

$\rightarrow$ basic point:

describe evolution of two domains

4.

X

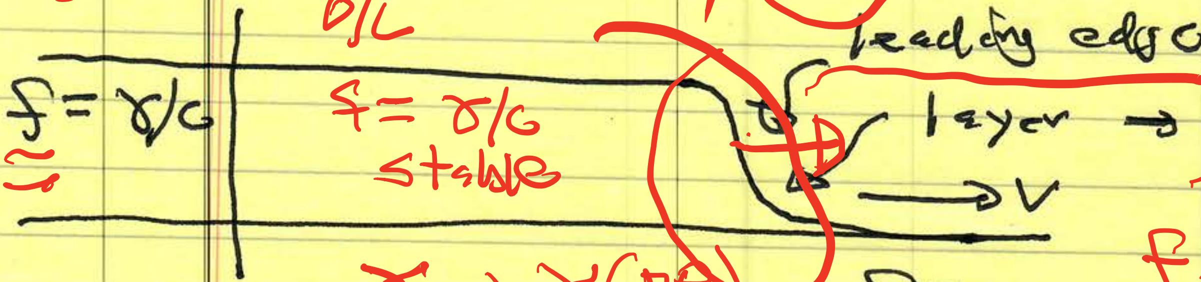
Fronts → $L \to \mathcal{L}$

Barrier build-up

$\gamma/L$

$\sim C$

Supercritical bifurcation front




$\gamma \to \gamma(\nabla P)$

$f = 0$

$f = 0$ unstable

i.e. advance of 1 domain into another

Murray

Von Saarloos

P.D. et. al. '75

Gürcan, P.D '04 → spreading

Garbet, et. al, '07

\+ many others.

$E_r$

$E_r'$ ~~$E_r$~~ $E_r'$

see:

K. Ida mid 2000

L. Schmitz → APS 2020 ☆

⇒ Analysis for Fisher.

# Fisher Fronts → Basic Paradigm

.) Pattern Formation Paradigms I : Epidemic Front Propagation – Fisher Equation

(i) Motivation

→ basic paradigm is nonlinear dynamics: {logistic problem}

map: $X_{n+1} = a\,\underset{\text{growth}}{X_n}\,(1-\underset{\text{saturation}}{X_n})$

continuous system: $\dfrac{dx}{dt} = \gamma_0 x - \alpha x^2$

Fixed pts: $\begin{cases} x_0 = 0 \\ x_0 = \gamma_0/\alpha \end{cases}$

stability: $\begin{cases} -i\omega = \gamma_0 - 2\alpha x_0 \\ \gamma = \gamma_0 - 2\alpha x_0 \end{cases}$

∴ Transition: $x_0 = 0$ (unstable) to $x_0 = \gamma_0/\alpha$ (stable) {i.e. growth to saturation of population}

→ in spatio-temporal generalization, allow diffusive dispersal of population $\rho$: 1D

$$\boxed{\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} = \gamma_0 \rho - \alpha \rho^2}$$ {Fisher Equation}

note: TDGL: $\dfrac{\partial \eta}{\partial t} - D\dfrac{\partial^2 \eta}{\partial x^2} = \overset{T-T_c}{a(T)}\,\eta - b\,\eta^3$

~~change~~



→ seek solutions of form:

$$\boxed{\rho = \rho(x - ct)}$$

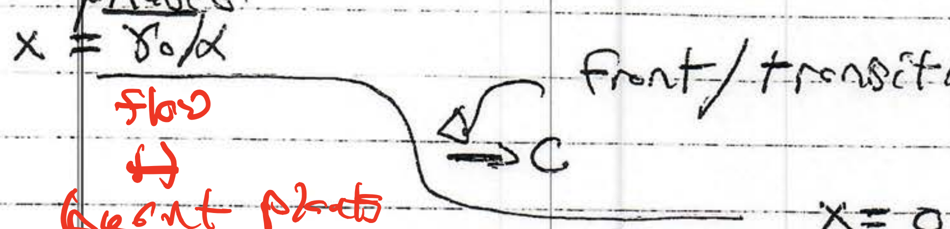
speed

i.e. [propagating solution of nonlinear equation

expect:

– propagation driven by $x_0 : 0 \to \gamma_0/\alpha$ transition instability

– solution to have form of front → domain wall separating regions of two phases




front / transition layer / domain wall

no flow → dried plants

i.e. ① similar phase transition
② combustion
etc.
fires

– seek:

① → structure of solution
*② → propagation speed $c$ → { what sets? physics of $\varepsilon_{min}$.
③ → stability of front

ii) Formulating problem:
if Fisher eqn:

$$\frac{\partial \rho}{\partial t} = k\rho(1-\rho) + D\frac{\partial^2 \rho}{\partial x^2}$$

8

$$t^* = \boxed{kt}$$
$$x^* = x(k/D)^{1/2}$$ and omitting $*$ $\Rightarrow$

$$\frac{\partial P}{\partial t} = P(1-P) + \frac{\partial^2 P}{\partial x^2}$$

$P = P(x-ct) \Rightarrow$ -1

-0

$$\left.\begin{array}{l} \boxed{P'' + cP' + P(1-P) = 0} \\ P(-\infty) = 1, \quad P(\infty) = 0 \end{array}\right\}$$

Now, can analyze via # of strategies: convert to

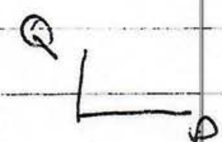
① dynamical system -3

$$\begin{cases} Q = P' \\ Q' = -cQ - P(1-P) \end{cases}$$

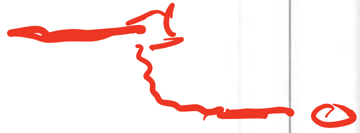
$\Rightarrow$

$$P' = Q$$
$$Q' = -cQ - P(1-P)$$

and




$$\left.\frac{dQ}{dP} = \frac{-cQ - P(1-P)}{Q}\right\}$$ phase plane trajectories



Observe similarity:

$P(x-ct)$

- Fisher Eqn. (generalized) and 1D mechanics

$u = \gamma P^2/2 - \frac{cP^3}{3}$

$$-D\frac{\partial^2 P}{\partial x^2} - c\frac{\partial P}{\partial x} = -\frac{\partial U(P)}{\partial P}$$

inertia    friction    force.

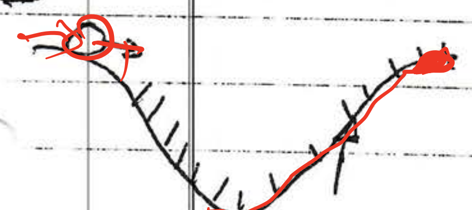
$$m\ddot{x} + \gamma\dot{x} = -\frac{\partial U(x)}{\partial x}$$

$P$

→ $c$ to stabilize transition in moving frame

→ $\gamma$ drag to balance force

i.e. ball motion



arrival here depends on i.c.'s due friction

$c \to$ dynamics

kink motion →

Front → marginally stable in co-moving frame.

∴ can expect:

→ sensitivity of trajectory to initial condition (i.e. push at $t=0$ to arrive at $X$?)

→ condition for propagation (over/under damping)

Now, trajectories have two critical points:

$$\rho = 0, \quad Q = 0 \qquad 2 \text{ F.P.}$$
$$\rho = 1 \quad Q = 0$$

can linearize about these:

$$-\gamma \tilde{\rho} = \tilde{Q}$$

$$-\gamma \tilde{Q} = -c\tilde{Q} - \tilde{\rho} + 2\rho_0 \tilde{\rho}$$

For $(0,0)$: $\quad -\gamma \tilde{\rho} = \tilde{Q}$

$$-\gamma \tilde{Q} = -c\tilde{Q} - \tilde{\rho}$$

$$0 = \begin{vmatrix} -\gamma, & -1 \\ 1, & -\gamma - c \end{vmatrix} \Rightarrow \begin{array}{l} \gamma(\gamma + c) + 1 = 0 \\ \gamma^2 + c\gamma + 1 = 0 \end{array}$$

$$\gamma = -\frac{c}{2} \pm \frac{1}{2}\left(c^2 - 4\right)^{1/2}$$

$\ast \longrightarrow$ $c \geq c_{min} = 2$ for non-negative definite $\rho$ (avoid oscillation)

population should not oscillate

For $(0,1)$:

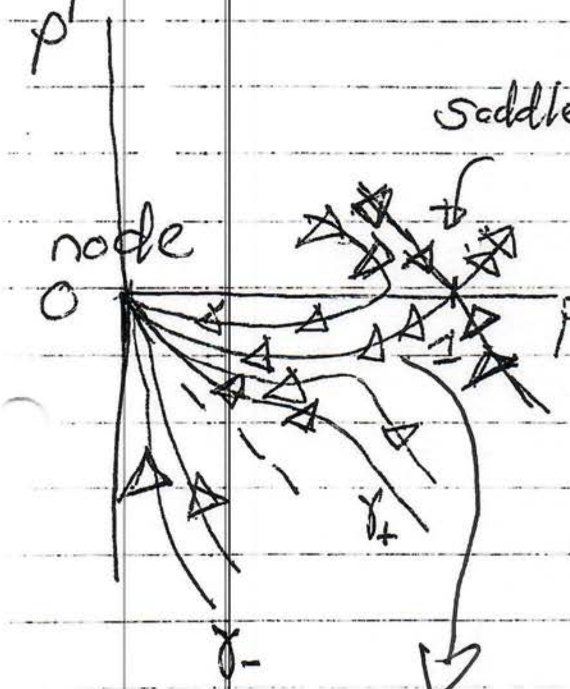
$$0 = \begin{vmatrix} -\gamma, & -1 \\ -1, & -\gamma - c \end{vmatrix} \Rightarrow \gamma(\gamma + c) - 1 = 0$$

$$\gamma = -\frac{c}{2} \pm \frac{1}{2}\left(c^2 + 4\right)^{1/2}$$

Thus, $(0,0)$ : stable node for $c^2 > 4$
stable focus (spiral) for $c^2 < 4$

$(0,1)$ : saddle point

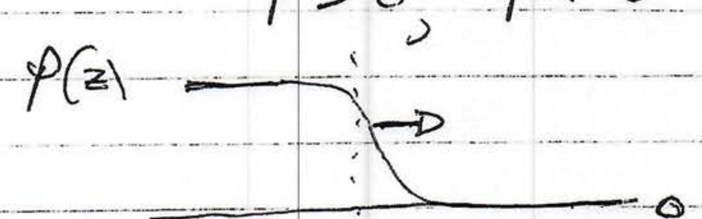
$\Rightarrow$ phase plane trajectories:




Clearly, $\exists$ a phase space trajectory from $(1,0) \rightarrow (0,0)$ which
* i.) falls in $p > 0$
ii.) $p' < 0$ (front)

for all wave speeds $c > 2$
$\Rightarrow$ front solution

Note:
- formally, travelling wave solutions exist for $c < c_{min} = 2$, but these are unphysical, as $p$ oscillates ($p < 0$ ?!)
- $c > c_{min}$ solution has $p > 0$, $p' < 0$ $\Rightarrow$ front




(physical solution)

$\therefore$ analysis establishes minimum speed for propagating front solution $c_{min} = 2(KD)^{1/2}$

## leading edge analysis — Key

- consider edge of evolving wave propagating from $-\infty$ to $+\infty$


$$\rho(x,0) \sim A e^{-\alpha x} \qquad x \to \infty$$

$e^{-\alpha x}$

- linearizing Fisher Eqn (about unstable fixed point):

Reaction-Diffusion

$$\frac{\partial}{\partial t}\rho = \rho + \frac{\partial^2 \rho}{\partial x^2}$$

$$\rho = A e^{-\alpha(x - ct)} \quad \text{(propagating leading edge)}$$

exponential

$$\boxed{\alpha c = 1 + \alpha^2} \Rightarrow \alpha^2 - \alpha c + 1 = 0$$

$$\boxed{\alpha = \frac{c}{2} \pm \frac{1}{2}\left(c^2 - 4\right)^{1/2}}$$

$c_{min}$

$\omega(k) = 0 \qquad k(\omega)$

$f(\omega, k) = 0$

$F(c, \alpha)$

$\therefore$ consistency with leading edge hypothesis structure forces $c > c_{min} = 2 \Rightarrow 2(kD)^{1/2}$

$c_{min} = 2 \rightarrow c = 2(kD)^{1/2}$, $2(\partial D)^{1/2}$

**Key Point:**

- in fixed frame, instability occurs at each point, as $\rho$ transitions $0 \to 1$

A
- $c_{min}$ specifies minimal speed such that marginal

$L^2$

Observe:

$$\boxed{C_{min} = 2(kD)^{1/2}}$$

$$\Delta x = (D/k)^{1/2}$$

kink width

[can sharpen kink via $D\downarrow$ or $k\uparrow$ (increase rate local instability)]

→ observe that with diffusion: $\Delta x$, $c_{min}$ emerge from marginal stability analysis

reaction

$$\gamma = k - k'^2 D$$

$\gamma = 0$   $\gamma \sim \gamma P + D\gamma P$

$$\gamma = 0 \Rightarrow k' \sim 1/\Delta x \sim \left(\frac{k}{D}\right)^{1/2}$$

→ $C_{min} \sim (kD)^{1/2}$   but diffusion → $D/L^2$

$$\frac{1}{\tau} \sim \frac{C}{L} \sim \left(\frac{kD}{L^2}\right)^{1/2}$$

local transition instability → $k$

$$\boxed{1/\tau_{trans}} \sim \left(\frac{D}{L^2} k\right)^{1/2}$$

→ [geometric mean of {diffusion, reaction, transition} time scale]

i.e. {propagation is synergism of local transition instability with diffusive coupling (spatially)}

$\gamma$

important physics!

## Why $C_{min}$?


stability maintained $\left( \partial/\partial t (\sim \gamma) \rightarrow -c \frac{\partial}{\partial x} \right)$.

- leading edge analysis illustrates wave speed dependence on conditions at $x = \pm\infty$.

→ Kolmogorov

Note: KPP $\boxed{\text{proved}}$ that if:

a.) $P(x,0)$ has compact support

b.) $P(x,0) = P_0(x) > 0$

$$P_0(x) = \begin{cases} 1, & x \leq x_1 \\ 0, & x \geq x_2 \end{cases} \qquad x_1 < x_2$$

$C_{min}$ → leading edge analysis

c.) $P_0(x)$ continuous $\quad x_1 < x < x_2$

(i.e. kink structure), then:

→ $\boxed{P(x,t) \text{ evolves to } P(x - C_{min} t)}$

Key issue: minimum speed is one selected

i.e. Counter-intuitive point is that pattern/front in Fisher Equation which is selected is one with minimum speed (marginal stability!?)

$c = 2(\gamma D)^{1/2}$

iii.) Front Stability

→ clearly, physically interesting solution should be stable

what learned from stability

→ while wave-front unstable to far-field perturbations, KPP thm. suggests insensitivity to i.e. perturbations with compact support.

Predator

$P \to U$

slow

∴ natural to investigate stability

For stability $\boxed{P = P(x-ct, t)}$

2 time scales

front prop. time dependence

instability

2 time dep

$K = \gamma = 1$

$\theta = 1$

$$\frac{\partial P}{\partial t} = P(1-P) + c\frac{\partial P}{\partial x} + \frac{\partial^2 P}{\partial x^2}$$

$$\boxed{P = \underbrace{P_0(x-ct)}_{Z} + \epsilon \tilde{P}(z, t)}$$

$(Z \equiv x-ct)$

$$\therefore \frac{\partial \tilde{P}}{\partial t} = \tilde{P} - 2P_0(z)\tilde{P} + c\frac{\partial \tilde{P}}{\partial z} + \frac{\partial^2 \tilde{P}}{\partial z^2}$$

$$\Rightarrow \frac{\partial \tilde{P}}{\partial t} = (1-2P_0(z))\tilde{P} + c\frac{\partial \tilde{P}}{\partial z} + \frac{\partial^2 \tilde{P}}{\partial z^2}$$

Now $\tilde{P} = \tilde{P}(z)e^{-\gamma t}$

$$\Rightarrow \boxed{\frac{\partial^2 \tilde{P}}{\partial z^2} + c\frac{\partial \tilde{P}}{\partial z} + (1+\gamma-2P_0(z))\tilde{P} = 0}$$

eigenmode equation

$\gamma > 0 \to$ stable

for $\gamma = 0$ have: $\boxed{\tilde{P}'' + c\tilde{P}' + (1-2P_0(z))\tilde{P} = 0}$

observe:

$P(x-ct)$

$P_0(z) \to P_0(z+\delta z)$



$$0 = \frac{\partial^2 \rho}{\partial z^2} + c\frac{\partial \rho}{\partial z} + \rho(1-\rho)$$

$\rho = \rho_0(z)$ is solution. Now, consider infinitesimal shift of solution:

$\rho_0(z + \delta z)$

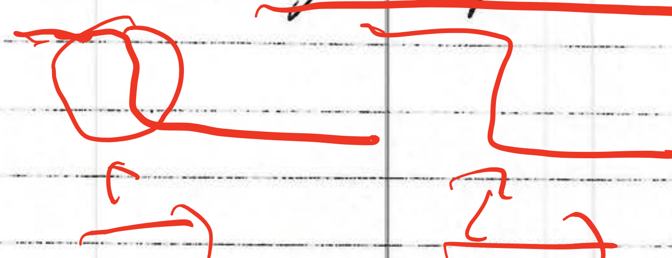
$\Rightarrow$

$$0 = \frac{\partial^2}{\partial z^2}\left(\rho_0(z) + \delta z \frac{d\rho_0(z)}{dz}\right) + c\frac{\partial}{\partial z}\left(\rho_0(z) + \delta z \frac{d\rho_0}{dz}\right)$$

$$+ \rho_0(1-\rho_0) + \left(\frac{d\rho_0}{dz} - 2\rho_0(z)\frac{d\rho_0}{dz}\right) + O(\delta z^2)$$

$$= (\rho_0')'' + c(\rho_0')' + (1 - 2\rho_0(z))\rho_0' \quad \text{eigenmode at } \gamma = 0.$$

$\therefore$ $\gamma = 0$ is "translation mode" $\Rightarrow$ related to translational invariance of system / momentum conservation of kink.



$\Rightarrow$ for stability, need:

$\gamma > 0 \Rightarrow \lim_{t\to\infty} \tilde{\rho} = 0$

$\gamma = 0 \Rightarrow \lim_{t\to\infty} \tilde{\rho} = \frac{d\rho_0}{dz}$ (front translation)

front speed $(D/\gamma)^{1/2}$

$\gamma = \gamma$ $d$ gap-jumping

$\gamma$ const. $\gamma(x)$

Now, substituting $\boxed{\tilde{\rho} \to \delta\rho\, e^{-cz/2}}$ $\Rightarrow$ S.E

$$\gamma\,\delta\rho'' + \left(\gamma - \left(2\rho_0(z) + \frac{c^2}{4} - 1\right)\right)\delta\rho = 0$$

$\delta\rho(\pm L) = 0$

$\gamma > 0$ $\quad e^{-\gamma t}$

$$\Rightarrow \gamma = \frac{1}{\left(\int \delta\rho^2 dz\right)} \left[\int dz \left(\frac{c^2}{4} - 1 + 2\rho_0(z)\right)\delta\rho^2 + \frac{1}{2}(\delta\rho')^2\right]$$

some $L$

$$\boxed{\gamma \geq 0 \Leftrightarrow c^2 \geq c^2_{min} = 4}$$

leading edge

$\leftarrow$ stab.

i.e. $c_{min}$ emerges from stability analysis for front.

$\to$ dyn. syst.

~~iv.) Asymptotic Analysis of Nonlinear Problem~~

~~$\to$ would be re-assuring to demonstrate viability of (leading edge) analysis – i.e. obtain analytic form for nonlinear front~~

~~$\to$ proceed via singular perturbation theory approach~~

$c$ ?

stab.

$c_{min} \to$ marginal stability in co-moving frame.

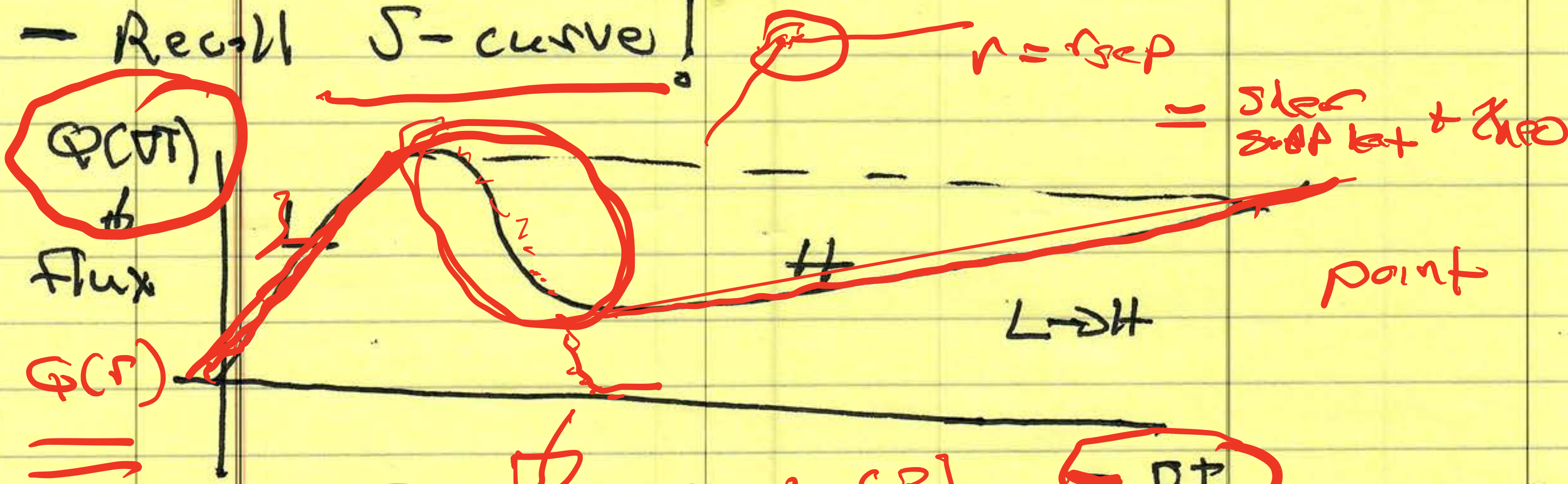
$\to$ ? $\to$ $c = c_{marg}$

X

17.

II.) **Bistability ?**

- Recall S-curve!




c.f. Hinton '91 Itohs (?) Cat a point!

exp. A. Hubbard early 2000

K. Ida 2010, 2003

⇓

how develop in space-time

Flux landscape

L → Q vs -∇T → steep

→ large $\chi_T$

H → Q vs -∇T → shallow

→ small $\chi_{neo}$.

X

N.B. → Fick → Bi-stable Fick

How? → $\boxed{\text{Shear suppression}}$

$$Q = -\frac{\chi_T \nabla T}{1+\alpha\left(V_E'^2/\gamma^2\right)} - \chi_{neo}\nabla T$$

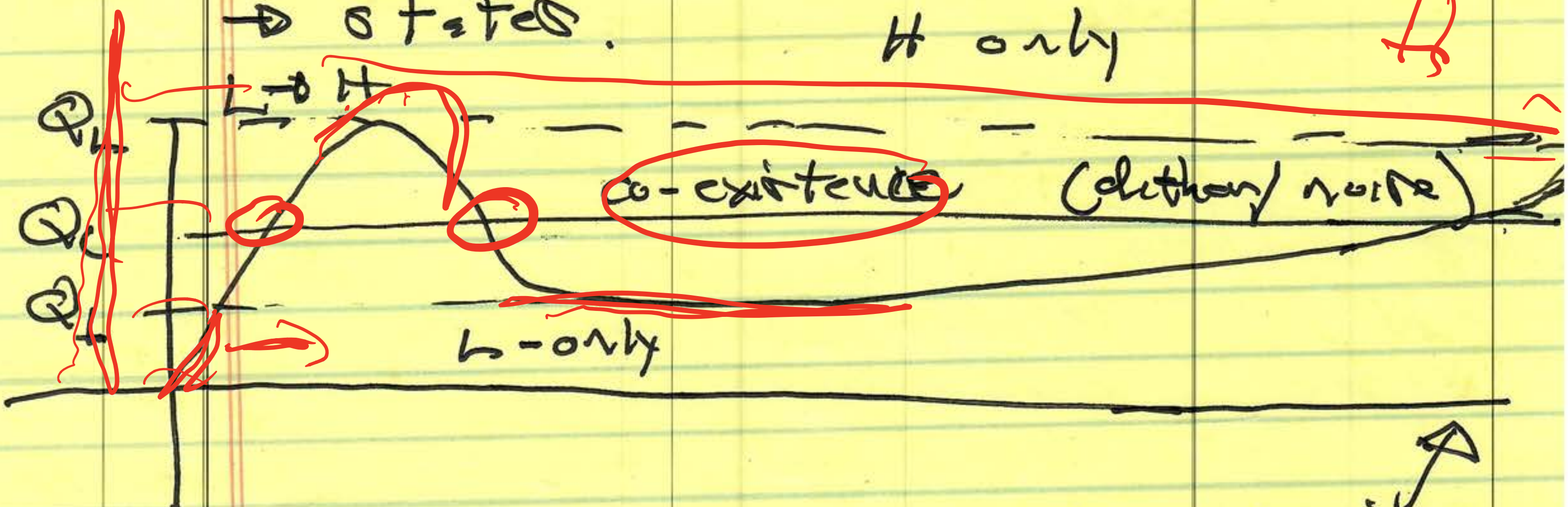
$V_E' \to T$
Radial F.B.

with radial force balance,
rolls over at larger $\nabla T$

$P_{in} \to Q_0$

$Q_0 \leftrightarrow$ power input

→ states.




other physics
may enter here
⇒ MHD stability (ELM)

X

Fisher → 1 stable

2 stable states

## Signature Feature : Bi – stability

Why?

$$\partial_t T = -\nabla Q + S_0$$

$S_0$ → heat source

$Q_0 = Q(\nabla T)$ roots

$$\partial_t \nabla T = -\nabla^2 Q$$

$\nabla T_0 \to$ roots

$$\boxed{\nabla T = (\nabla T)_0 + \delta \nabla T}$$

$(\nabla T)_0$: fixed pt root

$\delta \nabla T$: consider perturbation


const

const

$$\partial_t \left( \cancel{\nabla T_0} + (\delta \nabla T) \right) = -\nabla^2 \left( \cancel{Q(\nabla T_0)} + \frac{\partial Q}{\partial \nabla T}\, \delta(\nabla T) \right)$$

$$\boxed{\partial_t\, \delta(\nabla T) = -\nabla^2 \frac{\partial Q}{\partial \nabla T}\, \delta(\nabla T)}$$

$$= +\nabla^2 \frac{\partial Q}{\partial(-\nabla T)}\, \delta(\nabla T)$$

diffusion eqn

So observe:

$$\partial_t [\delta(\nabla T)] = \nabla^2 \left[\frac{\delta Q}{\delta(-\nabla T)}\right] [\delta(\nabla T)]$$

$\Updownarrow$ → perturbation domain

$$\partial_t f = \nabla^2 D_{eff} f$$

Diffusion Eqn.

$$D_{eff} = \delta Q / \delta(-\nabla T)$$

$\delta Q / \delta(-\nabla T) > 0 \rightarrow D_{eff} > 0$ stable

$\delta Q / \delta(-\nabla T) < 0 \rightarrow D_{eff} < 0$ unstable

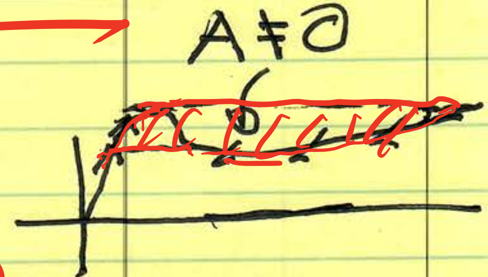
S-curve $Q(\nabla T) \rightarrow$ $\boxed{\text{slope} \rightarrow \text{sign of } D.}$

3 [redacted] roots; 2 stable, 1 unstable    bi-stable

1 root; 1 stable

Begs question

→ how under-stand co-existence of 2 stable roots?




→ thresholds, hysteresis

c.f. — Hinton '91
— Lebedev, P.D. '97 → fronts
" " et al '97 → flux Landscape
Malkov, P.D. '07 → analysis

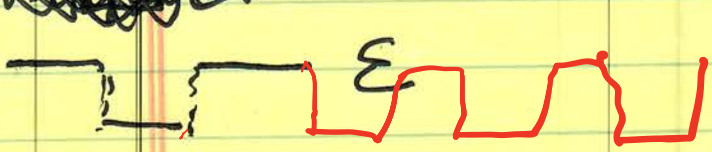
et seq.

⇒ Staircases (another discussion)

i.e. staircases = alternating sequence of bistable domains connected by jumps.

of bistability


Cahn-Hilliard Eqn



The proto-type:

$$\boxed{\frac{\partial u}{\partial t} = f(u) + D\,\partial_x^2 u}$$

Reaction-Diffusion Eqn.

$\rightsquigarrow$ Fitzhugh-Nagumo Eqn. (FN)

- n.b.: $f(u) = A\,(u-u_1)(u_2-u)(u-u_3)$

$A\,(u-u_1)(u-u_2)(u-u_3)$

$A \hookrightarrow \gamma$ 3 roots

↓ reaction function

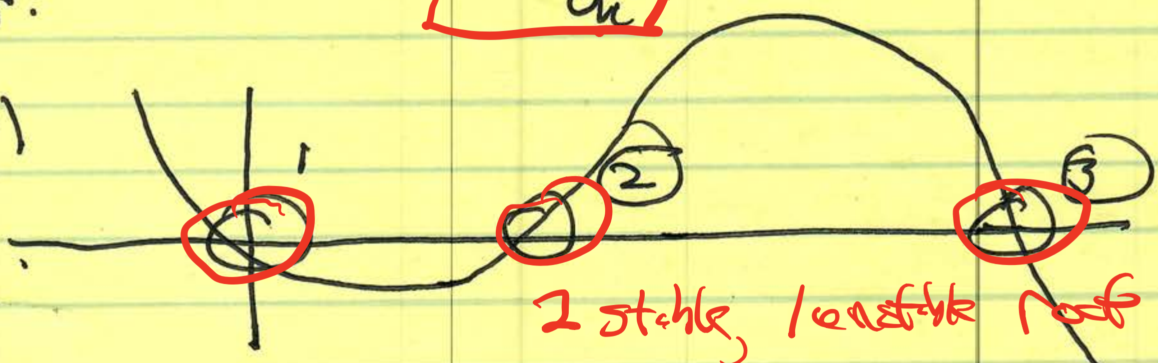
$A \sim \gamma \to$ rate

key pt: 3 fixed pts: $u_1, u_2, u_3$

stability: $\dfrac{\partial \tilde{u}}{\partial t} = \left.\dfrac{\partial f}{\partial u}\right|_{u_i} \tilde{u} + D\,\partial_x^2 \tilde{u}$

$\tilde{u} = u - u_i$
$i$th f.p.

$f(v)$




2 stable, 1 unstable root

pt: $\left.\dfrac{\partial f}{\partial u}\right|_{u_2} > 0 \to$ instability

$$\left.\frac{\partial f}{\partial u}\right|_{u_{1,0}, u_3} < 0 \rightarrow \text{stable}$$

→ WTFN? (General Culture)

- FN is 'toy' version of Hodgkin - Huxley model of neuron signals (muscle, etc ...)

Nobel Prize

→ switch $\begin{cases} \text{on} \\ \text{off} \end{cases}$ ⟺ bistability

on-off switch

c.f. at least a look at Hodgkin - Huxley paper (1952) is highly recommended.

FN system:

$$\frac{\partial v}{\partial t} = \underbrace{f(v)}_{\text{bi-stable}} - w + \overset{\text{current}}{I_a} + \frac{\partial^2 v}{\partial x^2}$$

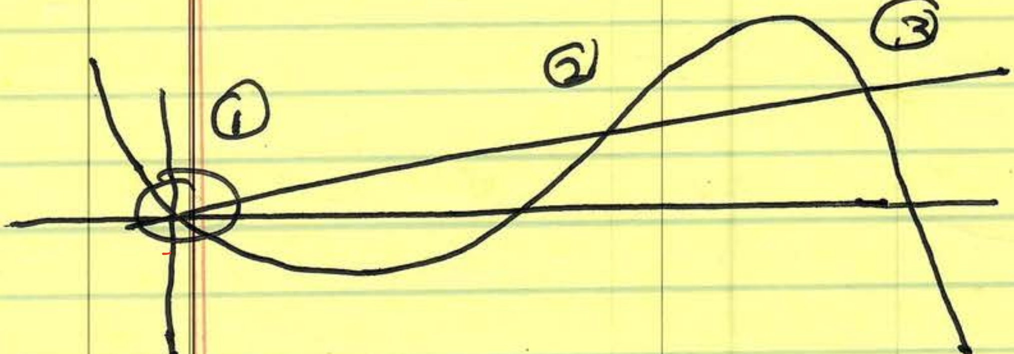
i.e. fast channel → sodium

$$\frac{\partial w}{\partial t} = bv - \gamma w$$

i.e. slow, as calcium

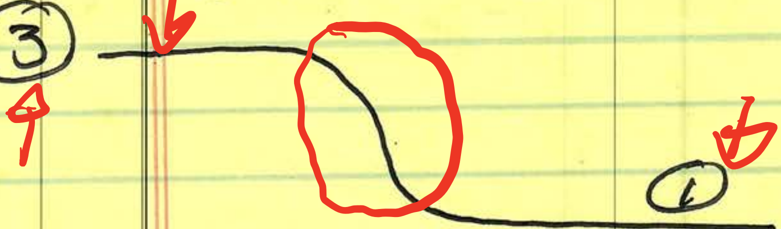
eq. for fixed points:

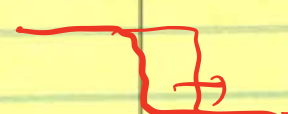
$$\left.\begin{aligned} w &= f(v) = w(v) \\ w &= bv/\gamma \end{aligned}\right\} \text{ f.p.}$$




evident: ①, ③ stable $f' < 0$
② unstable $f' > 0$

⇒ front can link/allow transition between 2 stable fixed points.

③ 

⇒ unstable root 'powers' front motion (ala' Fisher).

Can further simplify to FN equation:

$$\frac{\partial u}{\partial t} = D\,\partial_x^2 u + f(u)$$

$$f(u) = A\,(u-u_1)\,(u_2-u)\,(u-u_0)$$

$\Rightarrow$ { 'minimal' bistable model → switching of neurons
counterpart of Fisher

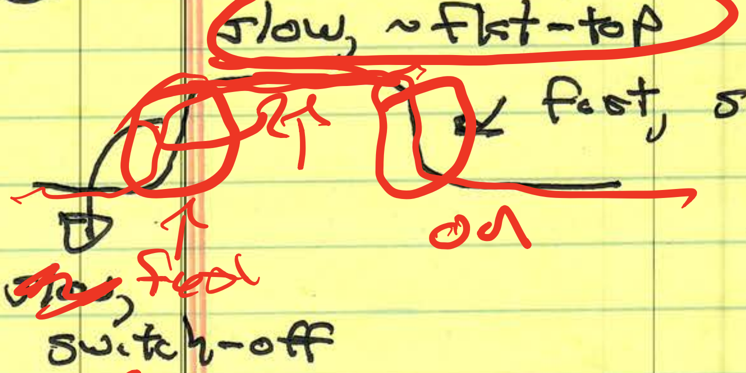
excitable media → threshold to activate

→ Note:

① One 'goal' of FN system is to describe pulses in excitable media

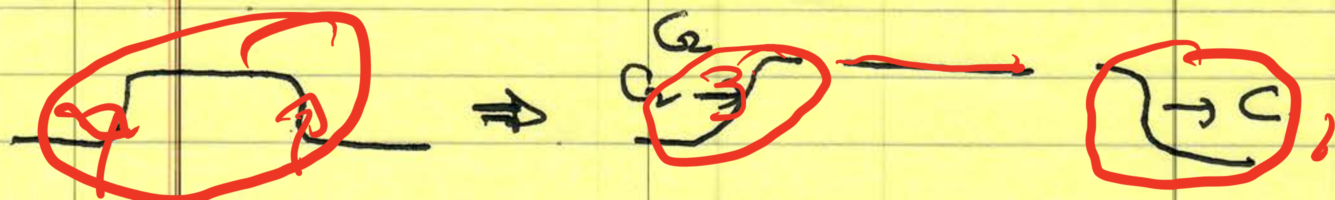
(MFE application?)

② What is a pulse?



X

Idea of FN: assemble 3 parts




to maintain coherence:

$$c_1 = c_3$$

~ key element of pulse is switch on-off front

~ switch-on starts from $u=0$ $\underline{\infty}$

$$\boxed{\partial_t u = D\,\partial_x^2 u + f(u)}$$

~ above.

③ F and FN:

leni

$$\partial_t u = D\,\partial_x^2 u + \gamma u(1-u)$$

↑ — reactions.

$$\partial_t u = D\,\partial_x^2 u + f(u)$$

bistable

⇒ reaction - diffusion equation

i.e. nonlinear reaction and diffusion separate.

In MFE: (also) "reaction-diffusion"

but:

$$\partial_t T = -\underline{\nabla} \cdot \underline{Q}(\nabla T) + S_0$$

$$Q = \frac{-\chi_T}{1 + \alpha {V_E'}^2} \nabla T - \chi_{neo} \nabla T$$

NL of $\delta f$

↑ → "reaction"

⇒ "reaction is in the diffusion"

i.e. nonlinear diffusion.

– harder!

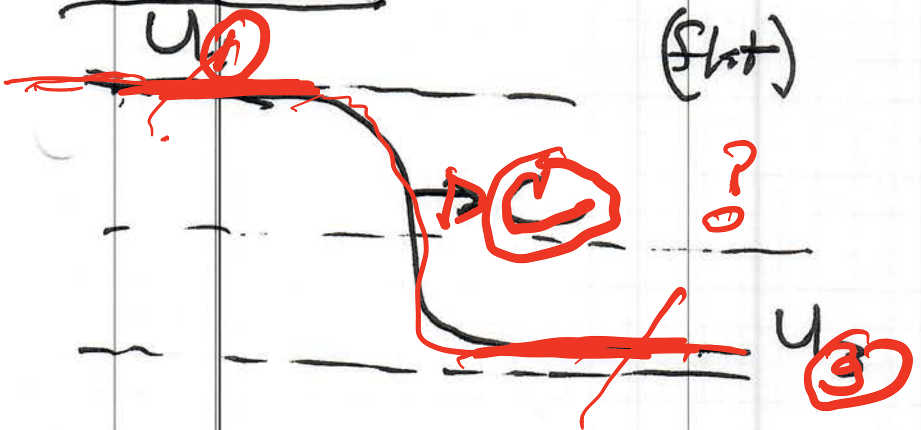
⇒ Cahn-Hilliard Equation (coming)

⇒ best understand FN first.

FN analysis →

Looking for
- Fronts.
- speed

As usual:




$c > 0$

$c < 0$

$c = 0 \to$ co-existence

As usual, look for speed $c$ via traveling wave:

$$u = u(x - ct)$$

$$\boxed{-cu' = Du'' + f(u)}$$

$$u'\left[-cu' = Du'' + f(u)\right]$$

so

$$-c\int_{x(u_3)}^{x(u_1)} u'^2\,dx = D\int_{x(u_3)}^{x(u_1)} u'u''\,dx + \int_{x(u_3)}^{x(u_1)} u' f(u)\,dx$$

① ② ③

$$② = \int_{x(u_3)}^{x(u_1)} u'u''\,dx = \int_{x(u_3)}^{x(u_1)} \frac{d}{dx}\left(u'^2/2\right)$$

$= 0$, as $u' = 0$ on both sides front.

② $$\int_{x(u_3)}^{x(u_1)} dx\, u' f(u) = \int_{x(u_3)}^{x(u_1)} du\, f(u)$$

$$= \int_{u_3}^{u_1} du\, f(u)$$

③ $$= \int_{u_3}^{u_1} du\, f(u)$$

$$\cancel{② = -c \int_{-\infty}^{+\infty} u' dx}$$

$\underbrace{\qquad\qquad}_{\text{non zero}}$

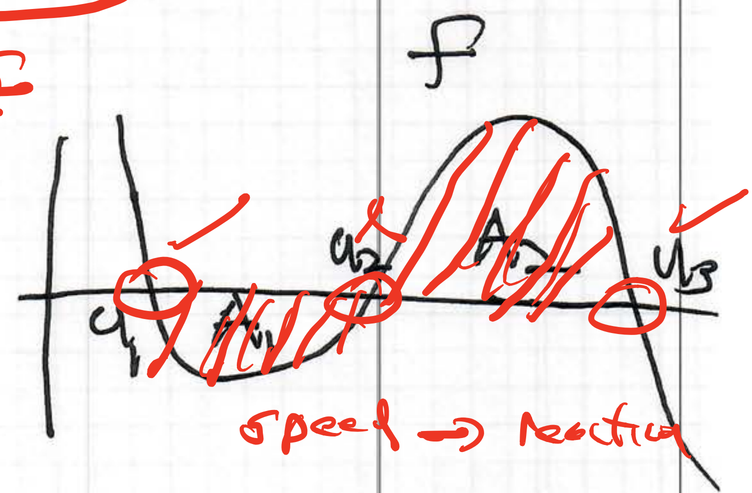
⑧ $$\boxed{c = -\int_{u_3}^{u_1} f(u)\, du \Big/ \int_{-\infty}^{+\infty} u'^2 dx} \Rightarrow \boxed{\text{front speed}}$$

front speed.

$>0$ factor

$$\boxed{c} \sim -\int_{u_3}^{u_1} f(u)\, du$$

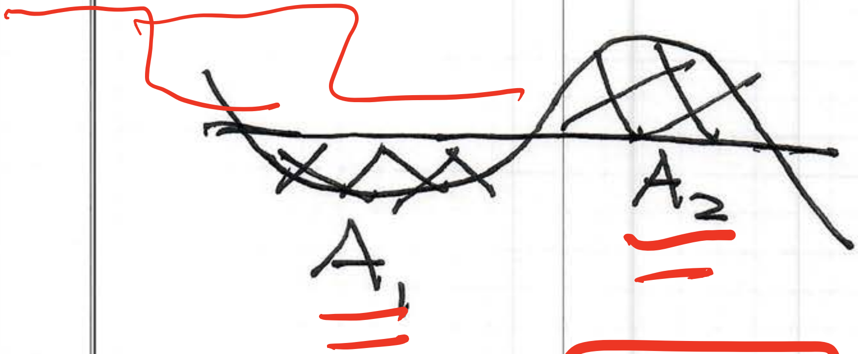
$$= \int_{u_1}^{u_3} du\, f(u)$$




speed → reaction

i.e. c linked to area under f → curve with ? zero crossings.

So




$A_2 > A_1 \to \boxed{c > 0}$ — $\boxed{\text{front advances}}$ toward $x>0$

$x_1 \to x_3$

$A_1 < A_2 \to c < 0$ — front advances toward $x < 0$

$x_1 \leftarrow x_3$

$A_1 = A_2 \to c = 0$ — front stationary — co-existence

→ Area rule is akin to Maxwell construction in thermodynamics. phase coexistence

Equal areas ⟺ co-existence of phases.

root - phases

→ Bistable dynamics is much richer than simple "Pac-Man" dynamics of Fisher.



HW Hinton → roots (Hinton)

Calculate $c$ ? → approximation ?

$$\frac{\partial u}{\partial t} = A(u-u_1)(u_2-u)(u-u_3) + D\frac{\partial^2 u}{\partial x^2}$$

$u = u(x-ct)$, $\quad u(-\infty) = u_3$
$\quad u(\infty) = u_1$ asymptotic behavior

$$L(u) = 0 = Du'' + cu' + A(u-u_1)(u_2-u)(u-u_3)$$

$u_1$ $u_3$

simple solution:

- assume $u$ satisfies

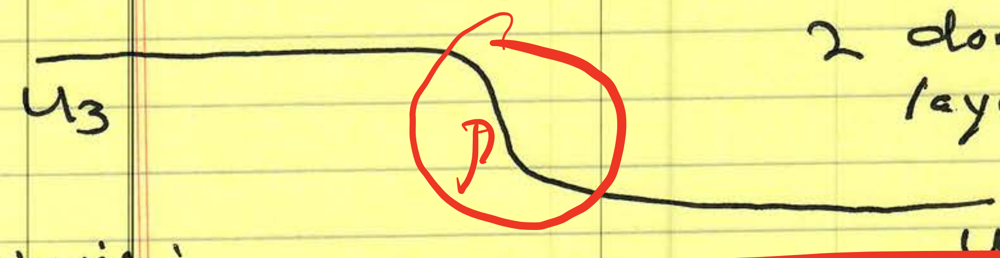
$$u' = a(u-u_1)(u-u_3)$$

→ trial determine $a$

why?

$u' = 0$ at $u_1, u_3$

i.e. has form:




2 domains + layer.

solving:

$$u(z) = \frac{u_3 + k\,u_1 \exp[a(u_3-u_1)z]}{1 + k\exp[a(u_3-u_1)z]}$$

Plugging into $0 = L(u)$,

$$L(u) = D\left[a(u-u_1)(u-u_3)\right]'$$
$$+ c\left[a(u-u_1)(u-u_3)\right]$$
$$+ A(u-u_1)(u_2-u)(u-u_3)$$

and repeat for $u'$ terms in above.

33.

$$L(u) = D\left(a(u-u_1)(u-u_3)\right)' + \left[c\,a\,(u-u_1)(u-u_3)\right]' + A(u-u_1)(u_2-u)(u-u_3)\Big]$$

$$= Da\left[u'(u-u_3) + (u-u_1)u'\right] + \left[\text{above}\right]$$

and plug in $u'$ again

$$= Da\left[a(u-u_1)(u-u_3)(u-u_3) + a(u-u_1)^2(u-u_3)\right] + \left[\quad\right] = 0$$

$A = 2Da^2$

re-grouping:

$$L(u) = (u-u_1)(u-u_3)\left[\boxed{(2Da^2 - A)u} - \left(Da^2(u_1+u_3) - ca - Au_3\right)\right]$$

speed $c$

so

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HW#2 → Translate FN to L→H problem

34.

so $L(u) = 0$

$\Rightarrow \quad 2Da^2 = A$

$$Da^2(u_1 + u_3) - Au_2 - ca = 0$$

or $a = (A/2D)^{1/2}$

and

speed !

$$\boxed{c = (AD/2)^{1/2}(u_1 - 2u_2 + u_3)}$$

values of 3 roots

$A \Leftrightarrow \gamma \quad \Rightarrow \quad \boxed{c \sim (D\gamma)^{1/2}}$, as before

speed

and $\boxed{c \cong \sqrt{D\gamma}\; c(u_1, u_2, u_3)}$

param. in reaction.

$$\boxed{u_2 = (u_1 + u_3)/2} \quad \rightarrow \quad \text{stationary front}$$

$c = 0$

Thus, speed set by parameters in reaction function. Weighted balance between these is the key.

$E'$ $F'$ $u \sim E'$

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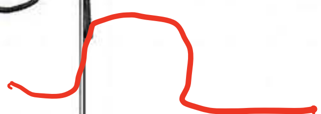
# Toward Pulses/Waves

Consider:


0BCD0

Hysteresis loop



"Excitable media"

threshold

$0<a<1$

$v_c$

$$\partial_t u = f(u) - v - D\partial_x^2 u$$

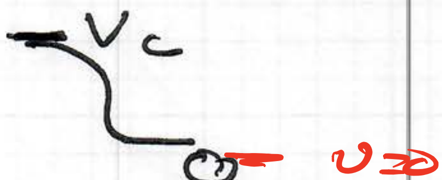
$$\partial_t v = bu - \gamma v$$


for slow eqn:

$$v_t = \epsilon(Lu - Mv)$$

Treat $v$ as fixed for fast bifurcations

$v_c$




if $v \to 0$ i.e.

$$\partial_t u = D u_{xx} + f(u)$$

$$f(u) = u(a-u)(u-1)$$

→ leading edge set at $v=0$

[redacted]

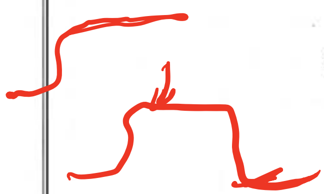
X

36.

if V finite:

$$\boxed{u_t = D u_{xx} + f(u) - V_c}$$

$\downarrow$ changes effective reaction function



so

$$c_- = \left(\frac{D}{2}\right)^{1/2}(u_c - 2u_p + u_0)$$

where $u_c, u_p, u_0$ roots of

$$f(u) = V_c$$

$\rightarrow$ speed is function of amplitude of V

$\downarrow$

like soliton


then pulse condition: coherence

$$\boxed{c_+ = c_-(V_c)}$$

guarantees that pulse will not disperse $\rightarrow$ i.e. forward and backward transitions propagate together at same speed. This sets a critical amplitude $V_c$ for

X

37.

pulse to be excited.

→ Slow evolution links the up, back transitions.

i.e.,

→ The FN model is basic simple model of pulse in excitable media.

→ so constructed from basic element of bistable front.

If time → spirals, etc. coming

In MFE — "Reaction is in the diffusion."

— More akin Cahn-Hilliard

coming

Staircase

Well separated

Ashourvan

2 step

turbulent QH

1-3 steps


$u_1$ → steel $u_3$

How build-up ?

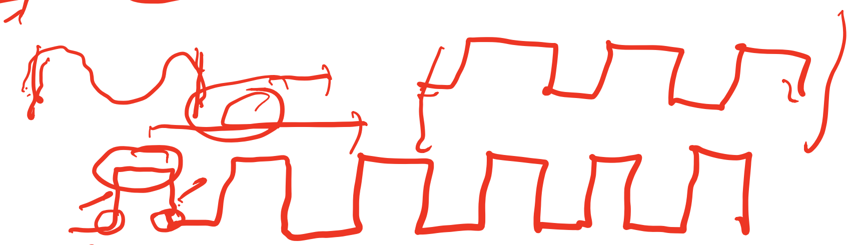
Speed pred-prey →

Feedback on ...

modulation + NL

bistable function

a ref



pulses

$\varepsilon$ $C_- = C_+$ chain pulse → stationary SCs

$C_- = C_+ = 0$ → stationary SCs

~ lattice of (small) carriers

Ideal cell → ELTs