

# PROFILES OF WIND DIRECTION AND SPEED IN TURBULENT EKMAN FLOW

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**ABSTRACT.** The profiles of wind speed and direction in turbulent Ekman flow are formulated based on asymptotic theory and data from direct numerical simulation. The profile of the stream-wise component follows the classical viscous, logarithmic and wake scaling. In the outer layer, the velocity component profiles can be described by an Ekman-spiral with adapted boundary conditions that result in a reduction of the spiral-like rotation. The span-wise component poses a conceptual challenge to the channel-flow analogy in the context of asymptotic matching; it exhibits a mixed scaling in the surface layer, but follows outer scaling for most of the outer layer.

## 1. INTRODUCTION

Wind veer away from its geostrophic direction is (i) due to direct frictional effects in the very vicinity of the surface and (ii) due to turbulence which exerts indirect frictional effects; these effects cause a slow-down of the mean wind reducing the Coriolis force thus turning the wind in favor of the pressure gradient force. Not only does the veering set the frame of reference for surface layer theory, it also has effects at small and large scales. From a large-scale perspective, the veering of wind across the planetary boundary layer determines the amount of cross-isobaric mass-flux, commonly referred to as 'Ekman pumping' (Ekman, 1905), and it is thus a key factor in the life-cycle of large-scale synoptic systems. At the small-scale end, the directional shear of the wind in the upper part of the surface layer may cause a systematic yaw for tall wind power generation devices where blades reach into the Ekman layer, i.e. that part of the boundary layer where the wind starts to turn.

Wind veer in the planetary boundary layer is commonly characterized by the surface veering angle  $\alpha$  defined as the angle between the negative surface shear stress  $\tau_{\text{sfc}}$  and the geostrophic wind. Surface veering  $\alpha$  and geostrophic drag  $Z \equiv u_{\star}/G$ , where  $u_{\star} \equiv \sqrt{\nu|\tau_{\text{sfc}}|}$ , uniquely determine the surface drag  $\tau_{\text{sfc}}$  in a turbulent Ekman flow. In any quantitative description of the surface layer, the friction velocity  $u_{\star}$  is the dynamic scale and  $\alpha$  defines the alignment of the frame of reference. Knowledge about  $u_{\star}$  and  $\alpha$  is thus a pre-requisite for any quantitative theory of the surface layer and a first attempt in theoretical quantification was undertaken by Rossby and Montgomery (1935) using integral relations in the boundary layer. Asymptotic similarity theory was later used by Tennekes (1973) and Blackadar and Tennekes (1968), and—based on his seminal direct numerical simulations (DNS) of Ekman flow—, Spalart, 1989 suggested a modification to take into account effects of low to intermediate Reynolds numbers. Later on, constants were re-evaluated with a focus on the atmospheric boundary layer based on observations (Högström, 1988;

Högström, 1996) and numerical modelling (Spalart et al., 2008; Spalart et al., 2009; Ansorge and Mellado, 2014; Ansorge, 2019).

Attempts were also undertaken to obtain profiles of the wind speed: One approach is to match the inner and outer layer at a reference height; Etling (2002) and Emeis (2018) (Sec. 21.10; Eq. 21.48) choose the Prandtl-layer height to match the wind speed profiles, which, however, requires to externally prescribe the veering  $\alpha$  at that height. A one-dimensional profile with constant veering is given by Emeis et al. (2007, Sec. 3, Eq. 3.1-3.19).

Gryning et al. (2007) present an extension of the wind-speed profile beyond the surface layer using a neutral reference profile and a stability correction; Kelly and Gryning, 2010, based on a probabilistic representation of stratification, develop a model for the long-term mean windspeed in the atmospheric boundary layer and compare this with observation at different sites; Kelly and Troen, 2016 demonstrate the effect of such improved model for wind-energy applications. In consideration of the large scale separation in geophysical flow, the rotation of the wind in the surface layer is often assumed negligible, and above investigations merely focus on the wind speed; that means, the veering of the wind with height is not described and there is little knowledge on the profile of the span-wise velocity component and the precise shape of the hodograph in the limit of a truly neutral Ekman boundary layer.

Ekman-layer models and two-layer models of the ABL take into account rotational effects at higher altitudes, for instance when the wind speed needs to be evaluated at heights on the order of 100-200 m, a particular concern when it comes to wind-power forecasting (Optis et al., 2014). Despite rotational effects being considered, the formulation of these models for the outer layer and analysis of their performance primarily focuses on wind speed. Still, in 2018, Jiang et al. recognized that the outer part of the Ekman boundary layer receives less attention in comparison with the surface layer and commence in studying the neutral problem by Large-Eddy simulation (LES). They focus on the wind speed and find an extended logarithmic layer when considering the wind speed instead of the shear-aligned component, and they eventually demonstrate by means of an analytical model that this vertical extension of the logarithmic layer may be explained by a transfer of stress to the spanwise velocity component where it is assumed that the shear vector  $\tau(z)$  and stress vectors  $(\partial_z U, \partial_z V)$  are aligned.

Turbulent Ekman flow is considered here as a conceptual model of the homogeneous, stationary atmospheric boundary layer over a flat surface under neutral stratification. Universal profiles of the wind vector for turbulent Ekman flow not only are a well-described limit for theoretical exploration or higher-order approaches taking into account possible effects of stratification, roughness or other physical complications encountered in the realgeophysical system. While, on first sight, the study of such a strongly idealized case appears as an academic problem, it contains the essence of surface similarity as it is used in most atmospheric models, be it conceptual or numeric ones. For instance, effects of roughness are commonly incorporated by a linear transformation of vertical scale involving the roughness parameter  $z_0$  and for larger roughness also a displacement height (Monin and Yaglom, 1975; Jacobs and Van Boxel, 1988; Högström, 1988); effects of stability are commonly accounted for by a linearization around the neutrally stratified profile (Monin, 1970; Monin and Yaglom, 1975; Högström, 1988; Högström, 1996; Sakagami et al., 2020). Furthermore, profiles can serve as better initial condition for numerical simulation of the flow,

to minimize the length of initial transient periods, or as benchmark for turbulence closures that can be tuned to reproduce the neutral limit case.

Despite the strong simplifications implied by our choice of set-up, there is no straightforward approach to solving this conceptually well-defined problem. Large-Eddy simulation not only needs to be tuned for the surface shear stress and veering angle, but it also relies on sub-grid closures that often assume alignment of the stress with gradients – a questionable pre-requisite when the wind rotates with height. Esau (2004) investigated the representation of the Ekman boundary layer by dynamical subgrid closures and Zikanov et al., 2003 proposed a closure for the wind profile using a linearized representation of the eddy viscosity. Despite advances in analysis of this simplified set-up (Jiang et al., 2018), there is yet insufficient understanding for a quantitative generalization of the results to arbitrary external forcing (manifest in variation of the Reynolds number) – and indeed the fundamental questions pertaining to such relatively simple dynamics of turbulence are not reflected in the research foci of LES for the ABL over the past 50 years (Stoll et al., 2020).

At the same time, an increasing amount of high-quality and high-resolution data from turbulence-resolving approaches is emerging due to recent advances in high-performance computing and its application to geophysical problem sets; the geophysical range of scale separation, however, is—and it will remain so for the foreseeable future—out of reach for such simulation (Dimotakis, 2005). Here, the routinely employed concept of Reynolds-number similarity can help. It postulates the existence of “*fully developed turbulence*” believed to occur for a sufficiently large but finite Reynolds number” (Barenblatt and Goldenfeld, 1995). (Already in 1998, this in fact lead Moin and Mahesh to the question *how high a Re is high enough?*) In fully developed turbulence certain statistics such as dissipation (Dimotakis, 2005) or profiles of mean velocity (Barenblatt, 1993) become independent of the Reynolds number when scaled appropriately while other statistics, such as the near-wall maximum in velocity fluctuation depend on Re (Baars and Marusic, 2020) and external interfaces may exert an impact on near-wall scaling (Silva et al., 2014). It appears that for certain statistics in Ekman flow, fully-developed turbulence is reached with the Reynolds numbers that became possible with the increasing capabilities of supercomputing over the past decades.

This paper exploits the robust features of mean velocity profiles from direct numerical simulation across a range of Reynolds numbers to formulate both the stream-wise and span-wise components of the mean velocity vector as a function of the Reynolds number.

## 2. PROBLEM FORMULATION AND NUMERICAL APPROACH

We consider here incompressible, turbulent Ekman flow, that is, the turbulent flow over a flat rotating plate, as a physical model for the truly neutral atmospheric boundary layer (ABL). The f-plane approximation is applied such that rotation only acts on horizontal velocity components; we thus neglect rotational effects on the horizontal components of velocity and dynamical effects due to latitudinal variation of the rate of rotation.

**2.1. Notation and Governing Equations.** The dimensional velocity vector of the numerical simulations is  $\underline{U} = (U_1, U_2, U_3) = (U, V, W)$  over the coordinate system

$Oxyz$ , where an approximate alignment (plus/ minus few degrees centigrade) of the direction  $Ox$  with the surface shear stress is achieved. The coordinate  $Oy$  points away from the wall, and  $Oz$  points in the span-wise direction normal to  $Oxy$ . For analysis of the results, we use two coordinate systems that are (i) exactly aligned with the surface shear stress and labelled by an upper index  $\alpha$  as in  $\underline{U}^\alpha$  for the velocity vector, and (ii) the coordinate system aligned with the free-atmosphere geostrophic wind labelled by an upper index  $G$  as in  $\underline{U}^G$ . We denote by  $u_*$  the modulus surface friction velocity and let  $Z_* = 1/u_*$  for brevity.

We consider the incompressible Navier–Stokes equations for the three velocity components on the f-plane in a framework that is governed by the physical parameters (i) geostrophic wind magnitude  $G = \sqrt{G_1^2 + G_3^2}$  (where  $\mathbf{G} = (G_1, 0, G_3)$  is the geostrophic wind vector), (ii) Coriolis parameter  $f$  (representing the angular rotation), and (iii) kinematic viscosity  $\nu$ . These yield a Reynolds number  $\text{Re}$  as non-dimensional parameter governing the system. We use the geostrophic wind as velocity and the Coriolis parameter  $f$  as time scale for the non-dimensional framework which implies the Rossby radius  $\Lambda_{\text{Ro}} = G/f$  as length scale, such that

$$(1) \quad \text{Re} = \frac{G\Lambda_{\text{Ro}}}{\nu}.$$

The non-dimensionalized governing equations read as

$$(2a) \quad \frac{\partial u_i}{\partial t} = \frac{\partial \pi}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j} + \epsilon_{i2j}(u_j - G_j) + \frac{1}{\text{Re}} \frac{\partial^2 u_i}{\partial x_j^2}$$

$$(2b) \quad \frac{\partial u_j}{\partial x_j} = 0,$$

where  $u_i = U_i/G$  are the non-dimensional components velocity,  $\pi$  is non-dimensional pressure, and  $g_j = G_j/G$  are non-dimensionalized components geostrophic wind (with  $g_1^2 + g_3^2 = 1$  by construction). These equations are solved inside a bounded cube of size  $L_x \times L_y \times L_z$  with periodic boundary conditions in the lateral directions  $Oxz$ , a no-slip–no-penetration boundary at  $y = 0$  and a no-penetration, free-slip boundary at  $y = L_y$ .

**2.2. Numerical Simulations.** The problem is solved numerically by tLab<sup>1</sup>, an open-source tool-suite to simulate and analyze turbulent flows. We use here a fourth-order–five-step Runge–Kutta integration and sixth-order compact schemes for spatial derivatives in all directions. The incompressibility constraint is enforced by a fractional step approach where the Poisson equation for the pressure field is soled to machine accuracy using a combined spectral/compact approach as described in Mellado and Ansorge (2012).

Simulations used here are shown in Tab. 1. We extend an existing set of simulations for  $\text{Re} \in \{125\,000, 281\,250, 500\,000\}$  (gray shading; cf. Ansorge and Mellado, 2014; Ansorge and Mellado, 2016) by new simulations at higher Reynolds numbers up to  $1.28 \times 10^6$ . In total, this yields one order of magnitude variation in terms of the scale separation in the boundary layer. [(new and old) Simulations]

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<sup>1</sup><https://github.com/turbulencia/tlab>

TABLE 1. Direct numerical simulation datasets used in this work

$\text{Re}_\Lambda$	$\text{Re}_D$	$L_{xy}/\Lambda$	$N_x \times N_y \times N_z$	$\Delta x^+$	$\Delta y^+$	$\Delta z^+ _{z=0}$
125 000	500	1.08	$2048 \times 2048 \times 192$	4.1	4.1	1.05
281 250	750	1.08	$3072 \times 3072 \times 384$	5.6	11.2	1.60
500 000	1 000	1.08	$3072 \times 6144 \times 512$	9.3	4.7	1.14
845 000	1 300	0.54	$2560 \times 5120 \times 640$	8.9	4.5	0.99
1 280 000	1 600	0.54	$3860 \times 7680 \times 960$	8.6	4.3	1.00

TABLE 2. Bulk characterization of the simulations

$\text{Re}$	$\delta^+ = \text{Re}_\tau$	$u_\star$	$\alpha$	$\delta_{95}/\delta$	$f u_\star^{-3} \int_0^\delta \epsilon dz$	$G^{-3} \int_0^\delta \epsilon dz$
500	479	0.0619	25.5	0.66	0.88	1.31
750	886	0.0561	21.0	0.65	0.90	1.34
1000	1403	0.0530	18.8	0.62	0.92	1.30
1300	2122	0.0501	17.9	0.59	0.85	1.24
1600	2978	0.0482	17.2	0.61	0.91	1.21

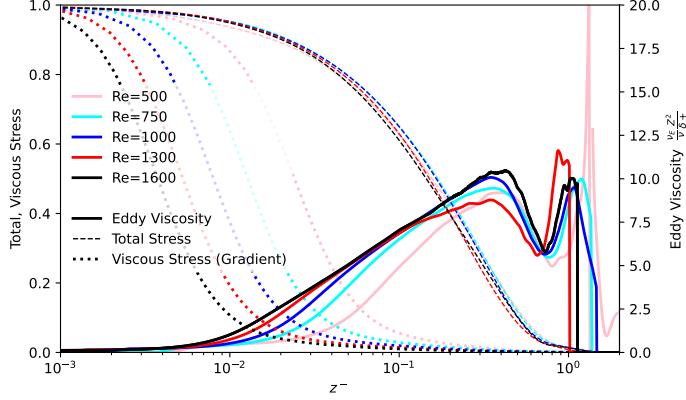


FIGURE 1. Reynolds stresses and eddy viscosity across the Ekman layer plotted in linear scaling for various Reynolds number

### 3. OVERVIEW OF THE NEW SIMULATIONS AT $\text{Re}_\tau$ UP TO 3000

Uniqueness and necessity of new simulations

PARAGRAPH ON BULK PARAMETERS AND SCALING (cf. 2)

PARAGRAPH ON VELOCITY AND STRESS PROFILES cf. 1

PARAGRAPH ON TURBULENCE STRUCTURE (2D SLICES, SPECTRA?)

cf. Fig. 2

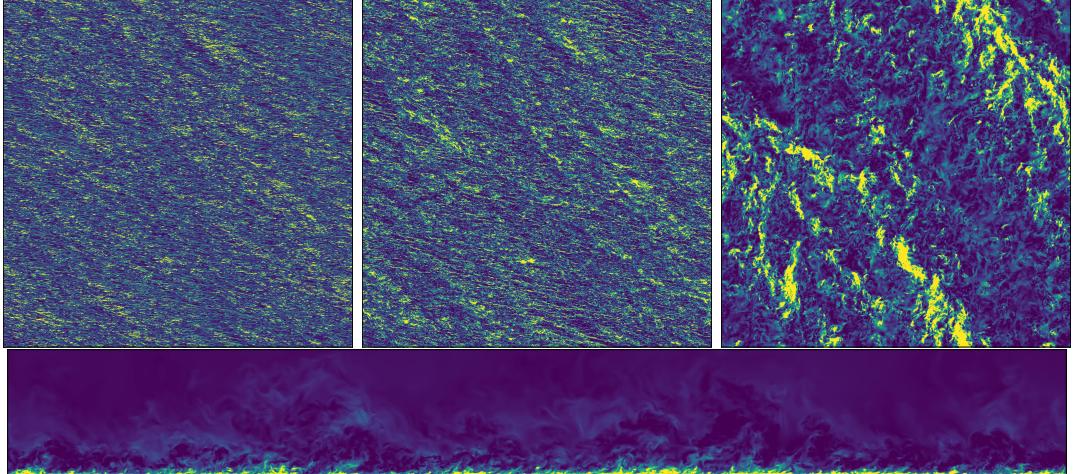


FIGURE 2. horizontal slices of turbulence kinetic energy in the Buffer layer ( $i=10$ ), Logarithmic Layer ( $i=100$ ) and outer layer ( $i=400$ ); coloring between percentiles 4 and 96 of the respective image. Lower panel: streamwise–vertical intersect through the domain

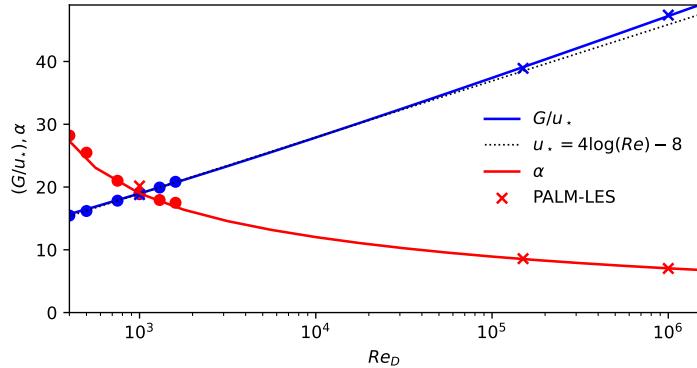


FIGURE 3. Variation of geostrophic drag,  $Z_*$ , and surface veering,  $\alpha$ , with Reynolds number according to the theory by Spalart et al. (1989) and as estimated from DNS data

#### 4. TOWARDS A UNIVERSAL VELOCITY PROFILE FOR THE TURBULENT EKMAN LAYER

Formulations for the outer layer that take into account the rotation, and thus deviation from the channel-flow analogy need to be matched to the framework of surface similarity, and it occurs that a smooth transition from the inner to the outer layer is not easily achieved. Optis et al., 2014, for instance, define an “effective geostrophic wind vector that has the same magnitude of the observed surface geostrophic wind and is rotated by the angle  $\alpha$  [their nomenclature]” to overcome the unsteady transition when coming approaching the Ekman layer from below. While such rotation of the wind vector is *a posteriori* justified by the observational data that the model outcomes are compared to, we believe that it is the manifestation of a mismatch between the theoretical treatment of the inner and outer layer of turbulent Ekman flow.

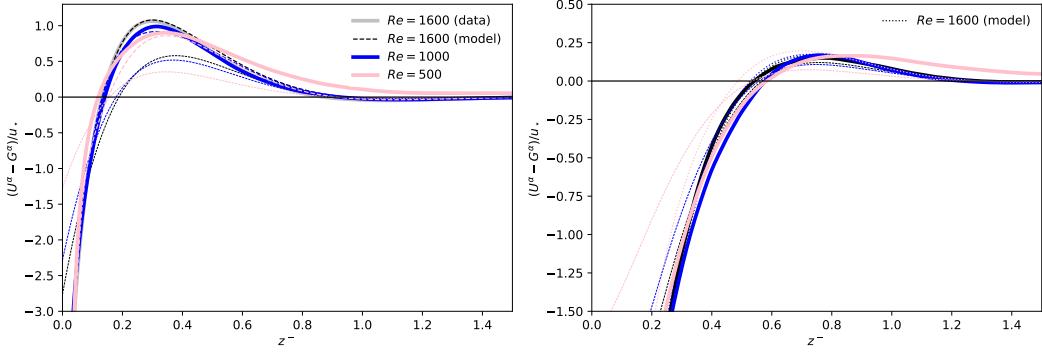


FIGURE 4

**4.1. Drag-Law.** A drag-law for Ekman flow determines two key parameters of the flow as a function of Reynolds number alone. These are, first, the modulus of the surface friction normalized by the geostrophic wind forcing,  $u_*$  and, second, the direction of surface shear stress,  $\alpha$ , also termed wind veering. A non-zero veering of the wind is a rather special case in comparison with most turbulent flows considered in an engineering context, and it confronts us with a situation where the most appropriate coordinate system for analysis (namely that aligned with the surface shear stress) is a priori unknown. We compare our DNS data against a semi-empirical drag-law based on integral consideration Spalart, 1989 and find, as demonstrated in previous work Ansorge and Mellado, 2014, excellent agreement in the range  $400 < \text{Re} < 1600$ , representing a factor of 16 in variation of viscosity.

We also find that the solution of the transient equation involved in estimation of  $u_*$  for a given Reynolds number  $Re_D$  is approximated reasonably by the formulation

$$(3) \quad u_* = 4 \log(Re_D) - 8$$

which quantifies the 'weak' dependence of  $u_*$  on the Reynolds number as an approximately logarithmic one, at least for problems with a scale separation on the order that is relevant to geophysical problems ( $Re_D < 10^8$ ).

**4.2. Profile in the outer layer.** The outer layer of Ekman flow is characterized by a turning of the wind velocity and the super-geostrophic maximum that is sustained by momentum convergence at the inflection point of the velocity profile. The super-geostrophic maximum of streamwise velocity and a secondary minimum aloft the bulk-turbulent part of the boundary layer are well-described by a classic Ekman spiral with adapted boundary conditions and a shift in reference height.

**4.3. Stream-wise Velocity Component.** For the stream-wise velocity profile (that in non-rotating flows due to the geometry is always aligned with the surface shear stress), well-established theories exist for various regimes according to their distance from the wall and the relative role of viscosity, turbulence and interaction with the outer region of the flow with the logarithmic law for the mean velocity as a central anchor point.

In immediate vicinity to the surface, local turbulent mixing cannot occur for the no-slip/no-penetration boundary condition, and the mean velocity is described

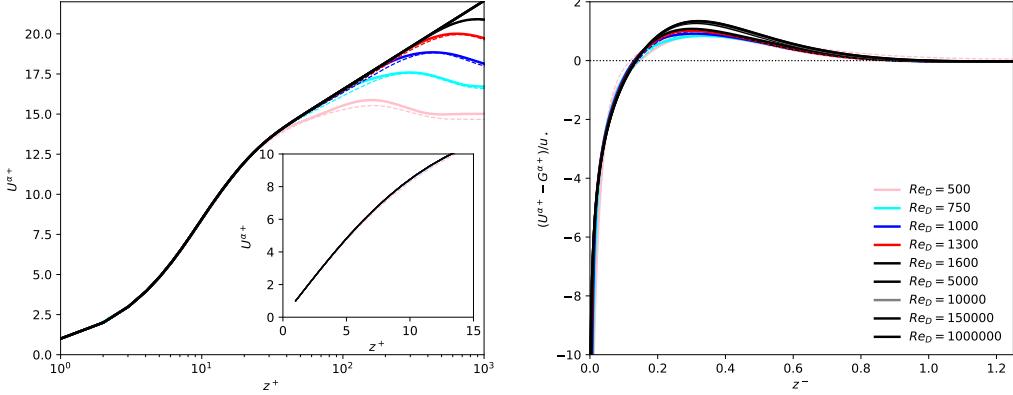


FIGURE 5. Shear-aligned profiles of velocity components  $U^{\alpha+}$  in inner (left) and outer (right) units.

by a viscous profile of the form

$$(4a) \quad (U, W)^{\alpha+} = (z^+, 0)$$

where the direction of the velocity points into the exact opposite direction of the wall shear stress  $\tau$ . In absence of roughness elements and for small roughness ( $z_0^+ < 5$ ), this linear regime is known as viscous sub-layer Foken, 2002; Foken et al., 1978. In fact, this law of the wall has no degree of freedom given the drag, i.e. once  $u_*$  and  $\alpha$  are defined. However, theoretical foundation is lacking for the exact shape of the velocity profile in the buffer layer; though crucial for turbulence production, it is commonly understood as a transition region between the linear profile at the surface and the logarithmic profile aloft. A pure blending from the linear velocity profile into the logarithmic one is, however, not reasonable as both the linear and logarithmic profile overestimate the velocity in the buffer layer. We therefore introduce a higher-order compensating term into the linear profile before blending it into the logarithmic law:

$$(4b) \quad U_{\text{inner}}^{\alpha+} = \frac{y^+ + \gamma_4(y^+)^4 + \gamma_6(y^+)^6}{1 + \gamma_6/u_{\text{ref}}(y^+)^6},$$

where the fourth- and sixth-order terms in the nominator are designed to fit the profile below  $y^+ \approx 10$  and the sixth-order correction in the denominator limits the growth of  $u_{\text{inner}}$  for large  $y^+$ .

In the logarithmic region, we use the profile

$$(4c) \quad U_{\log}^{\alpha+} = \frac{1}{\kappa} \log y^+ + C$$

with the von-Kármán constant  $\kappa = 0.416$  and the boundary condition  $C = 5.4605$ .

**4.4. Span-wise velocity.** The background rotation and associated veering of the surface wind implies a non-zero profile for the span-wise velocity which challenges the conventional assumptions related to the so-called channel-flow analogy: While the universal profiles in vicinity of the wall implies that the profile be zero or at least small with respect to the stream-wise component, the veering requires a value of  $V_{top} = U_G \sin \alpha$  in the free stream (and thus also at the top of the boundary layer if we assume that substantial velocity gradients are confined to the turbulent part of

the flow). This is commonly shown in terms of velocity hodographs aligned with the outer, geostrophic flow (cf. Fig. 7) and normalized by the geostrophic wind. Thus, the drag-law already implies three important properties of these hodographs, (i) the boundary conditions at the surface, (ii) the boundary condition at the top, and (iii) the inclination of the hodograph at the origin by the surface veering:

$$(5a) \quad \underline{U}^G(z=0) = 0,$$

$$(5b) \quad \underline{U}^G(z \mapsto \infty) = \underline{G}^G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(5c) \quad \left. \frac{\partial V^G}{\partial U^G} \right|_{z=0} = \sin \alpha$$

Outer scaling of the velocity profile further implies that the velocity deficit of the spanwise component  $V^\alpha - G^\alpha$  be a universal function of the outer height  $z^-$ , i.e.

$$(6) \quad V^\alpha - G^\alpha = f_V^\alpha(z^-).$$

Interpreting the shear-aligned spanwise velocity deficit  $f_V^\alpha(z^-)$  as a signature of outer rotation may lead one to conclude that this universal function is appropriate down to the surface—irrespective of the Reynolds number. This cannot be the case for the variation of  $f_V^\alpha(z^-)$  across the boundary layer (i.e. between  $0 < z^- < 1$ ) must fulfill the drag law and boundary conditions; at the same time, the weak dependence of  $\alpha$  and  $u_*$  on  $Re$  implies this range depends on  $Re$ .

**Outer Layer.** In the outer region of the flow (for  $z^- \mapsto 1$ ),  $f_V(z^-)$ , should govern the spanwise velocity profile which indeed is supported by our DNS data (Fig.6): The deficit is close to zero for  $z^- \gtrsim 1$ , and can be approximated by a logarithmic profile in the range  $0.15 \lesssim z^- \lesssim 0.4$ . The slope of this profile is best fit by  $\kappa_V = 0.836(1 + 150/Re_D)\kappa$ , where the correction term for low Reynolds number may be omitted for  $Re_D \gg 1000$ . Based on Fig. 6, we pick the offset as a constant fraction (0.906) of  $V_G^\alpha = Z_* \sin \alpha$ , i.e

$$(7a) \quad V_{\log}(z^-) = \frac{1}{\kappa_V} \log \left( \frac{z^-}{0.4} \right) + 0.906V_G^\alpha.$$

The eventually converged simulations at intermediate  $Re_D$  (where the DNS could be run over several inertial periods) exhibit a super-geostrophic maximum (positive velocity deficit) in the span-wise component. This maximum is not present or less pronounced for the simulations at higher  $Re$  where closer analysis reveals that the velocity profiles in the outer layer are still subject to a small though finite adaptation process on the inertial time scales  $2\pi/f$  due to the relatively short time span for which these cases could be simulated. While there is no physical model for this wake region that leads to an analytical velocity profile formulation in the span-wise direction, an error-function transition from the outer logarithmic law to the boundary condition provides a reasonable fit:

$$(7b) \quad V_{\text{outer}}(z^-) = (1 - w_V)V_{\log}(z^-) + w_V V_G^\alpha$$

with

$$(7c) \quad w_V = \frac{\text{erf}[\xi_{\text{outer}} \log(z/z_{\text{ref}})] + 1}{2}$$

where  $\xi_{\text{outer}}$  is a transition scale that defines the width of the wake region and  $z_{\text{ref}}$  defines the height. Based on the DNS data, we choose  $\xi_{\text{outer}} = 2$  and let  $z_{\text{ref}}$  such that  $V_{\log}(z_{\text{ref}}) = 0.98V_G^\alpha$ . There is, however, a degree of ambiguity in the exact

choice of these parameters due to the imperfect convergence of the DNS data to the statistical equilibrium. Inner Layer. It is important to note that the logarithmic layer observed here for the spanwise velocity does not coincide with the log-region of the streamwise velocity but is rather a 'continuation' through the outer layer. This is consistent with the recent finding of a second log-region in the outer region of Ekman flow (... et al. (incl. Sullivan), JAS 2019). Below this region, the gradients in spanwise velocity are rather small and the span-wise velocity monotonically approaches its surface boundary condition  $V(z = 0) = 0$ . While the stream-wise velocity follows a universal inner scaling that has acquired its universal,  $Re$ -independent shape for  $Re_D > \mathcal{O}(10^3)$ , the span-wise component that defines how the velocity vector veers when the surface is approached, does not collapse in inner units, and there is, most importantly no sign of convergence even at the highest Reynolds numbers for which simulations were carried out. Even though the simplest assumption  $V = 0$  is reasonable for the lower part of the surface surface layer ( $z^- < 10^{-3}$ ), it does not appropriately capture the profile in the rest of the surface layer:

First,  $V = 0$  implies a discontinuity in the velocity profile at  $z^- = 0.1$ , where the outer scaling found above yields a finite value at geophysical  $Re$ , i.e. there is non-zero veering in the upper part of the surface layer—as is well-known also from field observation. Second, the layer around  $z^- = 0.1$  is crucial to obtain the characteristic and well-established shape of the hodographs as the layer where  $V$  sets in marks the 'maximum' of  $V^-$  vs.  $U^-$ . This illustrates that the higher-order (in terms of the channel-flow analogy where  $V = 0$ ) correction follows a mixed scaling when the surface is approached which has been demonstrated for higher-order terms in other types of flow (Mellado et al., 2016).

The scale for the magnitude of the span-wise velocity component is  $u_* \sin \alpha$ . Based on our DNS data, we suggest that the Reynolds number scaling of this velocity-magnitude scale is captured by  $Re_\tau^{-1/2}$  which is indeed known from the generalization of higher-order statistics, such as turbulent fluxes in the inner layer (Marusic et al., 2013) that also follow a mixed scaling in the inner layer. We then parameterize the spanwise velocity at 10 wall units as anchor point in the inner layer:

$$(8a) \quad V_{10} \equiv V(z^+ = 10) = 750 \frac{u_* \sin \alpha}{\sqrt{Re_\tau}}.$$

This leaves us with three fixed points of the velocity profile in the inner layer, namely (i) the boundary condition  $V_0 = 0$ , (ii)  $V_{10}$  at  $z^+ = 10$ , and (iii) the lower end of the logarithmic profile at  $z^- = 0.1$  where the latter two are semi-empirically estimated from DNS data. In absence of well-established scaling considerations for the span-wise velocity, the choice of profile fits joining these three points is indeed arbitrary, but we can resort to the DNS data for an empirical approach and find that a square-root profile fits  $V(z^+)$  in the surface layer. A linear profile for  $V$  is employed in the viscous sub-layer. Based on the physical extent of the viscous sub-layer in Ekman flow around five wall units (Foken, 2002; Ansorge, 2019), we choose  $z^+ = 5$  to transition from one to the other and note that  $V$  is already very small at this height. The span-wise velocity profile in the surface layer is then estimated as

$$(8b) \quad V(z^+) \Big|_{\text{inner}} = \begin{cases} a_1 z^+ & ; \quad z^+ \leq 5 \\ b_1 + b_2 \sqrt{z^+} & ; \quad 5 < z^+ < Re_\tau/10 \end{cases},$$

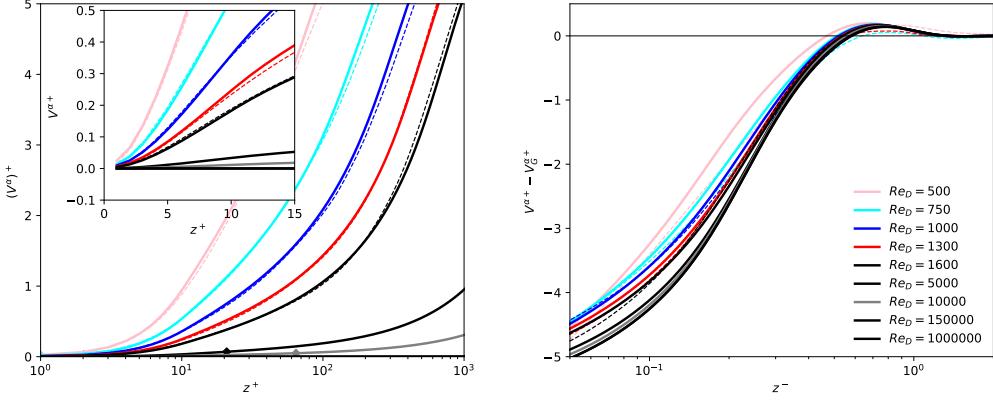


FIGURE 6. Profiles of shear-aligned span-wise velocity  $(W^\alpha)^+$  versus inner and outer height. Dashed lines show DNS data, thick, opaque lines are from the semi-empirical theory developed above.

with  $b_1$  and  $b_2$  estimated such that

$$(8c) \quad \begin{aligned} \frac{V(z^+ = 10)|_{\text{inner}}}{V(z^+ = Re_\tau/10)|_{\text{inner}}} &= \frac{V_{10}}{V_{\text{outer}}(0.1)} \Rightarrow \begin{cases} b_2 = \frac{V_{\text{outer}}(0.1) - V_{10}}{\sqrt{Re_\tau/10} - \sqrt{10}} \\ b_1 = V_{10} - \sqrt{10}b_2 \end{cases} \end{aligned}$$

We then estimate  $\alpha$  from the matching condition at  $z^+ = 5$ , i.e.

$$(8d) \quad 5a = b_1 + \sqrt{5}b_2 \Rightarrow a = \frac{1}{5} \left[ V_{10} + (\sqrt{5} - \sqrt{10}) \left( \frac{V_{\text{outer}} - V_{10}}{\sqrt{Re_\tau/10} - \sqrt{10}} \right) \right].$$

Matching region. While the profile composed of  $V_{\text{inner}}(z^+ \leq 0.1Re_\tau)$ ,  $V_{\text{outer}}(z^- > 0.1)$  is continuous, it is not smooth at  $z^- = 0.1$ , i.e. at the transition from power-law ( $V \propto \sqrt{z^+}$ ) to logarithmic scaling. To alleviate this issue, we use a second-order polynomial for transition from the inner to the outer layer in the range  $z_{\text{low}} < z < z_{\text{up}}$  such that

$$(8e) \quad V_{\text{trans}}(z^-) = V_{\text{inner}}(z_{\text{low}}^+) + \Delta V (az_{\text{arg}} + b(z_{\text{arg}})^2)$$

with  $\Delta V = V_{\text{outer}}(z_{\text{up}}^-) - V_{\text{inner}}(z_{\text{low}}^+)$  and  $z_{\text{arg}} = (z - z_{\text{low}})/(z_{\text{up}} - z_{\text{low}})$ . It is  $a + b = 1$  for  $V_{\text{trans}}(z_{\text{up}}^-) = V_{\text{outer}}(z_{\text{up}}^-)$ , and we constrain  $a$  by

$$(8f) \quad \left. \frac{\partial V_{\text{trans}}}{\partial z^-} \right|_{z=z_{\text{low}}} = \left. \frac{\partial V_{\text{inner}}}{\partial z^-} \right|_{z=z_{\text{low}}},$$

where we find that  $z_{\text{low}}^- = 0.06$  and  $z_{\text{up}}^- = 0.13$  yield satisfactory agreement with DNS data.

## 5. COMPARISON WITH OTHER THEORIES

**Compare** to Emeis (2002) and Gryning (2007); highlight explicit knowledge on veering-profile → directional sheer;

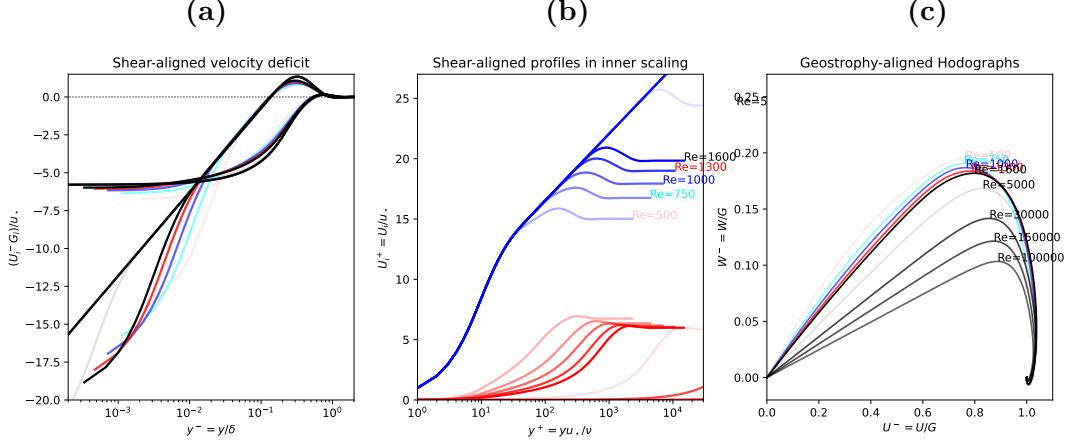


FIGURE 7. (a) Velocity deficit, (b) velocity profile in shear-aligned hodographs and (c) hodograph in geostrophy-aligned coordinates. Thick, solid lines show theory, dashed lines data from DNS.

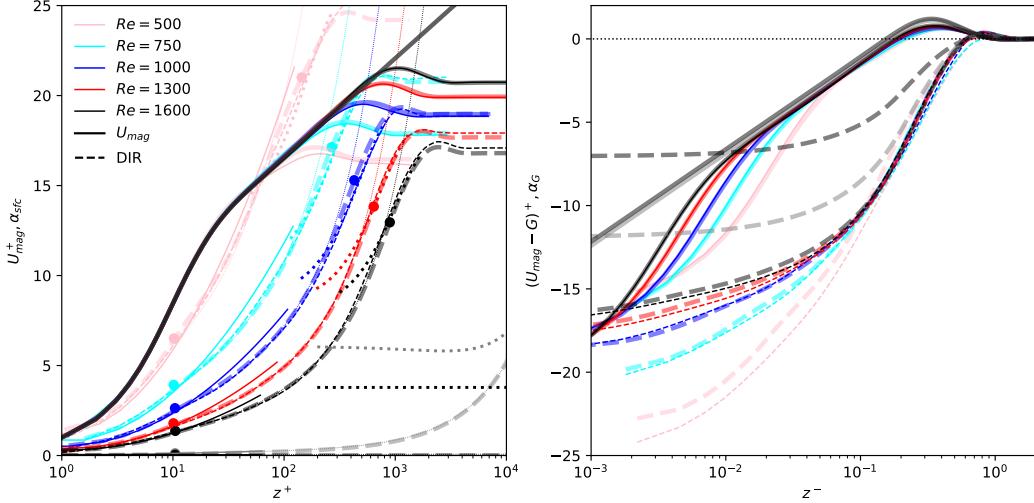


FIGURE 8. Total velocity and veering (in degrees) vs inner and outer height. Dashed lines show DNS data, thick lines are from semi-empirical theory.

Implications for **K-theory** (we now can consider that shear and stress are not necessarily perfectly aligned). → can we do something to infer a K-profile from these theoretical considerations?

## 6. CONCLUSIONS

Applications:

- reference-shear for neutral profile approaches(systematic!) → wind engineering!
- initial condition for LES/DNS to eliminate/minimize inertial oscillation in Benchmark simulations

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## APPENDIX A. LAMINAR EKMAN SOLUTION WITH CONSIDERATION OF INNER LAYER

$$(9a) \quad \begin{pmatrix} \partial_t U \\ \partial_t W \end{pmatrix} = \begin{pmatrix} fW & + \nu \partial_z^2 U \\ -f(U - G) & + \nu \partial_z^2 W \end{pmatrix}$$

$$(9b) \quad \Rightarrow \partial_t(U + iW) = f(W - i(U - G)) + \nu \partial_z^2(U + iW)$$

In stationary conditions, this system is solved by

$$(9c) \quad \hat{u}(z) = U_\infty + e^{-\gamma z} [A \cos \gamma z + B \sin \gamma z]$$

$$(9d) \quad \hat{w}(z) = W_\infty + e^{-\gamma z} [-A \sin \gamma z + B \cos \gamma z]$$

where the constants  $U_\infty$ ,  $W_\infty$  set the top boundary condition and  $A$  and  $B$  set the bottom boundary condition. The most common boundary condition for a surface Ekman layer is  $A = U_\infty = G$ ,  $B = 0$ , and  $W_\infty = 0$ . The lower boundary condition, however, neglects the existence of the surface layer, and it appears reasonable to define  $A = cG$  where  $c < 1$  is a constant that incorporates the increased shear in the surface layer. Given a 'matching height'  $z_{match}$  and normalized matching height  $\xi = \gamma z_{match}$  in the upper part of the inner layer, we can match the Ekman profile to the inner layer by letting

$$(10) \quad \begin{aligned} u(z_{match}) \equiv u_{match} &= U_\infty + e^{-\xi} [A \cos \xi + B \sin \xi] \\ w(z_{match}) \equiv w_{match} &= W_\infty + e^{-\xi} [-A \sin \xi + B \cos \xi] \end{aligned}$$

$$(11) \quad \Rightarrow \begin{pmatrix} u_{match} - U_\infty \\ w_{match} - W_\infty \end{pmatrix} = e^{-\xi} \begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} \cos \xi & +\sin \xi \\ -\sin \xi & +\cos \xi \end{pmatrix}$$

(12)

Matching the profile at  $\xi = 0$ , one obtains  $A = \Delta u_{match}$  and  $B = -\Delta w_{match}$ ; and when the direction  $Ox$  is aligned with the geostrophic wind, we obtain the textbook-case  $A = |\mathbf{G}|$  and  $B = 0$ .

Otherwise, choosing  $B \neq 0$  allows to introduce a phase shift of the Ekman rotation with respect to the decay of the wind spiral. As, however, in our context, the thickness and position of the spiral can already be controlled by the eddy viscosity and an offset in  $\zeta$ , here we let  $B = 0$ .

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