NOTES ON ETLING VELOCITY PROFILE FOR EKMAN/PRANDTL LAYER

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1. Basic velocity profiles

1.1. Velocity profile in the Prandtl layer.

$$u_p(z) = \frac{u_\star}{\kappa} log \frac{z^+}{z_0^+} \tag{1}$$

In coordinates of the geostrophic wind (for a veering angle of $\alpha_0 = \alpha(z=0)$

$$u_p(z) = \frac{u_\star}{\kappa} \log \frac{z^+}{z_0^+} \begin{pmatrix} \cos\alpha_0 \\ \sin\alpha_0 \end{pmatrix} \tag{2}$$

1.2. Velocity profile in the Ekman layer.

$$\begin{pmatrix} u_{\mathrm{ek}} \\ v_{\mathrm{ek}} \end{pmatrix}(z) = G\sqrt{2}\sin\alpha_0 \begin{pmatrix} \frac{1}{\sqrt{2}\sin\alpha_0} - & e^{-\tilde{z}}\cos(\tilde{z} + \frac{\pi}{4} - \alpha_0) \\ & e^{-\tilde{z}}\sin(\tilde{z} + \frac{\pi}{4} - \alpha_0) \end{pmatrix} \tag{3}$$

with the scaled Ekman height

$$\tilde{z} = \frac{z - z_P}{D},$$

where z_P is the Prandtl layer thickness (here assumed as 10% of the BL thicknes) and D the Ekman layer thickness. For D, there exists a matching based on the constant diffusivity within the Prandtl layer

$$D = \sqrt{2\frac{K_m}{f}} = \sqrt{2\frac{\kappa u_\star z_p}{f}} = \sqrt{2\kappa\delta z_p} \tag{4}$$

with $\delta = u_\star/f$. Using $z_p = 0.1\delta$, we obtain

$$D = \sqrt{0.2\kappa}\delta \approx 0.3\delta$$

2. Matching point

The profiles shall be matched at the upper end of the Prandtl layer ($z = z_P$). Given the geostrophic wind, Prandtl layer height z_p and height offset (in the case of a rough surface, the roughness height; for aerodynamically smooth surfaces a scaled version of the profile offset parameter), the profiles can be matched to yields the remaining parameters $u_\star$ and α_0.

$$\frac{u_\star}{\kappa}\log\frac{z_P}{z_0}\cos\alpha_0 = G\sqrt{2}\sin\alpha_0\left[\frac{1}{\sqrt{2}\sin\alpha_0} - e^{-\tilde{z}}\cos\phi\right] \tag{5a}$$

$$\frac{u_\star}{\kappa}\log\frac{z_P}{z_0}\sin\alpha_0 = G\sqrt{2}\sin\alpha_0 e^{-\tilde{z}}\sin\phi \tag{5b}$$

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where – utilizing the condition $z = z_P$, it is $\phi = \pi/4 - \alpha_0$. We can then eliminate $\sin\alpha_0$ from the equation for the span-wise component to obtain

$$\cos\alpha_0 = \frac{G}{u_\star}\sqrt{2}\lambda_z \sin\alpha_0 \left[\frac{1}{\sqrt{2}\sin\alpha_0} - \cos\phi\right] \tag{6a}$$

$$\frac{u_\star}{G} = \sqrt{2}\lambda_z \sin\phi \tag{6b}$$

where

$$\lambda_z = \frac{\kappa}{\log\frac{z_P}{z_0}}.$$

Next, we rewrite Eq. (6a) using Eq. (6b) as

$$\cos\alpha_0 = \sqrt{2}\lambda_z \frac{G}{u_\star}\left(\frac{1}{\sqrt{2}\sin\phi} - \frac{\sin\alpha_0\cos\phi}{\sin\phi}\right) \tag{7a}$$

$$\sin\phi\cos\alpha_0 + \cos\phi\sin\alpha_0 = \frac{1}{\sqrt{2}} \tag{7b}$$

The LHS of the latter equation can be rewritten as

$$\sin(\phi + \alpha_0) = \sin(\pi/4)$$

such that the equation holds for any α_0. Thus, α_0 becomes a parameter of the solution, and we can simply write

$$\frac{u_\star}{G} = \frac{\sqrt{2}\kappa}{\log(z_P/z_0)}\sin\left(\frac{\pi}{4} - \alpha_0\right) \tag{7c}$$

2.1. **Relation to the Reynolds number.** In a smooth flow, the scaling parameter of the logarithmic height is

$$\frac{1}{\kappa}\log z^+ + A = \frac{1}{\kappa}\left[\log z^+ - (-\kappa A)\right] = \frac{1}{\kappa}\left[\log z^+ - \log(e^{-\kappa A})\right] = \log(z^+/z_0^+)$$

where $z_0^+ = e^{-\kappa A} \approx 1/8$. This implies that $z_P/z_0 = z_P^+/z_0^+ = e^{\kappa A} z_P^+ = e^{\kappa A}\gamma_P Re_\tau$. Hence, the ratio z_P/z_0 implicitly defines the Reynolds number Re_τ as

$$Re_\tau = \frac{z_P}{z_0}\frac{1}{e^{\kappa A}\gamma_P} \sim Re_\tau. \tag{8}$$

2.2. **Linearization for α_0 in the trigonometric terms.** Based on a first guess for $u_\star$ and α_0, we can then estimate $u_\star/G$ from Eq. (6b) and α_0 from Eq. (6a) and iterate this procedure to arrive at a solution for α_0 and $u_\star$.

A fist guess may be obtained from a linearized version of the equations in the trigonometric terms. It is

$$\sin\phi = \sin(\pi/4 - \alpha_0) \approx \sin(\pi/4) - \cos(\pi/4)\alpha_0 = \frac{1}{\sqrt{2}}(1 - \alpha_0) \tag{9a}$$

$$\cos\phi = \cos(\pi/4 - \alpha_0) \approx \cos(\pi/4) + \sin(\pi/4)\alpha_0 = \frac{1}{\sqrt{2}}(1 + \alpha_0) \tag{9b}$$

$$\sin\alpha_0 \approx \sin(0) + \alpha_0 = \alpha_0. \tag{9c}$$

We thus obtain

$$-1 = \frac{G\lambda_z}{u_\star}\left[\frac{1}{\alpha_0} - (1 - \alpha_0)\right] \tag{10}$$

$$\Rightarrow 0 = \alpha_0 + \frac{G\lambda_z}{u_\star}\left[1 - (\alpha_0 - \alpha_0^2)\right] \tag{11}$$