HW 1

① a) $\boxed{a^{ij}(x_i + y_j)} = a^{ij}x_i + a^{ij}y_j$

C.EX: $\sum_{i=1}^{2}\sum_{j=1}^{3} a^{ij}(x_i + y_j) = a^{11}(x_1+y_1) + a^{21}(x_2+y_1) + a^{12}(x_1+y_2) + a^{22}(x_2+y_2) + a^{13}(x_1+y_3) + a^{23}(x_2+y_3)$

$a^{ij}x_i + a^{ij}y_j = \sum_{i=1}^{2} a^{ij}x_i + \sum_{j=1}^{3} a^{ij}y_j = a^{1j}x_1 + a^{2j}x_2 + a^{i1}y_1 + a^{i2}y_2 + a^{i3}y_3$

b) $\boxed{a^{ij}(x_i + y_i)} = \sum_{i=1}^{n} a^{ij}(x_i + y_i) = \sum_{i=1}^{n} a^{ij}x_i + \sum_{i=1}^{n} a^{ij}y_i = a^{ij}x_i + a^{ij}y_i$

② a) $\alpha(f) = \int_0^3 e^x f'(x)\,dx$, $f: \mathbb{R} \to \mathbb{R}$ contin.

$\alpha(a\cdot f + bg) = \int_0^3 e^x (af+bg)'(x)\,dx = \int_0^3 e^x (a\cdot f'(x) + b\cdot g'(x))\,dx =$

$= a\int_0^3 e^x f'(x)\,dx + b\int_0^3 e^x g'(x)\,dx = a\cdot\alpha(f) + b\cdot\alpha(g)$ ✓ LINEAR

b) $\beta(f,g) = f'(x)g(x)$ $f,g: \mathbb{R} \to \mathbb{R}$ cont. diff.

Lin. in first var: $\beta(af+bh, g) = (af+bh)'(x)g(x) = (af'(x) + bh'(x))g(x) = af'(x)g(x) + bh'(x)g(x) = a\beta(f,g) + b\beta(h,g)$

In second: $\beta(f, ag+bh) = f'(x)(ag+bh)(x) = af'(x)g(x) + bf'(x)h(x) = a\beta(f,g) + b\beta(f,h)$ ✓

$\Rightarrow \beta$ is Bilinear form

c) $\gamma(A,B) = tr(A^TB)$, where $A,B \in \mathbb{R}^{m\times n}$

$\gamma(aA + cC, B) = tr((aA+cC)^T B) = tr(aA^T B + cC^T B) =$

$= a\cdot tr(A^TB) + c\cdot tr(C^TB) = a\gamma(A,B) + c\gamma(C,B)$ ✓

Similar for the second component, as $tr(A^TB) = tr(B^TA)$

$\Rightarrow$ bilinear, as $A,B \in \mathbb{R}^{m\times n} = V$, $\gamma: V\times V \to \mathbb{R}$

d) $\delta(x,y) = \det(\underbrace{xy^T}_{\text{mtx of rang } 1})$, where $x,y \in \mathbb{R}^n$

$\Rightarrow$ if $n>1$, $\det(xy^T) = 0$

$\delta(x,y) = 0$ for all $x,y \in \mathbb{R}^n$; $n > 1$

$\Rightarrow$ is linear

$n=1$: $\delta(x,y) = x\cdot y$ is also linear.

$\Rightarrow \delta$ is bilinear form

3. τ linear form. $\exists\, a \in \mathbb{R}^n : \tau(x) = a^T x$

any x can be written as $\rightarrow$ standard basis for $\mathbb{R}^n$; $i = 1, \dots, n$

proof: $x = x_i e^i$

τ is linear : $\tau(x) = x_i \underbrace{\tau(e^i)}_{\text{scalar, since } \tau : \mathbb{R}^n \to \mathbb{R}} = [\tau(e_1) \cdots \tau(e_n)] \cdot x$

Then for us to know $\tau(x)$ for all x, we only need to know $\tau(e^i)$ for all i.

If we say $\tau(e^i) = a^i$; $a^T = [a_1^1\ a_2^2 \cdots a_n^n]$

$\tau(x) = x_i a^i = a^T x$

Notice that $a^T = [\tau(e_1) \cdots \tau(e_n)]$, so its components are equal to the mapping of basis vectors of the basis in which x is written.

4. $\boxed{\varphi(x,y) = x^T B y}$

$x = x_i e_m^i$ where e_m^i are vectors from st. b. of $\mathbb{R}^m$
$y = y_j e_n^j$ and e_n^j are from std. b. of $\mathbb{R}^n$.

$\tau(x,y) = x_i \tau(e_m^i, y) = x_i y_j \tau(e_m^i, e_n^j)$

linear in both variables

This double sum can be written as $x^T \underbrace{\begin{bmatrix} \tau(e_m^1, e_n^1) & \cdots & \tau(e_m^1, e_n^n) \\ \tau(e_m^2, e_n^1) & & \vdots \\ \vdots & & \\ & & \tau(e_m^m, e_n^n) \end{bmatrix}}_{B} y$,

let's note this mtx with B, then

$\tau(x,y) = x^T B y$ ∎

b) $\boxed{x^T A y = x^T B y \iff A = B}$ ($\Leftarrow$) trivial

($\Rightarrow$) We can write $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{m1} & & a_{mn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$, $x = \begin{bmatrix} x^1 \\ \vdots \\ x^m \end{bmatrix}$, $y = \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix}$

Then $x^T A y = x^i y^j a_{ij}$, $x^T B y = x^i y^j a_{ij}$.

For this two forms to be the same, $x^T A y = x^T B y$,

$x^i y^j a_{ij} - x^i y^j b_{ij} = 0$ needs to hold for all x, y.

If we select all posible combinations of basis vectors e_m^i, e_m^j for x, y, we will always get only one element of the double sum : others will be multiplied by 0:

$x^i y^j (a_{ij} - b_{ij}) = 0$;

$\Rightarrow 1 \cdot 1 (a_{ij} - b_{ij}) + 0 = 0 \quad \forall i,j$.

$\Rightarrow a_{ij} = b_{ij} \quad \forall i,j \rightarrow A = B$ ∎