HW 3+4

1. $p, q \in \mathbb{R}_2[x]$. $\quad g(p,q) = \int_0^2 p(x)\,q(x)\,dx$

a) g is an inner product:

- $g(p,q) = \int_0^2 p(x)\,q(x)\,dx = \int_0^2 q(x)\,p(x)\,dx = g(q,p)$ ✓
- $g(\alpha p + \beta r, q) = \int_0^2 (\alpha p(x) + \beta r(x))\,q(x)\,dx = \alpha \int_0^2 p(x)q(x)\,dx + \beta \int_0^2 r(x)q(x)\,dx$

$$= \alpha\, g(p,q) + \beta\, g(r,q)$$

$\Rightarrow$ linear in first component.
Because it's symetric it is also linear in 2. component.

- positive definite: $g(p,p) = \int_0^2 \underbrace{p^2(x)}_{\geq 0}\,dx \geq 0$

$g(p,p) = 0 \Leftrightarrow p(x) = 0$ everywhere on $(0,2)$

b) $\alpha(p) = \int_0^1 p(x)\,dx$ is a linear form

- $\alpha : \mathbb{R}_2[x] \to \mathbb{R}$ ✓
- α is linear: $\alpha(ap + bq) = \int_0^1 ap(x) + bq(x)\,dx = a\alpha(p) + b\alpha(q)$

b) Find an orthonormal basis for $\mathbb{R}_2[x]$ w.r.t. g. $1 \in B$.

$g(p,q) = \int_0^2 p(x)\,q(x)\,dx$

$$\begin{array}{c|ccc} & 1 & b_1 & b_2 \\ \hline 1 & 2 & 0 & 0 \\ b_1 & 0 & 1 & 0 \\ b_2 & 0 & 0 & 1 \end{array}$$

$b_1' = x - \frac{g(1,x)}{g(1,1)} = x - \frac{\int_0^2 x\,dx}{2} = x - 1$

$\boxed{b_1} = \frac{b_1'}{\sqrt{g(b_1', b_1')}} = \frac{x-1}{\sqrt{\int_0^2 (x-1)^2\,dx}} = \frac{x-1}{\sqrt{\frac{(x-1)^3}{3}\Big|_0^2}} = \boxed{\frac{\sqrt{3}}{\sqrt{2}}(x-1)}$

$b_2' = x^2 - \frac{g(x^2,1)}{g(1,1)} - g\left(x^2, \frac{\sqrt{3}}{\sqrt{2}}(x-1)\right) \cdot \frac{\sqrt{3}}{\sqrt{2}}(x-1)$

$g(x^2, 1) = \int_0^2 x^2\,dx = \frac{8}{3}$

$g\left(x^2, \frac{\sqrt{3}}{\sqrt{2}}(x-1)\right) = \frac{\sqrt{3}}{\sqrt{2}} \int_0^2 x^3 - x^2\,dx = \frac{\sqrt{3}}{\sqrt{2}} \cdot \left(\frac{16}{4} - \frac{8}{3}\right) = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{4}{3} = \frac{2\sqrt{2}}{\sqrt{3}}$

$b_2' = x^2 - \frac{8}{6} - \frac{2\cancel{\sqrt{2}}}{\cancel{\sqrt{3}}} \cdot \frac{\cancel{\sqrt{3}}}{\cancel{\sqrt{2}}}(x-1) = x^2 - 2x + \frac{2}{3}$

$\boxed{b_2} = \frac{b_2'}{\sqrt{g(b_2', b_2')}} = \boxed{\frac{\sqrt{45}}{\sqrt{8}}\left(x^2 - 2x + \frac{2}{3}\right)}$

$g(b_2', b_2') = \int_0^2 x^4 + 4x^2 + \frac{4}{9} - 4x^3 + \frac{4}{3}x^2 - \frac{8}{3}x\,dx =$

$= \frac{32}{5} + \frac{4 \cdot 8}{3} + \cancel{\frac{8}{8}} - \frac{\cancel{4} \cdot 16}{\cancel{4}} + \frac{5 \cdot 8}{9} - \frac{16}{3} = 6\frac{2}{5} + 5\frac{1}{3} - 16 + 4\frac{4}{9}$

$= \frac{2}{5} + \frac{1}{3} + \frac{4}{9} - 1 = \frac{1}{5} - \frac{2}{9} = \frac{18-10}{45} = \underline{\underline{\frac{8}{45}}}$

If we want the basis to contain 1, it is not ortonormal (as $g(1,1) = 2$)

Ortonormal basis would be $\left\{ \underbrace{\frac{1}{\sqrt{2}}}_{"b_0}, \underbrace{\frac{\sqrt{3}}{\sqrt{2}}(x-1)}_{"b_1}, \underbrace{\frac{\sqrt{45}}{\sqrt{8}}\left(x^2 - 2x + \frac{2}{3}\right)}_{"b_2} \right\}$

d) $r \in \mathbb{R}_2[x] \;:\; \alpha(p) = g(p, r) \quad \forall p \in \mathbb{R}_2[x]$.

$$\alpha(p) = {}_0\int^1 p(x)\,dx$$

$$g(p, r) = {}_0\int^2 p(x)\,r(x)\,dx$$

For ${}_0\int^1 p(x)\,dx = {}_0\int^2 p(x)\,r(x)\,dx$ to hold everywhere,

it's enough to check if it holds on basis.

$$r(x) = r_0 \cdot b_0 + r_1 \cdot b_1 + r_2 \cdot b_2$$

b_0 : ${}_0\int^1 \frac{1}{\sqrt{2}}\,dx = g\left(\frac{1}{\sqrt{2}}, r\right) = r_0 \cdot g\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) + r_1\, g\left(\frac{1}{\sqrt{2}}, b_1\right) + r_2\, g\left(\frac{1}{\sqrt{2}}, b_2\right)$

$$\frac{1}{\sqrt{2}} = r_0 \cdot 1$$

b_1 : ${}_0\int^1 \frac{\sqrt{3}}{\sqrt{2}}(x-1)\,dx = r_1 = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{(x-1)^2}{2}\Big|_0^1 = -\frac{\sqrt{3}}{2\sqrt{2}}$

b_2 : ${}_0\int^1 \frac{\sqrt{45}}{\sqrt{8}}\left(x^2 - 2x + \frac{2}{3}\right) = r_2 = \frac{\sqrt{45}}{\sqrt{8}} \cdot \left(\frac{x^3}{3} - \frac{2x^2}{2} + \frac{2}{3}x\right)\Big|_0^1 = \frac{\sqrt{45}}{\sqrt{8}} \cdot 0$

$$\Rightarrow \boxed{r(x)} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + -\frac{\sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{3}}{\sqrt{2}}(x-1) = \frac{1}{2} - \frac{3}{4}x + \frac{3}{4} = \boxed{\frac{5}{4} - \frac{3}{4}x}$$

2. $\alpha_1(p) = {}_0\int^1 p(x)\,dx \,,\quad \alpha_2(p) = {}_0\int^2 p(x)\,dx \quad$ in $(\mathbb{R}_1[x])^*$.

a) Prove that α_1 and α_2 form a basis in $(\mathbb{R}_1[x])^*$

- $\alpha_1, \alpha_2 : \mathbb{R}_1[x] \to \mathbb{R}$ ✓ $\Rightarrow \in (\mathbb{R}_1[x])^*$
- $\dim(\mathbb{R}_1[x]) = \dim(\mathbb{R}_1[x])^* = 2 = |\{\alpha_1, \alpha_2\}|$. ✓
- α_1, α_2 lineary indep.

$a_1 : \alpha_1(p) + a_2\alpha_2(p) = 0 \quad \forall p$

$$a_1 \cdot {}_0\int^1 p(x)\,dx + a_2 \cdot {}_0\int^2 p(x)\,dx = 0$$

1. $p(x) = 1$

$a_1 + 2a_2 = 0$

2. $p(x) = x$

$a_1 \cdot \frac{1}{2} + a_2 \cdot 2 = 0$

$\Rightarrow a_1, a_2 = 0 \Rightarrow \alpha_1, \alpha_2$ are lineary indep.

b) Find the dual basis $\{A^1, A^2\}$ to $\{\alpha_1, \alpha_2\}$

$$A^1(\alpha_1) = A^2(\alpha_2) = 1$$

$$A^1(\alpha_2) = A^2(\alpha_1) = 0$$

$$\alpha_1(\overset{ax+b}{p}) = \int_0^1 p(x)\,dx = \int_0^1 ax+b\,dx = \frac{a}{2} + b = \begin{bmatrix}\frac{1}{2} & 1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$$

$$\alpha_2(ax+b) = \int_0^2 p(x)\,dx = \int_0^2 ax+b\,dx = 2a+2b = \begin{bmatrix}2 & 2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$$

$$M = \begin{bmatrix}\frac{1}{2} & 2\\ 1 & 2\end{bmatrix} \quad \left[\begin{array}{cc|cc}\frac{1}{2} & 2 & 1 & 0\\ 1 & 2 & 0 & 1\end{array}\right] \sim \left[\begin{array}{cc|cc}1 & 0 & -2 & +2\\ 0 & 2 & 2 & -1\end{array}\right] \sim \left[\begin{array}{cc|cc}1 & 0 & -2 & 2\\ 0 & 1 & 1 & -\frac{1}{2}\end{array}\right]$$

$$\mathcal{B} = \{\alpha_1, \alpha_2\} \Rightarrow \mathcal{B}^* = \left\{\begin{bmatrix}-2 & 2\end{bmatrix}, \begin{bmatrix}1 & -\frac{1}{2}\end{bmatrix}\right\}$$

c) corresponding p_1, p_2 to A_1, A_2 via cann. isomorph.

$$p_1: \quad A_1(\alpha) = \alpha(p_1) \quad \forall\alpha \qquad \alpha = a_1\cdot\alpha_1 + a_2\cdot\alpha_2 = a_1\begin{bmatrix}\frac{1}{2} & 1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} + a_2\begin{bmatrix}2 & 2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$$

$$p_1 \to \begin{bmatrix}a\\b\end{bmatrix}$$

$$A_1(\alpha) = A_1(a_1\alpha_1 + a_2\alpha_2) = a_1$$

$$\alpha(p_1) = a_1\left(\frac{1}{2}a_{p_1} + b_{p_1}\right) + a_2\left(2a_{p_1} + 2b_{p_1}\right)$$

$$\forall a_1, a_2: \quad a_1\left(\frac{1}{2}a_{p_1} + b_{p_1} - 1\right) + a_2\left(2a_{p_1} + 2b_{p_1}\right) = 0$$

$$\Rightarrow \frac{1}{2}a_{p_1} + b_{p_1} - 1 = 0 \qquad 2a_{p_1} + 2b_{p_1} = 0$$

$$b_{p_1} = -a_{p_1}$$

$$-\frac{1}{2}a_{p_1} - 1 = 0$$

$$a_{p_1} = -2 \qquad b_{p_1} = 2$$

$$\Rightarrow p_1(x) = -2x + 2$$

$$A_2(\alpha) = \alpha(p_2)$$

$$a_2 = a_1\left(\frac{1}{2}a_{p_2} + b_{p_2}\right) + a_2\left(2a_{p_2} + 2b_{p_2}\right) = 0$$

$$\Rightarrow b_{p_2} = -\frac{1}{2}a_{p_2} \qquad 2a_{p_2} + 2b_{p_2} - 1 = 0$$

$$2a_{p_2} - a_{p_2} = 1$$

$$b_{p_2} = -\frac{1}{2} \qquad a_{p_2} = 1$$

$$\Rightarrow p_2(x) = x - \frac{1}{2}$$