MAT2 - homework 3

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Bounding the unbounded

We will prove that if we have a linear program (LP),

$$\min c^T x$$

$$Ax = b$$

$$x \geq 0,$$

where A is an $m \times n$ matrix, c, x are n-dimensional, and b is an m-dimensional vector. Assume also that the coefficients so A, b, c are integers and are (absolutely) bounded by U. Let $M = (mU)^m$. If LP is feasible, then there exists a feasible solution with all coordinates bounded by M. Also if LP has an optimal solution, then there exists an optimal solution with all coordinates bounded by M.

We will prove this proposition in steps:

1) We can observe that we have equality constraints in LP. We know that inequality constraints can be transformed to equalities using slack variables: if the above would be inequality, $Ax \leq b$, we would add $Ax + \gamma = b, \gamma \geq 0$. If we would have $Ax \geq b$ we would just change $\gamma \leq 0$. So, because we are proving this for any dimension, we have a general LP.
2) In the next step we will estimate the determinant of a square matrix $X \in R^{m \times m}$ in terms of its coefficients and its dimension. We will use Liebnitz formula, which sums over all permutations of m elements. If the coefficients are bounded by U, then each product in the formula is smaller or equal to U^m. We have $m!$ possible permutations, which means
$$det(X) \leq m! U^m \leq (mU)^m.$$
3) The condition that allows Cramer's rule application is, that the determinant of the matrix A from the linear equation is nonzero. Then the system has a unique solution
$$x_i = \frac{det A_i}{det A},$$
where A_i is the matrix where we replace the i-th column of A with b.
4) If the matrix is square, the linear system has a unique solution when its determinant is not equal to zero. If the matrix is not square, then we need to have less (or equal) unknowns than independent equations.
5) We want to show that the feasible solution with as many 0 coordinates as possible is unique. If we have x^o and $x^{o'}$, are both such solutions. If both satisfy $Ax = b$, then also all points that are on a line that goes through them also satisfy it. This line must have an intersection with one of coordinate planes. The point that lies on this intersection would have one 0 coordinate more, so we have a contradiction, and we know that such point is unique.
6) We now want to show that the optimal solution with as many 0 coordinates as possible is unique. We do the same, and see that if we define the line as $l(t) = x^o + t(x^{o'} - x^o)$, $l(t)$ is also feasible for small enough $|t|$ and optimal for $t \geq 0$. With the same conclusion as above, we came to contradiction and showed that such point is unique.
7) Now we can consider a feasible (optimal) solution with maximum number of zero coordinates. We write $x = (x_B, x_N)$, where x_N is zero, and x_B is the unique solution of $A_B x_B = b$, where A_B is the submatrix of A with columns indexed by nonzero coordinates in x. Then the coordinates of x_B, and therefore x are bounded by $(mU)^m$.

Analytic center

In this task we are looking for the analytic center of the following system of linear inequalities:

$$\begin{bmatrix} 2 & 2 \\ 3 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \leq \begin{bmatrix} 480 \\ 600 \\ 0 \\ 0 \end{bmatrix}.$$

Analytic center is defined as the vector x, that is a feasible solution of the upper inequality and minimizes $\Pi s(x)_i$, where $s(x) = b - Ax$. We can now write the function we're minimizing:

$$f(x) = (480 - 2x_1 - 2x_2)(600 - 3v_1 - 1v_2)v_1 v_2.$$

We found this minimum with online mathematical tools: $x = (x_1, x_2) = (55.3, 80.5)$. We can put this point into equations above and see that it is feasible. It is also unique, so it is the analytic center.

If we add an additional constraint to the above system, $2v_1 + 2v_2 \leq 600$, we know that it doesn't change the set of feasible solutions. But it changes the function we are minimizing. Now our function is equal to

$$f(x) = (480 - 2x_1 - 2x_2)(600 - 3v_1 - 1v_2)(600 - 2x_1 - 2x_2)v_1 v_2.$$

In the same way as before, we now find three local minima: $x \in (48.1, 63.8), (34.3, 422.2), (228, 52.5)$. We again check which of them are feasible, and see that the inequalities hold only for the first point. This means that again we only have one solution to our problem, so it is the analytic center.

We see that with additional constraint analytic center moved down and left. The analytic center minimizes the product of distances to the edges of our solution space, that is why when using central path, the all 1s vector is also the analytic center.

Vertices of matching polytope

First we need to show that matchings are extremal points of the matching polytope. A matching in graph G is a set of edges M which have no vertices in common. So it is a vector, that has binary values - 0 or 1. Extremal point is a point, that cannot be expressed as a proper convex combination of other points in the convex set we are studying.

The matching polytope is a feasible set of linearly relaxed maximal matching problem. All elements are vectors with components in $[0, 1]$. The first matching we will look at is 0. It is a matching point, and it is also extremal (no proper combination). Next are matchings e_i. Since we only have positive values of components, and proper combinations, we can express e_i only with vectors that also have all other components except i equal to 0. The only way to get a proper combination of two such points equal to ce_i would be if they would be both equal to

e_i, but that counters the definition of extremal point. We now showed that also matchings of shape e_i are extremal points. If this holds for each coordinate separately, it also holds for matchings that are sums of e_i. All the above arguments also hold in extended case.

We also want to show that there may exist extremal points of the matching polytope which are not matchings. Since we relaxed the MM problem, there can exist feasible and optimal solutions that have nonintegral coordinates, and those are not matchings. For example we can look at a triangular graph. The optimal solution is $[0.5, 0.5, 0.5]$, which is extremal, but is not a matching.