Homework 10

Exercise 11.4

Let X_i be IID and $\mu = E(X_1)$. Let variance of X_i be finite. Show that the mean of X_i, $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges in quadratic mean to μ.

For convergence in quadratic mean we need to calculate the limit

$$\begin{aligned}
\lim_{n\to\infty} E(|\bar{X}_n - \mu|^2) &= \lim_{n\to\infty} E(\bar{X}_n^2 - 2\mu\bar{X}_n + \mu^2) \\
&= \lim_{n\to\infty} E(\bar{X}_n^2) - \lim_{n\to\infty} E(2\mu\bar{X}_n) + \lim_{n\to\infty} E(\mu^2) \\
&= \lim_{n\to\infty} Var(\bar{X}_n) + \lim_{n\to\infty} E(\bar{X}_n)^2 - 2\mu \lim_{n\to\infty} E(\bar{X}_n) + \mu^2 \\
&= \lim_{n\to\infty} (\frac{\sigma^2}{n}) + \lim_{n\to\infty} ((\frac{n\mu}{n})^2) - 2\mu \lim_{n\to\infty} (\frac{n\mu}{n}) + \mu^2 \\
&= 0 + \mu^2 - 2\mu^2 + \mu^2 = 0.
\end{aligned}$$

We used the formula for expected value of a sum, and the rule $Var[X] = E[X^2] - E[X]^2$.