Homework 8

Exercise 8.3

Let X be a random vector of length k with $X_i \sim \mathbf{N}(0,1)$ and LL^* the Cholesky decomposition of a Hermitian positive-definite matrix A. Let μ be a vector of length k.

a) Find the distribution of the random vector $Y = \mu + LX$
If A is positive-definite, then A is non-singular, then also L is non-singular. Which means that (as we proved during lectures) $Y \sim \mathrm{N}(\mu, LL^*)$, so $Y \sim \mathrm{N}(\mu, A)$.

b) Find the Cholesky decomposition of $A = \begin{bmatrix} 2 & 1.2 \\ 1.2 & 1 \end{bmatrix}$. For Cholesky decomposition we first need to calculate:

$$L_1 = \begin{bmatrix} \sqrt{a_{11}} & 0 \\ \dfrac{a_{21}}{\sqrt{a_{11}}} & a_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ \dfrac{1.2}{\sqrt{2}} & 1 \end{bmatrix},$$

then we take $A_2 = a_{22} - \dfrac{1}{a_{11}} a_{21}^2 = 0.28$, which leads us to $L_2 = \sqrt{A_2} = \sqrt{0.28}$ and finally we put L_1 and L_2 together into

$$L = \begin{bmatrix} \sqrt{2} & 0 \\ \dfrac{1.2}{\sqrt{2}} & \sqrt{0.28} \end{bmatrix}.$$

Exercise 8.5

Let $X \sim \mathbf{N}(\mu, \Sigma)$, where $\mu = [2, 0, -1]^T$ and $\Sigma = \begin{bmatrix} 1 & -0.2 & 0.5 \\ -0.2 & 1.4 & -1.2 \\ 0.5 & -1.2 & 2 \end{bmatrix}$. Let A represent the first two random variables and B the third random variable.

d) Find the conditional distribution of $A|B = b$.
We know that $A|B = b \sim \mathrm{N}(\overline{\mu}, \overline{\Sigma})$. If we select $\mu_A = [2, 0]^T$, $\mu_B = -1$ and separate Σ into four blocks, $\begin{bmatrix} \Sigma_A & \Sigma_{AB} \\ \Sigma_{AB}^T & \Sigma_B \end{bmatrix}$, where upper left block is 2×2 matrix, we can then calculate:

$$\overline{\mu} = \mu_A + \Sigma_{AB}\Sigma_B^{-1}(b - \mu_B) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ -1.2 \end{bmatrix} \cdot \frac{1}{2} \cdot (b+1) = \begin{bmatrix} 2.25 + \dfrac{b}{4} \\ -0.6(b+1) \end{bmatrix}$$

$$\overline{\Sigma} = \Sigma_A - \Sigma_{AB}\Sigma_B^{-1}\Sigma_{AB}^T = \begin{bmatrix} 1 & -0.2 \\ -0.2 & 1.4 \end{bmatrix} - \begin{bmatrix} 0.5 \\ -1.2 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 0.5 & -1.2 \end{bmatrix} = \begin{bmatrix} 0.875 & 0.1 \\ 0.1 & 0.68 \end{bmatrix}$$