Homework 9

Exercise 9.2

Find the variance of the negative binomial distribution. Hint: Find the Taylor series of $(1-y)^{-r}$ at point 0

In labs we found the expected value of the negative binomial distribution, $E[X] = \frac{p \cdot (a-p)^r \cdot r}{(1-tp)^{r+1}}\Big|_{t=1} = \frac{pr}{1-p}$. To find the variance we need to derive the expected value (before inserting $t = 1$) again:

$$\frac{d^2}{dt^2}\alpha_x(t) = \frac{d}{dt}\frac{rp \cdot (a-p)^r}{(1-tp)^{r+1}} = p(1-p)^r r \frac{1}{(1-pt)^r}(r+1)\frac{-p}{(1-pt)^2}.$$

If we insert $t = 1$:

$$\frac{d^2}{dt^2}\alpha_x(t)\Big|_{t=1} = -p^2(1-p)^r r(r+1)\frac{1}{(1-pt)^{r+2}}\Big|_{t=1} = p^2 r(r+1)\frac{1}{(1-p)^2}.$$

And last, to get variance, we calculate

$$\begin{aligned} Var[X] &= \left|\frac{d^2}{dt^2}\alpha(t)\right|\Big|_{t=1} + E[X] - E[X]^2 = \frac{p^2 r(r+1)}{(1-p)^2} + \frac{pr}{1-p} - \frac{p^2 r^2}{(1-p)^2} \\ &= \frac{p^2 r(r+1-r) + pr(1-p)}{(1-p)^2} = \frac{pr(p+1-p)}{(1-p)^2} = \frac{pr}{(1-p)^2}. \end{aligned}$$