Homework 2

Exercise 2.2

Show that $\mathcal{C} = \sigma(\mathcal{C})$ if and only if $\mathcal{C}$ is a sigma algebra.

($\Leftarrow$) $\sigma(\mathcal{C})$ is the smallest σ-algebra that contains $\mathcal{C}$. So $\mathcal{C}$ is in $\sigma(\mathcal{C})$ by definition, and if $\mathcal{C}$ is already a σ-algebra, we don't need to add anything (and if we do, it isn't the smallest anymore).

($\Rightarrow$) $\sigma(X)$ is a σ-algebra for any set X, so if $\mathcal{C} = \sigma(X)$, $\mathcal{C}$ is a σ-algebra.

Exercise 2.6

Show that the Lebesgue measure of rational numbers on $[0, 1]$ is 0.

Between every two rational numbers, there is an irrational number $\Rightarrow \mathbb{Q}$ consists of singletons (and is countable), and measure of singletons is 0.
$\Rightarrow \lambda(\mathbb{Q}) = \sum_{i=1}^{\infty} \lambda(q_i) = \sum_{i=1}^{\infty} 0 = 0.$