Homework 6

Exercise 6.2

Find the Lebesgue integral of the following functions on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.

Lebesgue integral of a simple function is defined as $\int f(x)d\lambda = \sum a_i \lambda(Ai)$, where a_i are different values of f and A_i is a set of all x, where $f(x) = a_i$.

a)

$$f(\omega) = \begin{cases} \omega, & \textbf{for } \omega = 0, 1, ..., n \\ 0, & \textbf{elsewhere} \end{cases}$$

$$\int f(w)d\lambda = \sum_{w=0}^{n} w\lambda(w) = \sum_{w=0}^{n} w \cdot 0 = 0,$$

since Lebesgue measure of a singleton is 0.

b)

$$f(\omega) = \begin{cases} 1, & \textbf{for } \omega = \mathbb{Q}^c \cap [0, 1] \\ 0, & \textbf{elsewhere} \end{cases}$$

$$\int f(w)d\lambda = 1 \cdot \lambda(\mathbb{Q}^c \cap [0, 1]) = \lambda([0, 1]) - \lambda(\mathbb{Q} \cap [0, 1]) = 1 - 0 = 1$$

c)

$$f(\omega) = \begin{cases} n, & \textbf{for } \omega = \mathbb{Q}^c \cap [0, n] \\ 0, & \textbf{elsewhere} \end{cases}$$

$$\int f(w)d\lambda = n \cdot \lambda(\mathbb{Q}^c \cap [0, n]) = n \cdot (\lambda([0, n]) - \lambda(\mathbb{Q} \cap [0, n])) = n \cdot (n - 0) = n^2$$

Exercise 7.2

Let $X \sim$ **Binomial**(n, p)**.**

a) Find $E(X)$.

$$E[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} =$$

$$\sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} = \sum_{k=0}^{n-1} \frac{n!}{k!(n-k-1)!} p^{k+1} (1-p)^{n-k-1} =$$

$$np \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k!((n-1)-k)!} p^k (1-p)^{(n-1)-k} = np \cdot (p + (1-p))^{n-1} = np$$

We could change the start of the sequence to $k=1$, because the term for $k=0$ is equal to 0. Later we shifted it back to start at $k=0$ and end at $n-1$, with substituting k for $k+1$. In the end we saw that our sum is a sum of binomial sequence equal to $(p+(1-p))^{n-1}$.

b) Find $Var(X)$.

$$Var[X] = E[X^2] - E[X]^2$$

$$E[X^2] = \sum_{k=0}^{n} k^2 \cdot \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} k^2 \cdot \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} =$$

$$\sum_{k=1}^{n} k \cdot \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} =$$

$$\sum_{k=0}^{n-1} (k+1) \cdot \frac{n!}{k!(n-(k+1))!} p^{k+1} (1-p)^{n-(k+1)} =$$

$$\sum_{k=0}^{n-1} k \cdot \frac{n!}{k!(n-(k+1))!} p^{k+1} (1-p)^{n-(k+1)} +$$

$$\sum_{k=0}^{n-1} 1 \cdot \frac{n!}{k!(n-(k+1))!} p^{k+1} (1-p)^{n-(k+1)} =$$

$$\sum_{k=1}^{n-1} \frac{n!}{(k-1)!(n-(k+1))!} p^{k+1} (1-p)^{n-(k+1)} +$$

$$\sum_{k=0}^{n-1} 1 \cdot \frac{n!}{k!(n-(k+1))!} p^{k+1} (1-p)^{n-(k+1)} =$$

$$\sum_{k=0}^{n-2} \frac{n!}{k!(n-(k+2))!} p^{k+2} (1-p)^{n-(k+2)} + np =$$

$$p^2 n(n-1) \cdot \sum_{k=0}^{n-2} \frac{(n-2)!}{k!(n-2-k))!} p^k (1-p)^{n-2-k} + np =$$

$$(p^2 n^2 - p^2 n) \cdot (p + (1-p))^{n-2} + np = p^2 n^2 - p^2 n + np$$

$$Var[X] = E[X^2] - E[X]^2 = p^2 n^2 - p^2 n + np - (np)^2 = -p^2 n + np = np(1-p)$$

Again we used some of the same things as above. In the third line we come to the same sum as in a), so we just changed it to np.