Computational topology HW2

Urša Zrimšek

Theoretical problems

Triangulations

Let S be a set of n points in the plane, $n \in \mathbb{N}$:

a) *Show that for all $n \in \mathbb{N}$ there are at most $2^{\frac{n(n-1)}{2}}$ triangulations of S.*
There are $\frac{n(n-1)}{2}$ possible edges between n points. That means we can decide which edges we will take in $2^{\frac{n(n-1)}{2}}$ different ways - for each edge we decide whether it will be a part of our triangulation or not. Not all of these combinations are valid triangulations, but this also includes all valid ones, so there can't be more of them.

b) *The degree of a point in a triangulation T is the number of edges in T, incident to that point. For each $n \geq 3$ construct a set S such that all possible triangulations of S have a point of degree $n-1$.*
Such set S would be $n-1$ collinear points and one not on that line. That would mean that all triangles would need to have that non collinear point as their vertex, so this point would have degree $n-1$.

c) *If not all points in S are collinear, then any triangulation T of S has at most $3n-3$ edges. Use this fact to prove that any triangulation T of S has a point of degree 5 or less.*
Each edge increases the degree of two points, so with $3n-3$ edges, the summed degree of all points would be $6n-6$. If we want to maximize the minimal degree of all points, we would want all to have the same degree. But if we divide the summed degree, we can see that $\frac{6n-6}{n} < 6$, so we need to have at least one point with smaller degree, so 5 or less.

Vietoris-Rips Complex and Čech Complex

Let $S = \{(0,0),(2,0),(1,0.5),(1,1.5)\} \subset \mathbb{R}$. To simplify we will tag this points with A, B, C, D respectively.

a) *Build the Vietoris-Rips complex $VR_{2\epsilon}(S)$ and the Čech complex $\check{C}_\epsilon(S)$ for $\epsilon = 0.8$.*
$VR_{1.6}(S) = \check{C}_{0.8}(S) = \{A, B, C, D, AC, BC, CD\}$, with dimension 1 and $\chi = F - E + V = 1$.

b) *Build the Vietoris-Rips complex $VR_{2\epsilon}(S)$ and the Čech complex $\check{C}_\epsilon(S)$ for $\epsilon = 1$.*
$VR_2(S) = \{A, B, C, D, AB, AC, AD, BC, BD, CD, ABC, ABD, BCD, ACD, ABCD\}$, with dimension 3 and $\chi = 2$.
$\check{C}_1(S) = \{A, B, C, D, AB, AC, AD, BC, BD, CD, ABC, BCD, ACD\}$, with dimension 2 and $\chi = 1$.

Voronoi diagram

For all $n \in \mathbb{N}, n > 3$, find a configuration of n points in the plane such that their Voronoi diagram will have a cell with $n-1$ vertices.

Such configuration could be $n-1$ distinct points lying on a circle with a center in the remaining point. Then the cell containing the center point would have $n-1$ vertices.

Chessboard complex

The chessboard complex of a $m \times n$ chessboard is a simplicial complex $\Delta_{m\times n}$. The vertices of $\Delta_{m\times n}$ correspond to the squares of the chessboard. Simplices of $\Delta_{m\times n}$ correspond to non-taking placements of rooks (ie. placements where no two rooks are in the same column or in the same row).

a) *Show that the chessboard complex $\Delta_{3\times 2}$ is homeomorphic to the circle S^1*
$\Delta_{3\times 2}$ can be seen on Figure 1. There are no three points that would correspond to non-taking placements of rooks, so we only have 1D simplices. From the figure it is clear that it represents a connected 1-manifold with no boundary, so we know that it is homeomorphic to the circle.

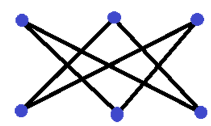

Figure 1. **Graphical representation of $\Delta_{3\times 2}$.**

b) *Show that the chessboard complex $\Delta_{4\times 3}$ is homeomorphic to a torus $S^1 \times S^1$.*
To show this we have to show that $\Delta_{4\times 3}$ is connected orientable surface (2-dim manifold) with Euler characteristic 0. First let's look at the simplices of this complex. Edges are all combinations of points that don't lie in the same column or row - each point is connected to 6 other points. That means we have $\frac{12\cdot 6}{2} = 36$ edges. Here we can also observe that the complex is connected. Triangles are all triplets of points where no 2 lie in the same row or column. Each point lies in 6 different triangles, so we have $\frac{12\cdot 6}{3} = 24$ triangles. From these triangles we observe star and link of each point, and see that it is equal to $\Delta_{3\times 2}$, which is homeomorphic to S^1. This proves that our complex is 2-manifold. Because we only have 3 columns, we can't have a 3-dimensional simplex. Now that we determined all simplices we can easily check that we can orient them consistently, and calculate the Euler characteristic: $\chi = 24-36+12 = 0$. If we have an orientable surface with $\chi = 0$ we know that it is homeomorphic to a torus.

c) *Show that the chessboard complex $\Delta_{n\times(n-1)}$ is a manifold for all n.*
Each vertex in $\Delta_{n\times(n-1)}$ is connected to al vertices that are not in the same row or column, which means it is connected to a subcomplex homeomorphic to $\Delta_{(n-1)\times(n-2)}$. From this we can inductively deduct that the link of each vertex is homeomorphic to S^{n-1} and thus $\Delta_{n\times(n-1)}$ is n-manifold for each n.

Programming problems

Line sweep triangulation

In this task we implemented line sweep algorithm, that returns lists of edges and triangles obtained by vertical or horizontal line sweep. To ensure points are in general position we added a small amount of noise to the input points before constructing the triangulation. The result of our algorithm can be seen on Figure 2.

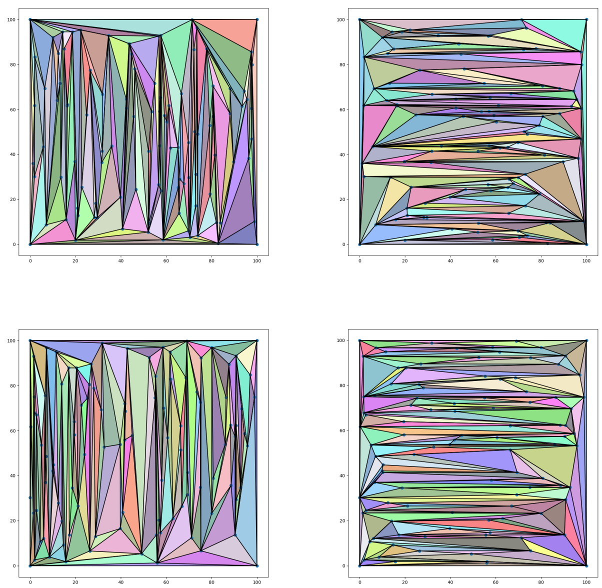

Figure 2. **Two test cases of line sweep on 100 random points.** On the left we can see result of vertical line sweep triangulation, where we sweep from left to right, and on the right horizontal line sweep, where we sweep from bottom to top. We also added corner points to the random ones.

Delauney triangulation

Next we implemented the edge-flip algorithm to produce Delauney triangulation from any other triangulation. On Figure 3 you can see the result of the algorithm.

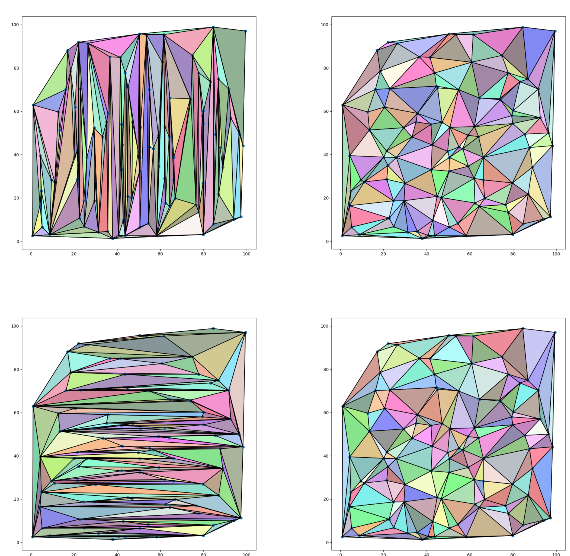

Figure 3. **Two test cases of Delauney triangulation.** On the left we can see result of line sweep triangulation, vertical on top, and horizontal on the bottom. On the right side we have the Delauney triangulation, obtained with edge-flipping of the triangulations on the left. Observe that we get the same triangulation from both initial ones.

Orientation of surfaces

We wrote a function that determined whether a 2-manifold given by its triangulation is orientable or not. We also found that orientation. We tested our functions on a torus, a Klein bottle, a sphere, a cylinder and a Moebius band, and found out that only Klein bottle and Moebius band are not orientable.