Setting

Consider the logistic regression model:

$$\ln\left\{\frac{\pi_i}{1-\pi_i}\right\} = \beta_0 + \beta_X X_i + \beta_Z Z_i,$$

where $\pi_i = \mathbb{P}(Y_i = 1|X_i, Z_i) = \mathbb{E}(Y_i|X_i, Z_i)$. Interest lies in testing $H_0 : \beta_X = 0$.

Model Components

Let $W_i = (1, X_i, Z_i)'$ and $\beta = (\beta_0, \beta_X, \beta_Z)'$ such that:

$$\ln\left\{\frac{\pi_i}{1-\pi_i}\right\} = W_i'\beta$$

The log-likelihood for β is:

$$\begin{aligned}\ell(\beta) &= \sum_{i=1}^{n} Y_i \ln \pi_i + (1-Y_i)\ln(1-\pi_i) \\ &= \sum_{i=1}^{n} Y_i e^{W_i'\beta} - \ln(1 + e^{W_i'\beta}).\end{aligned}$$

The score equation for β is:

$$\mathcal{U}_\beta = \sum_{i=1}^{n}(Y_i - \pi_i)W_i.$$

The information matrix is:

$$\mathcal{I}_{\beta\beta'} = \sum_{i=1}^{n} W_i \otimes W_i \cdot \pi_i(1-\pi_i),$$

where:

$$W_i \otimes W_i = \begin{pmatrix} 1 & X_i & Z_i \\ X_i & X_i^2 & X_i Z_i \\ Z_i & Z_i X_i & Z_i^2 \end{pmatrix}.$$

The information matrix converges in probability as:

$$\frac{1}{n}\mathcal{I}_{\beta\beta'} \xrightarrow{p} i_{\beta\beta'} \equiv \mathbb{E}\big\{W_i \otimes W_i \cdot \pi_i(1-\pi_i)\big\}$$

Let $\hat{\beta}$ denote the MLE of β. Asymptotically:

$$\sqrt{n}\big(\hat{\beta} - \beta\big) \rightsquigarrow N(0, i_{\beta\beta'}^{-1}).$$

Wald Test

Consider evaluating $H_0 : \beta_X = 0$. To obtain an expression for the asymptotic variance of $\hat{\beta}_X$, write $\beta = (\beta_X, \gamma)$ where $\gamma = (\beta_0, \beta_Z)$. In block matrix notation, the information for (β_X, γ) is:

$$
i_{(\beta_X,\gamma)(\beta_X,\gamma)'} = \begin{pmatrix} i_{\beta_X\beta_X'} & i_{\beta_X\gamma'} \\ i_{\gamma\beta_X'} & i_{\gamma\gamma'} \end{pmatrix},
$$

where:

$$
\begin{aligned}
i_{\beta_X\beta_X'} &= \mathbb{E}\big\{X_i^2 \cdot \pi_i(1-\pi_i)\big\}, \\
i_{\beta_X\gamma'} &= \mathbb{E}\left\{\begin{pmatrix} X_i & X_iZ_i \end{pmatrix} \cdot \pi_i(1-\pi_i)\right\}, \\
i_{\gamma\gamma'} &= \mathbb{E}\left\{\begin{pmatrix} 1 & Z_i \\ Z_i & Z_i^2 \end{pmatrix} \cdot \pi_i(1-\pi_i)\right\}
\end{aligned}
$$

Using block inversion, the asymptotic variance of β_X is:

$$
\mathbb{V}(\hat{\beta}_X) = \big(i_{\beta_X\beta_X'} - i_{\beta_X\gamma'}i_{\gamma\gamma'}^{-1}i_{\gamma\beta_X'}\big)^{-1}.
$$

The Wald test of $H_0 : \beta_X = 0$:

$$
T_W = n \cdot \hat{\beta}_X'(i_{\beta_X\beta_X'} - i_{\beta_X\gamma'}i_{\gamma\gamma'}^{-1}i_{\gamma\beta_X'})\hat{\beta}_X.
$$