Setting

Consider the model:

$$Y_i = X_i\beta_X + Z_i\beta_Z + \epsilon_i$$

where $\{(Y_i, X_i, Z_i)\}$ are IID, and $\mathbb{V}(\epsilon) = \mathbb{V}(Y|X,Z) = \sigma^2$. Suppose X, Y, Z have each been centered to have mean zero. Interest lies in testing the hypothesis $H_0 : \beta_X = 0$. Here we derive the power of the corresponding Wald test.

Model Components

Let $W_i = (X_i \; Z_i)$ and $\beta = (\beta_X, \beta_Z)$ such that:

$$Y_i = W_i\beta + \epsilon_i.$$

The score equation for $\boldsymbol{\beta}$ is:

$$\mathcal{U}_\beta = \frac{1}{\sigma^2}\boldsymbol{W}'(\boldsymbol{y} - \boldsymbol{W}\boldsymbol{\beta})$$

The information matrix for $\boldsymbol{\beta}$ is:

$$\mathcal{I}_{\beta\beta'} = \frac{1}{\sigma^2}\boldsymbol{W}'\boldsymbol{W} = \frac{1}{\sigma^2}\sum_{i=1}^{n} W_i \otimes W_i.$$

Since X, Z have mean zero:

$$\frac{1}{n}\mathcal{I}_{\beta\beta'} \xrightarrow{p} \frac{1}{\sigma^2}\mathbb{V}(W_i) = \frac{1}{\sigma^2}\left(\begin{array}{cc} \mathbb{V}(X_i) & \mathbb{C}(X_i, Z_i) \\ \mathbb{C}(Z_i, X_i) & \mathbb{V}(Z_i) \end{array}\right) \equiv i_{\beta\beta'}.$$

Here $i_{\beta\beta'}$ denotes the probability limit of $n^{-1}\mathcal{I}_{\beta\beta'}$. The efficient information for β_X is:

$$i_{X|Zs} = \frac{1}{\sigma^2}\big\{\mathbb{V}(X_i) - \mathbb{C}(X_i, Z_i)\mathbb{V}(Z_i)^{-1}\mathbb{C}(Z_i, X_i)\big\}.$$

Wald Test

The Wald test of $H_0 : \beta_X = 0$ is:

$$T_W = n \cdot \hat{\beta}_X' i_{X|Z}\hat{\beta}_X.$$

The expected value of the Wald statistic is:

$$\begin{aligned}\mathbb{E}(T_W) &= n \cdot \mathbb{E}\big\{\mathrm{tr}(\hat{\beta}_X' i_{X|Z}\hat{\beta}_X)\big\} = n \cdot \mathrm{tr}\big\{i_{X|Z}\mathbb{E}(\hat{\beta}_X \otimes \hat{\beta}_X)\big\} \\ &= n \cdot \mathrm{tr}\big\{i_{X|Z}(n^{-1}i_{X|Z}^{-1} + \beta_X \otimes \beta_X)\big\} = \dim(\beta_X) + n \cdot \beta_X' i_{X|Z}\beta_X.\end{aligned}$$

Observe that under $H_0 : \beta_X = 0$, the expectation of T_W is $\dim(\beta_X)$, which is the number of degrees of freedom. When $\beta_X \neq 0$, the expectation is increased by $n \cdot \beta_X' i_{X|Z}\beta_X$, which is the non-centrality parameter.

Power

The power of the Wald test at type I error level α is:

$$\gamma_\alpha(n) = \mathbb{P}\big\{\chi^2_k(\Delta) > \chi^2_{k,1-\alpha}(0)\big\},$$

where:

$$\Delta = n \cdot \beta_X' i_{X|Z} \beta_X = n \cdot \frac{1}{\sigma^2} \beta_X' \big\{\mathbb{V}(X_i) - \mathbb{C}(X_i, Z_i)\mathbb{V}(Z_i)^{-1}\mathbb{C}(Z_i, X_i)\big\}\beta_X.$$

The sample size needed to obtain a specified power π is:

$$\min\big\{n : \gamma_\alpha(n) \geq \pi\big\}.$$