Discrete-time Martingales

1.1 Introduction

Definition 1.1.1. A discrete-time stochastic process M_n is a **martingale** if:

$$\mathbb{E}\big(M_n|\mathscr{F}_{n-1}\big) = M_{n-1},$$

where $\mathscr{F}_n$ is a filtration including $\sigma(M_k : k \leq n)$, the *history of* M_n *at time* n. ■

Remark 1.1.1. Unless otherwise stated, the martingale is supposed to satisfy the initial condition $M_0 = 0$. These processes are described as *mean-zero martingales*. ♦

Example 1.1.1. The martingale property is equivalent to:

$$\mathbb{E}(M_n|\mathscr{F}_m) = M_m \text{ for } m < n.$$

To see this, suppose $m \leq n-1$, then by iterated expectation:

$$\mathbb{E}(M_n|\mathscr{F}_m) \overset{*}{=} \mathbb{E}\big\{\mathbb{E}(M_n|\mathscr{F}_{n-1})|\mathscr{F}_m\big\} = \mathbb{E}(M_{n-1}|\mathscr{F}_m).$$

Note that $\overset{*}{=}$ holds since $\mathscr{F}_m \subseteq \mathscr{F}_{n-1}$. If $m = n-1$, then the result has been shown. Otherwise, continue in like manner until:

$$\mathbb{E}(M_n|\mathscr{F}_m) = \cdots = \mathbb{E}(M_{m+1}|\mathscr{F}_m) = M_m.$$

♠

Proposition 1.1.1. Martingales have constant mean $\mathbb{E}(M_n) = \mathbb{E}(M_0)$. ♦

Proof.

$$\mathbb{E}(M_n) = \mathbb{E}\big\{\mathbb{E}(M_n|\mathscr{F}_0)\big\} = \mathbb{E}(M_0).$$

■

Proposition 1.1.2. Martingales have *uncorrelated increments*:

$$\mathbb{C}(M_m, M_n - M_m) = 0$$

♦

Proof.

$$\mathbb{C}(M_m, M_n - M_m) = \mathbb{E}\big\{M_m(M_n - M_m)\big\} - \mathbb{E}(M_m)\mathbb{E}(M_n - M_m).$$

The first term vanishes since:

$$\begin{aligned}
\mathbb{E}\big\{M_m(M_n - M_m)\big\} &= \mathbb{E}\big[\mathbb{E}\{M_m(M_n - M_m)|\mathscr{F}_m\}\big] \\
&= \mathbb{E}\big[M_m\{\mathbb{E}(M_n|\mathscr{F}_m) - M_m\}\big] \\
&= \mathbb{E}\big\{M_m(M_m - M_m)\big\} = 0.
\end{aligned}$$

The second term vanishes since:

$$\begin{aligned}
\mathbb{E}(M_n - M_m) &= \mathbb{E}\big\{\mathbb{E}(M_n - M_m|\mathscr{F}_m)\big\} \\
&= \mathbb{E}\big\{\mathbb{E}(M_n|\mathscr{F}_m) - M_m\big\} \\
&= \mathbb{E}(M_m - M_m) = 0.
\end{aligned}$$

■

Example 1.1.2. Suppose Z_i are independent and identically distributed (IID) random variables with $\mathbb{E}(Z_i) = 0$ and $\mathbb{V}(Z_i) = \sigma^2$. Define Y_n as the partial sum process $\sum_{i=1}^n Z_i$, and let $\mathscr{F}_n = \sigma(Y_k : k \leq n)$. Y_n is a mean-zero martingale. To see this, write:

$$Y_n = \sum_{i=1}^n Z_i = Z_n + \sum_{i=1}^{n-1} Z_i = Z_n + Y_{n-1}.$$

Taking the expectation conditional on the history:

$$\mathbb{E}(Y_n|\mathscr{F}_{n-1}) = \mathbb{E}(Z_n|\mathscr{F}_{n-1}) + \mathbb{E}(Y_{n-1}|\mathscr{F}_{n-1}) \overset{*}{=} \mathbb{E}(Z_n) + Y_{n-1} = Y_{n-1},$$

where $\overset{*}{=}$ holds because the Z_n is independent of $\mathscr{F}_{n-1}$. Finally, the marginal expectation of Y_n is:

$$\mathbb{E}(Y_n) = \sum_{i=1}^n E(Z_i) = 0.$$

♠

1.2 Martingale Difference Sequence

Definition 1.2.1. A discrete-time stochastic process ΔM_n is a **martingale difference sequence** if:

$$\mathbb{E}(\Delta M_n|\mathscr{F}_{n-1}) = 0.$$

■

Proposition 1.2.3. If M_n is a martingale, then $\Delta M_n = M_n - M_{n-1}$ is a martingale difference sequence. ♦

Proof.

$$\begin{aligned}\mathbb{E}(\Delta M_n|\mathscr{F}_{n-1}) &= \mathbb{E}(M_n - M_{n-1}|\mathscr{F}_{n-1}) \\ &= \mathbb{E}(M_n|\mathscr{F}_{n-1}) - M_{n-1} \\ &= M_{n-1} - M_{n-1} = 0.\end{aligned}$$

■

1.3 Variation Processes

Definition 1.3.1. Suppose M_n is a discrete-time mean-zero martingale. The **predictable variation** of M_n is the sum of the conditional variances of the increments:

$$\langle M\rangle_n = \sum_{m=1}^{n} \mathbb{V}(\Delta M_m|\mathscr{F}_{m-1}) = \sum_{m=1}^{n} \mathbb{E}\big\{(M_m - M_{m-1})^2|\mathscr{F}_{m-1}\big\}.$$

The **optional variation** of M_n is the sum of the squared martingale differences:

$$[M]_n = \sum_{m=1}^{n} \big(\Delta M_m\big)^2 = \sum_{m=1}^{n} \big(M_m - M_{m-1}\big)^2.$$

The predictable and optional variations are initialized at zero:

$$\langle M\rangle_0 = 0, \qquad [M]_0 = 0.$$

■

Proposition 1.3.4. Suppose M_n is a discrete-time mean-zero martingale. Then, the *compensated process* $M_n^2 - \langle M\rangle_n$ is also mean-zero martingale. ♦

Proof. Write the square M_n^2 as :

$$M_n^2 = (M_n - M_{n-1} + M_{n-1})^2 = (\Delta M_n)^2 + M_{n-1}^2 + 2(\Delta M_n)M_{n-1},$$

and the predictable variation as:

$$\langle M\rangle_n = \mathbb{E}\big\{(M_n - M_{n-1})^2|\mathscr{F}_{n-1}\big\} + \langle M\rangle_{n-1} = \mathbb{E}\big\{(\Delta M_n)^2|\mathscr{F}_{n-1}\big\} + \langle M\rangle_{n-1}.$$

The expectation of the difference satisfies:

$$\begin{aligned}&\mathbb{E}(M_n^2 - \langle M\rangle_n|\mathscr{F}_{n-1}) \\ &\quad= \mathbb{E}\big\{(\Delta M_n)^2 + M_{n-1}^2 + 2(\Delta M_n)M_{n-1} - (\Delta M_n)^2 - \langle M\rangle_{n-1}|\mathscr{F}_{n-1}\big\} \\ &\quad= \mathbb{E}(M_{n-1}^2 - \langle M\rangle_{n-1}|\mathscr{F}_{n-1}) + 2\mathbb{E}\big\{(\Delta M_n)M_{n-1}|\mathscr{F}_{n-1}\big\} \\ &\quad= M_{n-1}^2 - \langle M\rangle_{n-1} + 2\mathbb{E}(\Delta M_n|\mathscr{F}_{n-1})M_{n-1} = M_{n-1}^2 - \langle M\rangle_{n-1}.\end{aligned}$$

Finally, $M_n^2 - \langle M\rangle_n$ is mean-zero because:

$$\mathbb{E}\big\{M_n^2 - \langle M\rangle_n\big\} = \mathbb{E}\big\{M_0^2 - \langle M\rangle_0\big\} = \mathbb{E}(0) = 0.$$

■

Problem 1.3.1. Suppose M_n is a discrete-time mean-zero martingale. Prove that the compensated process $M_n^2 - [M]_n$ is also a mean-zero martingale. ♠

Proposition 1.3.5. Suppose M_n is a discrete-time mean-zero martingale, then:

$$\mathbb{V}(M_n) = \mathbb{E}\langle M\rangle_n = \mathbb{E}[M]_n.$$

♦

Proof. Since $\mathbb{E}(M_n) = 0$, the variance $\mathbb{V}(M_n) = \mathbb{E}(M_n^2)$. Now, since the compensated process $M_n^2 - \langle M\rangle_n$ is a mean-zero martingale:

$$\mathbb{E}\big\{M_n^2 - \langle M\rangle_n\big\} = 0 \implies \mathbb{E}(M_n^2) = \mathbb{E}\langle M\rangle_n.$$

Likewise for $M_n^2 - [M]_n$. ■

1.4 Covariation Processes

Definition 1.4.1. Suppose M_{1n} and M_{2n} are discrete-time mean-zero martingales. The **predictable covariation** of M_{1n} and M_{2n} is the sum of the conditional covariance of the increments:

$$\langle M_1, M_2\rangle_n = \sum_{m=1}^{n} \mathbb{C}(\Delta M_{1m}\Delta M_{2m}|\mathscr{F}_{m-1}) = \sum_{m=1}^{n} \mathbb{E}(\Delta M_{1m}\Delta M_{2m}|\mathscr{F}_{m-1}).$$

The **optional covariation** of M_{1n} and M_{2n} is the sum of the joint increments:

$$[M_1, M_2]_n = \sum_{m=1}^{n} \Delta M_{1m}\Delta M_{2m}.$$

■

Proposition 1.4.6. Suppose M_{1n} and M_{2n} are discrete-time mean-zero martingales. Then, the compensated process $M_{1n}M_{2n} - \langle M_1, M_2\rangle_n$ is also a mean-zero martingale. ♦

Proof. Write the product $M_{1n}M_{2n}$ as:

$$\begin{aligned} M_{1n}M_{2n} &= (\Delta M_{1n} + M_{1(n-1)})(\Delta M_{2n} + M_{2(n-1)}) \\ &= \Delta M_{1n}\Delta M_{2n} + \Delta M_{1n}M_{2(n-1)} + M_{1(n-1)}\Delta M_{2n} + M_{1(n-1)}M_{2(n-1)}, \end{aligned}$$

and the predictable covariation $\langle M_1, M_2\rangle_n$ as:

$$\langle M_1, M_2\rangle_n = \mathbb{E}\big(\Delta M_{1n}\Delta M_{2n}|\mathscr{F}_{n-1}\big) + \langle M_1, M_2\rangle_{n-1}.$$

Now the expectation of the difference is expressible as:

$$
\begin{aligned}
&\mathbb{E}\big(M_{1n}M_{2n} - \langle M_1, M_2\rangle_n | \mathscr{F}_{n-1}\big) \\
=&\mathbb{E}\big\{\Delta M_{1n}\Delta M_{2n} + \Delta M_{1n}M_{2(n-1)} + M_{1(n-1)}\Delta M_{2n} + M_{1(n-1)}M_{2(n-1)} \\
&\quad - \Delta M_{1n}\Delta M_{2n} - \langle M_1, M_2\rangle_{n-1} | \mathscr{F}_{n-1}\big\} \\
=&M_{1(n-1)}M_{2(n-1)} - \langle M_1, M_2\rangle_{n-1} \\
&\quad + \mathbb{E}(\Delta M_{1n}|\mathscr{F}_{n-1})M_{2(n-1)} + M_{1(n-1)}\mathbb{E}(\Delta M_{2n}|\mathscr{F}_{n-1}) \\
=&M_{1(n-1)}M_{2(n-1)} - \langle M_1, M_2\rangle_{n-1}.
\end{aligned}
$$

Finally, $M_{1n}M_{2n} - \langle M_1, M_2\rangle_n$ is mean-zero because:

$$
\mathbb{E}\big\{M_{1(n-1)}M_{2(n-1)} - \langle M_1, M_2\rangle_{n-1}\big\} = \mathbb{E}\big\{M_{10}M_{20} - \langle M_1, M_2\rangle_0\big\} = \mathbb{E}(0) = 0.
$$

■

Problem 1.4.2. Suppose M_{1n} and M_{2n} are discrete-time mean-zero martingales. Prove that the compensated process $M_{1n}M_{2n} - [M_1, M_2]_n$ is also a mean-zero martingale. ♠

Proposition 1.4.7. Suppose M_{1n} and M_{2n} are discrete-time mean-zero martingales, then:

$$
\mathbb{C}(M_{1n}, M_{2n}) = \mathbb{E}\langle M_1, M_2\rangle_n = \mathbb{E}[M_1, M_2]_n.
$$

♦

Proof. Since $\mathbb{E}(M_{1n}) = \mathbb{E}(M_{2n}) = 0$, the covariance $\mathbb{C}(M_{1n}, M_{2n}) = \mathbb{E}(M_{1n}M_{2n})$. Now, since $M_{1n}M_{2n} - \langle M_1, M_2\rangle_n$ is a is a mean-zero martingale:

$$
\mathbb{E}\big\{M_{1n}M_{2n} - \langle M_1, M_2\rangle_n\big\} = 0 \implies \mathbb{E}(M_{1n}M_{2n}) = \mathbb{E}\langle M_1, M_2\rangle_n.
$$

Likewise for $M_{1n}M_{2n} - [M_1, M_2]_n$. ■

1.5 Transformations

Definition 1.5.1. A discrete-time process H_n is **predictable** if H_n is measurable with respect to $\mathscr{F}_{n-1}$. ■

Definition 1.5.2. The **transformation** of a process X_n by H_n, written $Z = H \bullet X$, is the process:

$$
Z_n = \sum_{m=1}^{n} H_m \Delta X_m. \tag{1.5.1}
$$

■

Proposition 1.5.8 (**Martingale Transformation**)**.** The transformation of a discrete-time mean-zero martingale M_n by a predictable process remains a mean-zero martingale.

♦

Proof. We show that ΔZ_n is a martingale difference sequence.

$$\begin{aligned}\mathbb{E}(\Delta Z_n|\mathscr{F}_{n-1}) &= \mathbb{E}(Z_n - Z_{n-1}|\mathscr{F}_{n-1}) \\ &= \mathbb{E}(H_n \Delta M_n|\mathscr{F}_{n-1}) \\ &\overset{i.}{=} H_n \mathbb{E}(\Delta M_n|\mathscr{F}_{n-1}) \overset{ii.}{=} 0,\end{aligned}$$

where $\overset{i.}{=}$ holds because H_n is predictable, and $\overset{ii.}{=}$ holds because ΔM_n is a martingale difference sequence. ■

Proposition 1.5.9. The predictable variation of a transformation satisfies:

$$\langle H \bullet X \rangle = H^2 \bullet \langle X \rangle. \tag{1.5.2}$$

♦

Proof. First, observe that:

$$\begin{aligned}\Delta(H \bullet X)_n &= (H \bullet X)_n - (H \bullet X)_{n-1} \\ &= \sum_{m=1}^{n} H_m \Delta X_m - \sum_{m=1}^{n-1} H_m \Delta X_m = H_n \Delta X_n.\end{aligned}$$

Also:

$$\begin{aligned}\Delta\langle X \rangle_n &= \langle X \rangle_n - \langle X \rangle_{n-1} \\ &= \sum_{m=1}^{n} \mathbb{V}(\Delta M_m|\mathscr{F}_{m-1}) - \sum_{m=1}^{n-1} \mathbb{V}(\Delta M_m|\mathscr{F}_{m-1}) = \mathbb{V}(\Delta M_n|\mathscr{F}_{n-1}).\end{aligned}$$

Therefore:

$$\begin{aligned}\langle H \bullet X \rangle_n &= \sum_{m=1}^{n} \mathbb{V}\big\{\Delta(H \bullet X)_m|\mathscr{F}_{m-1}\big) \\ &= \sum_{m=1}^{n} \mathbb{V}\big(H_m \Delta X_m|\mathscr{F}_{m-1}\big) \\ &= \sum_{m=1}^{n} H_m^2 \mathbb{V}(\Delta X_m|\mathscr{F}_{m-1}) = \sum_{m=1}^{n} H_m^2 \Delta\langle X \rangle_m.\end{aligned}$$

■

Problem 1.5.3. Verify that the optional variation of a transformation satisfies:

$$[H \bullet X] = H^2 \bullet [X].$$

♠

Definition 1.5.3. A random variable τ is a **stopping time** if the event $\{\tau = t\}$ is decidable (measurable) with respect to $\mathscr{F}_t$. ■

Proposition 1.5.10. If M_n is a mean-zero martingale and τ is a stopping time, then the *stopped process* $M_n^\tau = M_{n\wedge\tau}$ is a mean-zero martingale. ♦

Proof. Define the predictable process:

$$H_n = \begin{cases} 0, & \tau \leq n-1, \\ 1, & \tau > n-1 \end{cases} = \mathbb{I}(\tau > n-1).$$

The stopped martingale is expressible as:

$$M_{n\wedge\tau} = \sum_{m=1}^{n} H_m \Delta M_m = H \bullet M.$$

To see this, note that if $\tau \geq n$

$$\sum_{m=1}^{n} \Delta M_m = (M_n - M_{n-1}) + (M_{n-1} - M_{n-2}) + \cdots + (M_2 - M_1) + M_1 = M_n.$$

If $\tau < n$ was the stopping time, then:

$$\sum_{m=1}^{n} H_m \Delta M_m = 0 + \cdots + (M_\tau - M_{\tau-1}) + \cdots + (M_2 - M_1) + M_1 = M_\tau.$$

Since the stopped martingale $M_{n\wedge\tau}$ is the transformation of a mean-zero martingale by a predictable process, $M_{n\wedge\tau}$ is again a mean-zero martingale. ■

Continuous-time Martingales

2.1 Introduction

Definition 2.1.1. A continuous-time stochastic process $X(t)$ is **adapted** to a *filtration* $\mathscr{F}(t)$ if for each t, $X(t)$ is measurable with respect to $\mathscr{F}(t)$. ■

Definition 2.1.2. A continuous-time stochastic process M_t is a **martingale** if:

$$\mathbb{E}\big\{M(t)|\mathscr{F}(s)\big\} = M(s) \text{ for } s \leq t,$$

where $\mathscr{F}(t)$ is a filtration including $\sigma\big\{M(s) : s \leq t\big\}$. ■

Remark 2.1.1. Each realization of a continuous-time stochastic process is called at **sample path**. Unless otherwise stated, continuous-time martingales are assumed to have *càdlàg* sample paths, to satisfy the initial condition $M(0) = 0$, and to have a finite time horization $\mathcal{T} = [0, \tau]$. Many of the properties of discrete-time martingales carry over to continuous-time. ♦

Discussion 2.1.1. The martingale property may be expressed in differential form as:

$$\mathbb{E}\big\{dM(t)|\mathscr{F}(t-)\big\} = 0,$$

where $dM(t)$ is the increment of the martingale over the time interval $[t, t + dt)$ and $\mathscr{F}(t-)$ is the history just before time t. Compare this with the discrete-time result:

$$\mathbb{E}(\Delta M_n|\mathscr{F}_{n-1}) = 0.$$

♠

Proposition 2.1.1. As in discrete-time, continuous-time martingales have constant mean and *uncorrelated increments*:

$$\mathbb{C}\big\{M(t) - M(s), M(v) - M(u)\big\} = 0,$$

for $s < t < u < v$. ♦

Proof. To demonstrate constant mean:

$$\mathbb{E}\big\{M(t)\big\} = \mathbb{E}\big[\mathbb{E}\{M(t)|\mathscr{F}(0)\}\big] = \mathbb{E}\big\{M(0)\big\}.$$

For uncorrelated increments, first observe that:

$$\mathbb{E}\big\{M(t) - M(s)\big\} = \mathbb{E}\big\{M(v) - M(u)\big\} = 0$$

due to constant mean. Thus:

$$\mathbb{C}\big\{M(t) - M(s), M(v) - M(u)\big\} = \mathbb{E}\big[\{M(t) - M(s)\}\{M(v) - M(u)\}\big].$$

By iterated expectation:

$$\begin{aligned}
\mathbb{E}\big[\{M(t) - M(s)\}\{M(v) - M(u)\}\big] &= \mathbb{E}\left(\mathbb{E}\big[\{M(t) - M(s)\}\{M(v) - M(u)\}|\mathscr{F}(u)\big]\right) \\
&= \mathbb{E}\left(\{M(t) - M(s)\}[\mathbb{E}\{M(v)|\mathscr{F}(u)\} - M(u)]\right) \\
&= \mathbb{E}\left[\{M(t) - M(s)\}\{M(u) - M(u)\}\right] = 0.
\end{aligned}$$

■

2.2 Variation Processes

Definition 2.2.1. Suppose $M(t)$ is a continuous-time mean-zero martingale. The **predictable variation** of $M(t)$ is:

$$\langle M\rangle(t) = \lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{V}\big\{\Delta M_k|\mathscr{F}(\delta_{k-1})\big\}, \tag{2.2.3}$$

where $\delta_k = (k/n)t$, and $\Delta M_k = M(\delta_k) - M(\delta_{k-1})$. Note that the δ_k partition the interval $[0, t]$ into n sub-intervals of length t/n.

The **optional variation** of $M(t)$ is:

$$[M](t) = \lim_{n\to\infty}\sum_{k=1}^{n}\big(\Delta M_k\big)^2, \tag{2.2.4}$$

where $\Delta M_k = M(\delta_k) - M(\delta_{k-1})$ as for the predictable variation. ■

Discussion 2.2.1. In differential form:

$$d\langle M\rangle(t) = \mathbb{V}\big\{dM(t)|\mathscr{F}(t-)\big\}.$$

Compare this with the discrete-time result, obtained in the proof of (1.5.2), that:

$$\Delta\langle M\rangle_n = \mathbb{V}(\Delta M_n|\mathscr{F}_{n-1}).$$

♠

Proposition 2.2.2. Suppose $M(t)$ is a continuous-time mean-zero martingale. Then, the following *compensated processes* are also mean-zero martingales:

$$M^2(t) - \langle M\rangle(t), \qquad\qquad M^2(t) - [M](t).$$

Consequently,

$$\mathbb{V}\big\{M(t)\big\} = \mathbb{E}\big\{M^2(t)\big\} = \mathbb{E}\langle M\rangle(t) = \mathbb{E}[M](t).$$

♦

2.3 Covariation Process

Definition 2.3.1. Suppose $M_1(t)$ and $M_2(t)$ are continuous-time mean-zero martingales. The **predictable covariation** of M_1 and M_2 is:

$$\langle M_1, M_2\rangle(t) = \lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{C}\big\{\Delta M_{1k}\Delta M_{2k}|\mathscr{F}(\delta_{k-1})\big\},$$

where $\delta_k = (k/n)t$, and $\Delta_{jk} = M_j(\delta_k) - M_j(\delta_{k-1})$. The **optional covariation** of M_1 and M_2 is:

$$[M_1, M_2](t) = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta M_{1k} \Delta M_{2k}.$$

■

Proposition 2.3.3. Suppose $M_1(t)$ and $M_2(t)$ are continuous-time mean-zero martingales. Then the following *compensated processes* are also mean-zero martingales:

$$M_1 M_2 - \langle M_1, M_2 \rangle, \qquad M_1 M_2 - [M_1 M_2].$$

Consequently,

$$\mathbb{C}\big\{M_1(t), M_2(t)\big\} = \mathbb{E}\big\{M_1(t) M_2(t)\big\} = \mathbb{E}\langle M_1, M_2\rangle(t) = \mathbb{E}[M_1, M_2](t).$$

♦

Proposition 2.3.4. Similar to the covariance operator for random variables, the predictable and optional covariation processes are bilinear:

$$\langle M_1 + M_2 \rangle = \langle M_1 \rangle + \langle M_2 \rangle + 2\langle M_1, M_2 \rangle,$$
$$[M_1 + M_2] = [M_1] + [M_2] + 2[M_1, M_2].$$

♦

2.4 Stochastic Integrals

Definition 2.4.1. A stochastic process $H(t)$ is **predictable** if $H(t)$ is adapted to the filtration $\mathscr{F}(t)$ and $H(t)$ has *left-continuous* sample paths. ■

Definition 2.4.2. The **stochastic integral** of a predictable process with respect to a martingale may be defined as:

$$I(t) = \int_0^t H(s) dM(s) = \lim_{n\to\infty} \sum_{k=1}^{n} H_k \Delta M_k,$$

where $\delta_k = (k/n)t$ partitions the interval $[0, t]$ into n sub-intervals of length t/n, H_k is $H(\delta_{k-1})$, and $\Delta M_k = M(\delta_k) - M(\delta_{k-1})$. ■

Theorem 2.4.1 (Martingale Transformation). The stochastic integral $I(t)$ of a predictable process $H(t)$ with respect to a mean-zero martingale $M(t)$ remains a mean-zero martingale. □

Remark 2.4.2. This is analogous to preservation of the martingale property under transformation $(H \bullet M)$ of a discrete time martingale M_n by a predictable process H_n. ♦

Proposition 2.4.5. The predictable and optional covariations of the stochastic integral of a predictable process with respect to a mean-zero martingale obeys the following rules:

$$\left\langle \int H_1(t)dM_1(t), \int H_2(t)dM_2(t) \right\rangle = \int H_1 H_2(t) d\langle M_1, M_2\rangle(t)$$
$$\left[\int H_1(t)dM_1(t), \int H_2(t)dM_2(t) \right] = \int H_1 H_2(t) d[M_1, M_2](t).$$

In the case that $H_1 = H_2 = H$:

$$\left\langle \int H(t)dM(t) \right\rangle = \int H^2(t) d\langle M\rangle(t)$$
$$\left[\int H(t)dM(t) \right] = \int H^2(t) d[M](t)$$

♦

2.5 Doob-Meyer Decomposition

Definition 2.5.1. A continuous-time stochastic process $N(t)$ is a **sub-martingale** if it tends to increase as time passes:

$$\mathbb{E}\big\{N(t)|\mathscr{F}(s)\big\} \geq N(s) \text{ for } t > s.$$

■

Theorem 2.5.2. A continuous-time sub-martingale $N(t)$ (e.g. a counting process) can be uniquely decomposed as:

$$N(t) = A(t) + M(t), \tag{2.5.5}$$

where $A(t)$ is a non-decreasing predictable process (the *compensator*), and $M(t)$ is a mean-zero martingale. □

Discussion 2.5.1. In differential form, the Doob-Meyer decomposition is expressible as:

$$dN(t) = dA(t) + dM(t).$$

Upon taking the conditional expectation:

$$\mathbb{E}\big\{dN(t)|\mathscr{F}(t-)\big\} = \mathbb{E}\big\{dA(t)|\mathscr{F}(t-)\big\} + \mathbb{E}\big\{dM(t)|\mathscr{F}(t-)\big\}.$$

Since $A(t)$ is a predictable process and $M(t)$ is a martingale:

$$\mathbb{E}\big\{dN(t)|\mathscr{F}(t-)\big\} = A(t).$$

The increments of the martingale are thus:

$$\begin{aligned} dM(t) &= dN(t) - dA(t) \\ &= dN(t) - \mathbb{E}\big\{dN(t)|\mathscr{F}(t-)\big\}. \end{aligned}$$

♠

2.6 Poisson Process

Example 2.6.1. Suppose $N(t)$ is the number of events by time t in a standard Poisson process with rate λ. Define the compensated process:

$$M(t) = N(t) - \lambda t.$$

By the independent increments property:

$$\mathbb{E}\big\{M(t) - M(s)|\mathscr{F}(s)\big\} = \mathbb{E}\big\{M(t) - M(s)\big\} = \mathbb{E}\big\{N(t) - N(s)\big\} - \lambda(t-s) = 0.$$

Therefore:

$$0 = \mathbb{E}\big\{M(t) - M(s)|\mathscr{F}(s)\big\} = \mathbb{E}\big\{M(t)|\mathscr{F}(s)\big\} - M(s).$$

Combined with $M(0) = 0$, this shows $M(t)$ is a mean-zero martingale. Now consider the square process:

$$M^2(t) = N^2(t) + (\lambda t)^2 - 2\lambda t N(t)$$

Consider the process $M^2(t) - \lambda t$:

$$\mathbb{E}\big\{M^2(t) - \lambda t|\mathscr{F}(s)\big\} = \mathbb{E}\big\{N^2(t)|\mathscr{F}(s)\big\} + (\lambda t)^2 - 2\lambda t\mathbb{E}\big\{N(t)|\mathscr{F}(s)\big\} - \lambda t.$$

The first expectation reduces as:

$$\begin{aligned} \mathbb{E}\big\{N^2(t)|\mathscr{F}(s)\big\} =& \mathbb{E}\big[\{N(t) - N(s) + N(s)\}^2|\mathscr{F}(s)\big] \\ =& \mathbb{E}\big[\{N(t) - N(s)\}^2|\mathscr{F}(s)\big] + N^2(s) + 2N(s)\mathbb{E}\big\{N(t) - N(s)|\mathscr{F}(s)\big\} \end{aligned}$$

By the independent increments property:

$$\begin{aligned} \mathbb{E}\big[\{N(t) - N(s)\}^2|\mathscr{F}(s)\big] =& \mathbb{E}\big[\{N(t) - N(s)\}^2\big] \\ =& \mathbb{V}\big\{N(t-s)\big\} + \mathbb{E}^2\big\{N(t-s)\big\} \\ =& \lambda(t-s) + \lambda^2(t-s)^2. \end{aligned}$$

The second expectation reduces to:

$$
\begin{aligned}
\mathbb{E}\big\{N(t)|\mathscr{F}(s)\big\} =& \mathbb{E}\big\{N(t) - N(s) + N(s)|\mathscr{F}(s)\big\} \\
=& \mathbb{E}\big\{N(t) - N(s)\big\} + N(s) \\
=& \lambda(t-s) + N(s)
\end{aligned}
$$

Overall:

$$
\begin{aligned}
\mathbb{E}\big\{M^2(t) - \lambda t|\mathscr{F}(s)\big\} =& \lambda(t-s) + \lambda^2(t-s)^2 + N^2(s) + 2\lambda(t-s)N(s) \\
& + (\lambda t)^2 - 2\lambda t\big\{\lambda(t-s) + N(s)\big\} - \lambda t \\
=& N^2(s) + \lambda^2 s^2 - 2\lambda s N(s) - \lambda s \\
=& M^2(s) - \lambda s.
\end{aligned}
$$

Conclude that $\langle M\rangle(t) = \lambda t$, which gives:

$$
\mathbb{E}\big\{M^2(t)\big\} = \mathbb{E}\langle M\rangle(t) = \lambda t.
$$

♠

2.7 Counting Process

Definition 2.7.1. A **counting process** is a continuous-time *càdlàg* process with increments of size 1 at event times. ■

Discussion 2.7.1. Since any counting process $N(t)$ is a sub-martingale, by the Doob-Meyer decomposition, there exists a unique predictable process $\Lambda(t)$, the *cumulative intensity*, such that $M(t) = N(t) - \Lambda(t)$ is a mean-zero martingale. If $\Lambda(t)$ is absolutely continuous, there exists a predictable *intensity* process such that:

$$
\Lambda(t) = \int_0^t \lambda(s)ds.
$$

In differential form:

$$
dM(t) = dN(t) - \lambda(t)dt.
$$

Upon taking the conditional expectation:

$$
\begin{aligned}
\mathbb{E}\big\{dM(t)|\mathscr{F}(t-)\big\} &= \mathbb{E}\big\{dN(t)|\mathscr{F}(t-)\big\} - \mathbb{E}\big\{\lambda(t)dt|\mathscr{F}(t-)\big\} \\
0 &= \mathbb{E}\big\{dN(t)|\mathscr{F}(t-)\big\} - \lambda(t)dt.
\end{aligned}
$$

Finally, since $N(t)$ has unit increments, $dN(t)$ is Bernoulli, and:

$$
\lambda(t)dt = \mathbb{E}\big\{dN(t)|\mathscr{F}(t-)\big\} = \mathbb{P}\big\{dN(t) = 1|\mathscr{F}(t-)\big\}.
$$

For the optional variation of a counting process martingale, note that as the partition of $[0, t]$ becomes infinitely fine, only the jumps $dN(t)$, which are of size 1, will contribute to (2.2.4). This motivates:

$$\big[M\big](t) = N(t).$$

For the predictable variation,

$$\begin{aligned} d\langle M\rangle(t) &= \mathbb{V}\big\{dM(t)|\mathscr{F}(t-)\big\} \\ &= \mathbb{V}\big\{dN(t) - \lambda(t)dt|\mathscr{F}(t-)\big\} \\ &\stackrel{*}{=} \mathbb{V}\big\{dN(t)|\mathscr{F}(t-)\big\}. \end{aligned}$$

where $\stackrel{*}{=}$ holds since $\lambda(t)$ is predictable. Now, since $dN(t)$ is Bernoulli:

$$d\langle M\rangle(t) = \lambda(t)dt\big\{1 - \lambda(t)dt\big\} = \lambda(t)dt + o(dt).$$

This motivates the relation:

$$\langle M\rangle(t) = \int_0^t \lambda(s)ds = \Lambda(t).$$

That is, the predictable variation of a counting process martingale $M(t) = N(t) - \Lambda(t)$ is its cumulative intensity. ♠

Theorem 2.7.3. If $N(t)$ is a counting process with cumulative intensity $\Lambda(t)$, then the following are mean-zero martingales:

$$\begin{gathered} M(t) = N(t) - \Lambda(t), \\ M^2(t) - \Lambda(t) \end{gathered}$$

□

Definition 2.7.2. Two martingales M_1 and M_2 are **orthogonal** if the predictable co-variation is zero:

$$\langle M_1, M_2\rangle(t) = 0.$$

Note that, for orthogonal martingales, the product $M_1(t)M_2(t)$ is itself a mean-zero martingale. ■

Proposition 2.7.6. If $N_1(t)$ and $N_2(t)$ are counting processes that cannot jump simultaneously, then the corresponding martingales $M_1(t)$ and $M_2(t)$ are orthogonal. ♦

Discussion 2.7.2. Suppose $N_1(t), \cdots, N_n(t)$ are counting processes, no two of which can jump simultaneously, then the corresponding martingales $M_1(t), \cdots M_n(t)$ are orthogonal. Since the predictable covariation obeys:

$$\left\langle \int H_1(t)dM_1(t), \int H_2(t)dM_2(t) \right\rangle = \int H_1 H_2(t) d\langle M_1, M_2 \rangle(t),$$

the corresponding stochastic integrals $\int H_1(t)dM_1(t), \cdots \int H_n(t)dM_n(t)$ are orthogonal. From bilinearity of the predictable variation:

$$\left\langle \sum_{i=1}^{n} \int_0^t H_i(s)dM_i(s) \right\rangle = \sum_{i=1}^{n} \int_0^t H_i^2(s) d\langle M_i \rangle(s) = \sum_{i=1}^{n} \int_0^t H_i^2(s)\lambda_i(s)ds.$$

For the optional variation:

$$\left[\sum_{i=1}^{n} \int_0^t H_i(s)dM_i(s) \right] = \sum_{i=1}^{n} \int_0^t H_i^2(s) d[M_i](s) = \sum_{i=1}^{n} \int_0^t H_i^2(s)dN_i(s).$$

♠

2.8 Martingale Central Limit Theorem

Definition 2.8.1. The **Brownian motion** process $W(t)$ has continuous sample paths and independent, stationary increments. The increment over the time interval $(s, t]$ is normally distributed with:

$$\mathbb{E}\big\{W(t) - W(s)\big\} = 0, \qquad \mathbb{V}\big\{W(t) - W(s)\big\} = t - s.$$

If $\alpha(t)$ is a strictly increasing and continuous remapping of time, with $\alpha(0) = 0$, then $U(t) = W\big\{\alpha(t)\big\}$ is again a Brownian motion. The increment over the time interval $(s, t]$ is normally distributed with:

$$\mathbb{E}\big\{U(t) - U(s)\big\} = 0, \qquad \mathbb{V}\big\{U(t) - U(s)\big\} = \alpha(t) - \alpha(s).$$

The process $U(t) = W\big\{\alpha(t)\big\}$ is a *Gaussian martingale.* ■

Proposition 2.8.7. Let $U(t) = W\big\{\alpha(t)\big\}$, where $\alpha(t)$ is a strictly increasing and continuous remapping of time, with $\alpha(0) = 0$. Then $U(t)$ is a mean-zero martingale, with predictable variation $\langle U \rangle(t) = \alpha(t)$. ♦

Proof.

$$\begin{aligned}
\mathbb{E}\big\{U(t)|\mathscr{F}(s)\big\} &= \mathbb{E}\big\{U(t) - U(s) + U(s)|\mathscr{F}(s)\big\} \\
&= \mathbb{E}\big\{U(t) - U(s)|\mathscr{F}(s)\big\} + U(s) \\
&= \mathbb{E}\big\{U(t) - U(s)\big\} + U(s) = U(s).
\end{aligned}$$

Thus $U(t)$ is a martingale. Note too that:

$$\mathbb{E}\big\{U(t) - U(s)|\mathscr{F}(s)\big\} = 0,$$

and that $\alpha(t)$ is deterministic. Now consider the process $U^2(t) - \alpha(t)$:

$$\begin{aligned}\mathbb{E}\big\{U^2(t) - \alpha(t)|\mathscr{F}(s)\big\} =& \mathbb{E}\big[\{U(t) - U(s) + U(s)\}^2|\mathscr{F}(s)\big] - \alpha(t) \\ =& \mathbb{E}\big[\{U(t) - U(s)\}^2|\mathscr{F}(s)\big] \\ & + U^2(s) + 2U(s)\mathbb{E}\big\{U(t) - U(s)|\mathscr{F}(s)\big\} - \alpha(t).\end{aligned}$$

From the above, the cross term vanishes. The second moment evaluates to:

$$\begin{aligned}\mathbb{E}\big[\{U(t) - U(s)\}^2|\mathscr{F}(s)\big] &= \mathbb{E}\big[\{U(t) - U(s)\}^2\big] \\ &= \mathbb{V}\big\{U(t) - U(s)\big\} + \mathbb{E}^2\big\{U(t) - U(s)\big\} \\ &= \alpha(t) - \alpha(s) + 0.\end{aligned}$$

Overall:

$$\mathbb{E}\big\{U^2(t) - \alpha(t)|\mathscr{F}(s)\big\} = \alpha(t) - \alpha(s) + U^2(s) - \alpha(t) = U^2(s) - \alpha(s).$$

Since $\alpha(0) = 0$, $U^2(t) - \alpha(t)$ is a mean-zero martingale, and:

$$\mathbb{V}\big\{U(t)\big\} = \mathbb{E}\big\{U^2(t)\big\} = \mathbb{E}\langle U\rangle(t) = \alpha(t).$$

$\blacksquare$

Theorem 2.8.4 (Martingale CLT). Suppose $M^{(n)}(t)$ is a sequence of mean-zero martingales defined on $[0, \tau]$, and for any $\epsilon > 0$ let $M_{\epsilon}^{(n)}(t)$ denote the martingale containing all jumps of $M^{(n)}(t)$ that are of size greater than ϵ. If the following conditions hold:

i. $\langle M^{(n)}\rangle \xrightarrow{p} \alpha(t)$ for all $t \in [0, \tau]$ as $n \to \infty$, where $\alpha(t)$ is a strictly increasing continuous function with $\alpha(0) = 0$.

ii. $\langle M_{\epsilon}^{(n)}\rangle \xrightarrow{p} 0$ for all $t \in [0, \tau]$ for any $\epsilon > 0$ as $n \to \infty$.

Then, $M^{(n)}(t)$ converges weakly to a Gaussian martingale:

$$M^{(n)}(t) \rightsquigarrow W\big\{\alpha(t)\big\}.$$

$\square$

Example 2.8.2. Suppose $N_1^{(n)}(t), \cdots, N_n^{(n)}(t)$ are counting processes, no two of which can jump simultaneously. Recall that the corresponding martingales $M_1^{(n)}(t), \cdots M_n^{(n)}(t)$ are orthogonal. Define the process:

$$U^{(n)}(t) = \sum_{i=1}^{n} U_i^{(n)}(t) = \sum_{i=1}^{n} \int_0^t H_i^{(n)}(t) dM_i^{(n)}(s),$$

where $H_i^{(n)}$ is a predictable process, and:

$$M_i^{(n)} = N_i^{(n)} - \int_0^t \lambda_i^{(n)}(s)ds$$

is a counting process martingale. The requirements for the martingale CLT are that:

$$\langle U^{(n)}\rangle(t) = \sum_{i=1}^{n}\int_0^t \big\{H_i^{(n)}(t)\big\}^2 d\lambda_i^{(n)}(s) \xrightarrow{p} \alpha(t) \text{ for } \forall t \in [0,\tau].$$

$$\langle U_\epsilon^{(n)}\rangle(t) = \sum_{i=1}^{n}\int_0^t \big\{H_i^{(n)}(t)\big\}^2 \mathbb{I}\big\{|H_i^{(n)}(s) > \epsilon\big\} d\lambda_i^{(n)}(s) \xrightarrow{p} 0 \text{ for } \forall \epsilon > 0, \forall t \in [0,\tau].$$

♠

Example 2.8.3. For $n \to \infty$, the path of $U^{(n)}(t)$ on the interval $[0,\tau]$ is approximated by that of a particle undergoing time-transformed Brownian motion $U(t) = W\big\{\alpha(t)\big\}$. For a continuous functional f, the behavior of $f \circ W\big\{\alpha(t)\big\}$ approximates that of $f \circ U^{(n)}(s)$:

$$\big\{f \circ U^{(n)}(s) : s \in [0,\tau]\big\} \rightsquigarrow \big\{f \circ U(s) : s \in [0,\tau]\big\}.$$

For example, the supremum of $\big|U^{(n)}(t)\big|$ may be approximated by that of $\big|U(t)\big|$:

$$\sup_{t\in[0,\tau]} \big|U^{(n)}(t)\big| \rightsquigarrow \sup_{t\in[0,\tau]} \big|U(t)\big|.$$

♠

Summary

- In discrete time, the **martingale property** is $\mathbb{E}(M_n|\mathscr{F}_{n-1}) = M_{n-1}$. In continuous time $\mathbb{E}\big\{M(t)|\mathscr{F}(s)\big\} = M(s)$ for $s \leq t$.

- Martingales have constant mean $\mathbb{E}(M_n) = M_0$ and *uncorrelated increments*:

$$\mathbb{C}\big\{M(t) - M(s), M(v) - M(u)\big\} = 0.$$

- In discrete-time, the predictable **covariation** of a mean-zero martingale M_n is defined as:

$$\langle M_1, M_2\rangle_n = \sum_{m=1}^{n} \mathbb{C}(\Delta M_{1m}\Delta M_{2m}|\mathscr{F}_{m-1}) = \sum_{m=1}^{n} \mathbb{E}(\Delta M_{1m}\Delta M_{2m}|\mathscr{F}_{m-1}).$$

  The optional **covariation** is defined as:

$$[M_1, M_2]_n = \sum_{m=1}^{n} \Delta M_{1m}\Delta M_{2m}.$$

  The predictable and optional **variations** are obtained by taking $M_1 = M_2 = M$. The continuous-time covariations at time t are obtained by taking the limit as the partition of $[0, t]$ becomes infinitely fine.

  The covariance of a martingale is the expectation of its predictable (or optional) variation. In continuous time:

$$\mathbb{C}\big\{M_1(t), M_2(t)\big\} = \mathbb{E}\big\{M_1(t)M_2(t)\big\} = E\langle M_1, M_2\rangle(t) = E[M_1, M_2](t).$$

  Taking $M_1 = M_2 = M$:

$$\mathbb{V}\big\{M(t)\big\} = \mathbb{E}\big\{M^2(t)\big\} = E\langle M\rangle(t) = E[M](t).$$

- In discrete-time, the **transformation** of a mean-zero martingale by a predictable process is:

$$Z_n = \sum_{m=1}^{n} H_m \Delta X_m.$$

  In continuous-time, by taking the limit as the partition of $[0, t]$ becomes infinitely fine, the transformation becomes a **stochastic integral**:

$$I(t) = \int_0^t H(s)dM(s).$$

  In either case, the transformation or stochastic integral of a mean-zero martingale by a predictable process remains a mean-zero martingale.

- The predictable covariation of stochastic integrals obeys:

$$\left\langle \int H_1(t)dM_1(t), \int H_2(t)dM_2(t) \right\rangle = \int H_1H_2(t)d\langle M_1, M_2\rangle(t).$$

  Similarly, for the optional variation:

$$\left[\int H_1(t)dM_1(t), \int H_2(t)dM_2(t) \right] = \int H_1H_2(t)d[M_1, M_2](t)$$

- A continuous-time stochastic process $N(t)$ is a **sub-martingale** if it tends to increase as time passes:

$$\mathbb{E}\big\{N(t)|\mathscr{F}(s)\big\} \geq N(s) \text{ for } t > s.$$

  (**Doob-Meyer Decomposition**) A continuous-time sub-martingale $N(t)$ can be uniquely decomposed as:

$$N(t) = A(t) + M(t), \tag{3.0.6}$$

  where $A(t)$ is a non-decreasing predictable process (the *compensator*), and $M(t)$ is a mean-zero martingale.

- A **counting process** is a continuous-time *càdlàg* process with increments of size 1 at event times. By Doob-Meyer, if $N(t)$ is a counting process, there exists a unique predictable process, termed the *cumulative intensity*, such that $M(t) = N(t) - \Lambda(t)$ is a mean-zero martingale.

  The optional variation of a counting process martingale is the counting process:

$$\big[M\big] = N(t).$$

  The predictable variation is its cumulative intensity:

$$\langle M\rangle(t) = \Lambda(t).$$

- Counting processes $N_1(t)$ and $N_2(t)$ are **orthogonal** if the predictable covariation of the corresponding martingales is zero:

$$\langle M_1, M_2\rangle(t) = 0.$$

  If two counting processes cannot jump simultaneously, then they are orthogonal.

- The **Brownian motion** process $W(t)$ has continuous sample paths, independent stationary increments, and the increment over the time interval $(s, t]$ is normally distributed with:

$$\mathbb{E}\big\{W(t) - W(s)\big\} = 0, \qquad \mathbb{V}\big\{W(t) - W(s)\big\} = t - s.$$

  If $\alpha(t)$ is a strictly increasing continuous remapping of time with $\alpha(0) = 0$, then $U(t) = W\big\{\alpha(t)\big\}$ is again Brownian motion, with increments:

$$\mathbb{E}\big\{U(t) - U(s)\big\} = 0, \qquad \mathbb{V}\big\{U(t) - U(s)\big\} = \alpha(t) - \alpha(s).$$

  The process $U(t) = W\big\{\alpha(t)\big\}$ is a *Gaussian martingale*.

- (**Martingale Central Limit Theorem**) Suppose $M^{(n)}(t)$ is a sequence of mean-zero martingales defined on $[0, \tau]$, and for any $\epsilon > 0$ let $M_\epsilon^{(n)}(t)$ denote the martingale containing all jumps of $M^{(n)}(t)$ that are of size greater than ϵ. If the following conditions hold:

  i. $\langle M^{(n)} \rangle \xrightarrow{p} \alpha(t)$ for all $t \in [0, \tau]$ as $n \to \infty$, where $\alpha(t)$ is a strictly increasing continuous function with $\alpha(0) = 0$.

  ii. $\langle M_\epsilon^{(n)} \rangle \xrightarrow{p} 0$ for all $t \in [0, \tau]$ for any $\epsilon > 0$ as $n \to \infty$.

  Then, $M^{(n)}(t)$ converges weakly to a Gaussian martingale:

$$M^{(n)}(t) \rightsquigarrow W\big\{\alpha(t)\big\}.$$