Poisson Process

Definition 1.0.1. A **Poisson process** may be characterized in the following ways:

i. By the sequence of arrival times (T_k), where T_k is the time of kth arrival.

ii. The sequence of inter-arrival times (Δ_k), where $\Delta_k = T_k - T_{k-1}$ is the time between the $(k-1)$st and kth arrivals.

iii. By the number of arrivals by time t:

$$N_t = \max\{n \in \mathbb{N} : T_n \leq t\}.$$

By hypothesis, the Poisson process has a **strong renewal property**: at each arrival and at each fixed time, the process restarts, independent of the past. ■

Discussion 1.0.1. The time of kth arrival and the sequence of inter-arrival times are related by:

$$T_n = \sum_{k=1}^{n} \Delta_k, \qquad \Delta_k = T_k - T_{k-1}.$$

The arrival time process T_n is linked with the counting process N_t by:

$$T_n = \inf\big\{t \in \mathbb{T} : N_t = n\big\}, \qquad N_t = \max\big\{n \in \mathbb{N} : T_n \leq t\big\}.$$

The number of arrivals by time t is at least n if and only if the time of nth arrival is at most t: $N_t \geq n \iff T_n \leq t$. ♠

1.1 Inter-arrival Times

Discussion 1.1.1. The first component of the strong renewal property states that the process restarts upon each arrival. This implies that the sequence of inter-arrival times is independent and identically distributed (IID). The second component of the strong renewal property is that the process restarts at each fixed time point. This implies that the inter-arrival time distribution is **memory-less**:

$$\mathbb{P}(T > s + t | T > s) = \mathbb{P}(T > t). \tag{1.1.1}$$

Among continuous-time distributions, the property of being memory-less uniquely characterizes the exponential distribution. Therefore, the inter-arrival times of the Poisson process are IID exponential random variables:

$$f(\delta) = \lambda e^{-\lambda\delta}, \text{ for } \delta > 0.$$

♠

Proposition 1.1.1. The exponential distribution has constant hazard. That is, if T follows an exponential distribution with rate λ, then:

$$\lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}(t \leq T < t + \delta | T \geq t) = \lambda. \tag{1.1.2}$$

♦

Proposition 1.1.2. If $\Delta_1 \sim \text{Exp}(\lambda_1)$ and $\Delta_2 \sim \text{Exp}(\lambda_2)$, then the probability that Δ_1 arrives first is:

$$\mathbb{P}(\Delta_1 \leq \Delta_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

♦

Proof. By iterated expectation:

$$\mathbb{P}(\Delta_1 \leq \Delta_2) = \mathbb{E}\big\{\mathbb{P}(\Delta_2 \geq \Delta_1 | \Delta_1)\big\} = \mathbb{E}\big(e^{-\lambda_2 \Delta_1}\big).$$

The remaining expectation evaluates to:

$$\mathbb{E}\big(e^{-\lambda_2 \Delta_1}\big) = \int_0^\infty e^{-\lambda_2 u} \cdot \lambda_1 e^{-\lambda_1 u} du = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

■

1.2 Time of nth Arrival

Proposition 1.2.3. Recall that the inter-arrival times are $\Delta_k \overset{\text{IID}}{\sim} \text{Exp}(\lambda)$. The time of nth arrival T_n follows a gamma distribution with shape n and rate λ:

$$f(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}, \text{ for } t > 0. \tag{1.2.3}$$

♦

Proposition 1.2.4. The arrival process has:

- **Stationary increments**: $T_n - T_m \overset{d}{=} T_{n-m}$ for $\forall (m \leq n)$.
- **Independent increments**: For $n_1 < n_2 < \cdots$,

$$(T_{n_1}, T_{n_2} - T_{n_1}, T_{n_3} - T_{n_2}, \cdots)$$

  are independent.

♦

Proof. For stationarity:

$$T_n - T_m = \sum_{k=1}^{n} \Delta_k - \sum_{l=1}^{m} \Delta_l = \sum_{k=m+1}^{n} \Delta_k \stackrel{d}{=} \text{Gamma}(n-m, \lambda) \stackrel{d}{=} T_{n-m}.$$

For independent increments, observe that each interval may be expressed as the sum of disjoint sets of IID inter-arrival times:

$$T_{n_1} = \sum_{k=1}^{n_1} \Delta_k, \qquad T_{n_2} - T_{n_1} = \sum_{k=n_1+1}^{n_2} \Delta_k, \qquad T_{n_3} - T_{n_2} = \sum_{k=n_3+1}^{n_2} \Delta_k, \cdots$$

■

1.3 Counting Process

Proposition 1.3.5. The number of arrivals N_t by time $t \in [0, \infty)$ follows a Poisson distribution with expectation λt:

$$\mathbb{P}(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \text{ for } n \in \{0, 1, \cdots\}.$$

♦

Proof. The number of arrivals by time t is at least n if and only if the time of nth arrival is no greater than t:

$$\begin{aligned}
\mathbb{P}(N_t \geq n) = \mathbb{P}(T_n \leq t) &= \frac{\lambda^n}{\Gamma(n)} \int_0^t s^{n-1} e^{-\lambda s} ds \\
&= \frac{\lambda^n}{\Gamma(n)} \int_0^{\lambda t} (\lambda^{-1} u)^{n-1} e^{-u} \lambda^{-1} du \\
&= \frac{1}{(n-1)!} \int_0^{\lambda t} u^{n-1} e^{-u} du.
\end{aligned}$$

Integrating by parts:

$$\begin{aligned}
\frac{1}{(n-1)!} \int_0^{\lambda t} u^{n-1} e^{-u} du &= \frac{1}{(n-1)!} \big\{ -u^{n-1} e^{-u} - (n-1) u^{n-2} e^{-u} - \cdots - (n-1)! e^{-u} \big\}_{u=0}^{u=\lambda t} \\
&= 1 - \left\{ \frac{(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!} + \frac{(\lambda t)^{n-2} e^{-\lambda t}}{(n-2)!} + \cdots + e^{-\lambda t} \right\} \\
&= 1 - \sum_{k=1}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!}
\end{aligned}$$

Finally:

$$\begin{aligned}
\mathbb{P}(N_t = n) &= \mathbb{P}(N_t \geq n) - \mathbb{P}(N_t \geq n+1) \\
&= \sum_{k=1}^{n} e^{-\lambda t} \frac{(\lambda t)^k}{k!} - \sum_{k=1}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.
\end{aligned}$$

■

Discussion 1.3.1. The counting process has:

- **Stationary increments**: $N_t - N_s \stackrel{d}{=} N_{t-s}$ for $s < t$.
- **Independent increments**: For $t_1 < t_2 < \cdots$:

$$(N_{t_1}, N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \cdots)$$

are independent.

These properties likewise follow from the strong renewal property of the Poisson process. The difference $N_t - N_s$ represents the number of arrivals occurring in the interval $(s, t]$. Since the process resets at time s, the number of arrivals in this interval is equivalent in distribution to the number of arrivals in the interval $(0, t - s]$:

$$\mathbb{P}(N_t - N_s = k) = \mathbb{P}(N_{t-s} = k) \text{ for } k \in \{0, 1, \cdots\}.$$

Moreover, since the process resets at each time point in the sequence $t_1 < t_2 < \cdots$, the number of arrivals occurring in the disjoint intervals $(0, t_1]$, $(t_1, t_2]$, $\cdots$ are independent.

♠

Proposition 1.3.6. The conditional distribution of the first arrival time T_1, given that $N_t = 1$, is uniform on $(0, t]$. ♦

Proof.

$$\begin{aligned}
\mathbb{P}(T_1 \leq s | N_t = 1) &= \frac{\mathbb{P}(T_1 \leq s, N_t = 1)}{\mathbb{P}(N_t = 1)} = \frac{\mathbb{P}(N_s = 1, N_t = 1)}{\mathbb{P}(N_t = 1)} \\
&= \frac{\mathbb{P}(N_s = 1, N_{t-s} = 0)}{\mathbb{P}(N_t = 1)} = \frac{\mathbb{P}(N_s = 1)\mathbb{P}(N_{t-s} = 0)}{\mathbb{P}(N_t = 1)} \\
&= \frac{e^{-s}(\lambda s)e^{-(t-s)}}{e^{-t}\lambda t} = \frac{s}{t} \text{ for } s \in (0, t].
\end{aligned}$$

■

Remark 1.3.1. More generally, the conditional distribution of the first n arrival times $(T_1, \cdots, T_n)$ given $N_t = n$ is the distribution of the order statistics for a sample of size n from the uniform $(0, t]$ distribution. ♦

Proposition 1.3.7. For $s < t$, the conditional distribution of the number of arrivals by time s, given that the number of arrivals by time t is $N_t = n$, is binomial with n trials and probability of success $p = s/t$:

$$N_s \big| (N_t = n) \sim \text{Binomial}\big(n, s/t\big). \tag{1.3.4}$$

♦

1.4 Thinned Process

Definition 1.4.1. Thinning a Poisson processes refers to classifying each arrival into one of a finite number of types. ■

Example 1.4.1. Suppose arrivals occur according to a Poisson process with intensity λ. Each arrival is independently classified as type 1 with probability π, and type 0 with probability $1-\pi$. Let I_k indicate the type of the kth arrival. The sequence of type indicators (I_k) forms a *Bernoulli* process with rate π.

Let X_k denote the (discrete) inter-arrival times of the Bernoulli process. X_k represents the number of trials between consecutive type 1 arrivals. Let U_n denote the (discrete) time of nth arrival from the Bernoulli process. U_n represents the trial number of the nth type 1 arrival. Finally, let Δ_{1k} denote the (continuous) time between the $(k-1)$st and kth type 1 arrival. Δ_{1k} is a geometric sum of independent $\text{Exp}(\lambda)$ random variables:

$$\Delta_{1k} = \sum_{i=U_{k-1}+1}^{U_k} \Delta_i.$$

Since the inter-arrival times (Δ_i) of the original process are IID, and the arrival times of the Bernoulli process are stationary:

$$\Delta_{1k} \overset{d}{=} \sum_{i=1}^{U_k-U_{k-1}} \Delta_i \overset{d}{=} \sum_{i=1}^{U_1} \Delta_i.$$

Finding the moment generating function of Δ_{1k}:

$$M(t) = \mathbb{E}\big\{e^{t\sum_{i=1}^{U_1}\Delta_i}\big\} = \mathbb{E}\big\{\mathbb{E}(e^{t\sum_{i=1}^{V_1}\Delta_i}|U_1)\big\}.$$

Recall that, given U_1, $\sum_{i=1}^{U_1}\Delta_i$ follows a Gamma distribution with shape U_1 and rate λ:

$$\begin{aligned} M(t) = \mathbb{E}\big\{(1-t/\lambda)^{-U_1}\big\} &= \sum_{v=1}^{\infty}(1-t/\lambda)^{-v}\cdot(1-\pi)^{v-1}\pi \\ &= \frac{\pi}{1-\pi}\sum_{v=1}^{\infty}(1-t/\lambda)^{-v}(1-\pi)^{v} \\ &= \frac{\pi}{1-\pi}\sum_{v=1}^{\infty}\left\{\left(\frac{\lambda}{\lambda-t}\right)(1-\pi)\right\}^{v}. \end{aligned}$$

The sum of the geometric series $\sum_{k=1}^{\infty} r^k$ is $r(1-r)^{-1}$, therefore:

$$\begin{aligned} M(t) &= \frac{\pi}{1-\pi}\cdot\frac{\left(\frac{\lambda}{\lambda-t}\right)(1-\pi)}{1-\left(\frac{\lambda}{\lambda-t}\right)(1-\pi)} = \frac{\pi\left(\frac{\lambda}{\lambda-t}\right)}{1-\left(\frac{\lambda}{\lambda-t}\right)(1-\pi)} \\ &= \frac{\lambda\pi}{(\lambda-t)-\lambda(1-\pi)} = \frac{\lambda\pi}{\lambda\pi-t} = \left(1-\frac{t}{\lambda\pi}\right)^{-1} \end{aligned}$$

The form of the moment generating function $M(t)$ identifies the distribution of Δ_{1k} as exponential with rate $\lambda\pi$. Since the inter-arrival times of the type 1 arrivals are IID exponential, the sequence of type 1 arrivals form a Poisson process with rate $\lambda\pi$. ♠

1.5 Superposition

Definition 1.5.1. Superposition refers to combining the arrivals from two or more distinct processes. ■

Example 1.5.2. The superposition of two independent Poisson processes with intensities λ_1 and λ_2 is again a Poisson processes with intensity $\lambda_1 + \lambda_2$. To see this, note that the time to first arrival of the combined process T_1 is the minimum of the time to first arrival from process 1, $T_1^{(1)}$, and the time to first arrival from process 2, $T_1^{(2)}$. The minimum of independent exponentials is again exponential:

$$T = \min\left(T_1^{(1)}, T_1^{(2)}\right) \sim \text{Exp}(\lambda_1 + \lambda_2).$$

Since the first and second Poisson processes each have the strong renewal property, both reset upon arrival T_1. Therefore, the distribution of inter-arrival times for the combined process is IID $\text{Exp}(\lambda_1 + \lambda_2)$. ♠

1.6 Compound Process

Definition 1.6.1. In a **compound** Poisson process, each arrival is accompanied by a real valued random variable that represents its value. ■

Proposition 1.6.8. Let V_k denote the value of the kth arrival in a compound Poisson process. Suppose the sequence (V_k) is IID. The partial sum process:

$$S_t = \sum_{k=1}^{N_t} V_k,$$

has *stationary* and *independent* increments. Moreover, the mean and variance of S_t are:

$$\mathbb{E}(S_t) = \mu \cdot \lambda t \qquad \mathbb{V}(S_t) = \left(\mu^2 + \sigma^2\right) \cdot \lambda t, \tag{1.6.5}$$

where $\mu = \mathbb{E}(V_k)$ and $\sigma^2 = \mathbb{V}(V_k)$. ♦

1.7 Non-homogeneous Process

Definition 1.7.1. In a **non-homogeneous** Poisson process, the intensity is allowed to vary across time. ■

Discussion 1.7.1. A non-homogeneous Poisson process lacks the strong renewal property because the intensity $\lambda(t)$ is time-dependent. Although the associated counting process N_t retains independents increments, the increments are no longer stationary. Instead, the distribution of the number of arrivals in the interval $(s, t]$, $N_t - N_s$, is Poisson with mean:

$$\mu_{(s,t]} = \int_s^t \lambda(u)du.$$

♠

1.8 Problems

1. Prove the following properties of the exponential distribution:

   i. The exponential distribution forms a scale family:

   $$\text{Exp}(\lambda) \stackrel{d}{=} \lambda^{-1}\text{Exp}(1).$$

   ii. The exponential distribution is memory-less (1.1.1).

   iii. The exponential distribution has constant hazard (1.1.2).

   iv. If (Δ_i) are independent exponential random variables, with rates (λ_i), then:

   $$\min(\Delta_1, \cdots, \Delta_n) \sim \text{Exp}(\lambda),$$

   where $\lambda = \sum_{k=1}^{n} \lambda_k$.

   v. If (Δ_i) are independent exponential random variables, with rates (λ_i), then:

   $$\mathbb{P}\big(\Delta_i \leq \Delta_j \text{ for } \forall j \neq i\big) = \frac{\lambda_i}{\sum_{j \neq i} \lambda_j}.$$

   Observe that:

   $$\mathbb{P}\big(\Delta_i \leq \Delta_j \text{ for } \forall j \neq i\big) = \mathbb{P}\left(\Delta_i \leq \min_{j \neq i} \Delta_j\right).$$

2. Prove the following properties of the gamma distribution:

   (a) The sum of IID exponential random variables with common rate λ forms a Gamma(n, λ) distribution (1.2.3).

(b) The sum of independent gamma distributions with shape parameters (n_1, n_2) and common rate λ forms a $\text{Gamma}(n_1 + n_2, \lambda)$ distribution.

3. Prove that:

$$\frac{\lambda^n}{\Gamma(n)} \int_0^t u^{n-1} e^{-\lambda u} du = 1 - \sum_{k=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

4. Prove that the conditional distribution of $N_s|(N_t = n)$ is binomial (1.3.4).

5. Consider the thinned Poisson process with rate λ and type 1 probability π. Find the conditional distribution of the total number of arrivals N_t given that the number of type 1 arrivals $N_{1t} = n$.

6. Prove that the cumulative value S_t of the compound Poisson process has:

   (a) Stationary and independent increments.

   (b) Mean and variance as given in (1.6.5).