Real Univariate Function

1.1 First Proof of Taylor's Theorem

Theorem 1.1.1 (Fermat). Suppose $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b). If f attains an extremum at $c \in (a, b)$, then $\dot{f}(c) = 0$. $\square$

Proof. Suppose $f(c)$ is a local maximum; the case of a minimum is analogous. Let c_n^+ denote a sequence of real numbers in (a, b) approaching c from above, such that $c_n^+ - c > 0$ for $\forall n \in \mathbb{N}$. Since $f(c)$ is the local maximum, $f(c_n^+) - f(c) \leq 0$ for $\forall n \in \mathbb{N}$, thus:

$$\dot{f}(c) = \lim_{n \to \infty} \frac{f(c_n^+) - f(c)}{c_n^+ - c} \leq 0.$$

Let c_n^- denote a sequence of real numbers in (a, b) approach c from below, such that $c_n^- - c < 0$ for $\forall n \in \mathbb{N}$. Since $f(c_n^-) - f(c) \leq 0$ for $\forall n \in \mathbb{N}$:

$$\dot{f}(c) = \lim_{n \to \infty} \frac{f(c_n^-) - f(c)}{c_n^- - c} \geq 0.$$

Conclude that $\dot{f}(c) = 0$. $\blacksquare$

Theorem 1.1.2 (Rolle). Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ such that $\dot{f}(c) = 0$. $\square$

Proof. Since f is continuous over the compact set $[a, b]$, f attains a minimum and a maximum value. If the minimum and maximum occur at the endpoints, then f is constant on $[a, b]$, and $\dot{f}(c) = 0$ for $\forall c \in (a, b)$. Suppose not, then either the minimum or maximum (or both) occur on the interior of $[a, b]$. Let c denote the point in (a, b) at which the extremum is attained. By Fermat's theorem, $\dot{f}(c) = 0$. $\blacksquare$

Theorem 1.1.3 (Differential Mean Value). Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ where:

$$\dot{f}(c) = \frac{f(b) - f(a)}{b - a}.$$

$\square$

Proof. Define:

$$g(x) = \left\{ \frac{f(b) - f(a)}{b - a} \right\} (x - a) + f(a),$$

and let:

$$h(x) \equiv f(x) - g(x).$$

Observe that $g(a) = f(b)$ and $g(b) = f(b)$ such that $h(a) = h(b) = 0$. Now by Rolle's theorem there exists $c \in (a, b)$ such that $\dot{h}(c) = 0$. That is:

$$0 = \dot{h}(c) = \dot{f}(c) - \left\{ \frac{f(b) - f(a)}{b - a} \right\}.$$

■

Lemma 1.1.1 (Extension to Rolle's). Suppose f is $(n-1)$ times continuously differentiable on (a, b), and that $f^{(n)}$ exists, with:

$$f(a) = \dot{f}(a) = \cdots = f^{(n-1)}(a) \text{ and } f(b) = 0.$$

Then, there exists $c \in (a, b)$ such that $f^{(n)}(c) = 0$. ■

Proof. Since $f(a) = f(b) = 0$, by Rolle's theorem, there exists $c_1 \in (a, b)$ such that $\dot{f}(c_1) = 0$. Now since $\dot{f}(a) = \dot{f}(c_1) = 0$, by Rolle's theorem there exists $c_2 \in (a, c_1)$ such that $\ddot{f}(c_2) = 0$. Continuing in like manner provides a finite sequence of critical points $a < c_n < c_{n-1} < \cdots < b$ such that:

$$f^{(k)}(c_k) = 0, \text{ for } k \in \{1, \cdots, n\}$$

In particular, at $c = c_n$, $f^{(n)}(c_n) = 0$ as desired. ■

Theorem 1.1.4 (Taylor's). Suppose $f : [a, b] \to \mathbb{R}$ is $(n-1)$ times continuously differentiable on (a, b), and that $f^{(n)}$ exists. There exists $c \in (a, b)$ such that:

$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n)}(c)}{n!}(b-a)^n. \tag{1.1.1}$$

□

Proof. Define $g(x) = \sum_{k=0}^{n} \alpha_k (x-a)^k$, and let:

$$h(x) \equiv f(x) - g(x).$$

Now, the kth derivative of h, evaluated at a, is:

$$h^{(k)}(a) = f^{(k)}(a) - \alpha_k k!.$$

Let $\alpha_k = f^{(k)}(a)/k!$, then:

$$h^{(k)}(a) = 0 \text{ for } k \in \{0, \cdots, n-1\}.$$

Suppose α_n is determined by the requirement that:

$$h(b) = f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k - \alpha_n (b-a)^n = 0.$$

Specifically, let:

$$\alpha_n = \frac{1}{(b-a)^n}\left\{f(b) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(b-a)^k\right\}.$$

By construction $h(a) = \dot{h}(a) = \cdots = h^{(n-1)}(a) = 0$ and $h(b) = 0$. From the extension to Rolle's theorem, there exists $c \in (a,b)$ such that $h^{(n)}(c) = 0$. That is:

$$0 = h^{(n)}(c) = f^{(n)}(c) - \alpha_n n! = f^{(n)}(c) - \frac{n!}{(b-a)^n}\left\{f(b) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(b-a)^k\right\}.$$

Upon rearranging:

$$f(b) = \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n)}(c)}{n!}(b-a)^n.$$

$\blacksquare$

1.2 Second Proof of Taylor's Theorem

Theorem 1.1.5 (Fundamental Theorem of Calculus). Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. Define:

$$g(x) = \int_a^x f(t)dt.$$

Then g is continuous on $[a,b]$ and differentiable on (a,b) with $\dot{g} = f$. $\square$

Proof. Observe that:

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt.$$

Since f is uniformly continuous on $[a,b]$, for $\forall(x \in [a,b], \epsilon > 0)$ there $\exists(\delta > 0)$ such that $|t - x| < \delta$ implies $|f(t) - f(x)| < \delta$. Take $0 < h < \delta$, then for $t \in [x, x+h)$:

$$f(x) - \epsilon \leq f(t) \leq f(x) + \epsilon.$$

Integrating over $[x, x+h)$:

$$\int_x^{x+h}\big\{f(x) - \epsilon\big\}dt \leq \int_x^{x+h} f(t)dt \leq \int_x^{x+h}\big\{f(x) + \epsilon\big\}dt,$$
$$h\big\{f(x) - \epsilon\big\} \leq \int_x^{x+h} f(t)dt \leq h\big\{f(x) + \epsilon\big\}.$$

Upon rearranging:

$$-\epsilon \leq \frac{1}{h}\int_x^{x+h} f(t)dt - f(x) \leq \epsilon.$$

Therefore:

$$\left|\frac{g(x+h)-g(x)}{h} - f(x)\right| \leq \epsilon \implies \lim_{h\downarrow 0}\frac{g(x+h)-g(x)}{h} = f(x).$$

Analogous reasoning shows that:

$$\lim_{h\uparrow 0}\frac{g(x+h)-g(x)}{h} = f(x).$$

Conclude that:

$$\dot{g}(x) = \lim_{h\to\infty}\frac{g(x+h)-g(x)}{h} = f(x).$$

$\blacksquare$

Corollary 1.1.1. If $\dot{f} : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ then:

$$f(b) = f(a) + \int_a^b \dot{f}(t)dt. \tag{1.1.2}$$

$\clubsuit$

Proof. Define $g(x) = \int_a^x \dot{f}(t)dt$ and let $h(x) = f(x) - g(x)$. Since f and g are each continuous on $[a,b]$, so too is h. By the FTC, $\dot{h}(x) = \dot{f}(x) - \dot{g}(x) = \dot{f}(x) - \dot{f}(x) = 0$ for $\forall x \in (a,b)$. Thus h is continuous on $[a,b]$ and constant on (a,b). Conclude that $h(b) = h(a)$. Consequently,

$$f(b) - \int_a^b \dot{f}(t)dt = h(b) = h(a) = f(a).$$

$\blacksquare$

Theorem 1.1.6 (Integral Mean Value). Suppose $f[a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then there $\exists c \in (a,b)$ such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(t)dt.$$

$\square$

Proof. Let $g(x) = \int_a^x f(t)dt$. By the FTC, g is continuous on $[a,b]$ and differentiable on (a,b). By the differential MVT, there exists $c \in (a,b)$ such that:

$$f(c) = \dot{g}(c) = \frac{g(b)-g(a)}{b-a} = \frac{1}{b-a}\int_a^b f(t)dt.$$

$\blacksquare$

Theorem 1.1.7 **(Taylor's).** Suppose $f : [a, b] \to \mathbb{R}$ is $(n-1)$ times continuously differentiable on (a, b) and that $f^{(n)}$ exists, then:

$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{(-1)^{n-1}}{(n-1)!} \int_a^b (t-b)^{n-1} f^{(n)}(t) dt. \tag{1.1.3}$$

□

Proof. By (1.1.2):

$$f(b) = f(a) + \int_a^b \dot{f}(t) dt.$$

Integrating by parts:

$$\begin{aligned} f(b) &= f(a) + \Big[t\dot{f}(t)\Big]_{t=a}^{t=b} - \int_a^b t\ddot{f}(t) dt \\ &= f(a) + b\dot{f}(b) - a\dot{f}(a) - \int_a^b t\ddot{f}(t) dt. \end{aligned} \tag{1.1.4}$$

Applying the FTC to $\dot{f}(x)$:

$$\dot{f}(b) = \dot{f}(a) + \int_a^b \ddot{f}(t) dt. \tag{1.1.5}$$

Substituting (1.1.5) into (1.1.4):

$$\begin{aligned} f(b) &= f(a) + b\left(\dot{f}(a) + \int_a^b \ddot{f}(t) dt\right) - a\dot{f}(a) - \int_a^b t\ddot{f}(t) dt \\ &= f(a) + (b-a)\dot{f}(a) + b\int_a^b \ddot{f}(t) dt - \int_a^b t\ddot{f}(t) dt \\ &= f(a) + (b-a)\dot{f}(a) - \int_a^b (t-b)\ddot{f}(t) dt. \end{aligned}$$

Integrating again by parts:

$$\begin{aligned} f(b) &= f(a) + (b-a)\dot{f}(a) - \frac{1}{2}\Big[(t-b)^2\ddot{f}(t)\Big]_{t=a}^{t=b} + \frac{1}{2}\int_a^b (t-b)^2 f^{(3)}(t) dt \\ &= f(a) + (b-a)\dot{f}(a) + \frac{1}{2}(b-a)^2\ddot{f}(a) + \frac{1}{2}\int_a^b (t-b)^2 f^{(3)}(t). \end{aligned}$$

Proceeding in like manner:

$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{(-1)^{n-1}}{(n-1)!} \int_a^b (t-b)^{n-1} f^{(n)}(t) dt.$$

■

Corollary 1.1.2 (Cauchy Remainder). Taylor's expansion is expressible as:

$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n)}(c)}{(n-1)!}(b-c)^{n-1}(b-a),$$

where $c \in (a, b)$. ♣

Proof. Consider the integral form of the Taylor remainder:

$$R_n = \frac{(-1)^{n-1}}{(n-1)!} \int_a^b (t-b)^{n-1} f^{(n)}(t) t dt.$$

By applying the integral MVT to the integral, there exists $c \in (a, b)$ such that:

$$R_n = \frac{(-1)^{n-1}}{(n-1)!} \cdot (c-b)^{n-1} f^{(n)}(c) \cdot (b-a) = \frac{f^{(n)}(c)}{(n-1)!}(b-c)^{n-1}(b-a).$$

■

Real Multivariate Function

Theorem 1.2.1. Suppose $f : \mathscr{U}(\boldsymbol{a}) \to \mathbb{R}$, where $\mathscr{U}(\boldsymbol{a}) \subset \mathbb{R}^J$ is a compact neighborhood of $\boldsymbol{a}$ on which all nth order partial derivatives of f exist and are continuous. Introduce the *multi-index* β with these properties:

$$|\beta| = \sum_{j=1}^{J} \beta_j, \qquad \beta! = \prod_{j=1}^{J} \beta_j!, \qquad \boldsymbol{x}^\beta = \prod_{j=1}^{J} x_j^{\beta_j}, \qquad D^\beta f = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \cdots \partial x_J^{\beta_J}}.$$

Now, there exists $g_\beta : \mathscr{U}(\boldsymbol{a}) \to \mathbb{R}$ such that:

$$f(\boldsymbol{x}) = \sum_{|\beta| \le n} \frac{D^\beta f(\boldsymbol{a})}{\beta!}(\boldsymbol{x} - \boldsymbol{a})^\beta + \sum_{|\beta| = n} g_\beta(\boldsymbol{x})(\boldsymbol{x} - \boldsymbol{a})^\beta.$$

and $\lim_{\boldsymbol{x} \to \boldsymbol{a}} g_\beta(\boldsymbol{x}) = 0$.

If in addition the $(n+1)$st order partials exist and are continuous, then:

$$f(\boldsymbol{x}) = \sum_{|\beta| \le n} \frac{D^\beta f(\boldsymbol{a})}{\beta!}(\boldsymbol{x} - \boldsymbol{a})^\beta + \sum_{|\beta| = n+1} \frac{D^\beta f(\boldsymbol{c})}{\beta!}(\boldsymbol{x} - \boldsymbol{a})^\beta,$$

where $\boldsymbol{c} = \boldsymbol{a} + \lambda(\boldsymbol{x} - \boldsymbol{a})$ for some $\lambda \in (0, 1)$.

□

Example 1.2.1. Suppose $f : \mathbb{R}^J \to \mathbb{R}$ satisfies the conditions the multivariate Taylor expansion. The 2nd order Taylor expansion is expressible as:

$$f(\boldsymbol{x}) = f(\boldsymbol{a}) + (\boldsymbol{x} - \boldsymbol{a})' \frac{\partial f}{\partial \boldsymbol{x}}(\boldsymbol{a}) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{a})' \frac{\partial^2 f}{\partial \boldsymbol{x} \partial \boldsymbol{x}'}(\boldsymbol{x} - \boldsymbol{a}) + R_3,$$

where:

$$R_3 = \frac{1}{3!} \sum_{|\beta|=3} \frac{D^\beta f(\boldsymbol{c})}{\beta!}(\boldsymbol{x} - \boldsymbol{a})^\beta.$$

In the context of asymptotic expansions, if the 3rd order partials of f remain bounded for $n \to \infty$, then the 2nd order Taylor expansion is often expressed as:

$$f(\boldsymbol{x}) = f(\boldsymbol{a}) + (\boldsymbol{x} - \boldsymbol{a})' \frac{\partial f}{\partial \boldsymbol{x}}(\boldsymbol{a}) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{a})' \frac{\partial^2 f}{\partial \boldsymbol{x} \partial \boldsymbol{x}'}(\boldsymbol{x} - \boldsymbol{a}) + \mathcal{O}\big(||\boldsymbol{x} - \boldsymbol{a}||_2^3\big).$$

♠