Ghosh Lin Estimation

1.1 Setup

Let D denote the time to a terminal event (e.g. death), C an independent censoring time, and $N^*(t)$ the number of recurrent events by time t. While censoring is assumed independent of D and $N^*(t)$, no restrictions are placed on the dependence of recurrent or terminal events. Due to censoring, an observation takes the form $\{N(\cdot), U, \delta\}$, where:

$$N(t) = N^*\big(t \wedge C\big), \qquad U = \min(C, D), \qquad \delta = \mathbb{I}\big(D \leq C\big).$$

The observed data $\mathcal{D} = \{N_i(\cdot), U_i, \delta_i\}$ are IID replicates of $\{N(\cdot), U, \delta\}$.

1.2 Mean Cumulative Function

Definition 1.2.1. The **mean cumulative function** (MCF) is defined as:

$$\mu(t) = \mathbb{E}\big\{N^*(t)\big\}.$$

■

Proposition 1.2.1. The MCF is expressible as:

$$\mu(t) = \int_0^t S(u) dR(u),$$

where $S(u) = \mathbb{P}(D > u)$ is the survival function, and $dR(u) = \mathbb{E}\{dN^*(t)|D \geq u\}$. ♦

Proof.

$$\begin{aligned}\mu(t) &= \int_0^t \mathbb{E}\{dN^*(u)\} = \int_0^t \mathbb{P}\{dN^*(u) = 1\} = \int_0^t \mathbb{P}\{dN^*(u) = 1, D \geq u\} \\ &= \int_0^t \mathbb{P}(D \geq u)\mathbb{P}\{dN^*(u) = 1|D \geq u\} = \int_0^t S(u)dR(u).\end{aligned}$$

■

Example 1.2.1. Suppose the gap times for the recurrent event process $N^*(t)$ are independent and exponentially distributed with arrival rate λ_A. In the absence of terminal events, $N^*(t)$ is a Poisson process with MCF $\mu(\tau) = \mathbb{E}\{N^*(\tau)\} = \lambda_A\tau$. Suppose death occurs independently according to an exponential distribution with arrival rate λ_D, then:

$$\mu(\tau) = \int_0^\tau e^{-\lambda_D t}\lambda_A dt = \frac{\lambda_A}{\lambda_D}\left\{1 - e^{-\lambda_D \tau}\right\}.$$

♠

Definition 1.2.2. The **Ghosh Lin** estimator [1] of the MCF is:

$$\hat{\mu}(t) = \int_0^t \hat{S}(u) d\hat{R}(u),$$

where $\hat{S}(u)$ is the Kaplan-Meier estimator for $\mathbb{P}(D > u)$ and $\hat{R}(u)$ is the Nelson-Aalen estimator of cumulative recurrence:

$$\hat{R}(t) = \int_0^t \frac{dN(u)}{Y(u)} = \sum_{i=1}^{n} \int_0^t \frac{dN_i(u)}{Y(u)}.$$

■

Discussion 1.2.1.

- In the recurrent events framework, a subject remains under observation after the occurrence of the event. The subject is only lost to observation due to censoring or a terminal event.
- In the absence of terminal events, $\hat{S}(u) \equiv 1$ and the Ghosh-Lin estimator reduces to the Nelson-Aalen estimator $\hat{R}(t)$.
- Regarding terminal events as censoring results in systematic overestimation of the MCF for t greater than or equal to the first terminal event time.
- If the event of interest can occur at most once per patient, then $\hat{\mu}(t)$ is the standard estimator of the cumulative incidence curve with death acting as a competing risk.

♠

Example 1.2.2 (Estimation). The Ghosh Lin curve may be tabulated as follows. Let $\tau_1 < \cdots < \tau_K$ denote the distinct observed event, censoring, or death times. Let $e_k = dN(\tau_k)$ denote the number of events occurring at τ_k, $d_k = dN^D(\tau_k)$ the number at deaths, and $n_k = Y(\tau_k)$ the number at risk. The instantaneous event rate at time τ_k is $\hat{r}_k = e_k/n_k$, and the instantaneous hazard is $\hat{h}_k = d_k/n_k$. The Kaplan-Meier estimate of the survival at time τ_k is the cumulative product:

$$\hat{S}_k = \hat{S}(\tau_k) = \prod_{j \leq k} \left(1 - \hat{h}_j\right).$$

Finally, the Ghosh-Lin estimate MCF at time τ_k is:

$$\hat{\mu}_k = \hat{\mu}(\tau_k) = \sum_{j \leq k} \hat{S}_j \cdot \hat{r}_j.$$

Consider the following table in which an event occurs at τ_1, followed by a death at τ_2, then a second event at τ_3, a censoring at τ_4, and a final event at τ_5. Note that the number at risk decreases following a terminal event (or censoring), but not after an event. For clarity, the number of censorings c_k is also tracked.

τ_k	c_k	e_k	d_k	n_k	$\hat{r}_k$	$\hat{h}_k$	$\hat{S}_k$	$\hat{\mu}_k$
τ_1	0	1	0	10	1/10	0	1	1/10
τ_2	0	0	1	10	0	1/10	9/10	1/10
τ_3	0	1	0	9	1/9	0	9/10	2/10
τ_4	1	0	0	9	0	0	9/10	2/10
τ_5	0	1	0	8	1/8	0	9/10	5/16
$\vdots$								$\vdots$

♠

1.3 1-Sample Asymptotics

Proposition 1.3.2 (Influence Function Expansion).

$$\sqrt{n}\{\hat{\mu}(t) - \mu(t)\} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi_i(t) + o_p(1), \tag{1.3.1}$$

where:

$$\psi_i(t) = \int_0^t \frac{\mu(u)}{y(u)}dM_i^D(u) - \mu(t)\int_0^t \frac{1}{y(u)}dM_i^D(u) + \int_0^t \frac{S(u)}{y(u)}dM_i(u), \tag{1.3.2}$$

is the influence function, where $y(u) = \mathbb{P}(U \geq u)$ is the probability limit of $n^{-1}Y(u)$,

$$M_i(t) = N_i(t) - \int_0^t Y_i(u)dR(u),$$

is the recurrent events martingale and:

$$M_i^D(t) = N_i^D(t) - \int_0^t Y_i(u)dA^D(u)$$

is the terminal event martingale.

♦

Proof. The normalized difference is expressible as:

$$\sqrt{n}\{\hat{\mu}(t) - \mu(t)\} = \sqrt{n}\int_0^t \hat{S}(u)d\hat{R}(u) - \sqrt{n}\int_0^t S(u)dR(u).$$

Adding and subtracting $\sqrt{n}\int_0^t \hat{S}(u)dR(u)$ gives:

$$\sqrt{n}\big\{\hat{\mu}(t) - \mu(t)\big\} = I_1(t) + I_2(t),$$

where:

$$\begin{aligned} I_1(t) &= \sqrt{n}\int_0^t \big\{\hat{S}(u) - S(u)\big\}dR(u) \\ I_2(t) &= \sqrt{n}\int_0^t \hat{S}(u)d\big\{\hat{R}(u) - R(u)\big\}. \end{aligned}$$

From:

$$\sqrt{n}\frac{\big\{\hat{S}(u) - S(u)\big\}}{-S(u)} = \sqrt{n}\big\{\hat{A}^D(u) - A^D(u)\big\} + o_p(1),$$

the first term is asymptotically equivalent to:

$$\begin{aligned} I_1(t) &= -\sqrt{n}\int_0^t \frac{\big\{\hat{S}(u) - S(u)\big\}}{-S(u)} \cdot S(u)dR(u) \\ &= -\int_0^t \sqrt{n}\big\{\hat{A}^D(u) - A^D(u)\big\} \cdot d\mu(u) + o_p(1), \end{aligned}$$

where $A^D(u)$ is the cumulative hazard for terminal events, and $\hat{A}^D(u)$ is the corresponding Nelson-Aalen estimator.

Integrating by parts:

$$\begin{aligned} -&\int_0^t \sqrt{n}\big\{\hat{A}^D(u) - A^D(u)\big\} \cdot d\mu(u) \\ &= -\sqrt{n}\Big[\{\hat{A}^D(u) - A^D(u)\}\mu(u)\Big]_{u=0}^{u=t} + \sqrt{n}\int_0^t \mu(u) \cdot d\big\{\hat{A}^D(u) - A^D(u)\big\} \\ &= -\sqrt{n}\{\hat{A}^D(t) - A^D(t)\}\mu(t) + \sqrt{n}\int_0^t \mu(u) \cdot d\big\{\hat{A}^D(u) - A^D(u)\big\}. \end{aligned}$$

Using the martingale expansion of the Nelson-Aalen estimator:

$$\sqrt{n}\{\hat{A}^D(t) - A^D(t)\} = \frac{1}{\sqrt{n}}\int_0^t \frac{dM^D(u)}{y(u)} + o_p(1),$$

$I_1(t)$ becomes:

$$I_1(t) = -\frac{\mu(t)}{\sqrt{n}}\int_0^t \frac{dM^D(u)}{y(u)} + \frac{1}{\sqrt{n}}\int_0^t \frac{\mu(u)}{y(u)}dM^D(u) + o_p(1).$$

Now consider $I_2(t)$. By uniform consistency of the Kaplan-Meier estimator:

$$I_2(t) = \sqrt{n}\int_0^t S(u)d\big\{\hat{R}(u) - R(u)\big\} + o_p(1).$$

From Ghosh Lin (A.8):

$$\sqrt{n}\big\{\hat{R}(t) - R(t)\big\} = \frac{1}{\sqrt{n}}\int_0^t \frac{dM(u)}{y(u)} + o_p(1),$$

where:

$$M(t) = N(t) - \int_0^t Y(u)dR(u).$$

Consequently:

$$\sqrt{n}\int_0^t S(u)d\big\{\hat{R}(u) - R(u)\big\} = \frac{1}{\sqrt{n}}\int_0^t \frac{S(u)}{y(u)}dM(u) + o_p(1),$$

and overall:

$$\sqrt{n}\big\{\hat{\mu}(t) - \mu(t)\big\} = \int_0^t \frac{\mu(u)}{y(u)}dM^D(u) - \mu(t)\int_0^t \frac{1}{y(u)}dM^D(u) + \int_0^t \frac{S(u)}{y(u)}dM(u) + o_p(1).$$

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Corollary 1.3.1. $\sqrt{n}\big\{\hat{\mu}(t) - \mu(t)\big\}$ converges weakly to a mean-zero Guassian process with covariance function:

$$\gamma(s,t) = \mathbb{E}\big\{\psi(s)\cdot\psi(t)\big\},$$

which is estimable by:

$$\hat{\gamma}(s,t) = \frac{1}{n}\sum_{i=1}^{n}\psi_i(s)\cdot\psi_i(t).$$

In particular:

$$\sqrt{n}\big\{\hat{\mu}(t) - \mu(t)\big\} \rightsquigarrow W\big\{\sigma^2_{\text{MCF}}(t)\big\},$$

with:

$$\sigma^2_{\text{MCF}}(t) = \mathbb{E}\big\{\psi^2(t)\big\},$$

♣

Discussion 1.3.1. Since $\mu(t)$ is non-negative, point-wise confidence intervals may be constructed via:

$$\sqrt{n}\big\{\ln\hat{\mu}(t) - \ln\mu(t)\big\} = \sqrt{n}\frac{\big\{\hat{\mu}(t) - \mu(t)\big\}}{\mu(t)} + o_p(1).$$

In particular, an asymptotic $(1-\alpha)$ level confidence interval is given by:

$$\hat{\mu}(t)\exp\left\{\pm\frac{z_{1-\alpha/2}\hat{\sigma}^2_{\text{MCF}}(t)}{\sqrt{n}\hat{\mu}(t)}\right\}.$$

♠

1.4 2-Sample Asymptotics

Definition 1.4.1. The two sample log-rank statistic for equality of two MCFs, $H_0 : \mu_1(t) = \mu_0(t)$ for all $t \in [0, \tau]$ is:

$$T_{\mathrm{LR}}(\tau) = \int_0^\tau \omega(t) d\big\{\hat{\mu}_1(t) - \hat{\mu}_0(t)\big\},$$

where:

$$\omega(t) = \frac{n}{n_1 n_0} \frac{Y_1(t) Y_0(t)}{Y_1(t) + Y_0(t)}.$$

■

Proposition 1.4.3. Under $H_0 : \mu_1(t) = \mu_0(t)$:

$$\left(\frac{n_1 n_0}{n}\right)^{1/2} T_{\mathrm{LR}} \xrightarrow{d} N\big(0, \sigma^2_{\mathrm{LR}}\big),$$

where:

$$\hat{\sigma}^2_{\mathrm{LR}} = \frac{n_0}{n n_1} \sum_{i=1}^{n_1} \left\{ \int_0^\tau \omega(t) d\psi_{1i}(t) \right\}^2 + \frac{n_1}{n n_0} \sum_{i=1}^{n_0} \left\{ \int_0^\tau \omega(t) d\psi_{0i}(t) \right\}^2$$

and $\psi_{ij}(t)$ is the influence function for subject i in arm j from (1.3.2). ♦

Proof. Consider the log-rank statistic:

$$\begin{aligned} \left(\frac{n_1 n_0}{n}\right)^{1/2} \hat{T}_{\mathrm{LR}} = & \left(\frac{n_1 n_0}{n}\right)^{1/2} \int_0^\tau \omega(t) d\big\{\hat{\mu}_1(t) - \mu_1(t)\big\} \\ & - \left(\frac{n_1 n_0}{n}\right)^{1/2} \int_0^\tau \omega(t) d\big\{\hat{\mu}_0(t) - \mu_0(t)\big\} \end{aligned}$$

From (1.3.1):

$$\sqrt{n_j}\big\{\hat{\mu}_j(t) - \mu_j(t)\big\} = \frac{1}{\sqrt{n_j}} \sum_{i=1}^{n_j} \psi_{ij}(t) + o_p(1).$$

Applied to the log-rank statistic:

$$\begin{aligned} \left(\frac{n_1 n_0}{n}\right)^{1/2} \hat{T}_{\mathrm{LR}} = & \left(\frac{n_0}{n}\right)^{1/2} \frac{1}{\sqrt{n_1}} \sum_{i=1}^{n_1} \int_0^\tau \omega(t) d\psi_{i1}(t) \\ & - \left(\frac{n_1}{n}\right)^{1/2} \frac{1}{\sqrt{n_0}} \sum_{i=1}^{n_0} \int_0^\tau \omega(t) d\psi_{i0}(t) + o_p(1). \end{aligned}$$

Suppose $n_1, n_0 \to \infty$ such that $n_0/n \to \pi_0$ and $n_1/n \to \pi_1$, the normalized log-rank statistic converges in distribution to a zero-mean random variable with variance:

$$\sigma^2_{\mathrm{LR}} = \pi_0 \cdot \mathbb{E}\left\{ \int_0^\tau \omega(t) d\psi_1(t) \right\}^2 + \pi_1 \cdot \mathbb{E}\left\{ \int_0^\tau \omega(t) d\psi_0(t) \right\}^2$$

■

Area Under the Mean Cumulative Function

2.1 Asymptotics

Definition 2.1.1. Define the area under the MCF as:

$$U(\tau) = \int_0^\tau \mu(t)dt = \int_0^\tau \left\{ \int_0^t S(u)dR(u) \right\} dt.$$

By exchanging the order of integration:

$$U(\tau) = \int_0^\tau \int_0^\tau \mathbb{I}(u \leq t)S(u)dR(u)dt = \int_0^\tau (\tau - u)S(u)dR(u).$$

■

Proposition 2.1.1 (Influence Function Expansion).

$$\sqrt{n}\big\{\hat{U}(\tau) - U(\tau)\big\} = \frac{1}{\sqrt{n}}\sum_{i=1}^n \psi_i(\tau) + o_p(1),$$

where:

$$\psi_i(\tau) = \int_0^\tau \frac{(\tau - u)S(u)}{y(u)}dM_i(u) - \int_0^\tau \frac{\int_u^\tau (\tau - s)d\mu(s)}{y(u)}dM_i^D(u),$$

is the influence function, $y(u) = \mathbb{P}(U \geq u)$ is the probability of being at risk,

$$M_i(t) = N_i(t) - \int_0^t Y_i(u)dR(u),$$

is the recurrent events martingale and:

$$M_i^D(t) = N_i^D(t) - \int_0^t Y_i(u)dA^D(u)$$

is the terminal event martingale.

♦

Proof. Consider the distribution of:

$$\sqrt{n}\big\{\hat{U}(\tau) - U(\tau)\big\} = \sqrt{n}\int_0^\tau (\tau - u)\hat{S}(u)d\hat{R}(u) - \sqrt{n}\int_0^\tau (\tau - u)S(u)dR(u).$$

Adding and subtracting $\sqrt{n}\int_0^\tau (\tau - u)\hat{S}(u)dR(u)$ gives:

$$\sqrt{n}\big\{\hat{U}(\tau) - U(\tau)\big\} = I_1(\tau) + I_2(\tau),$$

where:

$$I_1(\tau) = \sqrt{n}\int_0^\tau (\tau - u)\hat{S}(u)d\big\{\hat{R}(u) - R(u)\big\},$$
$$I_2(\tau) = \sqrt{n}\int_0^\tau (\tau - u)\big\{\hat{S}(u) - S(u)\big\}dR(u).$$

From Ghosh and Lin (A.8):

$$\sqrt{n}\big\{\hat{R}(t) - R(t)\big\} = \frac{1}{\sqrt{n}}\int_0^t \frac{dM(u)}{y(u)} + o_p(1),$$

the first integral is expressible as:

$$\begin{aligned}
I_1(\tau) &= \int_0^\tau (\tau - u)\hat{S}(u)\cdot\sqrt{n}d\big\{\hat{R}(u) - R(u)\big\} \\
&= \int_0^\tau (\tau - u)\hat{S}(u)\cdot\frac{1}{\sqrt{n}}\frac{dM(u)}{y(u)} + o_p(1) \\
&= \frac{1}{\sqrt{n}}\int_0^\tau \frac{(\tau - u)S(u)}{y(u)}dM(u) + o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^n\left\{\int_0^\tau \frac{(\tau - u)S(u)}{y(u)}dM_i(u)\right\} + o_p(1).
\end{aligned}$$

From:

$$\sqrt{n}\frac{\big\{\hat{S}(u) - S(u)\big\}}{-S(u)} = \sqrt{n}\big\{\hat{A}^D(u) - A^D(u)\big\} + o_p(1),$$

the second integral is expressible as:

$$\begin{aligned}
I_2(\tau) &= -\sqrt{n}\int_0^\tau (\tau - u)\frac{\big\{\hat{S}(u) - S(u)\big\}}{-S(u)}\cdot S(u)dR(u) \\
&= -\sqrt{n}\int_0^\tau (\tau - u)\big\{\hat{A}^D(u) - A^D(u)\big\}\cdot d\mu(u) + o_p(1).
\end{aligned}$$

Integrating by parts:

$$\begin{aligned}
&-\sqrt{n}\int_0^\tau (\tau - u)\big\{\hat{A}^D(u) - A^D(u)\big\}d\mu(u) \\
&\quad= -\Big[\sqrt{n}(\tau - u)\big\{\hat{A}^D(u) - A^D(u)\big\}\mu(u)\Big]_{u=0}^{u=\tau} + \sqrt{n}\int_0^\tau \mu(u)d\big\{(\tau - u)\big(\hat{A}^D(u) - A^D(u)\big)\big\}.
\end{aligned}$$

The first term vanishes, leaving:

$$I_2(\tau) = \sqrt{n}\int_0^\tau \mu(u)d\big\{(\tau - u)\big(\hat{A}^D(u) - A^D(u)\big)\big\}.$$

Expanding the differential:

$$I_2(\tau) = -\sqrt{n}\int_0^\tau \mu(u)\big\{\hat{A}^D(u) - A^D(u)\big\}du + \sqrt{n}\int_0^\tau \mu(u)(\tau - u)d\big\{\hat{A}^D(u) - A^D(u)\big\}.$$

Using the martingale expansion for the cumulative hazard:

$$\sqrt{n}\{\hat{A}^D(t) - A^D(t)\} = \frac{1}{\sqrt{n}}\int_0^t \frac{dM^D(u)}{y(u)} + o_p(1),$$

the second integral takes the form:

$$\begin{aligned}
I_2(\tau) &= -\int_0^\tau \mu(u)\cdot\sqrt{n}\{\hat{A}^D(u) - A^D(u)\}\cdot du + \int_0^\tau \mu(u)(\tau-u)\cdot\sqrt{n}d\{\hat{A}^D(u) - A^D(u)\} \\
&= -\frac{1}{\sqrt{n}}\int_0^\tau \mu(u)\left\{\int_0^u \frac{dM^D(s)}{y(s)}\right\}du + \frac{1}{\sqrt{n}}\int_0^\tau \frac{\mu(u)(\tau-u)}{y(u)}dM^D(u) + o_p(1).
\end{aligned}$$

These terms may be combined as follows:

$$\begin{aligned}
I_2(\tau) - o_p(1) &= -\frac{1}{\sqrt{n}}\int_0^\tau\int_0^\tau \mathbb{I}(s\le u)\frac{\mu(u)}{y(s)}dM^D(s)du + \frac{1}{\sqrt{n}}\int_0^\tau \frac{\mu(u)(\tau-u)}{y(u)}dM^D(u) \\
&= -\frac{1}{\sqrt{n}}\int_0^\tau\int_u^\tau \mu(s)ds\cdot\frac{dM^D(u)}{y(u)} + \frac{1}{\sqrt{n}}\int_0^\tau \mu(u)(\tau-u)\cdot\frac{dM^D(u)}{y(u)} \\
&= \frac{1}{\sqrt{n}}\int_0^\tau\left\{\mu(u)(\tau-u) - \int_u^\tau \mu(s)ds\right\}\frac{dM^D(u)}{y(u)}.
\end{aligned}$$

The term in braces {} simplifies by reversing integration by parts. In particular:

$$\begin{aligned}
\int_u^\tau (\tau-s)d\mu(s) &= \{(\tau-s)\mu(s)\}_{s=u}^{s=\tau} - \int_u^\tau \mu(s)d(\tau-s) \\
&= -(\tau-u)\mu(u) + \int_u^\tau \mu(s)ds,
\end{aligned}$$

therefore:

$$\mu(u)(\tau-u) - \int_u^\tau \mu(s)ds = -\int_u^\tau (\tau-s)d\mu(s).$$

The second integral is now expressible as:

$$\begin{aligned}
I_2(\tau) &= -\frac{1}{\sqrt{n}}\int_0^\tau\left\{\int_u^\tau(\tau-s)d\mu(s)\right\}\frac{dM^D(u)}{y(u)} \\
&= -\frac{1}{\sqrt{n}}\sum_{i=1}^n\int_0^\tau \frac{\int_u^\tau(\tau-s)d\mu(s)}{y(u)}dM_i^D(u) + o_p(1).
\end{aligned}$$

■

Example 2.1.1. For implementation, let $\tau_1 < \cdots < \tau_K$ denote the distinct observed event, censoring, or death times. For subject i, the martingale increments are approximated as:

$$\begin{aligned}
d\hat{M}_{ik} &= d\hat{M}_i(\tau_k) = dN_i(\tau_k) - Y_i(\tau_k)\cdot\hat{r}_k, \\
d\hat{M}_{ik}^D &= d\hat{M}_i^D(\tau_k) = dN_i(\tau_k) - Y_i(\tau_k)\cdot\hat{h}_k
\end{aligned}$$

where $\hat{r}_k$ and $\hat{h}_k$ are the sample event rate and hazard (of death) at time τ_k.

For a truncation time $\tau \geq \tau_K$, the first term of the influence function is:

$$\int_0^\tau \frac{(\tau - u)S(u)}{y(u)} dM_i(u) = \sum_{k=1}^{K} \frac{(\tau - \tau_k)\hat{S}(\tau_k)}{Y(\tau_k)/n} d\hat{M}_{ik}.$$

At time τ_k, the numerator in the second term of the influence function is:

$$\begin{aligned}
\nu_{ik} &= \int_{\tau_k}^{\tau} (\tau - u) d\mu(u) \\
&= \int_{\tau_k}^{\tau} (\tau - u) S(u) dR(u) \\
&= \int_0^{\tau} (\tau - u) S(u) dR(u) - \int_0^{\tau_k} (\tau - u) S(u) dR(u) \\
&= \sum_{j=1}^{K} (\tau - \tau_j)\hat{S}(\tau_j)\hat{r}_j - \sum_{j \leq k} (\tau - \tau_j)\hat{S}(\tau_j)\hat{r}_j.
\end{aligned}$$

Now, the second term of the influence function is:

$$\int_0^\tau \frac{\int_u^\tau (\tau - s) d\mu(s)}{y(u)} dM_i^D(u) = \sum_{k=1}^{K} \frac{\nu_{ik}}{Y(\tau_k)/n} d\hat{M}_{ik}^D.$$

Overall, the influence function is:

$$\hat{\psi}_i(\tau) = \sum_{k=1}^{K} \frac{(\tau - \tau_k)\hat{S}(\tau_k)}{Y(\tau_k)/n} d\hat{M}_{ik} - \sum_{k=1}^{K} \frac{\nu_{ik}}{Y(\tau_k)/n} d\hat{M}_{ik}^D.$$

♠

References

[1] Lin DY. “Non-parametric inference for cumulative incidence functions in competing risks studies”. In: *Statistics in Medicine* 16.3 (1997), pp. 901–910.