Cumulants

Definition 1.1.1. The **moment generating function** of a random variable X is:

$$M_X(s) = E(e^{Xs}) = \int e^{xs} f(x) dx.$$

The *moments* of X are the coefficients m_j in the expansion:

$$M_X(s) = \sum_{j=0}^{\infty} m_j \frac{s^j}{j!} = 1 + m_1 s + m_2 \frac{s^2}{2!} + m_3 \frac{s^3}{3!} + \cdots$$

The **cumulant generating function** is the logarithm of the MGF:

$$K_X(s) = \ln M_x(s).$$

The *cumulants* of X are defined by the coefficients κ_j in the expansion:

$$K_X(s) = \sum_{j=1}^{\infty} \kappa_j \frac{s^j}{j!} = \kappa_1 s + \kappa_2 \frac{s^2}{2!} + \kappa_3 \frac{s^3}{3!} + \cdots$$

∎

Example 1.1.1. Cumulants are related to moments via the identity:

$$\begin{aligned}\sum_{j=1}^{\infty} \kappa_j \frac{s^j}{j!} &= K_X(s) = \ln M_X(s) = \ln\left(\sum_{j=0}^{\infty} m_j \frac{s^j}{j!}\right) = \ln\left(1 + \sum_{j=1}^{\infty} m_j \frac{s^j}{j!}\right) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(\sum_{j=1}^{\infty} m_j \frac{s^j}{j!}\right)^k.\end{aligned}$$

Expanding the sums gives:

$$\begin{aligned}\kappa_1 s + \kappa_2 \frac{s^2}{2!} + \kappa_3 \frac{s^3}{3!} + \cdots = &+ \frac{1}{1}\left(m_1 s + m_2 \frac{s^2}{2!} + m_3 \frac{s^3}{3!} + m_4 \frac{s^4}{4!} + \cdots\right) \\ &- \frac{1}{2}\left(m_1 s + m_2 \frac{s^2}{2!} + m_3 \frac{s^3}{3!} + \cdots\right)^2 \\ &+ \frac{1}{3}\left(m_1 s + m_2 \frac{s^2}{2!} + \cdots\right)^3 - \frac{1}{4}\left(m_1 s + \cdots\right)^4 + \cdots\end{aligned}$$

Equating coefficients gives:

$$\begin{aligned}\kappa_1 &= m_1, \\ \kappa_2 &= m_2 - m_1^2, \\ \kappa_3 &= m_3 - 3m_1 m_2 + 2m_1^3, \\ \kappa_4 &= m_4 - 4m_1 m_3 - 3m_2^2 + 12 m_1^2 m_2 - 6m_1^4.\end{aligned}$$

Alternatively, the moments are related to the cumulants via the identity:

$$\sum_{j=0}^{\infty} m_j \frac{s^j}{j!} = M_X(s) = \exp\{K_X(s)\} = \exp\left\{\sum_{j=1}^{\infty} \kappa_j \frac{s^j}{j!}\right\} = \prod_{j=1}^{\infty} \exp\left\{\kappa_j \frac{s^j}{j!}\right\}$$
$$= \prod_{j=1}^{\infty} \left\{\sum_{l=0}^{\infty} \frac{1}{l!}\left(\kappa_j \frac{s^j}{j!}\right)^l\right\} = \prod_{j=1}^{\infty} \left\{1 + \kappa_j \frac{s^j}{j!} + \kappa_j^2 \frac{s^{2j}}{2!(j!)^2} + \cdots\right\}.$$

Expanding the sum and product gives:

$$1 + m_1 s + m_2 \frac{s^2}{2!} + m_3 \frac{s^3}{3!} + m_4 \frac{s^4}{4!} + \cdots = \left\{1 + \kappa_1 s + \kappa_1^2 \frac{s^2}{2!} + \kappa_1^3 \frac{s^3}{3!} + \kappa_1^4 \frac{s^4}{4!} + \cdots\right\}$$
$$\times \left\{1 + \kappa_2 \frac{s^2}{2!} + \kappa_2^2 \frac{s^4}{2!(2!)^2} + \cdots\right\}$$
$$\times \left\{1 + \kappa_3 \frac{s^3}{3!} + \cdots\right\} \times \left\{1 + \kappa_4 \frac{s^4}{4!} + \cdots\right\} \times \cdots$$

Equating coefficients gives:

$$\begin{aligned}
m_1 &= \kappa_1, \\
m_2 &= \kappa_1^2 + \kappa_2, \\
m_3 &= \kappa_1^3 + 3\kappa_1\kappa_2 + \kappa_3, \\
m_4 &= \kappa_1^4 + 6\kappa_1^2\kappa_2 + 3\kappa_2^2 + 4\kappa_1\kappa_3 + \kappa_4.
\end{aligned}$$

♠

Hermite Polynomials

2.1 Basic Properties

Definition 1.2.1. The **Hermite polynomials** $H_k(x)$ are defined by the series expansion of the generating function $G_H(x,t)$:

$$G_H(x,t) = e^{xt-t^2/2} = \sum_{k=0}^{\infty} H_k(x)\frac{t^k}{k!}. \tag{1.2.1}$$

The first several Hermite polynomials are:

$$\begin{aligned} H_1(z) &= z, \\ H_2(z) &= z^2 - 1, \\ H_3(z) &= z^3 - 3z, \\ H_4(z) &= z^4 - 6z^2 + 3. \end{aligned}$$

■

Proposition 1.2.1. The Hermite polynomials satisfy the recursion:

$$\dot{H}_k(x) = kH_{k-1}(x) \text{ for } k \geq 1.$$

♦

Proof. Differentiating the exponential representation of (1.2.1) w.r.t. x:

$$\begin{aligned} \frac{\partial}{\partial x} G_H(x,t) &= \frac{\partial}{\partial x} e^{xt-t^2/2} = te^{xt-t^2/2} = tG_H(x,t) \\ &= \sum_{k=0}^{\infty} H_k(x)\frac{t^{k+1}}{k!} = \sum_{k=0}^{\infty}(k+1)H_k(x)\frac{t^{k+1}}{(k+1)!} = \sum_{k=1}^{\infty} kH_{k-1}(x)\frac{t^k}{k!}. \end{aligned}$$

Differentiating the series representation of (1.2.1) w.r.t. x:

$$\frac{\partial}{\partial x} G_H(x,t) = \frac{\partial}{\partial x}\sum_{k=0}^{\infty} H_k(x)\frac{t^k}{k!} = \sum_{k=0}^{\infty} \dot{H}_k(x)\frac{t^k}{k!} = \dot{H}_0(x) + \sum_{k=1}^{\infty} \dot{H}_k(x)\frac{t^k}{k!}$$

Equating powers of t gives:

$$\dot{H}_k(x) = kH_{k-1}(x) \text{ for } k \geq 1.$$

■

Proposition 1.2.2. The Hermite polynomials satisfy the recursion:

$$H_{k+1}(x) = xH_k(x) - kH_{k-1}(x) \text{ for } k \geq 1.$$

♦

Proof. Differentiating the exponential representation of (1.2.1) w.r.t. t:

$$\begin{aligned}\frac{\partial}{\partial t}G_H(x,t) &= \frac{\partial}{\partial t}e^{xt-t^2/2} = (x-t)e^{xt-t^2/2} = (x-t)G_H(x,t) \\ &= \sum_{k=0}^{\infty} xH_k(x)\frac{t^k}{k!} - \sum_{k=0}^{\infty}(k+1)H_k(x)\frac{t^{k+1}}{(k+1)!} \\ &= \sum_{k=0}^{\infty} xH_k(x)\frac{t^k}{k!} - \sum_{k=0}^{\infty} kH_{k-1}(x)\frac{t^k}{k!}.\end{aligned}$$

Differentiating the series representation of (1.2.1) w.r.t. t:

$$\frac{\partial}{\partial t}G_H(x,t) = \frac{\partial}{\partial t}\sum_{k=0}^{\infty} H_k(x)\frac{t^k}{k!} = \sum_{k=1}^{\infty} H_k(x)\frac{t^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} H_{k+1}(x)\frac{t^k}{k!}$$

Equating powers of t gives:

$$H_{k+1}(x) = xH_x(x) - kH_{k-1}(x).$$

■

Result 1.2.1 (Rodrigues' Formula). The **Hermite polynomials** satisfy:

$$(-1)^k\frac{d^k}{dx^k}\phi(x) = H_k(x)\phi(x), \tag{1.2.2}$$

where $\phi(x)$ is the standard normal density. ♣

Proof. The Hermite generating function (1.2.1) is expressible as:

$$G_H(x,t) = e^{xt-t^2/2} = e^{x^2/2-(t-x)^2/2} = \sum_{k=0}^{\infty} H_k(x)\frac{t^k}{k!}.$$

The kth Hermite polynomial is given by:

$$\left\{\frac{\partial^k}{\partial t^k}e^{xt-t^2/2}\right\}_{t=0} = \left\{\frac{\partial^k}{\partial t^k}\sum_{l=0}^{\infty} H_l(x)\frac{t^k}{k!}\right\}_{t=0} = \left\{\sum_{l=k}^{\infty} H_l(x)\frac{t^{l-k}}{(l-k)!}\right\}_{t=0} = H_k(x).$$

Replacing the kernel of the generating function:

$$H_k(x) = \left\{\frac{\partial^k}{\partial t^k}e^{xt-t^2/2}\right\}_{t=0} = \left\{\frac{\partial^k}{\partial t^k}e^{x^2/2-(t-x)^2/2}\right\}_{t=0} = e^{x^2/2}\left\{\frac{\partial^k}{\partial t^k}e^{-(t-x)^2/2}\right\}_{t=0}.$$

Observe that:

$$\frac{\partial}{\partial t}e^{-(t-x)^2/2} = -(t-x)e^{-(t-x)^2/2} = -\frac{\partial}{\partial x}e^{-(t-x)^2/2}.$$

Replacing the derivatives in t with derivatives in x:

$$\begin{aligned} H_k(x) &= e^{x^2/2}\left\{\frac{\partial^k}{\partial t^k}e^{-(t-x)^2/2}\right\}_{t=0} \\ &= e^{x^2/2}\left\{(-1)^k\frac{\partial^k}{\partial x^k}e^{-(t-x)^2/2}\right\}_{t=0} = e^{x^2/2}(-1)^k\frac{d^k}{dx^k}e^{-x^2/2}. \end{aligned}$$

Multiplying through through by $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ gives:

$$\phi(x)H_k(x) = (-1)^k\frac{d^k}{dx^k}\phi(x).$$

■

Result 1.2.2 (Orthogonality). The Hermite polynomials form an orthogonal system to the *ϕ-weighted inner product*:

$$\langle H_m, H_n\rangle_\phi = \int_{-\infty}^{\infty} H_m(x)H_n(x)\phi(x)dx = \begin{cases} n!, & n = m, \\ 0, & n \neq m. \end{cases}$$

♣

Proof. Multiply two instances of the Hermite generating function (1.2.1):

$$G_H(x,s)G_H(x,t) = e^{xs-s^2/2}e^{xt-t^2/2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}H_m(x)H_n(x)\frac{s^m t^n}{m!n!}.$$

Multiplying the series representation by $e^{-x^2/2}$ and integrating gives:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m t^n}{m!n!}\int_{-\infty}^{\infty}H_m(x)H_n(x)e^{-x^2/2}dx.$$

Multiplying the exponential representation by $e^{-x^2/2}$ and integrating gives:

$$\int_{-\infty}^{\infty}e^{xs-s^2/2+xt-t^2/2-x^2/2}dx = e^{st}\int_{-\infty}^{\infty}e^{-(x-s-t)^2/2}dx = e^{st}\sqrt{2\pi} = \sqrt{2\pi}\sum_{k=0}^{\infty}\frac{(st)^k}{k!}$$

Since the two representations are equivalent:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m t^n}{m!n!}\int_{-\infty}^{\infty}H_m(x)H_n(x)e^{-x^2/2}dx = \sqrt{2\pi}\sum_{k=0}^{\infty}\frac{(st)^k}{k!}$$

Equating coefficients gives the result. ■

Fourier Transform

Definition 1.3.1. Define the **Fourier transform** as:

$$\mathscr{F}\{g(x)\} = \int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx,$$

and the **inverse Fourier transform** as:

$$\mathscr{F}^{-1}\{G(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega x}d\omega.$$

■

Result 1.3.1. $\phi(t)$ is an **eigenfunction** of the Fourier transform and its inverse:

$$\mathscr{F}\{\phi(x)\} = e^{-\omega^2/2} = \sqrt{2\pi}\phi(\omega),$$
$$\mathscr{F}^{-1}\{\sqrt{2\pi}\phi(\omega)\} = \mathscr{F}^{-1}\{e^{-\omega^2/2}\} = \phi(x).$$

♣

Proof. The Fourier transform of $\phi(x)$ is:

$$\begin{aligned}\mathscr{F}\{\phi(x)\} &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-(x+i\omega)^2/2-\omega^2/2}dx \\ &= \frac{1}{\sqrt{2\pi}}e^{-\omega^2/2}\int_{-\infty}^{\infty}e^{-u^2/2}du = e^{-\omega^2/2}.\end{aligned}$$

The inverse Fourier transform of $e^{-\omega^2/2}$ is:

$$\begin{aligned}\mathscr{F}^{-1}\left(e^{-\omega^2/2}\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\omega^2/2}e^{i\omega x}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-(\omega-ix)^2/2-x^2/2}d\omega \\ &= \frac{1}{2\pi}e^{-x^2/2}\int_{-\infty}^{\infty}e^{-u^2/2}du = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.\end{aligned}$$

■

Corollary 1.3.1. The Fourier transform of $x^k\phi(x)$ is:

$$\mathscr{F}\{x^k\phi(x)\} = i^k(-1)^kH_k(\omega)e^{-\omega^2/2}.$$

The inverse Fourier transform of $\omega^k e^{-\omega^2/2}$ is:

$$\mathscr{F}^{-1}\{\omega^k e^{-\omega^2/2}\} = i^kH_k(x)\phi(x).$$

♣

Proof. Observe that:

$$\frac{\partial}{\partial\omega}e^{-i\omega x} = (-i)xe^{-i\omega x}, \qquad xe^{-i\omega x} = \frac{1}{(-i)}\frac{\partial}{\partial\omega}e^{-i\omega x}, \qquad x^k e^{-i\omega x} = \frac{1}{(-i)^k}\frac{\partial^k}{\partial\omega^k}e^{-i\omega x}.$$

The Fourier transform of $x^k\phi(x)$ is:

$$\begin{aligned}\mathscr{F}\{x^k\phi(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}x^k e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\frac{1}{(-i)^k}\frac{\partial^k}{\partial\omega^k}e^{-i\omega x}dx \\ &= i^k\frac{d^k}{d\omega^k}\mathscr{F}\{\phi(x)\} = i^k\frac{d^k}{d\omega^k}e^{-\omega^2/2} = i^k(-1)^k H_k(\omega)e^{-\omega^2/2}.\end{aligned}$$

Similarly:

$$\begin{aligned}\mathscr{F}^{-1}\{\omega^k e^{-\omega^2/2}\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^k e^{-\omega^2/2}e^{i\omega x}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{i^k}\frac{\partial^k}{\partial x^k}e^{i\omega x}dx \\ &= (-i)^k\frac{d^k}{dx^k}\mathscr{F}^{-1}\{e^{-\omega^2/2}\} = (-i)^k\frac{d^k}{dx^k}\phi(x) = i^k H_k(x)\phi(x).\end{aligned}$$

■

3.1 Fourier-Hermite Expansion

Theorem 1.3.1. The Hermite polynomials form a *complete*, orthogonal basis for the Hilbert space $\mathcal{H}$ of square-integrable functions w.r.t. to the ϕ-*weighted* inner product:

$$\langle f, g\rangle_\phi = \int_{-\infty}^{\infty} f(x)g(x)\phi(x)dx.$$

□

Corollary 1.3.2. Any $h \in \mathcal{H}$ admits a **Fourier-Hermite expansion**:

$$h(x) = \sum_{k=0}^{\infty}\alpha_k H_k(x).$$

To determine the coefficients (α_k), take the inner product of h wrt the mth Hermite polynomial H_m:

$$\langle h, H_m\rangle_\phi = \sum_{k=0}^{\infty}\alpha_k\langle H_k, H_m\rangle_\phi = \alpha_m m!,$$

$$\alpha_m = \frac{1}{m!}\langle h, H_m\rangle_\phi = \frac{1}{m!}\int_{-\infty}^{\infty} h(x)H_m(x)\phi(x)dx.$$

Alternatively, consider the Hilbert space $\mathcal{L}^2$ of square integrable functions with the inner product:

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)dx.$$

The **Gram-Charlier expansion** with the normal kernel takes the form:

$$h(x) = \sum_{k=0}^{\infty} \alpha_k H_k(x)\phi(x). \tag{1.3.3}$$

Suppose $X \sim f_X$ with $f_X \in \mathcal{L}^2$. The coefficients of the *Gram-Charlier expansion* are:

$$\langle f, H_m \rangle = \sum_{k=0}^{\infty} \alpha_k \langle H_k \phi, H_m \rangle = \alpha_m m!,$$
$$\alpha_m = \frac{1}{m!}\langle h, H_m \rangle = \frac{1}{m!}\int_{-\infty}^{\infty} f(x) H_m(x) dx = \frac{1}{m!} E_X\{H_m(X)\}.$$

♣

Edgeworth Expansion

Definition 1.4.1. The **characteristic function** of a random variable X is the scaled inverse Fourier transform of the density f_X:

$$\psi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)e^{i\omega x}(x)dx.$$

■

Theorem 1.4.1 (Characteristic Function Inversion). The density f_X is recovered from the characteristic function $\psi(\omega)$ via:

$$f_X(x) = \frac{1}{2\pi}\int \psi_X(\omega)e^{-i\omega x}d\omega. \tag{1.4.4}$$

□

Proposition 1.4.1. Suppose $X_i \overset{\text{IID}}{\sim} F_X$. Define the standardized observation Z_i and the standardized sum S:

$$Z_i = \frac{(X_i - \mu)}{\sigma}, \qquad S = \sqrt{n}\frac{(\bar{X} - \mu)}{\sigma}.$$

The CGFs of S and Z are related via:

$$K_S(t) = nK_Z\left(\frac{t}{\sqrt{n}}\right).$$

♦

Proof. The MGFs are related by:

$$\begin{aligned}
M_S(t) = E\{e^{tS}\} = E\left[\exp\left\{t\cdot\frac{\sqrt{n}}{\sigma}(\bar{X}-\mu)\right\}\right] &= E\left[\exp\left\{t\cdot\frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{i=1}^{n}X_i - \mu\right)\right\}\right] \\
&= \exp\left[\exp\left\{t\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{(X_i-\mu)}{\sigma}\right\}\right] = E\left\{\exp\left(t\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Z_i\right)\right\} \\
&= E\left(\prod_{i=1}^{n}e^{\frac{t}{\sqrt{n}}Z_i}\right) = \prod_{i=1}^{n}E\left\{e^{\frac{t}{\sqrt{n}}Z_i}\right\} = \left\{M_Z\left(\frac{t}{\sqrt{n}}\right)\right\}^n.
\end{aligned}$$

Taking the logarithm gives the result. ■

Proposition 1.4.2. A formal expansion for the CDF of S is given by:

$$F_S(s) = \frac{1}{2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty} e^{-\omega^2/2 - i\omega t}\left\{1 + \sum_{k=1}^{\infty}\frac{1}{k!}\left(\sum_{j=3}^{\infty}\kappa_j\frac{(i\omega)^j}{j!n^{j/2-1}}\right)^k\right\}d\omega dt, \tag{1.4.5}$$

where κ_j is the jth cumulant of the standardized observation Z_i. ♦

Proof. From the inversion identity in (1.4.4):

$$F_S(s) = \int_{-\infty}^{s} f_S(t)dt = \frac{1}{2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty} \psi_S(\omega)e^{-i\omega t}d\omega dt,$$

where $\psi_S(\omega) = M_S(i\omega)$ is the characteristic function of S.

Expressing $\psi_S(\omega)$ in terms of the CGF for Z:

$$\begin{aligned} F_S(s) &= \frac{1}{2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty}\left\{M_Z\left(\frac{i\omega}{\sqrt{n}}\right)\right\}^n e^{-i\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty}\exp\left\{nK_Z\left(\frac{i\omega}{\sqrt{n}}\right) - i\omega t\right\}d\omega dt. \end{aligned}$$

Expanding the CGF as a power series:

$$F_S(s) = \frac{1}{2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty}\exp\left\{n\sum_{j=1}^{\infty}\kappa_j\frac{(i\omega)^j}{j!n^{j/2}} - i\omega t\right\}d\omega dt.$$

Since Z is standardized $\kappa_1 = 0$ and $\kappa_2 = 1$:

$$\begin{aligned} F_S(s) &= \frac{1}{2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty}\exp\left\{n\left(\frac{(i\omega)^2}{2!n} + \sum_{j=3}^{\infty}\kappa_j\frac{(i\omega)^j}{j!n^{j/2}}\right) - i\omega t\right\}d\omega dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty}\exp\left\{-\frac{\omega^2}{2} - i\omega t + \sum_{j=3}^{\infty}\kappa_j\frac{(i\omega)^j}{j!n^{j/2-1}}\right\}d\omega dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty}e^{-\omega^2/2 - i\omega t}\exp\left\{\sum_{j=3}^{\infty}\kappa_j\frac{(i\omega)^j}{j!n^{j/2-1}}\right\}d\omega dt. \end{aligned}$$

Taylor expanding the exponential term that depends on n gives:

$$F_S(s) = \frac{1}{2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty}e^{-\omega^2/2 - i\omega t}\left\{1 + \sum_{k=1}^{\infty}\frac{1}{k!}\left(\sum_{j=3}^{\infty}\kappa_j\frac{(i\omega)^j}{j!n^{j/2-1}}\right)^k\right\}d\omega dt.$$

■

Corollary 1.4.1. A formal expansion for the density of S is given by:

$$f_S(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\omega^2/2 - i\omega t}\left\{1 + \sum_{k=1}^{\infty}\frac{1}{k!}\left(\sum_{j=3}^{\infty}\kappa_j\frac{(i\omega)^j}{j!n^{j/2-1}}\right)^k\right\}d\omega dt. \tag{1.4.6}$$

♣

Remark 1.4.1. The **Edgeworth expansion** for the standardized sum S is obtained by evaluating the integrand of (1.4.5) term-wise. ♦

Proposition 1.4.3. The $\mathcal{O}(n^{-3/2})$ Edgeworth expansion for $F(s)$ is:

$$F_S(s) = \Phi(s) - \phi(s)\left\{\frac{\kappa_3}{6n^{1/2}}H_2(s) + \frac{\kappa_4}{24n}H_3(s) + \frac{\kappa_3^2}{72n}H_5(s)\right\} + \mathcal{O}(n^{-3/2}).$$

♦

Proof. The series from (1.4.5) is expressible as:

$$\left\{1 + \left(\kappa_3\frac{(i\omega)^3}{3!n^{1/2}} + \kappa_4\frac{(i\omega)^4}{4!n} + \cdots\right) + \frac{1}{2}\left(\kappa_3\frac{(i\omega)^3}{3!n^{1/2}} + \cdots\right)^2 + \cdots\right\}.$$

The $\mathcal{O}(1)$ term of the expansion is:

$$I_0 = \frac{1}{2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty} e^{-\omega^2/2 - i\omega} d\omega dt.$$

By completing the square:

$$e^{-\omega^2/2-i\omega} = e^{-t^2/2-(\omega+it)^2/2}. \tag{1.4.7}$$

Evaluating the $\mathcal{O}(1)$ term:

$$\begin{aligned} I_0 &= \frac{1}{2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty} e^{-t^2/2-(\omega+it)^2/2}d\omega dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{s} e^{-t^2/2}\int_{-\infty}^{\infty} e^{-(\omega+it)^2/2}d\omega dt = \int_{-\infty}^{s}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt = \Phi(s). \end{aligned}$$

The $\mathcal{O}(n^{-1/2})$ term is:

$$\frac{\kappa_3}{3!n^{1/2}}I_1 \equiv \frac{\kappa_3}{3!n^{1/2}\cdot 2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty} e^{-\omega^2/2}(i\omega)^3 e^{-i\omega t}d\omega dt.$$

Observe that:

$$(i\omega)e^{-i\omega t} = (-1)\frac{\partial}{\partial t}e^{-i\omega t}, \qquad (i\omega)^k e^{-i\omega t} = (-1)^k\frac{\partial^k}{\partial t^k}e^{-i\omega t}.$$

The integral in the $\mathcal{O}(n^{-1/2})$ term reduces to:

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty} e^{-\omega^2/2}(-1)^3\frac{\partial^3}{\partial t^3}e^{-i\omega t}d\omega dt = \frac{1}{2\pi}\int_{-\infty}^{s}\int_{-\infty}^{\infty}(-1)^3\frac{\partial^3}{\partial t^3}e^{-\omega^2/2-i\omega 2}d\omega dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{s}(-1)^3\frac{d^3}{dt^3}\int_{-\infty}^{\infty} e^{-\omega^2/2-i\omega t}d\omega dt = \frac{1}{2\pi}\int_{-\infty}^{s}(-1)^3\frac{d^3}{dt^3}\int_{-\infty}^{\infty} e^{-t^2/2-(\omega+it)^2/2}d\omega dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{s}(-1)^3\frac{d^3}{dt^3}e^{-t^2/2}\int_{-\infty}^{\infty} e^{-(\omega+it)^2/2}d\omega dt = \int_{-\infty}^{s}(-1)^3\frac{d^3}{dt^3}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt \\ &= \int_{-\infty}^{s}(-1)^3\frac{d^3}{dt^3}\phi(t)dt = (-1)^3\int_{-\infty}^{s}\frac{d^3}{dt^3}\phi(t)dt = (-1)^3\frac{d^2}{ds^2}\phi(s). \end{aligned}$$

Applying Rodrigues' formula (1.2.2):

$$I_1 = (-1)(-1)^2 \frac{d^2}{ds^2}\phi(s) = (-1)H_2(s)\phi(s).$$

Combining the $\mathcal{O}(1)$ and $\mathcal{O}(n^{-1/2})$ terms of the Edgeworth expansion:

$$F_S(s) = \Phi(s) - \frac{\kappa_3}{6n^{1/2}} H_2(s)\phi(s) + \mathcal{O}(n^{-1}). \tag{1.4.8}$$

Obtaining the $\mathcal{O}(n^{-1})$ term of the Edgeworth series requires consideration of the $k = 2$ term from (1.4.5). The contribution of the $k = 1$ order term from (1.4.5) is:

$$\frac{\kappa_4}{4!n \cdot 2\pi} \int_{-\infty}^{s} \int_{-\infty}^{\infty} e^{-\omega^2/2}(i\omega)^4 e^{-i\omega t} d\omega dt = -\frac{\kappa_4}{24n} H_3(s)\phi(s),$$

where evaluation of the integral follows the same procedure as for the $\mathcal{O}(n^{-1/2})$ term. The contribution of the $k = 2$ term from (1.4.5) is:

$$\frac{\kappa_3^2}{2!(3!n^{1/2})^2 \cdot 2\pi} \int_{-\infty}^{s} \int_{-\infty}^{\infty} e^{-\omega^2/2}(i\omega)^6 e^{-i\omega t} d\omega dt = -\frac{\kappa_3^2}{72n} H_5(s)\phi(s).$$

Combining the $\mathcal{O}(n^{-1})$ terms with (1.4.8) gives the result. ■

Corollary 1.4.2. An analogous derivation applied to (1.4.6) gives:

$$f_S(s) = \phi(s)\left\{1 + \frac{\kappa_3}{6n^{1/2}} H_3(s) + \frac{\kappa_4}{24n} H_4(s) + \frac{\kappa_3^2}{72n} H_6(s)\right\} + \mathcal{O}(n^{-3/2}).$$

♣