Estimation and Inference

1.1 Setting

Consider data of the form (U_i, δ_i), where U_i is the observation time, and δ_i indicates that an event occurred prior to censoring:

$$\delta_i = \mathbb{I}\big(T_i \le C_i\big).$$

Suppose the event times (T_i) follow a generalized Gamma distribution with *shape parameters* α and β, and *rate parameter* λ. The survival function is:

$$S(t) = \frac{1}{\Gamma(\alpha)}\Gamma\big\{\alpha, (\lambda t)^{\beta}\big\},\ t > 0.$$

The density of the generalized gamma distribution is:

$$f(t) = \frac{\beta\lambda}{\Gamma(\alpha)}(\lambda t)^{\alpha\beta-1}e^{-(\lambda t)^{\beta}},\ t > 0.$$

where $\Gamma(\cdot)$ is the standard gamma function, and $\Gamma(\cdot,\cdot)$ is the upper incomplete gamma function:

$$\Gamma(\alpha, t) = \int_t^{\infty} s^{\alpha-1}e^{-s}ds.$$

The hazard function for the generalized gamma distribution is:

$$h(t) = \frac{(\lambda t)^{\alpha\beta-1}e^{-(\lambda t)^{\beta}}}{\Gamma\big\{\alpha, (\lambda t)^{\beta}\big\}},\ t > 0.$$

1.2 Likelihood

The density contribution of the ith subject is:

$$f_i = \frac{\beta\lambda}{\Gamma(\alpha)}(\lambda u_i)^{\alpha\beta-1}e^{-(\lambda u_i)^{\beta}}.$$

taking the logarithm:

$$\ln f_i = \ln\beta + \ln\lambda - \ln\Gamma(\alpha) + (\alpha\beta - 1)\ln(\lambda u_i) - (\lambda u_i)^{\beta}.$$

The survival contribution of the ith subject is:

$$S_i = \frac{1}{\Gamma(\alpha)}\Gamma\big\{\alpha, (\lambda u_i)^{\beta}\big\}.$$

The log survival contribution is:

$$\ln S_i = \ln\Gamma\big\{\alpha, (\lambda u_i)^{\beta}\big\} - \ln\Gamma(\alpha).$$

The right censored likelihood is:

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f_i^{\delta_i} S_i^{1-\delta_i}.$$

The right censored log likelihood is:

$$\begin{aligned}\ell(\boldsymbol{\theta}) \propto &\sum_{i=1}^{n} \delta_i \Big(\ln\beta + \ln\lambda - \ln\Gamma(\alpha) + \alpha\beta\ln\lambda + \alpha\beta\ln u_i - \ln\lambda - \lambda^\beta u_i^\beta \Big) \\ &+ \sum_{i=1}^{n}(1-\delta_i)\Big(\ln\Gamma\big\{\alpha, (\lambda u_i)^\beta\big\} - \ln\Gamma(\alpha)\Big).\end{aligned}$$

Let $\Delta_n = \sum_{i=1}^{n}\delta_i$ denote the number of observed failures. The log likelihood becomes:

$$\begin{aligned}\ell(\boldsymbol{\theta}) \propto &\Delta_n\ln\beta - n\ln\Gamma(\alpha) + \Delta_n\alpha\beta\ln\lambda \\ &+ \alpha\beta\sum_{i=1}^{n}\delta_i\ln u_i - \lambda^\beta\sum_{i=1}^{n}\delta_i u_i^\beta + \sum_{i=1}^{n}(1-\delta_i)\ln\Gamma\big\{\alpha,(\lambda u_i)^\beta\big\}.\end{aligned}$$

When $\beta = 1$, the gamma log likelihood is recovered:

$$\begin{aligned}\ell(\alpha, \beta=1, \lambda) = &-n\ln\Gamma(\alpha) + \Delta_n\alpha\ln\lambda \\ &+ \alpha\sum_{i=1}^{n}\delta_i\ln u_i - \lambda\sum_{i=1}^{n}\delta_i u_i + \sum_{i=1}^{n}(1-\delta_i)\ln\Gamma\big\{\alpha,\lambda u_i\big\}.\end{aligned}$$

When evaluated at $\alpha = 1$, the upper incomplete gamma reduces to:

$$\ln\Gamma(\alpha=1, s) = \ln\int_s^\infty e^{-u}du = \ln e^{-s} = -s.$$

Consequently, when $\alpha = 1$, the Weibull log likelihood is recovered:

$$\begin{aligned}\ell(\alpha=1,\beta,\lambda) &= \Delta_n\ln\beta + \Delta_n\beta\ln\lambda + \beta\sum_{i=1}^{n}\delta_i\ln u_i - \lambda^\beta\sum_{i=1}^{n}\delta_i u_i^\beta - \sum_{i=1}^{n}(1-\delta_i)\lambda^\beta u_i^\beta \\ &= \Delta_n\ln\beta + \Delta_n\beta\ln\lambda + \beta\sum_{i=1}^{n}\delta_i\ln u_i - \lambda^\beta\sum_{i=1}^{n}u_i^\beta.\end{aligned}$$

1.3 Score Equations

The score equation for α is:

$$\mathcal{U}_\alpha = -n\psi(\alpha) + \Delta_n\beta\ln\lambda + \beta\sum_{i=1}^{n}\delta_i\ln u_i + \sum_{i=1}^{n}(1-\delta_i)\frac{\partial\ln\Gamma\big\{\alpha,(\lambda u_i)^\beta\big\}}{\partial\alpha}$$

The score equation for β is:

$$\begin{aligned}\mathcal{U}_\beta =& \frac{\Delta_n}{\beta} + \Delta_n \alpha \ln \lambda + \alpha \sum_{i=1}^{n} \delta_i \ln u_i - \lambda^\beta \ln \lambda \sum_{i=1}^{n} \delta_i u_i^\beta \\ & - \lambda^\beta \sum_{i=1}^{n} \delta_i u_i^\beta \ln u_i + \sum_{i=1}^{n} (1-\delta_i) \frac{\partial \ln \Gamma\big\{\alpha, (\lambda u_i)^\beta\big\}}{\partial(\lambda u_i^\beta)} \cdot \frac{\partial (\lambda u_i)^\beta}{\partial \beta}.\end{aligned}$$

The score equation for λ is:

$$\mathcal{U}_\lambda = \frac{\Delta_n \alpha \beta}{\lambda} - \beta \lambda^{\beta-1} \sum_{i=1}^{n} \delta_i u_i^\beta + \sum_{i=1}^{n} (1-\delta_i) \frac{\partial \ln \Gamma\big\{\alpha, (\lambda u_i)^\beta\big\}}{\partial (\lambda u_i)^\beta} \cdot \frac{\partial (\lambda u_i)^\beta}{\partial \lambda}$$

The partials of the upper incomplete gamma were obtained in the derivations for the (standard) gamma distribution.

1.4 Observed Information

The Hessian for α is:

$$\mathcal{H}_{\alpha\alpha} = -n\dot{\psi}(\alpha) + \sum_{i=1}^{n} (1-\delta_i) \Psi_2\big\{\alpha, (\lambda u_i)^\beta\big\}.$$

The Hessian for β is:

$$\begin{aligned}\mathcal{H}_{\beta\beta} =& -\frac{\Delta_n}{\beta^2} - \lambda^\beta \ln^2 \lambda \sum_{i=1}^{n} \delta_i u_i^\beta - 2\lambda^\beta \ln \lambda \sum_{i=1}^{n} \delta_i u_i^\beta \ln u_i - \lambda^\beta \sum_{i=1}^{n} \delta_i u_i^\beta \ln^2 u_i \\ & + \sum_{i=1}^{n} (1-\delta_i) \left(\frac{\partial^2 \ln \Gamma\big\{\alpha, (\lambda u_i)^\beta\big\}}{\partial \big\{(\lambda u_i)^\beta\big\}^2} \cdot \left\{ \frac{\partial (\lambda u_i)^\beta}{\partial \beta} \right\}^2 + \frac{\partial \ln \Gamma\big\{\alpha, (\lambda u_i)^\beta\big\}}{\partial (\lambda u_i)^\beta} \cdot \left\{ \frac{\partial^2 (\lambda u_i)^\beta}{\partial \beta^2} \right\} \right).\end{aligned}$$

The Hessian in λ is:

$$\begin{aligned}\mathcal{H}_{\lambda\lambda} =& -\frac{\Delta_n \alpha \beta}{\lambda^2} - \beta(\beta-1)\lambda^{\beta-2} \sum_{i=1}^{n} \delta_i u_i^\beta \\ & + \sum_{i=1}^{n} (1-\delta_i) \left(\frac{\partial^2 \ln \Gamma\big\{\alpha, (\lambda u_i)^\beta\big\}}{\partial \big\{(\lambda u_i)^\beta\big\}^2} \cdot \left\{ \frac{\partial (\lambda u_i)^\beta}{\partial \lambda} \right\}^2 + \frac{\partial \ln \Gamma\big\{\alpha, (\lambda u_i)^\beta\big\}}{\partial (\lambda u_i)^\beta} \cdot \left\{ \frac{\partial^2 (\lambda u_i)^\beta}{\partial \lambda^2} \right\} \right).\end{aligned}$$

The mixed partial w.r.t. α and β is:

$$\mathcal{H}_{\alpha\beta} = \Delta_n \ln \lambda + \sum_{i=1}^{n} \delta_i \ln u_i + \sum_{i=1}^{n} (1-\delta_i) \frac{\partial^2 \ln \Gamma\big\{\alpha, (\lambda u_i)^\beta\big\}}{\partial \alpha \partial (\lambda u_i)^\beta} \cdot \left\{ \frac{\partial (\lambda u_i)^\beta}{\partial \beta} \right\}.$$

The mixed partial w.r.t. α and λ is:

$$\mathcal{H}_{\alpha\lambda} = \frac{\Delta_n \beta}{\lambda} + \sum_{i=1}^{n} (1-\delta_i) \frac{\partial^2 \ln \Gamma\big\{\alpha, (\lambda u_i)^\beta\big\}}{\partial \alpha \partial (\lambda u_i)^\beta} \cdot \left\{ \frac{\partial (\lambda u_i)^\beta}{\partial \lambda} \right\}.$$

The mixed partial w.r.t. β and λ is:

$$\begin{aligned}\mathcal{H}_{\beta\lambda} =& \frac{\Delta_n \alpha}{\lambda} - \lambda^{\beta-1}\sum_{i=1}^{n}\delta_i u_i^{\beta} - \beta\lambda^{\beta-1}\ln\lambda\sum_{i=1}^{n}\delta_i u_i^{\beta} - \beta\lambda^{\beta-1}\sum_{i=1}^{n}\delta_i u_i^{\beta}\ln u_i \\ &+ \sum_{i=1}^{n}(1-\delta_i)\left(\frac{\partial^2 \ln\Gamma\big\{\alpha,(\lambda u_i)^{\beta}\big\}}{\partial\big\{(\lambda u_i)^{\beta}\big\}^2}\cdot\left\{\frac{\partial(\lambda u_i)^{\beta}}{\partial\beta}\right\}\left\{\frac{\partial(\lambda u_i)^{\beta}}{\partial\lambda}\right\}\right) \\ &+ \sum_{i=1}^{n}(1-\delta_i)\left(\frac{\partial \ln\Gamma\big\{\alpha,(\lambda u_i)^{\beta}\big\}}{\partial(\lambda u_i)^{\beta}}\cdot\left\{\frac{\partial^2(\lambda u_i)^{\beta}}{\partial\beta\partial\lambda}\right\}\right).\end{aligned}$$

1.5 Complete Data

1.5.1 Likelihood

Suppose all events are observed. The log likelihood reduces to:

$$\ell(\boldsymbol{\theta}) \propto n\ln\beta - n\ln\Gamma(\alpha) + n\alpha\beta\ln\lambda + \alpha\beta\sum_{i=1}^{n}\ln u_i - \lambda^{\beta}\sum_{i=1}^{n}u_i^{\beta}.$$

1.5.2 Score

The score for α is:

$$\mathcal{U}_{\alpha} = -n\psi(\alpha) + n\beta\ln\lambda + \beta\sum_{i=1}^{n}\ln u_i.$$

The score for β is:

$$\mathcal{U}_{\beta} = \frac{n}{\beta} + n\alpha\ln\lambda + \alpha\sum_{i=1}^{n}\ln u_i - \lambda^{\beta}\ln\lambda\sum_{i=1}^{n}u_i^{\beta} - \lambda^{\beta}\sum_{i=1}^{n}u_i^{\beta}\ln u_i.$$

The score for λ is:

$$\mathcal{U}_{\lambda} = \frac{n\alpha\beta}{\lambda} - \beta\lambda^{\beta-1}\sum_{i=1}^{n}u_i^{\beta}.$$

Solving $\mathcal{U}_{\lambda} \overset{\text{Set}}{=} 0$ for λ:

$$\hat{\lambda}(\alpha,\beta) = \left(\frac{1}{n\alpha}\sum_{i=1}^{n}u_i^{\beta}\right)^{-1/\beta}.$$

1.5.3 Hessian

The Hessian for α is:

$$\mathcal{H}_{\alpha\alpha} = -n\dot{\psi}(\alpha).$$

The Hessian for β is:

$$\mathcal{H}_{\beta\beta} = -\frac{n}{\beta^2} - \lambda^{\beta}\ln^2\lambda\sum_{i=1}^{n}u_i^{\beta} - 2\lambda^{\beta}\ln\lambda\sum_{i=1}^{n}u_i^{\beta}\ln u_i - \lambda^{\beta}\sum_{i=1}^{n}u_i^{\beta}\ln^2 u_i.$$

The Hessian for λ is:

$$\mathcal{H}_{\lambda\lambda} = -\frac{n\alpha\beta}{\lambda^2} - \beta(\beta-1)\lambda^{\beta-2}\sum_{i=1}^{n}u_i^{\beta}.$$

The mixed partials are:

$$\mathcal{H}_{\alpha\beta} = n\ln\lambda + \sum_{i=1}^{n}\ln u_i,$$
$$\mathcal{H}_{\alpha\lambda} = \frac{n\beta}{\lambda},$$
$$\mathcal{H}_{\beta\lambda} = \frac{n\alpha}{\lambda} - \beta\lambda^{\beta-1}\ln\lambda\sum_{i=1}^{n}u_i^{\beta} - \lambda^{\beta-1}\sum_{i=1}^{n}u_i^{\beta} - \beta\lambda^{\beta-1}\sum_{i=1}^{n}u_i^{\beta}\ln u_i.$$

1.5.4 Profiling

The profile log likelihood of (α,β) is:

$$\ell(\alpha,\beta) \propto n\ln\beta - n\ln\Gamma(\alpha) + n\alpha\ln(n\alpha) - n\alpha\ln\left(\sum_{i=1}^{n}u_i^{\beta}\right) + \alpha\beta\sum_{i=1}^{n}\ln u_i - n\alpha.$$

Taking the partial with respect to β:

$$\mathcal{U}_{\beta}^{\dagger} = \frac{n}{\beta} - n\alpha\frac{\sum_{i=1}^{n}u_i^{\beta}\ln u_i}{\sum_{i=1}^{n}u_i^{\beta}} + \alpha\sum_{i=1}^{n}\ln u_i.$$

This equation admits a solution in α:

$$\hat{\alpha}(\beta) = \frac{1}{\beta}\left(\frac{\sum_{i=1}^{n}u_i^{\beta}\ln u_i}{\sum_{i=1}^{n}u_i^{\beta}} - \frac{1}{n}\sum_{i=1}^{n}\ln u_i\right)^{-1}.$$

Substituting the MLE for α into the profile log likelihood of (α,β) gives a profile log likelihood for β alone.

Properties

Result 2.0.1. The kth moment of the generalized gamma distribution is:

$$\mathbb{E}(T^k) = \frac{\Gamma\big(\alpha + k/\beta\big)}{\lambda^k \Gamma(\alpha)}.$$

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Proof. Writing out the kth moment:

$$\mathbb{E}(T^k) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^k \big\{(\lambda t)^\beta\big\}^\alpha e^{-(\lambda t)^\beta} (\beta\lambda)(\lambda t)^{-1} dt.$$

Make the change of variables:

$$u = (\lambda t)^\beta, \qquad du = \beta\lambda(\lambda t)^{\beta-1} dt.$$

The inverse transformation is:

$$t = u^{1/\beta}/\lambda.$$

Expressing the change of measure as:

$$u^{-1} du = (\beta\lambda)(\lambda t)^{-1} dt,$$

the integral becomes:

$$\mathbb{E}(T^k) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{u^{k/\beta}}{\lambda^k} u^\alpha e^{-u} u^{-1} du = \frac{\Gamma(\alpha + k/\beta)}{\lambda^k \Gamma(\alpha)}.$$

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Corollary 2.0.1. The mean of the generalized gamma distribution is:

$$\mathbb{E}(T) = \frac{\Gamma(\alpha + 1/\beta)}{\lambda \Gamma(\alpha)}.$$

The variance of the generalized gamma distribution is:

$$\mathbb{V}(T) = \frac{1}{\lambda^2 \Gamma(\alpha)} \left\{ \Gamma(\alpha + 2/\beta) + \frac{\Gamma^2(\alpha + 1/\beta)}{\Gamma(\alpha)} \right\}.$$

♣

2.1 First Arrival

Suppose that T follows a generalized gamma (α, β, λ) distribution, and that C follows a Weibull (β, δ) distribution. The probability that T is censored by C is:

$$\mathbb{P}(T > C) = \mathbb{E}\mathbb{P}(T > C|T) = \mathbb{E}_T\Big\{1 - e^{-(\delta T)^\beta}\Big\} = \frac{\beta\lambda}{\Gamma(\alpha)}\int_0^\infty (\lambda t)^{\alpha\beta - 1} e^{-(\lambda t)^\beta}\Big(1 - e^{-(\delta t)^\beta}\Big)dt.$$

Rearranging the integrand:

$$\mathbb{P}(T > C) = 1 - \frac{\lambda^{\alpha\beta}}{\Gamma(\alpha)}\int_0^\infty t^{\alpha\beta} e^{-\left(\lambda^\beta + \delta^\beta\right)t^\beta} \cdot \beta t^{-1} dt.$$

Let $\theta = (\lambda^\beta + \delta^\beta)$, and make the change of variables:

$$u = \theta t^\beta, \qquad du = \beta\theta t^{\beta - 1}dt.$$

The inverse transformation is:

$$t = (u/\theta)^{1/\beta},$$

and the change of measure is expressible as:

$$\beta t^{-1}dt = u^{-1}du.$$

The probability of first arrival becomes:

$$\mathbb{P}(T > C) = 1 - \frac{\lambda^{\alpha\beta}}{\Gamma(\alpha)}\int_0^\infty \frac{u^\alpha}{\theta^\alpha} e^{-u} \cdot u^{-1}du = 1 - \frac{\lambda^{\alpha\beta}}{\theta^\alpha} = 1 - \frac{\left(\lambda^\beta\right)^\alpha}{\left(\lambda^\beta + \delta^\beta\right)^\alpha}.$$

Suppose we desire to generate independent exponential censoring times C such that the probability of censoring is π_C. The required rate for the censoring distribution is:

$$\delta = \lambda\left\{\frac{1 - (1 - \pi_C)^{1/\alpha}}{(1 - \pi_C)^{1/\alpha}}\right\}^{1/\beta}.$$