Summary

- Every idempotent matrix $\boldsymbol{P}^2 = \boldsymbol{P}$ represents a projection. If, in addition, the matrix is self-adjoint $\langle \boldsymbol{P}\boldsymbol{x}, \boldsymbol{y}\rangle = \langle \boldsymbol{x}, \boldsymbol{P}\boldsymbol{y}\rangle$, then the projection is orthogonal.
- If V is expressible as the direct sum of two subspaces $V_1 \oplus V_2$, then the projection onto V is the sum of the projections onto V_1 and onto V_2.
- The orthogonal projection $\hat{\boldsymbol{u}}$ of $\boldsymbol{u}$ onto a subspace V has two defining properties: (i.) $\hat{\boldsymbol{u}} \in V$ and (ii.) $\boldsymbol{u} - \hat{\boldsymbol{u}} \in V^{\perp}$.
- Every finite dimensional subspace V of a Hilbert space $\mathcal{H}$ is closed.
- If V is a closed subspace of $\mathcal{H}$, then for any $\boldsymbol{u} \in \mathcal{H}$, there exists a unique closest element in V to $\boldsymbol{u}$, and that element is the orthogonal projection $\hat{\boldsymbol{u}}$.

Direct Sums

Definition 2.0.1. Suppose V_1 and V_2 are linear subspaces. The sum $V = V_1 + V_2$ consists of those vectors $\{\boldsymbol{v}_1 + \boldsymbol{v}_2 : \boldsymbol{v}_1 \in V_1, \boldsymbol{v}_2 \in V_2\}$. If V_1 and V_2 are linearly independent, then V is described as the **direct sum** of V_1 and V_2, written $V = V_1 \oplus V_2$. ■

Proposition 2.0.1. If $V = V_1 \oplus V_2$, then any $\boldsymbol{v} \in V$ has a unique representation $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ where $\boldsymbol{v}_1 \in V_1$ and $\boldsymbol{v}_2 \in V_2$. ♦

Proof. Since V is the sum of V_1 and V_2, there exist $\boldsymbol{v}_1 \in V_1$ and $\boldsymbol{v}_2 \in V_2$ such that $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$. Suppose that there exists another representation $\boldsymbol{v} = \boldsymbol{u}_1 + \boldsymbol{u}_2$, then:

$$\boldsymbol{0} = \boldsymbol{v} - \boldsymbol{v} = \big(\boldsymbol{v}_1 - \boldsymbol{u}_1\big) + \big(\boldsymbol{v}_2 - \boldsymbol{u}_2\big).$$

But $\boldsymbol{v}_1 - \boldsymbol{u}_1 \in V_1$ and $\boldsymbol{v}_2 - \boldsymbol{u}_2 \in V_2$. By linear independence of V_1 and V_2, $\boldsymbol{v}_1 - \boldsymbol{u}_1 = \boldsymbol{0}$ and $\boldsymbol{v}_1 - \boldsymbol{u}_1 = \boldsymbol{0}$. Conclude the representation of $\boldsymbol{v}$ is unique. ■

Definition 2.0.2. A **basis** is a collection B of *linearly independent* vectors that *span* a linear space V. ■

Proposition 2.0.2. Suppose B_1 is a basis for V_1 and B_2 is a basis for V_2, then the union $B = B_1 \cup B_2$ is a basis for $V = V_1 \oplus V_2$. ♦

Proof. Any $\boldsymbol{v} \in V$ is uniquely expressible as $\boldsymbol{v}_1 + \boldsymbol{v}_2$, where $\boldsymbol{v}_1 \in V_1$ and $\boldsymbol{v}_2 \in V_2$. Since B_1 and B_2 are bases for V_1 and V_2, $\boldsymbol{v}_1$ has a unique representation with respect to B_1, and $\boldsymbol{v}_2$ has a unique representation with respect to B_2. Thus $\boldsymbol{v}$ is in the span of $B_1 \cup B_2$. Moreover, since V_1 and V_2 are linearly independent, and B_1 and B_2 are each bases, the collection of vectors $B_1 \cup B_2$ is linearly independent. ■

Corollary 2.0.1. If $V = V_1 \oplus V_2$, then $\dim(V) = \dim(V_1) + \dim(V_2)$. ♣

Projections

Definition 3.0.1. Suppose $V = V_1 \oplus V_2$, then every $\boldsymbol{v} \in V$ has a uniform representation $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ where $\boldsymbol{v}_1 \in V_1$ and $\boldsymbol{v}_2 \in V_2$. The mapping $\Pi(\boldsymbol{v}|V_1) = \boldsymbol{v}_1$ is the **projection** of $\boldsymbol{v}$ onto $\boldsymbol{V}_1$. Likewise, $\Pi(\boldsymbol{v}|V_2) = \boldsymbol{v}_2$ is the projection of $\boldsymbol{v}$ onto V_2. ■

Proposition 3.0.1. Suppose $V = V_1 \oplus V_2$, and $\boldsymbol{v} \in V$. The projection $\Pi(\boldsymbol{v}|V_1)$ of $\boldsymbol{v}$ onto V_1 is a linear mapping. ♦

Proof. Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are in V, then $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$ and $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$, with $\boldsymbol{u}_k, \boldsymbol{v}_k \in V_k$. Consider the sum $\boldsymbol{w} \equiv \boldsymbol{u} + \boldsymbol{v} = \big(\boldsymbol{u}_1 + \boldsymbol{v}_1\big) + \big(\boldsymbol{u}_2 + \boldsymbol{v}_2\big)$. Observe that $\boldsymbol{w}_k = \boldsymbol{u}_k + \boldsymbol{v}_k \in V_k$. Since this representation is unique:

$$\Pi(\boldsymbol{u} + \boldsymbol{v}|V_1) = \Pi(\boldsymbol{w}|V_1) = \boldsymbol{w}_1 = \boldsymbol{u}_1 + \boldsymbol{v}_1 = \Pi(\boldsymbol{u}|V_1) + \Pi(\boldsymbol{v}|V_1).$$

■

Corollary 3.0.1. $\Pi(\cdot|V_1)$ is expressible as a *projection matrix* $\boldsymbol{P}$. ♣

Proposition 3.0.2. Suppose $V = V_1 \oplus V_2$, and $\boldsymbol{v} \in V$. The projection $\Pi(\boldsymbol{v}|V_1)$ mapping is **idempotent**. Consequently, $\boldsymbol{P}^2 = \boldsymbol{P}$. ♦

Proof. The projection of $\boldsymbol{v}$ onto V_1 is:

$$\Pi(\boldsymbol{v}|V_1) = \boldsymbol{v}_1 = \boldsymbol{P}\boldsymbol{v}.$$

Since v_1 already belongs to V_1, upon projecting again:

$$\boldsymbol{P}\boldsymbol{P}\boldsymbol{v} = \boldsymbol{P}\boldsymbol{v}_1 = \Pi(\boldsymbol{v}_1|V_1) = \boldsymbol{v}_1 = \boldsymbol{P}\boldsymbol{v}.$$

Since $\boldsymbol{P}^2\boldsymbol{v} = \boldsymbol{P}\boldsymbol{v}$ holds for $\forall \boldsymbol{v} \in V$, conclude that $\boldsymbol{P}^2 = \boldsymbol{P}$. ■

Proposition 3.0.3. Suppose $V = V_1 \oplus V_2$, and $\boldsymbol{v} \in V$. Let $\boldsymbol{P}$ denote projection onto $\boldsymbol{V}_1$. Then, $\boldsymbol{Q} = \boldsymbol{I} - \boldsymbol{P}$ is projection onto V_2 and $\boldsymbol{P}\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{P} = \boldsymbol{0}$. ♦

Proof. The projection onto V_2 is:

$$\Pi(\boldsymbol{v}|V_2) = \boldsymbol{v}_2 = \boldsymbol{v} - \boldsymbol{v}_1 = \boldsymbol{v} - \Pi(\boldsymbol{v}|V_1) = \boldsymbol{v} - \boldsymbol{P}\boldsymbol{v} = \big(\boldsymbol{I} - \boldsymbol{P}\big)\boldsymbol{v}.$$

Let $\boldsymbol{Q} = \boldsymbol{I} - \boldsymbol{P}$, such that $\boldsymbol{Q}\boldsymbol{v} = \boldsymbol{v} - \boldsymbol{P}\boldsymbol{v} = \boldsymbol{v} - \boldsymbol{v}_1 = \boldsymbol{v}_2 = \Pi(\boldsymbol{v}|V_2)$. By direct calculation:

$$\boldsymbol{P}\boldsymbol{Q} = \boldsymbol{P}\big(\boldsymbol{I} - \boldsymbol{P}\big) = \boldsymbol{P} - \boldsymbol{P}^2 = \boldsymbol{P} - \boldsymbol{P} = \boldsymbol{0}.$$

Likewise, $\boldsymbol{Q}\boldsymbol{P} = \big(\boldsymbol{I} - \boldsymbol{P}\big)\boldsymbol{P} = \boldsymbol{P} - \boldsymbol{P}^2 = \boldsymbol{P} - \boldsymbol{P} = \boldsymbol{0}$. Thus:

$$\Pi\big\{\Pi(\boldsymbol{v}|V_1)|V_2\big\} = \boldsymbol{0} = \Pi\big\{\Pi(\boldsymbol{v}|V_2)|V_1\big\}$$

when V_1 is linearly independent of $\boldsymbol{V}_2$, or equivalently when $V_1 \cap V_2 = \{\boldsymbol{0}\}$. ■

Proposition 3.0.4. Suppose $\boldsymbol{v} \in V$ and $\boldsymbol{P}^2 = \boldsymbol{P}$, then $V = V_1 \oplus V_2$ where $V_1 = \text{im}(\boldsymbol{P})$ and $V_2 = \text{im}(\boldsymbol{I} - \boldsymbol{P})$. ♦

Proof. Any $\boldsymbol{v}$ in V is expressible as:

$$\boldsymbol{v} = \big(\boldsymbol{I} - \boldsymbol{P} + \boldsymbol{P}\big)\boldsymbol{v} = \big(\boldsymbol{I} - \boldsymbol{P}\big)\boldsymbol{v} + \boldsymbol{P}\boldsymbol{v}.$$

Thus $V = V_2 + V_1$. To verify that V is the direct sum, it suffices to show $V_1 \cap V_2 = \{\boldsymbol{0}\}$. Suppose $\boldsymbol{u} \in V_1 \cap V_2$. Since $\boldsymbol{u} \in V_2$, there exists a coefficient vector $\boldsymbol{\alpha}$ such that $\boldsymbol{u} = (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{\alpha}$. Moreover, since $\boldsymbol{u} \in V_1$, $\boldsymbol{u} = \boldsymbol{P}\boldsymbol{u}$. Now:

$$\boldsymbol{u} = \boldsymbol{P}\boldsymbol{u} = \boldsymbol{P}(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{\alpha} = (\boldsymbol{P} - \boldsymbol{P}^2)\boldsymbol{\alpha} = (\boldsymbol{P} - \boldsymbol{P})\boldsymbol{\alpha} = \boldsymbol{0}.$$

■

Proposition 3.0.5. All eigenvalues of a projection matrix are either 0 or 1. ♦

Proof. Suppose $\boldsymbol{P}$ is a projection matrix with eigenvector $\boldsymbol{u}$, then $\boldsymbol{P}\boldsymbol{u} = \lambda\boldsymbol{u}$ for some eigenvalue λ. Left multiplying by $\boldsymbol{P}$:

$$\boldsymbol{P}^2\boldsymbol{u} = \boldsymbol{P}\lambda\boldsymbol{u} = \lambda\boldsymbol{P}\boldsymbol{u} = \lambda^2\boldsymbol{u}.$$

Yet by idempotency of $\boldsymbol{P}$, $\boldsymbol{P}^2\boldsymbol{u} = \boldsymbol{P}\boldsymbol{u} = \lambda\boldsymbol{u}$. Thus $\lambda\boldsymbol{u} = \lambda^2\boldsymbol{u}$, so $\lambda \in \{0, 1\}$. ■

Hilbert Space

Definition 4.0.1 (Hilbert Space). A **Hilbert** space $\mathcal{H}$ is a *complete linear space* with an *inner product norm*:

$$||\boldsymbol{v}||^2 = \langle \boldsymbol{v}, \boldsymbol{v} \rangle, \text{ for } \boldsymbol{v} \in \mathcal{H}.$$

Euclidean space is a finite dimensional Hilbert space. ■

4.1 Orthogonal Complement

Definition 4.1.2. The **orthogonal complement** $V^{\perp}$ to a subspace V consists of those vectors orthogonal to each element of V:

$$V^{\perp} = \big\{\boldsymbol{u} \in \boldsymbol{\Omega} : \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0 \text{ for } \forall \boldsymbol{v} \in V\big\}.$$

■

Proposition 4.1.1. The orthogonal complement $V^{\perp}$ is (always) a closed subspace. ♦

Proof. Suppose $\boldsymbol{v} \in V$. If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two vectors in $V^{\perp}$, then any linear combination $\boldsymbol{u} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2$ remains in $V^{\perp}$ since:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \alpha_1 \langle \boldsymbol{u}_1, \boldsymbol{v} \rangle + \alpha_2 \langle \boldsymbol{u}_2, \boldsymbol{v} \rangle = 0.$$

Suppose $(\boldsymbol{u}_n)$ is a Cauchy sequence in $V^{\perp}$, then $(\boldsymbol{u}_n)$ converges to some limit $\boldsymbol{u}$. By continuity of the inner product:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \lim_{n \to \infty} \langle \boldsymbol{u}_n, \boldsymbol{v} \rangle = \boldsymbol{0}.$$

■

Proposition 4.1.2. The following inclusions hold:

i. $V \subseteq V^{\perp\perp}$.

ii. If $U \subseteq V$, then $V^{\perp} \subseteq U^{\perp}$.

♦

Proof. (i.) If $\boldsymbol{v} \in V$, then $\langle \boldsymbol{v}, \boldsymbol{u} \rangle = 0$ for $\forall \boldsymbol{u} \in V^{\perp}$, therefore $\boldsymbol{v} \in V^{\perp\perp}$.

(ii.) If $\boldsymbol{v}_{\perp} \in V^{\perp}$, then $\boldsymbol{v}_{\perp}$ is orthogonal to each $\boldsymbol{v}$ in V. Now $U \subseteq V$, so $\boldsymbol{v}_{\perp}$ is orthogonal to each $\boldsymbol{u} \in U$. Conclude $\boldsymbol{v}_{\perp} \in U^{\perp}$. ■

4.2 Finite Closure

Theorem 4.2.1. Suppose V is a *finite* subspace of a Hilbert space $\mathcal{H}$, then V is complete, and therefore closed. □

Proof. Suppose V is a finite dimensional subspace with basis $\boldsymbol{B} = (\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$, then every $\boldsymbol{v} \in V$ is expressible as $\boldsymbol{v} = \alpha_1 \boldsymbol{b}_1 + \cdots + \alpha_n \boldsymbol{b}_n$ for some coefficients $(\alpha_1, \cdots, \alpha_n)$. There exists a bijection $T : V \to \mathbb{F}^n$ such that $T(\boldsymbol{v}) = (\alpha_1, \cdots, \alpha_n)$, where $\boldsymbol{F}$ is a complete field. Moreover, this mapping is linear since:

$$T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = (\alpha_1 + \beta_1, \cdots, \alpha_n + \beta_n) = (\alpha_1, \cdots, \alpha_n) + (\beta_1, \cdots, \beta_n) = T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2).$$

Suppose $(\boldsymbol{v}_n)$ is a Cauchy sequence in V. For $\epsilon > 0$, define $\delta = \epsilon / ||T||$, where:

$$||T|| = \sup_{\{\boldsymbol{v} \neq \boldsymbol{0}\}} \frac{||T(\boldsymbol{x})||}{||\boldsymbol{x}||}.$$

Since $\boldsymbol{v}_n$ is Cauchy, there exist a threshold ν such that when $n, m \geq \nu$:

$$||\boldsymbol{v}_n - \boldsymbol{v}_m|| < \frac{\epsilon}{||T||}.$$

Now for $n, m \geq \nu$:

$$||T(\boldsymbol{v}_n) - T(\boldsymbol{v}_m)|| = ||T(\boldsymbol{v}_n - \boldsymbol{v}_m)|| \leq ||T|| \cdot ||\boldsymbol{v}_n - \boldsymbol{v}_m|| = \epsilon.$$

Conclude that $\{T_n \equiv T(\boldsymbol{v}_n)\}$ is Cauchy. Since $\mathbb{F}^n$ is complete, (T_n) converges to some limit $T_0 \in \mathbb{F}^n$, and:

$$\lim_{n\to\infty} ||T(\boldsymbol{v}_n) - T_0|| = 0.$$

Since T is a bijection, there exists $\boldsymbol{v}_0 \in V$ such that $T(\boldsymbol{v}_0) = T_0$. Now:

$$0 \leq \lim_{n\to\infty} ||\boldsymbol{v}_n - \boldsymbol{v}_0|| = \lim_{n\to\infty} ||T^{-1}T(\boldsymbol{v}_n) - T^{-1}T_0|| \leq ||T^{-1}|| \lim_{n\to\infty} ||T(\boldsymbol{v}_n) - T_0|| = 0.$$

Since $(\boldsymbol{v}_n)$ converges to a limit in V, conclude that V is complete. ■

Orthogonal Projection

Definition 5.0.1. Let V denote a closed subspace of a Hilbert space $\mathcal{H}$. The **orthogonal projection** of $\boldsymbol{u} \in \mathcal{H}$ onto V is the element $\Pi_0(\boldsymbol{u}|V)$ with these properties:

i. the projection resides in V: $\Pi_0(\boldsymbol{u}|V) \in V$.

ii. the residual is orthogonal to V: $\boldsymbol{u} - \Pi_0(\boldsymbol{u}|V) \in V^{\perp}$.

■

Remark 5.0.1. The notation Π_0 distinguishes the orthogonal projection from the potentially oblique projection Π. ♦

Lemma 5.0.1.

$$||\boldsymbol{x} + \boldsymbol{y}||^2 + ||\boldsymbol{x} - \boldsymbol{y}||^2 = 2||\boldsymbol{x}||^2 + 2||\boldsymbol{y}||^2. \tag{5.0.1}$$

Proof is by direct expansion. ■

Theorem 5.0.1 (Projection Theorem). Suppose V is a closed subspace of a Hilbert space $\mathcal{H}$. For any $\boldsymbol{u} \in \mathcal{H}$, there *exists* a *unique* closest element in V to $\boldsymbol{u}$:

$$\hat{\boldsymbol{u}} = \arg\min_{\boldsymbol{v} \in V} ||\boldsymbol{v} - \boldsymbol{u}||^2,$$

and $\hat{\boldsymbol{u}}$ is the *orthogonal projection* of $\boldsymbol{u}$ onto V. □

Proof. (Existence) If $\boldsymbol{u} \in V$, then $\hat{\boldsymbol{u}} = \boldsymbol{u}$. Suppose not. Let $\delta^2 = \inf_{\boldsymbol{v} \in V} ||\boldsymbol{v} - \boldsymbol{u}||^2$. Construct a sequence $\boldsymbol{v}_n \in V$ such that:

$$||\boldsymbol{v}_n - \boldsymbol{u}||^2 \leq \delta^2 + \frac{1}{n}.$$

By (5.0.1):

$$||(\boldsymbol{v}_n - \boldsymbol{u}) + (\boldsymbol{u} - \boldsymbol{v}_m)||^2 + ||(\boldsymbol{v}_n - \boldsymbol{u}) - (\boldsymbol{u} - \boldsymbol{v}_m)||^2 = 2||\boldsymbol{v}_n - \boldsymbol{u}||^2 + 2||\boldsymbol{u} - \boldsymbol{v}_m||^2.$$

Upon rearranging:

$$\begin{aligned} ||\boldsymbol{v}_n - \boldsymbol{v}_m||^2 + 4\left|\left|\frac{\boldsymbol{v}_n + \boldsymbol{v}_m}{2} - \boldsymbol{u}\right|\right|^2 &= 2||\boldsymbol{v}_n - \boldsymbol{u}||^2 + 2||\boldsymbol{u} - \boldsymbol{v}_m||^2, \\ ||\boldsymbol{v}_n - \boldsymbol{v}_m||^2 &= 2||\boldsymbol{v}_n - \boldsymbol{u}||^2 + 2||\boldsymbol{u} - \boldsymbol{v}_m||^2 - 4\left|\left|\frac{\boldsymbol{v}_n + \boldsymbol{v}_m}{2} - \boldsymbol{u}\right|\right|^2. \end{aligned}$$

Now, for all $(n, m) \in \mathbb{N}^2$, the midpoint $\boldsymbol{\mu}_{nm} = (\boldsymbol{v}_n + \boldsymbol{v}_m)/2$ is in V since V is convex. Therefore, $||\boldsymbol{\mu}_{nm} - \boldsymbol{u}||^2 \geq \delta^2$, and:

$$||\boldsymbol{v}_n - \boldsymbol{v}_m||^2 \leq 2||\boldsymbol{v}_n - \boldsymbol{u}||^2 + 2||\boldsymbol{v}_m - \boldsymbol{u}||^2 - 4\delta^2.$$

By construction $\lim_{n\to\infty} ||\boldsymbol{v}_n - \boldsymbol{u}||^2 = \delta^2$, thus:

$$\lim_{(n,m)\to\infty} ||\boldsymbol{v}_n - \boldsymbol{v}_m||^2 \leq 2\delta^2 + 2\delta^2 - 4\delta^2 = 0.$$

Conclude that $(\boldsymbol{v}_n)$ is a Cauchy sequence. Since V is a closed subspace of a Hilbert space, $(\boldsymbol{v}_n)$ converges to a limit in V. Call this limit $\hat{\boldsymbol{u}}$. By continuity of the norm:

$$||\hat{\boldsymbol{u}} - \boldsymbol{u}||^2 = \lim_{n\to\infty} ||\boldsymbol{v}_n - \boldsymbol{u}||^2 \leq \lim_{n\to\infty} \left(\delta^2 + \frac{1}{n}\right) = \delta^2.$$

Moreover since $\hat{\boldsymbol{u}} \in V$, $||\hat{\boldsymbol{u}} - \boldsymbol{u}||^2 \geq \delta^2$ by definition of the infimum. Conclude that $||\hat{\boldsymbol{u}} - \boldsymbol{u}||^2 = \delta^2$, and therefore $\hat{\boldsymbol{u}} = \arg\min_{\boldsymbol{v} \in V} ||\boldsymbol{v} - \boldsymbol{u}||^2$.

(Orthogonal projection) Since $\hat{\boldsymbol{u}}$ is in V, to establish $\hat{\boldsymbol{u}}$ is the orthogonal projection of $\boldsymbol{u}$ onto V, it remains to show that $\boldsymbol{u} - \hat{\boldsymbol{u}}$ is in $V^\perp$. Suppose to the contrary that there exists $\boldsymbol{v}$ in V not orthogonal to $\boldsymbol{u} - \hat{\boldsymbol{u}}$. wlog let $||\boldsymbol{v}||^2 = 1$, and set $\theta = \langle \boldsymbol{u} - \hat{\boldsymbol{u}}, \boldsymbol{v} \rangle$. Define $\tilde{\boldsymbol{u}} = \theta \boldsymbol{v} + \hat{\boldsymbol{u}}$, which is in V. Then:

$$\begin{aligned} ||\tilde{\boldsymbol{u}} - \boldsymbol{u}||^2 = ||\theta \boldsymbol{v} + \hat{\boldsymbol{u}} - \boldsymbol{u}||^2 &= ||\theta \boldsymbol{u}||^2 + ||\hat{\boldsymbol{u}} - \boldsymbol{u}||^2 + 2\theta\langle \boldsymbol{v}, \hat{\boldsymbol{u}} - \boldsymbol{u} \rangle \\ &= \theta^2 + ||\hat{\boldsymbol{u}} - \boldsymbol{u}||^2 - 2\theta^2 = ||\hat{\boldsymbol{u}} - \boldsymbol{u}||^2 - \theta^2 \leq ||\hat{\boldsymbol{u}} - \boldsymbol{u}||^2. \end{aligned}$$

Thus, if $\boldsymbol{u} - \hat{\boldsymbol{u}}$ is not in $V^\perp$, then $\hat{\boldsymbol{u}}$ is not the closest point in V to $\boldsymbol{u}$. Since $\hat{\boldsymbol{u}} \in V$ and $\boldsymbol{u} - \hat{\boldsymbol{u}} \in V^\perp$ if $\hat{\boldsymbol{u}} = \arg\min_{\boldsymbol{v} \in V} ||\boldsymbol{v} - \boldsymbol{u}||^2$, conclude that $\hat{\boldsymbol{u}} = \Pi_0(\boldsymbol{u}|V)$.

(Uniqueness) Suppose $\hat{\boldsymbol{u}}_1$ and $\hat{\boldsymbol{u}}_2$ are both orthogonal projections of $\boldsymbol{u}$ onto V, then $\boldsymbol{u} - \hat{\boldsymbol{u}}_1 \in V^\perp$ and $\boldsymbol{u} - \hat{\boldsymbol{u}}_2 \in V^\perp$. Thus $\big(\boldsymbol{u} - \hat{\boldsymbol{u}}_1\big) - \big(\boldsymbol{u} - \hat{\boldsymbol{u}}_2\big) = \hat{\boldsymbol{u}}_2 - \hat{\boldsymbol{u}}_1 \in V^\perp$. But $\hat{\boldsymbol{u}}_2 \in V$ and $\hat{\boldsymbol{u}}_1 \in V$, so $\hat{\boldsymbol{u}}_2 - \hat{\boldsymbol{u}}_1 \in V \cap V^\perp = \{\boldsymbol{0}\}$. Conclude $\hat{\boldsymbol{u}}_1 = \hat{\boldsymbol{u}}_2$. ∎

Corollary 5.0.1. If V is a closed linear subspace of a Hilbert space $\mathcal{H}$, then:

$$\mathcal{H} = V \oplus V^{\perp}.$$

♣

Proposition 5.0.1. Suppose V is a closed subspace of Hilbert space $\mathcal{H}$ and that V is the direct sum of two orthogonal subspaces: $V = V_0 \oplus V_1$ with $V_0 \perp V_1$. Then, for any $\boldsymbol{u} \in \mathcal{H}$, the projection onto V is the sum of the projections onto V_0 and onto V_1:

$$\Pi_0(\boldsymbol{u}|V) = \Pi_0(\boldsymbol{u}|V_0) + \Pi_0(\boldsymbol{u}|V_1).$$

♦

Proof. Let $\hat{\boldsymbol{u}}_0$ and $\hat{\boldsymbol{u}}_1$ denote orthogonal projection onto V_0 and V_1. Consider the sum $\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}_0 + \hat{\boldsymbol{u}}_1$. Since $V = V_0 \oplus V_1$, $\hat{\boldsymbol{u}} \in V$. Moreover, for any $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1 \in V$:

$$\begin{aligned}\langle \boldsymbol{u} - \hat{\boldsymbol{u}}, \boldsymbol{v} \rangle &= \langle \boldsymbol{u} - \hat{\boldsymbol{u}}_0 - \hat{\boldsymbol{u}}_1, \boldsymbol{v}_0 + \boldsymbol{v}_1 \rangle \\ &= \langle \boldsymbol{u} - \hat{\boldsymbol{u}}_0, \boldsymbol{v}_0 \rangle - \langle \hat{\boldsymbol{u}}_1, \boldsymbol{v}_0 \rangle + \langle \boldsymbol{u} - \hat{\boldsymbol{u}}_1, \boldsymbol{v}_1 \rangle - \langle \hat{\boldsymbol{u}}_0, \boldsymbol{v}_1 \rangle = \boldsymbol{0}.\end{aligned}$$

Since $\hat{\boldsymbol{u}} \in V$ and $\boldsymbol{u} - \hat{\boldsymbol{u}} \in V^{\perp}$, $\hat{\boldsymbol{u}}$ is the orthogonal projection onto V. ■

Proposition 5.0.2. Suppose $\boldsymbol{P}$ is the matrix for orthogonal projection onto a closed linear subspace V. $\boldsymbol{P}$ is **self-adjoint**:

$$\langle \boldsymbol{P}\boldsymbol{u}_1, \boldsymbol{u}_2 \rangle = \langle \boldsymbol{P}\boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2 \rangle = \langle \boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2 \rangle.$$

♦

Proof. Since $\boldsymbol{u}_2 = \boldsymbol{P}\boldsymbol{u}_2 + (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{u}_2$ with $\boldsymbol{P}\boldsymbol{u}_2 = \hat{\boldsymbol{u}}_2 \in V$ and $(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{u}_2 = \boldsymbol{u}_2 - \hat{\boldsymbol{u}}_2 \in V^{\perp}$,

$$\langle \boldsymbol{P}\boldsymbol{u}_1, \boldsymbol{u}_2 \rangle = \langle \boldsymbol{P}\boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2 + (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{u}_2 \rangle = \langle \boldsymbol{P}\boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2 \rangle.$$

Writing $\boldsymbol{u}_1 = \boldsymbol{P}\boldsymbol{u}_1 + (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{u}_1$ likewise gives:

$$\langle \boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2 \rangle = \langle \boldsymbol{P}\boldsymbol{u}_1 + (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2 \rangle = \langle \boldsymbol{P}\boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2 \rangle.$$

■

Corollary 5.0.2. For Euclidean space:

$$\boldsymbol{u}_1'\boldsymbol{P}'\boldsymbol{u}_2 = \langle \boldsymbol{P}\boldsymbol{u}_1, \boldsymbol{u}_2 \rangle = \langle \boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2 \rangle = \boldsymbol{u}_1'\boldsymbol{P}\boldsymbol{u}_2.$$

That is, $\boldsymbol{P}$ is symmetric. ♣

5.1 Euclidean Space

Proposition 5.1.3. Suppose $V_0 \subseteq V$ are closed linear subspaces of Euclidean space. Let $\boldsymbol{P}_0$ and $\boldsymbol{P}$ denote orthogonal projection onto V_0 and V, then:

$$\boldsymbol{P}\boldsymbol{P}_0 = \boldsymbol{P}_0 = \boldsymbol{P}_0\boldsymbol{P}.$$

That is, projection onto V_0 is equivalent to projection onto V, followed by projection onto $V_0 \subseteq V$. The operator $\boldsymbol{P}_0\boldsymbol{P}$ is an *iterated projection.* ♦

Proof. For any $\boldsymbol{v} \in \mathcal{H}$, since $\boldsymbol{P}_0\boldsymbol{v} \in V_0 \subseteq V$, $\boldsymbol{P}(\boldsymbol{P}_0\boldsymbol{v}) = \boldsymbol{P}_0\boldsymbol{v}$. Thus, $\boldsymbol{P}\boldsymbol{P}_0 = \boldsymbol{P}_0$. Further, since $\boldsymbol{P}_0$ and $\boldsymbol{P}$ are symmetric in Euclidean space:

$$\boldsymbol{P}_0 = \boldsymbol{P}_0' = (\boldsymbol{P}\boldsymbol{P}_0)' = \boldsymbol{P}_0'\boldsymbol{P}' = \boldsymbol{P}_0\boldsymbol{P}.$$

■

Proposition 5.1.4. Suppose $V_0 \subseteq V$ are closed linear subspaces of Euclidean space. Let $\boldsymbol{P}_0$ and $\boldsymbol{P}$ denote orthogonal projection onto V_0 and V. Define V_1 as the subspace of V that is orthogonal to V_0, $V_1 = V_0^{\perp} \cap V$. Then:

i. $V = V_0 \oplus V_1$ with $V_0 \perp V_1$.

ii. $\boldsymbol{P}_1 = \boldsymbol{P} - \boldsymbol{P}_0$, where $\boldsymbol{P}_1$ is orthogonal projection onto V_1.

♦

Proof. (i.) Suppose $\boldsymbol{v} \in V$, and let $\hat{\boldsymbol{v}}_0 = \boldsymbol{P}_0\boldsymbol{v}$. Since $\boldsymbol{v} - \hat{\boldsymbol{v}}_0 \in V$ and $\boldsymbol{v} - \hat{\boldsymbol{v}}_0 \in V_0^{\perp}$, $\boldsymbol{v} - \hat{\boldsymbol{v}}_0 \in V_1$. Writing $\boldsymbol{v} = \hat{\boldsymbol{v}}_0 + (\boldsymbol{v} - \hat{\boldsymbol{v}}_0)$ demonstrates $\boldsymbol{v} \in V_0 + V_1$. Conversely, since $V_0 \subseteq V$ and $V_1 \subseteq V$, $V_0 + V_1 \subseteq V$. Moreover, since $V_1 = V_0^{\perp} \cap V \subseteq V_0^{\perp}$, $V_1 \perp V_0$, and the sum $V = V_0 \oplus V_1$ is direct.

(ii.) Since V is the direct sum of two orthogonal subspaces, the orthogonal projection onto V is expressible as $\boldsymbol{P} = \boldsymbol{P}_0 + \boldsymbol{P}_1$. ■

Example 5.1.1. Suppose $\mathbb{E}^n$ is Euclidean space and $V \subset \mathbb{E}^n$ is a linear subspace, then $\mathbb{E}^n$ is expressible as $V \oplus V^{\perp}$. Every $\boldsymbol{u} \in \mathbb{E}^n$ is uniquely expressible as $\hat{\boldsymbol{u}} + \boldsymbol{u}_{\perp}$, where $\hat{\boldsymbol{u}} = \boldsymbol{P}\boldsymbol{u}$ is the orthogonal projection onto V, and $\boldsymbol{v}_{\perp} = (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{u}$ is the orthogonal projection onto $V^{\perp}$. The orthogonal projections are guaranteed to exist to by closure of V and $V^{\perp}$. Moreover, the two projections are orthogonal $\langle \hat{\boldsymbol{u}}, \boldsymbol{u}_{\perp} \rangle = 0$. ♠