Mathematical Analysis of the Buoyant Force in Non-Newtonian Fluids

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Abstract

This paper details the construction of a physical apparatus for measuring the position of an object submerged in liquid rising due to the buoyant force. Physical data is then analyzed alongside a differential equation derived from Newton's Second Law, which when solved, models the position of an object rising due to the buoyant force as a function of time. Subsequently, these methods are extended to the analysis of the buoyant force inside oobleck, a dilatant non-Newtonian fluid. A revised apparatus is constructed to allow for physical observation in spite of oobleck's opaque nature. Mathematical and physical theory to explain the abnormal behavior of oobleck's buoyant force is proposed and analyzed using second order differential equations.

Keywords: buoyancy, Archimedes' Principle, non-Newtonian, viscosity, differential equations

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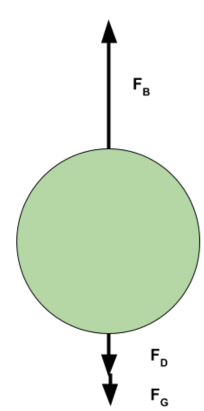
Establishing a Baseline Model with Water

Proposed by Greek Mathematician Archimedes of Syracuse, Archimedes' Principle is the foundation of modeling the position of objects submerged in fluids of higher density. It states that the buoyant force on an object is equal to the weight of the fluid it displaces:

$$F_B = -\rho g V$$

where ρ is the fluid density, g is the gravitational constant (9.81), and V is the volume of the object submerged in the fluid.

Force Diagram



When an object rises due to the buoyant force, it is subject to resistance in the form of drag. In this case, we will use Stoke's Law to approximate drag of our spherical object:

$$F_D = 6\pi R\eta v$$

where v is the velocity of the object, R is the radius, and n is viscosity.

In addition to the buoyant and drag forces, objects submerged in fluid continue to be subject to the gravitational force:

$$F_G = mg$$

where m is the mass of the object and g is the gravitational constant.

With these two forces established, one can apply Newton's Second Law:

$$\Sigma F \ = \ ma$$

$$F_B + \ F_G + F_D = \ ma$$

$$\rho gV \ - \ mg \ - \ 6\pi R\eta v \ = \ ma$$

Recognizing that our acceleration is constrained to movement in the vertical direction, we can substitute a for the second derivative of position (y) with respect to time and v for the derivative of y with respect to time:

$$\rho gV \ - \ mg \ - \ 6\pi R\eta\frac{dy}{dt} \ = \ m\frac{d^2y}{dt^2}$$

To find values for the constants in this equation, we must introduce the physical parameters of our tangible model. As shown in the image of *Apparatus I* on the right, our submerged object is a ping pong ball, providing a mass and volume of 0.0027 kg and 0.000033510 m^3. Also apparent in the image on the right, water is used to establish a baseline, so ρ = 997 kg/m^3. The radius of the ping pong ball is 0.02 m. The viscosity of water is 0.01 poise. Plugging these constants into our equation:

$$0.3277 \ - \ 0.0265 \ - 0.0038(\frac{dy}{dt}) \ = \ 0.0027\frac{d^2y}{dt^2}$$

Adding our initial conditions of zero position and velocity at our starting point, we establish the following initial value problem:

$$0.0027y'' \ + \ 0.0038y' \ = 0.3012; \ y'(0) \ = \ 0; \ y(0) \ = \ 0$$

Rewriting in standard form:

$$y'' \ + \ 1.4074y' \ = \ 111.5556; y'(0) \ = \ 0; \ y(0) \ = \ 0$$

The ODE above can be classified as linear, autonomous, and non-homogeneous. To solve, one should transform the second order ODE into a first order system.

$$\text{Let } y_1 = \ y \ and \ y_2 = y'$$

$$Y' = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y_2 \\ -1.4074y_2 \end{bmatrix}$$

$$Y' = AY + B = \begin{bmatrix} 0 & 1 \\ 0 & -1.4074 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 111.5556 \end{bmatrix} \qquad Y(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now we must find the eigenvalues and eigenvectors to construct the Wronskian matrix.

$$determinant \begin{bmatrix} -\lambda & 1 \\ 0 & -1.4074-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 + 1.4074\lambda = 0 \Rightarrow \lambda = -1.4074, 0$$

$$\begin{bmatrix} 1.4074 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_{11} = 1, v_{12} = -1.4074$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -1.4074 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_{21} = 1, v_{22} = 0$$

$$W = \begin{bmatrix} e^{-1.4074t} & 1 \\ -1.4074e^{-1.4074t} & 0 \end{bmatrix}$$

Using the Wronskian matrix, we can find our homogeneous solution.

$$y_h = c_1e^{-1.4074t} + c_2$$

To find our particular solution, we can deploy the variation of arbitrary coefficients formula:

$$Y_p = W(t) \int^t W^{-1}(s)B(s)\, ds$$

$$W^{-1} = \frac{1}{1.4074e^{-1.4074t}} \begin{bmatrix} 0 & -1 \\ 1.4074e^{-1.4074t} & e^{-1.4074t} \end{bmatrix} = \begin{bmatrix} 0 & -0.7105e^{1.4074t} \\ 1 & 0.7105 \end{bmatrix}$$

$$W^{-1}(s)B(s) = \begin{bmatrix} 0 & -0.7105e^{1.4074s} \\ 1 & 0.7105 \end{bmatrix} \begin{bmatrix} 0 \\ 111.5556 \end{bmatrix} = \begin{bmatrix} -79.2603e^{1.4074s} \\ 79.2603 \end{bmatrix}$$

$$Y_p = W(t) \int^t W^{-1}(s)B(s)\, ds = \begin{bmatrix} e^{-1.4074t} & 1 \\ -1.4074e^{-1.4074t} & 0 \end{bmatrix} \begin{bmatrix} -56.3168e^{1.4074t} \\ 79.2603t \end{bmatrix}$$

$$y_p = 79.2603t - 56.3168$$

Putting it all together:

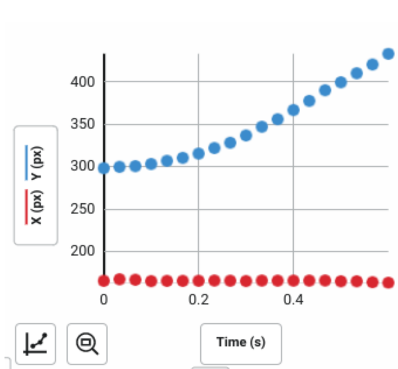
$$y = c_1e^{-1.4074t} + 72.2603t + c_2$$

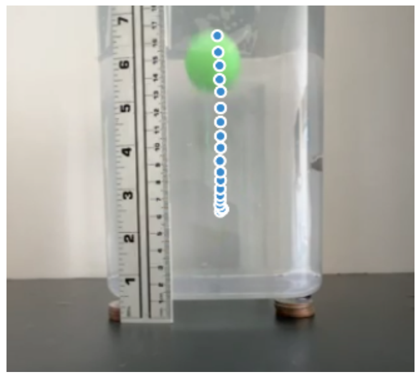
$$y' = -1.4074c_1e^{-1.4074t} + 72.2603 \Rightarrow 0 = -1.4074c_1 + 72.2603 \Rightarrow c_1 = 51.34$$

$$y = 51.34e^{-1.4074t} + 72.2603t + c_2 \Rightarrow 0 = 51.34 + c_2 \Rightarrow c_2 = -51.34$$

$$y = 51.34e^{-1.4074t} + 72.2603t - 51.34$$

Now that this theoretical model has been established, we can turn our attention to the physical model. At the initial equilibrium, the ping pong ball is held in place by a string of negligible mass that is sandwiched between magnets on opposite sides of the tank. The magnet on the bottom is attached to a string, which when pulled, releases the tension holding the ball in place, thus setting the ball on its upward path. In the following images and graphs, this process is documented using slow motion video at 240 fps and processed using video analysis software.





The image on the left shows pixel height as a function of time, and one can see that velocity increases initially, but seems to level off towards the end. While the coefficients used in the theoretical model cause it to not align completely with the experimental values, the general form of the curve is consistent. As viewed in the equation of y as a function of time above, the exponential term causes initial upward concavity that is later dominated by the linear term.

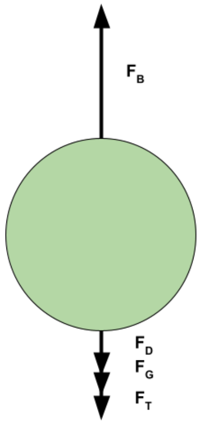
Oobleck

Named by Dr. Seuss, oobleck is a 3:2 mixture of cornstarch and water. The same properties that make the substance of interest to young kids and Dr. Seuss make oobleck a scientific anomaly. From a molecular standpoint, cornstarch particles are long and thin, causing them to tangle and make the substance behave like a solid when exposed to high amounts of stress. For example, if one were to form a fist and attempt to puncture the oobleck, it will feel solid on impact but will become less viscous and thus more liquid-like as pressure is continuously applied. Due to this tendency, oobleck is classified as a dilatant non-Newtonian fluid, meaning that its viscosity increases as stress increases. The batches of oobleck used in this experiment are displayed on the right. The density of the oobleck used in this experiment is 1195.5 kg/m^3.

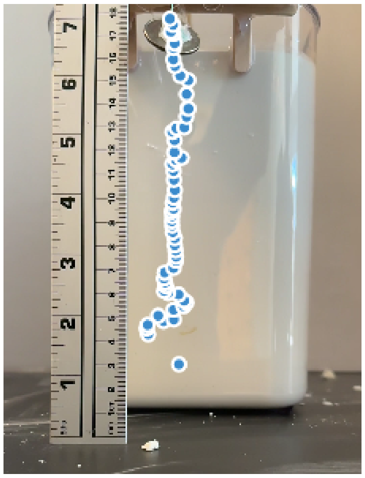
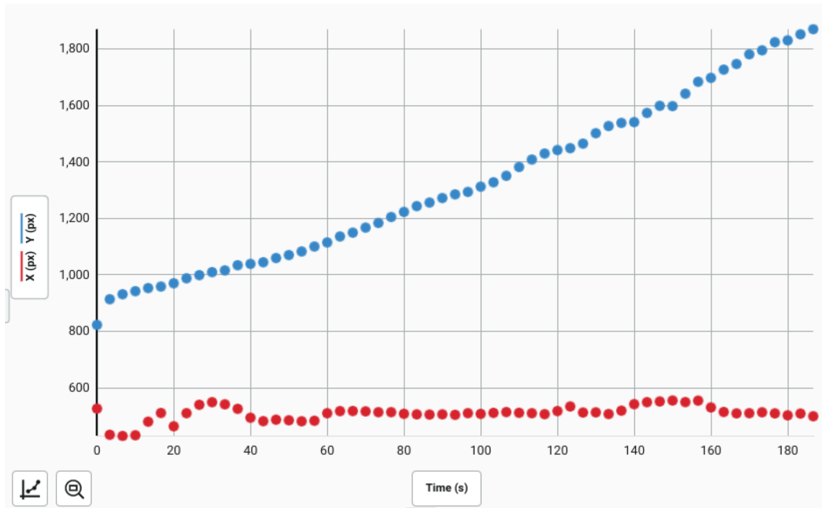
A Revised Apparatus

Oobleck's lack of transparency necessitates the construction of an apparatus that allows one to measure the change in the ball's position without being able to see the ball. *Apparatus II*, visible on the right, uses a pulley system to transfer the translational motion of the ball to an object hanging outside of the tank. The system contains three pulleys, one at the bottom of the tank, one directly above it, and another hanging above the ledge of the tank. Dental floss is utilized as a lightweight wire traveling on the pulleys. On one end of the wire is the ping pong ball and on the opposite end, hanging outside the tank, is a quarter. As the ball rises, the quarter rises at the same speed. For the sake of this experiment, the pulleys are analyzed as if they are ideal pulleys. However, there is likely a significant loss due to friction along the pulleys. Even with the assumption that the pulleys are ideal, one still must consider the force of tension, that is the downward force the string exerts on the ball, when constructing a model using Newton's Second Law. In this case, the tension force is equal to gravitational force exerted on the quarter.

Force Diagram



Data

To record data using *Apparatus II*, the quarter is held in place such that the ball is situated at the bottom of the tank. To launch a trial, the quarter is released and then the position is charted, as shown in the image above on the left. The pixel position is charted on the right, with y position is blue, x position in red, and time on the x axis. As one can see in the image and red points, there is increased variance in the x position due to the rotational motion of the quarter. Since this energy is transferred during the release of the quarter, its effects are considered negligible in the study of vertical position. Concerning the curve of vertical position with respect to time, overall, the curve is slightly concave up. One of its most distinguishing features is the sharp rise relative to the rest of the curve between the first and second data points. Additionally, it is worth noting that while there is an overall trend of slightly upwards concavity, the slope of the curve is very oscillatory. While the slope never becomes especially large, it appears to follow cycles of varying length and increasing amplitude where the slope bottoms out close to 0. Lastly, it is important to note that the ball took over 3 minutes to rise compared to the fraction of a second it took for the ball to rise in water.

Proposed Explanations and Corresponding Differential Equations

1. Immediate Jump

   To explain the immediate rise in vertical position, one need not look further than tangible observation. As previously discussed, in the event that an object impacts oobleck,

oobleck will behave as a solid. However, the entanglement of corn starch particles that initiates this extremely high viscosity behavior is not instantaneous or completely immune to impact. That being said, an object that attempts to move through oobleck, like the ping pong ball in this experiment, is able to make some initial headway before being brought to a halt due to oobleck's variable viscosity, thus explaining the immediate jump in vertical position.

2. Streamer Theory

As shown in the image on the right, a solid-like stream of oobleck follows the ping pong ball as it travels. While this image is taken outside of the body of oobleck, the same phenomenon occurs while the ping pong ball is submerged in the substance. To model the effects of this "streamer", one can treat it as additional mass that forms as time increases. For simplicity's sake, we will take drag out of the equation for this analysis.

$$\text{Let } m = 0.0027 + 0.01t$$

Like we did for the baseline with water, we will next derive a second order differential equation using Newton's Second Law.

$$\Sigma F = ma = F_B + F_G + F_T$$

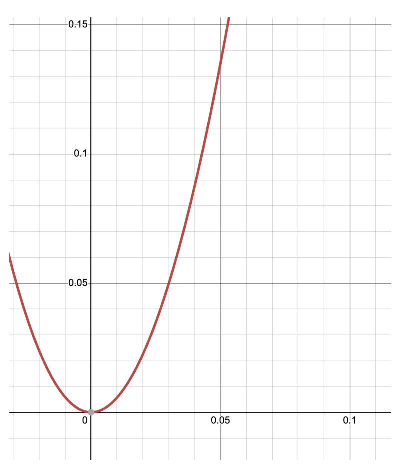
$$ma = \rho g V - mg - m_{quarter} g$$

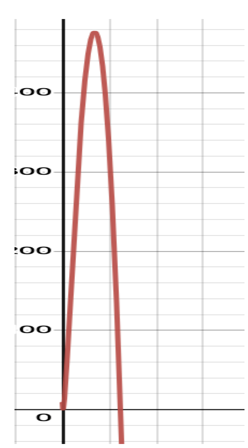
$$ma = (1195.5)(9.81)(0.00003351) - m(9.81) - 0.005671(9.81)$$

$$\frac{d^2y}{dt^2} = \frac{0.3374}{m} - 9.81$$

$$y'' = \frac{0.3374}{0.0027 + 0.01t} - 9.81;\ y'(0) = 0;\ y(0) = 0$$

The ODE above is linear, non-homogeneous, and non-autonomous. The absence of y and y' terms means that it can be solved simply by integrating twice. This procedure is not included for brevity purposes.





$$y(t) =$$

$$33.74t\ln(100t + 27) + 9.1098\ln(100t + 27) - 4.905t^2 - 144.9415t - 30.0244$$

3. Continuous Stratification

Previously, oobleck was assumed to have constant density. However, considering that it is made up of two substances of differing density and the cornstarch doesn't dissolve in water, it is worthwhile to consider the effects of treating oobleck as a continuously stratified fluid. If we treat the oobleck as linearly stratified, then the symmetry of the ping pong ball allows us to represent the average density of the fluid surrounding the ball in terms of vertical position. The following expression for density is somewhat arbitrarily defined to study the proposed phenomenon.

$$\rho = 1445.5 - (500/0.08)y$$

Using this expression, we can rewrite our equation, once again not including drag- this time to focus on the effects of stratification on the buoyant force.

$$\Sigma F = ma = F_B + F_G + F_T$$

$$ma = \rho gV - mg - m_{quarter}g$$

$$ma = (1445.5 - (500/0.08)y)(9.81)(0.00003351) - 0.0027(9.81) - 0.005671(9.81)$$

$$0.0027y'' = -2.055y + 0.4785 - 0.02649 - 0.05563$$

$$y'' = -761.1111y + 146.8$$

$$y'' + 761.1111y = 146.8;\ y'(0) = 0;\ y(0) = 0$$

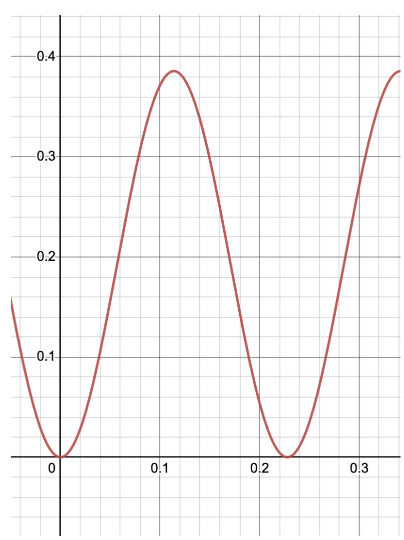
Note that this equation takes a similar form to the model of the ball rising in water with drag. This means that the stratification has an initial effect similar to that of drag- the decreased weight of the fluid displaced over time decreases the buoyant force. Below, the equation is solved using Laplace Transform.

$$L[y'' + 761.1111y](s) = L[146.8](s)$$

$$s^2F(s) + 761.1111F(s) = \frac{146.8}{s}$$

$$(s^2 + 761.1111)F(s) = \frac{146.8}{s}$$

$$F(s) = \frac{146.8}{s(s^2+761.1111)}$$



$$y(t) = L^{-1}\left[\frac{0.1929}{s} - \frac{0.1929s}{s^2+761.11111}\right](t)$$

$$y(t) = 0.1929 - 0.1929cos(27.5882t)$$

The oscillatory behavior of the solution occurs since eventually the gravitational force on the object becomes greater than the weight of the thinning fluid, causing the object to sink. Inside the apparatus used in this experiment, however, the density of the fluid will never become low enough that the object will sink.

4. Oscillatory Drag

   As mentioned in the results of the oobleck experiment, the velocity of the ball (and thus the quarter) appears to oscillate over time. In an attempt to model this phenomenon, one can introduce a drag term that has a sinusoidal component to our force equation. We will use constant mass and density in this analysis.

$$F_D = (1 + sin(t))v$$

$$\Sigma F = ma = F_B + F_G + F_T + F_D$$

$$ma = \rho gV - mg - m_{quarter}g - (1 + sin(t))v$$

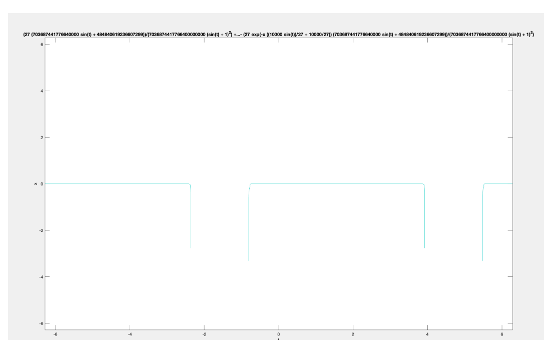
$$0.0027a = (1195.5)(9.81)(0.00003351) - 0.0027(9.81) - 0.005671(9.81) - (1 + sin(t))v$$

$$0.0027y'' = 0.393 - 0.0264 - 0.0556 - (1 + sin(t))y'$$

$$0.0027y'' = 0.311 - (1 + sin(t))y'$$

$$y'' + \frac{(1+sin(t))}{0.0027}y' = 115.1852;\ y'(0) = 0;\ y(0) = 0$$

While finding a solution of this equation is non-trivial, we can model it using numerical methods. The following MATLAB plot shows vertical position as a function of time.



Discussion

On their own, none of these propositions offer a clear-cut explanation of the behavior of the buoyant force in dilatant non-Newtonian fluids like oobleck. However, taken together, these analyses start to paint a picture of the mathematical and physical principles governing the abnormal behavior of an object rising in oobleck.

The concept of an immediate jump explains the sudden rise in vertical position once the ball is released from equilibrium. The ideas of "streamer theory" and continuous stratification start to explain why the rise of the ball takes such an extended period of time and with concavity less clearly upwards than the water model. Lastly, the idea of oscillatory behavior attempts to offer an explanation for the oscillation in velocity as the ball rises in oobleck.

To offer a more complete explanation of the behavior of an object as it rises through oobleck, one will need access to more complex research tools, including an ultrasound and machine learning and computer vision software designed to recognize patterns in the movement of oobleck.

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6. Additional tools used: Vernier Video Analysis, MATLAB, Desmos Graphing Calculator