DERIVING THE RIEMANN-SIEGEL FORMULA WITH THE METHOD OF STEEPEST DESCENT

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ABSTRACT. This paper presents a high-level derivation of the Riemann-Siegel formula using the method of steepest descent. Beginning with foundational integral representations, we apply asymptotic analysis techniques to systematically develop the formula. The significance of this approach within analytic number theory and its implications for computational number theory are discussed.

1. Introduction

The Riemann-Siegel formula provides a practical and efficient way to numerically evaluate the Riemann zeta function, $\zeta(s)$, especially along the critical line $\Re(s) = \frac{1}{2}$. Originating from Bernhard Riemann's seminal work, the formula was discovered and refined by Carl Ludwig Siegel. In this paper, we derive this formula using the method of steepest descent, highlighting its analytic power and elegance.

2. Background

2.1. The Riemann Zeta Function.

We recall the classical definition of the zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1. \tag{1}$$

Using Euler's Product Formula, one can write the zeta function as a sum over the primes:

$$\zeta(s) = \sum_{p} \frac{1}{1 - \frac{1}{p^s}} \tag{2}$$

Furthermore, a more detailed investigation (see Section 1.17 of [1]) shows that the zeros of the zeta function are directly linked to the distribution of the primes through Riemann's explicit formula:

$$\pi(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}\big(x^{\rho}\big) + \int_{x}^{\infty} \frac{dt}{t\,(t^2-1)\,\ln t} - \ln 2, \tag{3}$$

Here, $\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}$ is the logarithmic integral, the sum is over all non-trivial zeros ρ of $\zeta(s)$, and the integral from x to infinity together with the term $-\ln 2$ takes into account the pole at $s = 1$ and the trivial zeros.

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2.2. **The Generalized Factorial Function $\boldsymbol{\Pi(s) = \Gamma(s+1)}$.** We also recall that the factorial function generalized to the complex numbers can be represented as follows:

$$\Pi(s) = \int_0^\infty e^{-x}x^s dx, \quad \Re(s) > -1. \tag{4}$$

$\Pi(s)$ has several properties relevant to the subsequent derivations:

Proposition 2.1. *Properties of the generalized factorial function.*

(1) $\Pi(s) = \prod_{n=1}^\infty \frac{n^{1-s}(n+1)^s}{s+n} = \prod_{n=1}^\infty (1+\frac{s}{n})^{-1}(1+\frac{1}{n})^s$
(2) $\Pi(s) = s\Pi(s-1)$
(3) $\sin(\pi s) = \frac{\pi s}{\Pi(s)\Pi(-s)}$
(4) $\Pi(s) = 2^s\Pi(s/2)\Pi(\frac{s-1}{2})\pi^{-\frac{1}{2}}$
(5) $\int_0^\infty e^{-nx}x^{s-1}dx = \frac{\Pi(s-1)}{n^s}$

For $\Re(s) > 1$, write

$$\frac{1}{e^x-1} = \frac{e^{-x}}{1-e^{-x}} = \sum_{n=1}^\infty e^{-nx}, \qquad \int_0^\infty x^{s-1}e^{-nx}\,dx = \frac{\Pi(s-1)}{n^s}.$$

Thus

$$\begin{aligned}\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx &= \sum_{n=1}^\infty \int_0^\infty x^{s-1}e^{-nx}\,dx \\ &= \sum_{n=1}^\infty \frac{\Pi(s-1)}{n^s} = \Pi(s-1)\sum_{n=1}^\infty \frac{1}{n^s} = \Pi(s-1)\,\zeta(s).\end{aligned} \tag{5}$$

2.3. **Method of Steepest Descent.** The asymptotic integration technique of the Method of Steepest Descent ([3], [4]) is motivated by a desire to avoid oscillatory integrals, and instead (using the path independence granted by Cauchy's Theorem), select a contour with no x-dependent oscillations, yielding exponentially decaying Laplace type integrals. If this contour is selected to pass through saddle points ($\rho'(x^*) = 0$), then the integral confined to a small neighboring region of the saddle point dominates the global behavior as the large parameter is no longer exponentially suppressed.

In practice, given an integral $I(t) = \int_C h(x)e^{t\rho(x)}dx$, $t \to \infty$, where C denotes a contour in the complex-x plane, where $x = u+iv$ and $h(x) = f(u,v)+ig(u,v)$ and $\rho(x) = \phi(u,v) + i\psi(u,v)$ are complex analytic functions, one can deploy the Method of Steepest Descent by:

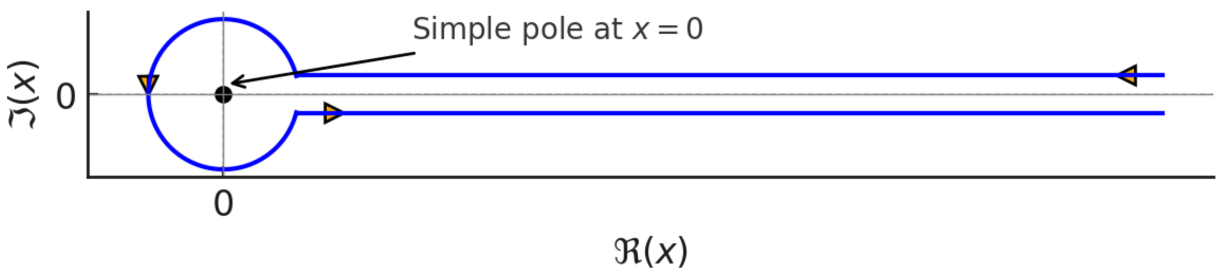
(1) Writing $I(t) = \int_C h(x)e^{t\phi}e^{it\psi}dx$ (generalized phase),
(2) Finding $\phi(u,v) = \Re(\rho(u,v)), \quad \psi(u,v) = \Im(\rho(u,v))$,
(3) Locating the saddle points: $\rho'(x^*) = 0$,
(4) Determining the possible contours by setting $\psi = c, \quad c \in \mathbb{R}$ and solving for c s.t. the contour passes through the saddle points,
(5) Expanding $\rho(x)$ about the saddle points using its Taylor Expansion,
(6) Identifying the descent sectors in the neighborhood of the saddle points,
(7) Integrating in the neighborhood of the saddle points along the tangent ray in the descent region, and
(8) Detecting any poles within the contour and applying the residue theorem to account for their contribution.

3. Brief Derivation of the Functional Representation of $\zeta(s)$

Following Section 1.4 of [1], consider

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} \tag{6}$$

where the bounds of integration denote the following contour on the complex plane



We proceed by breaking up this contour into three parts:

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} = \int_{+\infty}^{\delta} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} + \int_{|x|=\delta} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} + \int_{\delta}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} \tag{7}$$

the first of which describes the segment of the contour that extends down the positive real axis toward the origin, the second describes the circle of radius δ about the simple pole at the origin, and the last describes the segment which travels back to $+\infty$ along the real axis. First, we consider the behavior of the second segment as $\delta \to 0^+$.

$$\lim_{\delta \to 0^+} \int_{|x|=\delta} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} = \lim_{\delta \to 0^+} \int_0^{2\pi} \frac{(-\delta e^{i\theta})^s}{e^{\delta e^{i\theta}} - 1} \frac{i\delta e^{i\theta}}{\delta e^{i\theta}} = \lim_{\delta \to 0^+} i \int_0^{2\pi} \frac{(-\delta)^s e^{i\theta s}}{e^{\delta e^{i\theta}} - 1} d\theta$$

Using the Taylor Series expansion of the exponential term, the denominator approaches $\delta e^{i\theta}$ as $\delta \to 0^+$. Thus,

$$\lim_{\delta \to 0^+} \int_{|x|=\delta} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} = \lim_{\delta \to 0^+} i \int_0^{2\pi} (-1)^s (\delta e^{i\theta})^{s-1} d\theta = \lim_{\delta \to 0^+} i(-1)^s \delta^{s-1} \int_0^{2\pi} e^{i\theta(s-1)} d\theta$$

Since the integral on the left evaluates to 0 in all cases excluding when $s = 1$, we can assert that the piece of the contour surrounding the pole at the origin vanishes under the same condition. Therefore, for $s \neq 1$,

$$\begin{aligned} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} &= \lim_{\delta \to 0^+} \int_{+\infty}^{\delta} \frac{\exp(s \log x - i\pi)}{(e^x - 1)x} + \int_{\delta}^{+\infty} \frac{\exp(s \log x - i\pi)}{(e^x - 1)x} \\ &= (e^{i\pi s} - e^{-i\pi s}) \int_0^{\infty} \frac{x^{s-1} dx}{e^x - 1} \end{aligned}$$

Applying 5 and Prop. 2.1:

$$\begin{aligned}\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x-1}\frac{dx}{x} &= (e^{i\pi s} - e^{-i\pi s})\Pi(s-1)\zeta(s)\\ &= (\cos(\pi s) + i\sin(\pi s) - (cos(-\pi s) + i\sin(-\pi s))\Pi(s-1)\zeta(s)\\ &= 2i\sin(\pi s)\Pi(s-1)\zeta(s) = 2i\frac{\pi s}{\Pi(s)\Pi(-s)}\Pi(s-1)\zeta(s)\\ &= \frac{2\pi i s}{s\Pi(s-1)\Pi(-s)}\Pi(s-1)\zeta(s) = \frac{2\pi i\zeta(s)}{\Pi(-s)}\end{aligned}$$

Rearranging terms yields the functional representation of the zeta function:

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i}\int_{+\infty}^{+\infty}\frac{(-x)^s}{e^x-1}\frac{dx}{x} \tag{8}$$

4. Introducing $Z(t)$

Utilizing the fact that $\zeta(s)$ is real-valued on the critical line, the general strategy for locating the zeros of the zeta function along the line is to determine the values of t at which $\zeta(s = 1/2 + it)$ changes sign. To detect such points it is customary (per Section 6.5 of [1]) to rewrite the zeta function on the critical line as

$$\begin{aligned}\zeta(1/2+it) &= \frac{s}{2}\Pi\left(\frac{s}{2}-1\right)\pi^{-s/2}(s-1)\zeta(s) = \exp\left(\log\Pi\left(\frac{s}{2}-1\right)\right)\pi^{-s/2}\frac{s(s-1)}{2}\zeta(s)\\ &= \left(\exp\left(\Re\left(\log\Pi\left(\frac{s}{2}-1\right)\right)\right)\pi^{-1/4}\frac{-t^2-\frac{1}{4}}{2}\right)\left(\exp\left(i\Im\left(\log\Pi\left(\frac{s}{2}-1\right)\right)\right)\pi^{-it/2}\zeta\left(\frac{1}{2}+it\right)\right).\end{aligned} \tag{9}$$

Since the left term in 9 is negative real, the sign change of $\zeta(s = 1/2 + it)$ is completely determined by the right term (to which the sign of $\zeta(s = 1/2 + it)$ is opposite. Given this part of the expansion is essential to the study of the zeros of the zeta function along the critical line, it is customary to rewrite this term as

$$Z(t) = r(t)\zeta(1/2+it) = e^{i\theta(t)}\zeta(1/2+it), \tag{10}$$

where $\theta(t) = \Im\left(\log\Pi\left(\frac{it}{2}-\frac{3}{4}\right)\right) - \frac{t}{2}\log\pi$.

5. Derivation of the Riemann-Siegel Formula

The goal of the Riemann-Siegel Formula is to develop an asymptotic formula for $Z(t)$, which through 10 provides information on the approximate location of the zeros to the Riemann Zeta function along the critical line ($\Re(s) = \frac{1}{2}$) when t is large.

5.1. **Integral Representation of $\zeta(s)$.** To develop an integral representation suitable for asymptotic analysis, our first task is to find a way to split off finite sums from 8. As Edwards notes, there are two clear was to do this:

(1) Use $\frac{e^{-Nx}}{e^x-1} = \sum_{n=N+1}^{\infty} e^{-nx}$ to find

$$\zeta(s) = \sum_{n=1}^{N} n^{-s} + \frac{\Pi(-s)}{2\pi i}\int_{+\infty}^{+\infty}\frac{e^{-Nx}(-x)^s}{e^x-1}\frac{dx}{x} \tag{11}$$

(2) Introduce a new contour C_M which follows a similar path of Figure 3, but circles the disk of $|s| = (2M+1)\pi$ rather than the origin.

$$\zeta(s) = \Pi(-s)(2\pi)^{s-1}2\sin\frac{\pi s}{2}\sum_{n=1}^{M} n^{-(1-s)} + \frac{\Pi(-s)}{2\pi i}\int_{C_M}\frac{(-x)^s}{e^x-1}\frac{dx}{x} \tag{12}$$

Combining 11 and 12 yields

$$\zeta(s) = \sum_{n=1}^{N} n^{-s} + \Pi(-s)(2\pi)^{s-1}2\sin\frac{\pi s}{2}\sum_{n=1}^{M} n^{-(1-s)} + \frac{\Pi(-s)}{2\pi i}\int_{C_M}\frac{(-x)^s e^{-Nx}}{e^x-1}\frac{dx}{x} \tag{13}$$

To put 13 in a more symmetrical form, one must multiply the equation by $\frac{1}{2}s(s-1)\Pi(\frac{1}{2}s-1)\pi^{-\frac{s}{2}}$ and use properties of $\Pi(s)$ given in Proposition 2.1 to derive

$$\begin{aligned}\zeta(s) = (s-1)\Pi\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\sum_{n=1}^{N} n^{-s} + (-s)\Pi\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\sum_{n=1}^{M} n^{-(1-s)} + \\ + \frac{(-s)\Pi(1-\frac{s}{2})\pi^{-\frac{1-s}{2}}}{(2\pi)^{s-1}2\sin(\frac{\pi s}{2})2\pi i}\int_{C_M}\frac{(-x)^s e^{-Nx}}{e^x-1}\frac{dx}{x}, \quad \forall N, M, s\end{aligned} \tag{14}$$

Given the rationale underlying the Riemann Hypothesis, the case of greatest interest is that of $\zeta(\frac{1}{2}+it)$. Symmetry between s and $1-s$ motivates us to set $N = M$. For notational convenience, we write $f(t) := (-\frac{1}{2}+it)\Pi(\frac{\frac{1}{2}+it}{2})$. This gives us

$$\begin{aligned}\zeta\left(\frac{1}{2}+it\right) = f(t)\sum_{n=1}^{N} n^{-(\frac{1}{2})-it} + f(-t)\sum_{n=1}^{N} n^{-(\frac{1}{2})+it} + \\ + \frac{f(-t)}{(2\pi)^{\frac{1}{2}+it}2i\sin(\frac{1}{2}\pi(\frac{1}{2}+it))}\int_{C_M}\frac{-(-x)^{-\frac{1}{2}+it}e^{-Nx}}{e^x-1}\frac{dx}{x}\end{aligned} \tag{15}$$

From Section 4, we know $Z(t)$ satisfies $\zeta(\frac{1}{2}+it) = r(t)Z(t)$ where

$$\begin{aligned} r(t) &= \exp(\Re(\log\Pi(\frac{1}{2}s-1)))\pi^{-\frac{1}{4}}\frac{-t^2-\frac{1}{4}}{2} \\ &= \exp(\log\Pi(\frac{1}{2}s-1))\pi^{-\frac{1}{4}}\frac{s(s-1)}{2}\exp(-i\Im(\log\Pi(\frac{1}{2}s-1))) \\ &= \Pi(\frac{s}{2})(s-1)\pi^{-\frac{1}{4}}e^{-i\theta(t)}\pi^{-\frac{1}{2}it} = f(t)e^{-i\theta(t)}\end{aligned}$$

Canceling a factor of $r(t) = r(-t)$, using that θ is odd and $2i\sin(\frac{\pi s}{2}) = e^{-\frac{i\pi s}{2}}(e^{i\pi s}-1) = -e^{-i\frac{\pi}{4}}e^{i\frac{\pi}{2}}(1-ie^{-t\pi})$ produces the following formulation of $Z(t)$:

$$Z(t) = \sum_{n=1}^{N} n^{-\frac{1}{2}}2\cos(\theta(t)-t\log n) + \frac{e^{-i\theta(t)}e^{-t\frac{\pi}{2}}}{(2\pi)^{\frac{1}{2}}(2\pi)^{it}e^{-i\frac{\pi}{4}}(1-ie^{-t\pi})}\int_{C_N}\frac{(-x)^{\frac{1}{2}+it}e^{-Nx}dx}{e^x-1} \tag{16}$$

5.2. **Applying Steepest Descent.** We first note that the behavior of the integrand of 16 away from the singularities at $0, \pm 2\pi i, \pm 4\pi i 4, \cdots$ is dominated by the contribution from the numerator. Ergo, in application of the Method of Steepest Descent, it suffices to focus our efforts on the numerator. We can rewrite the numerator of the integrand in 16 as $e^{\phi(x)}$ where

$$\phi(x) = \Re\left(\left(-\frac{1}{2} + it\right)\log(-x) - Nx\right) \tag{17}$$

Let $\rho(t)$ be the entire (real and complex) phase function. In accordance with Section 2.3, we now seek to find the saddle point x^* in the x-complex plane:

$$\begin{aligned} 0 = \rho'(x^*) &= (-\frac{1}{2} + it)\frac{d}{dx^*}\log(-x^*) - N = \frac{-\frac{1}{2}+it}{x^*} - N \\ x^* &= \frac{-\frac{1}{2}+it}{N} \end{aligned}$$

In the neighborhood of x^*, we can use the local Taylor Expansions and the value of x^* to derive the following asymptotic expansion of 17:

$$\begin{aligned} \phi(x) &= \Re\left(\left(-\frac{1}{2}+it\right)\log(x^*) + \left(-\frac{1}{2}+it\right)\log\left(1+\frac{x-x^*}{x^*}\right) - Nx^* - N(x-x^*)\right) \\ &= \Re\left(\left(-\frac{1}{2}+it\right)\left(\left(\frac{x-x^*}{x^*}\right) - \frac{1}{2}\left(\frac{x-x^*}{x^*}\right)^2 + \frac{1}{3}\left(\frac{x-x^*}{x^*}\right)^3 + \cdots\right) - \frac{-\frac{1}{2}+it}{x^*}(x-x^*)\right) + c_1 \\ &= \Re\left(-\frac{1}{2}\left(-\frac{1}{2}+it\right)\left(\frac{x-x^*}{x^*}\right)^2 + \frac{1}{3}\left(-\frac{1}{2}+it\right)\left(\frac{x-x^*}{x^*}\right)^3 + \cdots\right) + c_1 \\ &= c_1 - \frac{1}{2}\Re\left(\frac{N^2(x-x^*)^2}{-\frac{1}{2}+it}\right) + \text{H.O.T.} \end{aligned}$$

where $c_1 \in \mathbb{R}$, H.O.T. is a colloquial abbreviation of the higher order terms of the series. Following Section 2.3, we seek a contour in which $\psi(x) = \Im(\rho(x))$ is constant and the contour passes through the saddle point x^*. Using 17, this gives us the condition that the contour passes along:

$$\Im(\log(x-x^*)) = \frac{1}{2}\Im\left(\log\left(-\frac{1}{2}+it\right)\right),$$

where

$$\frac{(x-x^*)^2}{-\frac{1}{2}+it} \in \mathbb{R}^+.$$

Concerning the angle of the contour ray as it passes through x^*, we parametrize small displacements from the saddle by $\delta x = re^{i\theta}$, yielding $(\delta x)^2 = r^2 e^{i2\theta}$. Imposing the condition that $(\delta x)^2/(-\frac{1}{2}+it)$ is positive real requires

$$\arg\left((\delta x)^2\right) = \arg\left(-\frac{1}{2}+it\right) \mod 2\pi.$$

Since for large t we have $\arg\left(-\frac{1}{2}+it\right) \approx \frac{\pi}{2}$, this gives

$$2\theta \approx \frac{\pi}{2} \mod 2\pi, \quad \text{so} \quad \theta \approx \frac{\pi}{4} \quad \text{or} \quad \theta \approx \frac{5\pi}{4}.$$

Thus, the contour must pass through x^* at an angle of $\pi/4$ or $5\pi/4$ relative to the real axis, corresponding to a slope of $+1$ (bottom right to top left) or -1 (top right to bottom left). In our case, we select the ray with slope $+1$ ascending from the bottom right to the top left, consistent with decay along the steepest descent path. Let R denote the contour ray which passes through the neighborhood of the saddle point.

Choice of N. We must choose N such that we ensure the ability for the contour to cross the imaginary axis near x^* between the poles at $2\pi N$ and $2\pi(N+1)$. One common choice, as presented in [1], to achieve this is to select $N = \left\lfloor (t/2\pi)^{\frac{1}{2}} \right\rfloor$.

5.3. **Emergence of the Riemann-Siegel Terms.** Section 7.3 of [1] carefully outlines the asymptotics to select the rest of the contour away from R such that contribution from these segments is negligible. For the sake of this investigation, we assume this to be the case and by Cauchy's Theorem assert

$$\int_{C_N} \frac{(-x)^{\frac{1}{2}+it} e^{-Nx} dx}{e^x - 1} \approx \int_R \frac{(-x)^{\frac{1}{2}+it} e^{-Nx} dx}{e^x - 1} \tag{18}$$

Rather than expanding the numerator in powers of $(x - x^*)$ as in Section 5.2, Edwards instead notes that it of greater convenience to avoid the dependence on the discrete variable N and the small real part of x^* and instead expand numerator in powers of $(x-a)$ where $a = i(2\pi t)^{\frac{1}{2}}$. Doing so yields, by means of Taylor Series expansion, the following representation of the numerator of the integrand:

$$\begin{aligned}
&\exp\left(\left(-\frac{1}{2} + it\right)\log(-a) + \left(-\frac{1}{2} + it\right)\log\left(1 + \frac{x-a}{a} - Na - N(x-a)\right)\right) \\
&\quad = (a)^{-(1/2)+it} e^{-Na} \exp\left(\left(\left(-\frac{1}{2} + it\right)a^{-1} - N\right)(x-a) - \left(-\frac{1}{2} + it\right)\frac{1}{2}(x-a)^2 a^{-2}\right)
\end{aligned}$$

After this expansion, the coefficient of $(x-a)$ becomes $(-1/2 + it)(1/a) - N = (-1/2 + it)(-i(2\pi t)^{-1/2}) - N \approx (t/2\pi)^{1/2} - N$ or the fractional part of $(t/2\pi)^{1/2}$ which the book represents as p. The coefficient of $(x-a)^2$ becomes $(-1/2 + it)(-1/2a^2) = 1/4a^2 - i(t/2a^2) \approx -i(t/2((i(2\pi t)^{1/2})^2) = i/4\pi$. Similar logic yields that the coefficient of a higher order term $(x-a)^n$ is approximately $\pm(1/n)(it)/(i(2\pi t)^{1/2})^n$, which describes decay of base t of increasing negative powers as n gets larger. Hence, for t large, the exponential of these terms tends to 1. From this we can approximate the numerator of the integrand of 18 as

$$(-a)^{-(1/2)+it} e^{-Na} e^{p(x-a)} e^{(i/4\pi)(x-a)^2} \tag{19}$$

Pulling the constant terms out of the expression, this greatly simplifies the integral we seek to evaluate:

$$\int_R \frac{e^{-Na} e^{p(x-a)} e^{(i/4\pi)(x-a)^2}}{e^x - 1} dx \tag{20}$$

In Riemann's records, which were interpreted by Siegel, Riemann was able to derive a closed form solution to this integral (20). While his methodologies are of too great depth and length to be included in this high-level work, Riemann deploys a combination of changing variables, manipulating contours, and Cauchy's integral formula to produce the first term of the approximation of the integral. This yields the following approximation to the remainder term of $Z(t)$ from the expansion given by 16:

$$Z(t) \sim \sum_{n=1}^{N} n^{-\frac{1}{2}} 2\cos(\theta(t) - t\log n) + (-1)^{N-1}\left(\frac{t}{2\pi}\right)^{-1/4} \frac{\cos 2\pi(p^2 - p - 1/16)}{\cos 2\pi p} \tag{21}$$

Additional terms in the approximation are recoverable by keeping higher order powers of $(x-a)$ which we truncated in 19. The exact coefficients of these terms are recoverable through the development of a recurrence relation via a long sequence of algebra and calculus steps (see [1], Section 7.5).

6. Discussion

As discussed in Section 4, $Z(t)$ has the opposite sign of the zeta function along the critical line. Accordingly, if we can detect that $Z(t)$ changes sign, it follows that $\zeta(1/2 + it)$ also change sign (in the opposite manner). For large t, numerical methods can be deployed to sample the Riemann-Siegel function near where it changes sign. Using Haselgrove's table [5] of the coefficients of the higher order powers we truncated in our formulation, one can detect points on either side of the sign change with extraordinary precision. Once one has detected said points, mere linear interpolation suffices to approximate the zero of the zeta function along the critical line. Given the integral relation of the zeros of the zeta function to the distribution of the prime numbers, as briefly alluded to in Section 2.1, the Riemann-Siegel Formula is of pivotal importance to numerical studies of prime distribution which are vital to computing and cryptography.

This paper also serves as testament to the shear elegance and analytical power of the Method of Steepest Descent. By selecting our contour that passed through the saddle point of the generalized phase function in the complex-x plane along the path of steepest descent, we reduce the problem to integrating along a local contour ray, dramatically simplifying computation.

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