A Chain of Knot Invariants

by

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ABSTRACT

In this paper, we will study a chain of knot invariants whose values bound one another. Starting with the way these invariants are defined in previous literature we will better understand a new way of interpreting these invariants in terms of a relatively new mathematical object, the bridge map. We will better understand how this object is computed and analyze and graph the bridge map of a handful of knot conformations. Lastly, we will study the combinatorial properties of the bridge map and how these properties can be used to better understand the chain of knot invariants it defines.

Acknowledgements

I would like to thank Colin Adams for choosing me as one of his Thesis students and guiding me through a year of mathematics research. I cannot thank him enough for all of his helpful conversations and feedback. Thank you.

CONTENTS

1. INTRODUCTION

Since the beginnings of knot theory, a crucial question has involved distinguishing two mathematical knots from one another. One can think of a mathematical knot as an equivalence class of embeddings of a circle, S^1, into 3-dimensional Euclidean Space, $\mathbb{R}^3$. Two different embeddings of a knot are equivalent up to continuous deformations that do not allow the knot to pass through itself. However, it is unknown how much one embedding might need to be deformed in order to demonstrate equivalence to the other embedding, therefore, in practice it is unreasonable to resort to deformations when trying to determine if two knot embeddings are equivalent.

A knot invariant is a quantity defined for a knot that does not depend on the particular embedding of the knot. Therefore, by comparing the invariant values of one knot to another we can successfully tell two knots apart when their invariant values differ. However, if their invariant values are the same, we cannot conclude the knots themselves are equivalent.

One can think of a knot invariant as a function that collapses all the information of a knot into a single value that captures a specific component of its complexity. However, computing the values of certain knot invariants has proven computationally difficult. For the sake of theory, one may continue defining computationally difficult knot invariants and prove properties relating to the invariant itself, however, such invariants may not be useful if we can never compute the value. This computational limitation for knot invariants has led to the study of the relationships between different knot invariants and what the value of one invariant implies about the value of another for a given knot equivalence class.

Specifically, in this paper we focus on invariants coming out of the geometric shape of the knot and the knot's curvature and torsion. Before we define these invariants, it will be crucial to understand the difference between a knot, knot conformation, and a knot projection. A knot, as has already been described, is the equivalence class of embeddings up to non-self-intersecting deformations. A knot conformation is a specific embedding of the knot in $\mathbb{R}^3$. As one continuously deforms a knot embedding, the conformation changes, however, the knot the conformation describes remains the same. The projection of a knot is the image of a knot conformation projected onto a plane. When projecting a knot conformation to a plane we will almost always introduce singular points where two or more points on the knot are projected to the same point. A majority of knot theory deals with projections in which each singular point is a double point, meaning the view of one strand of the knot only blocks a single other strand at any given point. However, one can imagine that a projection of a knot conformation yields singularities of degree greater than two. In this paper we will restrict ourselves to projections which only consist of double-point singularities.

As we will see the invariants that we focus on are typically described in terms of the collection of all knot conformations or all knot projections for a given knot. This ensures the invariant is defined for the knot's equivalence class. We will now introduce some of the relevant knot invariants studied in this paper.

Definition 1.1. *The* ***bridge number*** *of a knot projection, P_K, is*

$$b(P_K) = \min_{w \in S^1} \mu(w, P_K)$$

where $\mu(w, P_K)$ is the number of maxima of projection, P_K, in the direction of vector w in the projection plane. In other words, $b(P_K)$ is the least number of maxima in projection, P_K, in any direction.

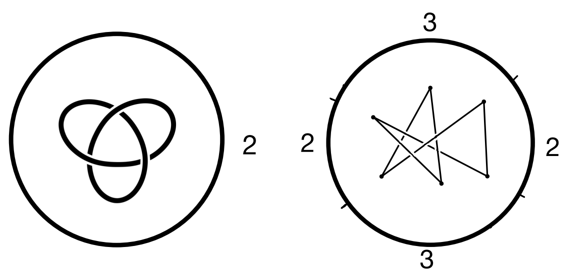

FIGURE 1. Two projections of 3_1. The left image is a smooth projection while the right image is a stick projection.

In Figure 1 we see two different projections of the trefoil knot, 3_1, and the respective number of maxima in each possible planar vector direction represented as the labels on the outer circle. In the stick projection since the number of maxima vary between 2 and 3 as we consider different vectors in the plane it follows that $b(P_{3_1}) = 2$ since 2 is the minimum value. For the smooth projection since the number of maxima is constant in all directions the minimum is 2 and therefore $b(P_{3_1}) = 2$.

It is important to note that if the bridge number of a projection is 1 then the following projection corresponds to the trivial knot. Alongside this fact, there is a nice symmetric property when computing the number of planar maxima of a given knot projection. Since a knot projection is an embedding of S^1 (i.e is a closed loop) onto a plane with singularity points it follows that for every maxima in one particular direction there must also be a corresponding minima. We can further convince ourselves of this by looking at the example projections above. If we look at the maxima values labelled on the circle around the projections in Figure 1 we will notice that the circle is symmetrically labelled. This relates to the statement above about minima because the number of minima of a particular direction is equal to the number of maxima of the opposite direction.

The bridge number of a knot K can now be defined as the minimum bridge number of a projection over all possible projections.

Definition 1.2. *The **bridge number** of a knot, K, is*

$$b(K) = \min_{P_K} b(P_K).$$

In Figure 1 the bridge number of both projections are the same, but this will not be the case for any two projections in general.

One can also think of the definition of bridge number in terms of the different 3-dimensional conformations of a knot.

Definition 1.3. *The **bridge number** of a knot conformation, C_K, is*

$$b(C_K) = \min_{v \in S^2} \min_{w \in S^1} \mu(w, P_v(C_K))$$

where $\mu(w, P_v(C_K))$ is the number of maxima in the direction of vector w with respect to a projection of C_K defined by vector v, namely $P_v(C_K)$.

Note that the corresponding value of the bridge number of a knot conformation is defined by two vectors. First, a vector that is required to project the 3-dimensional knot conformation onto a plane, creating a projection. Then a vector that is required for determining the number of maxima of that projection (i.e intuitively the same vector from Definition 1.1). This definition can be visualized in Figure 2.

One can take a similar approach as in Definition 1.2 and define the bridge number of a knot to be the minimum bridge number of a knot conformation over all possible conformations.

Definition 1.4. *The **bridge number** of a knot, K, is*

$$b(K) = \min_{C_K \in [C_K]} b(C_K)$$

where $b(C_K)$ is the bridge number of conformation C_K.

Note that the bridge number defined over all knot projections, Definition 1.2, is equivalent to the definition when defined over all knot conformations, Definition 1.4. However, the

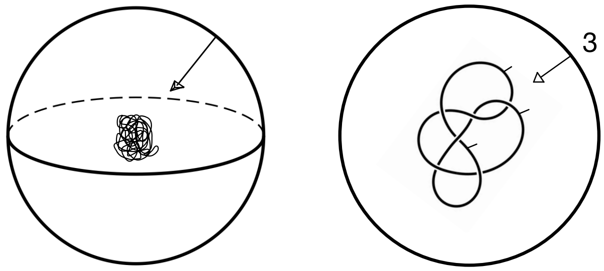

FIGURE 2. The first vector v corresponds to a point on S^2 and projects the conformation to a plane. The second vector w is a planar vector yielding a value corresponding to the number of maxima in that direction.

former definition only considers the 2-dimensional projections of the knot while the latter considers the 3-dimensional conformations of the knot. It turns out that knot theory studied purely in terms of projections is an equivalent theory to knot theory studied purely in terms of conformations. Later, we will see that there are some computational advantages for interpreting the bridge number in terms of the 3-dimensional conformations of a knot.

A similar knot invariant to bridge number is the projective super bridge number of a knot, denoted $psb(K)$. The projective super bridge number can also be defined in terms of projections and conformations; however, the projective super bridge number is concerned with the largest number of maxima in any given direction of a knot projection minimized over all the possible projections. For consistency we will first define the projective super bridge number of a knot projection.

Definition 1.5. *The* ***projective super bridge number*** *of a knot projection, P_K, is:*

$$psb(P_K) = \max_{w \in S^1} \mu(w, P_K)$$

where $\mu(w, P_K)$ is the number of maxima of projection, P_K, in the direction of vector w in the projection plane.

Looking back at Figure 1 we can see that $psb(P_K)$ for the smooth projection is 2 while the $psb(P_K)$ for the stick projection is 3. This is we are maximizing over the projection directions rather than minimizing. Similar to bridge number it follows that the projective super bridge number of a knot can be defined by minimizing $psb(P_K)$ over all possible projections.

Definition 1.6. *The* ***projective super bridge number*** *of a knot, K, is*

$$psb(K) = \min_{P_K} psb(P_K)$$

where $psb(P_K)$ is the maximum number of maxima over all vector directions of vector w in the projection plane.

As we might expect the projective super bridge number of a knot can also be defined in terms of a knot conformation.

Definition 1.7. *The* ***projective super bridge number*** *of a knot conformation, C_K, is:*

$$psb(C_K) = \min_{v \in S^2} \max_{w \in S^1} \mu(w, P_v(C_K))$$

where $\mu(w, P_v(C_K))$ is the number of maxima in the direction of vector w with respect to the projection of C_K defined by vector v, namely $P_v(C_K)$).

It follows that the projective super bridge number of a knot can be defined as the minimum projective super bridge number over all possible knot conformations.

Definition 1.8. *The* ***projective super bridge number*** *of a knot, K, is:*

$$psb(K) = \min_{C_K \in [C_K]} psb(C_K)$$

where $psb(C_K)$ is the maximum number of maxima minimized over all projections of conformation, C_K.

Another closely related knot invariant to $b(K)$ and $psb(K)$ is the super bridge number of a knot, denoted $sb(K)$. Unlike the other two invariants, the super bridge number can only be defined in terms of a knot's conformations for reasons we will discuss shortly. The super bridge number is concerned with the maximum number of maxima for a given knot conformation minimized over the set of all conformations. We will first define the super bridge number in terms of a knot's conformation.

Definition 1.9. *The* ***super bridge number*** *of a knot conformation C_K is*

$$sb(C_K) = \max_{v \in S^2} \max_{w \in S^1} \mu(w, P_v(C_K))$$

where $\mu(w, P_v(C_K))$ is the number of maxima in the direction of vector w with respect to the projection of C_K defined by vector v, namely $P_v(C_K)$.

Again generalizing to the equivalence class of the knot it follows that the super bridge number of a knot can be defined as the minimum super bridge number of a knot conformation over all knot conformations.

Definition 1.10. *The **super bridge number** of a knot, K, is:*

$$sb(K) = \min_{C_K \in [C_K]} sb(C_K)$$

where $sb(C_K)$ is the maximum number of maxima in conformation C_K.

The reason the super bridge number of a knot can't be defined in terms of projections is because we need to minimize a maximum value over all conformations and the set of conformations partitions the set of projections (since projections are derived from projecting a conformation). If we attempted to define the super bridge number of a knot over all projections, we wouldn't successfully be able to isolate and minimize over these smaller partitioned sets of projections that each conformation defines. Therefore, by defining the super bridge number of a knot in terms of the knot's conformations we are isolating the set of projections for each conformation by vector v (see Definition 1.9) and then minimizing over this set of projections.

When looking $b(K), psb(K)$, and $sb(K)$ defined in terms of the set of all knot conformations it is quite easy to see that $b(K) \leq psb(K) \leq sb(K)$.

$$b(K) = \min_{C_K \in [C_K]} \min_{v \in S^2} \min_{w \in S^1} \mu(w, P_v(C_K))$$

$$psb(K) = \min_{C_K \in [C_K]} \min_{v \in S^2} \max_{w \in S^1} \mu(w, P_v(C_K))$$

$$sb(K) = \min_{C_K \in [C_K]} \max_{v \in S^2} \max_{w \in S^1} \mu(w, P_v(C_K))$$

These definitions can be confusing and it is important to think of minimizing/maximizing from the inside out.

The bridge number and super bridge number are well established knot invariants and have been around for a while. The bridge number of a knot was first introduced in 1954 by Schubert, in [14]. In 1987 Kuiper, [11], introduced the super bridge number of a knot and proved that $b(K) < sb(K)$ for all knots using properties of a knot's curvature and torsion. In this paper, we will give a simplified version of this proof. The projective super bridge number of a knot was first introduced in 2010 by Colin Adams et al, [4], and hasn't been studied as much.

Below we will review a few results proved in Schubert and Kuiper's papers related to the introduction of $b(K)$ and $sb(K)$. Most of their results are for a specific category of knots called torus knots.

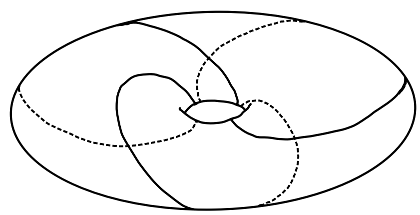

FIGURE 3. $(3, 2)$ torus knot which also corresponds to the trefoil knot 3_1. The dotted lines are parts of the knot that are on the backside of the torus.

The category of torus knots consists of knots that have a conformation that can be embedded on the surface of a torus with no crossings see Figure 3. Torus knots are defined by a (p, q)-curve ($p, q \in \mathbb{Z}_{>0}$) on a torus where p represents the number of times the curve wraps around the meridian and q represents the number of times the curve wraps around the longitude, we denote them by $T(p, q)$ [1]. When p and q are co-prime the resulting object is a knot while in the other case the resulting object is a link, which can be thought of as a chain of interlinked knots. When we refer to a (p, q)-torus knot, we will only be referring to cases where p and q are relatively prime and therefore that generate a knot.

It turns out that torus knots have a lot of nice properties, and many invariants can be defined by a general rule in terms of p and q. Schubert proved that the bridge number for a (p, q)-torus knot is equal to $\min\{p, q\}$ in [14]. However, a (p, q)-torus knot is equivalent to a (q, p) torus knot since we can associate the longitude of the first torus with the meridian of second torus and vice versa. As we see in Figure 3 every time the torus knot wraps around the meridian a maxima is created relative to the vector perpendicular to the xy-plane. Since the bridge number is concerned with the minimum number of maxima and one can continuously deform a (p, q)-torus knot into a (q, p)-torus knot it makes sense that $b(T(p, q)) \leq \min\{p, q\}$ since there at least exists conformations that realize both p and q maxima. However, demonstrating the lower bound is where most of the difficulty arises. In general, the standard notation of a (p, q)-torus knot is written with the larger number first $p > q$. This implies $b(T(p, q)) = q$. Kuiper proved that $sb(T(p, q)) = \min\{p, 2q\}$ in [11].

Unlike bridge number there is no immediate intuitive reasoning for this formula.

Another useful invariant is the stick number of a knot [1]. It turns out that embeddings of knots can be represented with sticks glued end-to-end. Although a knot conformation is typically thought to be smooth one can simply replace these smooth edges with straight sticks. By viewing the knot conformation as a stick knot conformation nothing changes about the complexity of the conformation itself other than the fact that it is made from sticks. Making knots out of sticks creates a lot of interesting new theory and it turns out that all knots require a minimal number of sticks in order to represent all of its complexity.

Definition 1.11. *The **stick number** of a knot, K, denoted $s(K)$ is the least number of straight sticks glued end-to-end needed to create a stick conformation of K.*

A minimum stick conformation of a knot K can also be thought of as the minimal closed polygonal path in 3-space that is equivalent to K. A given stick conformation can also be represented in projection form by projecting the given conformation onto a plane. The theory of stick knots can purely be studied in the plane; however, we must be cautious as not all projections that appear to define a knot in projection form correspond to a stick conformation with an equivalent number of straight sticks. It is quite easy to draw a stick knot projection that requires the sticks to bend in order for the object to be embedded into 3-space. Figure 4 represents a projection of 5_1 that appears to represent it with 5 sticks. However, this projection can't actually be embedded into 3-space with five sticks. Therefore, not all stick projection that appear to represent a certain knot can be mapped to a stick conformation of that knot with an equivalent number of sticks. The appropriate stick number of 5_1 is 8 and can be found by trial and error of all possible stick projections up to 8 sticks or by the fact that 5_1 is a $T(5,2)$ (i.e $(5,2)$ torus knot).

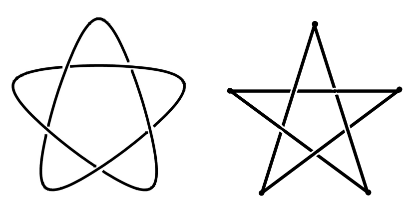

FIGURE 4. Smooth knot projection of 5_1 and an invalid stick projection that appears to represent 5_1.

It was been proven by Gyo Taek Jin in [10], that the stick number of a (p, q)-torus knot is $s(T(p, q)) = \min\{2p, 4q\}$. It is also known and trivial to show by brute force that any non-trivial knot must have a stick number of at least 6. One can prove this by demonstrating that all 5 stick projections represent the trivial knot. This is because we can view a 5 stick representation with a vector that is parallel to one of the sticks. This strategy will create the illusion that there are 4 sticks in a projection, but such a projection can only represent the trivial knot.

There also exists another lower bound on stick number in terms of the projective super bridge number, $psb(k)$, of a knot. Colin Adams et al. proved that $s(k) \geq 2psb(k) + 1$ in the previously mentioned paper [4]. This result makes sense and aligns nicely with another bound from Gyo Taek Jin $s(k) \geq 2sb(k)$ which follows from the fact that a stick conformation can have at most $s(K)/2$ local max. The composition of two factor knots K_1 and K_2 is a well-known operation that can be applied to any two knots.

Definition 1.12. *The* ***composite knot*** *of two factor knots, K_1 and K_2, denoted $K_1 \# K_2$ is the resulting knot obtained by cutting out an exterior arc of K_1 and K_2 and gluing the resulting endpoints together to form a single closed loop.*

It is a theorem that the composition of any two non-trivial knots is non-trivial [1]. Also, the composition of the unknot with any non-trivial knot yields the original non-trivial knot. A popular question in knot theory involves asking how knot invariants behave under composition.

Typically, it is trivial to get an upper bound on the corresponding invariant values of the composition knot by just composing two minimal invariant knot projections/conformations and computing the resulting invariant value of that projection/conformation. However, the main difficulty with defining such a rule for an invariant value under composition is providing a lower bound on the resulting invariant value for the composite knot. How does one know that given this composite projection/conformation one cannot continuously deform it resulting in a composite projection/conformation that yields a smaller invariant value?

It has been proven by Schubert and others that the bridge number of a knot is sub additive under composition [1]

$$b(K_1 \# K_2) = b(K_1) + b(K_2) - 1.$$

There are no explicit rules on how projective super bridge or super bridge behave under composition. However, bounds of the super bridge number under composition have been proven and there also exists specific rules for certain families of knots.

The stick number of a knot is also interesting because one can strategically compose two stick knot conformations to eliminate a handful of sticks. It is quite unlikely that there will be a single specific rule for the stick number under composition due to the number of special cases that exist for certain stick knot conformations. For example one can compose two trefoil knots, $3_1\#3_1$, and yield a stick number of 8 when $s(3_1) = 6 \implies s(3_1\#3_1) = s(3_1) + s(3_1) - 4$. However, if one were to compose the trefoil knot with 8_{20} where $s(8_{20}) = 8$ it would most likely follow from Clayton Shonkwiler's and Thomas Eddy's paper and computer program [7] [6] that $s(3_1\#8_{20}) = 11 = s(3_1) + s(8_{20}) - 3$. Thus, a general rule for stick number under composition seems too optimistic. It rather appears that when increasing the complexity of the composite knot there are strict threshold that are met forcing the composite knot's minimal stick number to bump up to a certain value.

We will now introduce several knot invariants which are defined for a specific subset of knot projections. First, it will be useful to better understand the following projections themselves.

Definition 1.13. *A* ***spiral projection*** *of a knot K denoted P_{spK} is a projection where the planar curvature of the knot always has the same sign.*

We can also think of a spiral projection as a projection of a knot such that there exists no inflection points (i.e points where the planar curvature of the knot switches signs). See the right image in Figure 5 for an example spiral projection of a knot.

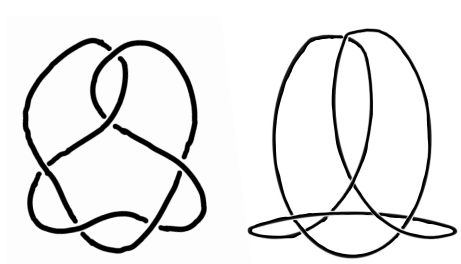

FIGURE 5. Two projections of 6_1. The left is a non-spiral projection while the right is a spiral projection.

Within the set of all spiral projections, we can also define the set of braid projections.

Definition 1.14. *A **braid projection** of a knot K denoted $P_{\mathcal{B}K}$ is a spiral projection that revolves around a single fixed point.*

It is important to note that the way we have defined a braid projection is slightly different than the traditional definition. The traditional definition does not require the projection to be spiral. However, it is true that we can isotope a braid projection (defined in the traditional way) to be an equivalent braid projection that is also a spiral projection.

Since a braid projection and spiral projection both require the knot projection to have constant sign of curvature and a braid projection additionally requires the strands to revolve around a fixed point it follows that the set of braid projections is a proper subset of the set of spiral projections. This property will define a nice relationship between the spiral number and braid number of a knot. Quickly, it will be worth defining a spiral and braid conformation as we will refer to such objects later in the paper. We can define a spiral conformation and braid conformation as follows.

Definition 1.15. *A **spiral conformation** of a knot, K, denoted C_{spK} is a conformation that realizes a spiral projection of K. Thus, there exists a vector $v \in S^2$ such that $P_v(C_K) = P_{spK}$ for some $P_{spK} \in [P_K]$.*

Definition 1.16. *A **braid conformation** of a knot, K, denoted $C_{\mathcal{B}K}$ is a conformation that realizes a braid projection of K. Thus, there exists a vector $v \in S^2$ such that $P_v(C_K) = P_{\mathcal{B}K}$ for some $P_{\mathcal{B}K} \in [P_K]$.*

Later we will see that there is a simple way to identify if a given knot conformation is a spiral and/or braid conformation.

As you may have noticed when looking at the spiral projection of knot 6_1 in Figure 5 it is true that projections that never change the sign of curvature have an equivalent number of maxima over the entire circle's worth of directions of the knot projection. This is because projections that have no inflection points are only allowed to traverse 360 degrees of curvature prior to adding an additional maximum in the starting direction. This observation combined with the fact that a knot is a closed loop ensures any projection with no inflection points has the same number of maxima in every direction of the projection plane.

The corresponding number of maxima in a spiral and/or braid projection is exactly the value the spiral and braid number of a projection defines.

Definition 1.17. *The **spiral number** of a spiral projection, P_{spK}, denoted $sp(P_{spK})$ is the number of maxima in any direction of the spiral projection.*

Definition 1.18. *The **braid number** of a braid projection, $P_{\mathcal{B}K}$, denoted $\mathcal{B}(P_{\mathcal{B}K})$ is the number of maxima in any direction of the braid projection.*

It follows that the spiral and braid number of a knot can be defined as the minimum spiral and braid number over all possible spiral and braid projections.

Definition 1.19. *The **spiral number** of a knot, K, is:*

$$sp(K) = \min_{P_{spK} \in [P_K]} sp(P_{spK})$$

Definition 1.20. *The **braid number** of a knot, K, is:*

$$\mathcal{B}(K) = \min_{P_{\mathcal{B}K} \in [P_K]} \mathcal{B}(P_{\mathcal{B}K})$$

Although we defined spiral and braid number as different invariants it isn't obvious that they realize different values for a knot. Since the set of spiral projections is a super set of the set of braid projections and since in both definition we are minimizing over the set of projections it immediately follows that $sp(K) \leq \mathcal{B}(K)$. However, in theory they could realize the same value for all knots.

It turns out that not requiring a spiral projection to wind around a fixed axis is enough to uniquely define each invariant. There is a special name for the class of knots where $sp(K) < \mathcal{B}(K)$ first defined by Colin Adams et. al. in previously cited paper [4].

Definition 1.21. *A knot K is **curly** if $sp(K) < \mathcal{B}(K)$. In particular, we say K is an n-curly knot if it is curly with $sp(K) = n$.*

Given the similarity of spiral and braid number one can ask how the ratio of curly knots to non-curly knots behaves as we consider knots of greater crossing number. Up to 9 crossings 40 of the first 81 prime knots are known to be curly, 26 are known to be not curly (i.e. $sp(K) = \mathcal{B}(K)$) and it is unknown for the remaining 15. Since braid number is a more well-established invariant and behaves nicely under composition the braid number is practically known for all tabulated knots. This implies for the 15 unknown curly knots we just haven't determined the spiral number.

Since the spiral number is simply defined to be the minimum number of maxima for projections that have a constant number of maxima in all directions, we know that the spiral number is bounded below by the projective super bridge number (i.e. $psb(K) \leq sp(K)$). This is the case because the projective super bridge number is the minimum maximum number of maxima over all projections and the set of spiral projections is a subset of the set of projections. Therefore, since we are minimizing over the maximum number of maxima it follows that the minimum value for the larger super set will be at least as small as the minimum value of the smaller subset.

Thus, $psb(K) \leq sp(K)$. It follows from previously demonstrated inequalities that $b(K) \leq psb(K) \leq sp(K) \leq \mathcal{B}(K)$. Note that super bridge number which is greater than or equal to $psb(K)$ is not in this chain of inequalities. That is because there is no well-known way of comparing the super bridge number to the spiral and braid number of a knot. Towards the end of this section, we will now fit one more invariant into this chain.

Returning to our discussion of spiral and braid number, it is a fair question to wonder how it was proven that there exist knots with a spiral number strictly less than its braid number. Obviously, one does not have the time nor technology to manually construct every possible spiral and braid projection of a given knot and minimize over these sets to get its invariant value. Fortunately, for families of knots that share a similar projection form there exists a clever method for demonstrating they are indeed curly, proving the uniqueness of the spiral and braid number. We won't get bogged down on too many of the details regarding these families of knots, however, it turns out that by constructing a shared spiral projection one can create a set of knots that have the same Seifert projection [4] [9].

Seifert's algorithm is a map from a 2d knot projection to a set of polygons with curved edges each of which has a respective orientation [1]. The orientation of the resulting polygons is dependent on the orientation of the projected knot. If we were to compute the Seifert map of a spiral projection it turns out that the Seifert circles tell us about the number of maxima of the knot. In particular the Seifert map of a spiral projection must consist of Seifert circles of the form of o-circles and i-circles which can be thought of as planar polygons with either strictly convex sides or strictly concave sides. The spiral number of the spiral projection can then be represented by the following formula: $sp(P_K) = O - I$ where O is the number of o-circles and I is the number of i-circles of the Seifert map.

By using the formula one can create a class of knots that share a projection who's seifert map has the same number of o- and i-circles and therefore share a projection with the same spiral number. Knots that share the following projection as in Figure 6 were proven to be 3-curly knots with $sp(K) = 3$ and $\mathcal{B}(K) = 4$. This was proven by first determining that knots that share this projection form are curly (i.e. $sp(K) < \mathcal{B}(K)$) and then demonstrating that there cannot exist any 1- or 2-curly knots.

We will quickly outline the result that there cannot exist a 1- or 2-curly knot. If $sp(K) = 1 \implies b(K) \leq 1$ which as previously mentioned is only true for the trivial knot and for any trivial knot $\mathcal{B}(K) = 1$. Thus, $sp(K)$ cannot be less than $\mathcal{B}(K)$ which implies there

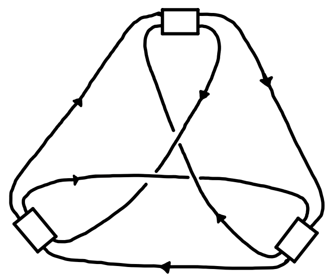

FIGURE 6. Spiral projection form that defines a family of 3-curly knots [4].

cannot exist a 1-curly knot. From a combinatorial argument one can show that the Seifert diagram of any curly projection of a knot must have at least one i-circle. Recall that an i-circle is a planar polygon from Seifert's map with strictly concave sides. In order for a planar polygon to consist of all concave sides it must have at-least 3 sides (since a two-sided polygon must have sides that curve towards each other). A small argument can demonstrate that any projection that realizes an i-circle with 4-sides must have more than two maxima in any given direction. Finally, in the case where the i-circle is a triangle (i.e. has 3 sides) it can be argued that the projection must have greater than 2 maxima demonstrating that a curly projection must have a spiral number strictly greater than 2. Therefore, there cannot exist a 2-curly knot. This argument concludes there cannot exist a 1- or 2-curly knot.

By adding alternating crossings to each of the boxes in Figure 6, the resulting projection will have a Seifert map demonstrating $sp(K) \leq 3$ due to the formula $sp(k) = O - I$. With an algorithmic and combinatorial argument we can demonstrate that any knot projections of this form can be put into a minimal projection demonstrating $\mathcal{B}(K) = 4$. Therefore, knots that share a projection of this form are curly. This combined with the result above that there cannot exist a $sp(K) = 1$ or $sp(K) = 2$ demonstrates these knots must be 3-curly. By choosing the number of crossings we add to each box we will either be left with a knot or link. All that we are stating is true for both knots and links, however, in this paper we will only be considering knots. The set of knots derived from this shared Seifert projection were the first knots found that demonstrated that spiral number can be distinct from braid number.

Claus Ernst [8] later introduced a handful of theorems and projection forms that define different classes of spiral knots. It turns out that all two-bridge knots (i.e $b(K) = 2$) with $\mathcal{B} \geq 4$ are curly. Since there cannot exist 1- or 2-curly knots this theorem also implies that for the class of two-bridge knots with $\mathcal{B}(K) \geq 4$, $sp(K) \geq 3$. Thus, in the case $\mathcal{B}(K) = 4$ the knot must have a spiral number of 3. This result alone allows us to compute a handful

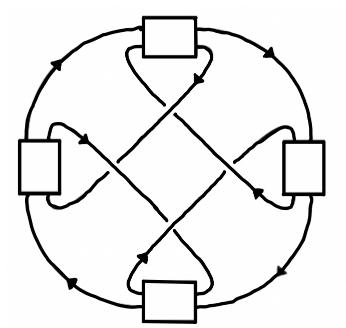

FIGURE 7. Spiral projection form that defines a family of 4-curly knots [9].

of spiral indices and create new bounds on the spiral number for such knots. Claus Ernst also proved the curly property (i.e. $sp(K) < \mathcal{B}(K)$) for the family of Montesinos Links by using a similar shared projection based argument, see Figure 7. The fact of whether a knot is curly or not mainly comes down to the observation of a shared projection that has $i - circles$ and that can algorithmically be translated into a braid projection demonstrating it is curly. Overall, the spiral number of a knot is still a very new knot invariant, however, it draws an interesting connection between the Seifert diagram of a knot projection and the number of maxima in that projection.

Lastly, we will define an older knot invariant first introduced in 1953 by John Milnor [12]. Milnor was most interested in characterizing the curvature and torsion of a knot. Before we define the curvature-torsion invariant, it will be useful to gain intuition about what the invariant describes. Milnor's invariant represents how much a knot conformation curves and twists in space. Since a knot conformation lives in 3-space from any given point the conformation can curve relative to two different planes. Namely the plane the point and previous curve lives in or the plane normal to that plane. Categorizing a value that represents the total amount of curving and twisting a knot conformation embodies is exactly the goal of this invariant. The curvature-torsion invariant is defined as follows.

$$(\kappa + \tau)(K) = \inf_{K \in [K]} \int_K (\kappa + |\tau|)$$

We won't go into too much detail about the actual definition above, however, essentially $(\kappa + \tau)(K)$ is the minimal curvature and torsion over all possible knot conformations. Therefore, if we were to associate a parameter corresponding to the tangledness of a knot conformation the curvature-torsion invariant's value would be the result of minimizing this parameter. It turns out with little work the curvature-torsion invariant fits nicely into our chain of invariants described above. It was proven in [4] that $psb(K) \leq (\kappa+\tau)(K)/2\pi \leq sp(K)$. It is important to note that in order to make $(\kappa + \tau)(K)$ fit into the chain of invariants we must divide it by 2π. This is meant to map the total angle of the knot conformation in space to a value related to the number of circles worth of curvature and torsion the conformation traverses. As we will see in later sections this value is closely related to the number of maxima the knot conformation realizes. We will not go into the details of this proof, however, we will gather some intuition for why $(\kappa + \tau)(K)/2\pi \leq sp(K)$. Recall, that the spiral number is associated to a projection that has no inflection points and therefore has an equal number of maxima in all directions. This implies that the planar curvature of a spiral projection that realizes the spiral number of a knot is $2\pi * sp(K)$. Since a projection of a knot has only curvature and zero torsion, if we were to map the knot projection to 3-space we would be able to lose curvature in exchange for torsion. Due to the properties of projecting down an object, the curvature will be able to decrease more than the increase we would see in torsion. Therefore, it is the case that the curvature plus torsion divided by 2π of the knot conformation must have at most the curvature that a spiral knot projection has divided by 2π. Thus, it follows $\frac{(\kappa+\tau)(K)}{2\pi} \leq \frac{2\pi * sp(K)}{2\pi} \implies (\kappa + \tau)(K)/2\pi \leq sp(K)$. These observations lead to the final chain of knot invariants that will guide us through the remainder of the paper.

$$b(k) \leq psb(k) \leq \frac{(k+\tau)[K]}{2\pi} \leq sp(k) \leq \mathcal{B}(k)$$

In terms of computing the curvature-torsion invariant there are greater technicalities when considering a smooth knot conformation. For a smooth knot we need to integrate the conformation to determine the total curvature and torsion. However, for a stick knot conformation this invariant is much easier to compute since from one stick to the next stick one can identify two angles that define how the conformation curves. Therefore, a stick knot conformation of n sticks only requires the computation and summation of $2n$ angles to determine the total curvature and torsion. Effectively a stick knot conformation partitions the conformation's curvature and torsion into smaller pieces defined by the angles each stick makes with its neighboring sticks. For this reason, the curvature-torsion invariant is much more straightforward when dealing with a stick knot conformation.

It turns out that the computation of the curvature-torsion invariant of a stick knot conformation is equivalent to the sum of the lengths of the tangent and binormal indicatrices. These curves represent the differential geometry of a closed space curve. The study of the relationship between knot conformations and the spherical indicatrices that the closed polygonal path of the conformation emits was first researched and categorized by Colin Adams, Dan

Collins, Katherine Hawkins, Charmaine Sia, Rob Silversmith, and Bena Tshishiku in [3]. This paper connected Milnor's curvature-torsion invariant to the lengths of the previously mentioned indicatrices as well as introduced the mathematics behind the defining theme for the remainder of the paper, the bridge map.

2. THE BRIDGE MAP

Now that we have defined the necessary knot invariants and framework needed for the rest of the paper, we can introduce the bridge map of a knot conformation which will play a crucial role for the remainder of our analysis. The bridge map, which we will formally define later, can be thought of as a partition of S^2 into regions that represent how the number of maxima of a knot conformation change when considering all projections of the conformation. It turns out that the bridge map isn't the only way of analyzing a given property of a knot conformation over the set of all possible projections. There also exists a crossing map first defined in [5] that partitions S^2 into regions that describe the number of crossings over all projections. The tangent indicatrix of a stick knot conformation graphs the occurrence of Reidemeister moves when considering different projections of the knot conformation. This method of analyzing a knot conformation over all projections can be generalized to any property of our liking and it tells us how the property changes when considering the set of projections of a knot conformation. First, we will define the general definition for analyzing a knot conformation in this aspect.

Definition 2.1. *The* ***map*** *of a knot conformation, C_K, is:*

$$m_{C_K} : S^2 \mapsto \mathbb{Z}$$

that associates a point on S^2 to the value of a computed property of the knot conformation. One can think of all points on S^2 as vectors infinite distance away from the knot conformation.

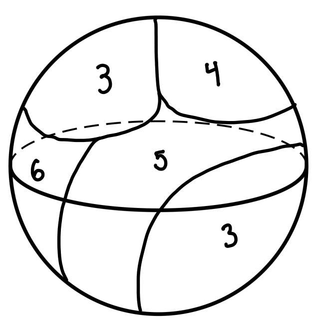

FIGURE 8. An example of the visualization of a map.

A reasonable way one can define such a property is by considering the projection that each vector around the knot conformation corresponds to. When graphing the map of a certain property, the knot conformation can be thought of as the center point of the sphere with radius infinity. When graphing the map of a property of a knot conformation we can also graph the stick knot conformation that corresponds to the map and visually see the projection corresponding to each point. Also, it is important to note that the map of a knot conformation is associated to a set of curves on S^2 that graph how the observed property changes with different vectors. This implies not all points on S^2 are mapped to values. However, since these curves have infinitely thin width, we will ignore this technicality in the definition above.

With this general object, the map of a property of a knot conformation, we can easily define the bridge map. There exists more than one way to define the bridge map. In the definition below we don't consider the projections of the knot conformation. Instead, we consider the projection of a knot conformation to an individual vector. We can think of this as a continuous overlapping closed path in $\mathbb{R}_{\geq 0}$. The bridge map is a map whose property of interest is the number of maxima of the knot conformation.

Definition 2.2. *The **bridge map** of a knot conformation, C_K, is:*

$$\mathfrak{B}_{C_K} : S^2 \mapsto \mathbb{Z}$$

that associates a point, $p \in S^2$ to the number of maxima of C_K when being projected to the vector defined by p.

The values on the bridge map are partitioned nicely according to the binormal indicatrix unioned with the anti-binormal indicatrix. We will refer to these curves as the binotrix and anti-binotrix which are related through an antipodal mapping. Recall that the length of the binotrix plus the length of the tangent indicatrix, tantrix, is equal to the curvature-torsion invariant value for the given knot conformation that is being graphed. We will later discuss the abstract properties of these curves embedded on S^2. Bounded regions on the bridge map represent regions that share the same value with respect to the map $\mathfrak{B}_{C_K}$. However, in general when working with the map of a specific property of a knot conformation, it is quite unlikely that there exists a nice algorithmic way of computing the boundary of the partitioned regions of the map. This algorithmic approach of computing the binotrix is exactly what makes this abstract object computationally reasonable.

Since we will be referring to the regions on a bridge map very often throughout the remainder of the paper it will be worthwhile defining a region formally.

Definition 2.3. *A **region** on a bridge map $\mathfrak{B}_{C_K}$ is a connected set of points on S^2 that share the same value. The binotrix and anti-binotrix define points where the number of maxima* *change as we project the knot conformation to a vector. Therefore, each region on the bridge map is bounded by the binotrix and/or anti-binotrix.*

A natural follow-up question is how does one compute the binotrix of a knot conformation. We will go into these details in Section 3. As previously mentioned, the computation of the binotrix is much simpler when working with a stick knot conformation. It turns out that graphing a minimal n-stick knot conformation allows us to make stronger claims about the combinatorial implications of the bridge map. For this reason, from here on out we will always refer to a stick knot conformation when discussing the bridge map of C_K.

As previously mentioned, the boundary of the regions of the bridge map is determined by the union of the binotrix and anti-binotrix. Since the anti-binotrix is simply the antipodal map of the binotrix we know that the bridge map's regions are symmetrical. This implies that by knowing the values of one hemisphere of points on the bridge map we can construct the values of the other hemisphere of points. Therefore, if one were to compute the bridge map in a brute force manner, one would only have to consider a single hemisphere's worth of vectors. The reason for this symmetrical property of the bridge map is somewhat intuitive. The bridge map is symmetrical since the number of maxima and minima in any given direction of a knot conformation is equivalent. Therefore, the label of a point p, corresponding to vector v is equivalent to the label of a point p' corresponding to vector $-v$, which is also equal to the number of minima with respect to v.

However, the bridge map can be viewed in a different way than what was described above. Instead of projecting a knot conformation to an individual vector and then counting the number of maxima with respect to that vector it turns out that we can identify points on S^2 to the planar maxima of a projection of the conformation. Mechanically when we look at a knot conformation from a certain angle we are seeing the projection of the knot conformation to the plane normal to our point of view. Also, we can project a knot conformation to a single vector, v, in two steps: first by projecting the conformation to the plane defined by the normal vector to v (i.e. the plane the vector lies in) and then by projecting the projection to the planar vector corresponding to v. Therefore, when viewing a projection of a knot conformation we are observing a planes worth of possible vectors that can be mapped to individual vectors on the bridge map. It follows that the number of the planar maxima of a knot projection relative to each vector $w \in S^1$ correspond to a single point on the bridge map. This observation leads to the following theorem which identifies what set of points the number of maxima of each planar vector of a knot projection correspond to.

Suppose we are viewing a knot conformation from vector $u \in S^2$. u can be mapped to a great circle on the sphere that corresponds to the intersection of it's normal plane and the sphere. This is exactly the points on S^2 that we would cut along to split the sphere into two even hemispheres. In other words the great circle defined by u, G_u, is the equator of S^2 if we were to think of u as the north or south pole. When viewing the knot conformation from u, we are actually viewing the projection of the knot conformation on the disk bounded by G_u. Therefore, projecting this projection to any planar vector and counting the number of maxima is equivalent to projecting the knot conformation to the vector in S^2 that lives in the disk bounded by G_u. This can be visualized in Figure 9 and also leads to the theorem below. Above we have laid out a majority of the foundation for the proof, therefore, it may seem slightly repetitive.

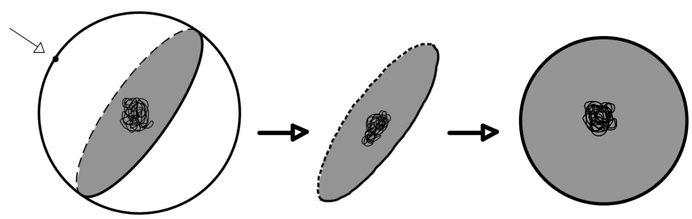

FIGURE 9. A point on S^2 defines a great circle which bounds a projection disc. The local maxima of this projection in each planar direction corresponds to the points on the original great circle of S^2.

Theorem 2.4. *The collection of local maxima values obtained from the planar vectors of a knot projection of C_K corresponds to a great circle's worth of points on the bridge map of C_K.*

Proof.
Recall that when projecting a knot conformation to a projection we can quantify the number of maxima in each planar direction by projecting this projection to a given vector in the plane. The combination of these two vector projections of a knot conformation C_K is equivalent to the two step process for projecting a knot conformation to single vector in S^2. Note that the initial vector required for projecting the knot conformation to a plane defines a great circle on the bridge map for each planar direction of the projection. Therefore, by projecting the projection to each of these planar direction we are essentially applying this two step process for projecting a knot conformation to a single vector for each vector on the great

circle defined by the projecting vector. Therefore, the local maxima of a knot projection of C_K correspond to a great circle's worth of points on the bridge map of C_K. $\square$

Corollary 2.5. *The bridge map describes the maxima for the set of all projections of a knot conformation.*

Proof.
This follows from Theorem 2.4 and the fact S^2 is a complete set of possible projection vectors of a knot conformation, C_K. $\square$

This result allows us to associate a planar projection of a knot conformation to a great circle on the bridge map. Using this theorem, we can better understand the relationship between our previously defined knot invariants and prove the equivalence of new definitions for these invariants in terms of the bridge map.

Recall that the bridge map is concerned with a specific conformation of a knot, C_K, therefore, we still must consider all the possible knot conformations when redefining such invariants. A great advantage of utilizing the bridge map is that we no longer need to manually compute and consider all the possible projections of a specific knot conformation. The bridge map itself already realizes the values of this set of projections. This approach allows us to partition the set of all projections by their corresponding knot conformation and purely analyze the bridge map corresponding to each conformation. This form of analysis can reduce the problem of computing an invariant's value into discovering a set of non-redundant knot conformations and algorithmically computing their associated maps. However, given how little we understand knot conformations this statement is quite unreasonable and most likely would be very computationally intensive.

Now we will redefine the bridge number of a knot. For clarity we will first redefine the bridge number in terms of an individual knot conformation. The relation between

Definition 2.6. *The **bridge number** of a knot conformation, C_K, is:*

$$b(C_K) = \min_{p \in S^2} \mathfrak{B}_{C_K}(p)$$

where p is a point on the bridge map. In other words, the bridge number of C_K is equal to the smallest labelled region on the bridge map.

Theorem 2.7. *Definition 2.6 is equivalent to Definition 1.3.*

Proof.
In Definition 1.3 we minimize the number of maxima of a knot conformation over two

vectors, first specifying the projection defining vector followed by a planar vector. However, due to Theorem 2.4 we can nicely relate a projection of a knot conformation to a great circle on the bridge map of that conformation. Therefore, by minimizing over all vectors within a projection we are minimizing over all points corresponding to a great circle of the bridge map and by minimizing over all projections of a conformation we are minimizing over all great circles of the bridge map. Since the set of all great circles of S^2 contains S^2, we are simply identifying the smallest labelled point on the entire bridge map which is exactly Definition 2.6. □

Another observation worth noting is that the original definition of the bridge number of a knot conformation, 1.3, is quite repetitive and recomputes every maximum infinitely many more times. Since the definition first projects the conformation onto a projection plane defined by $v \in S^2$ and then considers the number of maxima in each planar direction of this projection, for each point we are considering a great circle on the bridge map.

Since the original definitions consider all projection vectors in S^2 and each projection corresponds to a great circle on the bridge map one can easily convince themselves that by considering all possible projections and therefore all possible great circles the value of every point will be redundantly recomputed. With a little thought you can convince yourself that when considering a great circle of projection vectors, each of these projection vectors will define a great circle that all intersect at a shared point. Therefore, each point on the bridge map with this original definition will be recomputed for an S^1's worth of vectors in S^2.

To redefine the bridge number of a knot, K, we can minimize the smallest labeled region over all conformations of K. This new definition is equivalent to Definition 1.4 with $b(C_K)$ defined as in 2.6.

The bridge map also can be used to redefine the projective super bridge number of a knot. Again, we will first redefine the projective super bridge number in terms of an individual knot conformation.

Definition 2.8. *The* ***projective super bridge number*** *of a knot conformation, C_K, is:*

$$psb(C_K) = \min_{S^1 \subseteq S^2} \max_{p \in S^1} \mathfrak{B}_{C_K}(p)$$

where S^1 is a great circle on the bridge map and p is a point on that great circle of the bridge map. In other words the projective super bridge number of C_K is equal to the greatest value of the region a great circle on the bridge map intersects minimized over all possible great circles.

Theorem 2.9. *Definition 1.7 is equivalent to Definition 2.8.*

Proof. Recall for a given knot conformation the projective super bridge number is the minimum maximum number of maxima over all possible projections of the given conformation. From Theorem 2.4 we know the planar maxima of every projection of a knot conformation corresponds to a great circle on the bridge map. Therefore, we are minimizing the maximum point on an individual great circle over all possible great circles on the bridge map which is exactly Definition 2.8. □

In terms of actually identifying this value on the bridge map we will see later that there exists nice methods of graphing the bridge map that simplify finding the projective super bridge number of a knot conformation. Also, since the bridge map is symmetric by an antipodal mapping we know that every great circle will also be symmetrical since it consists of a proper subset of this antipodal relationship (i.e. all great circles consist of antipodal points). Therefore, we only have to know that values of a continuous semi-circle worth of points on the great circle to know the label of all the points on the great circle and therefore the maximum point on the great circle.

Similarly, one can redefine the projective super bridge number of a knot, K, by minimizing this value over all conformations. This is equivalent to Definition 1.8 with $psb(C_K)$ defined as in 2.8.

The bridge map can also be used to redefine the super bridge number of a knot. Again, we will first define the super bridge number in terms of a knot conformation.

Definition 2.10. *The* ***super bridge number*** *of a knot conformation, C_K, is:*

$$sb(C_K) = \max_{p \in S^2} \mathfrak{B}_{C_K}(p)$$

where p is a point on the bridge map. In other words, the super bridge number of C_K is equal to the largest labelled region on the bridge map.

Theorem 2.11. *Definition 1.9 is equivalent to Definition 2.10.*

Proof. In Definition 1.9 we maximize the number of maxima of a knot conformation over two vectors, first specifying the projection defining vector followed by a planar vector. By Theorem 2.4 this is equivalent to maximizing the maximum point on a great circle over all possible great circles. Since the set of all great circles of S^2 contains S^2, we are simply identifying the largest labelled point on the entire bridge map which is exactly Definition 2.10. □

Therefore, one can redefine the super bridge number of a knot, K, by minimizing this value over all conformations. This is equivalent to Definition 1.10 with $sb(C_K)$ defined as in 2.10.

We will now analyze how the spiral and braid number behave with respect to the bridge map of a knot conformation. The spiral number can be defined purely in terms of the bridge map; however, it remains unclear how one would go about doing this for the braid number.

Since the bridge map can define the spiral number, we know that it can at least provide a lower bound on the braid number (since $sp(K) \leq \mathcal{B}(K)$). As discussed in Section 1 the spiral and braid number require projections with very specific properties. Namely, they have planar curvature of the same sign along with an individual requirement of no fixed center point or a fixed center point. Since every knot conformation has a different set of projections associated to it, it is not always the case that a knot conformation realizes a projection that abides by these properties. Luckily, the bridge map immediately tells us when there exists a spiral projection of a knot conformation and, therefore, tells us if this conformation is a spiral conformation (recall Definition 1.15). Since the set of spiral projections contains the set of braid projections, any braid projection is also a spiral projection. As we will see in the next section, with not much extra work we can determine when a spiral conformation also represents a braid conformation.

Lemma 2.12. *The bridge map of a knot conformation corresponds to a spiral conformation if and only if there exists a region on the bridge map that contains a great circle.*

Proof. ($\Longrightarrow$) Recall that a spiral conformation is a knot conformation that has a spiral projection. Since a spiral projection requires the planar curvature to always have the same sign, we know the projection must have an equivalent number of maxima in all directions. By Theorem 2.4 we know that the spiral projection corresponds to a great circle on the bridge map with the same label. Since the bridge map is broken up into regions depending on their label this implies there exists a region on the bridge map that contains this great circle of points.

($\Longleftarrow$) If there exists a region on the bridge map that contains a great circle, by Theorem 2.4 we know there exists a projection with the same number of maxima in every direction. So as we travel along the projection we cannot have an inflection point since this will change the number of local maxima. Therefore, the projection cannot have any inflection points which implies the projection has an unchanging sign of curvature. Thus, the projection is a spiral projection and therefore the corresponding conformation is a spiral conformation. □

Figure 10 is an example of a bridge map that has a region that contains a great circle (the shaded region). Therefore, the knot conformation C_K corresponding to this bridge map is a spiral conformation and has a projection that realizes a spiral projection of K with spiral number 2. Since there exists no 1- or 2-curly knot this implies $\mathcal{B}(K) < 3$. Assuming K is non-trivial this implies $\mathcal{B}(K) = 2$ and $b(K) = 2$. Therefore, $2 = b(k) \leq psb(k) \leq \frac{(k+\tau)[K]}{2\pi} \leq sp(k) \leq \mathcal{B}(k) = 2$.

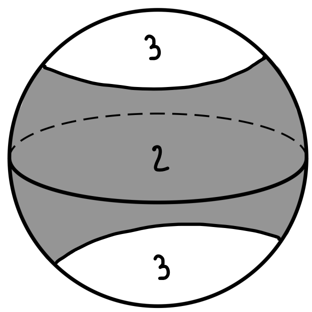

FIGURE 10. A bridge map that contains a great circle

With knowledge of the properties of the curves defining the boundary of the regions of the bridge map, we can better understand when exactly such a region can not contain a great circle. Recall that the bridge map of a knot conformation is divided into regions by the binotrix and anti-binotrix.

Lemma 2.13. *If the binotrix and anti-binotrix intersect, the bridge map cannot have a region that contains a great circle.*

Proof. Suppose the bridge map does have a region that contains a great circle. Then that great circle cannot intersect either the binotrix or anti-binotrix. In particular the binotrix must be entirely to one side of the great circle. Since the anti-binotrix is the set of all antipodal points to the binotrix, it must be to the opposite side of the great circle. Hence, the two cannot intersect one another. □

Figure 11 is an example of a bridge map whose binotrix and anti-binotrix intersect. Recall that the bridge map is symmetrical and therefore it is enough to consider a semicircle of a great circle that does not intersect the binotrix or anti-binotrix to determine if there exists a great circle with such a property. Clearly this is impossible when the binotrix and anti-binotrix intersect.

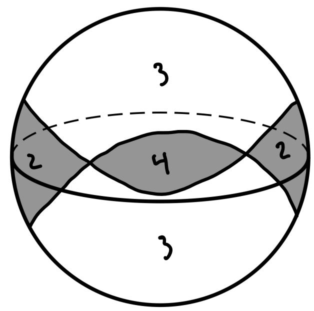

FIGURE 11. A bridge map whose binotrix and anti-binotrix intersect

It turns out that the contrapositive of Lemma 2.13 is not true in general. However, we will define a name for the special case when this is true.

Definition 2.14. *The **great ocean** of a bridge map $\mathfrak{B}_{C_K}$ is a region on the bridge map which contains a great circle. This can only occur when the binotrix and anti-binotrix do not intersect one another.*

A bridge map can only contain a single great ocean since any two great circles must intersect one another. With this definition we can now define the spiral number in terms of the bridge map. Since the spiral number only applies to spiral projections, we will have to restrict ourselves to considering spiral conformations in our definition.

Definition 2.15. *The **spiral number** of a spiral knot conformation, C_{spK}, is:*

$$sp(C_{spK}) = \textit{great ocean label of } \mathfrak{B}_{C_K}$$

Therefore, we can define the spiral number of a knot by minimizing over all possible knot conformations.

Definition 2.16. *The **spiral number** of a knot, K, is:*

$$sp(K) = \min_{C_{spK} \in [C_K]} sp(C_{spK})$$

Theorem 2.17. *Definition 1.19 is equivalent to Definition 2.16.*

Proof. In Definition 1.19 we minimize the spiral number over all spiral projections. Since spiral projections correspond to spiral conformations by minimizing over all spiral conformation we are effectively minimizing over all spiral projections. □

Defining the spiral number in terms of the bridge map doesn't really simplify the definition of spiral number as it did with the other invariants. This is the case because not all conformations realize spiral projections and the conformations that do realize spiral projections can only realize spiral projections of the same spiral number. Therefore, by defining the spiral number in terms of the bridge map we are not consolidating a whole lot of redundant computation. Despite this shortcoming, utilizing the bridge map allows us to quickly identify if a given knot conformation realizes a spiral projection which can be a very helpful tool.

As previously mentioned, we cannot define the braid number in terms of the bridge map, however, since the set of braid conformations is a subset of the set of spiral conformations it is possible that when realizing a spiral conformation we are also realizing a braid conformation. As we will see in the next section by graphing the bridge map of a spiral knot conformation we can manually view the spiral projection of the spiral conformation and determine whether or not it is also a braid projection and therefore is also a braid conformation. This quick check would provide an upper bound on the braid number if the conformation is indeed a braid conformation.

We will next become familiar with the computation of the bridge map and how we can utilize the definitions introduced in this section to determine invariant values for graphed stick conformations.

3. The Computation of the Bridge Map

Now that we have spent some time defining the bridge map and better understanding how it relates to the previously defined knot invariants, it will be worthwhile understanding how we can compute it. Computing the bridge map of a stick knot conformation only requires us to know the coordinates that define the particular conformation of whose bridge map we want to graph.

In our analysis we encode an n stick knot conformation by an ordered set of $n+1$ vertices in $\mathbb{R}^3$ where the first and last vertex in the ordered set are the same point. From the first vertex to the second in the ordered set we can graph a line segment that represents the first stick in our n stick knot conformation. We can do this for every pair of consecutive vertices resulting in a total of n sticks. The reason for appending the first coordinate to the end is for simplicity of computing the final stick in the knot conformation. However, how do we find the coordinates of a stick knot conformation? It turns out we can do this manually by first finding a stick projection of the knot we want to find coordinates for. Recall that manually finding a stick projection of a knot can be deceiving since in the plane we can draw 'sticks' that are required to bend in order for the planar representation to be embedded in $\mathbb{R}^3$. It turns out we can be more rigorous with this approach of finding a valid stick knot projection by associating a height label to each vertex: high, low, plane. By labelling two vertices defining a stick p we are defining the stick to lie in the plane of the projection. By labelling the next vertex h we are specifying that the stick adjacent to this stick lying in the plane tilts upwards to a point above the plane. By going through the cycle of vertices and labelling them successfully in a non-contradicting way we can demonstrate such a stick projection is valid. In general, it is good practice to start with labelling our first two chosen points (that define a stick) as p. An easy way to actually get the coordinate values once we find a proper planar stick projection, is by graphing this projection on a labelled grid and estimating the appropriate z-coordinate values based on the projection's labelling. By knowing we have a valid planar projection there will always exist a choice of z-coordinate values that will correspond to a stick conformation of the knot represented in the planar projection. However, it is possible that this is only the case for a very small interval of z-coordinates and therefore you may have to try a handful of values before getting a proper conformation.

Below we will go through an example of finding the coordinates of the trefoil knot. Figure 12 is a 6 stick projection of the trefoil knot with a valid labelling scheme. Since this projection has the property that every vertex is in the exterior region of the projection, for

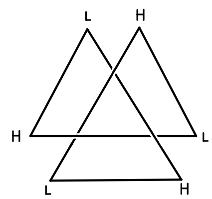

FIGURE 12. A 6-stick projection of the trefoil knot

symmetrical purposes we can evenly space these vertices along a unit circle to determine the x and y coordinates of the corresponding 6 stick conformation of the trefoil knot. We can now assign the z-coordinate of each vertex the value of -1 or 1 depending on if the respective label is L or H.

It turns out that evenly spacing the coordinates along the unit circle and picking heights of -1 and 1 won't actually properly create a conformation of the trefoil knot. By being purely symmetrical the resulting closed chain of sticks has singularity points rather than the necessary over/under crossing a stick knot conformation has. Fortunately, there is an easy fix to this by making the points along the unit circle to be symmetrical within some $\delta > 0$. By making the coordinates slightly non-symmetrical we can eliminate the self-intersections and create a proper conformation of the trefoil knot in $\mathbb{R}^3$.

Also one could have worked around the issue of evenly spaced vertices by adjusting the chosen z-coordinate values, however, I found that the coordinates from Figure 13 create a more visually appealing conformation when graphed.

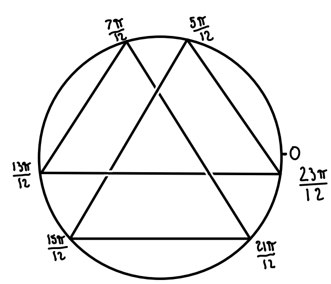

FIGURE 13. Example of using the unit circle to find valid x, y coordinates for Figure 12

By using $\cos$ and $\sin$ we can compute the numerical values of the x and y coordinates of each vertex spread along the unit circle. Combining this with the z-coordinate value we have the corresponding coordinates required to embed the following stick projection in Figure 12. With the data below we can see that the first coordinate in the list corresponds to the point $(\cos(\frac{-\pi}{12}), \sin(\frac{-\pi}{12}), -1)$ and the second coordinate is $(\cos(\frac{5\pi}{12}), \sin(\frac{5\pi}{12}), 1)$.

```
%Trefoil stick knot conformation coordinates
numx = [0.96592582628 0.2588190451 -0.70710678118 0.70710678118 -0.2588190451
    -0.96592582628 0.96592582628];
numy = [-0.2588190451 0.96592582628 -0.70710678118 -0.70710678118 0.96592582628
    -0.2588190451 -0.2588190451];
numz = [-1 1 -1 1 -1 1 -1];
```

With this list of coordinates, we can now graph the following stick knot conformation and thanks to MATLAB after graphing the conformation we can toggle the view to observe all projections of the conformation. In Figure 14 we can see some snapshots of the graphed trefoil stick knot conformation from the coordinates above.

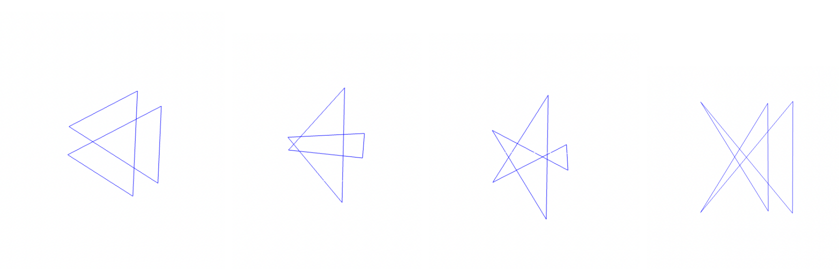

FIGURE 14. Different viewpoints of a trefoil stick knot conformation

Since a knot conformation is a closed loop in $\mathbb{R}^3$ how do we know which coordinate to pick as the starting coordinate and ending coordinate in our ordered list of vertices and how do we know which neighboring vertex to choose second? In terms of graphing the stick knot conformation the order of the vertices or choice of direction does not impact the resulting conformation. However, it turns out that this choice does have a slight impact when computing the binotrix of the stick knot conformation. Recall that the bridge map is described by the binotrix and anti-binotrix. It turns out that computing the binotrix of one orientation direction of vertices yields the anti-binotrix relative to the computation of the reversed direction of vertices. However, since the bridge map consists of both the binotrix and anti-binotrix and an antipodal map applied twice to a curve is the identity map, the orientation we denote when ordering the vertices doesn't affect the resulting bridge map.

To compute the binotrix of a stick knot you first must compute the coordinates of the tantrix. The coordinates of the tantrix of a stick knot conformation are the points on S^2 if you were to treat each stick in the knot conformation as an oriented vector that extends out to infinite. In Figure 15 the points where the extension of each stick in the closed polygonal path hit the sphere are the defining coordinates of the tantrix. For visual purposes Figure 15 is a simple 3 stick closed polygonal trivial loop, however, as you consider a more complex tangled loop this image will become increasingly complex. Actually, computing these coordinates on S^2 is fairly straightforward.

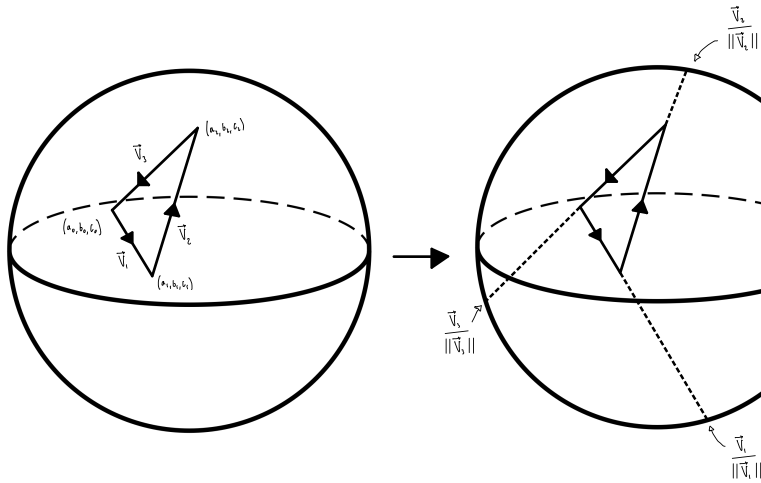

FIGURE 15. Coordinates of the tantrix of this simple 3 stick closed polygonal path

```
%computes tantrix coordinates
for i = 1:(length(zcoord)-1)
   xtantrix(i) = xcoord(i+1) - xcoord(i);
   ytantrix(i) = ycoord(i+1) - ycoord(i);
   ztantrix(i) = zcoord(i+1) - zcoord(i);
   magnitude = sqrt(sum([xtantrix(i); ytantrix(i); ztantrix(i)].^2,1));
   xtantrix(i) = xtantrix(i)/magnitude;
   ytantrix(i) = ytantrix(i)/magnitude;
   ztantrix(i) = ztantrix(i)/magnitude;
end
```

By using the endpoints of each stick in the stick knot conformation you can easily compute the vectors that correspond to each stick in the chain. Since we graph these coordinates on a unit sphere, we simply have to normalize the vector in order to get the corresponding point on S^2. This computation will give us an ordered set of points on the unit sphere corresponding to the tangent indicatrix. By connecting each pair of consecutive points on the unit sphere by geodesics, we can graph the curves that define the occurrence of a subset of reidemeister moves and would contribute to graphing the crossing map of a stick knot conformation. However, we are concerned with graphing the bridge map whose points are defined by the binormal vectors of each pair of sticks in this chain of normalized stick vectors. Therefore, by iterating through each consecutive pair of points on the tantrix we can compute their cross product and again normalize to find the points on the unit sphere corresponding to the binormal vector of each vertex of the knot conformation.

```
%computes binotrix coordinates
for i = 1:(length(ztantrix)-1)
   xbinotrix(i) = ytantrix(i)*ztantrix(i+1) - ytantrix(i+1)*ztantrix(i);
   ybinotrix(i) = xtantrix(i)*ztantrix(i+1) - xtantrix(i+1)*ztantrix(i);
   zbinotrix(i) = xtantrix(i)*ytantrix(i+1) - xtantrix(i+1)*ytantrix(i);
   magnitude = sqrt(sum([xbinotrix(i); ybinotrix(i); zbinotrix(i)].^2,1));
   xbinotrix(i) = xbinotrix(i)/magnitude;
   ybinotrix(i) = ybinotrix(i)/magnitude;
   zbinotrix(i) = zbinotrix(i)/magnitude;
end
```

By connecting the final ordered list of points on the binormal indicatrix we can graph the chain of geodesic curves connecting each pair of consecutive points to get the binotrix of the stick knot. To finish the computation of the bridge map we can graph the antipodal map of these coordinates and their corresponding geodesics.

What do the curves on the bridge map connected by these points defined by the binormal vector tell us about the knot conformation? It turns out that each geodesic arc represents a set of vectors that are normal to the stick connecting the two graphed vertices in the stick knot conformation corresponding to the points on the binormal indicatrix. Therefore, when projecting the knot conformation to this vector the stick that is normal to this vector will not count towards a maxima or minima. In Figure 16 we can see an example of what going over the bintorix/anti-binotrix on the bridge map represents in terms of the vector's view of the knot conformation. Specifically in this example we are moving from the larger labelled region to the smaller labelled region decreasing the number of maxima in that vector direction.

Since the points along a geodesic of the bridge map represent vectors normal to a stick in the conformation, we can technically assign values to the points along the binotrix and anti-binotrix. Since a completely horizontal stick does not count towards a maxima technically when directly on the binotrix/anti-binotrix you are already realizing less maxima. Therefore, the points along the binotrix/anti-binotrix have the value of the smaller labelled region and therefore belong to that region. However, in practice the binotrix and anti-binotrix can intersect one another and intersect itself. What does it mean in terms of the projection of the knot conformation when these curves have geodesics that intersect one another? Since two separate geodesics are required in order for an intersection to occur between the

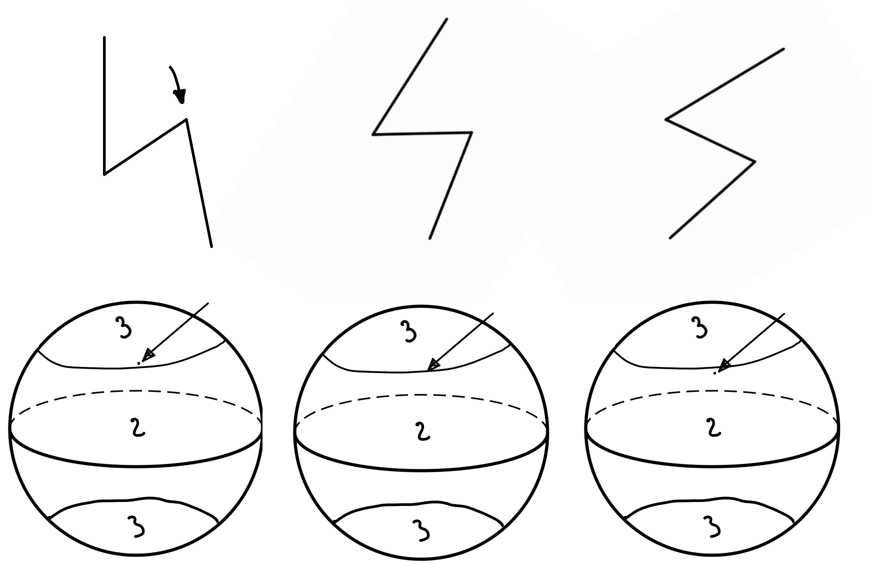

FIGURE 16. Visual representation of what the curves on the bridge map physically represent when projecting a knot conformation

binotrix/anti-binotrix and itself or the binotrix and anti-binotrix we know that the vector at this crossing point will represent a point defining a vector normal to two separate sticks in the knot conformation. This is the case since each individual geodesic arc represents a set of vectors normal to a single stick and at this intersection these two sets of vectors overlap and therefore correspond to a vector normal to two sticks at once. This implies the 4 regions adjacent to a crossing on a bridge map have must have two regions that differ by a gap of 2. With some thought you can convince yourself that all crossings on the bridge map can be expressed in the following form as in Figure 17 where two of the diagonal regions have the same value and the other two diagonal regions have values one greater and one less than the value of the other 2 regions.

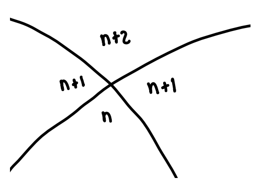

FIGURE 17. A general form of any crossing on the bridge map

The point of intersection of the two geodesic curves will have a value corresponding to the smallest labelled region since it is the first vector to be normal to these two sticks that no longer count towards a maxima. Therefore, in Figure 17 the $n + 2$ region is an open set with no boundary components since the points along the curve belong to the smaller labelled region. The $n + 1$ regions are two disjoint clopen sets with boundary along the $n + 2$ border belonging to each region respectively. Lastly the n region is a closed set with all its boundary belonging to its region. Note that if n is the smallest labelled region on the entire bridge map then this region will be a closed set with all its boundary belonging to it. In the opposite case the region on the bridge map corresponding to the greatest value will be an open set since none of its boundary will belong to its region. Unfortunately, I haven't had the chance to further analyze this observation in practice, however, it would be very interesting to better understand what it means to be an open, clopen, or closed region on the bridge map and if this tells us anything interesting about knot invariants in general.

We can now view the bridge map as the union of connected sets that can be open, clopen, or closed and share a common number of maxima when projecting the knot conformation from a vector defined by a point in that set.

Now that we have a better understanding of how to compute the bridge map of a stick knot conformation and what the points along the binotrix and anti-binotrix represent we will observe the binotrix of a handful of stick knots. However, manually computing the coordinates of stick knot conformations is quite tedious and can take a lot of time and trial and error. Fortunately, Clayton Shonkwiler and Thomas Eddy made a computer program [6] which randomly generated closed polygonal paths out of sticks and identified which knot these stick conformations corresponded to. The purpose of this program was to discover examples of new minimal stick knot conformations for knot's whose stick number was unknown. Along with the code for their program they included a folder with the coordinates for over 1000 stick knot conformations some of which are minimal stick conformations.

For the remainder of this section, we will graph and analyze the bridge map of a handful of stick knot conformations using a program I wrote [13] and discuss some interesting properties we noticed from analyzing these bridge maps. When graphing these knot conformations along with utilizing previously mentioned bounds we discovered new invariant values and bounds on invariant values for a handful of knots. This implies that this method of analyzing knot invariants is quite efficient and computationally feasible once finding stick conformation coordinates. As we will see later, graphing minimal stick knot conformations gives us better bounds on invariant values over graphing non-minimal stick knot conformations. Also, by finding more symmetric and/or 'nicer' stick knot conformations potentially we can combinatorially simplify the bridge map yielding stronger bounds on these invariants. Perhaps stick knot conformations that correspond to a smaller curvature-torsion invariant value have nicer bridge map properties.

Starting with the trefoil knot and the coordinates we computed above, the graphed bridge map is expressed in Figure 18. Unfortunately, with a screen shot of the graphed bridge map we can only view one hemisphere of the bridge map at a time, however, since one hemisphere of the bridge map is symmetrical to the other hemisphere through an antipodal mapping, we can essentially understand everything by observing only one hemisphere. Notice that the bridge map of the trefoil knot consists of two trivial simple closed curves embedded on S^2. These two disjoint curves represent the binotrix and anti-binotrix and since there exists a great ocean (recall Definition 2.14) we know that this knot conformation realizes a spiral projection by Lemma 2.12.

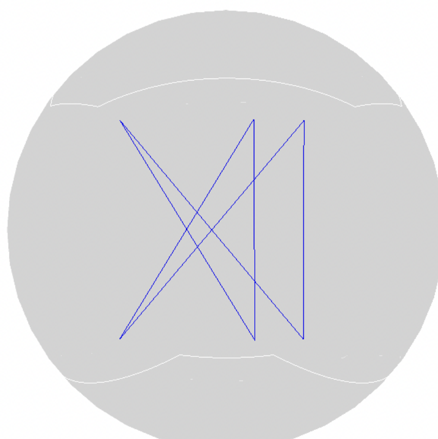

FIGURE 18. The graphed bridge map of a trefoil knot conformation

In order to actually compute the values of the regions in Figure 18 we can use Theorem 2.4 to relate the current projection we see to the points along the equator of the bridge map from our current angle of view. From this result we know the great circle defined by the projection vector has values equal to the local maxima of this projection relative to its planar maxima. Therefore, the northern and southern caps of the bridge map in Figure 18 each correspond to a value of 3 and the great ocean region corresponds to a value of 2 (since the planar maxima in the horizontal directions of the projection is 2). This also implies that the points along the binotrix and anti-binotrix belong to the region of label 2 (the inner region) since the boundary points always belong to the smaller region. Therefore, the bridge map of this graphed trefoil stick knot projection is the union of two disjoint open sets both with a value of 3 and a closed set with a value of 2. Also, immediately by graphing the bridge map corresponding to this stick conformation we can identify an upper bound on the value of the invariants defined by the bridge map. The computational benefit of using the bridge map is that without even observing all the projections of a knot conformation we can identify the best upper bounds for each invariant quite easily.

We will next analyze the bridge map of a stick knot conformation corresponding to the knot 7_2. The image on the left in Figure 19 is a image of one hemisphere of the bridge map. Although we cannot see the backside of the sphere, we know that this curve represents either the binotrix or anti-binotrix and therefore everything we would see if we rotated the sphere 180 degrees would be the antipodal map of this closed curve. Clearly from analyzing the upper hemisphere, the bridge map has a great ocean and therefore corresponds to a spiral conformation. Therefore, by viewing the knot conformation from the point that defines a great circle contained in the great ocean of the bridge map, we can manually view the projection of the knot conformation that corresponds to a spiral projection. Recall this is the case since the projection defined by this view has planar maxima defined by the constant labelled region containing the equator. By zooming in on the associated projection (the image on the right) we can see the spiral projection of this spiral knot conformation. It is also worth noting that this projection has a fixed center point and therefore also corresponds to a braid projection. This implies that the following graphed stick conformation is a braid conformation (recall braid conformations are a subset of spiral conformations).

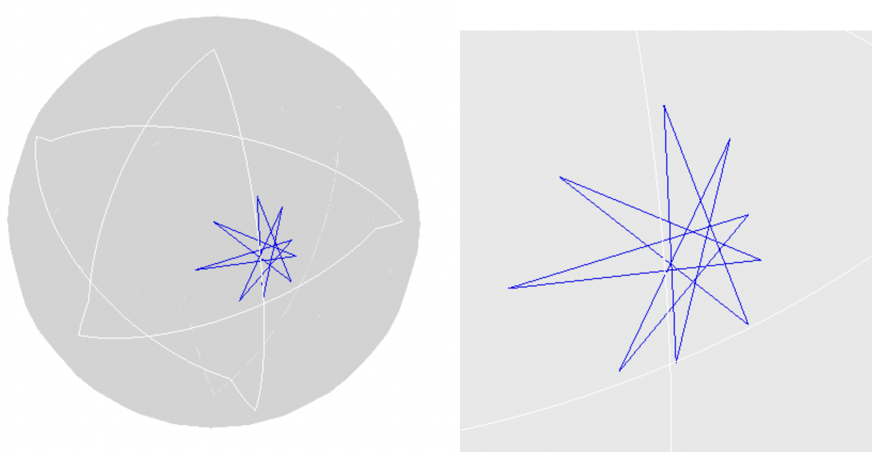

FIGURE 19. Bridge map of conformation of 7_2

Although we could not define the braid number in terms of the bridge map, by determining if the spiral projection of a spiral conformation has a fixed center point and therefore also represents a braid projection, we can realize the braid number of a stick conformation. Therefore, the union of information from the graphed bridge map and graphed stick knot conformation is enough to define the braid number. We will do this formally below.

Definition 3.1 (Romrell)**.** *The **braid number** of a braid knot conformation $C_{\mathcal{B}K}$ is*

$$\mathcal{B}(C_{\mathcal{B}K}) = \textit{great ocean label of } \mathfrak{B}_{C_K}.$$

Therefore, we can define the braid number of a knot by minimizing over all possible knot conformations.

Definition 3.2 (Romrell)**.** *The **braid number** of a knot, K, is:*

$$\mathcal{B}(K) = \min_{C_{\mathcal{B}K} \in [C_K]} \mathcal{B}(C_{\mathcal{B}K})$$

Unlike the case of a spiral conformation, we cannot identify a braid conformation purely from analyzing the bridge map. However, by using extra information we can do so and therefore formally define the braid number as above.

In Figure 19 the label of the great ocean corresponds to the value of 4 which implies that $\mathcal{B}(7_2) \leq 4$. It turns out that the $\mathcal{B}(7_2) = 4$. By altering the point of view of the bridge map and knot conformation we can identify the labels of the other regions. Each exterior leaf region of the star has a label of 3 and the interior region has a label of 2. Therefore, the bridge map suggests that $b(7_2) = 2$ since $b(K) = 1$ if and only if K is the trivial knot and 7_2 is not the trivial knot. Also, from the chain of inequalities we know $psb(7_2) \leq 4$, $sp(7_2) \leq 4$, and $sb(7_2) \leq 4$.

The last invariant we have yet to connect to the bridge map is Milnor's curvature-torsion invariant. Since the value of the curvature-torsion invariant is simply the length of the binotrix plus the length of the tantrix and all of these coordinates must be computed in order to compute the bridge map we can easily find the length of the geodesics with these coordinates as shown in the code below.

```
%a,b,c are lists of the x,y,z coordinates of a generic set of points
function val = findLength(a,b,c)
   total = 0;
   for i = 1:(length(c)-1)
      current = acos(a(i)*a(i+1) + b(i)*b(i+1) + c(i)*c(i+1));
      total = total + current;
   end
   val = total;
end
```

By passing in the computed tantrix and binotrix points we can utilize the function above to compute the curvature-torsion invariant value for a specific stick knot conformation and display this value divided by 2π.

```
%computation of Milnor's curvature-torsion invariant
len_tan = findLength(xtantrix, ytantrix, ztantrix);
len_bin = findLength(xbinotrix, ybinotrix, zbinotrix);
milnor = len_tan + len_bin;
display(milnor/(2*pi));
```

In this case of Figure 19 the computed $(\kappa + \tau)(C_K)/2\pi = 5.2569$ which implies $(\kappa + \tau)(7_2)/2\pi \leq 5.2569$, however, since $(\kappa + \tau)(7_2)/2\pi \leq sp(k) \leq 4$ the following computed value isn't the best bound the bridge map tells us. In fact, for a majority of the stick knot coordinates gathered from [6] the curvature-torsion invariant value divided by 2π does not provide any improved bounds. This is most likely due to the way the stick coordinates were generated by the computer program. It would be very interesting to try to generate stick knot conformations that do realize the minimal curvature-torsion value. Perhaps such conformations generate bridge maps that realize other invariant values.

We will next analyze the bridge map of a stick conformation corresponding to the knot 6_2. Figure 20 contains images from my program of the bridge map of the knot conformation from two different angles. Both images are different hemispheres from the same bridge map. Clearly by looking at the image on the left, the bridge map contains a great ocean and is therefore a spiral conformation. Using similar techniques from the previous examples we can analyze the projections and determine the labels of the regions and determine upper bounds on certain invariant values. However, this example also demonstrates a nice property that we discussed earlier in this section. That is what the curves on the bridge map represent and how they determine where the number of maxima change.

Recall from the computation steps described above the points along the binotrix and anti-binotrix actually belong to a certain region and represent vectors that are normal to certain sticks in the stick knot conformation. We can see this visually in the right image of Figure 20. Note that this bridge map represents a spiral conformation and therefore the binotrix and anti-binotrix do not intersect each other. However, as we can see the binotrix and anti-binotrix both have a single self-intersection point. In the image on the right, we are observing the projection when moving these singularity points to the equator of the bridge map. Therefore, there exists 2 different sticks that are normal to the vectors of both singularity points. We can visually see these two sticks on the graphed stick knot conformation.

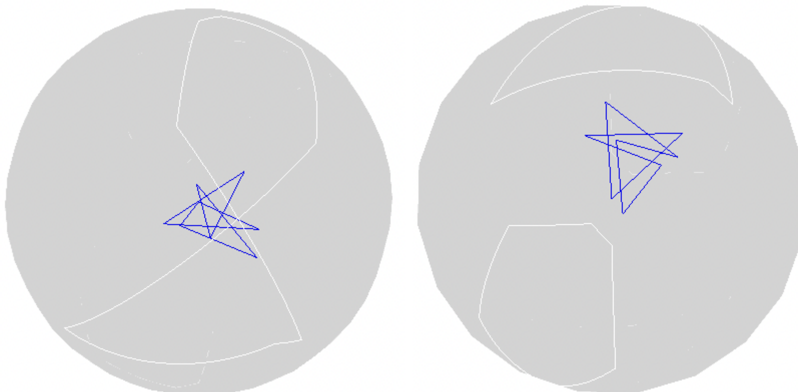

FIGURE 20. Two different viewpoints of the same bridge map describing a stick knot conformation of 6_2

In Figure 21 we will next analyze an example of a bridge map whose binotrix and anti-binotrix intersect. Such cases are not as interesting computationally since we cannot realize a bound on spiral or braid number, however, we will utilize this example to better understand how we can determine the projective super bridge number of a knot conformation from the bridge map. In this example we essentially have two oval-like binotrices that extend far enough to create a pocket of intersection at two points on the sphere. Recall that the bridge map is symmetrical and therefore the backside of the bridge map from Figure 21 is simply a rotated mirror image of what we see here. By moving around the knot conformation and counting the number of maxima we can determine the values of each region on the bridge map. The region that is defined by the intersection of the binotrix and anti-binotrix has a label of 2 while the regions that share its edges have labels of 3 and the regions that are diagonal from the crossings have label 4. Therefore, this bridge map demonstrates $b(7_4) = 2$ and $sb(7_4) \leq 4$.

Although we cannot realize a spiral or braid number from this bridge map we can realize a bound on $psb(7_4)$ which will end up actually equaling its value. Recall that the projective super bridge number of a knot conformation is the maximum region a great circle on the bridge map intersects minimized over all possible great circles. From analyzing Figure 21 it is clear that in order to minimize over all great circles we would ideally want to find such a great circle that does not intersect the region of label 4 (i.e the far east and west regions in the Figure). Also, it is slightly difficult to consider an entire great circle when only being able to view a hemisphere of the bridge map. However, due to the symmetrical properties of the bridge map it is enough to consider a semi-circle on a hemisphere of the

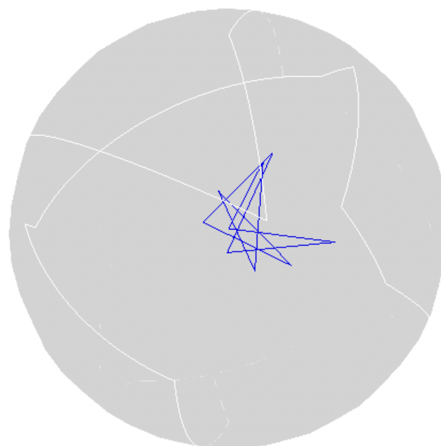

FIGURE 21. Bridge map of a conformation of 7_4 whose binotrix and anti-binotrix intersect

bridge map to compute the value of the projective super bridge number of the graphed conformation. Clearly there exists a semicircle that avoids these two east and west regions and goes through the regions bounded by the binotrix and anti-binotrix and their intersection. These regions correspond to values of label 2 and 3, therefore, the maximum of these two is 3 and $psb(7_4) \leq 3$. As mentioned it turns out that $psb(7_4) = 3$.

We will next analyze an example of two knot conformations who combinatorically realize the same bridge map, however, geometrically do not. This will lead to some interesting questions and a result proven in the Section 4. In Figure 22 we can see two bridge maps that visually look different however, are combinatorially equivalent. From my program it is difficult to see but, the bridge map of 9_{27} actually has a small triangular region at the intersection of each leaf region. On both bridge maps this small triangular region corresponds to a label of 2, each of the leaf regions attached to this triangular region correspond to a label of 3 and the great ocean region has a label of 4. This implies the invariant value bounds we can obtain from each bridge map are the same set of numbers for both knot conformations of 9_{27} and 9_{31}. However, geometrically it seems as if the graphed knot conformation of 9_{27} is more close to realizing a different combinatorial pattern. Perhaps if we wiggled the vertices defining the stick knot conformation of 9_{27} (preserving the knot the conformation it defines) we could shift around the points of the binotrix to create a different combinatorial pattern. With some sense of the size of the regions on the bridge map we can derive a notion of how close the graphed stick knot conformation is to realizing a different combinatorial pattern. Perhaps the vertices of a stick knot conformation can be shifted around to realize a different bridge map without changing the number of sticks.

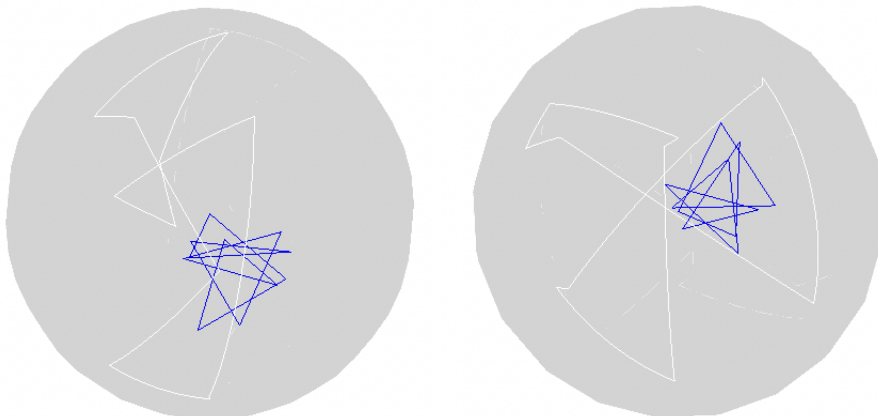

FIGURE 22. 9_{27} (left) and 9_{31} (right)

This distinction between combinatorial equivalence and geometric equivalence is quite interesting and presents questions for future areas of research. Perhaps we could use the bridge map to define an equivalence class of knot conformations that realize the same binotrix and are connected through valid knot preserving ambient isotopy moves from one stick conformation to another. However, obviously when working with a smaller number of sticks you will be more restricted with the knot preserving moves on the stick coordinates. You can fix this problem by adding in more sticks. Technically a smooth knot conformation can be thought of as a stick conformation with infinitely many sticks of infinitely small length. Since all smooth knot conformations of a knot are connected through such shifts, all stick knot conformations can also be connected through such shifts combined with the addition/subtraction of stick edges. This notion of an equivalence class of stick knot conformation defined by the combinatorial complexity of its corresponding bridge map would properly partition the set of all possible stick knot conformations. Each of these sets associated to a specific bridge map would have a stick conformation with a minimal number of sticks. Therefore, by only considering each of these minimal stick conformations for each possible bridge map we can compute the invariant values of a knot much more efficiently by ignoring redundant conformations. In theory this task is feasible, however, computationally it would require a lot of simulation and data analysis.

Lastly, we will consider some different knot conformations that share bridge maps that are combinatorially similar. In Figure 23 you can see three different bridge maps corresponding to minimal 9-stick representations of knots 7_4, 7_5, and 7_6. Below each image of the graphed stick knot conformation and corresponding bridge map, we can see a planar representation of the binotrix and anti-binotrix and how they intersect. As we will explore more in the next section it turns out that the labelling property of the bridge map can be expressed by an orientation on the binotrix and anti-binotrix. In Figure 23 the hand drawn planar representations have an orientation that signifies how the labels of the regions are related. The region to the right of the oriented curve has a greater number of maxima than the region to the left. We will explore this idea more in Section 4.

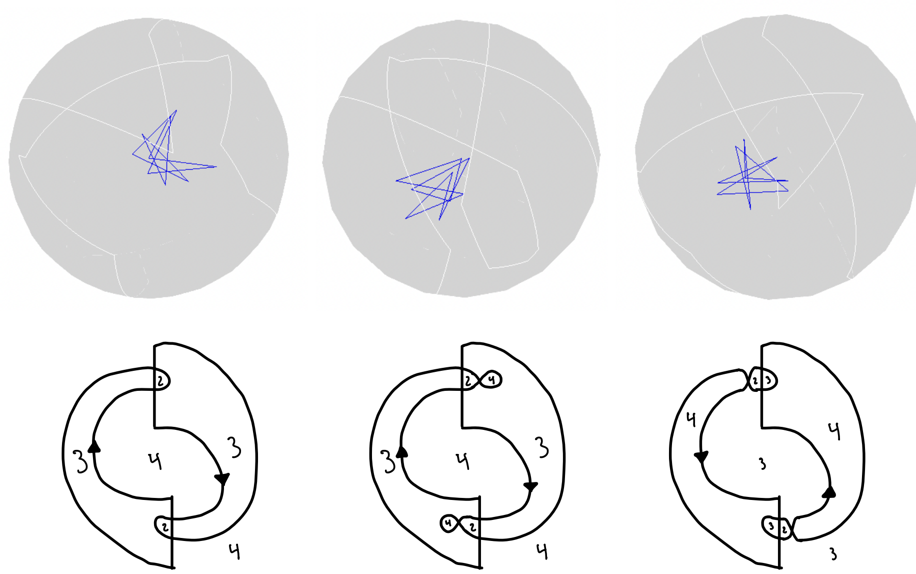

FIGURE 23. 7_4 (left), 7_5 (middle), 7_6 (right)

When looking at the actual invariant values for 7_4, 7_5, and 7_6 we know that

$$
\begin{gathered}
b(7_4) = b(7_5) = b(7_6) = 2 \\
psb(7_4) = psb(7_5) = psb(7_6) = 3 \\
(\kappa + \tau)(7_4)/2\pi = (\kappa + \tau)(7_5)/2\pi = (\kappa + \tau)(7_6)/2\pi = 3 \\
sp(7_4) = sp(7_5) = sp(7_6) = 3 \\
\mathcal{B}(7_5) = 3, \mathcal{B}(7_4) = \mathcal{B}(7_6) = 4 \\
sb(7_5) = sb(7_6) = 4 \text{ and } 3 \leq sb(7_4) \leq 4.
\end{gathered}
$$

Due to the great similarity between the invariant values, it almost seems as if there might exist shifts to the knot conformations that can combinatorially change the bridge map to each of the 3 different patterns. Assuming $sb(7_4) = 4$ the only invariant value they would differ by would be the braid index which as we have previously seen cannot be purely defined in terms of the bridge map. Perhaps through the shifting of vertices and addition/subtraction of sticks as mentioned earlier we could categorize conformation shifts that add such twist moves to the binotrix and anti-binotrix of the bridge map.

Now that we better understand how to compute the bridge map, what the curves on the bridge map represent and some interesting examples of bridge maps we will investigate and prove properties of the bridge map.

4. The Properties of the Bridge Map

What are the combinatorial properties of the bridge map and why does it suffice to compute the value of only two regions to know the value of the remaining regions? Since the bridge map consists of two closed curves, the binotrix and anti-binotrix, embedded on S^2, it will be useful to understand how closed curves with respect to the bridge map's labelling property behave. The pair of closed curves on the bridge map can be thought to each have an orientation which determines how the number of maxima change as you view the knot conformation from different angles. Recall the points along these curves belong to the region of smaller value. The absolute value of the difference of the number of maxima between any two adjacent regions on the bridge map is 1. Therefore, by finding the relationship between two adjacent regions we can figure out how the regions behave relative to one another. This will allow us to define a consistent labelling property for the entire curve. We will do this by associating an orientation to the closed curve. Once we find the respective orientation of the closed curve, we can use the orientation to compute the value of the remaining regions bounded by the closed curve.

However, on the bridge map we are dealing with two closed curves the binotrix and anti-binotrix. It turns out that by finding the orientation of one of the closed curves we can determine the orientation of the other closed curve through the antipodal mapping.

To better understand how the bridge map behaves we first consider how a single oriented closed curve behaves. It turns out that when analyzing a closed curve there exist different equivalence classes when you consider the curve to live in the plane versus on the sphere. We first analyze a closed curve in the plane and then relate the planar instances to spherical instances.

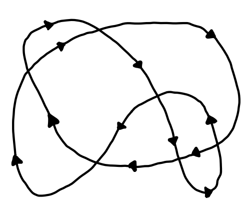

FIGURE 24. An arbitrary closed curve.

Consider the closed curve in Figure 24. To capture the relationship between regions on the bridge map we will associate an orientation to the closed curve. The orientation will define a right-hand rule which means that the region to the right of the orientation is equal to the value of its neighboring region $+1$. For simplicity suppose the exterior region has a value of n. By applying the right-hand rule to the oriented diagram, we can determine the value of the other regions in terms of n.

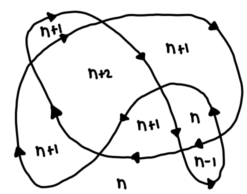

FIGURE 25. Labelling of Figure 24 with respect to right hand rule.

As you can see in Figure 25, after considering the orientation and labelling of the exterior region we can compute that the resulting diagram will consist of one region with label $n-1$, two regions with label n, four regions with label $n+1$, and one region with label $n+2$. The labels of these regions are defined relative to the value of the exterior region. However, what happens if we consider the values relative to a different region, perhaps the $n+2$ region in Figure 25? For clarity we will denote the new label with x. See Figure 26 for the new labelling scheme when initializing this new region with value x. We could have also determined the value of the remaining regions after this renormalization process by setting $x = n+2$ and solving for the regions' values in terms of x in the original diagram. Renormalizing the labelling scheme doesn't change the relative ordering of the labels and therefore doesn't change the combinatorial complexity of the object. Therefore, for consistency when analyzing planar diagrams, we will always initialize the exterior region to n.

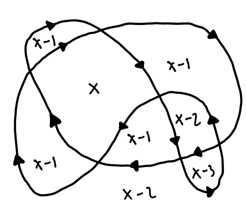

FIGURE 26. Renormalization of Figure 25.

What happens if we reverse the orientation of the closed curve and relabel the respective regions? As you can see in Figure 27 by flipping the orientation we change the sign of what was initially being added and subtracted in Figure 25. Intuitively this makes sense because when reversing the orientation, we are reversing the right hand-rule which now acts in the opposite direction. Therefore, by changing the orientation of the closed curve we mirror the difference of the labels with respect to the initialized region. Unlike renormalizing, when changing the orientation of the closed curve we are combinatorially changing the relative labelling.

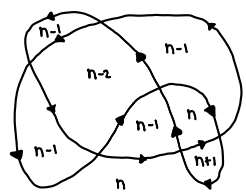

FIGURE 27. Reversed orientation of Figure 24 labelled with respect to the right hand rule.

What does the set of all unique planar closed curves that abide by this labelling property look like? Since up to a change of variables it does not matter which region, we label n, the only determining factor is the orientation of the closed curve. Therefore, this question can be reduced to categorizing all oriented planar closed curves with double points. However, since a closed curve has only two possible orientations one can simply categorize all closed curves with double-points and then duplicate the set and assign opposite orientations to each pair. It turns out that in some cases this does not create two unique diagrams. In the plane you can still have equivalence up to rotation.

As you can see in Figure 28 there are only 3 unique single crossing oriented closed curves. Note that the two one crossing "infinity sign" closed curves are equivalent up to rotation. This equivalence should also generalize to any odd number of crossing closed curves in this similar form. However, in the even crossing case the two opposite oriented closed curves of this form are unique.

Since the bridge map is concerned with closed curves embedded on a sphere, it is reasonable to ask how do oriented closed curves behave when embedded on a sphere. We will first try to better understand how an individual closed curve behaves when embedded on a sphere. For example, look at the oriented spherical closed curve in Figure 29.

On the sphere we can perform a sequence of moves expressed in Figure 30 to get from one representation to another.

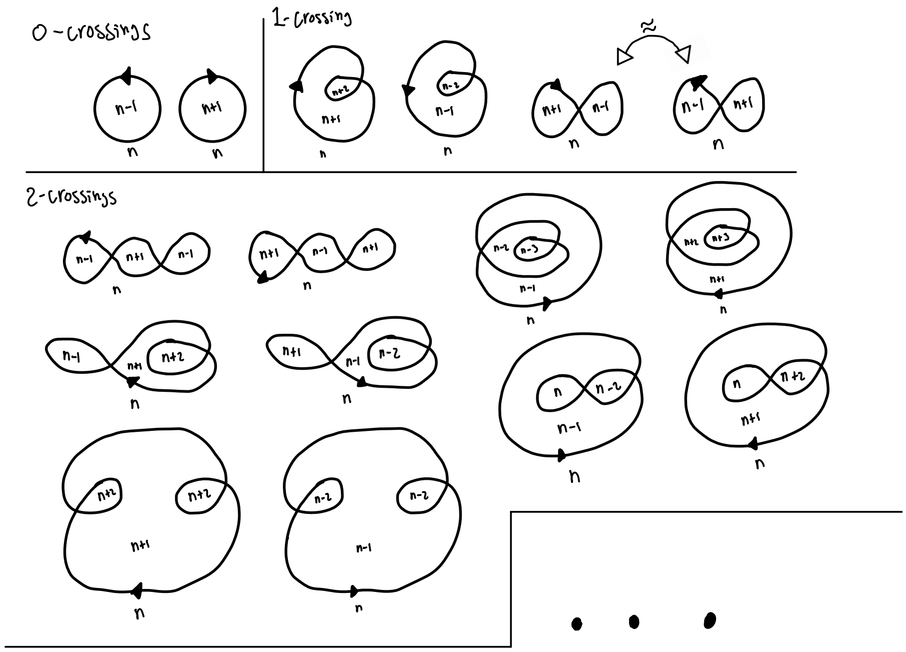

FIGURE 28. Combinatorially distinct planar closed curves with enumerated by crossing number.

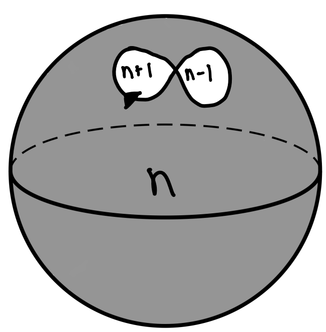

FIGURE 29. Spherical closed curve with the region labelled n being shaded in grey

This is the case since there is no exterior region on a sphere. Therefore, there is more than one way of representing an oriented closed curve embedded on S^2 in the plane. However, by utilizing what we know about oriented planar closed curves we can define an equivalence class of spherical oriented closed curves and enumerate all possible oriented spherical closed curves. We will do this by considering the set of planar oriented closed curves that exist

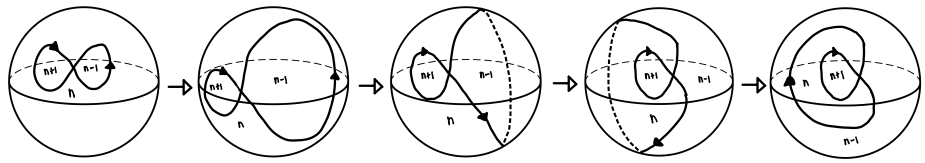

FIGURE 30. Sequence of moves preserving the combinatorial properties of a spherical oriented closed curve.

when specifying different exterior regions on the sphere. This way of interpreting oriented spherical closed curves isn't necessarily computationally efficient, however, can allow us to draw connections between the planar cases and spherical cases and serve as confirmation that we are successfully enumerating the set of unique oriented spherical closed curves correctly.

When defining a spherical oriented closed curve in this way the resulting object will be a set of at most n planar representations when there exists n regions of the spherical closed curve on S^2. We will see in some cases the planar representations when projecting the oriented spherical closed curve from two different regions are equivalent. Since the spherical oriented closed curve in Figure 29 has 3 regions there are 3 planar representations we can consider.

More rigorously we are essentially considering the stereographic projection of the sphere from a point in each of the regions bounded by the oriented spherical closed curve. This will create n planar projections for each of the n regions. In Figure 31 we can see the set of planar representations corresponding to Figure 29.

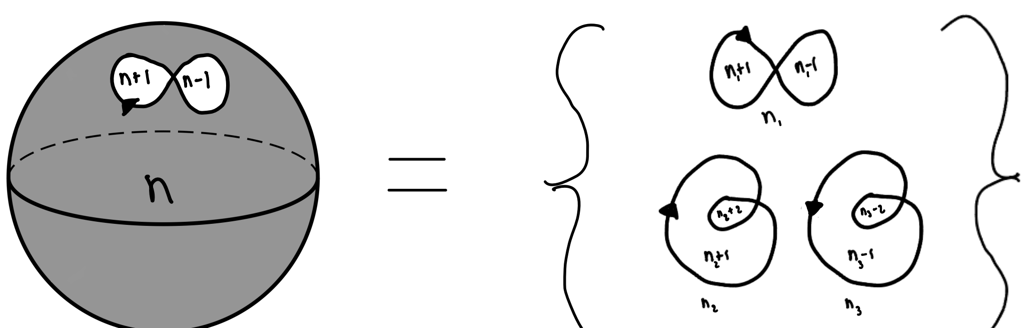

FIGURE 31. Example definition of an oriented spherical closed curve.

Note that with the planar representations we always label the exterior region n. However, when considering the different planar projections of an oriented spherical closed curve the exterior region will not always have the label of n. For example, in Figure 30 the original label of the exterior region if we were to interpret the curve in the plane is n, however, after pulling part of the curve around the backside of the sphere this region now has the value of $n-1$. Since up to renormalization these planar representations are equivalent, for consistency purposes we will always view the planar representations with respect to the exterior region. Since these exterior regions do not always share the same value, we will use a unique n_i for normalizing each planar representation in the set. This will allow us to equivalently categorize oriented spherical closed curves ignoring all the possible combinatorially equivalent labelling schemes. It is also worth noting that all the planar representations of an oriented spherical closed curve will have the same gap between smallest and largest labelled region.

It turns out that there is only one oriented spherical closed curve with zero crossings and only one with a single crossing. In Figure 32 we can see the set of planar representations that define the zero-crossing spherical closed curve.

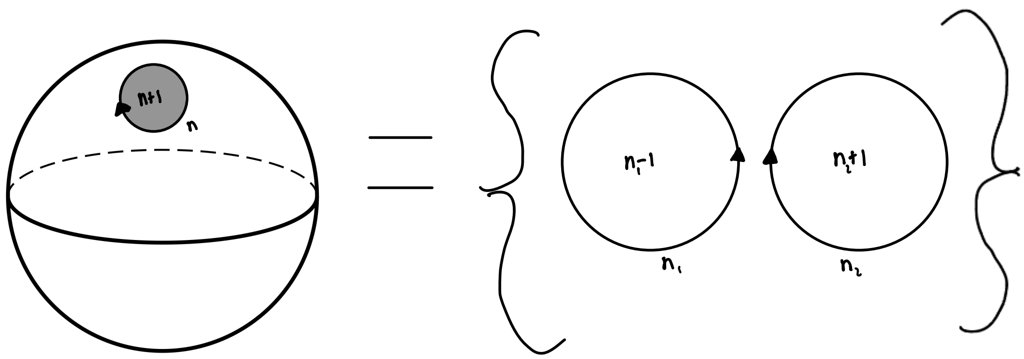

FIGURE 32. Set of planar representations for the zero crossing spherical closed curve

Looking back at the planar closed curves in Figure 28 we can see that the zero (Figure 32) and one crossing (Figure 31) spherical closed curves represent all the planar oriented closed curves of the same crossing number. It isn't until the 2-crossing case that we first see more than one unique oriented spherical closed curve of the same crossing number. As we can see in Figure 33 there are 3 unique 2-crossing oriented spherical closed curves.

Typically, we would expect each set to have 4 unique planar representations when the oriented spherical closed curve divides S^2 into 4 regions, however, in this case two of the sets only have 3 elements. This is the case because the starred planar representations in the upper two sets are the image of taking the stereographic projection of the oriented spherical

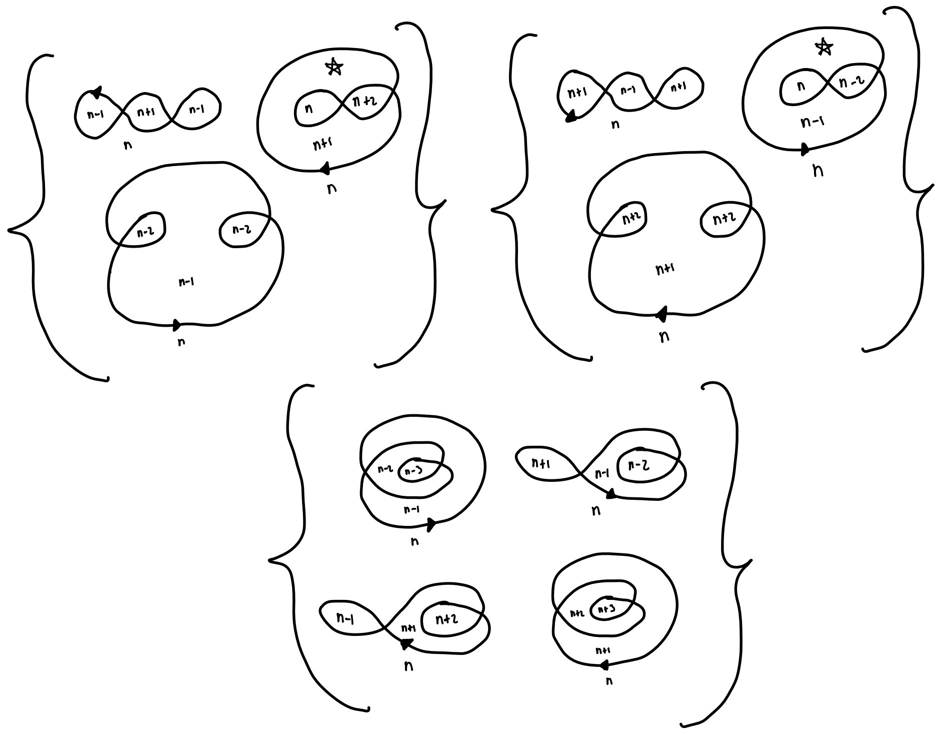

FIGURE 33. Planar sets of the 3 2-crossing oriented spherical closed curves.

closed curve from two different regions.

To better understand the set of possible bridge maps we can use the exact same approach as we did in the single closed curve case and extend it to the double closed curve case. However, since the closed curves on the bridge map are related through an antipodal map, we can restrict the set of double planar oriented closed curves we consider. Unfortunately, there doesn't exist a nice way of viewing an antipodal map in the plane. Therefore, it would be more difficult to construct the set of double planar closed curves that are going to correspond to the set of double spherical closed curves.

In the case where the binotrix and anti-binotrix do not intersect one another we only need to consider one of the curves and can view that curve in the plane since any moves described in Figure 30 would create a crossing with the antipodal map. Therefore, we can utilize our analysis from the single planar oriented closed curve case to better understand the combinatorial complexity of such bridge maps.

Recall that the number of geodesics that make up the binotrix and anti-binotrix is equal to the number of sticks used to graph the stick knot conformation that corresponds to the bridge map. Therefore, instead of considering a smooth oriented closed curve we can consider oriented closed polygonal chains. Each planar oriented closed curve as in Figure 28 requires a minimum number of sticks in order to be combinatorially represented. For example, any closed polygonal chain must have at least 3 sticks. This implies that we can derive a better understanding of the potential curves on the bridge map by knowing the number of sticks in the knot conformation we are analyzing.

In Figure 34 we can see the beginning of the enumeration of the possible closed polygonal chains starting with the minimum number of required sticks 3. Since 6 sticks is the minimum number of sticks needed to represent a nontrivial stick knot conformation, by analyzing all the possible oriented 6 stick closed polygonal chains we can identify the set of all possible non-intersecting bridge maps for a minimal stick knot conformation of the trefoil knot. Although it is true that all knots have a braid conformation and therefore a spiral conformation, it is unknown whether all knots have a minimal stick conformation that is also a spiral conformation. Therefore, it is possible that we would have to consider closed polygonal chains of greater stick number for a specific knot to realize a bridge map with a non-intersecting binotrix and anti-binotrix. However, from graphing an example of a minimal stick conformation of the trefoil knot we know that there exists a minimal stick representation that does realize a spiral form and therefore can be expressed as a 6-stick planar closed polygonal chain.

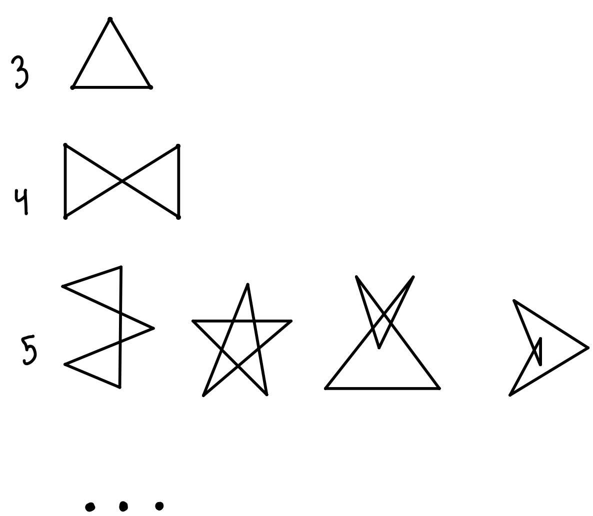

FIGURE 34. Enumeration of closed polygonal paths up to 5 sticks

Having a six-stick simple closed curve on the bridge map of a trefoil knot conformation that technically only requires 3 sticks to make emphasizes that greater stick numbers can redundantly create lower stick number closed curves (namely the ones in Figure 34). For the sake of conciseness, I did not include such representations in the figure. Therefore, the set of closed polygonal chains enumerated by their minimal stick number serves as a table of the increasingly complex patterns that can appear as the number of sticks of the graphed stick knot conformation increases.

Since these closed polygonal paths represent bridge maps whose binotrix and anti-binotrix do not intersect one another, we can analyze the greatest gap of the labelled regions after applying an orientation to each of the closed polygonal chains by simply looking at only one of the closed polygonal chains (because the bridge map is symmetrical). Since the smallest region on the bridge map corresponds to $b(C_K)$ and the largest region corresponds to $sb(C_K)$ this will provide a bound on the difference between these two knot invariants when $b(K) = b(C_K)$. Unfortunately, this will only work when $b(K)$ is realized because it is possible that a non-minimal stick knot conformation that does not realize $b(K)$ has a combinatorially more simple bridge map than a minimal stick knot conformation that realizes both $b(K)$ and $sb(K)$. However, since conformations made from fewer sticks reduce the combinatorial complexity of the bridge map it seems unlikely that this would occur. This analysis of the bridge gap in relation to stick number leads to the following theorem defining the largest possible gap that can occur when the binotrix and anti-binotrix do not intersect. However, prior to establishing this bound we must prove a couple useful lemmas and define a sub loop of a simple closed curve.

Definition 4.1. *A* ***sub loop*** *of a simple closed curve is a sub arc on the curve starting at a crossing that loops exactly* $360°$ *and then crosses a previous portion of the sub loop bounding a circle.*

In Figure 36 a sub loop is the curve bounding the $n + 2$ and $n + 3$ regions. Although the sub loop intersects different portions of the closed curve, these intersections occur before $360°$ has been traversed and therefore do not happen at a previous point on the sub loop.

Lemma 4.2. *An oriented closed curve with no inflection points realizes its largest gap (difference between the smallest and largest region) with respect to the labelling property of the bridge map when it is in braid form.*

Before jumping into the proof, a visualization of Lemma 4.2 can be seen in Figure 35.

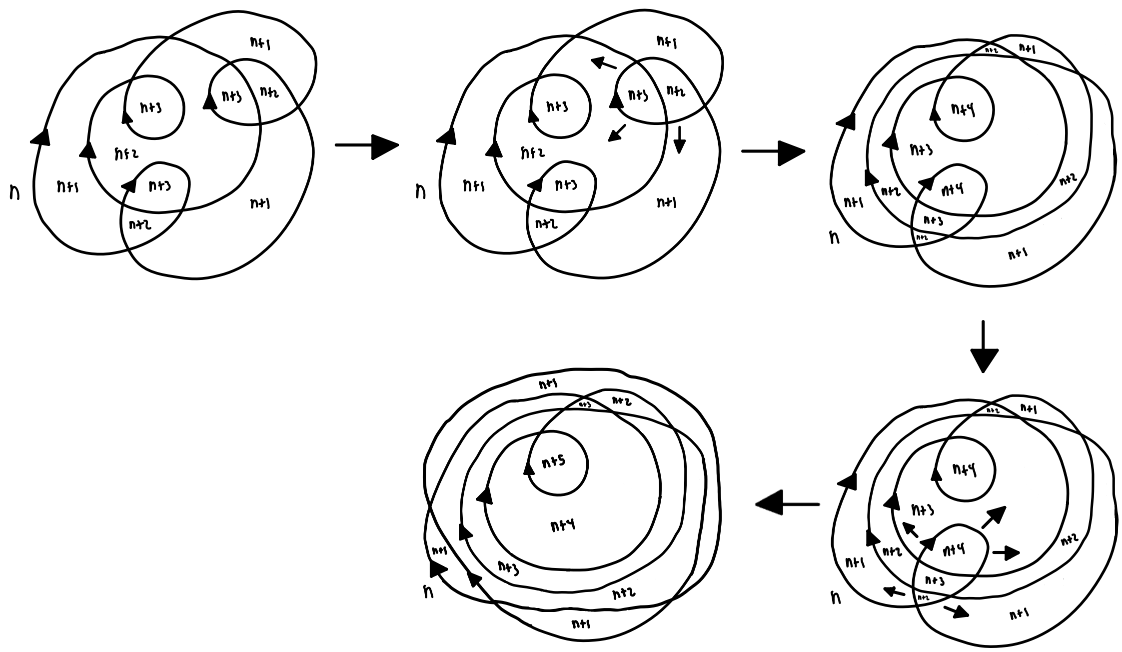

FIGURE 35. Transformation of an oriented closed curve with no inflection points to its corresponding curve with nested regions

Proof.
The closed curves defining the binotrix and anti-binotrix on the bridge map abide by the right-hand rule where the region to the right of the oriented curve differs by one in a fixed direction to the region on the left. When dealing with an oriented closed curve with no inflection points all regions bounded by a closed sub loop ($360°$ of the curve) have a label moving in the same direction with respect to the exterior regions of the closed sub loop. See Figure 36 for an example of this from a sub loop of the first oriented closed curve in Figure 35.

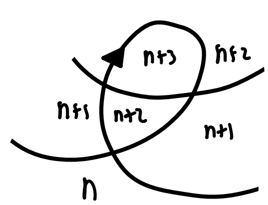

FIGURE 36. Sub loop from first oriented closed curve in Figure 35

Therefore, the label of the regions bounded by each sub loop are dependent on the values of their exterior regions. The value of the interior regions of the sub loop realizes their greatest gap with respect to the diagram when their exterior region's values realize their greatest gap. Therefore, if we start with an oriented closed curve with no inflection points and perform isotopy to make one of its closed sub loops contain the entire oriented closed curve, we are creating another layer between the original exterior region and the oriented closed curve potentially increasing the gap of the new exterior region relative to all the remaining interior regions. Therefore, by performing such moves to create nested interior regions you can only increase the gap of the oriented closed curve with no inflection points. However, how do we know we can always perform such moves without creating inflection points to create nested regions? This is the case since when you have no inflection points you can always make each of the sub loops point in the same inward direction, see Figure 37. Once you have an oriented closed curve with such a property you can simply expand or contract each of the sub loops to create a nesting without adding any inflection points. Therefore, you can always go from a simple closed curve with no inflection points to a simple closed curve in braid form without adding any inflection points.

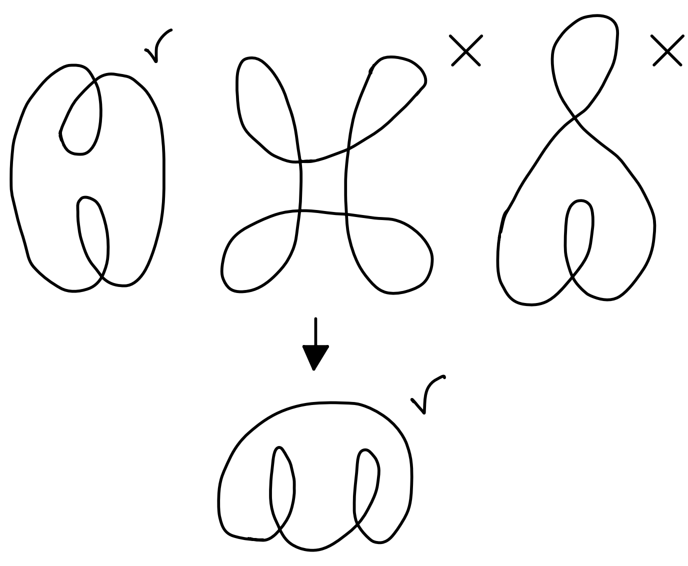

FIGURE 37. Oriented closed curve with no inflection points in the correct form (left), incorrect form with move to correct form (middle), and an oriented closed curve with inflection points (right)

□

We will now start to analyze cases and determine the greatest possible gap n sticks can create. Utilizing Lemma 4.2 we will first try to quantify the maximal gap a closed polygonal path with no inflection sticks can realize. In order to maximize such a gap, we now know that we want to create as many nested regions as possible.

Lemma 4.3. *Given an oriented closed polygonal path with no inflection sticks the largest possible gap is* $\lfloor \frac{n-1}{2} \rfloor$.

The intuition for this falls behind the premise of Figure 38. Namely the first observation is that it takes 3 sticks to create a closed loop. Followed by the observation that by adding two non-inflection sticks in such a manner we can increase the resulting gap by 1.

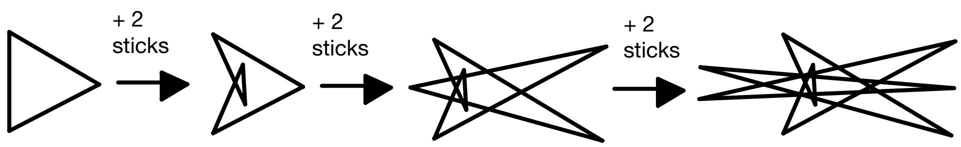

FIGURE 38. Example of closed polygonal path with no inflection sticks that realizes the gap describe in Lemma 4.3

Proof.
By Lemma 4.2 we know that a closed polygonal path with no inflection points will realize a larger gap when its regions are nested. By considering stick projections of the form expressed in Figure 38 we know that we can realize at least a gap of $\lfloor \frac{n-1}{2} \rfloor$ when n is odd. However, how do we know we cannot realize a larger gap? We can prove this by considering the maximum possible exterior angle each additional stick can contribute. Each additional stick can contribute $x°$ arbitrarily close to $180°$. Thus, as we see in Figure 39 with the addition of 2 sticks we can add $2x°$ arbitrarily close to $360°$.

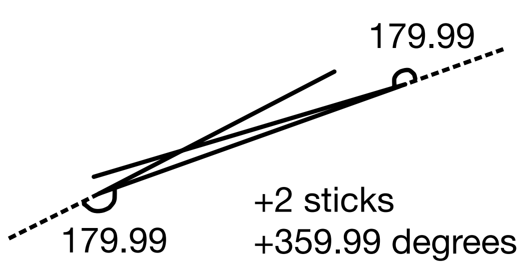

FIGURE 39

Since each interior region requires 360° worth of sticks we need an additional stick to account for the remaining $\delta^\circ > 0$. In the odd stick cases expressed in Figure 38 we utilize the same single stick to continually account for the remaining $\delta^\circ > 0$. Therefore, we need at least 2 sticks to create an additional bounded region (360° worth of sticks). In the even stick number case for a closed polygonal path in braid form it turns out that we will always be left with an additional stick not contributing to an interior region. We can see this by considering the maximum degrees an even number of sticks can represent. Each additional stick can add x° arbitrarily close to 180°. Therefore, with n sticks where n is even (i.e. $n = 2z$ for some $z \in \mathbb{Z}_{>1}$) we can add at most $y^\circ < z * 360^\circ$. This implies there are $< z$ possible bounded regions (since z bounded regions would require $y^\circ > z * 360^\circ$). However, $n - 1$ sticks is an odd number of sticks and therefore from the argument above can realize a gap of $\frac{n-2}{2} = \frac{n}{2} - 1$. This implies $n = 2z$ sticks can realize a max gap $\geq \frac{n}{2} - 1 = z - 1$ and $< z$. Thus, when n is even the maximum gap is $z - 1 = \frac{n-2}{2}$ which is equal to $\lfloor \frac{n-1}{2} \rfloor$. Therefore, for a closed polygonal chain in braid form the maximum gap is described by the formula $\lfloor \frac{n-1}{2} \rfloor$. However, how do we know we cannot realize a larger gap with a closed polygonal chain containing no inflection sticks not in braid form? Due to Lemma 4.2 we know that an oriented closed curve with no inflection sticks and n sub loops realizes it's largest gap when the n sub loops are nested. Since constructing n loops requires at least $n * 360^\circ$ we know from the previous argument that we still need at least $2n + 1$ sticks to create n loops and with these n loops not being nested we are missing out on a possible increase in gap (since unnested sub loops realizes a gap less than or equal to the gap of nested sub loops). Therefore, we cannot realize a larger gap when a closed polygonal chain with no inflection sticks is not in braid form. Thus, the largest gap is described by $\lfloor \frac{n-1}{2} \rfloor$. $\square$

This result leads to the following theorem regarding the largest gap of any closed polygonal path. The intuition behind this result was gained from trying to find an example of a 6-stick closed polygonal chain which has a gap greater than 2. From Lemma 4.3 we know that given a 6 stick closed polygonal chain with no inflection sticks the largest gap would be $\lfloor \frac{6-1}{2} \rfloor = 2$. However, from finding a counter example it turns out we can incorporate an inflection stick to create a 6-stick closed polygonal chain with a gap greater than 2. See Figure 40 for an example of a 6-stick closed polygonal chain that has a gap of 3. From Lemma 4.3 we know that such a closed polygonal chain must contain an inflection stick.

As we can see in Figure 40 the closed polygonal chain consists of a 5-stick component that contains no inflection sticks followed by a single inflection stick that contributes an

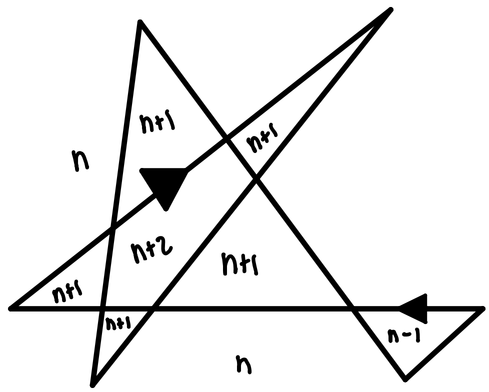

FIGURE 40

additional gap. From Lemma 4.3 we know that this 5-stick sub chain can contribute a gap of at most $\lfloor \frac{5-1}{2} \rfloor = 2$ which is exactly what we see in Figure 40. The addition of this single inflection stick reverses the orientation of the region it bounds creating a region moving in the opposite direction of the nested regions from the sub chain that contains no inflection sticks. Therefore, increasing the previous gap of 2 to 3.

In general, when working with the even stick case for a closed polygonal chain that contains no inflection sticks the additional stick we can add from the odd case ($n-1$ case) cannot increase the gap. However, when considering inflection sticks, we see a jump in the gap by 1 when adding a single inflection stick. This implies the single inflection stick case has a one-to-one increase in gap with respect to increase in stick number. What happens as we append more inflection sticks? Could we possibly get away with adding fewer sticks for a greater increase in the gap compared to the no inflection stick case? Since each interior region of this newly created inflection stick region requires $360°$ worth of sticks to create a new region from a similar angle argument we know that we need to add at least 2 sticks along with this initial inflection stick to further increase the gap. This observation leads to the following theorem.

Theorem 4.4. *Given an n-stick oriented closed polygonal chain the largest gap can be at most $\lfloor \frac{n}{2} \rfloor$.*

Proof.
From Lemma 4.3 we know the largest gap of an n-stick oriented closed polygonal chain with no inflection sticks is $\lfloor \frac{n-1}{2} \rfloor$.

In the odd stick case, it turns out that by substituting any number of non-inflection sticks for an inflection stick we cannot increase the gap (even in the 1 stick case). This is due to the property that when you change a single non-inflection stick to an inflection stick you know have a sub chain of $n = (2x + 1) - 1$ non inflection sticks which yield a smaller gap with respect to $\lfloor \frac{n-1}{2} \rfloor$. Therefore, the extra gap increase from this single inflection stick does provide any leverage on the gap. However, this trick relied on us adding the inflection stick to an exterior region. Perhaps we could add the inflection stick somewhere within the non-inflection stick closed polygonal path to create different behavior for the gap. This does not end up being an issue since the potential gap this new region can create is dependent on the regions it borders. Therefore, when this stick is added to the interior of this closed polygonal chain (namely when the boundary of the entire closed polygonal chain in the plane has no inflection points, see Figure 41 (left)) the decrease in gap happens in a region whose boundary regions already realize a nonzero gap. Therefore, this decrease can at most bound a region of relative gap 0 not increasing the total gap of the closed polygonal chain.

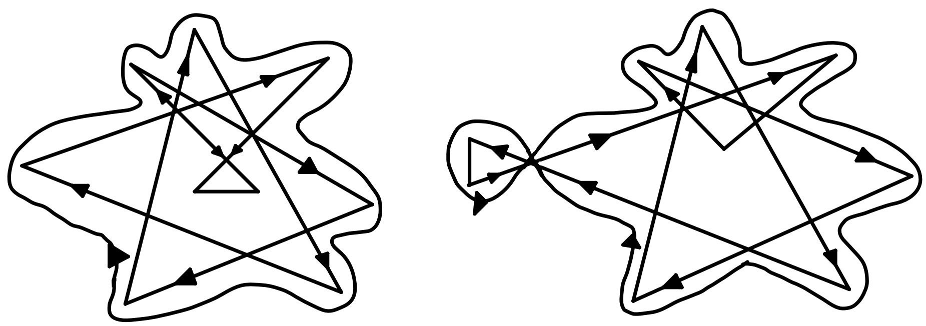

FIGURE 41. Closed polygonal chain with strictly interior regions and an inflection stick (left) closed polygonal chain with both an interior region and exterior regions with an exterior inflection stick (right)

Could we possibly increase the gap by adding more inflections sticks to either the interior or exterior region of the closed polygonal chain? In the case where we add more inflection sticks to a new exterior region (see Figure 42) we do not increase the gap because we already have a region realizing the first reversed orientation's level.

If we were to add more sticks to an already existing exterior region as we know from previous angle arguments, we would need to add 2 sticks for an increase of 1 in the gap. Therefore, the resulting odd stick closed polygonal path will have $n - 3$ contributing to one sub chain of no inflection sticks and 3 contributing to another sub chain of no inflection

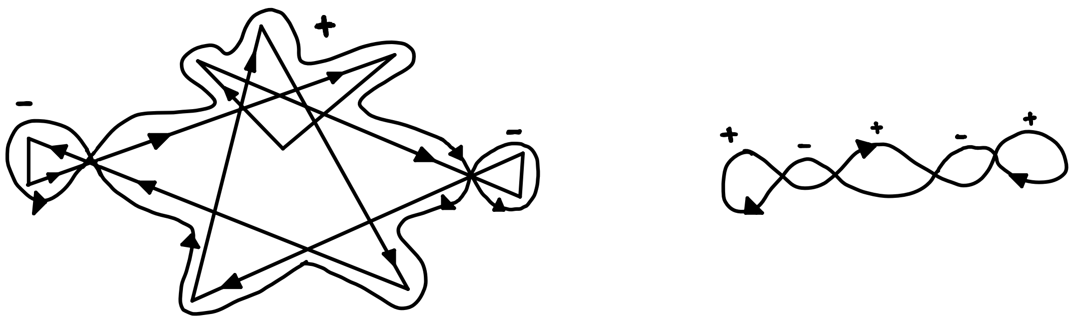

FIGURE 42. Closed polygonal chain with two exterior regions realizing the same gap

sticks (going in the opposite direction). Utilizing our bounds for these two specific cases implies $\lfloor\frac{n-4}{2}\rfloor + \lfloor\frac{3-1}{2}\rfloor \leq \lfloor\frac{n-1}{2}\rfloor$.

Before arguing the next case it is important to note that when referring to non-inflection sticks we are referring to the oriented sticks that contribute to the realized gap (i.e. the gap moves in their orientations direction) and the inflection sticks to be the smaller subset of sticks contributing in the opposite direction (in the case the gap is 0 the non-inflection sticks are the greater number of oriented sticks, tie breaks can be handled in either way) . If we were to switch more than one non-inflection stick to an inflection stick in the interior of a closed polygonal chain, we know that we would need at least two more sticks for the additional increase of one in the sub chain's gap. However, since we have two different moving sub chains stacked on top of one another a portion of their gap contributions cancels one another. Therefore, we will realize a less extreme gap.

Therefore, in the odd stick number case switching any number of sticks to inflection sticks does not increase the maximum gap and therefore the largest gap is $\lfloor\frac{n-1}{2}\rfloor$ which is equivalent to $\lfloor\frac{n}{2}\rfloor$ when n is odd.

It turns out that the even stick case is very similar, however, we can increase the gap by switching a single non-inflection stick to an inflection stick in the exterior region. This is because as previously mentioned 1 stick does not contribute to the gap in the even case. Therefore, changing his corresponding stick to an inflection stick in the exterior region makes the gap of an even stick closed polygonal chain realize $\lfloor\frac{n}{2}\rfloor$. However, how do we know we cannot change more sticks to inflection sticks to realize a greater gap? For very similar arguments as above adding more inflection sticks to the exterior region will realize the same gap if you are adding and removing 2 sticks at a time. Adding to a new exterior region will always be wasteful and realize a smaller gap. In the case of adding inflection sticks to the interior region we know this will also be counterproductive and require overcoming the self-cancellation of different oriented neighboring regions before contributing to a larger gap.

Therefore, in the even stick number case switching any number of sticks to inflection sticks does not increase the maximum gap and therefore the largest gap is equivalent to $\lfloor \frac{n}{2} \rfloor$ when n is even.

Thus, given an n-stick oriented closed polygonal chain the largest gap can be at most $\lfloor \frac{n}{2} \rfloor$. $\square$

In Figure 43 we can see an 8-stick closed polygonal curve that incorporates inflection sticks to realize a gap of 4. In this example 3 inflection sticks are used, however, as we know from the proof above the gap of $\lfloor \frac{n}{2} \rfloor$ can always be realized with just a single inflection stick.

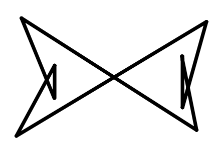

FIGURE 43. 8 stick closed polgonal path with a gap of 4

We can use this same approach of incorporating stick number bounds to better understand cases where the binotrix and anti-binotrix intersect one another, however, simply enumerating all these cases would get tedious and difficult to visualize. To use computers to help categorize all the possible combinatorially distinct bridge maps, one can create a program that randomly generates n points on the unit sphere and connects them with geodesics and then graphs its antipodal map. By doing this randomization thousands of times we could start to discover all the possible bridge maps for n-stick knot conformations which would give us a better understanding of how these knot invariants behave as the number of sticks required to graph a stick knot conformation increase. The value of these knot invariants for the set of all knot conformations behaves according to the set of all possible bridge maps and certain values are realized with a higher probability if they correspond to a bridge map that occurs with a higher probability. By analyzing all the possible bridge maps, we can get a better understanding whether it is possible for a knot to realize a different value for each of the invariants in the chain of knot invariants referenced earlier (which was the original guiding question of this thesis).

To close this section we will prove a nice property about the set of all possible bridge maps. Namely, the map from a particular stick knot conformation to the set of bridge maps is not one to one. In this setting we are considering two bridge maps distinct if they geometrically represent different objects. In other words, if the points defining the curves on the bridge map differ from one another then the objects are geometrically distinct. As we saw in Figure 22 geometric equivalence is much stronger than combinatorial equivalence. The general idea behind the proof relies on the computation steps described in Section 3.

Theorem 4.5. *Given two knots there exists a stick knot conformation for each knot that realize geometrically equivalent bridge maps.*

Proof.
Given a set of coordinates describing a stick knot conformation, recall the way of computing the defining points on the bridge map is first by computing the points of the tantrix of the knot conformation. As previously mentioned in Section 3, the points defining the tantrix of a stick knot conformation are simply the unit vectors of each stick given by the conformation's orientation. Since the unit vectors each stick describes does not depend on the length of the stick, we can apply a simple trick to make any two knot conformations realize the same tantrix (Ying Qing Wu , unpublished). Once we have computed the tantrix we compute the unit binormal vectors of each consecutive pair of points on the tantrix. Therefore, if two knot conformations realize the same tantrix then they realize the same bridge map. The strategy of making two stick knot conformations realize the same tantrix first requires us to find equal numbered stick knot conformations. Once we have such a stick knot conformation for each knot, we can apply the following trick expressed in Figure 44. Given a starting stick for each stick knot conformation, we alternate the sticks from each chain keeping their orientations from the previous conformations fixed. This will give us a closed curve in 3-space. Now, if we shrink the sticks for knot A to be very small, the resulting conformation represents knot B. And if we shrink the sticks for knot B we will be left with knot A. But in either case, since we did not change the direction of any stick, both of the resulting conformations will have the same geometric bridge map. □

As we can see in Figure 44 since the length of the sticks do not impact the unit vector the sticks define, both resulting stick knot conformations will have the same ordered set of coordinates that describe the tantrix. Therefore, they will also have the same ordered coordinates that describe the bridge map.

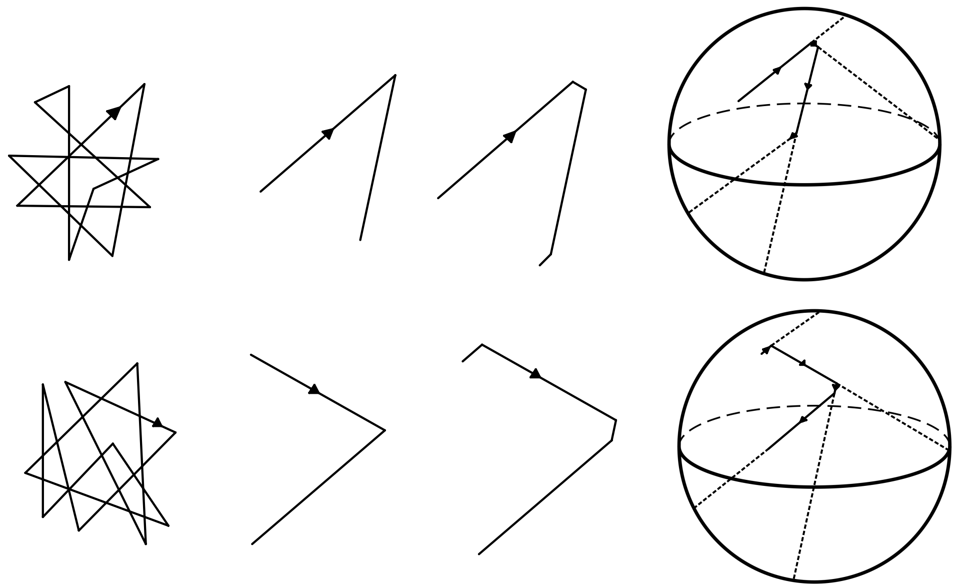

FIGURE 44. Two equal stick knot conformation and the process of alternating sticks and adjusting their lengths to realize new conformations corresponding to the same knots with a geometrically equivalent tantrix

In fact, one can extend this argument to more than two knots. The only requirement is that the resulting chain of sticks is equivalent for each stick knot conformation. However, by sufficiently choosing the lengths of each stick you can ensure that the resulting conformation describes the intended knot.

It is also important to note that the bridge map that these two knots will end up describing is very unlikely to realize any invariant value for either knot. Since the resulting knot conformations have twice as many sticks and therefore at least two times more geodesics than the number of geodesics corresponding to a minimal stick conformation for either knot, we know that the combinatorial complexity of the joint bridge map will likely be more complex and provide weaker bounds on the invariant's values. Better understanding how the shared bridge map combinatorially relates to each individual bridge map prior to utilizing this trick would be an interesting future question to research.

5. Conclusion

Primarily, knot theory has been studied using knot projections. This paper has served as a conglomerate of literature related to knot invariants regarding the geometric shape of the knot and to motivate the computational advantages of analyzing invariants in terms of knot conformations, ignoring the set of projections. From starting with the following chain of knot invariants, $b(k) \leq psb(k) \leq \frac{(k+\tau)[K]}{2\pi} \leq sp(k) \leq \mathcal{B}(k)$, we were able to better understand their relationship to this relatively new mathematical object, the bridge map. Along with the computational advantages of computing the coordinates defining the bridge map we were able to create a MATLAB program to visualize bridge maps in practice and further identify many interesting questions regarding the bridge map. The notion of geometric equivalence versus combinatorial equivalence of the bridge map is subtle, however, presents interesting questions such as why certain knot conformations are geometrically closer to realizing a combinatorially different bridge map versus others and what does this imply about the knot's invariant values. We also further analyzed the idea behind the stick number of knot conformations directly corresponding to the complexity of its bridge map. This property leads to a lot of questions and even suggests minimal stick knot conformations are more likely to be realize the minimal invariant values for the ones that can be defined in terms of the bridge map. Perhaps a minimal stick knot conformation does not realize the value of all the invariants, but there exists an algorithm to get to another conformation (through combinatorially adjusting the bridge map) that realizes the remaining invariant values. Such a discovery would make the process of computing these invariant values of a knot quite trivial.

Another question that is worthwhile thinking about that has not been mentioned in the paper involves analyzing how the bridge maps of two stick knot conformations relate to the bridge map of their composite knot. Is it true that two stick knot conformations that realize most of their invariant values compose to a composite stick knot conformation that also realizes a majority of its invariant values. Also, since in some cases we can compose two knot conformations in different ways to yield two different knots, it is true these different ways of composing yields different bridge maps and geometrically these two different composite knots are unique. Lastly, it would be very interesting to generalize this idea of a map to other invariants. Perhaps there exists other closely related invariants that can be described in terms of knot conformations in a similar way. The concept of a Genus map has been expressed in [2], however, it may not be easy to combine the tantrix and other curves that describe the seifert circles in order to graph and visualize a genus map of a knot conformation.

In conclusion with advancements in computational power and technology analyzing invariants in terms of knot conformations is becoming increasingly feasible. Such techniques also allow us to better understand how invariants that are traditionally studied through projections relate to one another from a 3-dimensional perspective. I have greatly enjoyed spending my year doing research in knot theory and look forward to following future work related to the bridge map.

6. APPENDIX

Below we have included a list of the currently known values for these invariants for knots up to 10 crossings. $(\kappa+\tau)(K)/2\pi = False$ mean the value has not yet determined.

k	b(k) <=	psb(k)	psb(k) <=	(k+t)[K]/2pi	<=sp(k)	sp(k)<=	B(k)	<=sb(k)	sb(k <=	<=s(k)	s(k)<=
3_1	2	2	2	2	2	2	2	3	0	6	6
4_1	2	3	3	3	3	3	3	3	0	7	7
5_1	2	2	2	2	2	2	2	4	0	8	8
5_2	2	3	3	3	3	3	3	3	4	8	8
6_1	2	3	3	3	3	3	4	3	4	8	8
6_2	2	3	3	3	3	3	3	3	4	8	8
6_3	2	3	3	3	3	3	3	3	4	8	8
7_1	2	2	2	2	2	2	2	4	0	9	9
7_2	2	3	3	3	3	3	4	3	4	9	9
7_3	2	3	3	3	3	3	3	3	4	9	9
7_4	2	3	3	3	3	3	4	3	4	9	9
7_5	2	3	3	3	3	3	3	4	0	9	9
7_6	2	3	3	3	3	3	4	4	0	9	9
7_7	2	3	3	3	3	3	4	4	0	9	9
8_1	2	3	4	FALSE	4	4	5	4	0	9	10
8_2	2	3	3	3	3	3	3	4	0	9	10
8_3	2	3	4	FALSE	4	4	5	4	0	9	10
8_4	2	3	3	3	3	3	4	3	4	9	10
8_5	3	3	3	3	3	3	3	4	4	9	10
8_6	2	3	3	3	3	3	4	4	0	9	10
8_7	2	3	3	3	3	3	3	4	0	9	10
8_8	2	3	3	3	3	3	4	4	0	9	10
8_9	2	3	3	3	3	3	3	3	4	9	10
8_10	3	3	3	3	3	3	3	4	0	9	10
8_11	2	3	3	3	3	3	4	4	0	9	10
8_12	2	3	4	FALSE	4	4	5	4	0	9	10
8_13	2	3	3	3	3	3	4	4	0	9	10
8_14	2	3	3	3	3	3	4	4	0	9	10
8_15	3	3	3	3	3	3	4	4	0	9	10
8_16	3	3	3	3	3	3	3	4	0	9	9
8_17	3	3	3	3	3	3	3	4	0	9	9
8_18	3	3	3	3	3	3	3	4	0	9	9
8_19	3	3	3	3	3	3	3	4	0	8	8
8_20	3	3	3	3	3	3	3	4	0	8	8
8_21	3	3	3	3	3	3	3	4	0	9	9
9_1	2	2	2	2	2	2	2	4	0	9	10
9_2	2	3	4	FALSE	4	4	5	4	5	9	10
9_3	2	3	3	3	3	3	3	4	4	9	10
9_4	2	3	3	3	3	3	4	4	4	9	10
9_5	2	3	4	FALSE	4	4	5	4	5	9	10
9_6	2	3	3	3	3	3	3	4	4	9	11
9_7	2	3	3	3	3	3	4	4	0	9	10
9_8	2	3	4	FALSE	4	4	5	4	5	9	10
9_9	2	3	3	3	3	3	3	4	4	9	10

9_10	2	3	3	3	3	3	4	4	5	9	10
9_11	2	3	3	3	3	3	4	4	4	9	10
9_12	2	3	4	FALSE	4	4	5	4	5	9	10
9_13	2	3	3	3	3	3	4	4	4	9	10
9_14	2	3	4	FALSE	4	4	5	4	5	9	10
9_15	2	3	4	FALSE	4	4	5	4	5	9	10
9_16	3	3	3	3	3	3	3	4	0	9	10
9_17	2	3	3	3	3	3	4	4	4	9	10
9_18	2	3	3	3	3	3	4	4	4	9	10
9_19	2	3	4	FALSE	4	4	5	4	5	9	10
9_20	2	3	3	3	3	3	4	4	0	9	10
9_21	2	3	4	FALSE	4	4	5	4	5	9	10
9_22	3	3	4	FALSE	3	4	4	4	4	9	10
9_23	2	3	3	3	3	3	4	4	4	9	11
9_24	3	3	3	3	3	3	4	4	5	9	10
9_25	3	4	4	4	4	4	5	4	4	9	10
9_26	2	3	3	3	3	3	4	4	0	9	10
9_27	2	3	3	3	3	3	4	4	4	9	10
9_28	3	3	3	3	3	3	4	4	0	9	10
9_29	3	3	4	FALSE	3	4	4	4	0	9	9
9_30	3	3	4	FALSE	3	4	4	4	4	9	10
9_31	2	3	3	3	3	3	4	4	4	9	10
9_32	3	3	4	FALSE	3	4	4	4	0	9	10
9_33	3	3	4	FALSE	3	4	4	4	0	9	10
9_34	3	3	4	FALSE	3	4	4	4	0	9	9
9_35	3	3	5	FALSE	4	5	5	4	0	9	9
9_36	3	3	4	FALSE	3	4	4	4	4	9	11
9_37	3	3	5	FALSE	4	5	5	4	5	9	10
9_38	3	3	4	FALSE	3	4	4	4	5	9	10
9_39	3	4	4	4	4	4	5	4	0	9	9
9_40	3	3	4	FALSE	3	4	4	4	0	9	9
9_41	3	4	4	4	4	4	5	4	0	9	9
9_42	3	3	4	FALSE	3	4	4	4	0	9	9
9_43	3	3	4	FALSE	3	4	4	4	0	9	9
9_44	3	3	4	FALSE	3	4	4	4	0	9	9
9_45	3	3	4	FALSE	3	4	4	4	0	9	9
9_46	3	4	4	4	4	4	4	4	0	9	9
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10_2	2	3	3	3	3	3	3	4	5	9	11
10_3	2	3	5	FALSE	3	5	6	4	5	9	11
10_4	2	3	4	FALSE	3	4	5	4	5	9	11
10_5	2	3	3	3	3	3	3	4	5	9	11

10_6	2	3	3	3	3	3	4	4	5	9	11
10_7	2	3	4	FALSE	3	4	5	4	5	9	11
10_8	2	3	3	3	3	3	4	4	5	9	10
10_9	2	3	3	3	3	3	3	4	5	9	11
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10_11	2	3	4	FALSE	3	4	5	4	5	9	11
10_12	2	3	3	3	3	3	4	4	5	9	11
10_13	2	3	6	FALSE	3	6	6	4	5	9	11
10_14	2	3	3	3	3	3	4	4	5	9	11
10_15	2	3	3	3	3	3	4	4	5	9	11
10_16	2	3	4	FALSE	3	4	5	4	5	9	10
10_17	2	3	3	3	3	3	3	4	5	9	10
10_18	2	3	4	FALSE	3	4	5	4	5	9	10
10_19	2	3	3	3	3	3	4	4	5	9	11
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10_95	3	3	4	FALSE	3	4	4	4	5	9	11

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10_98	3	3	4	FALSE	3	4	4	4	5	9	11
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10_109	3	3	3	3	3	3	3	4	5	9	10
10_110	3	3	5	FALSE	3	5	5	4	5	9	10
10_111	3	3	4	FALSE	3	4	4	4	5	9	10
10_112	3	3	3	3	3	3	3	4	5	9	10
10_113	3	3	4	FALSE	3	4	4	4	5	9	10
10_114	3	3	4	FALSE	3	4	4	4	5	9	10
10_115	3	3	5	FALSE	3	5	5	4	5	9	10
10_116	3	3	3	3	3	3	3	4	5	9	10
10_117	3	3	4	FALSE	3	4	4	4	5	9	10
10_118	3	3	3	3	3	3	3	4	5	9	10
10_119	3	3	4	FALSE	3	4	4	4	5	9	10
10_120	3	3	5	FALSE	3	5	5	4	5	9	10
10_121	3	3	4	FALSE	3	4	4	4	5	9	10
10_122	3	3	4	FALSE	3	4	4	4	5	9	10
10_123	3	3	3	3	3	3	3	4	5	9	11
10_124	3	3	3	3	3	3	3	5	0	10	10
10_125	3	3	3	3	3	3	3	4	5	9	10
10_126	3	3	3	3	3	3	3	4	5	9	10
10_127	3	3	3	3	3	3	3	4	5	9	10
10_128	3	3	4	FALSE	3	4	4	4	5	9	10
10_129	3	3	4	FALSE	3	4	4	4	5	9	10
10_130	3	3	4	FALSE	3	4	4	4	5	9	10
10_131	3	3	4	FALSE	3	4	4	4	5	9	10
10_132	3	3	4	FALSE	3	4	4	4	5	9	10
10_133	3	3	4	FALSE	3	4	4	4	5	9	10
10_134	3	3	4	FALSE	3	4	4	4	5	9	10
10_135	3	3	4	FALSE	3	4	4	4	5	9	10
10_136	3	3	4	FALSE	3	4	4	4	5	9	10
10_137	3	3	5	FALSE	3	5	5	4	5	9	10
10_138	3	3	5	FALSE	3	5	5	4	5	9	10
10_139	3	3	3	3	3	3	3	4	5	9	10
10_140	3	3	4	FALSE	3	4	4	4	5	9	10

10_141	3	3	3	3	3	3	3	4	5	9	10
10_142	3	3	4	FALSE	3	4	4	4	5	9	10
10_143	3	3	3	3	3	3	3	4	5	9	10
10_144	3	3	4	FALSE	3	4	4	4	5	9	10
10_145	3	3	4	FALSE	3	4	4	4	5	9	10
10_146	3	3	4	FALSE	3	4	4	4	5	9	10
10_147	3	3	4	FALSE	3	4	4	4	5	9	10
10_148	3	3	3	3	3	3	3	4	5	9	10
10_149	3	3	3	3	3	3	3	4	5	9	10
10_150	3	3	4	FALSE	3	4	4	4	5	9	10
10_151	3	3	4	FALSE	3	4	4	4	5	9	10
10_152	3	3	3	3	3	3	3	4	5	9	10
10_153	3	3	4	FALSE	3	4	4	4	5	9	10
10_154	3	3	4	FALSE	3	4	4	4	5	9	11
10_155	3	3	3	3	3	3	3	4	5	9	10
10_156	3	3	4	FALSE	3	4	4	4	5	9	10
10_157	3	3	3	3	3	3	3	4	5	9	10
10_158	3	3	4	FALSE	3	4	4	4	5	9	10
10_159	3	3	3	3	3	3	3	4	5	9	10
10_160	3	3	4	FALSE	3	4	4	4	5	9	10
10_161	3	3	3	3	3	3	3	4	5	9	10
10_162	3	3	4	FALSE	3	4	4	4	5	9	10
10_163	3	3	4	FALSE	3	4	4	4	5	9	10
10_164	3	3	4	FALSE	3	4	4	4	5	9	10
10_165	3	3	4	FALSE	3	4	4	4	5	9	10

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