FINA2204 Tutorial 4: Pricing futures contracts

Zirui Song

UWA Business School

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Agenda

- Problem 5.9
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Recap: arbitrage

- Definition: *A strategy that yields a positive cash flow at one point in time and non-negative cash flows at all other points in times.*

Strategy	CF_0	CF_1	CF_2
A	+2	+1	-1
B	0	+1	0
C	-1	+2	+4
D	-1	+5	0
E	+1	0	0

Noteworthily, with a 5% risk-free rate, we can borrow 1 at time 0, and lock in an arbitrage payoff of 5-1*1.05=3.95 at time 1 with 0 initial cost.

Recap: arbitrage

Arbitrage works with investment assets

- Investment asset vs. consumption asset
- Investment asset: An asset that is held solely for investment purposes.

1. Financial assets - stocks, bonds, currencies
2. Commodities held for investment purposes – gold, silver

- Consumption asset: An asset that is held primarily for consumption.
- Examples are commodities such as copper, crude oil, corn, and pork bellies.

Recap: COC model

- The relationship between futures prices and spot prices can be summarized in terms of **the cost of carry**.
- This measures the storage cost plus the interest that is paid to finance the asset less the income earned on the asset.
- Define the cost of carry as *c*. For an investment asset, the futures price is

$$F_0 = S_0 e^{cT} \tag{5.18}$$

Recap: COC model

- This model is most appropriate where assets are non-perishable, storable and **do not** provide a convenience yield.

Case	**COC Model**	**Examples**
No income	$F_0 = S_0 e^{rT}$	Single stock futures (non-dividend stock)
Known lump-sum income	$F_0 = (S_0 - I)e^{rT}$	Single stock futures (dividend-paying stock)
Known income yield	$F_0 = S_0 e^{(r-q)T}$ $F_0 = S_0 e^{(r-r_f)T}$	Stock index futures Foreign currency futures

Recap: COC model

- Asset owners incur 'storage costs' (*U*/*u*) holding commodity in inventory.

Case	COC Model	Examples
Known lump-sum storage costs	$F_0 = (S_0 + U)e^{rT}$	Commodities with fixed warehouse costs
Known % storage costs	$F_0 = S_0 e^{(r+u)T}$	Commodities that incur ongoing storage costs

Recap: COC model

- Convenience yields (y) measures the extent to which users of the commodity feel that ownership of the physical asset provides may **benefits** that are not obtained by the holders of the futures contract.
- These benefits may include the ability to profit from temporary local shortages or the ability to keep a production process running.
- **Takeaways**: For *consumption assets*, we can only obtain an **upper bound** for the futures price, but can not nail down an equality relationship between futures and spot prices.

Recap: COC model

- Oil refinery > crude oil (e.g., 1973 oil crisis)
- But these benefits are uncertain and hard-to-quantify.
- They are also firm-specific and context-specific.

Recap: COC model

- Convenience yields can be incorporated into the COC model in **ad hoc** fashion as follows:

Case	COC Model
Known lump-sum storage costs and % convenience yield	$F_0 = (S_0 + U)e^{(r-y)T}$
Known % storage costs and % convenience yield	$F_0 = S_0 e^{(r+u-y)T}$

Note: it's called ad hoc because y is added arbitrarily to fit observed prices, not derived from the first principles.

Recap: COC model

- Notation

S_0 = Spot price today

T = time until delivery date

I = PV of lump-sum income

U = PV of lump-sum storage costs

r_f = foreign riskfree rate

F_0 = Futures or forward price today

r = risk-free interest rate (for futures on currencies, domestic)

q = % income yield

u = % storage costs

y = % convenience yield

Problem 5.24

- *What is the* ***cost of carry*** *for (a) a non-dividend-paying stock, (b) a stock index, (c) a commodity with storage costs, and (d) a foreign currency?*

Answer 5.24

a) the risk-free rate (r)

b) the excess of the risk-free rate (r) over the dividend yield (q)

c) the risk-free rate (rf) plus the storage cost (u)

d) the excess of the domestic risk-free rate (r) over the foreign risk-free rate (rf).

Recap: no income

To generalize this example, we consider a forward contract on an investment asset with price S_0 that provides no income. Using our notation, T is the time to maturity, r is the risk-free rate, and F_0 is the forward price. The relationship between F_0 and S_0 is

$$F_0 = S_0 e^{rT} \tag{5.1}$$

If $F_0 > S_0 e^{rT}$, arbitrageurs can buy the asset and short forward contracts on the asset. If $F_0 < S_0 e^{rT}$, they can short the asset and enter into long forward contracts on it.[2] In our

RULES: *"buying the dip and selling the high"*

1. Actual futures price > the theorical futures price.

→ choose **Cash & carry strategy**

2. Actual futures price < the theorical futures price.

→ choose **reverse Cash & carry strategy**

Recap: no income

- Consider a long forward contract to purchase a non-dividend-paying stock in three months. Assume the current stock price is $40 and the three-month risk-free interest rate is 5% per annum *(we will normally measure interest rates with continuous compounding (c.c)).*

$$\boldsymbol{The\ theoretical\ price} = \mathbf{40 \times e^{5\% \times 3/12} = 40.50}$$

Forward Price = $43 Cash & carry	*Forward Price = $39* Reverse Cash & carry
Action now:	*Action now*:
Borrow $40 at 5% for 3 months	Short 1 unit of asset to realize $40
Buy one unit of asset	Invest $40 at 5% for 3 months
Enter into forward contract to sell asset in 3 months for $43	Enter into a forward contract to buy asset in 3 months for $39
Action in 3 months:	*Action in 3 months*:
Sell asset for $43	Buy asset for $39
Use $40.50 to repay loan with interest	Close short position
	Receive $40.50 from investment
Profit realized = $2.50	Profit realized = $1.50

Source: textbook, p.128

Recap: valuing forward contracts (no income)

- The value of a forward contract at inception is essentially **zero** because the forward price is set under the **no-arbitrage condition**, leaving neither party with an immediate gain or loss.
- Later, the contract may be worth more or less because the current forward price changes while the contract's delivery price is fixed;
- The gap is the contract's value (positive or negative).

Recap: valuing forward contracts (no income)

Using the notation introduced earlier, we suppose F_0 is the current forward price for a contract that was negotiated some time ago, the delivery date is T years from today, and r is the T-year risk-free interest rate. We also define:

K: Delivery price in the contract

f: Value of forward contract today

A general result, applicable to all long forward contracts (on both investment assets and consumption assets), is

$$f = (F_0 - K)e^{-rT} \qquad \textbf{(5.4)}$$

$$f = S_0 - Ke^{-rT} \qquad \textbf{(5.5)}$$

For the value of a short forward contract:

$$\boldsymbol{f = -(S_0 - Ke^{-rT})}$$

Problem 5.9

- *A one-year long forward contract on a non-dividend-paying stock is entered into when the stock price is \$40 and the risk-free rate of interest is 10% per annum with continuous compounding.*

(a) *What are the forward price and the initial value of the forward contract?*

(b) *Six months later, the price of the stock is \$45 and the risk-free interest rate is still 10%. What are the forward price and the value of the forward contract?*

Answer 5.9

a) The forward price, F_0 , is given by equation (5.1) as:

$$F_0 = 40e^{0.1\times 1} = 44.21$$

Or $44.21.

The initial value of the forward contract is **zero**.

The delivery price *K* in the contract is $44.21. The value of the contract, *f*, after six months is given by equation (5.5) as:

$$f = 45 - 44.21e^{-0.1\times 0.5} = 2.95$$

i.e., it is $2.95.

The forward price is:

$$45e^{0.1\times 0.5} = 47.31$$

Or $47.31.

Recap: Known income

We can generalize from this example to argue that, when an investment asset provides income with a present value of I during the life of a forward contract, we have

$$F_0 = (S_0 - I)e^{rT} \tag{5.2}$$

Similarly, using equation (5.4) in conjunction with (5.2) gives the following expression for the value of a long forward contract on an investment asset that provides a known income with present value I:

$$f = S_0 - I - Ke^{-rT} \tag{5.6}$$

For the value of a short forward contract:

$$\boldsymbol{f = -(S_0 - I - Ke^{-rT})}$$

Recap: Known income

- Consider a long forward contract to purchase a coupon-bear bond whose current price is $900. We will suppose that the forward contract matures in nine months. We will also suppose that a coupon payment of $40 is expected after four months. We assume the four-month and nine—month risk-free interest rates (c.c) are 3% and 4% per annum, respectively.
- $PV\ of\ coupon\ payment = 40e^{-0.03\times 4/12} = 39.6$
- $The\ theoretical\ price = (900 - 39.6)e^{0.04\times 9/12} = 886.60$

Recap: Known income

Forward price = $910	***Forward price = $870***
Action now:	*Action now*:
Borrow $900: $39.60 for 4 months and $860.40 for 9 months	Short 1 unit of asset to realize $900
Buy one unit of asset	Invest $39.60 for 4 months and $860.40 for 9 months
Enter into forward contract to sell asset in 9 months for $910	Enter into a forward contract to buy asset in 9 months for $870
Action in 4 months:	*Action in 4 months*:
Receive $40 of income on asset	Receive $40 from 4-month investment
Use $40 to repay first loan with interest	Pay income of $40 on asset
Action in 9 months:	*Action in 9 months*:
Sell asset for $910	Receive $886.60 from 9-month investment
Use $886.60 to repay second loan with interest	Buy asset for $870
	Close short position
Profit realized = $23.40	Profit realized = $16.60

Source: textbook, p.131

Problem 5.29

- *A stock is expected to pay a dividend of $1 per share in two months and in five months. The stock price is $50, and the risk-free rate of interest is 8% per annum with continuous compounding for all maturities. An investor has just taken a short position in a six-month forward contract on the stock.*

(a) *What are the forward price and the initial value of the forward contract?*

(b) *Three months later, the price of the stock is $48 and the risk-free rate of interest is still 8% per annum. What are the forward price and the value of the short position in the forward contract?*

Answer 5.29

a) The present value, *I* , of the income from the security is given by:

$$I = 1\times\mathrm{e}^{-0.08\times 2/12} + 1\times e^{-0.08\times 5/12} = 1.9540$$

From equation (5.2) the forward price, F_0, is given by:

$$F_0 = (50 - 1.9540)e^{0.08\times 0.5} = 50.01$$

Or $50.01.

The initial value of the forward contract is (by design) zero. The fact that the forward price is very close to the spot price should come as no surprise. When the compounding frequency is ignored the dividend yield on the stock equals the risk-free rate of interest.

Answer 5.29

b) In three months:

$$I = e^{-0.08\times 2/12} = 0.9868$$

The delivery price, K, is 50.01. From equation (5.6) the value of the short forward contract, f, is given by

$$f = -(48 - 0.9868 - 50.01e^{-0.08\times 3/12}) = 2.01$$

The forward price is

$$(48 - 0.9868)e^{0.08\times 3/12} = 47.96$$

Recap: Known yield

Define q as the average yield per annum on an asset during the life of a forward contract with continuous compounding. It can be shown (see Problem 5.20) that

$$F_0 = S_0 e^{(r-q)T} \qquad \textbf{(5.3)}$$

Finally, using equation (5.4) in conjunction with (5.3) gives the following expression for the value of a long forward contract on an investment asset that provides a known yield at rate q:

$$f = S_0 e^{-qT} - K e^{-rT} \qquad \textbf{(5.7)}$$

Problem 5.10

- *The risk-free rate of interest is 7% per annum with continuous compounding, and the dividend yield on a stock index is 3.2% per annum. The current value of the index is 150. What is the six-month futures price?*

Answer 5.10

Using equation (5.3) the six month futures price is

$$150e^{(0.07-0.032)\times 0.5} = 152.88$$

Or $152.88.

Problem 5.12

- *Suppose that the risk-free interest rate is 10% per annum with continuous compounding and that the dividend yield on a stock index is 4% per annum. The index is standing at 400, and the futures price for a contract deliverable in four months is 405. What arbitrage opportunities does this create?*

Answer 5.12

The theoretical futures price is

$$400e^{(0.10-0.04)\times 4/12} = 408.08$$

The actual futures price is only 405. This shows that the index futures price is too low relative to the index. The correct arbitrage strategy (reverse cash & carry) is

a) buy futures contracts

b) short the shares underlying the index.

Recap: futures on currencies

A foreign currency has the property that the holder of the currency can earn interest at the risk-free interest rate prevailing in the foreign country. For example, the holder can invest the currency in a foreign-denominated bond. We define r_f as the value of the foreign risk-free interest rate when money is invested for time T. The variable r is the domestic risk-free rate when money is invested for this period of time.

The relationship between F_0 and S_0 is

$$F_0 = S_0 e^{(r-r_f)T} \tag{5.9}$$

Recap: futures on currencies

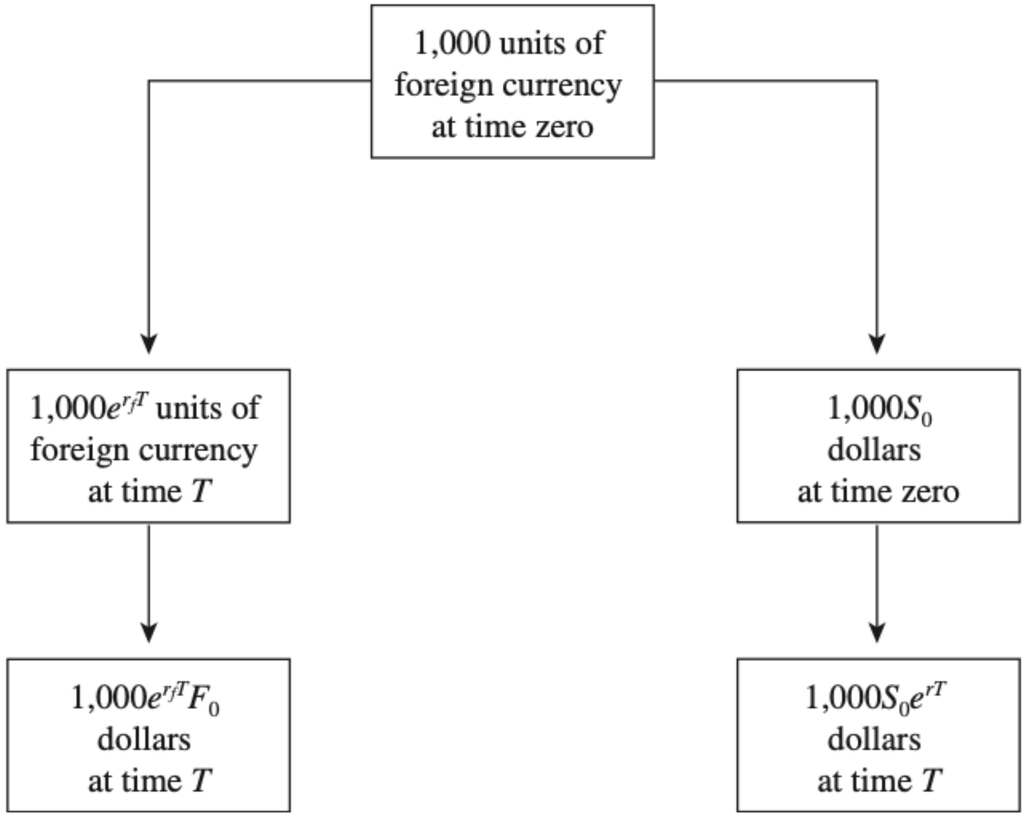
Two way of converting 1,000 units of foreign currency to dollars at time T.

1. Investing it for T years at r_f and entering into a forward contract to sell the proceeds for dollars at time T.
2. Exchanging the foreign currency for dollars in the spot market and investing the proceeds for T years at r.

$$1{,}000e^{r_f T}F_0 = 1{,}000S_0e^{rT}$$

So that

$$F_0 = S_0e^{(r-r_f)T}$$



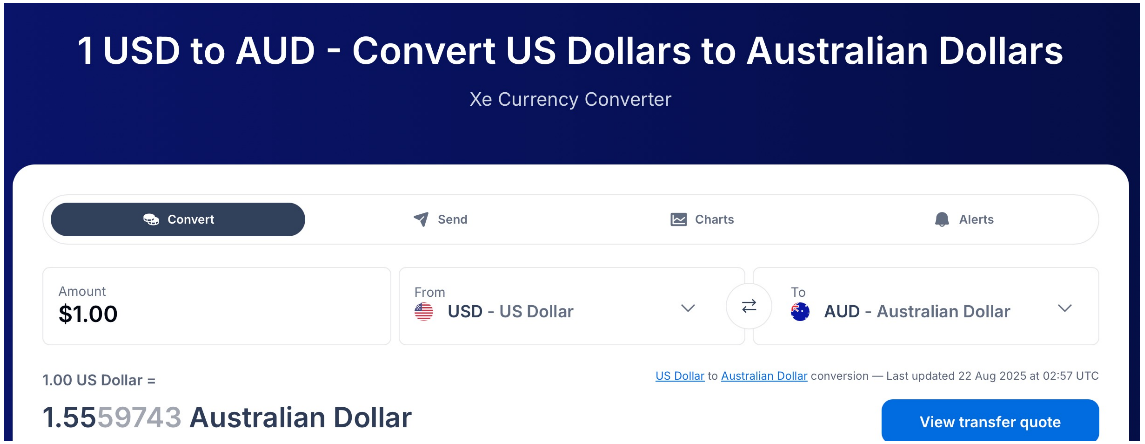
Source: textbook, p.139

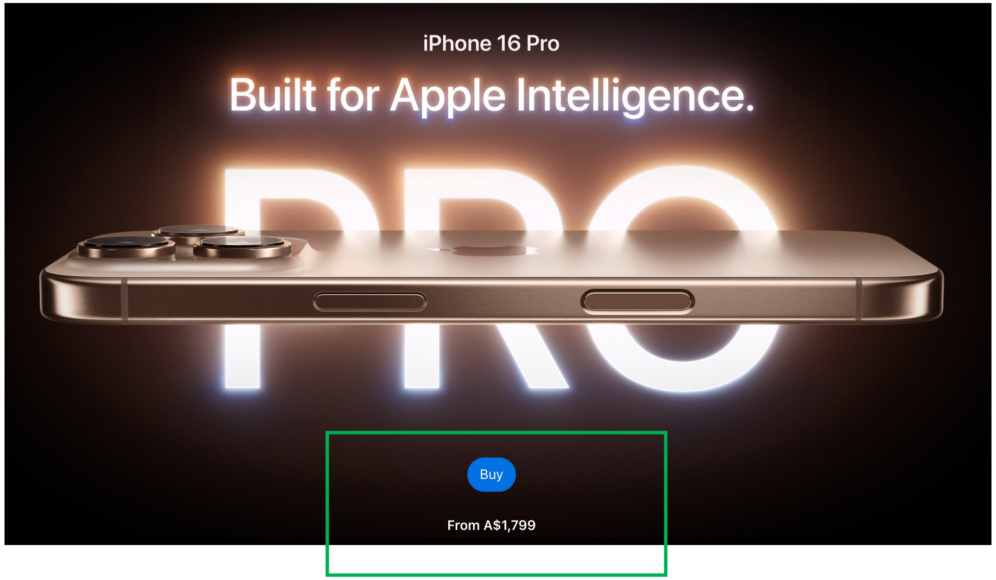
Recap: futures on currencies

Foreign

Domestic

Exchange rate quote: USD1 = AUD1.556





Price tag: 1 iPhone=1,799 auds

Problem 5.27

- *The current USD/euro exchange rate is 1.4000 dollar per euro. The six month forward exchange rate is 1.3950. The six month USD interest rate is 1% per annum continuously compounded. Estimate the six month euro interest rate.*

Answer 5.27

From the way the spot rate is expressed (€1 = US$1.4000), the Euro is the 'foreign currency'. If the six-month euro interest rate is r_f then

$$1.3950 = 1.4000e^{(0.01 - r_f)\times 0.5}$$

So that

$$0.01 - r_f = 2\ln\left(\frac{1.3950}{1.4000}\right) = -0.00716$$

And $r_f = 0.01716$. The six-month euro interest rate is 1.716%.

Recap: futures on commodities

Storage costs can be treated as negative income. If U is the present value of all the storage costs during the life of a forward contract, it follows from equation (5.2) that

$$F_0 = (S_0 + U)e^{rT} \tag{5.11}$$

Example 5.8 provides an application of this formula.

If the storage costs (net of income) incurred at any time are proportional to the price of the commodity, they can be treated as negative yield. In this case, from equation (5.3), we have

$$F_0 = S_0 e^{(r+u)T} \tag{5.12}$$

where u denotes the storage costs per annum as a proportion of the spot price net of any yield earned on the asset.

Problem 5.15

- *The current price of silver is \$30 per ounce. The storage costs are \$0.48 per ounce per year payable quarterly in advance. Assuming that interest rates are 10% per annum for all maturities, calculate the futures price of silver for delivery in nine months.*

Answer 5.15

The present value of the storage costs for nine months are

$$U = 0.12 + 0.12e^{-0.10\times0.25} + 0.12e^{-0.10\times0.5} = 0.351$$

Or \$0.351.

The futures price is from equation (5.11) given by F_0 where

$$F_0 = (30 + 0.351)e^{0.10\times0.75} = 32.72$$

i.e., it is \$32.72 per ounce.