Review

积分的应用--微元法

- $\rho=\rho(\theta),\theta\in[\alpha,\beta]$,向径扫过的面积 $S=\int_{\alpha}^{\beta}\frac{1}{2}\rho^{2}(\theta)\mathrm{d}\theta.$

- 曲线的弧长

$$L:\ x=x(t),y=y(t),z=z(t),\quad t\in[\alpha,\beta].$$

$$l=\int_{\alpha}^{\beta}\sqrt{(x'(t))^{2}+(y'(t))^{2}+(z'(t))^{2}}\,\mathrm{d}t.$$

- 旋转体的体积:

(1)$y=f(x), a\le x\le b$,绕 x 轴旋转

$$V(\Omega)=\pi\int_a^b f^2(x)\mathrm{d}x.$$

(2)$x=x(t), y=y(t), \alpha\le t\le\beta, x(t), y(t)\in C^1[\alpha,\beta]$,

绕 x 轴旋转

$$V(\Omega)=\left|\pi\int_\alpha^\beta y^2(t)x'(t)\mathrm{d}t\right|.$$

- 旋转面的面积

$y=f(x), a\le x\le b,$ 绕 x 轴旋转

$$S=2\pi\int_a^b |f(x)|\sqrt{1+(f'(x))^2}\,\mathrm{d}x.$$

$x=x(t), y=y(t), \alpha\le t\le \beta,$ 绕 x 轴旋转

$$S=2\pi\int_\alpha^\beta |y(t)|\sqrt{(x'(t))^2+(y'(t))^2}\,\mathrm{d}t.$$

- 积分在物理中的应用(功,质量,质心,引力)

§1.广义Riemann积分的概念

- 无穷限积分

Def.若 $\lim\limits_{A\to+\infty}\int_a^A f(x)\mathrm{d}x=I$,则称 f 在 $[a,+\infty)$ 上的广义积分收敛,称 I 为 f 在 $[a,+\infty)$ 上的广义积分(值),记作

$$\int_a^{+\infty} f(x)\mathrm{d}x=\lim_{A\to+\infty}\int_a^A f(x)\mathrm{d}x.$$

若 $\lim\limits_{A\to+\infty}\int_a^A f(x)\mathrm{d}x$ 不存在,则称广义积分 $\int_a^{+\infty} f(x)\mathrm{d}x$ 发散.

Def. $\int_{-\infty}^{a} f(x)\mathrm{d}x \triangleq \lim\limits_{A\to-\infty}\int_A^a f(x)\mathrm{d}x.$

Ex. $\int_0^{+\infty}\frac{1}{1+x^2}\mathrm{d}x=\lim_{A\to+\infty}\int_0^{A}\frac{1}{1+x^2}\mathrm{d}x=\lim_{A\to+\infty}\arctan x\Big|_0^{A}=\frac{\pi}{2}.$

$$\left(\int_0^{+\infty}\frac{1}{1+x^2}\mathrm{d}x=\arctan x\Big|_0^{+\infty}=\frac{\pi}{2}.\right)$$

(广义积分的Newton-Leibnitz公式)

Ex. $\int_1^{+\infty}\frac{\ln x}{x^2}\mathrm{d}x=\lim_{A\to+\infty}\int_1^{A}\frac{\ln x}{x^2}\mathrm{d}x=-\lim_{A\to+\infty}\int_1^{A}\ln x\,\mathrm{d}\frac{1}{x}$

$$=-\lim_{A\to+\infty}\left(\frac{\ln x}{x}\Big|_1^{A}-\int_1^{A}\frac{1}{x^2}\mathrm{d}x\right)=-\lim_{A\to+\infty}\left(\frac{\ln x}{x}+\frac{1}{x}\right)\Big|_1^{A}=1.$$

$$\left(\int_1^{+\infty}\frac{\ln x}{x^2}\mathrm{d}x=-\int_1^{+\infty}\ln x\,\mathrm{d}\frac{1}{x}=-\frac{\ln x}{x}\Big|_1^{+\infty}+\int_1^{+\infty}\frac{1}{x^2}\mathrm{d}x=0-\frac{1}{x}\Big|_1^{+\infty}=1.\right)$$

(广义积分的分部积分)

Ex. $\displaystyle\int_1^{+\infty}\frac{\mathrm{d}x}{x^2\sqrt{1+x^2}}=\lim_{a\to+\infty}\int_1^{a}\frac{\mathrm{d}x}{x^2\sqrt{1+x^2}}$

$$\underline{\underline{x=\tan t}}\ \lim_{a\to+\infty}\int_{\pi/4}^{\arctan a}\frac{\sec^2 t\mathrm{d}t}{\tan^2 t\sec t}=\lim_{a\to+\infty}\int_{\pi/4}^{\arctan a}\frac{\cos t\mathrm{d}t}{\sin^2 t}$$

$$=\lim_{a\to+\infty}\left(-\frac{1}{\sin t}\bigg|_{\pi/4}^{\arctan a}\right)=\sqrt{2}-1.$$

$$\left(\int_1^{+\infty}\frac{\mathrm{d}x}{x^2\sqrt{1+x^2}}=\int_{\pi/4}^{\pi/2}\frac{\cos t\mathrm{d}t}{\sin^2 t\sec t}=-\frac{1}{\sin t}\bigg|_{\pi/4}^{\pi/2}=\sqrt{2}-1.\right)$$

（广义积分的变量替换）

Ex. 讨论广义积分 $\int_1^{+\infty}\frac{1}{x^p}\mathrm{d}x$ 的敛散性.

解: $p=1$时, $\int_1^{+\infty}\frac{1}{x^p}\mathrm{d}x=\int_1^{+\infty}\frac{1}{x}\mathrm{d}x=\ln x\Big|_1^{+\infty}=+\infty$, 发散.

$$p\neq 1\text{时}, \int_1^{+\infty}\frac{1}{x^p}\mathrm{d}x=\frac{1}{1-p}x^{1-p}\Big|_1^{+\infty}=\begin{cases}\dfrac{1}{p-1}, & p>1,\\ +\infty, & p<1.\end{cases}$$

综上, $p>1$时, $\int_1^{+\infty}\frac{1}{x^p}\mathrm{d}x=\frac{1}{p-1}$; $p\leq 1$时, $\int_1^{+\infty}\frac{1}{x^p}\mathrm{d}x$ 发散. □

Ex. 讨论广义积分 $\int_e^{+\infty}\frac{1}{x(\ln x)^p}\mathrm{d}x$ 的敛散性.

解: $p=1$ 时, $\int_e^{+\infty}\frac{1}{x(\ln x)^p}\mathrm{d}x=\int_e^{+\infty}\frac{1}{\ln x}\mathrm{d}\ln x=\ln\ln x\Big|_e^{+\infty}=+\infty.$

$p\neq 1$ 时, $\int_e^{+\infty}\frac{1}{x(\ln x)^p}\mathrm{d}x=\int_e^{+\infty}\frac{1}{(\ln x)^p}\mathrm{d}\ln x$

$$=\frac{1}{1-p}(\ln x)^{1-p}\Big|_e^{+\infty}=\begin{cases}\frac{1}{p-1}, & p>1,\\ +\infty, & p<1.\end{cases}$$

综上, $\int_1^{+\infty}\frac{1}{x(\ln x)^p}\mathrm{d}x$ 当 $p>1$ 时收敛, 当 $p\leq 1$ 时发散. □

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Remark. 若$\int_a^{+\infty} f(x)\mathrm{d}x$与$\int_{-\infty}^{a} f(x)\mathrm{d}x$均收敛,则$\forall b\in\mathbb{R}$,

$$\int_b^{+\infty} f(x)\mathrm{d}x = \int_b^{a} f(x)\mathrm{d}x + \int_a^{+\infty} f(x)\mathrm{d}x \text{ 收敛,}$$

$$\int_{-\infty}^{b} f(x)\mathrm{d}x = \int_{-\infty}^{a} f(x)\mathrm{d}x - \int_b^{a} f(x)\mathrm{d}x \text{ 收敛.}$$

Def.若$\exists a\in\mathbb{R}$, $s.t.\int_a^{+\infty} f(x)\mathrm{d}x$与$\int_{-\infty}^{a} f(x)\mathrm{d}x$均收敛,则称广义积分$\int_{-\infty}^{+\infty} f(x)\mathrm{d}x$收敛,且

$$\int_{-\infty}^{+\infty} f(x)\mathrm{d}x \triangleq \int_{-\infty}^{a} f(x)\mathrm{d}x + \int_a^{+\infty} f(x)\mathrm{d}x = \lim_{\substack{\mathrm{A}\to-\infty\\ B\to+\infty}} \int_{\mathrm{A}}^{\mathrm{B}} f(x)\mathrm{d}x.$$

Ex. $\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{e^x + e^{2-x}} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{e^{2-x}(e^{2x-2}+1)}$

$$= \frac{1}{e}\int_{-\infty}^{+\infty} \frac{e^{x-1}\mathrm{d}x}{(e^{x-1})^2+1} = \frac{1}{e}\arctan e^{x-1}\Big|_{-\infty}^{+\infty} = \frac{\pi}{2e}.$$

Question.变上(下)限的广义积分如何求导?

$$\left(\int_{-\infty}^{x} f(t)\mathrm{d}t\right)' = \left(\int_{-\infty}^{a} f(t)\mathrm{d}t + \int_{a}^{x} f(t)\mathrm{d}t\right)' = f(x)$$

$$\left(\int_{\alpha(x)}^{+\infty} f(t)\mathrm{d}t\right)' = \left(\int_{0}^{+\infty} f(t)\mathrm{d}t - \int_{0}^{\alpha(x)} f(t)\mathrm{d}t\right)' = -f(\alpha(x))\cdot\alpha'(x)$$

Ex. $F(x)=e^{x^2/2}\int_x^{+\infty}e^{-t^2/2}\mathrm{d}t,\ x\in[0,+\infty).$

证明：(1) $\lim\limits_{x\to+\infty}F(x)=0$, (2) $F(x)$在$[0,+\infty)$上单减.

Proof. (1) $x>1$时, $0<\int_x^{+\infty}e^{-t^2/2}\mathrm{d}t\le\int_x^{+\infty}te^{-t^2/2}\mathrm{d}t$

$$=-e^{-t^2/2}\Big|_x^{+\infty}=e^{-x^2/2}\ (\to 0,\ x\to+\infty\text{时}.)$$

$$\lim_{x\to+\infty}F(x)=\lim_{x\to+\infty}\frac{\int_x^{+\infty}e^{-t^2/2}\mathrm{d}t}{e^{-x^2/2}}=\lim_{x\to+\infty}\frac{-e^{-x^2/2}}{-xe^{-x^2/2}}=\lim_{x\to+\infty}\frac{1}{x}=0.$$

(2) $F'(x)=xe^{x^2/2}\int_x^{+\infty}e^{-t^2/2}\mathrm{d}t-1\le e^{x^2/2}\int_x^{+\infty}te^{-t^2/2}\mathrm{d}t-1=0.$

故$F(x)$在$[0,+\infty)$上单减.□

●瑕积分(无界函数积分)

Def. f在$[a,b)$上定义,在b点附近无界(此时称$x=b$为f的一个瑕点),若$\forall\delta\in(0,b-a)$, $f\in R[a,b-\delta]$,且

$$\lim_{\delta\to 0^+}\int_a^{b-\delta} f(x)\mathrm{d}x = I,$$

则称f在$[a,b)$上的瑕积分收敛,称I为f在$[a,b)$上的瑕积分(值),记作

$$\int_a^b f(x)\mathrm{d}x = \lim_{\delta\to 0^+}\int_a^{b-\delta} f(x)\mathrm{d}x.$$

若$\lim\limits_{\delta\to 0^+}\int_a^{b-\delta} f(x)\mathrm{d}x$不存在,则称瑕积分$\int_a^b f(x)\mathrm{d}x$发散.

Def. f在(a,b)上定义, a,b为瑕点, 若$\exists c\in(a,b)$, s.t.瑕积分 $\int_a^c f(x)\mathrm{d}x$与$\int_c^b f(x)\mathrm{d}x$均收敛, 则

$$\int_a^b f(x)\mathrm{d}x \triangleq \int_a^c f(x)\mathrm{d}x+\int_c^b f(x)\mathrm{d}x.$$

此时, $\int_a^b f(x)\mathrm{d}x=\int_a^d f(x)\mathrm{d}x+\int_d^b f(x)\mathrm{d}x, \forall d\in(a,b)$;

$$\int_a^b f(x)\mathrm{d}x=\lim_{\alpha\to a^+}\int_\alpha^c f(x)\mathrm{d}x+\lim_{\beta\to b^-}\int_c^\beta f(x)\mathrm{d}x$$

$$=\lim_{\substack{\alpha\to a^+,\\ \beta\to b^-}}\int_\alpha^\beta f(x)\mathrm{d}x.$$

Ex. $\int_0^1 \ln x\mathrm{d}x = \lim_{\delta\to 0+}\int_\delta^1 \ln x\mathrm{d}x = \lim_{\delta\to 0+}\left(x\ln x - x\right)\Big|_\delta^1 = -1.$

$$\left(\int_0^1 \ln x\mathrm{d}x = x\ln x\Big|_0^1 - \int_0^1 x\cdot\frac{1}{x}\mathrm{d}x = -1.\right)$$

Ex. $\int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \lim_{\substack{\alpha\to(-1)^+\\ \beta\to 1^-}}\int_\alpha^\beta \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \lim_{\substack{\alpha\to(-1)^+\\ \beta\to 1^-}} \arcsin x\Big|_\alpha^\beta = \pi.$

$$\left(\int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x\Big|_{-1}^1 = \pi.\right)$$

Ex.讨论广义积分$\int_0^1 \frac{1}{x^p}\mathrm{d}x$的敛散性.

解: $p=1$时, $\int_0^1 \frac{1}{x^p}\mathrm{d}x = \int_0^1 \frac{1}{x}\mathrm{d}x = \ln x\Big|_0^1 = +\infty.$

$$p\neq 1\text{时}, \int_0^1 \frac{1}{x^p}\mathrm{d}x = \frac{1}{1-p}x^{1-p}\Big|_0^1 = \begin{cases} \dfrac{1}{1-p}, & p<1, \\ +\infty, & p>1. \end{cases}$$

综上, $p<1$时, $\int_0^1 \frac{1}{x^p}\mathrm{d}x = \frac{1}{p-1}$; $p\geq 1$时, $\int_0^1 \frac{1}{x^p}\mathrm{d}x$发散. □

Remark. $f\in R[a,b]$, 则$\int_a^b f(x)\mathrm{d}x = \lim\limits_{c\to b^-}\int_a^c f(x)\mathrm{d}x = \lim\limits_{c\to a^+}\int_c^b f(x)\mathrm{d}x.$

所以, $p\leq 0$时, 尽管$\int_0^1 \frac{1}{x^p}\mathrm{d}x$不是瑕积分, 我们也称其收敛.

Ex. $\int_0^{+\infty}\frac{1}{\sqrt{x}(1+x)}\mathrm{d}x$

$$=\int_0^1\frac{1}{\sqrt{x}(1+x)}\mathrm{d}x+\int_1^{+\infty}\frac{1}{\sqrt{x}(1+x)}\mathrm{d}x$$ 瑕积分+无穷限积分

$$=\int_0^1\frac{2}{1+(\sqrt{x})^2}\mathrm{d}\sqrt{x}+\int_1^{+\infty}\frac{2}{1+(\sqrt{x})^2}\mathrm{d}\sqrt{x}$$

$$=2\arctan\sqrt{x}\Big|_0^1+2\arctan\sqrt{x}\Big|_1^{+\infty}=\pi.$$

$$\left(\int_0^{+\infty}\frac{1}{\sqrt{x}(1+x)}\mathrm{d}x=\int_0^{+\infty}\frac{2}{1+(\sqrt{x})^2}\mathrm{d}\sqrt{x}=2\arctan\sqrt{x}\Big|_0^{+\infty}=\pi.\right)$$

Ex. 计算$\int_0^1 x^n \ln^n x\mathrm{d}x, n$为正整数.

解: $\int_0^1 x^n \ln^n x\mathrm{d}x = \frac{1}{n+1}\int_0^1 \ln^n x\mathrm{d}x^{n+1}$

$$= \frac{1}{n+1}x^{n+1}\ln^n x\Big|_0^1 - \frac{n}{n+1}\int_0^1 x^n \ln^{n-1} x\mathrm{d}x = -\frac{n}{n+1}\int_0^1 x^n \ln^{n-1} x\mathrm{d}x$$

$$= -\frac{n}{(n+1)^2}\int_0^1 \ln^{n-1} x\mathrm{d}x^{n+1} = \frac{n(n-1)}{(n+1)^2}\int_0^1 x^n \ln^{n-2} x\mathrm{d}x$$

$$= \cdots = (-1)^n \frac{n!}{(n+1)^n}\int_0^1 x^n\mathrm{d}x = (-1)^n \frac{n!}{(n+1)^{n+1}}. \square$$

Ex. $I=\int_0^{\pi/2}\ln(\cos x)\mathrm{d}x.$ 瑕积分

解: $I=\int_0^{\pi/2}\ln(\sin x)\mathrm{d}x$ (广义积分的变量替换)

$$=\int_0^{\pi/2}\ln 2\mathrm{d}x+\int_0^{\pi/2}\ln\sin\frac{x}{2}\mathrm{d}x+\int_0^{\pi/2}\ln\cos\frac{x}{2}\mathrm{d}x$$

$$=\frac{\pi}{2}\ln 2+2\int_0^{\pi/4}\ln\sin t\mathrm{d}t+2\int_0^{\pi/4}\ln\cos t\mathrm{d}t$$

$$=\frac{\pi}{2}\ln 2+2\int_0^{\pi/4}\ln\sin t\mathrm{d}t+2\int_{\pi/4}^{\pi/2}\ln\sin t\mathrm{d}t$$

$$=\frac{\pi}{2}\ln 2+2I,\qquad I=-\frac{\pi}{2}\ln 2.\square$$

Ex. $I=\int_0^{\pi/2}\sin x\cdot\ln\sin x\,dx$　　　瑕积分

分析:$I=-\int_0^{\pi/2}\ln\sin x\,d\cos x$

$$=-\cos x\ln\sin x\Big|_0^{\pi/2}+\int_0^{\pi/2}\frac{\cos^2 x}{\sin x}dx,$$ 无法计算

解：$I=\int_0^{\pi/2}\ln\sin x\,d(1-\cos x)$

$$=(1-\cos x)\ln\sin x\Big|_0^{\pi/2}-\int_0^{\pi/2}\frac{1-\cos x}{\sin x}\cos x\,dx$$

$$=0-\int_0^{\pi/2}\frac{\sin^2 x}{\sin x(1+\cos x)}\cos x\,dx=\int_0^{\pi/2}\frac{\cos x}{1+\cos x}d\cos x$$

$$=\int_1^0\frac{t}{1+t}dt=\int_0^1\left(\frac{1}{1+t}-1\right)dt=-1+\ln(1+t)\Big|_0^1=-1+\ln 2.\square$$

Lemma (Riemann-Lebesgue). f在$[a,b]$上可积或广义绝对可积(即f与$|f|$均在$[a,b]$上广义可积),则

$$\lim_{\lambda\to\infty}\int_a^b f(x)\cos\lambda x\mathrm{d}x=0,\quad \lim_{\lambda\to\infty}\int_a^b f(x)\sin\lambda x\mathrm{d}x=0.$$

Proof. 只证第一式,第二式同理.

Case1. 设f在$[a,b]$上可积,则f在$[a,b]$上有界,即

$$\exists M>0, s.t. |f(x)|\le M, \forall x\in[a,b].$$

任意给定$\lambda>1$,令$n=\lfloor\sqrt{\lambda}\rfloor$. n等分$[a,b]$:

$$x_i=a+(b-a)i/n,\quad i=0,1,2,\cdots,n.$$

$$\omega_i(f)=\sup\{f(\xi)-f(\eta):\xi,\eta\in[x_{i-1},x_i]\},\quad i=1,2,\cdots,n.$$

f在$[a,b]$上可积,则$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\omega_i(f)\Delta x_i=0.$ 于是

$$\left|\int_a^b f(x)\cos\lambda x\mathrm{d}x\right|=\left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)\cos\lambda x\mathrm{d}x\right|$$

$$\leq\left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left(f(x)-f(x_i)\right)\cos\lambda x\mathrm{d}x\right|+\left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x_i)\cos\lambda x\mathrm{d}x\right|$$

$$\leq\sum_{i=1}^{n}\omega_i(f)\Delta x_i+\sum_{i=1}^{n}\left|f(x_i)\right|\left|\int_{x_{i-1}}^{x_i}\cos\lambda x\mathrm{d}x\right|$$

$$\leq\sum_{i=1}^{n}\omega_i(f)\Delta x_i+\frac{2Mn}{\lambda}=\sum_{i=1}^{\lfloor\sqrt{\lambda}\rfloor}\omega_i(f)\Delta x_i+\frac{2M\lfloor\sqrt{\lambda}\rfloor}{\lambda}$$

$\to 0$, 当$\lambda\to+\infty$时.

Case2. f在$[a,b]$上广义绝对可积, 不妨设a为唯一的瑕点.

则$\forall \varepsilon > 0, \exists \delta > 0$, s.t., f在$[a+\delta, b]$上可积, 且

$$\int_a^{a+\delta} |f(x)|\,\mathrm{d}x < \varepsilon/2,$$

从而 $$\left|\int_a^{a+\delta} f(x)\cos\lambda x\,\mathrm{d}x\right| \le \int_a^{a+\delta} |f(x)|\,\mathrm{d}x < \varepsilon/2,$$

$$\lim_{\lambda\to+\infty}\int_{a+\delta}^{b} f(x)\cos\lambda x\,\mathrm{d}x = 0.$$

于是$\exists \Lambda > 0$, 当$\lambda > \Lambda$时, $\left|\int_{a+\delta}^{b} f(x)\cos\lambda x\,\mathrm{d}x\right| < \varepsilon/2$, 进而有

$$\left|\int_a^b f(x)\cos\lambda x\,\mathrm{d}x\right| \le \left|\int_a^{a+\delta} f(x)\cos\lambda x\,\mathrm{d}x\right| + \left|\int_{a+\delta}^{b} f(x)\cos\lambda x\,\mathrm{d}x\right|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \forall \lambda > \Lambda. \square$$

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作业：习题6.1

No.2(2,8),3(3,5),4(4,6)