Review

- 隐函数求导、参数函数求导

- **Thm.** 设 $f(x)$ 与 $g(x)$ 在点 x 处有 n 阶导数, $c\in\mathbb{R}$, 则

  (1) $(f+g)^{(n)}(x)=f^{(n)}(x)+g^{(n)}(x)$;

  (2) $(cf)^{(n)}(x)=c\cdot f^{(n)}(x)$;

  (3) $(f\cdot g)^{(n)}(x)=\sum_{k=0}^{n}C_n^k f^{(k)}(x)g^{(n-k)}(x)$. (Leibniz公式)

§ 1. 微分中值定理

- (Fermat Thm) x_0是f的极值点, $f'(x_0)$存在, 则$f'(x_0)=0$.
- $f,g\in C[a,b]$, f,g在(a,b)可导, 则

(Rolle)若$f(a)=f(b)$, 则$\exists\xi\in(a,b)$, $s.t.\ f'(\xi)=0$.

(Lagrange)$\exists\xi\in(a,b)$, $s.t.\ f'(\xi)=\dfrac{f(b)-f(a)}{b-a}$.

(Cauchy)若$g'(t)\neq 0$, 则$\exists\xi\in(a,b)$, $s.t.\ \dfrac{f'(\xi)}{g'(\xi)}=\dfrac{f(b)-f(a)}{g(b)-g(a)}$.

- (Darboux Thm) f在$[a,b]$上可导, $f_+'(a)\neq f_-'(b)$, 则对介于$f_+'(a)$与$f_-'(b)$之间的任意实数λ, $\exists\xi\in(a,b)$, $s.t.\ f'(\xi)=\lambda$.

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Def. f在x_0的邻域中有定义,若$\exists\rho>0, s.t.$

$$f(x)\geq(\leq)f(x_0),\quad \forall x\in U(x_0,\rho),$$

则称f在x_0取得极小(大)值,并称x_0是f的极小(大)值点.

若$\exists\rho>0, s.t.$

$$f(x)>(<)f(x_0),\quad \forall x\in U(x_0,\rho),$$

则称f在x_0取得严格极小(大)值,并称x_0是f的严格极小(大)值点.

Question. 极值与最值的区别与联系?

Thm.(Fermat) x_0是f的极值点, $f'(x_0)$存在,则$f'(x_0)=0$.

Proof. 不妨设x_0是f的极小值点,则$\exists\rho>0, s.t.$

$$f(x)\ge f(x_0),\quad \forall x\in U(x_0,\rho).$$

而$f'(x_0)$存在, 由极限的保序性,有

$$f'_-(x_0)=\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}\le 0,$$

$$f'_+(x_0)=\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}\ge 0.$$

故 $f'(x_0)=f'_-(x_0)=f'_+(x_0)=0.\square$

Def. 若$f'(x_0)=0$,则称x_0为f的驻点.

Thm.(Rolle) $f \in C[a,b]$, f在(a,b)可导.若$f(a)=f(b)$,则存在$\xi \in (a,b), s.t.\ f'(\xi)=0.$

Proof. $f \in C[a,b]$,则f在$[a,b]$上有最大值M和最小值m.

若$f(a)=f(b)=M=m$,则f在$[a,b]$上恒为常数,因而$\forall x \in (a,b)$,有$f'(x)=0.$

若M与m至少有一个不在端点取到, 不妨设$f(a)<M$.

则$\exists \xi \in (a,b), s.t.\ f(\xi)=M$. 由Fermat定理, $f'(\xi)=0.\square$

Question. Rolle定理的几何意义?

Thm.(Lagrange) $f\in C[a,b]$, f在(a,b)可导, 则$\exists\xi\in(a,b)$, s.t.

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}.$$

Proof. 令$h(x)=f(x)-\dfrac{f(b)-f(a)}{b-a}(x-a)$,

则$h\in C[a,b]$, h在(a,b)可导, $h(a)=h(b)=f(a)$. 由Rolle定理,

$\exists\xi\in(a,b)$, s.t. $h'(\xi)=f'(\xi)-\dfrac{f(b)-f(a)}{b-a}=0.$ □

Question. Lagrange中值定理的几何意义?

Remark. $f\in C[a,b]$, f在(a,b)可导,则

(1)$\exists\xi\in(a,b)$, s.t. $\quad f(b)-f(a)=f'(\xi)(b-a)$.

(2)$\forall x,x_0\in[a,b]$,$\exists$介于x与x_0之间的ξ, s.t.

$$f(x)-f(x_0)=f'(\xi)(x-x_0).$$

(3)$\forall x_0,x_0+\Delta x\in[a,b]$,$\exists\theta\in(0,1)$, s.t.

$$f(x_0+\Delta x)-f(x_0)=f'(x_0+\theta\Delta x)\cdot\Delta x.$$

Thm.(Cauchy) $f,g\in C[a,b]$, f,g在(a,b)可导,且$\forall t\in(a,b)$,

有$g'(t)\neq 0$. 则存在$\xi\in(a,b)$, s.t. $\dfrac{f'(\xi)}{g'(\xi)}=\dfrac{f(b)-f(a)}{g(b)-g(a)}$.

Proof. 由Lagrange中值定理, $\exists \eta \in (a,b), s.t.$

$$g(b)-g(a)=g'(\eta)(b-a)\neq 0.$$

令$h(x)=f(x)-\dfrac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))$, 则 $h\in C[a,b]$,

h在(a,b)可导, $h(a)=h(b)=f(a)$. 由Rolle定理,

$$\exists \xi \in (a,b), s.t.\ h'(\xi)=f'(\xi)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(\xi)=0.\square$$

Question. Cauchy中值定理的几何意义?

Remark. Lagrange中值定理是Cauchy中值定理的特殊情形.

Thm. f在$[a,b]$上可导.

(1)若$f'_+(a)f'_-(b)<0$,则$\exists\xi\in(a,b), s.t. f'(\xi)=0$.

(2)若$f'_+(a)\neq f'_-(b)$,则对介于$f'_+(a)$与$f'_-(b)$之间的任意实数λ, $\exists\xi\in(a,b), s.t. f'(\xi)=\lambda$. (Darboux)

Proof.(1)不妨设

$$f'_+(a)=\lim_{x\to a+}\frac{f(x)-f(a)}{x-a}>0,\ f'_-(b)=\lim_{x\to b-}\frac{f(x)-f(b)}{x-b}<0.$$

由极限的保序性,$\exists x_1,x_2\in(a,b), s.t. f(x_1)>f(a),\ f(x_2)>f(b)$.

于是 f在$[a,b]$上的最大值在某点$\xi\in(a,b)$处取得. 由Fermat定理, $f'(\xi)=0$.

(2) $f_+'(a)\neq f_-'(b)$,则对介于$f_+'(a)$与$f_-'(b)$之间的任意λ,令

$$g(x)=f(x)-\lambda x.$$

则 $g_+'(a)g_-'(b)=(f_+'(a)-\lambda)(f_-'(b)-\lambda)<0.$

由(1)中结论, $\exists\xi\in(a,b), s.t. g'(\xi)=0$,即$f'(\xi)=\lambda$.□

Corollary. 导函数没有第一类间断点. （思考题）

Ex. 若$f'(x)\equiv 0, x\in(a,b)\Leftrightarrow f(x)$在$(a,b)$上为常数.

Proof. $\Leftarrow$: 显然.

$\Rightarrow$: $\forall x_1,x_2\in(a,b)$,由Lagrange中值定理,$\exists$介于x_1,x_2之间的ξ,

s.t. $f(x_1)-f(x_2)=f'(\xi)(x_1-x_2)=0$. 故$f$在$(a,b)$上为常数.□

Ex. $\dfrac{x}{1+x} < \ln(1+x) < x \quad (x > -1, x \neq 0).$

Proof. 由Lagrange中值定理, $\forall x > -1, \exists \theta \in (0,1), s.t.$

$$\ln(1+x) = \ln(1+x) - \ln 1 = \frac{x}{1+\theta x}.$$

当$x > 0$时,

$$1 < 1+\theta x < 1+x, \ \frac{1}{1+x} < \frac{1}{1+\theta x} < 1, \ \ \frac{x}{1+x} < \frac{x}{1+\theta x} < x.$$

当$-1 < x < 0$时,

$$1 > 1+\theta x > 1+x, \ \frac{1}{1+x} > \frac{1}{1+\theta x} > 1, \ \ \frac{x}{1+x} < \frac{x}{1+\theta x} < x. \square$$

Remark. $\dfrac{1}{n+1} < \ln(1+\dfrac{1}{n}) < \dfrac{1}{n} \quad (n \in \mathbb{N}).$

Ex. f在$[0,+\infty)$上可导, $f'(x)$单调递减, $f(0)=0$,则

$$f(x_1+x_2)\le f(x_1)+f(x_2)\quad (x_1,x_2>0).$$

Proof. 不妨设$0<x_1\le x_2$,由$f(0)=0$,得

$$f(x_1+x_2)-f(x_1)-f(x_2)$$

$$=f(x_1+x_2)-f(x_2)-(f(x_1)-f(0))$$

$$=f'(x_2+\theta_1 x_1)x_1-f'(\theta_2 x_1)x_1\quad (0<\theta_1,\theta_2<1)$$

$$\le 0\quad (f'\text{单调递减}).\square$$

Ex. $f \in C[a,b]$, f在(a,b)上可导, $f(x)+f'(x)<\varepsilon\ (a<x<b)$

$f(a)<\varepsilon$. 证明: $f(x)<\varepsilon\ (a<x<b)$.

证法一:用Lagrange中值定理. 令$h(x)=e^x(f(x)-\varepsilon)$, 则

$\forall x\in(a,b), \exists \xi\in(a,x), s.t.$

$$h(x)=h(a)+e^{\xi}(f(\xi)+f'(\xi)-\varepsilon)(x-a)<h(a)<0,$$

故$f(x)<\varepsilon$.□

证法二:用Cauchy中值定理. $\forall x\in(a,b), \exists \xi\in(a,x), s.t.$

$$\frac{e^x f(x)-e^a f(a)}{e^x-e^a}=\frac{e^{\xi}(f(\xi)+f'(\xi))}{e^{\xi}}<\varepsilon,$$

故$e^x f(x)<e^a f(a)+\varepsilon(e^x-e^a)<\varepsilon e^x$, 从而$f(x)<\varepsilon$.□

Ex. $x^4+2x^3+6x^2-4x-5=0$恰有两个不同的实根.

Proof. 令$f(x)=x^4+2x^3+6x^2-4x-5$,则

$$\lim_{x\to\pm\infty} f(x)=+\infty.$$

于是$\exists a<0<b$, s.t. $f(a)>0$, $f(b)>0$. 而$f(0)=-5<0$,

由介值定理, $f(x)=0$至少有两个相异实根.

假设$f(x)=0$至少有3个相异实根. 由Rolle定理, $f'(x)$

至少有2个相异实根, $f''(x)$至少有1个实根. 但

$$f''(x)=12x^2+12x+12>0,$$

矛盾.故$f(x)=0$恰有两个相异实根.□

Ex. 求 $\lim\limits_{x\to e}\dfrac{\tan x^x-\tan e^x}{e^{x^x}-e^{e^x}}$.

解: 令 $f(x)=\tan x,\ g(x)=e^x$, 由Cauchy中值定理, 存在 ξ_x

介于 x^x 与 e^x 之间, 使得

$$\frac{\tan x^x-\tan e^x}{e^{x^x}-e^{e^x}}=\frac{f(x^x)-f(e^x)}{g(x^x)-g(e^x)}=\frac{f'(\xi_x)}{g'(\xi_x)}=\frac{\sec^2\xi_x}{e^{\xi_x}},$$

且 $\lim\limits_{x\to e}\xi_x=e^e$, 因此

$$\lim_{x\to e}\frac{\tan x^x-\tan e^x}{e^{x^x}-e^{e^x}}=\lim_{x\to e}\frac{\sec^2\xi_x}{e^{\xi_x}}=\frac{\sec^2 e^e}{e^{e^e}}.\square$$

Ex. $f,g\in C[a,b]$, f,g在(a,b)内可导, $f(a)=f(b)=0$.证明:

$\exists\xi\in(a,b), s.t.\ f'(\xi)+g'(\xi)f(\xi)=0.$

Proof.令$h(x)=f(x)e^{g(x)}$,则h在(a,b)内可导,$h(a)=f(b)=0.$

由Rolle定理,$\exists\xi\in(a,b), s.t.$

$$h'(\xi)=(f'(\xi)+g'(\xi)f(\xi))e^{g(\xi)}=0.$$

$$f'(\xi)+g'(\xi)f(\xi)=0.\square$$

Remark. 辅助函数.

Question. 什么辅助函数求导可出现

(1) $f'(x)+g'(x)f(x)$?　　(2) $f''(x)+2f'(x)+f(x)$?

(3) $f''(x)-2f'(x)+f(x)$?　　(4) $f''(x)-f(x)$?

(5) $f''(x)-f'(x)$?　　$\cdots$

$$(1)\left(f(x)e^{g(x)}\right)'=(f'(x)+f(x)g'(x))e^{g(x)},$$

$$(2)(3)\ (f(x)e^{\pm x})''=\left((f'(x)\pm f(x))e^{\pm x}\right)'$$

$$=(f''(x)\pm 2f'(x)+f(x))e^{\pm x},$$

$$(4)\ \left(e^{\pm x}(f'(x)\mp f(x))\right)'=e^{\pm x}(f''(x)-f(x))$$

$$(5)\ (f'(x)-f(x))'=f''(x)-f'(x).$$

$$(e^{-x}f'(x))'=e^{-x}\left(f''(x)-f'(x)\right).$$

Ex. f在$[0,1]$上二阶可导, $f(0)=f(1)=0$, 则$\exists\xi\in(0,1)$, $s.t.$

$$f''(\xi)=2f'(\xi)/(1-\xi).$$

分析: $\left(a(x)e^{b(x)}\right)'=\left(a'(x)+a(x)b'(x)\right)e^{b(x)}$.

取$a(x)=f'(x)$, 则$b'(x)=-2/(1-x)$, $b(x)=2\ln(1-x)$.

Proof. $f(0)=f(1)=0$, 则$\exists\eta\in(0,1)$, $s.t.\, f'(\eta)=0$.

令$h(x)=(1-x)^2 f'(x)$, 则h在$[0,1]$上可导, $h(\eta)=h(1)=0$,

由Rolle定理,$\exists\xi\in(\eta,1)\subset(0,1)$,s.t.

$$h'(\xi)=(1-\xi)^2 f''(\xi)-2(1-\xi)f'(\xi)=0.$$

$$f''(\xi)=2f'(\xi)/(1-\xi).\square$$

$$\left(\alpha(x)e^{\beta(x)}\right)' = (\alpha'(x)+\alpha(x)\beta'(x))e^{\beta(x)}$$

Ex. f在$[0,1]$上二阶可导, $f'(0)=0$.则$\exists\xi\in(0,1)$, $s.t.$

$$f'(\xi)-(\xi-1)^2 f''(\xi)=0.$$

Proof. 令$g(x)=\begin{cases} e^{1/(x-1)} f'(x), & x\neq 1 \\ 0, & x=1 \end{cases}$,则 $g\in C[0,1]$, g在$(0,1)$上可导, $g(0)=e^{-1}f'(0)=0=g(1)$,

$$g'(x)=\left(f''(x)-\frac{1}{(x-1)^2}f'(x)\right)e^{1/(x-1)}, x\in(0,1).$$

由Rolle定理,$\exists\xi\in(0,1)$, $s.t. g'(\xi)=0$. 从而有

$$f''(\xi)-\frac{1}{(\xi-1)^2}f'(\xi)=0. \square$$

Ex. f, g在$[a,b]$上二阶可导, $f(a)=f(b)=g(a)=g(b)=0$,

$g''(x)\neq 0(x\in(a,b))$. 则$\exists\xi\in(a,b)$, $s.t.$ $\dfrac{f(\xi)}{g(\xi)}=\dfrac{f''(\xi)}{g''(\xi)}$.

Proof. 令$h(x)=f'(x)g(x)-f(x)g'(x)$. 由$f(a)=f(b)=g(a)$ $=g(b)=0$, 得 $h(a)=h(b)=0$. 由Rolle定理, $\exists\xi\in(a,b)$, $s.t.$

$$h'(\xi)=f''(\xi)g(\xi)-f(\xi)g''(\xi)=0.$$

若$g(\xi)=0$, 因$g(a)=g(b)=0$, 由Roll定理, $\exists\xi_1\in(a,\xi)$, $\xi_2\in(\xi,b)$, s.t. $g'(\xi_1)=g'(\xi_2)=0$. $\exists\xi_3\in(\xi_1,\xi_2)$, s.t. $g''(\xi_3)=0$,

与 $g''(x)\neq 0(x\in(a,b))$ 矛盾. 故 $g(\xi)\neq 0$, $\dfrac{f(\xi)}{g(\xi)}=\dfrac{f''(\xi)}{g''(\xi)}$. □

Ex. f,g在$[a,b]$上可导, $g'(x)\neq 0(x\in(a,b))$. 则$\exists\xi\in(a,b)$, s.t.

$$\frac{f(a)-f(\xi)}{g(\xi)-g(b)}=\frac{f'(\xi)}{g'(\xi)}.$$

Proof. 令$h(x)=\big(f(a)-f(x)\big)\big(g(x)-g(b)\big)$, 则 $h(a)=h(b)=0$,

$$h'(x)=[g(x)-g(b)]'[f(a)-f(x)]$$
$$+[f(a)-f(x)]'[g(x)-g(b)].$$

由Rolle定理, $\exists\xi\in(a,b)$, s.t. $h'(\xi)=0$, 即

$$g'(\xi)[f(a)-f(\xi)]-f'(\xi)[g(\xi)-g(b)]=0$$

$$\Leftrightarrow\frac{f(a)-f(\xi)}{g(\xi)-g(b)}=\frac{f'(\xi)}{g'(\xi)}.\square$$

Ex. f在$[0,1]$上可导, $f(0)=0$, $f(x)>0(0<x<1)$, 则

$$\exists \xi \in (0,1), s.t.\ \frac{2f'(\xi)}{f(\xi)}=\frac{f'(1-\xi)}{f(1-\xi)}.$$

Proof. 令$h(x)=f^2(x)f(1-x)$. 由$f(0)=0$得$h(0)=h(1)=0$.

由Rolle定理, $\exists \xi \in (0,1), s.t.$

$$h'(\xi)=2f(\xi)f'(\xi)f(1-\xi)-f^2(\xi)f'(1-\xi)=0.$$

$$2f'(\xi)f(1-\xi)-f(\xi)f'(1-\xi)=0.$$

又 $f(x)>0(0<x<1)$, 故 $\frac{2f'(\xi)}{f(\xi)}=\frac{f'(1-\xi)}{f(1-\xi)}$.□

Ex. f在$[0,1]$上二阶可导, $f(x)>0(x\in[0,1]), f'(0)=f'(1)=0,$

则$\exists\xi\in(0,1), s.t. f(\xi)f''(\xi)-2\left(f'(\xi)\right)^2=0.$

Proof. 令$h(x)=f'(x)/f^2(x)$. 由$f'(0)=f'(1)=0$,得

$$h(0)=h(1)=0.$$

由Rolle定理,$\exists\xi\in(0,1), s.t.$

$$h'(\xi)=\frac{f''(\xi)f^2(\xi)-2f(\xi)\left(f'(\xi)\right)^2}{f^4(\xi)}=0.$$

$$f(\xi)f''(\xi)-2\left(f'(\xi)\right)^2=0.\square$$

作业：习题4.1

No. 2,5,11,12,14