1.5 线性映射的运算

回顾：$\mathcal{A}: \mathbb{R}^n \to \mathbb{R}^m$称为线性映射，如果$\mathcal{A}(\boldsymbol{v}+\boldsymbol{w}) = \mathcal{A}(\boldsymbol{v}) + \mathcal{A}(\boldsymbol{w}), \mathcal{A}(c\boldsymbol{v}) = c\mathcal{A}(\boldsymbol{v})$

$\boxed{\mathcal{A}(\boldsymbol{x}) = A\boldsymbol{x}}$

线性映射$\mathcal{A}$ ⟷ 矩阵$A = [\mathcal{A}(\boldsymbol{e}_1), \mathcal{A}(\boldsymbol{e}_2), \dots, \mathcal{A}(\boldsymbol{e}_n)] \in M_{m\times n}(\mathbb{R})$

线性映射的加法 ⟷ 矩阵的加法

线性映射的数乘 ⟷ 矩阵的数乘

线性映射的复合 ⟷ 矩阵的乘法 ⇐ 重点

主要内容：

1. 矩阵加法与数乘(√)
2. 矩阵的乘法

— 定义

— 举例

— 矩阵的转置及其性质

— 方阵的迹

— 一些特殊的矩阵：初等矩阵，对称矩阵等

— 矩阵乘法分块看法

回顾上次课:

定义线性映射的加法与数乘，通过线性映射与矩阵的对应关系得到矩阵的加法与数乘。

设$\mathcal{A},\mathcal{B}:\mathbb{R}^n\to\mathbb{R}^m$为线性映射，$c\in\mathbb{R}$,得到线性映射:

$$\mathcal{A}+\mathcal{B}:\ \mathbb{R}^n\to\mathbb{R}^m,\ \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x})+\mathcal{B}(\boldsymbol{x})\in\mathbb{R}^m.$$

$$c\mathcal{A}:\mathbb{R}^n\to\mathbb{R}^m,\boldsymbol{x}\mapsto(c\mathcal{A})(\boldsymbol{x})=\mathcal{A}(c\boldsymbol{x})=c\mathcal{A}(\boldsymbol{x})\in\mathbb{R}^m.$$

$$\boxed{\mathcal{A}(\boldsymbol{x})=A\boldsymbol{x}}$$

设$A=\left(a_{ij}\right)_{ij},B=\left(b_{ij}\right)_{ij}\in M_{m\times n}(\mathbb{R})$,定义 $A+B=(a_{ij}+b_{ij})_{1\le i\le m,1\le j\le n}$

设$A=\left(a_{ij}\right)_{ij}\in M_{m\times n}(\mathbb{R}),c\in\mathbb{R}$，定义 $cA=(ca_{ij})_{1\le i\le m,1\le j\le n}$

1. $\mathbb{R}^n\longrightarrow\mathbb{R}^m$所有线性映射构成的集合在加法和数乘运算下满足与$\mathbb{R}^n$中向量运算类似的8条性质

2. $m\times n$阶矩阵的集合在加法和数乘运算下满足与$\mathbb{R}^n$中向量运算类似的8条性质

线性映射的复合:

$\mathcal{A}: \mathbb{R}^n \to \mathbb{R}^m, \mathcal{B}: \mathbb{R}^p \to \mathbb{R}^n$ 为线性映射，定义

$\mathcal{B}$的陪域等于$\mathcal{A}$的定义域

$$\mathcal{A} \circ \mathcal{B}: \mathbb{R}^p \to \mathbb{R}^m, \boldsymbol{x} \mapsto \mathcal{A}\big(\mathcal{B}(\boldsymbol{x})\big).$$

$\mathcal{A} \circ \mathcal{B}$是线性映射.

问题：线性映射$\mathcal{A}, \mathcal{B}$的表出矩阵分别为A, B. $\mathcal{A} \circ \mathcal{B}$的表出矩阵是哪个矩阵?

$\mathcal{A} \circ \mathcal{B}$的表出矩阵为

$$C = \big[\mathcal{A} \circ \mathcal{B}(\boldsymbol{e}_1), \dots, \mathcal{A} \circ \mathcal{B}(\boldsymbol{e}_p)\big].$$

$$\mathcal{A} \circ \mathcal{B}(\boldsymbol{e}_j) = \mathcal{A}\left(\mathcal{B}(\boldsymbol{e}_j)\right) = \mathcal{A}(\boldsymbol{b}_j) = A\boldsymbol{b}_j.$$

$\boxed{\mathcal{A}(\boldsymbol{x}) = A\boldsymbol{x}}$

因此，$\mathcal{A} \circ \mathcal{B}$的表出矩阵为$C = \big[A\boldsymbol{b}_1, \dots, A\boldsymbol{b}_p\big]$.

矩阵乘法的定义:

B的行数等于A的列数 ⟺ $\mathcal{B}$的陪域等于$\mathcal{A}$的定义域

$$A=\left(a_{ij}\right)_{i,j}=\begin{bmatrix}\tilde{\boldsymbol{a}}_1^T\\ \vdots\\ \tilde{\boldsymbol{a}}_m^T\end{bmatrix}\in M_{m\times n}(\mathbb{R}),B=\left(b_{jk}\right)_{jk}=\left[\boldsymbol{b}_1,\dots,\boldsymbol{b}_p\right]\in M_{n\times p}(\mathbb{R}),\text{定义}$$

$$\boxed{AB=\left[A\boldsymbol{b}_1,\dots,A\boldsymbol{b}_p\right]\in M_{m\times p}(\mathbb{R})}$$

矩阵的乘法的性质：

矩阵加法与乘法满足如下性质：

设矩阵A, B, C使得下列等式中的运算可进行，则

(1)$A(B+C)=AB+AC$.

(2)$(A+B)C=AC+BC$.

(3)$(AB)C=A(BC)$.

证明：（1）$A(B+C)=[A(\boldsymbol{b}_1+\boldsymbol{c}_1),\dots,A(\boldsymbol{b}_n+\boldsymbol{c}_n)]$

$=[A\boldsymbol{b}_1+A\boldsymbol{c}_1,\dots,A\boldsymbol{b}_n+A\boldsymbol{c}_n]$

$=[A\boldsymbol{b}_1,\dots,A\boldsymbol{b}_n]+[A\boldsymbol{c}_1,\dots,A\boldsymbol{c}_n]$

$=AB+AC$.

（2）与（1）类次

(3) $(AB)C = A(BC)$.

证明：考查线性映射$\mathcal{A},\mathcal{B},\mathcal{C}$,

$$\mathbb{R}^p \xrightarrow{\mathcal{C}} \mathbb{R}^n \xrightarrow{\mathcal{B}} \mathbb{R}^m \xrightarrow{\mathcal{A}} \mathbb{R}^k$$

它们的表出矩阵分别为A,B,C.

$$(\mathcal{A}\circ\mathcal{B})\circ\mathcal{C} = \mathcal{A}\circ(\mathcal{B}\circ\mathcal{C})$$

表出矩阵为（$(AB)C$　　表出矩阵为$A(BC)$

因此，$(AB)C$与$A(BC)$为同一个线性映射的矩阵表出.

于是$(AB)C = A(BC)$.

矩阵的乘积

$$A=\left(a_{ij}\right)_{i,j}=\begin{bmatrix}\widetilde{\boldsymbol{a}}_1^T\\ \vdots\\ \widetilde{\boldsymbol{a}}_m^T\end{bmatrix}\in M_{m\times n}(\mathbb{R}), B=\left(b_{jk}\right)_{jk}=\left[\boldsymbol{b}_1,\dots,\boldsymbol{b}_p\right]\in M_{n\times p}(\mathbb{R}),$$

$$\boxed{AB=\left[A\boldsymbol{b}_1,\dots,A\boldsymbol{b}_p\right]\in M_{m\times p}(\mathbb{R}).}$$

AB的第i行第j列的元素:

$$(AB)_{ij}=\widetilde{\boldsymbol{a}}_i^T\boldsymbol{b}_j$$

$$=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}$$

$$=\sum_{k=1}^n a_{ik}b_{kj}.$$

矩阵的乘法

Hard to remember?

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & b_{1j} & b_{1p} \\ b_{21} & b_{22} & b_{2j} & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{nj} & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{1j} & a_{1p} \\ c_{21} & c_{22} & c_{2j} & c_{2p} \\ c_{i1} & c_{i2} & c_{ij} & c_{ip} \\ c_{m1} & c_{m2} & c_{mj} & c_{mp} \end{bmatrix}$$

$$A \bullet B \qquad\qquad C$$

$$m \times n \qquad n \times p \qquad\qquad m \times p$$

$$\boxed{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = c_{ij}}$$

矩阵的乘法

$$\begin{bmatrix} b_{11} & b_{12} & b_{1j} & b_{1p} \\ b_{21} & b_{22} & b_{2j} & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{nj} & b_{np} \end{bmatrix} \quad B$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{1j} & a_{1p} \\ c_{21} & c_{22} & c_{2j} & c_{2p} \\ c_{i1} & c_{i2} & c_{ij} & c_{ip} \\ c_{m1} & c_{m2} & c_{mj} & c_{mp} \end{bmatrix} \quad C = AB$$

$$A$$

$$\boxed{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = c_{ij}}$$

例: $(m\times 1)\cdot(1\times n)$

$$\begin{bmatrix}1\\0\\2\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2 & 3 & 4\\0 & 0 & 0 & 0\\2 & 4 & 6 & 8\end{bmatrix}$$

秩1矩阵

例: $(1 \times n) \cdot (n \times 1)$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$= 10$

例:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

因此, $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. 对角矩阵

例:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

在矩阵乘法中，$AB = BA$一般不成立！

A, B为同阶方程，若$AB = BA$,则称A, B**可交换**

例:

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

非零矩阵的乘积可能为零

例:

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A \qquad B \qquad\qquad\qquad A \qquad C$$

A非零矩阵，$AB = AC$一般不能得到$B = C$！

例: 左乘对角方阵

$A \in M_{n\times p}(\mathbb{R})$

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & \cdots & d_1a_{1p} \\ d_2a_{21} & d_2a_{22} & \cdots & d_2a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ d_na_{n1} & d_na_{n2} & \cdots & d_na_{np} \end{bmatrix}$$

例: 右乘对角方阵

$A \in M_{n\times p}(\mathbb{R})$

$$\begin{array}{c} \begin{array}{cccc} d_1 & d_2 & \quad & d_p \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{bmatrix} \end{array} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_p \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_2 a_{12} & \cdots & d_p a_{1p} \\ d_1 a_{21} & d_2 a_{22} & \cdots & d_p a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ d_1 a_{n1} & d_2 a_{n2} & \cdots & d_p a_{np} \end{bmatrix}$$

例: 矩阵的数乘可写为矩阵乘法

$$A \in M_{m\times n}(\mathbb{R})$$

$$cA = \begin{bmatrix} c & & \\ & \ddots & \\ & & c \end{bmatrix}_{m\times m} A = A \begin{bmatrix} c & & \\ & \ddots & \\ & & c \end{bmatrix}_{n\times n}.$$

例:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

$$\boldsymbol{e}_j = \begin{array}{c c} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{c} \\ \\ \\ j \\ \\ \\ \\ \end{array} \end{array} \in \mathbb{R}^n$$

$$A\boldsymbol{e}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^m.$$

取A的第j列

例:

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

$$\boldsymbol{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ i \\ \\ \\ \\ \end{matrix} \in \mathbb{R}^m$$

$$\boldsymbol{e}_i^T A = \quad [a_{i1}, a_{i2}, \dots, a_{in}]$$

取A的第i行

例:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

$$\boldsymbol{e}_i \in \mathbb{R}^m \qquad \boldsymbol{e}_j \in \mathbb{R}^n$$

$$\boldsymbol{e}_i^T A \boldsymbol{e}_j = \quad a_{ij}$$

取A的第i,j位置元素

方阵的幂：

设A为n阶方阵，$k \geq 1$, 记

$$A^k = \underbrace{A \cdot A \cdots\cdots A}_{k\text{个}} \Leftarrow \underbrace{\mathcal{A} \circ \mathcal{A} \circ \cdots \circ \mathcal{A}}_{k\text{个}}$$

记 $A^0 = I_n$.

A, B为同阶方阵$(A+B)^2 = ?$

$(A+B)^2 = A^2 + AB + BA + B^2$

$(A+B)^2 = A^2 + 2AB + B^2$当且仅当$A, B$可交换

例:

$$\begin{bmatrix}1 & 2 & 1\\0 & 0 & 0\\2 & 4 & 2\end{bmatrix} = \begin{bmatrix}1\\0\\2\end{bmatrix}\begin{bmatrix}1 & 2 & 1\end{bmatrix}$$

秩1方阵

$$\begin{bmatrix}1 & 2 & 1\\0 & 0 & 0\\2 & 4 & 2\end{bmatrix}^{2022} = 3^{2021}\begin{bmatrix}1 & 2 & 1\\0 & 0 & 0\\2 & 4 & 2\end{bmatrix}$$

秩为1的方阵容易计算方幂

例：$\begin{bmatrix}5 & -4\\ 2 & -1\end{bmatrix}^3 = ?$

对角矩阵

$$\begin{bmatrix}5 & -4\\ 2 & -1\end{bmatrix} = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}3 & \\ & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}$$

$$\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix} = \begin{bmatrix}1 & \\ & 1\end{bmatrix}$$

$\begin{bmatrix}5 & -4\\ 2 & -1\end{bmatrix}$ 称为**可对角化矩阵**

$$\begin{bmatrix}5 & -4\\ 2 & -1\end{bmatrix}^3 = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}3 & \\ & 1\end{bmatrix}^3\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}$$

小结：

1. 矩阵乘法与数的乘法的不同

在矩阵乘法中$AB=BA$一般不成立

非零矩阵的乘积可能为零

矩阵的乘法没有消去率

2. 秩1方阵和可对角化方阵容易计算方幂

矩阵的转置（transpose）：

$A=\left(A_{ij}\right)_{ij}\in M_{m\times n}(\mathbb{R})$,定义$A$ 的转置$A^T\in M_{n\times m}(\mathbb{R})$为$(A^T)_{ij}=A_{ji}$.

例:

$$A=\begin{bmatrix}1\\2\\3\end{bmatrix}\qquad A^T=[1,2,3]\qquad (A^T)^T=A$$

$$A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\qquad A^T=\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}\qquad (A^T)^T=A$$

例：

(1) $I_n^T = I_n$. D对角矩阵, $D^T = D$.

(2) 上三角矩阵的转置是下三角矩阵，下三角矩阵的转置是上三角矩阵.

(3) $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, 点积$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^T \boldsymbol{w} = \boldsymbol{w}^T \boldsymbol{v} \in \mathbb{R}$.

问题：矩阵的转置 $\overset{?}{\longleftrightarrow}$ 线性映射

矩阵的转置的性质：

(1)$(A^T)^T = A$.

(2)$(A+B)^T = A^T + B^T$

(3)$c \in \mathbb{R}, (cA)^T = cA^T$.

(4)$(AB)^T = B^T A^T$

证明：(1)$\left((A^T)^T\right)_{ij} = (A^T)_{ji} = A_{ij}$.

(2)$(A+B)^T_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = A^T_{ij} + B^T_{ij}$

(3)$(cA)^T_{ij} = (cA)_{ji} = cA_{ji} = (cA^T)_{ij}$.

(4)$(AB)^T = B^T A^T$.

证法一：直接验证：

$$(AB)_{ij}^T = (AB)_{ji} = \sum_k A_{jk} B_{ki}$$

$$= \sum_k B_{ki} A_{jk} = \sum_k (B^T)_{ik} (A^T)_{kj}$$

$$= (B^T A^T)_{ij}.$$

证法二（用到矩阵的分块乘法）：

如果$B = \boldsymbol{x} = [x_1, \dots, x_n]^T$ 为一列向量, $A = [\boldsymbol{a}_1, \dots, \boldsymbol{a}_n]$

$$(A\boldsymbol{x})^T = (\boldsymbol{a}_1 x_1 + \boldsymbol{a}_2 x_2 + \cdots + \boldsymbol{a}_n x_n)^T = (\boldsymbol{a}_1 x_1)^T + (\boldsymbol{a}_2 x_2)^T + \cdots + (\boldsymbol{a}_n x_n)^T$$

$$= x_1 \boldsymbol{a}_1^T + x_2 \boldsymbol{a}_2^T + \cdots + x_n \boldsymbol{a}_n^T = [x_1, x_2, \dots, x_n] \begin{bmatrix} \boldsymbol{a}_1^T \\ \vdots \\ \boldsymbol{a}_n^T \end{bmatrix}$$

$$= \boldsymbol{x}^T A^T.$$

一般的，$B=[\boldsymbol{b}_1,\ldots,\boldsymbol{b}_p]$

$$(AB)^T=[A\boldsymbol{b}_1,\ldots,A\boldsymbol{b}_p]^T=\begin{bmatrix}(A\boldsymbol{b}_1)^T\\ \vdots\\ (A\boldsymbol{b}_p)^T\end{bmatrix}$$

$$=\begin{bmatrix}\boldsymbol{b}_1^TA^T\\ \vdots\\ \boldsymbol{b}_p^TA^T\end{bmatrix}=\begin{bmatrix}\boldsymbol{b}_1^T\\ \vdots\\ \boldsymbol{b}_p^T\end{bmatrix}A^T$$

$$=B^TA^T.$$

方阵的迹（trace）：

$A=\left(a_{ij}\right)_{ij}\in M_{n\times n}(\mathbb{R})$,定义$A$ 的迹$trace(A)=\sum_i a_{ii}$

一些性质：

A,B同阶方阵$trace(A+B)=trace(A)+trace(B)$

$trace(A^T)=trace(A)$

$trace(AB)=trace(BA)$

更多的性质见本周作业

对称矩阵与(反)斜对称矩阵：

(i) 矩阵S满足$S^T = S$，则称为对称矩阵（symmetric matrix).

(ii) 矩阵A满足$A^T = -A$，则称为反对称或斜对称矩阵(anti-symmetric or skew symmetric).

$S = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$为对称矩阵

$A = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$为反对称矩阵

对角线元素=0

注意：对称矩阵与反对称矩阵均为方阵

对称实矩阵在研究一般实矩阵时有重要作用:

$A^T A$， $n \times n$ 阶方阵

$A: m \times n$阶实矩阵

AA^T， $m \times m$ 阶方阵

$A^T A$与 AA^T 为对称实矩阵, 且**对角线元素**≥ 0.

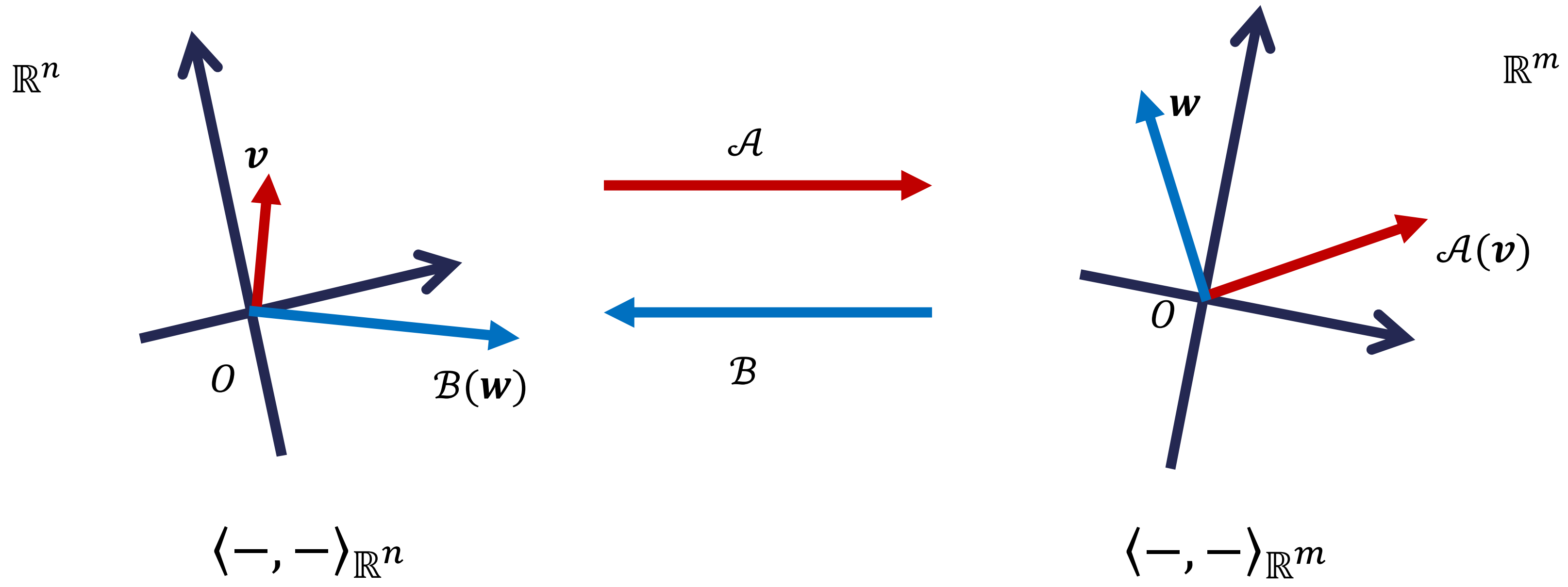
$$(AA^T)_{ii} = \sum_{k=1}^{n} A_{ik} (A^T)_{ki} = \sum_k A_{ik} A_{ik} \geq 0.$$

$trace(AA^T) = 0 \Rightarrow A = O$

$A^T A$与 AA^T 将在研究A的性质时扮演重要作用(见第六章).

矩阵转置与线性映射:

令$A \in M_{m\times n}(\mathbb{R})$ 对应线性映射$\mathcal{A}: \mathbb{R}^n \to \mathbb{R}^m$



定义$\mathcal{B}: \mathbb{R}^m \to \mathbb{R}^n, \boldsymbol{w} \mapsto A^T\boldsymbol{w}$

$\mathcal{B}$满足$\boxed{\langle \mathcal{B}(\boldsymbol{w}), \boldsymbol{v}\rangle_{\mathbb{R}^n} = \langle \boldsymbol{w}, \mathcal{A}(\boldsymbol{v})\rangle_{\mathbb{R}^m}}, \forall \boldsymbol{w} \in \mathbb{R}^m$， $\boldsymbol{v} \in \mathbb{R}^n$

矩阵转置与线性映射:

定义$\mathcal{B}: \mathbb{R}^m \to \mathbb{R}^n, \boldsymbol{w} \mapsto A^T \boldsymbol{w}$

$\mathcal{B}$满足$\langle \mathcal{B}(\mathbf{w}), \boldsymbol{v} \rangle_{\mathbb{R}^n} = \langle \boldsymbol{w}, \mathcal{A}(\boldsymbol{v}) \rangle_{\mathbb{R}^m}, \forall \boldsymbol{w} \in \mathbb{R}^m$， $\boldsymbol{v} \in \mathbb{R}^n$

$\langle \mathcal{B}(\boldsymbol{w}), \boldsymbol{v} \rangle = \langle A^T \boldsymbol{w}, \boldsymbol{v} \rangle = \boldsymbol{v}^T (A^T \boldsymbol{w})$

$\langle \boldsymbol{w}, \mathcal{A}(\boldsymbol{v}) \rangle = (A\boldsymbol{v})^T \boldsymbol{w} = (\boldsymbol{v}^T A^T) \boldsymbol{w}.$

矩阵乘法的分块看法:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & b_{1j} & b_{1p} \\ b_{21} & b_{22} & b_{2j} & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{nj} & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{1j} & a_{1p} \\ c_{21} & c_{22} & c_{2j} & c_{2p} \\ c_{i1} & c_{i2} & c_{ij} & c_{ip} \\ c_{m1} & c_{m2} & c_{mj} & c_{mp} \end{bmatrix}$$

视为 1×1

视为 $1 \times p$

视为 $1 \times p$

$$A \qquad [\boldsymbol{b}_1 \ \ \boldsymbol{b}_2 \qquad \boldsymbol{b}_p] = [\boldsymbol{c}_1 \ \ \boldsymbol{c}_2 \qquad \boldsymbol{c}_p]$$

$$\boxed{A\boldsymbol{b}_i = \boldsymbol{c}_i}$$

矩阵乘法的分块：

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & b_{1j} & b_{1p} \\ b_{21} & b_{22} & b_{2j} & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{nj} & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{1j} & a_{1p} \\ c_{21} & c_{22} & c_{2j} & c_{2p} \\ c_{i1} & c_{i2} & c_{ij} & c_{ip} \\ c_{m1} & c_{m2} & c_{mj} & c_{mp} \end{bmatrix}$$

视为 $m \times 1$ 视为 1×1 视为 $m \times 1$

$$\begin{bmatrix} \tilde{\boldsymbol{a}}_1^T \\ \tilde{\boldsymbol{a}}_2^T \\ \vdots \\ \tilde{\boldsymbol{a}}_m^T \end{bmatrix} \quad B \quad = \begin{bmatrix} \tilde{\boldsymbol{c}}_1^T \\ \tilde{\boldsymbol{c}}_2^T \\ \vdots \\ \tilde{\boldsymbol{c}}_m^T \end{bmatrix}$$

$$\boxed{\tilde{\boldsymbol{a}}_i^T B = \tilde{\boldsymbol{c}}_i^T}$$

应用：初等矩阵

定义：对单位矩阵I_n做一次<u>初等行变换</u>得到的矩阵称为**初等矩阵**

1. 对换矩阵：互换I_n的i,j行，得到对换矩阵P_{ij}

$$P_{ij}=\begin{array}{cc}
\begin{bmatrix}
\ddots & & & & & & & & \\
 & 1 & & & & & & & \\
 & & 0 & & & & 1 & & \\
 & & & 1 & & & & & \\
 & & & & \ddots & & & & \\
 & & & & & 1 & & & \\
 & & 1 & & & & 0 & & \\
 & & & & & & & 1 & \\
 & & & & & & & & \ddots
\end{bmatrix}
&
\begin{array}{c}
\\ \\ i \\ \\ \\ \\ j \\ \\ \\
\end{array}
\\
\begin{array}{ccccccccc}
 & & i & & & & j & &
\end{array}
&
\end{array}$$

例: 左乘对换矩阵

$$A=[\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n]=\begin{bmatrix}\widetilde{\boldsymbol{a}}_1^T\\ \vdots\\ \widetilde{\boldsymbol{a}}_m^T\end{bmatrix}$$

$$P_{ij}A=\left[P_{ij}\boldsymbol{a}_1,P_{ij}\boldsymbol{a}_2,\dots,P_{ij}\boldsymbol{a}_n\right]=\begin{bmatrix}\vdots\\ \widetilde{\boldsymbol{a}}_j^T\\ \vdots\\ \widetilde{\boldsymbol{a}}_i^T\\ \vdots\end{bmatrix}$$

例: 右乘对换矩阵

$$A = \begin{array}{ccccc} [\dots, & \boldsymbol{a}_i, & \dots, & \boldsymbol{a}_j, & \dots] \\ & i & & j & \end{array} = \begin{bmatrix} \tilde{\boldsymbol{a}}_1^T \\ \vdots \\ \tilde{\boldsymbol{a}}_m^T \end{bmatrix} \qquad P_{ij} \in M_{n\times n}(\mathbb{R})$$

$$AP_{ij} = \begin{bmatrix} \tilde{\boldsymbol{a}}_1^T \\ \vdots \\ \tilde{\boldsymbol{a}}_m^T \end{bmatrix} P_{ij} = \begin{bmatrix} \tilde{\boldsymbol{a}}_1^T P_{ij} \\ \vdots \\ \tilde{\boldsymbol{a}}_m^T P_{ij} \end{bmatrix} = \begin{array}{ccccc} [\dots, & \boldsymbol{a}_j, & \dots, & \boldsymbol{a}_i, & \dots] \\ & i & & j & \end{array}$$

互换A的i，j列

例: 初等矩阵

2.倍乘矩阵：I_n的第i行乘以非零常数k，得到倍乘矩阵$E_{ii}(k)$

$$E_{ii}(k)=\begin{array}{cc} \begin{bmatrix} \ddots &&&& \\ & 1 &&& \\ && k && \\ &&& 1 & \\ &&&& \ddots \end{bmatrix} & \begin{matrix} \\ \\ i \\ \\ \\ \end{matrix} \\ \begin{matrix} \quad & \quad & i & \quad & \quad \end{matrix} & \end{array}$$

例: 左乘倍乘矩阵

$$A = [\ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_j, \ldots] = \begin{bmatrix} \widetilde{\boldsymbol{a}}_1^T \\ \vdots \\ \widetilde{\boldsymbol{a}}_m^T \end{bmatrix} \qquad E_{ii}(k) \in M_{m\times m}(\mathbb{R})$$

$$E_{ii}(k)A = [E_{ii}(k)\boldsymbol{a}_1, E_{ii}(k)\boldsymbol{a}_2, \ldots, E_{ii}(k)\boldsymbol{a}_n] = \begin{array}{cc} \begin{bmatrix} \vdots \\ k\widetilde{\boldsymbol{a}}_i^T \\ \vdots \end{bmatrix} & \begin{array}{c} \\ i \\ \\ \end{array} \end{array}$$

第i行乘以k倍

例: 右乘倍乘矩阵

$$A = [\dots, \boldsymbol{a}_i, \dots, \boldsymbol{a}_j, \dots] = \begin{bmatrix} \widetilde{\boldsymbol{a}}_1^T \\ \vdots \\ \widetilde{\boldsymbol{a}}_m^T \end{bmatrix} \qquad E_{ii}(k) \in M_{n\times n}(\mathbb{R})$$

$$AE_{ii}(k) = \begin{bmatrix} \widetilde{\boldsymbol{a}}_1^T \\ \vdots \\ \widetilde{\boldsymbol{a}}_m^T \end{bmatrix} E_{ii}(k) = \begin{bmatrix} \widetilde{\boldsymbol{a}}_1^T E_{ii}(k) \\ \vdots \\ \widetilde{\boldsymbol{a}}_m^T E_{ii}(k) \end{bmatrix} = \underset{i}{[\dots, k\boldsymbol{a}_i, \dots]}$$

第i列乘以k倍

例: 初等矩阵

3.倍加矩阵：把I_n的第i行的k倍加到第j行上，得到倍加矩阵$E_{ji}(k)$

$$E_{ji}(k)=\begin{array}{cc} \begin{bmatrix} \ddots & & & & \\ & 1 & & & \\ & & \ddots & & \\ & k & & 1 & \\ & & & & \ddots \end{bmatrix} (j>i) & \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \\ \begin{array}{ccccc} \quad & i & \quad & j & \quad \end{array} & \end{array}$$

$$E_{ji}(k)=\begin{array}{cc} \begin{bmatrix} \ddots & & & & \\ & 1 & & k & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix} (j<i) & \begin{array}{c} \\ j \\ \\ i \\ \\ \end{array} \\ \begin{array}{ccccc} \quad & j & \quad & i & \quad \end{array} & \end{array}$$

例: 左乘倍加矩阵

$$A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n] = \begin{bmatrix} \widetilde{\boldsymbol{a}}_1^T \\ \vdots \\ \widetilde{\boldsymbol{a}}_m^T \end{bmatrix} \qquad E_{ji}(k) \in M_{m\times m}(\mathbb{R}) \;\; j > i$$

$$E_{ji}(k)A = \left[E_{ji}(k)\boldsymbol{a}_1, E_{ji}(k)\boldsymbol{a}_2, \dots, E_{ji}(k)\boldsymbol{a}_n\right] = \begin{array}{cc} \begin{bmatrix} \vdots \\ \widetilde{\boldsymbol{a}}_i^T \\ \vdots \\ k\widetilde{\boldsymbol{a}}_i^T + \widetilde{\boldsymbol{a}}_j^T \\ \vdots \end{bmatrix} & \begin{matrix} \\ i \\ \\ j \\ \\ \end{matrix} \end{array}$$

第j行$\widetilde{\boldsymbol{a}}_j^T$变为$\widetilde{\boldsymbol{a}}_j^T + k\widetilde{\boldsymbol{a}}_i^T$

例: 初等矩阵(elementary matrix)

$$A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n] = \begin{bmatrix} \tilde{\boldsymbol{a}}_1^T \\ \vdots \\ \tilde{\boldsymbol{a}}_m^T \end{bmatrix} \qquad E_{ji}(k) \in M_{m\times m}(\mathbb{R}) \ \ j > i$$

$$AE_{ji}(k) = \left[\cdots, \underset{i}{\boldsymbol{a}_i + k\boldsymbol{a}_j}, \dots, \underset{j}{\boldsymbol{a}_j}, \dots\right]$$

第i列$\boldsymbol{a}_i$变为$\boldsymbol{a}_i + k\boldsymbol{a}_j$

例：初等行变换与初等列变换

$$E_{21}(-3) = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} \tilde{\boldsymbol{a}}_1^T \\ \tilde{\boldsymbol{a}}_2^T \\ \tilde{\boldsymbol{a}}_3^T \end{bmatrix}$$

$$E_{21}(-3)A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{a}}_1^T \\ \tilde{\boldsymbol{a}}_2^T \\ \tilde{\boldsymbol{a}}_3^T \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{a}}_1^T \\ -3\tilde{\boldsymbol{a}}_1^T + \tilde{\boldsymbol{a}}_2^T \\ \tilde{\boldsymbol{a}}_3^T \end{bmatrix}$$

例：初等行变换与初等列变换

$$E_{21}(-3)=\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}, A=[\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3]$$

$$AE_{21}(-3)=[\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3]\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}=[\boldsymbol{a}_1-3\boldsymbol{a}_2,\boldsymbol{a}_2,\boldsymbol{a}_3]$$

小结:

1. 分块: $A=\begin{bmatrix}\tilde{\boldsymbol{a}}_1^T\\ \tilde{\boldsymbol{a}}_2^T\\ \vdots\\ \tilde{\boldsymbol{a}}_m^T\end{bmatrix}\in M_{m\times n}(\mathbb{R}), B=\left[\boldsymbol{b}_1,\dots,\boldsymbol{b}_p\right]\in M_{n\times p}(\mathbb{R})$

$$
\begin{aligned}
AB &= A\left[\boldsymbol{b}_1,\boldsymbol{b}_2,\dots,\boldsymbol{b}_p\right]=\left[A\boldsymbol{b}_1,\dots,A\boldsymbol{b}_p\right]\\
&=\begin{bmatrix}\tilde{\boldsymbol{a}}_1^T\\ \tilde{\boldsymbol{a}}_2^T\\ \vdots\\ \tilde{\boldsymbol{a}}_m^T\end{bmatrix}B=\begin{bmatrix}\tilde{\boldsymbol{a}}_1^T B\\ \tilde{\boldsymbol{a}}_2^T B\\ \vdots\\ \tilde{\boldsymbol{a}}_m^T B\end{bmatrix}\in M_{m\times p}(\mathbb{R})
\end{aligned}
$$

2. 左乘初等矩阵: 对矩阵进行初等**行**变换

右乘初等矩阵: 对矩阵进行初等**列**变换

本节小结：

1. 矩阵的运算：加法、数乘、乘法和转置
2. 通过线性映射的加法与数乘定义矩阵的加法与数乘
3. $M_{m\times n}(\mathbb{R})$在加法与数乘下满足向量空间的8条性质
4. 线性映射的复合给出矩阵的乘法
5. 几种运算之间的关系
6. 特殊的矩阵：初等矩阵，对称矩阵等