



Review

定积分的计算

- $\int_a^b F'(x)dx = F(x)\Big|_{x=a}^b$

- $\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$

$f \in C[a, b], \varphi(\alpha) = a, \varphi(\beta) = b, a \leq \varphi(t) \leq b, \varphi \in C^1[\alpha, \beta].$

- $\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx.$

带积分余项的Taylor公式

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{n!} \int_{x_0}^x (x - t)^n f^{(n+1)}(t) dt.$$



§ 7. 积分的应用 --微元法

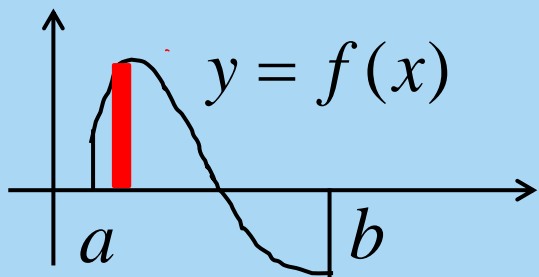
- 平面区域的面积
- 曲线的弧长
- 平面曲线的曲率
- 旋转体的体积
- 旋转面的面积
- 积分在物理中的应用



● 平面图形的面积

1) $y = f(x)$, $y = 0$, $x \in [a, b]$. Riemann积分四部曲:

分割、取点、近似和、极限



$$\lim_{|T| \rightarrow 0} \sum_{i=1}^n |f(\xi_i)| \Delta x_i = S = \int_a^b |f(x)| dx.$$

微元法: $[x, x + \Delta x]$ 对应窄条的面积 $\Delta S \approx |f(x)| \Delta x$,

$$f \in C[a, b] \Rightarrow |\Delta S - |f(x)| \Delta x| \leq \left(\max_{x \leq t \leq x + \Delta x} f(t) - \min_{x \leq t \leq x + \Delta x} f(t) \right) \Delta x$$

$$\Delta S = |f(x)| \Delta x + o(\Delta x), \Delta x \rightarrow 0.$$

$$dS = |f(x)| dx, \quad S = \int_a^b |f(x)| dx.$$



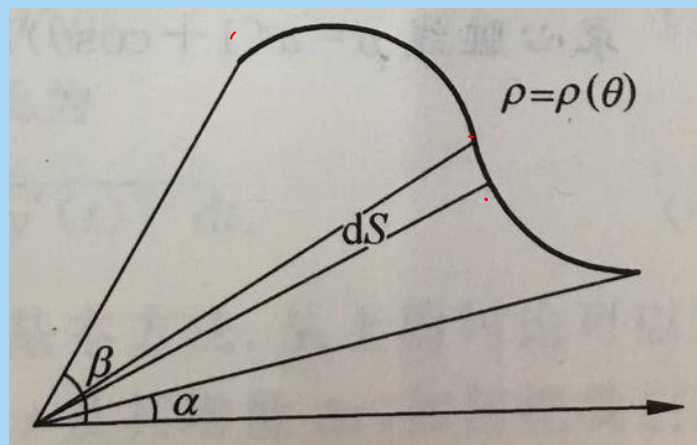
$$2) y = f_1(x), y = f_2(x), x \in [a, b].$$

$$S = \int_a^b |f_1(x) - f_2(x)| dx$$

$$3) \rho = \rho(\theta), \theta \in [\alpha, \beta].$$

(注意 ρ 和 θ 的几何意义!)

$$\text{微元法: } \Delta S \approx \frac{1}{2} \rho^2(\theta) \Delta \theta$$



$$(\rho(\theta) \in C[\alpha, \beta] \Rightarrow) \Delta S = \frac{1}{2} \rho^2(\theta) \Delta \theta + o(\Delta \theta), \Delta \theta \rightarrow 0.$$

$$dS = \frac{1}{2} \rho^2(\theta) d\theta, \quad S = \int_{\alpha}^{\beta} \frac{1}{2} \rho^2(\theta) d\theta.$$



Ex. 求 $x^{2/3} + y^{2/3} = a^{2/3} (a > 0)$ 所围区域的面积.

解法一: 曲线关于 x 轴和 y 轴对称, 因此 $S = 4 \int_0^a y(x) dx$.

第一象限中曲线有参数方程

$$x = a \sin^3 t, y = a \cos^3 t, t \in [0, \frac{\pi}{2}].$$

故

$$\begin{aligned} S &= 4 \int_0^{\pi/2} a \cos^3 t \cdot 3a \sin^2 t \cos t \, dt \\ &= 3a^2 \int_0^{\pi/2} \sin^2 2t \cos^2 t \, dt \\ &= \frac{3}{4} a^2 \int_0^{\pi/2} (1 - \cos 4t)(1 + \cos 2t) \, dt = \frac{3}{8} \pi a^2. \end{aligned}$$



解法二：曲线有参数方程 $x = a \sin^3 \theta$, $y = a \cos^3 \theta$, $\theta \in [0, 2\pi]$.

$$\begin{aligned}\rho^2(\theta) &= x^2(\theta) + y^2(\theta) = a^2(\sin^6 \theta + \cos^6 \theta) \\ &= a^2(\sin^4 \theta - \sin^2 \theta \cos^2 \theta + \cos^4 \theta) \\ &= a^2\left(1 - \frac{3}{4}\sin^2 2\theta\right)\end{aligned}$$

$$S \neq \frac{1}{2} \int_0^{2\pi} \rho^2(\theta) d\theta = 2a^2 \int_0^{\pi/2} \left(1 - \frac{3}{4}\sin^2 2\theta\right) d\theta$$

$$= \pi a^2 - \frac{3}{4} a^2 \int_0^{\pi/2} (1 - \cos 4\theta) d\theta = \frac{5}{8} \pi a^2. \quad \text{X}$$

参数方程中的 θ 并非辐角！



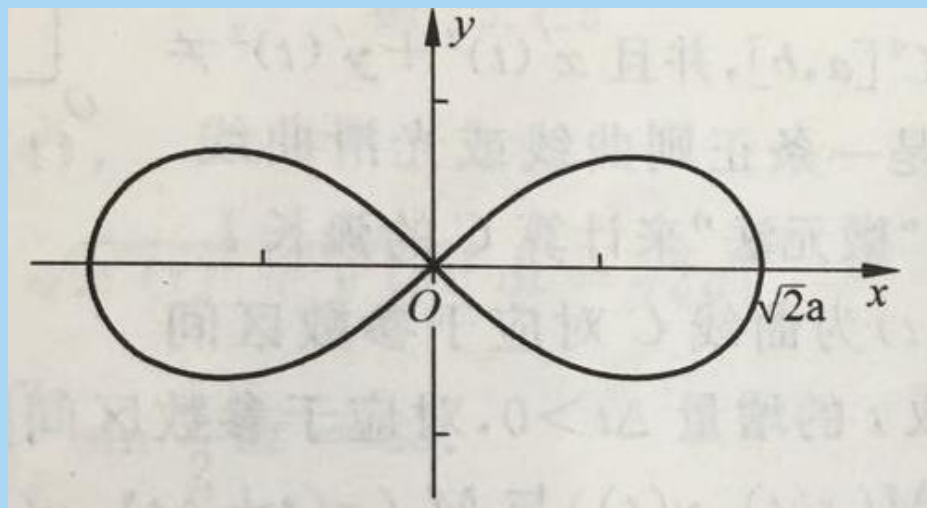
Ex. 求双纽线 $\rho^2 = 2a^2 \cos 2\theta$ 所围区域的面积 S .

解: $\rho(-\theta) = \rho(\theta)$, $\rho(\pi - \theta) = \rho(\theta)$, 故图像关于 x, y 轴对称.

$$2a^2 \cos 2\theta = \rho^2 \geq 0$$

$$\Rightarrow \cos 2\theta \geq 0$$

$$\Rightarrow \text{第一象限中 } \theta \in [0, \frac{\pi}{4}]$$



$$\begin{aligned} S &= 4S_1 = 4 \int_0^{\pi/4} \frac{1}{2} \rho^2(\theta) d\theta = 4 \int_0^{\pi/4} a^2 \cos 2\theta d\theta \\ &= 2a^2 \sin 2\theta \Big|_0^{\pi/4} = 2a^2. \square \end{aligned}$$



Ex. 求心脏线 $\rho = a(1 + \cos \theta)$ 所围区域的面积 S .

解: $a(1 + \cos \theta) = \rho \geq 0$

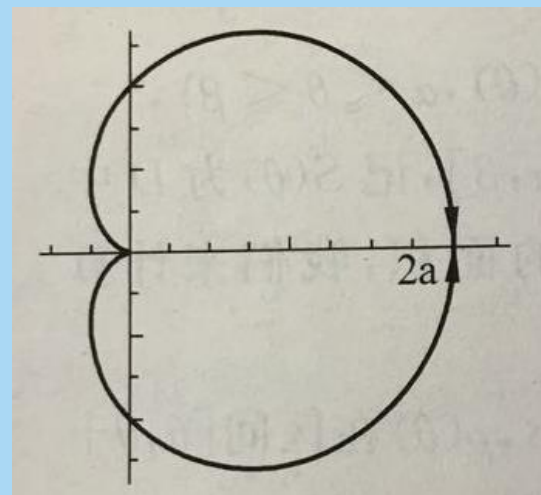
$$\Rightarrow \theta \in [-\pi, \pi].$$

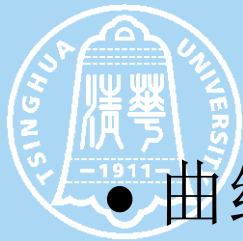
$$S = \int_{-\pi}^{\pi} \frac{1}{2} \rho^2(\theta) d\theta$$

$$= \int_0^{\pi} a^2 (1 + \cos \theta)^2 d\theta$$

$$= \int_0^{\pi} a^2 \left(1 + 2\cos \theta + \frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= \frac{3}{2} \pi a^2. \square$$





● 曲线的弧长 $L: x = x(t), y = y(t), z = z(t), t \in [\alpha, \beta],$
 $x(t), y(t), z(t) \in C^1[\alpha, \beta].$

考虑 $[t, t + \Delta t]$ 对应的弧段

$$\begin{aligned}\Delta l &\approx \sqrt{(x(t + \Delta t) - x(t))^2 + (y(t + \Delta t) - y(t))^2 + (z(t + \Delta t) - z(t))^2} \\ &= \sqrt{(x'(\xi))^2 + (y'(\eta))^2 + (z'(\zeta))^2} \Delta t \\ &\approx \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \Delta t.\end{aligned}$$

弧长微元 $dl = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt.$

$$l = \int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt.$$



Remark. • 平面曲线 $L: x = x(t), y = y(t), \alpha \leq t \leq \beta$, 的弧长

$$l = \int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

• 曲线 $L: y = f(x), a \leq x \leq b$, 的弧长 $l = \int_a^b \sqrt{1 + (f'(x))^2} dx$.

• $\rho = \rho(\theta), \alpha \leq \theta \leq \beta$, 的弧长 (注意 ρ 和 θ 的几何意义!)

$$x = \rho(\theta) \cos \theta, y = \rho(\theta) \sin \theta,$$

$$\sqrt{(x'(\theta))^2 + (y'(\theta))^2} = \sqrt{(\rho'(\theta))^2 + (\rho(\theta))^2}$$

$$l = \int_{\alpha}^{\beta} \sqrt{(\rho'(\theta))^2 + (\rho(\theta))^2} d\theta.$$



Ex. 求心脏线 $\rho = a(1 + \cos \theta)$, $-\pi \leq \theta \leq \pi$, 的弧长 L .

解: $x = \rho(\theta) \cos \theta = a(1 + \cos \theta) \cos \theta,$

$$y = \rho(\theta) \sin \theta = a(1 + \cos \theta) \sin \theta, \quad \theta \in [-\pi, \pi].$$

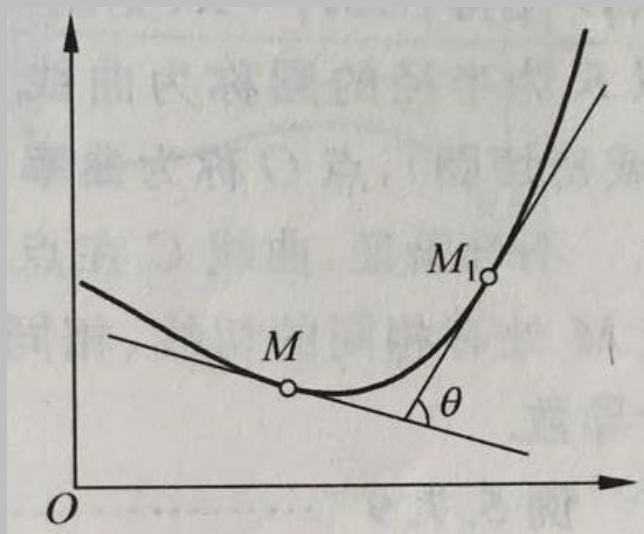
$$L = \int_{-\pi}^{\pi} \sqrt{(x'(\theta))^2 + (y'(\theta))^2} d\theta$$

$$= \int_{-\pi}^{\pi} \sqrt{(\rho'(\theta))^2 + (\rho(\theta))^2} d\theta$$

$$= 2a \int_{-\pi}^{\pi} \cos \frac{\theta}{2} d\theta = 4a \sin \frac{\theta}{2} \Big|_{-\pi}^{\pi} = 8a. \square$$



平面曲线的曲率



$$L: x = x(t), y = y(t), \quad t \in [\alpha, \beta].$$

$$M(x(t), y(t)), M_1(x(t + \Delta t), y(t + \Delta t)),$$

$$\sigma = MM_1 = \int_t^{t+\Delta t} \sqrt{(x'(\tau))^2 + (y'(\tau))^2} d\tau$$

$$\theta = \arctan \frac{y'(t + \Delta t)}{x'(t + \Delta t)} - \arctan \frac{y'(t)}{x'(t)}$$

$$\begin{aligned} \text{曲率 } k &\triangleq \lim_{\Delta t \rightarrow 0} \frac{|\theta|}{\sigma} = \lim_{\Delta t \rightarrow 0} \frac{|\theta / \Delta t|}{\sigma / \Delta t} = \frac{\left| \frac{d}{dt} \left(\arctan \frac{y'(t)}{x'(t)} \right) \right|}{\sqrt{(x'(t))^2 + (y'(t))^2}} \\ &= \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\left((x'(t))^2 + (y'(t))^2 \right)^{3/2}}. \end{aligned}$$



Remark. $y = f(x)$ 在点 x 处的曲率为 $k = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$.

Ex. 求 $x = R \cos t, y = R \sin t, 0 \leq t \leq 2\pi$ 的曲率.

解:
$$k = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{((x'(t))^2 + (y'(t))^2)^{3/2}} = \frac{1}{R}. \square$$

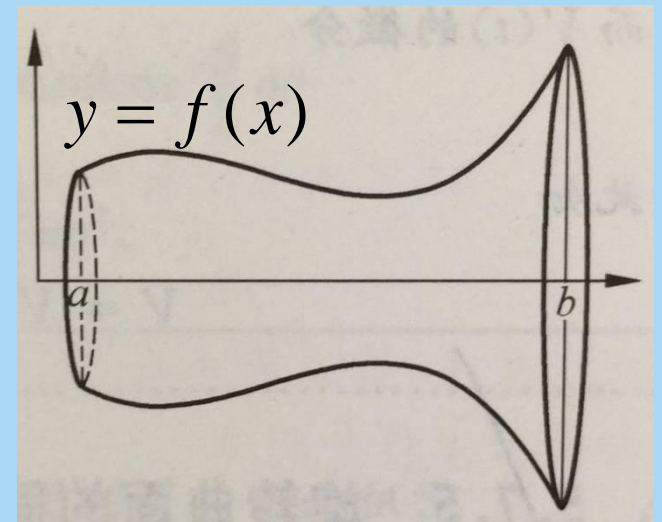
Def. 曲线 C 在点 M 处的曲率 k 的倒数 $R = 1/k$ 称为 C 在点 M 的曲率半径. C 的凹侧与 C 相切的半径为 R 的圆称为 C 在点 M 的曲率圆(密切圆), 曲率圆的圆心称为曲率中心.

Remark. 曲线的曲率圆与曲线在切点处有相同的切线、曲率与二阶导数.



- 旋转体的体积

曲线 $y = f(x)$, $a \leq x \leq b$, 绕 x 轴旋转得旋转体 Ω . 求 $V(\Omega)$.



微元法: $[x, x + \Delta x]$ 对应的薄片体积

$$(f \in C[a, b] \Rightarrow) \Delta V = \pi f^2(x) \Delta x + o(\Delta x), \Delta x \rightarrow 0.$$

$$V(\Omega) = \pi \int_a^b f^2(x) dx.$$

Remark. $x = f(y)$, $c \leq y \leq d$, 绕 y 轴旋转得旋转体 Ω 的体积

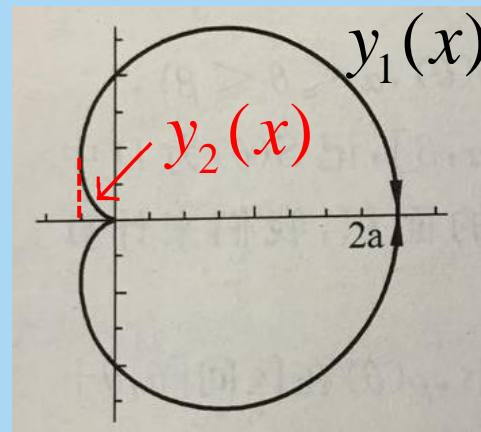
$$V(\Omega) = \pi \int_c^d f^2(y) dy.$$



Ex. 求心脏线 $\rho = a(1 + \cos \theta)$ 绕 x 轴旋转所得旋转体的体积.

解: 心脏线关于 x 轴对称, 因此只需考虑 $0 \leq \theta \leq \pi$.

$$\begin{aligned} x &= a(1 + \cos \theta) \cos \theta \\ &= a \left(\left(\cos \theta + \frac{1}{2} \right)^2 - \frac{1}{4} \right) \geq -\frac{1}{4} a. \end{aligned}$$



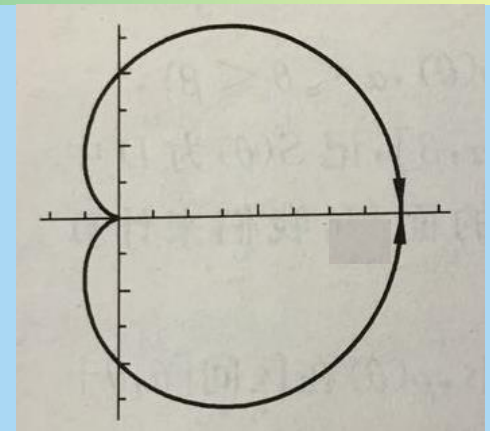
$$\begin{aligned} V &= \int_{-a/4}^{2a} \pi y_1^2(x) dx - \int_{-a/4}^0 \pi y_2^2(x) dx \\ &= \pi \int_{2\pi/3}^0 y_1^2(\theta) x'(\theta) d\theta - \pi \int_{2\pi/3}^{\pi} y_2^2(\theta) x'(\theta) d\theta \\ &= \pi \int_{\pi}^0 a^2 (1 + \cos \theta)^2 \sin^2 \theta \cdot a(\cos \theta + \cos^2 \theta)' d\theta = \frac{8}{3} \pi a^3. \quad \square \end{aligned}$$



(2) 曲线 $x = x(t), y = y(t), \alpha \leq t \leq \beta, x(t), y(t) \in C^1[\alpha, \beta]$, 绕 x 轴旋转得旋转体 Ω .

$$\Delta x = x'(t)\Delta t + o(\Delta t), \Delta t \rightarrow 0,$$

$\Delta t > 0$ 不能保证 $\Delta x \geq (\leq) 0$.



微元法: $[t, t + \Delta t]$ 对应薄片的 (有向) 体积

$$\Delta V = \pi y^2(t) \Delta x + o(\Delta x), \Delta x \rightarrow 0$$

$$= \pi y^2(t) x'(t) \Delta t + o(\Delta t), \Delta t \rightarrow 0$$

$$V(\Omega) = \left| \pi \int_{\alpha}^{\beta} y^2(t) x'(t) dt \right|.$$

Remark. 上例的计算在参数形式下可简化.



Ex. 求心脏线 $\rho = a(1 + \cos \theta)$ 绕 x 轴旋转所得旋转体的体积.

解: 心脏线关于 x 轴对称, 因此只需考虑 $0 \leq \theta \leq \pi$.

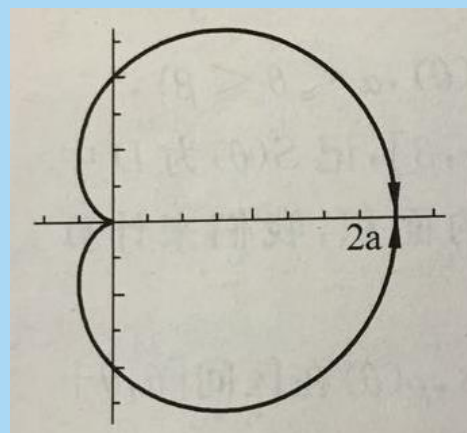
$$x = a(1 + \cos \theta) \cos \theta$$

$$y = a(1 + \cos \theta) \sin \theta$$

$$V = \left| \int_0^\pi \pi y^2(\theta) x'(\theta) d\theta \right|$$

$$= \left| \pi \int_0^\pi a^2 (1 + \cos \theta)^2 \sin^2 \theta \cdot a(\cos \theta + \cos^2 \theta)' d\theta \right|$$

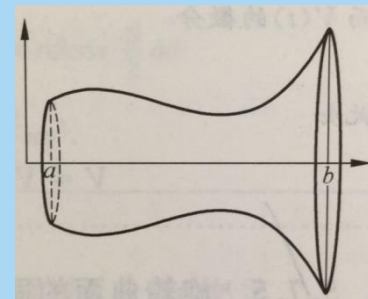
$$= \frac{8}{3} \pi a^3. \square$$





● 旋转面的面积

(1) 曲线 $y = f(x)$, $a \leq x \leq b$, $f \in C^1[a, b]$,
绕 x 轴旋转得旋转面面积



Question. 用圆柱侧面积近似旋转面面积微元:

$[x, x + \Delta x]$ 对应的面积微元 $dS = 2\pi f(x)dx$? ✗

- 曲线是几乎垂直于 x 轴的直线段?
- 曲线高频振动, 曲面褶皱很多? $f(x) = x^4 \sin^2 \frac{1}{x}$, $f(0) = 0$.

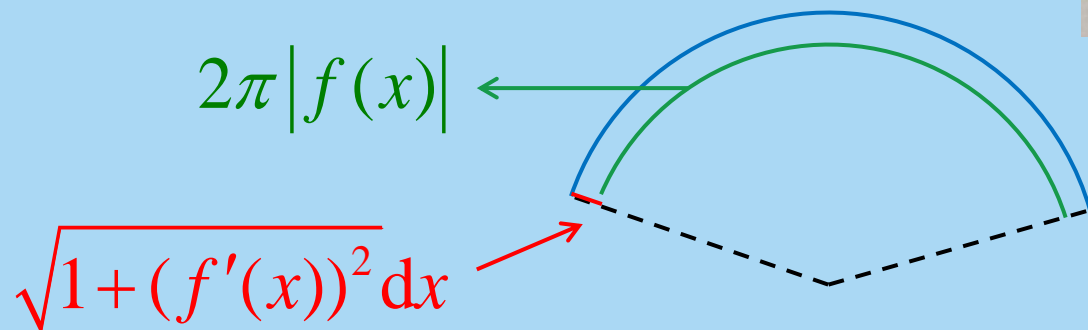
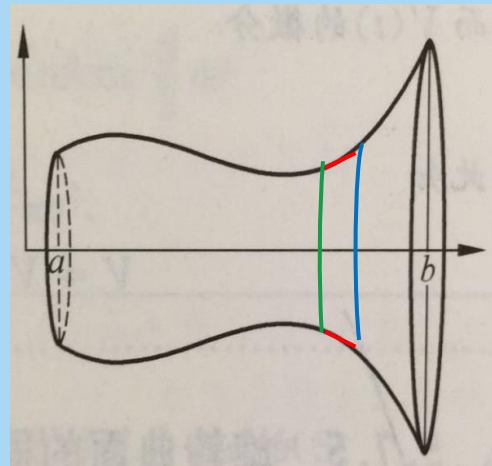
$\Delta S(0) \neq 2\pi f(0)\Delta x + o(\Delta x)$, $\Delta x \rightarrow 0$ 时.

Question. 用圆台侧面积近似旋转面面积微元? ✓



曲线 $y = f(x)$ 上 $[x, x + \Delta x]$ 对应弧段 σ 的弧长微元为 $\sqrt{1 + (f'(x))^2} dx$,

平展 $[x, x + \Delta x]$ 对应的旋转面:



旋转面面积微元 $dS = 2\pi |f(x)| \sqrt{1 + (f'(x))^2} dx$.

$$S = 2\pi \int_a^b |f(x)| \sqrt{1 + (f'(x))^2} dx.$$



(2) 曲线 $x = x(t)$, $y = y(t)$, $\alpha \leq t \leq \beta$, $x(t), y(t) \in C^1[a, b]$, 绕 x 轴旋转得旋转面 Σ 的面积 $S(\Sigma)$.

微元法: $[t, t + \Delta t]$ 对应段弧 σ 的弧长为

$$dl = \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

σ 绕 x 轴旋转所得曲面面积

$$\Delta S \approx 2\pi |y(t)| \sqrt{(x'(t))^2 + (y'(t))^2} \Delta t$$

$$\Delta S = 2\pi |y(t)| \sqrt{(x'(t))^2 + (y'(t))^2} \Delta t + o(\Delta t), \Delta t \rightarrow 0.$$

$$S = 2\pi \int_{\alpha}^{\beta} |y(t)| \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$



Ex. 求心脏线 $\rho = a(1 + \cos \theta)$, $a > 0$, $0 \leq \theta \leq 2\pi$ 绕 x 轴旋转一周所得旋转面的面积 S .

解: 心脏线关于 x 轴对称, 只需考虑 $0 \leq \theta \leq \pi$.

$$x = \rho \cos \theta, y = \rho \sin \theta.$$

$$\begin{aligned} S &= 2\pi \int_0^\pi |y(\theta)| \sqrt{(x'(\theta))^2 + (y'(\theta))^2} d\theta \\ &= 2\pi \int_0^\pi \rho \sin \theta \sqrt{(\rho'(\theta))^2 + (\rho(\theta))^2} d\theta \\ &= 2\pi \int_0^\pi a(1 + \cos \theta) \sin \theta \cdot 2a \cos \frac{\theta}{2} d\theta \\ &= 16\pi a^2 \int_0^\pi \cos^4 \frac{\theta}{2} \sin \frac{\theta}{2} d\theta = \frac{32\pi a^2}{5}. \quad \square \end{aligned}$$



- 积分在物理中的应用(功,质量,质心,引力)

Ex. $C: x = x(t), y = y(t), a \leq t \leq b$. C 上点 $M(x(t), y(t))$ 处密度为 $\rho(t)$. 求 C 的质量 m 与重心坐标 (\bar{x}, \bar{y}) .

分析: 平面质点系 $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ 的质量分别为 m_1, m_2, \dots, m_n . 其重心坐标 (\bar{x}, \bar{y}) 为

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}, \quad \bar{y} = \frac{\sum_{i=1}^n m_i y_i}{\sum_{i=1}^n m_i}.$$

对 C 进行分割, 近似为有限个质点.



解: 分析 $[t, t + \Delta t]$ 对应的段弧.

$$\text{弧长微元 } dl = \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

$$\text{质量微元 } dm = \rho(t)dl = \rho(t)\sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

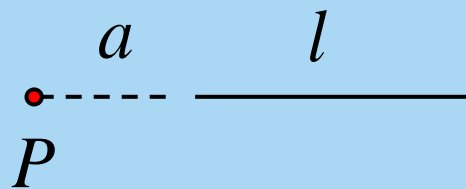
因而
$$m = \int_a^b \rho(t)\sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

$$\bar{x} = \frac{\int_a^b x(t)\rho(t)\sqrt{(x'(t))^2 + (y'(t))^2} dt}{m},$$

$$\bar{y} = \frac{\int_a^b y(t)\rho(t)\sqrt{(x'(t))^2 + (y'(t))^2} dt}{m}. \quad \square$$



Ex. 质量为 M 长度为 l 的均匀细杆, 对其延长线上距离 a 处质量为 m 的质点 P 的引力 F .



解: 取 P 为坐标原点, 细杆所在直线为 x 轴.

考虑细杆上一小段 $[x, x + \Delta x]$, 视之为质点, 其质量为 $\frac{M}{l} \Delta x$, 它对 P 的引力为 $\Delta F \approx k \frac{Mm \Delta x}{lx^2}$, k 为引力常数, 故

$$F = k \int_a^{a+l} \frac{Mm}{lx^2} dx = - \frac{kMm}{lx} \Big|_a^{a+l} = \frac{kMm}{a(a+l)}. \quad \square$$



作业：习题5.7

No.2(5), 3(1), 7(4), 8(2), 9(2).