Review

Thm.(确界原理) 非空有上界的集合必有上确界.

Thm.(单调收敛原理) 单调有界列必收敛.

Thm.(闭区间套定理) 若闭区间列$[a_n, b_n]$满足条件:

(1) $[a_{n+1}, b_{n+1}] \subset [a_n, b_n] (n = 1, 2, \cdots)$,

(2) $\lim\limits_{n\to\infty}(b_n - a_n) = 0$,

$$\boxed{\bigcap_{n\ge1}(0, 1/n] = \phi}$$

则$\exists!\, \xi \in \mathbb{R}$, $s.t.$ $\xi \in \bigcap\limits_{n\ge1}[a_n, b_n]$; $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \xi$.

Thm.(Bolzano-Weirstrass定理) 有界列必有收敛子列.

Thm.(Cauchy收敛原理) 收敛列$\Leftrightarrow$ Cauchy列.

§1. 函数的极限

$$N(x_0,\delta) := (x_0-\delta, x_0+\delta),$$

$$U(x_0,\delta) := N(x_0,\delta)\setminus\{x_0\}$$

Def.(函数在一点的极限) 设f在$U(x_0,\rho)$中有定义, $A\in\mathbb{R}$.

若$\forall \varepsilon>0, \exists \delta\in(0,\rho), s.t.$

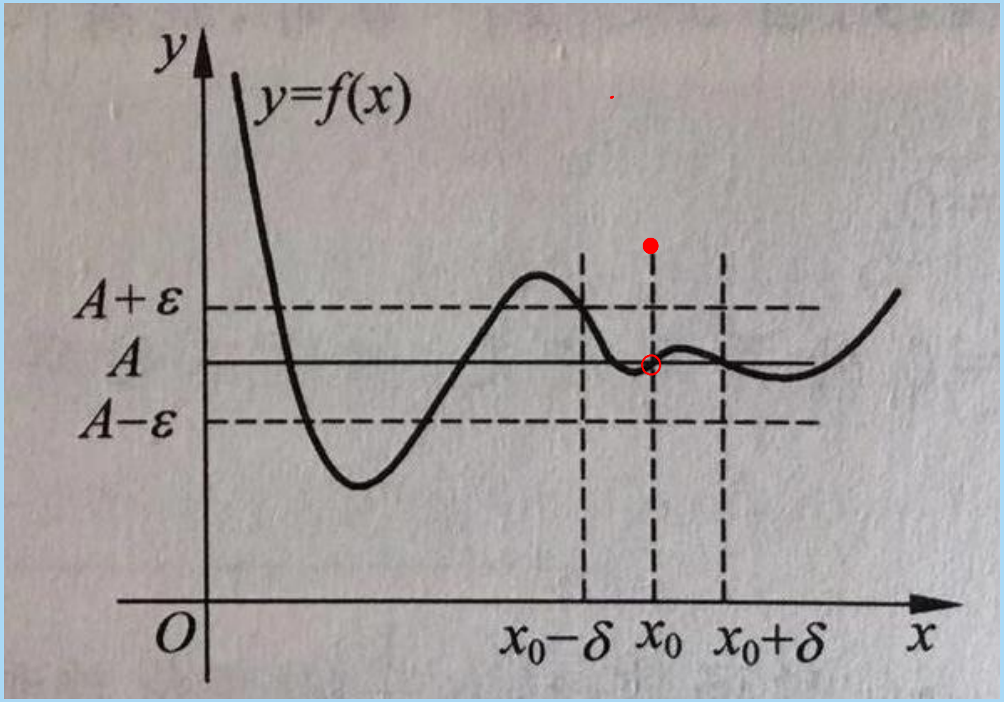
$$|f(x)-A|<\varepsilon, \quad \forall x\in U(x_0,\delta),$$

则称$f(x)$在点x_0处有极限A,或者当x趋于x_0时, $f(x)$趋于A.

记作 $\lim\limits_{x\to x_0} f(x)=A$, 或 $f(x)\to A(x\to x_0)$.

Remark. $\lim\limits_{x\to x_0} f(x)$与$f$在$x_0$的定义无关.

Question. $\lim\limits_{x\to x_0} f(x)=\mathrm{A}$的几何意义?



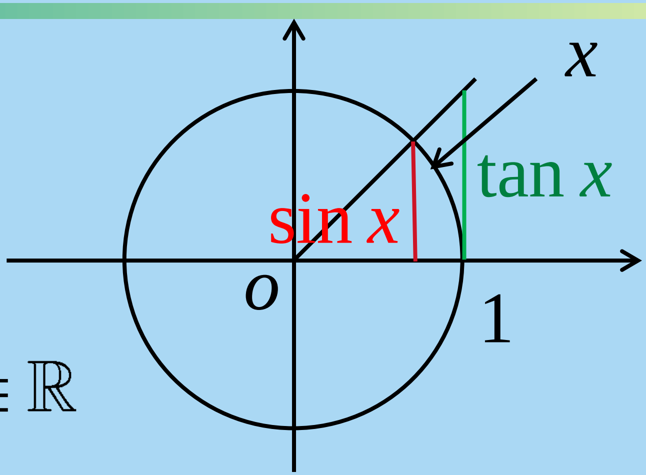
Question. 如何用$\varepsilon-\delta$语言描述$\lim\limits_{x\to x_0} f(x)\neq \mathrm{A}$?

$$|\sin x| \le |x|, \forall x \in \mathbb{R}.$$

$$|x| \le |\tan x|, \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

$$|\sin x - \sin y| \le |x - y|, \forall x, y \in \mathbb{R}$$

$$|\cos x - \cos y| \le |x - y|, \forall x, y \in \mathbb{R}$$



Ex. $\lim\limits_{x \to x_0} \cos x = \cos x_0$.

Proof. $\forall \varepsilon > 0, \exists \delta = \varepsilon$, 当$0 < |x - x_0| < \delta$时, 有

$$|\cos x - \cos x_0| \le |x - x_0| < \delta = \varepsilon. \square$$

Ex. $\lim\limits_{x \to x_0} \sin x = \sin x_0$.

Ex. $\lim\limits_{x\to 0} x\sin\frac{1}{x} = \underline{\quad 0 \quad}$.

$$\left|x\sin\frac{1}{x}\right| \le |x|.$$

Ex. $\lim\limits_{x\to 1} \frac{x^2-3x+2}{x^2-x} = \underline{\quad -1 \quad}$.

分析: $\frac{x^2-3x+2}{x^2-x} = \frac{(x-1)(x-2)}{x(x-1)} = \frac{x-2}{x}, \forall x \ne 1.$

Proof. 当 $|x-1| < \frac{1}{2}$ 时, $|x| > \frac{1}{2}$. $\forall \varepsilon > 0, \exists \delta = \min\{\frac{1}{2}, \frac{\varepsilon}{4}\} > 0, s.t.$

$$\left|\frac{x^2-3x+2}{x^2-x} - (-1)\right| = 2\left|\frac{x-1}{x}\right| < 4|x-1| < \varepsilon,$$

$$\forall 0 < |x-1| < \delta. \square$$

Def.(右极限) 设f在$(x_0, x_0+\rho)$中有定义,$A\in\mathbb{R}$.若$\forall\varepsilon>0$,

$\exists\delta\in(0,\rho)$, s.t.

$$|f(x)-A|<\varepsilon,\qquad \forall x_0 < x < x_0+\delta,$$

则称$f(x)$在点x_0处有右极限A,或者当x趋于x_0^+时,$f(x)$趋于A.记作$\lim\limits_{x\to x_0^+} f(x)=A$,或$f(x)\to A(x\to x_0^+)$.

Def.(左极限) 设f在$(x_0-\rho, x_0)$中有定义,$A\in\mathbb{R}$.若$\forall\varepsilon>0$,

$\exists\delta\in(0,\rho)$, s.t.

$$|f(x)-A|<\varepsilon,\qquad \forall x_0-\delta < x < x_0,$$

则称$f(x)$在点x_0处有左极限A,或者当x趋于x_0^-时,$f(x)$趋于A.记作$\lim\limits_{x\to x_0^-} f(x)=A$,或$f(x)\to A(x\to x_0^-)$.

Thm. $\lim\limits_{x\to x_0} f(x) = A \Leftrightarrow \lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^-} f(x) = A.$

Proof. 略.

Ex. $\operatorname{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$

$\lim\limits_{x\to 0^+} \operatorname{sgn}(x) \underline{\ =1\ }, \lim\limits_{x\to 0^-} \operatorname{sgn}(x) \underline{\ =-1\ }, \lim\limits_{x\to 0} \operatorname{sgn}(x)$ $\underline{\text{不存在}}$.

Def. $\lim\limits_{x\to x_0} f(x)=+\infty$:

$$\forall M>0, \exists\delta>0, \text{使得} \forall x\in U(x_0,\delta), \text{有 } f(x)>M.$$

Question. 如何定义 $\lim\limits_{x\to x_0^+} f(x)=+\infty$, $\lim\limits_{x\to x_0^-} f(x)=+\infty$,

$\lim\limits_{x\to x_0} f(x)=-\infty$, $\lim\limits_{x\to x_0^+} f(x)=-\infty$, $\lim\limits_{x\to x_0^-} f(x)=-\infty$,

$\lim\limits_{x\to x_0} f(x)=\infty$, $\lim\limits_{x\to x_0^+} f(x)=\infty$, $\lim\limits_{x\to x_0^-} f(x)=\infty$?

Thm. $\lim\limits_{x\to x_0} f(x)=+\infty \Leftrightarrow \lim\limits_{x\to x_0^+} f(x)=\lim\limits_{x\to x_0^-} f(x)=+\infty$.

Ex. $\lim\limits_{x\to 0^-} e^{\frac{1}{x}}$ $\underline{=0}$, $\lim\limits_{x\to 0^+} e^{\frac{1}{x}}$ $\underline{=+\infty}$, $\lim\limits_{x\to 0} e^{\frac{1}{x}}$ $\underline{\text{不存在}}$.

Def.(函数在无穷远点的极限)

(1)设$|x|>a$时f有定义,$\mathrm{A}\in\mathbb{R}$.若$\forall\varepsilon>0,\exists M>0,s.t.$

$$|f(x)-\mathrm{A}|<\varepsilon,\quad \forall |x|>M,$$

则称当x趋于∞时,$f(x)$有极限A.记作

$$\lim_{x\to\infty} f(x)=\mathrm{A},\ 或\ f(x)\to\mathrm{A}(x\to\infty).$$

(2)设$x>a$时f有定义,$\mathrm{A}\in\mathbb{R}$.若$\forall\varepsilon>0,\exists M>0,s.t.$

$$|f(x)-\mathrm{A}|<\varepsilon,\quad \forall x>M,$$

则称当x趋于$+\infty$时,$f(x)$有极限A.记作

$$\lim_{x\to+\infty} f(x)=\mathrm{A},\ 或\ f(x)\to\mathrm{A}(x\to+\infty).$$

(3)设$x<a$时f有定义,$\mathrm{A}\in\mathbb{R}$.若$\forall\varepsilon>0,\exists M>0,s.t.$

$$\left|f(x)-\mathrm{A}\right|<\varepsilon,\quad \forall x<-M,$$

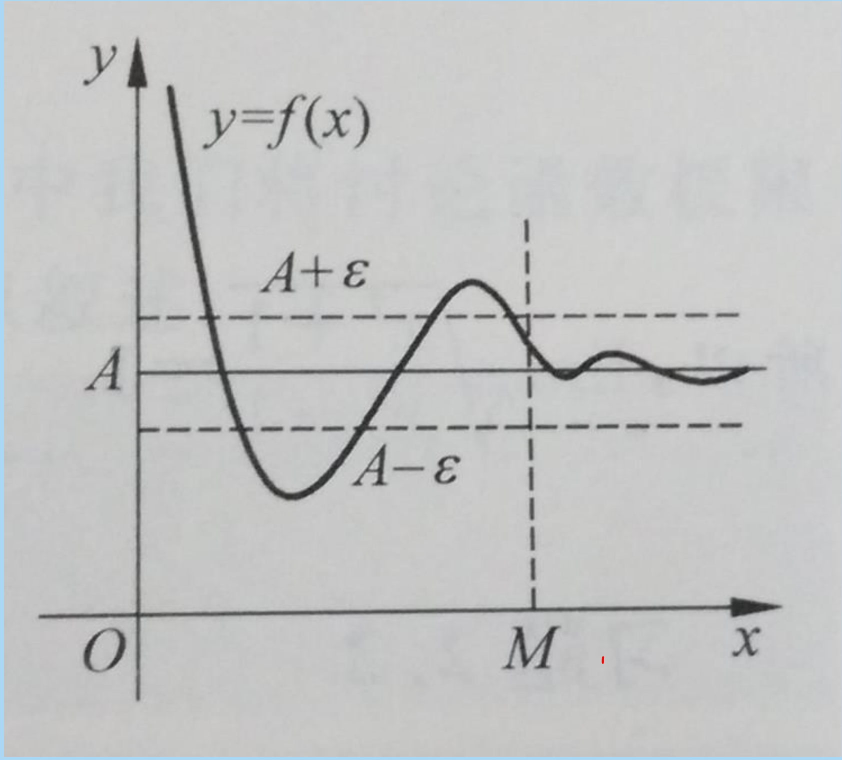
则称当x趋于$-\infty$时,$f(x)$有极限A.记作

$$\lim_{x\to-\infty}f(x)=\mathrm{A},\ \text{或}\ f(x)\to\mathrm{A}(x\to-\infty).$$

Question. 如何用$\varepsilon-\delta$语言描述$\lim\limits_{x\to+\infty}f(x)\neq\mathrm{A}$?

$$\exists\varepsilon>0,\forall M>0,\ \exists x>M,s.t.\left|f(x)-\mathrm{A}\right|>\varepsilon.$$

Question. $\lim\limits_{x\to+\infty} f(x) = A$的几何意义?



Ex. $\lim\limits_{x\to+\infty}\sqrt{\dfrac{x^2+1}{x^2-1}} = \underline{\quad 1 \quad}$

Proof. $\forall\varepsilon>0, \exists M=\max\{\sqrt{2},\dfrac{2}{\varepsilon}\}>0, s.t.$

$$\left|\sqrt{\frac{x^2+1}{x^2-1}}-1\right| = \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2-1}}$$

$$=\frac{2}{\sqrt{x^2-1}\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)} < \frac{2}{|x|} < \varepsilon, \forall x>M.$$

故 $\lim\limits_{x\to+\infty}\sqrt{\dfrac{x^2+1}{x^2-1}}=1.\square$

Question. 如何定义

$$\lim_{x\to\infty} f(x)=\infty,\ \lim_{x\to+\infty} f(x)=\infty,\ \lim_{x\to-\infty} f(x)=\infty,$$

$$\lim_{x\to\infty} f(x)=-\infty,\ \lim_{x\to+\infty} f(x)=-\infty,\ \lim_{x\to-\infty} f(x)=-\infty,$$

$$\lim_{x\to\infty} f(x)=+\infty,\ \lim_{x\to+\infty} f(x)=+\infty,\ \lim_{x\to-\infty} f(x)=+\infty\,?$$

Remark. 函数极限的24种定义.

§ 2. 函数极限的性质

$$\lim_{x\to x_0} f(x),\quad \lim_{x\to x_0+} f(x),\quad \lim_{x\to x_0-} f(x),$$

$$\lim_{x\to\infty} f(x),\quad \lim_{x\to+\infty} f(x),\quad \lim_{x\to-\infty} f(x).$$

以 $\lim\limits_{x\to x_0} f(x)$ 为例叙述函数极限的性质, 其他情形类似.

Prop1. 若 $\lim\limits_{x\to x_0} f(x)$ 存在, 则极限值唯一.

Prop2. 若 $\lim\limits_{x\to x_0} f(x)$ 存在, 则 $\exists\delta>0, M>0, s.t.$

$$|f(x)|<M,\quad \forall x\in U(x_0,\delta).$$ (局部有界)

Prop3.(保序性) $\lim\limits_{x\to x_0} f(x)=\mathrm{A}, \lim\limits_{x\to x_0} g(x)=\mathrm{B}.$

(1)若$\mathrm{A}>\mathrm{B}$,则$\exists\delta>0, s.t.$

$$f(x)>g(x),\quad \forall x\in U(x_0,\delta).$$

(2)若$\exists\delta>0, s.t.$

$$f(x)\ge g(x),\quad \forall x\in U(x_0,\delta),$$

则$\mathrm{A}\ge\mathrm{B}.$

Prop4.(四则运算) $\lim\limits_{x\to x_0} f(x)=\mathrm{A}, \lim\limits_{x\to x_0} g(x)=\mathrm{B}.$

(1) $\forall c\in\mathbb{R}, \lim\limits_{x\to x_0} cf(x)=c\mathrm{A};$

(2) $\lim\limits_{x\to x_0}\big(f(x)\pm g(x)\big)=\mathrm{A}\pm\mathrm{B};$

(3) $\lim\limits_{x\to x_0}\big(f(x)\cdot g(x)\big)=\mathrm{AB};$

(4) $\mathrm{B}\neq 0$时, $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{\mathrm{A}}{\mathrm{B}}.$

Remark. A, B可取 $+\infty,-\infty$或∞, 只要右端运算有意义.

Ex. $\lim\limits_{x\to x_0}\tan x=\tan x_0,\quad \lim\limits_{x\to x_0}\cot x=\cot x_0,$

$\lim\limits_{x\to x_0}\sec x=\sec x_0,\quad \lim\limits_{x\to x_0}\csc x=\csc x_0.$

条件?

Prop5.(夹挤原理)若

$$\left.\begin{array}{l} f(x) \le g(x) \le h(x), \quad \forall x \in U(x_0, \rho) \\ \lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} h(x) = \mathrm{A} \end{array}\right\} \Rightarrow \lim_{x \to x_0} g(x) = \mathrm{A}.$$

Ex. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1.$ $\quad \dfrac{0}{0}$型极限

Proof. $\forall 0 < |x| < \dfrac{\pi}{2}$, 有 $|\sin x| \le |x| \le |\tan x|$,

$$\cos x \le \frac{|\sin x|}{|x|} \le 1, \qquad \cos x \le \frac{\sin x}{x} = \frac{|\sin x|}{|x|} \le 1.$$

而 $\lim\limits_{x \to 0} \cos x = 1$, 由夹挤原理, $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1.$ □

Prop6.(单调收敛原理)

(1) f在(a,b)上单增有上界, 则 $\lim\limits_{x\to b^-} f(x) = \sup\limits_{a<x<b} f(x)$;

(2) f在(a,b)上单减有下界, 则 $\lim\limits_{x\to b^-} f(x) = \inf\limits_{a<x<b} f(x)$;

(3) f在(a,b)上单增有下界, 则 $\lim\limits_{x\to a^+} f(x) = \inf\limits_{a<x<b} f(x)$;

(4) f在(a,b)上单减有上界, 则 $\lim\limits_{x\to a^+} f(x) = \sup\limits_{a<x<b} f(x)$.

Proof. 只证(1),其它情形同理可证. $\{f(x): x\in(a,b)\}$

非空有上界, 从而有上确界

$$\mathrm{A} = \sup\{f(x): x\in(a,b)\} \in \mathbb{R}.$$

由上确界的定义,

$$\forall \varepsilon > 0, \exists x_1 \in (a,b), s.t.\ \ f(x_1) > \mathrm{A} - \varepsilon,$$

且

$$f(x) \le \mathrm{A}, \quad \forall x \in (a,b).$$

$f\uparrow$, 则$\forall x \in (x_1, b)$, 有

$$\mathrm{A} - \varepsilon < f(x_1) \le f(x) \le \mathrm{A}.$$

故 $\lim\limits_{x \to b^-} f(x) = \mathrm{A}.\square$

Corollary. (a,b)上的单调函数在每一点处左右极限都存在.

Prop7. $\left.\begin{array}{l}\lim\limits_{x\to x_0} g(x)=u_0\\ \lim\limits_{u\to u_0} f(u)=\mathrm{A}\\ g(x)\neq u_0, \forall x\neq x_0\end{array}\right\} \Rightarrow \lim\limits_{x\to x_0} f(g(x))=\mathrm{A}=\lim\limits_{u\to u_0} f(u).$

(复合函数的极限)

Proof. $\lim\limits_{u\to u_0} f(u)=\mathrm{A}$, 则$\forall\varepsilon>0, \exists\delta>0, s.t.$

$$\left|f(u)-\mathrm{A}\right|<\varepsilon,\quad \forall 0<\left|u-u_0\right|<\delta.$$

对此$\delta>0$, 因 $g(x)\neq u_0, \forall x\neq x_0, \lim\limits_{x\to x_0} g(x)=u_0, \exists\eta>0, s.t.$

$$0<\left|g(x)-u_0\right|<\delta,\quad \forall 0<\left|x-x_0\right|<\eta,$$

从而 $$\left|f(g(x))-\mathrm{A}\right|<\varepsilon,\quad \forall 0<\left|x-x_0\right|<\eta.$$

由函数极限定义, 有 $\lim\limits_{x\to x_0} f(g(x))=\mathrm{A}.\square$

Remark. 复合函数的极限运算可以理解为函数极限运算的变量替换法.

Question. 条件“$g(x)\neq u_0, \forall x\neq x_0$”是否可去?反例?

否. 反例：$f(u)=\begin{cases}1 & u\neq 0\\ 0 & u=0\end{cases}$, $\quad \lim\limits_{u\to 0} f(u)=1$,

$$g(x)=x\sin\frac{1}{x},\qquad g\left(\frac{1}{k\pi}\right)=0, \forall k\in\mathbb{Z}\setminus\{0\},$$

$$f(g(x))=\begin{cases}1 & x\neq 0, \dfrac{1}{k\pi}\\ 0 & x=\dfrac{1}{k\pi}\end{cases},\qquad \lim_{x\to 0} f(g(x))\text{不存在.}$$

Question. 条件 "$g(x) \neq u_0, \forall x \neq x_0$" 何时可去?

Remark. $$\left.\begin{array}{l} \lim\limits_{x\to x_0} g(x) = u_0 \\ \lim\limits_{u\to u_0} f(u) = f(u_0) \end{array}\right\} \Rightarrow \lim_{x\to x_0} f(g(x)) = f(u_0)$$

$$= \lim_{u\to u_0} f(u) = f(\lim_{x\to x_0} g(x)).$$

Remark. $$\left.\begin{array}{l} \lim\limits_{x\to x_0} g(x) = \pm\infty \\ \lim\limits_{u\to \pm\infty} f(u) = \mathrm{A} \end{array}\right\} \Rightarrow \lim_{x\to x_0} f(g(x)) = \mathrm{A}$$

$$= \lim_{u\to \pm\infty} f(u).$$

Ex. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}=e.$ （常用以处理1^{∞}型极限）

Proof. $$\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor}\le\left(1+\frac{1}{x}\right)^{x}\le\left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1},\ \forall x>1.$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1}=\lim_{x\to+\infty}\left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor}\lim_{x\to+\infty}\left(1+\frac{1}{\lfloor x\rfloor}\right)=e,$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor}=\frac{\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor+1}}{\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{\lfloor x\rfloor+1}\right)}=e,$$

由夹挤原理, $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^{x}=e.$

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{x\to-\infty}\left(\frac{x}{1+x}\right)^{-x}$$

$$=\lim_{x\to-\infty}\left(1+\frac{1}{-(1+x)}\right)^{-(x+1)}\cdot\lim_{x\to-\infty}\frac{x}{1+x}=\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y}=e.$$

综上, $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}=e.\square$

Remark. $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e.$

Thm. f在$U(x_0,\rho)$中有定义,则以下命题等价:

(1)$\forall\varepsilon>0,\exists\delta>0,\forall x,y\in U(x_0,\delta)$,有$|f(x)-f(y)|<\varepsilon$;

(2)$\exists A\in\mathbb{R}$,对$U(x_0,\rho)$中任意收敛到x_0的点列$\{x_n\}$,有

$$\lim_{n\to\infty}f(x_n)=A;$$

(3)$\lim\limits_{x\to x_0}f(x)=A.$

Remark. $(1)\Leftrightarrow(3)$ (函数极限的Cauchy收敛原理)

Remark. $(2)\Leftrightarrow(3)$ (用数列的极限来研究函数的极限)

Proof: $(1)\Rightarrow(2)\Rightarrow(3)\Rightarrow(1).$

$(1) \Rightarrow (2)$: 设$x_n \in U(x_0,\rho), \lim\limits_{n\to\infty} x_n = x_0$. 由(1), $\forall \varepsilon > 0, \exists \delta > 0$, s.t.

$$|f(x) - f(y)| < \varepsilon, \quad \forall x, y \in U(x_0,\delta).$$

对此δ,因$\lim\limits_{n\to\infty} x_n = x_0, \exists N, s.t. \quad x_n \in U(x_0,\delta), \quad \forall n > N.$

于是 $$|f(x_n) - f(x_m)| < \varepsilon, \qquad \forall n, m > N.$$

故$\{f(x_n)\}$为Cauchy列,收敛, $\exists \mathrm{A} \in \mathbb{R}, s.t. \lim\limits_{n\to\infty} f(x_n) = \mathrm{A}.$

设$y_n \in U(x_0,\rho), \lim\limits_{n\to\infty} y_n = x_0$,同理$\lim\limits_{n\to\infty} f(y_n) = \mathrm{B}$. 只要证

$\mathrm{A} = \mathrm{B}$. 构造$\{z_n\}$: $z_{2n-1} = x_n,\ z_{2n} = y_n$, 则$\lim\limits_{n\to\infty} z_n = x_0$,$\{f(z_n)\}$

收敛, 且 $\mathrm{A} = \lim\limits_{n\to\infty} f(z_{2n-1}) = \lim\limits_{n\to\infty} f(z_n) = \lim\limits_{n\to\infty} f(z_{2n}) = \mathrm{B}.$

$(2) \Rightarrow (3)$:

设 $\lim\limits_{x\to x_0} f(x) \neq \mathrm{A}$. 则 $\exists \varepsilon_0 > 0, \forall n \in \mathbb{N}, \exists x_n \in U(x_0, \frac{1}{n}), s.t.$

$$\left| f(x_n) - \mathrm{A} \right| > \varepsilon_0.$$

此时, $\lim\limits_{n\to\infty} x_n = x_0$, 但 $\lim\limits_{n\to\infty} f(x_n) \neq \mathrm{A}$, 与(2)矛盾.

$(3) \Rightarrow (1)$: 略.□

Remark. $x_n \neq x_0, y_n \neq x_0, \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = x_0$, 则

- $\lim\limits_{n\to\infty} f(x_n) = \mathrm{A} \neq \mathrm{B} = \lim\limits_{n\to\infty} f(y_n) \quad \Rightarrow \lim\limits_{x\to x_0} f(x)$不存在;
- $\lim\limits_{n\to\infty} f(x_n)$不存在 $\Rightarrow \lim\limits_{x\to x_0} f(x)$不存在.

Ex. Dirichlet函数$D(x)=\begin{cases}1, & x\in\mathbb{Q},\\ 0, & x\notin\mathbb{Q}.\end{cases}$则$\forall x_0\in\mathbb{R}$,

$\lim\limits_{x\to x_0^-}D(x)$ <u>不存在</u>, $\lim\limits_{x\to x_0^+}D(x)$ <u>不存在</u>, $\lim\limits_{x\to x_0}D(x)$ <u>不存在</u>.

Ex. $\lim\limits_{x\to 0}\sin\dfrac{1}{x}$ <u>不存在</u>.

Proof. $x_n=\dfrac{1}{2n\pi}, y_n=\dfrac{1}{\left(2n+\frac{1}{2}\right)\pi}, \lim\limits_{n\to+\infty}x_n=\lim\limits_{n\to+\infty}y_n=0,$

而$\lim\limits_{n\to+\infty}\sin\dfrac{1}{x_n}=0, \lim\limits_{n\to+\infty}\sin\dfrac{1}{y_n}=1,$故$\lim\limits_{x\to 0}\sin\dfrac{1}{x}$不存在.□

Ex.(1) $\lim\limits_{x\to x_0} e^x = e^{x_0}$, (2) $\lim\limits_{x\to x_0} \ln x = \ln x_0 \, (x_0 > 0)$.

Proof. $\forall \{x_n\}, x_n \to x_0$, 有 $\lim\limits_{n\to\infty} e^{x_n} = e^{x_0}$, $\lim\limits_{n\to\infty} \ln x_n = \ln x_0$. □

Remark. $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = \lim\limits_{x\to 0} \ln(1+x)^{\frac{1}{x}} = \ln e = 1$.

Ex. $\lim\limits_{x\to x_0} u(x) = a$, $\lim\limits_{x\to x_0} v(x) = b$, a^b有意义, 则 $\lim\limits_{x\to x_0} u(x)^{v(x)} = a^b$.

Proof. $\lim\limits_{x\to x_0} u(x)^{v(x)} = \lim\limits_{x\to x_0} e^{v(x)\ln u(x)}$

$$= e^{\lim\limits_{x\to x_0}\left(v(x)\ln u(x)\right)} = e^{\lim\limits_{x\to x_0} v(x)\cdot \lim\limits_{x\to x_0} \ln u(x)} = e^{b\ln a} = a^b. \square$$

question. $\lim\limits_{x\to x_0} u(x)=a, \lim\limits_{x\to x_0} v(x)=b,$ 则

$$\lim_{x\to x_0} u(x)^{v(x)} = \lim_{x\to x_0} a^{v(x)}\,?\ \lim_{x\to x_0} u(x)^{v(x)} = \lim_{x\to x_0} u(x)^{b}\,?$$

否！ 反例：

$$e=\lim_{x\to 0}(1+x)^{1/x} \neq \lim_{x\to 0} 1^{1/x} = 1.$$

$$0=\lim_{x\to 0^+} x = \lim_{x\to 0^+}\left(x^{1/x}\right)^{x} \neq \lim_{x\to 0^+}\left(x^{1/x}\right)^{0} = 1.$$

Remark. a^b无意义的情形：$1^\infty, \infty^0, 0^0$均为未定型！

1^∞：$\lim_{x\to0}(1+x)^{1/x} = e, \quad \lim_{x\to0}(1+x)^{2/x} = e^2,$

$$\lim_{x\to0^+}(1+x)^{1/x^2} = \lim_{x\to0^+}\exp\{\frac{1}{x}\cdot\frac{\ln(1+x)}{x}\} = e^{+\infty\cdot1} = +\infty.$$

$$\lim_{x\to0^-}(1+x)^{1/x^2} = \lim_{x\to0^-}\exp\{\frac{1}{x}\cdot\frac{\ln(1+x)}{x}\} = e^{-\infty\cdot1} = 0.$$

0^0：$\lambda\in\mathbb{R}, \lim_{x\to0}(e^{\frac{-1}{x^2}})^{\lambda\sin^2 x} = \lim_{x\to0} e^{-\lambda\frac{\sin^2 x}{x^2}} = e^{-\lambda}.$

∞^0：可以通过0^0的例子改写.

Question. $x \to +\infty$时, $x^b, a^x, \ln x, x^x$的增长速度? $(a>1, b>0)$

Ex. $\lim\limits_{x\to+\infty} \dfrac{\log_a x}{x^b} = 0 \ (a>1, b>0)$.

Proof. $0 < \dfrac{\ln x}{x} \le \dfrac{\ln(\lfloor x \rfloor + 1)}{\lfloor x \rfloor} \le \dfrac{\ln 2}{\lfloor x \rfloor} + \dfrac{\ln \lfloor x \rfloor}{\lfloor x \rfloor}, \quad \forall x > 1.$

$$\lim_{x\to+\infty}\left(\frac{\ln 2}{\lfloor x \rfloor} + \frac{\ln\lfloor x \rfloor}{\lfloor x \rfloor}\right) = \lim_{x\to+\infty}\frac{\ln 2}{\lfloor x \rfloor} + \lim_{x\to+\infty}\frac{\ln\lfloor x \rfloor}{\lfloor x \rfloor} = 0.$$

由夹挤原理, $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x} = 0.$

$$\lim_{x\to+\infty}\frac{\log_a x}{x^b} = \lim_{y\to+\infty}\frac{\log_a y^{1/b}}{y} = \frac{1}{b\ln a}\lim_{y\to+\infty}\frac{\ln y}{y} = 0. \square$$

Remark. $\lim_{x\to 0^+} x^b \log_a x = 0\ (a>1, b>0)$.

Ex. $\lim_{x\to+\infty} \frac{x^b}{a^x} = 0\ (a>1, b>0)$.

Proof. $0 < \frac{x^b}{a^x} \le \frac{\left(\lfloor x\rfloor + 1\right)^b}{a^{\lfloor x\rfloor}} \le \frac{\left(2\lfloor x\rfloor\right)^b}{a^{\lfloor x\rfloor}} \le \frac{2^b \lfloor x\rfloor^b}{a^{\lfloor x\rfloor}}, \forall x > 1.$

$\lim_{x\to+\infty} \frac{2^b \lfloor x\rfloor^b}{a^{\lfloor x\rfloor}} = 2^b \lim_{x\to+\infty} \frac{\lfloor x\rfloor^b}{a^{\lfloor x\rfloor}} = 0.$ 由夹挤原理, $\lim_{x\to+\infty} \frac{x^b}{a^x} = 0.\square$

Ex. $\lim_{x\to+\infty} \frac{a^x}{x^x} = 0\ (a>0, a\ne 1)$.

Proof. $\lim_{x\to+\infty} \frac{a^x}{x^x} = \lim_{x\to+\infty} e^{x(\ln a - \ln x)} = e^{(+\infty)\cdot(-\infty)} = e^{-\infty} = 0.\square$

1^{∞}型极限

Ex. $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}}$.

证法一. $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}} = \lim\limits_{x\to 0}\left(1-2\sin^2\dfrac{x}{2}\right)^{\frac{1}{x^2}}$

$$= \lim_{x\to 0}\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{-2\sin^2\frac{x}{2}}\cdot\frac{-2\sin^2\frac{x}{2}}{x^2}}$$

$$= \left\{\lim_{x\to 0}\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{-2\sin^2\frac{x}{2}}}\right\}^{-\frac{1}{2}\lim\limits_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2} = e^{-\frac{1}{2}}.$$

证法二. $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0}\exp\left\{\frac{1}{x^2}\ln\cos x\right\}$

$$= \exp\left\{\lim_{x\to 0}\frac{1}{x^2}\ln(1-2\sin^2\frac{x}{2})\right\}$$

$$= \exp\left\{\lim_{x\to 0}\frac{\ln(1-2\sin^2\frac{x}{2})}{-2\sin^2\frac{x}{2}}\cdot\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{(\frac{x}{2})^2}\cdot(-\frac{1}{2})\right\}$$

$$= \exp\left\{1\cdot 1\cdot(-\frac{1}{2})\right\} = e^{-\frac{1}{2}}. \square$$

推荐证法二！

清華大學

作业：

习题2.2 No. 3(4)(7),7,8

习题2.3 No. 6(8)(11)(13)(14)(17),
7(4)(12),8(2)(6),9(1).