



Review

- $\inf A$, $\sup A$, $\min A$, $\max A$

- $\lim_{n \rightarrow \infty} a_n = A$ 的 ε - N 语言描述

$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{s.t. 当 } n > N \text{ 时, 有 } a_n \in (A - \varepsilon, A + \varepsilon)$

$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{s.t. 当 } n \geq N \text{ 时, 有 } A - \varepsilon \leq a_n \leq A + \varepsilon$

$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{s.t. 当 } n > N \text{ 时, 有 } |a_n - A| < \varepsilon / 2$

$\Leftrightarrow \forall \varepsilon \in (0, 1), \exists N \in \mathbb{N}, \text{s.t. 当 } n \geq N \text{ 时, 有 } |a_n - A| \leq 2\varepsilon.$

$\Leftrightarrow \forall k \in \mathbb{N}, \exists N = N(k) \in \mathbb{N}, \text{s.t. 当 } n \geq N \text{ 时, 有 } |a_n - A| \leq \frac{1}{2^k}.$



- ε -N语言中N的选取.

(1) 放缩法求解不等式 $|a_n - A| < \varepsilon$

$$|a_n - A| < \cdots < \boxed{n \text{ 的简单表达式}} < \varepsilon$$

(2) 分段法确定N

$$N = \max\{N_1, N_2, \cdots, N_k\}$$

- 记住一些基本结论.

$$(1) \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1,$$

$$(2) \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0,$$



(3) $\lim_{n \rightarrow \infty} a_n = a \in [-\infty, +\infty]$, 则

$$\lim_{n \rightarrow \infty} e^{a_n} = e^{\lim_{n \rightarrow \infty} a_n} = e^a = \begin{cases} +\infty, & a = +\infty, \\ e^a, & a \in \mathbb{R}, \\ 0, & a = -\infty; \end{cases}$$

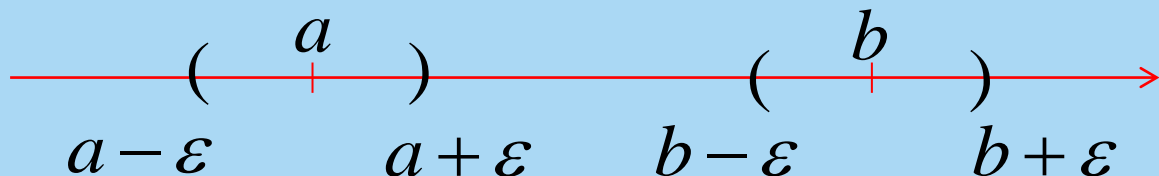
(4) 若 $a_n > 0$, $\lim_{n \rightarrow \infty} a_n = a \in [0, +\infty]$, 则

$$\lim_{n \rightarrow \infty} \ln a_n = \ln(\lim_{n \rightarrow \infty} a_n) = \ln a = \begin{cases} +\infty, & a = +\infty, \\ \ln a, & 0 < a \in \mathbb{R}, \\ -\infty, & a = 0. \end{cases}$$



§ 3. 收敛列的性质

Prop1. 收敛列的极限唯一.



Proof. 假设 a, b 均为 $\{a_n\}$ 的极限, 且 $a \neq b$. $\forall 0 < \varepsilon < \frac{|a-b|}{2}$,

由 $\lim_{n \rightarrow \infty} a_n = a$, $\exists N_1$, 当 $n > N_1$ 时, $|a_n - a| < \varepsilon$,

由 $\lim_{n \rightarrow \infty} a_n = b$, $\exists N_2$, 当 $n > N_2$ 时, $|a_n - b| < \varepsilon$.

令 $N = \max\{N_1, N_2\} + 1$, 则 $N > N_i, i = 1, 2$,

$$\begin{aligned} |a-b| &= |(a - a_N) - (b - a_N)| \leq |a - a_N| + |b - a_N| \\ &< 2\varepsilon < |a-b|, \text{ 矛盾. } \square \end{aligned}$$



$$\lim_{n \rightarrow \infty} a_n = A \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{s.t. } \forall n > N, \text{有 } |a_n - A| < \varepsilon.$$

$$\lim_{n \rightarrow \infty} a_n \neq A \Leftrightarrow \exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n > N, \text{s.t. } |a_n - A| > \varepsilon.$$

Prop2. 在数列中添加、删除有限项, 或者改变有限项的值, 不改变数列的敛散性与极限值.

Def. $0 < n_1 < n_2 < \dots < n_k < \dots$ 为一列自然数, 称 $\{a_{n_k}\}$ 为 $\{a_n\}$ 的一个子列.

Prop3. (收敛列的任意子列收敛到原极限)

$$\lim_{n \rightarrow \infty} a_n = a \Rightarrow \lim_{k \rightarrow \infty} a_{n_k} = a. \quad (n_k \geq k)$$

Corollary. $\lim_{k \rightarrow \infty} a_{n_k} = a \neq b = \lim_{k \rightarrow \infty} a_{m_k} \Rightarrow \{a_n\}$ 发散.



Ex. $\{(-1)^n\}$ 发散.

Question. $\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} a_{2n+1} = A \stackrel{?}{\Rightarrow} \lim_{n \rightarrow \infty} a_n = A.$ *Yes!*

Prop4. 收敛列一定有界.

Proof. 设 $\lim_{n \rightarrow \infty} a_n = a$. 对 $\varepsilon = 1, \exists N$, 当 $n > N$ 时, $|a_n - a| < 1$.

因此, $|a_n| = |(a_n - a) + a| \leq |a_n - a| + |a| < |a| + 1, \forall n > N$.

令 $M = \max\{|a_1|, |a_2|, \dots, |a_N|, |a| + 1\}$, 则 $|a_n| \leq M, \forall n \in \mathbb{N}.$ \square

Question. 有界列是否必为收敛列? *No!*

Def. 若 $\lim_{n \rightarrow \infty} a_n = 0$, 则称 $\{a_n\}$ 为无穷小数列.




Prop5. $\{a_n\}$ 为无穷小列, $\{b_n\}$ 为有界列, 则 $\{a_n b_n\}$ 为无穷小列.

Prop6. (极限的保序性) $\lim_{n \rightarrow \infty} a_n = a, \lim_{n \rightarrow \infty} b_n = b.$

(1) 若 $a < b$, 则 $\exists N$, 当 $n > N$ 时有 $a_n < b_n$.

(2) 若 $\exists N$, 当 $n > N$ 时有 $a_n \leq b_n$, 则 $a \leq b$.

Proof. (1) 

(2) 反设 $a > b$. 由(1)中结论, $\exists N_1$, 当 $n > N_1$ 时有 $a_n > b_n$. 矛盾. \square

Question.
$$\left. \begin{array}{l} \lim_{n \rightarrow \infty} a_n = a, \lim_{n \rightarrow \infty} b_n = b \\ a_n < b_n, \quad \forall n \end{array} \right\} \not\Rightarrow a < b \quad (\times)$$



Prop7. (极限的四则运算) 若 $\{a_n\}$ 与 $\{b_n\}$ 都收敛, 则

$$(1) \forall c \in \mathbb{R}, \lim_{n \rightarrow \infty} (ca_n) = c \lim_{n \rightarrow \infty} a_n;$$

$$(2) \lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n;$$

$$(3) \lim_{n \rightarrow \infty} (a_n \cdot b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n;$$

$$(4) \lim_{n \rightarrow \infty} b_n \neq 0 \text{ 时, } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}.$$

Remark. 可以推广到 $\lim_{n \rightarrow \infty} a_n, \lim_{n \rightarrow \infty} b_n$ 为 ∞ 或 $\pm\infty$ 的情形, 但要右端运算 **有意义**。

Remark. (3) $a > 0, a \cdot \pm\infty = \pm\infty$; $0 \cdot \infty, 0 \cdot \pm\infty$ 没有意义。

(4) $a \neq \infty, \frac{a}{\infty} = 0$; $a \neq 0, \frac{a}{0} = \infty$; $\frac{0}{0}, \frac{(\pm)\infty}{(\pm)\infty}$ 没有意义。



Proof. (3) $\{b_n\}$ 收敛, 则有界, $\exists M > 0, s.t.$

$$|b_n| < M, \quad \forall n.$$

因 $\lim_{n \rightarrow \infty} a_n = a, \lim_{n \rightarrow \infty} b_n = b, \forall \varepsilon > 0, \exists N_1, N_2, s.t.$

$$|a_n - a| < \varepsilon, \quad \forall n > N_1,$$

$$|b_n - b| < \varepsilon, \quad \forall n > N_2.$$

当 $n > N = \max\{N_1, N_2\}$ 时,

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n - a| |b_n| + |a| |b_n - b| \\ &\leq (M + |a|) \varepsilon. \end{aligned}$$



(4) $\lim_{n \rightarrow \infty} b_n = b \neq 0$, 对 $\varepsilon_0 = |b|/2$, $\exists N_1$, 当 $n > N_1$ 时, $|b_n - b| < \varepsilon_0$.

$$\begin{aligned}\text{因此, } |b_n| &= |b_n - b + b| \geq |b| - |b_n - b| \\ &> |b| - \varepsilon_0 = |b|/2, \quad \forall n > N_1.\end{aligned}$$

因 $\lim_{n \rightarrow \infty} a_n = a$, $\lim_{n \rightarrow \infty} b_n = b$, $\forall \varepsilon > 0$, $\exists N_2, N_3$, s.t.

$$|a_n - a| < \varepsilon, \quad \forall n > N_2; \quad |b_n - b| < \varepsilon, \quad \forall n > N_3.$$

当 $n > N = \max\{N_1, N_2, N_3\}$ 时,

$$\begin{aligned}\left| \frac{a_n}{b_n} - \frac{a}{b} \right| &= \frac{|ba_n - ab_n|}{|bb_n|} \leq \frac{2}{b^2} |ba_n - ab + ab - ab_n| \\ &\leq \frac{2}{b^2} (|b||a_n - a| + |a||b - b_n|) \leq \frac{2}{b^2} (|b| + |a|) \varepsilon. \quad \square\end{aligned}$$



Prop8. (夹挤原理) $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = a$, 且 $\exists n_0, s.t. \forall n > n_0$,

有 $a_n \leq x_n \leq b_n$, 则 $\lim_{n \rightarrow \infty} x_n = a$.

Proof. 因 $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = a, \forall \varepsilon > 0, \exists N_1, N_2, s.t.$

$$-\varepsilon < a_n - a < \varepsilon, \quad \forall n > N_1,$$

$$-\varepsilon < b_n - a < \varepsilon, \quad \forall n > N_2.$$

$n > n_0$ 时, $a_n \leq x_n \leq b_n$, 令 $N = \max\{n_0, N_1, N_2\}$, 则

$$-\varepsilon < a_n - a \leq x_n - a \leq b_n - a < \varepsilon, \quad \forall n > N.$$

因此 $\lim_{n \rightarrow \infty} x_n = a. \square$

Question. 能否用极限的保序性直接得到夹挤原理? 否!



Ex. 设 $b > 0, a > 1$. 证明: $\lim_{n \rightarrow \infty} \frac{n^b}{a^n} = 0$.

Proof. 法一: $\ln \frac{n^b}{a^n} = \ln \frac{(\sqrt[n]{n})^{nb}}{(\sqrt[b]{a})^{nb}} = nb \left(\ln \sqrt[n]{n} - \ln \sqrt[b]{a} \right)$

$$\begin{aligned} \lim_{n \rightarrow \infty} \ln \frac{n^b}{a^n} &= \lim_{n \rightarrow \infty} nb \cdot \lim_{n \rightarrow \infty} \left(\ln \sqrt[n]{n} - \ln \sqrt[b]{a} \right) \\ &= +\infty \cdot (\ln 1 - \ln \sqrt[b]{a}) \\ &= -\infty \end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{n^b}{a^n} = \lim_{n \rightarrow \infty} \exp \left(\ln \frac{n^b}{a^n} \right) = \exp \left(\lim_{n \rightarrow \infty} \ln \frac{n^b}{a^n} \right) = \exp(-\infty) = 0.$$



法二: (1) $b = k$ 为整数时. 令 $d = a - 1$, 则 $d > 0$, 当 $n \geq 2k$ 时

$$\begin{aligned} 0 < \frac{n^k}{a^n} &= \frac{n^k}{(1+d)^n} < \frac{n^k}{C_n^{k+1} d^{k+1}} = \frac{n^k (k+1)!}{n(n-1)\cdots(n-k)d^{k+1}} \\ &= \frac{(k+1)!}{n(1-\frac{1}{n})\cdots(1-\frac{k}{n})d^{k+1}} \leq \frac{2^k (k+1)!}{nd^{k+1}} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{2^k (k+1)!}{nd^{k+1}} = 0, \text{ 由夹挤原理, } \lim_{n \rightarrow \infty} \frac{n^k}{a^n} = 0.$$



(2) $b > 0$ 不为整数时. $0 < \frac{n^b}{a^n} \leq \frac{n^{\lceil b \rceil}}{a^n}$.

由(1)中结论, $\lim_{n \rightarrow \infty} \frac{n^{\lceil b \rceil}}{a^n} = 0$. 由夹挤原理, $\lim_{n \rightarrow \infty} \frac{n^b}{a^n} = 0$. \square

Ex. $a > 0$, 证明: $\lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$.

Proof. 当 $a \geq 1$ 时, $\forall n > \lceil a \rceil$, 有 $1 \leq \sqrt[n]{a} < \sqrt[n]{n}$, 而 $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$,

由夹挤原理, $\lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$.

当 $0 < a < 1$ 时, $\lim_{n \rightarrow \infty} \sqrt[n]{a} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{1/a}} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{1/a}} = 1$. \square



Question. 错在哪里？

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \underbrace{\left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} \right)}_{n \uparrow} \\ &= \underbrace{\lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} \frac{1}{n} + \cdots + \lim_{n \rightarrow \infty} \frac{1}{n}}_{n \uparrow} = 0 + 0 + \cdots 0 = 0 \end{aligned}$$

无穷个0相加



Question. 谁对谁错? 错在哪里?

$$\begin{aligned} (1) \lim_{n \rightarrow \infty} \frac{1}{n\sqrt[n]{n!}} &= \lim_{n \rightarrow \infty} \underbrace{\left(\sqrt[n]{1} \cdot \sqrt[n]{\frac{1}{2}} \cdot \sqrt[n]{\frac{1}{3}} \cdots \sqrt[n]{\frac{1}{n}} \right)}_{n \uparrow} \\ &= \underbrace{\lim_{n \rightarrow \infty} \sqrt[n]{1} \cdot \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{2}} \cdots \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n}}}_{n \uparrow} = 1 \cdot 1 \cdots 1 = 1 \end{aligned}$$

$$(2) \quad n! > \left(\frac{n}{2}\right)^{n/2}, 0 < \frac{1}{n\sqrt[n]{n!}} < \frac{1}{\sqrt{n/2}}, \lim_{n \rightarrow \infty} \frac{1}{n\sqrt[n]{n!}} = 0.$$



作业：习题1.3 No. 4(单),6,8

$$\text{No.8} \quad \frac{1}{2\sqrt{n}} \leq \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$