Review

- 单调收敛原理：单调有界列必收敛.
- $a_n \uparrow \quad \Rightarrow \quad \lim\limits_{n\to\infty} a_n = \sup\{a_n\} \in (-\infty, +\infty]$
- $a_{2n} \uparrow A,\ a_{2n+1} \downarrow A \quad \Rightarrow \quad \lim\limits_{n\to\infty} a_n = A.$
- 重要极限 $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e;\qquad \lim\limits_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = 1;$

$$\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1;\qquad \lim_{n\to\infty}\frac{\ln\left(1-\frac{1}{n}\right)}{-\frac{1}{n}} = 1.$$

- Stolz定理

(1) $$\left.\begin{array}{r}\{b_n\}\text{严格}\uparrow \\ \lim\limits_{n\to\infty} b_n = +\infty \\ \lim\limits_{n\to\infty}\dfrac{a_n - a_{n-1}}{b_n - b_{n-1}} = A\end{array}\right\} \Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n} = A;$$

(2) $$\left.\begin{array}{r}\{b_n\}\text{严格}\downarrow \\ \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0 \\ \lim\limits_{n\to\infty}\dfrac{a_n - a_{n-1}}{b_n - b_{n-1}} = A\end{array}\right\} \Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n} = A.$$

§ 5. 实数系的几个基本定理

Thm.(确界原理) 非空有上界的集合必有上确界.

Thm.(单调收敛原理) 单调有界列必收敛.

Thm.(闭区间套定理) 若闭区间列$[a_n, b_n]$满足条件:

(1) $[a_{n+1}, b_{n+1}] \subset [a_n, b_n] (n = 1, 2, \cdots)$,

(2) $\lim\limits_{n\to\infty}(b_n - a_n) = 0$,

则$\exists!\, \xi \in \mathbb{R}$, $s.t.$ $\xi \in \bigcap\limits_{n\geq 1}[a_n, b_n]$; $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \xi$.

Thm.(Bolzano-Weirstrass定理) 有界列必有收敛子列.

Thm.(Cauchy收敛原理) 收敛列 $\Leftrightarrow$ Cauchy列.

以上五个定理相互等价

•确界原理 $\Rightarrow$ 单调收敛原理 (前一节已证).

•单调收敛原理 $\Rightarrow$ 闭区间套定理:

存在性. 由$[a_{n+1},b_{n+1}]\subset[a_n,b_n]$, $a_n\uparrow, b_n\downarrow$, 且

$$a_1\le a_n\le b_n\le b_1,\qquad \forall n.$$

由单调收敛原理, $\lim\limits_{n\to\infty}a_n$与$\lim\limits_{n\to\infty}b_n$存在. 又$\lim\limits_{n\to\infty}(b_n-a_n)=0$, 故

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n-\lim_{n\to\infty}(b_n-a_n)=\lim_{n\to\infty}b_n\triangleq\xi.$$

若$\exists a_k>\xi$. 由$\{a_n\}$单增, 有$a_n\ge a_k,\forall n>k$. 令$n\to+\infty$, 有

$\xi=\lim\limits_{n\to\infty}a_n\ge a_k>\xi$, 矛盾. 所以$a_n\le\xi,\forall n$. 同理, $\xi\le b_n,\forall n$.

故

$$a_n\le\xi\le b_n,\forall n.$$

唯一性. 若η满足$a_n \le \eta \le b_n, \forall n$. 由极限的保序性, 有

$\lim\limits_{n\to\infty} a_n \le \eta \le \lim\limits_{n\to\infty} b_n$. 而$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \xi$, 故$\eta = \xi$.□

•闭区间套定理 $\Rightarrow$ Bolzano-Weirstrass定理(列紧性定理):

设$\{x_n\}$为有界列. 则$\exists a_1 < b_1, s.t.\ x_n \in [a_1, b_1], \forall n$. 用中点

$\dfrac{a_1+b_1}{2}$将$[a_1,b_1]$分为两个区间, 其中至少有一个含有$\{x_n\}$

中无穷多项,记之为$[a_2,b_2]$. 用中点$\dfrac{a_2+b_2}{2}$将$[a_2,b_2]$分为

两个区间, 其中至少有一个含有$\{x_n\}$中无穷多项,记之为

$[a_3,b_3]$. 如此继续,得到一列区间$[a_n,b_n], n=1,2,\cdots,$满足

$$[a_{n+1},b_{n+1}]\subset[a_n,b_n](n=1,2,\cdots),$$

$$\lim_{n\to\infty}(b_n-a_n)=\lim_{n\to\infty}\frac{b_1-a_1}{2^{n-1}}=0.$$

由闭区间套定理, $\exists!\,\xi\in\bigcap_{n\geq1}[a_n,b_n]$, 且 $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=\xi$.

$[a_1,b_1]$中包含$\{x_n\}$的无穷多项, 因此$\exists x_{n_1}\in[a_1,b_1]$. $[a_2,b_2]$中包含$\{x_n\}$的无穷多项, 因此 $\exists x_{n_2}\in[a_2,b_2]$, 且$n_2>n_1$. 依此推, $\exists x_{n_{k+1}}\in[a_{k+1},b_{k+1}]$, 且$n_{k+1}>n_k$. 由此得到$\{x_n\}$的子列$\left\{x_{n_k}\right\}$, 满足$a_k\leq x_{n_k}\leq b_k$, $\forall k$. 令$k\to\infty$, 由夹挤原理得

$$\lim_{k\to\infty}x_{n_k}=\lim_{k\to\infty}a_k=\lim_{k\to\infty}b_k=\xi.\square$$

Def.(Cauchy列)

$\{x_n\}$为Cauchy列

$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n, m > N,$ 有 $|x_n - x_m| < \varepsilon.$

$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n > N, \forall p \in \mathbb{N},$ 有 $|x_n - x_{n+p}| < \varepsilon.$

$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n \geq N, \forall p \in \mathbb{N},$ 有 $|x_n - x_{n+p}| \leq \varepsilon.$

$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n, m > N,$ 有 $|x_n - x_m| < 2\varepsilon$

Remark. $\{x_n\}$不是Cauchy列

$\Leftrightarrow \exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n, m > N, s.t. |x_n - x_m| > \varepsilon.$

$\Leftrightarrow \exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n > N, \exists p > 0, s.t. |x_{n+p} - x_n| > \varepsilon.$

Lemma. Cauchy列必为有界列.

Proof. 设$\{x_n\}$为Cauchy列. 则对$\varepsilon=1, \exists N\in\mathbb{N}, s.t. \forall n>N$, 有$|x_n-x_{N+1}|<1$. 于是

$$|x_n|\le \max\{|x_1|,|x_2|,\cdots,|x_N|,|x_{N+1}|+1\}, \quad \forall n. \square$$

•Bolzano-Weirstrass定理 $\Rightarrow$ Cauchy收敛原理:

先证Cauchy列必为收敛列. 设 $\{x_n\}$ 为Cauchy列. $\forall \varepsilon > 0$, $\exists N, s.t.$

$$|x_n - x_m| < \varepsilon, \quad \forall n, m > N.$$

Cauchy列 $\{x_n\}$ 为有界列, 由Bolzano-Weirstrass定理, $\exists$ 收敛子列 $\{x_{n_k}\}$, 设 $\lim\limits_{k\to\infty} x_{n_k} = a$. 对前面的 $\varepsilon > 0, \exists K > N, s.t.$

$$|x_{n_k} - a| < \varepsilon, \quad \forall k \geq K.$$

于是

$$|x_n - a| \leq |x_n - x_{n_K}| + |x_{n_K} - a| < 2\varepsilon, \quad \forall n > N.$$

故 $\lim\limits_{n\to\infty} x_n = a$.

再证收敛列必为Cauchy列.

设$\lim\limits_{n\to\infty} x_n = a$.则$\forall \varepsilon > 0, \exists N, s.t.$

$$|x_n - a| < \frac{\varepsilon}{2}, \quad \forall n > N.$$

于是

$$|x_n - x_m| \le |x_n - a| + |x_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n, m > N.$$

故$\{x_n\}$为Cauchy列.□

- Cauchy收敛原理 $\overset{?}{\Rightarrow}$ 确界原理

Hint:二分区间法构造闭区间套, 进而取Cauchy列.

Ex. $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $\{x_n\}$发散.

Proof. $|x_{2n} - x_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{1}{2n} \cdot n = \frac{1}{2}$,

$\{x_n\}$不为Cauchy列, 故$\{x_n\}$发散.□

Ex. $x_n = \sum_{k=1}^{n} \frac{(-1)^k}{k^2}$, $\{x_n\}$收敛.

Proof. $|x_n - x_{n+p}| \le \sum_{k=n+1}^{n+p} \frac{1}{k^2} \le \sum_{k=n+1}^{n+p} \frac{1}{k(k-1)} = \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n}$.

$\forall \varepsilon > 0, \exists N = \lceil \varepsilon^{-1} \rceil$, s.t., $\forall n > N, \forall p \in \mathbb{N}$, 有$|x_n - x_{n+p}| < \varepsilon$.

$\{x_n\}$为Cauchy列, 故$\{x_n\}$收敛.□

Question. 相较于定义, 利用Cauchy收敛原理判别数列敛散性的优势?

Question. $\lim\limits_{n\to\infty}\left|x_n - x_{n+p}\right| = 0, \forall p\in\mathbb{N} \overset{?}{\Rightarrow} \{x_n\}$为Cauchy列

No! 反例:$\{\sqrt{n}\},\{\ln n\},\left\{\sum\limits_{k=1}^{n}\frac{1}{k}\right\}$.

Question. $\left|x_n - x_{n+p}\right| \le \frac{p}{n}, \forall p, n\in\mathbb{N} \overset{?}{\Rightarrow} \{x_n\}$为Cauchy列.

No! 反例:$\left\{\sum\limits_{k=1}^{n}\frac{1}{k}\right\}$.

Question. $\left|x_n - x_{n+p}\right| \le \frac{p}{n^2}, \forall p, n \in \mathbb{N} \overset{?}{\Rightarrow} \{x_n\}$为Cauchy列. Yes!

Proof. $\left|x_n - x_{n+p}\right| \le \frac{p}{n^2}, \forall p, n \in \mathbb{N}$, 则$\left|x_n - x_{n+1}\right| \le \frac{1}{n^2}, \forall n.$

于是

$$\begin{aligned}\left|x_n - x_{n+p}\right| &\le \left|x_n - x_{n+1}\right| + \left|x_{n+1} - x_{n+2}\right| + \cdots + \left|x_{n+p-1} - x_{n+p}\right| \\ &\le \frac{1}{n^2} + \cdots + \frac{1}{(n+p-1)^2} \\ &\le \frac{1}{n(n-1)} + \cdots + \frac{1}{(n+p-1)(n+p-2)} \\ &= \frac{1}{n-1} - \frac{1}{n+p-1} < \frac{1}{n-1}, \quad \forall p, \forall n > 1. \square\end{aligned}$$

Ex. $\exists M>0, s.t. \sum_{k=1}^{n}\left|x_{k+1}-x_{k}\right| \leq M, \forall n \Rightarrow\{x_{n}\}$为Cauchy列.

Proof. 令$y_{n}=\sum_{k=1}^{n}\left|x_{k+1}-x_{k}\right|, n \in \mathbb{N}$. $\{y_{n}\}$单增有上界,$\{y_{n}\}$收敛,

$\{y_{n}\}$为Cauchy列. $\forall \varepsilon>0, \exists N, s.t.$

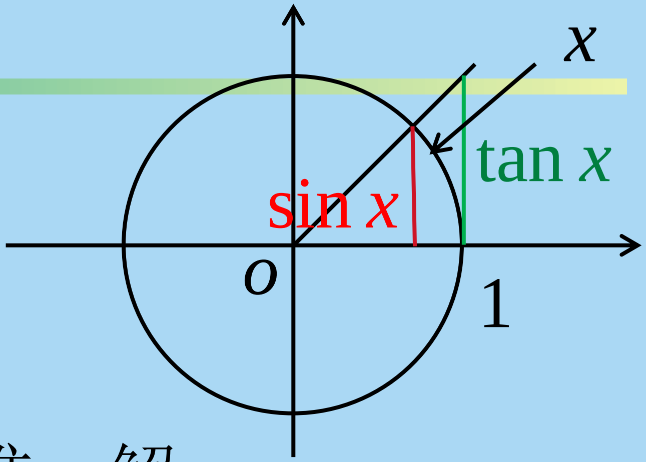
$$0 \leq y_{n+p}-y_{n}<\varepsilon, \quad \forall n>N, \forall p.$$

从而有

$$\begin{aligned}\left|x_{n+p}-x_{n}\right| &\leq\left|x_{n+p}-x_{n+p-1}\right|+\cdots+\left|x_{n+1}-x_{n}\right| \\ &=y_{n+p-1}-y_{n-1}<\varepsilon, \quad \forall n>N+1, \forall p. \square\end{aligned}$$

$$|\sin x| \le |x|, \forall x \in \mathbb{R}.$$

$$|x| \le |\tan x|, \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2})$$



Ex. $0 < a < 1, b \in \mathbb{R}$, 则$x - a\sin x = b$有唯一解.

Proof. 存在性. 令$x_0 = b$, $x_{n+1} = b + a\sin x_n$, 则

$$|x_{n+p} - x_n| = a|\sin x_{n+p-1} - \sin x_{n-1}|$$

$$= 2a\left|\sin\frac{x_{n+p-1} - x_{n-1}}{2}\cos\frac{x_{n+p-1} + x_{n-1}}{2}\right| \le a|x_{n+p-1} - x_{n-1}|$$

$$\le \cdots \le a^n|x_p - x_0| = a^{n+1}|\sin x_{p-1}| \le a^{n+1}.$$

$0 < a < 1$, 故$\{x_n\}$为Cauchy列. 设$\lim\limits_{n\to+\infty} x_n = \xi$.

$$\left|\sin x_n - \sin \xi\right| \le \left|x_n - \xi\right|,$$

由夹挤原理得

$$\lim_{n\to\infty} \sin x_n = \sin \xi.$$

在$x_{n+1} = b + a\sin x_n$中令$n \to \infty$,得 $\xi = b + a\sin \xi.$

唯一性. 设$\eta = b + a\sin \eta$,则

$$\left|\xi - \eta\right| = a\left|\sin \xi - \sin \eta\right| \le a\left|\xi - \eta\right|.$$

由$0 < a < 1$得$\xi = \eta$.□

Ex. $0 \le x_{n+m} \le x_n + x_m$, 则 $\inf\limits_{n\ge1}\{\frac{x_n}{n}\}$ 存在, 且 $\lim\limits_{n\to\infty}\frac{x_n}{n} = \inf\limits_{n\ge1}\{\frac{x_n}{n}\}$.

Proof. $0 \le \frac{x_n}{n} \le x_1$, 则 $\inf\limits_{n\ge1}\{\frac{x_n}{n}\} = \mathrm{A}$ 存在. $\forall \varepsilon > 0, \exists m, s.t.$

$$\mathrm{A} \le \frac{x_m}{m} < \mathrm{A} + \varepsilon.$$

$\forall n > m$, 有 $n = km + r, k, r \in \mathbb{Z}, 0 \le r < m$. 记 $x_0 = 0$, 由已知条件得

$$\mathrm{A} \le \frac{x_n}{n} \le \frac{kx_m + x_r}{n} = \frac{kx_m}{km+r} + \frac{x_r}{n} \le \frac{x_m}{m} + \frac{x_r}{n} \le \mathrm{A} + \varepsilon + \frac{x_r}{n}.$$

$\exists N > m, s.t. \max\limits_{0\le r<m}\{\frac{x_r}{n}\} < \varepsilon, \forall n > N$. 于是,

$$\mathrm{A} \le \frac{x_n}{n} \le \mathrm{A} + 2\varepsilon, \forall n > N. \square$$

Ex. $a_1=1, a_{n+1}=a_n+\dfrac{1}{a_1+a_2+\cdots+a_n}$, 则 $\lim\limits_{n\to\infty}\dfrac{a_n}{\sqrt{2\ln n}}=1.$

Proof. $a_{n+1}>a_n\ge 1$, $(a_1+\cdots+a_n)>n$, 由Stolz定理,

$$\lim_{n\to+\infty}\frac{a_n^2}{2\ln n}=\lim_{n\to+\infty}\frac{n(a_{n+1}^2-a_n^2)}{2n\ln(1+1/n)}=\lim_{n\to+\infty}\frac{n(a_{n+1}^2-a_n^2)}{2}$$

$$=\lim_{n\to+\infty}\frac{na_n}{(a_1+\cdots+a_n)}+\lim_{n\to+\infty}\frac{n}{2(a_1+\cdots+a_n)^2}$$

$$=\lim_{n\to+\infty}\frac{na_n}{(a_1+\cdots+a_n)}\overset{\text{Stolz}}{=\!=\!=}\lim_{n\to+\infty}\frac{(n+1)a_{n+1}-na_n}{a_{n+1}}$$

$$=\lim_{n\to+\infty}\left(n+1-\frac{n}{1+1/(a_n(a_1+\cdots+a_n))}\right)$$

$$1<n+1-\frac{n}{1+1/\left(a_n(a_1+\cdots+a_n)\right)}$$

$$<n+1-\frac{n}{1+1/(na_n)}=1+\frac{1/a_n}{1+1/(na_n)}<1+\frac{1}{a_n}.$$

$$a_{n+1}^2-a_n^2=\frac{2a_n}{a_1+\cdots+a_n}+\frac{1}{(a_1+\cdots+a_n)^2}>\frac{2a_n}{a_1+\cdots+a_n}>\frac{2}{n}.$$

$$a_{2n}^2-a_n^2=(a_{2n}^2-a_{2n-1}^2)+\cdots+(a_{n+1}^2-a_n^2)\geq\frac{2}{2n-1}+\cdots+\frac{2}{n}>1.$$

故$\left\{a_n^2\right\}$非Cauchy列,$\lim\limits_{n\to\infty}a_n^2=+\infty$, $\lim\limits_{n\to\infty}a_n=+\infty$,由夹挤原理,

$$\lim_{n\to+\infty}\frac{a_n^2}{2\ln n}=\lim_{n\to+\infty}\left(n+1-\frac{n}{1+1/\left(a_n(a_1+\cdots+a_n)\right)}\right)=1.\square$$

作业：习题1.5

No. 2(4)(5),3(2)(3),8