Review

- $\int_a^b \big(\alpha f(x)+\beta g(x)\big)\mathrm{d}x = \alpha\int_a^b f(x)\mathrm{d}x + \beta\int_a^b g(x)\mathrm{d}x.$
- $a<b<c$,则 $f\in R[a,c] \Leftrightarrow f\in R[a,b] \,\&\, f\in R[b,c].$

  此时 $\int_a^c f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x.$
- $f(x)\le g(x) \;\Rightarrow\; \int_a^b f(x)\mathrm{d}x \le \int_a^b g(x)\mathrm{d}x.$
- $f\in R[a,b] \Rightarrow |f|\in R[a,b]$,且 $\left|\int_a^b f(x)\mathrm{d}x\right| \le \int_a^b |f(x)|\mathrm{d}x.$
- $f,g\in R[a,b] \Rightarrow fg\in R[a,b].$

- $f \in R[a,b],\ f \geq 0 \Rightarrow \sqrt{f} \in R[a,b].$

- $f \in R[a,b], |f| \geq \lambda > 0 \Rightarrow \dfrac{1}{f} \in R[a,b].$

- Cauchy不等式

$$\left(\int_a^b f(x)g(x)\mathrm{d}x\right)^2 \leq \int_a^b f^2(x)\mathrm{d}x \cdot \int_a^b g^2(x)\mathrm{d}x.$$

- 积分第一中值定理

$f \in C[a,b], g \in R[a,b]$, g不变号, 则$\exists \xi \in [a,b]$, s.t.

$$\int_a^b f(x)g(x)\mathrm{d}x = f(\xi)\int_a^b g(x)\mathrm{d}x.$$

§3.微积分基本定理—Newton-Leibniz公式

Def. 称F为f在区间I上的一个原函数,若

$$F'(x)=f(x),\quad \forall x\in I.$$

Remark. 求原函数与求导互为逆运算.

Def. f在区间I上有原函数,称f在I上的原函数全体为f在I上的不定积分,记为$\int f(x)\mathrm{d}x$.

Remark. F和G都为f在区间I上的原函数,则

$$\big(F(x)-G(x)\big)'=F'(x)-G'(x)=f(x)-f(x)=0.$$

$$F(x)-G(x)\equiv const.$$

Remark.

(1) F 为 f 在区间 I 上的原函数, 则 $\int f(x)\mathrm{d}x = F(x) + C$,

C 为任意常数.

(2) $\int [\alpha f(x) + \beta g(x)]\mathrm{d}x = \alpha \int f(x)\mathrm{d}x + \beta \int g(x)\mathrm{d}x$,

(3) $\dfrac{\mathrm{d}}{\mathrm{d}x}\int f(x)\mathrm{d}x = f(x), \int f'(x)\mathrm{d}x = f(x) + C$.

$$\int \cos x\mathrm{d}x = \sin x + C, \qquad \int \sin x\mathrm{d}x = -\cos x + C,$$

$$\int \sec^2 x\mathrm{d}x = \tan x + C, \qquad \int \csc^2 x\mathrm{d}x = -\cot x + C,$$

$$\int \sec x \tan x\mathrm{d}x = \sec x + C, \qquad \int \csc x \cot x\mathrm{d}x = -\csc x + C,$$

$$\int x^{\alpha}\mathrm{d}x = \frac{x^{\alpha+1}}{\alpha+1} + C \quad (\alpha \neq -1), \qquad \int \frac{1}{x}\mathrm{d}x = \ln|x| + C,$$

$$\int a^{x}\mathrm{d}x = \frac{a^{x}}{\ln a} + C \quad (a > 0, a \neq 1),$$

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C = -\arccos x + C,$$

$$\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C = -\operatorname{arc}\cot x + C,$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C.$$

$$\text{Ex.}\int\frac{2x^2}{1+x^2}\mathrm{d}x=\int\left(2-\frac{2}{1+x^2}\right)\mathrm{d}x$$

$$=2\int\mathrm{d}x-2\int\frac{1}{1+x^2}\mathrm{d}x=2x-2\arctan x+C.$$

$$\text{Ex.}\int\frac{1}{\cos^2 x\cdot\sin^2 x}\mathrm{d}x=\int\frac{\cos^2 x+\sin^2 x}{\cos^2 x\cdot\sin^2 x}\mathrm{d}x$$

$$=\int\frac{\mathrm{d}x}{\cos^2 x}+\int\frac{\mathrm{d}x}{\sin^2 x}=\tan x-\cot x+C.$$

$$\text{Ex.}\int\frac{\mathrm{d}x}{a^2-x^2}=\frac{1}{2a}\int\left(\frac{1}{a-x}+\frac{1}{a+x}\right)\mathrm{d}x=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C.$$

Thm.(微积分基本定理)

$$f\in R[a,b], F(x)=\int_a^x f(t)\mathrm{d}t\ \ (a\le x\le b),\text{则}$$

(1)$F\in C[a,b]$;

(2)若f在$x_0\in[a,b]$连续,则F在x_0可导,且$F'(x_0)=f(x_0)$;

(3)若$f\in C[a,b]$,则F是f在$[a,b]$上的一个原函数. 若G为f的任一个原函数,则

$$\int_a^b f(t)\mathrm{d}t=G(b)-G(a)\triangleq G(x)\Big|_a^b.$$ (Newton-Leibniz)

Proof. (1) $f\in R[a,b]$, 则$\exists M>0, s.t. \left|f(x)\right|\le M\,(a\le x\le b)$.

$$\left|F(x)-F(x_0)\right|=\left|\int_{x_0}^x f(t)\mathrm{d}t\right|\le\left|\int_{x_0}^x \left|f(t)\right|\mathrm{d}t\right|\le M\left|x-x_0\right|.$$

(2) f在$x_0\in[a,b]$连续, 则$\forall\varepsilon>0,\exists\delta>0,s.t.$

$$\left|f(x)-f(x_0)\right|<\varepsilon,\quad \forall\left|x-x_0\right|<\delta,x\in[a,b].$$

于是, $\forall 0<\left|x-x_0\right|<\delta,x\in[a,b]$, 有

$$\left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right|=\left|\frac{1}{x-x_0}\int_{x_0}^{x}f(t)\mathrm{d}t-f(x_0)\right|$$

$$=\left|\frac{1}{x-x_0}\int_{x_0}^{x}\left[f(t)-f(x_0)\right]\mathrm{d}t\right|$$

$$\leq\frac{1}{\left|x-x_0\right|}\left|\int_{x_0}^{x}\left|f(t)-f(x_0)\right|\mathrm{d}t\right|\leq\varepsilon.$$

(3) $f \in C[a,b]$, 由(2)知, F是f在$[a,b]$上的一个原函数.

设G是f在$[a,b]$上的任一原函数, 则

$$\big(G(x)-F(x)\big)' = G'(x)-F'(x) = f(x)-f(x) \equiv 0, x \in [a,b].$$

因此, $\exists$常数C, s.t. $G(x) \equiv F(x)+C$, 从而

$$\int_a^x f(t)\mathrm{d}t = F(x) = F(x)-F(a) = G(x)-G(a), \forall x \in [a,b].$$

特别地, $\int_a^b f(t)\mathrm{d}t = G(b)-G(a)$.□

Question. $f \in R[a,b]$, 是否有$\left(\int_a^x f(t)\mathrm{d}t\right)' = f(x), \forall x \in [a,b]$?

Ex. 求$\int e^{|x|}\mathrm{d}x$.

解: $e^{|x|}$在$\mathbb{R}$上连续,因而$\forall x\in\mathbb{R}$, $F(x)=\int_0^x e^{|t|}\mathrm{d}t$ 有定义, 且

$$F'(x)=e^{|x|}=\begin{cases} e^x, & x\ge 0 \\ e^{-x}, & x<0 \end{cases},\quad F(x)=\begin{cases} e^x+C_1, & x\ge 0 \\ -e^{-x}+C_2, & x<0 \end{cases}.$$

$F(x)$在$x=0$处连续,则 $1+C_1=-1+C_2$,

$$\int e^{|x|}\mathrm{d}x=F(x)+C=C+\begin{cases} e^x, & x\ge 0 \\ -e^{-x}+2, & x<0 \end{cases}.\square$$

Thm. f连续, u,v可导, $G(x)=\int_{v(x)}^{u(x)} f(t)\mathrm{d}t$, 则

$$G'(x)=f(u(x))\cdot u'(x)-f(v(x))\cdot v'(x).$$

解: 令$F(u)=\int_a^u f(t)\mathrm{d}t$, 则$F'(u)=f(u)$.

$$G(x)=\int_a^{u(x)} f(t)\mathrm{d}t-\int_a^{v(x)} f(t)\mathrm{d}t$$

$$=F(u(x))-F(v(x))$$

$$G'(x)=F'(u(x))\cdot u'(x)-F'(v(x))\cdot v'(x)$$

$$=f(u(x))\cdot u'(x)-f(v(x))\cdot v'(x).\square$$

Ex. f连续, $F(x)=\int_a^x (x-t)f(t)\mathrm{d}t$, 求$F''(x)$.

解: $F(x)=x\int_a^x f(t)\mathrm{d}t-\int_a^x tf(t)\mathrm{d}t,$

$$F'(x)=\int_a^x f(t)\mathrm{d}t+xf(x)-xf(x)=\int_a^x f(t)\mathrm{d}t,\ F''(x)=f(x).\square$$

Ex. $$\lim_{x\to+\infty}\frac{\left(\int_0^x e^{t^2}\mathrm{d}t\right)^2}{\int_0^x e^{2t^2}\mathrm{d}t}=\lim_{x\to+\infty}\frac{2e^{x^2}\int_0^x e^{t^2}\mathrm{d}t}{e^{2x^2}}$$

$$=\lim_{x\to+\infty}\frac{2\int_0^x e^{t^2}\mathrm{d}t}{e^{x^2}}=\lim_{x\to+\infty}\frac{2e^{x^2}}{2xe^{x^2}}=0.\square$$

Ex.k,n为非负整数,$a\in\mathbb{R}$,则

$$\int_a^{a+2\pi}\sin kt\mathrm{d}t=0;$$

$$\int_a^{a+2\pi}\cos kt\mathrm{d}t=\begin{cases}0, & k\neq 0\\ 2\pi & k=0\end{cases};$$

$$\int_a^{a+2\pi}\sin kt\cdot\cos nt\mathrm{d}t=0;$$

$$\int_a^{a+2\pi}\sin kt\cdot\sin nt\mathrm{d}t=\begin{cases}0, & k\neq n\\ \pi, & k=n\neq 0\end{cases};$$

$$\int_a^{a+2\pi}\cos kt\cdot\cos nt\mathrm{d}t=\begin{cases}0, & k\neq n\\ \pi, & k=n\end{cases}.$$

Ex. $\sigma_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{n+n}$, 求 $\lim\limits_{n\to+\infty} \sigma_n$.

解: $\sigma_n = \dfrac{1}{n}\left[\dfrac{1}{1+\frac{1}{n}} + \dfrac{1}{1+\frac{2}{n}} + \cdots + \dfrac{1}{1+\frac{n}{n}}\right]$ 是函数 $\dfrac{1}{1+x}$ 在[0,1]上

关于n等分分割的一个Riemann和. 而 $\dfrac{1}{1+x}$ 在[0,1]上可积,

$\ln(1+x)$是 $\dfrac{1}{1+x}$ 的一个原函数, 故

$$\lim_{n\to+\infty} \sigma_n = \int_0^1 \frac{1}{1+x}\,\mathrm{d}x = \ln(1+x)\Big|_0^1 = \ln 2. \square$$

Ex. $\sigma_n = \dfrac{1}{n+1} + \dfrac{1}{n+3} + \cdots + \dfrac{1}{n+(2n+1)}$, 求 $\lim\limits_{n\to+\infty} \sigma_n$.

法一: $\lim\limits_{n\to+\infty} \sigma_n = \lim\limits_{n\to+\infty} \dfrac{1}{2} \sum\limits_{k=1}^{n} \dfrac{2}{n} \cdot \dfrac{1}{1+\dfrac{2k-1}{n}} + \lim\limits_{n\to+\infty} \dfrac{1}{n+(2n+1)}$

$$= \frac{1}{2}\int_0^2 \frac{1}{1+x}\,\mathrm{d}x + 0 = \frac{1}{2}\ln(1+x)\Big|_0^2 = \frac{1}{2}\ln 3. \square$$

法二: $\lim\limits_{n\to+\infty} \sigma_n = \lim\limits_{n\to+\infty} \sum\limits_{k=1}^{n} \dfrac{1}{n} \cdot \dfrac{1}{1+\dfrac{2(k-1/2)}{n}} + \lim\limits_{n\to+\infty} \dfrac{1}{n+(2n+1)}$

$$= \int_0^1 \frac{1}{1+2x}\,\mathrm{d}x + 0 = \frac{1}{2}\ln(1+2x)\Big|_0^1 = \frac{1}{2}\ln 3. \square$$

Ex. $\int_0^x e^t \mathrm{d}t = xe^{\theta(x)x}, \theta(x) \in (0,1)$. 求证: $\lim\limits_{x\to+\infty} \theta(x) = 1$.

Proof. $xe^{\theta(x)x} = \int_0^x e^t \mathrm{d}t = e^t\Big|_{t=0}^{x} = e^x - 1.$

$$\lim_{x\to+\infty} \theta(x) = \lim_{x\to+\infty} \frac{\ln(e^x-1) - \ln x}{x}$$

$$= \lim_{x\to+\infty} \left(\frac{e^x}{e^x-1} - \frac{1}{x} \right)$$

$$= 1. \square$$

Ex. f 可导, $f'(0)\neq 0$, $\int_0^x f(t)\mathrm{d}t = f(\xi(x))x, \xi(x)\in(0,x)$, 求 $\lim\limits_{x\to 0}\dfrac{\xi(x)}{x}$.

解: 令 $F(x)=\int_0^x f(t)\mathrm{d}t$, 则 $F'(x)=f(x), F''(x)=f'(x)$,

$$F(x)=F(0)+F'(0)x+\frac{1}{2!}F''(0)x^2+o(x^2), x\to 0.$$

于是 $$f(\xi(x))x=f(0)x+\frac{1}{2}f'(0)x^2+o(x^2),$$

$$\lim_{x\to 0}\frac{f(\xi(x))-f(0)}{x}=\lim_{x\to 0}\left(\frac{1}{2}f'(0)+o(1)\right)=\frac{1}{2}f'(0),$$

$$\lim_{x\to 0}\frac{\xi(x)}{x}=\frac{\lim\limits_{x\to 0}\dfrac{f(\xi(x))-f(0)}{x}}{\lim\limits_{x\to 0}\dfrac{f(\xi(x))-f(0)}{\xi(x)}}=\frac{\frac{1}{2}f'(0)}{f'(0)}=\frac{1}{2}.\square$$

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Ex. $f\in C^2[-1,1],\ f(0)=0$, 则 $\exists\xi\in[-1,1],\ s.t.\ f''(\xi)=3\int_{-1}^{1}f(x)\mathrm{d}x.$

证法一: $f(0)=0$, 则 $f(x)=f'(0)x+\dfrac{f''(\xi_x)}{2}x^2$, ξ_x介于0与x之间.

$f\in C^2[-1,1]$, 记$M=\max\limits_{-1\le x\le 1}f''(x),\ m=\min\limits_{-1\le x\le 1}f''(x)$, 则

$$f'(0)x+\frac{m}{2}x^2\le f(x)\le f'(0)x+\frac{M}{2}x^2,\ x\in[-1,1].$$

$$\frac{m}{3}=\frac{m}{2}\int_{-1}^{1}x^2\mathrm{d}x\le\int_{-1}^{1}f(x)\mathrm{d}x\le\frac{M}{2}\int_{-1}^{1}x^2\mathrm{d}x=\frac{M}{2}\cdot\frac{1}{3}x^3\Big|_{-1}^{1}=\frac{M}{3}.$$

$f\in C^2[-1,1]$, 由介值定理, $\exists\xi\in[-1,1],\ s.t.\ f''(\xi)=3\int_{-1}^{1}f(x)\mathrm{d}x.$

证法二: $F(x)=\int_0^x f(t)\mathrm{d}t$, 带Lagrange余项的 2阶Taylor公式.□

作业：习题5.3 No.4,5,7,
12(6),13(1),14,15