Review

- 收敛列的性质

1. 收敛列的极限唯一.
2. 改变有限项,不改变数列的敛散性与极限值.
3. 收敛列的任意子列收敛到原极限
4. 收敛列必为有界列.
5. $\lim_{n\to\infty} a_n = 0, \{b_n\}$为有界列,则$\lim_{n\to\infty} a_n b_n = 0.$
6. 极限的保序性
7. 极限的四则运算
8. 夹挤原理

- 重要极限

$$\lim_{n\to+\infty}\frac{n^b}{a^n}=0\,(a>1, b\in\mathbb{R}),$$

$$\lim_{n\to+\infty}\sqrt[n]{a}=1\,(a>0)$$

§ 4. 单调数列

Def. 称$\{a_n\}$单调递增, 若$\forall n$, 有$a_{n+1} \geq a_n$;

称$\{a_n\}$严格单调递增, 若$\forall n$, 有$a_{n+1} > a_n$;

称$\{a_n\}$单调递减, 若$\forall n$, 有$a_{n+1} \leq a_n$;

称$\{a_n\}$严格单调递减, 若$\forall n$, 有$a_{n+1} < a_n$.

Thm.(单调收敛原理) 单调有界列必收敛.

Proof. 我们来证明:

(1)若$\{a_n\}$单调递增且有上界, 则$\lim\limits_{n\to\infty} a_n = \sup\{a_n\}$;

(2)若$\{a_n\}$单调递减且有下界, 则$\lim\limits_{n\to\infty} a_n = \inf\{a_n\}$.

(1)设$\{a_n\}\uparrow$,有上界,由确界原理,$\xi=\sup\{a_n\}\in\mathbb{R}$.

下证$\lim\limits_{n\to\infty}a_n=\xi$. 由上确界定义,有

$$a_n\le\xi,\quad \forall n;$$

$$\forall\varepsilon>0,\exists a_k,s.t.\ \ \xi-\varepsilon<a_k.$$

而$\{a_n\}\uparrow$,因此

$$\xi-\varepsilon<a_k\le a_n\le\xi,\quad \forall n>k.$$

故$\lim\limits_{n\to\infty}a_n=\xi=\sup\{a_n\}$.

(2) 同理可证,$\{a_n\}\downarrow$有下界$\Rightarrow\lim\limits_{n\to\infty}a_n=\inf\{a_n\}$.□

Remark. $\{a_n\}$有上界,从某一项后单增$\Rightarrow\{a_n\}$收敛;

$\{a_n\}$有下界,从某一项后单减$\Rightarrow\{a_n\}$收敛.

Remark. $\{a_n\}\uparrow$,无上界$\Rightarrow \lim\limits_{n\to\infty} a_n = +\infty$;

$\{a_n\}\downarrow$, 无下界$\Rightarrow \lim\limits_{n\to\infty} a_n = -\infty$.

Lemma.(Bernoulli不等式) 设$x \geq -1, n$为正整数,则

$$(1+x)^n \geq 1+nx.$$

Proof. 数学归纳法, 略.$\square$

Ex. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ 存在.

Proof. 由单调收敛原理, 只要证$a_n=\left(1+\frac{1}{n}\right)^n$ 单增有上界.

$$\frac{a_n}{a_{n-1}}=\left(\frac{n+1}{n}\right)^n\cdot\left(\frac{n-1}{n}\right)^{n-1}=\left(\frac{n^2-1}{n^2}\right)^{n-1}\cdot\frac{n+1}{n}$$

$$=\left(1-\frac{1}{n^2}\right)^{n-1}\cdot\frac{n+1}{n}\geq\left(1-\frac{n-1}{n^2}\right)\cdot\frac{n+1}{n}$$

$$=\frac{n^2-n+1}{n^2}\cdot\frac{n+1}{n}=\frac{n^3+1}{n^3}>1,\quad \text{故} a_n\uparrow.$$

$$a_n = \left(1+\frac{1}{n}\right)^n = 1+\sum_{k=1}^{n} C_n^k \frac{1}{n^k}$$

$$= 1+\sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k} \le 1+\sum_{k=1}^{n}\frac{1}{k!}$$

$$\le 1+1+\frac{1}{2}+\frac{1}{6}+\sum_{k=4}^{n}\frac{1}{2^{k-1}} = 2+\frac{1}{2}+\frac{1}{6}+\frac{\frac{1}{2^3}(1-\frac{1}{2^{n-3}})}{1-\frac{1}{2}}$$

$$< 2+\frac{1}{2}+\frac{1}{6}+\frac{1}{4} = 2+\frac{11}{12} < 3,$$

a_n有上界.□

Remark. (1) $\lim\limits_{n\to+\infty}\left(1+\dfrac{1}{n}\right)^n = e \in (2, 2\dfrac{11}{12}] \subset (2,3)$;

(2) $\lim\limits_{n\to+\infty} n\ln\left(1+\dfrac{1}{n}\right) = 1$;

(3) $\lim\limits_{n\to+\infty} \dfrac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1$; (4) $\lim\limits_{n\to+\infty} \dfrac{\ln\left(1-\frac{1}{n}\right)}{-\frac{1}{n}} = 1.$

Proof of (4):

$$\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{-n}=\lim_{n\to+\infty}\left(\frac{n}{n-1}\right)^{n}=\lim_{n\to+\infty}\left(1+\frac{1}{n-1}\right)^{n}$$

$$=\lim_{n\to+\infty}\left(1+\frac{1}{n-1}\right)^{n-1}\left(\frac{n}{n-1}\right)=\lim_{n\to+\infty}\left(1+\frac{1}{n-1}\right)^{n-1}\cdot\lim_{n\to+\infty}\left(\frac{n}{n-1}\right)$$

$$=e\cdot 1\ =e.$$

因此 $\lim\limits_{n\to+\infty}\dfrac{\ln\left(1-\frac{1}{n}\right)}{-\frac{1}{n}}=1.\square$

Remark. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\dfrac{1}{k!}=e.$ 如何证明?

Hint. 记$a_n=\left(1+\frac{1}{n}\right)^n, b_n=\sum_{k=0}^{n}\frac{1}{k!}$. $b_n\uparrow, a_n\le b_n<3$(上例已证),

故b_n有极限, 设为b, 由极限的保序性得 $e\le b$. 另一方面,

任意固定$(2<)m\in\mathbb{N}$, $\forall n>m$, 有

$$a_n=1+\sum_{k=1}^{n}C_n^k\frac{1}{n^k}>2+\sum_{k=2}^{m}\frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k}$$

$$=2+\frac{1}{2!}(1-\frac{1}{n})+\cdots+\frac{1}{m!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{m-1}{n})\triangleq c_n.$$

令$n\to+\infty$, 得 $e=\lim\limits_{n\to\infty}a_n\ge\lim\limits_{n\to\infty}c_n=2+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{m!}=b_m.$

再令$m\to+\infty$, 得 $e\ge b$.□

Ex. 设$b\in\mathbb{R}, a>1$.证明:$\lim\limits_{n\to\infty}\dfrac{n^b}{a^n}=0$.

Proof. 令$x_n=\dfrac{n^b}{a^n}$,则

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\frac{1}{a}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^b=\frac{1}{a}\lim_{n\to\infty}\exp\left\{b\ln\frac{n+1}{n}\right\}=\frac{1}{a}<1.$$

由极限的保序性,$\exists N$, s.t. $\dfrac{x_{n+1}}{x_n}<1, \forall n>N$. $\{x_n\}$有下界0, 从第N项后单减,故$\{x_n\}$收敛,设$\lim\limits_{n\to\infty}x_n=x$. 又 $x_{n+1}=\dfrac{1}{a}\left(\dfrac{n+1}{n}\right)^b x_n$,

两边取极限得 $x=\dfrac{x}{a}$. 由$a>1$得$x=0$.□

Question. 能否去掉极限存在性的证明？否!考虑$\{(-1)^n\}$.

Remark. $a_{2n}\uparrow A,\ a_{2n+1}\downarrow A \Rightarrow \lim\limits_{n\to\infty} a_n = A.$（自证）

Ex. $a_1 = 1,\ a_{n+1} = 1 + \dfrac{1}{a_n}$, 证明 $\lim\limits_{n\to\infty} a_n = \dfrac{1+\sqrt{5}}{2}.$

Proof. $a_{n+1} = 1 + \dfrac{1}{a_n},\ a_1 = 1,\ a_2 = 2, a_3 = \dfrac{3}{2},\ a_4 = \dfrac{5}{3}.$

归纳可证$a_{2n}\downarrow, a_{2n+1}\uparrow$. 又$1\le a_n \le 2$,由单调收敛原理可设

$$\lim_{n\to\infty} a_{2n} = a,\ \lim_{n\to\infty} a_{2n+1} = b.$$

由极限的保序性,$1\le a\le 2, 1\le b\le 2.$

$$a_{n+2} = 1 + \frac{1}{a_{n+1}} = 1 + \frac{a_n}{1+a_n},$$

令$n = 2m \to \infty$,得

$$a = 1 + \frac{a}{1+a}, \qquad a = \frac{1+\sqrt{5}}{2}, a = \frac{1-\sqrt{5}}{2}(舍).$$

同理, 令$n = 2m+1 \to \infty$,得 $b = a = \dfrac{1+\sqrt{5}}{2}$,即

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = \frac{1+\sqrt{5}}{2}.$$

故$\displaystyle\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}$.□

Thm.(Stolz定理)

(1) (*/∞型)

$$\left.\begin{array}{r} \{b_n\}\text{严格}\uparrow \\ \lim\limits_{n\to\infty} b_n = +\infty \\ \lim\limits_{n\to\infty} \dfrac{a_n - a_{n-1}}{b_n - b_{n-1}} = \mathrm{A} \end{array}\right\} \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \mathrm{A};$$

(2) (0/0型)

$$\left.\begin{array}{r} \{b_n\}\text{严格}\downarrow \\ \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0 \\ \lim\limits_{n\to\infty} \dfrac{a_n - a_{n-1}}{b_n - b_{n-1}} = \mathrm{A} \end{array}\right\} \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \mathrm{A}.$$

Proof. $\lim\limits_{n\to\infty}\dfrac{a_n-a_{n-1}}{b_n-b_{n-1}}=A$,则 $\lambda_n\triangleq\dfrac{a_n-a_{n-1}}{b_n-b_{n-1}}-A\to 0.$

$\forall\varepsilon>0,\exists N(\varepsilon)\in\mathbb{N},\forall n>N$,有$|\lambda_n|<\varepsilon$. $\forall n>m>N$,下式

$$a_k-Ab_k=a_{k-1}-Ab_{k-1}+\lambda_k(b_k-b_{k-1})$$

对k从$m+1$到n累加,得

$$a_n-Ab_n=a_m-Ab_m+\lambda_n(b_n-b_{n-1})+\cdots+\lambda_{m+1}(b_{m+1}-b_m).$$

因为$\{b_n\}$单调,所以有

(i) $|a_n-Ab_n|\le|a_m-Ab_m|+\varepsilon|b_n-b_m|,\quad \forall n>m>N;$

(ii) $|a_m-Ab_m|\le|a_n-Ab_n|+\varepsilon|b_n-b_m|,\quad \forall n>m>N.$

(1)($*/\infty$型) 由(i),得

$$\left|\frac{a_n}{b_n}-\mathrm{A}\right| \le \frac{|a_m-\mathrm{A}b_m|}{|b_n|}+\varepsilon\frac{|b_n-b_m|}{|b_n|}, \quad \forall n>m>N.$$

$\lim\limits_{n\to\infty} b_n=+\infty$, 则对前述$\varepsilon$和$N$,取定$m>N$, $\exists N_1>m, s.t.$

$$\frac{|a_m-\mathrm{A}b_m|}{|b_n|}<\varepsilon, \quad \frac{|b_n-b_m|}{|b_n|}\le 1+\frac{|b_m|}{|b_n|}<2, \quad \forall n>N_1.$$

于是,$\left|\frac{a_n}{b_n}-\mathrm{A}\right|\le 3\varepsilon, \forall n>N_1$. 由极限的定义知$\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\mathrm{A}$.

$$\boxed{\text{(ii)}\left|a_m - \mathrm{A}b_m\right| \le \left|a_n - \mathrm{A}b_n\right| + \varepsilon\left|b_n - b_m\right|, \quad \forall n > m > N.}$$

(2)(0/0型) 由(ii),得

$$\left|\frac{a_m}{b_m} - \mathrm{A}\right| \le \frac{\left|a_n - \mathrm{A}b_n\right|}{\left|b_m\right|} + \varepsilon\left(\frac{\left|b_n\right|}{\left|b_m\right|} + 1\right), \quad \forall n > m > N.$$

对前述ε和N,任意取定$m > N$, 因为 $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0,$

$\exists N_2 > m, s.t.$ $\quad \dfrac{\left|a_n - \mathrm{A}b_n\right|}{\left|b_m\right|} < \varepsilon, \ \dfrac{\left|b_n\right|}{\left|b_m\right|} < 1, \ \forall n > N_2.$

从而有$\left|\dfrac{a_m}{b_m} - \mathrm{A}\right| \le 3\varepsilon, \ \forall m > N.$ 由极限定义知 $\lim\limits_{m\to\infty} \dfrac{a_m}{b_m} = \mathrm{A}.$□

(1)另证:利用Toeplitz数表.

令$a_0 = b_0 = 0, t_{nk} = \dfrac{b_k - b_{k-1}}{b_n}, \forall k, n,$ 则 $\sum_{k=1}^{n} t_{nk} = 1,$

$\lim_{n\to\infty} b_n = +\infty,$ 则 $\lim_{n\to\infty} t_{nk} = 0, \forall k.$ 又

$$\frac{a_n}{b_n} = \sum_{k=1}^{n} t_{nk} \frac{a_k - a_{k-1}}{b_k - b_{k-1}}, \qquad \lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = A,$$

故有 $\lim_{n\to\infty} \dfrac{a_n}{b_n} = A.$

(2)另证:$\lim_{n\to\infty} \dfrac{a_n - a_{n-1}}{b_n - b_{n-1}} = A,$ 则 $\forall \varepsilon > 0, \exists N, s.t.$

$$\mathrm{A}-\varepsilon<\frac{a_n-a_{n-1}}{b_n-b_{n-1}}<\mathrm{A}+\varepsilon,\qquad \forall n>N.$$

$\{b_n\}$严格$\downarrow$,则

$$(\mathrm{A}-\varepsilon)(b_{n-1}-b_n)<a_{n-1}-a_n<(\mathrm{A}+\varepsilon)(b_{n-1}-b_n),\forall n>N.$$

$\forall m>N,k>0,$ 上式对n从$m+1$到$m+k$累加,得

$$(\mathrm{A}-\varepsilon)(b_m-b_{m+k})<a_m-a_{m+k}<(\mathrm{A}+\varepsilon)(b_m-b_{m+k}).$$

任意固定$m>N$,令$k\to+\infty$,由$\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n=0$,得

$$(\mathrm{A}-\varepsilon)b_m\le a_m\le(\mathrm{A}+\varepsilon)b_m,\qquad \forall m>N.$$

b_m严格$\downarrow 0$, 则$b_m>0$,且$\mathrm{A}-\varepsilon\le\dfrac{a_m}{b_m}\le\mathrm{A}+\varepsilon$, $\forall m>N.\square$

Remark. Stolz定理与L'Hospital法则.

Remark. Stolz定理中其它条件不变,

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\mathrm{A} \quad \overset{?}{\Rightarrow} \quad \lim_{n\to\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}}=\mathrm{A}.$$

Hint. 考虑 $\lim\limits_{n\to\infty}\dfrac{\sin n}{n}$, $\lim\limits_{n\to\infty}\dfrac{n+\sin n}{n}$.

Ex. $\lim\limits_{n\to\infty} a_n = A$.证明： $\lim\limits_{n\to\infty} \dfrac{a_1+a_2+\cdots+a_n}{n} = A$.

Proof. 令$x_n = a_1+a_2+\cdots+a_n, y_n = n$,则

$$y_n\text{严格}\uparrow +\infty,\quad \lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} a_n = A.$$

由Stolz定理,

$$\lim_{n\to\infty}\frac{a_1+a_2+\cdots+a_n}{n} = \lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}} = A.\square$$

Ex. $x_n = \dfrac{1}{\ln n}\left(1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}\right)$, 求 $\lim\limits_{n\to\infty} x_n$.

解：令 $a_n = 1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}, b_n = \ln n$, 则 b_n 严格 $\uparrow +\infty$,

$$\lim_{n\to\infty}\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to\infty}\frac{1/n}{\ln n - \ln(n-1)} = \lim_{n\to\infty}\frac{-1/n}{\ln(1-1/n)} = 1.$$

由Stolz定理,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = 1. \square$$

Remark. $\left\{1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}-\ln n\right\}$收敛, 其极限称为Euler常数.

Ex. k为正整数, $x_n = \dfrac{1^k + 2^k + \cdots n^k}{n^{k+1}}$, 求$\lim\limits_{n\to\infty} x_n$.

解: 令$a_n = 1^k + 2^k + \cdots n^k, b_n = n^{k+1}$, 则$b_n \uparrow +\infty$.

由Stolz定理,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to\infty}\frac{n^k}{n^{k+1} - (n-1)^{k+1}}$$

$$= \lim_{n\to\infty}\frac{n^k}{n^k + n^{k-1}(n-1) + n^{k-2}(n-1)^2 + \cdots + (n-1)^k}$$

$$= \lim_{n\to\infty}\frac{1}{1 + \left(1-\frac{1}{n}\right) + \left(1-\frac{1}{n}\right)^2 + \cdots + \left(1-\frac{1}{n}\right)^k} = \frac{1}{k+1}. \square$$

Ex. $\lim\limits_{n\to\infty} a_n \sum\limits_{k=1}^{n} a_k^2 = 1$, 则 $\lim\limits_{n\to\infty} \sqrt[3]{3n}\,a_n = 1$.

Proof. 令 $T_n = \sum\limits_{k=1}^{n} a_k^2$, 则 $T_n \uparrow$, $\lim\limits_{n\to\infty} a_n T_n = 1$. 下证 $\lim\limits_{n\to\infty} T_n = +\infty$. 若不然, 则 $\lim\limits_{n\to\infty} T_n = \mathrm{A} \in (0,+\infty)$, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{a_n T_n}{T_n} = \dfrac{\lim\limits_{n\to\infty} a_n T_n}{\lim\limits_{n\to\infty} T_n} = \dfrac{1}{\mathrm{A}} \neq 0.$

对 $\varepsilon_0 = \dfrac{1}{2\mathrm{A}}$, $\exists N$, $s.t. \left| a_n - 1/\mathrm{A} \right| < \varepsilon_0$, $\forall n > N$. 于是

$$a_n > 1/\mathrm{A} - \varepsilon_0 = \varepsilon_0, \quad \forall n > N.$$

$$T_n = \sum_{k=1}^{n} a_k^2 \geq \sum_{k=N+1}^{n} a_k^2 \geq \frac{n}{2}\varepsilon_0^2, \quad \forall n > 2N.$$

令 $n \to \infty$, 得 $\lim\limits_{n\to\infty} T_n = +\infty$, 矛盾.

由$\lim\limits_{n\to\infty} T_n = +\infty$, $\lim\limits_{n\to\infty} a_n T_n = 1$, 得

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n T_n \Big/ \lim_{n\to\infty} T_n = 0.$$

又 $T_n^3 - T_{n-1}^3 = (T_n - T_{n-1})(T_n^2 + T_n T_{n-1} + T_{n-1}^2)$

$$= a_n^2\left(T_n^2 + T_n(T_n - a_n^2) + (T_n - a_n^2)^2\right)$$

$$= 3a_n^2 T_n^2 - 3a_n^4 T_n + a_n^6$$

于是, $\lim\limits_{n\to\infty} \dfrac{1}{3na_n^3} = \lim\limits_{n\to\infty} \dfrac{T_n^3}{3n} \Big/ \lim\limits_{n\to\infty} a_n^3 T_n^3 = \lim\limits_{n\to\infty} \dfrac{T_n^3}{3n}$

$$\underline{\underline{\text{Stolz}}} \lim_{n\to\infty} \frac{T_n^3 - T_{n-1}^3}{3} = \frac{1}{3} \lim_{n\to\infty}\left(3a_n^2 T_n^2 - 3a_n^4 T_n + a_n^6\right) = 1. \square$$

作业: 习题1.4

No. 3,4(2),5(2),12(1)(4),16