Review

Thm. f,g在$(x_0,x_0+\rho)$中可导, $g'(x)\neq 0$, $\lim\limits_{x\to x_0+}\dfrac{f'(x)}{g'(x)}=A$,则

(1)(0/0型) $\lim\limits_{x\to x_0+}f(x)=\lim\limits_{x\to x_0+}g(x)=0\Rightarrow\lim\limits_{x\to x_0+}\dfrac{f(x)}{g(x)}=A.$

(2)($*/\infty$型) $\lim\limits_{x\to x_0+}g(x)=\infty\Rightarrow\lim\limits_{x\to x_0+}\dfrac{f(x)}{g(x)}=A.$

Thm. f,g在$(a,+\infty)$中可导, $g'(x)\neq 0$, $\lim\limits_{x\to+\infty}\dfrac{f'(x)}{g'(x)}=\mathrm{A}$,则

(1)(0/0型) $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty} g(x)=0 \Rightarrow \lim\limits_{x\to+\infty}\dfrac{f(x)}{g(x)}=\mathrm{A}.$

(2)(*/∞型) $\lim\limits_{x\to+\infty} g(x)=\infty \Rightarrow \lim\limits_{x\to+\infty}\dfrac{f(x)}{g(x)}=\mathrm{A}.$

- 运用L'Hospital法则时注意适时分离与等价因子替换.

§ 3.Taylor公式

f在x_0可导,则有

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+o(x-x_0),\quad (x\to x_0)$$

Question. f在 x_0 处 n 阶可导,是否有更高精度的近似?
是否有 n 次多项式近似?

Question. 当$x \to x_0$时,

$$f(x) \approx P_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n,$$

系数$a_0, a_1, \cdots, a_n$应满足什么条件? 若要求

$$f(x_0) = P_n(x_0),\ f'(x_0) = P_n'(x_0),\ \cdots,\ f^{(n)}(x_0) = P_n^{(n)}(x_0),$$

则有

$$a_0 = f(x_0),\ f'(x_0) = a_1,\ \cdots,\ f^{(n)}(x_0) = n!a_n.$$

Def. f在x_0处有n阶导数, 称

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$$

为f在x_0处的n阶Taylor多项式.

Thm.(带Peano余项的Taylor公式)

f在x_0处有n阶导数,则当$x \to x_0$时,

$$f(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(x_0)(x-x_0)^k+o\left((x-x_0)^n\right).$$

$x_0=0$时,称之为Maclaurin公式.

Proof. f在x_0处有n阶导数,则f在x_0的邻域中$n-1$阶可导.

令

$$R_n(x)=f(x)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(x_0)(x-x_0)^k,$$

则$R_n(x)$在x_0的邻域中$n-1$阶可导,在x_0处n阶可导, 且

$$R_n(x_0)=R_n'(x_0)=R_n''(x_0)=\cdots=R_n^{(n)}(x_0)=0.$$

$$\lim_{x\to x_0}R_n(x)=\lim_{x\to x_0}R_n'(x)=\cdots=\lim_{x\to x_0}R_n^{(n-1)}(x)=0.$$

应用$n-1$次L'Hospital法则,得

$$\lim_{x\to x_0}\frac{R_n(x)}{(x-x_0)^n}=\lim_{x\to x_0}\frac{R_n'(x)}{n(x-x_0)^{n-1}}=\cdots=\lim_{x\to x_0}\frac{R_n^{(n-1)}(x)}{n!(x-x_0)}$$

$$=\lim_{x\to x_0}\frac{R_n^{(n-1)}(x)-R_n^{(n-1)}(x_0)}{n!(x-x_0)}=\frac{R_n^{(n)}(x_0)}{n!}=0.\square$$

Question.以上证明中为什么不用n次L'Hospital法则?

Thm.(带Lagrange余项的Taylor公式) f在(a,b)上$n+1$阶可导, $f^{(n)}\in C[a,b], x_0, x\in[a,b]$,则存在介于$x_0$与$x$之间的$\xi$, s.t.

$$f(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(x_0)(x-x_0)^k+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

Proof. 令$R_n(x)=f(x)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(x_0)(x-x_0)^k$,则 $R_n(x_0)=$

$R_n'(x_0)=R_n''(x_0)=\cdots=R_n^{(n)}(x_0)=0.$ 由Cauchy中值定理,

$$\frac{R_n(x)}{(x-x_0)^{n+1}}=\frac{R_n(x)-R_n(x_0)}{(x-x_0)^{n+1}-(x_0-x_0)^{n+1}}=\frac{R_n'(\xi_1)}{(n+1)(\xi_1-x_0)^n}$$

$$=\cdots=\frac{R_n^{(n+1)}(\xi_{n+1})}{(n+1)!}=\frac{f^{(n+1)}(\xi_{n+1})}{(n+1)!}.\square$$

Thm.(Taylor多项式的唯一性) f 在 x_0 处有 n 阶导数,存在不高于n 次的多项式$Q_n(x)$, s.t.

$$f(x)=Q_n(x)+o\left((x-x_0)^n\right)\quad (x\to x_0),$$

则$Q_n(x)=P_n(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(x_0)(x-x_0)^k.$

Proof.由带Peano余项的Taylor公式,

$$f(x)=P_n(x)+o\left((x-x_0)^n\right)\quad (x\to x_0).$$

因而 $Q_n(x)-P_n(x)=o\left((x-x_0)^n\right).$

记$Q_n(x)-P_n(x)=b_0+b_1(x-x_0)+\cdots+b_n(x-x_0)^n$,则

$$b_0+b_1(x-x_0)+\cdots+b_n(x-x_0)^n=o\left((x-x_0)^n\right)\quad(x\to x_0).$$

令$x\to x_0$,得 $b_0=0$,

$$b_1(x-x_0)+\cdots+b_n(x-x_0)^n=o\left((x-x_0)^n\right)\quad(x\to x_0),$$

$$b_1+b_2(x-x_0)+\cdots+b_n(x-x_0)^{n-1}=o\left((x-x_0)^{n-1}\right)\quad(x\to x_0).$$

令$x\to x_0$,得 $b_1=0$,

$$b_2(x-x_0)+\cdots+b_n(x-x_0)^{n-1}=o\left((x-x_0)^{n-1}\right)\quad(x\to x_0).$$

依此类推, 得

$$b_0=b_1=\cdots=b_n=0,$$

$$Q_n(x)-P_n(x)=0.\square$$

Remark. 若 $\lim\limits_{x\to x_0} g(x)=t_0$, f 在 t_0 处有 n 阶Taylor公式

$$f(t)=P_n(t)+o\left((t-t_0)^n\right),\ (t\to t_0)$$

则 $$f(g(x))=P_n(g(x))+o\left((g(x)-t_0)^n\right), x\to x_0.$$

Proof. $\forall\varepsilon>0$, 由 f 在 t_0 处有 n 阶Taylor公式, $\exists\delta>0$, s.t.

$$\left|f(t)-P_n(t)\right|\le\varepsilon\left|(t-t_0)^n\right|,\quad \forall\left|t-t_0\right|<\delta.$$

而 $\lim\limits_{x\to x_0} g(x)=t_0$, 因此对此 $\delta>0$, $\exists\eta>0$, s.t.

$$\left|g(x)-t_0\right|<\delta,\quad \forall 0<\left|x-x_0\right|<\eta.$$

从而有 $\left|f(g(x))-P_n(g(x))\right|\le\varepsilon\left|g(x)-t_0\right|^n, \forall 0<\left|x-x_0\right|<\eta.$ □

Ex. $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + o(x^n), \quad x \to 0.$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^{\xi}}{(n+1)!} x^{n+1}, \xi\text{介于}0, x\text{之间}.$$

Ex. $\sin x = x - \dfrac{x^3}{3!} + \cdots + (-1)^{n-1} \dfrac{x^{2n-1}}{(2n-1)!} + o(x^{2n}), \quad x \to 0.$

$(2n\text{阶})$

$$\sin x = x - \frac{x^3}{3!} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + o(x^{2n-1}), \quad x \to 0.$$

$(2n-1\text{阶})$

$$= x - \frac{x^3}{3!} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \frac{\sin(\xi + \frac{2n+1}{2}\pi)}{(2n+1)!} x^{2n+1}, \quad x \to 0.$$

Ex. $\cos x = 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n}), \quad x \to 0.$

$$\cos x = 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1}), \quad x \to 0.$$

Ex. $\ln(1+x) = x - \frac{x^2}{2} + \cdots + (-1)^{n-1} \frac{x^n}{n} + o(x^n), \quad x \to 0.$

Ex. $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \cdots$

$$+ \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n + o(x^n), \quad x \to 0.$$

Ex. $\dfrac{1}{1+x}=1-x+x^2+\cdots+(-1)^n x^n+o(x^n),\quad x\to 0.$

Ex. $\dfrac{1}{1-x}=1+x+x^2+\cdots+x^n+o(x^n),\quad x\to 0.$

Ex. $\dfrac{1}{2x-x^2}$在$x_0=1$处带Peano余项的Taylor公式及$f^{(100)}(1)$.

解: $f(x)=\dfrac{1}{1-(x-1)^2}$

$$=1+(x-1)^2+(x-1)^4+\cdots+(x-1)^{2n}+o((x-1)^{2n}),\quad x\to 1.$$

$$f^{(100)}(1)=100!.$$

Remark.间接展开法求Taylor公式.

Ex. $f(x)=e^{\sin^2 x},\ x_0=0$, 4阶Peano.

$$\sin x = x-\frac{1}{6}x^3+\cdots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+o(x^{2n}) \qquad (x\to 0),$$

$$e^t=1+t+\frac{t^2}{2!}+\cdots+\frac{t^n}{n!}+o(t^n)\quad (t\to 0).$$

$$e^{\sin^2 x}=1+\sin^2 x+\frac{\sin^4 x}{2!}+o(\sin^4 x)$$

$$=1+\left(x-\frac{1}{6}x^3+o(x^3)\right)^2+\frac{1}{2!}\left(x+o(x)\right)^4+o(x^4)$$

$$=1+x^2-\frac{1}{3}x^4+o(x^4)+\frac{1}{2}x^4+o(x^4)$$

$$=1+x^2+\frac{1}{6}x^4+o(x^4)\quad (x\to 0).\square$$

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Ex. $\lim\limits_{x\to 0+}\dfrac{e^{\sin^2 x}-\cos 2\sqrt{x}-2x}{x^2}$

Question.展开到哪一阶?

解: $\cos 2\sqrt{x}=1-\dfrac{4x}{2!}+\dfrac{16x^2}{4!}+o(x^2)\quad (x\to 0)$

$$e^{\sin^2 x}=1+\sin^2 x+o(\sin^2 x)\qquad (x\to 0)$$

$$=1+\left(x+o(x)\right)^2+o(x^2)\qquad (x\to 0)$$

$$=1+x^2+o(x^2)\qquad (x\to 0)$$

$$\text{原式}=\lim_{x\to 0+}\frac{1+x^2-(1-2x+\frac{2}{3}x^2)-2x+o(x^2)}{x^2}=\frac{1}{3}.\square$$

Ex. $\lim\limits_{x\to 0}\dfrac{e^{ax^k}-\cos x^2}{x^8}$ 存在, 求 a,k 及极限值.

解: $x\to 0$ 时, $\cos x^2=1-\dfrac{x^4}{2!}+\dfrac{x^8}{4!}+o(x^8)$,

$$e^{ax^k}=1+ax^k+\frac{1}{2!}a^2x^{2k}+o(x^{2k}).$$

$$e^{ax^k}-\cos x^2=ax^k+\frac{x^4}{2!}+\frac{1}{2!}a^2x^{2k}-\frac{x^8}{4!}+o(x^8)+o(x^{2k})$$

原极限存在, 则 $ax^k+\dfrac{x^4}{2!}=0$, $k=4, a=-\dfrac{1}{2}$,

$$\text{原极限}=\lim_{x\to 0}\frac{\frac{1}{8}x^8-\frac{1}{4!}x^8+o(x^8)}{x^8}=\frac{1}{12}.\square$$

Ex. $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin^2 x}-\dfrac{1}{x^2}\right)$

$$=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}=\lim_{x\to 0}\frac{x^2-\left(x-\frac{1}{6}x^3+o(x^3)\right)^2}{x^4}$$

$$=\lim_{x\to 0}\frac{x^2-\left(x^2-\frac{1}{3}x^4+o(x^4)\right)}{x^4}$$

$$=\lim_{x\to 0}\frac{\frac{1}{3}x^4+o(x^4)}{x^4}=\frac{1}{3}.\square$$

Ex. $\lim\limits_{x\to\infty}(1+1/x)^{x^2}e^{-x}$

$$=\lim_{x\to\infty}\exp\left\{x^2\ln(1+1/x)-x\right\}$$

$$=\lim_{x\to\infty}\exp\left\{x^2\left(\frac{1}{x}-\frac{1}{2x^2}+o(\frac{1}{x^2})\right)-x\right\}$$

$$=\lim_{x\to\infty}\exp\left\{-\frac{1}{2}+o(1)\right\}$$

$$=e^{-1/2}.$$

Ex. $a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{(-1)^{n-1}}{n}$, 求 $\lim\limits_{n\to\infty} a_n$

解法一. $\ln(1+x) = x - \frac{x^2}{2} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \frac{(-1)^n x^{n+1}}{(n+1)(1+\xi_n)^{n+1}}$.

令 $x=1$, 得 $\quad \ln 2 = a_n + \frac{(-1)^n}{(n+1)(1+\xi_n)^{n+1}}, \xi_n \in (0,1)$.

$$\lim_{n\to\infty} a_n = \ln 2 - \lim_{n\to\infty} \frac{(-1)^n}{(n+1)(1+\xi_n)^{n+1}} = \ln 2.$$

解法二. 令 $b_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$, 由 $\frac{1}{n+1} \le \ln(1+\frac{1}{n}) \le \frac{1}{n}$

可得

$$b_{n+1}-b_n=\frac{1}{n+1}-\ln(1+\frac{1}{n})<0,$$

$$b_n>\ln 2+\ln\frac{3}{2}+\ln\frac{4}{3}+\cdots+\ln\frac{n+1}{n}-\ln n=\ln(n+1)-\ln n>0.$$

由单调收敛原理，$\{b_n\}$收敛，设$\lim\limits_{n\to\infty}b_n=\gamma$. 于是

$$a_{2n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}-2(\frac{1}{2}+\frac{1}{4}+\cdots\frac{1}{2n})$$

$$=b_{2n}+\ln(2n)-b_n-\ln n\ \to\ln 2,n\to\infty\text{时}.$$

而$\lim\limits_{x\to\infty}a_{2n+1}=\lim\limits_{x\to\infty}(a_{2n}+\frac{1}{2n+1})=\lim\limits_{x\to\infty}a_{2n}=\ln 2$，故$\lim\limits_{x\to\infty}a_n=\ln 2$.□

Ex. 证明e是无理数.

Proof. 反设$e=\dfrac{m}{n}, m,n>0$, 互质.

$$e^t=1+t+\frac{t^2}{2!}+\cdots+\frac{t^n}{n!}+\frac{e^{\theta t}t^{n+1}}{(n+1)!},\quad 0<\theta<1.$$

令$t=1$, 得 $e=2+\dfrac{1}{2!}+\cdots+\dfrac{1}{n!}+\dfrac{e^{\theta}}{(n+1)!}>2,\quad 0<\theta<1.$

于是, $\dfrac{e^{\theta}}{n+1}=n!\left(e-2-\dfrac{1}{2!}-\cdots-\dfrac{1}{n!}\right)$为正整数.

而$\theta\in(0,1), e^{\theta}\in(1,e)$, 所以$n+1=2, n=1, e=m\in\mathbb{Z}$,

与$e=\lim\limits_{k\to\infty}(1+\dfrac{1}{k})^k\in(2,3)$矛盾.□

Ex. f 在$[-1,1]$上三阶可导, $f(1)=1,\ f(-1)=0,\ f'(0)=0$, 则 $\exists\xi\in(-1,1), s.t.\ f'''(\xi)=3.$

Proof. $\exists\xi_1\in(0,1),\xi_2\in(-1,0), s.t.$

$$1=f(1)=f(0)+f'(0)+\frac{1}{2!}f''(0)+\frac{1}{3!}f'''(\xi_1)$$

$$0=f(-1)=f(0)-f'(0)+\frac{1}{2!}f''(0)-\frac{1}{3!}f'''(\xi_2)$$

两式相减,由$f'(0)=0$得

$$3=\frac{1}{2}\left(f'''(\xi_1)+f'''(\xi_2)\right).$$

由Darboux定理,$\exists\xi\in[\xi_2,\xi_1]\subset(-1,1), s.t.\ f'''(\xi)=3.$ □

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Ex. $(1)\forall x\in\mathbb{R},\left|f(x)\right|\le M_0,\left|f''(x)\right|\le M_2$,则$\left|f'(x)\right|\le\sqrt{2M_0M_2}$.

$(2)\forall c\in(0,1),\left|f(c)\right|\le M_0,\left|f''(c)\right|\le M_2$,则$\left|f'(c)\right|\le 2M_0+\dfrac{1}{2}M_2$.

Proof. $(1)\forall x\in\mathbb{R},\forall h>0,$

$$f(x+h)=f(x)+f'(x)h+\frac{f''(\xi_1)}{2}h^2,$$

$$f(x-h)=f(x)-f'(x)h+\frac{f''(\xi_2)}{2}h^2.$$

两式相减,得

$$2f'(x)h=f(x+h)-f(x-h)+\frac{h^2}{2}\left(f''(\xi_2)-f''(\xi_1)\right)$$

$$|f'(x)| \le \frac{|f(x+h)-f(x-h)|}{2h} + \frac{h}{4}|f''(\xi_2)-f''(\xi_1)|$$

$$\le \frac{M_0}{h} + \frac{h}{2}M_2, \qquad \forall x \in \mathbb{R}, \forall h > 0.$$

令$h=\sqrt{2M_0/M_2}$,得 $|f'(x)| \le \sqrt{2M_0M_2}, \quad \forall x \in \mathbb{R}.$

(2) $\forall c \in (0,1), \forall \delta \in (0,\frac{1}{2})$,有

$$f(1-\delta) = f(c) + f'(c)(1-\delta-c) + \frac{f''(\xi_1)}{2}(1-\delta-c)^2,$$

$$f(\delta) = f(c) + f'(c)\cdot(\delta-c) + \frac{f''(\xi_2)}{2}\cdot(\delta-c)^2,$$

两式相减,得

$$(1-2\delta)f'(c)=f(1-\delta)-f(\delta)-\frac{1}{2}\left(f''(\xi_1)(1-\delta-c)^2-f''(\xi_2)(\delta-c)^2\right),$$

$$|f'(c)|\le\frac{1}{1-2\delta}|f(1-\delta)-f(\delta)|$$

$$+\frac{1}{2(1-2\delta)}\left(|f''(\xi_1)|(1-\delta-c)^2+|f''(\xi_2)|(\delta-c)^2\right)$$

$$\le\frac{2M_0}{1-2\delta}+\frac{M_2}{2(1-2\delta)}\left((1-\delta-c)^2+(\delta-c)^2\right)$$

令$\delta\to 0^+$,得

$$|f'(c)|\le 2M_0+\frac{M_2}{2}\left((1-c)^2+c^2\right)\le 2M_0+\frac{M_2}{2},\forall c\in(0,1).\square$$

Ex. 已知$xy-e^x+e^y=0$确定了隐函数$y=y(x)$，求$y(x)$在$x_0=0$处的2阶Maclaurin展开式.

解: $$y(x)=y(0)+y'(0)x+\frac{y''(0)}{2!}x^2+o(x^2).$$

由$xy-e^x+e^y=0$得 $y(0)=0.$

求导得 $y+xy'-e^x+e^y\cdot y'=0,\ y'(0)=1.$

再求导得

$$2y'+xy''-e^x+e^y\cdot\left(y'\right)^2+e^y\cdot y''=0,\ y''(0)=-2.$$

故$y(x)=x-x^2+o(x^2)$.□

Ex. $f(x)=f(0)+f'(0)x+\frac{1}{2}f''(\theta(x)x)x^2, \theta(x)\in(0,1).$

若$f'''(0)\neq 0$,则$\lim\limits_{x\to 0}\theta(x)=1/3.$

Proof. $f(x)=f(0)+f'(0)x+\frac{1}{2}f''(\theta(x)x)x^2,$

$$f(x)=f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+\frac{1}{3!}f'''(0)x^3+o(x^3), x\to 0.$$

于是,

$$\lim_{x\to 0}\frac{f''(\theta(x)x)-f''(0)}{x}=\lim_{x\to 0}\left(\frac{f'''(0)}{3}+o(1)\right)=\frac{f'''(0)}{3}.$$

而$\lim\limits_{x\to 0}\dfrac{f''(\theta(x)x)-f''(0)}{\theta(x)x}=f'''(0)\neq 0$, 故$\lim\limits_{x\to 0}\theta(x)=\dfrac{1}{3}.\square$

作业：习题4.3

No.5(2)(4),7(1),8,10