Review

- $\int_a^b f(x)\mathrm{d}x = \mathrm{I}: \forall\varepsilon>0, \exists\delta>0,$ 当 $|T|<\delta$ 时，无论 $\xi_i\in[x_{i-1},x_i]$ 如何取，都有 $$\left|\sum_{i=1}^{n} f(\xi_i)\Delta x_i - \mathrm{I}\right| < \varepsilon.$$

- Darboux上和 $U(f,T)=\sum_{i=1}^{n} M_i\Delta x_i, \qquad M_i \triangleq \sup\limits_{x\in[x_{i-1},x_i]} f(x),$

  Darboux下和 $L(f,T)=\sum_{i=1}^{n} m_i\Delta x_i, \qquad m_i \triangleq \inf\limits_{x\in[x_{i-1},x_i]} f(x).$

  Riemann和 $\sigma(f,T,\{\xi_i\})=\sum_{i=1}^{n} f(\xi_i)\Delta x_i.$

  $$m(b-a)\le L(f,T)\le \sigma(f,T,\{\xi_i\})\le U(f,T)\le M(b-a).$$

- 在T中加入k个新分点得到T_k, 则

$$0 \le U(f,T) - U(f,T_k) \le k|T|(M-m);$$

$$0 \le L(f,T_k) - L(f,T) \le k|T|(M-m).$$

- $L(f,T_1) \le U(f,T_2).$

- Darboux上积分 : $\overline{\int_a^b} f(x)\mathrm{d}x = \inf\{U(f,T) : T为[a,b]的分割\},$

Darboux下积分 : $\underline{\int_a^b} f(x)\mathrm{d}x = \sup\{L(f,T) : T为[a,b]的分割\}.$

- $L(f,T) \le \underline{\int_a^b} f(x)\mathrm{d}x \le \overline{\int_a^b} f(x)\mathrm{d}x \le U(f,T).$

- f在$[a,b]$有界,则

$$f \in R[a,b];$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists [a,b]\text{的分割}T, s.t.\ U(f,T) - L(f,T) < \varepsilon;$$

$$\Leftrightarrow \underline{\int_a^b} f(x)\mathrm{d}x = \overline{\int_a^b} f(x)\mathrm{d}x.$$

- $[a,b]$上的可积函数类.

Remark. 修改有限个点处的函数值,不改变有界闭区间上函数的Riemann可积性.

Remark. 有界闭区间上的无界函数不是Riemann可积的. 我们将来会讨论无界函数的广义Riemann可积性.

§2.Riemann积分的性质

Prop1. (线性性质)

$$f,g\in R[a,b],\alpha,\beta\in\mathbb{R}\Rightarrow \alpha f+\beta g\in R[a,b],\text{且}$$

$$\int_a^b\big(\alpha f(x)+\beta g(x)\big)\mathrm{d}x=\alpha\int_a^b f(x)\mathrm{d}x+\beta\int_a^b g(x)\mathrm{d}x.$$

Proof. $\lim\limits_{|T|\to 0}\sigma(\alpha f+\beta g,T,\{\xi_i\})$

$$=\lim_{|T|\to 0}\alpha\sigma(f,T,\{\xi_i\})+\lim_{|T|\to 0}\beta\sigma(g,T,\{\xi_i\}).\square$$

Prop2. (积分区间的可加性) $a<b<c$,则

$$f\in R[a,c]\Leftrightarrow f\in R[a,b]\,\&\, f\in R[b,c].$$

此时$\int_a^c f(x)\mathrm{d}x=\int_a^b f(x)\mathrm{d}x+\int_b^c f(x)\mathrm{d}x.$

Proof. $\Leftarrow$:设$f\in R[a,b], f\in R[b,c]$.$\forall\varepsilon>0,\exists[a,b]$的分割$T_1$, $[b,c]$的分割T_2, s.t.

$$U(f,T_1)-L(f,T_1)<\varepsilon,\quad U(f,T_2)-L(f,T_2)<\varepsilon.$$

合并T_1,T_2的分点得到$[a,c]$的分割T,则

$$U(f,T)-L(f,T)$$
$$=U(f,T_1)-L(f,T_1)+U(f,T_2)-L(f,T_2)<2\varepsilon.$$

$\Rightarrow$:设$f \in R[a,c]$. 则$\forall \varepsilon > 0, \exists [a,c]$的分割$T_0$, s.t.

$$U(f,T_0) - L(f,T_0) < \varepsilon.$$

在T_0中添加分点b得到$[a,c]$的分割T, 则

$$U(f,T) - L(f,T) \le U(f,T_0) - L(f,T_0) < \varepsilon.$$

T在$[a,b],[b,c]$上的限制分别记为T_1, T_2, 则

$$U(f,T_i) - L(f,T_i) \le U(f,T) - L(f,T) < \varepsilon, \quad i = 1,2.$$

故$f \in R[a,b], f \in R[b,c]$.

至此,我们证明了$f \in R[a,c] \Leftrightarrow f \in R[a,b]$ & $f \in R[b,c]$.

下证 $\int_a^c f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x.$

设T_1, T_2分别为$[a,b],[b,c]$上的分割,合并T_1, T_2的分点得到$[a,c]$上的分割T. 则 $|T| \to 0 \Leftrightarrow |T_i| \to 0, i=1,2,$ 且

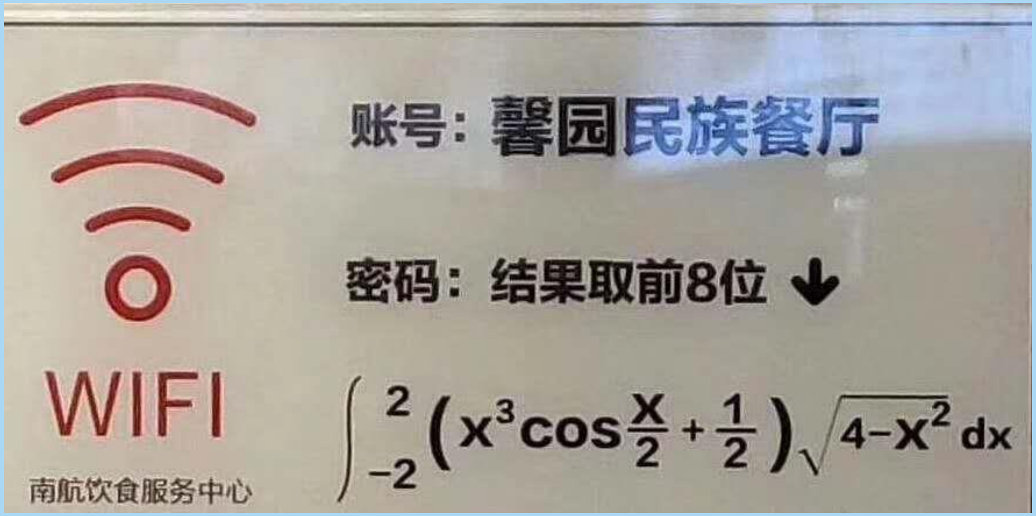
$$\int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x$$

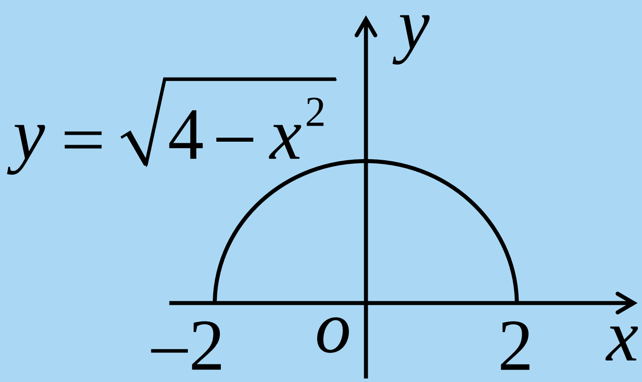
$$= \lim_{|T_1|\to 0} \sigma(f, T_1, \{\xi_i\}) + \lim_{|T_2|\to 0} \sigma(f, T_2, \{\eta_i\})$$

$$= \lim_{|T|\to 0} \sigma(f, T, \{\xi_i\} \cup \{\eta_i\}) = \int_a^c f(x)\mathrm{d}x. \square$$

Prop3. $f\in R[-a,a]$, f是奇函数, 则$\int_{-a}^{a} f(x)\mathrm{d}x=0$;

$f\in R[-a,a]$, f是偶函数, 则$\int_{-a}^{a} f(x)\mathrm{d}x=2\int_{0}^{a} f(x)\mathrm{d}x$.





$$=0+\frac{1}{2}\int_{-2}^{2}\sqrt{4-x^2}\,\mathrm{d}x=\pi=3.1415926\cdots$$

Prop4. (单调性) $f, g \in R[a,b]$, 且$f(x) \le g(x)$,则

$$\int_a^b f(x)\mathrm{d}x \le \int_a^b g(x)\mathrm{d}x.$$

特别地,若$m \le f(x) \le M\,(a \le x \le b)$,则

$$m(b-a) \le \int_a^b f(x)\mathrm{d}x \le M(b-a).$$

Proof. $g(x) - f(x) \ge 0 (a \le x \le b)$,则

$$\int_a^b g(x)\mathrm{d}x - \int_a^b f(x)\mathrm{d}x = \int_a^b (g(x) - f(x))\mathrm{d}x$$

$$= \lim_{|T|\to 0} \sigma(g-f, T, \{\xi_i\}) \ge 0.\square$$

Prop5.(积分估值) $f\in R[a,b]\Rightarrow |f|\in R[a,b]$, 且

$$\left|\int_a^b f(x)\mathrm{d}x\right|\le\int_a^b|f(x)|\mathrm{d}x.$$

Proof. $U(|f|,T)-L(|f|,T)\le U(f,T)-L(f,T)$.$\square$

Question. $|f|\in R[a,b]\overset{?}{\Rightarrow} f\in R[a,b]$

$$f=\begin{cases}1, & x\in\mathbb{Q}\bigcap[0,1],\\ -1, & x\in[0,1]\setminus\mathbb{Q}.\end{cases}\qquad f\notin R[0,1].$$

$$|f|=1,\forall x\in[0,1],\quad |f|\in R[0,1].$$

Prop6. $f, g \in R[a,b] \Rightarrow fg \in R[a,b]$.

Proof. 1)设 $f \geq 0, g \geq 0$. 记$M = \sup\limits_{a \leq x \leq b} f(x), N = \sup\limits_{a \leq x \leq b} g(x)$.

任给$T: a = x_0 < x_1 < \cdots < x_n = b$, 记

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x),\quad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x),$$

$$N_i = \sup_{x_{i-1} \leq x \leq x_i} g(x),\quad n_i = \inf_{x_{i-1} \leq x \leq x_i} g(x).$$

$$U(fg,T) - L(fg,T) \leq \sum_{i=1}^{n} (M_i N_i - m_i n_i) \Delta x_i$$

$$= \sum_{i=1}^{n} M_i (N_i - n_i) \Delta x_i + \sum_{i=1}^{n} n_i (M_i - m_i) \Delta x_i$$

$$\leq M\left[U(g,T) - L(g,T)\right] + N\left[U(f,T) - L(f,T)\right].$$

2)对任意 f, 记$f^{\pm}(x)=\frac{1}{2}\left[|f(x)|\pm f(x)\right]$,则

$$f^{\pm}\geq 0,\quad f^{\pm}\in R[a,b],\quad f=f^{+}-f^{-},$$

于是 $$fg=(f^{+}-f^{-})(g^{+}-g^{-})$$

$$=f^{+}g^{+}-f^{+}g^{-}-f^{-}g^{+}+f^{-}g^{-}\in R[a,b].\square$$

Prop7. $f\in R[a,b],|f|\geq\lambda>0\Rightarrow\frac{1}{f}\in R[a,b].$

Proof. $$\left|\frac{1}{f(x)}-\frac{1}{f(y)}\right|=\left|\frac{f(x)-f(y)}{f(x)f(y)}\right|\leq\frac{1}{\lambda^{2}}|f(x)-f(y)|.$$

$$U(\frac{1}{f},T)-L(\frac{1}{f},T)\leq\frac{1}{\lambda^{2}}\big(U(f,T)-L(f,T)\big).\ \square$$

Prop8. $f \in R[a,b],\ f \geq 0 \Rightarrow \sqrt{f} \in R[a,b].$

Proof. $\left(U(\sqrt{f},T)-L(\sqrt{f},T)\right)^2=\left(\sum_{i=1}^{n}(\sqrt{M_i}-\sqrt{m_i})\sqrt{\Delta x_i}\sqrt{\Delta x_i}\right)^2$

$$\leq \sum_{i=1}^{n}(\sqrt{M_i}-\sqrt{m_i})^2\Delta x_i \cdot \sum_{i=1}^{n}\Delta x_i$$

$$=(b-a)\sum_{i=1}^{n}(\sqrt{M_i}-\sqrt{m_i})^2\Delta x_i$$

$$\leq (b-a)\sum_{i=1}^{n}(M_i-m_i)\Delta x_i=(b-a)\left(U(f,T)-L(f,T)\right).\square$$

Ex. 设$f \in C[a,b]$, $f(x) \ge 0$, $\int_a^b f(x)\mathrm{d}x = 0$. 求证: $f(x) \equiv 0$.

Proof. 反证. 设$f(x)$不恒为0, 则$\exists x_0 \in [a,b]$, s.t. $f(x_0) > 0$. 不妨设$x_0 \in (a,b)$. $f \in C[a,b]$, 则$\exists \delta > 0$, s.t. $N_\delta(x_0) \subset [a,b]$, 且

$$f(x) > f(x_0)/2 > 0, \quad \forall x \in [x_0 - \delta, x_0 + \delta].$$

而$f(x) \ge 0$, 于是

$$0 = \int_a^b f(x)\mathrm{d}x = \int_a^{x_0-\delta} f(x)\mathrm{d}x + \int_{x_0-\delta}^{x_0+\delta} f(x)\mathrm{d}x + \int_{x_0+\delta}^b f(x)\mathrm{d}x$$

$$\ge 0 + \int_{x_0-\delta}^{x_0+\delta} \frac{f(x_0)}{2}\mathrm{d}x + 0 \ge f(x_0)\delta > 0,$$ 矛盾. □

Remark. 设$f, g \in C[a,b]$, $f(x) \geq g(x)$, $f \neq g$, 则

$$\int_a^b f(x)\mathrm{d}x > \int_a^b g(x)\mathrm{d}x.$$

Thm.(Cauchy不等式) $f, g \in R[a,b]$, 则

$$\left(\int_a^b f(x)g(x)\mathrm{d}x\right)^2 \leq \int_a^b f^2(x)\mathrm{d}x \cdot \int_a^b g^2(x)\mathrm{d}x.$$

Proof. 令$A = \int_a^b f^2(x)\mathrm{d}x$, $B = \int_a^b f(x)g(x)\mathrm{d}x$, $C = \int_a^b g^2(x)\mathrm{d}x$.

则 $0 \leq \int_a^b [tf(x) + g(x)]^2 \mathrm{d}x = At^2 + 2Bt + C, \quad \forall t \in \mathbb{R}.$

故 $(2B)^2 - 4AC \leq 0.$ □

证法二: $\left(\sigma(fg, T, \{\xi_i\})\right)^2 \leq \sigma(f^2, T, \{\xi_i\}) \cdot \sigma(g^2, T, \{\xi_i\}).$

Thm.(积分第一中值定理) $f\in C[a,b], g\in R[a,b]$, g不变号,

则$\exists\xi\in[a,b]$, s.t. $\int_a^b f(x)g(x)\mathrm{d}x = f(\xi)\int_a^b g(x)\mathrm{d}x.$

特别地, $g(x)\equiv 1$时, $\int_a^b f(x)\mathrm{d}x = f(\xi)(b-a).$

Proof.不妨设$g\geq 0$. $f\in C[a,b]$,则f在$[a,b]$上有最大值M与最小值m,从而

$$m\int_a^b g(x)\mathrm{d}x \leq \int_a^b f(x)g(x)\mathrm{d}x \leq M\int_a^b g(x)\mathrm{d}x.$$

因此, $\exists\lambda\in[m,M]$, s.t. $\int_a^b f(x)g(x)\mathrm{d}x = \lambda\int_a^b g(x)\mathrm{d}x.$

由连续函数的介值定理, $\exists\xi\in[a,b]$, s.t. $f(\xi)=\lambda$. □

Question. g变号, 反例?

提示: 构造$\int_a^b g(x)\mathrm{d}x=0, \int_a^b f(x)g(x)\mathrm{d}x>0.$

Ex. $f\in R[a,b]$, $f\geq 0$, $\int_a^b xf(x)\mathrm{d}x=0$, 则

$$\int_a^b x^2 f(x)\mathrm{d}x\leq -ab\int_a^b f(x)\mathrm{d}x.$$

Proof. $(x-a)(x-b)f(x)\leq 0, \forall x\in[a,b].$

$$0\geq \int_a^b (x-a)(x-b)f(x)\mathrm{d}x$$
$$=\int_a^b x^2 f(x)\mathrm{d}x-(a+b)\int_a^b xf(x)\mathrm{d}x+ab\int_a^b f(x)\mathrm{d}x$$
$$=\int_a^b x^2 f(x)\mathrm{d}x+ab\int_a^b f(x)\mathrm{d}x. \square$$

Ex. $x \to 0$时, $f(x)=\int_{\sin x}^{x} \ln(1+e^t)\mathrm{d}t$ 与x^p是同阶无穷小, 则

$p=$ ________, 此时$\lim\limits_{x\to 0} f(x)/x^p =$ ________.

解: 由积分中值定理, 存在介于x与$\sin x$之间的ξ, s.t.

$$f(x)=(x-\sin x)\ln(1+e^{\xi})=\left(\frac{x^3}{3!}+o(x^3)\right)\ln(1+e^{\xi}), x\to 0.$$

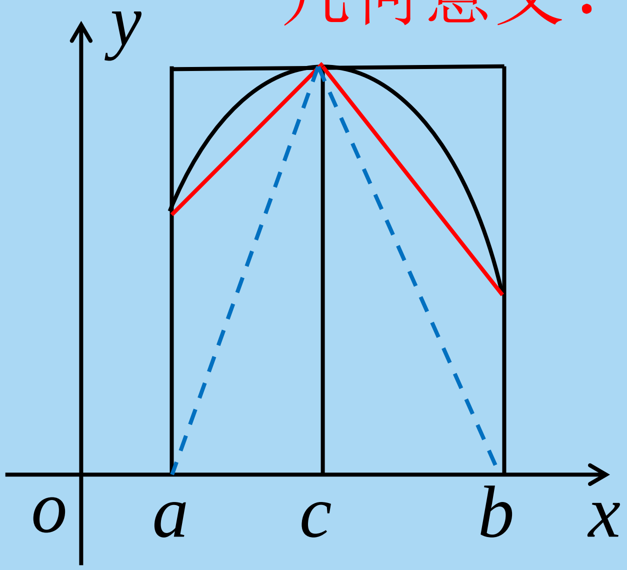
$f(x)$与x^p是同阶无穷小, 则

$$\lim_{x\to 0}\frac{f(x)}{x^p}=\ln 2\cdot\lim_{x\to 0}\frac{\frac{x^3}{3!}+o(x^3)}{x^p} \text{存在且非0,}$$

因而$p=3$, $\lim\limits_{x\to 0}\dfrac{f(x)}{x^p}=\dfrac{\ln 2}{6}$.□

Ex. $f\geq 0,\ f''\leq 0\Rightarrow \max\limits_{a\leq x\leq b} f(x)\leq \dfrac{2}{b-a}\int_a^b f(x)\mathrm{d}x.$

几何意义？



Proof.不妨设$f(c)=\max\limits_{a\leq x\leq b} f(x), c\in(a,b).$

$f''\leq 0$,则 f 上凸, 因此, $\forall x\in[a,c]$,有

$$f(x)\geq f(a)+\frac{f(c)-f(a)}{c-a}(x-a).$$

两边在$[a,c]$上积分,再利用$f\geq 0$,得

$$\int_a^c f(x)\mathrm{d}x\geq \frac{1}{2}(c-a)(f(a)+f(c))\geq \frac{1}{2}(c-a)f(c).$$

同理, $\int_c^b f(x)\mathrm{d}x\geq \dfrac{1}{2}(b-c)f(c)$, 从而

$$\int_a^b f=\int_a^c f+\int_c^b f\geq \frac{1}{2}(b-a)f(c)=\frac{1}{2}(b-a)\max_{a\leq x\leq b} f(x).\square$$

Ex. f在$[0,1]$可导, $f(1)=4\int_0^{1/4} e^{1-x^3} f(x)\mathrm{d}x$.则$\exists \xi\in(0,1)$, $s.t.$

$$f'(\xi)=3\xi^2 f(\xi).$$

Proof. 由积分第一中值定理, $\exists \eta\in[0,1/4]\subset[0,1)$, $s.t.$

$$f(1)=4\int_0^{1/4} e^{1-x^3} f(x)\mathrm{d}x = e^{1-\eta^3} f(\eta).$$

令$g(x)=e^{1-x^3} f(x)$, 则 $g(\eta)=g(1)=f(1)$,

$$g'(x)=e^{1-x^3}(f'(x)-3x^2 f(x)).$$

由Rolle定理, $\exists \xi\in(\eta,1)\subset(0,1)$, $s.t. g'(\xi)=0$, 从而

$$f'(\xi)=3\xi^2 f(\xi).\square$$

Ex. $f \in C[a,b]$, 则

$$\left(\int_a^b f(x)\cos x\mathrm{d}x\right)^2+\left(\int_a^b f(x)\sin x\mathrm{d}x\right)^2 \le \left(\int_a^b |f(x)|\mathrm{d}x\right)^2.$$

Proof. 由积分估值及Cauchy不等式,

$$\left(\int_a^b f(x)\cos x\mathrm{d}x\right)^2 \le \left(\int_a^b \sqrt{|f(x)|}\cdot\sqrt{|f(x)|}\,|\cos x|\mathrm{d}x\right)^2$$

$$\le \int_a^b |f(x)|\mathrm{d}x\cdot\int_a^b |f(x)|\cos^2 x\mathrm{d}x.$$

$$\left(\int_a^b f(x)\sin x\mathrm{d}x\right)^2 \le \int_a^b |f(x)|\mathrm{d}x\cdot\int_a^b |f(x)|\sin^2 x\mathrm{d}x.$$

两式相加即证所需结论.□

Ex. $\lim\limits_{n\to+\infty}\int_0^1 x^n\mathrm{d}x = 0.$

Proof. 令$a_n=\int_0^1 x^n\mathrm{d}x$,则$a_n$单调下降有下界0, $\lim\limits_{n\to+\infty} a_n$存在.

$\forall \varepsilon\in(0,1)$,

$$0\le \int_0^1 x^n\mathrm{d}x = \int_0^{1-\varepsilon} x^n\mathrm{d}x + \int_{1-\varepsilon}^1 x^n\mathrm{d}x \le (1-\varepsilon)^{n+1}+\varepsilon.$$

因此, $0\le \lim\limits_{n\to+\infty}\int_0^1 x^n\mathrm{d}x \le \varepsilon$. 由$\varepsilon$的任意性, $\lim\limits_{n\to+\infty}\int_0^1 x^n\mathrm{d}x = 0$.□

Question. $\lim\limits_{n\to+\infty}\int_0^1 x^n\mathrm{d}x = \lim\limits_{n\to+\infty}\xi^n = 0$,是否正确?

错误! $\xi=\xi_n$不是常数, ξ_n^n不一定趋于0.

作业：习题5.2

No.6(1),7(2),9,10