Review

- 有理函数 $\dfrac{p(x)}{q(x)}$ $(p,q$为多项式$)$ 的积分
- 三角有理式$R(\sin x,\cos x)$ 万能变换 $t=\tan\dfrac{x}{2}$
- 可化为有理式的简单无理式

1) $\int R(x,\sqrt[n]{\dfrac{ax+b}{cx+d}})\mathrm{d}x \quad (ad-bc\neq 0)$ 令$t=\sqrt[n]{\dfrac{ax+b}{cx+d}}$

2) $\int R(x,\sqrt{ax^2+bx+c})\mathrm{d}x,(a\neq 0)$

三角变换开根号、Euler变换

§ 6.定积分的计算

Newton-Leibniz公式 $\int_a^b F'(x)\mathrm{d}x = F(x)\Big|_{x=a}^{b}$

Thm.(换元法) $f \in C[a,b], \varphi \in C^1[\alpha,\beta]$(或$\varphi \in C^1[\beta,\alpha]$), $\varphi(\alpha) = a, \varphi(\beta) = b, a \le \varphi(t) \le b$, 则 $\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\mathrm{d}t.$

Proof. $f \in C[a,b]$, 则f在$[a,b]$上有原函数$F(x)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\varphi(t)) = f(\varphi(t))\varphi'(t), \quad \forall t \in [\alpha,\beta].$$

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)\mathrm{d}t = F(\varphi(t))\Big|_\alpha^\beta = F(b) - F(a) = \int_a^b f(x)\mathrm{d}x. \square$$

Remark. $\varphi \in C^1[\alpha,\beta]$, 不要求$x = \varphi(t)$可逆.

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Question.“$\varphi(t)\in[a,b]$”可去吗？

可改为“f在$\varphi(t)$的值域连续”.

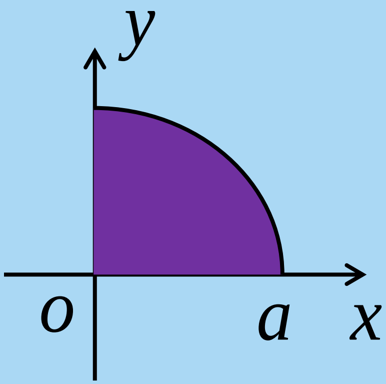
Question.不定积分的第一、二换元法以及定积分换元法中，对$x=\varphi(t)$分别有什么要求？

Ex.判断正误 $\int_{-1}^{1}\frac{1}{1+x^2}\mathrm{d}x=\arctan x\Big|_{-1}^{1}=\frac{\pi}{2}$ （√）

$$\int_{-1}^{1}\frac{1}{1+x^2}\mathrm{d}x=-\int_{-1}^{1}\frac{1}{1+\frac{1}{x^2}}\mathrm{d}\frac{1}{x}=-\arctan\frac{1}{x}\Bigg|_{-1}^{1}=-\frac{\pi}{2}.\quad (\times)$$

$$\left(-\arctan\frac{1}{x}\Bigg|\text{在}x=0\text{处不连续};\ x=\frac{1}{t}\text{在}t=0\text{处不连续}\right)$$

Ex. $\int_0^a \sqrt{a^2 - x^2}\,\mathrm{d}x \ (a > 0)$ 几何意义？



解：令 $x = a\sin t, t \in [0, \pi/2]$,

$$\int_0^a \sqrt{a^2 - x^2}\,\mathrm{d}x = \int_0^{\pi/2} a^2 \cos^2 t\,\mathrm{d}t$$

$$= \frac{1}{2}a^2 \int_0^{\pi/2} (1 + \cos 2t)\,\mathrm{d}t = \frac{1}{2}a^2 \left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{\pi/2} = \frac{\pi}{4}a^2. \square$$

Question. 取 $t \in [0, 5\pi/2]$ 得到 $\int_0^a \sqrt{a^2 - x^2}\,\mathrm{d}x = \frac{5}{4}\pi a^2$？

$$\int_0^a \sqrt{a^2 - x^2}\,\mathrm{d}x = \int_0^{5\pi/2} a^2 |\cos t| \cos t\,\mathrm{d}t = \frac{\pi}{4}a^2.$$

Ex. $f \in C[-a,a]$为偶函数,则$\int_{-a}^{a}\frac{f(x)}{1+e^x}\mathrm{d}x=\int_0^a f(x)\mathrm{d}x.$

Proof. $\int_{-a}^{a}\frac{f(x)}{1+e^x}\mathrm{d}x=\int_0^a\frac{f(x)}{1+e^x}\mathrm{d}x+\int_{-a}^{0}\frac{f(x)}{1+e^x}\mathrm{d}x$

$$=\int_0^a\frac{f(x)}{1+e^x}\mathrm{d}x-\int_a^0\frac{1}{1+e^{-t}}f(-t)\mathrm{d}t \qquad (x=-t)$$

$$=\int_0^a\frac{f(x)}{1+e^x}\mathrm{d}x+\int_0^a\frac{e^t}{1+e^t}f(t)\mathrm{d}t \qquad (f\text{偶})$$

$$=\int_0^a\frac{f(x)}{1+e^x}\mathrm{d}x+\int_0^a\frac{e^x}{1+e^x}f(x)\mathrm{d}x=\int_0^a f(x)\mathrm{d}x.\square$$

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Ex. $f \in C[0,a],\ f(x)+f(a-x) \neq 0$, 则 $\int_0^a \frac{f(x)}{f(x)+f(a-x)}\mathrm{d}x = \frac{a}{2}$.

Proof. $I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\mathrm{d}x$

(令 $x = a-t$)

$$= -\int_a^0 \frac{f(a-t)}{f(a-t)+f(t)}\mathrm{d}t = \int_0^a \frac{f(a-t)}{f(a-t)+f(t)}\mathrm{d}t$$

(令 $t = x$)

$$= \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}\mathrm{d}x$$

$$2I = \int_0^a \left(\frac{f(x)}{f(x)+f(a-x)} + \frac{f(a-x)}{f(a-x)+f(x)} \right)\mathrm{d}x = \int_0^a \mathrm{d}x = a. \square$$

Ex. $f \in C[1,+\infty), a>1,$ 则 $\int_1^a f(x^2+\frac{a^2}{x^2})\frac{\mathrm{d}x}{x}=\int_1^a f(x+\frac{a^2}{x})\frac{\mathrm{d}x}{x}.$

Proof. $\int_1^a f(x^2+\frac{a^2}{x^2})\frac{\mathrm{d}x}{x}=\frac{1}{2}\int_1^{a^2} f(t+\frac{a^2}{t})\frac{\mathrm{d}t}{t}$ $\quad (t=x^2)$

$$=\frac{1}{2}\int_1^{a} f(t+\frac{a^2}{t})\frac{\mathrm{d}t}{t}+\frac{1}{2}\int_a^{a^2} f(t+\frac{a^2}{t})\frac{\mathrm{d}t}{t}\triangleq\frac{1}{2}(I_1+I_2).$$

令 $s=\frac{a^2}{t}$，则

$$I_2=-\int_a^1 f(s+\frac{a^2}{s})\frac{\mathrm{d}s}{s}=\int_1^a f(s+\frac{a^2}{s})\frac{\mathrm{d}s}{s}=I_1.$$

故 $\int_1^a f(x^2+\frac{a^2}{x^2})\frac{\mathrm{d}x}{x}=\int_1^a f(x+\frac{a^2}{x})\frac{\mathrm{d}x}{x}.$ □

Ex. (1) $f\in C[a,b]$, 则 $\int_a^b f(x)\mathrm{d}x=\int_a^b f(a+b-x)\mathrm{d}x$;

几何意义?

$$(2)\ I=\int_{\pi/6}^{\pi/3}\frac{\cos^2 x}{x(\pi-2x)}\mathrm{d}x=\frac{1}{\pi}\ln 2.$$

Proof. (1) $\int_a^b f(a+b-x)\mathrm{d}x \overset{t=a+b-x}{=\!=\!=\!=} -\int_b^a f(t)\mathrm{d}t=\int_a^b f(t)\mathrm{d}t$

$$=\int_a^b f(x)\mathrm{d}x.$$

(2) 由(1), $I=\int_{\pi/6}^{\pi/3}\frac{\sin^2 x}{x(\pi-2x)}\mathrm{d}x$

$$=\int_{\pi/6}^{\pi/3}\frac{\cos^2 x}{x(\pi-2x)}\mathrm{d}x=\frac{1}{2}\int_{\pi/6}^{\pi/3}\frac{\cos^2 x+\sin^2 x}{x(\pi-2x)}\mathrm{d}x$$

$$=\frac{1}{\pi}\int_{\pi/6}^{\pi/3}\left(\frac{1}{2x}+\frac{1}{\pi-2x}\right)\mathrm{d}x=\frac{1}{2\pi}\ln\frac{2x}{\pi-2x}\bigg|_{\pi/6}^{\pi/3}=\frac{1}{\pi}\ln 2.\ \square$$

Ex. $I=\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x.$

解: $I=\int_0^{\pi/4}\ln(1+\tan t)\mathrm{d}t \qquad (t=\arctan x)$

$$=\int_0^{\pi/4}\ln(\sin t+\cos t)\mathrm{d}t-\int_0^{\pi/4}\ln(\cos t)\mathrm{d}t=I_1-I_2.$$

$$I_1=\int_0^{\pi/4}\left(\ln\sqrt{2}+\ln\sin(t+\frac{\pi}{4})\right)\mathrm{d}t=\frac{\pi}{8}\ln 2+\int_0^{\pi/4}\ln\cos(\frac{\pi}{4}-t)\mathrm{d}t$$

$$=\frac{\pi}{8}\ln 2-\int_{\pi/4}^{0}\ln\cos u\mathrm{d}u=\frac{\pi}{8}\ln 2+I_2.$$

$$I=\frac{\pi}{8}\ln 2.\square$$

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Thm.(定积分的分部积分法) $u,v\in C^1[a,b]$, 则

$$\int_a^b u(x)v'(x)\mathrm{d}x = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\mathrm{d}x.$$

Proof. $\big(u(x)v(x)\big)' = u'(x)v(x) + u(x)v'(x)$

$$\Rightarrow \int_a^b u'(x)v(x)\mathrm{d}x + \int_a^b u(x)v'(x)\mathrm{d}x = u(x)v(x)\Big|_a^b$$

$$\Rightarrow \int_a^b u(x)v'(x)\mathrm{d}x = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\mathrm{d}x. \square$$

Ex. 证明$I_n \triangleq \int_0^{\pi/2} \sin^n x\mathrm{d}x = \int_0^{\pi/2} \cos^n x\mathrm{d}x$,并求$I_n$.

Proof. 令$t = \dfrac{\pi}{2} - x$,则

$$\int_0^{\pi/2} \sin^n x\mathrm{d}x = -\int_{\pi/2}^{0} \sin^n\left(\frac{\pi}{2} - t\right)\mathrm{d}t = \int_0^{\pi/2} \cos^n t\mathrm{d}t.$$

$$I_n = -\int_0^{\pi/2} \sin^{n-1} x \,\mathrm{d}\cos x$$

$$= -\sin^{n-1} x \cos x\Big|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2} x\mathrm{d}x$$

$$= (n-1)\int_0^{\pi/2} (1 - \sin^2 x)\sin^{n-2} x\mathrm{d}x$$

$$= (n-1)I_{n-2} - (n-1)I_n.$$

$$I_n = \frac{n-1}{n} I_{n-2}.$$

$$I_0 = \int_0^{\pi/2} \mathrm{d}x = \frac{\pi}{2},$$

$$I_1 = \int_0^{\pi/2} \sin x \mathrm{d}x = -\cos x\Big|_0^{\pi/2} = 1,$$

$$I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \cdots \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2},$$

$$I_{2n-1} = \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdot \cdots \cdot \frac{2}{3} \cdot 1 = \frac{(2n-2)!!}{(2n-1)!!}. \square$$

Ex. $f,g\in C[a,b],\int_a^x f(t)\mathrm{d}t\ge\int_a^x g(t)\mathrm{d}t\ \ (a\le x\le b),$

$$\int_a^b f(x)\mathrm{d}x=\int_a^b g(x)\mathrm{d}x,\text{则}\int_a^b xf(x)\mathrm{d}x\le\int_a^b xg(x)\mathrm{d}x.$$

Proof. 令$F(x)=\int_a^x f(t)\mathrm{d}t,G(x)=\int_a^x g(t)\mathrm{d}t,$则

$$F(a)=G(a)=0,F(b)=G(b),F(x)\ge G(x)(a\le x\le b).$$

$$\int_a^b x(f(x)-g(x))\mathrm{d}x=\int_a^b x\mathrm{d}\left(F(x)-G(x)\right)$$

$$=x(F(x)-G(x))\Big|_a^b-\int_a^b(F(x)-G(x))\mathrm{d}x$$

$$=-\int_a^b(F(x)-G(x))\mathrm{d}x\ \le 0.\square$$

Ex. $f\in C^1[a,b],\ f(a)=0$, 则

$$\int_a^b f^2(x)\mathrm{d}x\le\frac{(b-a)^2}{2}\int_a^b\left(f'(x)\right)^2\mathrm{d}x-\frac{1}{2}\int_a^b(x-a)^2\left(f'(x)\right)^2\mathrm{d}x.$$

Proof. $f(a)=0$, 则 $f^2(x)=\left(\int_a^x 1\cdot f'(t)\mathrm{d}t\right)^2\le(x-a)\int_a^x\left(f'(t)\right)^2\mathrm{d}t.$

$$\int_a^b f^2(x)\mathrm{d}x\le\int_a^b\left(\int_a^x\left(f'(t)\right)^2\mathrm{d}t\right)\mathrm{d}\frac{(x-a)^2}{2}$$

$$=\frac{(x-a)^2}{2}\int_a^x\left(f'(t)\right)^2\mathrm{d}t\Bigg|_{x=a}^{b}-\int_a^b\frac{(x-a)^2}{2}\left(f'(x)\right)^2\mathrm{d}x$$

$$=\frac{(b-a)^2}{2}\int_a^b\left(f'(x)\right)^2\mathrm{d}x-\frac{1}{2}\int_a^b(x-a)^2\left(f'(x)\right)^2\mathrm{d}x.\square$$

Ex. $f\in C^1[a,b],\ f(a)=f(b)=0,\ \int_a^b f^2(x)\mathrm{d}x=1$, 则

$$\int_a^b \left(f'(x)\right)^2\mathrm{d}x\cdot\int_a^b x^2 f^2(x)\mathrm{d}x>\frac{1}{4}.$$

Proof. 若 $f'(x)+\lambda x f(x)\equiv 0$, 则

$$\left(f(x)e^{\frac{1}{2}\lambda x^2}\right)'=\left(f'(x)+\lambda x f(x)\right)e^{\frac{1}{2}\lambda x^2}\equiv 0,$$

$$f(x)e^{\frac{1}{2}\lambda x^2}\equiv C,\qquad f(x)=Ce^{\frac{-1}{2}\lambda x^2},$$

$f(a)=f(b)=0$, 则 $C=0$, $f(x)\equiv 0$, 与 $\int_a^b f^2(x)\mathrm{d}x=1$ 矛盾.

故 $\forall\lambda\in\mathbb{R},\ \int_a^b\left(f'(x)+\lambda x f(x)\right)^2\mathrm{d}x>0$, 即

$$\int_a^b \left(f'(x)\right)^2 \mathrm{d}x + 2\lambda\int_a^b xf(x)f'(x)\mathrm{d}x + \lambda^2\int_a^b x^2 f^2(x)\mathrm{d}x > 0.$$

于是$\int_a^b \left(f'(x)\right)^2 \mathrm{d}x \cdot \int_a^b x^2 f^2(x)\mathrm{d}x > \left(\int_a^b xf(x)f'(x)\mathrm{d}x\right)^2.$

而$\int_a^b xf(x)f'(x)\mathrm{d}x = \frac{1}{2}\int_a^b x\,\mathrm{d}f^2(x)$

$$= \frac{1}{2}xf^2(x)\Big|_a^b - \frac{1}{2}\int_a^b f^2(x)\,\mathrm{d}x = -\frac{1}{2}$$

故$\int_a^b \left(f'(x)\right)^2 \mathrm{d}x \cdot \int_a^b x^2 f^2(x)\mathrm{d}x > \frac{1}{4}.$ □

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Ex. $\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1\cdot 3\cdot 5\cdots(2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}$.

Proof. $T_n \triangleq 1\cdot 3\cdot 5\cdots(2n-1)$,

$$2\ln T_n = 2\sum_{k=2}^{n}\ln(2k-1) < \sum_{k=2}^{n}\int_{2k-1}^{2k+1}\ln x\,\mathrm{d}x$$

$$= \int_3^{2n+1}\ln x\,\mathrm{d}x = x\ln x\Big|_3^{2n+1} - \int_3^{2n+1} x\cdot\frac{1}{x}\,\mathrm{d}x$$

$$= (x\ln x - x)\Big|_3^{2n+1} = x\ln\frac{x}{e}\Big|_3^{2n+1} < (2n+1)\ln\frac{2n+1}{e}.$$

同理, $2\ln T_n > \sum_{k=2}^{n}\int_{2k-3}^{2k-1}\ln x\,\mathrm{d}x = \int_1^{2n-1}\ln x\,\mathrm{d}x = (2n-1)\ln\frac{2n-1}{e} + 1.$ □

Thm.(带积分余项的Taylor公式) $f\in C^{n+1}[a,b], x_0\in[a,b]$,

则$\forall x\in[a,b]$,有

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\mathrm{d}t.$$

Proof. $n=0$时, 即Newton-Leibniz公式.

假设$n=m-1$时,定理成立, 即

$$f(x)=\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{1}{(m-1)!}\int_{x_0}^{x}(x-t)^{m-1} f^{(m)}(t)\mathrm{d}t.$$

对余项分部积分,得

$$\frac{1}{(m-1)!}\int_{x_0}^{x}(x-t)^{m-1}f^{(m)}(t)\mathrm{d}t=\frac{-1}{m!}\int_{x_0}^{x}f^{(m)}(t)\mathrm{d}(x-t)^m$$

$$=-\frac{1}{m!}f^{(m)}(t)(x-t)^m\Big|_{t=x_0}^{x}+\frac{1}{m!}\int_{x_0}^{x}(x-t)^m f^{(m+1)}(t)\mathrm{d}t$$

$$=\frac{1}{m!}f^{(m)}(x_0)(x-x_0)^m+\frac{1}{m!}\int_{x_0}^{x}(x-t)^m f^{(m+1)}(t)\mathrm{d}t.$$

故 $f(x)=\sum_{k=0}^{m}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{1}{m!}\int_{x_0}^{x}(x-t)^m f^{(m+1)}(t)\mathrm{d}t,$

即 $n=m$ 时,定理成立.□

作业：习题5.6

No.1(2,11,14),2(2,4,8)

3(1,4,8),5,8,11,13.

Thm.(Wallis公式) $\lim_{n\to\infty}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2\frac{1}{2n+1}=\frac{\pi}{2}.$

Proof. 由$\int_0^{\pi/2}\sin^{2n+1}x\mathrm{d}x\le\int_0^{\pi/2}\sin^{2n}x\mathrm{d}x\le\int_0^{\pi/2}\sin^{2n-1}x\mathrm{d}x$, 得

$$\frac{(2n)!!}{(2n+1)!!}\le\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}\le\frac{(2n-2)!!}{(2n-1)!!},$$

从而 $a_n\triangleq\left[\frac{(2n)!!}{(2n-1)!!}\right]^2\frac{1}{2n+1}\le\frac{\pi}{2}\le\left[\frac{(2n)!!}{(2n-1)!!}\right]^2\frac{1}{2n}\triangleq b_n.$

$$0\le\frac{\pi}{2}-a_n\le b_n-a_n=a_n\cdot\frac{1}{2n}\le\frac{\pi}{2}\cdot\frac{1}{2n}\to 0, n\to\infty \text{ 时}. \square$$

Thm.(Stirling公式) $n! \sim (\frac{n}{e})^n \sqrt{2n\pi}\ (n \to \infty)$.

更精细地,有 $\sqrt{2n\pi}n^n e^{-n} < n! < \sqrt{2n\pi}n^n e^{-n}(1+\frac{1}{4n})$.

Proof. $[k,k+1]$ 上 $y=\ln x$ 的下方图形面积 $S_k = \int_k^{k+1} \ln x \mathrm{d}x$,

两端点间割线下方梯形面积 $\underline{S}_k = \frac{1}{2}[\ln k + \ln(k+1)]$,

$x = k + \frac{1}{2}$ 处切线下方梯形面积 $\bar{S}_k = \ln(k+\frac{1}{2})$.

$(\ln x)'' < 0, \ln x$ 为严格上凸函数. 因此 $\underline{S}_k < S_k < \bar{S}_k$.

$$\lambda_n \triangleq \int_1^n \ln x \, \mathrm{d}x - \sum_{k=1}^{n-1} \underline{S}_k$$

$$= n\ln n - n + 1 - \sum_{k=1}^{n-1} \frac{1}{2}[\ln k + \ln(k+1)]$$

$$= n\ln n - n + 1 - \left[\ln n! - \frac{1}{2}\ln n\right]$$

则 $\ln n! = 1 - \lambda_n + (n + \frac{1}{2})\ln n - n,\quad n! = e^{1-\lambda_n} n^{(n+\frac{1}{2})} e^{-n}.$

下证 λ_n (严格)单增有上界.

$$0 < S_k - \underline{S}_k < \overline{S}_k - \underline{S}_k = \ln(k+\frac{1}{2}) - \frac{1}{2}\ln k - \frac{1}{2}\ln(k+1)$$

$$= \frac{1}{2}\ln(1+\frac{1}{2k}) - \frac{1}{2}\ln(1+\frac{1}{2k+1})$$

$$< \frac{1}{2}\ln(1+\frac{1}{2k}) - \frac{1}{2}\ln(1+\frac{1}{2k+2})$$

$$0 < \lambda_n = \int_1^n \ln x \, \mathrm{d}x - \sum_{k=1}^{n-1} \underline{S}_k = \sum_{k=1}^{n-1} (S_k - \underline{S}_k)$$

$$\leq \frac{1}{2}\ln\frac{3}{2} - \frac{1}{2}\ln(1+\frac{1}{2n}) < \frac{1}{2}\ln\frac{3}{2}.$$

故 λ_n 严格单增有上界,因而收敛,设 $\lim\limits_{n\to\infty}\lambda_n=\lambda(>0)$.

$$\text{由}\lambda_m-\lambda_n=\sum_{k=n}^{m-1}(\lambda_{k+1}-\lambda_k)=\sum_{k=n}^{m-1}(S_k-\underline{S}_k)<\frac{1}{2}\ln(1+\frac{1}{2n}),$$

$$\text{得}\lambda-\lambda_n=\lim_{m\to\infty}(\lambda_m-\lambda_n)\le\frac{1}{2}\ln(1+\frac{1}{2n}).$$

$\mu_n\triangleq e^{1-\lambda_n}$, 则 μ_n 严格单减,且 $\lim\limits_{n\to\infty}\mu_n=e^{1-\lambda}\triangleq\mu$.

$$1<\frac{\mu_n}{\mu}=e^{\lambda-\lambda_n}\le e^{\frac{1}{2}\ln(1+\frac{1}{2n})}=\sqrt{1+\frac{1}{2n}}<1+\frac{1}{4n}.$$

$$\text{因此}\quad \mu n^{(n+\frac{1}{2})}e^{-n}<n!=e^{1-\lambda_n}n^{(n+\frac{1}{2})}e^{-n}<\mu n^{(n+\frac{1}{2})}e^{-n}(1+\frac{1}{4n}).$$

只要证$\mu=\sqrt{2\pi}$.

由Wallis公式 $\lim\limits_{n\to\infty}\left[\dfrac{(2n)!!}{(2n-1)!!}\right]^2\dfrac{1}{2n+1}=\dfrac{\pi}{2}$,有

$$\lim_{n\to\infty}\frac{[(2n)!!]^2}{(2n)!}\frac{1}{\sqrt{n}}=\sqrt{\pi},\quad \lim_{n\to\infty}\frac{(n!)^2 2^{2n}}{(2n)!}\frac{1}{\sqrt{n}}=\sqrt{\pi}.$$

而 $n!=e^{1-\lambda_n}n^{(n+\frac{1}{2})}e^{-n}=\mu_n n^{(n+\frac{1}{2})}e^{-n}$, 因此

$$\sqrt{\pi}=\lim_{n\to\infty}\frac{\left[\mu_n n^{(n+\frac{1}{2})}e^{-n}\right]^2 2^{2n}}{\mu_{2n}(2n)^{(2n+\frac{1}{2})}e^{-2n}}\frac{1}{\sqrt{n}}=\lim_{n\to\infty}\frac{\mu_n^2}{\sqrt{2}\mu_{2n}}=\frac{\mu}{\sqrt{2}}.\square$$