回顾上节课内容：

(a) 内积

(b) 子空间正交与正交补

定义：

$\mathcal{M},\mathcal{N}$为$\mathbb{R}^m$的两个子集，如果$\boldsymbol{v}^T\boldsymbol{w}=0,\forall\boldsymbol{v}\in\mathcal{M},\forall\boldsymbol{w}\in\mathcal{N}$
称$\mathcal{M}$与$\mathcal{N}$正交，记为$\mathcal{M}\perp\mathcal{N}$

特别的，我们常会用到$\mathcal{M},\mathcal{N}$为$\mathbb{R}^m$的子空间的情形

例:

如果$\boldsymbol{v}$与$\boldsymbol{w}_1, \dots, \boldsymbol{w}_n$都正交，则$\boldsymbol{v} \perp Span(\boldsymbol{w}_1, \dots, \boldsymbol{w}_n)$。

证明:

$$\boldsymbol{w} \in Span(\boldsymbol{w}_1, \dots, \boldsymbol{w}_n), \boldsymbol{w} = c_1\boldsymbol{w}_1 + \cdots + c_n\boldsymbol{w}_n, c_1, \dots, c_n \in \mathbb{R}$$

$$\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{v}^{\mathrm{T}}(c_1\boldsymbol{w}_1 + \cdots + c_n\boldsymbol{w}_n) = c_1\boldsymbol{v}^T\boldsymbol{w}_1 + \cdots + c_n\boldsymbol{v}^T\boldsymbol{w}_n = 0$$

例:

$\mathbb{R}^3$中$\boldsymbol{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$与$yz$平面$= Span\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$正交

$$Span\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \cap Span\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \{\boldsymbol{0}\}$$

命题：

$A \in M_{m\times n}(\mathbb{R})$. $\mathcal{N}(A) \perp \mathcal{R}\left(A^T\right), \mathcal{N}\left(A^T\right) \perp \mathcal{R}(A)$.

证明：只需证明$\mathcal{N}(A) \perp \mathcal{R}\left(A^T\right)$.

$$A = \begin{bmatrix} \tilde{\boldsymbol{a}}_1^T \\ \vdots \\ \tilde{\boldsymbol{a}}_m^T \end{bmatrix}, A^T = [\tilde{\boldsymbol{a}}_1, \dots, \tilde{\boldsymbol{a}}_m].$$

$$\boldsymbol{x} \in \mathcal{N}(A) \Longrightarrow \tilde{\boldsymbol{a}}_1^T \boldsymbol{x} = \cdots = \tilde{\boldsymbol{a}}_m^T \boldsymbol{x} = 0$$

因此，$\boldsymbol{x} \perp Span(\tilde{\boldsymbol{a}}_1, \dots, \tilde{\boldsymbol{a}}_m) = \mathcal{R}\left(A^T\right)$. 即$\mathcal{N}(A) \perp \mathcal{R}\left(A^T\right)$

命题：

$\mathcal{M},\mathcal{N}$为$\mathbb{R}^n$的子空间，如果$\mathcal{M} \perp \mathcal{N}$，则$\mathcal{M} \cap \mathcal{N} = \{\mathbf{0}\}$.

证明：如果$\boldsymbol{v} = [x_1, \ldots, x_n]^T \in \mathcal{M} \cap \mathcal{N}$，则

$0 = \boldsymbol{v}^T \boldsymbol{v} = x_1^2 + \cdots + x_n^2$

于是$x_1 = \cdots = x_n = 0, \boldsymbol{v} = \mathbf{0}$.

我们有$\mathcal{N}(A) \cap \mathcal{R}\left(A^T\right) = \{\mathbf{0}\}$，$\mathcal{N}\left(A^T\right) \cap \mathcal{R}(A) = \{\mathbf{0}\}$.

正交补的定义：

给定$\mathbb{R}^n$的子空间$\mathcal{M}$，则$\mathbb{R}^n$的子集$\mathcal{M}^\perp := \{\boldsymbol{v} \in \mathbb{R}^n | \boldsymbol{v} \perp \mathcal{M}\}$称为$\mathcal{M}$的**正交补**.

命题:

$\mathcal{M}^\perp$为$\mathbb{R}^n$子空间。

证明：首先$\mathbf{0} \in \mathcal{M}^\perp$, $\mathcal{M}^\perp$非空。

其次，容易验证$\mathcal{M}^\perp$对加法和数乘封闭。

例:

$\mathbb{R}^3$中x轴$=Span\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$与$yz$平面$=Span\left(\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$正交

且$Span\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)^{\perp}=Span\left(\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$

$$Span\left(\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)^{\perp}=Span\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

关于正交补的性质，我们需要下列关键命题：

$A \in M_{m\times n}(\mathbb{R})$. 则$\mathcal{N}(A) = \mathcal{R}\left(A^T\right)^{\perp}$,同样的$\mathcal{N}\left(A^T\right) = \mathcal{R}(A)^{\perp}$。

证明：

$$A = \begin{bmatrix} \tilde{\boldsymbol{a}}_1^T \\ \vdots \\ \tilde{\boldsymbol{a}}_m^T \end{bmatrix}, A^T = [\tilde{\boldsymbol{a}}_1, \dots, \tilde{\boldsymbol{a}}_m]. \mathcal{N}(A) \perp \mathcal{R}\left(A^T\right)$$

$$\forall \boldsymbol{x} \in \mathcal{N}(A), \boldsymbol{x} \perp \mathcal{R}\left(A^T\right) \quad \Longrightarrow \mathcal{N}(A) \subseteq \mathcal{R}\left(A^T\right)^{\perp}$$

反过来，如果$\boldsymbol{v} \in \mathcal{R}\left(A^T\right)^{\perp}$,$\boldsymbol{v} \perp \mathcal{R}\left(A^T\right) = Span(\tilde{\boldsymbol{a}}_1, \dots, \tilde{\boldsymbol{a}}_m)$

特别地，$\tilde{\boldsymbol{a}}_1^T\boldsymbol{v} = \cdots = \tilde{\boldsymbol{a}}_m^T\boldsymbol{v} = 0$

因此，$A\boldsymbol{v} = \begin{bmatrix} \tilde{\boldsymbol{a}}_1^T \\ \vdots \\ \tilde{\boldsymbol{a}}_m^T \end{bmatrix} \boldsymbol{v} = \begin{bmatrix} \tilde{\boldsymbol{a}}_1^T\boldsymbol{v} \\ \vdots \\ \tilde{\boldsymbol{a}}_m^T\boldsymbol{v} \end{bmatrix} = \boldsymbol{0}$,即$\boldsymbol{v} \in \mathcal{N}(A)$。

正交补的性质：

对$\mathbb{R}^n$的子空间$\mathcal{M}$，有

（1）$\mathcal{M} \cap \mathcal{M}^{\perp} = \{\mathbf{0}\}$

（2）$dim\mathcal{M}^{\perp} = n - dim\mathcal{M}$

(3)（$\mathcal{M}^{\perp}$）$^{\perp} = \mathcal{M}$，即$\mathcal{M}$与$\mathcal{M}^{\perp}$互为正交补

（4）对任意$\boldsymbol{v} \in \mathbb{R}^n$，存在唯一的$\boldsymbol{v}_1 \in \mathcal{M}$，$\boldsymbol{v}_2 \in \mathcal{M}^{\perp}$使得$\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$。

$\boldsymbol{v}_1, \boldsymbol{v}_2$分别称为$\boldsymbol{v}$向$\mathcal{M}$，$\mathcal{M}^{\perp}$的**正交投影。**

证明：

回顾：$\mathcal{M} \perp \mathcal{N} \Rightarrow \mathcal{M} \cap \mathcal{N} = \{\mathbf{0}\}$。

因此，由$\mathcal{M} \perp \mathcal{M}^{\perp}$得到(1)

(2) $dim\mathcal{M}^{\perp} = n - dim\mathcal{M}$

证明:

取$\mathcal{M} \subseteq \mathbb{R}^n$的一组基$\tilde{\boldsymbol{a}}_1, \dots, \tilde{\boldsymbol{a}}_r, r = dim\mathcal{M}$

令$A = \begin{bmatrix} \tilde{\boldsymbol{a}}_1^T \\ \vdots \\ \tilde{\boldsymbol{a}}_r^T \end{bmatrix}, A^T = [\tilde{\boldsymbol{a}}_1, \dots, \tilde{\boldsymbol{a}}_r]$

$\mathcal{M} = \mathcal{R}\left(A^T\right), dim\mathcal{R}\left(A^T\right) = r$

于是$\mathcal{M}^{\perp} = \mathcal{R}\left(A^T\right)^{\perp} = \mathcal{N}(A)$

因此，$dim\mathcal{M}^{\perp} = dim\mathcal{N}(A) = n - r = n - dim\mathcal{M}$

(3) $\left(\mathcal{M}^{\perp}\right)^{\perp} = \mathcal{M}$，即$\mathcal{M}$与$\mathcal{M}^{\perp}$互为正交补

证明：

由定义$\mathcal{M} \subseteq \left(\mathcal{M}^{\perp}\right)^{\perp}$

$$dim\left(\mathcal{M}^{\perp}\right)^{\perp} = n - dim\mathcal{M}^{\perp} = n - (n - dim\mathcal{M}) = dim\mathcal{M}$$

因此，$\mathcal{M} = \left(\mathcal{M}^{\perp}\right)^{\perp}$

(4) 对任意$\boldsymbol{v} \in \mathbb{R}^n$，存在唯一的$\boldsymbol{v}_1 \in \mathcal{M}, \boldsymbol{v}_2 \in \mathcal{M}^\perp$,使得$\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$。$\boldsymbol{v}_1, \boldsymbol{v}_2$分别称为$\boldsymbol{v}$向$\mathcal{M}$，$\mathcal{M}^\perp$的**正交投影。**

证明:

由维数公式，$dim\left(\mathcal{M} + \mathcal{M}^\perp\right) = dim\mathcal{M} + dim\mathcal{M}^\perp - dim(\mathcal{M} \cap \mathcal{M}^\perp) = n$

因此，$\mathcal{M} + \mathcal{M}^\perp = \mathbb{R}^n$。

即对任意$\boldsymbol{v} \in \mathbb{R}^n$，存在$\boldsymbol{v}_1 \in \mathcal{M}, \boldsymbol{v}_2 \in \mathcal{M}^\perp$，使得$\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$.

分解的唯一性：如果有$\boldsymbol{v}_1, \boldsymbol{v}_1' \in \mathcal{M}$, $\boldsymbol{v}_2, \boldsymbol{v}_2' \in \mathcal{M}^\perp$满足$\boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{v}_1' + \boldsymbol{v}_2'$

那么$\boldsymbol{v}_1 - \boldsymbol{v}_1' = \boldsymbol{v}_2' - \boldsymbol{v}_2 \in \mathcal{M} \cap \mathcal{M}^\perp$=$\{\boldsymbol{0}\}$.

于是$\boldsymbol{v}_1 = \boldsymbol{v}_1', \boldsymbol{v}_2 = \boldsymbol{v}_2'$.

例:

设$\mathcal{M}$为$\mathbb{R}^m$的一个子空间，则有线性映射$\mathcal{B}$使得$\mathcal{N}(\mathcal{B}) = \mathcal{M}$。

因此，任意子空间也可以用一个线性映射的零空间得到。

设$\mathcal{M} = Span(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$

令$\mathcal{M}^\perp = Span(\boldsymbol{b}_1, \dots, \boldsymbol{b}_t)$

可取$B = \begin{bmatrix} \boldsymbol{b}_1^T \\ \vdots \\ \boldsymbol{b}_t^T \end{bmatrix}$, 则$\mathcal{N}(B) = \mathcal{M}$。

首先有$\mathcal{M} \subseteq \mathcal{N}(B)$。

计算维数，$\dim \mathcal{N}(B) = m - \dim \mathcal{R}\left(B^T\right) = m - \dim \mathcal{M}^\perp = \dim \mathcal{M}$

应用:

$A \in M_{m\times n}(\mathbb{R})$, 则$\mathcal{R}\left(A^T A\right) = \mathcal{R}\left(A^T\right)$。

因此，对任意$\boldsymbol{b}$，方程$A^T A\boldsymbol{x} = A^T\boldsymbol{b}$有解。

证明:

回顾对于实矩阵，我们有$\mathcal{N}\left(A^T A\right) = \mathcal{N}(A)$

因此，$\mathcal{R}\left(A^T A\right) = \mathcal{N}\left(\left(A^T A\right)^T\right)^{\perp} = \mathcal{N}\left(A^T A\right)^{\perp} = \mathcal{N}(A)^{\perp} = \mathcal{R}\left(A^T\right)$

方程 $\boxed{A^T A\boldsymbol{x} = A^T\boldsymbol{b}}$ 称为**正规方程**

将在本章起到重要作用

线性代数基本定理v.1.0:

$A \in M_{m\times n}(\mathbb{R})$

(1) $rank(A) \leq \min(m,n)$.

(2) 四个基本子空间的维数

行空间$\mathcal{R}(A^T)$： $dim\mathcal{R}\left(A^T\right) = rank(A)$

零空间$\mathcal{N}(A)$： $dim\mathcal{N}(A) = n - rank(A)$

列空间$\mathcal{R}(A)$： $dim\mathcal{R}(A) = rank(A)$

左零空间$\mathcal{N}(A^T)$:$dim\mathcal{N}\left(A^T\right) = m - rank(A)$

(3)四个基本子空间的一组基

由$rref(A)$的非零行给出

由规范基础解系给出

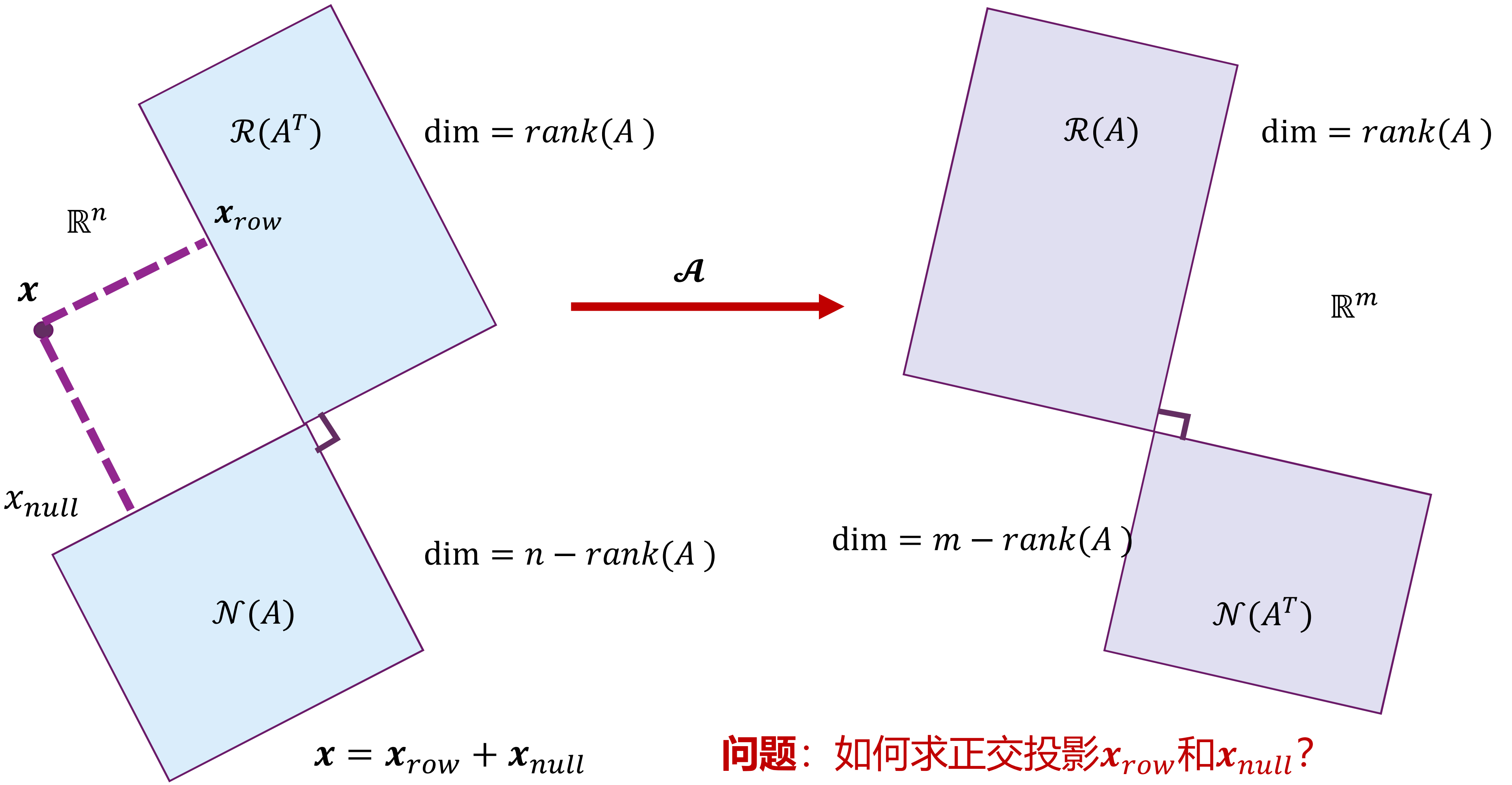
由A的主元列给出

由$EA = rref(A)$中E的后$m - rank(A)$行给出

(4)空间分解： $\mathbb{R}^n = \mathcal{N}(A) + \mathcal{R}\left(A^T\right), \mathbb{R}^m = \mathcal{N}\left(A^T\right) + \mathcal{R}(A)$

(5)$\mathcal{N}(A) = \mathcal{R}\left(A^T\right)^{\perp}, \mathcal{N}\left(A^T\right) = \mathcal{R}(A)^{\perp}$

空间分解:

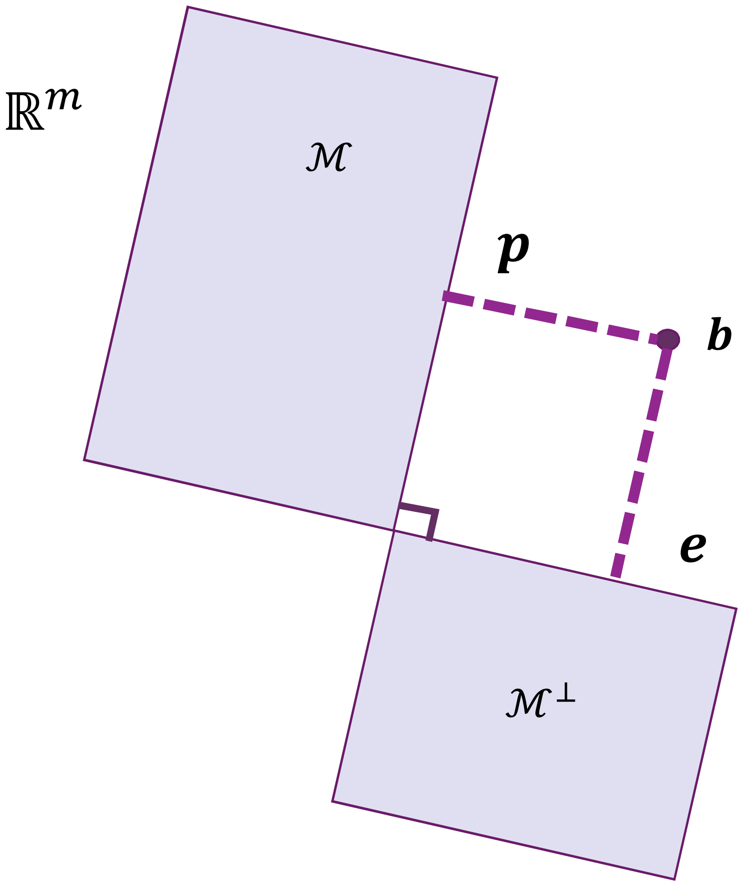


$\boldsymbol{x} = \boldsymbol{x}_{row} + \boldsymbol{x}_{null}$

问题：如何求正交投影$\boldsymbol{x}_{row}$和$\boldsymbol{x}_{null}$？

(c) 正交投影:

为了之后使用方便，我们在$\mathbb{R}^m$上考虑正交投影问题



$\mathcal{M}$为$\mathbb{R}^m$的子空间, $\mathcal{M}^\perp$是$\mathcal{M}$的正交补

$\boldsymbol{b} \in \mathbb{R}^m$, 存在唯一的向量$\boldsymbol{p} \in \mathcal{M}$, $\boldsymbol{e} \in \mathcal{M}^\perp$使得

$$\boldsymbol{b} = \boldsymbol{p} + \boldsymbol{e}$$

projection　　error

$\boldsymbol{p}$，$\boldsymbol{e}$分别为$\boldsymbol{b}$向$\mathcal{M}$，$\mathcal{M}^\perp$的正交投影

命题:

给定子空间$\mathcal{M}$，定义正交投影映射$P_{\mathcal{M}}: \mathbb{R}^m \to \mathbb{R}^m$

$$\boldsymbol{b} \longmapsto P_{\mathcal{M}}(\boldsymbol{b}) = \boldsymbol{p}$$

$P_{\mathcal{M}}$是一个线性变换。

证明:

$\boldsymbol{b}_1, \boldsymbol{b}_2 \in \mathbb{R}^m$，$\boldsymbol{b}_1 = \boldsymbol{p}_1 + \boldsymbol{e}_1, \boldsymbol{b}_2 = \boldsymbol{p}_2 + \boldsymbol{e}_2$，$\boldsymbol{p}_1, \boldsymbol{p}_2 \in \mathcal{M}, \boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathcal{M}^{\perp}$

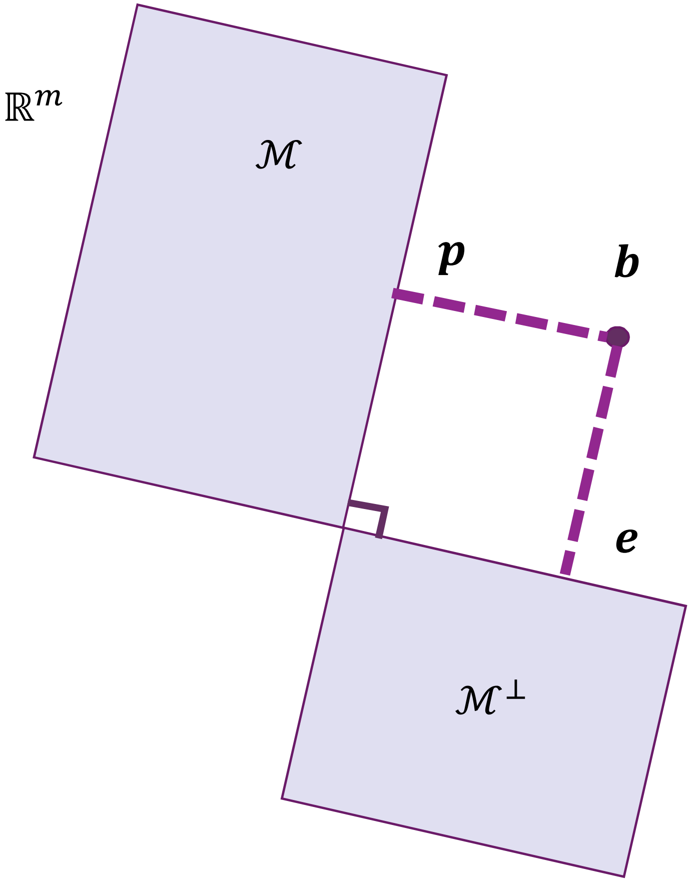
$\boldsymbol{b}_1 + \boldsymbol{b}_2 = (\boldsymbol{p}_1 + \boldsymbol{p}_2) + (\boldsymbol{e}_1 + \boldsymbol{e}_2), \boldsymbol{p}_1 + \boldsymbol{p}_2 \in \mathcal{M}$，$\boldsymbol{e}_1 + \boldsymbol{e}_2 \in \mathcal{M}^{\perp}$

因此，$P_{\mathcal{M}}(\boldsymbol{b}_1 + \boldsymbol{b}_2) = \boldsymbol{p}_1 + \boldsymbol{p}_2 = P_{\mathcal{M}}(\boldsymbol{b}_1) + P_{\mathcal{M}}(\boldsymbol{b}_2)$

类似的方法可证明$P_{\mathcal{M}}(c\boldsymbol{b}) = cP_{\mathcal{M}}(\boldsymbol{b})$

问题:

如何求$P_{\mathcal{M}}$ 的表出矩阵?



(1) 如果$\boldsymbol{v} \in \mathcal{M}$，$P_{\mathcal{M}}(\boldsymbol{v}) = \boldsymbol{v}$

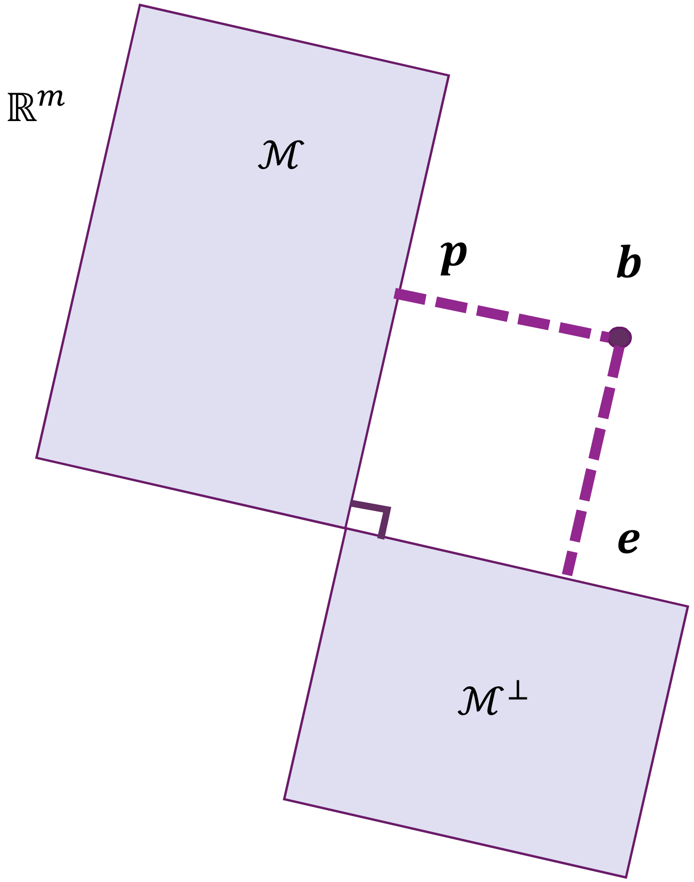
(2) 如果$\boldsymbol{v} \in \mathcal{M}^{\perp}$，$P_{\mathcal{M}}(\boldsymbol{v}) = \boldsymbol{0}$

由于$\mathbb{R}^m = \mathcal{M} + \mathcal{M}^{\perp}$，$P_{\mathcal{M}}$由这两条决定，即如果有线性映射满足(1) (2)，则它就是$\mathcal{M}$上的正交投影映射

线性映射$\boldsymbol{b} \mapsto \boldsymbol{e}$给出正交投影变换

$P_{\mathcal{M}^{\perp}}: \mathbb{R}^m \to \mathbb{R}^m$

求解正交投影矩阵的关键



$\boldsymbol{b} = \boldsymbol{p} + \boldsymbol{e}$

由分解$\mathbb{R}^m = \mathcal{M} + \mathcal{M}^\perp$的唯一性

$\boldsymbol{p}$是唯一的$\mathcal{M}$中向量满足

$\boldsymbol{b} - \boldsymbol{p} \in \mathcal{M}^\perp$

问题:

对$\mathbb{R}^m$的任意子空间$\mathcal{M}$, 如何用矩阵的方法求任意向量在$\mathcal{M}$上的投影? 正交投影映射的矩阵表示?

(1) 联系子空间和矩阵.

取$\mathcal{M}$的一组生成元$\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n$, 即

$\mathcal{M} = Span\,(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)$.

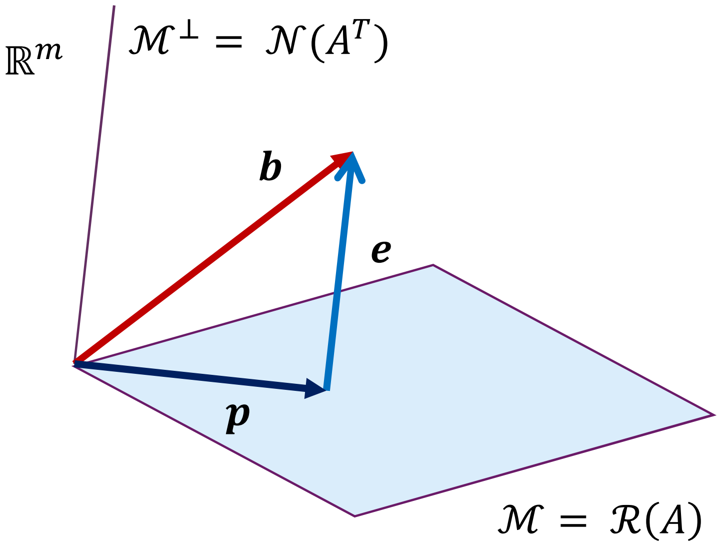
考虑$m \times n$阶矩阵$A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n]$

因此$\mathcal{M} = \mathcal{R}(A)$.

如果取$\mathcal{M}$的一组基$\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n$, 则$A$是**列满秩**矩阵

(2) 运用空间分解$\mathbb{R}^m = \mathcal{R}(A) + \mathcal{N}(A^T)$

对任意$\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{b} = \boldsymbol{p} + \boldsymbol{e}$ 其中, $\boldsymbol{p} \in \mathcal{R}(A)$, $\boldsymbol{e} \in \mathcal{N}(A^T)$



(3) 运用$\boldsymbol{e} \in \mathcal{N}(A^T)$

$\boldsymbol{p} \in \mathcal{R}(A)$可写成$\boldsymbol{p} = A\boldsymbol{x}$.

$\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} \in \mathcal{N}(A^T)$, 即$A^T\boldsymbol{e} = \boldsymbol{0}$

因此, 得到方程

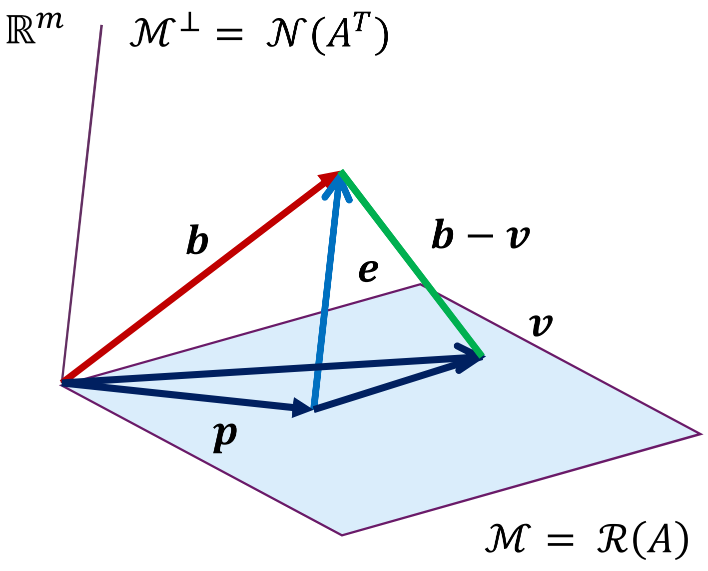
$$A^T(\boldsymbol{b} - A\boldsymbol{x}) = \boldsymbol{0}$$

$$\boxed{A^T A\boldsymbol{x} = A^T\boldsymbol{b}}$$

正规方程，总有解！

小结:

$\boldsymbol{b}$在子空间$\mathcal{M} = \mathcal{R}(A)$上的投影为 $\boldsymbol{p} = A\boldsymbol{x}$, 其中$\boldsymbol{x}$是正规方程



$$\boxed{A^T A\boldsymbol{x} = A^T\boldsymbol{b}}$$

的任意一个解.

如果$\boldsymbol{x}_0$是一个特解,

所有解 $= \{\boldsymbol{x}_0 + \mathcal{N}(A^T A)\}$

$= \{\boldsymbol{x}_0 + \mathcal{N}(A)\}$

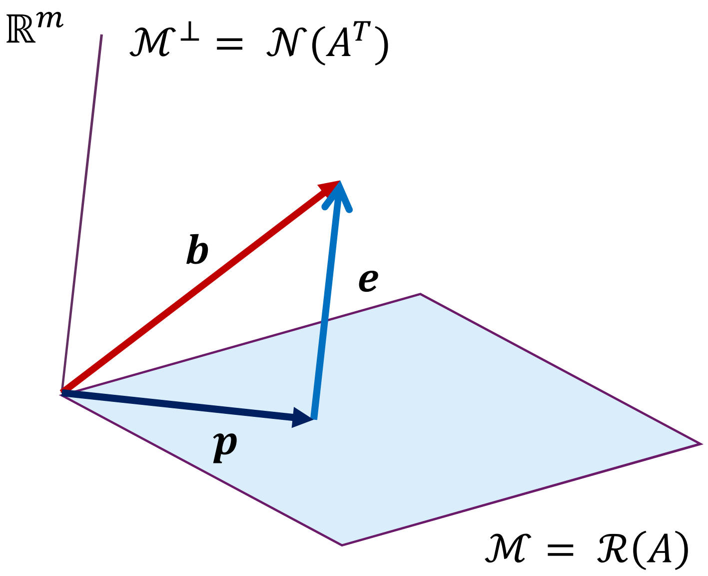
$\boldsymbol{p}$满足$\| \boldsymbol{b}-\boldsymbol{p} \| \leq \| \boldsymbol{b}-\boldsymbol{v} \|, \forall \boldsymbol{v} \in \mathcal{R}(A)$

问题: 如何求正交投影矩阵?

π

如果A是**列满秩**矩阵，

A^TA可逆，正规方程$A^TA\boldsymbol{x} = A^T\boldsymbol{b}$有唯一解



$$\boxed{\hat{\boldsymbol{x}} = \left(A^TA\right)^{-1}A^T\boldsymbol{b}}$$

$\boldsymbol{b}$在子空间$\mathcal{M} = \mathcal{R}(A)$上的投影为

$\boldsymbol{p} = A\hat{\boldsymbol{x}} = A\left(A^TA\right)^{-1}A^T\boldsymbol{b}$

正交投影映射为

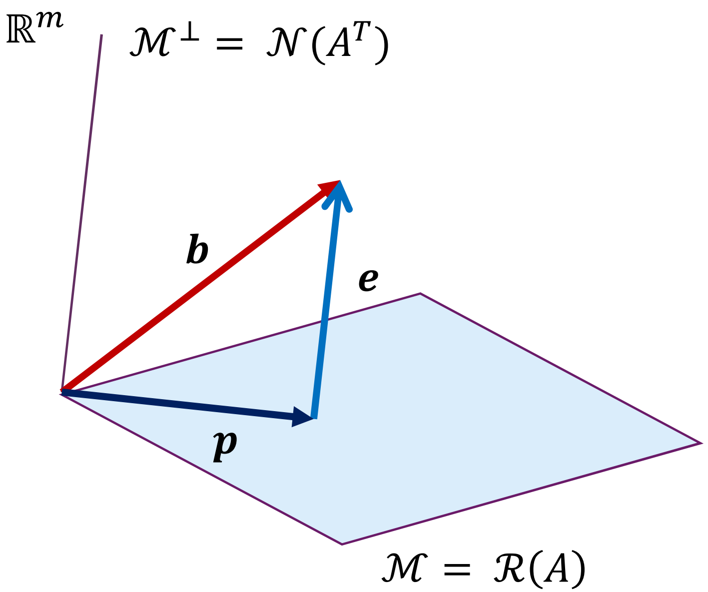
$$\boldsymbol{b} \mapsto A\left(A^TA\right)^{-1}A^T\boldsymbol{b}$$

正交投影矩阵

$$P_{\mathcal{R}(A)} = A\left(A^TA\right)^{-1}A^T$$

$P_{\mathcal{R}(A)}$ 也记为P_A

问题：对任意矩阵A，如何求$\mathcal{R}(A)$上正交投影矩阵?



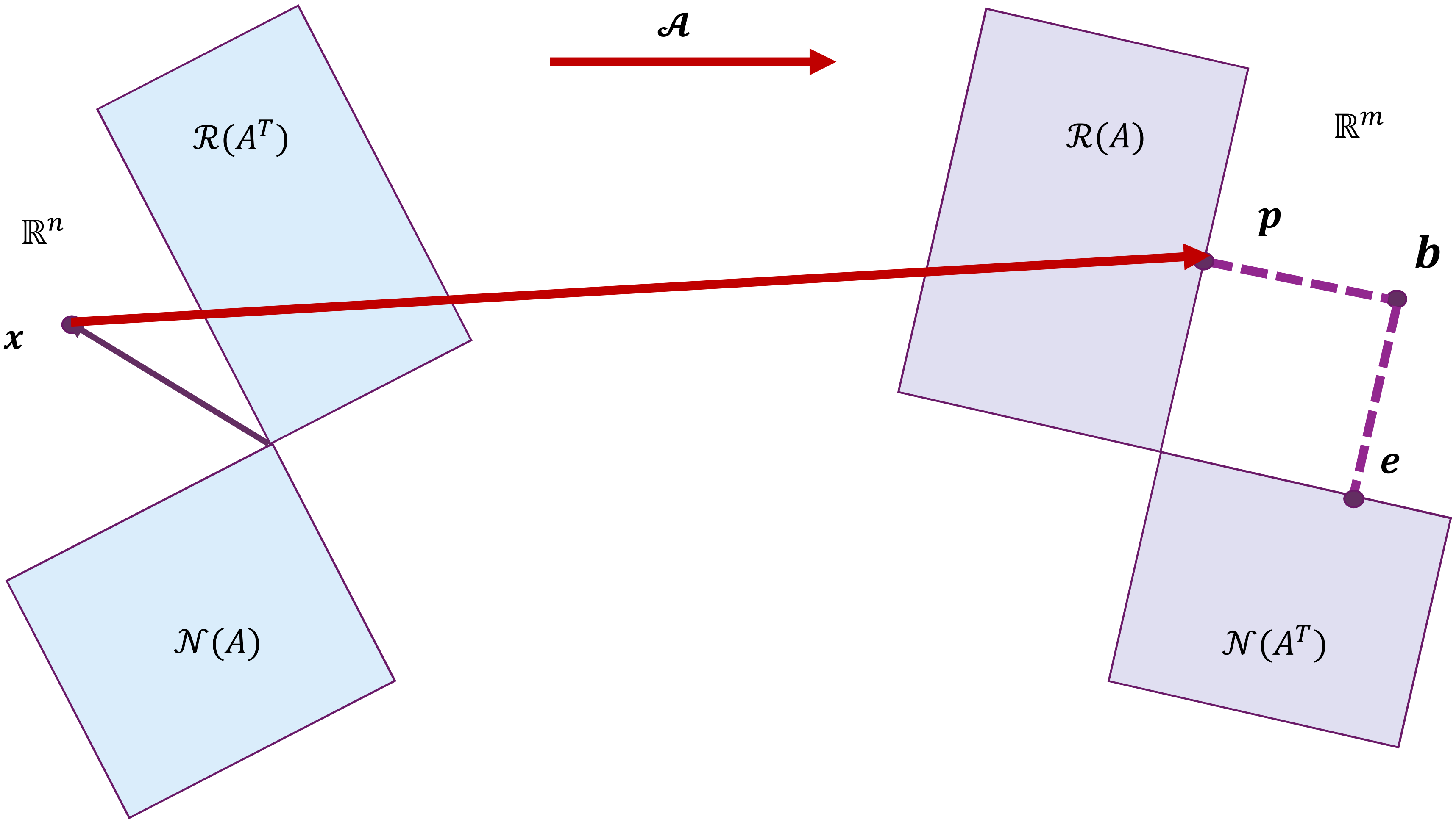
(1) 取A的列空间的一组基构成列满秩矩阵B（这里用到第二章内容）

(2) $\boldsymbol{b}$在子空间$\mathcal{R}(A) = \mathcal{R}(B)$上的正交投影为

$$\boldsymbol{p} = B\hat{\boldsymbol{x}} = B\left(B^T B\right)^{-1} B^T \boldsymbol{b}$$

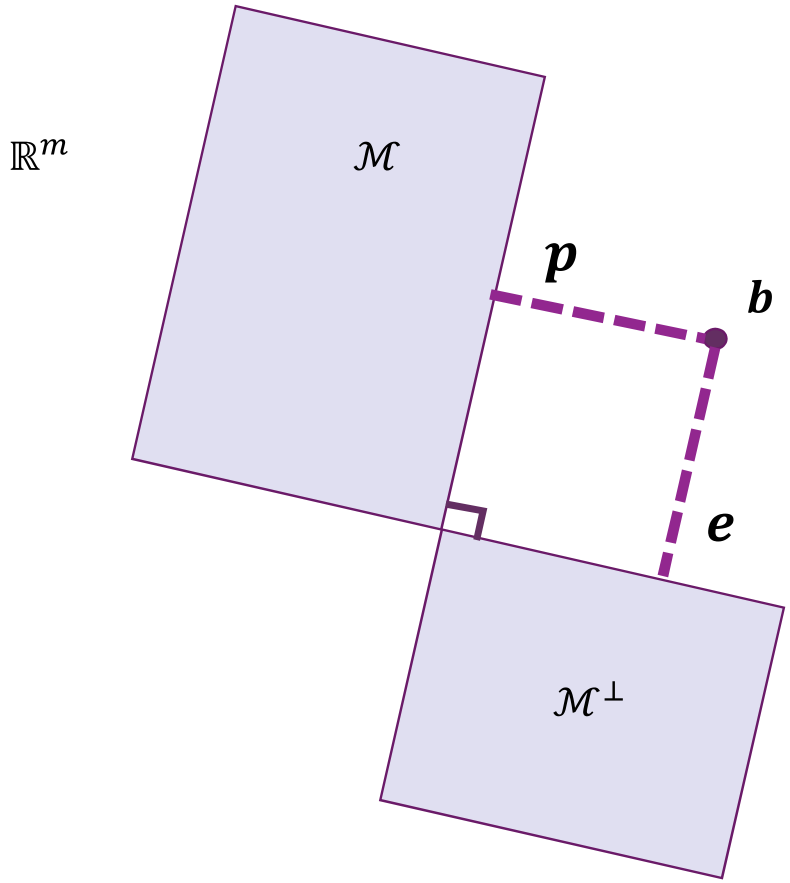
正交投影矩阵 $P_A = B\left(B^T B\right)^{-1} B^T$

正交投影:



$\boldsymbol{A}$
$\mathcal{R}(A^T)$
$\mathbb{R}^n$
$\boldsymbol{x}$
$\mathcal{N}(A)$
$\mathcal{R}(A)$
$\mathbb{R}^m$
$\boldsymbol{p}$
$\boldsymbol{b}$
$\boldsymbol{e}$
$\mathcal{N}(A^T)$

例:



$P_{\mathcal{M}}, P_{\mathcal{M}^\perp}: \mathbb{R}^m \to \mathbb{R}^m$

$P_{\mathcal{M}} + P_{\mathcal{M}^\perp} = ?$

$P_{\mathcal{M}}(\boldsymbol{b}) + P_{\mathcal{M}^\perp}(\boldsymbol{b}) = \boldsymbol{p} + \boldsymbol{e} = \boldsymbol{b}$

因此,

$P_{\mathcal{M}} + P_{\mathcal{M}^\perp} = I$ 为恒等映射