Review

- $\lim\limits_{x\to x_0} f(x) = \mathrm{A} \in [-\infty, +\infty]$的定义与几何意义

$$\lim_{x\to x_0^{\pm}} f(x),\ \lim_{x\to\infty} f(x),\ \lim_{x\to\pm\infty} f(x)$$

- 极限的性质

唯一性, 局部有界性, 保序性, 四则运算,

夹挤原理, 单调收敛原理, 复合函数的极限

- 重要不等式

$$|\sin x| \le |x|, \forall x \in \mathbb{R}. \qquad |x| \le |\tan x|, \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

- 重要极限

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e, \qquad \lim_{x\to 0}(1+x)^{1/x} = e,$$

$$\lim_{x\to+\infty}\frac{\log_a x}{x^b} = 0, \lim_{x\to 0^+} x^b \log_a x = 0 \ (a>1, b>0),$$

$$\lim_{x\to+\infty}\frac{x^b}{a^x} = 0 \ (a>1, b\in\mathbb{R}), \qquad \lim_{x\to+\infty}\frac{a^x}{x^x} = 0 \ (a>0, a\neq 1),$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \qquad \lim_{x\to x_0} e^x = e^{x_0}, \qquad \lim_{x\to x_0}\ln x = \ln x_0,$$

$$\lim_{x\to x_0} u(x)^{v(x)} = \left(\lim_{x\to x_0} u(x)\right)^{\lim\limits_{x\to x_0} v(x)} \ (\text{成立的条件?})$$

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•Thm. f在$U(x_0,\rho)$中有定义,则以下命题等价:

(1)$\forall\varepsilon>0,\exists\delta>0,\forall x,y\in U(x_0,\delta)$,有$|f(x)-f(y)|<\varepsilon$;

(2)$\exists A\in\mathbb{R}$,对$U(x_0,\rho)$中任意收敛到x_0的点列$\{x_n\}$,有

$$\lim_{n\to\infty} f(x_n)=A;$$

(3) $\lim\limits_{x\to x_0} f(x)=A.$

Remark. (1) $\Leftrightarrow$ (3) (函数极限的Cauchy收敛原理)

Remark. (2) $\Leftrightarrow$ (3) (用数列的极限来研究函数的极限)

§ 4. 无穷小量与无穷大量

Def.（无穷小量与无穷大量）

(1)若 $\lim\limits_{x\to x_0} f(x)=0$,则称$x\to x_0$时, $f(x)$是无穷小量, 记作

$$f(x)\to 0(x\to x_0);$$

(2)若 $\lim\limits_{x\to x_0} f(x)=\infty$,则称$x\to x_0$时, $f(x)$是无穷大量, 记作

$$f(x)\to \infty(x\to x_0);$$

(3)若 $\lim\limits_{x\to x_0} f(x)=\pm\infty$,则称$x\to x_0$时, $f(x)$是正（负）无穷大量, 记作$f(x)\to \pm\infty(x\to x_0)$.

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Def. 设$x \to x_0$时, $f(x)$与$g(x)$都是无穷大量.

(1)若$\lim\limits_{x \to x_0} f(x)/g(x) = 0$, 则称$x \to x_0$时, $f(x)$是$g(x)$的低阶无穷大量, 记作$f(x) = o(g(x))\ (x \to x_0)$;

(2)若$\lim\limits_{x \to x_0} f(x)/g(x) = c \neq 0$, 则称$x \to x_0$时, $f(x)$与$g(x)$是同阶无穷大量;特别地, 当$c = 1$时, 称$x \to x_0$时, $f(x)$与$g(x)$是等价无穷大量, 记作 $f(x) \sim g(x)\ \ (x \to x_0)$;

(3)若$\exists M > 0, \delta > 0$, 当$0 < |x - x_0| < \delta$时, 有$|f(x)/g(x)| < M$, 则称$x \to x_0$时, $f(x)$被$g(x)$控制, 记为$f(x) = O(g(x))\ (x \to x_0)$.

Def. 设$x \to x_0$时, $f(x)$与$g(x)$都是无穷小量, 且$g(x) \neq 0$.

(1)若$\lim\limits_{x \to x_0} f(x)/g(x) = 0$, 则称$x \to x_0$时, $f(x)$是$g(x)$的高阶无穷小量, 记作$f(x) = o(g(x))\ (x \to x_0)$;

(2)若$\lim\limits_{x \to x_0} f(x)/g(x) = c \neq 0$, 则称$x \to x_0$时, $f(x)$与$g(x)$是同阶无穷小量; 特别地, 当$c = 1$时, 称$x \to x_0$时, $f(x)$与$g(x)$是等价无穷小量, 记作 $f(x) \sim g(x)\ \ (x \to x_0)$;

(3)若$\exists M > 0, \delta > 0$, 当$0 < |x - x_0| < \delta$时, 有$|f(x)/g(x)| < M$, 则称$x \to x_0$时, $f(x)$被$g(x)$控制, 记为$f(x) = O(g(x))\ (x \to x_0)$.

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Question. 无穷小量$g(x)$在x_0的任意小去心邻域中都不满足$g(x)\neq 0$,如何修改高阶无穷小、等价无穷小等概念?

Def. 设$x\to x_0$时, $f(x)$与$g(x)$都是无穷小量, $\cancel{且g(x)\neq 0}$.

(1)若$\forall\varepsilon>0,\exists\delta>0, s.t.$

$$|f(x)|\leq\varepsilon|g(x)|, \forall 0<|x-x_0|<\delta,$$

则称$x\to x_0$时, $f(x)$是$g(x)$的高阶无穷小量, 记作

$$f(x)=o(g(x))\ (x\to x_0);$$

(2) $c \neq 0$. 若 $\forall \varepsilon > 0, \exists \delta > 0, s.t.$

$$|f(x) - cg(x)| \leq \varepsilon |g(x)|, \forall 0 < |x - x_0| < \delta,$$

则称 $x \to x_0$ 时, $f(x)$ 与 $g(x)$ 是同阶无穷小量; 特别地, 当 $c = 1$ 时, 称 $x \to x_0$ 时, $f(x)$ 与 $g(x)$ 是等价无穷小量, 记作

$$f(x) \sim g(x) \quad (x \to x_0);$$

(3) 若 $\exists M > 0, \delta > 0, s.t.$

$$|f(x)| \leq M |g(x)|, \ \forall 0 < |x - x_0| < \delta,$$

则称 $x \to x_0$ 时, $f(x)$ 被 $g(x)$ 控制, 记为 $f(x) = O(g(x)) \ (x \to x_0)$.

(4)若 $\lim\limits_{x\to x_0}\dfrac{f(x)}{(x-x_0)^k}=c\neq 0$, 称 $x\to x_0$ 时, $f(x)$ 是 k 阶无穷小量.

Question. $f(x)\to 0(x\to x_0)$, 是否一定存在 $k>0$, s.t. $x\to x_0$ 时, $f(x)$ 为 k 阶无穷小量?

否! 试考虑 $f(x)=x\sin\dfrac{1}{x}$.

Prop. $x\to x_0$ 时, $f(x)=o(1)$, $g(x)=O(1)$, 则 $f(x)g(x)=o(1)$.

Thm. 当$x \to 0$时：

$$(1)\ \sin x \sim \tan x \sim x; \qquad (2)\ 1-\cos x \sim \frac{1}{2}x^2; \qquad (3)\ \ln(1+x) \sim x;$$

$$(4)\ e^x - 1 \sim x,\ a^x - 1 \sim x\ln a\ (a>0); \qquad (5)\ (1+x)^\alpha - 1 \sim \alpha x.$$

Proof.(1) $$\lim_{x\to 0}\frac{\sin x}{x} = 1,\ \lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x} = 1.$$

(2) $$\lim_{x\to 0}\frac{1-\cos x}{\frac{1}{2}x^2} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\frac{1}{2}x^2} = \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = 1.$$

$$(3)\ \lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=\ln e=1.$$

(4) 令$u=e^x-1$，则$x=\ln(1+u)$，$x\to 0$等价于$u\to 0$，

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{u\to 0}\frac{u}{\ln(1+u)}=1.$$

$$a^x-1=e^{x\ln a}-1\sim x\ln a\quad (x\to 0).$$

$$(5)\ \frac{(1+x)^\alpha-1}{\alpha x}=\frac{e^{\alpha\ln(1+x)}-1}{\alpha\ln(1+x)}\cdot\frac{\ln(1+x)}{x}\to 1\ (x\to 0).$$

Remark. 设$x \to x_0$时, $f(x)$与$g(x)$是等价无穷小量, 则

$$f(x) = g(x) + o(g(x)), \quad x \to x_0.$$

Proof. $x \to x_0$时, $f(x)$与$g(x)$是等价无穷小量, 则

$$\forall \varepsilon > 0, \exists \delta > 0, s.t.$$

$$|f(x) - g(x)| \le \varepsilon |g(x)|, \forall 0 < |x - x_0| < \delta.$$

也即

$$f(x) - g(x) = o(g(x)), \quad x \to x_0. \square$$

Remark.

$$\sin x \sim x (x \to 0) \quad \Rightarrow \quad \sin x = x + o(x)(x \to 0)$$

$$\tan x \sim x (x \to 0) \quad \Rightarrow \quad \tan x = x + o(x)(x \to 0)$$

$$1 - \cos x \sim \frac{1}{2} x^2 (x \to 0) \Rightarrow 1 - \cos x = \frac{1}{2} x^2 + o(x^2)(x \to 0)$$

$$\ln(1+x) \sim x \quad \Rightarrow \quad \ln(1+x) = x + o(x)(x \to 0)$$

$$e^x - 1 \sim x (x \to 0) \quad \Rightarrow \quad e^x - 1 = x + o(x)(x \to 0)$$

$$a^x - 1 \sim x \ln a (x \to 0) \quad \Rightarrow \quad a^x - 1 = x \ln a + o(x)(x \to 0)$$

$$(1+x)^\alpha - 1 \sim \alpha x (x \to 0) \quad \Rightarrow \quad (1+x)^\alpha - 1 = \alpha x + o(x)(x \to 0)$$

Remark. $x \to 0$时,

$$o(x)+o(x)=o(x);$$

$$c \in \mathbb{R}, c \neq 0, \text{则} o(cx)=o(x);$$

$$o(x)+o(x^2)=o(x);$$

$$o(x^2)=o(x);$$

$$o(x) \neq o(x^2);$$

$$\vdots$$

Ex. $\lim\limits_{x\to 0^+}(\sin x)^{1/\ln x}$ $\left(0^0\text{型}\right)$

$$= \lim_{x\to 0^+} e^{(\ln\sin x)/\ln x} = \lim_{x\to 0^+} e^{\left(\ln\frac{\sin x}{x}+\ln x\right)/\ln x} = e.\square$$

Remark. 1)指数-对数变换. 2)利用极限典式.

Ex. $\lim\limits_{x\to 0^+} x^{\sin x} = \lim\limits_{x\to 0^+} e^{\sin x\ln x} = \lim\limits_{x\to 0^+} e^{\frac{\sin x}{x}x\ln x} = e^0 = 1.\square$

$\left(0^0\text{型}\right)$

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Ex. $\lim\limits_{x\to 0^+}(e^x+2x)^{1/x} = \lim\limits_{x\to 0^+}(1+e^x+2x-1)^{\frac{1}{e^x+2x-1}\cdot\frac{e^x+2x-1}{x}}$

$$= e^{\lim\limits_{x\to 0^+}\frac{e^x+2x-1}{x}} = e^{2+\lim\limits_{x\to 0^+}\frac{e^x-1}{x}} = e^3.$$

$(1^\infty$型$)$

另解: $\lim\limits_{x\to 0^+}(e^x+2x)^{1/x} = \lim\limits_{x\to 0^+}\exp\{\frac{1}{x}\ln(1+e^x+2x-1)\}$

$$= \exp\{\lim_{x\to 0^+}\frac{\ln(1+e^x+2x-1)}{e^x+2x-1}\cdot\lim_{x\to 0^+}\frac{e^x+2x-1}{x}\}$$

$$= \exp\{1\cdot\lim_{x\to 0^+}(\frac{e^x-1}{x}+2)\} = e^3.\square$$

Remark.极限运算中$o(\cdot)$的运用有时能简化计算.

Ex. $\lim\limits_{x\to1}\left(\dfrac{m}{1-x^m}-\dfrac{n}{1-x^n}\right)$, m,n为正整数.

解: 令$x=1+t$,则$x\to1$等价于$t\to0$.

$$\frac{m}{1-x^m}-\frac{n}{1-x^n}=\frac{n}{(1+t)^n-1}-\frac{m}{(1+t)^m-1}$$

$$=\frac{n}{nt+\dfrac{n(n-1)}{2}t^2+o(t^2)}-\frac{m}{mt+\dfrac{m(m-1)}{2}t^2+o(t^2)}$$

$$=\frac{\dfrac{mn(m-n)}{2}t^2+o(t^2)}{mnt^2+o(t^2)}\to\frac{m-n}{2}(t\to0)\square$$

Ex. $\lim\limits_{x\to1}\dfrac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}$

解：令$x=1+t$，则$x\to1$等价于$t\to0$.

$$\lim_{x\to1}\frac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}=\lim_{t\to0}\frac{(1+t)^{1/m}-1}{(1+t)^{1/n}-1}$$

$$=\lim_{t\to0}\frac{(1+t)^{1/m}-1}{t/m}\cdot\lim_{t\to0}\frac{t/n}{(1+t)^{1/n}-1}\cdot\lim_{t\to0}\frac{t/m}{t/n}$$

$$=\lim_{t\to0}\frac{t/m}{t/n}=\frac{n}{m}.$$

Remark. 等价因子替换法. $t\to0$时，$(1+t)^{1/m}-1\sim\dfrac{t}{m}$，

$(1+t)^{1/n}-1\sim\dfrac{t}{n}$，则 $\lim\limits_{t\to0}\dfrac{(1+t)^{1/m}-1}{(1+t)^{1/n}-1}=\lim\limits_{t\to0}\dfrac{t/m}{t/n}$.

Ex. $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^2\ln(1+x)}$

解法一: $x\to 0$时, $\tan x\sim x$, $\sin x\sim x$, $\ln(1+x)\sim x$.

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^2\ln(1+x)}=\lim_{x\to 0}\frac{x-x}{x^2\cdot x}=0.$$ 是否正确? ×

解法二:

$$\frac{\tan x-\sin x}{x^2\ln(1+x)}=\frac{\sin x(1-\cos x)}{x^2\cos x\ln(1+x)}$$

$$=\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2/2}\cdot\frac{x}{\ln(1+x)}\cdot\frac{1}{2\cos x}\to\frac{1}{2}\quad(x\to 0).$$

解法三:

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^2\ln(1+x)}=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^2\cos x\ln(1+x)}=\lim_{x\to 0}\frac{x\cdot\frac{1}{2}x^2}{x^2\cos x\cdot x}=\frac{1}{2}.\square$$

Question.何时不需强调因子亦可进行等价无穷小替换?

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^2\ln(1+x)}=\lim_{x\to 0}\frac{x+o(x)-\left(x+o(x)\right)}{x^2\cdot\left(x+o(x)\right)}$$

$$=\lim_{x\to 0}\frac{o(x)}{x^3+o(x^3)}$$ 无法继续计算!

用x近似$\sin x$, $\tan x$, 过于粗糙,因此

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^2\ln(1+x)}\cancel{=}\lim_{x\to 0}\frac{x-x}{x^2\cdot x}$$

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\ln(1+x)}=\lim_{x\to 0}\frac{x+o(x)-\left(x+o(x)\right)}{x+o(x)}=\lim_{x\to 0}\frac{o(x)}{x+o(x)}$$

$$=\lim_{x\to 0}\frac{\frac{o(x)}{x}}{1+\frac{o(x)}{x}}=\frac{\lim\limits_{x\to 0}\frac{o(x)}{x}}{1+\lim\limits_{x\to 0}\frac{o(x)}{x}}=\frac{0}{1}=0.$$

用x近似$\sin x, \tan x$,精度足够,因此

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\ln(1+x)}\neq\lim_{x\to 0}\frac{x-x}{x}.$$

$$\lim_{t\to 0}\frac{(1+t)^{1/m}-1}{(1+t)^{1/n}-1}=\lim_{t\to 0}\frac{t/m+o(t)}{t/n+o(t)}$$

$$=\lim_{t\to 0}\frac{\frac{1}{m}+\frac{o(t)}{t}}{\frac{1}{n}+\frac{o(t)}{t}}=\frac{\frac{1}{m}+\lim\limits_{t\to 0}\frac{o(t)}{t}}{\frac{1}{n}+\lim\limits_{t\to 0}\frac{o(t)}{t}}=\frac{n}{m}$$

用$t/m, t/n$近似$(1+t)^{1/m}-1, (1+t)^{1/n}-1$，精度足够，因此

$$\lim_{t\to 0}\frac{(1+t)^{1/m}-1}{(1+t)^{1/n}-1}=\lim_{t\to 0}\frac{t/m}{t/n}.$$

Remark. 带小$o(\cdot)$运算可以避免不恰当的无穷小替换！

Ex. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{\cos x}}{\cos\sqrt{x}-1+x}$ $\left(\dfrac{0}{0}\text{型}\right)$

解: $1-\sqrt{\cos x}=1-e^{\frac{1}{2}\ln\cos x}\sim-\dfrac{1}{2}\ln\cos x=-\dfrac{1}{2}\ln(1-2\sin^2\dfrac{x}{2})$

$$\sim\sin^2\frac{x}{2}\sim\frac{x^2}{4}\quad(x\to 0).$$

$$\lim_{x\to 0}\frac{1-\sqrt{\cos x}}{\cos\sqrt{x}-1+x}=\frac{1}{4}\lim_{x\to 0}\frac{x^2}{\cos\sqrt{x}-1+x}$$

$$=\frac{1}{4}\lim_{x\to 0}\frac{x}{\dfrac{\cos\sqrt{x}-1}{x}+1}=\frac{1}{4}\cdot\frac{0}{-\dfrac{1}{2}+1}=0.\square$$

Ex. $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}=\lim\limits_{x\to+\infty}\dfrac{\sqrt{1+\sqrt{\dfrac{1}{x}+\sqrt{\dfrac{1}{x^3}}}}}{\sqrt{1+\dfrac{1}{x}}}=1.$

Ex. $\lim\limits_{x\to+\infty}\dfrac{\ln(2+\sqrt{x})}{\ln(6+\sqrt[6]{x})}=\lim\limits_{x\to+\infty}\dfrac{\ln\sqrt{x}+\ln(1+2/\sqrt{x})}{\ln\sqrt[6]{x}+\ln(1+6/\sqrt[6]{x})}\overset{?}{=}\lim\limits_{x\to+\infty}\dfrac{\ln\sqrt{x}}{\ln\sqrt[6]{x}}=3.$ □

$$\ln\sqrt{x}+\ln(1+2/\sqrt{x})\sim\ln\sqrt{x}\ \ (x\to+\infty)$$

$$\ln\sqrt[6]{x}+\ln(1+6/\sqrt[6]{x})\sim\ln\sqrt[6]{x}\ \ (x\to+\infty)$$

Remark.等价无穷大因子替换！

Ex. $\lim\limits_{x\to+\infty}\left(\sqrt{x^2+2x}-\sqrt[3]{x^3-x^2}\right)$

解法一: $\lim\limits_{x\to+\infty}(\sqrt{x^2+2x}-x)$

$$=\lim_{x\to+\infty}\frac{2x}{\sqrt{x^2+2x}+x}=\lim_{x\to+\infty}\frac{2}{\sqrt{1+2/x}+1}=1,$$

$\lim\limits_{x\to+\infty}(\sqrt[3]{x^3-x^2}-x)$

$$=\lim_{x\to+\infty}\frac{-x^2}{\left(\sqrt[3]{x^3-x^2}\right)^2+x\cdot\sqrt[3]{x^3-x^2}+x^2}=-\frac{1}{3}.$$

$$原式=\lim_{x\to+\infty}(\sqrt{x^2+2x}-x)-\lim_{x\to+\infty}(\sqrt[3]{x^3-x^2}-x)=\frac{4}{3}.$$

解法二：令$y=1/x$. 则$x\to+\infty$时, $y\to 0$, 且

$$\lim_{x\to+\infty}\left(\sqrt{x^2+2x}-\sqrt[3]{x^3-x^2}\right)=\lim_{y\to 0}\frac{(1+2y)^{1/2}-(1-y)^{1/3}}{y}$$

$$=\lim_{y\to 0}\left(\frac{(1+2y)^{1/2}-1}{y}-\frac{(1-y)^{1/3}-1}{y}\right)$$

$$=\lim_{y\to 0}\frac{(1+2y)^{1/2}-1}{y}-\lim_{y\to 0}\frac{(1-y)^{1/3}-1}{y}$$

$$=2\cdot\frac{1}{2}-(-1)\cdot\frac{1}{3}=\frac{4}{3}.\square$$

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Ex. $\lim\limits_{x\to+\infty}\left(\sqrt[n]{(x^2+1)(x^2+2)\cdots(x^2+n)}-x^2\right)$

$$=\lim_{x\to+\infty}x^2\left(\sqrt[n]{(1+\frac{1}{x^2})(1+\frac{2}{x^2})\cdots(1+\frac{n}{x^2})}-1\right)$$

$$=\lim_{x\to+\infty}x^2\left(e^{\frac{1}{n}\sum_{k=1}^{n}\ln(1+k/x^2)}-1\right)$$

$$=\lim_{x\to+\infty}\frac{x^2}{n}\sum_{k=1}^{n}\ln(1+k/x^2)\qquad\left(\lim_{x\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\ln(1+k/x^2)=0\right)$$

$$=\lim_{x\to+\infty}\frac{1}{n}\sum_{k=1}^{n}k\ln(1+k/x^2)^{x^2/k}=\frac{1}{n}\sum_{k=1}^{n}k=\frac{n+1}{2}.\square$$

Ex. $\lim\limits_{x\to0^+}\left(2\sin\sqrt{x}+\sqrt{x}\sin\frac{1}{x}\right)^x$

解：$\lim\limits_{x\to0^+}\frac{\sin\sqrt{x}}{\sqrt{x}}=1$, 故$\exists\delta>0, s.t.\ \frac{3}{4}<\frac{\sin\sqrt{x}}{\sqrt{x}}\le1, \forall 0<|x|<\delta.$

于是 $\frac{1}{2}\sqrt{x}\le 2\sin\sqrt{x}+\sqrt{x}\sin\frac{1}{x}\le 3\sqrt{x}, \ \forall 0<|x|<\delta.$

$$\lim_{x\to0^+}\left(\frac{1}{2}\sqrt{x}\right)^x=\frac{\lim\limits_{x\to0^+}(\sqrt{x})^x}{\lim\limits_{x\to0^+}2^x}=\lim_{x\to0^+}(\sqrt{x})^x=\lim_{x\to0^+}e^{\frac{1}{2}x\ln x}=e^0=1.$$

$$\lim_{x\to0^+}(3\sqrt{x})^x=\lim_{x\to0^+}3^x\cdot\lim_{x\to0^+}(\sqrt{x})^x=1\cdot1=1.$$

由夹挤原理, $\lim\limits_{x\to0^+}\left(2\sin\sqrt{x}+\sqrt{x}\sin\frac{1}{x}\right)^x=1.\square$

Ex. f在$(0,+\infty)$上单调, $\lim\limits_{x\to+\infty}\dfrac{f(2x)}{f(x)}=1, a>0$, 则 $\lim\limits_{x\to+\infty}\dfrac{f(ax)}{f(x)}=1.$

Proof. 对任意正整数n, 有

$$\lim_{x\to+\infty}\frac{f(2^n x)}{f(x)}=\lim_{x\to+\infty}\frac{f(2^n x)}{f(2^{n-1}x)}\cdot\frac{f(2^{n-1}x)}{f(2^{n-2}x)}\cdot\cdots\cdot\frac{f(2x)}{f(x)}=1.$$

若$a\ge1$, 则$\exists n>0, s.t.\ 2^0\le a<2^n$, f单调, 从而

$$1=\frac{f(2^0 x)}{f(x)}\le(\ge)\frac{f(ax)}{f(x)}\le(\ge)\frac{f(2^n x)}{f(x)},\quad \forall x>1.$$

由夹挤原理, $\lim\limits_{x\to+\infty}\dfrac{f(ax)}{f(x)}=1.$

若$0<a<1$, 则 $\lim\limits_{x\to+\infty}\dfrac{f(ax)}{f(x)}=\lim\limits_{t\to+\infty}\dfrac{f(t)}{f(t/a)}=1.$ □

Ex. $\lim\limits_{x\to 0} f(x)=0, \lim\limits_{x\to 0}\dfrac{f(x)-f(x/2)}{x}=0$, 则 $\lim\limits_{x\to 0}\dfrac{f(x)}{x}=0.$

Proof. $\underline{\forall \varepsilon>0}$, 由 $\lim\limits_{x\to 0}\dfrac{f(x)-f(x/2)}{x}=0$, $\underline{\exists \delta>0, s.t.}$

$$\left|f(x)-f(x/2)\right|<\varepsilon|x|,\ \forall 0<|x|<\delta.$$

$$\left|f(x)\right|\le\sum_{k=1}^{n}\left|f\left(\frac{x}{2^{k-1}}\right)-f\left(\frac{x}{2^{k}}\right)\right|+\left|f\left(\frac{x}{2^{n}}\right)\right|$$

$$\le\sum_{k=1}^{n}\frac{\varepsilon|x|}{2^{k-1}}+\left|f\left(\frac{x}{2^{n}}\right)\right|<2\varepsilon|x|+\left|f\left(\frac{x}{2^{n}}\right)\right|,\quad \forall 0<|x|<\delta.$$

令 $n\to+\infty$, 由 $\lim\limits_{x\to 0} f(x)=0$ 得 $\underline{\left|f(x)\right|\le 2\varepsilon|x|,\ \forall 0<|x|<\delta}.$ $\square$

Ex. $\lim\limits_{x\to\infty}(\sin\sqrt{x^2+1}-\sin\sqrt{x^2-1})$

解: $\left|\sin\sqrt{x^2+1}-\sin\sqrt{x^2-1}\right|$

$$=2\left|\cos\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{2}\sin\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{2}\right|$$

$$\le\left|\sqrt{x^2+1}-\sqrt{x^2-1}\right|=\left|\frac{1}{\sqrt{x^2+1}+\sqrt{x^2-1}}\right|\to 0\ (x\to\infty).$$

故 $\lim\limits_{x\to\infty}(\sin\sqrt{x^2+1}-\sin\sqrt{x^2-1})=0.\square$

$$\text{Ex.}\lim_{x\to 0}\left(\frac{e^x+e^{2x}+\cdots+e^{nx}}{n}\right)^{1/x}$$

1^∞型极限

$$=\lim_{x\to 0}\left(1+\left(\frac{e^x+e^{2x}+\cdots+e^{nx}}{n}-1\right)\right)^{\frac{1}{\frac{e^x+e^{2x}+\cdots+e^{nx}}{n}-1}\cdot\frac{\frac{e^x+e^{2x}+\cdots+e^{nx}}{n}-1}{x}}$$

$$=\exp\left\{\lim_{x\to 0}\left(\frac{e^x+e^{2x}+\cdots+e^{nx}}{n}-1\right)\Big/x\right\}$$

$$=\exp\left\{\lim_{x\to 0}\frac{(e^x-1)/x+(e^{2x}-1)/x+\cdots+(e^{nx}-1)/x}{n}\right\}$$

$$=e^{(1+2+\cdots+n)/n}=e^{(n+1)/2}.\square$$

解法二. $\lim\limits_{x\to 0}\left(\dfrac{e^x+e^{2x}+\cdots+e^{nx}}{n}\right)^{1/x}$

$$=\lim_{x\to 0}\left(1+\frac{(e^x-1)+(e^{2x}-1)+\cdots+(e^{nx}-1)}{n}\right)^{1/x}$$

$$=\lim_{x\to 0}\left(1+\frac{x+o(x)+2x+o(x)+\cdots+nx+o(x)}{n}\right)^{1/x}$$

$$=\lim_{x\to 0}\left(1+\frac{(n+1)x}{2}+o(x)\right)^{1/x}=e^{\lim\limits_{x\to 0}\left(\frac{(n+1)x}{2}+o(x)\right)\Big/x}=e^{(n+1)/2}.\square$$

Ex. $\lim\limits_{n\to+\infty} n^2\left(\sqrt[n]{x}-\sqrt[n+1]{x}\right) \quad (x>0)$

$$=\lim_{n\to+\infty} n^2 x^{1/(n+1)}\left(x^{1/n(n+1)}-1\right)$$

$$=\lim_{n\to+\infty}\left(x^{1/(n+1)}\cdot\frac{x^{1/n(n+1)}-1}{1/n(n+1)}\cdot\frac{n^2}{n(n+1)}\right)$$

$$=\lim_{n\to+\infty} x^{1/(n+1)}\cdot\lim_{n\to+\infty}\frac{x^{1/n(n+1)}-1}{1/n(n+1)}\cdot\lim_{n\to+\infty}\frac{n^2}{n(n+1)}=\ln x.\square$$

Remark. $a^t-1\sim t\ln a\ (t\to 0) \Leftrightarrow \lim\limits_{t\to 0}\dfrac{a^t-1}{t}=\ln a.$

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Ex. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\tan x}-\sqrt{1-\tan x}}{e^x-1}=1$

Proof. $\dfrac{\sqrt{1+\tan x}-\sqrt{1-\tan x}}{e^x-1}=\dfrac{\sqrt{1+\tan x}-1}{e^x-1}-\dfrac{\sqrt{1-\tan x}-1}{e^x-1},$

$$\frac{\sqrt{1+\tan x}-1}{e^x-1}=\frac{1}{2}\cdot\frac{(1+\tan x)^{1/2}-1}{\frac{1}{2}\tan x}\cdot\frac{\tan x}{x}\cdot\frac{x}{e^x-1}\to\frac{1}{2}\quad(x\to 0)$$

同理, $\dfrac{\sqrt{1-\tan x}-1}{e^x-1}\to-\dfrac{1}{2}\quad(x\to 0).\square$

Remark.上例可用等价因子替换法:

$$\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1-\tan x}}{e^x-1}$$

$$=\lim_{x\to 0}\frac{\sqrt{1+\tan x}-1}{e^x-1}-\lim_{x\to 0}\frac{\sqrt{1-\tan x}-1}{e^x-1}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2}\tan x}{x}-\lim_{x\to 0}\frac{-\frac{1}{2}\tan x}{x}=\frac{1}{2}-(-\frac{1}{2})=1.$$

采用$o(\cdot)$更为简洁:

$$\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1-\tan x}}{e^x-1}$$

$$=\lim_{x\to 0}\frac{1+\frac{1}{2}\tan x+o(\tan x)-\left(1-\frac{1}{2}\tan x+o(\tan x)\right)}{x+o(x)}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2}x+o(x)-\left(-\frac{1}{2}x+o(x)\right)}{x+o(x)}=1.$$

Ex.(1) $\lim\limits_{x\to 0+}\dfrac{(a^x-b^x)^2}{a^{x^2}-b^{x^2}}(a,b>0,a\neq b)$, (2) $\lim\limits_{x\to a}\dfrac{a^{a^x}-a^{x^a}}{a^x-x^a}(a>0)$

解.(1) $\lim\limits_{x\to 0+}\dfrac{(a^x-b^x)^2}{a^{x^2}-b^{x^2}}=\lim\limits_{x\to 0+}\dfrac{b^{2x}((a/b)^x-1)^2}{b^{x^2}((a/b)^{x^2}-1)}$

$$=\lim_{x\to 0+}\frac{((a/b)^x-1)^2}{(a/b)^{x^2}-1}=\lim_{x\to 0+}\frac{(x\ln(a/b))^2}{x^2\ln(a/b)}=\ln\frac{a}{b}.$$

(2) $\lim\limits_{x\to a}\dfrac{a^{a^x}-a^{x^a}}{a^x-x^a}=\lim\limits_{x\to a}\dfrac{a^{x^a}(a^{a^x-x^a}-1)}{a^x-x^a}$

$$=\lim_{x\to a}a^{x^a}\lim_{x\to a}\frac{(a^{a^x-x^a}-1)}{a^x-x^a}=a^{a^a}\ln a.\square$$

Ex.(1) $\lim\limits_{x\to 0+} x\ln x$, (2) $\lim\limits_{x\to 0+}(x^x-1)\ln x$, (3) $\lim\limits_{x\to 0+} x^{x^x-1}$

解:(1) $\lim\limits_{x\to 0+} x\ln x = \lim\limits_{x\to 0+}\dfrac{-\ln(1/x)}{1/x} = \lim\limits_{y\to +\infty}\dfrac{-\ln y}{y} = 0.$

(2) $\lim\limits_{x\to 0+}(x^x-1)\ln x = \lim\limits_{x\to 0+}(e^{x\ln x}-1)\ln x$

$$= \lim_{x\to 0+} x(\ln x)^2 = \lim_{x\to 0+}(\sqrt{x}\ln x)^2 = 0.$$

(3) $\lim\limits_{x\to 0+} x^{x^x-1} = \lim\limits_{x\to 0+} e^{(x^x-1)\ln x} = e^{\lim\limits_{x\to 0+}(x^x-1)\ln x} = e^0 = 1.\square$

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Question.以下解答是否正确?

$$\lim_{x\to+\infty}\frac{e^x}{\left(1+\frac{1}{x}\right)^{x^2}}=\lim_{x\to+\infty}\frac{e^x}{\left(\left(1+\frac{1}{x}\right)^x\right)^x}$$

$$=\lim_{x\to+\infty}\frac{e^x}{\left(\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x\right)^x}=\lim_{x\to+\infty}\frac{e^x}{e^x}=1. \qquad (\times)$$

不能局部取极限!

正确结果? 正确解答? $\sqrt{e}$, L'Hospital 或Taylor展开.

作业：习题2.4
No. 8,9(1)(2)(6)(8)(11),12

思考题（不交）：

1. 无穷多个无穷小量的乘积是否一定是无穷小量?
   证明或给反例。

2. $\alpha \geq 1, n \to +\infty$时，$n$个$o(\frac{1}{n^{\alpha}})$之和是否为$o(\frac{1}{n^{\alpha-1}})$?
   证明或给反例。