回顾上节课内容：

(1) 给定$\mathbb{R}^m$的子空间$\mathcal{M}$，则$\mathbb{R}^m$的子集$\mathcal{M}^\perp := \{\boldsymbol{v} \in \mathbb{R}^m | \boldsymbol{v} \perp \mathcal{M}\}$称为$\mathcal{M}$的**正交补**。$\mathcal{M}^\perp$也是$\mathbb{R}^m$的子空间。

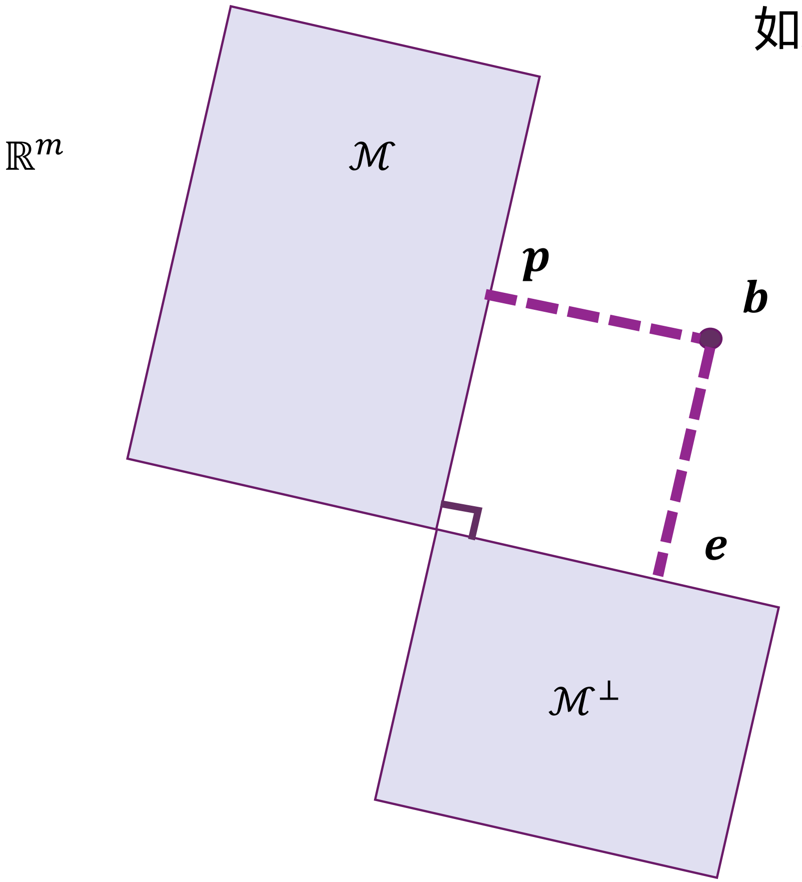
(2) $dim\mathcal{M}^\perp = m - dim\mathcal{M}$; $(\mathcal{M}^\perp)^\perp = \mathcal{M}$

(3) $\mathbb{R}^m = \mathcal{M} + \mathcal{M}^\perp$;

(4) 对任意$\boldsymbol{v} \in \mathbb{R}^m$，存在唯一的$\boldsymbol{v}_1 \in \mathcal{M}$，$\boldsymbol{v}_2 \in \mathcal{M}^\perp$使得$\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$。$\boldsymbol{v}_1, \boldsymbol{v}_2$分别称为$\boldsymbol{v}$向$\mathcal{M}$，$\mathcal{M}^\perp$的**正交投影。**

(5) $A \in M_{m\times n}(\mathbb{R})$. $\mathcal{N}\left(A^T\right)$与$\mathcal{R}(A)$在$\mathbb{R}^m$中互为正交补；$\mathcal{N}(A)$与$\mathcal{R}(A^T)$在$\mathbb{R}^n$中互为正交补

正交投影:



设$\mathcal{M} = Span\ (\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)$.
考虑$m \times n$阶矩阵$A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n]$，因此$\mathcal{M} = \mathcal{R}(A)$.
如果取$\mathcal{M}$的一组基$\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n$，则$A$是列满秩矩阵

令$\boldsymbol{p} = A\boldsymbol{x} \in \mathcal{R}(A)$.
$\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \boldsymbol{b} - A\boldsymbol{x} \in \mathcal{N}(A^T)$, 即$A^T\boldsymbol{e} = \boldsymbol{0}$
因此, 得到方程

$$A^T(\boldsymbol{b} - A\boldsymbol{x}) = \boldsymbol{0}$$

$$\boxed{A^T A\boldsymbol{x} = A^T\boldsymbol{b}}$$

正规方程，
总有解！

如果A是列满秩矩阵，A^TA可逆
$\boldsymbol{b}$在子空间$\mathcal{M} = \mathcal{R}(A)$上的投影为

$$\boldsymbol{p} = A\hat{\boldsymbol{x}} = A\left(A^TA\right)^{-1}A^T\boldsymbol{b}$$

正交投影矩阵

$$P_A = P_{\mathcal{R}(A)} = A\left(A^TA\right)^{-1}A^T$$

注意： $\boldsymbol{p}$是$\mathcal{M}$中距离$\mathbf{b}$最近的向量，即
$\|\boldsymbol{b} - \boldsymbol{p}\| \le \|\boldsymbol{b} - \boldsymbol{v}\|, \forall \boldsymbol{v} \in \mathcal{R}(A)$

例：向一维子空间投影

设$\boldsymbol{u} \in \mathbb{R}^m$为非零向量。求$\boldsymbol{b} \in \mathbb{R}^m$向一维子空间$\mathbb{R}\boldsymbol{u}$的正交投影矩阵。

$$\hat{\boldsymbol{x}} = \left(\boldsymbol{u}^T\boldsymbol{u}\right)^{-1}\boldsymbol{u}^T\boldsymbol{b} = \frac{\boldsymbol{u}^T\boldsymbol{b}}{\boldsymbol{u}^T\boldsymbol{u}}$$

$$\boldsymbol{p} = \boldsymbol{u}\left(\boldsymbol{u}^T\boldsymbol{u}\right)^{-1}\boldsymbol{u}^T\boldsymbol{b} = \frac{\boldsymbol{u}\boldsymbol{u}^T}{\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{b} = \frac{\boldsymbol{u}^T\boldsymbol{b}}{\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{u} = \hat{\boldsymbol{x}}\boldsymbol{u}.$$

正交投影矩阵$P_{\boldsymbol{u}} = \frac{1}{\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{u}\boldsymbol{u}^T$，这里$\boldsymbol{u}\boldsymbol{u}^T$是秩为1的$m$ 阶方阵。

如果$\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$，$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$,那么

$$\hat{\boldsymbol{x}} = \frac{1}{m}\boldsymbol{u}^T\boldsymbol{b} = \frac{b_1 + \cdots + b_m}{m}, \qquad \boldsymbol{p} = \frac{b_1 + \cdots + b_m}{m}\boldsymbol{u}.$$

对m个数值$b_1, \dots, b_m$做平均

例：向超平面投影

求向超平面$\mathcal{N}\left(\boldsymbol{u}^T\right)$投影的正交投影矩阵：

由于$\mathcal{N}\left(\boldsymbol{u}^T\right)$是$\mathbb{R}\boldsymbol{u}$的正交补，向超平面$\mathcal{N}\left(\boldsymbol{u}^T\right)$投影的正交投影矩阵为

$$I - P_{\boldsymbol{u}} = I - \frac{1}{\boldsymbol{u}^T\boldsymbol{u}}\boldsymbol{u}\boldsymbol{u}^T$$

如果$\boldsymbol{u}^T\boldsymbol{u} = 1$，即$\boldsymbol{u}$为单位向量，向超平面$\mathcal{N}\left(\boldsymbol{u}^T\right)$投影的正交投影矩阵为$I - \boldsymbol{u}\boldsymbol{u}^T$.

例：矩阵列空间上的正交投影

$A=\begin{bmatrix}1 & 2 & 1\\ 2 & 2 & 0\\ -1 & 0 & 1\end{bmatrix}=[\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3]$. 求子空间$\mathcal{R}(A)$上的正交投影矩阵。

注意$\boldsymbol{a}_2-\boldsymbol{a}_1=\boldsymbol{a}_3$，于是,

$\mathcal{R}(A)=Span(\boldsymbol{a}_1,\boldsymbol{a}_2)=Span(\boldsymbol{a}_1,\boldsymbol{a}_3)=Span(\boldsymbol{a}_2,\boldsymbol{a}_3)$

(a) 使用列满秩矩阵$B=[\boldsymbol{a}_1\ \ \boldsymbol{a}_2]$计算

(b) 使用列满秩矩阵$C=[\boldsymbol{a}_1\ \ \boldsymbol{a}_3]$计算

例：矩阵列空间上的正交投影

$A=\begin{bmatrix}1&2&1\\2&2&0\\-1&0&1\end{bmatrix}=[\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3]$.求子空间$\mathcal{R}(A)$上的正交投影矩阵。

(a)令$B=[\boldsymbol{a}_1,\boldsymbol{a}_2]=\begin{bmatrix}1&2\\2&2\\-1&0\end{bmatrix}$. $B^TB=\begin{bmatrix}1&2&-1\\2&2&0\end{bmatrix}\begin{bmatrix}1&2\\2&2\\-1&0\end{bmatrix}=\begin{bmatrix}6&6\\6&8\end{bmatrix}$

$$\left(B^TB\right)^{-1}=\frac{1}{48-36}\begin{bmatrix}8&-6\\-6&6\end{bmatrix}=\frac{1}{6}\begin{bmatrix}4&-3\\-3&3\end{bmatrix}.$$

$$P=B\left(B^TB\right)^{-1}B^T=\begin{bmatrix}1&2\\2&2\\-1&0\end{bmatrix}\frac{1}{6}\begin{bmatrix}4&-3\\-3&3\end{bmatrix}\begin{bmatrix}1&2&-1\\2&2&0\end{bmatrix}=\begin{bmatrix}2/3&1/3&1/3\\1/3&2/3&-1/3\\1/3&-1/3&2/3\end{bmatrix}$$

例：矩阵列空间上的正交投影

$A=\begin{bmatrix}1&2&1\\2&2&0\\-1&0&1\end{bmatrix}=[\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3]$. 求子空间$\mathcal{R}(A)$上的正交投影矩阵。

(b)令$C=\begin{bmatrix}1&1\\2&0\\-1&1\end{bmatrix}$. 注意$\boldsymbol{a}_1\perp\boldsymbol{a}_3, C^TC=\begin{bmatrix}1&2&-1\\1&0&1\end{bmatrix}\begin{bmatrix}1&1\\2&0\\-1&1\end{bmatrix}=\begin{bmatrix}6&0\\0&2\end{bmatrix}$.

$(C^TC)^{-1}=\begin{bmatrix}1/6&0\\0&1/2\end{bmatrix}$

$$P=C(C^TC)^{-1}C^T=\begin{bmatrix}1&1\\2&0\\-1&1\end{bmatrix}\begin{bmatrix}1/6&0\\0&1/2\end{bmatrix}\begin{bmatrix}1&2&-1\\1&0&1\end{bmatrix}=\begin{bmatrix}2/3&1/3&1/3\\1/3&2/3&-1/3\\1/3&-1/3&2/3\end{bmatrix}$$

取子空间的一组正交基可以简化投影的计算！

命题:

设$\boldsymbol{u}_1,\dots,\boldsymbol{u}_n\in\mathbb{R}^m$均非零且$\boldsymbol{u}_i^T\boldsymbol{u}_j=0, i\neq j$。

(1) $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n$线性无关，因此$[\boldsymbol{u}_1,\dots,\boldsymbol{u}_n]$是列满秩矩阵；

(2) 设P为$\mathbb{R}^m$在子空间$Span(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)$上的正交投影矩阵，则$P=P_{\boldsymbol{u}_1}+\cdots+P_{\boldsymbol{u}_n}$，

其中，$P_{\boldsymbol{u}_i}$为在子空间$Span(\boldsymbol{u}_i)=\mathbb{R}\boldsymbol{u}_i$上的正交投影矩阵。

证明:

(1) 如果$\sum_i c_i\boldsymbol{u}_i=\boldsymbol{0}$，$\boldsymbol{0}=\boldsymbol{u}_j^T\left(\sum_i c_i\boldsymbol{u}_i\right)=c_j\boldsymbol{u}_j^T\boldsymbol{u}_j$. 因此, $c_j=0$.

(2) 令$A=[\boldsymbol{u}_1,\dots,\boldsymbol{u}_n]. A^TA=D\left(\boldsymbol{u}_1^T\boldsymbol{u}_1,\dots,\boldsymbol{u}_n^T\boldsymbol{u}_n\right)$为对角阵

$$P=A\left(A^TA\right)^{-1}A^T=[\boldsymbol{u}_1,\dots,\boldsymbol{u}_n]D\left(\boldsymbol{u}_1^T\boldsymbol{u}_1,\dots,\boldsymbol{u}_n^T\boldsymbol{u}_n\right)^{-1}\begin{bmatrix}\boldsymbol{u}_1^T\\ \vdots\\ \boldsymbol{u}_n^T\end{bmatrix}$$

$$=\frac{\boldsymbol{u}_1\boldsymbol{u}_1^T}{\boldsymbol{u}_1^T\boldsymbol{u}_1}+\cdots+\frac{\boldsymbol{u}_n\boldsymbol{u}_n^T}{\boldsymbol{u}_n^T\boldsymbol{u}_n}=P_{\boldsymbol{u}_1}+\cdots+P_{\boldsymbol{u}_n}.$$

命题:

设$\mathcal{M} \subseteq \mathbb{R}^m$是子空间。正交投影矩阵$P_{\mathcal{M}}$满足

(1) $P_{\mathcal{M}}$为对称矩阵，即$P_{\mathcal{M}}^T = P_{\mathcal{M}}$

(2) $P_{\mathcal{M}}^2 = P_{\mathcal{M}}$

(3) $\mathcal{R}(P_{\mathcal{M}}) = \mathcal{M}$,即$P_{\mathcal{M}}$的列空间为$\mathcal{M}$。

证明:

取$\mathcal{M}$的一组基$\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$构成列满秩矩阵$A = [\boldsymbol{a}_1, \dots, \boldsymbol{a}_n]$

$P_{\mathcal{M}} = P_A = A\left(A^T A\right)^{-1} A^T$

(1) $P_{\mathcal{M}}^T = \left(A\left(A^T A\right)^{-1} A^T\right)^T = A\left(A^T A\right)^{-1} A^T = P_{\mathcal{M}}$

(2) $P_{\mathcal{M}}^2 = A(A^T A)^{-1} A^T A\left(A^T A\right)^{-1} A^T = A\left(A^T A\right)^{-1} A^T = P_{\mathcal{M}}$

(3) 映射$P_{\mathcal{M}}$的值域等于$\mathcal{M}$。于是$\mathcal{R}(P_{\mathcal{M}}) = \mathcal{M}$

定义:

m阶方阵P称为**正交投影矩阵**，如果$P^T = P$且$P^2 = P$。

命题:

如果m阶方阵P为正交投影矩阵，则$P = P_{\mathcal{R}(P)}$.

证明:

只需说明若$\boldsymbol{b} \in \mathcal{R}(P), P\boldsymbol{b} = \boldsymbol{b}$; 若$\boldsymbol{b} \in \mathcal{R}(P)^{\perp}$, $P\boldsymbol{b} = \boldsymbol{0}$.

(1) $\boldsymbol{b} \in \mathcal{R}(P)$,存在$\boldsymbol{v} \in \mathbb{R}^m$使得$\boldsymbol{b} = P\boldsymbol{v}$. $P\boldsymbol{b} = P(P\boldsymbol{v}) = P^2\boldsymbol{v} = P\boldsymbol{v} = \boldsymbol{b}$.

(2) $\boldsymbol{b} \in \mathcal{R}(P)^{\perp} = N\left(P^T\right), P\boldsymbol{b} = P^T\boldsymbol{b} = \boldsymbol{0}$.

本节小结

1. 正交补的性质。行空间和零空间互为正交补，列空间和左零空间互为正交补

2. 正交投影。

   取列满秩矩阵计算；

   取正交向量组计算（Gram-Schmidt正交化，见下节）