Thermal Noise of Resistors

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1 Purpose

In this laboratory exercise, we took various measurements of electronic circuit elements to investigate the statistical and stochastic nature of thermal noise and to confirm the Nyquist Theorem.

2 Theory and Methods

Measurements involving electronic circuits are fundamentally limited by the discrete particles which carry charge and their random thermal motion. For an ensemble of a sufficiently large number of charge particles, in this case electrons occupying microstates bound to the nuclei of atoms in a metal conductor, spontaneous fluctuations of their motion at equilibrium show small changes in the macroscopic properties current and potential over time giving rise to a noise spectrum. Taking the Fourier transform of the noise spectrum returns a power spectrum. That is, for any particular frequency, the magnitude of its contribution to the infinite series of periodic functions which approximate the noise spectrum, its Fourier series, gives the power of the signal. The time-dependence of the signal is shifted into the range from the domain. The power spectrum that results gives a Gaussian distribution of the thermal fluctuations in terms of their frequency. This distribution is statistically useful. Also, since it is independent of time, it does not focus particularly on the kinetic energy of the fluctuating particles, which is in turn dependent on tempurature, but on the underlying statistics of the random fluctuations.

Beginning with Ohm's Law and taking into account that the macroscopic observables, current and potential, are averages of the random motion of charge carriers, we get a relation[1] between average potential and the stochastic velocity of electrons. Since the power spectrum is the mean-square voltage vs. frequency, we next consider the power in a frequency bandwidth as the product of the spectral density times the bandwidth. This spectral density can be derived from the time-correlation function of the trajectory of the stochastic variable, in this case the potential, via the Weiner-Khintchine Theorem[2]. Using these

[1]See procedure, equation (3).
[2]See procedure, equation (6).

two relations and accounting for the physical properties of the materials being analyzed, we arrive at the Nyquist Theorem:

$$\langle V^2 \rangle = 4k_B T R \Delta f, \tag{1}$$

where k_B is Boltzmann's constant, T is temperature, R is resistance, and Δf is the frequency bandwidth.

To confirm the Nyquist Theorem, we performed various measurements hoping to linearly fit the data with a factor of Boltzmann's constant. Using a computer program connected to a digital oscilloscope, we took measurements of the root-mean-square (RMS) voltage as a function of frequency[3]. First confirmed that the RMS voltage from a function generator was not frequency-dependent. Then we took several measurements of the RMS voltage of a sine function passed through an amplifier at different frequencies toget the frequency-dependent gain characteristics of the amplifier. Finally, we took several measurements of the RMS voltage from various resistors passed through the amplifier.

3 Data and Plots

3.1 Temperature

The temperature was measured as 295 ± 2 K.

3.2 Function Generator and Amplifier Noise

We measured the noise from the function generator over seven trials as 55.9 mV with a standard of deviation of less than 1% of the average value.

The amplifier output for various frequencies was measured over three trials to a standard of deviation of less than 4% of the average value and is ploted below.

[3]Though the raw data was technically a noise spectrum in time, and was accompanied by a power spectrum in frequency, the calculations needed to confirm the Nyquist Theorem only require the RMS output, and so these spectra are not included in this report.

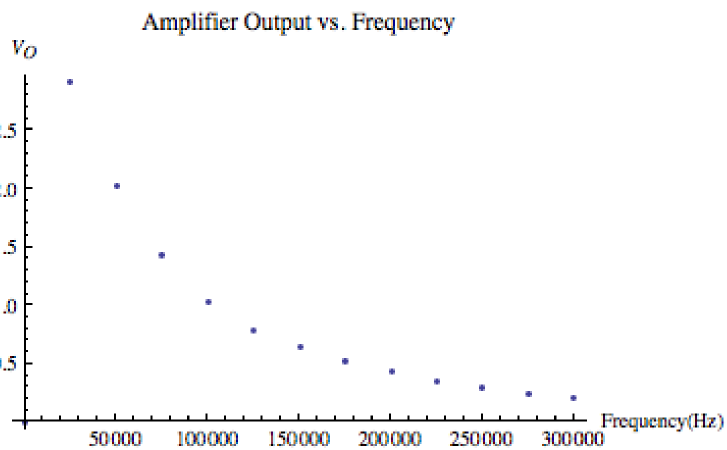

Figure 1: The average amplifier gain characteristics (V_O in volts) measured at frequency intervals of 25 kHz from 0 to 300 kHz.

3.3 Resistor Noise

We measured the reference value of amplified resistor noise at 50 Ω over three trials as 17.9 mV with a standard of deviation of less than 2% of the average value.

The amplified noise of various resistors was measured over three trials to a standard of deviation of less that 4% of the average value and is plotted below.

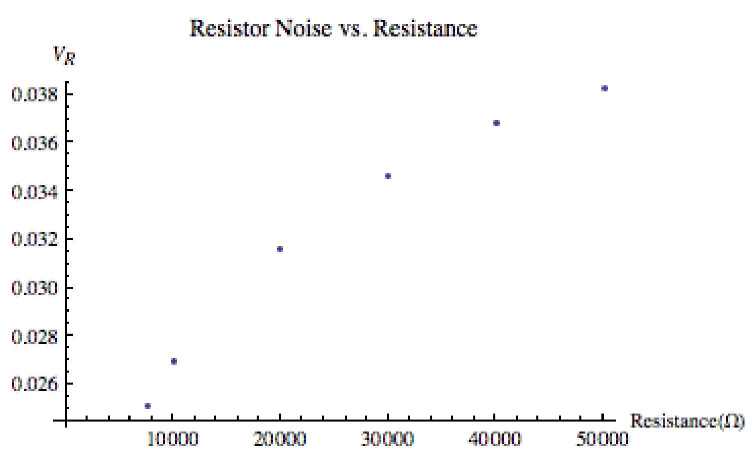

Figure 2: The average root-mean-square amplified resistor noise measured for resistances of 7.5, 10, 20, 30, 40, and 50 $k\Omega$ from a reference of 50 Ω.

4 Analysis

In order to get the frequency bandwidth Δf as a function of resistance R exclusive of the gain from the amplifier, we integrate numerically using the trapezoid rule:

$$\Delta f(R) = \int_0^\infty \frac{(V_O(f)/V_i)^2}{1+(2\pi fRC)^2} df. \tag{2}$$

Here, V_O is comes from the amplifier gain characterisic shownin Figure 1, V_i is the constant noise from the function generator, f is frequency, and C is the capacitance of the cable used to connect our circuit components.

The mean-square voltage as a function of resistance is given by

$$\langle V^2 \rangle = (\alpha V_R)^2 - (\alpha V_G)^2, \tag{3}$$

where V_R is the amplified resistor noise from Figure 2, $V_G$is the reference resistance, and $\alpha = 0.01$ is a scale factorneeded to account for amplification. This is plotted below.

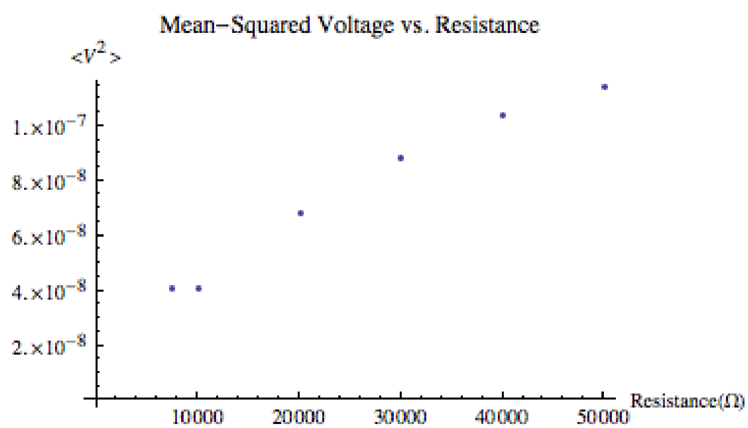

Figure 3: The mean-square voltage of the resistors adjusted for 40 dB amplification.

According to equation (1) above, we expect a linear relation between $\langle V^2 \rangle/\Delta f$ and frequency with a slope of $4k_BT$. The linear fit obtained is

$$\langle V^2 \rangle \Delta f(R) = 8.69 \times 10^{-17} + 1.82 \times 10^{-20} R. \tag{4}$$

The data points and linear fit are plotted below. This fit gives a value of $(1.82 \times 10^{-20} V s \Omega^{-1}) \div (4 \cdot 295K) = 1.54 \times 10^{-23} JK^{-1}$ for the Boltzmann constant, a deviation from the accepted value of $1.38 \times 10^{-23} JK^{-1}$ of 11.7%.

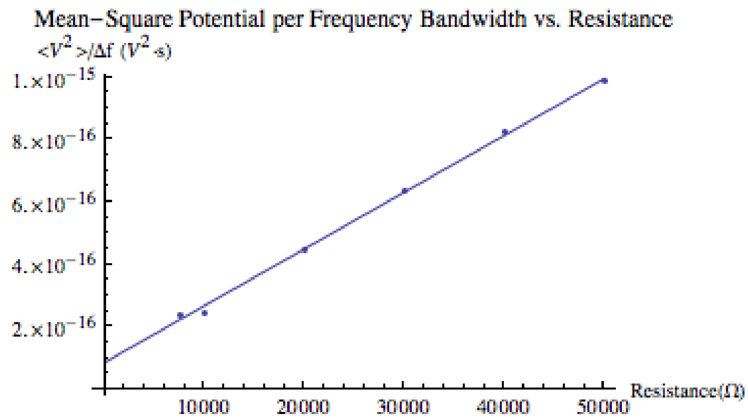

Figure 4: The mean-square potential per frequency bandwidth calculated using equations (2) and (3) at resistances of 7.5, 10, 20, 30, 40, and 50 $k\Omega$ from a reference of 50 Ω.

5 Discussion

The average value from the calculations above for the Boltzmann constant is significantly above the accepted value. In terms of data, this might be due to the number of frequencies at which we measured the noise. We could have gotten a more precise gain characteristic and value of Δf if we had taken more measurements. Indeed, integrating equation (2) using Simpson's rule instead of the trapezoid rule gives an even higher value[4]. Also, during initial measurements of resistor noise, we were getting an unwanted periodic signal, perhaps from some electrical signal in the lab which we were not able to discover. This extra "noise" would tend to give a higher value of RMS voltage than expected. Perhaps similar ambient noise remained when we took the gain characteristic of the amplifier, and so we got high measurements for V_O. This would give an inflated value of Δf, and I think that based on the shape of the gain characteristic, would give a larger slope than expected in the fit of mean-square voltage per frequency bandwidth vs. resistance from which we calculated Boltzmann's constant.

To minimize the uncertainty caused by resistor noise in any measurement involving electronics, we would need to minimize the resistance by various tricks. For example, we could operate the electronics at low temperatures. For a fixed resistance, though, this lab demonstrates that there is always a minimum abount of noise fundamental to circuits.

This lab reveals another statistical-mechanical way of measuring the Boltzmann constant, which is obviously fundamental to the statistical-mechanics and thermodynamics, of chemistry. Noise in circuit elements probably has heuristic parallels in measurements of gases; for example, deviation from ideal-gas

[4]This calculation excluded.

behavior could be thought of as “noise” due to random fluctuation caused by interparticle forces.