The Hyperbolic Structure of Corals

Saralin Zassman[1]

***Abstract*— While mathematicians struggled for centuries to accept the idea of a non-Euclidean geometry, several organisms in nature are representations of the hyperbolic plane. Many corals, sea slugs, and other kinds of marine life exhibit a fascinating hyperbolic structure. In this paper, we describe hyperbolic geometry and how it has been modelled in graphical terms, as well as in three-dimensional Euclidean space. The hyperbolic structure of corals is then discussed, as well as the modelling of coral reefs and coral in computer graphics. To conclude, we present areas for future work on hyperbolic and procedurally generated coral.**

I. Hyperbolic Geometry

From the houses we live in to the streets we drive on, the world around us consists primarily of straight lines and rectilinear forms. In nature, there are a myriad of organisms that do not follow this Euclidean geometry. The many types of corals, kelps, sponges, and sea slugs in the ocean represent a geometric form known as the hyperbolic plane. In order to grasp the unique structure of these living creatures, we must understand hyperbolic geometry and how it has been modelled.

Euclidean geometry is the type geometry we are taught in high school, where the angles of a triangle sum to $180°$ and the circumference of a circle is $2\pi r$. It precisely describes the behaviour of points and lines on a flat two-dimensional plane, and generalises to three dimensions as a mathematical model of the physical world [1]. Euclidean geometry been attributed to Euclid, who presented a description of the geometry and its five postulates in his monumental *Elements*. The fifth postulate, described below, sparked one of the most intense controversies in scientific history:

> For every line *l* and for every point P that does not lie on *l* there exists a unique line *m* through P that is parallel to *l*.

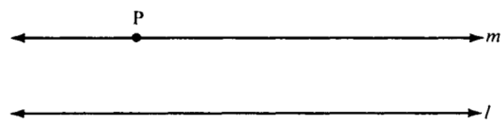

Fig. 1. Parallel lines *l* and *m* in Euclidean space.

Despite seeming quite intuitive, Euclid's final postulate cannot be proven like the other four. This is because when working with finite segments, there is no empirical way to show that two lines are parallel (i.e., do not intersect). Two segments that look parallel may appear to converge when extended further. Mathematicians tried to verify the parallel postulate for almost 2,000 years [1]. The issue was in assumption that geometry can only be Euclidean. Once they accepted the possibility of a geometry without the fifth postulate, the concept of hyperbolic geometry came to fruition. In the nineteenth century, Bolyai, Gauss, and Lobachevski independently determined that the fifth postulate is not necessarily true (i.e., it cannot be proven) [2]. A system that follows all five postulates is Euclidean, but other geometries exist that do not follow the parallel postulate. They each proposed a system in which multiple lines parallel to *l* pass through the point *P*, as seen in Figure 2. This non-Euclidean geometry is now recognized as hyperbolic geometry, and is built around the following version of the parallel postulate:

> Given a line *l*, for every point P that does not lie on *l* there exists at least two distinct lines through P that are parallel to *l*.

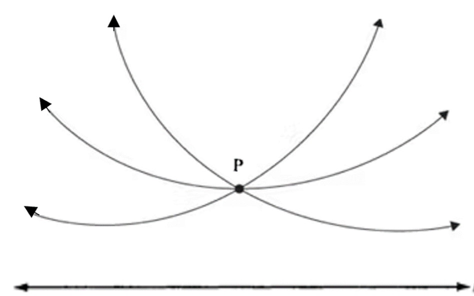

Fig. 2. The lines that pass through the point *P* are parallel to the line *l* in hyperbolic space.

Upon its initial discovery, several mathematicians refused to believe in an "imaginary geometry" [3]. Euclid's five postulates were seen as the truth for so long that naturally, the idea of an alternate geometry came with a fair amount of resistance. Once it became more widely accepted, mathematicians began to look for ways to model the hyperbolic plane. However, this proved to be a very difficult task given the nature of hyperbolic geometry. Hyperbolic forms extend indefinitely and deflect before they can close in on themselves [4]. They have a negative curvature, whereas Euclidean planes have no curvature and spheres have a positive curvature.

II. Models of Hyperbolic Geometry

A. Graphical Models

Non-euclidean geometry is crucial to many areas of physics and mathematics [5]. As a result, several attempts

[1] Saralin Zassman is with the School of Computer Science, University of Waterloo, 200 University Avenue, Waterloo, Ontario, Canada N2L 3G1.

in computer graphics have been made to display such geometries. For hyperbolic geometry, one of the most well-known graphical models is the Poincaré disk model, which represents the entire hyperbolic plane in two-dimensional Euclidean space. A line in the Poincaré disk model is a circle that intersects the edge of the disk at a right angle, as seen in Figure 3b. The angles in the Poincaré model are the actual angles in hyperbolic space (i.e., the angles are not distorted). Scale, however, is distorted because we view the model in Euclidean space. Therefore, although the shapes in Figure 3a appear to decrease in size near the border of the disc, they are all the same size in hyperbolic space. While graphical models like the Poincaré disk model are able to represent the hyperbolic plane, none of them provide a clear visualization of its shape or behaviour [3]. The hyperbolic plane can be understood more intuitively with the use of physical, three-dimensional models.

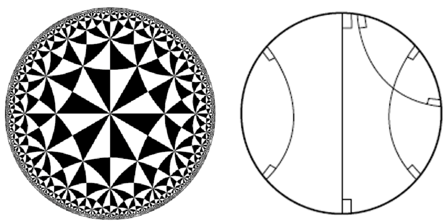

Fig. 3. (a) The Poincaré disk model; (b) geodesics (i.e., the smallest distance between two points) in the Poincaré disk model.

B. Paper and Tape Models

For over a century, mathematicians attempted to find a physical representation of a hyperbolic plane. William Thurston, an American mathematician, made the first paper model, which later became known as the Annular model [4]. The model is made of crescent-shaped annular strips (i.e., the area between two concentric circles), where the outer circle of one annulus is taped to the inner circle of another. The smaller the strips, the more accurate the model is in representing the hyperbolic plane.

Another paper model of hyperbolic geometry is the Hyperbolic soccer mall, which is based on the tiling pattern of a soccer ball. A tiling or tessellation is a collection on tiles (i.e., geometric shapes) that cover a plane with no gaps or overlaps [2]. Consider the hexagonal tiling of the Euclidean plane, as seen in Figure 4. When the dark hexagons are switched with pentagons, the plane closes in on itself and transforms into a sphere with a positive curvature. When the hexagons are replaced with heptagons, the surface forms a negative curvature by extending outward, and a hyperbolic plane is achieved.

Although paper models provide a clear depiction of the hyperbolic plane, they are difficult to make and fragile. They must be handled with care as they can easily tear or lose their shape, especially when being transported between locations. It is also difficult to draw parallel lines on them, which is a technique used (often with crochet models) to understand the hyperbolic plane.

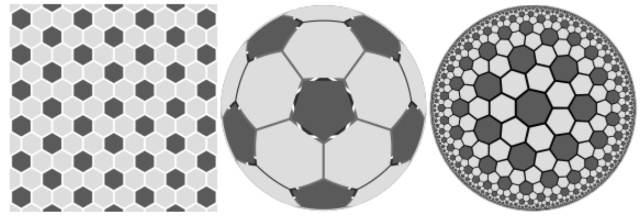

Fig. 4. From left to right, Euclidean hexagonal tiling, soccer ball, hyperbolic soccer ball in the Poincaré disk model. Figure reproduced from [6].

C. Crochet Models

In 1997, Daina Taimina encountered the Annular model in a workshop at Cornell. She discovered a way to make hyperbolic forms out of crochet, thus creating the first practical and physical model of a hyperbolic surface. Crochet models are not only easier to create, but also more durable than paper models in terms of touch and interaction. They can be folded, stretched, and handled in a variety of ways without losing their shape.

Crochet is a technique similar to knitting that uses a single hooked needle rather than a pair of long needles. Taimina was able to create hyperbolic shapes with crochet by increasing the number of stitches in each row. After the first row, an extra stitch is added every k stitches. This causes the number of stitches per row to grow exponentially [4]. The rate of stitch increase must stay constant in order to produce an accurate model. The value of k determines the curvature as well the radius of the hyperbolic plane. The smaller the value of k, the faster the frilly, hyperbolic shape develops.

Taimina first tried to model the hyperbolic plane with knitting, only to realize that crochet would be a much better option [7]. With knitting, the stitches are kept on one of the two needles at all times. As the number of stitches increases, it becomes difficult to place all of the stitches on one needle. With crocheting, only one stitch is attached to the hook at a time so there is no limit on the number of stitches in a row.

The crochet model remains the only practical and effective way to represent the hyperbolic plane in three-dimensional space [4]. It does, however, require the ability to crochet, which some may need to learn beforehand. Unlike paper and tape, most people do not already own yarn and a hook needle. Crochet models can also be very time consuming to make because they expand at such a fast rate.

The Crochet Coral Reef, created by Christine and Margaret Wertheim, is a project inspired by the hyperbolic nature of coral reefs. The art installation is promoted as a response to climate change, and consists of crochet coral reefs made by the Wertheim sisters and contributors from around the world. Margaret Wertheim has a background in mathematics, whereas Christine Wertheim is primarily an artist and curator. The Crochet Coral Reef has been exhibited at 27 museums and art galleries since its debut in 2007, and is recognized as one of the world's largest collaborative art and science efforts [8]. To raise awareness about the negative impact of environmental stress on coral reefs, the project includes crochet pieces that resemble bleached corals (i.e., corals that have lost their colour due to rising ocean temperatures), as well as a reef made entirely out of crocheted plastic.

When crocheting coral reefs, the rate of stitch increase does not typically stay constant (i.e., it varies throughout the model). Several reef crafters use Taimina's technique as a basis for their crochet work. In order to capture the diverse and imperfect nature of corals, a certain amount of ”liveliness” must be incorporated into the design process [9]. This means straying away from accuracy and structure and toward mistakes, impulsiveness, and creativity.

III. Coral Reefs and Coral

Coral reefs are among the world's most biologically and physically complex ecosystems . Also known as the ”rain-forests of the sea”, they provide habitat for over a million species as well as ecosystem services (e.g., food, coastal protection) for people throughout the tropic and subtropics regions [10]. Unfortunately, coral reefs are particularly sensitive to human activity and climate change. Due to factors such as overfishing and pollution, 50% of coral reefs have died in the last 20 years, and 90% are expected to die by 2050 [11].

Corals are invertebrates made of polyps, which are small organisms that are closely related to sea anemones and jellyfish. Despite existing in a such wide variety of shapes and sizes, corals are divided into two main categories: hard corals and soft corals. The polyps in hard corals produce calcium carbonate, and hundreds of hard corals together form a coral reef. Soft corals often resemble plants and do not have any reef-building properties. Several coral species get their colour from algae that reside in the coral polyps. Zooxanthellae, a specific type of algae, convert thermal energy into food for the coral host. When ocean temperatures are too high, algae are expelled by the polyps, thus depriving the coral of its primary food source. This is also what causes corals to lose their colour in a process known as coral bleaching.

Several reef organisms and species of coral have a hyperbolic structure. Hyperbolic forms are ideal when large surface areas as beneficial, such as for filter feeding animals (i.e., animals that feed off of food particles suspended in water). The frilly nature of the hyperbolic plane also serves an advantage in terms of coral growth. Corals need free space to expand and grow, which is particularly scarce in the reef environment [12]. Some species of coral have developed quite intricate survival strategies to compete against other corals for space. Hyperbolic shapes allow organisms to have large surface areas (and benefit from those large surface areas) without actually taking up that much space in the ocean.

IV. Models of Coral Reefs and Coral

There are several existing works on the modelling and simulation of coral reefs. Computational models are used to better understand and predict the dynamics of coral reef ecosystems. They often demonstrate the influence of multiple human-induced stressors and climate change. SEAMANCORE [13] is a two-dimensional model that shows the effect of simultaneous stressors on coral reefs. The model is presented as a means to investigate the impact of bleaching, eutrophication, and fishing. Simulation models have also been used to develop more effective management techniques [14], [15] . For example, the Atlantic Ecosystem Model [16] determined that water quality management is more influential in terms of coral reef preservation than reduced fishing.

Unlike coral reefs, there is very little formal research on the modelling of corals. It can be difficult to maintain corals in a standard laboratory, as they need a constant flow of high quality water to survive [17]. They are typically kept in labs with easy access to natural saltwater since the proper growth conditions are expensive to maintain otherwise. Corals also only reproduce once a year, thus making the study of coral larvae, embryos, and overall development particularly difficult.

The Smithsonian 3D digitization program [18] developed a collection of digitized coral and reef organisms using three-dimensional capture technology. The models are created by putting together images of a real coral that are taken from several different angles. The project preserves the organisms in a digital format for future research (i.e., if certain species become endangered or extinct). While there are over six thousand species of coral, the digital collection contains only 88 models. Each model includes information such as the date and location that the coral specimen was collected. The models are organized into five lessons to promote a better understanding of the different types of coral.

Several existing works use an L-system to procedurally generate corals that have recursive, tree-like structures. An L-system is a framework originally developed to simulate the growth of simple multicellular organisms [19]. L-system applications have since been expanded to include more complex branching structures, with two major areas being the production of fractals and plant modelling. L-systems use a set of rewriting rules to successively replace the individual units (i.e., modules) of an organism with zero, one, or more of the same or modified units. The rewriting rules can be applied sequentially, one module at a time, or in parallel, ”to capture the simultaneous development of different parts of the organism” [20].

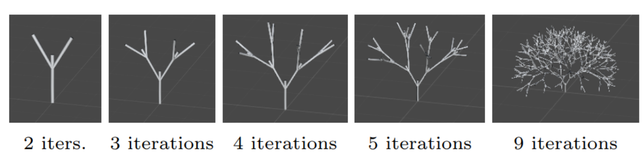

Fig. 5. A hard coral generated using an L-system. Figure reproduced from [21].

Coralize [21] is a tool that uses generative algorithms to model corals and sponges in an underwater scene. The hard corals are created with an L-system, where the user can specify the number of iterations with which the rewriting rules are applied. The higher the number of iterations, the more branches that appear in the coral, as seen in Figure 5. The thickness of the coral and variations in thickness can also be modified. Since many species of soft corals look like plants, the soft corals are modelled using an algorithm that generates veins on 2D leaves. The algorithm was modified to work in three dimensions where the veins are represented as cylinders of varying thickness.

Kaandorp and Kubler [20] evaluated several approaches to modelling coral, including L-systems. They obtained the best results with the accretive growth model. This technique simulates coral development by adding layers of material on top of existing layers. It is capable of producing a broad range of morphologies and takes into account environmental factors such as light and water current. However, since the model is quite computationally intensive, it can only generate one coral at a time (as opposed to an entire reef). Meister [22] developed an L-system to model corals with the goal of simulating reef growth. He was able to generate 3D models that quite closely resembled the coral *Lophelia pertusa* with which he was studying.

V. Future Work

Since corals come in many distinct shapes and sizes, modelling efforts tend to focus on more identifiable, non-hyperbolic species of coral. In order to develop realistic models and simulations of hyperbolic corals, there needs to be a greater understanding of the various species of corals that have the hyperbolic shape. We propose three research questions that could help advance the fields of hyperbolic modelling and procedural coral generation: (1) Are hyperbolic structure unique to a specific type of coral?; (2) How does the hyperbolic structure form during coral growth?; (3) How do we classify the structure of a coral as hyperbolic?. It would also be interesting to study how hyperbolic corals benefit from their structure, with metrics such as life-span, growth rate, and the number of organisms that live in the coral.

To develop realistic simulations of coral, there needs to be more research on external factors that impact coral development. The coral *Lophelia pertusa* exhibit a wide range of growth patterns as a result of variations in temperature, salinity, and current flow across the ocean [22]. However, due to a lack of quantitative data, Meister was unable to incorporate the impact of these factors into his model. Different materials could also be used to produce more realistic corals. Existing models [21], [22] tend to focus more on the structure of coral than the texture. Many corals have quite rough surfaces, and some species (e.g., the *Diploria labyrinthiformis* brain coral) have very distinct patterns. Bump maps, as suggested by Abela et al., [21], could be used to create realistic coral textures.

VI. Conclusion

Several types of coral are naturally occurring models of the hyperbolic plane. To the best of our knowledge, there is no formal research on the hyperbolic structure of coral. In this work, we draw a unique connection between mathematics, biology, and computer graphics, and we hope to inspire more research on corals, sea slugs, and other examples of hyperbolic geometry in nature.

References

[1] Marvin J. Greenberg, *Euclidean and non-Euclidean geometries: Development and History*, W.H. Freeman and Company, 3 edition, 1974.

[2] Craig S. Kaplan, *Computer Graphics and Geometric Ornamental Design*, Ph.D. thesis, University of Washington, 2002.

[3] J. S. Bardi, *The fifth postulate: How unraveling a two-thousand-year-old mystery unraveled the universe*, John Wiley & Sons, 2009.

[4] Shelby Frazier, "A brief look at the evolution of modeling hyperbolic space," *The Mathematics Enthusiast*, vol. 14, no. 1, 2017.

[5] Charlie Gunn, "Visualizing hyperbolic space," in *Computer Graphics and Mathematics*, Berlin, Heidelberg, 1992, pp. 299–311, Springer Berlin Heidelberg.

[6] Eryk Kopczyński and Dorota Celińska-Kopczyńska, "Virtual crocheting of euclidean planes in a 3-sphere," 07 2018.

[7] D. W. Henderson and Daina Taimiņa, *Crocheting the Hyperbolic Plane*, vol. 23, pp. 17–28, The Mathematical Intelligencer, 2001.

[8] Margaret Wertheim, "Science + art project: Crochet coral reef"," `https://www.margaretwertheim.com/crochet-coral-reef`.

[9] Sophia Roosth, "Evolutionary Yarns in Seahorse Valley: Living Tissues, Wooly Textiles, Theoretical Biologies," *Differences*, vol. 23, no. 3, pp. 9–41, 12 2012.

[10] Ove Hoegh-Guldberg, Emma V. Kennedy, Hawthorne L. Beyer, Caleb McClennen, and Hugh P. Possingham, "Securing a long-term future for coral reefs," *Trends in Ecology Evolution*, vol. 33, no. 12, pp. 936–944, 2018.

[11] Gaia Vince, "Why there is hope that the world's coral reefs can be saved," `https://www.theguardian.com/environment/2020/oct/17/why-there-is-hope-that-the-worlds-coral-reefs-can-be-saved`, Oct 2020.

[12] S. Salcedo-Sanz, L. Cuadra, and M.J.A. Vermeij, "A review of computational intelligence techniques in coral reef-related applications," *Ecological Informatics*, vol. 32, pp. 107–123, 2016.

[13] Sara Miñarro, Johannes Leins, Esteban Acevedo-Trejos, Elizabeth A. Fulton, and Hauke Reuter, "Seamancore: A spatially explicit simulation model for assisting the local management of coral reefs," *Ecological Modelling*, vol. 384, pp. 296–307, 2018.

[14] Glen Holmes and Ron Johnstone, "Modelling coral reef ecosystems with limited observational data," *Ecological Modelling*, vol. 221, 2010.

[15] Georgina Gurney, Jessica Melbourne-Thomas, Rollan Geronimo, Perry Aliño, and Craig Johnson, "Modelling coral reef futures to inform management: Can reducing local-scale stressors conserve reefs under climate change?," *PloS one*, vol. 8, 11 2013.

[16] Mariska Weijerman, Elizabeth Fulton, Isaac Kaplan, Rebecca Gorton, Rik Leemans, Wolf Mooij, and Russell Brainard, "An integrated coral reef ecosystem model to support resource management under a changing climate," *PloS one*, vol. 10, pp. e0144165, 12 2015.

[17] Elizabeth Hambleton, Annika Guse, and John Pringle, "Similar specificities of symbiont uptake by adults and larvae in an anemone model system for coral biology," *The Journal of experimental biology*, vol. 217, 02 2014.

[18] Smithsonian 3D Digitization, "Corals," `https://3d.si.edu/corals`.

[19] Przemyslaw Prusinkiewiczy, Przemyslaw Hananz, Mark Hammely, and Radomir Mech, "L-systems: From the theory to visual models of plants," *2nd CSIRO Symposium on Computational Challenges in Life Sciences*, 01 1966.

[20] Jaap Kaandorp and Janet Kübler, *The Algorithmic Beauty of Seaweeds, Sponges, and Corals*, 01 2001.

[21] Ryan Abela, Antonios Liapis, and Georgios N. Yannakakis, "A constructive approach for the generation of underwater environments," in *Proceedings of the FDG workshop on Procedural Content Generation in Games*, 2015.

[22] Martin Meister, *Interactive Visualization in Interdisciplinary Applications*, Ph.D. thesis, University of Erlangen-Nuremberg, 2008.