Computer Graphics I

Lecture 20: Soft-body simulation

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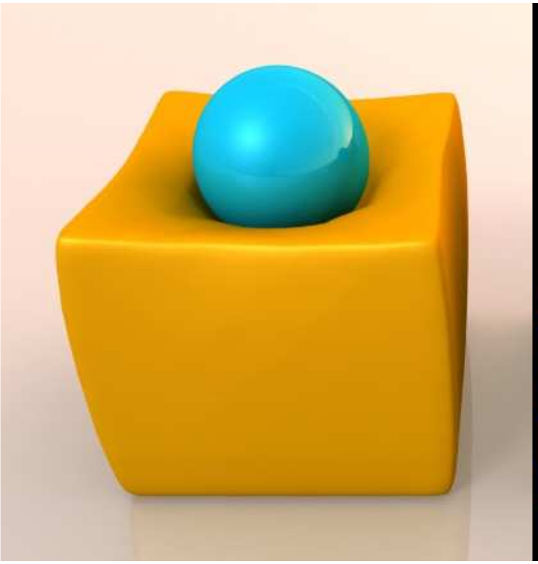
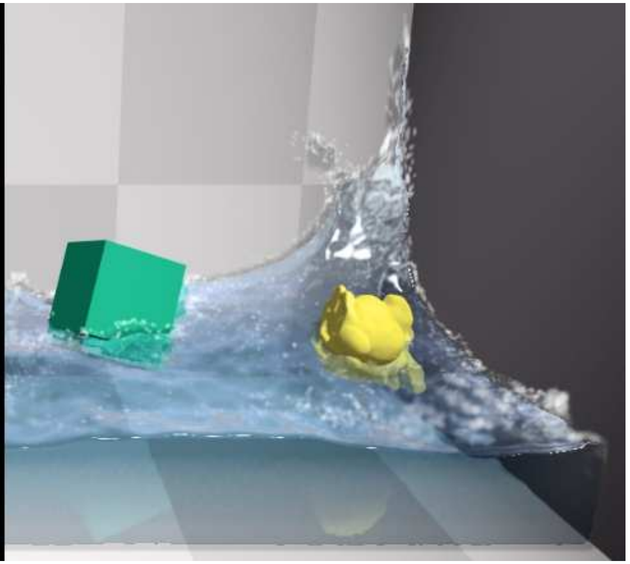
What is a soft body?

- **Unlike in simulation of rigid body**
  - Shape of soft bodies can change
  - The relative distance of two points on the object is not fixed





Soft body dynamics

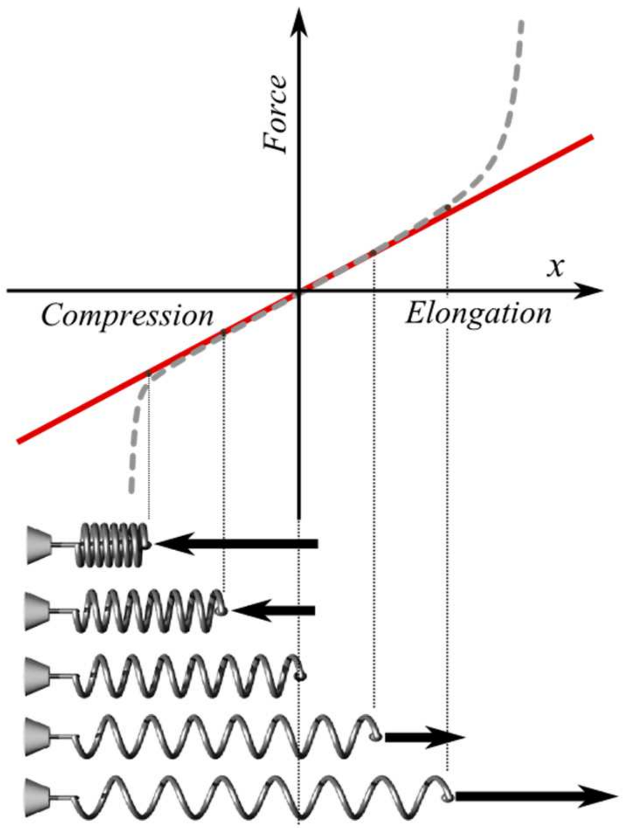
- **A field of computer graphics**
  - Focus on visually realistic physical simulations of the motion and properties of deformable objects
  - The scope of soft body dynamics is quite broad
    - Simulation of soft organic materials: muscle, fat, hair and vegetation
    - Other deformable materials: cloth, fabric, etc.
  - Type of simulation methods
    - Mass-spring model
    - Finite-element simulation
    - Energy minimization methods
    - Position-based dynamics, etc.

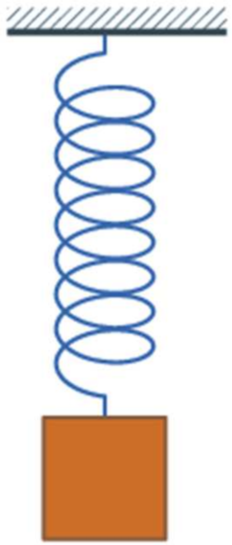
1. Mass-spring model

Mass-spring model

- **Mass-spring system**
  - The spring has a non-negligible mass m
- **Hook's law**

$$\vec{F} = -k\vec{x}$$





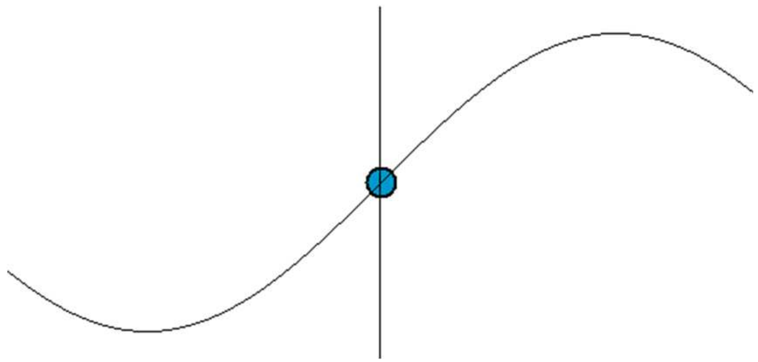
Mass-spring model

- **Harmonic oscillator**
  - Without damping

$$F = ma = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = m\ddot{x} = -kx$$

    - Analytical solution

$$x(t) = A\cos(\omega t + \phi)$$



Mass-spring model

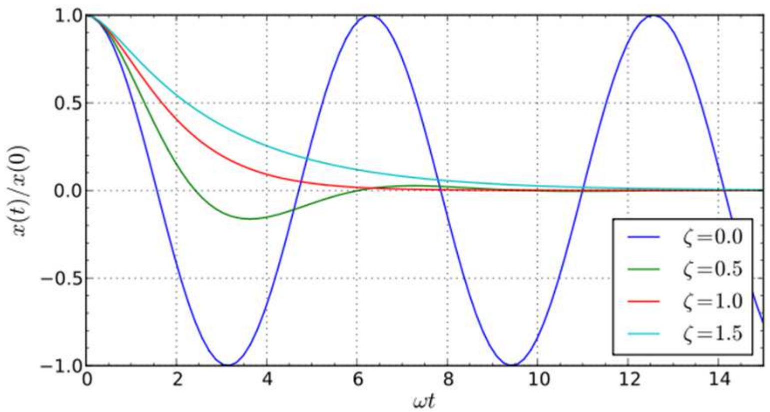
- **Harmonic oscillator**
  - With damping

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\zeta\omega_0\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^{\,2}x = 0,$$

where

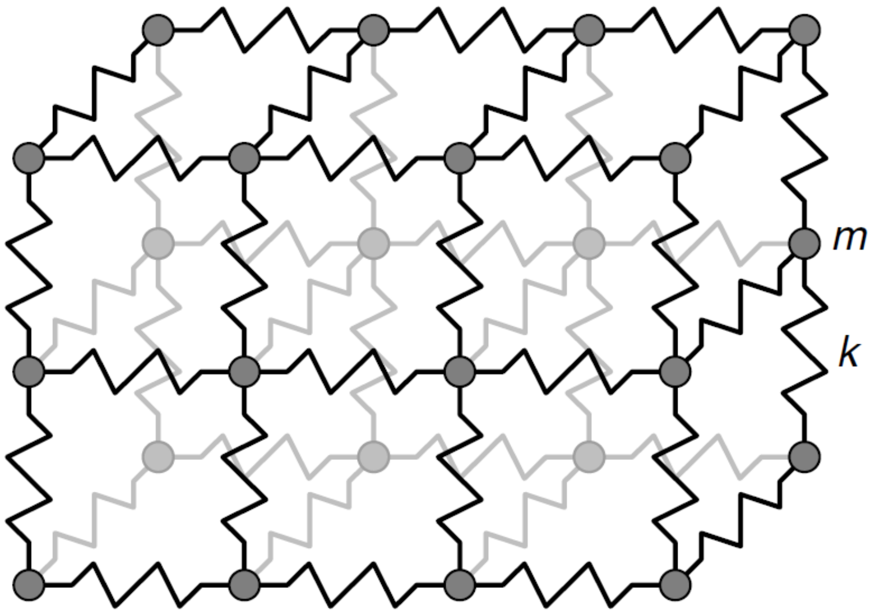
$\omega_0 = \sqrt{\frac{k}{m}}$ is called the 'undamped angular frequency of the oscillator' and

$\zeta = \frac{c}{2\sqrt{mk}}$ is called the 'damping ratio'.



Mass-spring model

- **Mass-spring system**
  - Physically-based technique for modeling deformable objects
  - An object is modeled as point masses connected by springs



Mass-spring model

- **Mass-spring system**
  - The spring can be linear (Hook's law)
  - Non-linear spring can also be used (model tissues such as human skin)
  - **Governing dynamic equation**
    - Using linear spring model

$$m_i \ddot{\mathbf{x}}_i = -\gamma_i \dot{\mathbf{x}}_i + \sum_j \mathbf{g}_{ij} + \mathbf{f}_i$$

$\mathbf{g}_{ij}$: forces exerted on mass i by spring between masses i and j

Mass-spring model

- **Mass-spring system**
  - Writing the equation for entire system

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}$$

M: mass matrix, diagonal
C: damping matrix, diagonal
K: stiffness matrix, encodes spring forces
from nearby connected springs

  - Re-expression in first-order system

$$\begin{aligned} \dot{\mathbf{v}} &= \mathbf{M}^{-1}\left(-\mathbf{C}\mathbf{v} - \mathbf{K}\mathbf{x} + \mathbf{f}\right) \\ \dot{\mathbf{x}} &= \mathbf{v} \end{aligned}$$

velocity of mass point

Mass-spring model

- **Mass-spring system**
  - Numerical solution: explicit Euler

$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}\\ \dot{\mathbf{x}}\end{pmatrix} = \frac{d}{dt}\begin{pmatrix}\mathbf{x}\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}\mathbf{v}\\ \mathbf{M}^{-1}\mathbf{f}(\mathbf{x},\mathbf{v})\end{pmatrix}$$

$\mathbf{x}_0 = \mathbf{x}(t_0)$ and $\mathbf{v}_0 = \mathbf{v}(t_0)$

$\Delta\mathbf{x} = \mathbf{x}(t_0 + h) - \mathbf{x}(t_0)$ and $\Delta\mathbf{v} = \mathbf{v}(t_0 + h) - \mathbf{v}(t_0)$

$$\begin{pmatrix}\Delta\mathbf{x}\\ \Delta\mathbf{v}\end{pmatrix} = h\begin{pmatrix}\mathbf{v}_0\\ \mathbf{M}^{-1}\mathbf{f}_0\end{pmatrix}$$

Mass-spring model

- **Mass-spring system**
  - Numerical solution: implicit Euler

$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}\\ \dot{\mathbf{x}}\end{pmatrix} = \frac{d}{dt}\begin{pmatrix}\mathbf{x}\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}\mathbf{v}\\ \mathbf{M}^{-1}\mathbf{f}(\mathbf{x},\mathbf{v})\end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix}\Delta\mathbf{x}\\ \Delta\mathbf{v}\end{pmatrix} = h\begin{pmatrix}\mathbf{v}_0+\Delta\mathbf{v}\\ \mathbf{M}^{-1}\mathbf{f}(\mathbf{x}_0+\Delta\mathbf{x},\mathbf{v}_0+\Delta\mathbf{v})\end{pmatrix}$$

$$\Downarrow \quad \mathbf{f}(\mathbf{x}_0+\Delta\mathbf{x},\mathbf{v}_0+\Delta\mathbf{v}) = \mathbf{f}_0 + \frac{\partial\mathbf{f}}{\partial\mathbf{x}}\Delta\mathbf{x} + \frac{\partial\mathbf{f}}{\partial\mathbf{v}}\Delta\mathbf{v}$$

$$\begin{pmatrix}\Delta\mathbf{x}\\ \Delta\mathbf{v}\end{pmatrix} = h\begin{pmatrix}\mathbf{v}_0+\Delta\mathbf{v}\\ \mathbf{M}^{-1}(\mathbf{f}_0+\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\Delta\mathbf{x}+\frac{\partial\mathbf{f}}{\partial\mathbf{v}}\Delta\mathbf{v})\end{pmatrix}$$

$$\Downarrow$$

$$\Delta\mathbf{v} = h\mathbf{M}^{-1}\left(\mathbf{f}_0 + \boxed{\frac{\partial\mathbf{f}}{\partial\mathbf{x}}}h(\mathbf{v}_0+\Delta\mathbf{v}) + \boxed{\frac{\partial\mathbf{f}}{\partial\mathbf{v}}}\Delta\mathbf{v}\right)$$

Mass-spring model

- **Mass-spring system**
  - Numerical solution: implicit Euler
    - Regrouping

$$\Delta \mathbf{v} = h\mathbf{M}^{-1}\left(\mathbf{f}_0 + \frac{\partial \mathbf{f}}{\partial \mathbf{x}} h(\mathbf{v}_0 + \Delta \mathbf{v}) + \frac{\partial \mathbf{f}}{\partial \mathbf{v}} \Delta \mathbf{v}\right)$$

$$\Downarrow$$

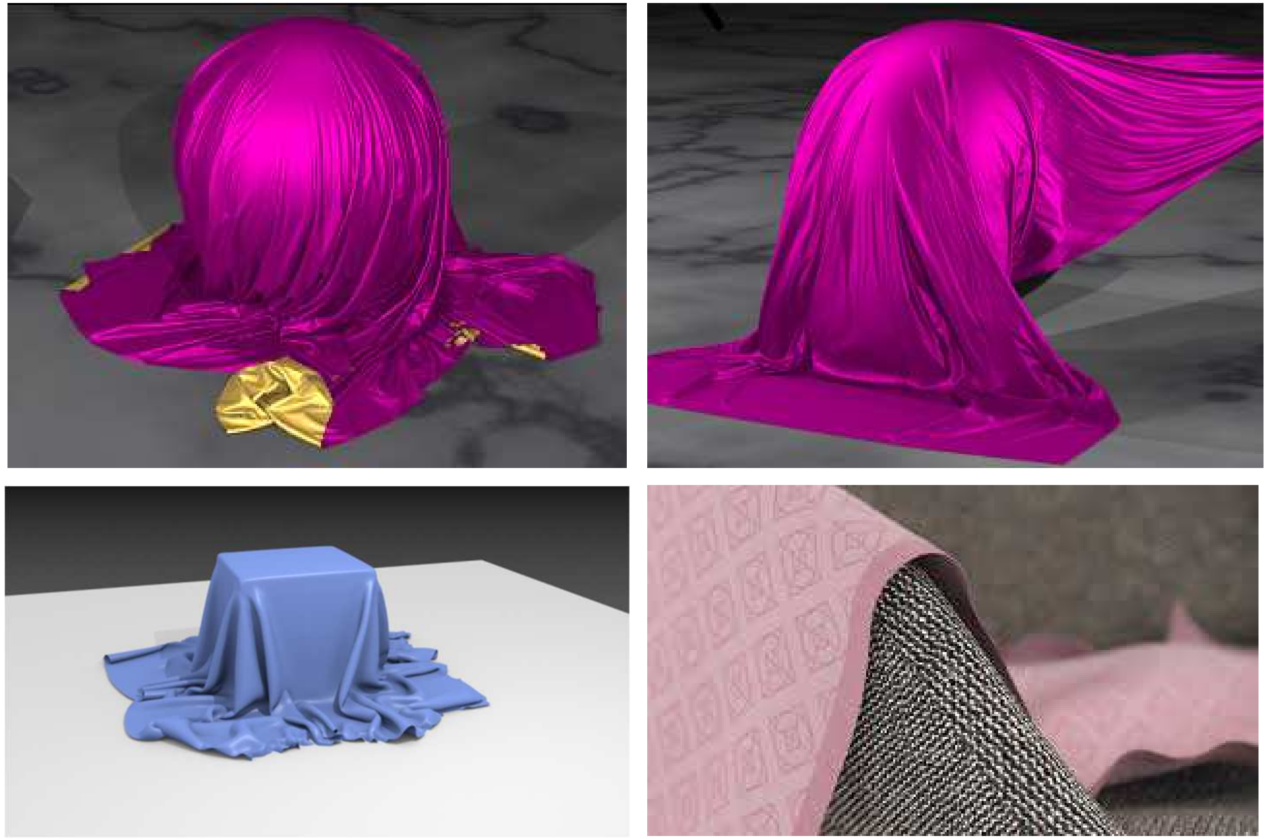
$$\left(\mathbf{I} - h\mathbf{M}^{-1}\frac{\partial \mathbf{f}}{\partial \mathbf{v}} - h^2\mathbf{M}^{-1}\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\Delta \mathbf{v} = h\mathbf{M}^{-1}\left(\mathbf{f}_0 + h\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{v}_0\right)$$

  - We then solve for $\Delta \mathbf{v}$ (sparse linear system, conjugate gradient)
  - Given $\Delta \mathbf{v}$, we then compute $\Delta \mathbf{x} = h(\mathbf{v}_0 + \Delta \mathbf{v})$

2. Cloth simulation

Cloth simulation

- **Simulate the motion of a cloth**
  - Modeling internal forces within cloth



Cloth simulation

- **Forces**
  - Cloth's material behavior
    - Customarily described in terms of a scalar potential energy function E(**x**)
  - Force f arising from this energy

$$\mathbf{f} = -\partial E / \partial \mathbf{x}$$

    - Impractical as a single monolithic function
  - Internal behavior by a vector condition **C**(**x**)
    - We want **C**(**x**) to be zero
    - Associated energy with stiffness constant k

$$E_{\mathbf{C}}(\mathbf{x}) = \frac{k}{2}\mathbf{C}(\mathbf{x})^T\mathbf{C}(\mathbf{x})$$

Cloth simulation

- **Forces and force derivatives**
  - **Force evaluation**
    - Each element $\mathbf{f}_i$ is a vector in R^3

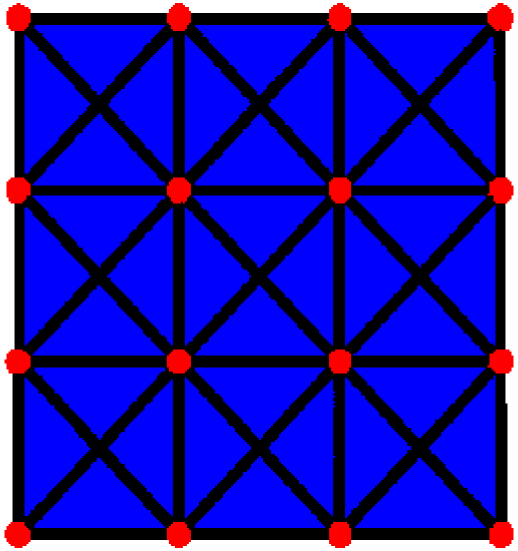
$$\mathbf{f}_i = -\frac{\partial E_{\mathbf{C}}}{\partial \mathbf{x}_i} = -k\frac{\partial \mathbf{C}(\mathbf{x})}{\partial \mathbf{x}_i}\mathbf{C}(\mathbf{x})$$

  - **Force derivative**
    - Derivative matrix

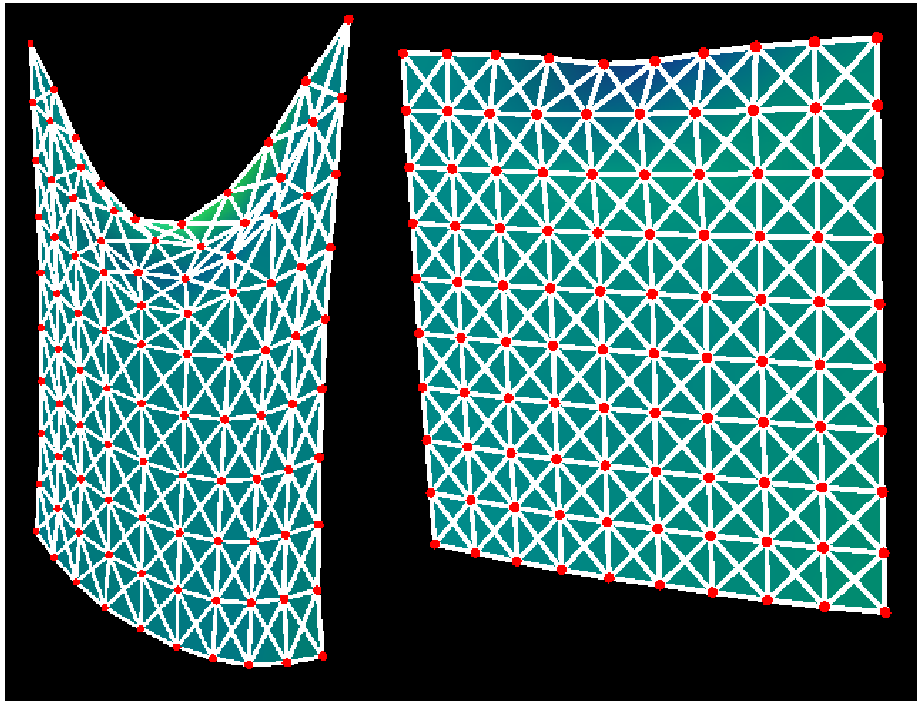
$$\mathbf{K}_{ij} = \frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_j} = -k\left(\frac{\partial \mathbf{C}(\mathbf{x})}{\partial \mathbf{x}_i}\frac{\partial \mathbf{C}(\mathbf{x})}{\partial \mathbf{x}_j}^T + \frac{\partial^2 \mathbf{C}(\mathbf{x})}{\partial \mathbf{x}_i \partial \mathbf{x}_j}\mathbf{C}(\mathbf{x})\right)$$

Cloth simulation

- **Rest vs. deformed shapes**



Rest shape: no external force



Deformed shape: external force applied

Cloth simulation

- **Stretch force**
  - Like texture mapping, a single continuous function **w**(u,v) mapping plane coordinates to world space
  - Stretch can be measure by

$$\mathbf{w}_u = \partial \mathbf{w}/\partial u \text{ and } \mathbf{w}_v = \partial \mathbf{w}/\partial v$$

  - Stretch energy

$$(\mathbf{w}_u^T \mathbf{w}_u - 1)^2$$

  - Nearby vertices (triangle meshes, vertices i,j,k)

$$\Delta \mathbf{x}_1 = \mathbf{x}_j - \mathbf{x}_i \text{ and } \Delta \mathbf{x}_2 = \mathbf{x}_k - \mathbf{x}_i \quad \Delta u_1 = u_j - u_i \quad \Delta u_2 = u_k - u_i$$

$$\Downarrow$$

$$\Delta \mathbf{x}_1 = \mathbf{w}_u \Delta u_1 + \mathbf{w}_v \Delta v_1 \text{ and } \Delta \mathbf{x}_2 = \mathbf{w}_u \Delta u_2 + \mathbf{w}_v \Delta v_2$$

Cloth simulation

- **Stretch force**
  - Nearby vertices (triangle meshes, vertices i,j,k)

$$\Delta \mathbf{x}_1 = \mathbf{w}_u \Delta u_1 + \mathbf{w}_v \Delta v_1 \text{ and } \Delta \mathbf{x}_2 = \mathbf{w}_u \Delta u_2 + \mathbf{w}_v \Delta v_2$$

$$\begin{pmatrix} \mathbf{w}_u & \mathbf{w}_v \end{pmatrix} = \begin{pmatrix} \Delta \mathbf{x}_1 & \Delta \mathbf{x}_2 \end{pmatrix} \begin{pmatrix} \Delta u_1 & \Delta u_2 \\ \Delta v_1 & \Delta v_2 \end{pmatrix}^{-1}$$

  - Condition for stretch energy

$$\mathbf{C}(\mathbf{x}) = a \begin{pmatrix} \|\mathbf{w}_u(\mathbf{x})\| - b_u \\ \|\mathbf{w}_v(\mathbf{x})\| - b_v \end{pmatrix} \qquad b_u = b_v = 1$$

a is the triangle's area in uv coordinates

Cloth simulation

- **Shear and bend forces**
  - Sheared in a triangle by considering the inner product

$$\mathbf{w}_u^T \mathbf{w}_v$$

  - Shear energy

$$C(\mathbf{x}) = a\mathbf{w}_u(\mathbf{x})^T \mathbf{w}_v(\mathbf{x})$$

  - Bending energy
    - Adjacent triangles with normals $\mathbf{n}_1$ and $\mathbf{n}_2$

$$\sin\theta = (\mathbf{n}_1 \times \mathbf{n}_2) \cdot \mathbf{e} \qquad \cos\theta = \mathbf{n}_1 \cdot \mathbf{n}_2$$

$$C(\mathbf{x}) = \theta$$

Cloth simulation

- **Damping**
  - Robust dynamic cloth simulation critically dependent on well-chosen damping forces
    - A function of both position and velocity
  - Should depend on the component of the system's velocity in the direction

$$\partial \mathbf{C}(\mathbf{x}) / \partial \mathbf{x}$$

  - Damping force formulation

$$\mathbf{d} = -k_d \frac{\partial \mathbf{C}(\mathbf{x})}{\partial \mathbf{x}} \dot{\mathbf{C}}(\mathbf{x})$$

Cloth simulation

- **Damping force derivative**

$$\Delta \mathbf{v} = h\mathbf{M}^{-1}\left(\mathbf{f}_0 + \boxed{\frac{\partial \mathbf{f}}{\partial \mathbf{x}}} h(\mathbf{v}_0 + \Delta \mathbf{v}) + \boxed{\frac{\partial \mathbf{f}}{\partial \mathbf{v}}} \Delta \mathbf{v}\right)$$

  - Spatial derivative

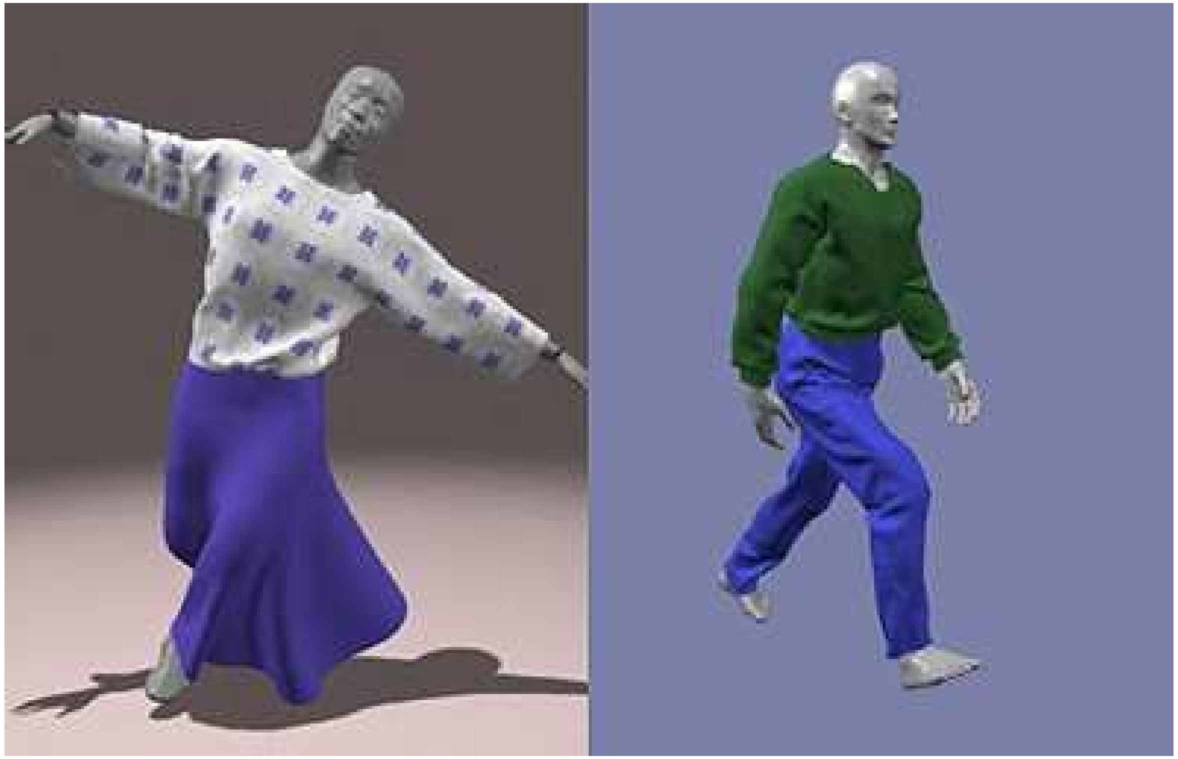
$$\frac{\partial \mathbf{d}_i}{\partial \mathbf{x}_j} = -k_d \left(\frac{\partial \mathbf{C}(\mathbf{x})}{\partial \mathbf{x}_i} \frac{\partial \dot{\mathbf{C}}(\mathbf{x})}{\partial \mathbf{x}_j}^T + \frac{\partial^2 \mathbf{C}(\mathbf{x})}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \dot{\mathbf{C}}(\mathbf{x}) \right)$$

  - Velocity derivative

$$\frac{\partial \mathbf{d}_i}{\partial \mathbf{v}_j} = -k_d \frac{\partial \mathbf{C}(\mathbf{x})}{\partial \mathbf{x}_i} \frac{\partial \dot{\mathbf{C}}(\mathbf{x})}{\partial \mathbf{v}_j}^T = -k_d \frac{\partial \mathbf{C}(\mathbf{x})}{\partial \mathbf{x}_i} \frac{\partial \mathbf{C}(\mathbf{x})}{\partial \mathbf{x}_j}^T$$

Cloth simulation results

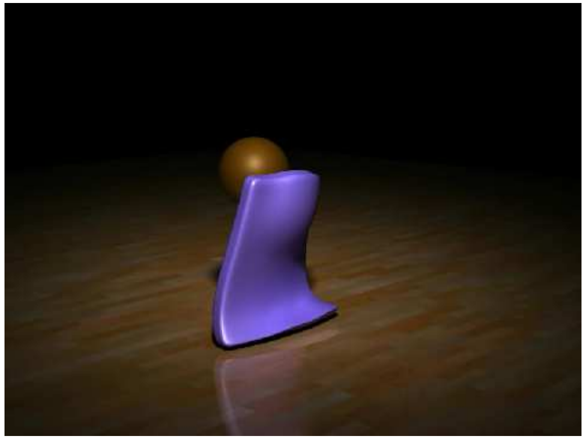
- **Constraints**
  - Interaction with solids (collision detection)



3. Simulation of solid deformable object

Solid deformable objects

- **A wide variety of behaviors**
  - Stretch and compress
  - Twist, curl and knot
  - Rip under large deformations
  - Form complex wrinkles and buckle under pressure

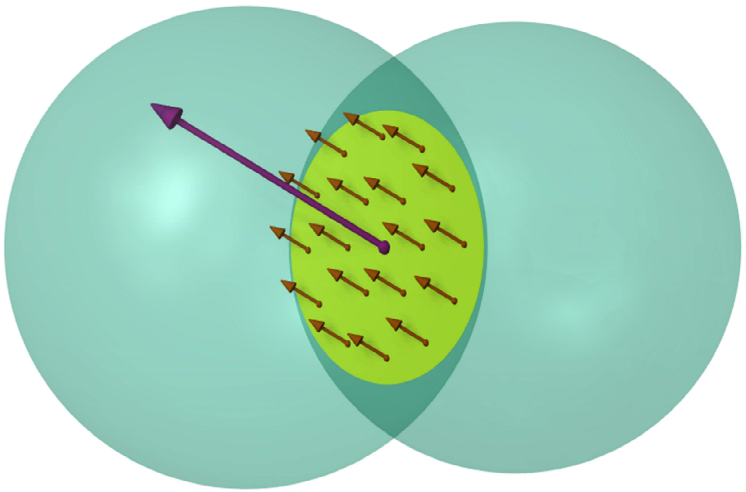




Continuum mechanics

- **A branch of mechanics**
  - Modeled as a continuous mass rather than as discrete particles
  - The matter in the body is continuously distributed
  - A continuum is a body that can be continually sub-divided into infinitesimal elements
    - Derivatives are available to compute
  - Deal with deformable bodies
    - As opposed to ideal rigid bodies
    - Analyzing internal force of rigid bodies should consider deformation (very small)

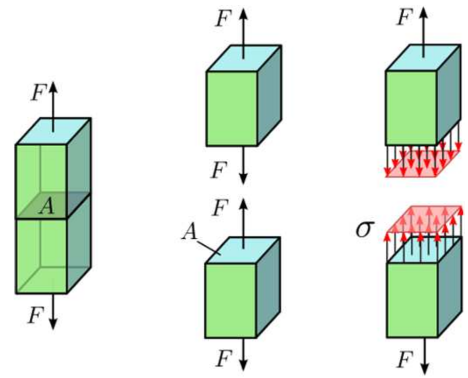
Stress of a material

- **A physical quantity of material**
  - Internal forces that neighboring particles exert on each other
  - Defined as the force across a "small" boundary per unit area of that boundary

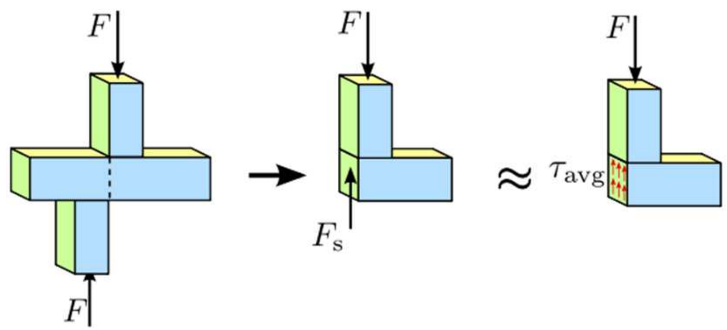


Stress of a material

- **Stress may be regarded as the sum of two components**
  - Normal stress
    - The stress component perpendicular to the surface (compression or tension)
  - Shear stress
    - The stress component parallel to the surface



Normal stress



Shear stress

Cauchy stress tensor

- **General stress**
  - Mechanical bodies experience more than one type of stress at the same time (combined stress)
  - Combined stresses cannot be described by a single vector
- **Cauchy's observation**
  - The stress vector across a surface will always be a linear function of the surface's normal

$$T = \boldsymbol{\sigma}(n) \qquad\qquad \boldsymbol{\sigma}(\alpha u + \beta v) = \alpha\boldsymbol{\sigma}(u) + \beta\boldsymbol{\sigma}(v)$$

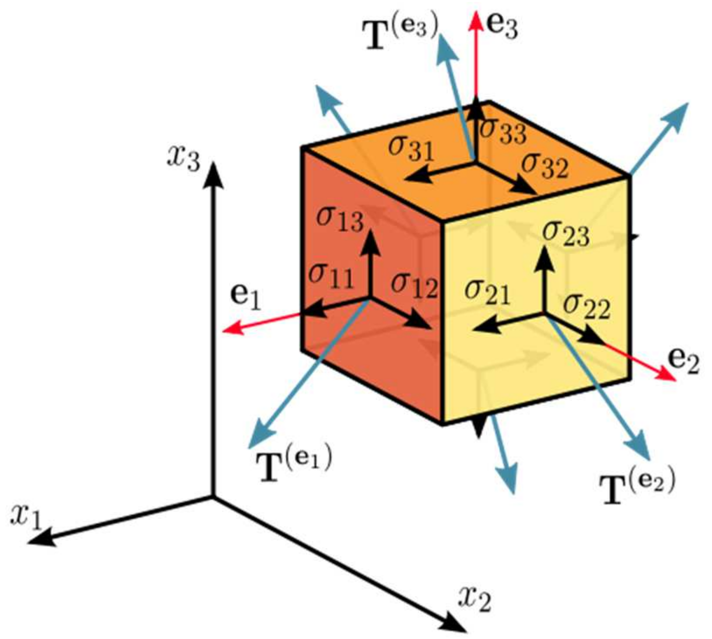
Cauchy stress tensor

- **Definition**
  - A 3x3 matrix

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

$$T = n \cdot \boldsymbol{\sigma}$$

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$



Cauchy stress tensor

- **Symmetric stress tensor**
  - Conservation of angular momentum implies that the stress tensor is symmetric

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

    - Normal stresses

$$\sigma_x, \sigma_y, \sigma_z$$

    - Shear stresses

$$\tau_{xy}, \tau_{xz}, \tau_{yz}$$

Deformation of material

- **Physical view of deformation**
  - Transformation of a body from a reference configuration to the current configuration
  - A configuration is a set containing the positions of all particles of the body

- **Causes of deformation**
  - External loads (usually on the exterior surfaces)
  - Body forces (volumetric force within the whole body, e.g., gravity force)

Types of deformation

- **Elastic deformations**
  - Deformations are recovered after the stress field has been removed
- **Irreversible deformation**
  - Deformations remain even after stresses have been removed
  - Plastic deformation
    - Occurs in material bodies after stresses have attained a certain threshold value (elastic limit or yield stress)

Strain of material

- **What is a strain**
  - A description of deformation in terms of relative displacement of particles
  - The relation between stresses and induced strains is expressed by constitutive equations
    - E.g., Hooke's law for linear elastic materials
- **Formulation**
  - A general deformation of a body can be expressed in the form

$$\mathbf{x} = \boldsymbol{F}(\mathbf{X})$$

    - **X** is the reference position of material points in the body

Strain of material

- **Formulation**
  - Such a measure does not distinguish between rigid body motions and changes in shape of the body
  - Mathematical definition

$$\boldsymbol{\varepsilon} \doteq \frac{\partial}{\partial \mathbf{X}} (\mathbf{x} - \mathbf{X}) = \boldsymbol{F}' - \boldsymbol{I}$$

- Strains measure how much a given deformation differs locally from a rigid-body deformation

Strain of material

- **Strain tensor**

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\gamma_{xy} = \gamma_{yx} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

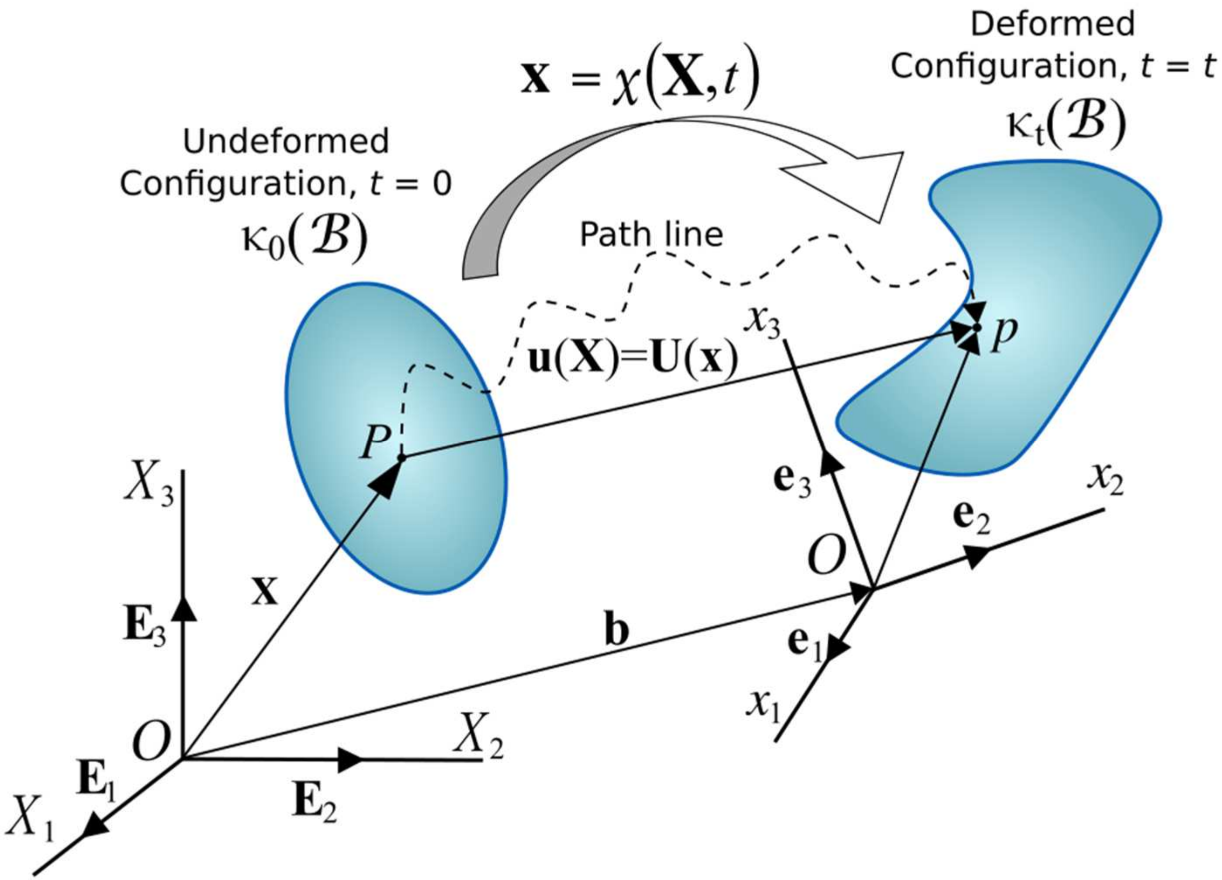
$$\gamma_{yz} = \gamma_{zy} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

$$\gamma_{zx} = \gamma_{xz} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_{zz} \end{bmatrix}$$

Deformation of a continuum body

- **Motion and deformation of a continuum body**



Continuous model

- **Define material coordinates**

$$\boldsymbol{u} = [u, v, w]^\mathsf{T}$$

- **Deformation of the material**

$$\boldsymbol{x}(\boldsymbol{u}) = [x, y, z]^\mathsf{T}$$

  - In areas where material exists, **x**(**u**) is continuous
  - Except across a finite number of surfaces within the volume that correspond to fractures

Continuous model

- **Green's strain tensor**
  - Measure the local deformation of the material

$$\epsilon_{ij} = \left(\frac{\partial \boldsymbol{x}}{\partial u_i} \cdot \frac{\partial \boldsymbol{x}}{\partial u_j} \right) - \delta_{ij}$$

    - Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & : \quad i = j \\ 0 & : \quad i \neq j \end{cases}$$

    - This strain metric only measures deformation
    - Invariant with respect to rigid body transformations

Continuous model

- **Strain rate tensor**
  - Measure the rate at which the strain is changing
  - Take the time derivative of strain

$$\nu_{ij} = \left(\frac{\partial \boldsymbol{x}}{\partial u_i} \cdot \frac{\partial \dot{\boldsymbol{x}}}{\partial u_j} \right) + \left(\frac{\partial \dot{\boldsymbol{x}}}{\partial u_i} \cdot \frac{\partial \boldsymbol{x}}{\partial u_j} \right)$$

  - Strain and strain rate tensors provide the raw information to compute internal elastic and damping forces
  - Do not take into account the properties of the material

Notation

- **Partial differentiation**

$$\mathbf{x}_{,i} \equiv \partial \mathbf{x} / \partial \theta_i, \; \mathbf{u}_{,ik} \equiv \frac{\partial^2 \mathbf{u}}{\partial \theta_i \, \partial \theta_k}$$

- **Operators**
  - Vector dot product: dot $(\cdot)$
  - Vector cross product: cross $(\times)$
  - Tensor double contraction: colon $(:)$

Linear elasticity

Linear elasticity

- **Commonly used in computer graphics**
  - Relatively simple formulation
  - Resulting efficient simulations

- **Three essential parts**
  - Geometry: study of deformation a body can undergo
  - Internal and external forces: how they affect an object's equilibrium or dynamics
  - Constitutive relation: how deforming geometry relates to internal forces

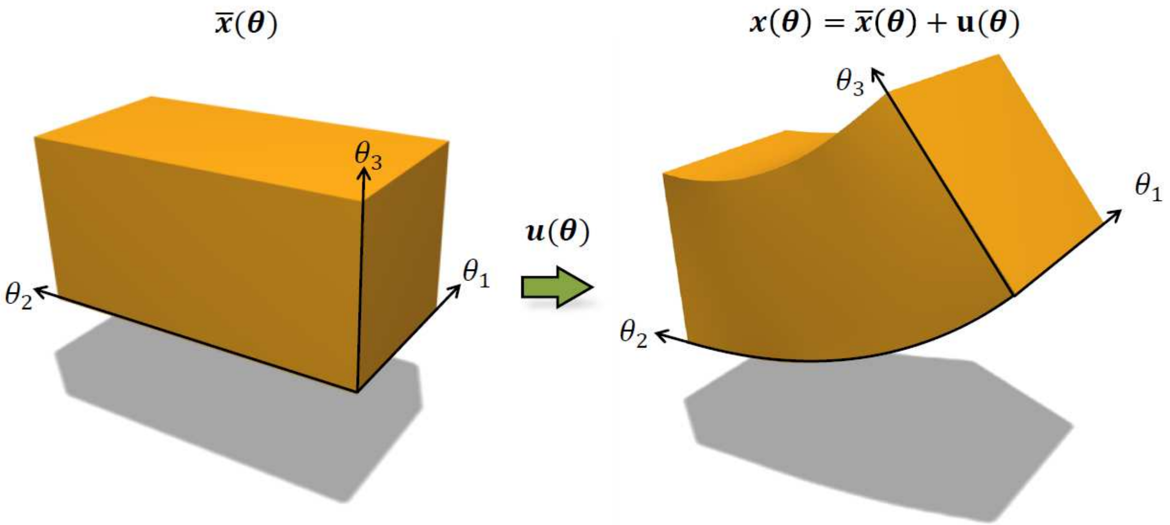
Linear elasticity

- **Geometry**
  - Restrict to Lagrangian description
    - Undeformed positions

$$\bar{\mathbf{x}}(\boldsymbol{\theta}) \qquad \boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T$$

    - Undergo a deformation

$$\mathbf{x}(\boldsymbol{\theta}) = \bar{\mathbf{x}}(\boldsymbol{\theta}) + \mathbf{u}(\boldsymbol{\theta})$$



Linear elasticity

- **Geometry**
  - Cauchy strain
    - Assuming only small displacements

$$\epsilon_{ij} = \frac{1}{2}\left(\mathbf{u}_{,i} \cdot \bar{\mathbf{x}}_{,j} + \bar{\mathbf{x}}_{,i} \cdot \mathbf{u}_{,j}\right)$$

    - The main diagonal of the tensor
      - The amount of stretch in the three (normal) spatial directions
    - Off-diagonal values
      - Amount of shear in the according planes
    - Linearization of the more general (non-linear) strain

Linear elasticity

- **Forces, equilibrium and dynamics**
  - Cauchy stress
    - Introduce a virtual cut plane with normal **n**
    - Force distribution at a point can compactly be described by the product of the Cauchy stress tensor with the plane normal

$$\sigma \cdot \mathbf{n} = \mathbf{f}_{\mathbf{n}}$$

    - Main diagonal
      - Normal stress
    - Off-diagonal
      - Shear stress

Linear elasticity

- **Forces, equilibrium and dynamics**
  - Total force
    - Summing up all traction forces on the side of the corresponding infinitesimal cube
    - Using the divergence (Gauss) theorem

$$\sigma \cdot \mathbf{n} = \mathbf{f}_{\mathbf{n}}$$

$$\Big\downarrow \text{ Volume integral}$$

$$\nabla \cdot \sigma + \mathbf{f} = 0$$

Linear elasticity

- **Forces, equilibrium and dynamics**
  - Energy
    - Internal forces in an elastic body are conservative
    - Related to an underlying scalar energy potential characterizing the amount of work for deformation

$$W_{int}(\mathbf{u}) = \int_{\Omega} \epsilon(\mathbf{u}) : \sigma(\mathbf{u})\, d\Omega$$

Strain (length) Stress (force)

- Conservative forces

$$\mathbf{f}_{int} = -\frac{\partial W_{int}(\mathbf{u})}{\partial \mathbf{u}}$$

Linear elasticity

- **Forces, equilibrium and dynamics**
  - Static equilibrium
    - All internal and external forces need to cancel each other

$$\nabla \cdot \sigma = \mathbf{f}_{ext}$$

    - Making use of the elastic potential

$$-\frac{\partial W_{int}(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_{ext}$$

      - Better suited for setting up the corresponding discrete problems

Linear elasticity

- **Forces, equilibrium and dynamics**
  - Equations of motion
    - If an object is not in static equilibrium: difference between internal and external forces results in net forces
    - Acceleration of the material according to Newton's second law

$$\rho\,\ddot{\mathbf{u}} + \mathbf{f}_d(\dot{\mathbf{u}}) + \nabla \cdot \sigma \;=\; \mathbf{f}_{ext}$$

      - $\mathbf{f}_d(\dot{\mathbf{u}})$: a damping force
    - Making use of the elastic energy potential

$$\rho\,\ddot{\mathbf{u}} + \mathbf{f}_d(\dot{\mathbf{u}}) - \frac{\partial W_{int}(\mathbf{u})}{\partial \mathbf{u}} \;=\; \mathbf{f}_{ext}$$

Linear elasticity

- **Constitutive relation**
  - Establish the relation between internal deformation and force
  - The simplest is the linear relation
    - a Hookean material

$$\sigma = \mathbf{C} : \epsilon$$

    - Young modulus: material's stiffness
    - Poisson's ratio: how much (linearized) change in volume (compression) is penalized during deformation

4. Resultant-based formulations

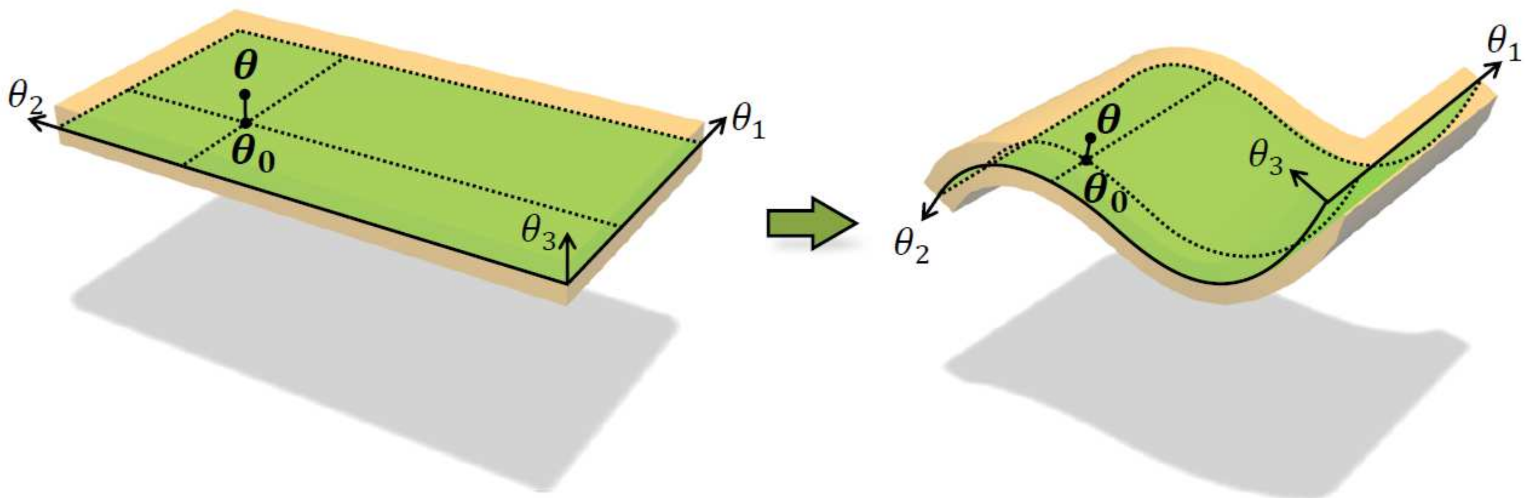
Resultant-based formulations

- **Thin geometries**
  - Specialized variants of the theory
  - More efficient and numerically better suited

- **Resultant-based models**
  - Material has only small extent in certain spatial directions
  - Simplified by making certain assumption on how the material can deform in these directions
  - Thin shell theory: reduction along one direction
  - Rod theory: reduction along two directions

Resultant-based formulations

- **Shells**
  - Consider a volumetric surface-like solid
  - Extent along tangent directions is much greater than along normal direction



Resultant-based formulations

- **Shells**
  - Strain about middle surface
    - Middle surface parameterized by the material-domain surface

$$\boldsymbol{\theta}_0 = (\theta_1, \theta_2, 0)$$

    - Assuming the shell to be sufficiently thin in normal direction

$$\begin{aligned} \bar{\mathbf{x}}(\boldsymbol{\theta}) &\approx \bar{\mathbf{x}}(\boldsymbol{\theta}_0) + \theta_3 \, \bar{\mathbf{x}}_{,3}(\boldsymbol{\theta}_0) \\ \mathbf{u}(\boldsymbol{\theta}) &\approx \mathbf{u}(\boldsymbol{\theta}_0) + \theta_3 \, \mathbf{u}_{,3}(\boldsymbol{\theta}_0) \end{aligned}$$

Resultant-based formulations

- **Shells**
  - In the view of the linear elasticity theory
    - Substitute into the linear Cauchy strain

$$\epsilon_{ij} = \frac{1}{2}\left(\mathbf{u}_{,i}\cdot\bar{\mathbf{x}}_{,j} + \bar{\mathbf{x}}_{,i}\cdot\mathbf{u}_{,j}\right)$$

$$\begin{aligned}\bar{\mathbf{x}}(\boldsymbol{\theta}) &\approx \bar{\mathbf{x}}(\boldsymbol{\theta}_0) + \theta_3\,\bar{\mathbf{x}}_{,3}(\boldsymbol{\theta}_0)\\ \mathbf{u}(\boldsymbol{\theta}) &\approx \mathbf{u}(\boldsymbol{\theta}_0) + \theta_3\,\mathbf{u}_{,3}(\boldsymbol{\theta}_0)\end{aligned}$$

$$\boldsymbol{\epsilon}(\boldsymbol{\theta}) \approx \boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \theta_3\,\boldsymbol{\beta}^3(\boldsymbol{\theta}_0)$$

- Membrane strain $\alpha_{ij} = \frac{1}{2}\left(\mathbf{u}_{,i}\cdot\bar{\mathbf{x}}_{,j} + \bar{\mathbf{x}}_{,i}\cdot\mathbf{u}_{,j}\right)$
- Bending strain $\beta_{ij}^k = \frac{1}{2}\left(\mathbf{u}_{,ik}\cdot\bar{\mathbf{x}}_{,j} + \bar{\mathbf{x}}_{,i}\cdot\mathbf{u}_{,jk} + \mathbf{u}_{,i}\cdot\bar{\mathbf{x}}_{,jk} + \bar{\mathbf{x}}_{,ik}\cdot\mathbf{u}_{,j}\right)$

Resultant-based formulations

- **Shells**
  - Energy integration
    - Elastic energy of the volumetric shell model

$$W_{tot}(\mathbf{u}) = \frac{1}{2}\int_{\Omega}(\boldsymbol{\epsilon}(\mathbf{u}) : \mathbf{C} : \boldsymbol{\epsilon}(\mathbf{u}) - \mathbf{f}_{ext} \cdot \mathbf{u})\, d\Omega \qquad \boldsymbol{\epsilon}(\boldsymbol{\theta}) \approx \boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \theta_3\, \boldsymbol{\beta}^3(\boldsymbol{\theta}_0)$$

$$\Downarrow$$

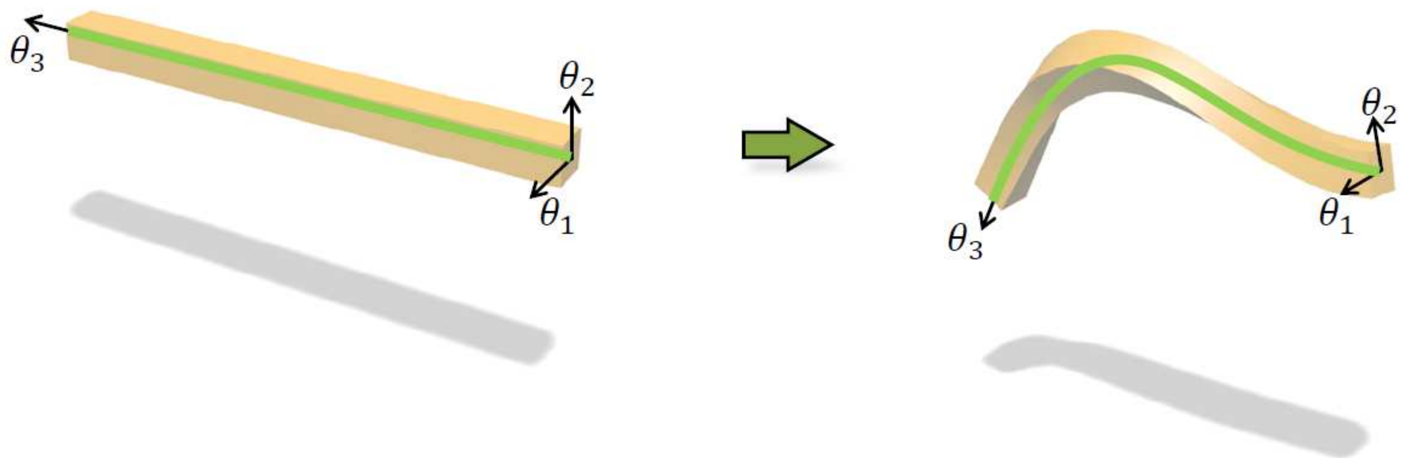
$$W = \frac{1}{2}\int_{\Omega}\Big(\boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \theta_3\, \boldsymbol{\beta}^3(\boldsymbol{\theta}_0)\Big) : \mathbf{C} : \Big(\boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \theta_3\, \boldsymbol{\beta}^3(\boldsymbol{\theta}_0)\Big)\, \mathrm{d}\Omega$$

    - Integration in normal direction can be performed analytically

$$W = \frac{h_3}{2}\int_{\mathcal{S}} \boldsymbol{\alpha} : \mathbf{C} : \boldsymbol{\alpha} + \frac{h_3^2}{12}\, \boldsymbol{\beta}^3 : \mathbf{C} : \boldsymbol{\beta}^3\, \mathrm{d}\mathcal{S}$$

Resultant-based formulations

- **Rods**
  - A volumetric curve-like solid
  - Extent along tangent direction is much greater than along normal directions



Resultant-based formulations

- **Rods**
  - Strain about centerline
    - Let the centerline curve Γ be parameterized by

$$\boldsymbol{\theta}_0 = (\theta_1, 0, 0)$$

    - For small extents along both normals

$$\begin{aligned} \bar{\mathbf{x}}(\boldsymbol{\theta}) &\approx \bar{\mathbf{x}}(\boldsymbol{\theta}_0) + \theta_2 \, \bar{\mathbf{x}}_{,2}(\boldsymbol{\theta}_0) + \theta_3 \, \bar{\mathbf{x}}_{,3}(\boldsymbol{\theta}_0) \\ \mathbf{u}(\boldsymbol{\theta}) &\approx \mathbf{u}(\boldsymbol{\theta}_0) + \theta_2 \, \mathbf{u}_{,2}(\boldsymbol{\theta}_0) + \theta_3 \, \mathbf{u}_{,3}(\boldsymbol{\theta}_0) \end{aligned}$$

    - The same steps for the derivation of the small strain

$$\boldsymbol{\epsilon}(\boldsymbol{\theta}) \approx \boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \theta_2 \, \boldsymbol{\beta}^2(\boldsymbol{\theta}_0) + \theta_3 \, \boldsymbol{\beta}^3(\boldsymbol{\theta}_0)$$

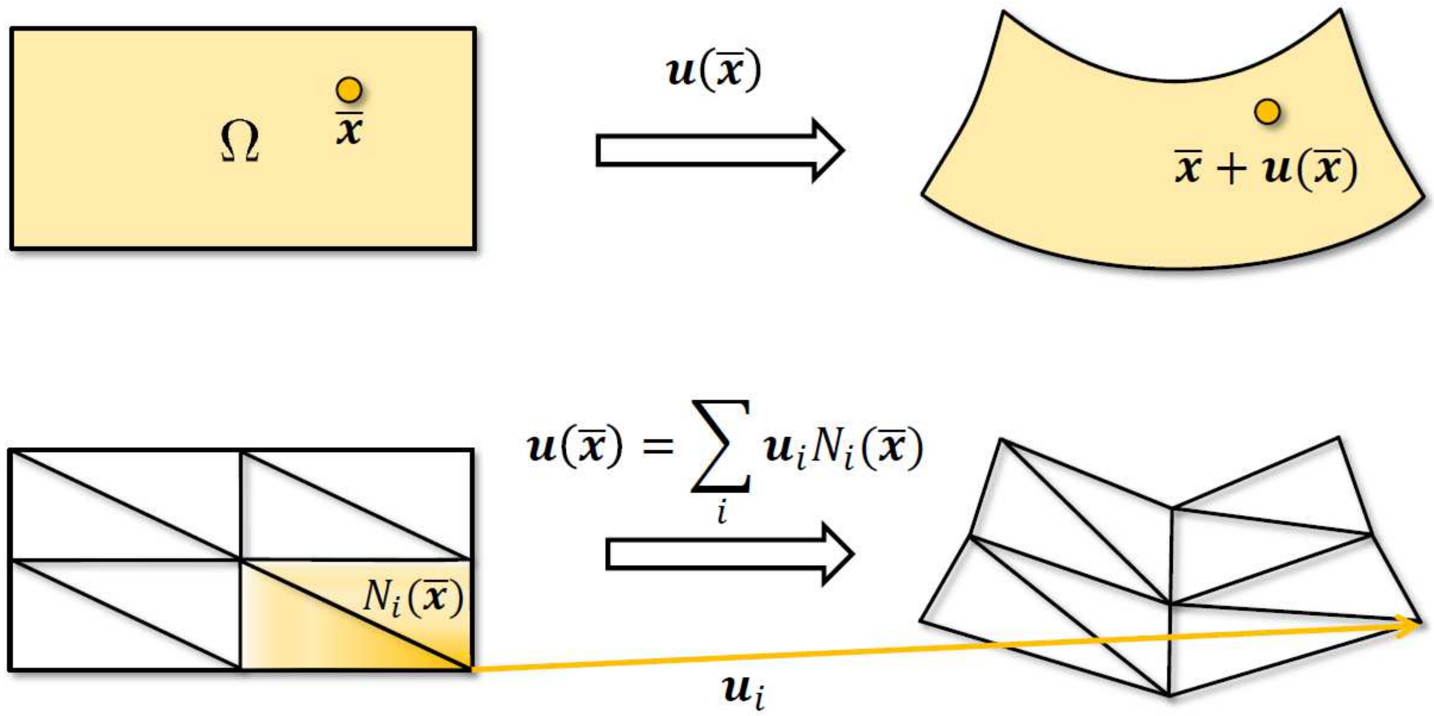
Resultant-based formulations

- **Rods**
  - Energy integration
    - Analytic integration in the normal directions yields the one-dimensional integral of axial energy density over the rod's centerline

$$W = \frac{h_2 h_3}{2} \int_{\Gamma} \boldsymbol{\alpha} : \mathbf{C} : \boldsymbol{\alpha} + \frac{h_2^2}{12} \boldsymbol{\beta}^2 : \mathbf{C} : \boldsymbol{\beta}^2 + \frac{h_3^2}{12} \boldsymbol{\beta}^3 : \mathbf{C} : \boldsymbol{\beta}^3 \, \mathrm{d}\Gamma$$

Discrete formulation

- **Finite-element method (FEM)**



Discrete formulation

- **Discrete formulation**
  - Representation of the finite dimensional space
    - Using a basis of shape functions

$$\mathbf{u}_N(\bar{\mathbf{x}}) = \sum_i^N \mathbf{u}_i N_i(\bar{\mathbf{x}}) \in V_N$$

    - Basis functions must also fulfill the completeness property
      - Constant reproduction: In order to represent arbitrary translations of the body
        » Partition of unity $\sum_i N_i(\bar{\mathbf{x}}) = 1$
      - Linear reproduction: necessary for a basis to represent constant strain fields as well as arbitrary rigid body motions

Space discretization for linear formulations

- **Energy-based approach**
  - Taking gradient to yield equation to solve
    - Static case

$$\mathbf{K}\mathbf{u} = \mathbf{f}_{ext}$$

    - Dynamic case

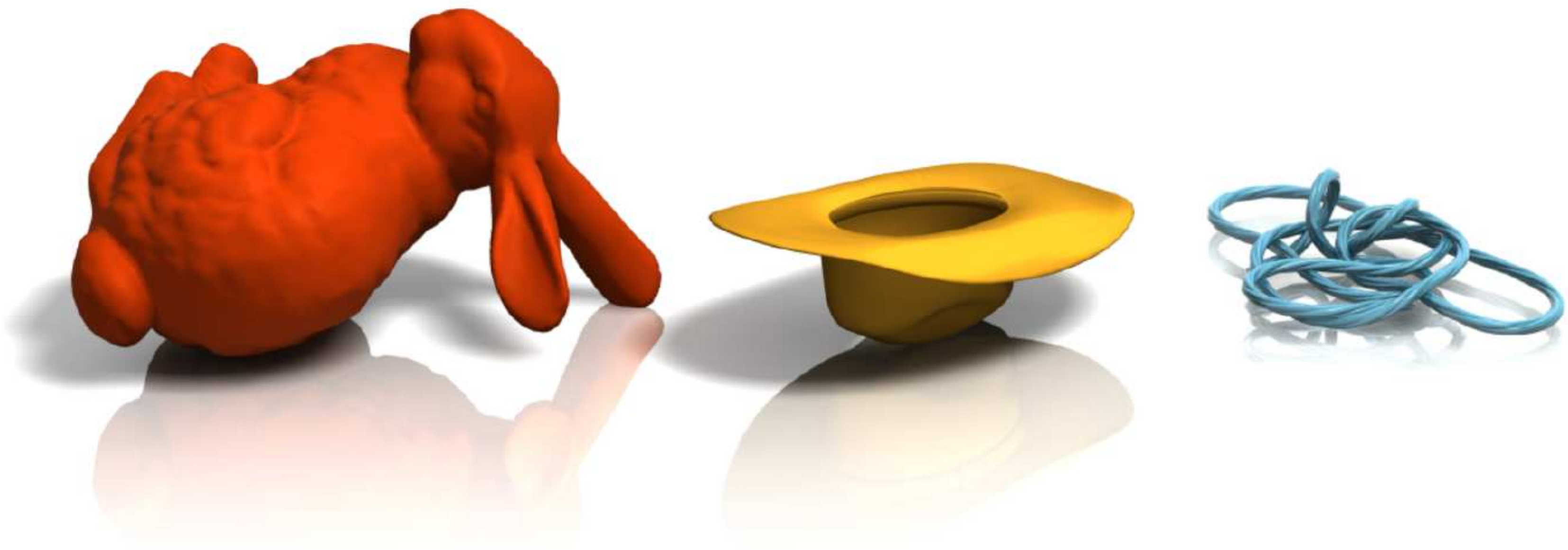
$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}_{ext}$$

$$\mathbf{M}_{ij} = \mathbf{I} \cdot \int_{\Omega} \rho N_i N_j \, d\Omega$$

4. Unifying resultant-based models

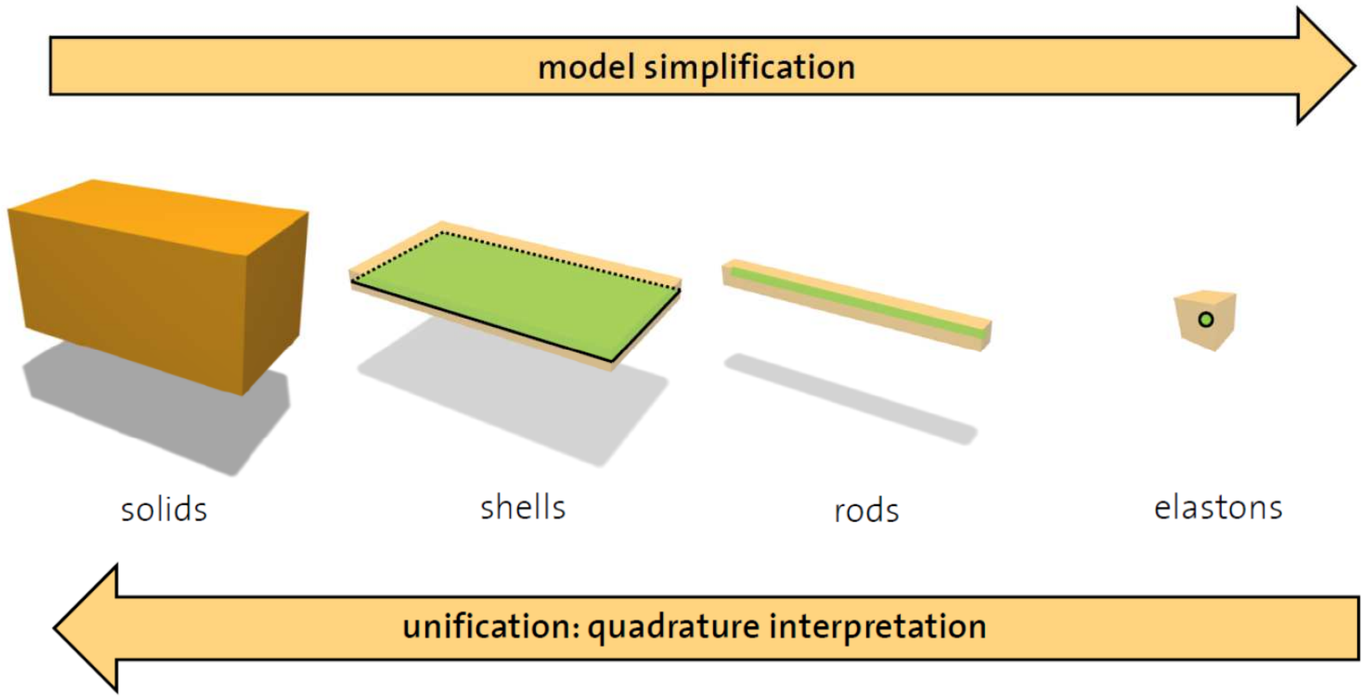
Unifying resultant-based models

- **Resultant-based models**
  - Only valid for single types of geometry (thin shell or rod)
  - Handle all three types in an unified manner?



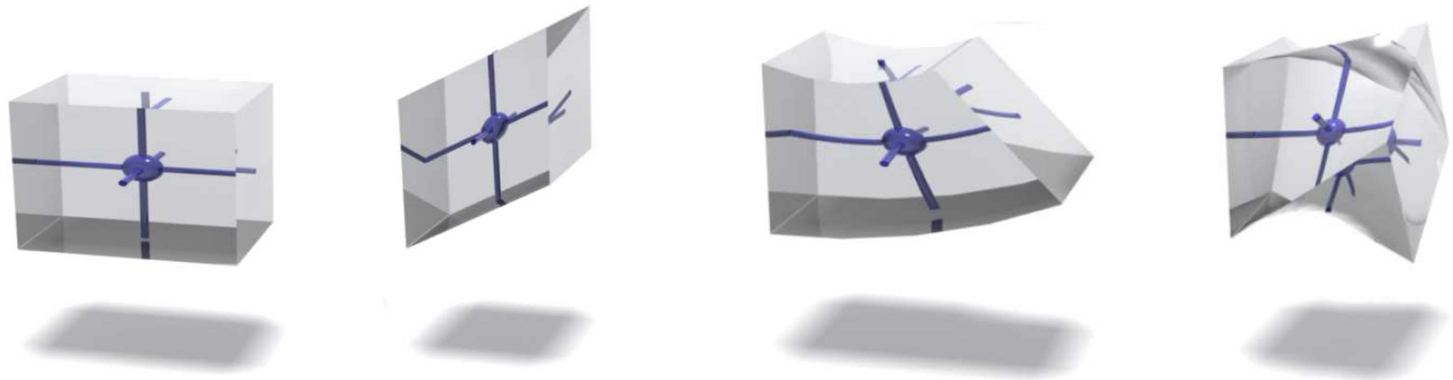
Elastons

- **Basic building blocks**
  - For assembling the elastic energy of any deformable object, independent of its form



Elastons

- **Consider a volumetric point-like solid**
  - Extent along all three directions is small
  - Strain in the vicinity of this point will measure
    - Linear deformations: stretch and shear at the center
    - Quadratic deformations: bending and twist along all three “normal” directions



Elastons

- **Linearizing strain**
  - Employ curvilinear coordinates describe an elaston centered at

$$\boldsymbol{\theta}_0 = (0,0,0)$$

  - First-order Taylor approximation of positions and displacements
    - In all three normal directions

$$\begin{aligned}
\bar{\mathbf{x}}(\boldsymbol{\theta}) &\approx \bar{\mathbf{x}}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \, \bar{\mathbf{x}}_{,k}(\boldsymbol{\theta}_0) \\
\mathbf{u}(\boldsymbol{\theta}) &\approx \mathbf{u}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \, \mathbf{u}_{,k}(\boldsymbol{\theta}_0)
\end{aligned}$$

Elastons

- **Linearizing strain**
  - Strain centered about the elaston

$$\epsilon_{ij} = \frac{1}{2}\left(\mathbf{u}_{,i} \cdot \bar{\mathbf{x}}_{,j} + \bar{\mathbf{x}}_{,i} \cdot \mathbf{u}_{,j}\right) \quad + \quad \begin{aligned} \bar{\mathbf{x}}(\boldsymbol{\theta}) &\approx \bar{\mathbf{x}}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \bar{\mathbf{x}}_{,k}(\boldsymbol{\theta}_0) \\ \mathbf{u}(\boldsymbol{\theta}) &\approx \mathbf{u}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \mathbf{u}_{,k}(\boldsymbol{\theta}_0) \end{aligned}$$

$$\Downarrow$$

$$\epsilon(\boldsymbol{\theta}) \approx \boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \boldsymbol{\beta}^k(\boldsymbol{\theta}_0)$$

- Naturally generalizes its shell and rod analogues
- Capture stretching, shearing, bending, and twisting along all three axes

Elastons

- **Energy integration**
  - Integral over the elaston's volume

$$\boldsymbol{\epsilon}(\boldsymbol{\theta}) \approx \boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \boldsymbol{\beta}^k(\boldsymbol{\theta}_0) \qquad \boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\epsilon} \qquad W_{int}(\mathbf{u}) = \int_{\Omega} \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\sigma}(\mathbf{u})\, d\Omega$$

$$W = \frac{1}{2} \int_{\Omega_e} \left(\boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \boldsymbol{\beta}^k(\boldsymbol{\theta}_0) \right) : \mathbf{C} : \left(\boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \boldsymbol{\beta}^k(\boldsymbol{\theta}_0) \right) \mathrm{d}\Omega$$

  - All three directions have thin extent
    - Analytically integrate

$$W = \frac{V}{2} \left(\boldsymbol{\alpha}(\boldsymbol{\theta}_0) : \mathbf{C} : \boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \frac{h_k^2}{12} \boldsymbol{\beta}^k(\boldsymbol{\theta}_0) : \mathbf{C} : \boldsymbol{\beta}^k(\boldsymbol{\theta}_0) \right)$$

$$V = h_1 h_2 h_3$$

Elastons

- **Summing up: a new integration rule**
  - The classical goal of resultant-based model
    - Reduce the dimensionality of the model
    - Simplify its numerical treatment
    - Energy integration can be performed analytically
  - Elastons offer the most general integration rule
    - Approximate the stored elastic energy of rods, shells, or solids, respectively

$$W_{int}(\mathbf{u}) = \int_{\Omega} \epsilon(\mathbf{u}) : \sigma(\mathbf{u})\, d\Omega$$

$$\Downarrow$$

$$W = \sum_{e \in \mathcal{E}} \frac{V^e}{2} \left(\boldsymbol{\alpha}^e : \mathbf{C} : \boldsymbol{\alpha}^e + \sum_{k=1}^{3} \frac{(h_k^e)^2}{12} \boldsymbol{\beta}^{ke} : \mathbf{C} : \boldsymbol{\beta}^{ke} \right)$$

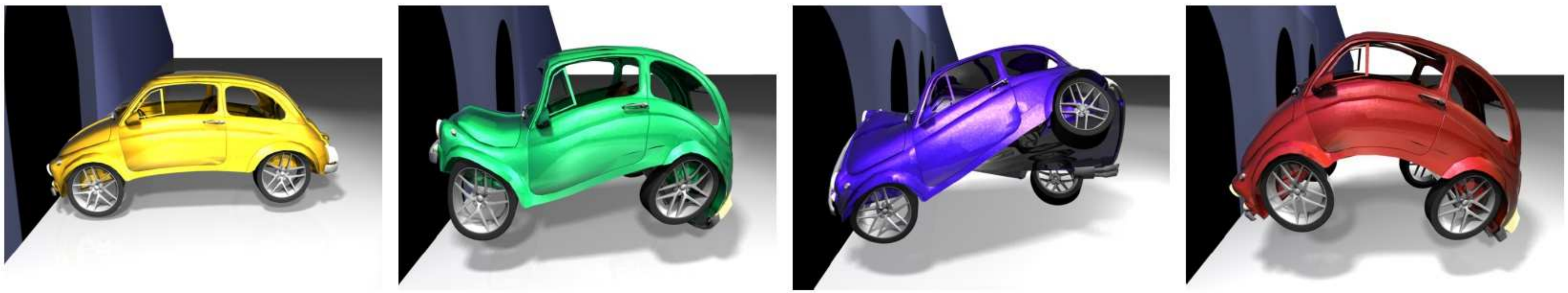
Results

[redacted]

5. Example-based elastic materials

Art-directable elastic potentials

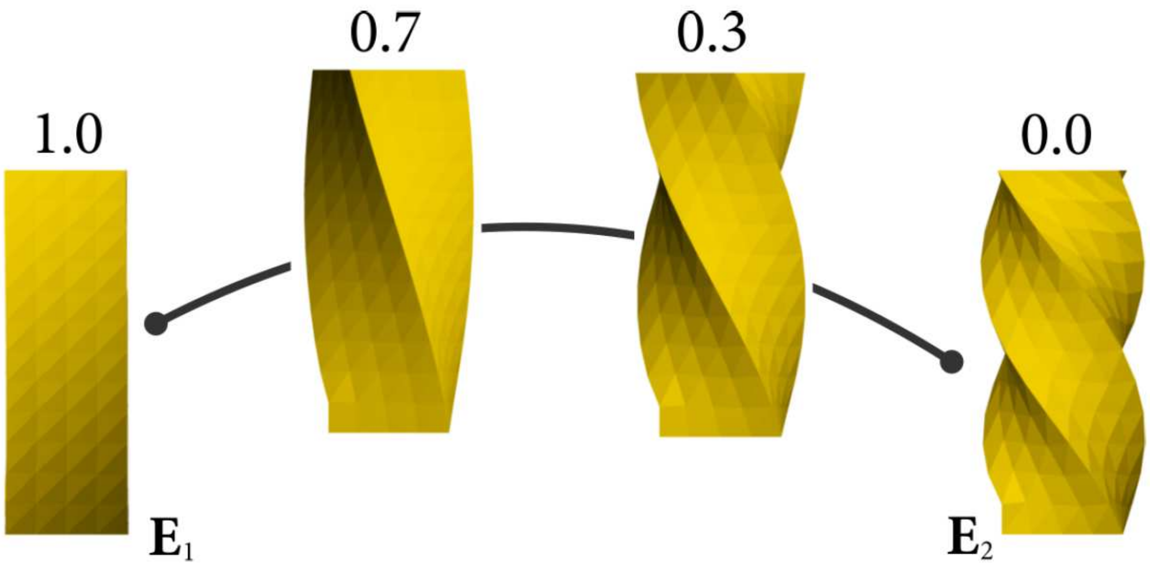
- **Control the deformation**
  - Classical deformation mainly driven by the chosen material model and its parameter values
  - Often, an artist starts with a vision on how an object should deform
  - Setting up a specialized elastic model to follow the preferred example poses



Example Manifold

- **Example manifold by example interpolation**
  - Interpolate between these examples by interpolating their descriptors

$$\mathbf{E}(w) = (1 - w)\mathbf{E}_1 + w\mathbf{E}_2$$



Results

[redacted]

Next lecture: Fluid simulation