Computer Graphics I

Lecture 21: Fluid simulation

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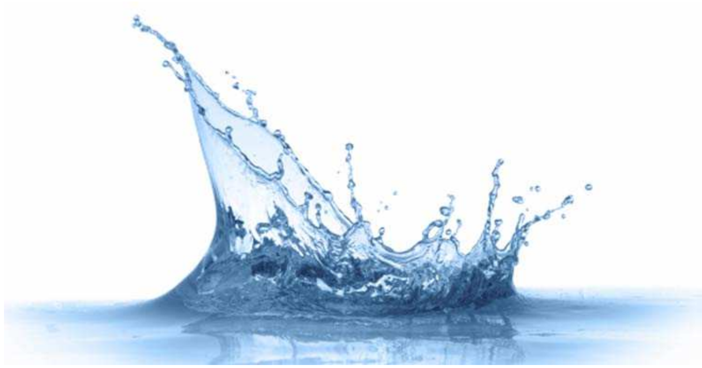
What is a fluid?

- **Definition**
  - A substance that continually deforms (flows) under an applied stress

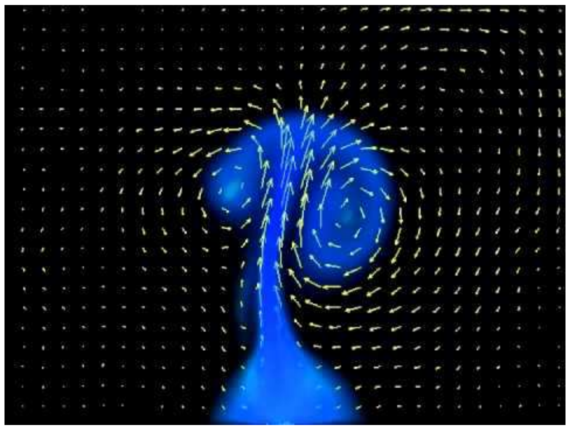


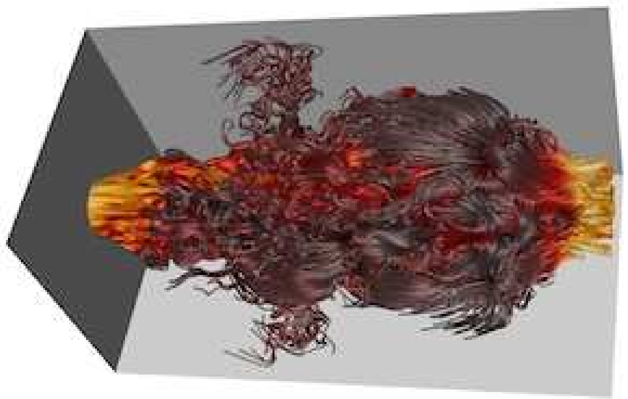

Fluid simulation

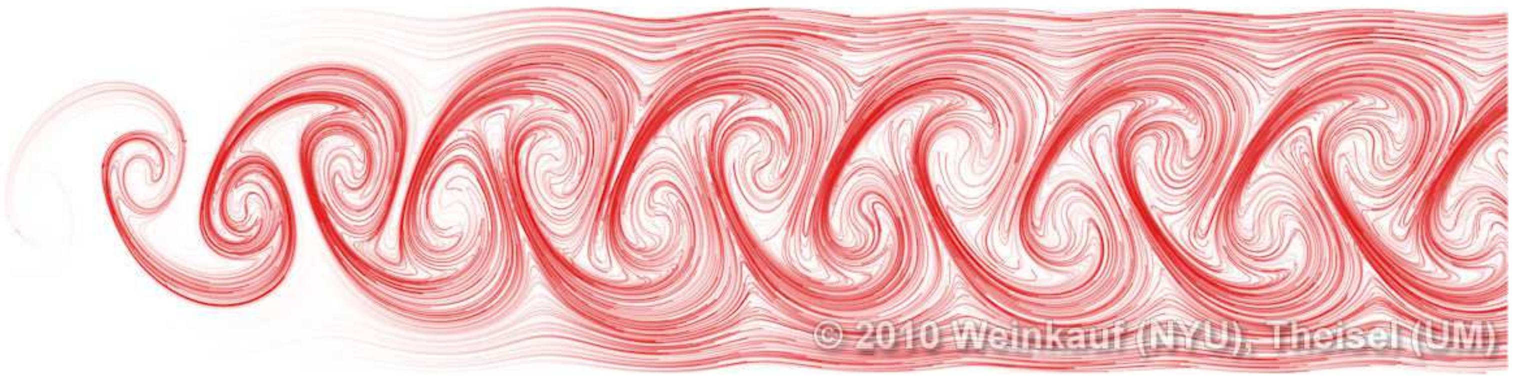
- **Simulate the behavior of fluid flows**
  - Modeling the fluid behavior
  - Solve the fluid equations numerically
  - Rendering the simulation result

Physical quantities

- **Velocity field u**
  - A spatially and temporally varying vector field of velocity

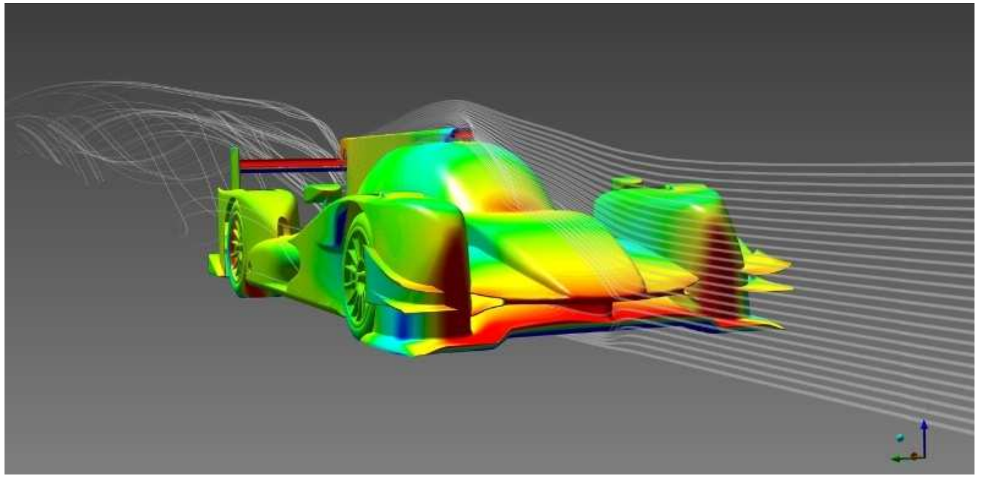






Physical quantities

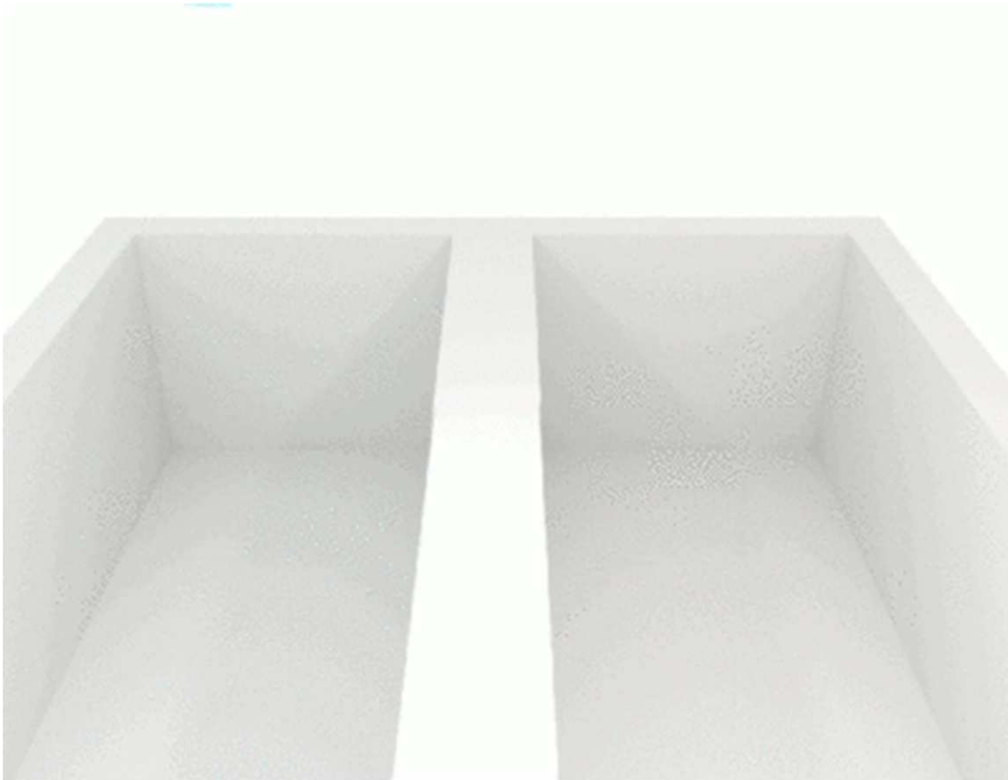
- **Likewise**
  - Pressure field *p*
    - A scalar field of pressure
  - Density field ρ
    - A scalar field of density
  - Temperature field T
    - A scalar field of temperature



Pressure field on a car surface

Physical quantities

- **Viscosity ν**
  - A measure of its resistance to gradual deformation
  - Small viscosity implies easier (large) deformation



Divergence

- **The divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume**

$$\operatorname{div} \mathbf{F}\big|_p = \lim_{V \to \{p\}} \iint_{S(V)} \frac{\mathbf{F} \cdot \hat{\mathbf{n}}}{|V|}\, dS$$

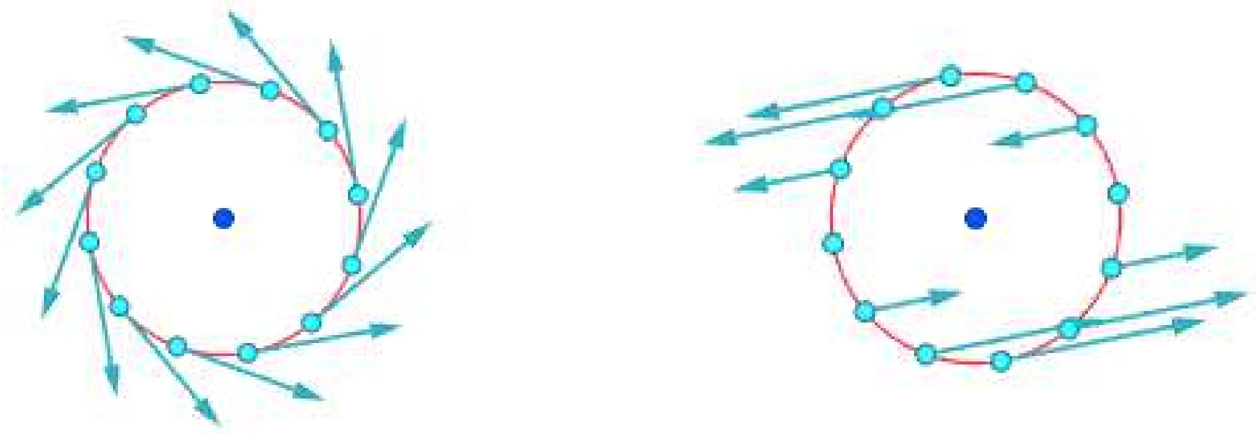
$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (U, V, W) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}.$$

Vorticity

- **Describes the local spinning motion**

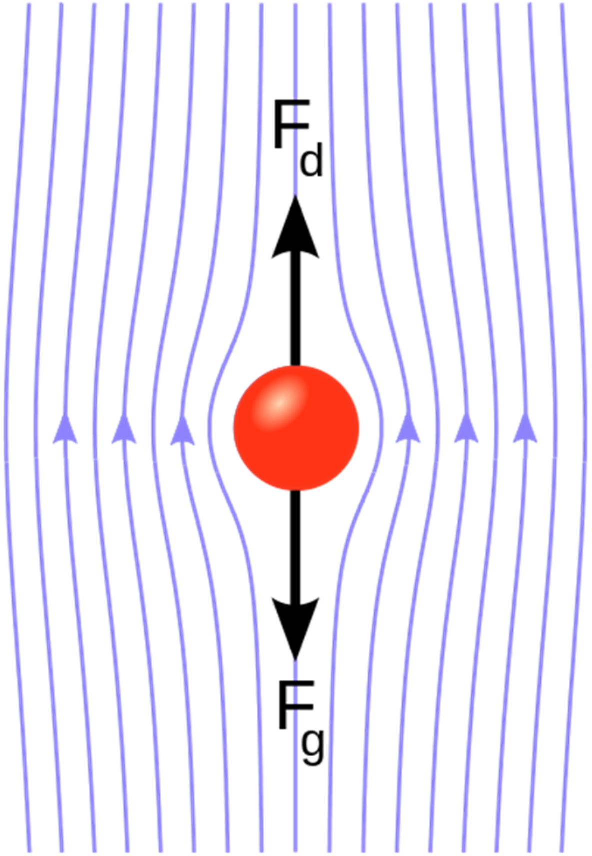
$$\vec{\omega} = \nabla \times \vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (v_x, v_y, v_z)$$

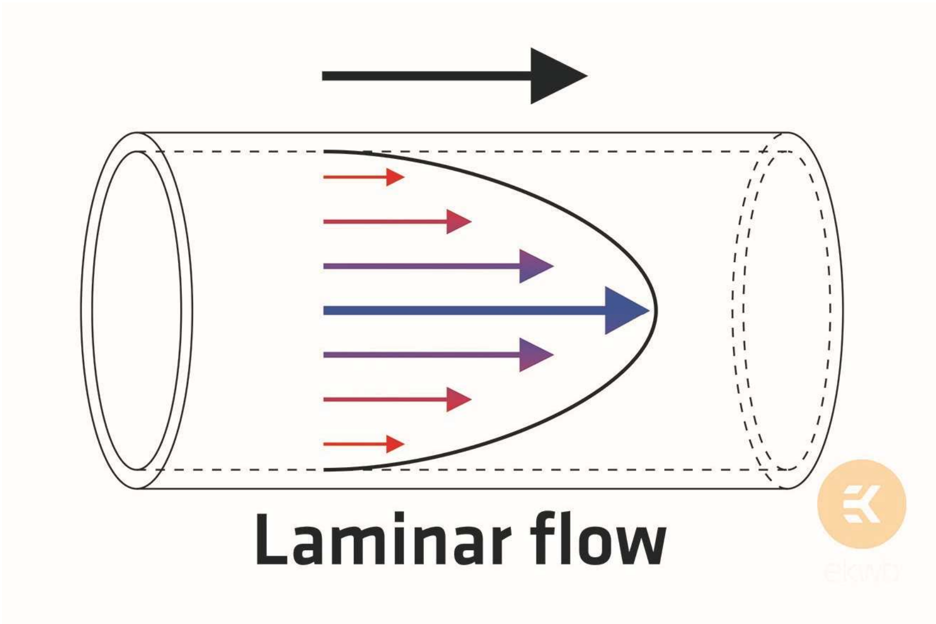
$$= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$



Laminar flow

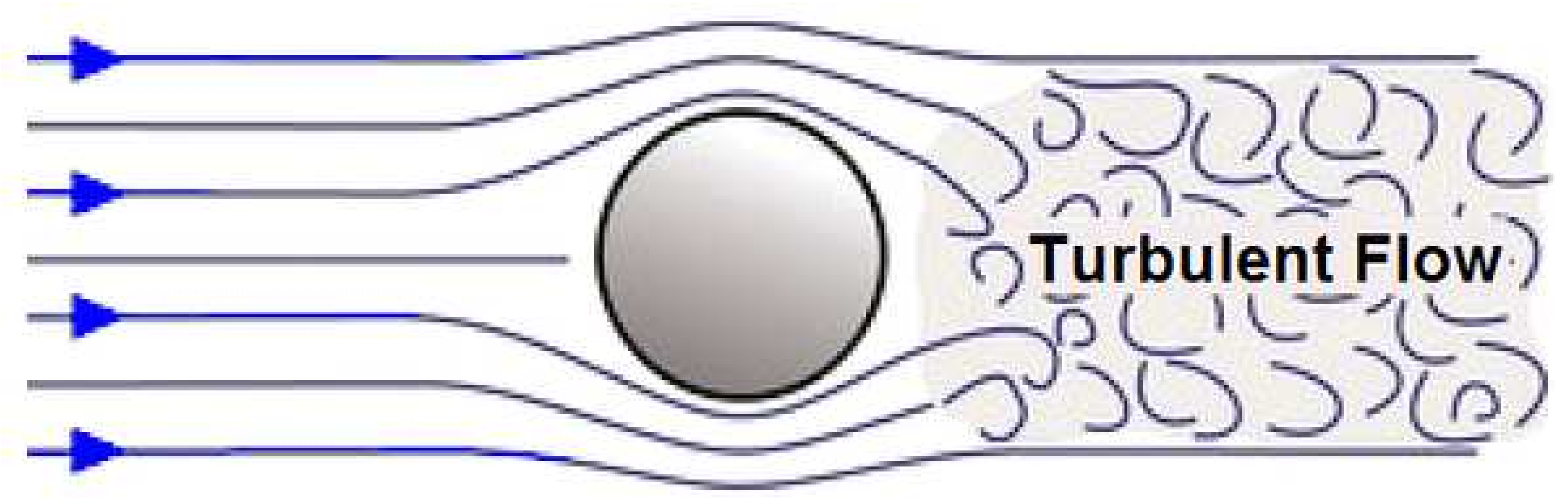
- **When a fluid flows in parallel layers**
  - No disruption between the layers





Turbulence flow

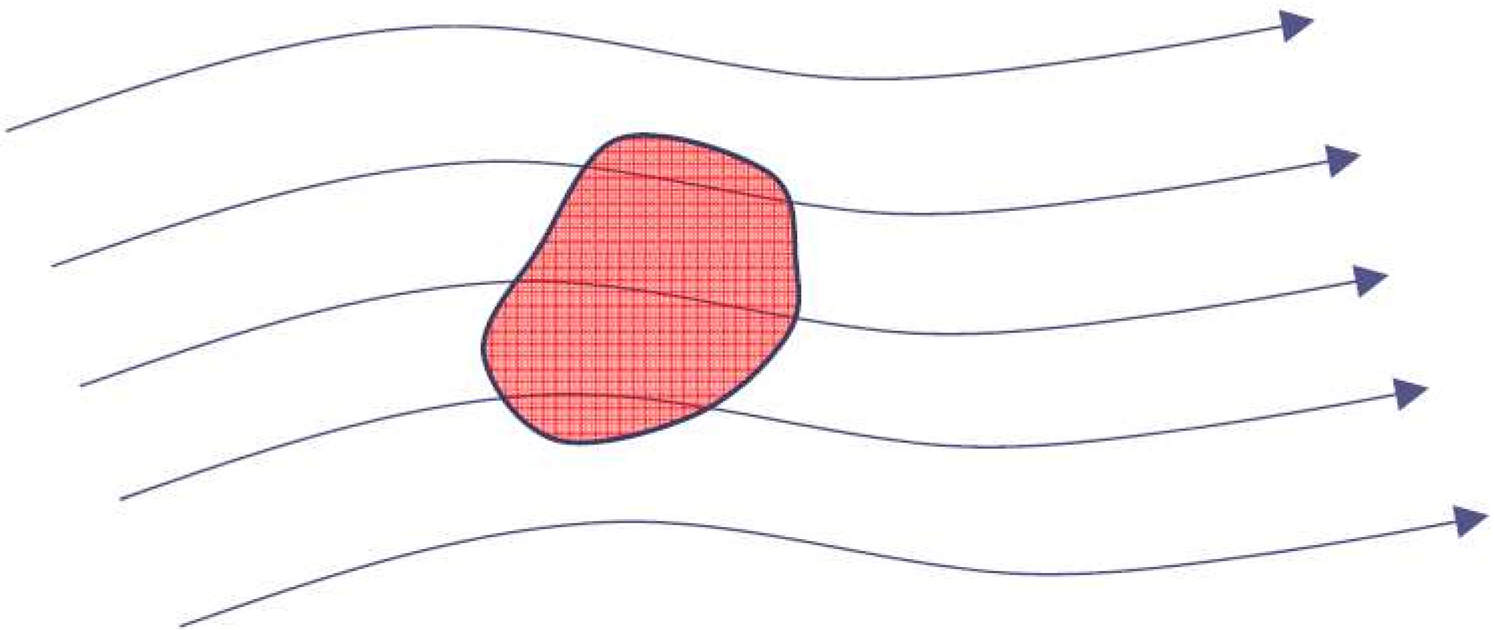
- **A flow in chaotic behavior**



1. Modeling for Fluids

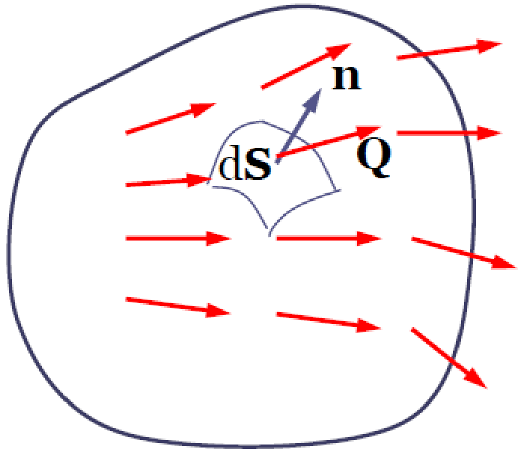
Control volume

- **An enclosed volume with arbitrary shape**

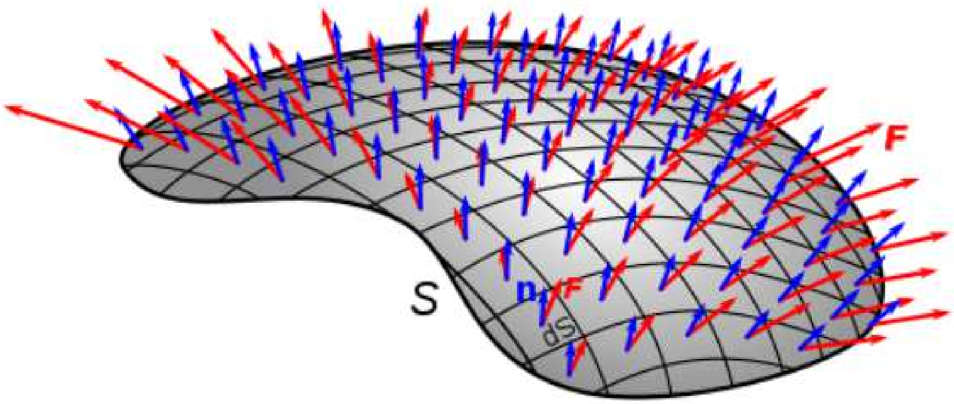


Flux

- **Any effect that passes through a surface or substance**



$$\varphi = \oint_S \mathbf{Q} \cdot d\mathbf{s}$$



Fluid dynamics

- **Conservation laws**
  - **Conservation of mass**
    - The rate of change of mass = mass flux rate
  - **Conservation of momentum**
    - The rate of change of momentum = internal + external force

Conservation of mass

- **Continuity equation**
  - Integral form

$$\frac{\partial}{\partial t}\oint_V \rho dV = -\oint_s \rho \mathbf{u} \cdot d\mathbf{s}$$

  - Differential form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Conservation of momentum

- **Momentum equation**
  - Apply Newton's second law
  - Integral form

$$\frac{\partial}{\partial t}\oint_V \rho \mathbf{u} dV + \oint_s (\rho \mathbf{u} \cdot d\mathbf{s})\mathbf{u} = -\oint_s p d\mathbf{s} + \oint_s \boldsymbol{\tau}_{shear} \cdot d\mathbf{s} + \oint_V \rho \mathbf{g} dV$$

  - Differential form

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla(\rho \mathbf{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau}_{shear} + \rho \mathbf{g}$$

Governing equations

- **Integral form**

$$\frac{\partial}{\partial t}\oint_V \rho dV = -\oint_s \rho \mathbf{u} \cdot d\mathbf{s}$$

$$\frac{\partial}{\partial t}\oint_V \rho \mathbf{u} dV + \oint_s (\rho \mathbf{u} \cdot d\mathbf{s})\mathbf{u} = -\oint_s p d\mathbf{s} + \oint_s \boldsymbol{\tau}_{shear} \cdot d\mathbf{s} + \oint_V \rho \mathbf{g} dV$$

Governing equations

- **Differential form**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla (\rho \mathbf{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau}_{shear} + \rho \mathbf{g}$$

Incompressible fluids

- **The volume and density do not change**
  - Differential form (constant density)

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \boldsymbol{\tau}_{shear} + \mathbf{g}$$

Newtonian fluids

- **Linear shear-stress relation**

$$\boldsymbol{\tau}_{shear} = -\nu' \rho (\nabla \cdot \mathbf{u})\mathbf{I} + 2\rho\nu\mathbf{S}$$

  - Strain rate (deformation rate)

$$\mathbf{S} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$

Incompressible Navier-Stokes equations

- Apply Newtonion fluid relation in incompressible governing equations

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{g}$$

Incompressible Navier-Stokes equations

- **Fluid equations in computer graphics**
  - Navier-Stokes equations in isothermal case

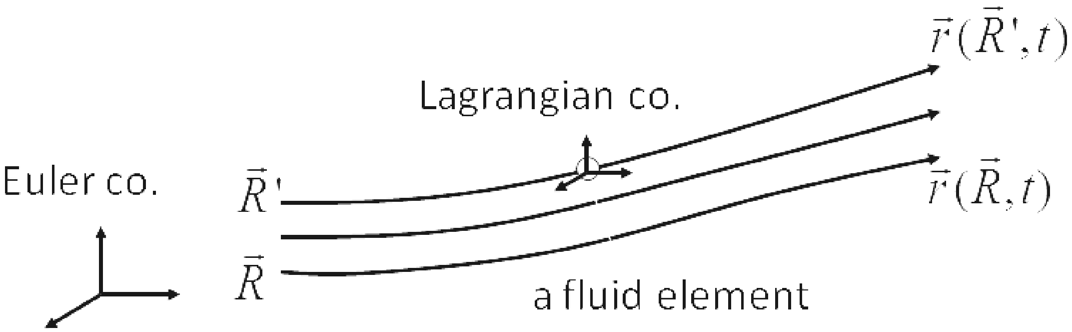
$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \frac{1}{\rho} \nabla p = \vec{g} + \nu \nabla \cdot \nabla \vec{u}$$

$$\nabla \cdot \vec{u} = 0$$

Incompressibility constraint

Governing equations

- **Lagrangian view**
  – Coordinate is moving with fluid



$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \frac{1}{\rho} \nabla p = \vec{g} + \nu \nabla \cdot \nabla \vec{u}$$

$$\downarrow$$

$$\frac{D\vec{u}}{Dt} + \frac{1}{\rho} \nabla p = \vec{g} + \nu \nabla \cdot \nabla \vec{u}$$

Governing equations

- **Pressure equation**
  - Taking divergence of the momentum equation

$$\nabla \cdot \frac{\partial \vec{u}}{\partial t} + \nabla \cdot (\vec{u} \cdot \nabla \vec{u}) + \nabla \cdot \frac{1}{\rho} \nabla p = \nabla \cdot (\vec{g} + \nu \nabla \cdot \nabla \vec{u})$$

  - Remove zero terms

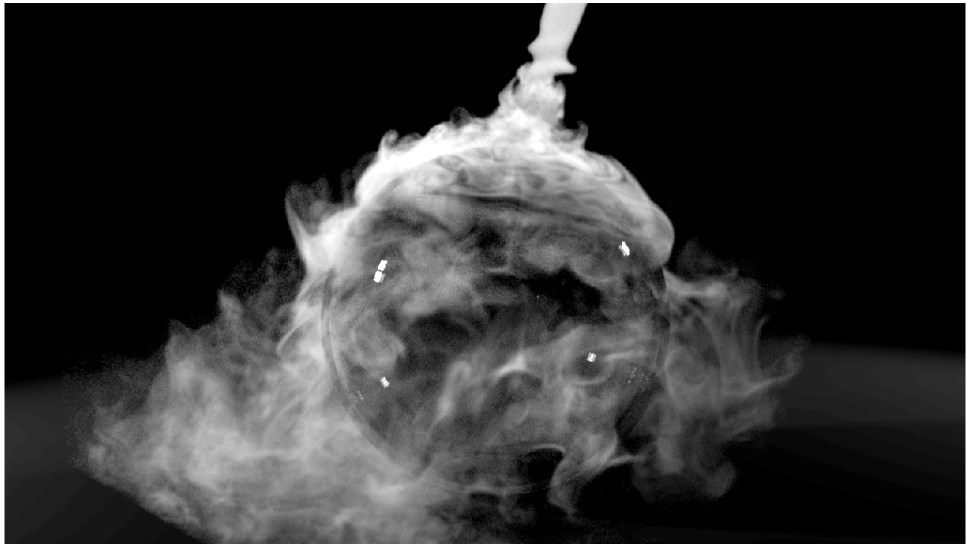
$$\nabla \cdot \vec{u} = 0 \quad \Longrightarrow \quad \nabla \cdot \frac{1}{\rho} \nabla p = \nabla \cdot (-\vec{u} \cdot \nabla \vec{u} + \vec{g} + \nu \nabla \cdot \nabla \vec{u})$$

Governing equations

- **Ideal inviscid flows**
  - No viscosity or very small viscosity
  - Drop the viscosity term

$$\frac{D\vec{u}}{Dt} + \frac{1}{\rho}\nabla p = \vec{g}$$

$$\nabla \cdot \vec{u} = 0$$



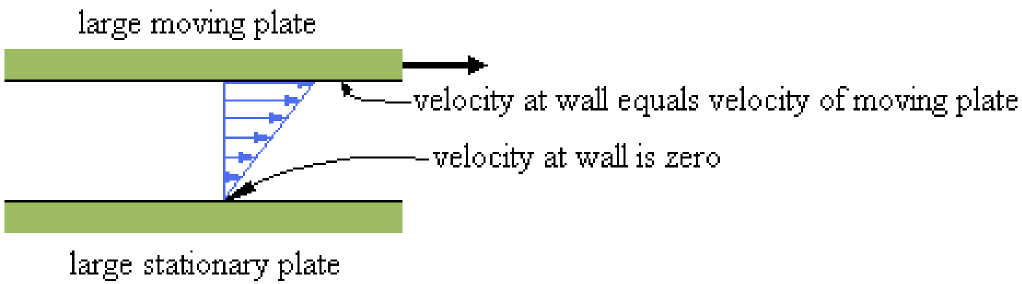
Boundary conditions

- **Solid wall boundary**
  - The flow cannot penetrate
    - Slip boundary

$$\vec{u} \cdot \hat{n} = \vec{u}_{\text{solid}} \cdot \hat{n}$$

    - No-slip (Neumann) boundary

$$\vec{u} = \vec{u}_{\text{solid}}$$



2. Unconditionally stable semi-Lagrangian method

Finite difference method

- **Derivative approximation**
  - First derivative
    - Taylor expansion

$$u(x+\Delta x) = u(x) + \Delta x \frac{\partial u(x)}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u(x)}{\partial x^2} + \frac{(\Delta x)^3}{3!} \frac{\partial^3 u(x)}{\partial x^3} + \cdots$$

    - First order approximation

$$\frac{u(x+\Delta x) - u(x)}{\Delta x} = \frac{\partial u(x)}{\partial x} + \frac{\Delta x}{2} \frac{\partial^2 u(x)}{\partial x^2} + \cdots = \frac{\partial u(x)}{\partial x} + \mathrm{O}(\Delta x)$$

Finite difference method

- **Derivative approximation**
  - First derivative
    - Second order approximation

$$u_{i+1} = u_i + \Delta x\left(\frac{\partial u}{\partial x}\right)_i + \frac{\Delta x^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{\Delta x^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \frac{\Delta x^4}{4!}\left(\frac{\partial^4 u}{\partial x^4}\right)_i + \cdots$$

$$u_{i-1} = u_i - \Delta x\left(\frac{\partial u}{\partial x}\right)_i + \frac{\Delta x^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{\Delta x^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \frac{\Delta x^4}{4!}\left(\frac{\partial^4 u}{\partial x^4}\right)_i + \cdots$$

$$\Downarrow$$

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + \mathrm{O}(\Delta x^2)$$

Splitting

- **How to solve the fluid equations?**
  - Split the complex equations into simpler ones

- **Splitting in general PDE**

$$\frac{dq}{dt} = f(q) + g(q)$$

$$\Downarrow$$

$$\begin{aligned} \tilde{q} &= q^n + \Delta t f(q^n) \\ q^{n+1} &= \tilde{q} + \Delta t g(\tilde{q}) \end{aligned}$$

Splitting

- **Splitting in general PDE**
  - Is splitting with the same order as original? Yes!

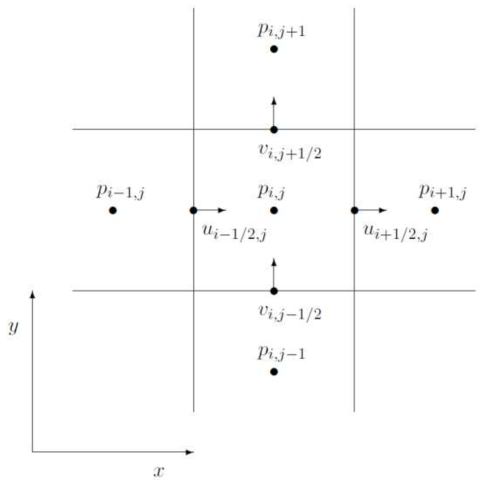
$$\begin{aligned}
q^{n+1} &= (q^n + \Delta t f(q^n)) + \Delta t g(q^n + \Delta t f(q^n)) \\
&= q^n + \Delta t f(q^n) + \Delta t \left(g(q^n) + O(\Delta t)\right) \\
&= q^n + \Delta t(f(q^n) + g(q^n)) + O(\Delta t^2) \\
&= q^n + \frac{dq}{dt}\Delta t + O(\Delta t^2)
\end{aligned}$$

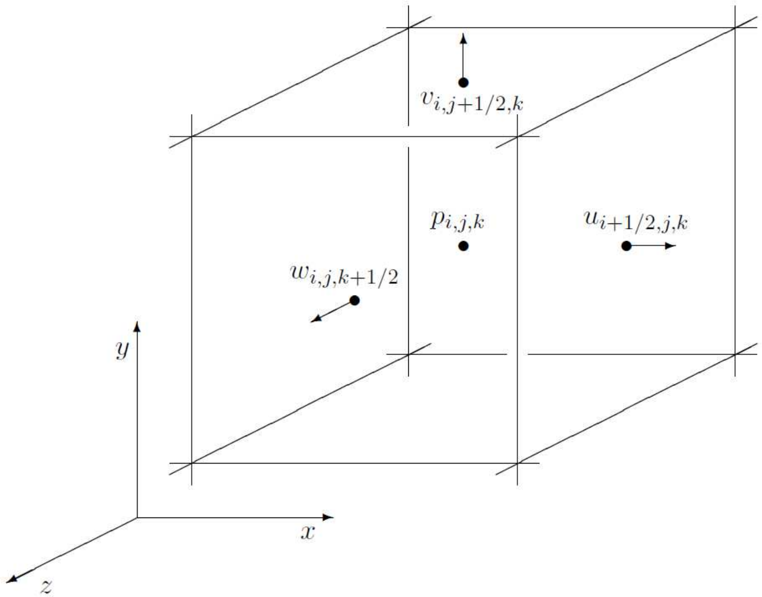
- **Splitting for Navier-Stokes equations**

$$\mathbf{w}_0(\mathbf{x}) \overbrace{\longrightarrow}^{\text{add force}} \mathbf{w}_1(\mathbf{x}) \overbrace{\longrightarrow}^{\text{advect}} \mathbf{w}_2(\mathbf{x}) \overbrace{\longrightarrow}^{\text{diffuse}} \mathbf{w}_3(\mathbf{x}) \overbrace{\longrightarrow}^{\text{project}} \mathbf{w}_4(\mathbf{x})$$

MAC grid

- **Staggered grid**
  - Staggered arrangement of velocity and pressure
  - Face center: velocity samples
  - Cell center: pressure samples
  - Avoid high frequency artifacts





Advection

- **What is an advection?**
  - Transport of a substance

$$Dq/Dt = 0$$

  - Numerical solver
    - The simple discretization: explicit Euler

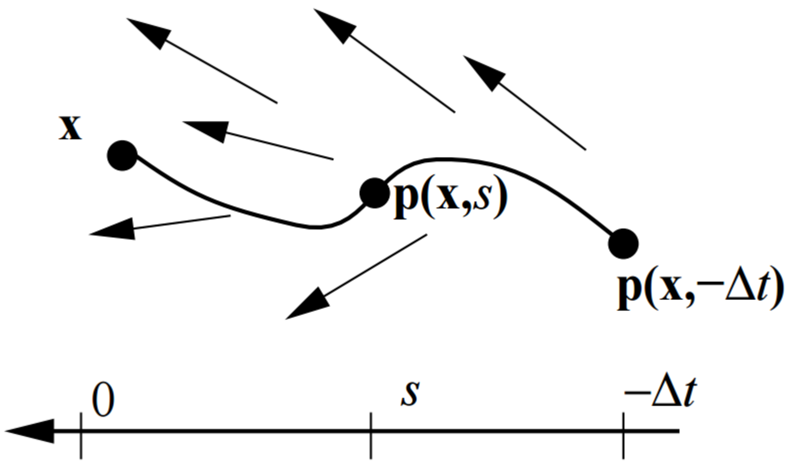
$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = 0 \quad \Rightarrow \quad \frac{q_i^{n+1} - q_i^n}{\Delta t} + u_i^n \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x} = 0$$

Unconditionally unstable! $\longrightarrow$ $$q_i^{n+1} = q_i^n - \Delta t u_i^n \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x}$$

Semi-Lagrangian advection

- **How to make stable solution?**
  - Implicit formulation?
    - Difficult to solve; hard for non-linear advection
- **Unconditionally stable yet simple solution?**
  - Semi-Lagrangian scheme
    - Jos Stam, Stable Fluids, SIGGRAPH 1999

$$\frac{d\vec{x}}{dt} = \vec{u}$$



Unconditionally stable simulation

- **Helmholtz-Hodge Decomposition**
  - Any vector field can be uniquely decomposed into
    - Divergence free field
    - Divergent field

$$\mathbf{w} = \mathbf{u} + \nabla q \qquad \nabla \cdot \mathbf{u} = 0$$

- **How can this be applied to fluid equations?**
  - Define an operator **P**: project any vector field **w** onto its divergence free part **u**

$$\nabla \cdot \mathbf{w} = \nabla^2 q \quad \Longrightarrow \quad \mathbf{u} = \mathbf{P}\mathbf{w} = \mathbf{w} - \nabla q$$

Unconditionally stable simulation

- **Apply to Navier-Stokes equation**
  - Apply to both sides of momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

$$\mathbf{P}\mathbf{u} = \mathbf{u} \quad \mathbf{P}\nabla p = 0 \qquad \Big\downarrow \ \textbf{P}$$

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{u} + \mathbf{f}\right)$$

$$\Big\downarrow$$

$$\mathbf{w}_0(\mathbf{x}) \overbrace{\longrightarrow}^{\text{add force}} \mathbf{w}_1(\mathbf{x}) \overbrace{\longrightarrow}^{\text{advect}} \mathbf{w}_2(\mathbf{x}) \overbrace{\longrightarrow}^{\text{diffuse}} \mathbf{w}_3(\mathbf{x}) \overbrace{\longrightarrow}^{\text{project}} \mathbf{w}_4(\mathbf{x})$$

Unconditionally stable simulation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{u} + \mathbf{f}\right)$$

- **Add force** $\mathbf{w}_1(\mathbf{x}) = \mathbf{w}_0(\mathbf{x}) + \Delta t\ \mathbf{f}(\mathbf{x}, t)$
- **Advect** $\mathbf{w}_2(\mathbf{x}) = \mathbf{w}_1(\mathbf{p}(\mathbf{x}, -\Delta t))$
- **Diffuse** $\left(\mathbf{I} - \nu\Delta t\nabla^2\right)\mathbf{w}_3(\mathbf{x}) = \mathbf{w}_2(\mathbf{x})$
- **Project** $\nabla^2 q = \nabla\cdot\mathbf{w}_3$ $\qquad \mathbf{w}_4 = \mathbf{w}_3 - \nabla q$

Vorticity confinement

- **Compensate for numerical diffusion**
  - Add a "vorticity confinement" force
  - Disturb the flow from current vorticity
  - Definition of vorticity
  - **Find vortices** $\vec{\omega} = \nabla \times \vec{u}$
    - The local axes of vorticity $\vec{N} = \dfrac{\nabla|\vec{\omega}|}{||\nabla|\vec{\omega}|||}$
  - **Construct disturbing force**

$$f_{\text{conf}} = \epsilon \Delta x (\vec{N} \times \vec{\omega})$$

Vorticity confinement

- **Implementation**
  - Computation of vorticity

$$\vec{\omega}_{i,j,k} = \left(\frac{w_{i,j+1,k} - w_{i,j-1,k}}{2\Delta x} - \frac{v_{i,j,k+1} - v_{i,j,k-1}}{2\Delta x}, \right.$$
$$\frac{u_{i,j,k+1} - u_{i,j,k-1}}{2\Delta x} - \frac{w_{i+1,j,k} - w_{i-1,j,k}}{2\Delta x},$$
$$\left. \frac{v_{i+1,j,k} - v_{i-1,j,k}}{2\Delta x} - \frac{u_{i,j+1,k} - u_{i,j-1,k}}{2\Delta x} \right)$$

  - Computation of vorticity gradient

$$\nabla|\vec{\omega}|_{i,j,k} = \left(\frac{|\vec{\omega}|_{i+1,j,k} - |\vec{\omega}|_{i-1,j,k}}{2\Delta x}, \quad \frac{|\vec{\omega}|_{i,j+1,k} - |\vec{\omega}|_{i,j-1,k}}{2\Delta x}, \quad \frac{|\vec{\omega}|_{i,j,k+1} - |\vec{\omega}|_{i,j,k-1}}{2\Delta x} \right)$$

  - Normalize this to get N

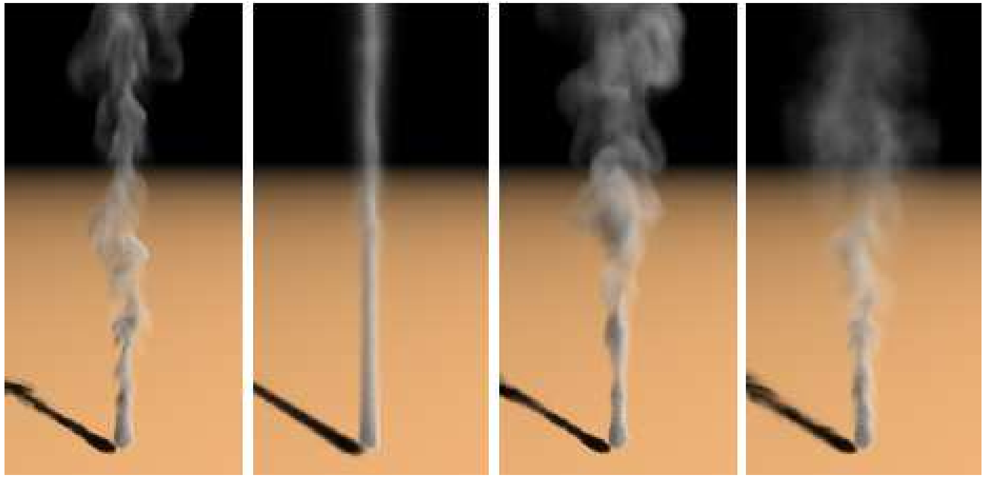
$$\vec{N}_{i,j,k} = \frac{\nabla|\vec{\omega}|_{i,j,k}}{||\nabla|\vec{\omega}|_{i,j,k}|| + 10^{-20}}$$

Smoke simulation

- **Evolve additional quantities**
  - Temperature *T* and smoke concentration *s*

$$\frac{DT}{Dt} = 0 \qquad \frac{Ds}{Dt} = 0$$

  - Buoyancy force $f_{\text{buoy}} = (0, -\alpha s + \beta(T - T_{\text{amb}}), 0)$



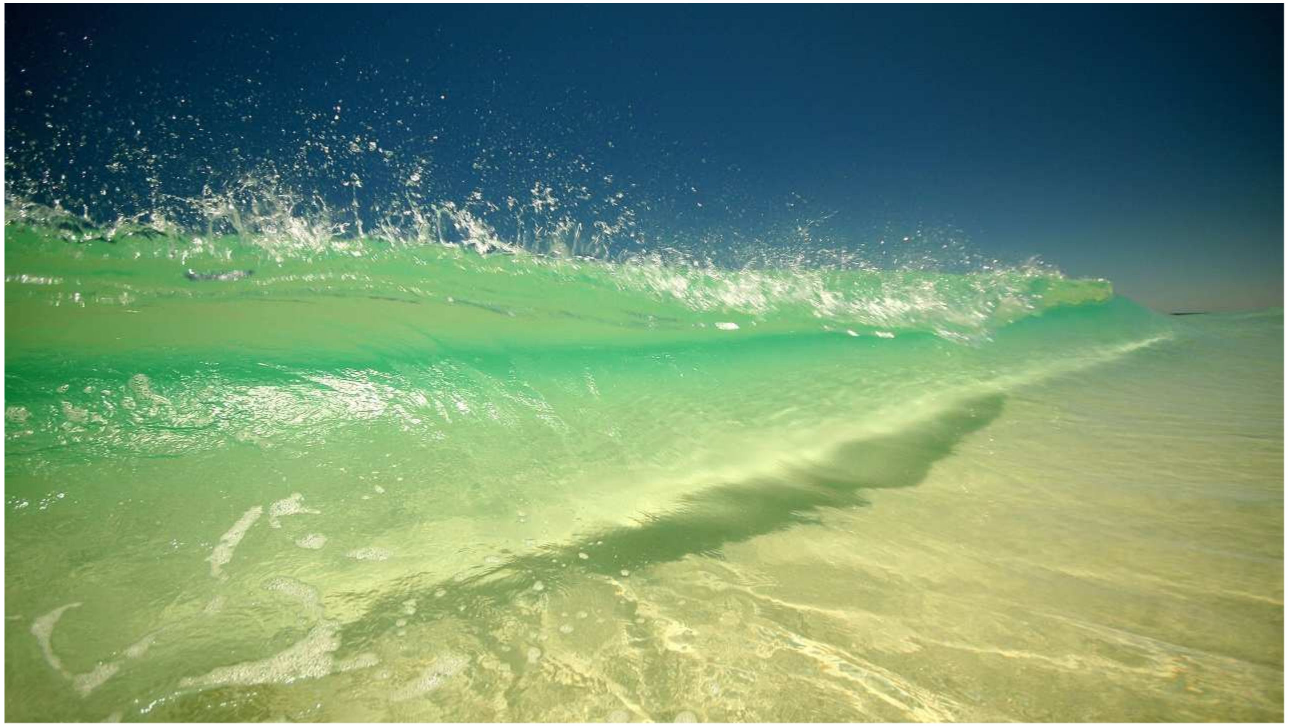
Fedkiw et al., Visual Simulation of Smoke, SIGGRAPH 2001


An Advection-Reflection Solver
for Detail-Preserving Fluid Simulation
Jonas Zehnder
Université de Montréal
Rahul Narain
Indian Institute of Technology Delhi
University of Minnesota
Bernhard Thomaszewski
Université de Montréal

4. Liquid Simulation

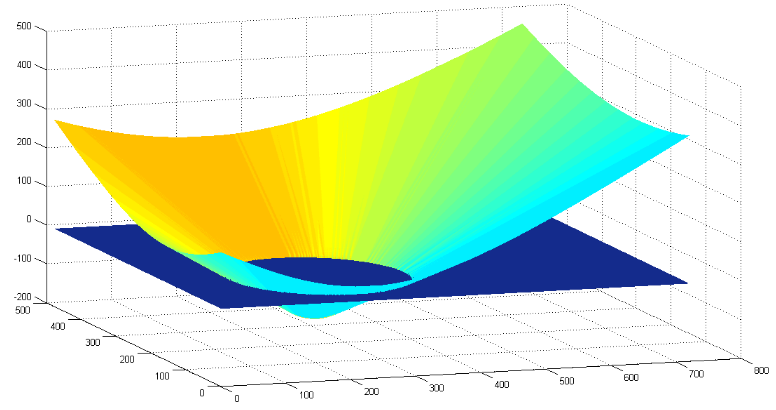
Liquid simulation

- **Characteristics of liquids**
  - Interface dynamics
  - Tracking the motion of interface



Liquid simulation

- **Implicit surface & level set**
  - Implicit surface
    - A surface in Euclidean space defined by an equation
      $$F(x, y, z) = 0.$$
    - Why? Easy for complex surfaces
  - Level set surface



Liquid simulation

- **Interface representation**
  - Implicit surface
  - liquid volume as one side of an isocontour of an implicit function

$$\phi \leq 0$$

  - Interface
    - Isosurface (isocontuour)

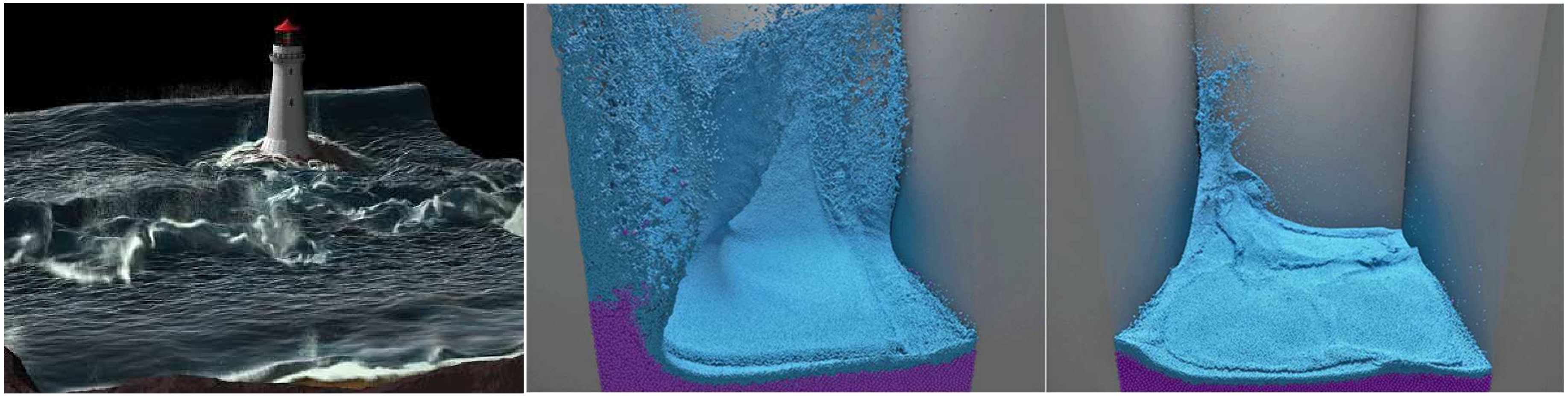
$$\phi = 0$$

  - Interface advection
    - Level set equation

$$\phi_t + \vec{u} \cdot \nabla \phi = 0$$

Problem with mesh-based method

- **How to simulate complex free surface?**
  - Complex free boundary condition
  - Water splashes

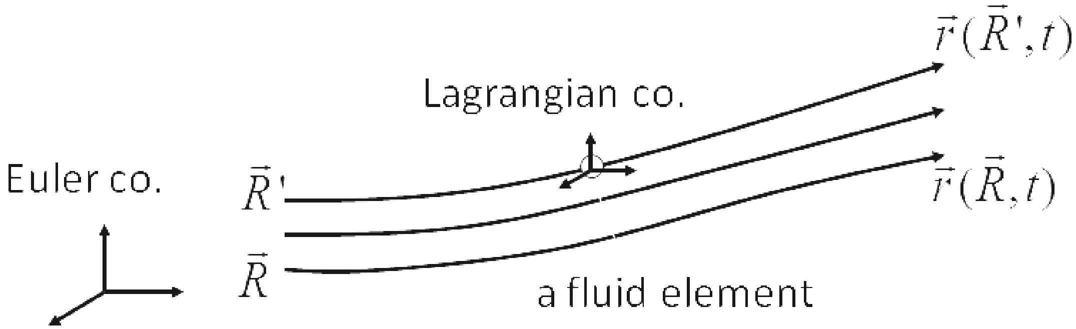


  - Particle methods are preferred

5. Smoothed Particle Hydrodynamics

Lagrangian solver for fluids

- **Coordinate is moving with fluid**



$$\boxed{\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}} + \frac{1}{\rho}\nabla p = \vec{g} + \nu \nabla \cdot \nabla \vec{u}$$

$$\boxed{\frac{D\vec{u}}{Dt}} + \frac{1}{\rho}\nabla p = \vec{g} + \nu \nabla \cdot \nabla \vec{u}$$

Smoothed-particle hydrodynamics

- **Function approximation with SPH**
  - Problem setting
    - Reconstructing an (unknown) function f from a set of irregular samples $f_i = f(x_i)$
    - Using the Dirac-delta function, we can rewrite f(x) as a convolution

$$f(\mathbf{x}) = \int_{\mathbf{x}'} f(\mathbf{x}')\delta(\|\mathbf{x} - \mathbf{x}'\|)\,\mathrm{d}V$$

    - Replace delta function with a kernel function w_h

$$\tilde{f}(\mathbf{x}) = \int_{\mathbf{x}'} f(\mathbf{x}')\omega_h(\|\mathbf{x} - \mathbf{x}'\|)\,\mathrm{d}V \qquad \int \omega_h = 1$$

Smoothed-particle hydrodynamics

- **Function approximation with SPH**
  - Discretize the integral into a sum over all sample points to obtain the SPH approximation

$$\tilde{f}(\mathbf{x}) = \int_{\mathbf{x}'} f(\mathbf{x}')\omega_h(\|\mathbf{x}-\mathbf{x}'\|)\,\mathrm{d}V \quad \Rightarrow \quad \langle f\rangle(\mathbf{x}) = \sum_i f_i \omega_h(\|\mathbf{x}_i - \mathbf{x}\|)V_i$$

  - How to compute volume V_i for each sample?
    - Associate with mass m_i

$$V_i = \frac{m_i}{\rho_i}$$

Smoothed-particle hydrodynamics

- **Function approximation with SPH**
  - How to compute density estimation?

$$\rho_i = \langle \rho \rangle (\mathbf{x}_i) = \sum_j \omega_h(\|\mathbf{x}_i - \mathbf{x}_j\|)\, \rho_j V_j \quad + \quad V_i = \frac{m_i}{\rho_i}$$

$$\Downarrow$$

$$\begin{aligned}
\rho_i = \langle \rho \rangle (\mathbf{x}_i) &= \sum_j \omega_h(\|\mathbf{x}_i - \mathbf{x}_j\|)\, \rho_j V_j \\
&= \sum_j \omega_h(\|\mathbf{x}_i - \mathbf{x}_j\|)\, \rho_j \frac{m_j}{\rho_j} \\
&= \sum_j \omega_h(\|\mathbf{x}_i - \mathbf{x}_j\|)\, m_j
\end{aligned}$$

Smoothed-particle hydrodynamics

- **Kernel functions**
  - Admissible kernel functions: they must be normalized

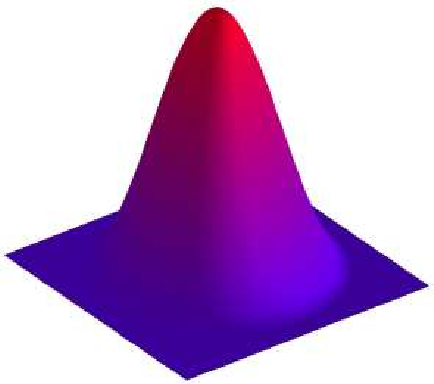
$$\int_{\mathbf{x}} \omega_h(||\mathbf{x}||)\mathrm{d}V = 1$$

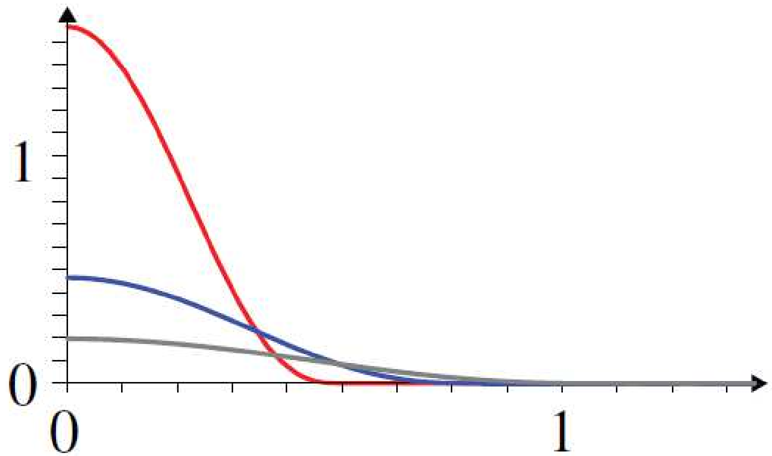
  - Smoothing parameter h
    - Allowing control over how far the influence of each sample point reaches (local support)
    - Too large values of h produce unnecessarily smooth reconstructions
    - Kernel function converges to a Dirac-delta function as h goes to zero

Smoothed-particle hydrodynamics

- **Kernel functions**
  - A good polynomial kernel function

$$\omega_h(d) = \begin{cases} \frac{315}{64\pi h^3}\left(1 - \frac{d^2}{h^2}\right)^3 & d < h, \\ 0 & \text{otherwise} \end{cases}$$





Smoothed-particle hydrodynamics

- **Approximation of differential operators**
  - Apply SPH approximations to the solution of partial differential equations
    - Not only a reconstruction of the continuous function f , but also the derivatives of the function
  - Sample values f_i are constants, we can write approximation of gradient as

$$\langle \nabla f \rangle (\mathbf{x}) = \sum_i f_i \nabla \omega_h(\|\mathbf{x} - \mathbf{x}_i\|) V_i$$

$$\nabla \omega_h(\|\mathbf{x} - \mathbf{x}_i\|) = \frac{\mathbf{x} - \mathbf{x}_i}{\|\mathbf{x} - \mathbf{x}_i\|} \, \omega_h'(\|\mathbf{x} - \mathbf{x}_i\|)$$

Smoothed-particle hydrodynamics

- **Approximation of differential operators**
  - Other linear operators can be treated similarly

$$\langle \Delta f \rangle (\mathbf{x}) = \sum_i f_i \Delta \omega_h(\|\mathbf{x} - \mathbf{x}_i\|) V_i$$

$$\langle \nabla \cdot \mathbf{f} \rangle (\mathbf{x}) = \sum_i \mathbf{f}_i \cdot \nabla \omega_h(\|\mathbf{x} - \mathbf{x}_i\|) V_i$$

  - Accuracy of the approximations of derivative
    - Strongly depends on the distribution of sample points within the support region
    - For highly irregular sample distributions, the differential properties can be very noisy

Smoothed-particle hydrodynamics

- **Approximation of differential operators**
  - Problem with previous estimation
    - Gradient approximation can yield non-zero values even if the function is constant
  - How to rectify?
    - Enforce a zero gradient for constant functions by subtracting the constant f_i

$$\begin{aligned}\nabla f(\mathbf{x}_i) &\approx \langle \nabla [f - f_i] \rangle (\mathbf{x}_i) \\ &= \sum_j (f_j - f_i) \nabla \omega_h (\|\mathbf{x}_i - \mathbf{x}_j\|) V_j\end{aligned}$$

Smoothed-particle hydrodynamics

- **Approximation of differential operators**
  - Same reasoning applied to the divergence and Laplace operators

$$\langle \nabla \cdot \mathbf{f} \rangle (\mathbf{x}_i) = \sum_j \left(\mathbf{f}_j - \mathbf{f}_i\right) \cdot \nabla \omega_h(\|\mathbf{x}_i - \mathbf{x}_j\|)\, V_j$$

$$\langle \Delta f \rangle (\mathbf{x}_i) = \sum_j \left(f_j - f_i\right) \Delta \omega_h(\|\mathbf{x}_i - \mathbf{x}_j\|)\, V_j$$

Smoothed-particle hydrodynamics

- **Approximation of differential operators**
  - Another variation of the gradient approximation important in particular for fluid simulation

$$
\nabla\left[\frac{f}{\rho}\right] = \frac{\rho\nabla f - f\nabla\rho}{\rho^2}
$$

$$
\Rightarrow \quad \nabla f = \rho\left(\nabla\left[\frac{f}{\rho}\right] + \frac{f\nabla\rho}{\rho^2}\right)
$$

$$
\Downarrow
$$

$$
\begin{aligned}
\nabla f(\mathbf{x}_i) &\approx \rho_i\left(\left\langle\nabla\left[\frac{f}{\rho}\right]\right\rangle(\mathbf{x}_i) + \frac{f_i\,\langle\nabla\rho\rangle(\mathbf{x}_i)}{\rho_i^2}\right) \\
&= \rho_i\sum_j m_j\left(\frac{f_j}{\rho_j^2} + \frac{f_i}{\rho_i^2}\right)\nabla\omega_h(\|x_i - x_j\|)
\end{aligned}
$$

Fluid simulation using SPH

- **For compressible fluids**
  - The momentum equation can be written as a combination of pressure, viscosity, and external forces

$$\frac{D\mathbf{v}}{Dt} = \frac{1}{\rho}\left(\mathbf{f}_{\mathrm{p}} + \mathbf{f}_{\mathrm{v}} + \mathbf{f}_{\mathrm{e}}\right)$$

  - $\mathbf{f}_e$ are external forces acting on the fluid
  - The pressure force $\mathbf{f}_p$ is a function of the pressure field p
  - Viscous forces smooth the velocity field

$$\mathbf{f}_{\mathrm{p}} = -\nabla p$$

$$\mathbf{f}_{\mathrm{v}} = \mu \nabla \cdot \nabla \mathbf{v} = \mu \Delta \mathbf{v}$$

Fluid Simulation using SPH

- **For compressible fluids**
  - How to compute pressure
    - Equation of state
    - Common choice: Tait equation

$$p = K\left(\left(\frac{\rho}{\rho_0}\right)^{\gamma} - 1\right)$$

  - The most straightforward spatial discretization
    - Consider Lagrangian coordinates

$$\frac{\partial \mathbf{v}_i}{\partial t} = -\frac{\langle \nabla p \rangle (\mathbf{x}_i)}{\rho_i} + \mu \langle \Delta \mathbf{v} \rangle (\mathbf{x}_i) + \frac{\mathbf{f}_{\mathrm{e}}(\mathbf{x}_i)}{\rho_i}$$

Fluid Simulation using SPH

- **Pressure forces**
  - Particle accelerations due to pressure forces

$$\frac{\mathbf{f}_p(\mathbf{x}_i)}{\rho(\mathbf{x}_i)} = -\frac{\langle \nabla p \rangle (\mathbf{x}_i)}{\rho_i}$$

  - Implementation

$$\frac{\mathbf{f}_p(\mathbf{x}_i)}{\rho(\mathbf{x}_i)} = -\sum_j \nabla \omega_h^{ij} p_j \frac{m_j}{\rho_j \rho_i} = -\sum_j \mathbf{a}_p^{ji}$$

Fluid Simulation using SPH

- **Pressure forces**
  - Problem
    - Observation

$$\mathbf{a}_p^{ji} + \mathbf{a}_p^{ij} \neq 0$$

    - Linear and angular momentum may not be conserved
    - Low resolution problem is more significant (visual artifacts)
  - Symmetrize

$$\nabla f(\mathbf{x}_i) \approx \rho_i \left(\left\langle \nabla \left[\frac{f}{\rho} \right] \right\rangle (\mathbf{x}_i) + \frac{f_i \langle \nabla \rho \rangle (\mathbf{x}_i)}{\rho_i^2} \right) = \rho_i \sum_j m_j \left(\frac{f_j}{\rho_j^2} + \frac{f_i}{\rho_i^2} \right) \nabla \omega_h(\|x_i - x_j\|)$$

$$\Rightarrow \quad \frac{\mathbf{f}_p(\mathbf{x}_i)}{\rho(\mathbf{x}_i)} = -\sum_j \nabla \omega_h^{ij} \left(\frac{p_j}{\rho_j^2} + \frac{p_i}{\rho_i^2} \right) m_j$$

Fluid Simulation using SPH

- **Viscosity forces**
  - Discretizing the acceleration on particles due to viscosity leads to the expression

$$\frac{\mathbf{f}_v(\mathbf{x}_i)}{\rho_i} = \mu \sum_j \Delta\omega_h^{ij} (\mathbf{v}_j - \mathbf{v}_i) \frac{m_j}{\rho_i \rho_j} = \mu \sum_j \mathbf{a}_v^{ji}$$

  - The force is symmetric:

$$\mathbf{a}_v^{ji} + \mathbf{a}_v^{ij} = 0$$

Fluid Simulation using SPH

- **Viscosity forces**
  - Problem
    - Higher-order derivatives depends strongly on the distribution of sample points
    - Quite noisy especially for small support radius of the kernel function
  - Smoothing operator
    - Create artificial viscosity

$$\begin{aligned}\tilde{\mathbf{v}}_i &= \xi \langle \mathbf{v} \rangle (\mathbf{x}_i) + (1-\xi)\mathbf{v}_i \\ &= \xi \left[\sum_j \omega_h^{ij} \mathbf{v}_j \frac{m_j}{\rho_j} \right] + (1-\xi)\mathbf{v}_i \end{aligned} \qquad 0 \le \xi \le 1$$

Fluid Simulation using SPH

- **Time discretization and simulation loop**
  - Using a simple explicit integration scheme

$$
\begin{aligned}
\mathbf{v}_i(t+\Delta t) &= \mathbf{v}_i(t) + \Delta t \frac{D\mathbf{v}_i}{Dt} \\
\mathbf{x}_i(t+\Delta t) &= \mathbf{x}_i(t) + \Delta t \mathbf{v}_i(t+\Delta t)
\end{aligned}
$$

  - Higher order time-integrator can be applied

2. Incompressible SPH

Incompressible SPH

- **Prediction-correction scheme**
  - PCISPH, Solenthaler et al. SIGGRAPH 2009
  - The density at a point in time t + 1

$$
\begin{aligned}
\rho_i(t+1) &= m\sum_j W(\mathbf{x}_i(t+1) - \mathbf{x}_j(t+1)) \\
&= m\sum_j W(\mathbf{x}_i(t) + \Delta\mathbf{x}_i(t) - \mathbf{x}_j(t) - \Delta\mathbf{x}_j(t)) \\
&= m\sum_j W(\mathbf{d}_{ij}(t) + \Delta\mathbf{d}_{ij}(t))
\end{aligned}
$$

where $\mathbf{d}_{ij}(t) = \mathbf{x}_i(t) - \mathbf{x}_j(t)$, and $\Delta\mathbf{d}_{ij}(t) = \Delta\mathbf{x}_i(t) - \Delta\mathbf{x}_j(t)$

Incompressible SPH

- **Prediction-correction scheme**
  - Assuming that Δd_{ij} is relatively small, the first order Taylor approximation can be applied

$$
\begin{aligned}
\rho_i(t+1) &= m \sum_j W(\mathbf{d}_{ij}(t)) + \nabla W(\mathbf{d}_{ij}(t)) \cdot \Delta \mathbf{d}_{ij}(t) \\
&= m \sum_j W(\mathbf{x}_i(t) - \mathbf{x}_j(t)) + \\
&\quad m \sum_j \nabla W(\mathbf{x}_i(t) - \mathbf{x}_j(t)) \cdot (\Delta \mathbf{x}_i(t) - \Delta \mathbf{x}_j(t)) \\
&= \rho_i(t) + \Delta \rho_i(t).
\end{aligned}
$$

    - The term $\Delta\rho_i(t)$ is unknown, a function of p which we are looking for

Incompressible SPH

- **Prediction-correction scheme**
  - Density error predictor

$$\rho^*_{err_i} = \rho^*_i - \rho_0$$

  - Pressure corrector

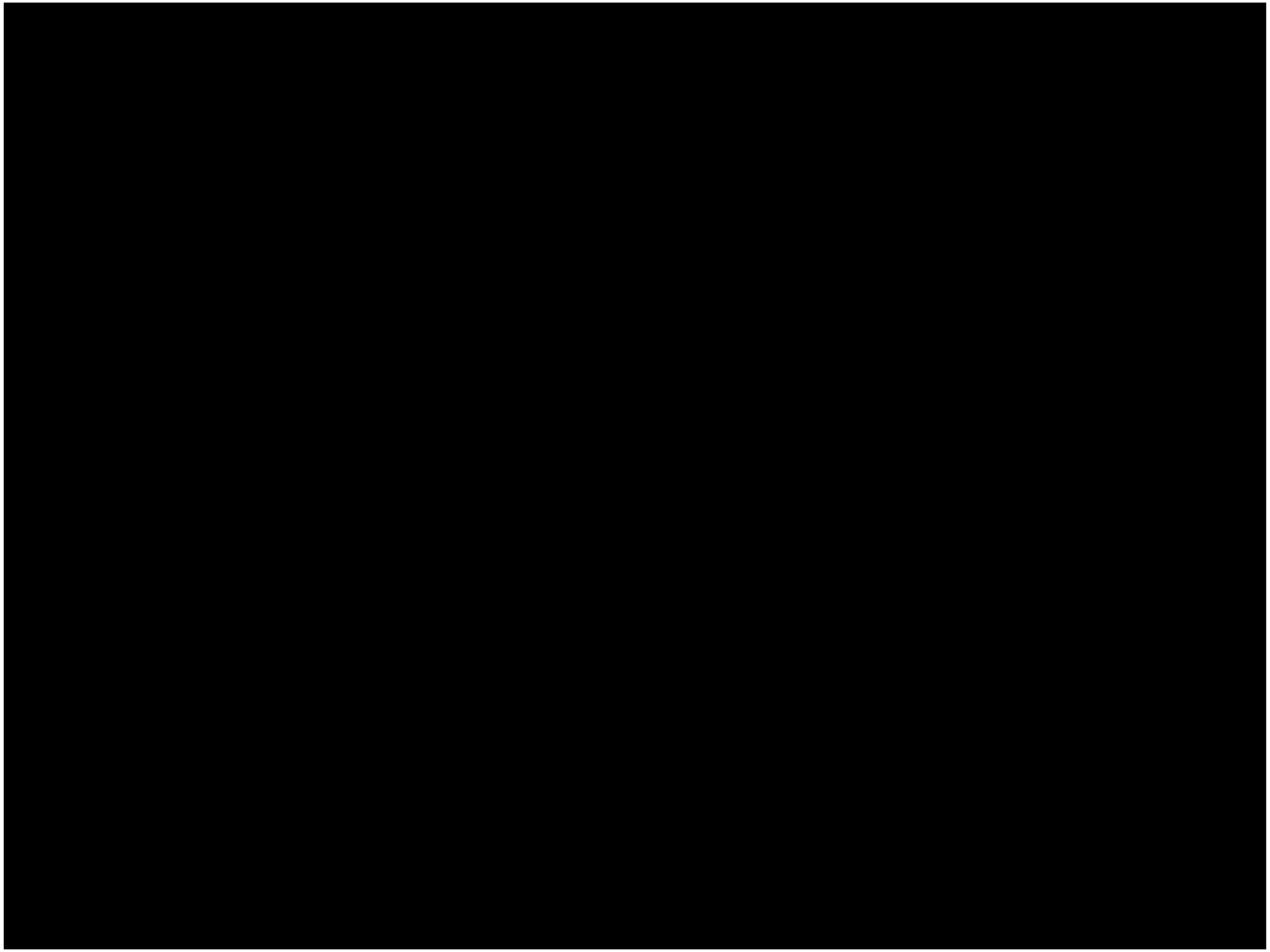
$$\delta = \frac{-1}{\beta(-\sum_j \nabla W_{ij} \cdot \sum_j \nabla W_{ij} - \sum_j (\nabla W_{ij} \cdot \nabla W_{ij}))}$$

$$\tilde{p}_i = \delta \rho^*_{err_i}$$

$$p_i \mathrel{+}= \tilde{p}_i$$

Incompressible SPH

Particle solver coupled with rigid bodies



End of the lecture