

Computer Graphics I

Lecture 11: Numerical integration

Xiaopei LIU

School of Information Science and Technology
ShanghaiTech University

Rendering equation

- **The fundamental rendering equation**

- Reflection equation

- Describe how an incident distribution of light at a point is transformed into an outgoing distribution

$$L_o(p, \omega_o) = \int_{\mathcal{S}^2} f(p, \omega_o, \omega_i) L_i(p, \omega_i) |\cos \theta_i| d\omega_i$$

- Scattering equation

- More complex integral equation

$$L_o(p_o, \omega_o) = \int_A \int_{\mathcal{H}^2(n)} S(p_o, \omega_o, p_i, \omega_i) L_i(p_i, \omega_i) |\cos \theta_i| d\omega_i dA$$

- **We need to evaluate the integral**

- Accurately
 - Efficiently

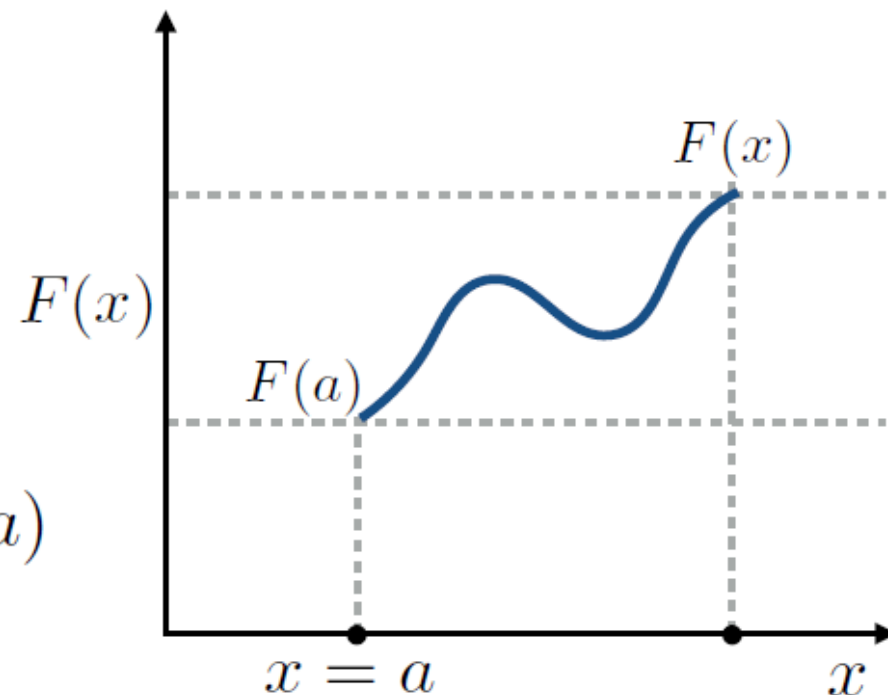
1. Traditional numerical integration

Review: fundamental theorem of calculus

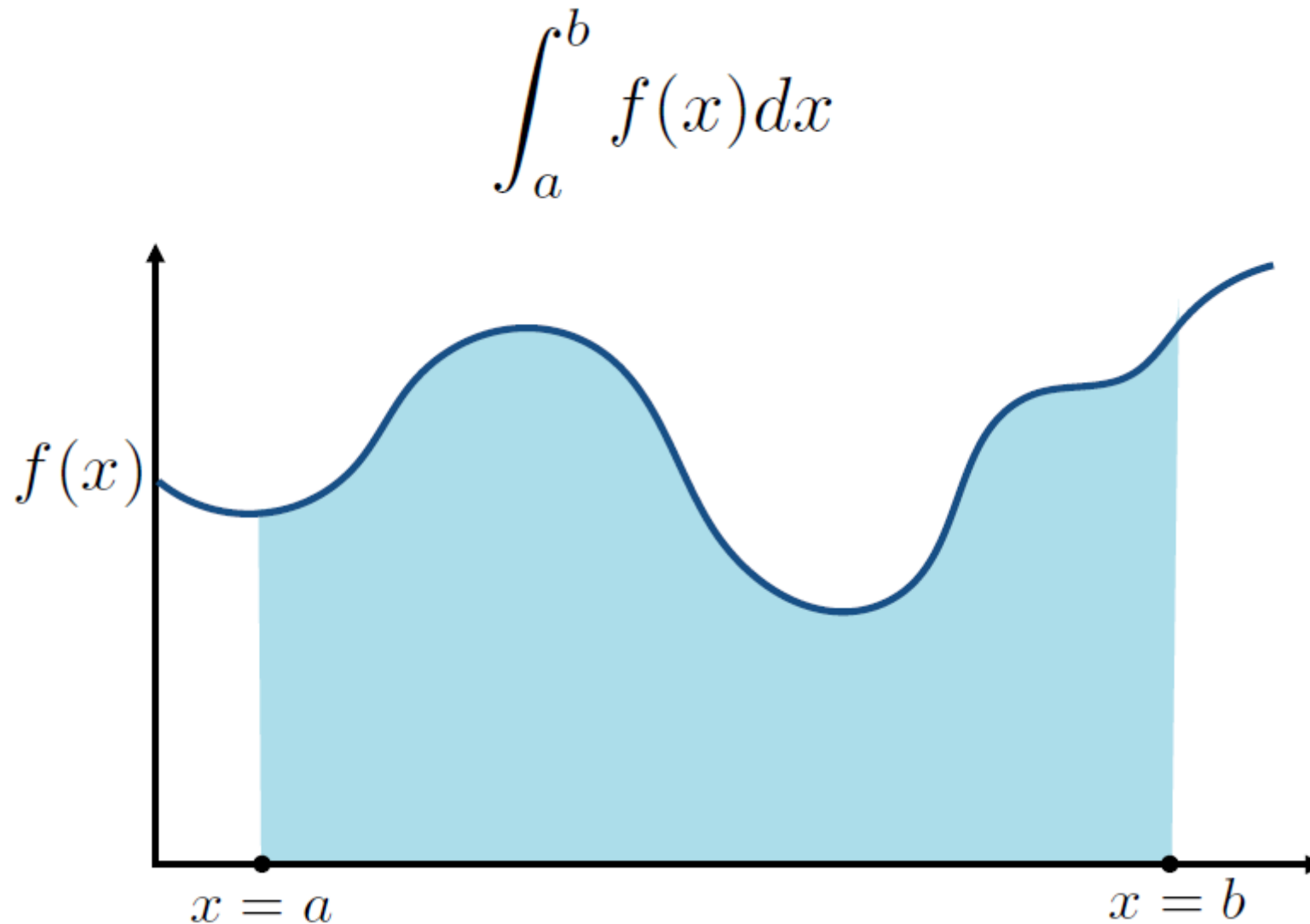
$$\int_a^b f(x) dx = F(b) - F(a)$$

$$f(x) = \frac{d}{dx} F(x)$$

$$\int_a^x f(t) dt = F(x) - F(a)$$

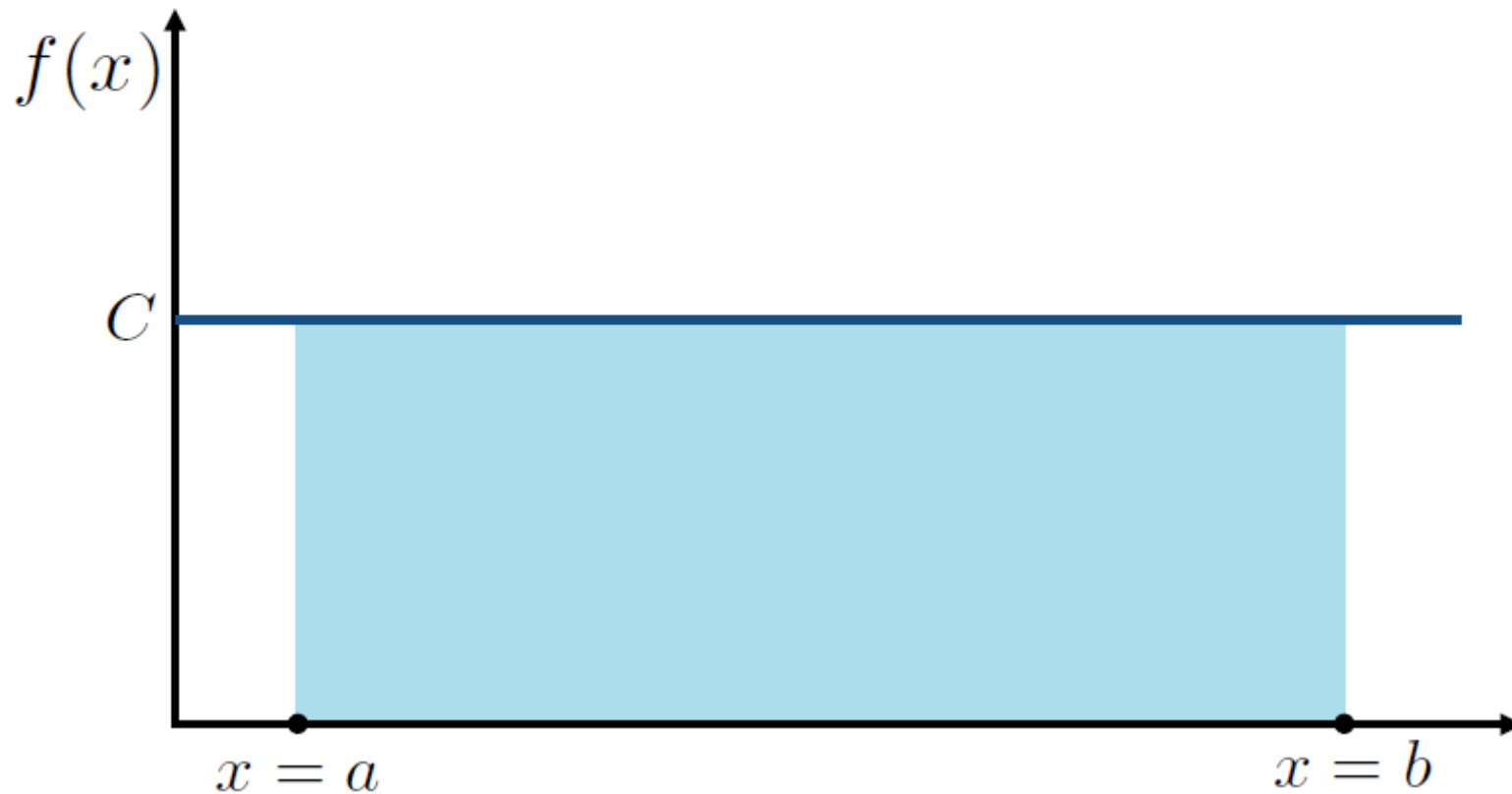


Definite integral as “area under curve”



Simple case: constant function

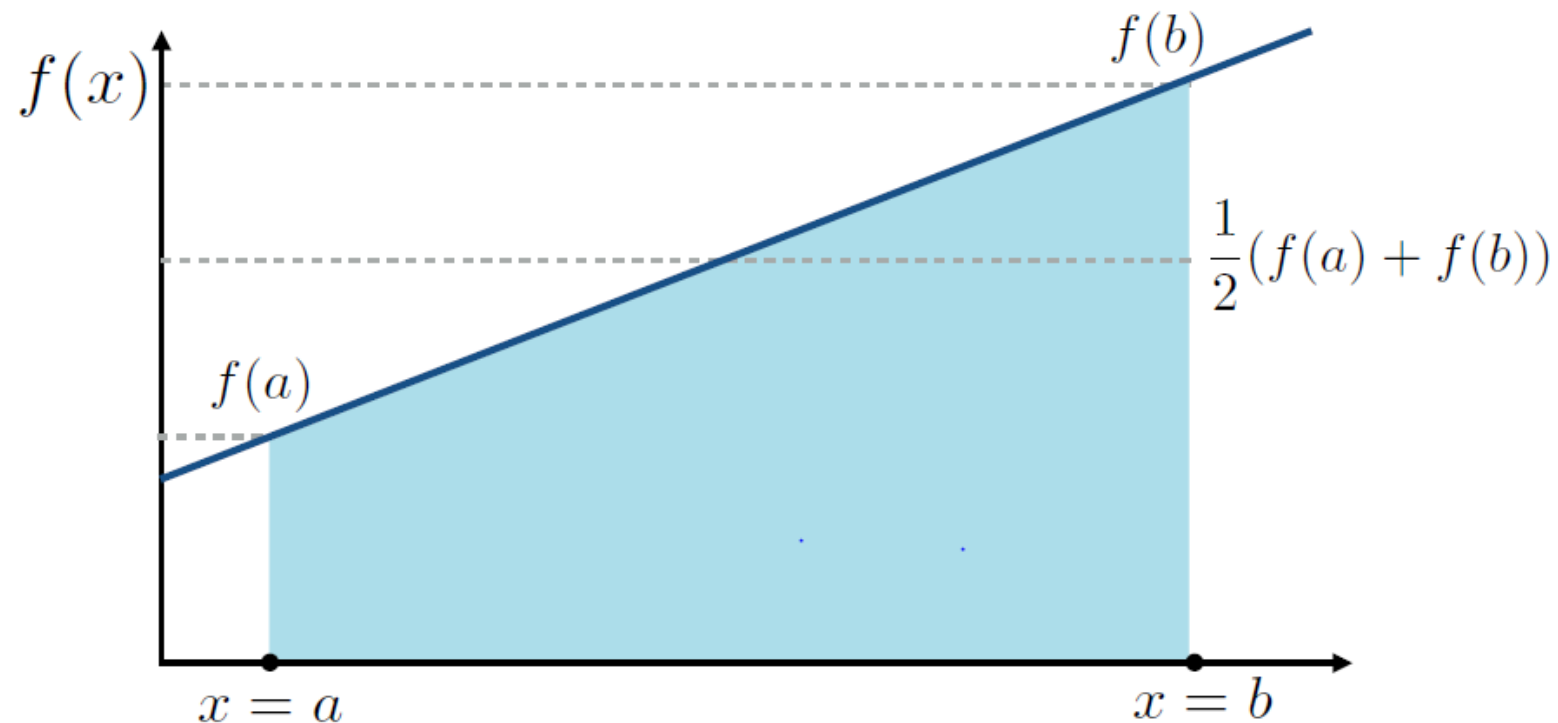
$$\int_a^b C dx = (b - a)C$$



Linear affine function

Affine function: $f(x) = cx + d$

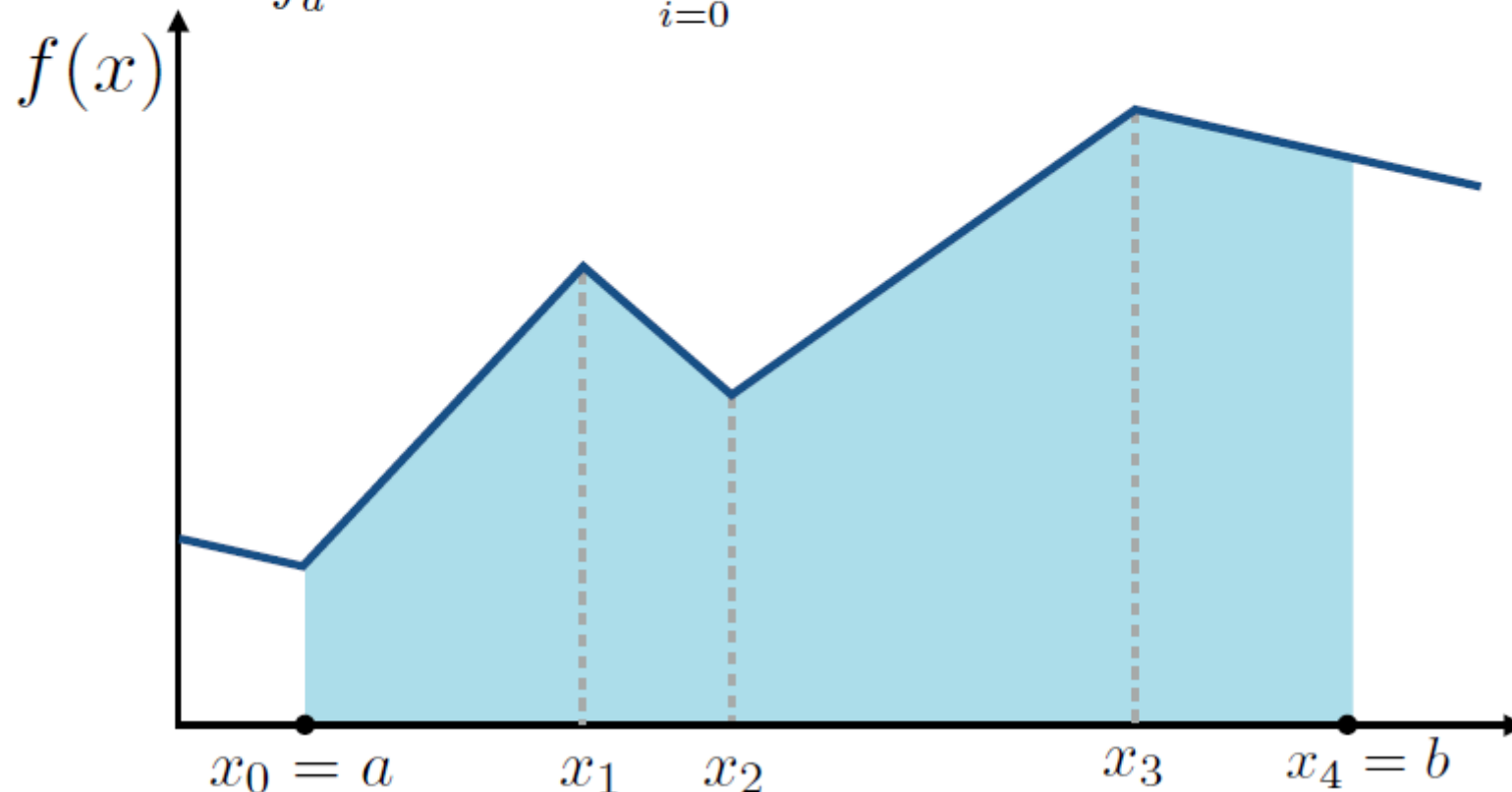
$$\int_a^b f(x) dx = \frac{1}{2}(f(a) + f(b))(b - a)$$



Piecewise affine function

Sum of integrals of individual affine components

$$\int_a^b f(x)dx = \frac{1}{2} \sum_{i=0}^{n-1} (x_{i+1} - x_i)(f(x_i) + f(x_{i+1}))$$

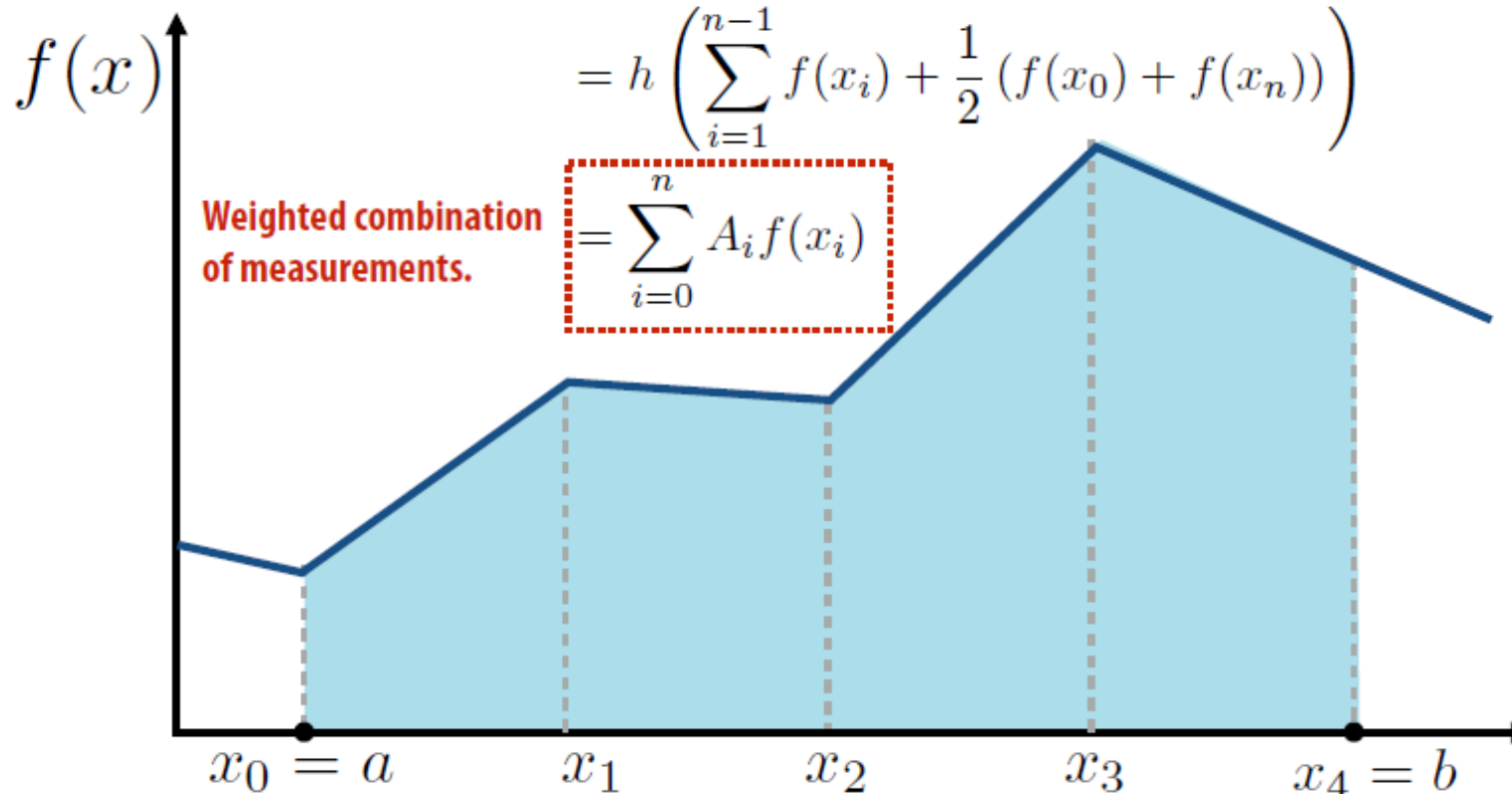


Piecewise affine function

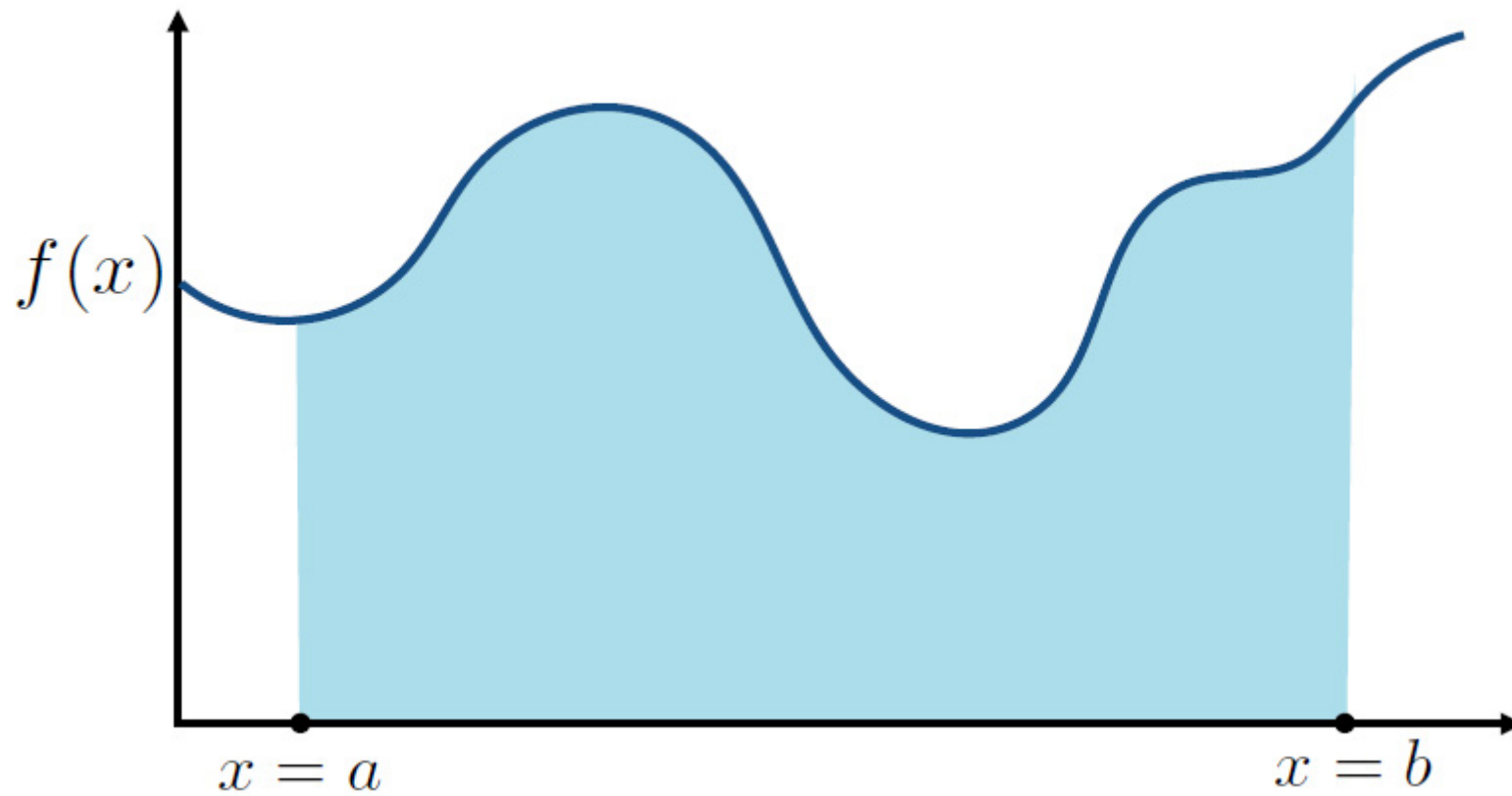
If $N-1$ segments are of equal length: $h = \frac{b-a}{n-1}$

$$\int_a^b f(x)dx = \frac{h}{2} \sum_{i=0}^{n-1} (f(x_i) + f(x_{i+1}))$$

$$= h \left(\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2} (f(x_0) + f(x_n)) \right)$$



Polynomials?



Aside: interpolating polynomials

Consider $n+1$ measurements of a function $f(x)$

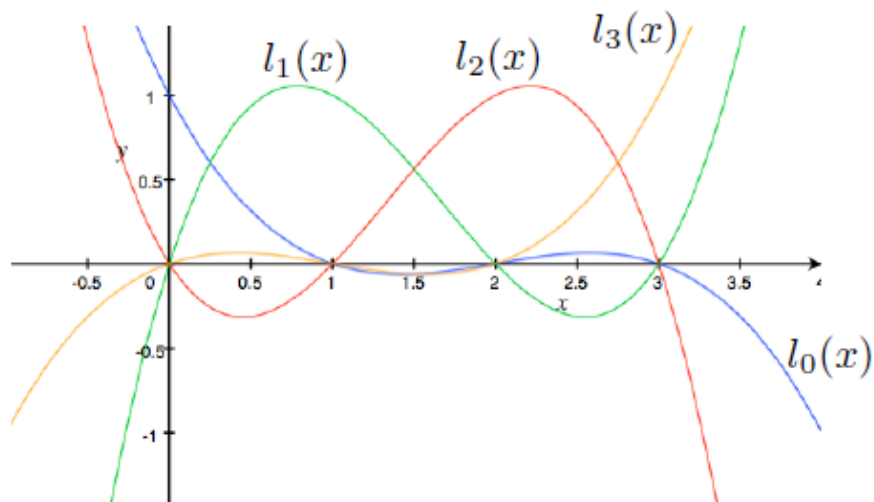
$$f(x_0), f(x_1), f(x_2), \dots, f(x_n)$$

There is a unique degree $\leq n$ polynomial that interpolates the points:

$$\begin{aligned} p(x) &= \sum_{i=0}^n f(x_i) \prod_{j \neq i, j=0}^n \left(\frac{x - x_j}{x_i - x_j} \right) \\ &= \sum_{i=0}^n f(x_i) l_i(x) \end{aligned}$$

Lagrange polynomial

Note: $l_i(x)$ is 1 at x_i and 0 at all other measurement points



Gaussian quadrature theorem

If $f(x)$ is a polynomial of degree of up to $2n+1$, then its integral over $[a,b]$ is computed exactly by a weighted combination of $n+1$ measurements in this range.

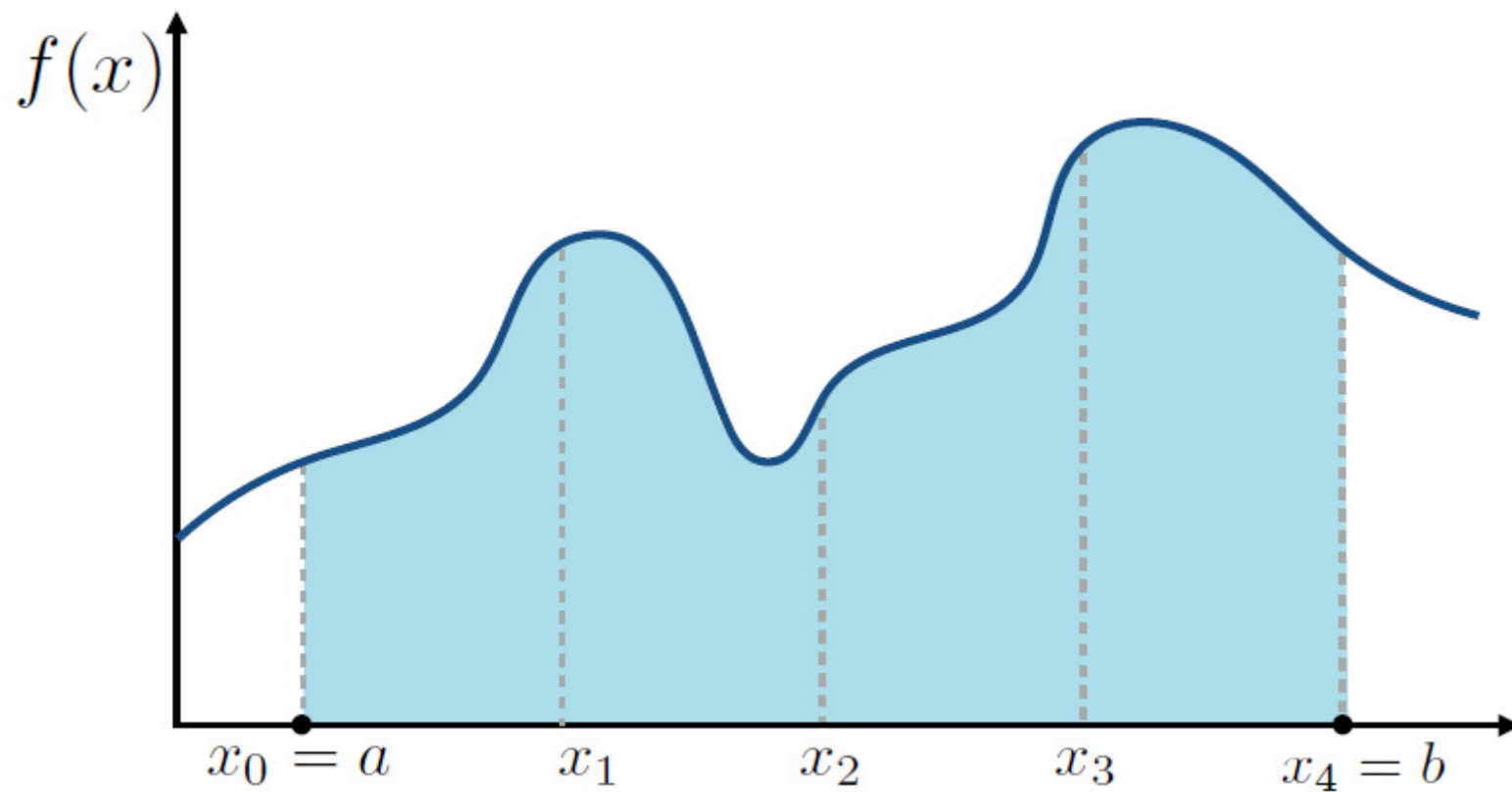
$$\int_a^b f(x)dx = \sum_{i=0}^n A_i f(x_i) \qquad A_i = \int_a^b l_i(x)dx$$

Where are these points?

Roots of degree $n+1$ polynomial $q(x)$ where:

$$\int_a^b x^k q(x)dx = 0 \qquad 0 \leq k \leq n$$

Arbitrary function $f(x)$?

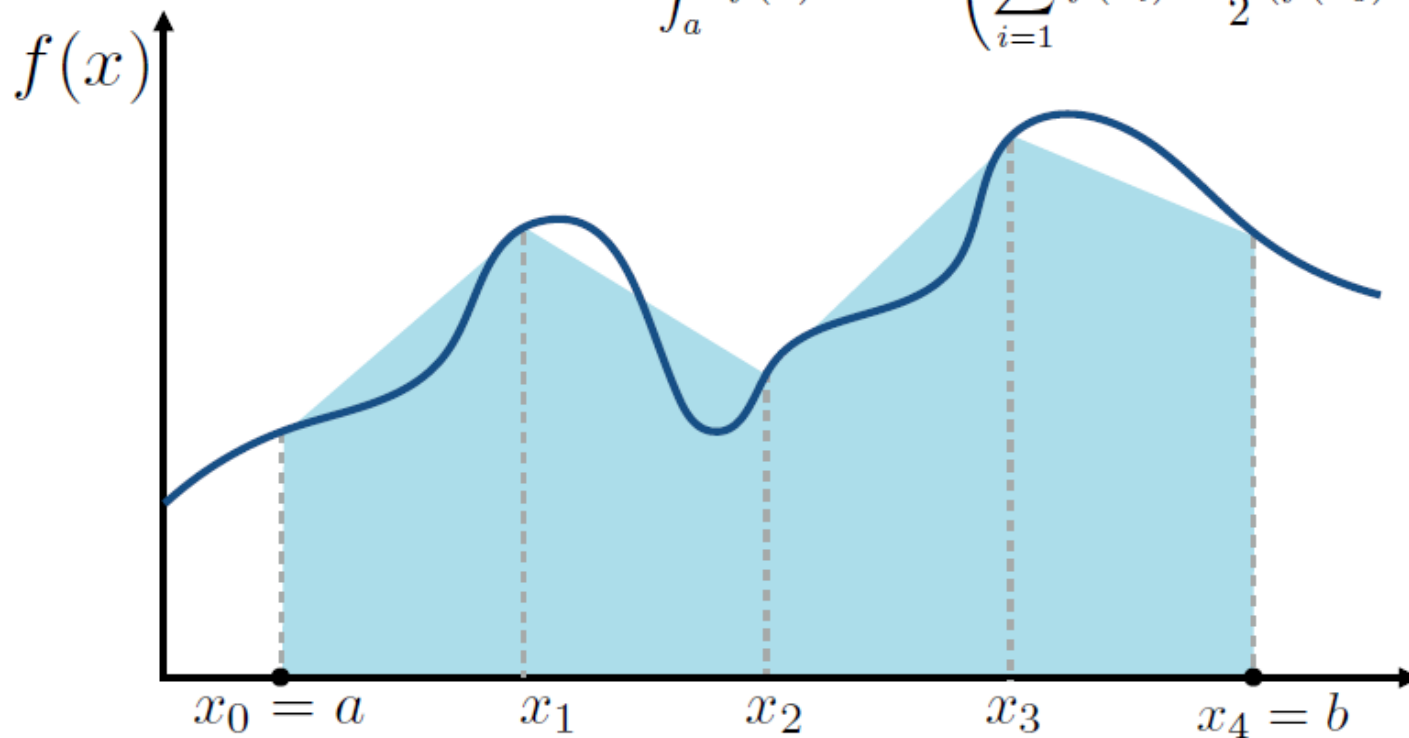


Trapezoidal rule

Approximate integral of $f(x)$ by assuming function is piecewise linear

For equal length segments: $h = \frac{b - a}{n - 1}$

$$\int_a^b f(x) dx = h \left(\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2} (f(x_0) + f(x_n)) \right)$$

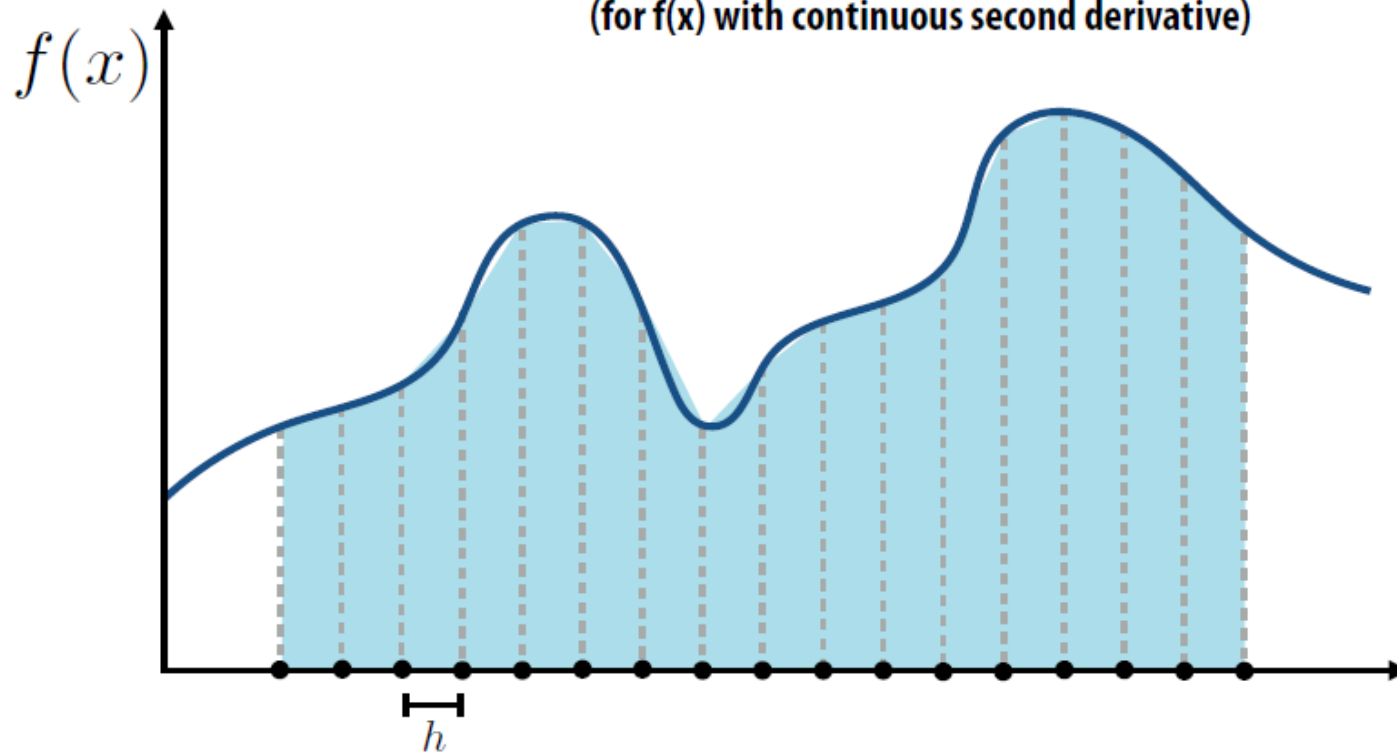


Trapezoidal rule

Consider cost and accuracy of estimate as $n \rightarrow \infty$ (or $h \rightarrow 0$)

Work: $O(n)$

Error can be shown to be: $O(h^2) = O(\frac{1}{n^2})$
(for $f(x)$ with continuous second derivative)



Integration in 2D

Consider integrating $f(x, y)$ using the trapezoidal rule
(apply rule twice: when integrating in x and in y)

$$\begin{aligned}\int_{a_y}^{b_y} \int_{a_x}^{b_x} f(x, y) dx dy &= \int_{a_y}^{b_y} \left(O(h^2) + \sum_{i=0}^n A_i f(x_i, y) \right) dy && \text{First application of rule} \\ &= O(h^2) + \sum_{i=0}^n A_i \int_{a_y}^{b_y} f(x_i, y) dy \\ &= O(h^2) + \sum_{i=0}^n A_i \left(O(h^2) + \sum_{j=0}^n A_j f(x_i, y_j) \right) && \text{Second application} \\ &= O(h^2) + \sum_{i=0}^n \sum_{j=0}^n A_i A_j f(x_i, y_j)\end{aligned}$$

Errors add, so error still: $O(h^2)$

But work is now: $O(n^2)$

($n \times n$ set of measurements)

Must perform much more work in 2D to get
same error bound on integral!

In K-D, let $N = n^k$

Error goes as: $O\left(\frac{1}{N^{2/k}}\right)$

Look at the rendering equation again

- The reflection and scattering equations

$$L_o(p, \omega_o) = \int_{\mathbb{S}^2} f(p, \omega_o, \omega_i) L_i(p, \omega_i) |\cos \theta_i| d\omega_i$$

$$L_o(p_o, \omega_o) = \int_A \int_{\mathcal{H}^2(\mathbf{n})} S(p_o, \omega_o, p_i, \omega_i) L_i(p_i, \omega_i) |\cos \theta_i| d\omega_i dA$$

- Very high dimensional
 - Consider the tracing process: infinite dimensional
 - Conventional numerical integration becomes prohibitive in computation

How to do realistic rendering?

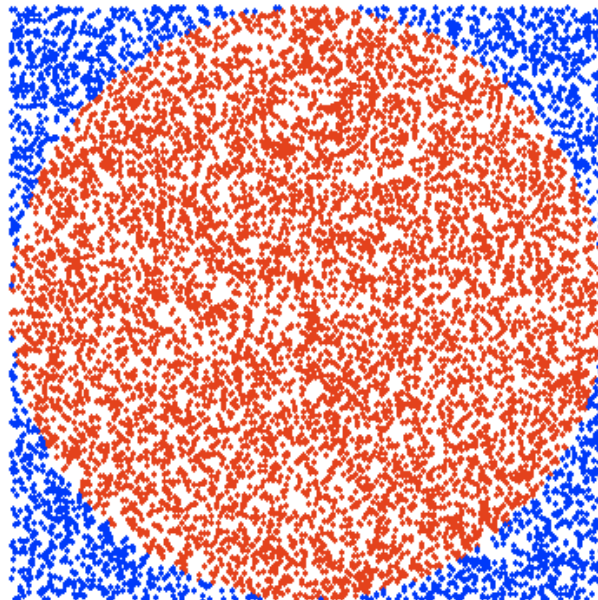
- **How to evaluate the integral efficiently?**
 - Rendering equations are usually high dimensional, hard to directly evaluate
 - Sampling? How many samples needed?
 - Convergence?



2. Monte-Carlo integration

Monte-Carlo integration

- **A technique for numerical integration**
 - Using random numbers (probabilistic rather than deterministic)
 - Algorithm gives the correct value of integral “on average”
 - Particularly useful for higher-dimensional integrals
 - Statistically very similar to the true answer



Review of probability

- **Random variable X**
 - A variable whose value is chosen by a random process
 - Applying function f to a random variable X results in a new random variable $Y = f(X)$
- **Probability \Pr**
 - The measure of the likelihood that an event will occur
- **Cumulative distribution function (CDF)**

$$P(x) = \Pr\{X \leq x\}$$

Review of probability

- **Continuous random variables x**
 - A random variable taking values over ranges of continuous domains
- **Probability density function (PDF)**
 - The relative likelihood for the random variable to take on a given value (non-negative)

$$p(x) = \frac{dP(x)}{dx}$$

- For uniform random variables

$$p(x) = \begin{cases} 1 & x \in [0, 1) \\ 0 & \text{otherwise} \end{cases}$$

Review of probability

- **Computing probability from PDF**

- The probability that a random variable lies inside the interval

$$P(x \in [a, b]) = \int_a^b p(x) \, dx$$

- **Expected value**

- Average value of a function over some distribution of values $p(x)$ over its domain

$$E_p[f(x)] = \int_D f(x) p(x) \, dx$$

Review of probability

- **Variance**

- The expected deviation of the function from its expected value

$$V[f(x)] = E \left[(f(x) - E[f(x)])^2 \right]$$

- Properties

$$E[af(x)] = aE[f(x)]$$

$$E \left[\sum_i f(X_i) \right] = \sum_i E[f(X_i)]$$

$$V[af(x)] = a^2 V[f(x)]$$

$$V[f(x)] = E \left[(f(x))^2 \right] - E[f(x)]^2$$

$$\sum_i V[f(X_i)] = V \left[\sum_i f(X_i) \right]$$

Review of probability

- **Joint distribution function**

- Give the probability that each of X, Y, \dots falls in any particular range of values specified for that variable

- **Marginal density function**

- The probabilities of various values of the variables in the subset without reference to the values of the other variables

$$p(x) = \int p(x, y) \, dy$$

- **Conditional probability**

- A measure of the probability of an event given that another event has occurred

$$p(y|x) = \frac{p(x, y)}{p(x)}$$

Monte-Carlo estimator

- **Approximate the value of an arbitrary integral**

- The foundation of the light transport algorithms

- **1D evaluation**

- A one-dimensional integral $\int_a^b f(x) \, dx$

- Given a supply of uniform random variables $X_i \in [a, b]$, the expected value of the integral estimator

$$F_N = \frac{b-a}{N} \sum_{i=1}^N f(X_i)$$

Monte-Carlo estimator

- **Expected value**
 - Equal to the integral

$$\begin{aligned} E[F_N] &= E \left[\frac{b-a}{N} \sum_{i=1}^N f(X_i) \right] \\ &= \frac{b-a}{N} \sum_{i=1}^N E[f(X_i)] \\ &= \frac{b-a}{N} \sum_{i=1}^N \int_a^b f(x) p(x) dx \\ &= \frac{1}{N} \sum_{i=1}^N \int_a^b f(x) dx \\ &= \int_a^b f(x) dx. \end{aligned}$$

Monte-Carlo estimator

- **More general**

- An arbitrary non-zero distribution function $f(x)$
- Random variables X_i drawn from arbitrary PDF $p(x)$

$$F_N = \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)}$$

$$\begin{aligned} E[F_N] &= E \left[\frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)} \right] \\ &= \frac{1}{N} \sum_{i=1}^N \int_a^b \frac{f(x)}{p(x)} p(x) \, dx \\ &= \frac{1}{N} \sum_{i=1}^N \int_a^b f(x) \, dx \\ &= \int_a^b f(x) \, dx. \end{aligned}$$

Monte-Carlo estimator

- **Multi-dimensional function estimation**
 - N samples X_i are taken from a multidimensional PDF
 - The estimator is applied as in 1D
- **Consider 3D integral**

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x, y, z) \, dx \, dy \, dz$$

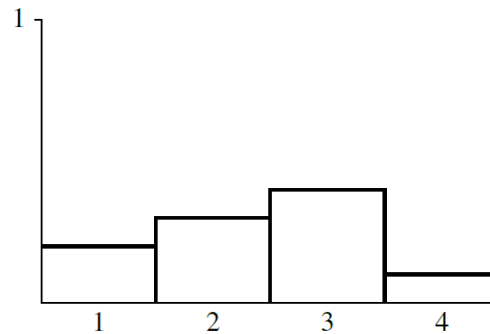
- Assuming separable joint distribution
 - The estimator

$$\frac{(x_1 - x_0)(y_1 - y_0)(z_1 - z_0)}{N} \sum_i f(X_i)$$

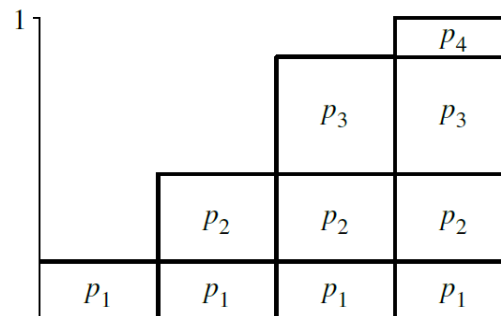
3. Sampling of random variables

Basic sampling of random variables

- **Inversion method**
 - Discrete case
 - Probability function



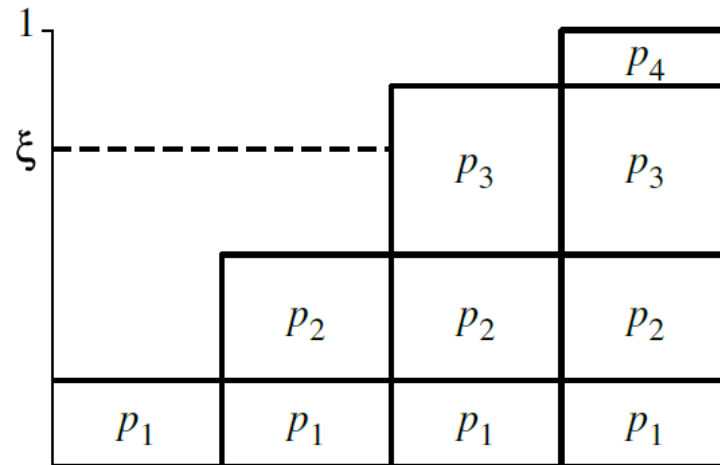
- Cumulative distribution function



Basic sampling of random variables

- **Inversion method**

- A canonical uniform random variable on vertical axis



- The inverse based on ξ value conforms to the desired distribution

Basic sampling of random variables

- **Inversion method**

- Application to continuous random variables

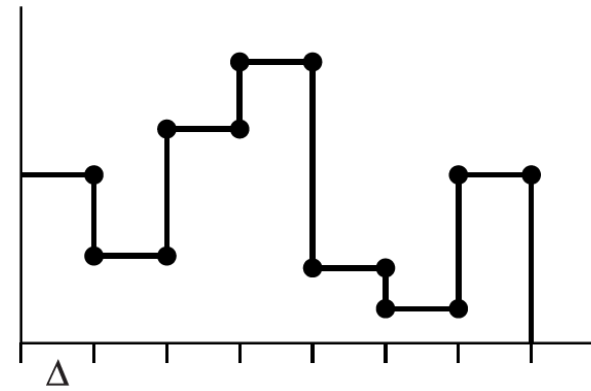
- 1. Compute the CDF $P(x) = \int_0^x p(x') dx'$
 - 2. Compute the inverse $P^{-1}(x)$
 - 3. Obtain a uniformly distributed random number ξ
 - 4. Compute $X_i = P^{-1}(\xi)$

Basic sampling of random variables

- **Inversion method**

- Piecewise-constant 1D functions over $[0,1]$

$$f(x) = \begin{cases} v_0 & x_0 \leq x < x_1 \\ v_1 & x_1 \leq x < x_2 \\ \vdots & \end{cases}$$



- The integral $\int_0^1 f(x) dx$

$$c = \int_0^1 f(x) dx = \sum_{i=0}^{N-1} \Delta v_i = \sum_{i=0}^{N-1} \frac{v_i}{N} \quad \rightarrow \quad p(x) = f(x)/c$$

Basic sampling of random variables

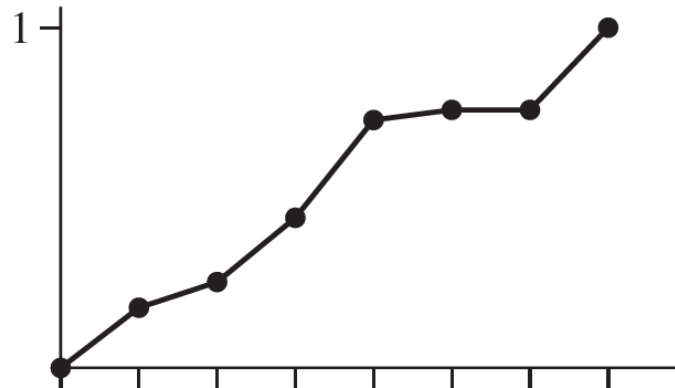
- **Inversion method**
 - Computing cumulative distribution function

$$P(x_0) = 0$$

$$P(x_1) = \int_{x_0}^{x_1} p(x) \, dx = \frac{v_0}{N_c} = P(x_0) + \frac{v_0}{N_c}$$

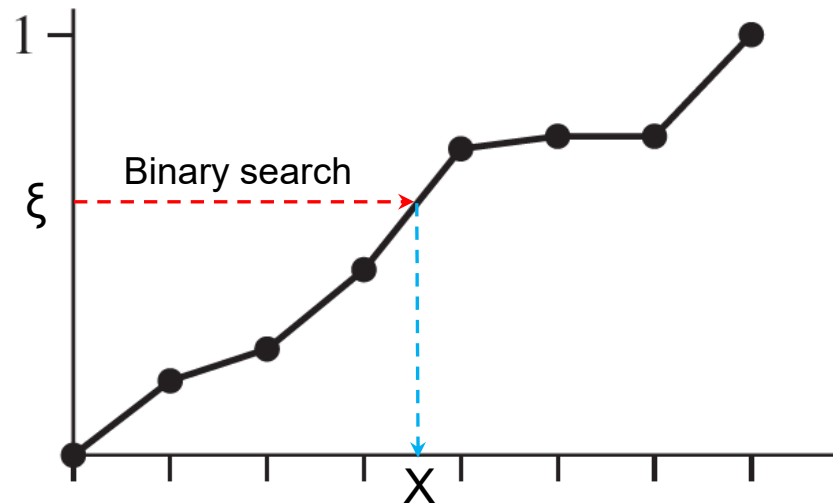
$$P(x_2) = \int_{x_0}^{x_2} p(x) \, dx = \int_{x_0}^{x_1} p(x) \, dx + \int_{x_1}^{x_2} p(x) \, dx = P(x_1) + \frac{v_1}{N_c}$$

$$P(x_i) = P(x_{i-1}) + \frac{v_{i-1}}{N_c}$$



Basic sampling of random variables

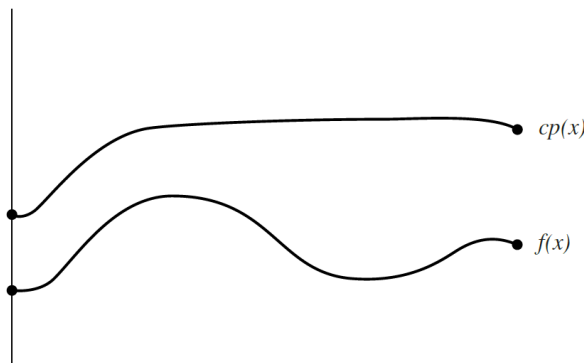
- **Inversion method**
 - Compute the inverse



Rejection method

- **Problem with inversion method**
 - Sometimes difficult to compute the CDF integral
 - Sometimes unable to obtain function inverse
- **Rejection method**
 - A dart-throwing approach
 - Find a PDF $p(x)$ from which we know how to sample
 - $p(x)$ must satisfy

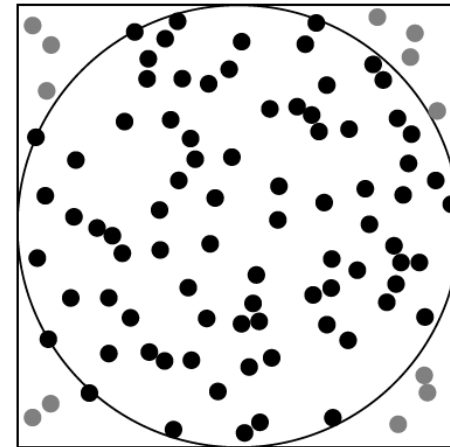
$$f(x) < c p(x)$$



Rejection method

- **Random sample generation**

- Start loop
 - Sample x from p 's distribution
 - Choose a random variable ξ
 - If $\xi < f(x)/(c p(x))$ then
Return x



- Efficiency

- Depends on how close $c p(x)$ bounds $f(x)$

- Rejection method isn't used in Monte-Carlo method for rendering

Metropolis sampling

- **A sampling technique with remarkable property**
 - Generate samples from any non-negative function f
 - Distributed proportional to f 's value
 - Only require the ability to evaluate f
 - Can efficiently generate samples from a wider variety of functions

Metropolis sampling

- **Basic algorithm**

- Generate a set of samples X_i from a function f defined over an arbitrary dimensional space Ω
 - Select the first sample X_0
 - Each sample X_i is generated using a random mutation to X_{i-1} to compute a proposed sample X'
 - In order to compute X' , we must compute a tentative transition function $T(X \rightarrow X')$: the transition probability
 - Compute the acceptance probability $a(X \rightarrow X')$

$$a(X \rightarrow X') = \min \left(1, \frac{f(X') T(X' \rightarrow X)}{f(X) T(X \rightarrow X')} \right) \quad \longrightarrow \quad a(X \rightarrow X') = \min \left(1, \frac{f(X')}{f(X)} \right)$$

Symmetric proposal

Metropolis sampling

- **Basic sampling pseudo-code**

```
X = X0
for i = 1 to n
  X' = mutate(X)
  a = accept(X, X')
  if (random() < a)
    X = X'
  record(X)
```

- The recorded X sequence will be used for integration

Metropolis sampling

- **Choosing mutation strategies**

- More freedom
- Subject to being able to compute the tentative transition density $T(X \rightarrow X')$

Several choices

- Local perturbation

$$x'_i = x_i \pm s \xi \qquad x'_i = x_i \pm b e^{-\log(b/a) \xi}$$

- Global uniform random

$$x_i = \xi$$

- Match some part of the function being sampled

$$T(X \rightarrow X') = p(X')$$

Estimating integrals with Metropolis sampling

- **We can apply Metropolis algorithm**

- Evaluate integrals

$$\int f(x)g(x) \, d\Omega$$

- Standard Monte-Carlo estimator

$$\int_{\Omega} f(x)g(x) \, d\Omega \approx \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)g(X_i)}{p(X_i)}$$

- Apply Metropolis sampling to generate samples from a density function that is proportional to $f(x)$

$$\int_{\Omega} f(x)g(x) \, d\Omega \approx \left[\frac{1}{N} \sum_{i=1}^N g(X_i) \right] \cdot \int_{\Omega} f(x) \, d\Omega$$

Transforming between distributions

- **Function of a random variable**

- Suppose we are given random variable X_i with PDF $p_x(x)$
- Given $Y_i=y(X_i)$, the following equality satisfies

$$Pr\{Y \leq y(x)\} = Pr\{X \leq x\} \quad \longrightarrow \quad P_y(y) = P_y(y(x)) = P_x(x)$$

- Differentiating

$$p_y(y) \frac{dy}{dx} = p_x(x) \quad \longrightarrow \quad p_y(y) = \left(\frac{dy}{dx} \right)^{-1} p_x(x)$$

- Usually we know $p_y(y)$ (and $P(y)$), how to sample y ?

$$y(x) = P_y^{-1} (P_x(x))$$

Transforming between distributions

- **Transformation in multiple dimensions**
 - Let n-dimensional random variable X with density function $p_x(x)$
 - Let $Y=T(X)$, T is a bijective mapping

$$p_y(y) = p_y(T(x)) = \frac{p_x(x)}{|J_T(x)|}$$

- Jacobian:

$$\begin{pmatrix} \partial T_1 / \partial x_1 & \cdots & \partial T_1 / \partial x_n \\ \vdots & \ddots & \vdots \\ \partial T_n / \partial x_1 & \cdots & \partial T_n / \partial x_n \end{pmatrix}$$

Transforming between distributions

- **Example**

- Polar coordinates

$$x = r \cos \theta$$

$$y = r \sin \theta$$

- Suppose we draw samples from some density $p(r, \theta)$
- Computing the Jacobian

$$J_T = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

- The determinant: $r (\cos^2 \theta + \sin^2 \theta) = r$
- So

$$p(x, y) = p(r, \theta)/r \quad \longrightarrow \quad p(r, \theta) = r p(x, y)$$

Transforming between distributions

- **Example**

- Spherical coordinates

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

- Computing the Jacobian determinant: $|J_T| = r^2 \sin \theta$
- The corresponding density function

$$p(r, \theta, \phi) = r^2 \sin \theta p(x, y, z)$$

- Solid angle defined with spherical coordinates $d\omega = \sin \theta d\theta d\phi$
- If we have a density function defined over a solid angle

$$p(\theta, \phi) d\theta d\phi = p(\omega) d\omega \rightarrow p(\theta, \phi) = \sin \theta p(\omega)$$

2D sampling with multi-dimensional transformations

- **2D joint density function $p(x,y)$**

- We wish to draw samples (X,Y) from

- **Sometimes separable**

$$p(x, y) = p_x(x) p_y(y)$$

- Random variable (X,Y) can be found independently

- Many useful densities aren't separable

- **Basic idea**

- Compute the marginal density to isolate one particular variable, and draw sample with 1D technique

- Compute the conditional probability and draw a sample from that distribution

2D sampling with multi-dimensional transformations

- **Example**

- Sampling a unit disk uniformly

- Wrong approach: $r = \xi_1, \theta = 2\pi \xi_2$
- PDF $p(x, y)$ by normalization is: $p(x, y) = 1/\pi$
- Transform into polar coordinate: $p(r, \theta) = r/\pi$ $p(r, \theta) = r p(x, y)$
- Compute the marginal and conditional densities

$$p(r) = \int_0^{2\pi} p(r, \theta) d\theta = 2r$$

$$p(\theta|r) = \frac{p(r, \theta)}{p(r)} = \frac{1}{2\pi}$$

- Integrating and inverting to find $P(r)$, $P^{-1}(r)$, $P(\theta)$, and $P^{-1}(\theta)$

$$r = \sqrt{\xi_1}$$

$$\theta = 2\pi \xi_2$$

3D sampling with multi-dimensional transformations

- **Example**

- Uniformly sampling a hemisphere

- Uniform sampling means $p(\omega) = c$

- Normalization:

$$\int_{\mathcal{H}^2} p(\omega) d\omega = 1 \Rightarrow c \int_{\mathcal{H}^2} d\omega = 1 \Rightarrow c = \frac{1}{2\pi} \quad \text{---} \quad p(\omega) = 1/(2\pi) \quad \text{---} \quad p(\theta, \phi) = \sin \theta / (2\pi)$$

$p(\theta, \phi) = \sin \theta p(\omega)$

- Consider sampling θ :

$$p(\theta) = \int_0^{2\pi} p(\theta, \phi) d\phi = \int_0^{2\pi} \frac{\sin \theta}{2\pi} d\phi = \sin \theta$$

- Compute the conditional density for ϕ :

$$p(\phi|\theta) = \frac{p(\theta, \phi)}{p(\theta)} = \frac{1}{2\pi}$$

3D sampling with multi-dimensional transformations

- **Example**

- Uniformly sampling a hemisphere
 - Use 1D inversion technique to sample:

$$P(\theta) = \int_0^\theta \sin \theta' d\theta' = 1 - \cos \theta$$

$$P(\phi|\theta) = \int_0^\phi \frac{1}{2\pi} d\phi' = \frac{\phi}{2\pi}$$

- Inversion is straightforward

$$\begin{array}{ll} \theta = \cos^{-1} \xi_1 & \longrightarrow x = \sin \theta \cos \phi = \cos(2\pi \xi_2) \sqrt{1 - \xi_1^2} \\ \phi = 2\pi \xi_2 & y = \sin \theta \sin \phi = \sin(2\pi \xi_2) \sqrt{1 - \xi_1^2} \\ & z = \cos \theta = \xi_1 \end{array}$$

3D sampling with multi-dimensional transformations

- **Cosine-weighted hemisphere sampling**

- It is useful to have a cosine distribution over the hemisphere (the incident cosine term)
- We require: $p(\omega) \propto \cos \theta$
- Derive as before:

$$\int_{\mathcal{H}^2} c p(\omega) d\omega = 1$$

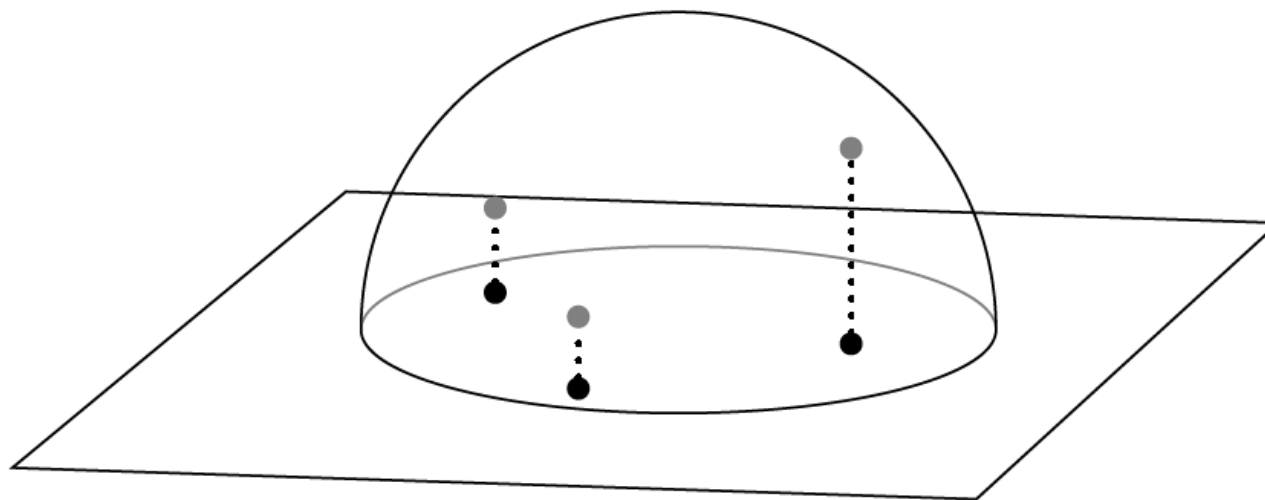
Jacobi 行列式
 $d\omega = \sin \theta d\theta d\phi$ $p(\theta, \phi) = \sin \theta p(\omega)$

$$\int_0^{2\pi} \int_0^{\pi/2} c \cos \theta \sin \theta d\theta d\phi = 1$$
$$c 2\pi \int_0^{\pi/2} \cos \theta \sin \theta d\theta = 1$$
$$c = \frac{1}{\pi}$$

→ $p(\theta, \phi) = \frac{1}{\pi} \cos \theta \sin \theta$

3D sampling with multi-dimensional transformations

- **Cosine-weighted hemisphere sampling**
 - Malley's method
 - Sampling a unit disk and project onto the sphere

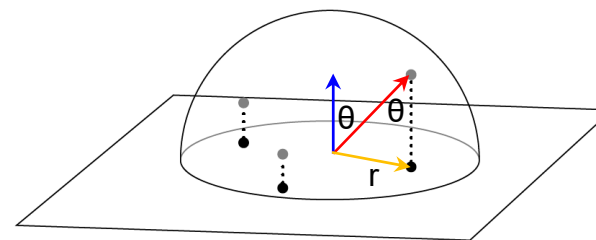


3D sampling with multi-dimensional transformations

- **Why Malley's method works?**

- Let (r, ϕ) be polar coordinates on disk, we know

$$p(r, \phi) = r/\pi$$



- Vertical projection gives: $\sin \theta = r$
- To complete the $(r, \phi) \rightarrow (\sin \theta, \phi)$ transformation, we need the determinant of the Jacobian

$$|J_T| = \begin{vmatrix} \cos \theta & 0 \\ 0 & 1 \end{vmatrix} = \cos \theta$$

- Therefore:

$$p(\theta, \phi) = |J_T| p(r, \phi) = \cos \theta r / \pi = (\cos \theta \sin \theta) / \pi$$

4. Sampling efficiency

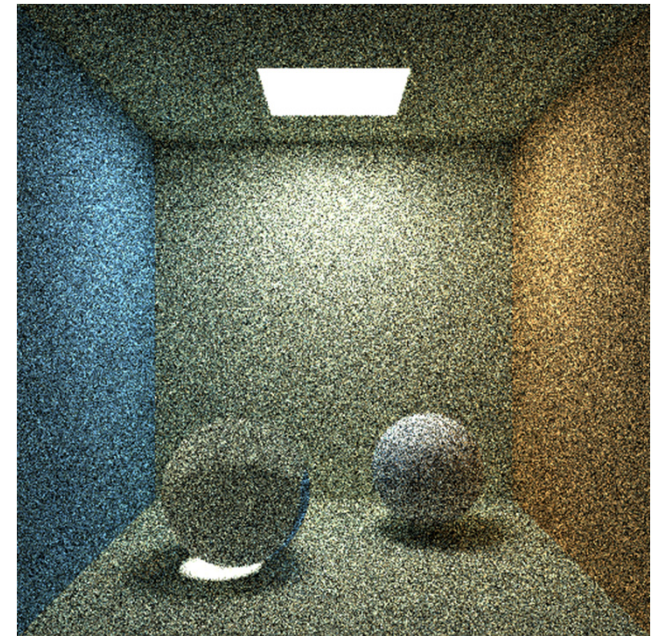
Estimating efficiency

- **Variance in Monte-Carlo ray tracing**
 - Manifest as image noise

- **Efficiency of an estimator**

$$\epsilon[F] = \frac{1}{V[F]T[F]}$$

- $V[F]$: sampling variance
- $T[F]$: running time to compute



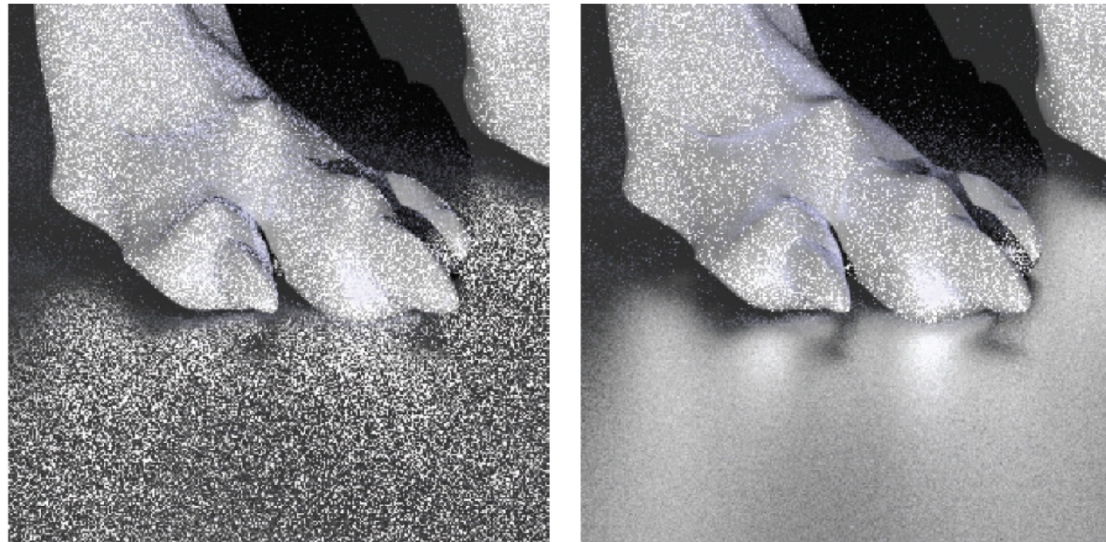
Different sampling rate

- **Improve efficiency**
 - Importance sampling

Stratified sampling

- **Stratified sampling**

- Subdivide the integration domain into n non-overlapping regions Λ_i
- We draw n_i samples from each region Λ_i according to density p_i inside each region
- Suffer from “curse of dimensionality”



Quasi Monte-Carlo sampling

- **Low-discrepancy sampling**
 - Poisson disk / best-candidate sampling
 - Foundation of a branch of Monte-Carlo sampling
- **Advantage**
 - Quasi Monte-Carlo converges asymptotically faster
 - Generally better for smooth integrand
- **Disadvantage**
 - Asymptotic convergence rate is not applicable to discontinuous integrand

Sampling bias

- **Another approach of variance reduction**
 - Introduce bias into the computation
 - Sacrifice for larger error in expected value for variance reduction
- **Unbiased estimator**
 - The expected value is equal to the correct answer
- **Bias estimation**

$$\beta = E[F] - \int f(x) \, dx$$

Sampling bias

- **Why bias is sometimes desirable?**
 - Consider computing estimation of the mean value of a distribution $X_i \sim p$ over $[0,1]$
 - Two estimators
 - 1. $\frac{1}{N} \sum_{i=1}^N X_i$
 - 2. $\frac{1}{2} \max(X_1, X_2, \dots, X_N)$
 - The first estimator has variance $O(N^{-1})$

Sampling bias

- **Why bias is sometimes desirable?**
 - The second estimator's expected value

$$0.5 \frac{N}{N+1} \neq 0.5$$

- It is biased, but its variance is $O(N^{-2})$
 - For large value of N , the second estimator is preferred

Sampling bias

- **Consider again image reconstruction**

- Consider a Monte-Carlo estimate of

$$I(x, y) = \iint f(x - x', y - y') L(x', y') \, dx' \, dy'$$

- Assume we take image samples uniformly: constant probability density (unbiased, larger variance):

$$I(x, y) \approx \frac{1}{N p_c} \sum_{i=1}^N f(x - x_i, y - y_i) L(x_i, y_i)$$

- The practical realization (biased, less variance):

$$I(x, y) = \frac{\sum_i f(x - x_i, y - y_i) L(x_i, y_i)}{\sum_i f(x - x_i, y - y_i)}$$

5. Importance sampling

Importance sampling

- **Importance sampling is a variance reduction technique**

- Monte-Carlo estimator

$$F_N = \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)}$$

- The fact

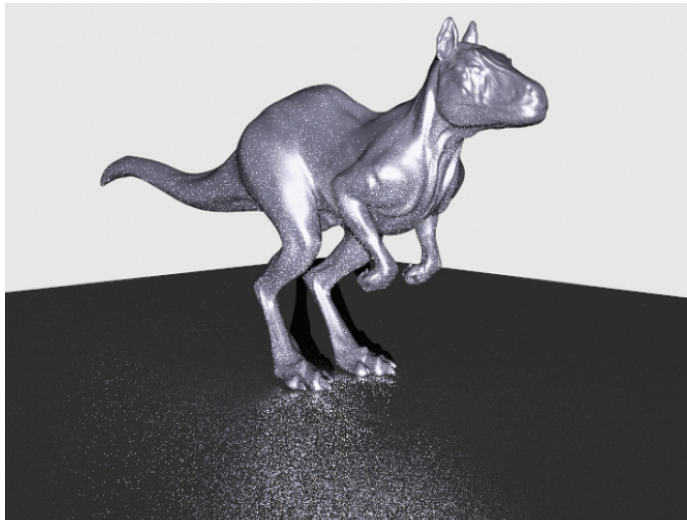
- If samples are taken from distribution $p(x)$ that is similar to function $f(x)$, the convergence will be much faster
- Can increase variance if $p(x)$ is bad

- Basic idea

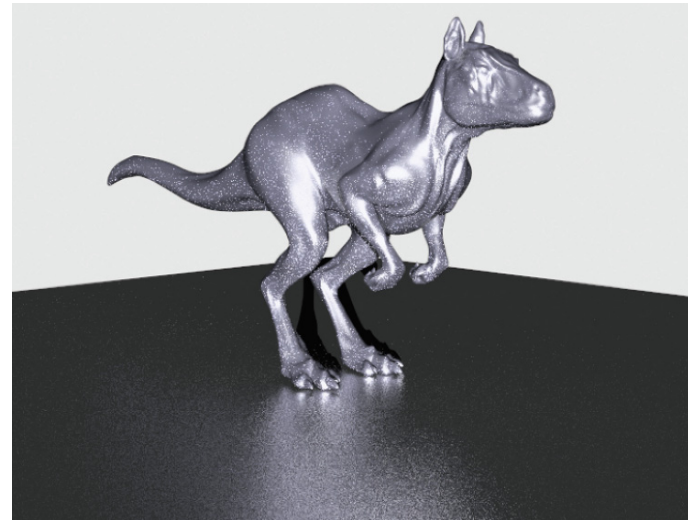
- Concentrate work where the values of the integrand is relatively high

Importance sampling

- **The practical case**
 - The integrand is the product of more than one function
 - Finding $p(x)$ similar to one of the multiplicands can be helpful
 - Especially important in rendering



Stratified uniform sampling



Importance sampling based on BRDF

Multiple importance sampling

- **We are frequently faced with integrals with two or more function**

$$\int f(x)g(x) dx$$

- Importance sampling strategy for both $f(x)$ and $g(x)$, which to choose?
- Assume we are not able to combine two sampling to compute a PDF proportional to $f(x)g(x)$
- A bad choice of sampling distribution can be much worse than uniform distribution

Multiple importance sampling

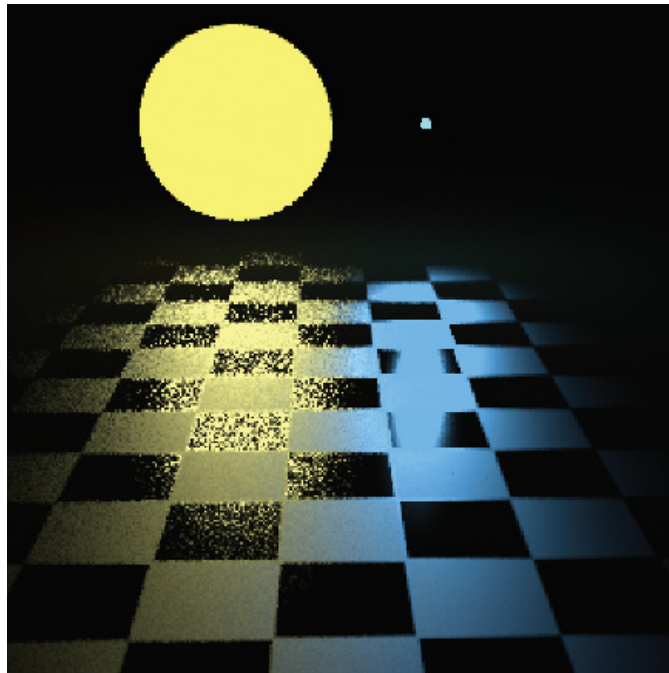
- Consider direct lighting integral evaluation

$$L_o(p, \omega_o) = \int_{\mathbb{S}^2} f(p, \omega_o, \omega_i) L_d(p, \omega_i) |\cos \theta_i| d\omega_i$$

- We can perform importance sampling based on either L_d or f , one of these will often perform poorly

Multiple importance sampling

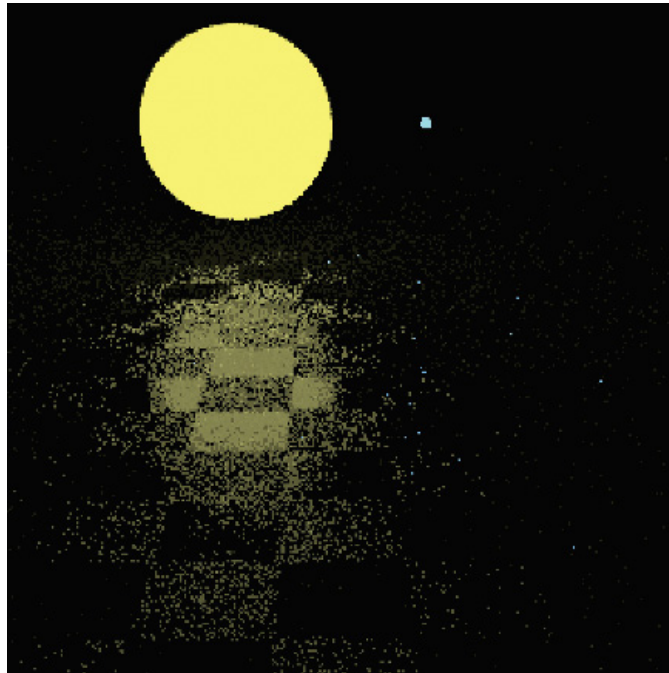
- **Consider a near-mirror BRDF**
 - The value of integrand will be close to 0 for angles off the reflection angle
 - Sampling L_d will lead to large variance



Sampling from light distribution

Multiple importance sampling

- **Consider a near-mirror BRDF**
 - Sampling BRDF could be much better
 - However, for diffuse and glossy BRDFs, sampling from BRDF will lead to similar problem



Sampling from BRDF

Multiple importance sampling

- **How to solve?**

- Try to match either of them
- Weighting scheme

- If two sampling distributions p_f and p_g are used to estimate the value of

$$\int f(x)g(x) dx$$

- The new Monte-Carlo estimator is given by

$$\frac{1}{n_f} \sum_{i=1}^{n_f} \frac{f(X_i)g(X_i)w_f(X_i)}{p_f(X_i)} + \frac{1}{n_g} \sum_{j=1}^{n_g} \frac{f(Y_j)g(Y_j)w_g(Y_j)}{p_g(Y_j)}$$

- n_f : number of samples taken from p_f distribution
- n_g : number of samples taken from p_g distribution

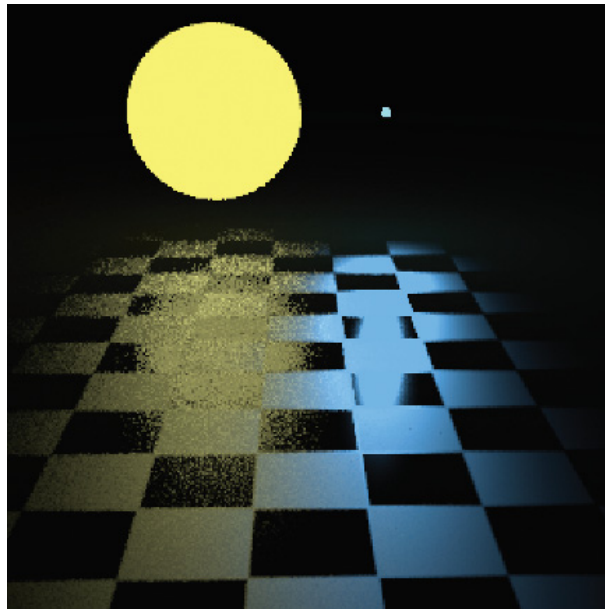
Multiple Importance Sampling

- **Weighting function**

- Balance heuristic

$$w_s(x) = \frac{n_s p_s(x)}{\sum_i n_i p_i(x)}$$

- Effectively proven to reduce variance



Next lecture: Sampling and reconstruction