Introduction to Machine Learning, Fall 2023
Homework 2

(Due Tuesday Nov. 14 at 11:59pm (CST))

November 5, 2023

1. [10 points] [Convex Optimization Basics]

   (a) Proof any norm $f : \mathbb{R}^n \to \mathbb{R}$ is convex. [2 points]

   (b) Determine the convexity (i.e., convex, concave or neither) of $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{>0}$. [2 points]

   (c) Determine the convexity of $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{>0}$. [2 points]

   (d) Recall Jensen's inequality $f(\mathbb{E}(X)) \leq \mathbb{E}(f(X))$ if f is convex for any random variable X. Proof the log sum inequality:

$$\sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i \right) \log \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

   where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are positive numbers. Hints: $f(x) = x \log x$ is strictly convex. [4 points]

Solution:

(a) Since f is a norm function, we have the properties of norm functions that,
1. $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$.
2. $\forall \mathbf{x} \in \mathbb{R}^n, \forall a \in \mathbb{R}, f(ax) = |a| f(\mathbf{x})$.

So we have, $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \forall \lambda \in [0, 1]$.
From property 1., we can get that

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leq f(\lambda x) + f((1 - \lambda)\mathbf{y})$$

From property 2., we can get that

$$f(\lambda x) = |\lambda| f(\mathbf{x}) \textit{ and } f((1 - \lambda)\mathbf{y}) = |1 - \lambda| f(\mathbf{y})$$

Since $\lambda \in [0, 1]$, so we have $|\lambda| = \lambda$ and $|1 - \lambda| = 1 - \lambda$,
So we can get that

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y})$$

So above all, from the defination, we can get that f is a convex function.

(b) Since $f(x_1, x_2) = \dfrac{x_1^2}{x_2}$, so we have the Hessain matrix of f is

$$H = \nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{2}{x_2} & -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix}$$

Since $x_2 > 0$, so $|H| = \dfrac{2}{x_2} \cdot \dfrac{2x_1^2}{x_2^3} - (-\dfrac{2x_1}{x_2^2})^2 = \dfrac{4x_1^2}{x_2^4} - \dfrac{4x_1^2}{x_2^4} = 0 \geq 0$
So we can get that $\nabla^2 f(x_1, x_2) \succeq 0$, so f is a convex function.

So above all, f is a convex function.

(c) Since $f(x_1, x_2) = \dfrac{x_1}{x_2}$, so we have the Hessain matrix of f is

$$H = \nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}$$

Since $x_2 > 0$, so $|H| = \dfrac{2x_1}{x_2^3} \cdot 0 - (-\dfrac{1}{x_2^2})^2 = -\dfrac{1}{x_2^4} \leq 0$
So we can get that $\nabla^2 f(x_1, x_2) \preceq 0$, so f is a concave function.

So above all, f is a concave function.

(d) We can construct a distribution X s.t.
the domain of X is $\dfrac{a_i}{b_i}, i \in \{1, 2, \cdots, n\}$, and the PMF of X is $P(X = \dfrac{a_i}{b_1}) = \dfrac{b_i}{\sum\limits_{i=1}^{n} b_i}$.

Since $\forall i \in \{1, 2, \cdots, n\}, a_i > 0, b_i > 0$,
So $P(X = \dfrac{a_i}{b_i}) > 0$, and $\sum\limits_{i=1}^{n} P(X = \dfrac{a_i}{b_i}) = 1$.
So it's a valid distribution.
So

$$\mathbb{E}(X) = \sum_{i=1}^{n} (\frac{a_i}{b_i}) \cdot P(X = \frac{a_i}{b_i}) = \sum_{i=1}^{n} \frac{a_i}{b_i} \cdot \frac{b_i}{\sum\limits_{k=1}^{n} b_k} = \frac{\sum\limits_{i=1}^{n} a_i}{\sum\limits_{i=1}^{n} b_i}$$

And since $f(x) = x \log x$ is strictly convex, so from the Jensen's inequality, we can get that

$$f(\mathbb{E}(X)) \leq \mathbb{E}(f(X))$$

i.e.

$$\left(\frac{\sum\limits_{i=1}^{n} a_i}{\sum\limits_{i=1}^{n} b_i} \right) \log \left(\frac{\sum\limits_{i=1}^{n} a_i}{\sum\limits_{i=1}^{n} b_i} \right) \leq \sum_{i=1}^{n} P(X = \frac{a_i}{b_i}) \cdot f(\frac{a_i}{b_i})$$

$$\left(\frac{\sum\limits_{i=1}^{n} a_i}{\sum\limits_{i=1}^{n} b_i} \right) \log \left(\frac{\sum\limits_{i=1}^{n} a_i}{\sum\limits_{i=1}^{n} b_i} \right) \leq \sum_{i=1}^{n} \frac{b_i}{\sum\limits_{k=1}^{n} b_k} \cdot (\frac{a_i}{b_i}) \log(\frac{a_i}{b_i})$$

Since $b_i > 0$, so $\sum\limits_{i=1}^{n} b_i > 0$, so appointment $\sum\limits_{i=1}^{n} b_i$ on both sides simultaneously, we can get that

$$\left(\sum_{i=1}^{n} a_i \right) \log \left(\frac{\sum\limits_{i=1}^{n} a_i}{\sum\limits_{i=1}^{n} b_i} \right) \leq \sum_{i=1}^{n} a_i \log \left(\frac{a_i}{b_i} \right)$$

i.e.

$$\sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i \right) \log \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

So above all, with such construction, we have proved the inequality

$$\sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i \right) \log \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

2. [10 points] [Linear Methods for Classification] Consider the "Multi-class Logistic Regression" algorithm. Given training set $\mathcal{D} = \{(x^i, y^i) \mid i = 1, \dots, n\}$ where $x^i \in \mathbb{R}^{p+1}$ is the feature vector and $y^i \in \mathbb{R}^k$ is a one-hot binary vector indicating k classes. We want to find the parameter $\hat{\beta} = [\hat{\beta}_1, \dots, \hat{\beta}_k] \in \mathbb{R}^{(p+1)\times k}$ that maximize the likelihood for the training set. Introducing the softmax function, we assume our model has the form

$$p(y_c^i = 1 \mid x^i; \beta) = \frac{\exp(\beta_c^\top x^i)}{\sum_{c'} \exp(\beta_{c'}^\top x^i)},$$

where y_c^i is the c-th element of y^i.

(a) Complete the derivation of the conditional log likelihood for our model, which is

$$\ell(\beta) = \ln \prod_{i=1}^{n} p(y_t^i \mid x^i; \beta) = \sum_{i=1}^{n} \sum_{c=1}^{k} \left[y_c^i (\beta_c^\top x^i) - y_c^i \ln \left(\sum_{c'} \exp(\beta_{c'}^\top x^i) \right) \right].$$

For simplicity, we abbreviate $p(y_t^i = 1 \mid x^i; \beta)$ as $p(y_t^i \mid x^i; \beta)$, where t is the true class for x^i. [4 points]

(b) Derive the gradient of $\ell(\beta)$ w.r.t. β_1, i.e.,

$$\nabla_{\beta_1} \ell(\beta) = \nabla_{\beta_1} \sum_{i=1}^{n} \sum_{c=1}^{k} \left[y_c^i (\beta_c^\top x^i) - y_c^i \ln \left(\sum_{c'} \exp(\beta_{c'}^\top x^i) \right) \right].$$

Remark: Log likelihood is always concave; thus, we can optimize our model using gradient ascent. (The gradient of $\ell(\beta)$ w.r.t. $\beta_2, \dots, \beta_k$ is similar, you don't need to write them) [6 points]

Solution:

(a)

(b)

3. [10 points] [Probability and Estimation] Suppose $\mathcal{D} = \{x_1, x_2, \ldots, x_n\}$ are i.i.d. samples from exponential distribution with parameter $\lambda > 0$, i.e., $X \sim \text{Expo}(\lambda)$. Recall the PDF of exponential distribution is

$$p(x \mid \lambda) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

(a) To derive the posterior distribution of λ, we assume its prior distribution follows gamma distribution with parameters $\alpha, \beta > 0$, i.e., $\lambda \sim \text{Gamma}(\alpha, \beta)$ (since the range of gamma distribution is also $(0, +\infty)$, thus it's a plausible assumption). The PDF of λ is given by

$$p(\lambda \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\beta},$$

where $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt,\ \alpha > 0$. Show that the posterior distribution $p(\lambda \mid \mathcal{D})$ is also a gamma distribution and identify its parameters. Hints: Feel free to drop constants. [4 points]

(b) Derive the maximum a posterior (MAP) estimation for λ under $\text{Gamma}(\alpha, \beta)$ prior. [3 points]

(c) For exponential distribution $\text{Expo}(\lambda)$, $\sum_{i=1}^n x_i \sim \text{Gamma}(n, \lambda)$ and the inverse sample mean $\frac{n}{\sum_{i=1}^n x_i}$ is the MLE for λ. Argue that whether $\frac{n-1}{n}\hat{\lambda}_{MLE}$ is unbiased ($\mathbb{E}(\frac{n-1}{n}\hat{\lambda}_{MLE}) = \lambda$). Hints: $\Gamma(z+1) = z\Gamma(z)$, $z > 0$. [3 points]

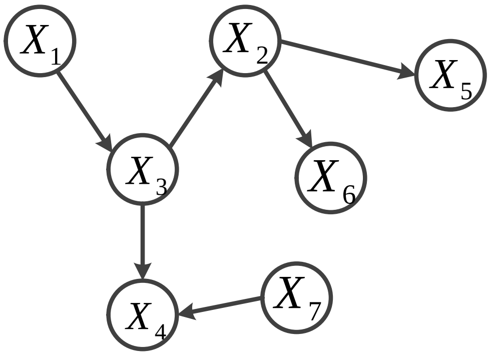
Solution:

(a)

(b)

(c)

4. [10 points] [Graphical Models] Given the following Bayesian Network,



answer the following questions.

(a) Factorize the joint distribution of $X_1, \cdots, X_7$ according to the given Bayesian Network. [2 points]

(b) Justify whether $X_1 \perp X_5 \mid X_2$? [2 points]

(c) Justify whether $X_5 \perp X_7 \mid X_3, X_4$? [2 points]

(d) Justify whether $X_5 \perp X_7 \mid X_4$? [2 points]

(e) Write down the variables that are in the Markov blanket of X_3. [2 points]

Solution:

(a)

(b)

(c)

(d)

(e)