



Divide and Conquer 2

CS240

Spring 2024

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Counting inversions

- Given a sequence a_1, \dots, a_n , the pair (a_i, a_j) is an **inversion** if $i < j$ but $a_i > a_j$.
- **Ex** In 5,2,6,1,4,3, there are 9 inversions.
- Can count number of inversions in $O(n^2)$ time.
- Use divide and conquer to solve in $O(n \log n)$ time.
- **Basic idea** Divide sequence in half.
 - Count the number of inversions in both halves.
 - Count number of inversions **between** the halves.



Counting inversions

- Let L and R be the left and right halves of a sequence.
- **Observation** No matter how we permute L and R , the number of inversions between L and R **stays the same**.
 - **Ex** There are 7 inversions between 5,2,6 and 1,3,4.
There are also 7 inversions between 2,5,6 and 1,3,4.
- Counting inversions between halves is easy if the halves are **sorted** in nondecreasing order.
 - Keep a pointer i for L , j for R , initially both 0.
 - If $L_i > R_j$, increment j .
 - Also increment number of inversions by $|L| - i$, because $L_k > R_j$ for all $k \geq i$, because L and R are sorted.
 - Otherwise increment i .
 - Just like merging L and R .
 - Takes $O(n)$ time, where $n = |L| + |R|$.

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.
- Conquer: recursively count inversions in each half.
- **Combine**: count inversions where a_i and a_j are in different halves, and return sum of three quantities.

1	5	4	8	10	2	6	9	12	11	3	7
---	---	---	---	----	---	---	---	----	----	---	---

Divide: $O(1)$.

1	5	4	8	10	2	6	9	12	11	3	7
---	---	---	---	----	---	---	---	----	----	---	---

5 blue-blue inversions

8 green-green inversions

Conquer: $2T(n / 2)$

9 blue-green inversions

5-3, 4-3, 8-6, 8-3, 8-7, 10-6, 10-9, 10-3, 10-7

Combine: ???

Total = $5 + 8 + 9 = 22$.

Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
- Combine two sorted halves into sorted whole.

$i = 6$



3	7	10	14	18	19
---	---	----	----	----	----



2	11	16	17	23	25
---	----	----	----	----	----

two sorted halves

--	--	--	--	--	--	--	--	--	--	--	--

auxiliary array

Total:

Merge and Count

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2	11	16	17	23	25
---	----	----	----	----	----

two sorted halves

6

2											
---	--	--	--	--	--	--	--	--	--	--	--

auxiliary array

Total: 6

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3	7	10	14	18	19
---	---	----	----	----	----



2	11	16	17	23	25
---	----	----	----	----	----

two sorted halves

6

2											
---	--	--	--	--	--	--	--	--	--	--	--

auxiliary array

Total: 6

Merge and Count

Merge and count step.

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- Combine two sorted halves into sorted whole.

$i = 6$



two sorted halves

6



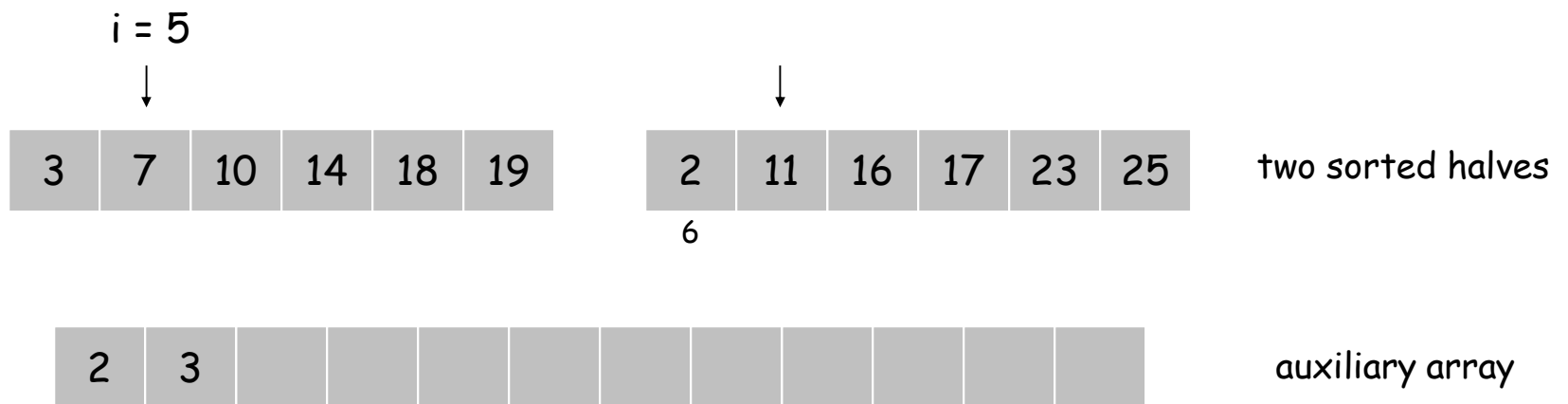
auxiliary array

Total: 6

Merge and Count

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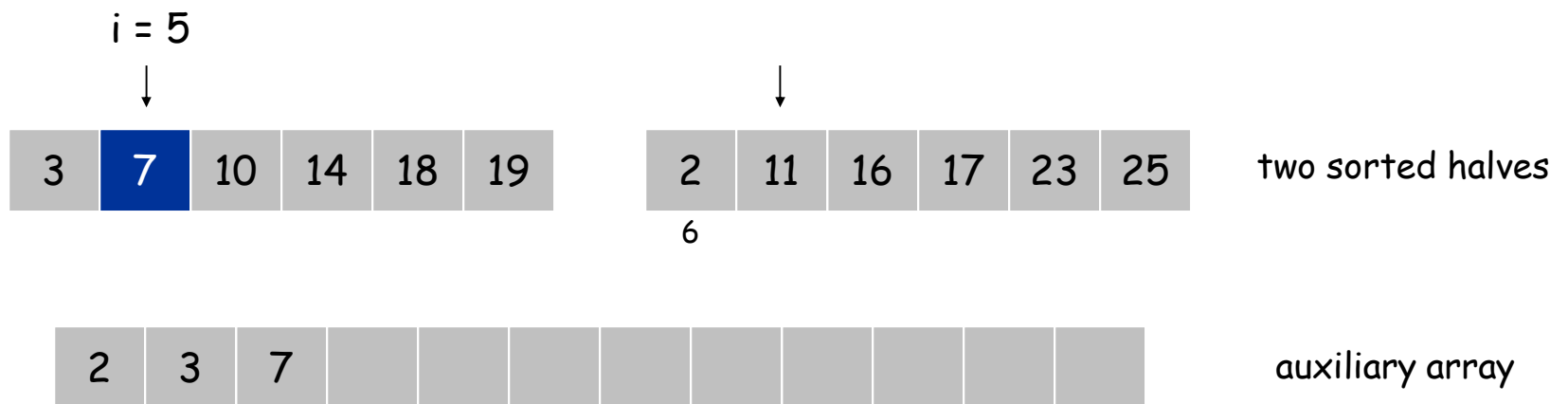


Total: 6

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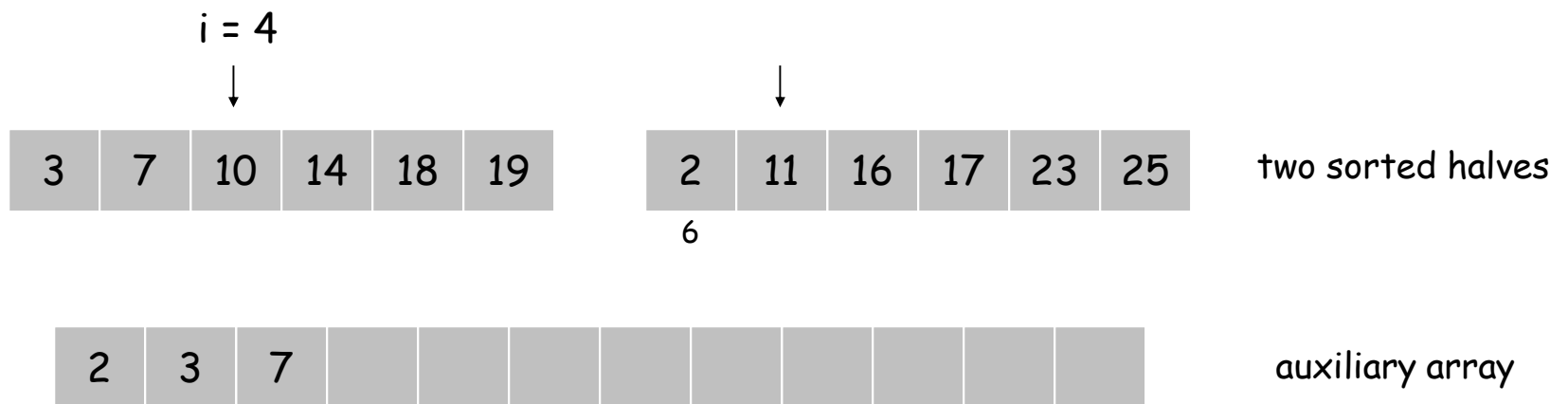


Total: 6

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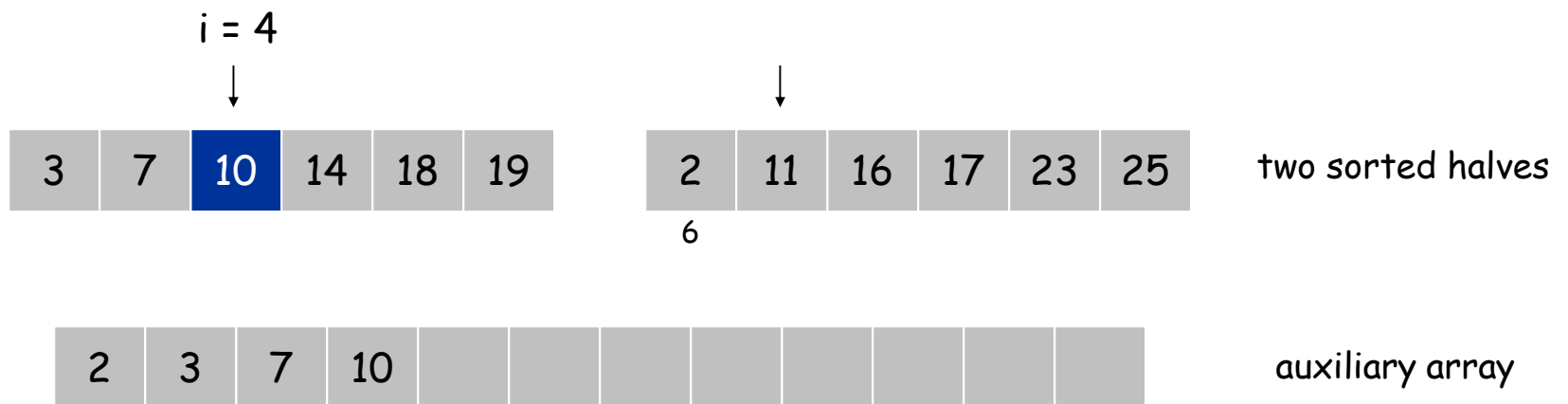


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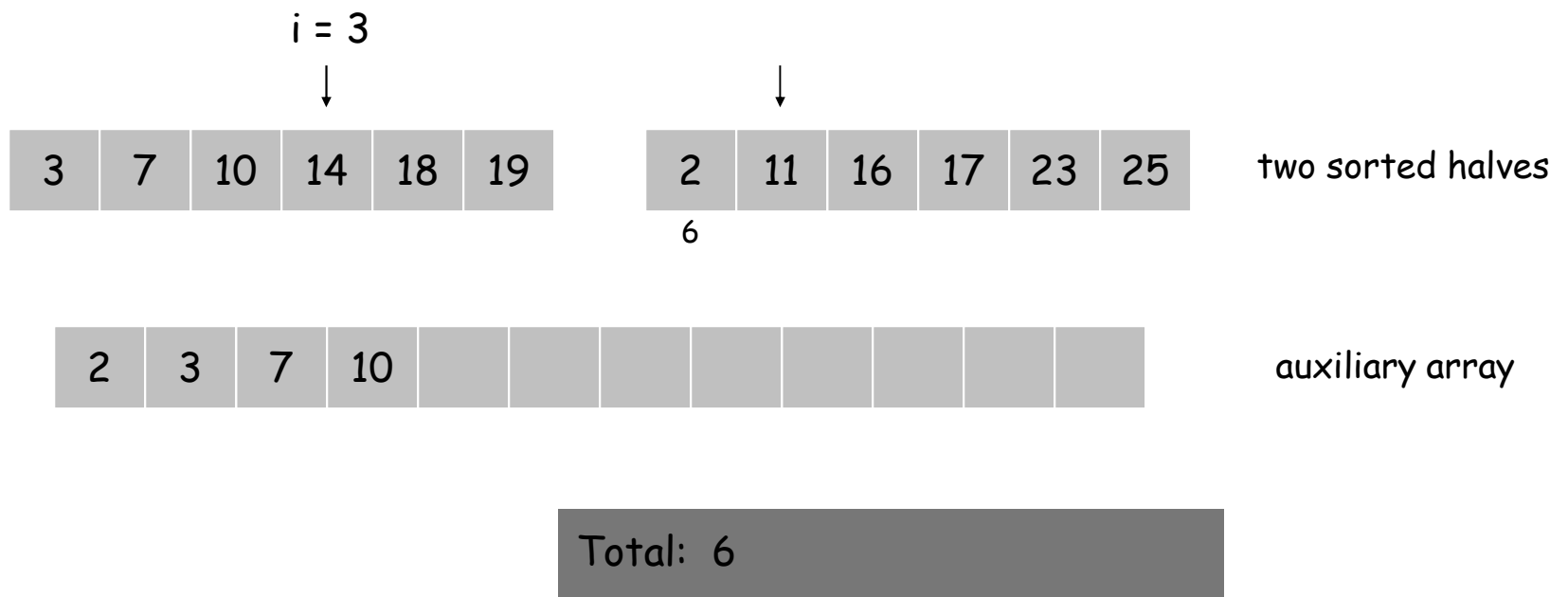


Total: 6

Merge and Count

Merge and count step.

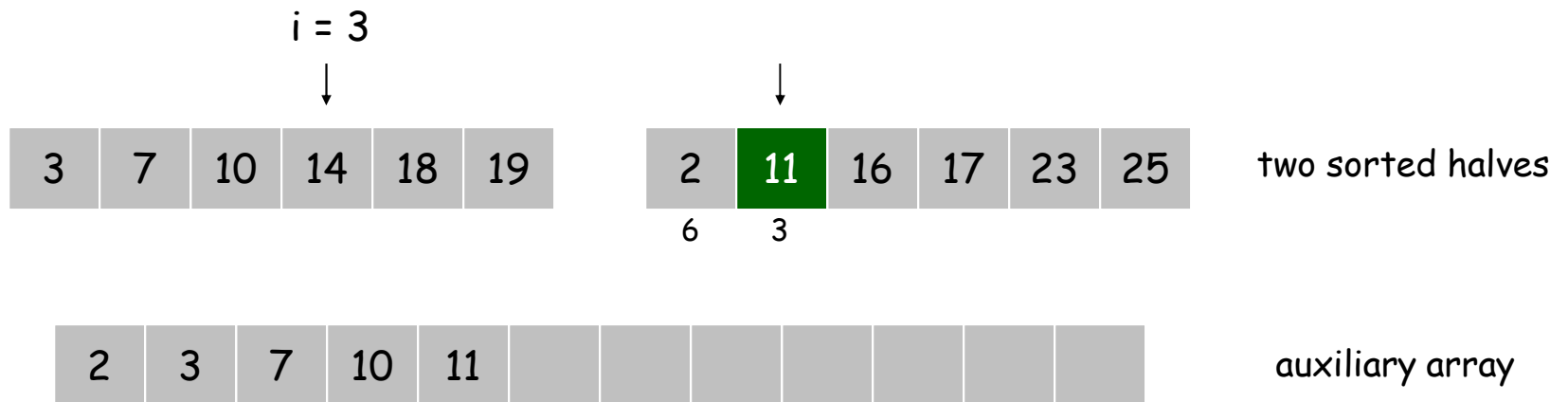
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Merge and Count

Merge and count step.

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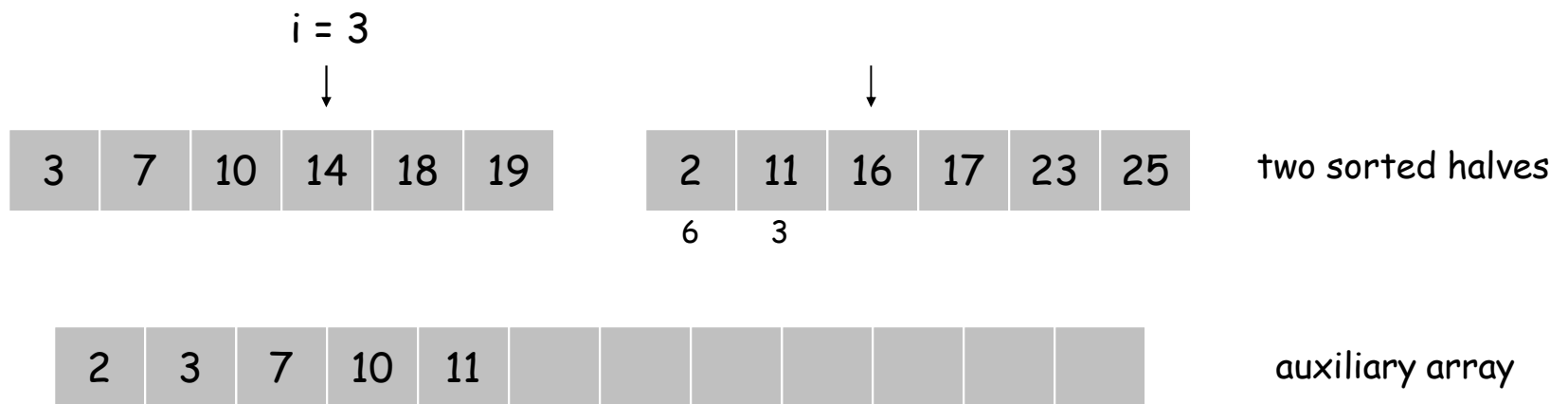


Total: 6 + 3

Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
- Combine two sorted halves into sorted whole.

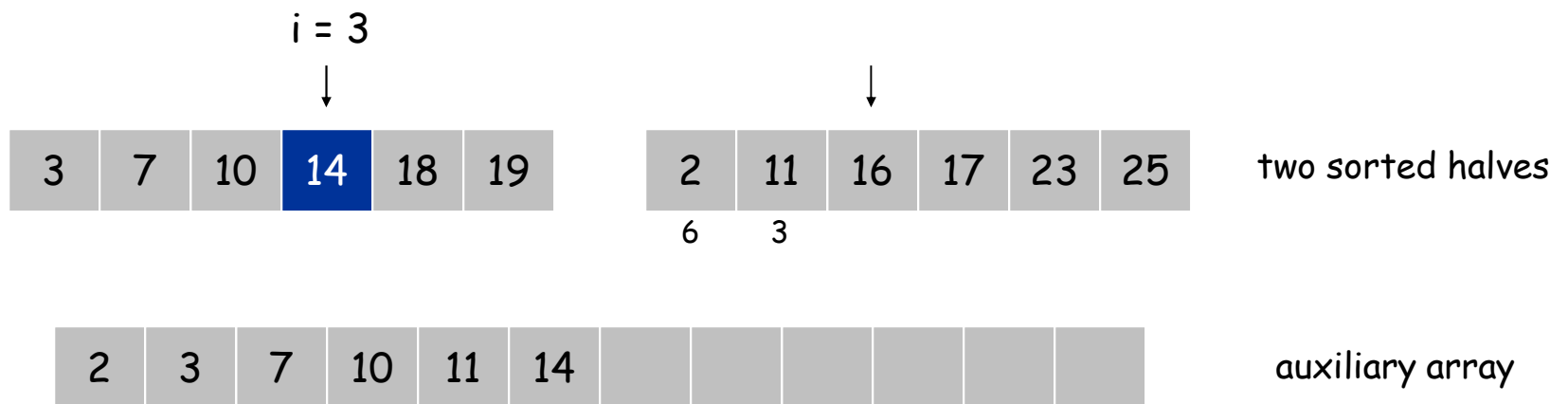


Total: 6 + 3

Merge and Count

Merge and count step.

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- Combine two sorted halves into sorted whole.

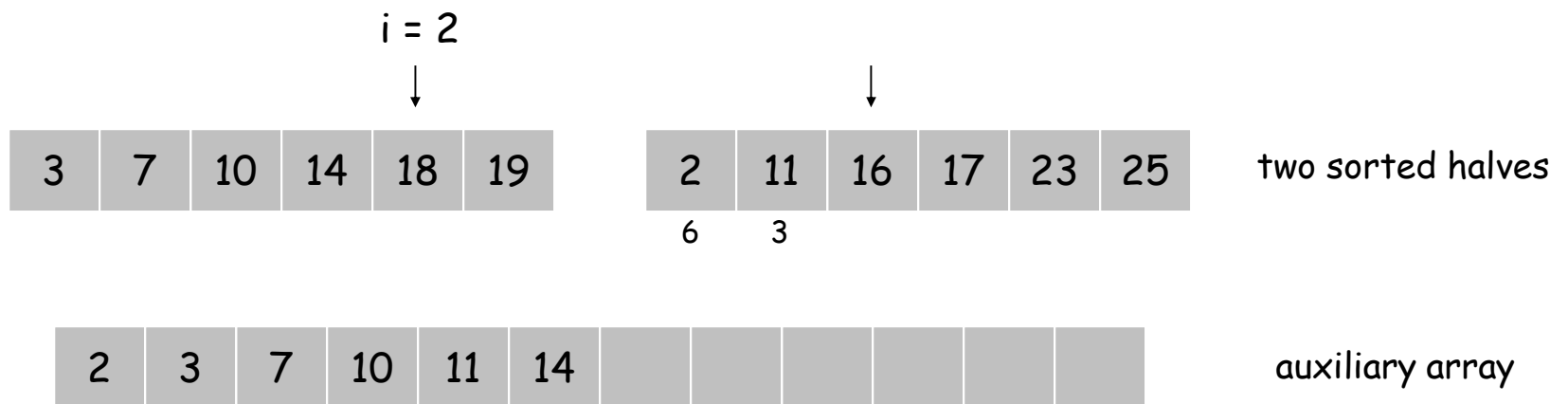


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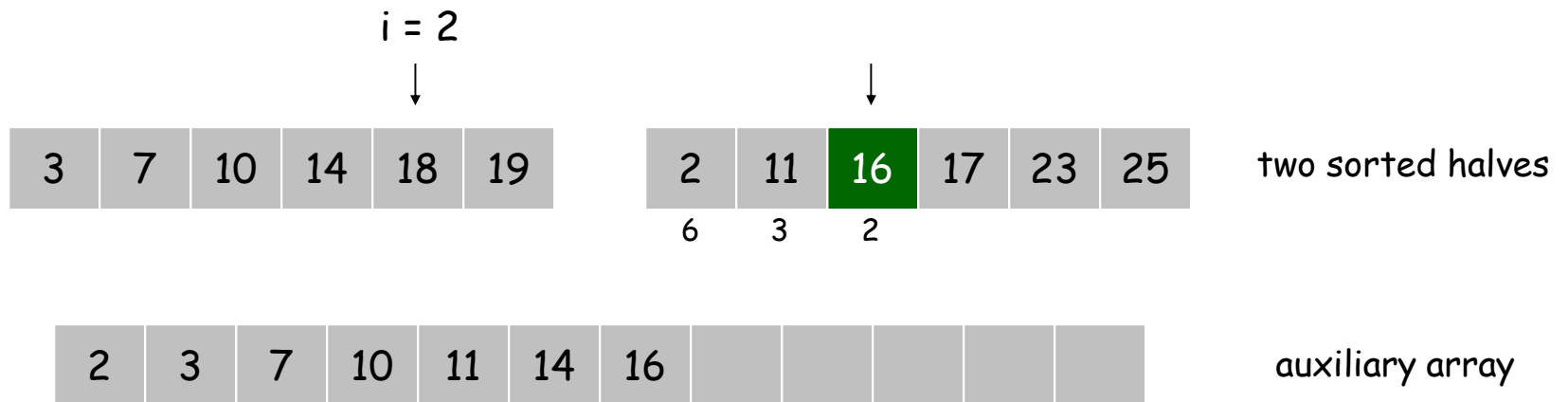


Total: 6 + 3

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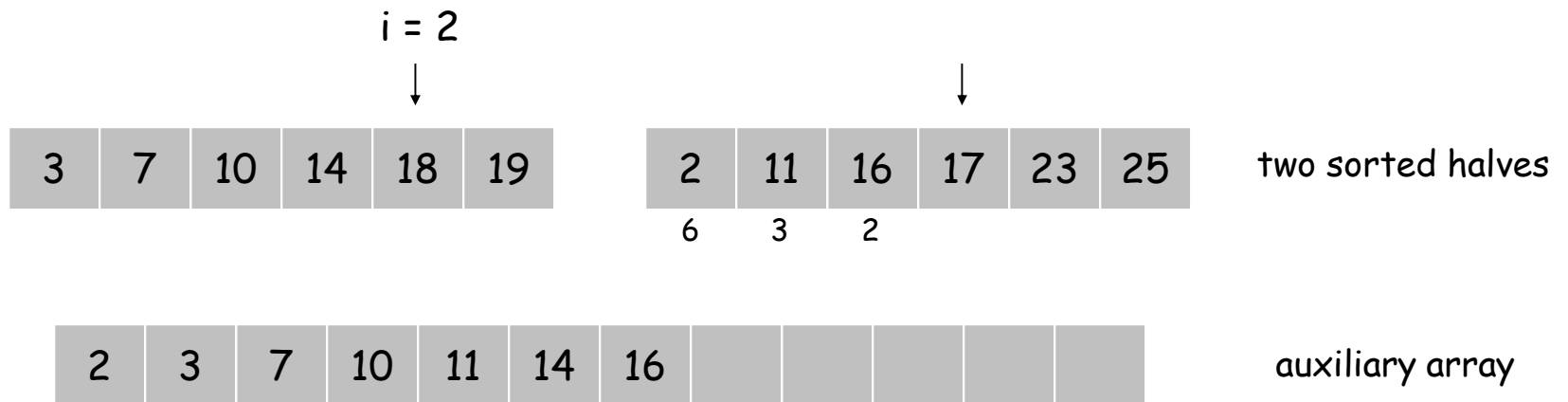


Total: $6 + 3 + 2$

Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
- Combine two sorted halves into sorted whole.

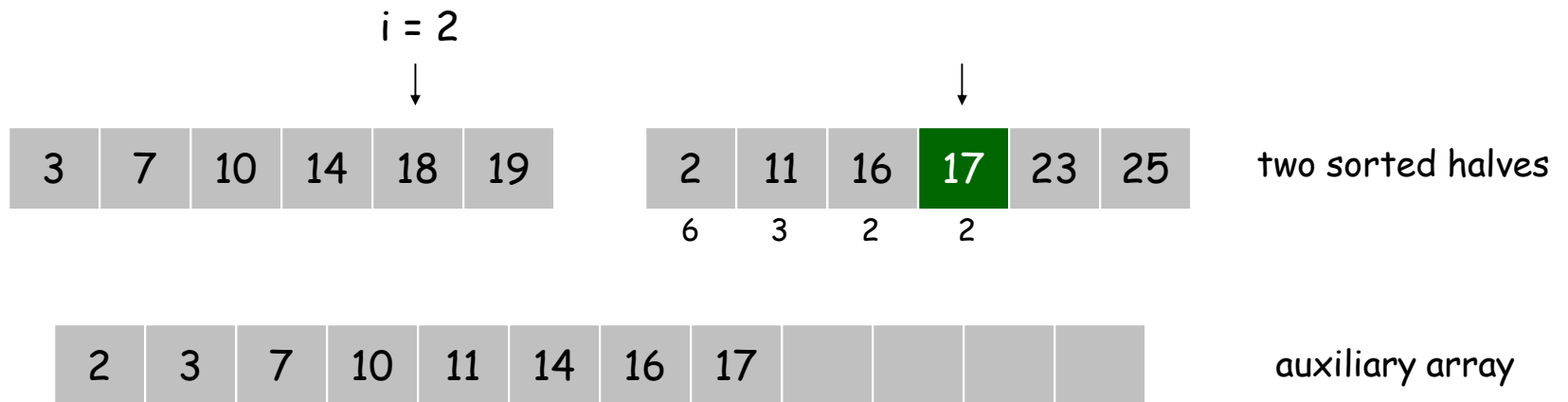


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Merge and Count

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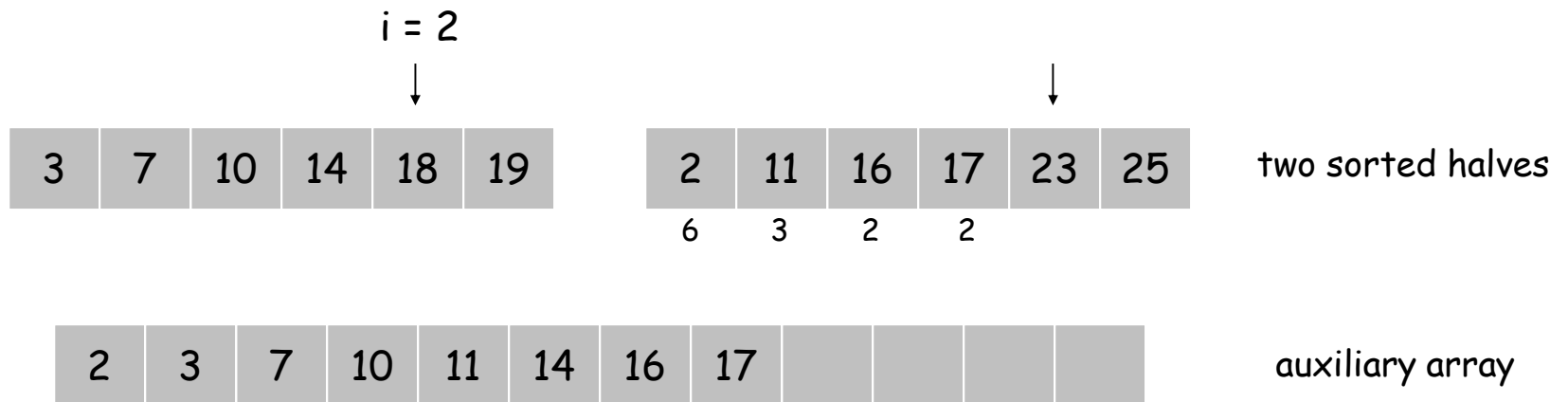


Total: $6 + 3 + 2 + 2$

Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
- Combine two sorted halves into sorted whole.

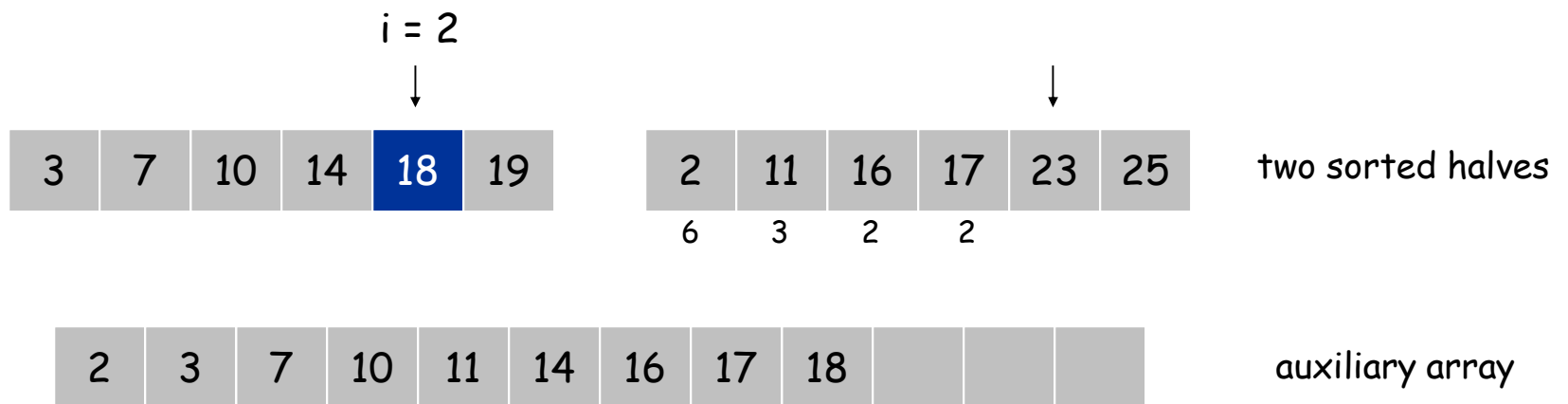


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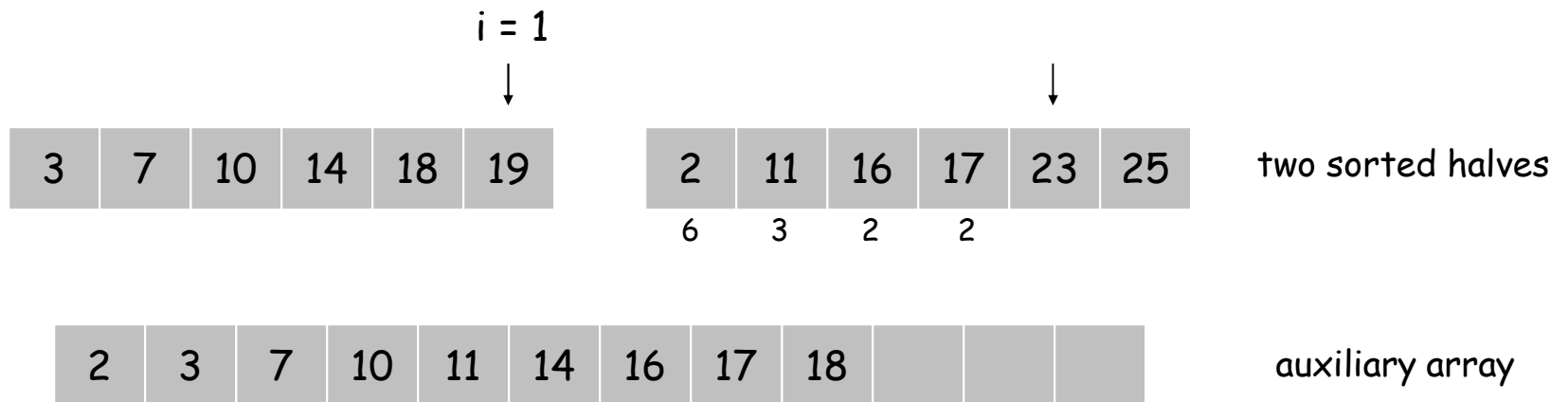


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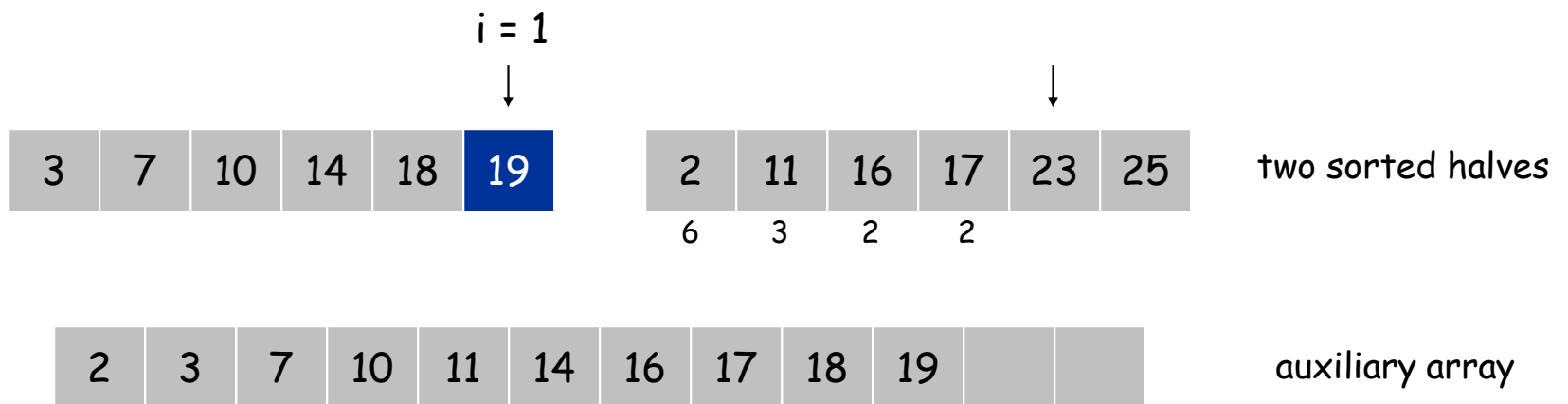


Total: $6 + 3 + 2 + 2$

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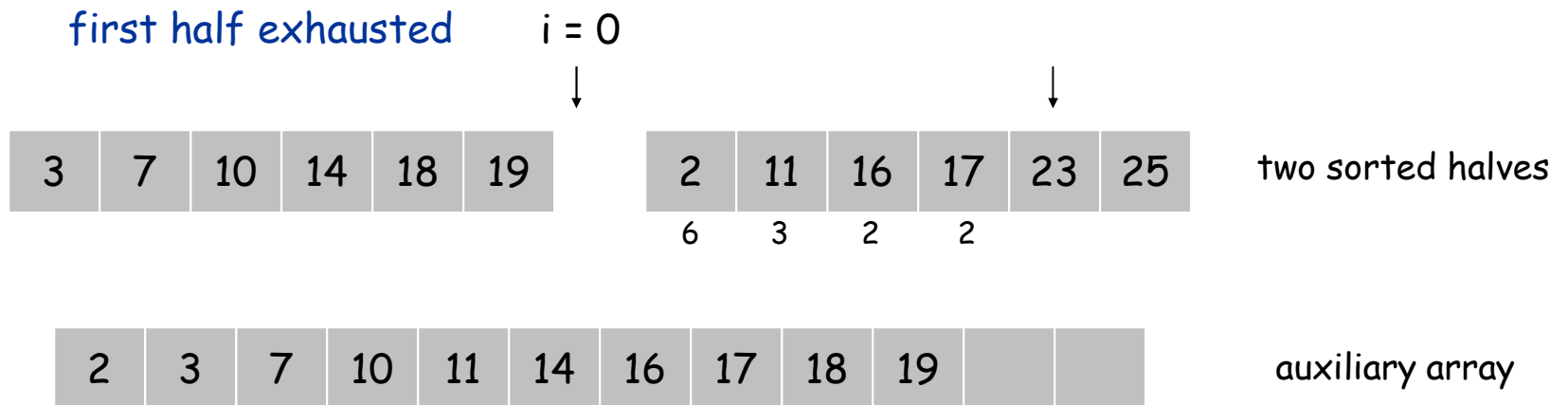


Total: $6 + 3 + 2 + 2$

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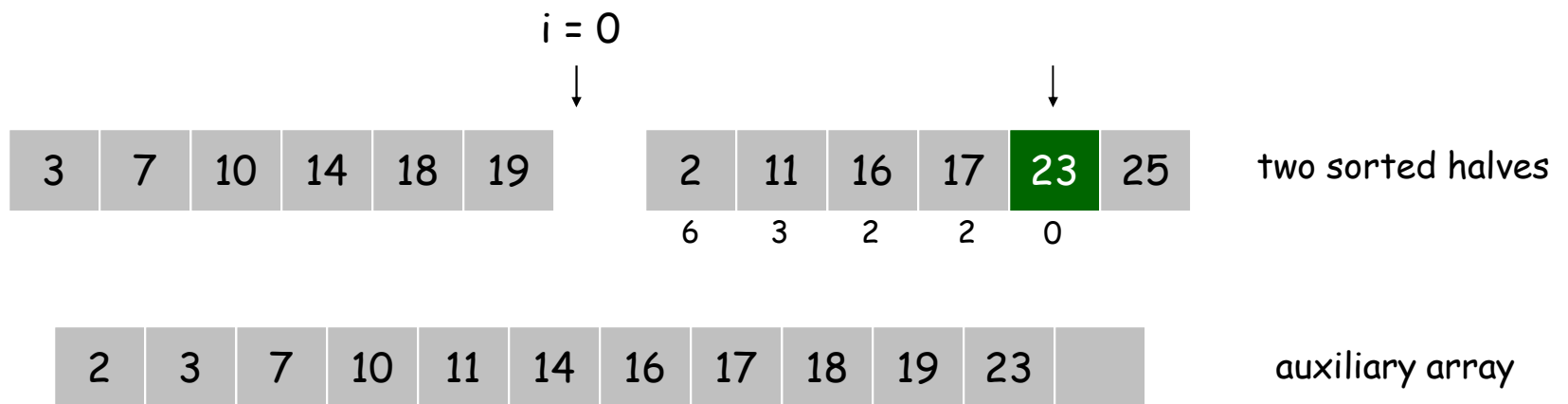


Total: $6 + 3 + 2 + 2$

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Merge and count step.

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- Combine two sorted halves into sorted whole.

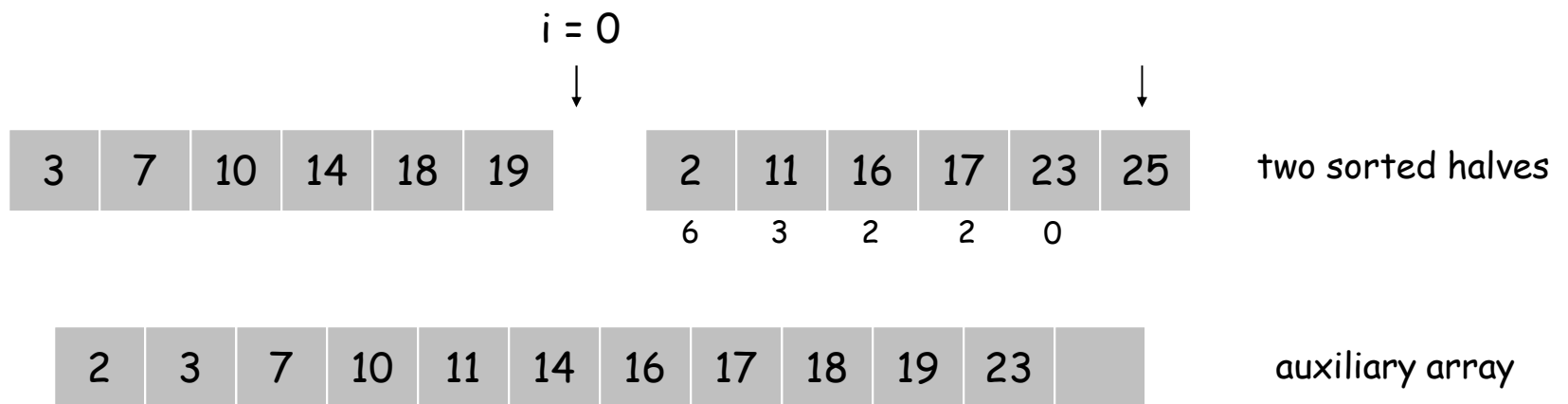


Total: $6 + 3 + 2 + 2 + 0$

Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
- Combine two sorted halves into sorted whole.

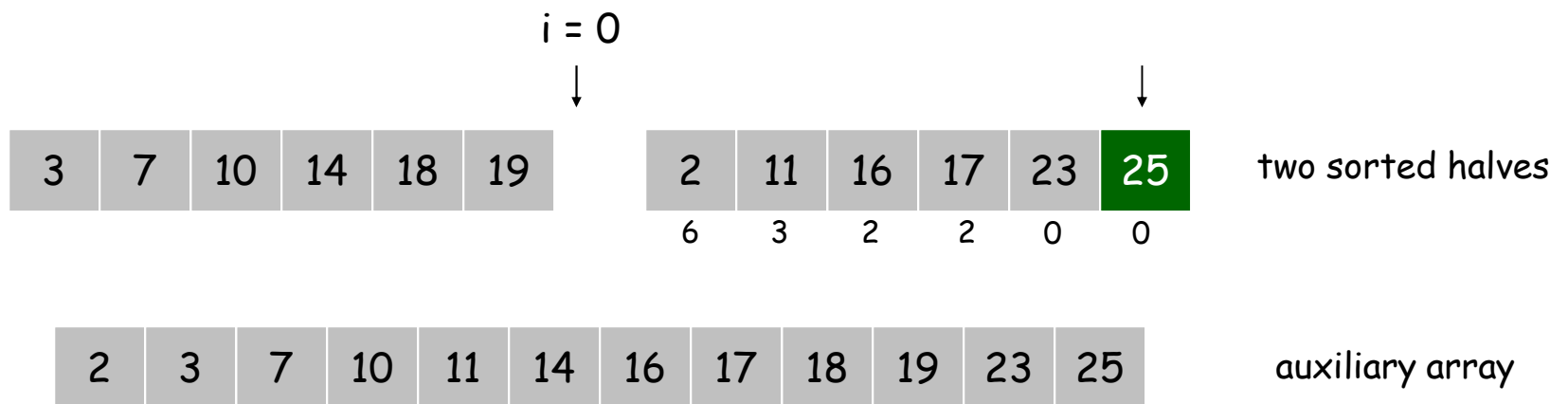


Total: $6 + 3 + 2 + 2 + 0$

Merge and Count

Merge and count step.

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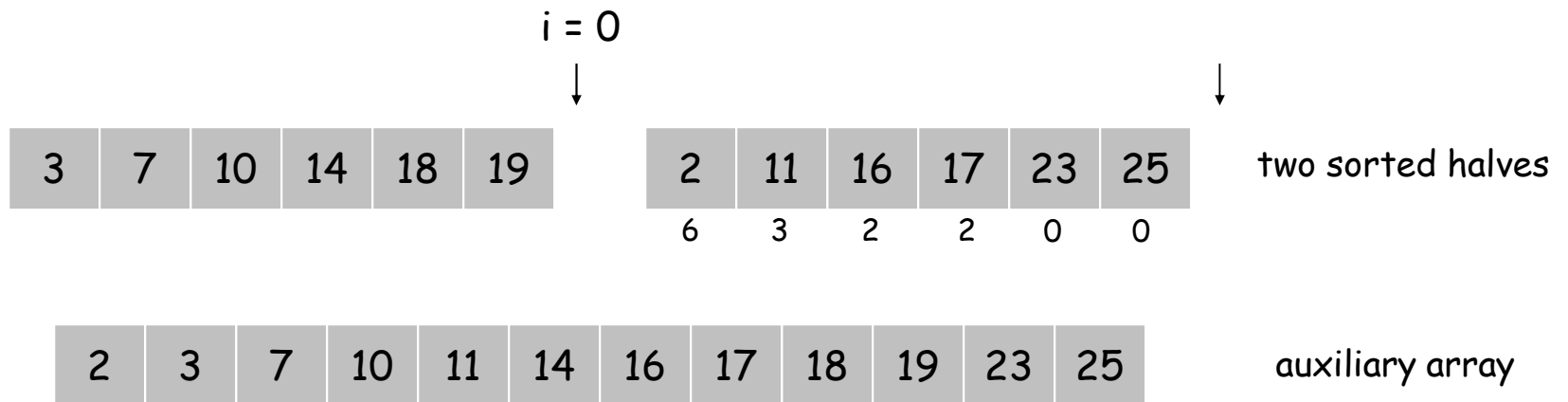


Total: $6 + 3 + 2 + 2 + 0 + 0$

Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
- Combine two sorted halves into sorted whole.



Total: $6 + 3 + 2 + 2 + 0 + 0 = 13$

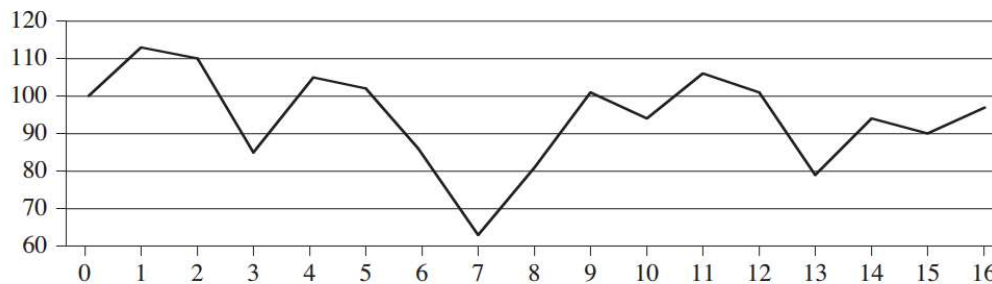


Counting inversions

- Let A be a sequence with left / right halves L / R .
- Function C counts the number of inversions in A , and returns A in sorted order.
 - Compute $x = C(L)$, $y = C(R)$.
 - After this, L and R are sorted.
 - Merge L and R , while counting number of inversions z .
 - Return $x+y+z$, and the merged sequence.
- Let T be the time complexity for C .
 - $T(n) = 2T\left(\frac{n}{2}\right) + O(n)$.
 - Thus, $T(n) = O(n \log n)$.

Maximum subarray

- **Motivation** Make money on stocks by buying and selling on days with largest price difference.

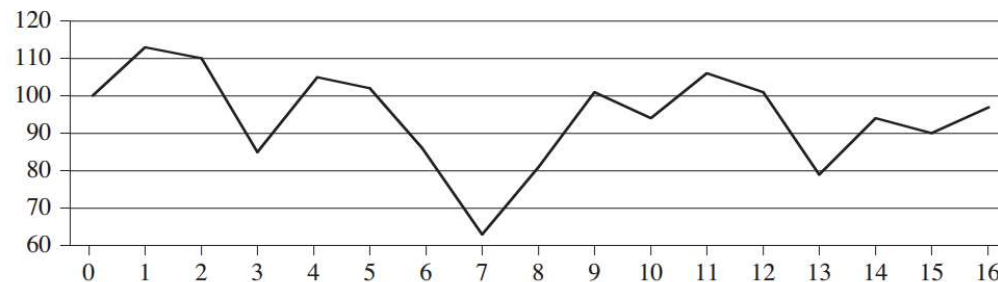


Day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Price	100	113	110	85	105	102	86	63	81	101	94	106	101	79	94	90	97
Change		13	-3	-25	20	-3	-16	-23	18	20	-7	12	-5	-22	15	-4	7

Source: *Introduction to Algorithms*
Cormen et al

- **Ex** Buy on day 7, sell on day 11, make \$106 - \$63 = \$43.
- If there are n days, can compute price difference of all $O(n^2)$ pairs of days and take the max.
- Is there a faster way?

Maximum subarray



Day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Price	100	113	110	85	105	102	86	63	81	101	94	106	101	79	94	90	97
Change		13	-3	-25	20	-3	-16	-23	18	20	-7	12	-5	-22	15	-4	7

- Let P be the array of stock prices.
- **Goal** Find $i < j$ such that $P[j] - P[i]$ is maximum.
- We first compute the price change on consecutive days.
 - **Ex** On day 4, the price change is $\$105 - \$85 = \$20$.
 - Call the array of changes A .
 - So $A[i] = P[i] - P[i - 1]$, for $i = 1, \dots, n$.

Maximum subarray

Day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Price	100	113	110	85	105	102	86	63	81	101	94	106	101	79	94	90	97

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A	13	-3	-25	20	-3	-16	-23	18	20	-7	12	-5	-22	15	-4	7

maximum subarray

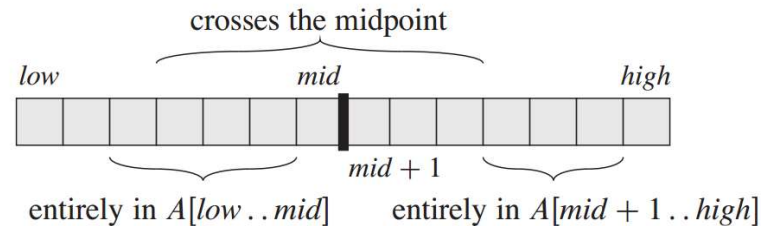
- **Observation** Finding $i < j$ with $\max P[j] - P[i]$ is the same as finding $i < j$ with $\max \sum_{k=i+1}^j A[k]$.
 - **Ex** $P[11] - P[7] = 43 = A[8] + A[9] + A[10] + A[11]$.
- Thus, we want to find a **subarray** of A with the maximum sum.
 - I.e. want to find a continuous set of elements of A with the largest sum.
 - **Ex** For A above, it's the 8th to 11th elements.



Maximum subarray

- **Goal** Given array A , find $i < j$ with max $\sum_{k=i+1}^j A[k]$.
- Seems no easier than initial problem... Still $O(n^2)$ pairs i, j to consider.
 - In fact, computing $\sum_{k=i+1}^j A[k]$ takes $O(n)$ time, so finding max subarray seems to take $O(n^3)$ time!
 - Actually, can find $\sum_{k=i+1}^j A[k]$ for all pairs i, j in $O(n^2)$ time. How?
- But with divide and conquer, can find max subarray in $O(n \log n)$ time.

A divide and conquer algorithm



Observation

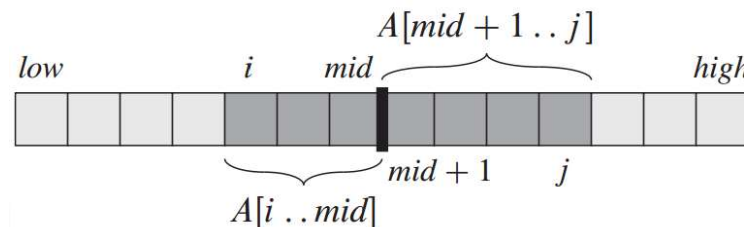
- Divide A down the middle. Then a max subarray of A either
 - Lies entirely in the left half.
 - Lies entirely in the right half.
 - Crosses the midpoint.

Algorithm

- Break A into left and right halves.
- Compute the max subarrays in each half.
- Compute the max subarray crossing the midpoint.
- Return max of these three subarrays.

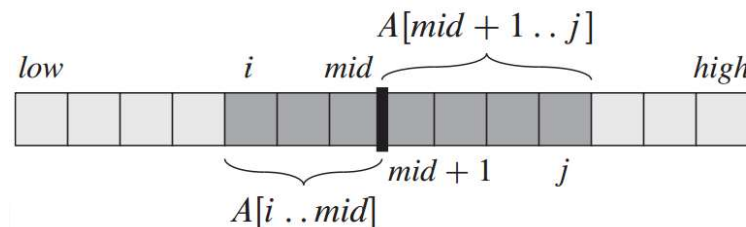
- **Analysis** $S(n) = 2S(n/2) + T(n) + O(1)$.
 - Finding max subarray in each half takes $S(n/2)$ time.
 - $T(n)$ = time to find max subarray crossing midpoint.
- We will show $T(n) = O(n)$.
- So $S(n) = O(n \log n)$.

Max crossing subarray



- **Goal** Find max subarray crossing the midpoint.
- **Solution** Find the max leftwards subarray from the midpoint.
 - I.e. find a subarray containing the midpoint and lying to the left, that has the max sum.
 - Also find the max rightwards subarray from the midpoint.
 - Combine them and return this.
- **Ex** $A = [3, 2, -8, 1, 6, 7, -4, 2, 8, 2, -4, 1, -2, 3, 1]$.
 - Max leftwards subarray from 2 is $[1, 6, 7, -4, 2]$.
 - Max rightwards subarray from 2 is $[2, 8, 2]$.
 - Max crossing subarray is $[1, 6, 7, -4, 2, 8, 2]$.

Max crossing subarray



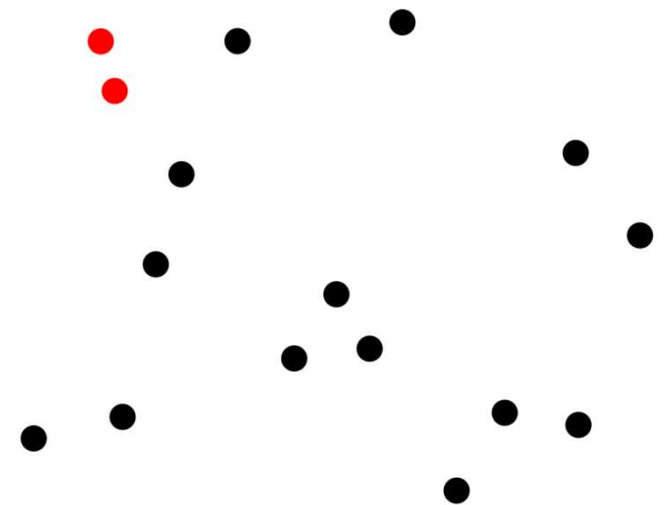
- **Algorithm** To find max leftwards subarray, sum array elements leftwards starting from midpoint.
 - Whenever sum exceeds current max, remember the index as the current max.
 - Similar for rightwards subarray.
- **Analysis** Scan through once to left and right. $O(n)$ time.

FIND-MAX-CROSSING-SUBARRAY($A, low, mid, high$)

```
1 left-sum =  $-\infty$ 
2 sum = 0
3 for i = mid downto low
4     sum = sum + A[i]
5     if sum > left-sum
6         left-sum = sum
7         max-left = i
8 right-sum =  $-\infty$ 
9 sum = 0
10 for j = mid + 1 to high
11     sum = sum + A[j]
12     if sum > right-sum
13         right-sum = sum
14         max-right = j
15 return (max-left, max-right, left-sum + right-sum)
```

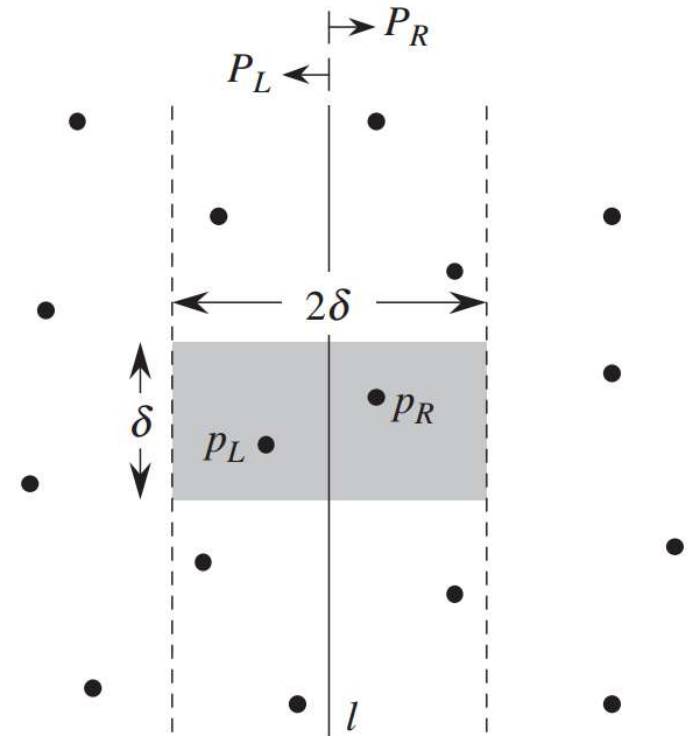
Closest point pair

- Given a set of n points in the plane, find the pair that's closest.
- Naive algorithm computes distances between all $O(n^2)$ pairs of points and chooses min.
- Use divide and conquer to improve complexity to $O(n \log n)$.



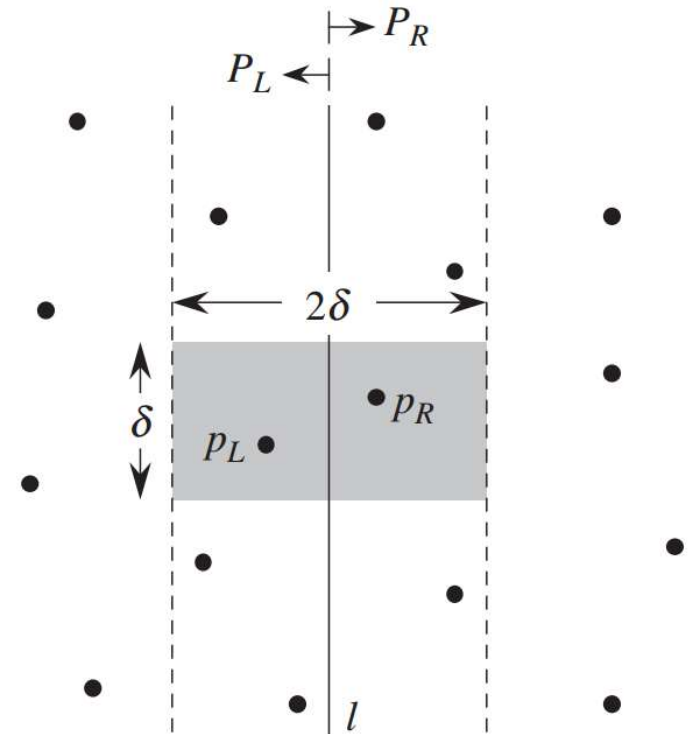
Closest point pair

- Split the points evenly using a vertical line, i.e. half the points lie on the left and half on the right.
- **Observation** The closest pair of points either
 - Both lie in the left half
 - Both lie in the right half, or
 - Straddles the line, i.e. one point on each side.
- This suggests the following algorithm.



Closest point pair

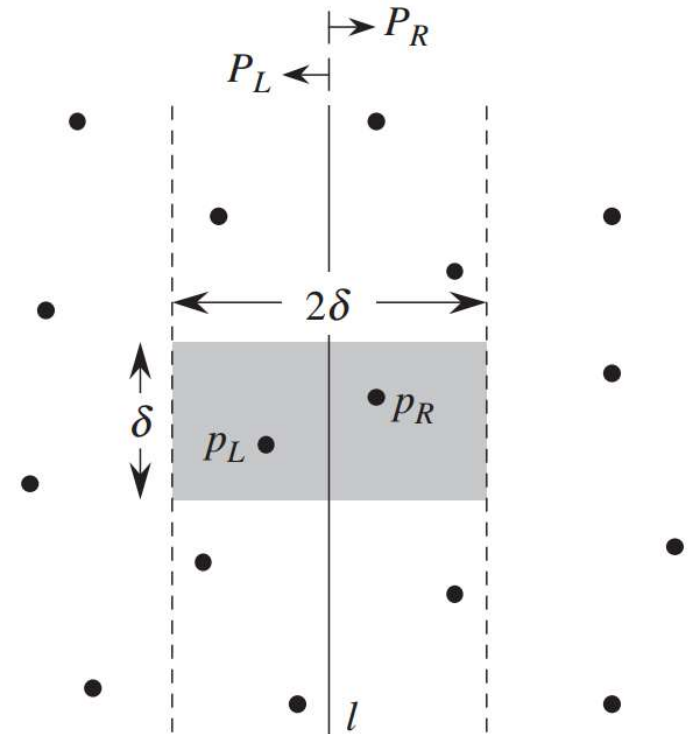
- Divide points evenly using vertical line.
- Recursively find closest point pair in left half and right half.
 - Let the **min distance** between any point pair in either half be δ .
- Look for closest pair of points straddling line with **distance** $< \delta$.
 - Don't need to consider straddling pairs with distance $\geq \delta$, since we already found such pairs on the left or right.
- If pair exists, return their distance.
- Else return δ .



Algorithm analysis

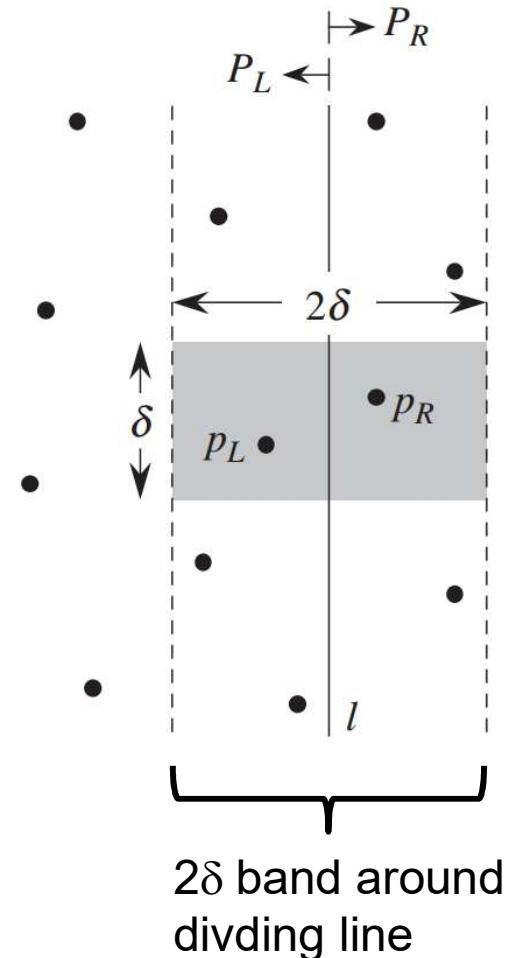
- Let $S(n)$ be time to find closest point pair of n points.
- $S(n) = 2S(n/2) + O(n)$
 - Can divide the points in $O(n)$ time.
 - Details on slide 46.
 - $2S(n/2)$ time to recursively find closest point pair in both halves.
 - Can find closest straddling pair in $O(n)$ time.
 - Details next slide.
- $S(2) = O(1)$.
 - If only two points, they're the closest pair.
- So $S(n) = O(n \log n)$.

- Divide the points evenly.
- Recursively find closest pair on left and right.
- Find closest straddling pair.
- Return the min of the three.



Closest straddling point pair

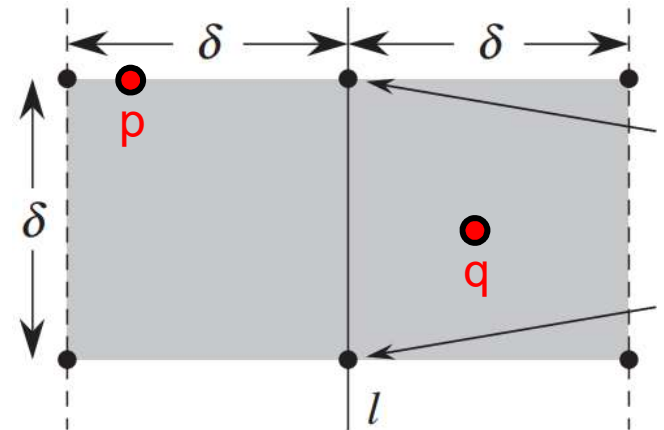
- **Goal** Find closest straddling pair, assuming their distance is $< \delta$.
- Only need to consider points within a band of **width 2δ** centered on dividing line.
 - Pairs outside band can't be closer than δ .
- Let B be set of points in band.
 - To form B, iterate through all points in any order, pick ones within distance δ from line.
 - Takes $O(n)$ time.
- Assume points in B **sorted** by y coordinate, i.e. from top to bottom.
 - By iterating in the right order when forming B, can get this property “for free”, without actually sorting B.
 - Details later.
- Now, use following lemma to find closest straddling pairs.



Sparsity lemma

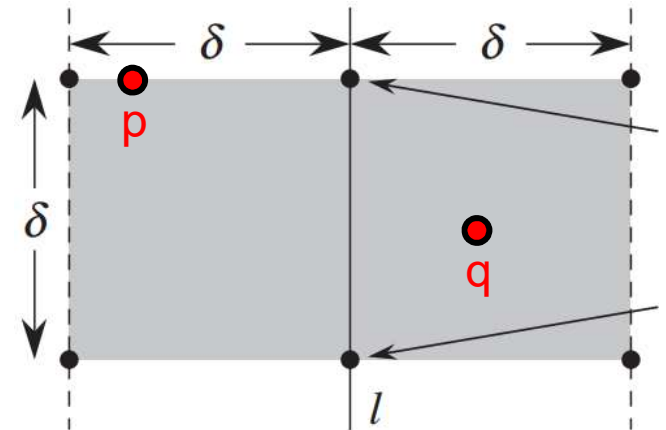
■ **Lemma** Let $p, q \in B$. Suppose q is below p , and has distance $< \delta$ from p . Then

1. q lies in a $\delta \times 2\delta$ rectangle centered on the dividing line, and with p on the top edge.
2. The rectangle contains at most 6 points from B (including p and q).
3. If we list the points in order B from top to bottom, the points in the rectangle immediately follow p in the ordering.



Sparsity lemma proof

1. Any point below the rectangle is $> \delta$ distance from p .
2. Any two points in rectangle on same side of the line are distance $\geq \delta$ apart.
 - ❖ Because δ is the min distance between any pair of points on either side.
 - ❖ So, at most 6 points in B fit in the rectangle.
 - ❖ **Ex** The 6 points can fit in the corners and the middle, as shown.
3. Points in the rectangle precede any points below it in y ordering.





Closest straddling point pair

- **Algorithm** Sweep through points in B from top to bottom.
 - For each point p, check next 5 points in R below it.
 - Let δ_p be distance to nearest one.
 - After sweeping through all points in B, return the minimum δ_p value or δ , whichever is smaller.
- **Correctness** By sparsity lemma, only next 5 points in B below p can be distance $< \delta$ from p.
 - Since we return the closest pair among these 5 points, we find overall closest straddling pair.
 - If no straddling pairs have distance $< \delta$, we return δ .
- **Analysis** Algorithm takes $O(n)$ time.
 - B contains $O(n)$ points.
 - For each point in B, check its distance to 5 other points.



Dividing points evenly

- At the beginning of the algorithm, sort all points horizontally and store in an array H .
 - Takes $O(n \log n)$ time.
- Assume at some level of recursion, input array is sorted horizontally.
- Then points to the left / right of dividing line are points in the first / second half of array.
 - Outputting either half takes $O(n)$ time.
- These points are sorted horizontally, for the next level of recursion.
 - So at every level of recursion, can get points in sorted order in $O(n)$ time.
- Add this $O(n \log n)$ preprocessing time to algorithm's running time.
 - Algorithm still $O(n \log n)$.



Sorting R points by y coordinate

- At the beginning of the algorithm, also sort all the points vertically. Store them in a separate array V .
 - Takes $O(n \log n)$ time.
- Points in H and V have pointers to each other.
 - I.e. given p in H , its pointer gives p 's index in V . Similarly given p in V , we can get p 's index in H .
- When picking out points left (or right) of dividing line using H , mark them in V by following the pointers.
- Next, iterate through V (in vertical order) and pick out marked points.
 - These points are again sorted vertically.
 - Takes $O(n)$ time.
- Add this $O(n \log n)$ preprocessing time to algorithm's running time. Algorithm still $O(n \log n)$.