ShanghaiTech University

CS240 Algorithm Design and Analysis

Spring 2024

Problem Set 3

Due: 23:59, April 30, 2024

1. Submit your solutions to the course Gradescope.
2. If you want to submit a handwritten version, scan it clearly.
3. Late homeworks submitted within 24 hours of the due date will be marked down 25%. Homeworks submitted more than 24 hours after the due date will not be accepted unless there is a valid reason, such as a medical or family emergency.
4. You are required to follow ShanghaiTech's academic honesty policies. You are allowed to discuss problems with other students, but you must write up your solutions by yourselves. You are not allowed to copy materials from other students or from online or published resources. Violating academic honesty can result in serious penalties.

Problem 1:

Given a network $G(V, E, s, t)$, give a polynomial time algorithm to determine whether G has a unique minimum $s - t$ cut.

Problem 2:

We are given an $n \times m$ array of cells. Denote the cell in the r'th row and the c'th column as (r, c), where $1 \leq r \leq n$ and $1 \leq c \leq m$. A robot is at cell $(1, 1)$ and wants to reach cell (n, m). From cell (x, y), the robot can only move to $(x, y+1)$ or $(x+1, y)$. In addition, some cells contain impassable obstacles and the robot cannot move to these cells. Your goal is to put additional obstacles in the minimum number of cells (other than cell (n, m)) such that the robot cannot reach (n, m). Find an algorithm based on max flow.

Problem 3:

Consider a restaurant with m different menu items. Each customer to the restaurant has a set of items they are willing to order. The restaurant makes d_i amount of item i, for $1 \leq i \leq m$, so that at most d_i customers can order item i. Given n customers, where the j'th customer has a list C_j of items they are willing to order, design an efficient algorithm to maximize the number of customers you can serve.

Problem 4:

Consider an $n \times n$ grid containing n rows and n columns of vertices. Each vertex (i, j) has edges to four neighbors $(i-1, j)$, $(i+1, j)$, $(i, j-1)$, $(i, j+1)$, except for the boundary vertices, which have edges to two or three neighbors. Each vertex and edge also has a positive integer capacity. We are given m start vertices, and for each vertex we want to find a path which connects the vertex to an arbitrary vertex on the grid's boundary. Furthermore, we need to ensure that the number of paths which pass through each vertex or edge does not exceed its capacity. Give an algorithm to determine whether this is possible.

Hint: First transform the graph to convert the vertex capacities to edge capacities.

Problem 5:

Given a set C, a collection D of subsets of C, and a value k, the k-SET-PACKING problem asks if there exist k subsets from D which are mutually disjoint, i.e. no two of the subsets share an element. Show that the SET-PACKING problem is in NP.

Problem 6:

Consider the following problem about scheduling courses. The problem is defined by 3 sets C, S and R. Here, C represents a set of courses, S repreesents the available time slots for the courses, and R contains a collection of course requests from students, where each request is a subset of C representing the the courses that a particular student wishes to take. The goal is to schedule all the courses without any conflicts.

For example, if $C = \{a, b, c, d\}, S = \{1, 2, 3\}, R = \{\{a, b, c\}, \{a, b, d\}, \{b, d\}\}$, then it is possible to schedule the courses without conflicts by assigning course a to time slot 1, b to slot 2 and c, d to slot 3. However, if $C = \{a, b, c, d\}, S = \{1, 2, 3\}, R = \{\{a, b, c\}, \{a, c\}, \{b, c, d\}\}$, then it is not possible to schedule the courses without any conflicts. Prove that the problem of determining whether a conflict-free schedule exists is NP-complete.

Hint: Use a reduction from the 3-COLOR problem.