Divide and conquer 3
Linear time sorting

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Polynomial multiplication

- Let $A(x) = \sum_{j=0}^{n-1} a_j x^j$ and $B(x) = \sum_{j=0}^{n-1} b_j x^j$ be two polynomials.
- Compute the product $C(x) = A(x)B(x)$.

$$
\begin{array}{rrrrrrrl}
 & & & 6x^3 & +\ 7x^2 & -\ 10x & +\ 9 & \quad A(x) \\
 & & & -\ 2x^3 & & +\ 4x & -\ 5 & \quad B(x) \\
\hline
 & & & -\ 30x^3 & -\ 35x^2 & +\ 50x & -\ 45 & \\
 & & 24x^4 & +\ 28x^3 & -\ 40x^2 & +\ 36x & & \\
-\ 12x^6 & -\ 14x^5 & +\ 20x^4 & -\ 18x^3 & & & & \\
\hline
-\ 12x^6 & -\ 14x^5 & +\ 44x^4 & -\ 20x^3 & -\ 75x^2 & +\ 86x & -\ 45 &
\end{array}
$$

- $C(x)$'s degree is at most 2n-2.
- $C(x) = \sum_{j=0}^{2n-2} c_j x^j$, where $c_j = \sum_{k=0}^{j} a_k b_{j-k}$.
- The naive method takes $O(n^2)$ time.
- Using FFT and divide & conquer, we'll do it in $O(n \log n)$ time.

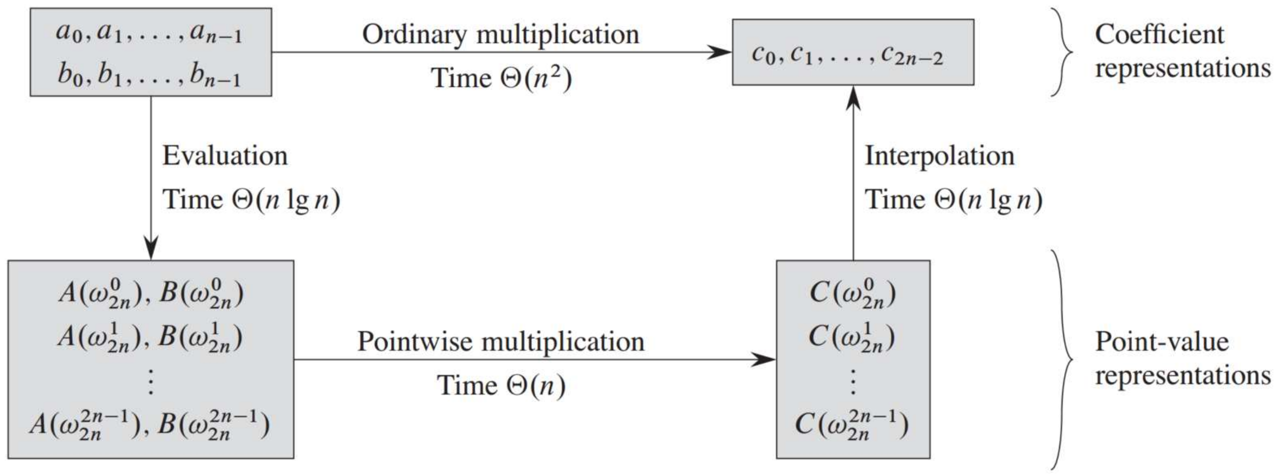
Polynomial representations

- The coefficient representation of a polynomial is $A(x) = \sum_{j=0}^{n-1} a_j x^j$.
- The point-value representation is $\{(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1})\}$, where $y_k = A(x_k)$.
- If we have $A(x), B(x)$ in point-value form, computing $C(x) = A(x)B(x)$ can be done in $O(n)$ time.
  - Pick $2n$ points $x_0, \dots, x_{2n-1}$. Then $C(x_k) = A(x_k)B(x_k)$ for $k = 0, \dots, 2n-1$.
  - In coefficient form, naive multiplication takes $O(n^2)$ time.
- Given $A(x)$ represented in point-value form, we can reconstruct the coefficient representation.

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

- Matrix V is the Vandermonde matrix. It's nonsingular and invertible.
  - So $a = V^{-1}y$.
  - Can compute a more efficiently than general matrix inversion and multiplication.

Fast polynomial multiplication

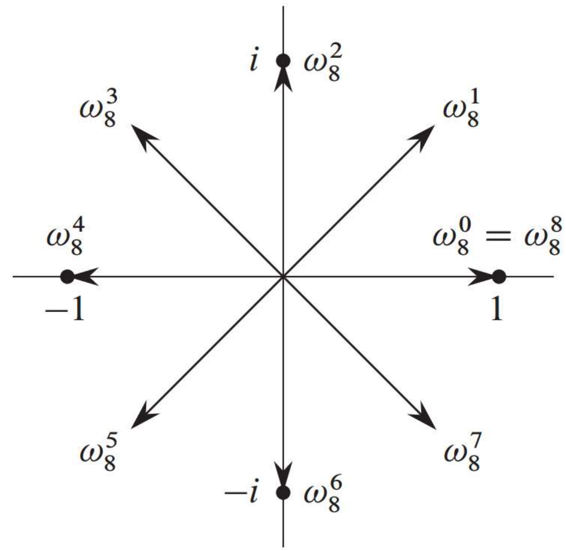


Source: Introduction to Algorithms, Cormen et al

- We want to multiply polynomials A and B quickly.
- Evaluate A and B at $2n$ points to get point-value representations.
  - Evaluation can be at any points. But we'll do it at the $2n$'th roots of unity.
  - Evaluating A and B at $2n$'th roots of unity by FFT takes $O(n \log n)$ time.
- Pointwise multiply A and B to get point-value representation for C.
  - This takes $O(n)$ time.
- Transform point-value representation of C to coefficient form.
  - This is done using inverse FFT in $O(n \log n)$ time.

Roots of unity

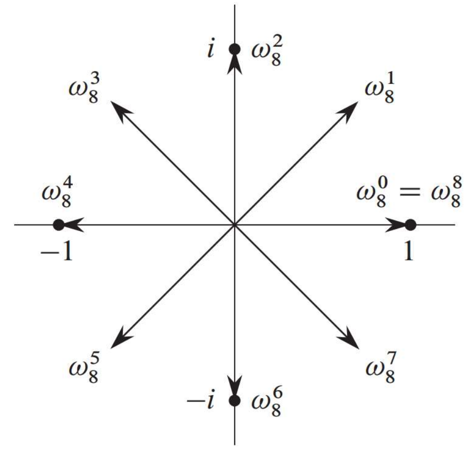
- An n'th root of unity is a complex number ω s.t. $\omega^n = 1$.
- There are n n'th roots of unity, and they have the form $e^{\frac{2\pi i k}{n}}$, for $0 \leq k \leq n-1$.
  - Write $\omega_n = e^{\frac{2\pi i}{n}}$, so that the n'th roots of unity are $\omega_n^0, \omega_n^1, \ldots, \omega_n^{n-1}$.
- Below, assume n is a power of 2.
- Fact 1 $\left(\omega_n^k\right)^n = 1$ for any $0 \leq k \leq n-1$.
- Fact 2 $\omega_n^{k+\frac{n}{2}} = -\omega_n^k$.
- Fact 3 $\left(\omega_n^k\right)^2 = \omega_n^{2k} = \omega_{n/2}^k$.
- Fact 4 For any $0 \leq k \leq n-1$, $\sum_{j=0}^{n-1}\left(\omega_n^k\right)^j = 0$.



Discrete Fourier Transform

- Given a degree $n-1$ polynomial $A(x)=a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n-1}$, DFT computes $A(\omega_n^0), A(\omega_n^1), \ldots, A(\omega_n^{n-1})$.
- FFT is a fast algorithm for DFT that runs in $O(n\log n)$ time using divide and conquer.
  - Assume n is a power of 2.
- FFT has a vast number of applications in CS and EE.
- One of IEEE's top 10 most important algorithms of the 20th century.
  - Popularized by Cooley and Tukey in 1965.
  - But variants known to Gauss in 1805!

Fast Fourier Transform



- Let $A^{[0]} = a_0 + a_2x^1 + a_4x^2 + \cdots + a_{n-2}x^{\frac{n}{2}-1}$

  $A^{[1]} = a_1 + a_3x^1 + a_5x^2 + \cdots + a_{n-1}x^{\frac{n}{2}-1}$.
- Then $A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2)$.
- So can compute $A(\omega_n^0), A(\omega_n^1), \dots, A(\omega_n^{n-1})$ by computing $A^{[0]}$ and $A^{[1]}$ at $(\omega_n^0)^2, (\omega_n^1)^2, \dots, (\omega_n^{n-1})^2$, and multiplying some terms by $\omega_n^0, \omega_n^1, \dots, \omega_n^{n-1}$.
- But $\left(\omega_n^{k+n/2}\right)^2 = \omega_n^{2k+n} = \omega_n^{2k} = \left(\omega_n^k\right)^2$ by Fact 1.
  - So $\{(\omega_n^0)^2, (\omega_n^1)^2, \dots, (\omega_n^{n-1})^2\} = \left\{(\omega_n^0)^2, (\omega_n^1)^2, \dots, \left(\omega_n^{n/2-1}\right)^2\right\}$, i.e. only need to evaluate $A^{[0]}$ and $A^{[1]}$ at $n/2$ points instead of n!
- Also, $\left(\omega_n^k\right)^2 = \omega_n^{2k} = \omega_{n/2}^k$.
  - So $\left\{(\omega_n^0)^2, (\omega_n^1)^2, \dots, \left(\omega_n^{n/2-1}\right)^2\right\} = \left\{\omega_{n/2}^0, \omega_{n/2}^1, \dots, \omega_{n/2}^{n/2-1}\right\}$, i.e. only need to evaluate $A^{[0]}$ and $A^{[1]}$ at the $(n/2)$'th roots of unity.

Fast Fourier Transform

- Thus, computing $A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2)$ for $x \in \{\omega_n^0, \omega_n^1, \ldots, \omega_n^{n-1}\}$ requires
  - Computing for $A^{[0]}(x)$ and $A^{[1]}(x)$ for $x \in \left\{\omega_{n/2}^0, \omega_{n/2}^1, \ldots, \omega_{n/2}^{n/2-1}\right\}$.
  - These are also DFT's, so can be done recursively using two $n/2$-point FFT's.
- For $0 \leq k \leq \frac{n}{2} - 1$
  - $A\left(\omega_n^k\right) = A^{[0]}\left(\omega_{n/2}^k\right) + \omega_n^k \, A^{[1]}\left(\omega_{n/2}^k\right)$
  - $A\left(\omega_n^{k+n/2}\right) = A^{[0]}\left(\omega_{n/2}^{k+n/2}\right) + \omega_n^{k+n/2} \, A^{[1]}\left(\omega_{n/2}^{k+n/2}\right)$
    $= A^{[0]}\left(\omega_{n/2}^k\right) - \omega_n^k \, A^{[1]}\left(\omega_{n/2}^k\right)$

FFT algorithm

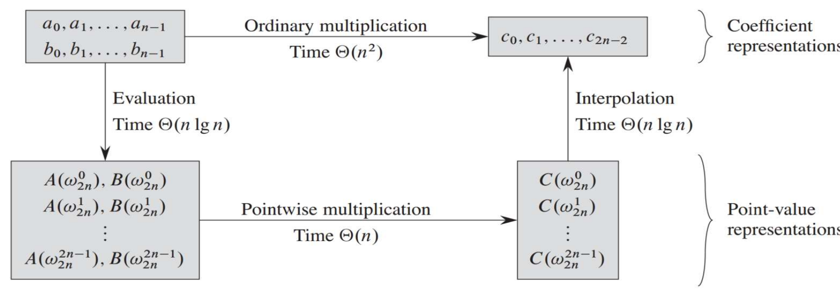
```
FFT(n, a_0,a_1,…,a_{n-1}) {
   if (n == 1) return a_0

   (e_0,e_1,…,e_{n/2-1}) ← FFT(n/2, a_0,a_2,a_4,…,a_{n-2})
   (d_0,d_1,…,d_{n/2-1}) ← FFT(n/2, a_1,a_3,a_5,…,a_{n-1})

   for k = 0 to n/2 - 1 {
      ω^k ← e^{2πik/n}
      y_k ← e_k + ω^k d_k
      y_{k+n/2} ← e_k - ω^k d_k
   }
   return (y_0,y_1,…,y_{n-1})
}
```

- Let $T(n)$ be the time to compute a size n FFT. Then $T(n) = 2T\left(\frac{n}{2}\right) + O(n)$, so $T(n) = \Theta(n \log n)$.

Inverse FFT



- So far we have
  - Computed A, B at the $2n$'th roots of unity.
  - Pointwise multiplied A, B to get point-value representation of C.
- To multiply A and B, the last step is to convert C back to coefficient representation using inverse FFT.
- Inverse FFT takes a polynomial represented in point-value form and computes the polynomial's coefficient form.
  - I.e. given $[y_0\ y_1 \ldots y_{n-1}]^T$ below, it computes $[a_0\ a_1\ \ldots\ a_{n-1}]^T$.

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

- For the polynomial multiplication problem, the a vector represents C.
  - $a = V_n^{-1} y$.
- Need to know the inverse of Vandermonde matrix V_n.

Inverse FFT

- Thm For $j, k = 0, \dots, n-1$, the (j, k) entry of V_n^{-1} is ω_n^{-kj}/n.

$$V_n^{-1} = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

- Proof

$$[V_n^{-1} V_n]_{jj'} = \sum_{k=0}^{n-1} \left(\frac{\omega_n^{-kj}}{n}\right)\left(\omega_n^{kj'}\right) = \frac{1}{n}\sum_{k=0}^{n-1} \omega_n^{k(j'-j)}$$

- By Fact 4, (j, j') entry is 1 exactly when $j = j'$, and 0 otherwise. So $V_n^{-1} V_n$ is the identity matrix.

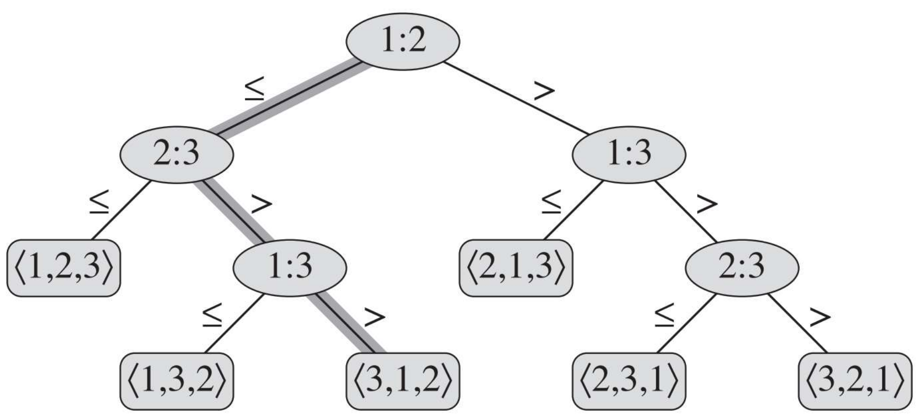
Inverse FFT

$$V_n^{-1} = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

- Since $[a_0\ a_1\ \ldots a_{n-1}]^T = V_n^{-1}[y_0\ y_1\ \ldots y_{n-1}]$, we have $a_j = \frac{1}{n}\sum_{k=0}^{n-1} y_k \omega_n^{-kj}$.
- Consider the degree $n-1$ polynomial $Y(x) = \frac{1}{n}(y_0 + y_1 x + y_2 x^2 + \ldots + y_{n-1}x^{n-1})$.
  - We want to compute Y at $1, \omega_n^{-1}, \omega_n^{-2}, \ldots, \omega_n^{-n}$.
- This is just another DFT, which we can do in $O(n \log n)$ time.
- Thus, we can multiply two degree $n-1$, or more generally, do $n-1$ point convolution in $O(n \log n)$ time.

Comparison sorts

- In insertion sort, mergesort and Quicksort, the output order only depended on comparisons between the input values.
  - Ex In insertion sort, we compare a value with other values to determine its sorted position.
  - These algorithms are comparison sorts.
  - They all take $\Omega(n \log n)$ time to sort n inputs.
- There is a general $\Omega(n \log n)$ lower bound on the time complexity of any comparison sort algorithm.
  - Any algorithm in which the output is only determined by comparisons between input values takes at least $\Omega(n \log n)$ steps.



Beyond comparison sorts

- To sort faster than $\Omega(n \log n)$ time, we need to use other operations besides comparison.
- An algorithm can sort in $O(n)$ linear time by reading the value of inputs, and using these for array indexing, comparing digits, etc.
- $O(n)$ time is asymptotically the best possible, since we need to read all n inputs.
- We'll look at counting and radix sort.

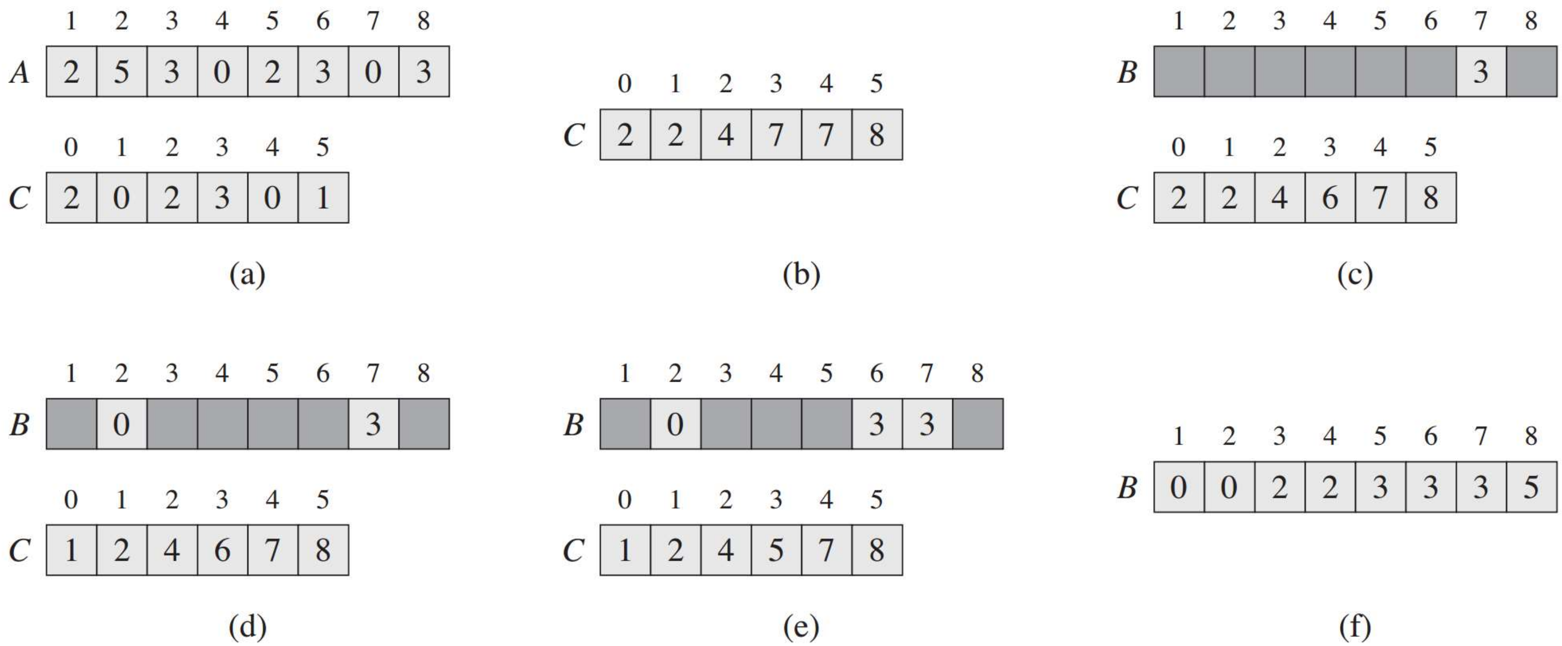
Counting sort

- Counting sort assumes all input values are integers in the range 0 to k, for some k.
- The algorithm runs in $\Theta(n)$ time when k is small.
  - As k gets larger, the algorithm becomes increasingly inefficient.
  - Counting sort is used as a stand-alone algorithm, and also as a subroutine in other algorithms, e.g. radix sort.
  - Radix sort ensures k is small, so counting sort is fast.

Counting sort

- Since we assume all inputs are in $[0, k]$, we use a size k array C to store how many inputs have each value.
  - $C[i] = c$ if there are c inputs with value i.
- Iterate through input array A.
  - For input value $A[i]$, increment $C[A[i]]$ to record an additional occurrence of value $A[i]$.
- Once we know number of occurrences of each value, we know the sorted output.
  - Ex If $C = [2,1,3,0,0,1]$, then output is $0,0,1,2,2,2,5$.
- Value i occurs $C[i]$ times.
  - It appears after values $0,1, \ldots, i-1$.
  - There are $\sum_{j=0}^{i-1} C[j]$ values $0,1, \ldots, i-1$.
  - So first occurrence of i is at index $\sum_{j=0}^{i-1} C[j] + 1$, and last occurrence is at $\sum_{j=0}^{i} C[j]$.
- After computing count array C, compute prefix sum C' of C.
  - $C'[i] = \sum_{j=0}^{i} C[j]$. Also, set $C'[-1] = 0$.
  - Value i occurs in indices $[C'[i-1] + 1, C'[i]]$ of output.

Example



Source: Introduction to Algorithms, Cormen et al.

- In (a), C contains number of occurrences of each input value in A.
- In (b), compute the prefix sum of C.
- In (c)-(f), iterate through A in reverse order.
  - Put value $A[i]$ in position $C[A[i]]$ of output, then decrement $C[A[i]]$ (because we output one more copy of $A[i]$).

Pseudocode and complexity

```
COUNTING-SORT(A, B, k)
let C[0..k] be a new array
for i = 0 to k
    C[i] = 0
for j = 1 to A.length
    C[A[j]] = C[A[j]] + 1
// C[i] now contains the number
for i = 1 to k
    C[i] = C[i] + C[i − 1]
// C[i] now contains the number
for j = A.length downto 1
    B[C[A[j]]] = A[j]
    C[A[j]] = C[A[j]] − 1
```

- First for loop resets all counts.
  - $O(k)$ time.
- Second for loop counts occurrences of each value.
  - $O(n)$ time.
- Third loop computes prefix sum.
  - $O(k)$ time.
- Last for loop uses prefix sum to scatter inputs to output positions.
  - $O(n)$ time.
- Overall complexity $\Theta(n + k)$.
- If $k = O(n)$, then complexity is $\Theta(n)$.

Stability

- A useful property of counting sort is that it's stable.
  - If two inputs are equal, then their order in the output is the same as their order in the input.
  - This is why we iterated through A in reverse order when producing B.
- Ex If input is $4, 1_A, 5, 2_A, 1_B, 2_B, 1_C$, then output is $1_A, 1_B, 1_C, 2_A, 2_B, 4, 5$.
- Stability is necessary when counting sort is used as a subroutine in other sorts, e.g. radix sort.

Radix sort

- Sort digit by digit, from least to most significant digit.
- Take list of input values, sort them on the singles digit.
- Take the new list, sort values on the tens digit.
- Take the new list, sort values on hundreds digit. Etc.
- The sorting algorithm for each digit must be stable.
- We will use counting sort.
  - It's stable.
  - Since we sort a digit at a time, the values being sorted are between 0 and 9.
  - Sorting n inputs takes $O(n)$ time.

Example

329	720	720	329
457	355	329	355
657	436	436	436
839	457	839	457
436	657	355	657
720	329	457	720
355	839	657	839

Notice due to stability, after sorting by the 10’s digit, 436 and 839 (for example) keep the same order they had after sorting by the 1’s digit.

Correctness

- Lemma 1 Let x and y be two inputs to radix sort.
  - Let k be the most significant digit on which they differ.
  - Suppose k'th digit of x is less than k'th digit of y.

  After sorting the k'th digit (in nondecreasing order), x will come before y in the remainder of the execution.
- Proof x comes before y right after sorting the k'th digit.
  - x and y are equal on all higher digits.
  - So when sorting on higher digits, x and y are always tied.
  - Since the sort is stable, x stays before y from the k'th sort onwards.

Complexity

- Lemma 2 Suppose we sort n d-digit numbers, where each digit is between 0 to $k-1$. Then radix sort takes $O(d(n+k))$ time.
- Proof Since each digit is between 0 and $k-1$, then counting sort takes $O(n+k)$ time per digit. So the total time is $O(d(n+k))$.
- Ex Sorting n d-digit binary numbers takes $O(dn)$ time.

Complexity

- Lemma 3 Given n b-bit numbers and $r \leq b$. Radix sort takes $O\left(\frac{b}{r}(n + 2^r)\right)$ time.
- Proof Break the b bits into blocks of r digits, having values between 0 and $2^r - 1$.
  - Ex For $b = 6, r = 2$, break the value 100111 into blocks 10, 01 and 11.
  - There are $d = \lceil b/r \rceil$ such blocks.
  - We can think of each b-bit number as a d digit number, where each digit has value between 0 and $2^r - 1$.
  - The lemma follows from Lemma 2.

Complexity

- Lemma 4 Setting $r = \min(\lfloor \log n \rfloor, b)$ minimizes the running time $\Theta\left(\frac{b}{r}(n + 2^r)\right)$.
- Proof If $b < \lfloor \log n \rfloor$, then for any $r \leq b$, we have $n + 2^r = \Theta(n)$. So we set $r = b$ to minimize b/r.
  - If $b \geq \lfloor \log n \rfloor$, setting $r = \lfloor \log n \rfloor$ makes running time $\Theta\left(\frac{bn}{\log}\right)$.
  - We show $r > \lfloor \log n \rfloor$ and $r < \lfloor \log n \rfloor$ both result in slower running times.
  - If $r > \lfloor \log n \rfloor$, then $2^r > n$, and 2^r in numerator increases faster than r in denominator, so running time increases.
  - If $r < \lfloor \log n \rfloor$, then the b/r term increases, but the $n + 2^r$ remains $\Theta(n)$.
- In other words, radix sort is efficient when there are many short numbers, but not when there are a few long numbers.