



NP and NP-completeness

CS240

Spring 2024

Rui Fan

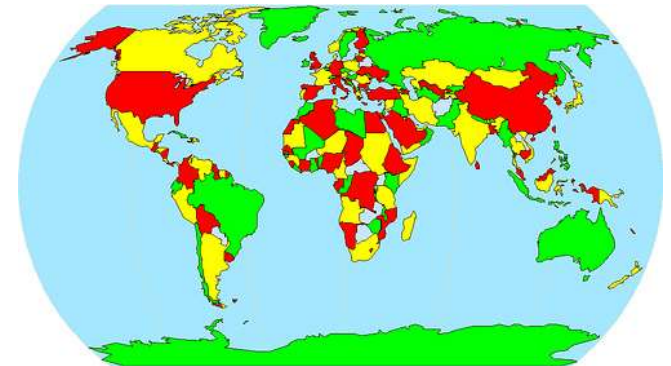


Limits of efficiency

- What is the fastest way to solve a problem?
 - E.g. sorting n numbers takes $O(n \log n)$ operations using mergesort.
 - Is there a different algorithm that sorts n numbers faster, say in $O(n)$ time?
 - No. Sorting n numbers takes $\Omega(n \log n)$ time if the algorithm can only compare numbers.

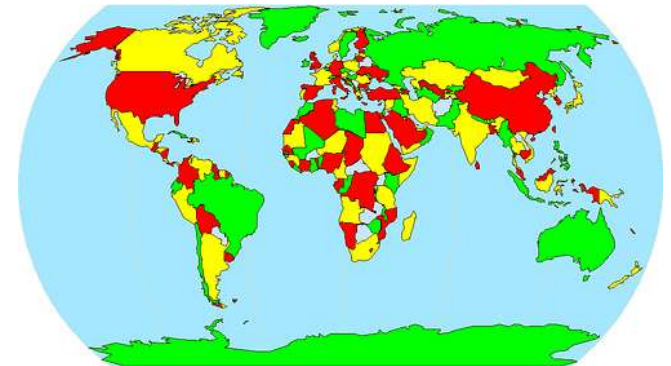
Limits of efficiency

- How much time does it take to solve a more difficult problem, like coloring?
- Four Color Theorem says every map can be colored with 4 colors, s.t. adjacent regions have different colors.
- So given a map, we can always efficiently answer the question whether the map can be 4-colored. Namely, yes.
- But sometimes 3 colors are enough to color the map.
- Can we efficiently determine whether a map can be 3-colored?
- If there are n regions, we can find a 3-coloring by trying all possible colorings.
 - There are 3^n possible colorings.
 - There are 195 countries in the world, and 1.1×10^{93} possible colorings!



Limits of efficiency

- Is there a much more efficient algorithm?
- Nobody knows of one. And almost everybody thinks no such algorithm exists.
- But **no one can prove** it doesn't exist either.
- The theory of **NP-completeness** is a mathematical attempt to prove some problems have no efficient solutions.
 - So far, it's led to more questions than answers...
- We'll define P, NP, and NP-completeness.





The class P

- A **polynomial time (polytime)** algorithm is one that runs in $O(n^k)$ time, for some constant k , when input has size n .
- P is the set of all problems that can be solved by a polytime algorithm.
 - These problems are called “efficiently computable”, because a polytime algorithm is considered efficient.
 - In practice though, an e.g. $O(n^3)$ algorithm is quite slow, even for moderate sized n .
- If a problem takes $\omega(n^k)$ time, for any constant k , it's considered not efficiently solvable.
 - **Ex** An $\Omega(2^n)$ time or $\Omega(n!)$ time algorithm isn't efficient.
 - We only know how to 3-color a map in $\Omega(3^n)$ time (more or less), so 3-coloring (currently) can't be solved efficiently.
 - An $\Omega(3^n)$ time algorithm is much slower than an $O(n^3)$ algorithm.
 - **Ex** If $n=10000$, then $n^3 = 10^{12}$, but $3^n = 1.6 \times 10^{4771}$.



The class NP

- NP = Nondeterministic polynomial time.
- Def An **instance** of a problem consists of an input for the problem.
 - Ex An instance of the sorting problem is a set {3,1,2,4} that we want to sort.
 - Ex An instance of the SSSP problem is a weighted graph along with a source node.
- P is the class of problems for which all instances can be **solved** in polynomial time by some algorithm.
- NP is the class of problems for which the solvability of an instance can be **verified** in polynomial time.
 - The verification is done by a “**verifier**” algorithm.
 - The verifier needs an additional “hint” to work correctly.
 - The hint is also called a “witness” or “**certificate**”.
 - The verifier doesn’t find a solution to a problem instance, but only checks that the instance has been solved.



The class NP

- The verifier has the following properties.
 - The verifier's input is a problem instance x , and a certificate y .
 - The verifier's output is either "accept" or "reject".
 - If x has a solution, then if y is a "good" certificate, the verifier will output accept.
 - If y is not a "good" certificate, the verifier can either accept or reject.
 - If x has no solution, the verifier rejects no matter what y is.
 - Intuitively, the certificate y indicates x is solvable.
 - For example, y can be a solution to x .
 - But y can also be an indirect representation of a solution.
 - The verifier is efficient, i.e. runs in polynomial time.

The class NP, formally

- **Def** A **decision problem** is a problem with a yes / no answer.
 - **Ex** Given a graph, is there a path from node s to t ?
 - **Ex** Given a map, is there a way to 3-color it?
 - **Ex** Given a number, is it prime?
- **Def** Given a decision problem, the set of **yes** (resp. **no**) **instances** are the instances of the problem for which the answer is yes (resp. no).
 - **Ex** 11 is a yes instance to the prime problem, 10 is a no instance.
- **Def** Given a decision problem A , a **polynomial time verifier** V for A is an algorithm that does the following
 - V 's input is an instance x of A , and a certificate string y .
 - $V(x, y) \in \{0, 1\}$, representing "reject" and "accept", resp.
 - If x is a yes instance, there exists a y for which V outputs 1, i.e. $\exists y: V(x, y) = 1$.
 - If x is a no instance, every y makes V output 0, i.e. $\forall y: V(x, y) = 0$.
 - V runs in polynomial time.
- **NP** is the set of all decision problems with polytime verifiers.

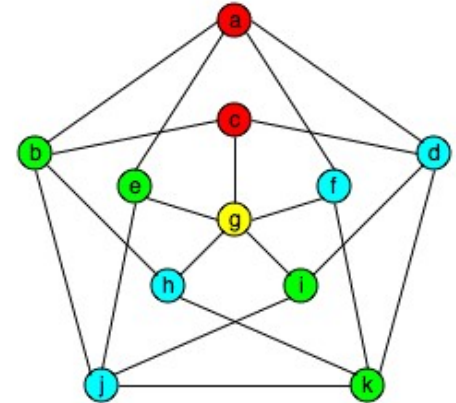


Showing a problem is in P or NP

- To show a problem is in P, give an algorithm solving the problem that runs in polynomial time.
- To show a decision problem is in NP, give a polynomial time verifier for the problem satisfying the properties on the previous slide.
 - This requires specifying what the certificates are, and how the verifier operates, given an instance of the problem and a certificate.

4-coloring is in NP

- Given a graph, can we assign each vertex one of 4 colors, such that adjacent vertices have different colors?
- **Verifier**
 - Certificate y is an assignment of colors to the vertices of graph x .
 - Check y uses at most 4 colors. If not, output no.
 - Go through all edges of x , and check endpoints of each edge have different colors.
 - If true for all edges, output 1. Else output 0.
- **If x is yes instance**
 - Then x is 4-colorable.
 - So there's way to assign each vertex one of 4 colors s.t. endpoints of each edge have different colors.
 - Let y be this assignment, and give y to V .
 - Clearly V outputs 1.
- **If x is no instance**
 - Then x is not 4-colorable.
 - So no matter how we assign 4 colors to vertices of x , some edge has endpoints with the same color.
 - So V outputs 0, for any input y .
- **V runs in polytime.**
 - If x has n vertices, then it has $O(n^2)$ edges, so V runs in $O(n^2)$ time.



Factoring is in NP

- Given an integer x , does it have a factor $y \neq 1, x$.

- **Verifier**

- ☐ Certificate y is a number.
- ☐ Check y divides x , and $y \neq 1, x$.
- ☐ If so, output 1, else output 0.

$$999999866000004473 = 999999929 \times 999999937$$

- **If x is yes instance**

- ☐ Then x has a nontrivial factor y .
- ☐ Give y to V , and V outputs 1.

- **If x is no instance**

- ☐ Then every factor of x is either 1 or x .
- ☐ So for any $y \neq 1, x$ given to V , V outputs 0.

- **V runs in polytime.**

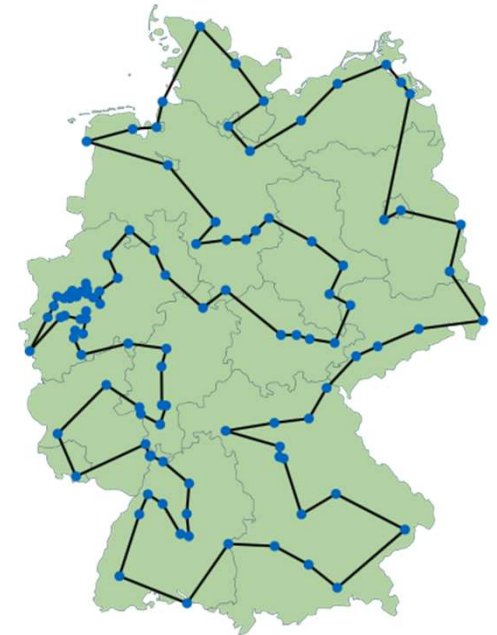
- ☐ Dividing x by y takes polynomial time.

- **However, factoring does not seem to be in P .**

- ☐ Given an n digit number, there's no known way determine if it has a nontrivial factor in $O(n^k)$ time, for any constant k .

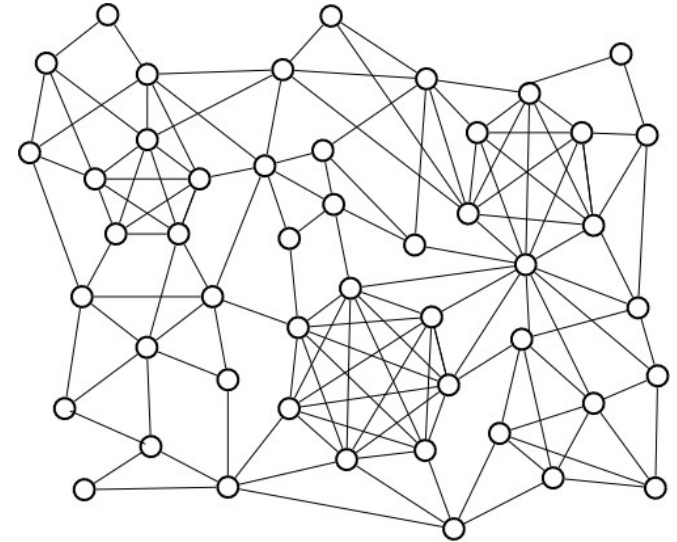
Traveling salesman is in NP

- Given a set of n cities, and distances between each pair of cities, is there a path visit each city exactly once, and has distance at most D , for a given D ?
- **Verifier**
 - Certificate y is a path through the graph.
 - Check y goes through every vertex once, and total length of y is $\leq D$. If so, output 1, else output 0.
- **If x is yes instance**
 - Then there is a path going through each vertex once with total length $\leq D$.
 - Call the path y and give it to V .
 - Clearly V outputs 1.
- **If x is no instance**
 - Then no matter what path y you use, either y doesn't go through each city once, or y has length $> D$.
 - So V outputs 0, no matter what y it gets.
- **V runs in polytime.**
 - If the graph has n vertices, then all of V 's checks can be done in $O(n)$ time.



k-Clique is in NP

- Given a graph with n nodes and a number k , are there k nodes that form a clique, i.e. that are all connected to each other?
- **Verifier**
 - Certificate y is a set of k nodes in x .
 - Check each pair of the k nodes is connected by an edge. If so, output 1. Otherwise output 0.
- **If x is yes instance**
 - Then there are k nodes that are mutually connected.
 - Call this set y and give it to V .
 - Clearly V outputs 1.
- **If x is no instance**
 - Then in any set of k nodes, some 2 nodes aren't connected.
 - So V outputs 0, no matter what set of k nodes it gets.
- **V runs in polytime.**
 - Checking k nodes are mutually connected takes $O(k^2)$ time.





All problems in P are in NP

- Let A be a problem in P. I.e. there's a polytime algorithm S s.t. on every instance x of A
 - If x has a solution, S returns a solution.
 - If x has no solution, S returns fail.
- **Verifier**
 - V runs S. If S finds a solution, V outputs 1. Otherwise V outputs 0.
- **If x is yes instance**
 - S finds a solution, so V outputs 1.
- **If x is no instance**
 - S returns fail, so V outputs 0.
- **V runs in polytime.**
 - Because V just runs S, which runs in polytime.
- Notice that for problems in P, V doesn't need a certificate y.
 - For problems in P, it's easy to determine if they're solvable or not.
- But for hard problems (not in P), V isn't powerful enough to determine solvability by itself.
 - So it needs a hint / witness / certificate.
- **Ex** In factoring, a polytime verifier isn't powerful enough to find a nontrivial factor of an input.
 - But if it's given a nontrivial factor, it can check the factor works in polytime, and therefore verify the input is composite.

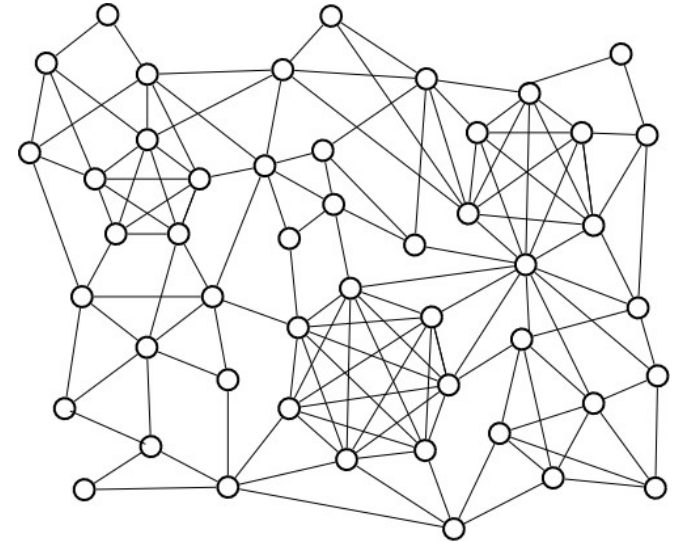


Primes is in NP

- Proving a problem is in NP isn't always so easy...
- Given a number x , is x prime?
- Verifier
 - What should the certificate y be?
 - If y is a single number s.t. $y \nmid x$, then y doesn't certify that x is prime.
 - Suppose y a vector giving $\frac{x}{y}$ for every $y \leq \sqrt{x}$.
 - V returns 1 if all these aren't integers, and 0 otherwise.
- If x is yes instance
 - I.e., x is prime. Then all the quotients are non-integer, so V returns 1.
- If x is no instance
 - Then x is composite, so x has a factor $y \leq \sqrt{x}$, so $\frac{x}{y}$ is integer, and V outputs 0.
- V runs in polytime.
 - No it doesn't!
 - Say x has n digits. Then there are $\sim 2^{n/2}$ numbers $\leq \sqrt{x}$, so y has size $O(n2^{n/2})$.
 - Since V has to check all values in y , it doesn't run in $\text{poly}(n)$ time.
- So, this verifier is incorrect. This verifier does not show Primes is in NP.
- That doesn't mean $\text{Primes} \notin \text{NP}$, it just means our verifier doesn't work.
- We can show Primes is in NP using another verifier and some number theory. This is called Pratt's Theorem, and is beyond our scope.

Incorrect verifiers

- We showed k-Clique is in NP by giving a correct verifier.
- Let's see some **incorrect verifiers**.
 - None of these verifiers can be used to prove k-Clique is in NP.
- **Verifier 1** Always outputs 1, regardless of y .
 - Wrong, because when graph doesn't contain a k-clique, V is supposed to output 0.
- **Verifier 2** Always output 0, regardless of y .
 - Wrong, because when the graph does contain a k-clique, V is supposed to output 1, for some y .
- **Verifier 3** Check all subsets of k nodes. If any form a clique, output 1, else output 0.
 - Seems OK. When x has a k-clique, V outputs 1, and when x doesn't, it outputs 0.
 - But V is still wrong, because it doesn't run in polytime.
 - There are $O(n^k)$ subsets of k nodes, and V checks all of them.





P vs NP

- Does $P=NP$?
 - I.e. suppose there's a problem for which we can **verify solvability** in polynomial time. Does that mean we can actually **find a solution** in polynomial time?
- This is the arguably the most important question in computer science.
 - The other would be to produce general AI.
- Many real-world problems are in NP. If $P=NP$, we can solve them efficiently. If $P \neq NP$, then we can't.
- Every P problem is in NP, as we saw. So $P \subseteq NP$.
- Is every NP problem in P, i.e. $NP \subseteq P$?
- After 50 years, nobody knows.
 - Most, but not all researchers think not all NP problems are in P.
 - There are probably problems we can efficiently verify but not efficiently solve.
 - **Ex** Factoring is something we can efficiently verify, but not solve.
- If you can prove $P \neq NP$, or even better, $P = NP$, then
 - you \geq Newton \geq Einstein \geq ...
 - You also get \$1M from the Clay Math Institute.
- Answering this question has vast and profound implications for CS, AI, math, physics, etc.



NP-completeness

- Out of all the NP problems, there's a subset of NP problems called **NP-complete** (NPC) problems that are the “hardest” NP problems.
- To determine whether $P=NP$, it suffices to know whether $P=NPC$.
 - If the hardest problems can be solved in polytime, then all NP problems can be solved in polytime.
I.e. $P=NP$.
- So the study of P vs NP focuses on NPC problems.



Hardness and reductions

- What does it mean to say problem B is harder than problem A?
- It means if you can solve B, you can also solve A.
 - Ex Algebra is harder than arithmetic, because if you can do algebra, you can also do arithmetic.
 - So if I have an algorithm for solving B, I can use it to solve A.
- We say A **reduces to** B.
 - Write $A \leq_R B$.
 - Read this as “A is equally or less difficult than B”.



Example

- FACTOR-ALL(n) finds all the factors of a number n .
- FACTOR-1(n) finds one factor.
- Of course, $\text{FACTOR-1} \leq_R \text{FACTOR-ALL}$.
 - If we can find all the factors, we can certainly find one.
- $\text{FACTOR-ALL} \leq_R \text{FACTOR-1}$.
 - We use FACTOR-1(n) to find one factor m of n .
 - Then divide n by m , and run FACTOR-1 on the result, to find another factor of n .
 - Keep repeating the previous steps until we get all the factors.
- The hard part about factoring a number, is just to find one factor.
 - Since $\text{FACTOR-1} \leq_R \text{FACTOR-ALL}$, $\text{FACTOR-ALL} \leq_R \text{FACTOR-1}$, these problems have the same hardness.



Reductions, formally

- Let A and B be two decision problems.
- Let X and Y be the set of yes instances for A and B , resp.
- **Ex** Say $A = \text{PRIME}$ and $B = k\text{-CLIQUE}$.
 - X is the set of prime numbers.
 - Y is the set of graphs containing a k -clique.
- Let f be a function that maps instances of A to instances of B .
- **Def** A **reduces to** B if there exists $f: A \rightarrow B$ s.t. for all instances x of A , $x \in X \Leftrightarrow f(x) \in Y$.
 - We write $A \leq_R B$.
- To show $A \leq_R B$, just give the mapping f .
- If $A \leq_R B$, then we can use an algorithm for B to solve A .
 - To solve an instance of A , first map it to an instance of B using f .
 - Then run the B algorithm.
 - Return the same answer for A as the B algorithm gives.
 - By definition, A is true $\Leftrightarrow f(A)$ is true.



Example

- Suppose we want to show $\text{PRIME} \leq_R \text{k-CLIQUE}$.
- This means there's some mapping f such that.
 - Given an instance of PRIME, i.e. a number n .
 - $f(n)$ is an instance of k-CLIQUE, i.e. $f(n)$ is a graph G .
 - n is prime if and only if $f(n)$ contains a k -clique.
- If we have an algorithm to solve k-CLIQUE, we can use it solve PRIME.
 - To tell if n is prime, map n to a graph G and run the k-CLIQUE algorithm on G .
 - If it returns true, n is prime. Otherwise n isn't.

Polynomial time reductions

- If the mapping function from A to B runs in polynomial time, then it's a **polynomial time reduction**, and we write $A \leq_p B$.
 - **Ex** If we're reducing PRIME to k-CLIQUE, then the function to generate a graph from a number must run in polytime.
- **Thm 1** Let A, B and C be three problems, and suppose $A \leq_p B$ and $B \leq_p C$. Then $A \leq_p C$.
- **Proof** Since $A \leq_p B$, there's a polytime mapping f from instances of A to instances of B.
 - Since $B \leq_p C$, there's a polytime mapping g from instances of B to instances of C.
 - Given an instance X of A, let $Y = f(X)$, and $Z = g(Y) = g(f(X))$.
 - Then X is a yes instance of A \Leftrightarrow Y is a yes instance of B \Leftrightarrow Z is a yes instance of C.
 - So $g \circ f$ is a valid mapping of A to C.
 - Since f and g are both polytime, $g \circ f$ is also polytime.



NP-completeness

- **Def** A problem A is **NP-complete** (NPC) if the following are true.
 - $A \in NP$.
 - Given any other problem $B \in NP$, $B \leq_p A$.
- Thus, a NP-complete problem is an NP problem that can be used to solve any other NP problem.
 - It's a “hardest” NP problem.



NP-completeness and SAT

- Do NP-complete problems really exist?
 - Can we really find an NP problem that can be used to solve every other NP problem?
 - One problem to rule them all?
- Yes! Steve Cook and Leonid Levin proved around 1970 that SAT is NP-complete.
- **SAT** = satisfiable Boolean formulas.
 - Given a Boolean formula, is there any setting for the variables which makes the formula true?
 - **Ex** $(A \vee B \vee \neg C) \wedge (A \vee \neg B \vee C) \wedge (\neg A \vee \neg B \vee \neg D) \in SAT$.
 - Setting $A=B=C=\text{true}$, $D=\text{false}$ makes the formula true.
 - **Ex** $A \wedge \neg A \notin SAT$.
 - The formula's false for all settings of A.

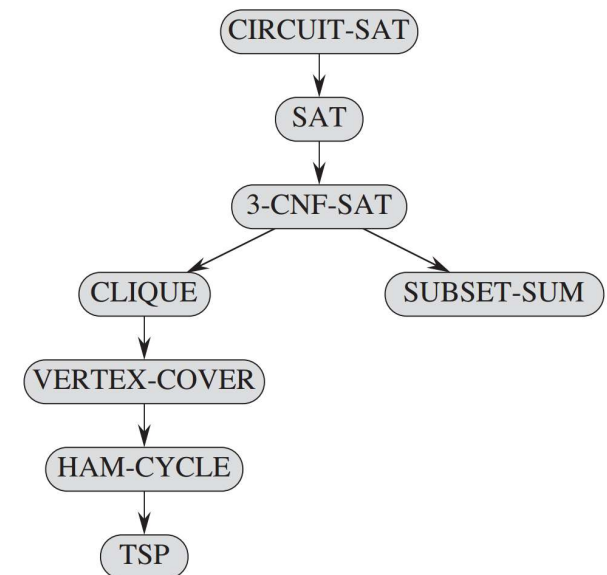


NP-completeness and SAT

- **Cook-Levin theorem** says 2 things.
 - $SAT \in NP$.
 - Prove this yourself.
 - Every NP problem reduces to SAT. I.e. every problem A in NP can be mapped to an SAT formula ϕ in polytime, such that
 - If A is true, then ϕ is satisfiable.
 - If A is false, then ϕ is not satisfiable.
- Basic idea of the theorem is to use the logical operations in a SAT formula to emulate the logical operations in any algorithm.
 - Any NP problem X has a polytime verifier V . The Cook-Levin theorem uses a SAT formula ϕ to emulate the verifier's operations.
 - For a yes instance of X , there's some certificate making V return 1.
 - The certificate can be transformed to a satisfying truth setting for ϕ .
 - Any certificate making V return 0 corresponds to a non-satisfying truth setting for ϕ .
 - So $\phi \in SAT$ if and only if X is a yes instance, and $X \leq_p SAT$.

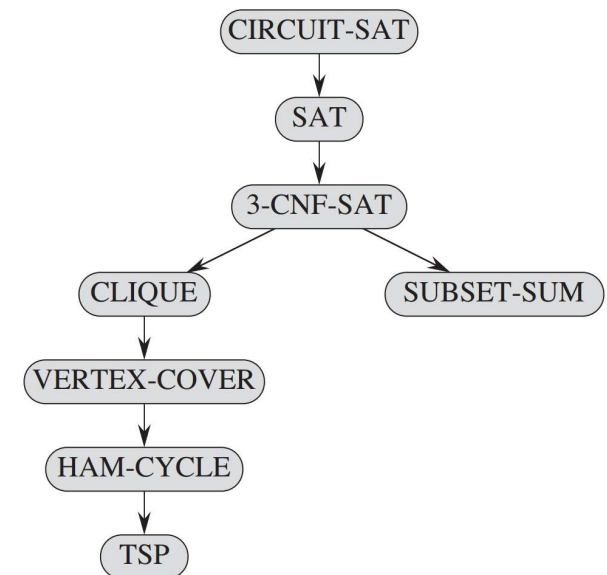
The web of NP-completeness

- For every problem in the picture, if A points to B, it means $A \leq_p B$.
 - So A can be solved using B.
- CIRCUIT-SAT was the original problem that Cook-Levin proved was NP-complete.
- So every problem in NP can be solved using CIRCUIT-SAT.
- But CIRCUIT-SAT can be solved using SAT, because $\text{CIRCUIT-SAT} \leq_p \text{SAT}$.
 - So every problem in NP can be solved using SAT.
 - So SAT is also NP-complete!
- SAT can be solved using 3-CNF-SAT.
 - So every NP problem can be solved using 3-CNF-SAT.
 - So 3-CNF-SAT is also NP-complete.
- All problems in the diagram are NP-complete.
- Of course, each of the reductions requires a proof, which is sometimes tricky.
 - We'll see some reduction proofs next lecture.
- There are thousands of other NPC problems.



The web of NP-completeness

- **Thm** Given two NP problems A and B, suppose A is NP-complete, and $A \leq_p B$. Then B is also NP-complete.
- **Proof** Let C be any NP problem. Then $C \leq_p A$, since A is NP-complete.
 - Since $A \leq_p B$, then by Theorem 1, we have $C \leq_p A \leq_p B$.
 - Since also $B \in NP$, then B is NPC.
- To prove a problem B is NP-complete
 - Take a problem A you know is NPC, and prove $A \leq_p B$.
 - E.g., A can be any problem in the previous diagram.
 - To prove $A \leq_p B$, you need to give a polytime reduction from A to B.
 - This can sometimes be quite challenging.
 - You also have to prove $B \in NP$, but that's usually not hard.



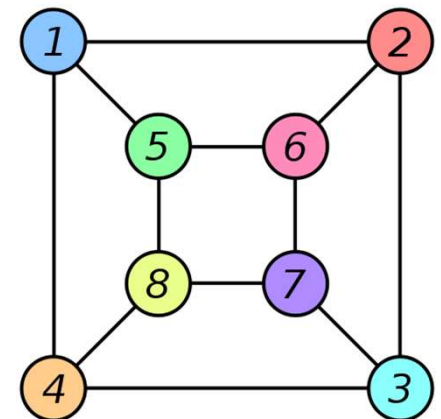
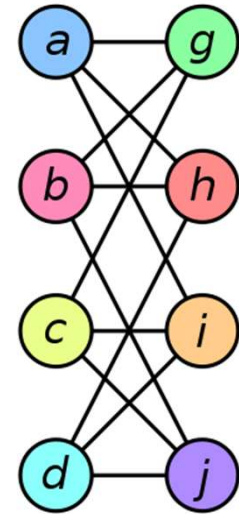


NP-completeness and P vs NP

- **Thm 2** Suppose a problem A is NP-complete, and $A \in P$. Then $P=NP$.
- **Proof** Consider any other NP problem B . We'll show $B \in P$.
 - Since A is NPC, there's a polytime mapping f from B to A .
 - Given an instance X of B , run f on X to get an instance Y of A .
 - Since $A \in P$, there's a polytime algorithm g to solve A .
 - Run $g(Y)$, and return the same answer for X .
 - By the definition of \leq_P , $g(Y)$ is true $\Leftrightarrow X$ is true.
 - Running f and g both take polytime. So we can solve B in polytime.
- **Cor** Suppose a problem A is NP-complete, and $A \notin P$. Then for any NP-complete problem B , $B \notin P$.
 - If $B \in P$, then since B is NPC, we have $P = NP$ by Theorem 2. So since $A \in NP$, we have $A \in P$, a contradiction.
- To prove $P \neq NP$ (which is what most people think), it's enough to show one NPC problem is not solvable in polytime, by the corollary.
 - But after 50 years, no one has any such proof.
 - Nor has anyone shown a polytime algorithm for any NPC problem.

Beyond NP

- NP includes many important and practical problems.
- But not all problems are in NP.
- In the **graph isomorphism** (GRAPH-ISO) problem, we ask whether two graphs simply relabelings of each other.
 - I.e. Given two graphs $G = (V, E)$ and $G' = (V, E')$, is there a permutation $f: V \rightarrow V$ s.t. $(u, v) \in E \Leftrightarrow (f(u), f(v)) \in E'$.
 - This is in NP, because on “yes” instances, we just give the relabeling to the verifier.
- The **graph non-isomorphism** (GRAPH-NONISO) problem is the opposite: are two graphs really different, and not relabelings of each other.
 - We don't know if GRAPH-NONISO \in NP.
 - If two graphs are really different, how do we produce a certificate to prove this to a polytime verifier?
 - We could give the verifier a list of all possible relabelings, and show the graphs are different under each.
 - But this isn't polytime because there are $n!$ relabelings.
 - We also don't know if GRAPH-NONISO \in P.





More complexity classes

- **co-NP** All problems whose “complement” is in NP.
 - E.g. $\text{GRAPH-ISO} \in \text{NP}$, so $\text{GRAPH-NONISO} \in \text{co-NP}$.
- **PSPACE** All problems whose computation takes polynomial amount of space.
 - Includes all problems in P.
- **EXPTIME** All problems whose computation takes at most exponential amount of time.
 - Includes all problems in P and NP.
- **NEXPTIME** All problems whose correct answer can be verified in at most exponential amount of time.
 - NEXPTIME is to EXPTIME what NP is to P.
- Each of these is called a **complexity class**.
 - Many, many other complexity classes, some very obscure.
 - “Complexity zoo”:
https://complexityzoo.net/Complexity_Zoo

Complexity theory

- A central goal of **complexity theory** and theoretical computer science is to study the relationship between complexity classes.
- We know some trivial things, like $P \subseteq NP$, or $NP \subseteq EXPTIME$.
- We know a few nontrivial things, like $PSPACE = NPSPACE = IP$, and $NL = coNL$.
- Beyond this, we really know hardly anything!
- $P ?= NP$, $P ?= co-NP$, $NP ?= co-NP$, $NP ?= PSPACE$, $NP ?= EXPTIME$.
- In the last 50 years, we haven't gotten much closer.
- Maybe our techniques are wrong?
- Understanding any of the relationships would have many profound implications.

