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① $P_{(X,Y)}[f(X)\neq Y] = \min(\pi(X), 1-\pi(X))$

② $\forall h(X) \quad P_{(X,Y)}[h(X)\neq Y] \geq P_{(X,Y)}(f(X)\neq Y)$

$$E_{(X,Y)}[f(X)\neq Y] = P_{(X,Y)}[f(X)\neq Y]$$

$y\in\{-1,+1\}$

$$\pi(x) = P(Y=+1\mid x), \quad f(x) = \begin{cases} +1, & \pi(x)\geq \frac{1}{2} \\ -1, & \pi(x)<\frac{1}{2} \end{cases}$$

for a fixed x

① $P_{(X,Y)}[f(X)\neq Y] = \cancel{P_{(X,Y)}[f(X)=+1, Y=-1] + P_{(X,Y)}[f(X)=-1, Y=+1]}$

$$= \begin{cases} \text{if } Y=+1, f(x)=-1 \Rightarrow P(Y\neq f(x)) = \pi(x), & \pi(x)<\frac{1}{2} \\ \text{if } Y=-1, f(x)=+1 \Rightarrow P(Y\neq f(x)) = 1-\pi(x), & \pi(x)\geq\frac{1}{2} \end{cases}$$

So above all

when $\pi(x)<\frac{1}{2}$, $P_{(X,Y)}[f(X)\neq Y] = \pi(x) < 1-\pi(x)$

when $\pi(x)\geq\frac{1}{2}$, $P_{(X,Y)}[f(X)\neq Y] = 1-\pi(x) < \pi(x)$

i.e. $P_{(X,Y)}[f(X)\neq Y] = \min(\pi(X), 1-\pi(X))$

②. for a fixed x, $\forall h(x)$

$p = P(h(x)=+1\mid x)$ is constant. no matter $h(x)$ is deterministic or not

$$P_{(X,Y)}[h(X)\neq Y] \overset{\text{LOTP}}{=\!=} P_{(X,Y)}[Y=1\mid x]\cdot P(h(x)=-1\mid x) + P_{(X,Y)}[Y=-1\mid x]\cdot P(h(x)=+1\mid x)$$

$$= \pi(x)\cdot p + (1-\pi(x))\cdot(1-p)$$

$$\geq \min(\pi(x), 1-\pi(x))\cdot p + \min(\pi(x), 1-\pi(x))\cdot(1-p)$$

$$= \min(\pi(X), 1-\pi(X))$$

$$\overset{①}{=} P_{(X,Y)}[f(X)\neq Y]$$

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So we have proved that $\forall h(x)$

$$P_{(X,Y)}[h(X)\neq Y] \geq P_{(X,Y)}[f(X)\neq Y]$$