Fundamentals of Information Theory

Homework 4

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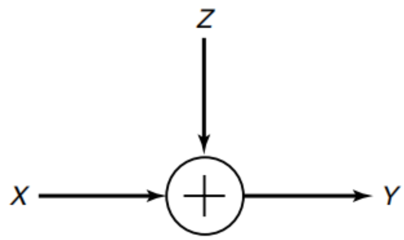
Problem 1

7.2 Additive noise channel. Find the channel capacity of the following discrete memoryless channel:



where $\Pr\{Z=0\} = \Pr\{Z=a\} = \frac{1}{2}$. The alphabet for x is $\boldsymbol{X} = \{0,1\}$. Assume that Z is independent of X. Observe that the channel capacity depends on the value of a.

Solution

We have $\mathcal{X} = \{0,1\}, \mathcal{Z} = \{0,1\}, \mathcal{Y} = \{0,1,a,a+1\}, Y = X+Z$.

So we need to discuss the value of a.

<1> When $a=0$, then $P(Z=0) = P(Z=a) = \frac{1}{2}$, which is not a suitable probability distribution.

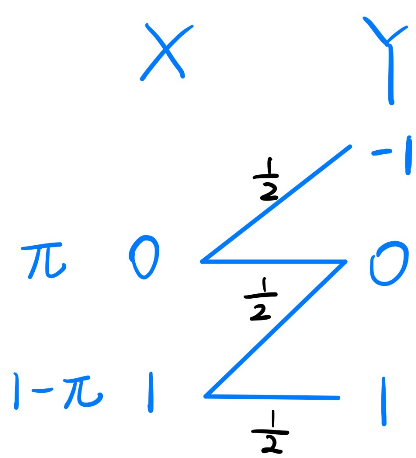
If we correct the probability into $P(Z=0)=1$, then we have

$$
\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) \\
&= H(X) \qquad \text{(since } Y = X\text{, so when given } Y\text{, } X \text{ is deterministic)} \\
&\leq \log|\mathcal{X}| \\
&= 1
\end{aligned}
$$

When $p(x) = \left(\frac{1}{2}, \frac{1}{2}\right)$, the inequality takes the equality.

So the channel capacity is $C = \max_{p(x)} I(X;Y) = 1$ bit per transmission.

<2> When $a=-1$, then $\mathcal{Y} = \{0,1,-1\}$, the DMC is as follows:



If we define $p(X=0) = \pi, p(X=1) = 1-\pi$, then

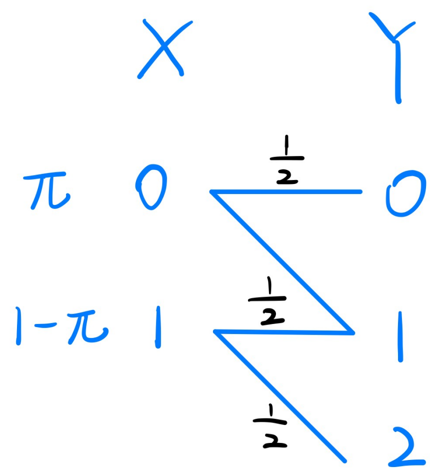
$$P(Y=0) = P(Y=0|X=0)P(X=0) + P(Y=0|X=1)P(X=1) = \frac{1}{2}\pi + \frac{1}{2}(1-\pi) = \frac{1}{2}$$

Also, we can get that $P(X=0|Y=0) = \dfrac{P(Y=0|X=0)P(X=0)}{P(Y=0)} = \pi, P(X=1|Y=0) = 1-\pi$.
So we have:

$$
\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) \\
&= H(X) - \sum_y H(X|Y=y)P(Y=y) \\
&= H(X) - H(X|Y=0)p(Y=0) \qquad \text{(since when } Y=\pm 1,\ X \text{ is deterministic, } H(X|Y=-1)=H(X|Y=1)=0) \\
&= H(\pi, 1-\pi) - \frac{1}{2}H(\pi, 1-\pi) \\
&= \frac{1}{2}H(\pi, 1-\pi) \\
&\le \frac{1}{2}\log 2 = \frac{1}{2}
\end{aligned}
$$

If $\pi = \frac{1}{2}$, i.e. $p(x) = \left(\frac{1}{2}, \frac{1}{2}\right)$, the inequality takes the equality.
So the channel capacity is $C = \max_{p(x)} I(X;Y) = \frac{1}{2}$ bit per transmission.

<3> When $a=1$, then $\mathcal{Y} = \{0,1,2\}$, the DMC is as follows:



Which is quite similar to the case when $a=-1$.
i.e. With the quite similar process, we can get that

$$P(Y=1) = \frac{1}{2}$$

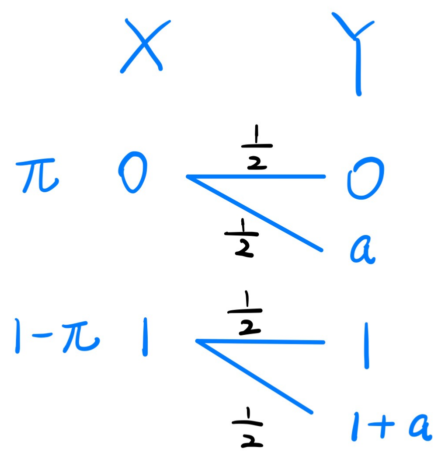
$$P(X=0|Y=1) = \pi$$

$$
\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) \\
&= H(X) - \sum_y H(X|Y=y)P(Y=y) \\
&= H(X) - H(X|Y=1)p(Y=1) \ \text{(When } Y=0,2,\ X \text{ is deterministic, } H(X|Y=0)=H(X|Y=2)=0) \\
&= H(\pi, 1-\pi) - \frac{1}{2}H(\pi, 1-\pi) \\
&= \frac{1}{2}H(\pi, 1-\pi) \le \frac{1}{2}\log 2 = \frac{1}{2}
\end{aligned}
$$

If $\pi = \frac{1}{2}$, i.e. $p(x) = \left(\frac{1}{2}, \frac{1}{2}\right)$, the inequality takes the equality.

So the channel capacity is $C = \max_{p(x)} I(X;Y) = \frac{1}{2}$ bit per transmission.

<4> When $a \neq 0, a \neq \pm 1$, then the channel is as follows:



Which is actually a noisy channel with non-overlapping outputs.

$$\begin{aligned} I(X;Y) &= H(X) - H(X|Y) \\ &= H(X) \qquad \text{(When given } Y\text{, } X \text{ is deterministic)} \\ &\leq \log|\mathcal{X}| = 1 \end{aligned}$$

So the channel capacity is $C = \max_{p(x)} I(X;Y) = 1$ bit per transmission.

So above all, the channel capacity of the following discrete memoryless channel is as follows:

$$C = \begin{cases} 1 & \text{, if } a = 0 \text{ , and the probablity is corrected into } P(X=0) = 1 \\ \frac{1}{2} & \text{, if } a = \pm 1 \\ 1 & \text{, otherwise} \end{cases}$$

Problem 2

7.3 Channels with memory have higher capacity. Consider a binary symmetric channel with $Y_i = X_i \oplus Z_i$, where $\oplus$ is mod 2 addition, and $X_i, Y_i \in \{0,1\}$. Suppose that $\{Z_i\}$ has constant marginal probabilities $\Pr\{Z_i = 1\} = p = 1 - \Pr\{Z_i = 0\}$, but that $Z_1, Z_2, \ldots, Z_n$ are not necessarily independent. Assume that Z^n is independent of the input X^n. Let $C = 1 - H(p, 1-p)$. Show that

$$\max_{p(x_1,x_2,\ldots,x_n)} I\left(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots Y_n\right) \geq nC$$

Solution

Since $Y_i = X_i \oplus Z_i \Rightarrow Z_i = X_i \oplus Y_i$, so $p(X^n|Y^n) = p(Z^n|Y^n)$. Then we have

$$\begin{aligned}
I(X^n; Y^n) &= H(X^n) - H(X^n|Y^n) \\
&= H(X^n) - H(Z^n|Y^n) \qquad (p(X^n|Y^n) = p(Z^n|Y^n)) \\
&\geq H(X^n) - H(Z^n) \qquad \text{(Conditioning reduces entropy)} \\
&= H(X^n) - \sum_{i=1}^{n} H(Z_i|Z_1, \ldots, Z_{i-1}) \\
&\geq H(X^n) - \sum_{i=1}^{n} H(Z_i) \qquad \text{(Conditioning reduces entropy)} \\
&= H(X^n) - nH\left(p, 1-p\right)
\end{aligned}$$

And since $|\mathcal{X}| = 2$, so we have

$$H(X^n) = \sum_{i=1}^{n} H(X_i|X_1, \ldots, X_{i-1}) \leq \sum_{i=1}^{n} H(X_i) \leq \sum_{i=1}^{n} \log|\mathcal{X}| = n$$

When X_i are independent, and $P(X_i = 0) = P(X_i = 1) = \dfrac{1}{2}$, the inequality holds the equality.
So

$$\begin{aligned}
\max_{p(x_1,x_2,\ldots,x_n)} I\left(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots Y_n\right) &\geq \max_{p(X^n)} \left(H(X^n) - nH\left(p, 1-p\right)\right) \\
&= n - nH\left(p, 1-p\right) \\
&= n\left(1 - H\left(p, 1-p\right)\right) \\
&= nC
\end{aligned}$$

So above all, we have proved that when $X_i \overset{i.i.d.}{\sim} \text{Bern}\left(\dfrac{1}{2}\right)$, the 'max' could be taken, and

$$\max_{p(x_1,x_2,\ldots,x_n)} I\left(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots Y_n\right) \geq nC$$

Problem 3

7.4 Channel capacity. Consider the discrete memoryless channel $Y = X + Z (\text{mod } 11)$, where

$$Z = \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

and $X \in \{0, 1, \ldots, 10\}$. Assume that Z is independent of X.

(a) Find the capacity.

(b) What is the maximizing $p^*(x)$?

Solution

(a) Since $Y = X + Z(\text{mod } 11)$, i.e. $Z = Y - X + 11(\text{mod } 11)$. And $X \perp Z$, so

$$\begin{aligned} H(Y|X) = H(Z|X) = H(Z) = H\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \log 3 \\ I(X;Y) = H(Y) - H(Y|X) \\ = H(Y) - \log 3 \\ \leq \log 11 - \log 3 \end{aligned}$$

When $Y \sim DUnif(11)$, the inequality takes equality. And we can prove that Y could get such distribution with certern $p^*(x)$ in (b).

So the channel capacity is $C = \max_{p(x)} I(X;Y) = \log \frac{11}{3}$ bits per transmission.

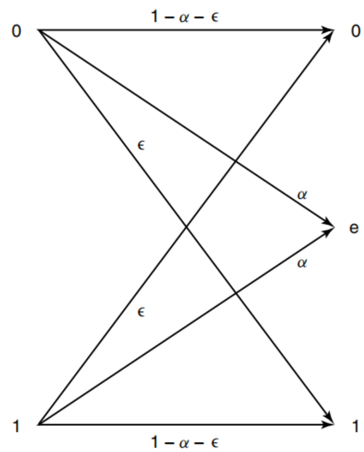
(b) Then we prove that we can construct $p^*(x)$, s.t. $\forall y = 0, \ldots, 10$, $P(Y = y) = \frac{1}{11}$. We define $p_x = P(X = x) = \frac{1}{11}$. Then we have

$$\begin{aligned} P(Y = y) &= \sum_x P(X = x) P(Y = y | X = x) \\ &= \frac{1}{3} p_{(y+8)\text{mod } 11} + \frac{1}{3} p_{(y+9)\text{mod } 11} + \frac{1}{3} p_{(y+10)\text{mod } 11} \\ &= 3 * \frac{1}{3} * \frac{1}{11} \\ &= \frac{1}{11} \end{aligned}$$

So above all, we have proved that $p^*(x) = \frac{1}{11}, \forall x = 0, \ldots, 10$ is the maximizing $p(x)$.

Problem 4

7.13 Erasures and errors in a binary channel. Consider a channel with binary inputs that has both erasures and errors. Let the probability of error be ϵ and the probability of erasure be α, so the channel is follows:



(a) Find the capacity of this channel.

(b) Specialize to the case of the binary symmetric channel ($\alpha = 0$).

(c) Specialize to the case of the binary erasure channel ($\epsilon = 0$).

Solution

(a) Let $p(X = 0) = \pi, p(X = 1) = 1 - \pi$, then we have

$$
\begin{aligned}
p(Y = 0) &= \sum_x p(Y = 0|X = x)p(X = x) = \pi(1 - \alpha - 2\epsilon) + \epsilon \\
p(Y = 1) &= \sum_x p(Y = 1|X = x)p(X = x) = \pi(2\epsilon + \alpha - 1) + 1 - \alpha - \epsilon \\
P(Y = e) &= \alpha
\end{aligned}
$$

So we have

$$
\begin{aligned}
I(X;Y) &= H(Y) - H(Y|X) \\
&= H(Y) - \sum_x p(X = x)H(Y|X = x) \\
&= H\left(\alpha, \pi(1 - \alpha - 2\epsilon) + \epsilon, \pi(2\epsilon + \alpha - 1) + 1 - \alpha - \epsilon\right) - H\left(1 - \alpha - \epsilon, \alpha, \epsilon\right) \\
&= -\left[\pi(2\epsilon + \alpha - 1) + 1 - \alpha - \epsilon\right] \log\left[\pi(2\epsilon + \alpha - 1) + 1 - \alpha - \epsilon\right] - \left[\pi(1 - \alpha - 2\epsilon) + \epsilon\right] \log\left[\pi(1 - \alpha - 2\epsilon) + \epsilon\right] \\
&\quad + \epsilon \log \epsilon + (1 - \alpha - \epsilon) \log (1 - \alpha - \epsilon) \\
&\triangleq f(\pi) \\
f'(\pi) &= -(2\epsilon + \alpha - 1) \log\left[\pi(2\epsilon + \alpha - 1) + 1 - \alpha - \epsilon\right] - (1 - \alpha - 2\epsilon) \log\left[\pi(1 - \alpha - 2\epsilon) + \epsilon\right] \\
f''(\pi) &= -(2\epsilon + \alpha - 1)^2 \log_e 2 \left[\frac{1}{\pi(2\epsilon + \alpha - 1) + 1 - \alpha - \epsilon} + \frac{1}{\pi(1 - \alpha - 2\epsilon) + \epsilon}\right]
\end{aligned}
$$

Then we need to discuss the sign of $2\epsilon + \alpha - 1$ to determine the sign of $f''(\pi)$. Since $\pi \in [0, 1]$, so:

1. If $2\epsilon + \alpha - 1 = 0$, then $f''(\pi) = 0$.

2. If $2\epsilon + \alpha - 1 > 0$, then

$$[\pi(2\epsilon + \alpha - 1) + 1 - \alpha - \epsilon]_{\min} = [\pi(2\epsilon + \alpha - 1) + 1 - \alpha - \epsilon]\Big|_{\pi=0} = 1 - \alpha - \epsilon \geq 0$$

$$[\pi(1 - \alpha - 2\epsilon) + \epsilon]_{\min} = [\pi(1 - \alpha - 2\epsilon) + \epsilon]\Big|_{\pi=1} = 1 - \alpha - \epsilon \geq 0$$

So $f''(\pi) \leq 0$.

3. If $2\epsilon + \alpha - 1 < 0$, then

$$[\pi(2\epsilon + \alpha - 1) + 1 - \alpha - \epsilon]_{\min} = [\pi(2\epsilon + \alpha - 1) + 1 - \alpha - \epsilon]\Big|_{\pi=1} = \epsilon \geq 0$$

$$[\pi(1 - \alpha - 2\epsilon) + \epsilon]_{\min} = [\pi(1 - \alpha - 2\epsilon) + \epsilon]\Big|_{\pi=0} = \epsilon \geq 0$$

So $f''(\pi) \leq 0$.

So above all, we have proved that $f''(\pi) \leq 0$ always holds, so $f(\pi)$ is concave.

In order to get that maximum value of $f(\pi)$, we have

$$f'(\pi^*) = 0 \Rightarrow \pi^* = \frac{1}{2}$$

Then we have

$$C = \max_{p(x)} I(X;Y) = f(\pi^*) = H\left(\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2}\right) - H(1 - \alpha - \epsilon, \alpha, \epsilon)$$

And since we have

$$H\left(\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2}\right) = -\alpha \log \alpha - 2 * \frac{1-\alpha}{2} \log \frac{1-\alpha}{2} = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha) + 1 - \alpha = H(\alpha, 1-\alpha) + 1 - \alpha$$

So above all, when $p(x) = \left(\frac{1}{2}, \frac{1}{2}\right)$, the inequality takes the equality, i.e.

$$C = H(\alpha, 1 - \alpha) - H(1 - \alpha - \epsilon, \alpha, \epsilon) + 1 - \alpha$$

(b) When $\alpha = 0$, $C = 1 - H(\epsilon, 1 - \epsilon)$.

(c) When $\epsilon = 0$, $C = 1 - \alpha$

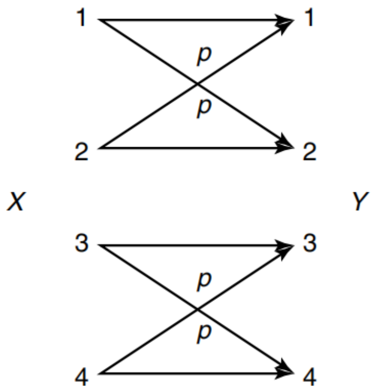
Problem 5

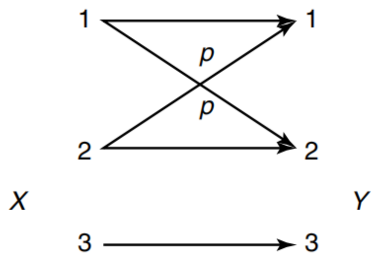
7.34 Capacity. Find the capacity of

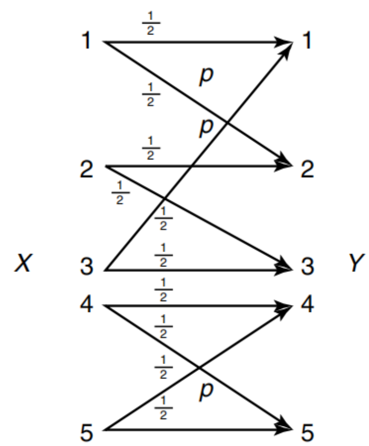
(a) Two parallel BSCs:

(b) BSC and a single symbol:

(c) BSC and a ternary channel:







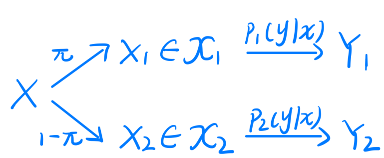
(d) Ternary channel:

$$p(y|x) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Solution

Since for (a), (b), (c): the channels are all combined with two parallel channels, so we can introduce two Lemmas.

Lemma1: If the channel is combined with two parallel channels with probability π to the first channel, and $1-\pi$ to the second channel, which is as followed.



Suppose the channel capacity of the total channel is C, and the capacity of the two channels are C_1 and C_2 respectively. Then we have

$$2^C = 2^{C_1} + 2^{C_2}$$

Proof:

Let $\Theta(X)$ be the auxiliary variable, indicating that X is send to the $\Theta(X)$'s channel. Then we have

$$\Theta(X) = \begin{cases} 1 & , X \in \mathcal{X}_1 \\ 2 & , X \in \mathcal{X}_2 \end{cases}$$

$$C_1 = \max_{p_1(x)} I(X;Y_1)$$

$$C_2 = \max_{p_2(x)} I(X;Y_2)$$

And we also have that $p(\Theta=1)=p(X\in\mathcal{X}_1)=\pi, p(\Theta=2)=p(X\in\mathcal{X}_2)=1-\pi$. And since when given Y or given X, we could know Θ, so we have $I(X;\Theta|Y)=H(\Theta|Y)-H(\Theta|X,Y)=0$. i.e.

$$\begin{aligned} I(X;Y,\Theta) &= I(X;\Theta)+I(X;Y|\Theta) \\ &= I(X;Y)+I(X;\Theta|Y)=I(X;Y) \end{aligned}$$

Then we have

$$\begin{aligned} I(X;Y) &= I(X;\Theta)+I(X;Y|\Theta) \\ &= H(\Theta)-H(\Theta|X)+I(X;Y|\Theta) \\ &= H(\Theta)+I(X;Y|\Theta) && (\Theta \text{ is deterministic when } X \text{ is given}) \\ &= H(\pi,1-\pi)+I(X;Y|\Theta) \end{aligned}$$

As for $I(X;Y|\Theta)$, we have

$$\begin{aligned} I(X;Y|\Theta) &= H(X|\Theta)-H(X|Y,\Theta) \\ &= \sum_\theta p(\Theta=\theta)H(X|\Theta=\theta)-\sum_\theta p(\Theta=\theta)H(X|Y,\Theta=\theta) \\ &= \sum_\theta p(\Theta=\theta)\left[H(X|\Theta=\theta)-H(X|Y,\Theta=\theta)\right] \\ &= \sum_\theta p(\Theta=\theta)I(X;Y|\Theta=\theta) \\ &= \pi I(X_1;Y_1)+(1-\pi)I(X_2;Y_2) \\ &\le \pi C_1+(1-\pi)C_2 && (C_1 \text{ and } C_2 \text{ are the capacity of the two channels}) \end{aligned}$$

So we have

$$I(X;Y)\le H(\pi,1-\pi)\pi C_1+(1-\pi)C_2=-\pi\log\pi-(1-\pi)\log(1-\pi)+\pi C_1+(1-\pi)C_2\triangleq f(\pi)$$

Since we want to maximize $f(\pi)$, so it is obviously that $\pi=0$ or $\pi=1$ is not the optimal solution. And from the property of capcity, we have $C_1,C_2\ge 0$, so we have

$$\begin{aligned} f'(\pi) &= \log(1-\pi)-\log\pi+C_1-C_2 \\ f''(\pi) &= -\log_e 2\left(\frac{1}{1-\pi}+\frac{1}{\pi}\right)<0 \end{aligned}$$

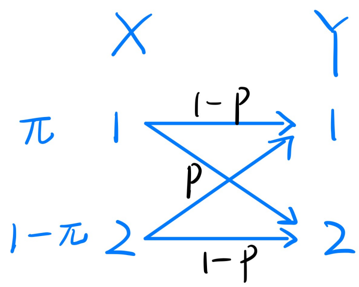
So $f(\pi)$ is a concave function, so the optimal solution is

$$f'(\pi^*)=0\Rightarrow\pi^*=\frac{2^{C_1}}{2^{C_1}+2^{C_2}}$$

So we have

$$\begin{aligned} C &= \max_{p(x)} I(X;Y)=\max_\pi f(\pi)=f(\pi^*) \\ &= \log\left(2^{C_1}+2_2^C\right) \end{aligned}$$

So we have proved that $2^C=2^{C_1}+2^{C_2}$. $\square$



Lemma2: For a BSC. The probability of error is p, then the capacity is $1 - H(p, 1-p)$.

$$\begin{aligned}
I(X;Y) &= H(Y) - H(Y|X) \\
&= H(Y) - \sum_x p(x) H(Y|X=x) \\
&= H(Y) - H(p, 1-p) \\
&\leq 1 - H(p, 1-p)
\end{aligned}$$

And we have when $P(X=1) = P(X=2) = \frac{1}{2}$, then $P(Y=0) = \sum_x P(X=x)P(Y=1|X=x) = \frac{1}{2}$, and so does $P(Y=1)$.

So the inequality takes the equality, so we have when $p(x) = \left(\frac{1}{2}, \frac{1}{2}\right)$

$$C = \max_{p(x)} I(X;Y) = 1 - H(p, 1-p)$$

So with the two Lemmas, we could solve the problem easier.

(a) The two channels are paralleled BSC, so we have $C_1 = C_2 = 1 - H(p, 1-p)$, so we have

$$C = \log\left(2^{1-H(p,1-p)} + 2^{1-H(p,1-p)}\right) = 2 - H(p, 1-p)$$

Where $\pi^* = \dfrac{2^{C_1}}{2^{C_1} + 2^{C_2}} = \dfrac{1}{2}$.
When $p(x) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$, $I(X;Y)$ could take the maximum value.
(b) The first channel is a BSC, so $C_1 = 1 - H(p, 1-p)$. And for the second channel, $Y_2 = X_2 = 3$, so

$$I(X_2;Y_2) = H(X_2) - H(X_2|Y_2) = H(X_2) = 0$$

i.e. $C_2 = 0$. So we have

$$C = \log\left(2^{1-H(p,1-p)} + 2^0\right) = \log\left(2^{1-H(p,1-p)} + 1\right)$$

Where $\pi^* = \dfrac{2^{C_1}}{2^{C_1} + 2^{C_2}} = \dfrac{2}{2 + 2^{H(p,1-p)}}$.
When $p(x) = \left(\dfrac{1}{2 + 2^{H(p,1-p)}}, \dfrac{1}{2 + 2^{H(p,1-p)}}, \dfrac{2^{H(p,1-p)}}{2 + 2^{H(p,1-p)}}\right)$, $I(X;Y)$ could take the maximum value.

(c) The second channel is a BSC, and $p = \frac{1}{2}$, so $C_2 = 1 - H(p, 1-p) = 0$.
And for the first channel:

$$\begin{aligned} I(X;Y) &= H(Y) - H(Y|X) \\ &= H(Y) - \sum_x p(x)H(Y|X=x) \\ &= H(Y) - \sum_x p(x)H\left(\frac{1}{2}, \frac{1}{2}\right) \\ &= H(Y) - 1 \\ &\le \log 3 - 1 \end{aligned}$$

When $p(X=1) = p(X=2) = p(X=3) = \frac{1}{3}$, we have

$$P(Y=1) = \sum_x P(X=x)P(Y=1|X=x) = \frac{1}{3}$$

Similarly, $P(Y=2) = P(Y=3) = \frac{1}{3}$. So the inequality takes the equality, so we have $C_1 = \log 3 - 1$.
So we have

$$C = \log\left(2^0 + 2^{\log 3 - 1}\right) = \log 5 - 1$$

Where $\pi^* = \frac{2^{C_1}}{2^{C_1} + 2^{C_2}} = \frac{3}{5}$ bit per transmission.
When $p(x) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$, $I(X;Y)$ could take the maximum value.
(d)

$$\begin{aligned} I(X;Y) &= H(Y) - H(Y|X) \\ &= H(Y) - \sum_x p(x)H(Y|X=x) \\ &= H(Y) - H\left(\frac{1}{3}, \frac{2}{3}\right) \\ &\le \log 3 - H\left(\frac{1}{3}, \frac{2}{3}\right) \end{aligned}$$

When $p(X=1) = p(X=2) = p(X=3) = \frac{1}{3}$, we have

$$P(Y=1) = \sum_x P(X=x)P(Y=1|X=x) = \frac{1}{3}$$

Similarly, $P(Y=2) = P(Y=3) = \frac{1}{3}$. So the inequality takes the equality.
So we have $C = \log 3 - H\left(\frac{1}{3}, \frac{2}{3}\right)$.
And we have

$$H\left(\frac{1}{3}, \frac{2}{3}\right) = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} = \log 3 - \frac{2}{3}$$

So above all, we have $C = \frac{2}{3}$ bit per transmission.

Problem 6

7.37 Joint typicality. Let (X_i, Y_i, Z_i) be i.i.d. according to $p(x, y, z)$. We will say that (x^n, y^n, z^n) is jointly typical [written $(x^n, y^n, z^n) \in A_\epsilon^{(n)}$] if

- $p\left(x^n\right) \in 2^{-n(H(X) \pm \epsilon)}$.
- $p\left(y^n\right) \in 2^{-n(H(Y) \pm \epsilon)}$.
- $p\left(z^n\right) \in 2^{-n(H(Z) \pm \epsilon)}$.
- $p\left(x^n, y^n\right) \in 2^{-n(H(X,Y) \pm \epsilon)}$.
- $p\left(x^n, z^n\right) \in 2^{-n(H(X,Z) \pm \epsilon)}$.
- $p\left(y^n, z^n\right) \in 2^{-n(H(Y,Z) \pm \epsilon)}$.
- $p\left(x^n, y^n, z^n\right) \in 2^{-n(H(X,Y,Z) \pm \epsilon)}$.

Now suppose that $\left(\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n\right)$ is drawn according to $p\left(x^n\right) p\left(y^n\right) p\left(z^n\right)$. Thus, $\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n$ have the same marginals as $p\left(x^n, y^n, z^n\right)$ but are independent. Find (bounds on) $\Pr\left\{\left(\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n\right) \in A_\epsilon^{(n)}\right\}$ in terms of the entropies $H(X), H(Y), H(Z), H(X,Y), H(X,Z)$, $H(Y,Z)$, and $H(X,Y,Z)$.

Solution

From the definition of jointly typicality, we have in probability:

$$
\begin{gathered}
-\frac{1}{n}\log p(x^n) \to H(X), -\frac{1}{n}\log p(y^n) \to H(Y), -\frac{1}{n}\log p(z^n) \to H(Z), -\frac{1}{n}\log p(x^n, y^n) \to H(X,Y) \\
-\frac{1}{n}\log p(x^n, z^n) \to H(X,Z), -\frac{1}{n}\log p(y^n, z^n) \to H(Y,Z), -\frac{1}{n}\log p(x^n, y^n, z^n) \to H(X,Y,Z)
\end{gathered}
$$

i.e. $\exists N_1, \ldots, N_7 \in \mathbb{N}$, s.t.

$$
\begin{aligned}
\Pr\left[\left|-\frac{1}{n}\log p(x^n) - H(X)\right| \geq \epsilon\right] &< \delta = \frac{\epsilon}{7} \qquad \text{for } n \geq N_1 \\
\Pr\left[\left|-\frac{1}{n}\log p(y^n) - H(Y)\right| \geq \epsilon\right] &< \delta = \frac{\epsilon}{7} \qquad \text{for } n \geq N_2 \\
\Pr\left[\left|-\frac{1}{n}\log p(z^n) - H(Z)\right| \geq \epsilon\right] &< \delta = \frac{\epsilon}{7} \qquad \text{for } n \geq N_3 \\
\Pr\left[\left|-\frac{1}{n}\log p(x^n, y^n) - H(X,Y)\right| \geq \epsilon\right] &< \delta = \frac{\epsilon}{7} \qquad \text{for } n \geq N_4 \\
\Pr\left[\left|-\frac{1}{n}\log p(x^n, z^n) - H(X,Z)\right| \geq \epsilon\right] &< \delta = \frac{\epsilon}{7} \qquad \text{for } n \geq N_5 \\
\Pr\left[\left|-\frac{1}{n}\log p(y^n, z^n) - H(Y,Z)\right| \geq \epsilon\right] &< \delta = \frac{\epsilon}{7} \qquad \text{for } n \geq N_6 \\
\Pr\left[\left|-\frac{1}{n}\log p(x^n, y^n, z^n) - H(X,Y,Z)\right| \geq \epsilon\right] &< \delta = \frac{\epsilon}{7} \qquad \text{for } n \geq N_7
\end{aligned}
$$

Then let $N = \max\{N_1, \ldots, N_7\}$, we have $\forall n \geq N$:

$$
\begin{gathered}
p\left(x^n \notin A_\epsilon^{(n)}(p_X)\right) < \frac{\epsilon}{7}, \quad p\left(y^n \notin A_\epsilon^{(n)}(p_Y)\right) < \frac{\epsilon}{7}, \quad p\left(z^n \notin A_\epsilon^{(n)}(p_Z)\right) < \frac{\epsilon}{7}, \quad p\left((x^n, y^n) \notin A_\epsilon^{(n)}(p_{X,Y})\right) < \frac{\epsilon}{7} \\
p\left((x^n, z^n) \notin A_\epsilon^{(n)}(p_{X,Z})\right) < \frac{\epsilon}{7}, \quad p\left((y^n, z^n) \notin A_\epsilon^{(n)}(p_{Y,Z})\right) < \frac{\epsilon}{7}, \quad p\left((x^n, y^n, z^n) \notin A_\epsilon^{(n)}(p_{X,Y,Z})\right) < \frac{\epsilon}{7}
\end{gathered}
$$

Then we have

$$\begin{aligned}
p\left[(x^n,y^n,z^n)\in A_\epsilon^{(n)}(P_{X,Y,Z})\right] &= 1-p\left[(x^n,y^n,z^n)\notin A_\epsilon^{(n)}(P_{X,Y,Z})\right]\\
&\geq 1-p\left(x^n\notin A_\epsilon^{(n)}(P_X)\right)-p\left(y^n\notin A_\epsilon^{(n)}(P_Y)\right)-p\left(z^n\notin A_\epsilon^{(n)}(P_Z)\right)\\
&\quad -p\left((x^n,y^n)\notin A_\epsilon^{(n)}(P_{X,Y})\right)-p\left((x^n,z^n)\notin A_\epsilon^{(n)}(P_{X,Z})\right)-p\left((y^n,z^n)\notin A_\epsilon^{(n)}(P_{Y,Z})\right)\\
&\quad -p\left((x^n,y^n,z^n)\notin A_\epsilon^{(n)}(P_{X,Y,Z})\right)\ \text{(against any of the requirements of jointly typical set)}\\
&\geq 1-\frac{\epsilon}{7}\times 7 = 1-\epsilon\to 1 \qquad \text{as } n\to\infty
\end{aligned}$$

With $p\left[(x^n,y^n,z^n)\in A_\epsilon^{(n)}(P_{X,Y,Z})\right]\geq 1-\epsilon$, we can get the lower bound of $\left|A_\epsilon^{(n)}(P_{X,Y,Z})\right|$:

$$\begin{aligned}
1-\epsilon &\leq p\left[(x^n,y^n,z^n)\in A_\epsilon^{(n)}(P_{X,Y,Z})\right]\\
&= \sum_{(x^n,y^n,z^n)\in A_\epsilon^{(n)}(P_{X,Y,Z})} p\left(x^n,y^n,z^n\right)\\
&\leq \left|A_\epsilon^{(n)}(P_{X,Y,Z})\right| 2^{-n(H(X,Y,Z)-\epsilon)}
\end{aligned}$$

So the lower bound of $\left|A_\epsilon^{(n)}(P_{X,Y,Z})\right|$ is

$$(1-\epsilon)2^{n(H(X,Y,Z)-\epsilon)}\leq \left|A_\epsilon^{(n)}(P_{X,Y,Z})\right|$$

As for the Upper bound of $\left|A_\epsilon^{(n)}(P_{X,Y,Z})\right|$, we have

$$\begin{aligned}
1 &= \sum_{(x^n,y^n,z^n)} p\left(x^n,y^n,z^n\right)\\
&\geq \sum_{(x^n,y^n,z^n)\in A_\epsilon^{(n)}(P_{X,Y,Z})} p\left(x^n,y^n,z^n\right)\\
&\geq \left|A_\epsilon^{(n)}(P_{X,Y,Z})\right| 2^{-n(H(X,Y,Z)+\epsilon)}
\end{aligned}$$

So the upper bound of $\left|A_\epsilon^{(n)}(P_{X,Y,Z})\right|$ is

$$2^{n(H(X,Y,Z)+\epsilon)}\geq \left|A_\epsilon^{(n)}(P_{X,Y,Z})\right|$$

Therefore, we have

$$2^{n(H(X,Y,Z)-\epsilon)}\leq \left|A_\epsilon^{(n)}(P_{X,Y,Z})\right|\leq 2^{n(H(X,Y,Z)+\epsilon)}$$

Then for the bound of $\Pr\left\{\left(\tilde{X}^n,\tilde{Y}^n,\tilde{Z}^n\right)\in A_\epsilon^{(n)}(P_{X,Y,Z})\right\}$, we have

$$\begin{aligned}
\Pr\left\{\left(\tilde{X}^n,\tilde{Y}^n,\tilde{Z}^n\right)\in A_\epsilon^{(n)}(P_{X,Y,Z})\right\} &= \sum_{(x^n,y^n,z^n)\in A_\epsilon^{(n)}(P_{X,Y,Z})} p\left(x^n\right)p\left(y^n\right)p\left(z^n\right) = \left|A_\epsilon^{(n)}(P_{X,Y,Z})\right| p\left(x^n\right)p\left(y^n\right)p\left(z^n\right)\\
&\in \Big[(1-\epsilon)2^{n(H(X,Y,Z)-\epsilon)}\cdot 2^{-n(H(X)+\epsilon)}\cdot 2^{-n(H(Y)+\epsilon)}\cdot 2^{-(nH(Z)+\epsilon)},\\
&\qquad 2^{n(H(X,Y,Z)+\epsilon)}\cdot 2^{-n(H(X)-\epsilon)}\cdot 2^{-n(H(Y)-\epsilon)}\cdot 2^{-(nH(Z)-\epsilon)}\Big]\\
&= \left[(1-\epsilon)2^{n(H(X,Y,Z)-H(X)-H(Y)-H(Z)-4\epsilon)}, 2^{n(H(X,Y,Z)-H(X)-H(Y)-H(Z)+4\epsilon)}\right]
\end{aligned}$$

So above all, the bound of $\Pr\left\{\left(\tilde{X}^n,\tilde{Y}^n,\tilde{Z}^n\right)\in A_\epsilon^{(n)}(P_{X,Y,Z})\right\}$ is

$$(1-\epsilon)2^{n(H(X,Y,Z)-H(X)-H(Y)-H(Z)-4\epsilon)}\leq \Pr\left\{\left(\tilde{X}^n,\tilde{Y}^n,\tilde{Z}^n\right)\in A_\epsilon^{(n)}(P_{X,Y,Z})\right\}\leq 2^{n(H(X,Y,Z)-H(X)-H(Y)-H(Z)+4\epsilon)}$$