Fundamentals of Information Theory

Homework 1

Name: Zhou Shouchen
Student ID: 2021533042

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Problem 1

2.1 Coin flips. A fair coin is flipped until the first head occurs. Let X denote the number of flips required.

(a) Find the entropy $H(X)$ in bits. The following expressions may be useful:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad \sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2}$$

(b) A random variable X is drawn according to this distribution. Find an "efficient" sequence of yes-no questions of the form, "Is X contained in the set S ?" Compare $H(X)$ to the expected number of questions required to determine X.

Solution

(a)

(b)

Problem 2

2.3 Minimum entropy. What is the minimum value of $H\left(p_{1}, \ldots, p_{n}\right)=H(\boldsymbol{p})$ as $\boldsymbol{p}$ ranges over the set of n-dimensional probability vectors? Find all $\boldsymbol{p}$'s that achieve this minimum.

Solution

Problem 3

2.5 Zero conditional entropy. Show that if $H(Y \mid X) = 0$, then Y is a function of X [i.e., for all x with $p(x) > 0$, there is only one possible value of y with $p(x, y) > 0$].

Solution

Problem 4

2.12 Example of joint entropy. Let $p(x, y)$ be given by

$X \backslash Y$	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

Find:

(a) $H(X), H(Y)$.

(b) $H(X \mid Y), H(Y \mid X)$.

(c) $H(X, Y)$.

(d) $H(Y) - H(Y \mid X)$.

(e) $I(X; Y)$.

(f) Draw a Venn diagram for the quantities in parts (a) through (e).

Solution

(a)

(b)

(c)

(d)

(e)

Problem 5

2.14 Entropy of a sum. Let X and Y be random variables that take on values $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$, respectively. Let $Z = X + Y$.

(a) Show that $H(Z \mid X) = H(Y \mid X)$. Argue that if X, Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus, the addition of independent random variables adds uncertainty.

(b) Give an example of (necessarily dependent) random variables in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.

(c) Under what conditions does $H(Z) = H(X) + H(Y)$?

Solution

(a)

(b)

(c)