1. (18 points) A discrete-time system has a unit-impulse response $h[n]$.

(a) Let the input to the discrete-time system be a pulse $x[n] = u[n] - u[n-4]$. Compute the output of the system **in terms of the impulse response.**

(b) Let $h[n] = 0.5^n u[n]$. What would be the response $y[n]$ of the system to $x[n]$? Show whether the system is causal and stable?

(c) Use the convolution sum to verify your response $y[n]$. Plot the output $y[n]$ for $n \in [-2,4]$.

a) $u[n] = \sum_{k=0}^{\infty} \delta[n-k] \rightarrow$

$x[n] = \sum_{k=0}^{\infty} \delta[n-k] - \sum_{k=4}^{\infty} \delta[n-k] = \sum_{k=0}^{3} \delta[n-k]$

$y[n] = \sum_{k=0}^{3} h[n-k]$

b). $y[n] = \sum_{k=0}^{3} 0.5^{n-k} u[n-k] \xrightarrow{n-k=m,\ k\in[0,3],\ m\in[n-3,n]}$

$= \sum_{m=n-3}^{n} 0.5^m u[m]$

➢ For any system whose input-output relationship is defined by

$y[n] = f(x[n])$

the impulse response $h[n]$ is calculated as

$h[n] = f(\delta[n])$ replace $x[n]$ by $\delta[n]$

□ If $h[n] = 0$ for $n < 0$, or $h(t) = 0$ for $t < 0$, the system is causal

$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ absolutely summable

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$= \sum_{m=n-3}^{n} 0.5^m u[m]$

$= 0.5^n [u[n] + 2u[n-1] + 4u[n-2] + 8u[n-3])$

$\sum_{k=0}^{\infty} |h[k]| = \sum_{k=0}^{\infty} 0.5^k \longrightarrow \frac{1-0.5^{n+1}}{1-0.5} \rightarrow \frac{1}{0.5} = 2. < \infty$

c) $y[n] = \sum_{k=-\infty}^{\infty} h[n-k]\, x[k] = 0.5^n \sum_{k=-\infty}^{\infty} 0.5^{-k} u[n-k]\, u[k] \rightarrow \begin{cases} k \geq 0 \\ n-k \geq 0 \end{cases}$ $0 \leq k \leq n$

$= 0.5^n \sum_{k=0}^{n} 2^k = 0.5^n (2^{n+1} - 1)\, \underline{u[n]}$

zero otherwise

$y[n] = 0.5^n (2^{n+1} - 1) u[n] - 0.5^{n-4} (2^{n+1-4} - 1) u[n-4]$

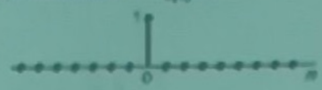
$$\begin{cases} 0 & n < 0 \\ 2 - 0.5^n & 0 \leq n \leq 3 \\ 15 \cdot (0.5)^n & n \geq 4 \end{cases}$$

2. (12 points) A causal, linear, time-invariant (LTI) system is governed by the difference equation

$$y[n] = -y[n-1] + x[n] - x[n-1]$$

(a) Is the step response bounded? Motivate your answer.
(b) Is the system stable? Motivate your answer.

- The impulse is the first difference of the step
  $\delta[n] = u[n] - u[n-1]$
- Unit impulse (unit sample) is defined as $\delta[n] = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$



- If $h[n] = 0$ for $n < 0$, or $h(t) = 0$ for $t < 0$, the system is causal
- The unit step response, $s[n]$, corresponding to the output with input $x[n] = u[n]$

$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ absolutely summable

$$h[n] = s[n] - s[n-1]$$

2. (a) causal $y[n] = 0$, $n < 0$

$s[n] = -s[n-1] + u[n] - u[n-1]$.

$n<0$, $s[n] = 0$

$n=1$, $s[0] = -s[-1] + 1 = 1$

$n=2$, $s[1] = -s[0] = -1$.

$n=3$, $s[2] = -s[1] = 1$.

$n \geq 1 \rightarrow s[n] = -s[n-1]$

$s[n] = (-1)^n u[n]$

$|s[n]| = 1 < \infty$

$h[n] = [u[n]]$

$s[n] = f(u[n])$

$\sum$

h

b) $h[n] = s[n] - s[n-1] = (-1)^n u[n] - (-1)^{n-1} u[n-1]$

$$h[n] = \begin{cases} 0, & n < 0 \\ 1, & n = 1 \\ 2(-1)^n, & \text{otherwise} \end{cases}$$

$\sum_{k=-\infty}^{\infty} |h[k]| = 0 + 1 + \sum_{k=2}^{\infty} 2 \rightarrow \infty$

3. (18 points)

(a) Determine the Fourier transform of

$$x(t) = \cos(\omega_0 t)\sin(\omega_1 t);$$

(b) A periodic signal $x(t)$ has the Fourier series

$$x(t) = \sum_{k=1}^{\infty} \frac{2}{k^2}\cos\left(\frac{3kt}{2}\right).$$

Compute the Fourier transform, $X(j\omega)$.

(c) Use the duality theorem to prove the following Fourier transform result (Hint: for $a > 0$, $\mathcal{F}[e^{-at}u(t)] = \frac{1}{a+j\omega}$):

$$x(t) = \frac{1}{t^2 + a^2}, a > 0 \quad \overset{FT}{\leftrightarrow} \quad X(j\omega) = \frac{\pi}{a}e^{-a|j\omega|}$$

$$r(t) = s(t)p(t) \overset{\mathcal{F}}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j(\omega))]$$

➢For $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ $\quad X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$

$$\boxed{X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-j\omega)}$$

Example

❑ For $a > 0$ $\quad e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} 1/(a + j\omega)$

$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} 2a/(a^2 + \omega^2)$

❑ use FT properties

$$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) = 2E_v\{e^{-at}u(t)\}$$

$$E_v\{e^{-at}u(t)\} \overset{\mathcal{F}}{\leftrightarrow} R_e\left\{\frac{1}{a+j\omega}\right\}$$

$$\mathcal{F}\{e^{-a|t|}\} = 2R_e\left\{\frac{1}{a+j\omega}\right\} = \frac{2a}{a^2+\omega^2}$$

(a) $\mathcal{F}[\cos(\omega_0 t)] = \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

$\mathcal{F}[\sin(\omega_0 t)] = \frac{\pi}{j}[\delta(\omega-\omega_1) - \delta(\omega-\omega_1)]$

$X(j\omega) = j\frac{\pi}{2}[\delta(\omega-\omega_0+\omega_1) + \delta(\omega+\omega_0+\omega_1) - \delta(\omega-\omega_0-\omega_1) - \delta(\omega+\omega_0-\omega_1)]$

b) $\omega_0 = \frac{3}{2}$.

$$a_k = \begin{cases} 0, & k=0 \\ \frac{1}{k^2}, & k \neq 0 \end{cases}$$

$$X(j\omega) = \sum_{k=1}^{\infty} 2\pi\frac{1}{k^2}\delta(\omega - \tfrac{3}{2}k) + \sum_{k=-\infty}^{-1} 2\pi\frac{1}{k^2}\delta(\omega - \tfrac{3}{2}k).$$

$\rightarrow k = -k.$

$$= 2\pi\sum_{k=1}^{\infty}\frac{1}{k^2}[\delta(\omega - \tfrac{3}{2}k) + \delta(\omega + \tfrac{3}{2}k)]$$

c) $e^{-a|t|} \overset{ft}{\rightarrow} \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{2a}{a^2+\omega^2}$ ✓.

a

❑ use FT properties

$$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) = 2E_v\{e^{-at}u(t)\}$$

$$E_v\{e^{-at}u(t)\} \overset{\mathcal{F}}{\leftrightarrow} R_e\left\{\frac{1}{a+j\omega}\right\}$$

$$\mathcal{F}\{e^{-a|t|}\} = 2R_e\left\{\frac{1}{a+j\omega}\right\} = \frac{2a}{a^2+\omega^2}$$

4. (16 points) Given the following properties of a discrete-time signal $x[n]$:
(1) $x[n]$ is real and even.

(2) The period of $x[n]$ is $N = 10$.

(3) Its Fourier series is denoted as a_k, and $a_{11} = 5$.

(4) $\frac{1}{10}\sum_{n=0}^{9} |x[n]|^2 = 50$

(a) Prove that $x[n] = A\cos(Bn + C)$, and calculate the constants A, B and C.

(b) If $x[n]$ is real and odd, $a_{11} = j5$ and the other properties (2) and (4) unchanged, find the sinosoidal expression of $x[n]$.

Solution:

(a) From 1), we know that a_k is real and even; **(1 points)**

From 2) and 3), we know that $a_{11} = a_1 = 5$, and therefore $a_{-1} = 5$; **(2 points)**

From 4), according to the Parseval's relation, we know that $\sum_{n=0}^{9} |a_k|^2 = 50$, as $|a_1|^2 + |a_{-1}|^2 = 50$, $a_k = 0$ for $k \neq rN + 1$ and $k \neq rN - 1$. **(2 points)**

Therefore, $x[n] = a_1 e^{j\frac{2\pi}{N}n} + a_{-1}e^{-j\frac{2\pi}{N}n} = 10\cos(\frac{\pi}{5}n)$ **(2 points)**

So $A = 10, B = \frac{\pi}{5}$ and $C = 0$. **(2 points)**

(b) From 2) and 3), we know that $a_{11} = a_1 = j5$, and therefore $a_{-1} = -j5$; **(2 points)**

From 4), $a_k = 0$ for $k \neq rN + 1$ and $k \neq rN - 1$. **(2 points)**

Therefore, $x[n] = a_1 e^{j\frac{2\pi}{N}n} + a_{-1}e^{-j\frac{2\pi}{N}n} = -10\sin(\frac{\pi}{5}n)$ **(2 points)**

5. (16 points) A signal and its Fourier transform are denoted as $x(t)$ and $X(j\omega)$, respectively. Prove that:

(a) If $x(t)$ is real and even, $X(j\omega)$ is real and even.

(b) If $x(t)$ is real and odd, $X(j\omega)$ is pure imaginary and odd.

(c)
$$x_e(t) \xleftrightarrow{\mathcal{F}} Re[X(j\omega)]$$
$$x_o(t) \xleftrightarrow{\mathcal{F}} j \cdot Im[X(j\omega)]$$

(Note that $x(t) = x_e(t) + x_o(t)$, where $x_e(t)$ is even, and $x_o(t)$ is odd. $Re[a]$ and $Im[a]$ are the real and imaginary parts of the complex number a, respectively.

(a) $x(t) = x(-t)$. even

$$\left.\begin{aligned} X(j\omega) &= X(-j\omega) \\ X(j\omega) &= X^*(j\omega) \end{aligned}\right\} \Rightarrow \underbrace{X(j\omega) = X^*(j\omega)}_{\text{real.}}$$

Conjugation and Conjugate Symmetry

- Conjugation property $\quad x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Longrightarrow \boxed{x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)}$

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\right]^* = \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}dt$$

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t)e^{-j\omega t}dt = \mathcal{F}\{x^*(t)\}$$

- Conjugation Symmetry

$$\boxed{X(-j\omega) = X^*(j\omega) \qquad [x(t)\ \text{real}]}$$

For a real-valued signal, the FT need only to be specified for positive frequencies.

Time reversing $\quad x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Longrightarrow \boxed{x(-t) \xleftrightarrow{\mathcal{F}} X(-j\omega)}$

- $x(t)$ even $\Longrightarrow X(j\omega) = X(-j\omega)$, $x(t)$ real $\Longrightarrow X(-j\omega) = X^*(j\omega)$
- $x(t)$ real and even $\Longrightarrow X(j\omega)$ real and even
- $x(t)$ real and odd $\Longrightarrow X(j\omega)$ purely imaginary and odd
- If $x(t)$ real

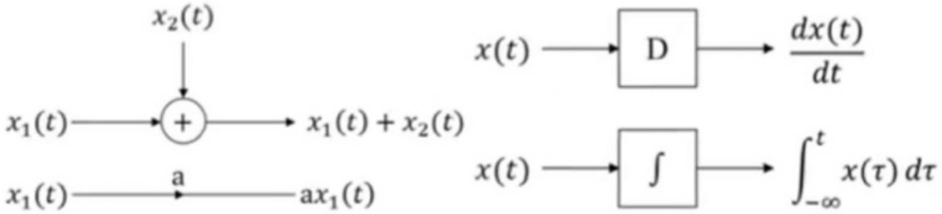
$$\left.\begin{aligned} x(t) &= x_e(t) + x_o(t) \\ \mathcal{F}\{x(t)\} &= \mathcal{F}\{x_e(t)\} + \mathcal{F}\{x_o(t)\} \end{aligned}\right\} \Leftrightarrow \begin{cases} E_v\{x(t)\} \xleftrightarrow{\mathcal{F}} R_e\{X(j\omega)\} \\ O_d\{x(t)\} \xleftrightarrow{\mathcal{F}} j \cdot I_m\{X(j\omega)\} \end{cases}$$

6. (20 points) Let G represent a causal and stable system that is described by the following differential equation

$$\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} - x(t)$$

Where $x(t)$ represents the input signal and $y(t)$ represents the output signal.

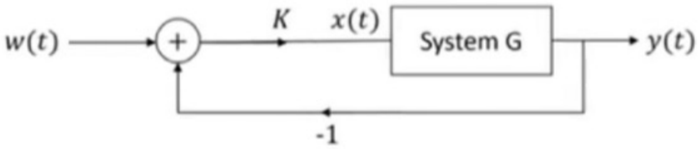
(a) Draw the block diagram of the differential equation using the basic elements illustrated as bellows.



(b) Determine the output of G when the input is

$$x_1(t) = \begin{cases} e^{-6t}; & t \geq 0 \\ 0; & otherwise \end{cases}$$

(c) Now consider a feedback loop that contains the G system described above.

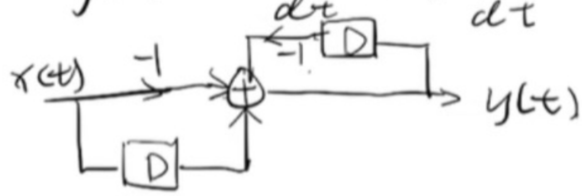


Determine a differential equation that relates $w(t)$ to $y(t)$ when $K = 10$ and find the frequency response $H(j\omega)$ of this system. The differential equation should not contain references to $x(t)$

$$\mathcal{F}\left\{\frac{dy(t)}{dt} + ay(t)\right\} = \mathcal{F}\{x(t)\}$$

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega)$$

6. a) $y(t) = \frac{dx(t)}{dt} - \frac{dy(t)}{dt} - x(t)$



b) $\frac{dy(t)}{dt} = j\omega Y(j\omega)$

$j\omega Y(j\omega) + Y(j\omega) = j\omega X(j\omega) - X(j\omega)$

$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega - 1}{j\omega + 1}$

$X_1(j\omega) = \frac{1}{j\omega + 6}$

$\Rightarrow Y_1(j\omega) = \frac{j\omega - 1}{(j\omega + 6)(j\omega + 1)}$

$\Rightarrow Y_1(j\omega) = \frac{j\omega - 1}{(j\omega + 6)(j\omega + 1)} \Rightarrow -\frac{2}{5}\frac{1}{j\omega + 1} + \frac{7}{5}\frac{1}{j\omega + 6}$

$\xrightarrow{FT^{-1}} y_1(t) = -\frac{2}{5}e^{-t}u(t) + \frac{7}{5}e^{-6t}u(t)$