Lab 3 Analysis of Periodic Signals in the Frequency Domain

Objective

- Explore the relationship between the time domain and the frequency domain.
- Understand the Fourier series of periodic signals.
- Master the expression of the signal spectrum.

Content

Fourier Series of Periodic Signal

Fourier analysis contains Fourier series and Fourier transform. The former is used to represent a periodic signal by a discrete sum of complex exponentials, while the latter is used to represent an aperiodic signal by a continuous superposition or integral of complex exponentials.

According to the theory of Fourier analysis, when satisfy the Dirichlet conditions, a periodic signal with period T_1 can be decomposed into sinusoid signals with different frequencies, magnitudes and phases.
The trigonometric Fourier series is shown as follows:

$$\mathrm{f}(t) = a_{0+}\sum_{n=1}^{\infty}(a_n \cos n\omega_1 t + b_n \sin n\omega_1 t)$$

$$a_0 = \frac{1}{T_1}\int_{t_0}^{t_0+T1} f(t)\,dt$$

$$a_n = \frac{2}{T_1}\int_{t_0}^{t_0+T1} f(t)\cos n\omega_1 t\,dt$$

$$b_n = \frac{2}{T_1}\int_{t_0}^{t_0+T1} f(t)\sin n\omega_1 t\,dt$$

$$\omega_1 = \frac{2\pi}{T_1}$$

ω_1 is called the fundamental angular frequency. $n\omega_1$ is called Nth harmonic. What do you find?
Example: The periodic rectangular pulse is shown in Figure 1. Expand it into a Fourier series.

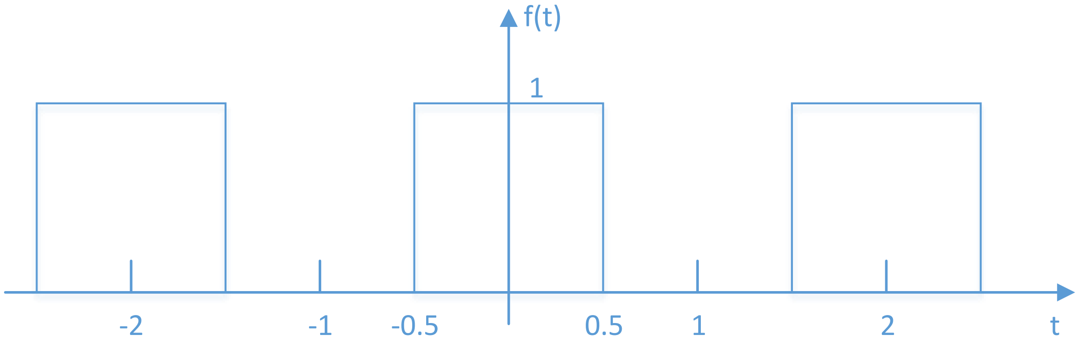

Figure 1 Periodic Rectangular Pulse

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{2}\int_{-1}^{1} f(t)dt = \frac{1}{2}[\int_{-1}^{-0.5} 0dt + \int_{-0.5}^{0.5} 1dt + \int_{0.5}^{1} 0dt] = \frac{1}{2}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t dt = \frac{2}{2}\int_{-1}^{1} f(t)\cos n\omega t dt$$

$$= \int_{-1}^{-0.5} 0 \times \cos n\pi t\, dt + \int_{-0.5}^{0.5} 1 \times \cos n\pi t\, dt + \int_{0.5}^{1} 0 \times \cos n\pi t\, dt$$

$$= \frac{1}{n\pi}\sin n\pi t \Big|_{-0.5}^{0.5} = \frac{2}{n\pi}\sin\frac{n\pi}{2} = Sa(\frac{n\pi}{2})$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t dt = \frac{2}{2}\int_{-1}^{1} f(t)\sin n\omega t dt$$

$$= \int_{-1}^{-0.5} 0 \times \sin n\pi t\, dt + \int_{-0.5}^{0.5} 1 \times \sin n\pi t\, dt + \int_{0.5}^{1} 0 \times \sin n\pi t\, dt$$

$$= -\frac{1}{n\pi}\cos n\pi t \Big|_{-0.5}^{0.5} = 0$$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} Sa(\frac{n\pi}{2})\cos n\pi t\,, \omega = \frac{2\pi}{T} = \pi$$

Realize Fourier Series with Numeric Method

Fourier Series with Loop Structure:

```
% numeric method: loop function, trapz
clear;clf;
T = 2; f = 1/T; w1 = 2*pi*f;
dt = 0.01;
t = -3:dt:3;
tao = -1:dt:1;
ft = 0.5+0.5*square(pi*(tao+0.5),50);
a0 = trapz(tao,ft)/T;
```

```
f = a0;
N = input('N=');
an = zeros(1,N);
bn = zeros(1,N);
for n = 1:N
    fcos = ft.*cos(n*w1*tao); an(n)=trapz(tao,fcos)*2/T;
    fsin = ft.*sin(n*w1*tao); bn(n)=trapz(tao,fsin)*2/T;
    f = f+ an(n)*cos(n*w1*t)+bn(n)*sin(n*w1*t);
end
plot(t,f);xlabel('t(s)');ylabel('ft'); grid on;
title(['Numeric Loop with N=' num2str(N)]);
```

Fourier Series with Matrix Operation:

Except loop structure, we can also calculate the Fourier Series by the advantage of matrix.

For $a_n = \frac{2}{T_1}\int_{t_0}^{t_0+T1} f(t)\cos n\omega_1 t\,dt$, let ω_n be $n\omega_1$, let $\tau \in [t_0, t_0 + \mathrm{T}]$ and τ_m be $(\mathrm{m}-1) * \mathrm{d}\tau$ $(\mathrm{d}\tau = \mathrm{dt})$.

Then we can change a_n to a matrix like this:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \frac{2}{\mathrm{T}_1}\int_{t_0}^{t_0+T_1} \begin{bmatrix} f(\tau)\bullet\cos w_1\tau \\ f(\tau)\bullet\cos w_2\tau \\ \vdots \\ f(\tau)\bullet\cos w_n\tau \end{bmatrix} d\tau$$

$$= \frac{2}{\mathrm{T}_1}\int_{t_0}^{t_0+T_1} \begin{bmatrix} f(\tau_1)\bullet\cos w_1\tau_1 & f(\tau_2)\bullet\cos w_1\tau_2 & \cdots & f(\tau_m)\bullet\cos w_1\tau_m \\ f(\tau_1)\bullet\cos w_2\tau_1 & f(\tau_2)\bullet\cos w_2\tau_2 & \cdots & f(\tau_m)\bullet\cos w_2\tau_m \\ \vdots & \vdots & \ddots & \vdots \\ f(\tau_1)\bullet\cos w_n\tau_1 & f(\tau_2)\bullet\cos w_n\tau_2 & \cdots & f(\tau_m)\bullet\cos w_n\tau_m \end{bmatrix} d\tau$$

$$= \frac{2}{\mathrm{T}_1}\int_{t_0}^{t_0+\mathrm{T}_1} \begin{bmatrix} f(\tau_1) & f(\tau_2) & \cdots & f(\tau_m) \end{bmatrix} \cdot * \cos\left(\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} * \begin{bmatrix} \tau_1 & \tau_2 & \cdots & \tau_m \end{bmatrix} \right) d\tau$$

When calculating $\sum_{n=1}^{\infty}(a_n cos w_n t)$, let $\omega_n = n\omega_1$ and $t_k = (k-1) * dt$,we can also change it to the matrix calculation, like:

$$\sum_{n=1}^{\infty}\left(a_n \cos w_n t\right)$$

$$=\left[a_1 \cos w_1 t + a_2 \cos w_2 t + \cdots a_n \cos w_n t\right]$$

$$=\begin{bmatrix} a_1 \cos w_1 t_1 + a_2 \cos w_2 t_1 + \cdots a_n \cos w_n t_1 \\ a_1 \cos w_1 t_2 + a_2 \cos w_2 t_2 + \cdots a_n \cos w_n t_2 \\ \vdots \\ a_1 \cos w_1 t_k + a_2 \cos w_2 t_k + \cdots a_n \cos w_n t_k \end{bmatrix}'$$

$$=\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} * \begin{bmatrix} \cos w_1 t_1 & \cos w_1 t_2 & \cdots & \cos w_1 t_k \\ \cos w_2 t_1 & \cos w_2 t_2 & \cdots & \cos w_2 t_k \\ \vdots & \vdots & \ddots & \vdots \\ \cos w_n t_1 & \cos w_n t_2 & \cdots & \cos w_n t_k \end{bmatrix}$$

$$=\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} * \cos\left(\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} * \begin{bmatrix} t_1 & t_2 & \cdots & t_k \end{bmatrix}\right)$$

```
% numeric method: matrix calculation
clear;clf;
T = 2; f = 1/T; w1 = 2*pi*f;
dt = 0.01;
t = -3:dt:3;
tao = -1:dt:1;
ft = 0.5+0.5*square(pi*(tao+0.5),50);
a0 = trapz(tao,ft)/T;
N = input('N=');
n = 1:N;
fcos = ft.*cos(n'*w1*tao); an = trapz(tao,fcos,2)*2/T;
fsin = ft.*sin(n'*w1*tao); bn = trapz(tao,fsin,2)*2/T;
f = a0 + an'*cos(n'*w1*t) + bn'*sin(n'*w1*t);
plot(t,f); xlabel('t(s)');ylabel('ft'); grid on;
title(['Numeric Matrix with N=' num2str(N)]);
```

Realize the Fourier Series with Symbolic Method

Calculate the Fourier Series with MATLAB (symbolic method):

```
% Symbolic method: loop function, int
clear; clf;
N = input('N=');
syms t a0 an bn n
T1 = 2; freq = 1/T1; w1 = 2*pi*freq;
range = [-1.5,0.5];          % take one cycle range
```

```
ft = 0.5+0.5*sign(t+0.5);   % the expression within the cycle
a0 = 1/T1*int(ft,t,range);
f = a0;
for n=1:N
    an = 2/T1*int(ft*cos(n*w1*t),t,range);
    bn = 2/T1*int(ft*sin(n*w1*t),t,range);
    f = f+an*cos(n*w1*t)+bn*sin(n*w1*t);
end
fplot(f); xlabel('t');ylabel('y(t)');
title(['Symbolic Loop with N=' num2str(N)]);
axis([-3,3,-0.2,1.2]); grid on;
```

1. Try to compare the three results by plotting them.
2. Change the value of N. What can you find?

Gibbs Phenomenon

Generally, a signal can be reconstructed with a small number of Fourier series if the original signal is smooth. For discontinuous signals, plenty of high-frequency components are required to reconstruct the signal accurately.
For signals with discontinuities, which are infinite in the frequency domain, only a portion of the Fourier series will be superimposed during reconstruction. As a result, there will a substantial overshoot near these discontinuities. As N increases, overshoot will occur at smaller and smaller intervals. However, the increase in N does not reduce the magnitude of the overshoot. This is called the Gibbs phenomenon, which is shown in figure 2.

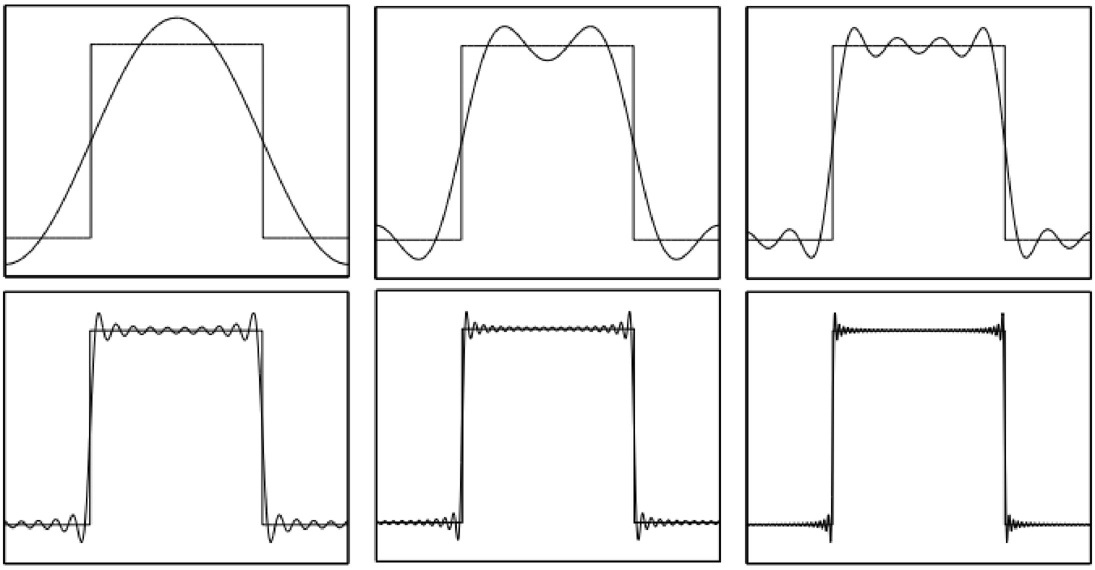

Figure 2 Gibbs Phenomenon

Frequency Analysis of Periodic Signal

Another format of trigonometric Fourier series is like:

$$\begin{aligned}
\mathrm{f}(t) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_1 t + b_n \sin n\omega_1 t) \\
&= a_0 + \sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2}\left[\frac{a_n}{\sqrt{a_n^2+b_n^2}}cos\omega_1 t - \frac{-b_n}{\sqrt{a_n^2+b_n^2}}sin\omega_1 t\right] \\
&= a_0 + \sum_{n=1}^{\infty} c_n(cos\varphi_n cosn\omega_1 t - sin\varphi_n sinn\omega_1 t) \\
&= c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_1 t + \varphi_n)
\end{aligned}$$

The corresponding relationship between the coefficients is as follows:

$$c_0 = a_0$$
$$c_n = \sqrt{a_n^2+b_n^2}, n = 1,2, \dots \dots$$
$$\varphi_n = -\tan^{-1}\frac{b_n}{a_n}$$

The unilateral spectrum of the signal can be calculated by a trigonometric Fourier series.

Periodic signals can also be expanded into a Fourier series of complex exponential form, which is like:

$$\begin{aligned}
\mathrm{f}(t) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_1 t + b_n \sin n\omega_1 t) \\
&= a_0 + \sum_{n=1}^{\infty}\left(a_n \cdot \frac{1}{2}\left[e^{jn\omega_1 t} + e^{-jn\omega_1 t}\right] + b_n \cdot \frac{-j}{2}\left[e^{jn\omega_1 t} + e^{-jn\omega_1 t}\right]\right) \\
&= a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - jb_n}{2}e^{jn\omega_1 t} + \frac{a_n + jb_n}{2}e^{-jn\omega_1 t}\right)
\end{aligned}$$

since $a_n = a_{-n}, b_n = -b_{-n}$

$$\mathrm{f}(t) = a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - jb_n}{2}e^{jn\omega_1 t} + \frac{a_{-n} - jb_{-n}}{2}e^{-jn\omega_1 t}\right)$$

let $F_n = \frac{a_n - jb_n}{2}, F_{-n} = \frac{a_{-n} - jb_{-n}}{2}$

then we have:

$$\mathrm{f}(t) = a_0 + \sum_{n=1}^{\infty}\left(F_n e^{jn\omega_1 t} + F_{-n}e^{-jn\omega_1 t}\right) = a_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(F_n e^{jn\omega_1 t}\right)$$

$$F_n = \frac{1}{2}\left[\frac{2}{T_1}\int_{t_0}^{t_0+T_1} f(t)(cosn\omega_1 t - jsinn\omega_1 t)dt\right] = \frac{1}{T_1}\int_{t_0}^{t_0+T1} f(t)e^{-jn\omega_1 t}\,dt$$

$$F_0 = \frac{1}{T_1}\int_{-T_1/2}^{T_1/2} f(t)dt = a_0$$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_1 t}$$

Different from the trigonometric form of the Fourier series, the exponential form displays the spectral in a bilateral spectral mode.

The relationship of complex exponential Fourier series and trigonometric Fourier series is:

$$F_0 = a_0 = c_0$$

$$F_n = |F_n| e^{j\varphi_n} = \frac{1}{2}(a_n - jb_n) = \frac{1}{2} c_n e^{j\varphi_n}$$

$$|F_n| = \frac{1}{2}\sqrt{{a_n}^2 + {b_n}^2} = \frac{1}{2} c_n$$

$$\varphi_n = -\tan^{-1}\frac{b_n}{a_n}$$

So F_n is the complex spectrum of $f(t)$. From F_n, we can find out the amplitude-frequency and phase-frequency characteristics by using function **abs**() and **angle**().

To find out the spectrum of the signal in Figure 1, we can use both symbolic method and numeric method.

```
clear; clf;
N = input('N=');
T1 = 2; freq = 1/T1; w1 = 2*pi*freq;

% Symbolic method: loop function, int
syms t1 a0 an bn n
range = [-1.5,0.5];
ft = 0.5+0.5*sign(t1+0.5);
Wn = (-N:N)*w1;
k = 1;
for n=-N:N
    Fn(k) = 1/T1*int(ft*exp(-1j*n*w1*t1),t1,range);
    k = k+1;
end
subplot(2,2,1);stem(Wn/(2*pi),abs(Fn));
xlabel('f(Hz)');ylabel('Amplitude');title('Symbolic Amplitude');
subplot(2,2,2);stem(Wn/(2*pi),angle(Fn)*180/pi);
xlabel('f(Hz)');ylabel('Phase(angle)');title('Symbolic Phase');

% numeric method: loop function, trapz
dt = 0.01;
t = -3:dt:3;
tao = -1:dt:1;
ft = 0.5+0.5*square(pi*(tao+0.5),50);
Wn = (-N:N)*w1;
Fn = zeros(1,2*N+1);
i = 1;
```

```
for n = -N:N
    F =ft.*exp(-j*n*w1*tao);
    Fn(i) =  trapz(tao,F)/T1;
    i = i+1;
end
subplot(2,2,3);stem(Wn/(2*pi),abs(Fn));
xlabel('f(Hz)');ylabel('Amplitude');title('Numeric Amplitude');
subplot(2,2,4);stem(Wn,angle(Fn).*(abs(Fn)>=1e-10));
xlabel('f(Hz)');ylabel('Phase(radian)');title('Numeric Phase');
```

Tips: when we use the numeric method to find out frequency-phase characteristics with function **angle( )**, since only a finite length of data can be calculated, the introduction of error cannot be avoided. Usually, the data less than 1e-10 can be ignored. So we can solve this problem by multiplying a logical value that determines whether the value is less than 1e-10, like:

`Fang = angle(Fn).*(abs(Fn)>=1e-10);`

We can also do it with matrix calculation.

For $F_n = \frac{1}{T_1}\int_{t_0}^{t_0+T1} f(t)e^{-jn\omega_1 t}\,dt$, let ω_n be $n\omega_1$, let $\tau \in [t_0, t_0 + \mathrm{T}]$ and τ_m be $(\mathrm{m}-1) * \mathrm{d}\tau$ $(\mathrm{d}\tau = \mathrm{dt})$.

$$\begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} = \frac{1}{\mathrm{T}_1}\int_{t_0}^{t_0+T_1} \begin{bmatrix} f(\tau)\bullet e^{-jw_1\tau} \\ f(\tau)\bullet e^{-jw_2\tau} \\ \vdots \\ f(\tau)\bullet e^{-jw_n\tau} \end{bmatrix} d\tau$$

$$= \frac{1}{\mathrm{T}_1}\int_{t_0}^{t_0+T_1} \begin{bmatrix} f(\tau_1)\bullet e^{-jw_1\tau_1} & f(\tau_2)\bullet e^{-jw_1\tau_2} & \cdots & f(\tau_m)\bullet e^{-jw_1\tau_m} \\ f(\tau_1)\bullet e^{-jw_2\tau_1} & f(\tau_2)\bullet e^{-jw_2\tau_2} & \cdots & f(\tau_m)\bullet e^{-jw_2\tau_m} \\ \vdots & \vdots & \ddots & \vdots \\ f(\tau_1)\bullet e^{-jw_n\tau_1} & f(\tau_2)\bullet e^{-jw_n\tau_2} & \cdots & f(\tau_m)\bullet e^{-jw_n\tau_m} \end{bmatrix} d\tau$$

$$= \frac{1}{\mathrm{T}_1}\int_{t_0}^{t_0+\mathrm{T}_1} [\,f(\tau_1) \quad f(\tau_2) \quad \cdots \quad f(\tau_m)]\cdot * e^{-j\left(\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} * [\tau_1 \quad \tau_2 \quad \cdots \quad \tau_m]\right)} d\tau$$

For $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_1 t}$, let $\omega_n = n\omega_1$ and $t_k = (k-1) * dt$,we can also change it to the matrix calculation, like:

$$\sum_{n=-\infty}^{\infty} F_n e^{jnw_1 t}$$

$$= \left[F_1 e^{jw_1 t} + F_2 e^{jw_2 t} + \cdots F_n e^{jw_n t} \right]$$

$$= \begin{bmatrix} F_1 e^{jw_1 t_1} + F_2 e^{jw_2 t_1} + \cdots F_n e^{jw_n t_1} \\ F_1 e^{jw_1 t_2} + F_2 e^{jw_2 t_2} + \cdots F_n e^{jw_n t_2} \\ \vdots \\ F_1 e^{jw_1 t_k} + F_2 e^{jw_2 t_k} + \cdots F_n e^{jw_n t_k} \end{bmatrix}'$$

$$= \begin{bmatrix} F_1 & F_2 & \cdots & F_n \end{bmatrix} * \begin{bmatrix} e^{jw_1 t_1} & e^{jw_1 t_2} & \cdots & e^{jw_1 t_k} \\ e^{jw_2 t_1} & e^{jw_2 t_2} & \cdots & e^{jw_2 t_k} \\ \vdots & \vdots & \ddots & \vdots \\ e^{jw_n t_1} & e^{jw_n t_2} & \cdots & e^{jw_n t_k} \end{bmatrix}$$

$$= \begin{bmatrix} F_1 & F_2 & \cdots & F_n \end{bmatrix} * e^{j\left(\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} * \begin{bmatrix} t_1 & t_2 & \cdots & t_k \end{bmatrix} \right)}$$

Note: Since Fn is a sequence of complex numbers, when it is transposed by “ ’ ”, we will get its conjugate numbers. To avoid this, use “ .’ ” to do the transposition. Try the following examples.

```
a = [1+1j, 2+2j; 3+3j, 4+4j]
a'
a.'
```