Exercies 1. (20 pts)

Compute the following convolutions:

(a) $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3]$ and $h[n] = 2\delta[n+1] + 2\delta[n-1]$

(b) $x[n] = 0.5^n u[n]$ and $h[n] = u[n+3]$

(c) $x[n] = 3^n u[-n-1] + \left(\frac{1}{3}\right)^n u[n]$ and $h[n] = \left(\frac{1}{4}\right)^n u[n+3]$

(d) $x[n]$ and h[n] are in Figure P1

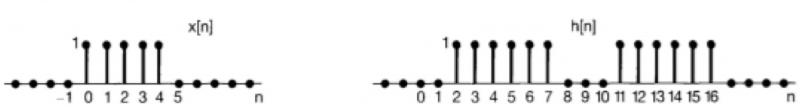


Figure 1: P1

Excercise 1

Solutions:

a) $$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

$$= h[-1]\, x[n+1] + h[1]\, x[n-1]$$

$$= 2x[n+1] + 2x[n-1]$$

$$= 2(\delta[n+1] + 2\delta[n] - \delta[n-2]) + 2(\delta[n-1] + 2\delta[n-2] - \delta[n-4])$$

$$= 2\delta[n+1] + 4\delta[n] + 2\delta[n-2] + 2\delta[n-1] - 2\delta[n-4]$$

b) $$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} (0.5)^k u[n]\, u[n-k+3]$$

$$= \sum_{k=0}^{n+3} (0.5)^k, \quad n \geq -3$$

$$S_n = a_1 \cdot \frac{1-q^n}{1-q}$$

$$= 1 \cdot \frac{1-(0.5)^{n+4}}{1-\frac{1}{2}}\, u[n+3]$$

$$= \left[2 - (0.5)^{n+3}\right] u[n+3]$$

(c) $x[n] = 3^n u[-n-1] + \left(\frac{1}{3}\right)^n u[n]$ and $h[n] = \left(\frac{1}{4}\right)^n u[n+3]$ $\quad -n-1 \geq 0 \Rightarrow n \leq -1$.

(d) $x[n]$ and h[n] are in Figure P1

$n-k+3 \geq 0$

Figure 1: P1

c) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$ $\quad (n \geq 0)$

$$= \underbrace{\sum_{k=-\infty}^{-1} 3^k \cdot u[-k-1] \left(\frac{1}{4}\right)^{n-k} u[n-k+3]}_{①} + \underbrace{\sum_{k=-\infty}^{n+3} \left(\frac{1}{3}\right)^k u[k] \left(\frac{1}{4}\right)^{n-k} \cdot u[n-k+3]}_{②}$$

① $\xrightarrow{k' = -k-1} \sum_{k'=0}^{\infty} 3^{-k'+1} u[k'] \left(\frac{1}{4}\right)^{n+k'+1} u[n+k'+4]$

$(k = -k'+1)$

$$= \frac{1}{12} \sum_{k'=0}^{\infty} \left(\frac{1}{3}\right)^{k'} \cdot \left(\frac{1}{4}\right)^{n+k'} u[n+k'+4]$$

$$= \begin{cases} \frac{1}{12} \sum_{k'=-(n+4)}^{\infty} \left(\frac{1}{3}\right)^{k'} \left(\frac{1}{4}\right)^{n+k'}, & n \leq -4 \\ \frac{1}{12} \sum_{k'=0}^{\infty} \left(\frac{1}{3}\right)^{k'} \left(\frac{1}{4}\right)^{n+k'}, & n > -4 \end{cases}$$

$$= \begin{cases} \frac{1}{12} \sum_{k'=0}^{\infty} \left(\frac{1}{12}\right)^{k'} \cdot \left(\frac{1}{4}\right)^{n} - \frac{1}{12} \sum_{k'=0}^{-(n+5)} \left(\frac{1}{12}\right)^{k'} \cdot \left(\frac{1}{4}\right)^{n}, & n \leq -4 \\ \frac{1}{11} \cdot \left(\frac{1}{4}\right)^{n}, & n > -4 \end{cases}$$

$$= \begin{cases} \frac{1}{11} \cdot \left(\frac{1}{4}\right)^{n} - \frac{1}{11} \cdot \left(\frac{1}{4}\right)^{n} + \frac{12^4}{11} \cdot 3^n, & n \leq -4 \\ \frac{1}{11} \cdot \left(\frac{1}{4}\right)^{n}, & n > -4 \end{cases}$$

② $\sum_{k=0}^{n+3} \left(\frac{1}{3}\right)^k \cdot \left(\frac{1}{4}\right)^{n-k} u[n-k+3] = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n+3} \left(\frac{4}{3}\right)^k = -3 \cdot \left(\frac{1}{4}\right)^n + \frac{256}{27}\left(\frac{1}{3}\right)^n, \quad n \geq -3$

(or multiply $u[n+3]$)

$$y[n] = \begin{cases} \frac{124}{11}(3)^n, & n \leq -4 \\ \frac{1}{11} \cdot \left(\frac{1}{4}\right)^n - 3 \cdot \left(\frac{1}{4}\right)^n + \frac{256}{27} \cdot \left(\frac{1}{3}\right)^n, & n > -4 \end{cases}$$

(d) $\boldsymbol{x[n]}$ and h[n] are in Figure P1

$x[n] = 1, \ [0,4]$

$h[n] = 1, \ [2,7] \cup [11,16]$

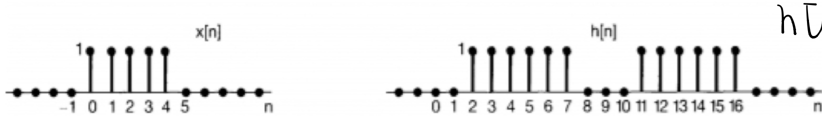


Figure 1: P1

d) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

$x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3] + x[4]h[n-4]$

$= h[n] + h[n-1] + h[n-2] + h[n-3] + h[n-4]$

$= (u[n-2] - u[n-8] + u[n-11] - u[n-17]) +$

$(u[n-3] - u[n-9] + u[n-12] - u[n-18]) +$

$(u[n-4] - u[n-10] + u[n-13] - u[n-19]) +$

$(u[n-5] - u[n-11] + u[n-14] - u[n-15]) +$

$(u[n-6] - u[n-12] + u[n-15] - u[n-16])$

$$= v[n-2] + v[n-3] + v[n-4] + v[n-5] + v[n-6]$$
$$- v[n-8] - v[n-9] - v[n-10] + v[n-13] + v[n-14]$$
$$+ v[n-15] - v[n-17] - v[n-18] - v[n-19] - v[n-20] - v[n-21]$$

Exercies 2. (20 pts)

$$x(t) = u(t-3) - u(t-5) \text{ and } h(t) = e^{-3t}u(t).$$

(a) Compute $y(t) = x(t) * h(t)$ and sketch the result.

(b) Compute $g(t) = \frac{dx(t)}{dt} * h(t)$.

Exercise 2.

a) $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} u(\tau-3) e^{-3(t-\tau)} u(t-\tau)\, d\tau - \int_{-\infty}^{\infty} u(\tau-5) e^{-3(t-\tau)} u(t-\tau)\, d\tau$

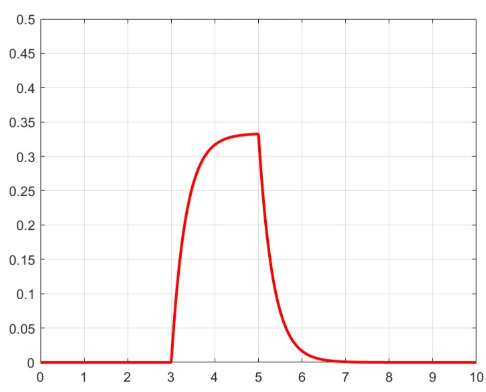
$= \int_{3}^{t} e^{-3t} \cdot e^{3\tau}\, d\tau - \int_{5}^{t} e^{-3t} \cdot e^{3\tau}\, d\tau$

$= e^{-3t} \cdot \frac{1}{3}(e^{3t} - e^{9})u(t-3) - e^{-3t} \cdot \frac{1}{3}(e^{3t} - e^{9})u(t-5)$

$= \frac{1}{3}[1 - e^{-3(t-3)}]u(t-3) - \frac{1}{3}[1 - e^{-3(t-5)}]u(t-5)$

To sketch the result,

$$y(t) = \begin{cases} 0, & t \le 3 \\ \frac{1}{3}[1 - e^{-3(t-3)}], & 3 < t \le 5 \\ \frac{1}{3}(1 - e^{-6}) e^{-3(t-5)}, & t > 5 \end{cases}$$



b) $\frac{dx(t)}{d(t)} = \delta(t-3) - \delta(t-5)$

$$g(t) = \frac{dx(t)}{dt} * h(t) = [\delta(t-3) - \delta(t-5)] * e^{-3t} u(t)$$

$$= e^{-3(t-3)} u(t-3) - e^{-3(t-5)} u(t-5)$$

Exercies 3. (20 pts)

Consider the cascade interconnection of three causal LTI systems, illustrated in Figure P2(a). The impulse response $h_2[n]$ is

$$h_2[n] = u[n] - u[n-2]$$

and the overall impulse response is as shown in Figure P2(b).

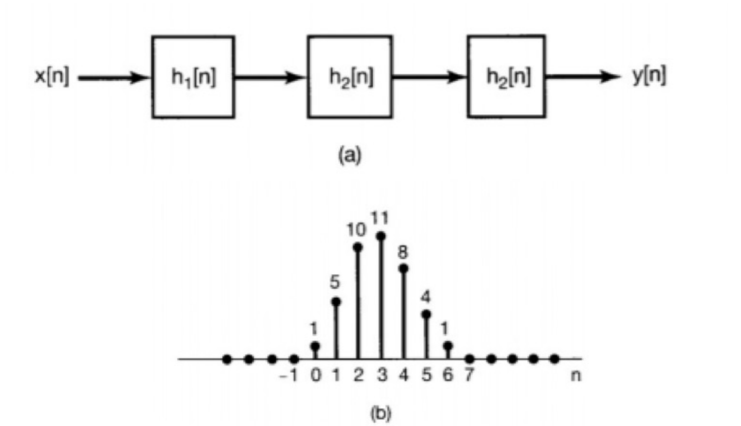


Figure 2: P2

(a) Find the impulse response $h_1[n]$.

(b) Find the response of the overall system to the input

$$x[n] = \delta[n] - \delta[n-1]$$

Exercise 3

a) $h_2[n] = (u[n] - u[n-1]) + (u[n-1] - u[n-2])$

$= \delta[n] + \delta[n-1]$

$h_2[n] * h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2] \Rightarrow$ shift for $h_1[n]$

Let $h_1[n] * h_2[n] * h_2[n] = h[n]$

$= h_1[n] + 2h_1[n-1] + h_1[n-2]$

$$h[0] = h_1[0] = 1$$

$$h[1] = h_1[1] + 2h_1[0] = 5 \Rightarrow h_1[1] = 3$$

$$h[2] = h_1[2] + 2h_1[1] + h_1[0] = 10 \Rightarrow h_1[2] = 3$$

$$h[3] = h_1[3] + 2h_1[2] + h_1[1] = 11 \Rightarrow h_1[3] = 2$$

$$h[4] = h_1[4] + 2h_1[3] + h_1[2] = 8 \Rightarrow h_1[4] = 1$$

$$h[5] = h_1[5] + 2h_1[4] + h_1[3] = 4 \Rightarrow h_1[5] = 0$$

$$h[6] = h_1[6] + h_1[4] = 1 \Rightarrow h_1[6] = 0$$

$$h[7] = h_1[7] = 0$$

result:

$$h_1[n] = \begin{cases} 1, & n=0 \\ 3, & n=1 \\ 3, & n=2 \\ 2, & n=3 \\ 1, & n=4 \\ 0, & n<0 \text{ or } n \geq 5 \end{cases}$$

b) $$y[n] = x[n] * h[n] = h[n] - h[n-1] = \begin{cases} 1, & n=0 \\ 4, & n=1 \\ 5, & n=2 \\ 1, & n=3 \\ -3, & n=4 \\ -4, & n=5 \\ -3, & n=6 \\ -1, & n=7 \\ 0, & \text{others} \end{cases}$$

Exercies 4. (20 pts)

Determine whether each of the following statements concerning LTI systems is true or false. Justify your answers(if True, please prove briefly; if False, please give a counter-example).

(a) The inverse of a causal LTI system is always causal.

(b) If $|h[n]| \le K$ for each n, where K is a given number, then the LTI system with $h[n]$ as its impulse response is stable.

(c) If a discrete-time LTI system has an impulse response $h[n]$ of finite duration, the system is stable.

(d) The cascade of a noncausal LTI system with a causal one is necessarily noncausal.

(e) A continuous-time LTI system is stable iff its step response $s(t)$ is absolutely integrable —— that is, if and only if

$$\int_{-\infty}^{\infty} |s(t)| dt < \infty$$

a) F. $h[n] = \delta[n-k] \xrightarrow{\text{inverse}} g[n] = \delta[n+k]$

b) F. $h[n] = u[n] \rightarrow \sum_{n=\infty}^{\infty} |h[n]| = \infty$

c) T. $n_1 \le n \le n_2 \Rightarrow h[n] \ne 0$ and finite long.

$\sum_{k=n_1}^{n_2} |h[k]| < \infty \Rightarrow$ stable

d) F. $h_1[n] = \delta[n-1]$, $h_2[n] = \delta[n+1] \Rightarrow h[n] = \delta[n]$

e) F. $h(t) = e^{-t}u(t)$. $s(t) = (1-e^{-t})u(t)$

$\int_0^\infty |1-e^{-t}| dt = t + e^{-t}\Big|_0^\infty = \infty \Rightarrow$ integrable

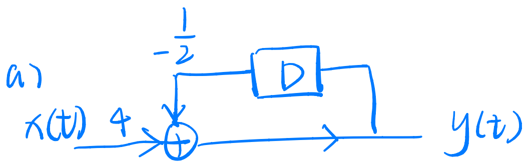
Exercies 5. (20 pts)

Draw block diagram representations for causal LTI systems described by the following differential equations, and determine the system output $y[n]$ or $y(t)$

(a) $y(t) = -(\frac{1}{2})\frac{dy(t)}{dt} + 4x(t)$ when $x(t) = e^{3t}u(t)$

(b) $y[n] = \frac{1}{3}y[n-1] + \frac{1}{2}x[n-1]$ when $x[n] = K\delta[n]$

a)



3'

Since causal, $\frac{dy(0)}{dt} = 0$

Let $y(t) = y_p(t) + y_n(t)$

Since $x(t) = e^{3t}u(t)$

when $t > 0$, $y_p(t) = ke^{3t}$

$\Rightarrow ke^{3t} = -\frac{1}{2} \cdot 3k\, e^{3t} + 4e^{3t}$

$\Rightarrow k = \frac{8}{5}$ 2'

$\Rightarrow y_p(t) = \frac{8}{5}e^{3t}$

Let $y_n(t) = Ae^{st}$.

$\Rightarrow Ae^{st} + \frac{1}{2}Ase^{st} = 0$

$\Rightarrow (1+\frac{1}{2}s)Ae^{st} = 0$

$\Rightarrow s = -2$ 2'

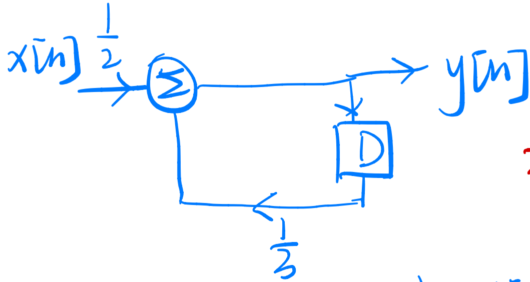
$\Rightarrow y(t) = \frac{8}{5}e^{3t} + Ae^{-2t}$ $(t \geq 0)$

Since causal, $y(0) = 0$

$\Rightarrow y(0) = \frac{8}{5} + A = 0$

$\Rightarrow A = -\frac{8}{5}$ 2'

$\Rightarrow y(t) = \frac{8}{5}(e^{3t} - e^{-2t})u(t)$ 1' b)



3'

$x[n] = k\delta[n] \Rightarrow x[0] = k,\ x[1] = x[2] = \cdots = 0$ 3'

Causal. $y[0]=0$

$y[1] = \frac{1}{3}y[0] + \frac{1}{2}k\delta[0] = \frac{1}{2}k$

$y[2] = \frac{1}{3}y[1] + \frac{1}{2}k\delta[1] = \frac{1}{2}\cdot\frac{1}{3}k = \frac{1}{6}k$ 2'

$y[3] = \frac{k}{18}$

$\vdots$

$y[n] = \frac{1}{2}\cdot\left(\frac{1}{3}\right)^{n-1} k$ 2'