

Signals and Systems

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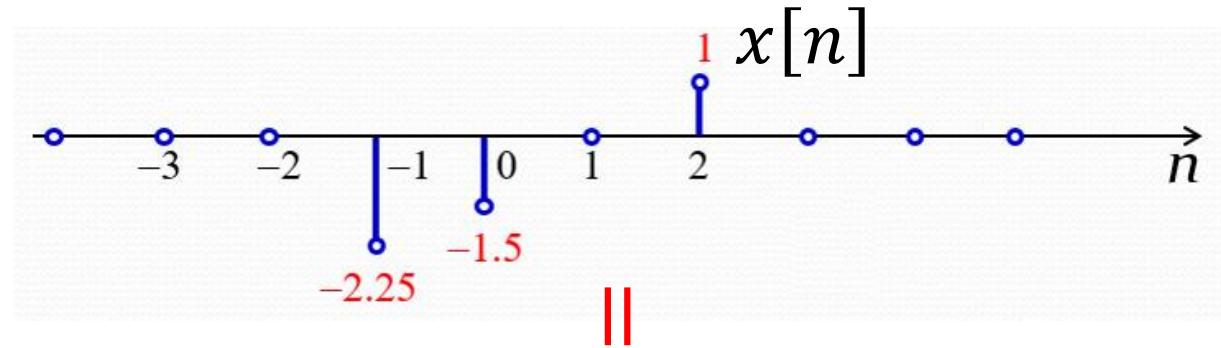
Chapter 2: Linear Time-Invariant Systems

- Discrete-Time LTI Systems
- Continuous-Time LTI Systems
- Properties of LTI Systems
- Differential or Difference Equations

线性时不变系统 LTI

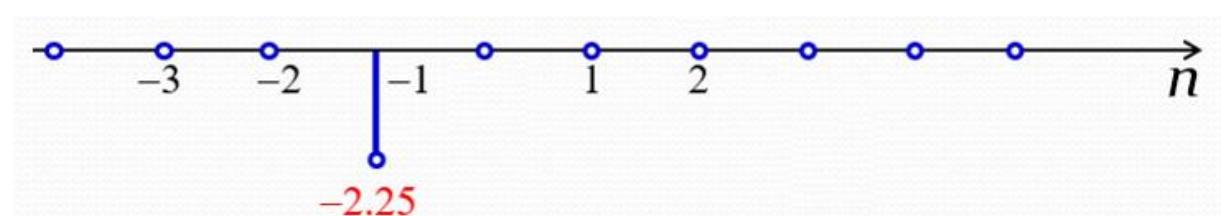
Discrete-Time LTI Systems

Representation of Discrete-Time Signals in Terms of Impulse

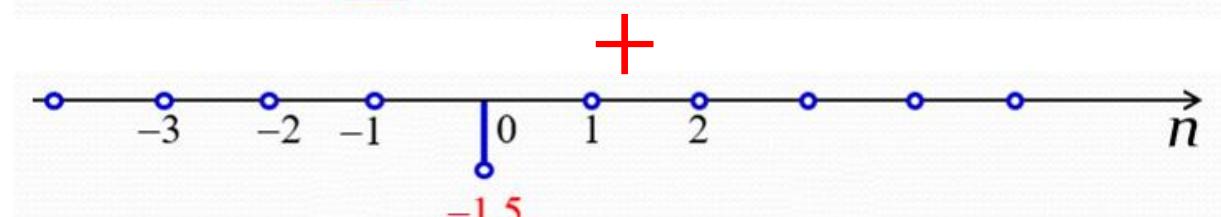


$$x[n]$$

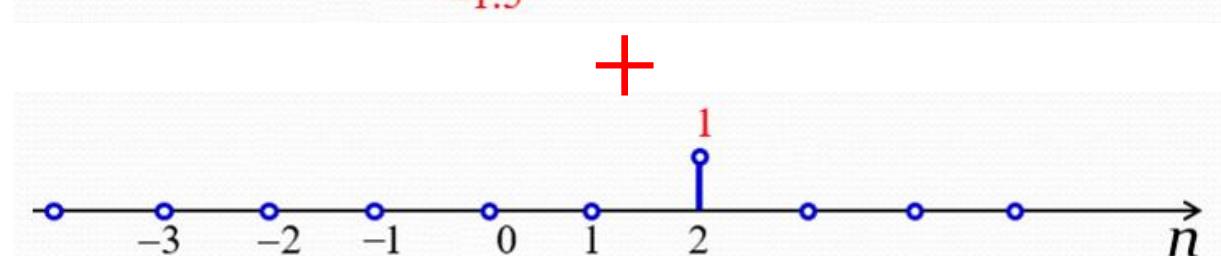
$\|$



$$\leftrightarrow x_1[n] = -2.25 \times \delta[n + 1]$$



$$\leftrightarrow x_2[n] = -1.5 \times \delta[n]$$



+

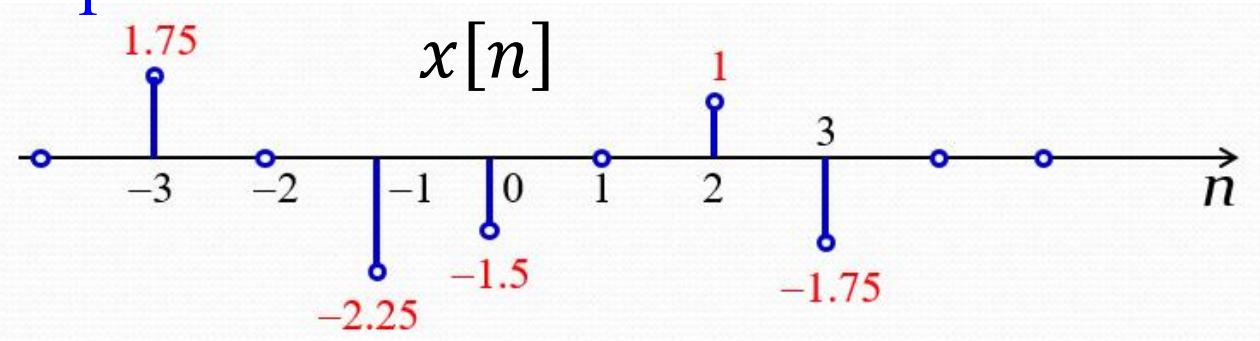
$$\leftrightarrow x_3[n] = 1 \times \delta[n - 2]$$



Discrete-Time LTI Systems

Representation of Discrete-Time Signals in Terms of Impulse

- An arbitrary sequence can be represented as the weighted sum of shifted unit impulses



$$x[n] = 1.75\delta[n+3] - 2.25\delta[n+1] - 1.5\delta[n] + \delta[n-2] - 1.75\delta[n-3]$$

- A general form

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Sifting property of $\delta[n]$

筛选特性

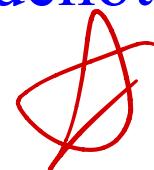


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Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- The response of a system to a unit impulse sequence $\delta[n]$ is called impulse response, denoted by $h[n]$



近：冲激响应
离：单位脉冲 ~



Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- How to calculate the impulse response of a system
- For any system whose input-output relationship is defined by

$$y[n] = f\{x[n]\}$$

the impulse response $h[n]$ is calculated as

$$h[n] = f\{\delta[n]\} \quad \text{replace } x[n] \text{ by } \delta[n]$$



Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

□ How to calculate the impulse response of a system

➤ Examples: a system is defined as

$$y[n] = a_1 \textcolor{red}{x}[n] + a_2 \textcolor{red}{x}[n - 1] + a_3 \textcolor{red}{x}[n - 2] + a_4 \textcolor{red}{x}[n - 3]$$

its impulse response $h[n]$ is

$$h[n] = a_1 \delta[n] + a_2 \delta[n - 1] + a_3 \delta[n - 2] + a_4 \delta[n - 3]$$



Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

□ How to calculate the impulse response of a system

➤ Examples: a system is defined as

$$y[n] = \sum_{k=-\infty}^n x[k]$$

its impulse response $h[n]$ is

$$h[n] = \sum_{k=-\infty}^n \delta[k]$$



Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

□ How to calculate the impulse response of a system

➤ Examples: a system is defined as

$$y[n] = x[n - 1] + \frac{1}{2}(x[n - 2] + x[n])$$

its impulse response $h[n]$ is

$$h[n] = \delta[n - 1] + \frac{1}{2}(\delta[n - 2] + \delta[n])$$



Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

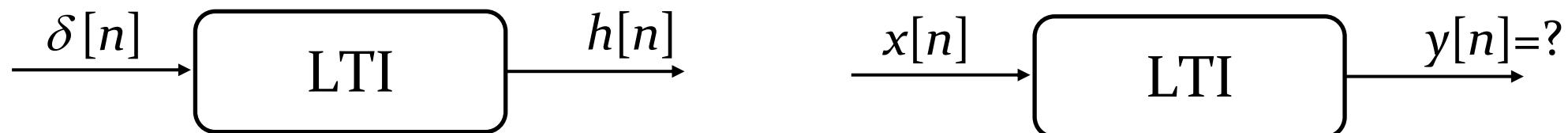
- An LTI discrete system is completely characterized by its impulse response
- In other words, knowing the impulse response one can compute the output of the LTI system for an arbitrary input



Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- The impulse response completely characterizes an LTI system



- Recall, an arbitrary input $x[n]$ can be expressed as a linear combination of shifted unit impulses

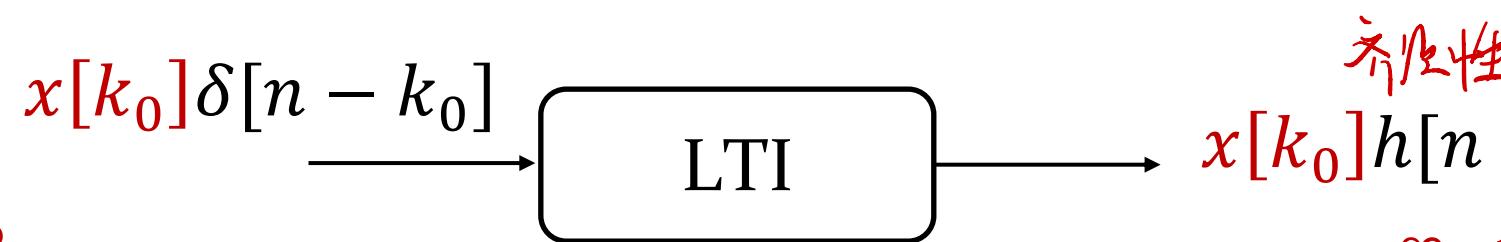
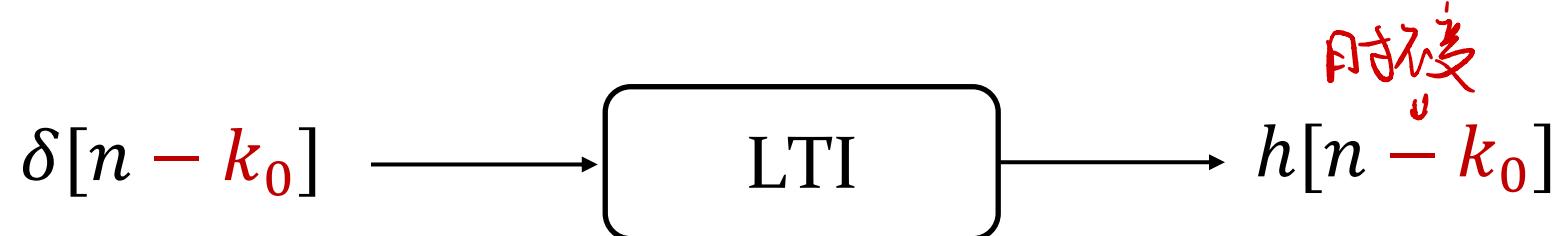
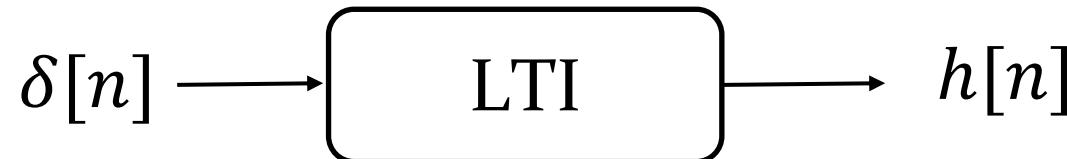
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k]$$



Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

□ For any $k = k_0$



$$\rightarrow x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k] \quad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

The equation shows the convolution sum $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k]$ being processed by an LTI system to produce the output $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$. The term $x[k]h[n - k]$ is bracketed with a curly brace and labeled "线性" (Linearity) in red. The entire process is also labeled with "线性" (Linearity) in red.

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

- $\sum_{k=-\infty}^{\infty} x[k]h[n - k]$ is referred as to the **convolution-sum** 卷积和

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = x[n] * h[n]$$

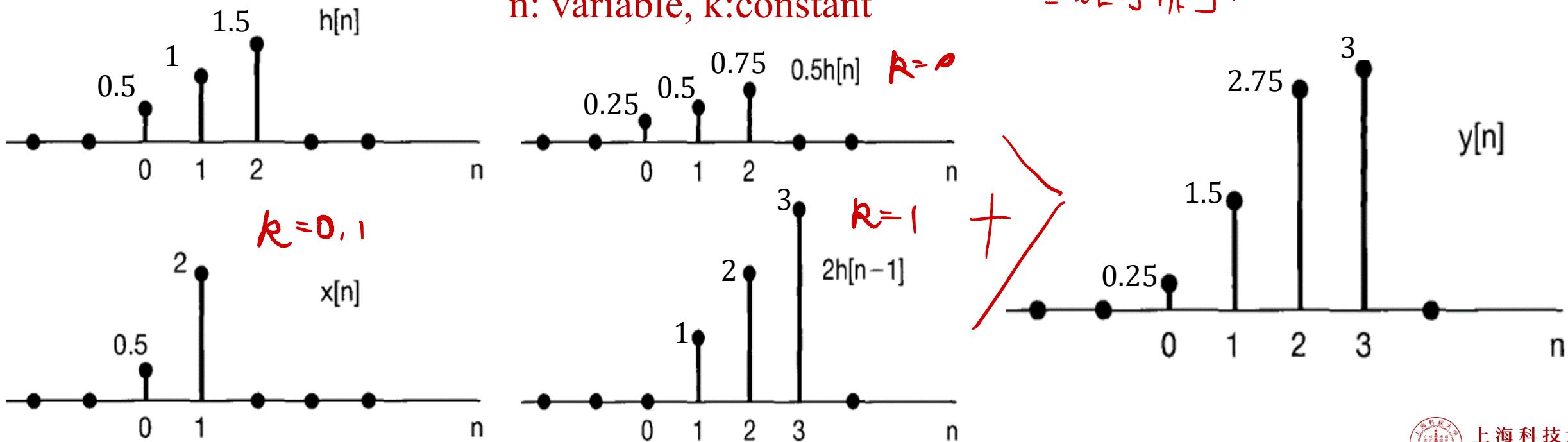
Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- Convolution-Sum calculation – Method 1: sum of k shifted and scaled $h[n]$

$$x[n] \rightarrow \text{LTI} \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

n: variable, k:constant



Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- Convolution-Sum calculation – Method 2: calculate $y[n]$ for each n

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

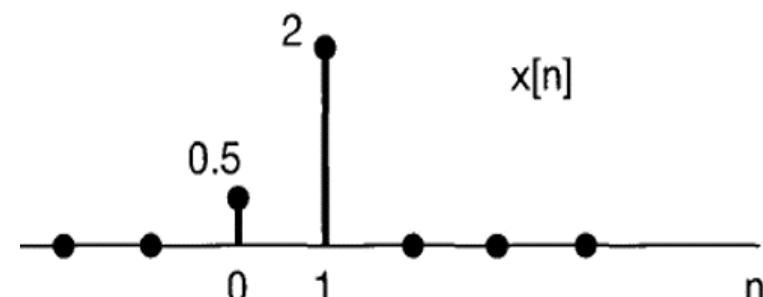
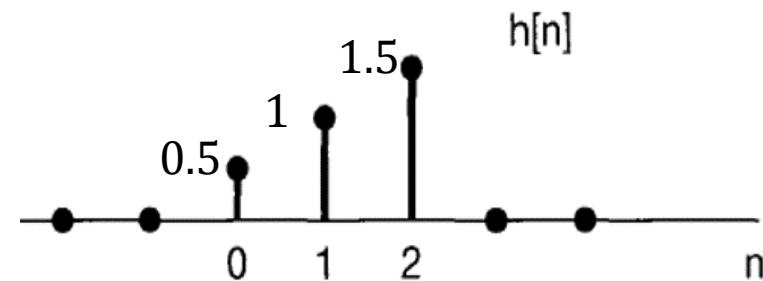
- Step 1: determine the range of k

$$k \in \{0,1\}$$

- Step 2: determine the range of n

$$[n - k] \in \{0,1,2\} \leftrightarrow n \in \{0,1,2,3\},$$

For other n , $y[n]=0$



Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- Convolution-Sum calculation – Method 2: calculate $y[n]$ for each n

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

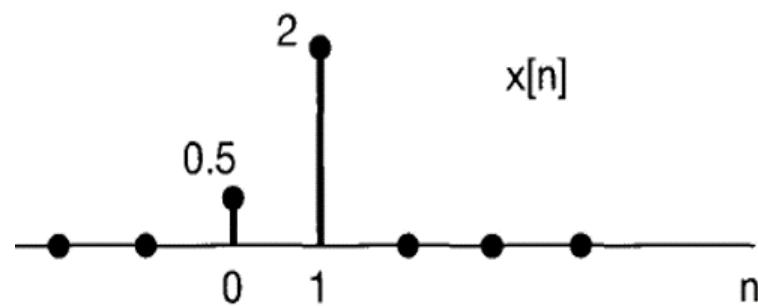
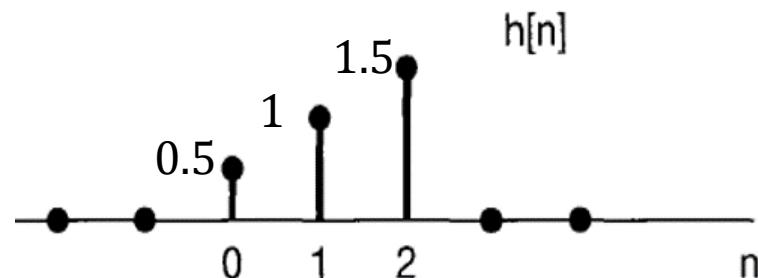
➤ Step 3: calculate $y[n]$ for each n

$$y[0] = \sum_{k=0}^1 x[k]h[0-k] = x[0]h[0] + x[1]h[-1] = 0.25$$

$$y[1] = \sum_{k=0}^1 x[k]h[1-k] = x[0]h[1] + x[1]h[0] = 1.5$$

$$y[2] = \sum_{k=0}^1 x[k]h[2-k] = x[0]h[2] + x[1]h[1] = 2.75$$

$$y[3] = \sum_{k=0}^1 x[k]h[3-k] = x[0]h[3] + x[1]h[2] = 3$$



Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

□ Convolution-Sum calculation – Method 3

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

For each n :

- Step 1: change time variables $x[n] \rightarrow x[k]$, $h[n] \rightarrow h[k]$,
and reverse $h[k] \rightarrow h[-k]$
- Step 2: Shift $h[-k] \rightarrow h[n-k]$, n is considered as a constant
- Step 3: multiply $x[k] \cdot h[n-k]$
- Step 4: Summation $\sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$

Change n , repeat step 1 to 4, calculate another $y[n]$

Discrete-Time LTI Systems

The Convolution-Sum

□ Convolution-Sum calculation

– Method 3

- If the lengths of the two sequences are M and N , then the sequence generated by the convolution is of length $M+N-1$

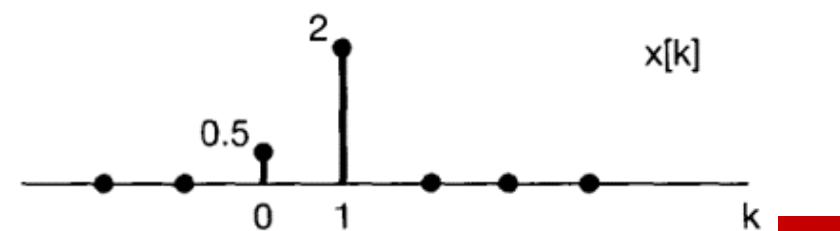
$$y[0] = \sum_{k=0}^1 x[k]h[0 - k]$$

$$y[1] = \sum_{k=0}^1 x[k]h[1 - k]$$

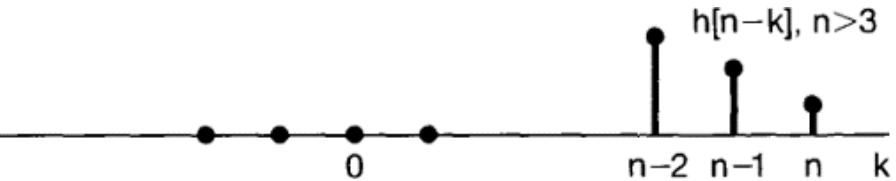
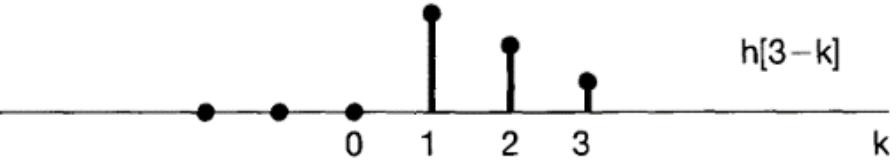
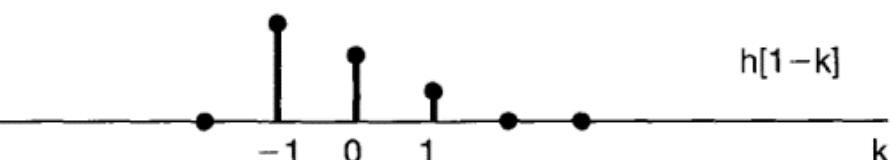
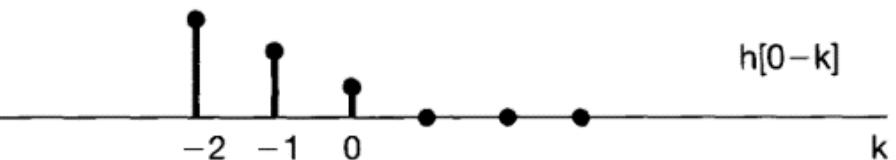
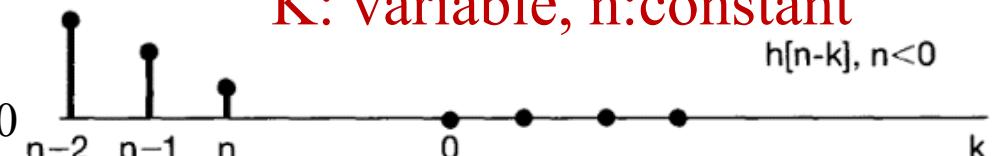
$$y[2] = \sum_{k=0}^1 x[k]h[2 - k]$$

$$y[3] = \sum_{k=0}^1 x[k]h[3 - k]$$

$$y[n] = 0, \text{ for } n > 3$$



K: variable, n:constant



Discrete-Time LTI Systems

The Convolution-Sum

□ Examples

$$y[n] = x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n]$$

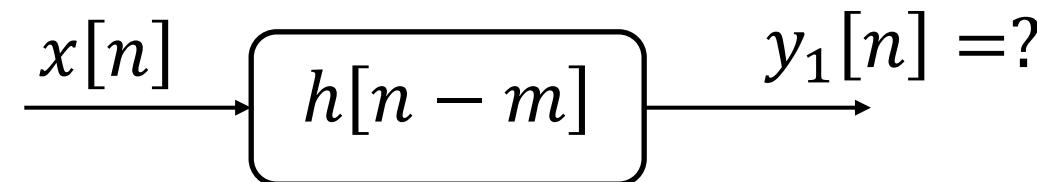
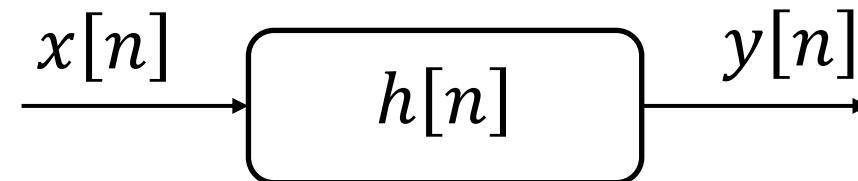
$$\begin{aligned} y[n] &= x[n] * \delta[n-d] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k-d] \quad \text{Let } k+d = k' \\ &= \sum_{k'= -\infty}^{\infty} x[k'-d]\delta[n-k'] \\ &= x[n-d] * \delta[n] = x[n-d] \end{aligned}$$



Discrete-Time LTI Systems

The Convolution-Sum

□ Examples



$$\begin{aligned}y_1[n] &= x[n] * h[n - m] = \sum_{k=-\infty}^{\infty} x[k]h[n - k - m] \quad \text{Let } k + m = k' \\&= \sum_{k'= -\infty}^{\infty} x[k' - m]h[n - k'] \\&= x[n - m] * h[n] = y[n - m]\end{aligned}$$



Chapter 2: Linear Time-Invariant Systems

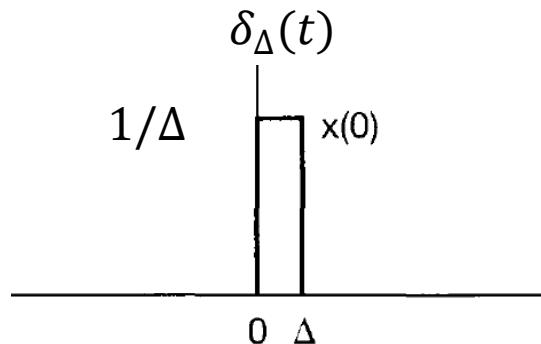
- Discrete-Time LTI Systems
- Continuous-Time LTI Systems
- Properties of LTI Systems
- Differential or Difference Equations

Continuous-Time LTI Systems

Continuous-Time Signals in Terms of Impulse

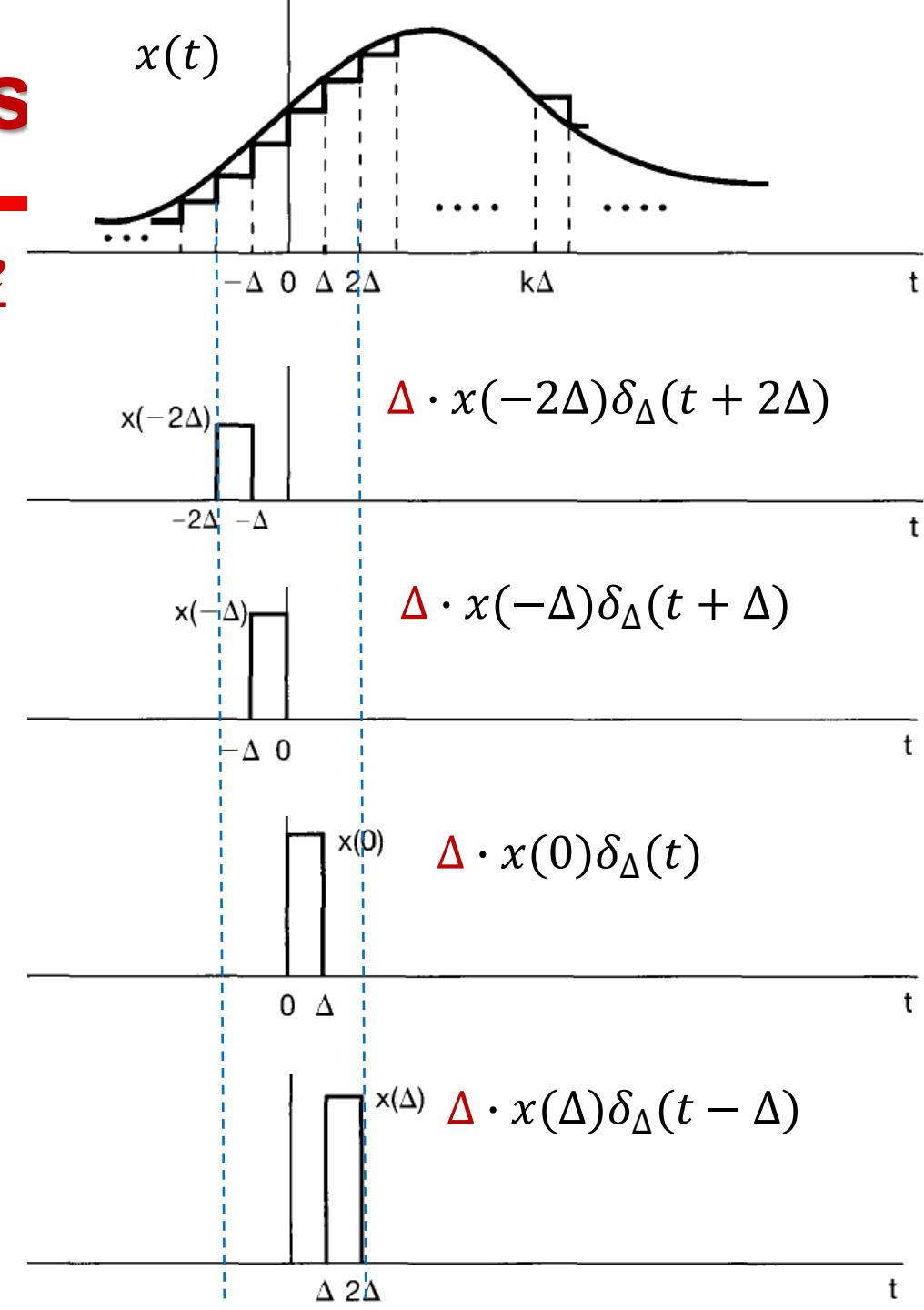
- “staircase” approximation of $x(t)$

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \leq t \leq \Delta \\ 0, & \text{otherwise} \end{cases}$$



$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) \delta_\Delta(t - k\Delta) \cdot \Delta$$

$$x(t) = \lim_{\Delta \rightarrow 0} \hat{x}(t)$$



Continuous-Time LTI Systems

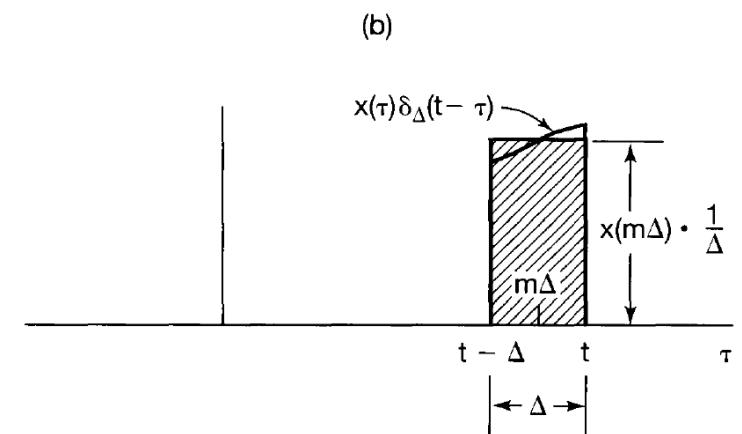
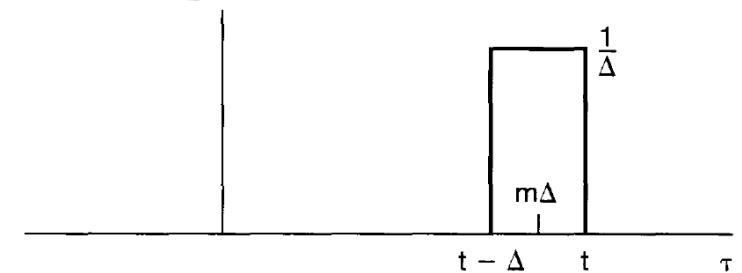
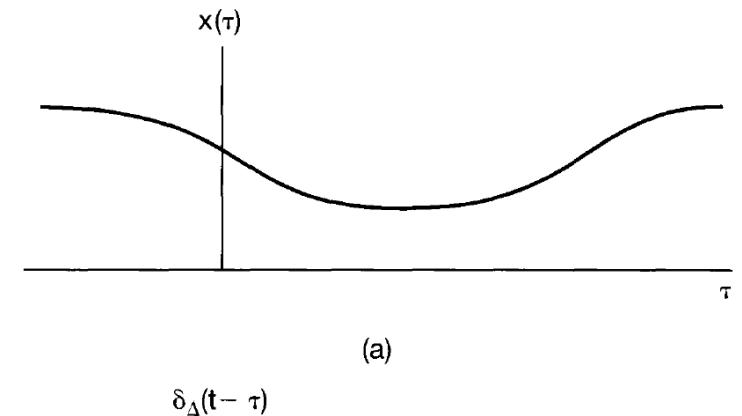
Continuous-Time Signals in Terms of Impulse

- “staircase” approximation of $x(t)$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) \delta_{\Delta}(t - k\Delta) \cdot \Delta$$

$$x(t) = \lim_{\Delta \rightarrow 0} \hat{x}(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$$

Sifting property of $\delta(t)$



Continuous-Time LTI System

Continuous-Time Signals in Terms of Impulses

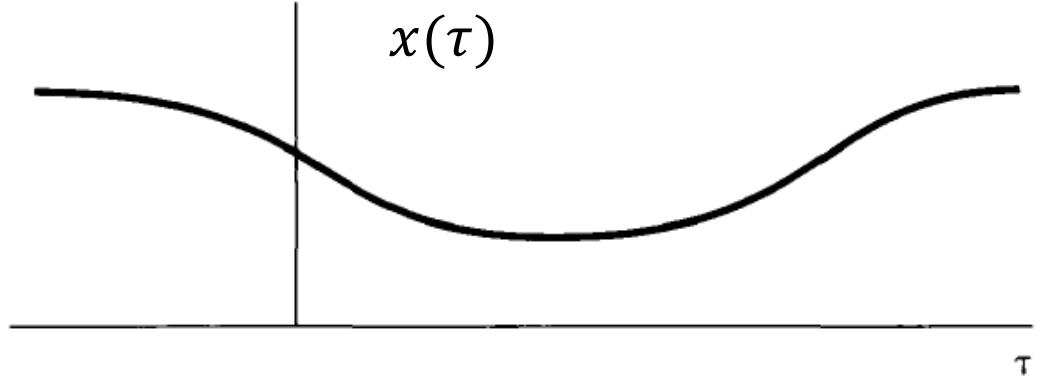
- Using sampling property of $\delta(t)$

$$\int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau = ?$$

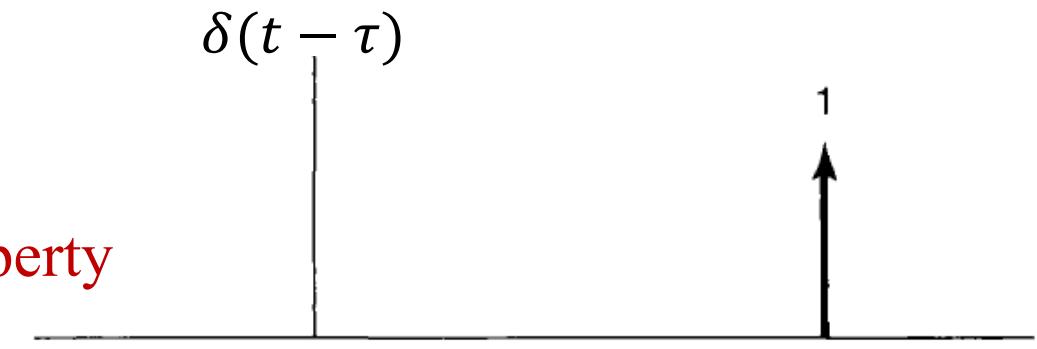
$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$ sampling property

$x(\tau)\delta(t - \tau) = x(t)\delta(t - \tau)$ t : constant

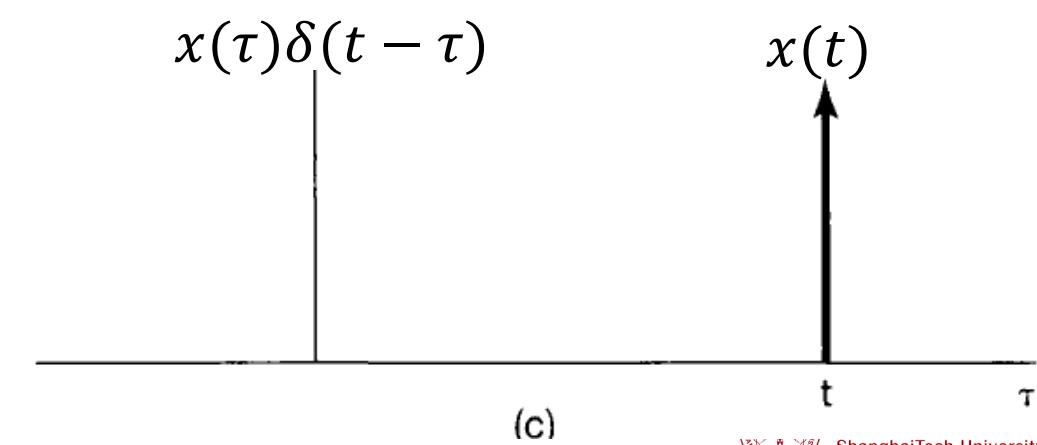
$$\begin{aligned}\therefore \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau &= \int_{-\infty}^{\infty} x(t) \delta(t - \tau) d\tau \\ &= x(t) \int_{-\infty}^{\infty} \delta(t - \tau) d\tau \\ &= x(t)\end{aligned}$$



(a)



(b)



(c)



Continuous-Time LTI Systems

Continuous-Time Signals in Terms of Impulse

- An example

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$$

$$u(t) = \int_{-\infty}^{\infty} u(\tau) \delta(t - \tau) d\tau = \int_0^{\infty} \delta(t - \tau) d\tau$$

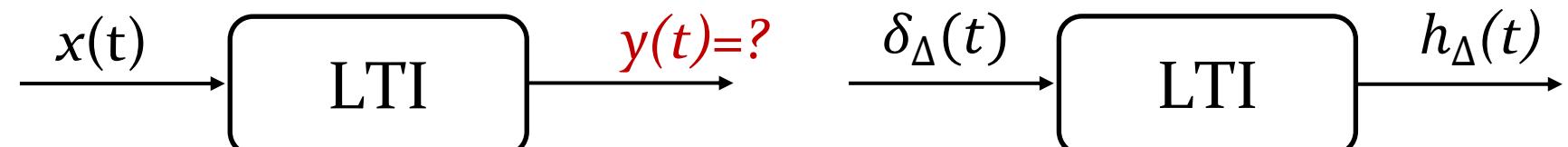
Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

□ Continuous-Time Unit Impulse Response



□ What about



$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) \delta_{\Delta}(t - k\Delta) \cdot \Delta \quad \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) h_{\Delta}(t - k\Delta) \cdot \Delta$$

$$x(t) = \lim_{\Delta \rightarrow 0} \hat{x}(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau \quad y(t) = \lim_{\Delta \rightarrow 0} \hat{y}(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

□ Continuous-Time Unit Impulse Response



$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

Integral of weighted and shift impulses

□ Convolution integral 卷积积分

$$\int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = x(t) * h(t)$$

Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

□ Computation convolution integral

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau$$

- Change time variables $x(t) \rightarrow x(\tau)$, $h(t) \rightarrow h(\tau)$, and reverse $h(\tau) \rightarrow h(-\tau)$
- Shift $h(-\tau) \rightarrow h(t - \tau)$
- Multiply $x(\tau) \cdot h(t - \tau)$
- Integral $\int_{-\infty}^{\infty} x(\tau) \cdot h(t - \tau) d\tau$



Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

- Computation convolution integral: examples

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau$$

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau) d\tau = x(t)$$

$\tau' = \tau + t_0$

$$x(t) * \delta(t - t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau - t_0) d\tau = \int_{-\infty}^{\infty} x(\tau)\delta(t - (\tau + t_0)) d\tau$$

$y(t) = x(t - t_0)\delta(t - t_0 - t_1)$

$$\begin{aligned}y'(t) &= y(t + t_0) \\&= x(t) * \delta(t + t_0) \\&= x(t + t_1) * \delta(t) \\&\vdots y(t - t_1)\end{aligned}$$

$$y(t + t_0) = x(t + t_1) * x(t - t_0) = x(t - t_0)$$



Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

- Computation convolution integral: examples

$$x(t) = e^{-at}u(t), \quad h(t) = u(t), \quad a > 0 \quad x(t) * h(t) = ?$$

$$y(t) = \int_{-\infty}^{\infty} e^{-a\tau} u(\tau) \cdot u(t - \tau) d\tau$$

$$\text{For } t < 0 \quad x(\tau) \cdot h(t - \tau) = 0 \quad \Rightarrow \quad y(t) = 0$$

$$\text{For } t \geq 0 \quad y(t) = \int_0^t e^{-a\tau} d\tau = \frac{-1}{a} e^{-at} \Big|_0^t = \frac{1}{a} (1 - e^{-at})$$



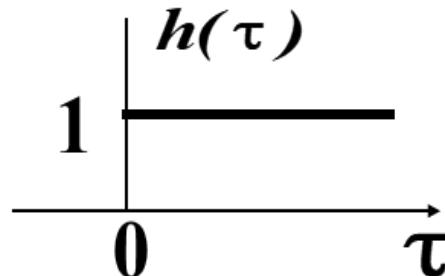
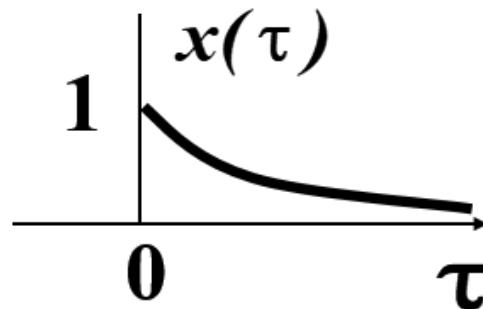
Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

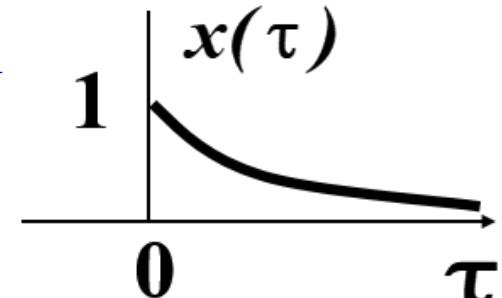
- Computation convolution integral: Graphical Solution

$$x(t) = e^{-at} u(t), \quad h(t) = u(t), \quad a > 0$$

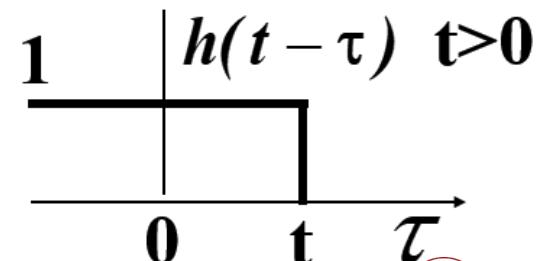
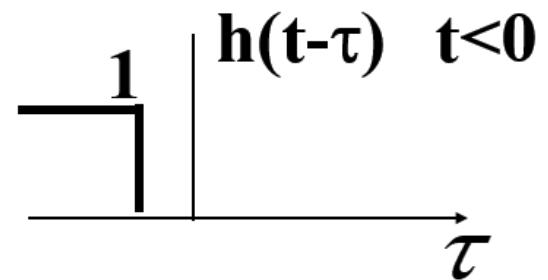
$$x(t) * h(t) = ?$$



$$y(t) = \int_0^t e^{-a\tau} d\tau = \frac{-1}{a} e^{-at} \Big|_0^t = \frac{1}{a} (1 - e^{-at})$$



τ : variable, t : constant



Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

- Computation convolution integral: examples

$$x(t) = e^{2t}u(-t) \quad h(t) = u(t - 3) \quad x(t) * h(t) = ?$$

$$-\tau > 0 \quad \tau < 0$$

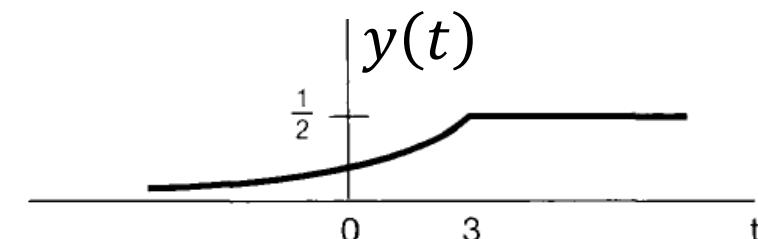
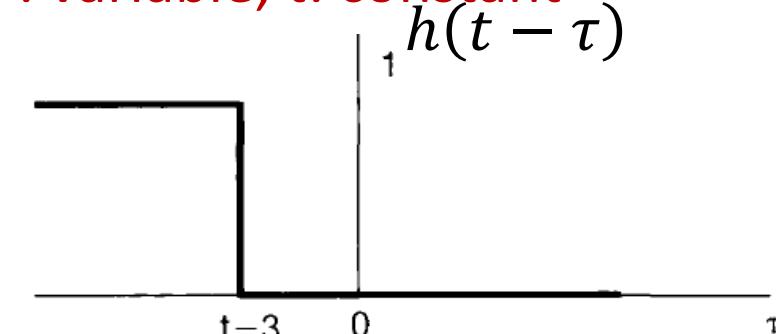
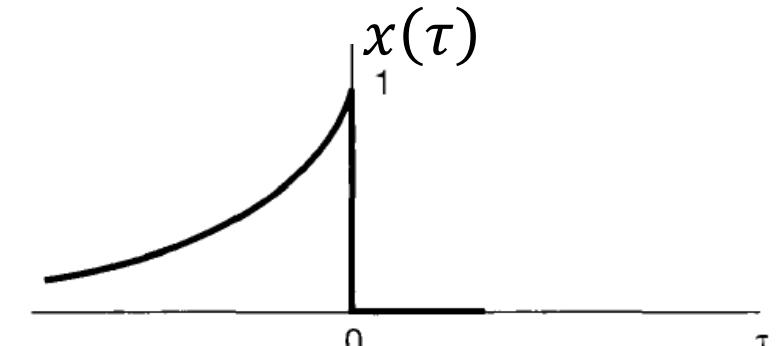
$$t - \tau - 3 > 0 \quad \tau < t - 3$$

$$t \leq 3: \text{ 原式} = \int_{-\infty}^{t-3} e^{2\tau} d\tau$$

$$= \frac{1}{2} e^{2\tau} \Big|_{-\infty}^{t-3} = \frac{1}{2} e^{2t-6}$$

$$t > 3 \quad \text{原式} = \int_{-\infty}^0 e^{2\tau} d\tau = \frac{1}{2} e^{2\tau} \Big|_{-\infty}^0 = \frac{1}{2}$$

$$\therefore x(t) * h(t) = \begin{cases} \frac{1}{2} e^{2t-6}, & t \leq 3 \\ \frac{1}{2}, & t > 3 \end{cases}$$

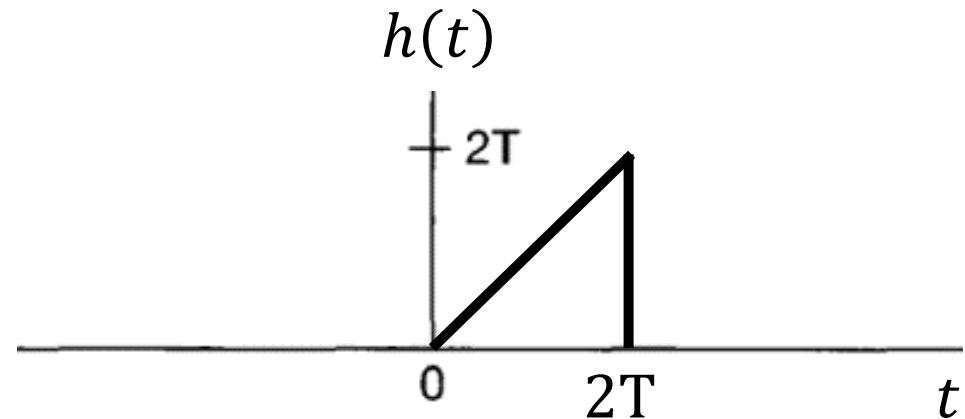
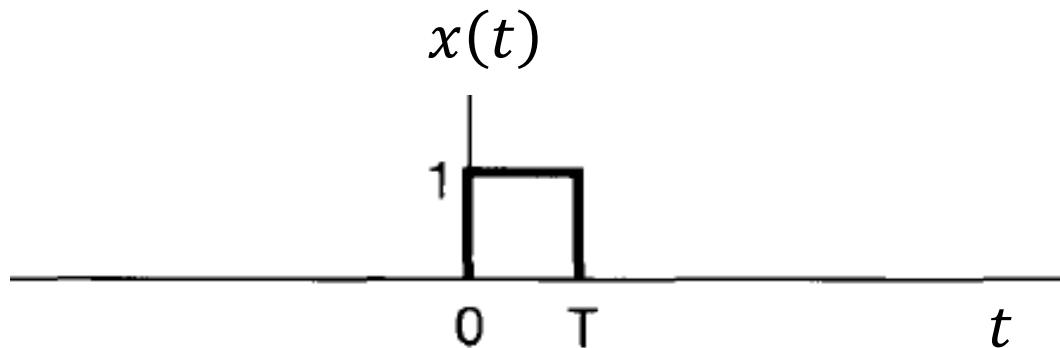


Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

- Computation convolution integral: examples

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases} \quad h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$



$$x(t) * h(t) = ?$$

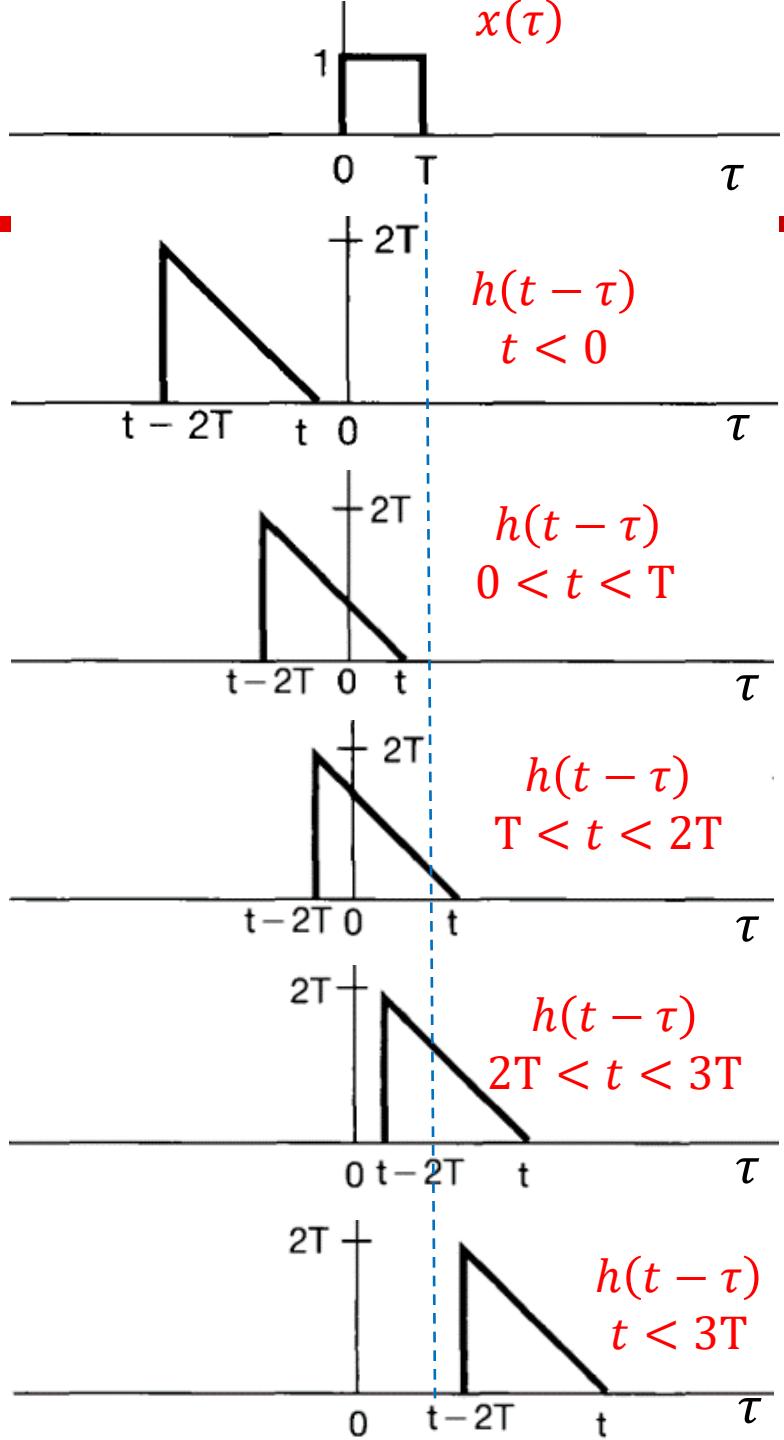


Continuous-Time LTI Systems

Convolution Integral

□ Computation: examples

$$y(t) = \begin{cases} 0, & t < 0 \\ \int_0^t (t - \tau) d\tau = \frac{1}{2}t^2, & 0 < t < T \\ \int_0^T (t - \tau) d\tau = Tt - \frac{1}{2}T^2, & T < t < 2T \\ \int_{t-2T}^T (t - \tau) d\tau = -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & t > 3T \end{cases}$$



Chapter 2: Linear Time-Invariant Systems

- Discrete-Time LTI Systems
- Continuous-Time LTI Systems
- Properties of LTI Systems
- Differential or Difference Equations

Properties of LTI Systems

The commutative property

交换律

□ Discrete-time

$$x[n] * h[n] = h[n] * x[n]$$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[\cancel{n} - k] \stackrel{n-k=m}{=} \sum_{m=-\infty}^{\infty} h[m]x[\cancel{n} - m] = h[n] * x[n]$$

□ Continuous-time

$$x(t) * h(t) = h(t) * x(t)$$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau \stackrel{t - \tau = \tau'}{=} \int_{-\infty}^{\infty} h(\tau')x(t - \tau')d\tau' = h(t) * x(t)$$



Properties of LTI Systems

The distribute property

分配律

□ Discrete-time

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

□ Proof

$$\begin{aligned} x[n] * (h_1[n] + h_2[n]) &= \sum_{k=-\infty}^{\infty} x[k] (h_1[n-k] + h_2[n-k]) \\ &= \sum_{k=-\infty}^{\infty} x[k] h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k] h_2[n-k] \\ &= x[n] * h_1[n] + x[n] * h_2[n] \end{aligned}$$



Properties of LTI Systems

The distribute property

□ Continuous-time

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$

□ Proof

$$x(t) * (h_1(t) + h_2(t)) = \int_{-\infty}^{\infty} x(\tau)(h_1(t - \tau) + h_2(t - \tau))d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)h_1(t - \tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(t - \tau)d\tau$$

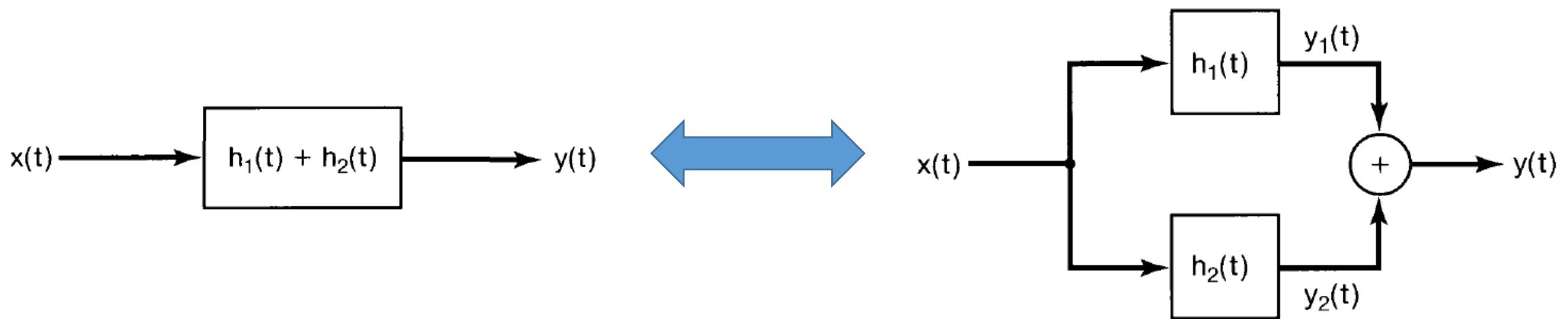
$$= x(t) * h_1(t) + x(t) * h_2(t)$$

Properties of LTI Systems

The distribute property

□ Continuous-time

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$



Properties of LTI Systems

The associative property

□ Discrete-time

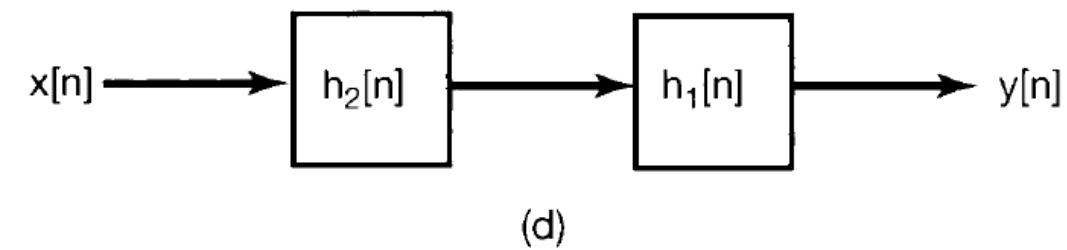
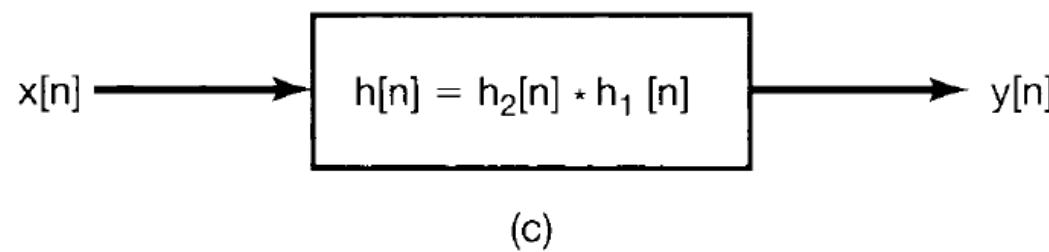
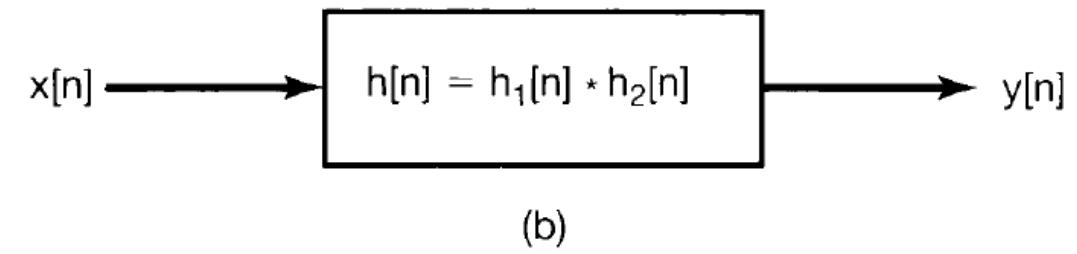
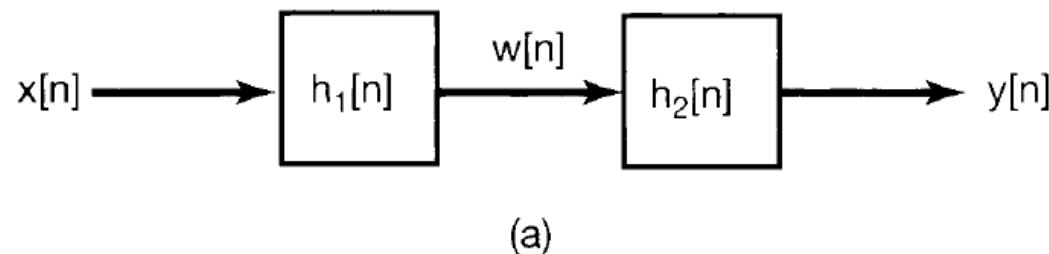
$$\begin{aligned} x[n] * (h_1[n] * h_2[n]) &= (x[n] * h_1[n]) * h_2[n] \quad \text{Associativity} \\ x[n] * (h_1[n] * h_2[n]) &= x[n] * y[n], \quad y[n] = \sum_{m=-\infty}^{\infty} h_1[m]h_2[n-m] \\ &= \sum_{k=-\infty}^{\infty} x[k]y[n-k] = \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} h_1[m]h_2[n-k-m] \\ \text{Let } k+m=l \\ &= \sum_{k=-\infty}^{\infty} x[k] \sum_{l=-\infty}^{\infty} h_1[l-k]h_2[n-l] \\ &= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]h_1[l-k]h_2[n-l] \\ &= \sum_{l=-\infty}^{\infty} (x[l] * h_1[l])h_2[n-l] = (x[n] * h_1[n]) * h_2[n] \end{aligned}$$



Properties of LTI Systems

The associative property

□ Discrete-time



Properties of LTI Systems

The associative property

□ Continuous-time

$$x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$$

$$x(t) * (h_1(t) * h_2(t)) = x(t) * \int_{-\infty}^{\infty} h_1(\tau)h_2(t - \tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau') \int_{-\infty}^{\infty} h_1(\tau)h_2(t - \tau' - \tau)d\tau d\tau'$$

Let $\tau' + \tau = \tau''$

$$= \int_{-\infty}^{\infty} x(\tau') \int_{-\infty}^{\infty} h_1(\tau'' - \tau')h_2(t - \tau'')d\tau'' d\tau'$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau')h_1(\tau'' - \tau')d\tau' h_2(t - \tau'')d\tau''$$

$$= \int_{-\infty}^{\infty} x(\tau'') * h_1(\tau'') h_2(t - \tau'')d\tau'' = (x(t) * h_1(t)) * h_2(t)$$



Properties of LTI Systems

LTI systems with and without memory

- Discrete-time system without memory only if $h[n] = 0$ for all $n \neq 0$

$$h[n] = h[0]\delta[n] = k\delta[n] \quad y[n] = kx[n] \quad \text{Why?}$$

- Continuous-time system without memory only if $h(t) = 0$ for all $t \neq 0$

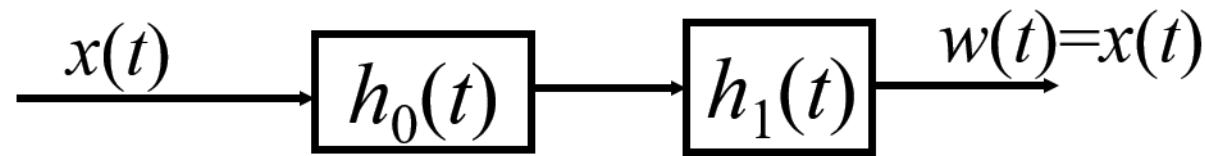
$$h(t) = h(0)\delta(t) = k\delta(t) \quad y(t) = kx(t)$$



Properties of LTI Systems

Invertibility for LTI systems

- If $h_0(t) * h_1(t) = \delta(t)$, the system with impulse response $h_1(t)$ is the **inverse of the system** with impulse response $h_0(t)$



- Similarly, if $h_0[n] * h_1[n] = \delta[n]$, the system with impulse response $h_1[n]$ is the **inverse of the system** with impulse response $h_0[n]$

Properties of LTI Systems

Invertibility for LTI systems

□ Examples

Consider $h_0[n] = u[n]$, determine the inverse system $h_1[n]$

$$\because h_0[n] * h_1[n] = u[n] * h_1[n] \stackrel{hold}{=} \delta[n]$$

$$\delta[n] = u[n] - u[n-1] = u[n] * (\underbrace{\delta[n] - \delta[n-1]}_{})$$

$$\therefore h_1[n] = \delta[n] - \delta[n-1]$$

Properties of LTI Systems

Invertibility for LTI systems

□ Examples

Consider the LTI system consisting of a pure time shift

$$y(t) = x(t - t_0), \quad \begin{array}{ll} t_0 > 0 & \text{延时} \\ t_0 < 0 & \text{超前} \end{array}$$

determine the inverse system.

$$\begin{aligned} h(t) &= \delta(t - t_0) \\ y(t) &= x(t) * h(t) \end{aligned}$$

$$\begin{aligned} y(t) &= x(t - t_0) = x(t) * \delta(t - t_0) \\ h(t) * \delta(t - t_0) &= \delta(t) \\ \Downarrow \\ h_1(t - t_0) &= \delta(t) \\ h_1(t') &= \delta(t' + t_0) \\ h_1(t) &= \delta(t + t_0) \end{aligned}$$



Properties of LTI Systems

Causality for LTI systems

- If $h[n] = 0$ for $n < 0$, or $h(t) = 0$ for $t < 0$, the system is causal
等价于初始静止条件
- Equivalent to the condition of initial rest: if $\underbrace{t \leq t_0, x(t) = 0}_{\text{(边界不考虑)}}$, then $y(t_0) = 0$

$$y[n] = \sum_{k=-\infty}^n x[k]h[n-k] \quad \text{or} \quad y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

y[n] is not related with x[n] ($n>k$)

$k>n \quad h[n-k]=0$
 $\therefore \quad n<0, h[n]=0$

$$y(t) = \int_{-\infty}^t x(\tau)h(t-\tau) d\tau \quad \text{or} \quad y(t) = \int_0^{\infty} h(\tau)x(t-\tau)d\tau$$

Properties of LTI Systems

Causality for LTI systems

□ Examples

- Accumulator: $y[n] = \sum_{l=-\infty}^n x[l]$ Causal LTI system

$$h[n] = \sum_{l=-\infty}^n \delta[l] = u[n] \quad h[n] = 0 \text{ for } n < 0$$

- Factor 2 interpolator: $y[n] = x_u[n] + \frac{1}{2}(x_u[n - 1] + x_u[n + 1])$
Non-Causal LTI system

$$h[n] = \delta[n] + \frac{1}{2}(\delta[n - 1] + \delta[n + 1])$$
$$h[n] \neq 0 \text{ for } n = -1 < 0$$



Properties of LTI Systems

Stability for LTI systems

- A discrete LTI system is stable if $\underline{h[n]}$ is absolutely summable 绝对可和
- A continuous LTI system is stable if $h(t)$ is absolutely integrable 绝对可积

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \quad \text{absolutely summable}$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \quad \text{absolutely integrable}$$



Properties of LTI Systems

Stability for LTI systems

□ Proof: “if and only if” (Sufficient and necessary condition)

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| = \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]|$$

$$\therefore |y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]|$$

If $|x[n-k]| \leq B_x$

$$|y[n]| \leq \underbrace{B_x}_{\text{finite}} \sum_{k=-\infty}^{\infty} |h[k]| \underbrace{< \infty}_{\text{finite}}$$

If and only if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ $|y[n]| < \infty$



Properties of LTI Systems

Stability for LTI systems

□ Proof: continuous case

If $|x(t - \tau)| \leq B_x$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau \right| \leq \int_{-\infty}^{\infty} |h(\tau)| \cdot |x(t - \tau)| d\tau \leq B_x \int_{-\infty}^{\infty} |h(\tau)| d\tau$$

If and only if $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$ $|y(t)| < \infty$



Properties of LTI Systems

Stability for LTI systems

□ Examples

$$y[n] = x[n - n_0]$$

$$h[n] = \delta[n - n_0]$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n - n_0]| = 1$$

Properties of LTI Systems

Stability for LTI systems

LTI ?

□ Examples

$$h[n] = \underbrace{\alpha^n u[n]}_{\text{LTI?}} \quad |\alpha| \leq 1$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\alpha^n| u[n] = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1 - |\alpha|}$$

If $|\alpha| < 1$, the system is stable

If $|\alpha| = 1$, the system is unstable



Properties of LTI Systems

The unit step response of LTI systems

- The unit step response, $s[n]$, corresponding to the output with input $x[n] = u[n]$

$$s[n] = u[n] * h[n] = \sum_{-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^n h[k]$$

$$u[n] = \sum_{k=-\infty}^n \delta[k]$$

$$s[n] = \sum_{k=-\infty}^n h[k]$$

$$h[n] = s[n] - s[n-1]$$

Properties of LTI Systems

The unit step response of LTI systems

- The unit step response, $s(t)$, corresponding to the output with input $x(t)=u(t)$

$$s(t) = u(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)u(t - \tau)d\tau = \int_{-\infty}^t h(\tau)d\tau$$

$$u(t) = \int_{-\infty}^t \delta(\tau)d\tau \quad s(t) = \int_{-\infty}^t h(\tau)d\tau$$

$$h(t) = \frac{ds(t)}{dt} = s'(t)$$



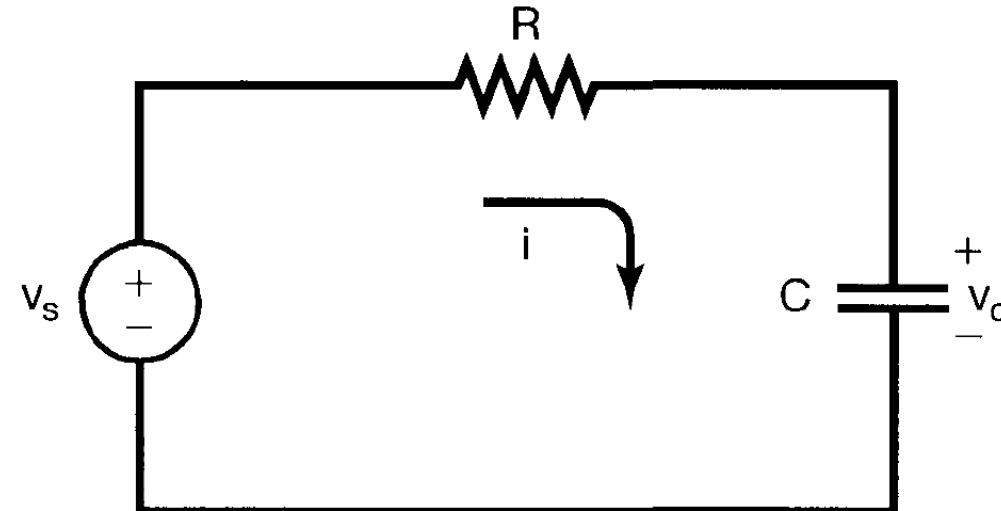
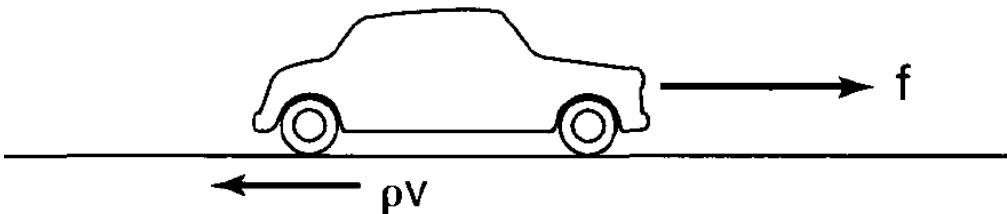
Chapter 2: Linear Time-Invariant Systems

- Discrete-Time LTI Systems
- Continuous-Time LTI Systems
- Properties of LTI Systems
- Differential or Difference Equations

Differential or Difference Equations

Differential equation

□ First order system



$$\frac{dv(t)}{dt} + \frac{\rho}{m} v(t) = \frac{1}{m} f(t)$$

$$\frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$$

□ In general:

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

Differential or Difference Equations

Differential equation

- Linear constant-coefficient DE

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

- Describes an implicit relationship between the input and the output
- Can not completely characterize a LTI system

附加条件

- Auxiliary conditions are required to solve the DE: causal (initial rest condition)

Differential or Difference Equations

Differential equation

- First order system: example

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

If $x(t) = Ke^{3t}u(t)$ $y(t) = ?$

- Solution:

$$y(t) = y_p(t) + y_h(t)$$

$y_p(t)$: particular solution, *forced response (same form as input)*

$y_h(t)$: Homogenous solution

齐次解(通解)
自然响应

$$\frac{dy(t)}{dt} + 2y(t) = 0$$



Differential or Difference Equations

Differential equation

- First order system: example

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

If $x(t) = Ke^{3t}u(t)$ $y(t) = ?$ 形式相同 Ye^{3t}

- Particular solution:

$$3Ye^{3t} + 2Ye^{3t} = Ke^{3t} \longrightarrow Y = K/5 \longrightarrow y_p(t) = \frac{K}{5}e^{3t} \text{ for } t > 0$$

- Homogenous solution: Let $y_h(t) = Ae^{\textcolor{red}{s}t}$, for $t > 0$

$$Ase^{\textcolor{red}{s}t} + 2Ae^{\textcolor{red}{s}t} = 0 \longrightarrow s = -2 \longrightarrow y_h(t) = Ae^{-2t}$$

$$y(t) = \textcolor{red}{A}e^{-2t} + \frac{K}{5}e^{3t}, \text{ for } t > 0$$

Differential or Difference Equations

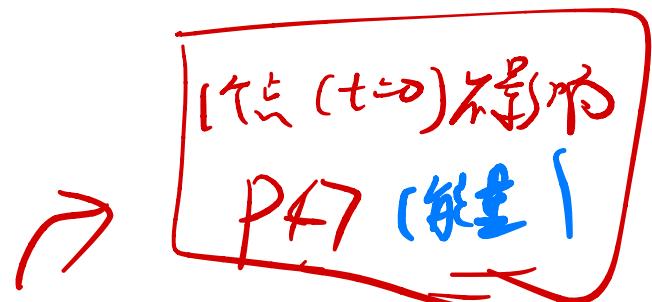
Differential equation

$$y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \text{ for } t > 0$$

- Auxiliary condition is required to determine A
- Initial rest as auxiliary condition for causal LTI systems: $y(0) = 0$

$$A + \frac{K}{5} = 0 \implies A = -\frac{K}{5} \implies y(t) = \frac{K}{5}(e^{3t} - e^{-2t}), \text{ for } t > 0$$

$$= \frac{K}{5}(e^{3t} - e^{-2t})u(t)$$



Differential or Difference Equations

Differential equation

- General case: N th-order linear constant-coefficient differential equation

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

- Particular solution + Homogenous solution: $y(t) = y_p(t) + y_h(t)$

- $y_p(t)$: *forced response (same form as input)*

- $y_h(t)$: Natural response, $\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = 0$

- Initial rest as auxiliary condition, that is if $x(t) = 0$ for $t \leq t_0$,

$$y(t_0) = \frac{dy(t_0)}{dt} = \dots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0$$

(因果系統)



Differential or Difference Equations

Difference equation

- General case: N th-order linear constant-coefficient difference equation

$$\sum_{k=0}^N a_k y[n - k] = \sum_{k=0}^M b_k x[n - k]$$

- Particular solution + Homogenous solution: $y[n] = y_p[n] + y_h[n]$
 - $y_p[n]$: *forced response (same form as input)*
 - $y_h[n]$: Natural response, $\sum_{k=0}^N a_k y[n - k] = 0$
- Initial rest as auxiliary condition, that is if $x[n] = 0$ for $n \leq n_0$,

$$y[n_0] = y[n_0-1] = \cdots = y[n_0-(N-1)] = 0$$



Differential or Difference Equations

Difference equation

□ Recursive solution:

$$y[n] = \frac{1}{a_0} \left\{ \sum_{k=0}^M b_k x[n-k] - \sum_{k=1}^N a_k y[n-k] \right\}$$

➤ Particular case $N=0$

$$y[n] = \frac{1}{a_0} \sum_{k=0}^M b_k x[n-k]$$

Non-recursive equation

$$h[n] = \frac{1}{a_0} \sum_{k=0}^M b_k \delta[n-k]$$

有限冲激响应
Finite impulse response (FIR) system

Differential or Difference Equations

Difference equation

□ Recursive solution: example

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

- Consider $x[n] = K\delta[n]$ and take initial rest: $y[-1] = 0$

$$y[0] = x[0] + \frac{1}{2}y[-1] = K$$

$$y[1] = x[1] + \frac{1}{2}y[0] = \frac{1}{2}K$$

$$y[2] = x[2] + \frac{1}{2}y[1] = \left(\frac{1}{2}\right)^2 K \quad \dots \quad y[n] = x[n] + \frac{1}{2}y[n-1] = \left(\frac{1}{2}\right)^n K$$

$$\therefore h[n] = \left(\frac{1}{2}\right)^n u[n]$$

Infinite impulse response (IIR) system

无限冲激响应

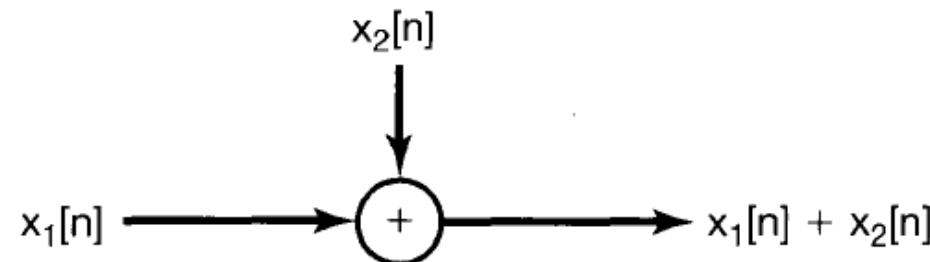
□ Generally $\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$ $\begin{cases} N = 0, \text{FIR system} \\ N > 0, \text{IIR system} \end{cases}$ Not always!



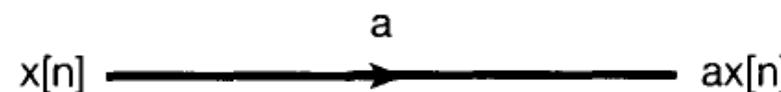
Differential or Difference Equations

Block Diagram Representations

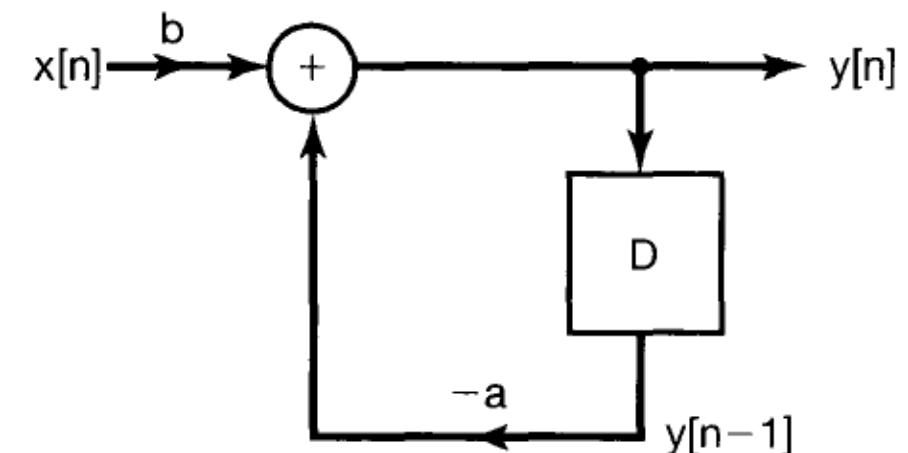
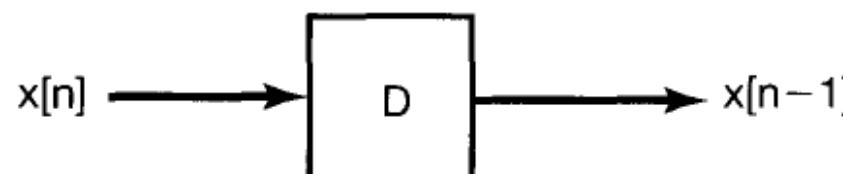
- Basic elements: discrete-time



$$y[n] + ay[n - 1] = bx[n]$$



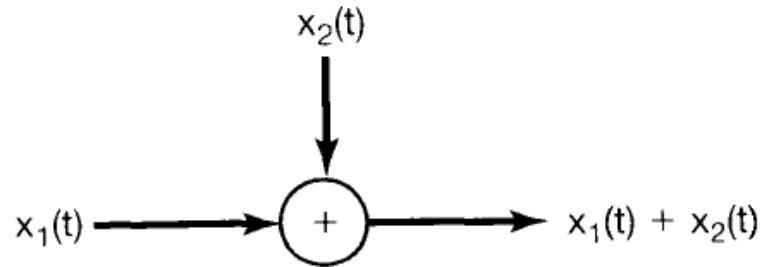
$$y[n] = -ay[n - 1] + bx[n]$$



Differential or Difference Equations

Block Diagram Representations

- Basic elements: continuous-time



$$x(t) \xrightarrow{a} ax(t)$$

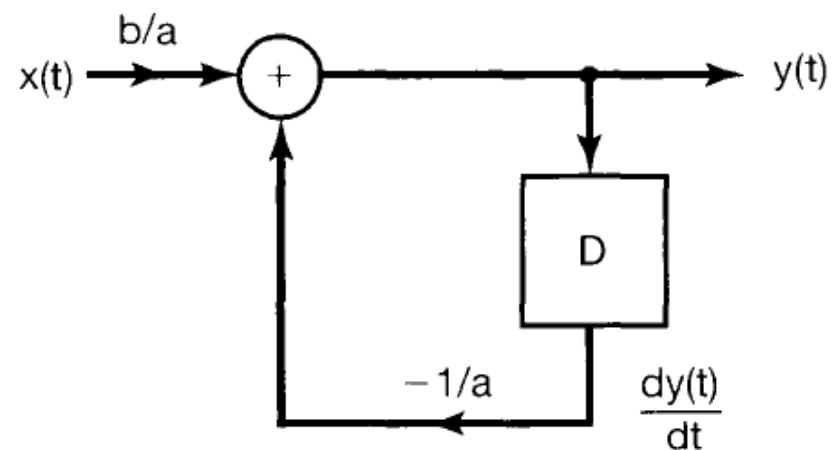
$$x(t) \rightarrow D \rightarrow \frac{dx(t)}{dt}$$

$$x(t) \rightarrow \int \rightarrow \int_{-\infty}^t x(\tau) d\tau$$

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

$$y(t) = -\frac{1}{a} \frac{dy(t)}{dt} + \frac{b}{a} x(t)$$

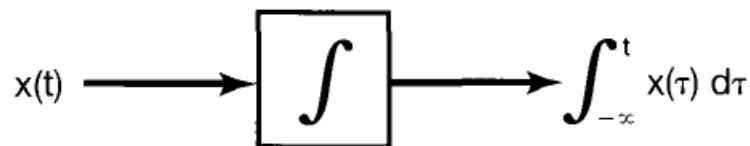
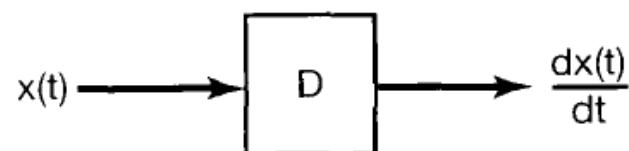
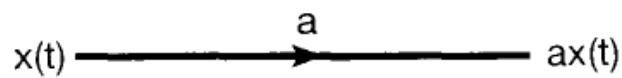
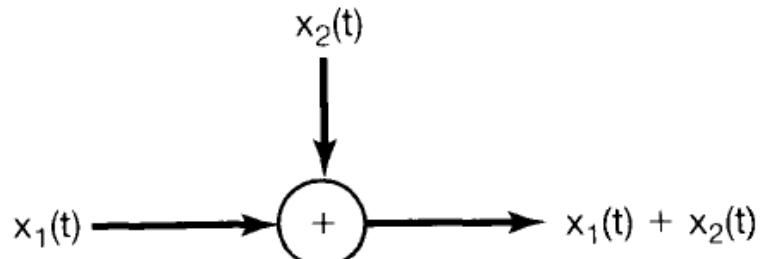
$x[n] \rightarrow D \rightarrow x[n-1]$



Differential or Difference Equations

Block Diagram Representations

- Basic elements: continuous-time



$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

$$\frac{dy(t)}{dt} = -ay(t) + bx(t)$$

$$y(t) = \int_{-\infty}^t [bx(\tau) - ay(\tau)] d\tau$$

