HW5 Solution

1.

1(a)

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

$$= \sum_{n=0}^{+\infty} \left(\left(\tfrac{1}{2}\right)^n \cos\left(\tfrac{\pi n}{2}\right) \right) e^{-jwn}$$

$$= \sum_{n=0}^{+\infty} \left(\tfrac{1}{2}\right)^n \frac{e^{j\frac{n}{2}\pi} + e^{-j\frac{n}{2}\pi}}{2} e^{jwn}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{+\infty} \left(\tfrac{1}{2}\right)^n e^{jn(\frac{\pi}{2}-w)} + \sum_{n=0}^{+\infty} \left(\tfrac{1}{2}\right)^n e^{-jn(\frac{\pi}{2}+w)} \right)$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}e^{j(\frac{\pi}{2}-w)}} + \frac{1}{1-\frac{1}{2}e^{-j(\frac{\pi}{2}+w)}} \right]$$

$$= \frac{1}{2-e^{j(\frac{\pi}{2}-w)}} + \frac{1}{2-e^{-j(\frac{\pi}{2}+w)}}$$

1(b)

$$X(e^{jw}) = a \cdot \sum_{n=-N_1}^{N_1} e^{jwn}$$

$$= a \sum_{m=0}^{2N_1} e^{-jw(m-N_1)}$$

$$= a e^{jwN_1} \sum_{m=0}^{2N_1} e^{-jwm}$$

$$= a e^{jwN_1} \left(\frac{1-e^{-jw(2N_1+1)}}{1-e^{jw}} \right)$$

$$= \frac{a \cdot e^{-jw/2} \cdot \left(e^{jw(N_1+\frac{1}{2})} - e^{jw(N_1+\frac{1}{2})}\right)}{e^{-jw/2} \cdot \left(e^{jw/2} - e^{-jw/2}\right)}$$

$$= \frac{a \sin[w(N_1+\frac{1}{2})]}{\sin(w/2)}$$

FT pairs

$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$	Fourier transform (FT)
$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$	Inverse Fourier transform

2)

(a)

$$nx[n] \stackrel{F}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

$$x[n]=(\tfrac{1}{3})^{|n|}, \quad X(e^{j\omega})=\sum_{-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{+\infty}(\tfrac{1}{3})^n e^{-j\omega n}+\sum_{n=1}^{+\infty}(\tfrac{1}{3})^n e^{j\omega n}$$

$$= \frac{1}{1-\frac{1}{3}e^{-j\omega}}+\frac{\frac{1}{3}e^{j\omega}}{1-\frac{1}{3}e^{j\omega}}$$

$$= \frac{4}{5-3\cos\omega}$$

$$(\tfrac{1}{3})^{|n|} \longleftrightarrow \frac{1-(\frac{1}{3})^2}{1-2\times\frac{1}{3}\cos\omega+\frac{1}{9}}=\frac{4}{5-3\cos\omega}$$

$$n(\tfrac{1}{3})^{|n|} \longleftrightarrow j\frac{d}{d\omega}\left(\frac{4}{5-3\cos\omega}\right)=\frac{-12j\sin\omega}{(5-3\cos\omega)^2}$$

$$X(e^{j\omega})=\frac{-12j\sin\omega}{(5-3\cos\omega)^2}-\frac{4}{5-3\cos\omega}$$

(b)

let $x_1[n]=\dfrac{\sin(\pi n/5)}{\pi n}$, so $X_1(e^{j\omega})=\begin{cases}1, & |\omega|\le\frac{1}{5}\pi\\ 0, & \pi/5<|\omega|<\pi\end{cases}$

$$x_2[n]=\cos(\tfrac{7}{2}\pi n)=(-1)^n\cos(\tfrac{1}{2}\pi n)=e^{j\pi n}\cos\tfrac{1}{2}\pi n$$

$$X_2(e^{j\omega})=\pi[\delta(\omega+\tfrac{1}{2}\pi)+\delta(\omega-\tfrac{1}{2}\pi)], \quad 0\le|\omega|<\pi$$

$$X(e^{j\omega})=\frac{1}{2\pi}\int_{-\pi}^{+\pi}X_2(e^{j\theta})X_1(e^{j(\omega-\theta)})\,d\theta$$

$$=\begin{cases}\frac{1}{2}, & \frac{3\pi}{10}<|\omega|<\frac{7}{10}\pi\\ 0, & 0\le|\omega|\le\frac{3\pi}{10},\ \frac{7}{10}\pi\le|\omega|<\pi,\end{cases}$$

$$y[n]=x_1[n]x_2[n] \stackrel{F}{\longleftrightarrow} Y(e^{j\omega})=\frac{1}{2\pi}\int_{2\pi}X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})\,d\theta$$

3.

For real signal $x[n]$, we have

$$Od\{x[n]\} = \frac{1}{2}[x[n]-x[-n]] \xleftrightarrow{FT} j\,Im\{X(e^{j\omega})\}$$

$$j\,Im(X(e^{j\omega})) = j\sin\omega - j\sin 2\omega = \frac{1}{2}(e^{j\omega}-e^{-j\omega}-e^{j2\omega}+e^{-j2\omega})$$

Then we have $\frac{1}{2}[x[n]-x[-n]] = \frac{1}{2}(\delta[n+1]-\delta[n-1]-\delta[n+2]+\delta[n-2])$

Since $x[n]=0$ for $n>0$, $x[n]=\delta[n+1]-\delta[n+2]$ $(n<0)$

As for $x[0]$, by Parseval's relation $(x[-1])^2+(x[0])^2+(x[-2])^2=6$

Since $x[0]<0$, $x[0]=-2$

Therefore, $x[n] = -\delta[n+2]+\delta[n+1]-\delta[n]$

Parseval's relation

$$\sum_{n=-\infty}^{+\infty}|x[n]|^2 = \frac{1}{2\pi}\int_{2\pi}|X(e^{j\omega})|^2 d\omega$$

Conjugation and Conjugate Symmetry

- Conjugation property

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \implies x^*[n] \xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega})$$

- Conjugation Symmetry

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \quad [x[n] \text{ real}] \implies \begin{cases} \mathcal{E}v\{x[n]\} \xleftrightarrow{\mathcal{F}} \mathcal{R}e\{X(e^{j\omega})\} \\ \mathcal{O}d\{x[n]\} \xleftrightarrow{\mathcal{F}} j\mathcal{I}m\{X(e^{j\omega})\} \end{cases}$$

4.

Answer:

$$\left(\frac{1}{2}\right)^{|n|} \xleftrightarrow{FT} \frac{1-\frac{1}{4}}{1-\cos\omega+\frac{1}{4}} = \frac{3}{5-4\cos\omega}$$

Then use the Fourier transform analysis equation to write

$$\frac{3}{5-4\cos\omega} = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^n e^{-j\omega n}$$

Let $\omega=-2\pi t$ in this equation, and replace n by k

$$\frac{1}{5-4\cos(2\pi t)} = \sum_{k=-\infty}^{\infty}\frac{1}{3}\cdot\left(\frac{1}{2}\right)^{|k|}e^{j2\pi kt}$$

Summary FS and FT expressions

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

By comparing this with the continuous-time Fourier series synthesis equation, it is apparent that $a_k = \frac{1}{3}\left(\frac{1}{2}\right)^{|k|}$ are the Fourier series coefficients of the signal $1/(5-4\cos(2\pi t))$

5.

Answer:

(a) $H(e^{jw}) = \dfrac{b+e^{-jw}}{1-ae^{-jw}}$

$\because |H(e^{jw})| = 1 \quad \therefore |b+e^{-jw}| = |1-ae^{-jw}|$

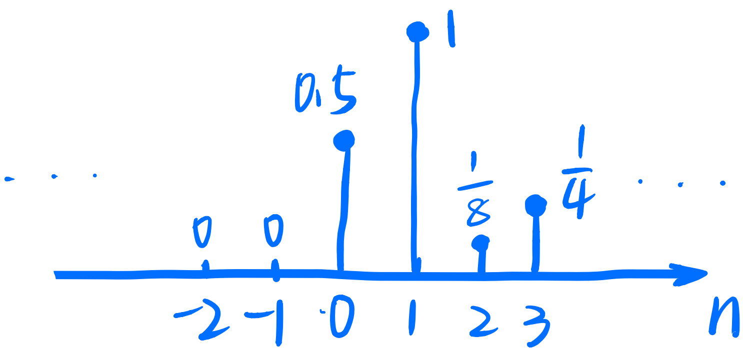
$\therefore \quad 1+b^2+2b\cos w = 1+a^2-2a\cos w,$ for all w

$\Rightarrow \quad b = -a$

(b) $X(e^{jw}) = \dfrac{1}{1-\frac{1}{2}e^{-jw}}$

$Y(e^{jw}) = X(e^{jw}) \cdot H(e^{jw}) = \dfrac{\frac{5}{4}}{1-\frac{1}{2}e^{-jw}} - \dfrac{\frac{3}{4}}{1+\frac{1}{2}e^{-jw}}$

$y[n] = \left[\frac{5}{4}\left(\frac{1}{2}\right)^n - \frac{3}{4}\left(-\frac{1}{2}\right)^n\right] u[n]$



6. (a) $H(e^{jw}) = Y(e^{jw}) / X(e^{jw}) = \dfrac{\frac{3}{2}-\frac{1}{2}e^{-jw}}{1-\frac{3}{4}e^{-jw}+\frac{1}{8}e^{-j2w}}$

$\Rightarrow y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = \frac{3}{2}x[n] - \frac{1}{2}x[n-1]$

(b) $H(e^{jw}) = 2 + \left(\frac{-7}{4}\right) \cdot \dfrac{1}{1+\frac{1}{2}e^{-jw}}$

$h[n] = 2\delta[n] - \frac{7}{4}\left(\frac{1}{2}\right)^n u[n]$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$