EE150 Signal and System
Homework 3

Due on 20 Oct 23:59 UTC+8

Exercies 1. (20pt)

(a) For the continuous-time periodic signal

$$x(t) = 2 + \cos(\frac{2\pi}{3}t) + 4\sin(\frac{5\pi}{3}t)$$

determine the fundamental frequency ω_0 and the Fourier series coefficients a_k such that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

(b) Let $x(t)$ be a periodic signal whose period is T and Fourier series coefficients are

$$a_k = \begin{cases} 0 & , k = 0 \\ j\left(\frac{1}{2}\right)^{|k|} & , otherwise \end{cases}$$

Use Fourier series properties to answer the following questions:

(1) Is $x(t)$ real?

(2) Is $x(t)$ even?

(3) Is $\int_{-\infty}^{t} x(\tau)d\tau$ even? And determine its Fourier series.

(4) Is $x(\alpha t)$ $\alpha > 0$(periodic with period $\frac{T}{\alpha}$) even? And determine its Fourier series.

(a) $x(t) = 2 + \frac{1}{2}e^{j\frac{2\pi}{3}t} + \frac{1}{2}e^{-j\frac{2\pi}{3}t} + \frac{4}{2j}e^{j\frac{5\pi}{3}t} - \frac{4}{2j}e^{-j\frac{5\pi}{3}t}$

so, $w_0 = \frac{\pi}{3}$

$a_0 = 2$, $a_1 = \frac{1}{2}$, $a_{-1} = \frac{1}{2}$, $a_{-5} = -\frac{4}{2j} = 2$, $a_5 = \frac{4}{2j} = -2$

(b), (1) If $x(t)$ is real, then $x(t) = x^*(t)$, so $a_k = a_{-k}^*$, so, not real

(2) If $x(t)$ is even, then $x(t) = x(-t)$, so $a_k = a_{-k}$, and we have $a_k = \begin{cases} 0 & k=0 \\ j(\frac{1}{2})^{|k|} & k \neq 0 \end{cases}$

so $a_k = a_{-k}$, it's even

(3) Let $g(t) = \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FS} b_k$. so $b_k = \frac{a_k}{jkw_0} = \frac{a_k}{jk\frac{2\pi}{T}}$ so

$b_k = \begin{cases} 0 & k=0 \\ \frac{T(\frac{1}{2})^{|k|}}{k \cdot 2\pi} & k \neq 0 \end{cases}$, not even

(4) Let $g(t) = x(\alpha t) \xleftrightarrow{FS} b_k$, so $b_k = a_k$, since $a_k = a_{-k}$, so, it's even

Exercies 2. (30pt)

(a) Suppose we are given the following information about a signal $x[n]$:

1. $x[n]$ is a real and even signal.
2. $x[n]$ has period $N = 10$ and Fourier coefficients a_k
3. $a_{11} = 5$
4. $\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = 50$

Show that $x[n] = A\cos(Bn + C)$, and specify numerical values for the constants A, B and C.

(b) Determine whether the following periodic signals can be represented in Fourier series form

(1) $x(t) = \tan(2\pi t)$

(2) $x(t) = 2\cos(\frac{2\pi}{t}) + \sin(\frac{2\pi}{t}) \quad 0 < t \leq 1$

(3)

$$x(t) = \begin{cases} 0 & t \notin Q \\ 1 & t \in Q \end{cases} \qquad for \quad 0 < t \leq 1$$

(a) from 1, $a_k = a_{-k}$

from 2, $a_k = a_{k+10}$, and $w_0 = \frac{2\pi}{10} = \frac{\pi}{5}$

from 3, we know $a_{11} = a_1 = 5$ and by 1. $a_1 = a_{-1} = 5$

from 4 and by Parseval, we have $\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = \sum_{k=-1}^{8}|a_k|^2 = 50$

so, $\begin{cases} a_k = 0 & \text{for other k in the same period} \\ a_1 = 5 \\ a_{-1} = 5. \end{cases}$

so, $x[n] = 5e^{j\frac{\pi}{5}n} + 5e^{-j\frac{\pi}{5}n}$

$= 10\cos(\frac{\pi}{5}n)$

$A = 10. \quad B = \frac{\pi}{5}. \quad C = 0.$

(b) (1) $x(t)$ is not absolute integrable, then it's $FS \neq x(t)$.
therefore, it can't.

(2) $x(t)$ has infinite maxima and minima, then it's $FS \neq x(t)$,
therefore, it can't.

(3) $x(t)$ has infinite discontinuities, therefore it's $FS \neq x(t)$,
therefore, it can't.

Exercies 3. (10pt)

Let

$$x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \end{cases}$$

be a periodic signal with fundamental period $T = 2$ and Fourier coefficients a_k.

(a) Determine the value of a_0.

(b) Determine the Fourier series representation of $\frac{dx(t)}{dt}$.

(c) Use the result of part (b) and the differentiation property of the continuous-time Fourier series to help determine the Fourier series coefficients of $x(t)$

(a) $a_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{1}{2}\int_0^1 t\,dt + \frac{1}{2}\int_1^2 (2-t)\,dt$

$= \frac{1}{4} + 1 + \frac{1}{4} - 1$

$= \frac{1}{2}$

(b) let $g(t) = \frac{dx(t)}{dt} = \begin{cases} 1 & 0 \le t < 1 \\ -1 & 1 \le t \le 2 \end{cases}$

Let $g(t) \overset{FS}{\longleftrightarrow} b_k$. so $b_0 = \frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt = 0$

$b_k = \frac{1}{2}\int_0^1 e^{-jk\frac{2\pi}{2}t}\,dt - \frac{1}{2}\int_1^2 e^{-jk\frac{2\pi}{2}t}\,dt$

$= \frac{1-e^{-jk\pi}}{jk\pi} \quad k \neq 0.$

(c) $g(t) \overset{FS}{\longleftrightarrow} b_k = jk\omega_0 a_k = jk\pi\, a_k.$

so, $a_k = \frac{b_k}{jk\pi} = \frac{e^{-jk\pi}-1}{k^2\pi^2}$

Exercies 4. (20pt)

Consider a continuous-time LTI system with impulse response

$$h(t) = e^{-4|t|}$$

Find the Fourier series representation of the output $y(t)$ for each of the following inputs:

(a) $x(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$

(b) $x(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t-n)$

(a) $H(jw) = \int_{-\infty}^{+\infty} h(t)\, e^{-jwt}\, dt$

$= \int_{-\infty}^{0} e^{4t} \cdot e^{-jwt} dt + \int_{0}^{+\infty} e^{-4t} \cdot e^{-jwt} dt$

$= \frac{1}{4-jw} + \frac{1}{4+jw}$

$= \frac{8}{16+w^2}$

$a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \delta(t) \cdot e^{-jk2\pi t} dt = 1, \quad w_0 = \frac{2\pi}{T} = 2\pi$

$b_k = a_k H(jkw_0) = H(jk2\pi) = \frac{8}{16+4\pi^2 k^2}$

(b) $T=2, \quad w_0 = \frac{2\pi}{2} = \pi.$

$a_k = \frac{1}{2}\int_0^2 [\delta(t) - \delta(t-1)]\, e^{-jkw_0 t} dt$

$= \frac{1}{2}(1-e^{-jk\pi}) = \frac{1}{2}[1-(-1)^k] = \begin{cases} 0, & k \text{ is even} \\ 1, & k \text{ is odd} \end{cases}$

so $b_k = a_k H(jk\pi) = \begin{cases} 0 & k \text{ is even} \\ \frac{8}{16+\pi^2 k^2} & k \text{ is odd} \end{cases}$

Exercies 5. (20pt)

Consider a causal discrete-time LTI system whose input $x[n]$ and output $y[n]$ are related by the following difference equation:

$$y[n] - \frac{1}{4}y[n-1] = x[n]$$

Find the Fourier series representation of the output $y[n]$ for each of the following inputs:

(a) $x[n] = \sin(\frac{3\pi}{4}n)$

(b) $x[n] = \cos(\frac{\pi}{4}n) + 2\cos(\frac{\pi}{2}n)$

and plot phase angle and magnitude of the Fourier Series.

(a) $H(e^{jw})e^{jwn} - \frac{1}{4}H(e^{jw})e^{jw(n-1)} = e^{jwn}$

$$H(e^{jw}) = \frac{1}{1-\frac{1}{4}e^{-jw}}$$

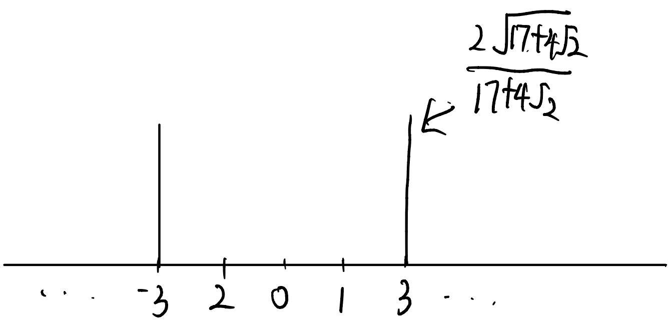
$x[n] = \sin(\frac{3\pi}{4}n) = \frac{1}{2j}e^{j\frac{3\pi}{4}n} - \frac{1}{2j}e^{-j\frac{3\pi}{4}n}$, $w_0 = \frac{\pi}{4}$

$a_3 = \frac{1}{2j}$, $a_{-3} = -\frac{1}{2j}$. $a_k = 0$ for other k in same period.

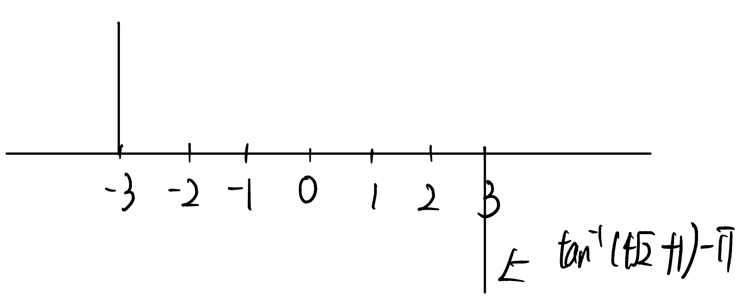
$b_3 = a_3 H(e^{j\frac{3\pi}{4}}) = \frac{1}{2j} \cdot \frac{1}{1-\frac{1}{4}e^{-j\frac{3\pi}{4}}}$

$b_{-3} = a_3 H(e^{j\frac{3\pi}{4}}) = -\frac{1}{2j} \cdot \frac{1}{1-\frac{1}{4}e^{j\frac{3\pi}{4}}}$

magnitude:



phase:



(b) $x[n] = \frac{1}{2}\left(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}\right) + 2\times\frac{1}{2}\left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right)$

$= \frac{1}{2}e^{j\frac{\pi}{4}n} + \frac{1}{2}e^{-j\frac{\pi}{4}n} + e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}$

so, $a_1 = a_{-1} = \frac{1}{2}$. $a_2 = a_{-2} = 1$. $a_k = 0$ for other k in same period.

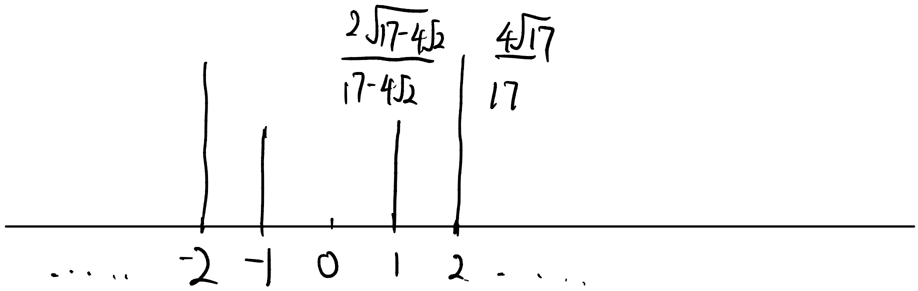
$b_1 = a_1 H(e^{j\frac{\pi}{4}}) = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{4}e^{-j\frac{\pi}{4}}}$

$b_{-1} = a_{-1} H(e^{-j\frac{\pi}{4}}) = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{4}e^{j\frac{\pi}{4}}}$

$b_2 = a_2 H(e^{j\frac{\pi}{2}}) = \frac{1}{1-\frac{1}{4}e^{-j\frac{\pi}{2}}}$

$b_{-2} = a_{-2} H(e^{j-\frac{\pi}{2}}) = \frac{1}{1-\frac{1}{4}e^{j\frac{\pi}{2}}}$

magnitude



phase