

Sampling (ch.7)

- Representation of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- Reconstruction of a Signal from Its Samples Using Interpolation
- The Effect of Undersampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time Signals

The Sampling Theorem

❑ What is sampling?

Converting continuous-time signals to discrete-time signals

❑ Why sampling?

To use the well-developed digital technology

❑ But, a signal could not always be uniquely specified by equally-spaced samples

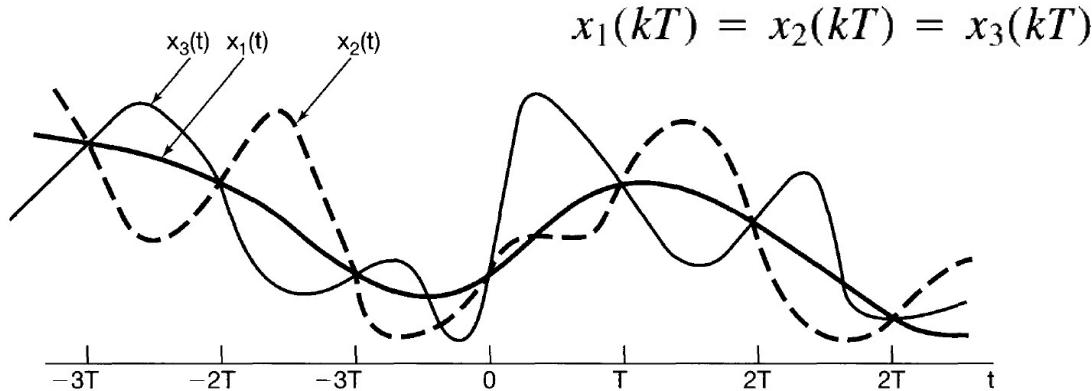
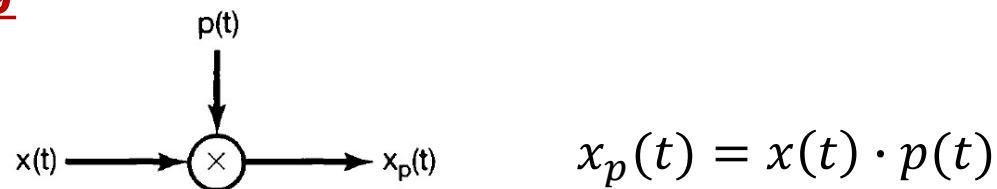


Figure 7.1 Three continuous-time signals with identical values at integer multiples of T .

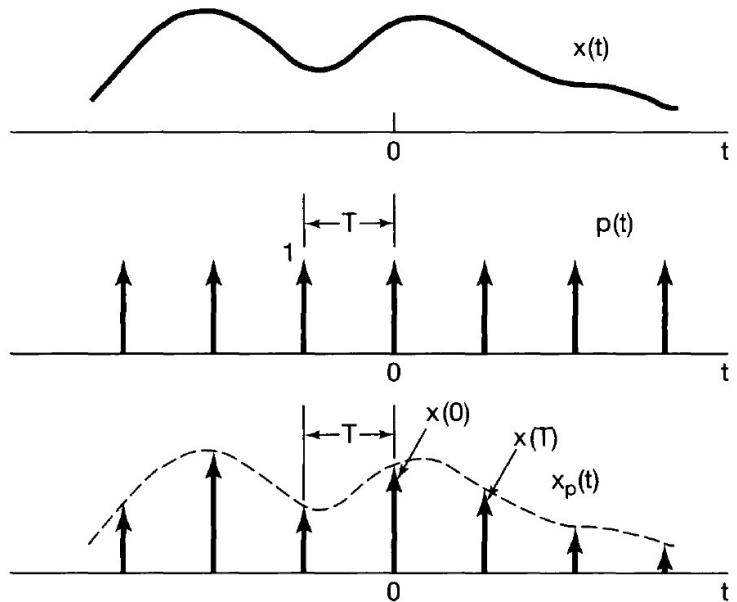
❑ The sampling theorem should be satisfied

The Sampling Theorem

Impulse-Train Sampling



□ Time domain

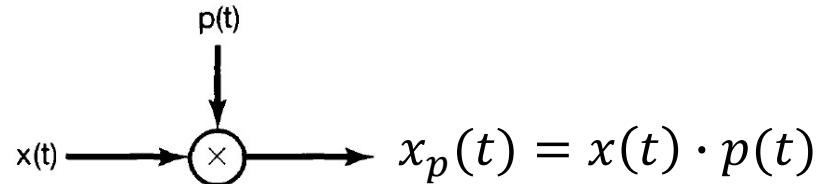


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)$$

The Sampling Theorem

Impulse-Train Sampling

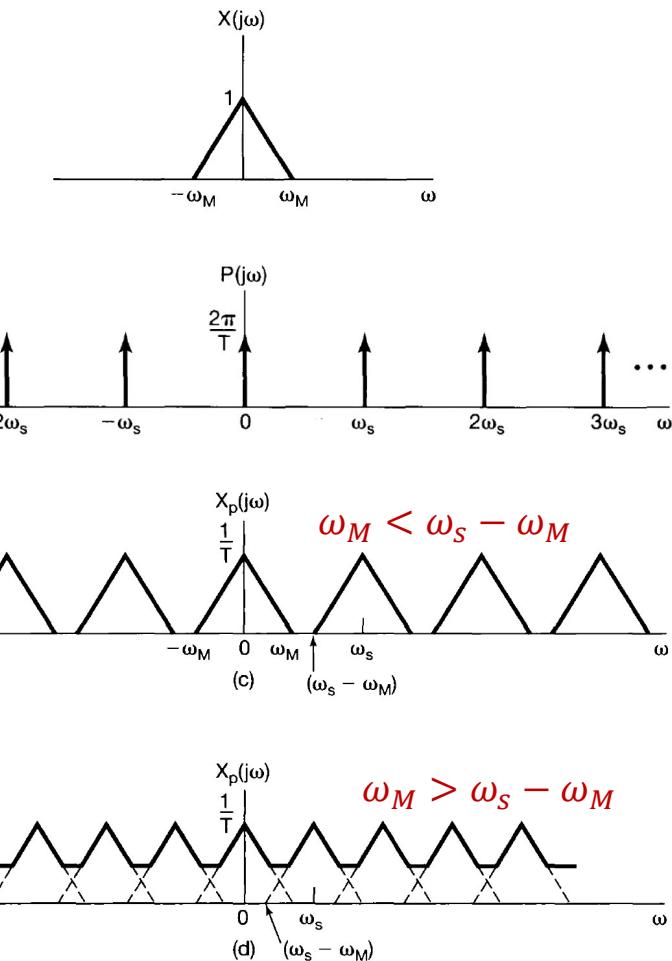


□ Frequency domain

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$P(j\omega) = \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta(\omega - k\omega_s) = \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta\left(\omega - k \frac{2\pi}{T}\right)$$

$$X_p(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(\omega - \theta)) d\theta = \frac{1}{T} \sum_{K=-\infty}^{\infty} X\left(j(\omega - k \cdot \omega_s)\right)$$





The Sampling Theorem

Sampling Theorem

Sampling Theorem:

Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \dots$, if

$$\omega_s > 2\omega_M,$$

where

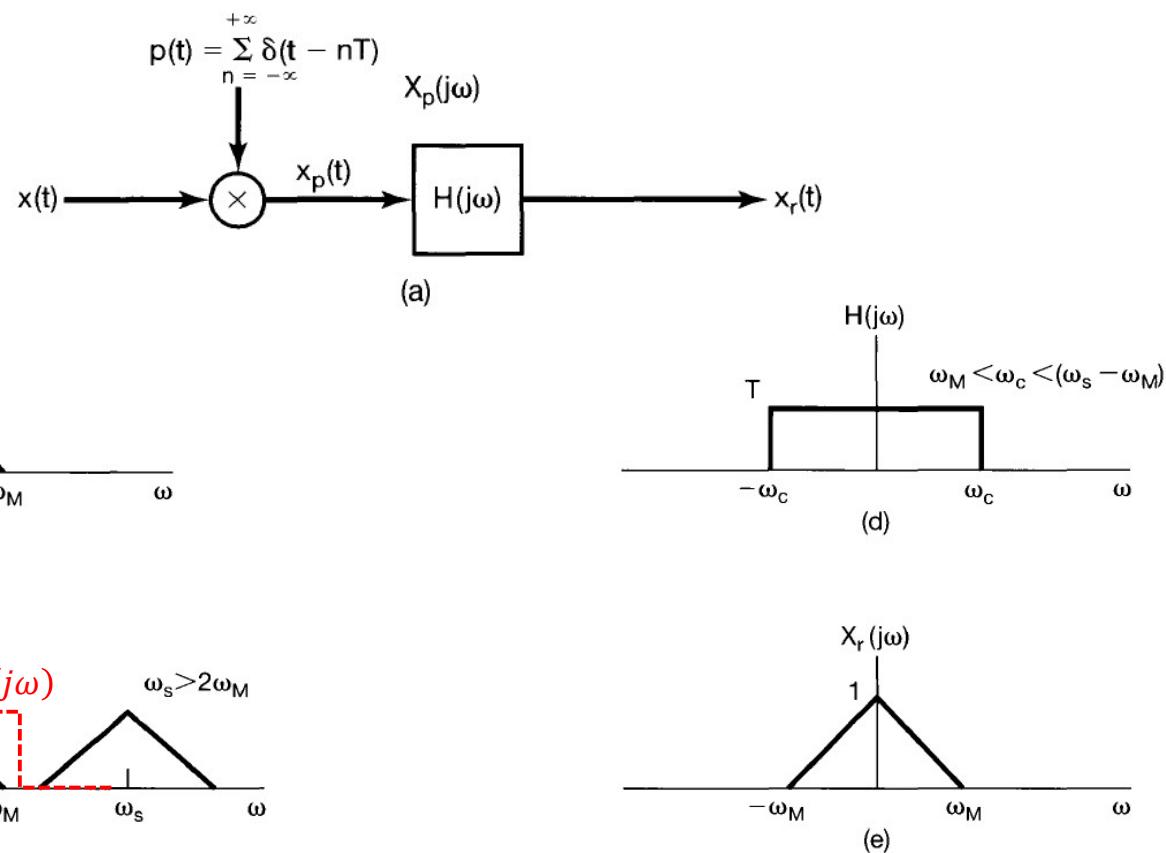
$$\omega_s = \frac{2\pi}{T}.$$

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain T and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal $x(t)$.

The Sampling Theorem

Recovery of the CT signal

□ Ideal low-pass filtering

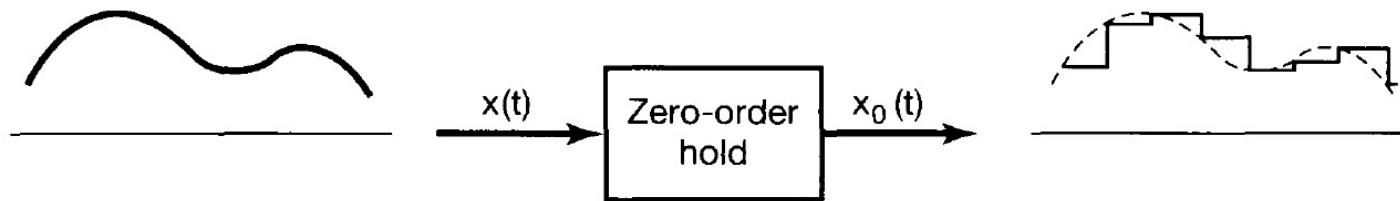


The Sampling Theorem



Sampling with a Zero-order Hold

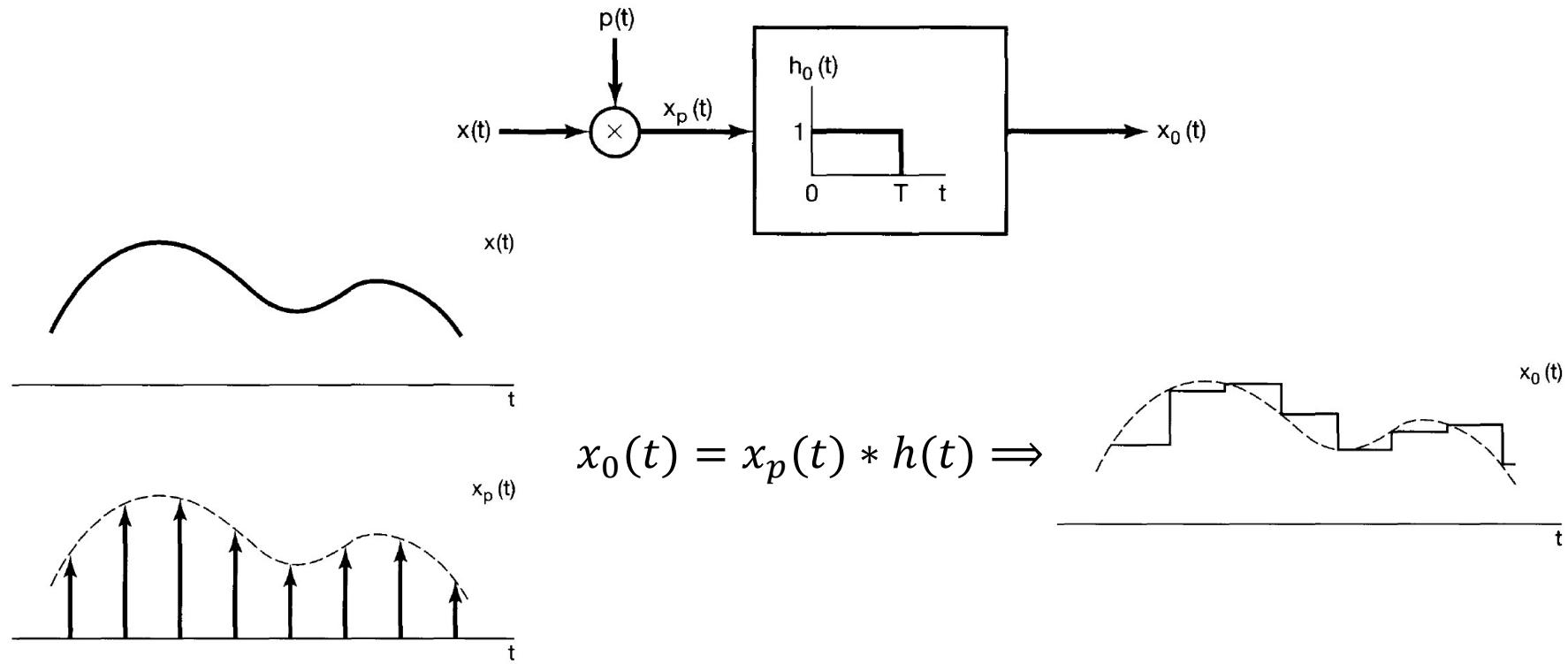
- Why: Impulse-train is difficult to generate
- Principle: Samples $x(t)$ at a given instant and holds that value until the next instant



The Sampling Theorem

Sampling with a Zero-order Hold 零阶保持采样

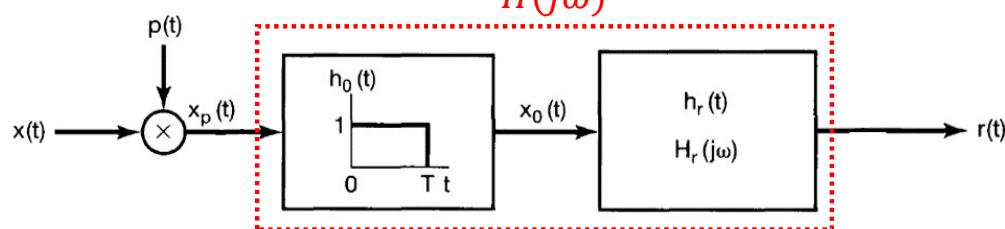
□ Equivalent: Impulse-train sampling + an LTI system with a rectangular impulse response



The Sampling Theorem

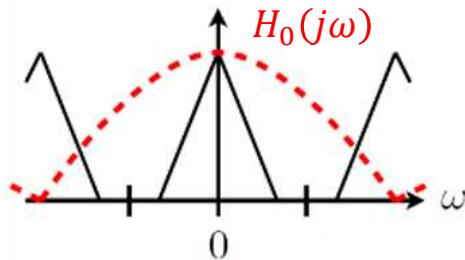
Sampling with a Zero-order Hold

□ Compensation filter 修正滤波器

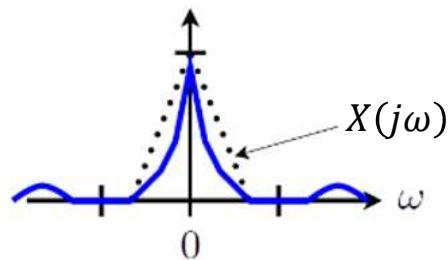


$$H_0(j\omega) = e^{-j\omega T/2} \left[\frac{2 \sin \omega T}{\omega} \right]$$

$X_p(j\omega)$

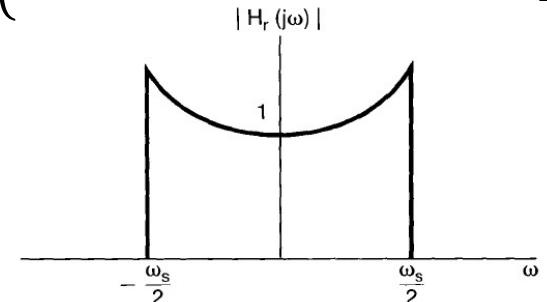


$X_0(j\omega)$

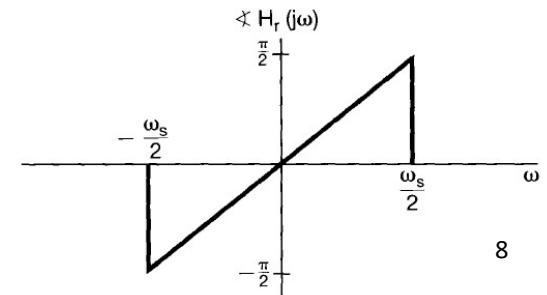


Let $H_0(j\omega)H_r(j\omega) = H(j\omega)$

$$H_r(j\omega) = \begin{cases} e^{j\omega T/2} / \left[\frac{2 \sin \omega T}{\omega} \right], & |\omega| \leq \frac{\omega_s}{2} \\ 0, & |\omega| > \frac{\omega_s}{2} \end{cases}$$



$\Rightarrow H_r(j\omega)$



Sampling (ch.7)

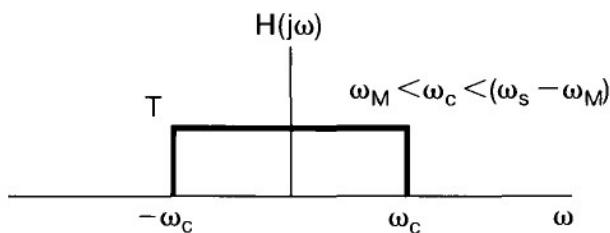
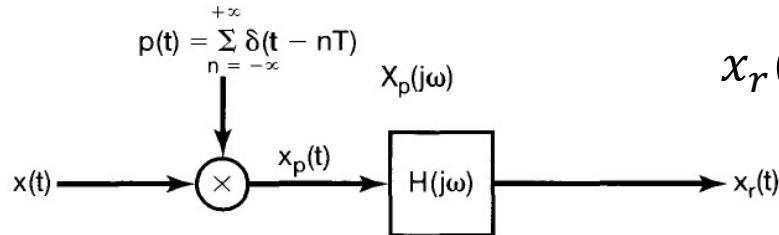
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Reconstruction of a Signal Using Interpolation



Band-limited interpolation: (ideal low-pass filter)



$$h(t) = \frac{T\omega_c}{\pi} \frac{\sin \omega_c t}{\omega_c t}$$

shifted (nT)
and weighted
 $x(nT)$ sum of hat)

$$x_r(t) = x_p(t) * h(t) = \left[\sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT) \right] * h(t)$$

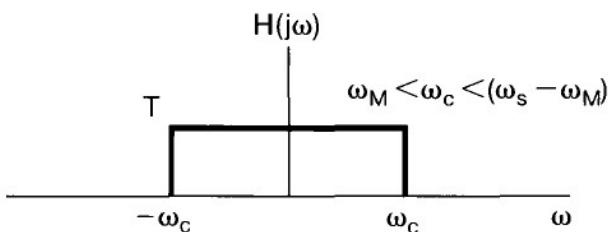
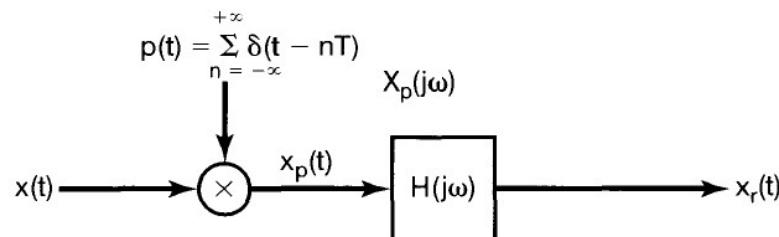
$$= \sum_{n=-\infty}^{\infty} x(nT) [\delta(t - nT) * h(t)]$$

$$= \sum_{n=-\infty}^{\infty} x(nT) h(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \frac{T\omega_c}{\pi} \frac{\sin \omega_c (t - nT)}{\omega_c (t - nT)}$$

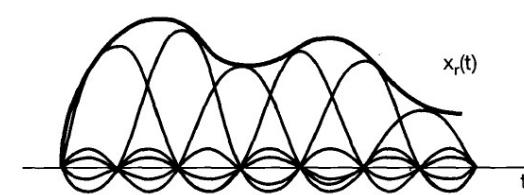
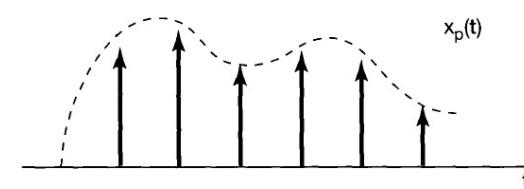
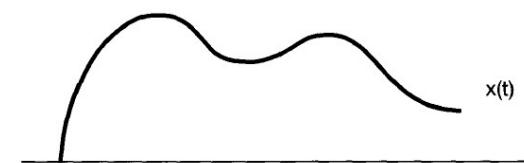
Reconstruction of a Signal Using Interpolation

Band-limited interpolation: (ideal low-pass filter)



$$h(t) = \frac{T\omega_c}{\pi} \frac{\sin \omega_c t}{\omega_c t}$$

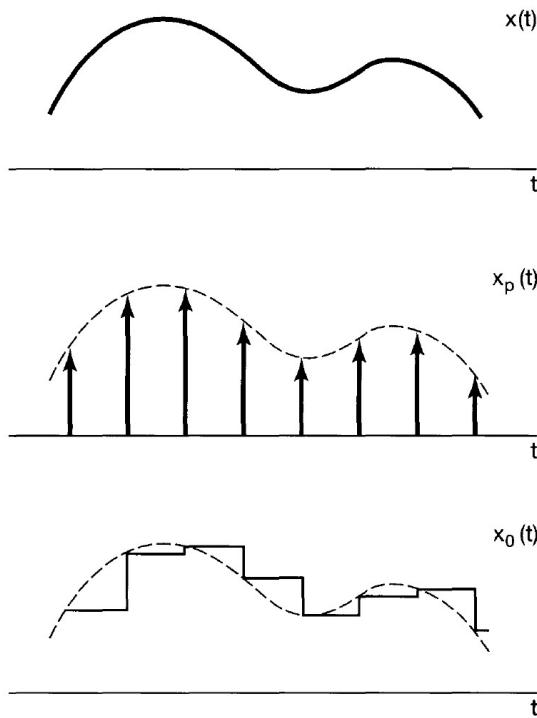
$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{T\omega_c}{\pi} \frac{\sin \omega_c (t - nT)}{\omega_c (t - nT)}$$



Reconstruction of a Signal Using Interpolation

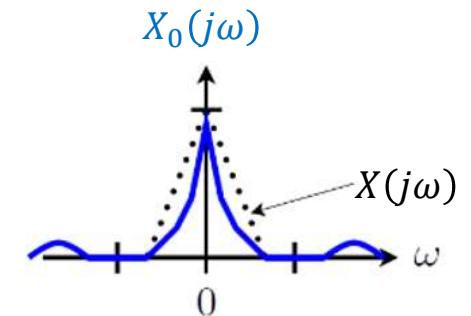
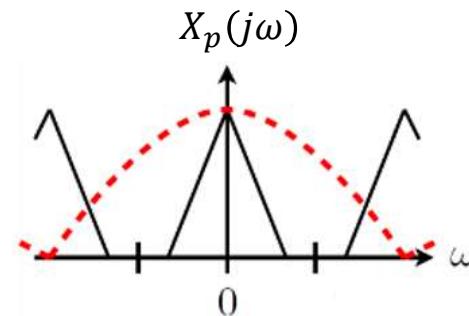
Zero-order hold

□ Time domain



□ Frequency domain

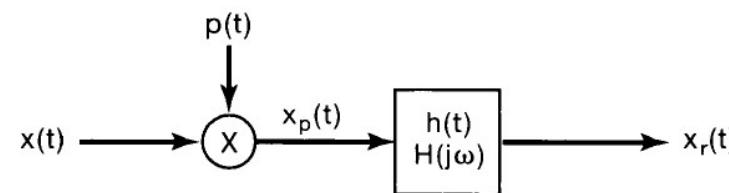
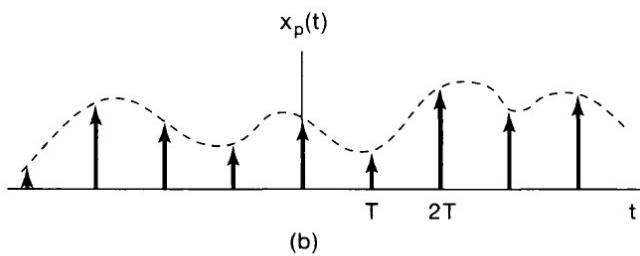
$$H_0(j\omega) = e^{-j\omega T/2} \left[\frac{2 \sin \omega T}{\omega} \right]$$



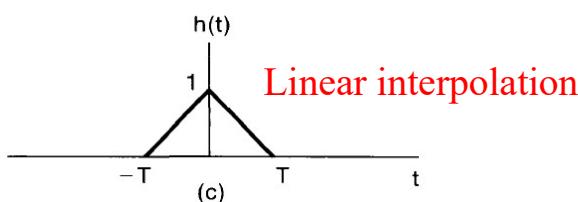
Reconstruction of a Signal Using Interpolation

First-order hold: Impulse-train sampling + an LTI system with a triangular impulse response

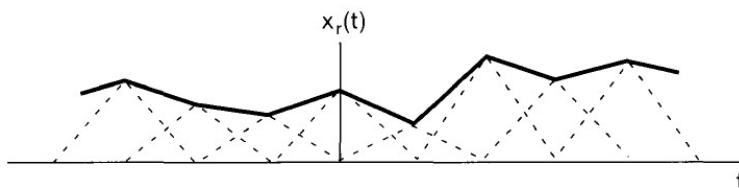
□ Time domain



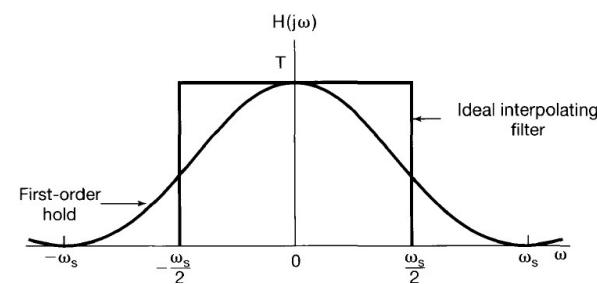
□ Frequency domain



线性插值



$$H(j\omega) = \frac{1}{T} \left[\frac{\sin(\omega T/2)}{\omega/2} \right]^2$$



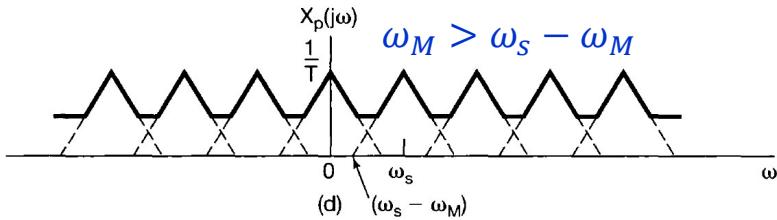
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Aliasing

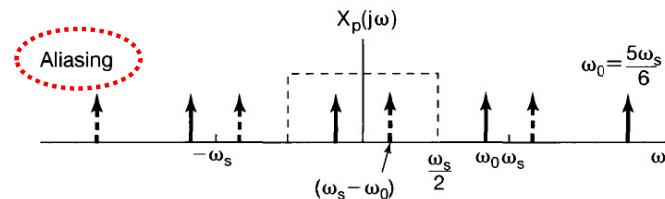
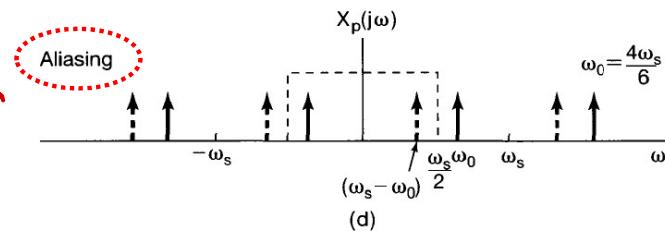
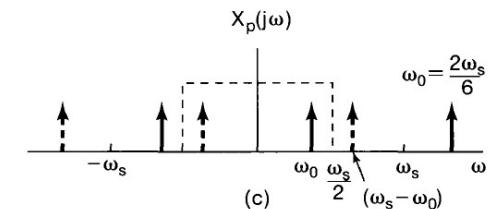
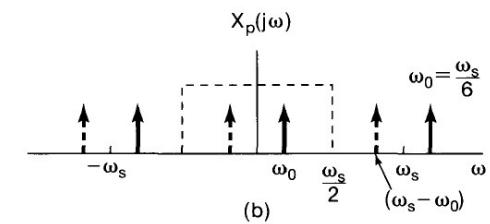
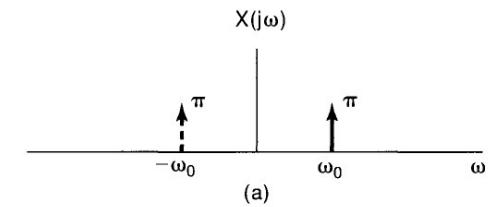
Aliasing 混淆

- When $\omega_s < 2\omega_M$, the individual spectrums overlap



- Consider original signal is $x(t) = \cos \omega_0 t$, with different ω_0 but sampled at same ω_s

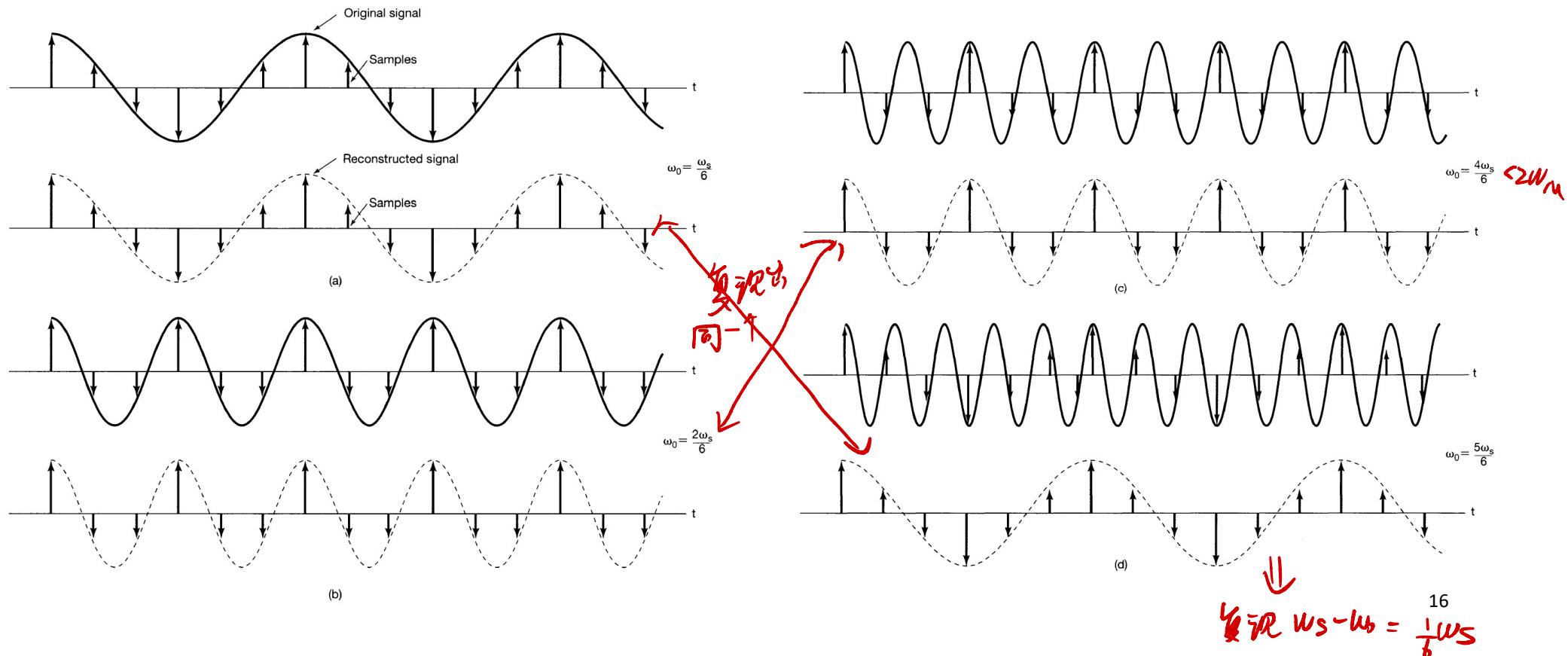
- When aliasing occurs, the original frequency ω_0 takes on the identity of lower frequency $(\omega_s - \omega_0)$.



Aliasing

Aliasing

$$x(t) = \cos \omega_0 t \text{ Time domain}$$



Aliasing



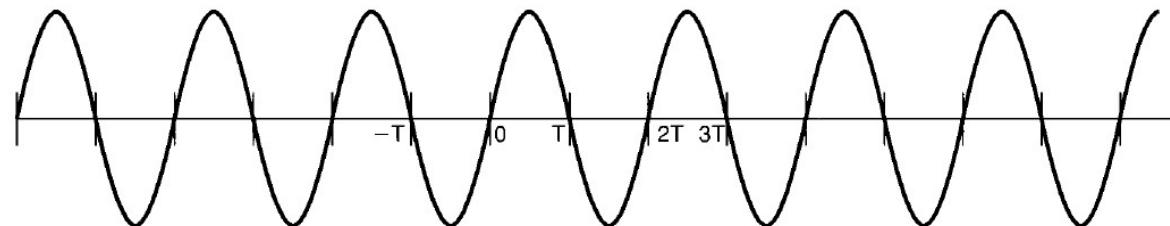
Aliasing

❑ $\omega_s = 2\omega_M$ is not sufficient to avoid aliasing

- Consider a signal $x(t) = \cos(\omega_0 t + \phi)$ is sampled using impulse sampling with $\omega_s = 2\omega_0$
- The reconstructed signal using ideal low-pass filter is

$$x_r(t) = \cos(\phi) \cos(\omega_0 t) = x(t) \text{ only if } \phi = 2k\pi$$

- Particularly, if $\phi = -\pi/2$, then $x(t) = \sin \omega_0 t$ and $x_r(t) = 0$



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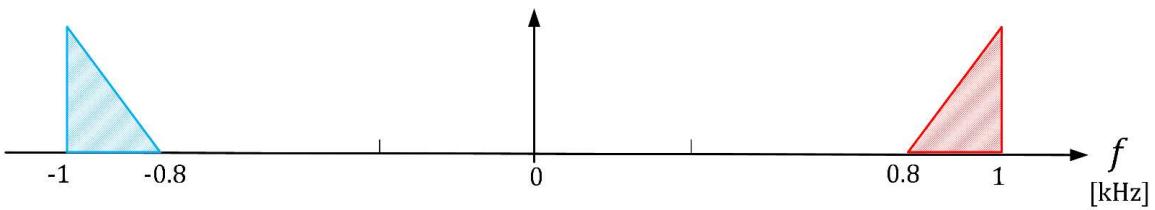
$$x(t) = \cos \omega_0 t + \cos \phi - \underline{\sin \omega_0 t \sin \phi} \rightarrow t = nT \Rightarrow$$

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t - nT) = \sum_{n=-\infty}^{+\infty} \cos \omega_0 nT + \cos \phi \Big|_{t=nT} \delta(t - nT)$$

Aliasing

Aliasing

- For signal with $f_c > B/2$, where $f_c = (f_h + f_l)/2$ and $B = f_h - f_l$



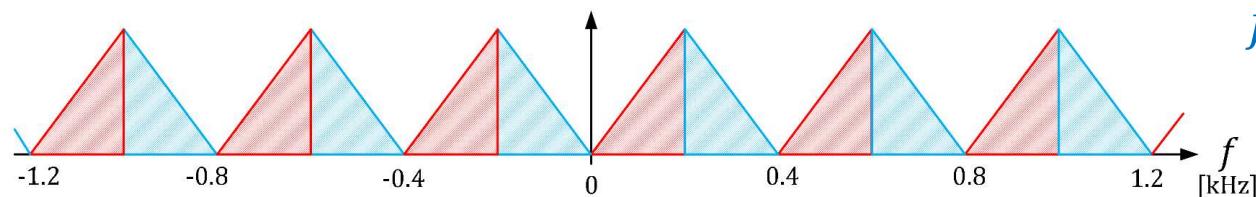
$$f_l = 800 \text{ Hz}, f_h = 1000 \text{ Hz}$$

Determine the lowest f_s with no aliasing

$\omega_s > 2\omega_m (f_s > f_m)$ ✓ 但有浪费

只需大于带宽的2倍即可

$$f_s = 400 \text{ Hz}$$

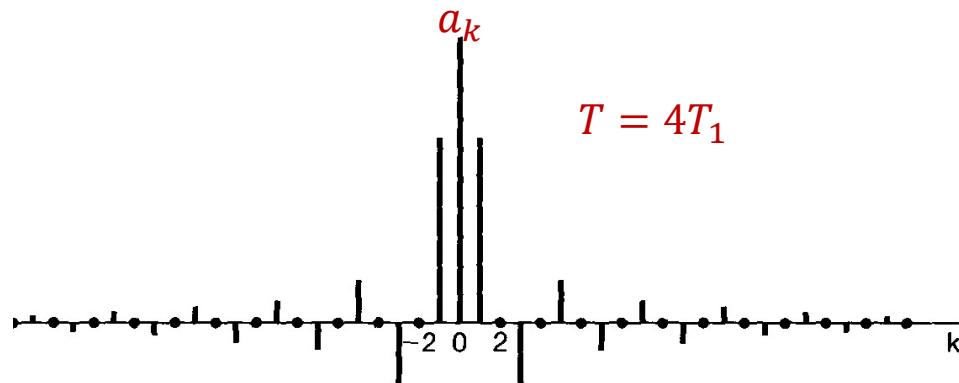
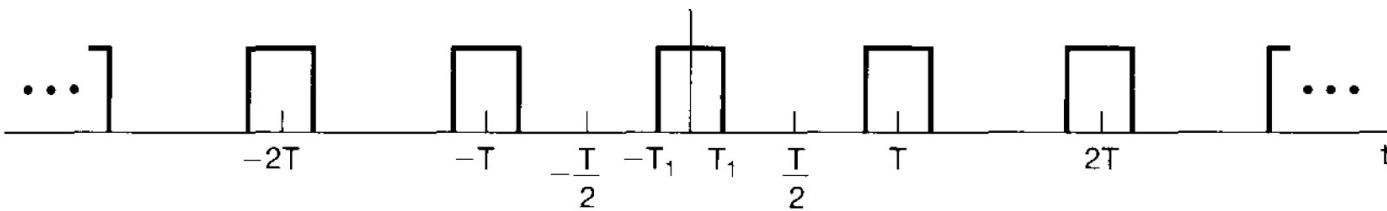


Q: What about $f_l = 850 \text{ Hz}$?

Aliasing

Aliasing

- For harmonic related signal, e.g., a square wave



- $\omega_s > 2K\omega_0$, with K the k th harmonics you want to include
- Low-pass filtering before sampling

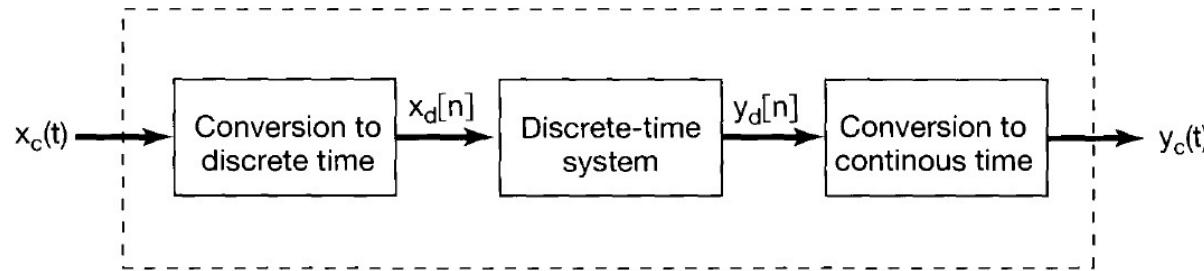
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- ❑ **Discrete-Time Processing of Continuous-Time Signals**
- ❑ Sampling of Discrete-Time signals

Discrete-Time Processing of Continuous-Time Signals



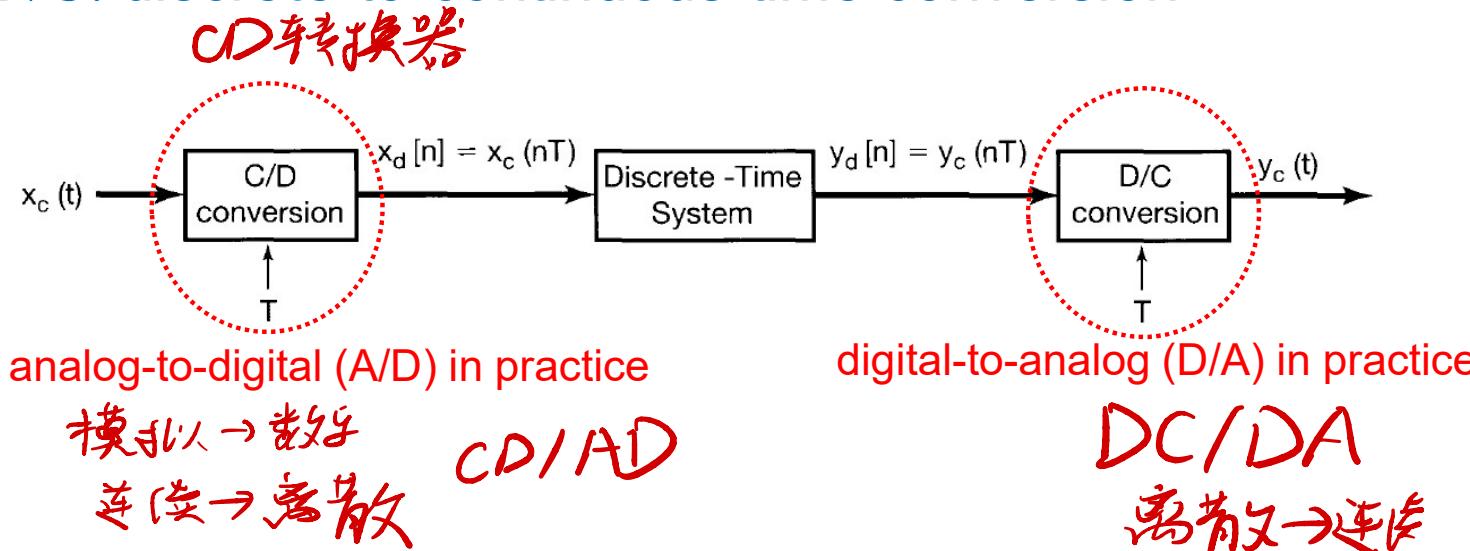
General scheme



$$x_d[n] = x_c(nT)$$

$$\underline{y_d[n] = y_c(nT)}$$

- C/D: continuous-to-discrete-time conversion
- D/C: discrete-to-continuous-time conversion

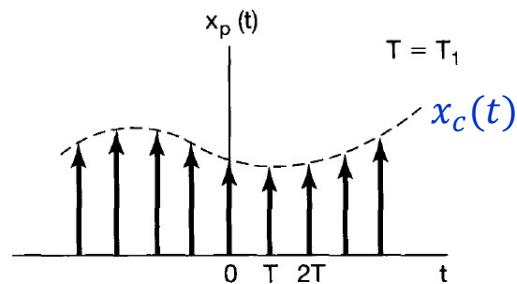
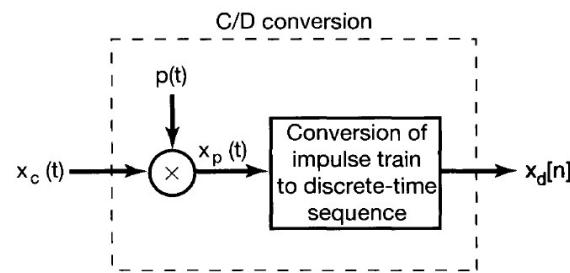


Discrete-Time Processing of Continuous-Time Signals

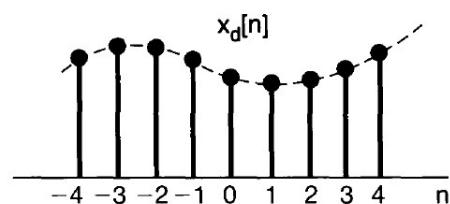


C/D conversion

Time domain



$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \cdot \delta(t - nT)$$



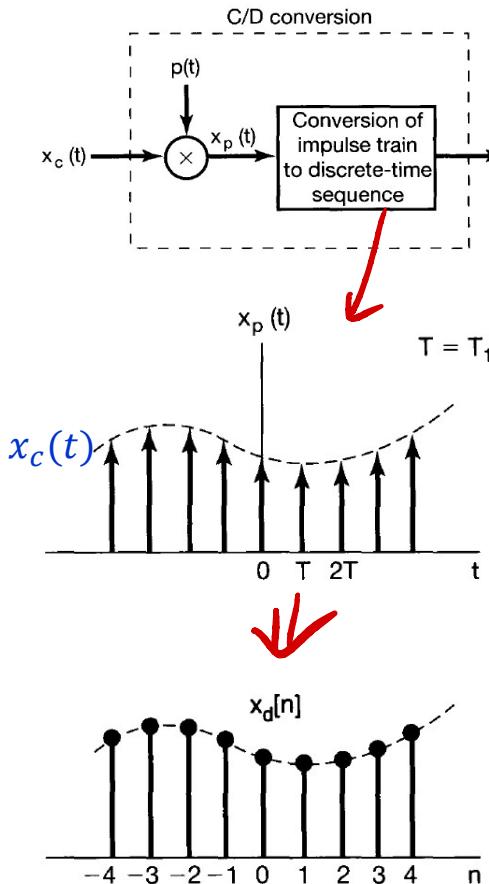
$$x_d[n] = x_c(nT)$$

Discrete-Time Processing of Continuous-Time Signals



C/D conversion

Frequency domain: ω for continuous time and Ω for discrete time



□ Spectrum of $x_d[n]$

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n] e^{-jn\Omega} = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-jn\Omega}$$

$= \int_{-\infty}^{+\infty} x_p(t) e^{-j\omega t} dt$

□ Spectrum of $x_p(t)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \cdot \delta(t - nT) \Rightarrow X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT) \cdot e^{-jn\Omega T}$$

$\Omega = \omega T$

□ If $\omega = \Omega/T$, $X_d(e^{j\Omega}) = X_p(j\Omega/T)$

□ The spectrum of $x_d[n]$ can be obtained from $X_p(j\omega)$ by replacing ω with Ω/T .

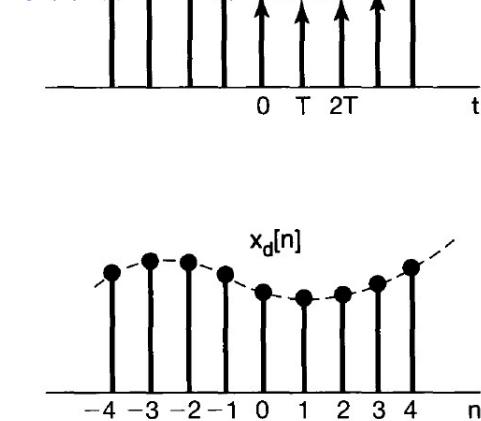
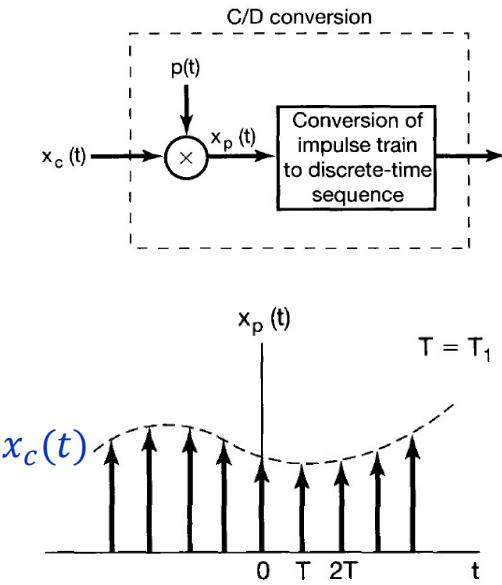
$\omega_m T = 2\pi$

Discrete-Time Processing of Continuous-Time Signals



C/D conversion

Frequency domain: ω for continuous time and Ω for discrete time



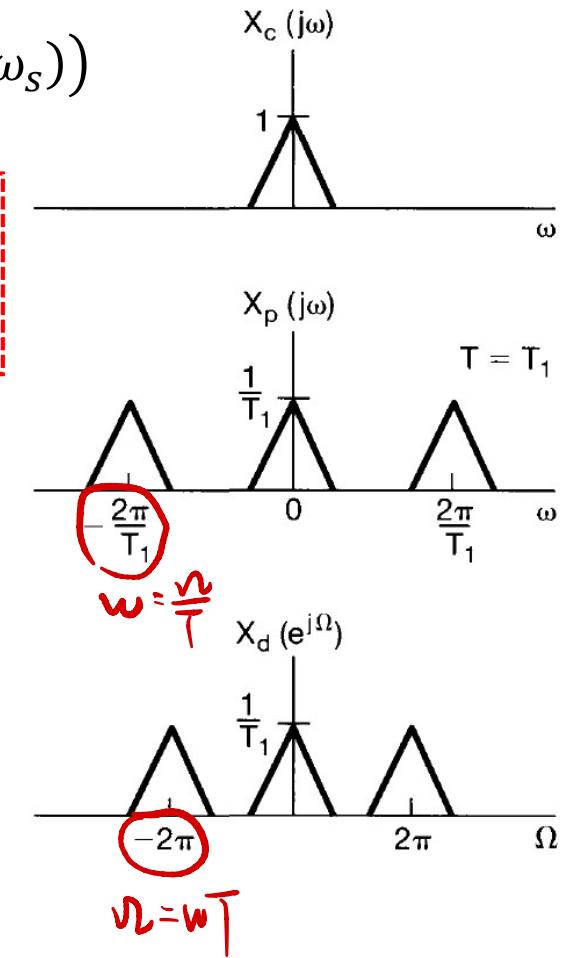
□ Recall: $X_p(j\omega) = \frac{1}{T} \sum_{K=-\infty}^{\infty} X(j(\omega - k \cdot \omega_s))$

□
$$\therefore X_d(e^{j\Omega}) = \frac{1}{T} \sum_{K=-\infty}^{\infty} X(j(\Omega - 2k\pi)/T)$$

□ $X_d(e^{j\Omega})$ is a frequency-scaled version of $X_p(j\omega)$, and is periodic with period of 2π

□ Informally

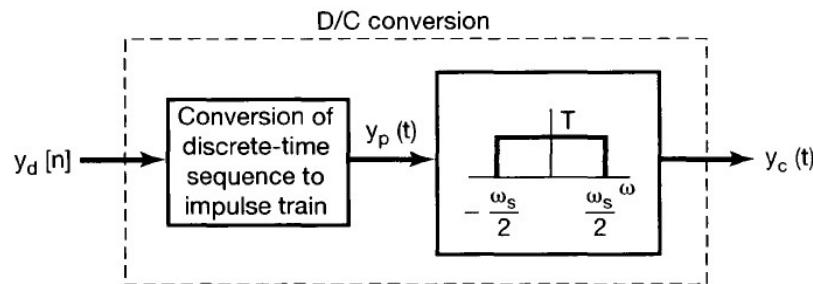
- t to n : time scaling by $1/T$
- ω to Ω : frequency scaling by T



Discrete-Time Processing of Continuous-Time Signals



D/C conversion

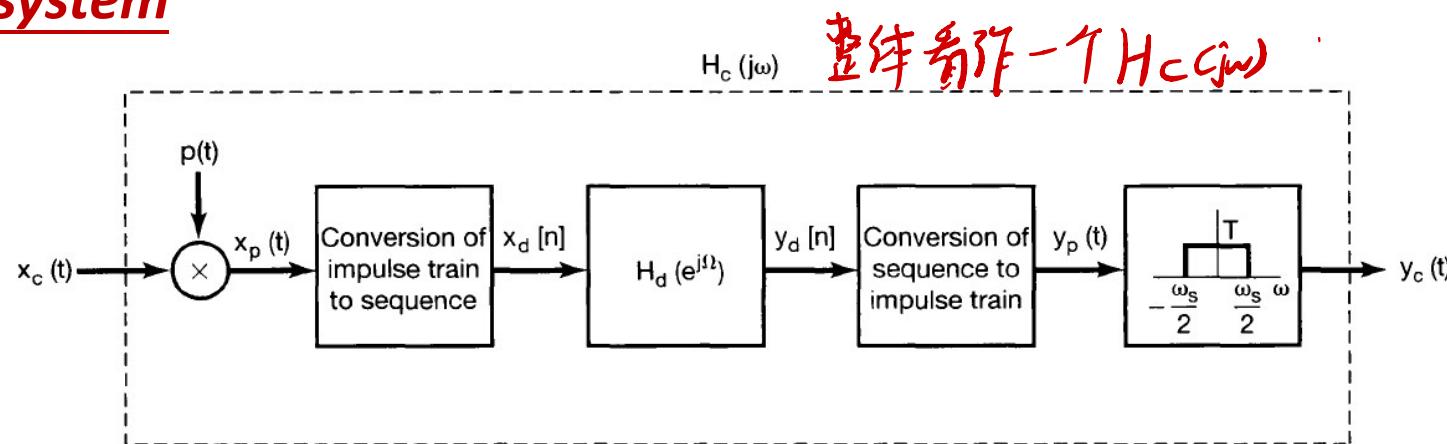


- $Y_d(e^{j\Omega})$: Spectrum of $y_d[n]$
- $Y_p(j\omega)$: Spectrum of $y_p(t)$
- $Y_p(j\omega)$ can be obtained from $Y_d(e^{j\Omega})$ by replacing Ω with ωT .

Discrete-Time Processing of Continuous-Time Signals



Overall system

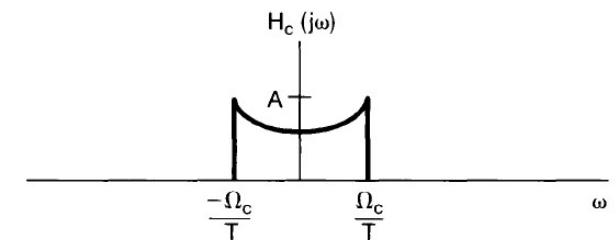
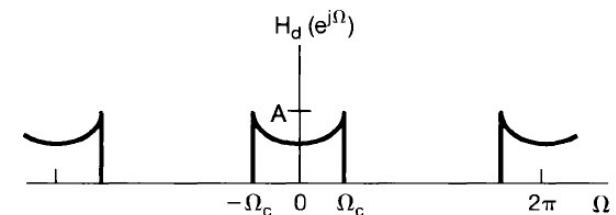
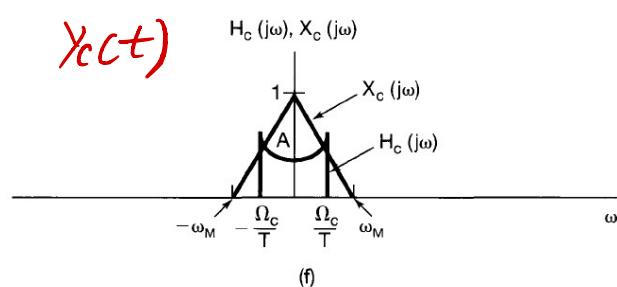
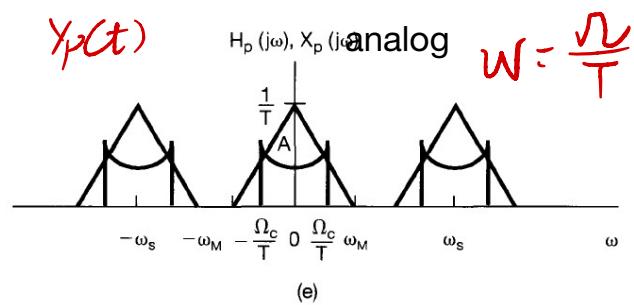
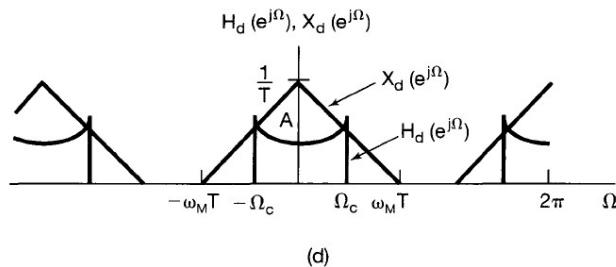
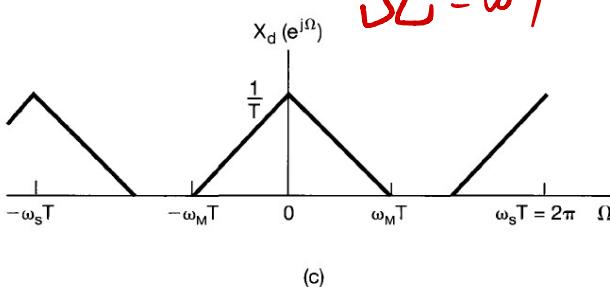
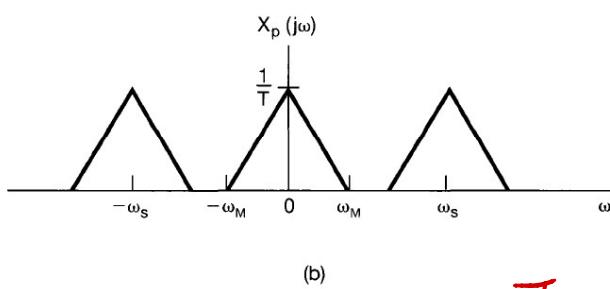
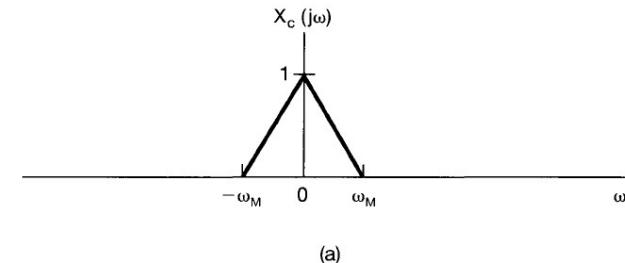


- $x_c(t)$: input
- $y_c(t)$: output
- The overall system is equivalent to a continuous-time system with frequency response $H_c(j\omega)$
- $H_c(j\omega) = ?$

Discrete-Time Processing of Continuous-Time Signals



Overall system 整个系统

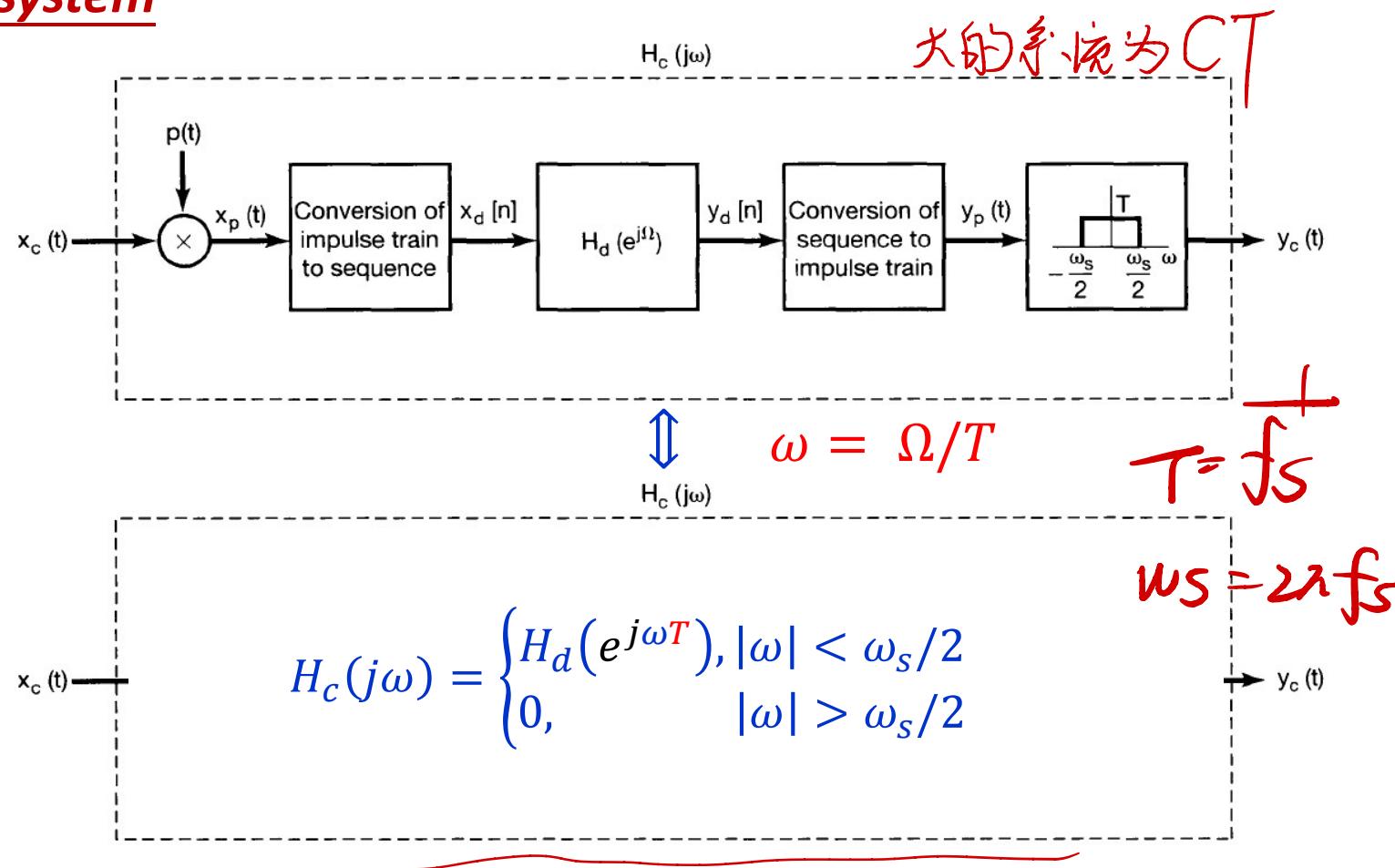


$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}$$

Discrete-Time Processing of Continuous-Time Signals



Overall system



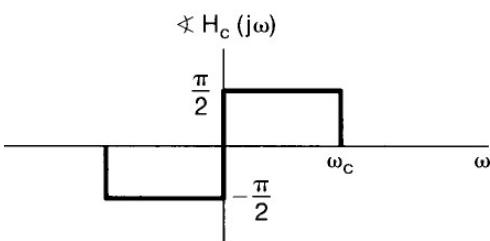
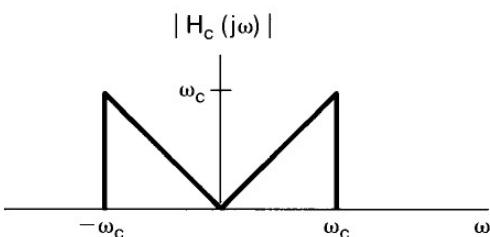
Discrete-Time Processing of Continuous-Time Signals



Digital differentiator: frequency response

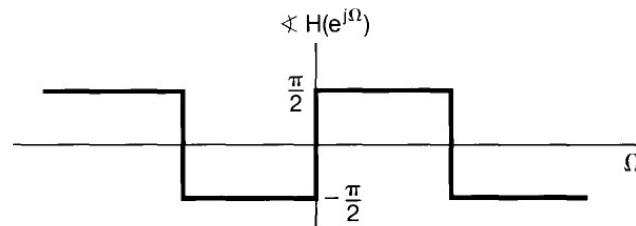
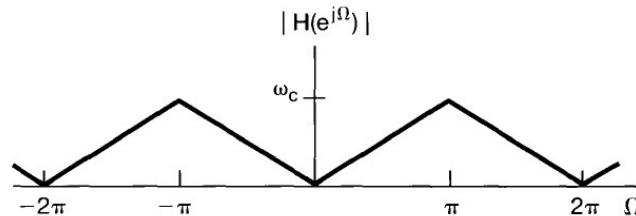
□ Band-limited CT differentiator \Leftrightarrow □ Corresponding DT differentiator

$$H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



$$H_d(e^{j\Omega}) = j \frac{\Omega}{T}, |\Omega| < \pi$$

$$\omega_c = \omega_s/2$$

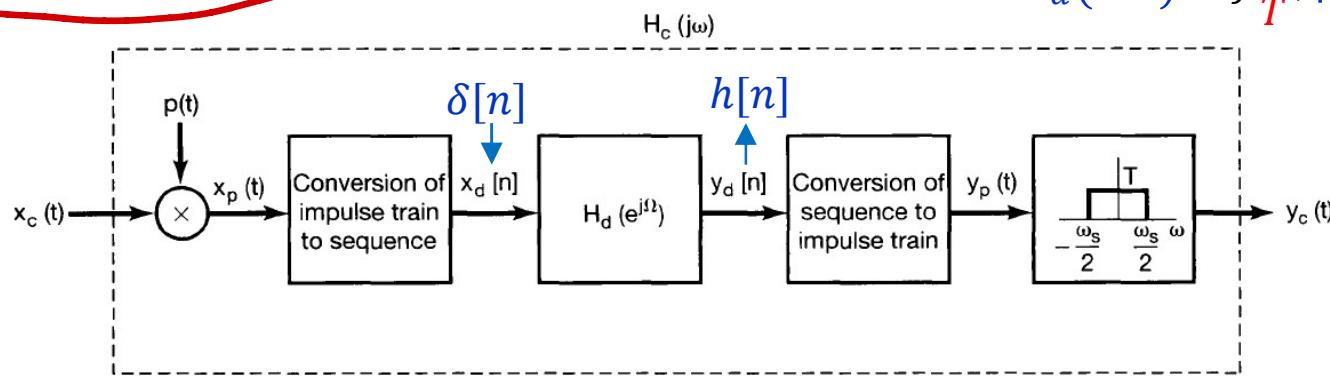


Discrete-Time Processing of Continuous-Time Signals



Digital differentiator; impulse response

$$H_d(e^{j\Omega}) = j \frac{\Omega}{T}, |\Omega| < \pi$$



$\square x_c(t) = \frac{\sin(\pi t/T)}{\pi t} \Rightarrow x_d[n] = x_c(nT) = \frac{1}{T} \delta[n]$ ↪ $\frac{\sin(n\pi)}{n\pi T} = \begin{cases} +, n=0 \\ 0, n \neq 0 \end{cases}$

$\square y_d[n] = y_c(nT)$ $y_c(t) = \frac{d}{dt} x_c(t) = \frac{\cos(\pi t/T)}{Tt} - \frac{\sin(\pi t/T)}{\pi t^2}$

$\square y_d[n] = \begin{cases} \frac{(-1)^n}{nT^2}, n \neq 0 \\ 0, n = 0 \end{cases}$ $\Rightarrow \therefore h_d[n] = \begin{cases} \frac{(-1)^n}{nT}, n \neq 0 \\ 0, n = 0 \end{cases}$

Sampling (ch.7)

- Representation of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- Reconstruction of a Signal from Its Samples Using Interpolation
- The Effect of Undersampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time signals**

Sampling of Discrete-Time Signals

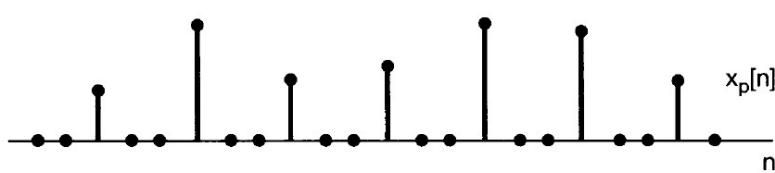
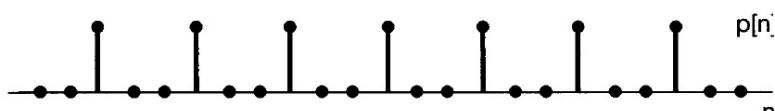
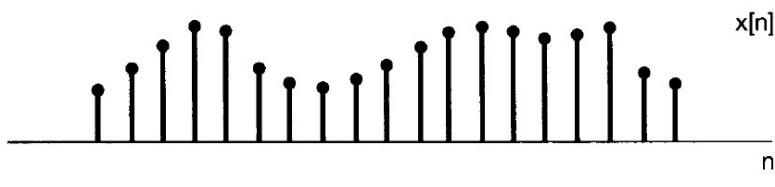
Impulse train sampling

Time domain



$$p[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$

N: sampling period



$$x_p[n] = x[n]p[n] = \sum_{-\infty}^{\infty} x[kN]\delta[n - kN]$$

$$= \begin{cases} x[n], & \text{if } n \text{ is an integer multiple of } N \\ 0, & \text{otherwise} \end{cases}$$

Sampling of Discrete-Time Signals

Impulse train sampling

Frequency domain

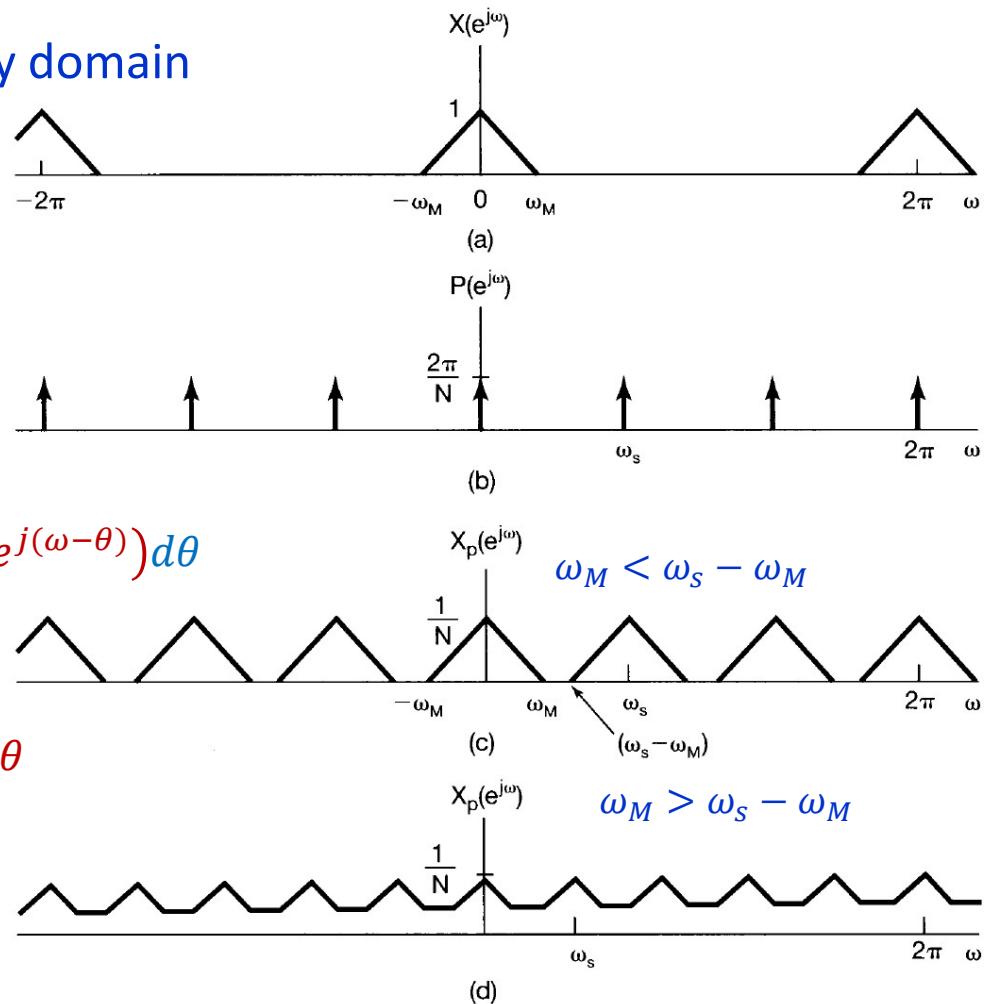
$$P(e^{j\omega}) = \frac{2\pi}{N} \sum_{K=-\infty}^{\infty} \delta(\omega - k\omega_s) \quad \omega_s = \frac{2\pi}{N}$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} P(e^{j\theta}) X(e^{j(\omega-\theta)}) d\theta$$

$$= \frac{1}{2\pi} \cdot \frac{2\pi}{N} \int_{2\pi} \left[\sum_{K=-\infty}^{\infty} \delta(\theta - k\omega_s) \right] X(e^{j(\omega-\theta)}) d\theta$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} \int_{2\pi} \delta(\theta - k\omega_s) X(e^{j(\omega-\theta)}) d\theta$$

$$X_p(e^{j\omega}) = \frac{1}{N} \sum_{K=0}^{N-1} X(e^{j(\omega-k\omega_s)})$$

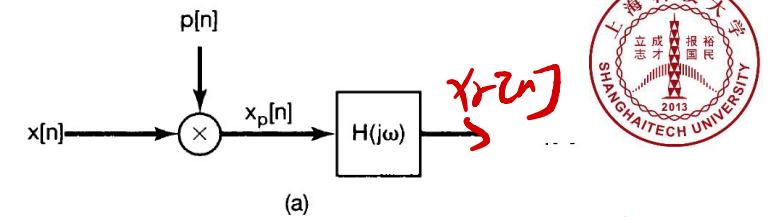


Sampling of Discrete-Time Signals

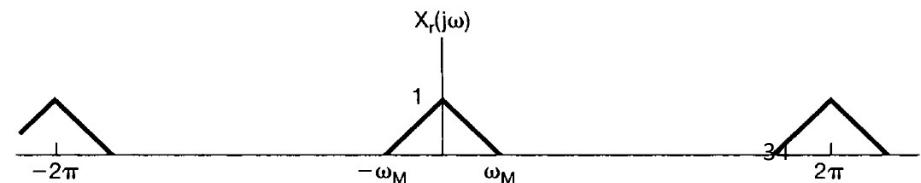
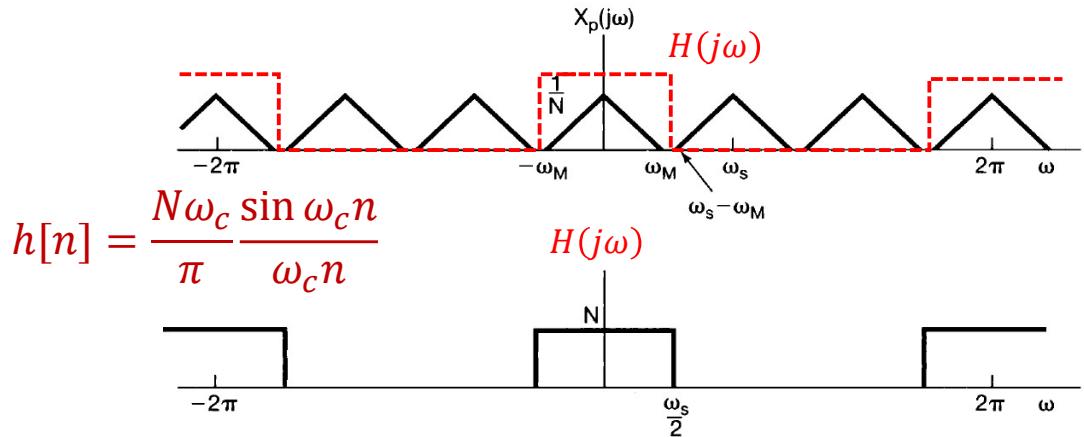
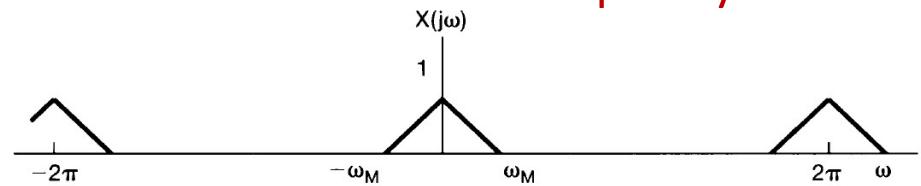
Impulse train sampling Reconstruction of $x[n]$

$$\begin{aligned}
 x_r[n] &= x_p[n] * h[n] \quad \text{Time domain} \\
 &= \left[\sum_{k=-\infty}^{\infty} x[kN] \cdot \delta[n - kN] \right] * h[n] \\
 &= \sum_{k=-\infty}^{\infty} x[kN] [\delta[n - kN] * h[n]] \\
 &= \sum_{k=-\infty}^{\infty} x[kN] h[n - kN]
 \end{aligned}$$

$$x_r[n] = \sum_{k=-\infty}^{\infty} x[kN] \frac{N\omega_c}{\pi} \frac{\sin \omega_c(n - kN)}{\omega_c(n - kN)}$$



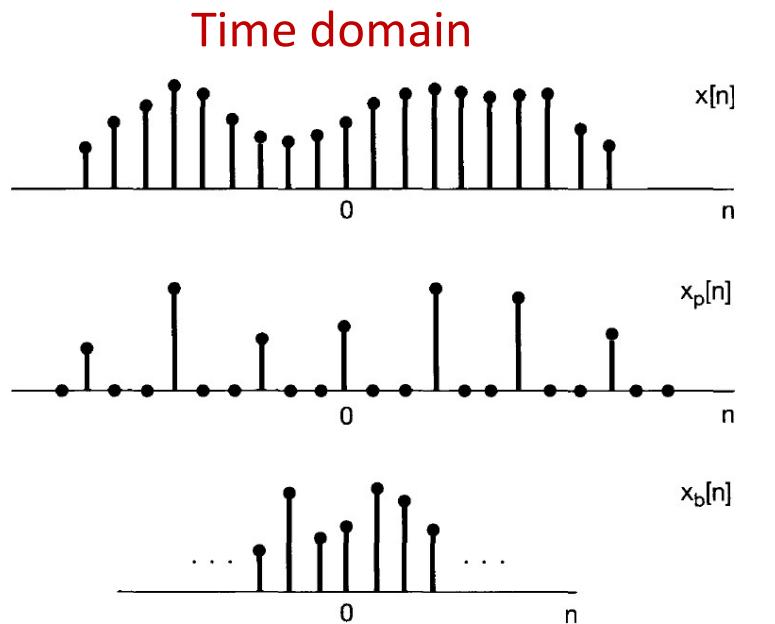
Frequency domain



Sampling of Discrete-Time Signals

抽样

Decimation (sample rate decrease, SRD)



$$x_b[n] = x_p[nN]$$

$$x_b[n] = x[nN]$$

采样率下降

Frequency domain

$$X_b(e^{j\omega}) = \sum_{K=-\infty}^{\infty} x_b[k] e^{-j\omega k}$$

$$X_b(e^{j\omega}) = \sum_{K=-\infty}^{\infty} x_p[kN] e^{-j\omega k} \quad n = kN$$

$$X_b(e^{j\omega}) = \sum_{\substack{n=\text{integer} \\ \text{number of } N}}^{\infty} x_p[n] e^{-j\omega n/N}$$

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_p[n] e^{-j\omega n/N}$$

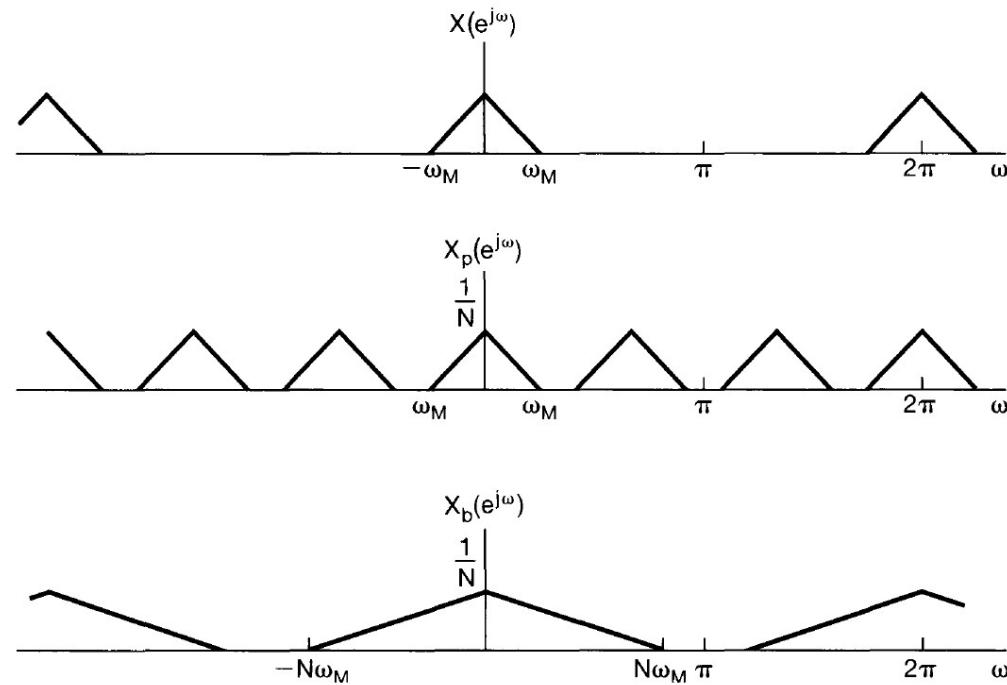
$$X_b(e^{j\omega}) = X_p(e^{j\omega/N})$$

35

$$\omega = \frac{\nu}{N} \quad x_b(e^{j\omega}) = X_p(e^{j\omega}) = X_p(e^{j\frac{\nu}{N}})$$

Sampling of Discrete-Time Signals

Decimation



$$x_b[n] = x_p[nN]$$

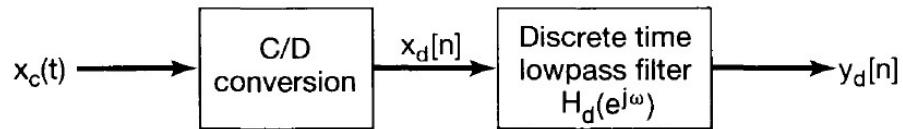
$$X_b(e^{j\omega}) = X_p(e^{j\omega/N})$$

Down-sampling if $N\omega_M > \pi$

下采样 \Rightarrow 减少存储量

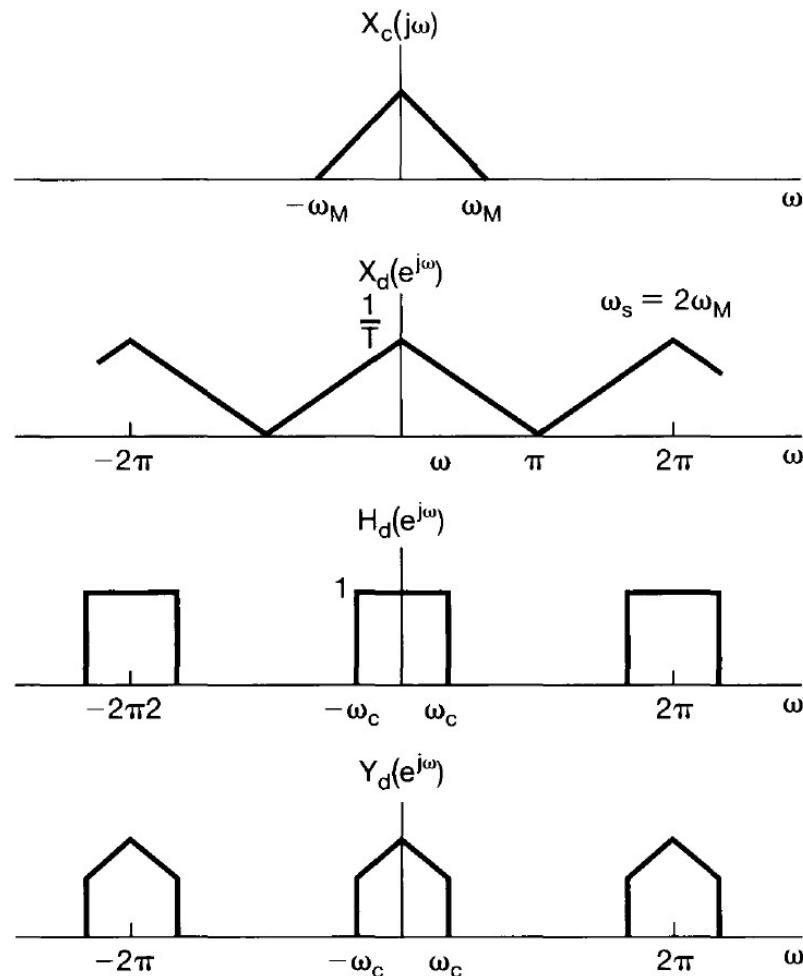
(频谱拉伸)

Sampling of Discrete-Time



Decimation

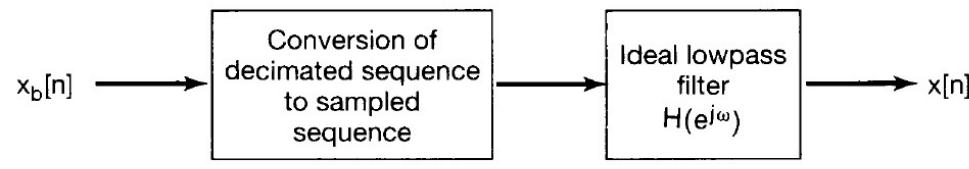
- Prevent aliasing by LPF in front of SRD \Rightarrow Decimator
抽取滤波器
Con-pass filter



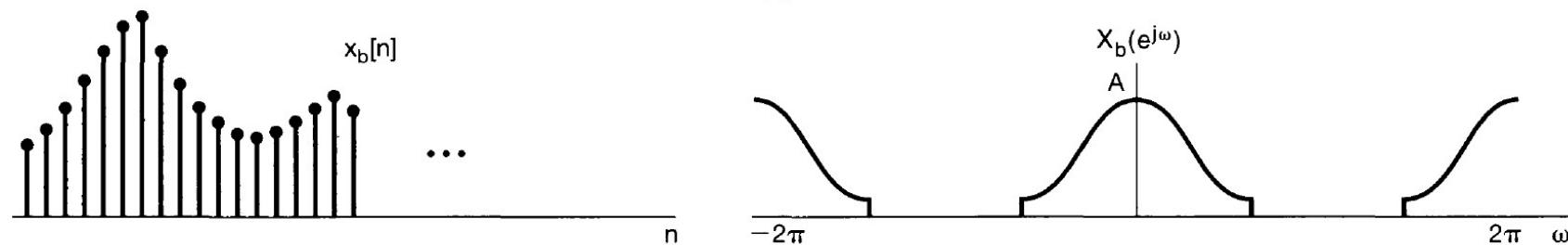
Sampling of

Interpolation (SRI)

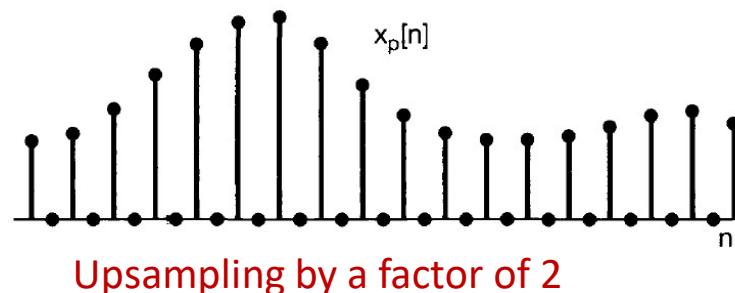
Sample rate
increase



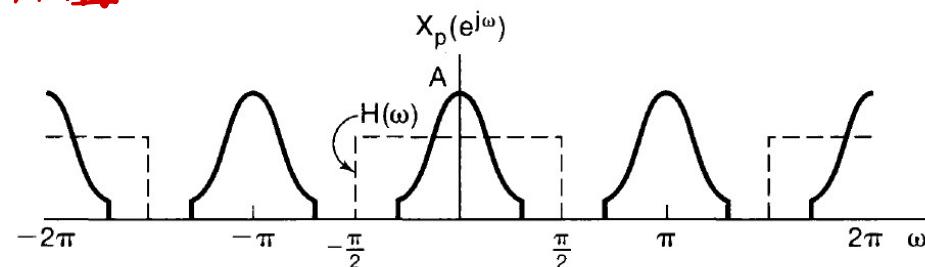
(a)



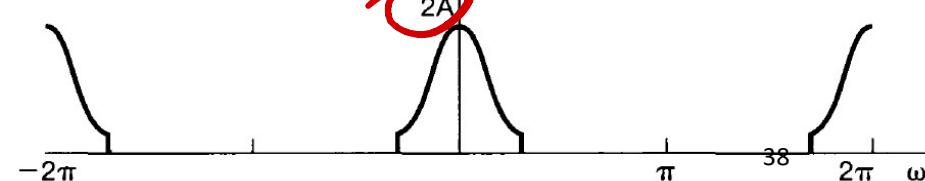
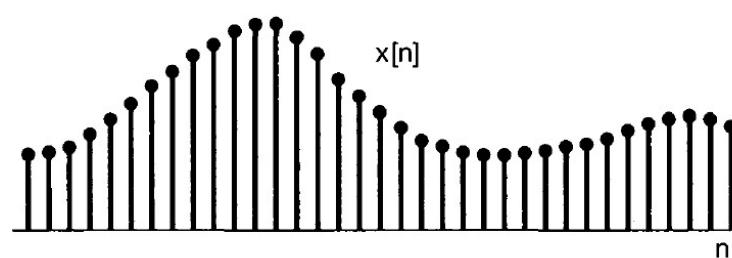
- Prevent mirrors by LPF after SRI \Rightarrow Interpolator 插值器



Upsampling by a factor of 2



38 2π ω



时域拉伸 → 频域收缩

Sampling of Discrete-

Interpolation (SRI)

Sample rate increase

$X(j\omega)$ 下采样 M 倍
 $\rightarrow \frac{1}{M} X(\frac{1}{M} j\omega)$

$X(j\omega)$ 采样 M 倍
 $\rightarrow M X(M j\omega)$

- Overall, SRI by factor of 4.5

