一、

The output $y(t)$ of a causual LTI system is related to the input $x(t)$ by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

20'

(a) Determine the frequency response

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

of the system, and sketch its Bode plot.

(b) Specify, as a function of frequency, the group delay associated with this system.

(c) If the input has its Fourier transform as follows, determine $Y(j\omega)$(the Fourier transform of the output) and the output $y(t)$

(i) $X(j\omega) = \frac{1+j\omega}{2+j\omega}$

(ii) $X(j\omega) = \frac{2+j\omega}{1+j\omega}$

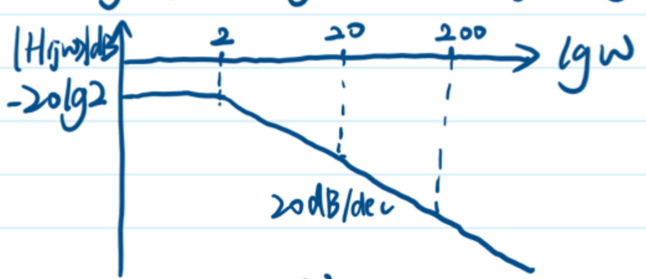
(iii) $X(j\omega) = \frac{1}{(2+j\omega)(1+j\omega)}$

(a) Taking the Fourier transform of both sides of the given differential equation, we obtain

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{2+j\omega}$$

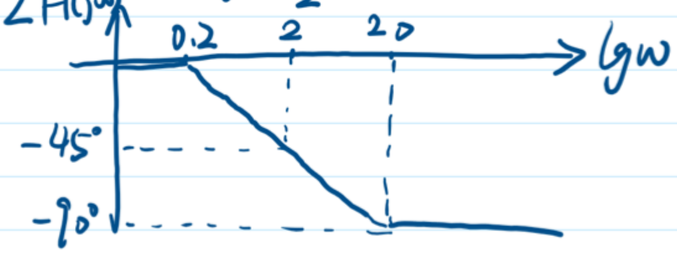
2'

$$20\lg|H(j\omega)| = -10\lg(4+w^2) = -20\lg 2 - 10\lg\left(1+\frac{w^2}{4}\right)$$



2'

$$\angle H(j\omega) = -\tan^{-1}\frac{w}{2}$$



2'

(b) From the expression for $H(j\omega)$ we obtain

$$\angle H(j\omega) = -\tan^{-1}(\omega/2).$$ 1'

Therefore,

$$\tau(\omega) = \frac{d\angle H(j\omega)}{d\omega} = \underline{\frac{2}{4+\omega^2}}$$ 2'

.

(c) (i) Here,

$$Y(j\omega) = \frac{1+j\omega}{(2+jw)^2}.$$ 1'

Taking the inverse Fourier transform of the partial fraction expansion of $Y(j\omega)$, we obtain

$$y(t) = e^{-2t}u(t) - te^{-2t}u(t).$$ 2'

(ii) Here,

$$Y(j\omega) = \frac{1}{(1+j\omega)}$$ 1'

Taking the inverse Fourier transform of $Y(j\omega)$, we obtain

$$y(t) = e^{-t}u(t)$$ 2'

.

(iii) Here,

$$Y(j\omega) = \frac{1}{(1+j\omega)(2+j\omega)^2}.$$ 2'

Taking the inverse Fourier transform of the partial fraction expansion of $Y(j\omega)$, we obtain

$$y(t) = e^{-t}u(t) - e^{-2t}u(t) - te^{-2t}u(t).$$ 3'

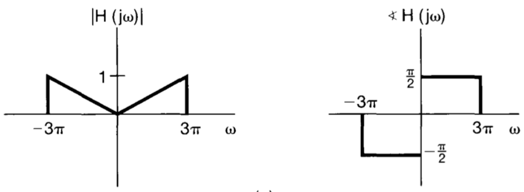
(一) 15'

Shown in Figure P6.22(a) is the frequency response $H(j\omega)$ of a continuous-time filter referred to as a lowpass differentiator. For each of the input signals $x(t)$ below, determine the filtered output signal $y(t)$.

10

(a) $x(t) = \cos(2\pi t + \theta) + \cos(4\pi t + \theta)$

~~**(b)** $x(t) = \cos(4\pi t + \theta)$~~

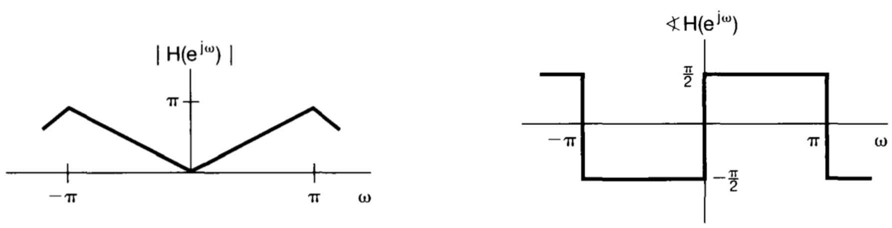


(a)

(二).

Shown in Figure P6.35 is the frequency response $H(e^{j\omega})$ of a discrete-time differentiator. Determine the output signal $y[n]$ as a function of ω_0 if the input $x[n]$ is

$$x[n] = \cos[\omega_0 n + \theta]. \quad \omega_0 \in [-\pi, \pi].$$



5'

Solution:

(一).

We know that $H(j\omega) = \begin{cases} \frac{j\omega}{3\pi} & -3\pi \le \omega \le 3\pi \\ 0 & , \text{otherwise} \end{cases}$ 3'

$x(t) = \cos(2\pi t + \theta) + \cos(4\pi t + \theta)$

$\therefore X(j\omega) = e^{j\theta}\pi\delta(\omega - 2\pi) + e^{-j\theta}\pi\delta(\omega + 2\pi)$ 2'

$\therefore Y(j\omega) = H(j\omega)X(j\omega) = \frac{j\omega}{3\pi} \cdot X(j\omega)$ 1'

$\therefore y(t) = \frac{1}{3\pi} \frac{d\, x(t)}{dt} = -\frac{2}{3}\sin(2\pi t + \theta)$

$X(j\omega) = e^{j\theta}\pi\delta(\omega - 4\pi) + e^{-j\theta}\pi\delta(\omega + 4\pi)$

$Y(j\omega) = H(j\omega)X(j\omega) = 0 \quad \therefore y(t) = 0$ 1'

$\therefore Y(t) = -\frac{2}{3}\sin(2\pi t + \theta)$

3'

(二)

$H(j\omega) = j\omega \quad \omega \in [-\pi, \pi]$

$$X(e^{j\omega}) = \pi \sum_{-\infty}^{+\infty} (e^{j\theta} \delta(\omega - \omega_0 - 2\pi i) + e^{-j\theta} \delta(\omega + \omega_0 - 2\pi i)]$$ 2'

$$Y(e^{j\omega}) = \pi \sum_{-\infty}^{+\infty} (e^{j\theta} j\omega_0 \delta(\omega - \omega_0 - 2\pi i) + e^{-j\theta} j\omega_0 \delta(\omega + \omega_0 - 2\pi i)]$$ 2'

$\therefore y[n] = -\omega_0 \sin(\omega_0 n + \theta)$. 1'

三. (一) 10'

Two signals $x_1(t) = cos20\pi t$ and $x_2(t) = cos100\pi t$ are sampled with the sampling frequency $40Hz$. Obtain the asscoiated time signals $x_1[n]$ and $x_2[n]$ and compare them and briefly explain it (Hint: think about the aliasing effect and the cause of it)

Solution:

1. Given the signal $x_1(t) = cos20\pi t$, we compare it with the standard form $x_1(t) = cos2\pi f_1 t$,so we have $f_1 = 10Hz$,now we replace the t in $x_1(t)$ by $\frac{n}{f_s}$ and here the sampling frequency $f_s = 40Hz$,therefore $x_1[n] = cos2\pi f_1 \cdot \frac{n}{f_s} = cos\frac{\pi}{2}n$ ~~8~~ 4
2. Given the signal $x_2(t) = cos100\pi t$, we compare it with the standard form $x_2(t) = cos2\pi f_2 t$,so we have $f_2 = 50Hz$,now we replace the t in $x_2(t)$ by $\frac{n}{f_s}$ and here the sampling frequency $f_s = 40Hz$,therefore $x_2[n] = cos2\pi f_2 \cdot \frac{n}{f_s} = cos\frac{5\pi}{2}n = cos\frac{\pi}{2}n$ ~~8~~ 4

$x_1[n]$ and $x_2[n]$ are the same, this is because when sampling $x_2[n]$,the frequency contained in $x_2(t)$ should be less than or equal to $\frac{f_s}{2}$,which means the frequency should be less than or equal to $20Hz$, However this is not the case in $x_2(t)$. So,aliasing happens, therefore $x_1[n] = x_2[n]$ due to the aliasing effect ~~4~~ 2

15'

(二) Find two different continous-time signals that would produce the sequence below when sampled at the frequency $f = 500Hz$. The discrete-time sequency is $x[n] = \cos[n\pi/5]$.

Answer：分有混叠与无混叠两种情况讨论

$T = \frac{1}{f_s} = \frac{2\pi}{\omega_s} \Rightarrow \omega_s = 2\pi f_s = 1000\pi \ (rad/s)$ 3'

$x_d[n] = \cos(\frac{\pi}{5}n) = \frac{1}{2}e^{j\frac{\pi}{5}n} + \frac{1}{2}e^{-j\frac{\pi}{5}n}$

$X_d(e^{j\Omega}) = \frac{1}{2}\sum_{l=-\infty}^{+\infty} 2\pi\delta(\Omega - \frac{\pi}{5} - 2\pi l) + \frac{1}{2}\sum_{l=-\infty}^{+\infty} 2\pi\delta(\Omega + \frac{\pi}{5} - 2\pi l)$

$= \pi\sum_{l=-\infty}^{+\infty}[\delta(\Omega + \frac{\pi}{5} - 2\pi l) + \delta(\Omega - \frac{1}{5}\pi - 2\pi l)]$

$= \pi[\delta(\Omega + \frac{\pi}{5}) + \delta(\Omega - \frac{1}{5}\pi)] \quad (-\pi \le \Omega \le \pi)$ 3'

$X_p(j\omega) = \pi[\delta(\omega T + \frac{\pi}{5}) + \delta(\omega T - \frac{\pi}{5})] \quad (-\frac{\pi}{T} \le \omega \le \frac{\pi}{T})$

$= \pi[\delta(\omega + 100\pi) + \delta(\omega - 100\pi)] \quad (-500\pi \le \omega \le 500\pi)$ 3'

① If there is no aliasing,

$H_c(j\omega) = \begin{cases} 1, & |\omega| < \frac{1}{2}\omega_s \\ 0, & \text{otherwise} \end{cases}$ $\quad X_c(j\omega) = H_c(j\omega)\cdot X_p(j\omega) = \pi[\delta(\omega + 100\pi) + \delta(\omega - 100\pi)]$

$x_c(t) = \frac{1}{2}e^{j100\pi t} + \frac{1}{2}e^{-j100\pi t} = \cos(100\pi t)$ 3'

② Otherwise, $100\pi = \omega_s - \omega_0$, $\omega_0 = 900\pi$

then $x_c(t) = \frac{1}{2}e^{j900\pi t} + \frac{1}{2}e^{-j900\pi t} = \cos(900\pi t)$ 3'

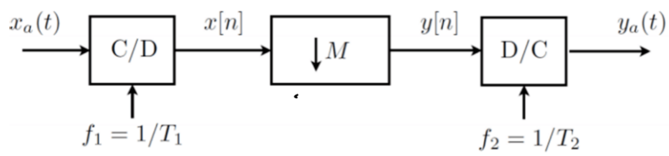
4.

20'

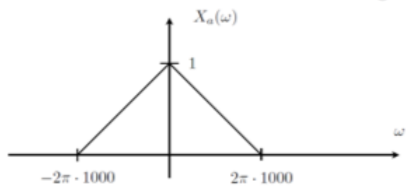
Consider the system shown in **Figure 5**.
Assume that the input is bandlimited, $X_a(\omega) = 0$ for $|\omega| > 2\pi \cdot 1000$.

(a) What constraints must be placed on M, T_1 and T_2 in order for $y_a(t)$ to be equal to $x_a(t)$?

(b) If $f_1 = f_2 = 20$kHz and $M = 4$, find an expression for $y_a(t)$ in terms of $x_a(t)$.



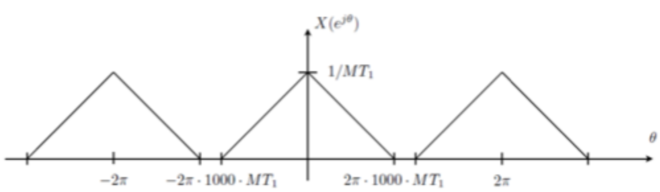
(a) Suppose that $x_a(t)$ has a Fourier transform as shown in the figure below. Because $y(n) =$



$x(Mn) = x_a(nMT_1)$, in order to prevent $x(n)$ from being aliased, it is necessary that

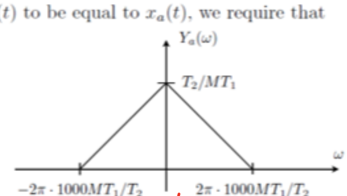
$$MT_1 < \frac{1}{2000}$$ 2'

If this constraint is satisfied, the output of the down-sampler has a FTD as shown below.



Going through the D/C converter produces signal $y_a(t)$, which has the Fourier transform shown below.

Therefore, in order for $y_a(t)$ to be equal to $x_a(t)$, we require that



1. $MT_1 \le 1/2000$ in order to avoid aliasing. 4'
2. $T_2 = MT_1$ to prevent frequency scaling. 4'

(b) With $T_1 = T_2 = 1/20000$ and $M = 4$, note that

$$MT_1 = \frac{1}{5000} < \frac{1}{2000}$$ 2'

Therefore, there is no aliasing. Thus, as we see from the figure above,

$$Y_a(\omega) = \frac{1}{4} X_a\left(\frac{\omega}{4}\right)$$ 4'

or

$$y_a(t) = x_a(4t)$$ 4'

5. 20'

A major problem in the recording of electrocardiograms (ECGs) is the appearance of unwanted 50 Hz interference in the output. The causes of this power line interference include magnetic induction, displacement currents in the leads on the body of the patient, and equipment interconnections. Assume that the bandwidth of the signal of interest is 1 kHz, that is,

$$X_a(f) = 0 \quad |f| > 1000\,\text{Hz}$$

The analog signal is converted into a discrete-time signal with an ideal A/D converter operating using a sampling frequency f_s. The resulting signal $x[n] = x_a(nTs)$ is then processed with a discrete-time system that is described by the difference equation

$$y[n] = x[n] + ax[n-1] + bx[n-2]$$

The filtered signal, $y[n]$, is then converted back into an analog signal using an ideal D/A converter. Design a system for removing the 50 Hz interference by specifying values for f_s, a, and b so that a 50 Hz signal of the form

00

$$w_a(t) = A\sin(120\pi t)$$

will not appear in the output of the D/A converter.

Exercise 4

The signal from which the 50[Hz] noise has to be removed is bandlimited to 1000[Hz]. Therefore, in order to avoid aliasing when the signal is sampled, we require a sampling frequency

$$f_s \geq 2000 \qquad 2'$$

Using the minimum rate of 2000 Hz, note that a 50-Hz signal $w_a(t) = \sin(100\pi t)$ becomes

$$w[n] = w_a(nT_s) = \sin\left(\frac{100\pi n}{2000}\right) = \sin(n\theta_0) \qquad 4$$

where $\theta_0 = 0.05\pi$. Recall that complex exponentials are eigenfunctions of linear shift-invariant systems. Therefore, if the input to an LSI system is $x[n] = e^{jn\theta_0}$, the output is

$$y[n] = H(e^{j\theta_0})e^{jn\theta_0}$$

Because

$$w[n] = \frac{e^{jn\theta_0} - e^{-jn\theta_0}}{2j} \qquad 4'$$

w[n] will be removed from x[n] if we design a filter so that $H(e^{j\theta})$ is equal to zero at $\theta = \pm\theta_0$. Because $H(e^{j\theta})$ is a second-order filter with a frequency response

$$H(e^{j\theta}) = 1 + ae^{-j\theta} + be^{-j2\theta} \qquad 4'$$

it may be factored as follows: $H(e^{j\theta}) = (1 - \alpha e^{-j\theta})(1 - \alpha^* e^{-j\theta})$, with $\alpha = e^{j\theta_0}$ and so we can write:

$$H(e^{j\theta}) = 1 - 2\Re e\{\alpha\}e^{-j\theta} + |\alpha|^2 e^{-j2\theta} \equiv 1 + ae^{-j\theta} + be^{-j2\theta}$$

From this it follows: $a = -2\Re e\{\alpha\} = -2\cos(\theta_0)$ with $\theta_0 = 0.05\pi$ and $b = |\alpha|^2 = 1$. Furthermore we have $f_s = 2000$.

2' 2' 2'