EE150 Signals and Systems - Spring 2018 - Midterm - Suggested Solutions

2 pages, 5 questions and 100 points in total.

2018-04-12

1. (10+10 points)

   a) For each statement, state (in the following table) if they are true or false.

      i) A memoryless system is definitely a causal system.

      ii) A causal system is definitely a memoryless system.

      iii) A system is invertible if distinct inputs lead to distinct outputs.

      iv) A system is time invariant if a time shift in the input signal produces no change in the output signal.

      v) Given a zero input to a linear system, it is impossible to have non-zero output.

i)	ii)	iii)	iv)	v)

   b) Consider the system $y(t) = e^{-(x(t)-1)^2}$. State (in the following table) if the system is: causal, linear, time-invariant, invertible, stable.

causal	linear	time-invariant	invertible	stable

Solution:

a) True, False, True, False, True

b) Causal, non-linear, time-invariant, non-invertible, stable.

2. (10+10 points) Consider a system whose input $x(t)$ and output $y(t)$ satisfy

$$y(t) = \int_{t-2}^{t} x(s)ds.$$

   a) Is the system linear? Is the system time-invariant? (Justify your answer.)

   b) Find the impulse response $h(t)$ of this system. (Justify your answer.)

Solution:

a) The system is linear, since

$$\int_{t-2}^{t} \big(ax_1(s) + bx_2(s)\big)ds = a\int_{t-2}^{t} x_1(s)ds + b\int_{t-2}^{t} x_2(s)ds = ay_1(t) + by_2(t).$$

The system is time-invariant. Let $x_1(t) = x(t - t_0)$ and $y_1(t)$ be its output. Then

$$\begin{aligned} y_1(t) = \int_{t-2}^{t} x_1(s)ds &= \int_{t-2}^{t} x(s - t_0)ds \\ &= \int_{t-t_0-2}^{t-t_0} x(w)d(w + t_0) \qquad // \ w = s - t_0 \\ &= \int_{t-t_0-2}^{t-t_0} x(w)dw. \end{aligned}$$

On the other hand

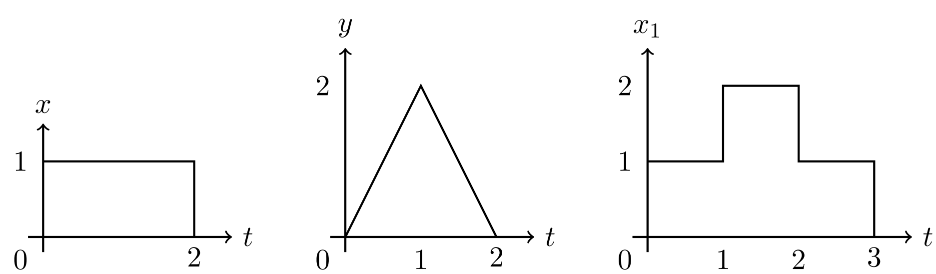
$$y(t - t_0) = \int_{t-t_0-2}^{t-t_0} x(s)ds.$$

Thus $y_1(t) = y(t - t_0)$. By definition, the system is time-invariant.

b) By definition, the impulse response $h(t)$ is the output to the input $\delta(t)$. Hence

$$\begin{aligned} h(t) = \int_{t-2}^{t} \delta(s)ds &= \begin{cases} 1, & \text{if } 0 \in [t-2, t] \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } t - 2 \le 0 \le t \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

3. (10+10 points) The following figure contains three signals: $x(t)$, $y(t)$ and $x_1(t)$.



Input $x(t)$. Output $y(t)$. Input $x_1(t)$.

Now consider an LTI system such that when the input is x, its output is y.

a) Compute the output of the same system to the input $x_1(t)$. (You can draw the figure.)

b) Compute the impulse response $h(t)$ of this system. (You can draw the figure.)

Solution: Since the system is LTI, we can do time-shifting and linear combinations.

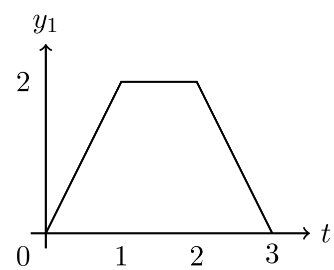
a) Notice that

$$x_1(t) = x(t) + x(t-1),$$

and since the system is LTI, we have

$$y_1(t) = y(t) + y(t-1).$$

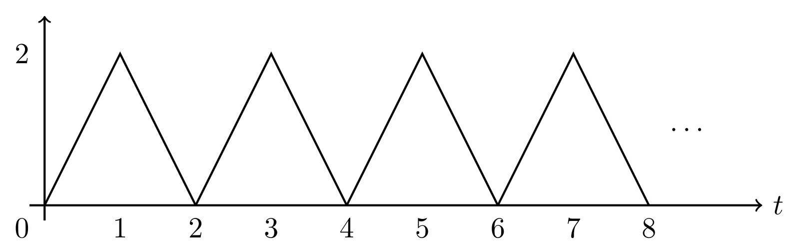
The figure is



Output $y_1(t)$.

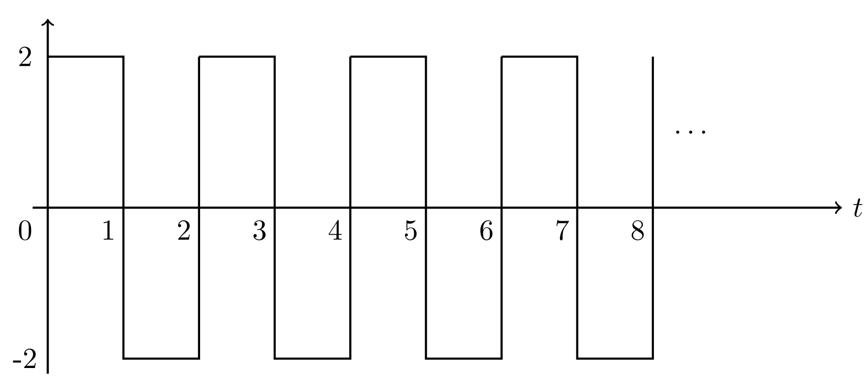
b) By definition, the impulse response $h(t)$ is the output to the input $\delta(t)$. Here we first compute the output to $u(t)$, then do the differentiation.

Notice $u(t) = x(t) + x(t-2) + x(t-4) + x(t-6) + \cdots$, the output to $u(t)$ is $y(t) + y(t-2) + y(t-4) + y(t-6) + \cdots$ The figure is



Output to $u(t)$.

Now one do differentiation to this output to obtain the impulse response:



Impulse response $h(t)$.

4. (10+10+10+5 points)

a) Compute the Fourier series a_k of the following signal with fundamental period 1:

$$x_1(t) = \begin{cases} -1, & -\frac{1}{2} < t < 0 \\ 1, & 0 \le t \le \frac{1}{2} \end{cases}.$$

b) Show that:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \qquad \text{(Hint: Apply Parseval's Identity to 4.a).)}$$

c) Compute the Fourier series b_k of the following signal with fundamental period 2: $x_2(t) = |t|$, $t \in [-1, 1]$.

d) Compute $\sum_{k=1}^{\infty} k^{-4}$.

Solution:

a) Let c_k be the Fourier series to the derivative of $x_1(t)$. In the interval $(-\frac{1}{2}, \frac{1}{2}]$, the derivative of $x_1(t)$ is $2\delta(t) - 2\delta(t - \frac{1}{2})$. Hence the Fourier series is

$$c_k = \frac{1}{1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(2\delta(t) - 2\delta(t - \frac{1}{2})\right) e^{jkw_0 t} dt = 2e^0 - 2e^{jkw_0 \frac{1}{2}}, \qquad k \neq 0.$$

From the differentiation property, we have $c_k = jkw_0 a_k$, hence $a_k = \frac{c_k}{jkw_0}$. Since $T_0 = 1$, we have $w_0 = 2\pi$ and

$$a_k = \frac{2 - 2e^{jk\pi}}{jk2\pi} = \frac{1 - (-1)^k}{jk\pi} = \begin{cases} \frac{2}{jk\pi}, & k \text{ is odd} \\ 0, & k \text{ is even}, k \neq 0 \end{cases}.$$

And $a_0 = \frac{1}{1} \int_{-\frac{1}{2}}^{\frac{1}{2}} x(s) ds = 0$. To summarize

$$a_k = \begin{cases} \frac{2}{jk\pi}, & k \text{ is odd} \\ 0, & k \text{ is even} \end{cases}.$$

b) By Parseval's Identity, $\langle x_1, x_1 \rangle = \sum_k |a_k|^2$. For each term, we have

$$\langle x_1, x_1 \rangle = \frac{1}{1} \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 ds = 1.$$

$$\sum_k |a_k|^2 = \sum_{k \text{ odd}} \frac{4}{k^2\pi^2} = 2 \cdot \sum_{k \ge 1, k \text{ odd}} \frac{4}{k^2\pi^2}.$$

Hence

$$\sum_{k \ge 1, k \text{ odd}} \frac{1}{k^2} = \frac{\pi^2}{8}.$$

Now let $A = \sum_{k\geq 1} \frac{1}{k^2}$, we have

$$
\begin{aligned}
A = \sum_{k\geq 1} \frac{1}{k^2} &= \sum_{k\geq 1, k \text{ odd}} \frac{1}{k^2} + \sum_{k\geq 1, k \text{ even}} \frac{1}{k^2} \\
&= \frac{\pi^2}{8} + \sum_{m\geq 1} \frac{1}{(2m)^2} \\
&= \frac{\pi^2}{8} + \frac{1}{4}A. \\
\implies \quad A &= \frac{4}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{6}.
\end{aligned}
$$

c) The fundamental period is $T_0 = 2$, and $w_0 = \pi$. Notice that the derivative of $x_2(t)$ is $x_1(2t)$ and the Fourier series of $x_1(2t)$ is also a_k (since the scaling factor 2 is positive), it is straightforward to obtain

$$
b_k = \frac{a_k}{jkw_0}, \quad k \neq 0.
$$

For $k = 0$, we have

$$
b_0 = \frac{1}{2} \int_{-1}^{1} |t| dt = \frac{1}{2}.
$$

Combining them together

$$
b_k = \begin{cases} -\frac{2}{k^2\pi^2}, & k \text{ is odd} \\ 0, & k \text{ is even}, k \neq 0 \\ \frac{1}{2}, & k = 0 \end{cases}.
$$

d) Similarly, $\langle x_2, x_2 \rangle = \sum_k |b_k|^2$. For each term

$$
\langle x_2, x_2 \rangle = \frac{1}{2} \int_{-1}^{1} t^2 dt = \frac{1}{3}.
$$

$$
\sum_k |b_k|^2 = \frac{1}{4} + \sum_{k \text{ odd}} \frac{4}{k^4\pi^4} = \frac{1}{4} + 2 \cdot \sum_{k\geq 1, k \text{ odd}} \frac{4}{k^4\pi^4}
$$

Hence

$$
\sum_{k\geq 1, k \text{ odd}} \frac{1}{k^4} = \frac{\pi^4}{8} \cdot (\frac{1}{3} - \frac{1}{4}) = \frac{\pi^4}{96}.
$$

Now let $B = \sum_{k\geq 1} k^{-4}$, we have

$$
\begin{aligned}
B &= \sum_{k\geq 1, k \text{ odd}} \frac{1}{k^4} + \sum_{k\geq 1, k \text{ even}} \frac{1}{k^4} = \sum_{k\geq 1, k \text{ odd}} \frac{1}{k^4} + \frac{1}{2^4}B. \\
\implies \quad B &= \frac{16}{15} \cdot \frac{\pi^4}{96} = \frac{\pi^4}{90}.
\end{aligned}
$$

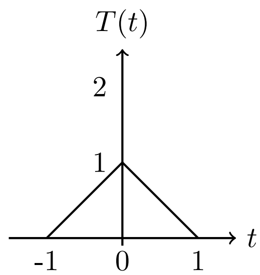
5. (5 points) Consider an LTI system, and the following (triangular) input

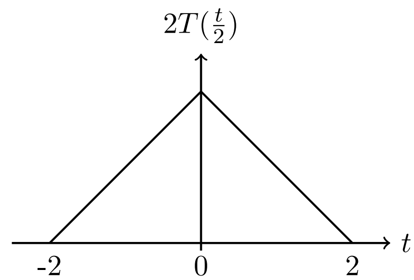
$$T(t) = \begin{cases} 1 - |t|, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}.$$

Suppose the output to the input $T(t)$ is $z(t)$, find the output to $T(\frac{t}{4})$.

Solution: First notice that

$$2T(\frac{t}{2}) = T(t+1) + 2 \cdot T(t) + T(t-1).$$





Let $y(t) = 2T(\frac{t}{2})$, then similarly

$$4T(\frac{t}{4}) = y(t+2) + 2y(t) + y(t-2).$$

Hence

$$\begin{aligned} 4T(\frac{t}{4}) &= 2T(\frac{t+2}{2}) + 4T(\frac{t}{2}) + 2T(\frac{t-2}{2}) \\ &= \Big[T(t+3) + 2 \cdot T(t+2) + T(t+1)\Big] \\ &+ 2\Big[T(t+1) + 2 \cdot T(t) + T(t-1)\Big] \\ &+ \Big[T(t-1) + 2 \cdot T(t-2) + T(t-3)\Big] \\ &= T(t+3) + 2T(t+2) + 3T(t+1) + 4T(t) + 3T(t-1) + 2T(t-2) + T(t-3) \\ &= \sum_{k=-3}^{3} (4 - |k|) \cdot T(t+k). \end{aligned}$$

Finally the output to $T(\frac{t}{4})$ is

$$\frac{1}{4} \cdot \sum_{k=-3}^{3} (4 - |k|) \cdot z(t+k).$$