Week 3
The Concept of Filtering

A DJ has to be familiar with signal processing !

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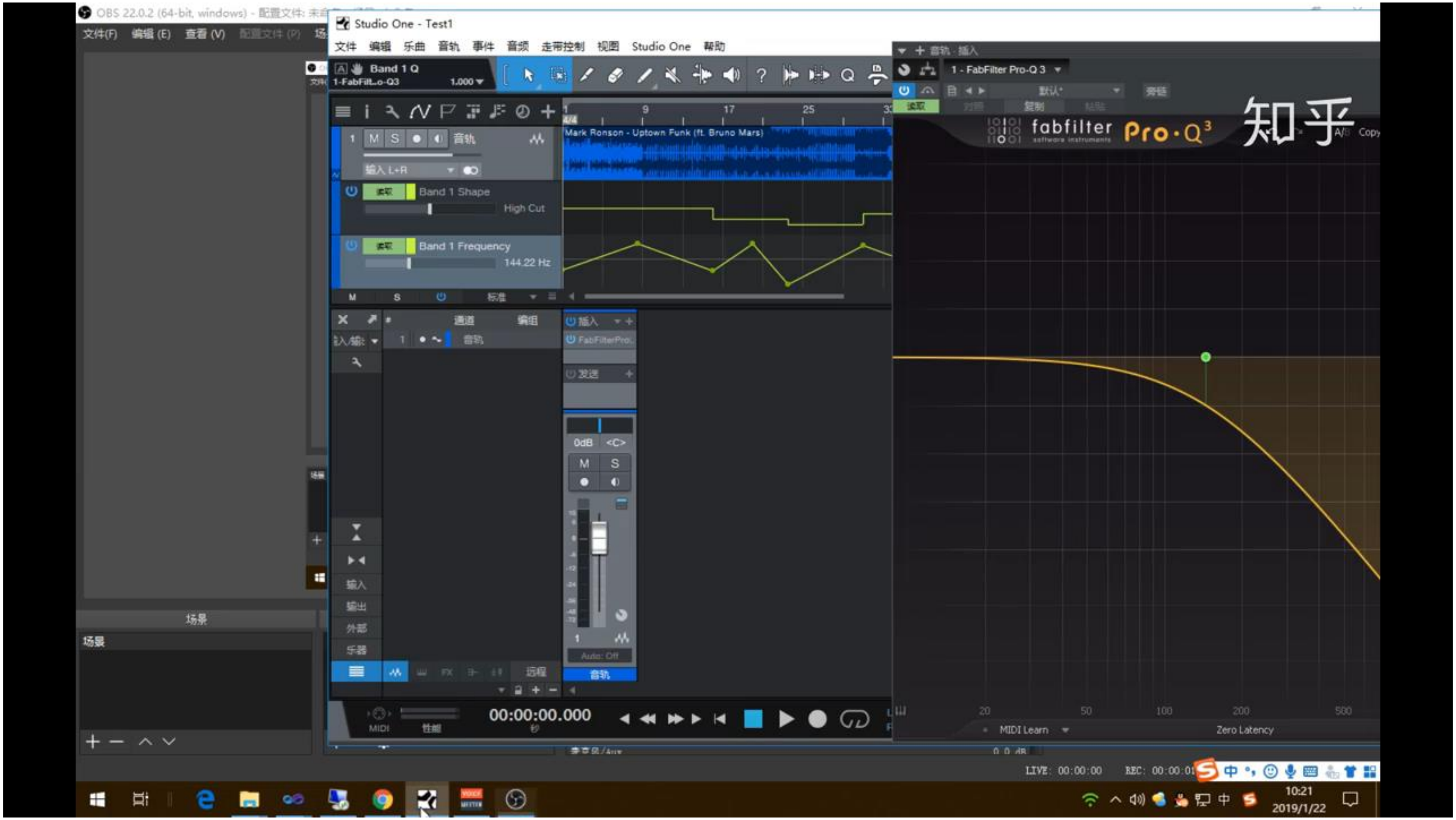
Equalizer

To adjust the balance of frequency components.

How Equalizer Works?

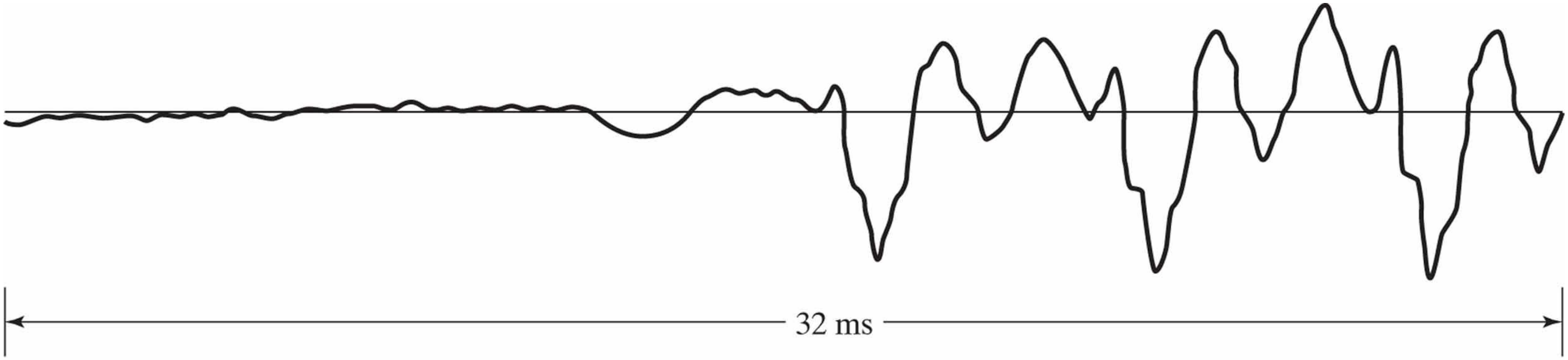
- Based on filters or filter banks



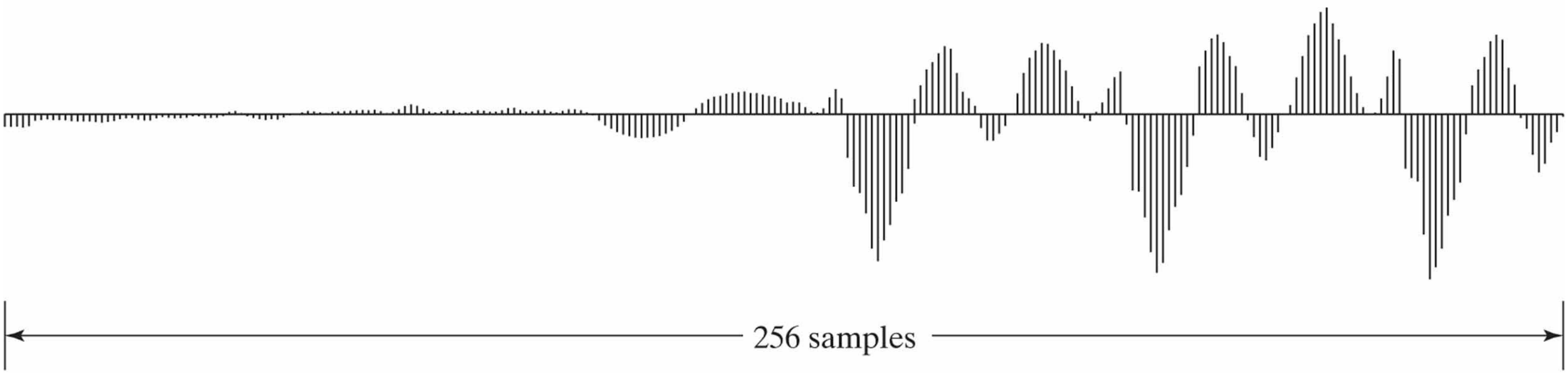
https://zhuanlan.zhihu.com/p/55543887

Some basic definitions

Discrete Time (DT) Signal



(a)



(b)

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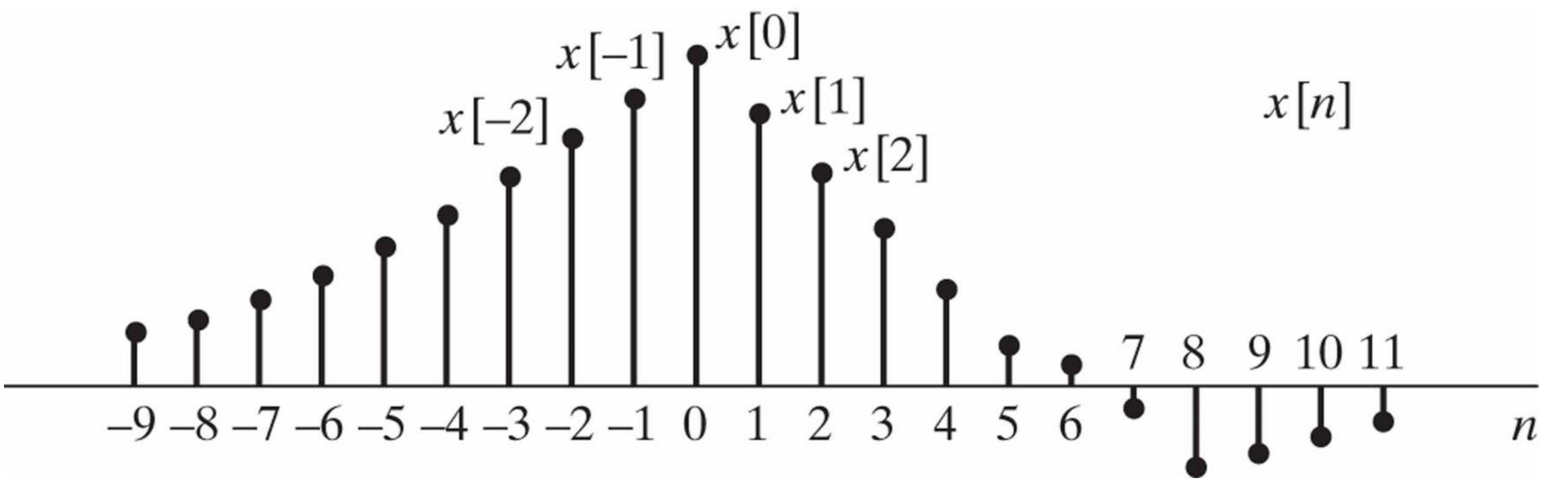
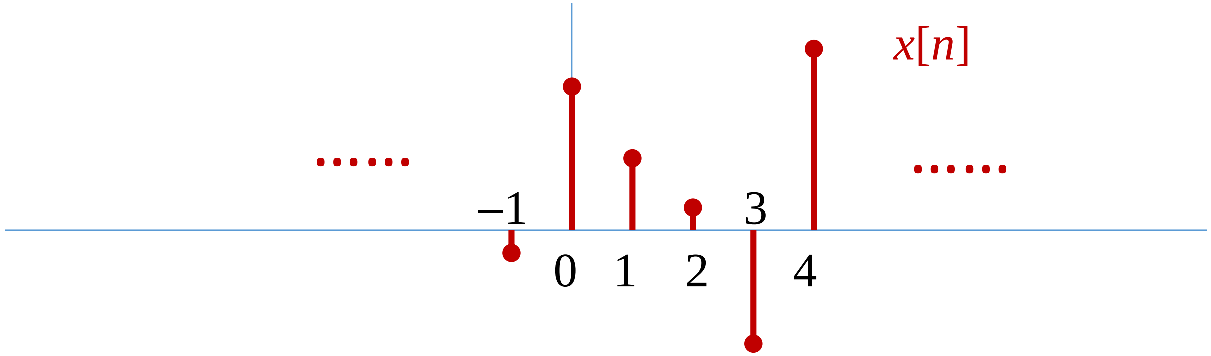
DT Signal

- Graphical representation of a discrete-time signal with real-valued samples



DT Signal

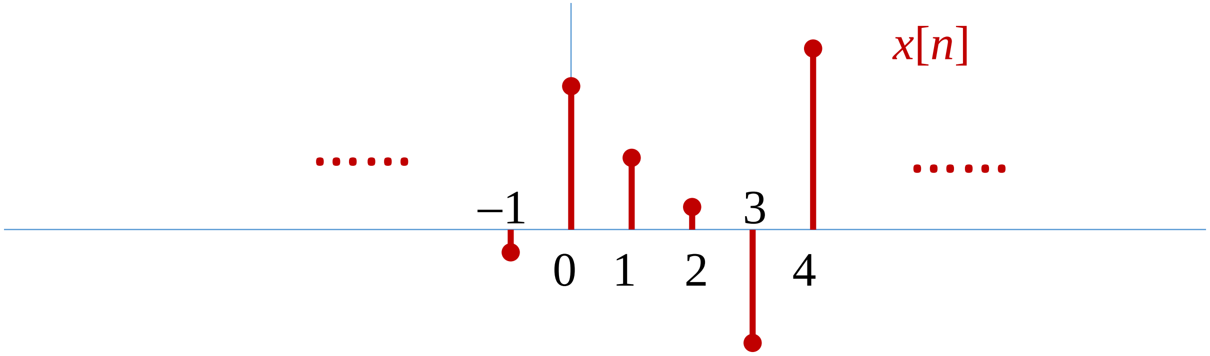
- Signals represented as sequences of numbers, called samples
- Sample value of a typical signal is denoted by $x[n]$ with n being an integer
- $x[n]$ is called the n^{th} sample of the sequence



DT Signal

- DT signals are defined only for integer values of n and undefined for non-integer values of n
- DT signals may also be written as a sequence of numbers inside braces

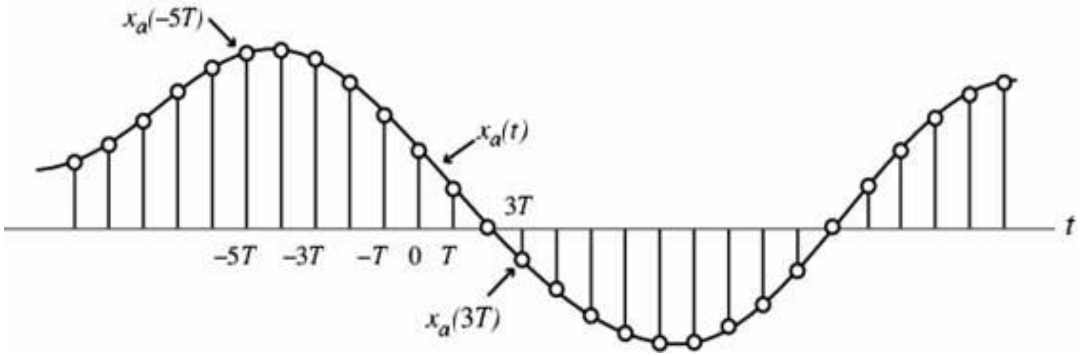
$$\{x[n]\} = \{\ldots, -0.2, 2.2, 1.1, 0.2, -1.9, 2.9, \ldots\}$$



DT Signal

- Samples of a continuous-time signal

$$x[n] = x_a(nT),\ n = \ldots, -1, 0, 1, 2, \ldots$$



- The spacing T between two consecutive samples is called the sampling interval or sampling period
- Reciprocal of sampling interval T_s, denoted as f_s , is called the sampling frequency:

$$f_s = 1/T_s$$

Relationship Between Frequencies

- The frequency we familiar with f, *in Hertz or Hz*
- For a signal with period T, we have

$$f = 1/T$$

- Angular frequency

$$\Omega = 2\pi f$$

- Digital frequency

$$\omega = 2\pi f / f_s$$

Fs is the sampling frequency

Relationship Between Frequencies

- Sampling frequency or f_s is the bridge between analog frequency and digital frequency

$$\omega = 2\pi f / f_s$$

$$f_s \to 2\pi$$

A Quick Example

- If $f_s = 44.1\text{K}$ $\quad \omega = 2\pi f / f_s$

Frequency (Hz)	1209	1336	1477
697	1	2	3
770	2	5	6
852	7	8	9
941	*	0	#

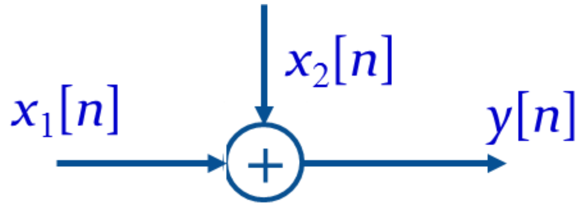
The two digital frequencies of 3 are 0.0316π and $0.0670\,\pi$

Elementary Operations

- Multiplication operation:
  - Multiplier

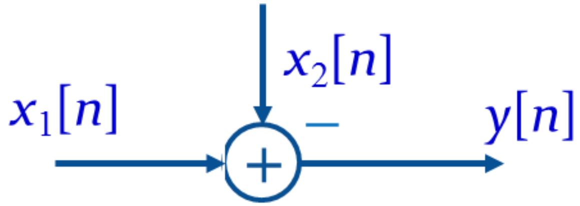
$$y[n] = \alpha x[n]$$

- Addition operation:
  - Adder



$$y[n] = x_1[n] + x_2[n]$$

- Subtraction operation:
  - Subtractor



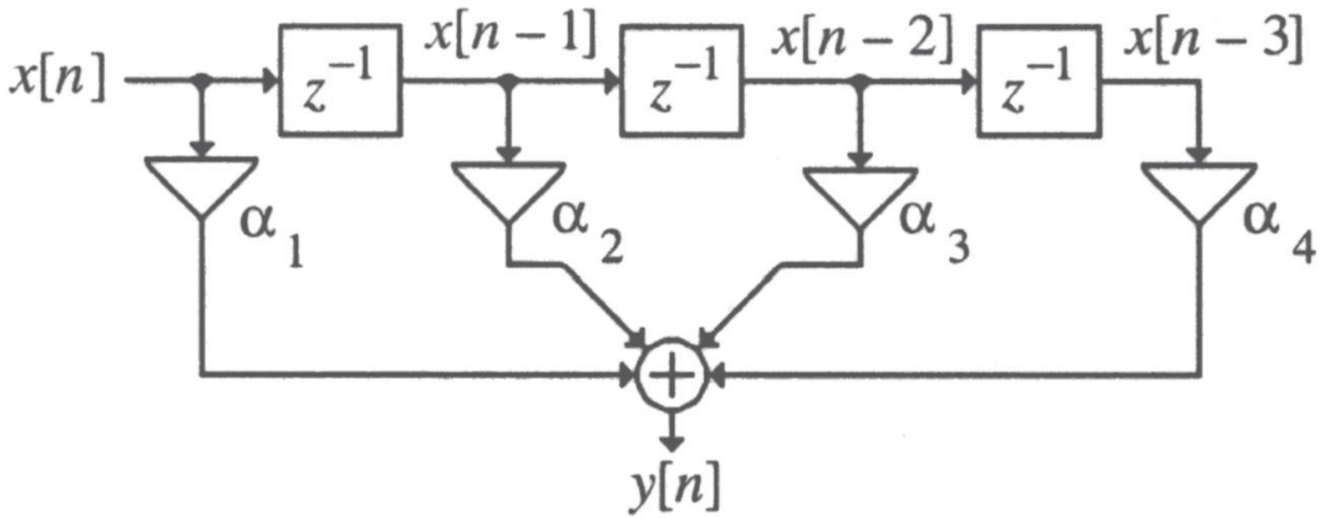
$$y[n] = x_1[n] - x_2[n]$$

Elementary Operations

- **Time-shifting** operation: $y[n] = x[n - n_0]$, where n_0 is an integer
- If $n_0 > 0$, it is delaying operation
  - Unit delay $x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow y[n]$ $\quad y[n] = x[n-1]$

Combinations of Basic Operations

- Example

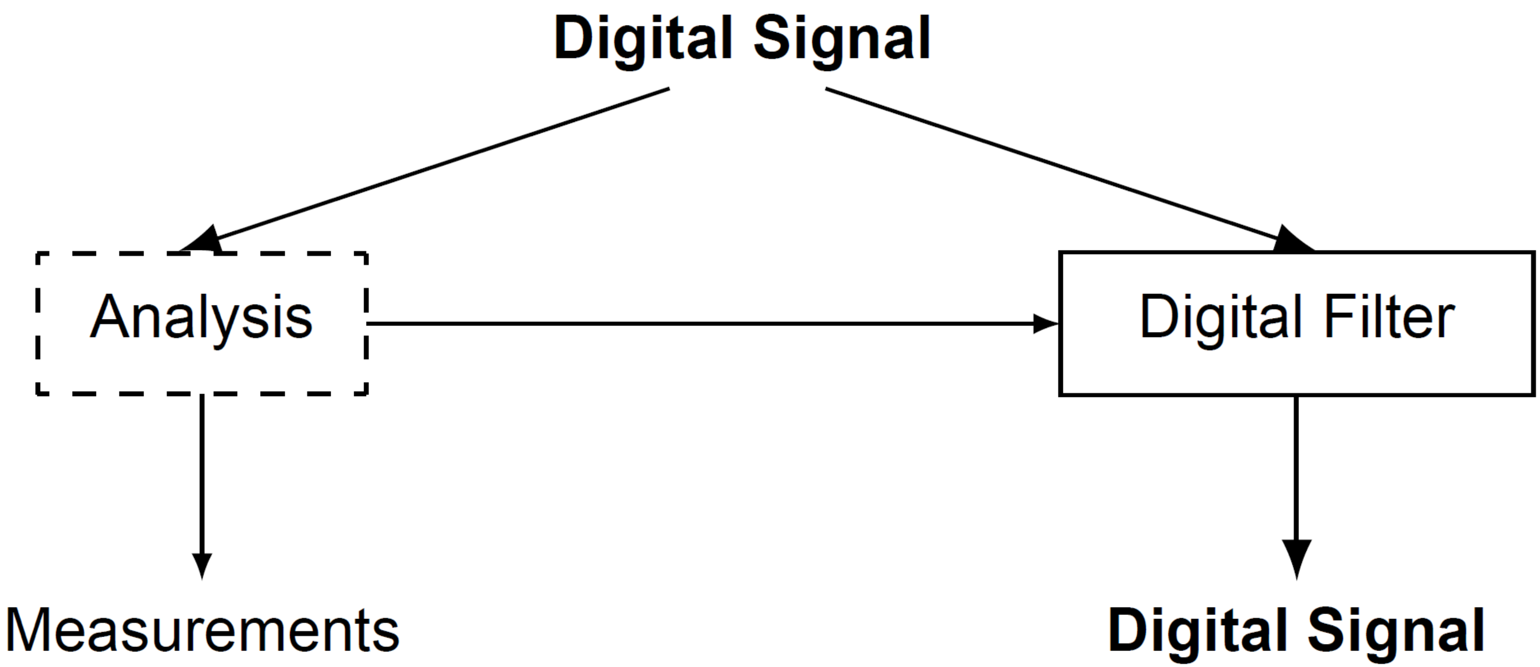


$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

The Concept of Filtering

The Objective of Signal Processing

- The objective of signal processing

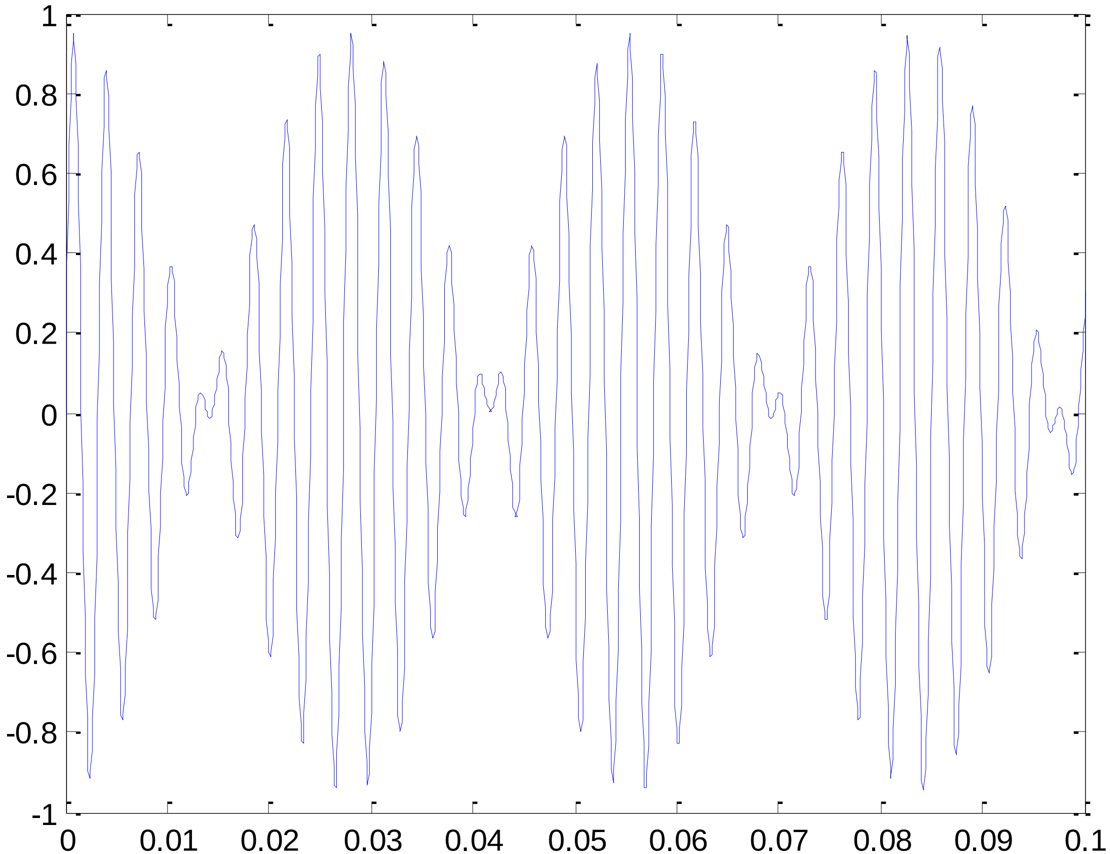


The Concept of Filtering

- To pass certain frequency components in an input signal without any distortion (is possible) and to block other frequency components

Back to Where We Begin

- Time domain



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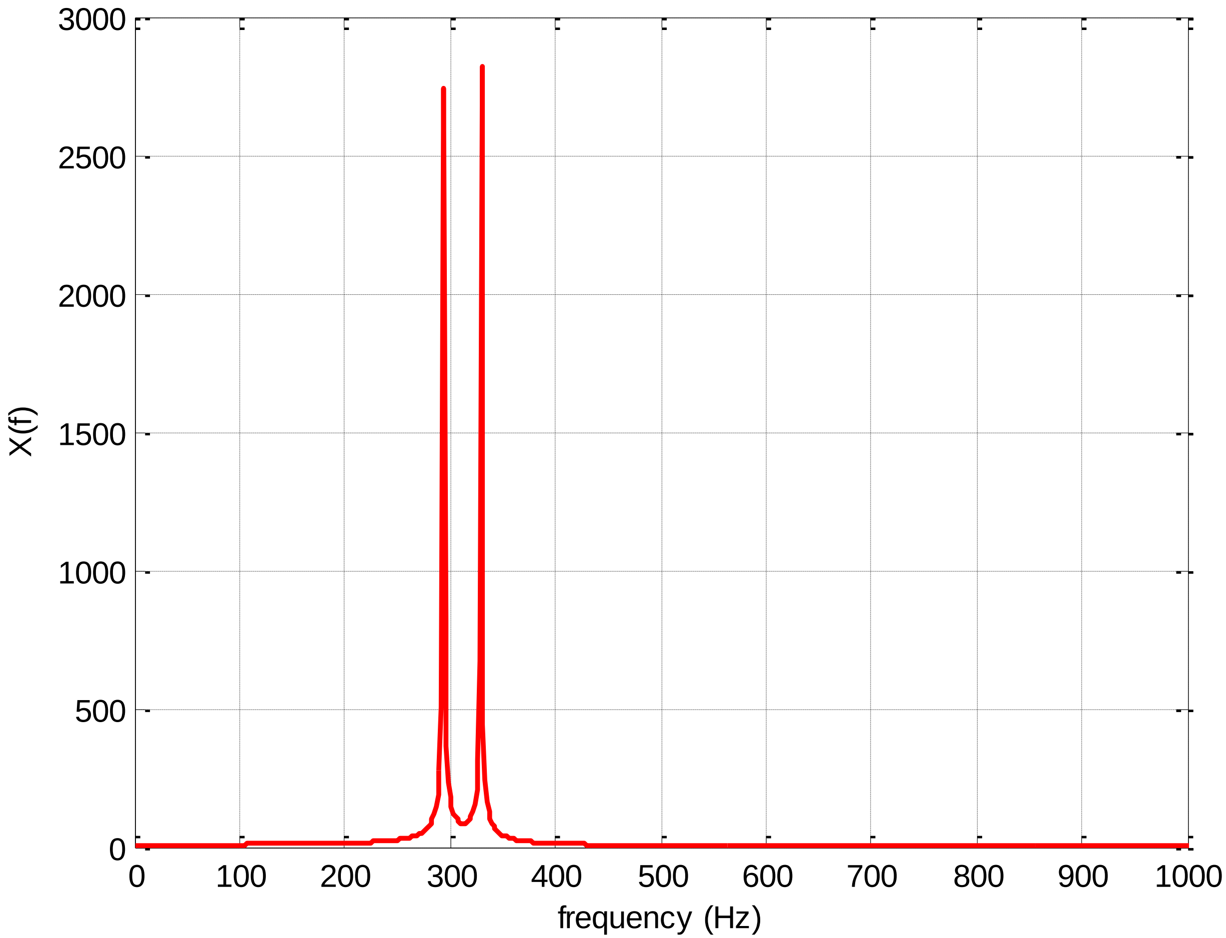
Back to Where We Begin

- Frequency domain



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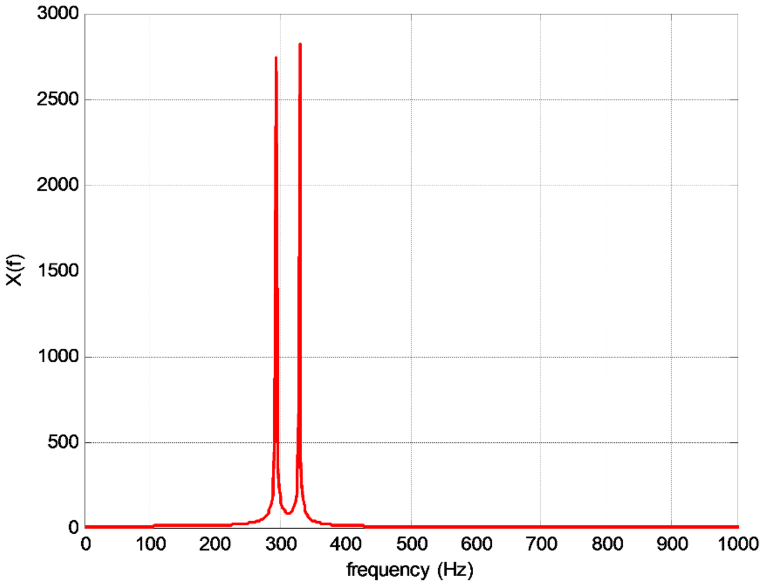
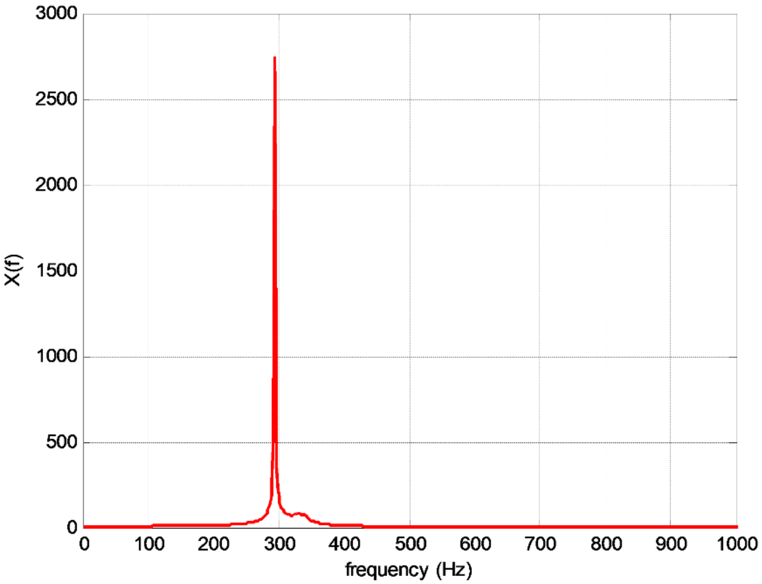
Back to Where We Begin

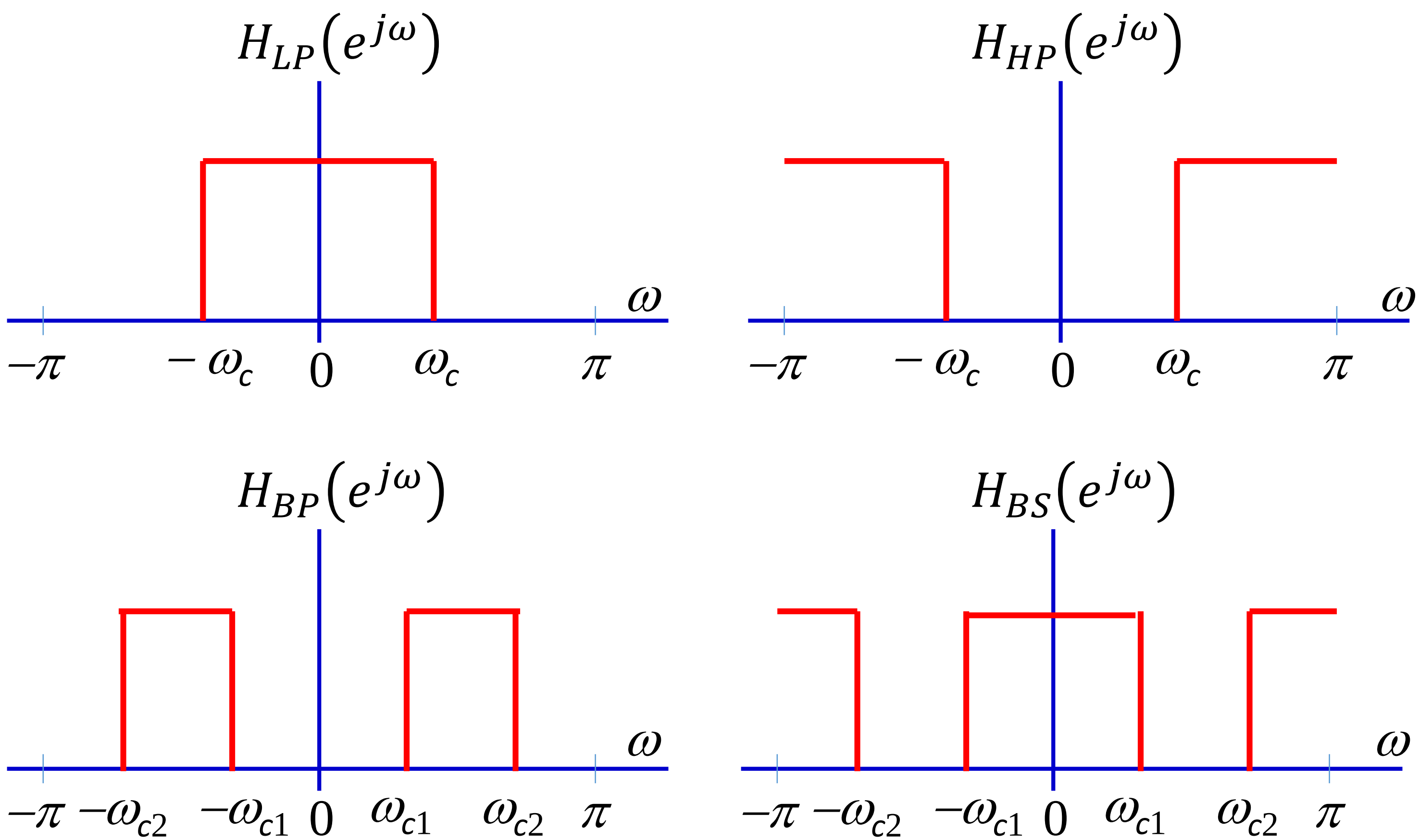
- Frequency domain





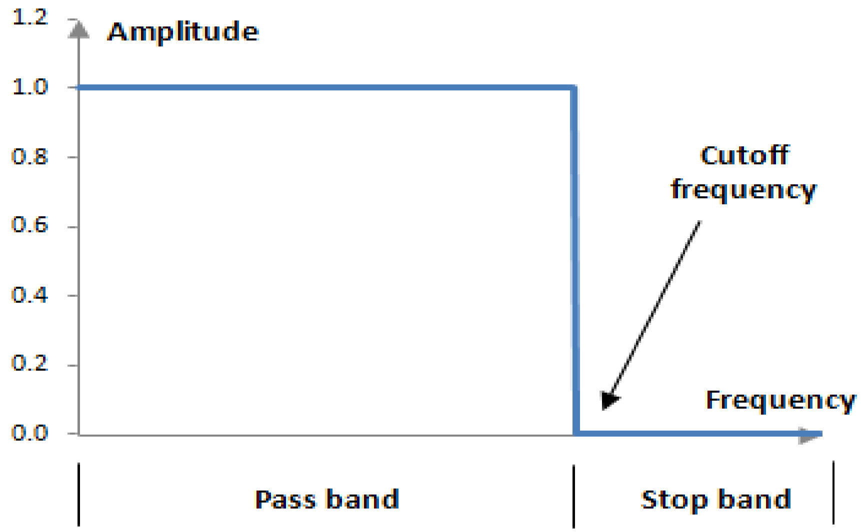
Magnitude Characteristics

- Digital Filter with Ideal Magnitude Responses

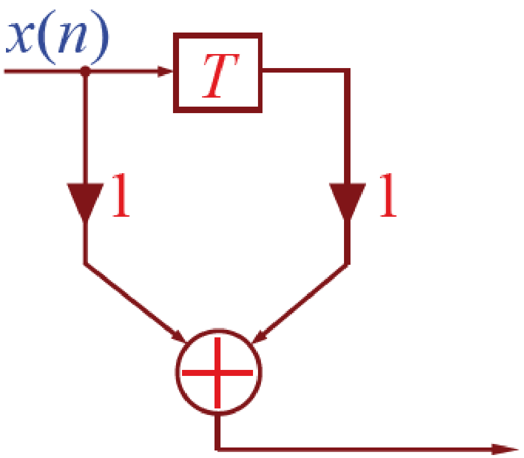


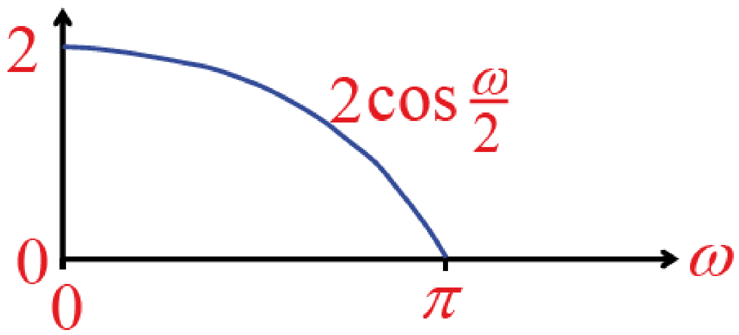
Passband and Stopband

- Passband
  - The range of frequencies that is allowed to pass through the filter
- Stopband
  - the range of frequencies that is blocked by the filter



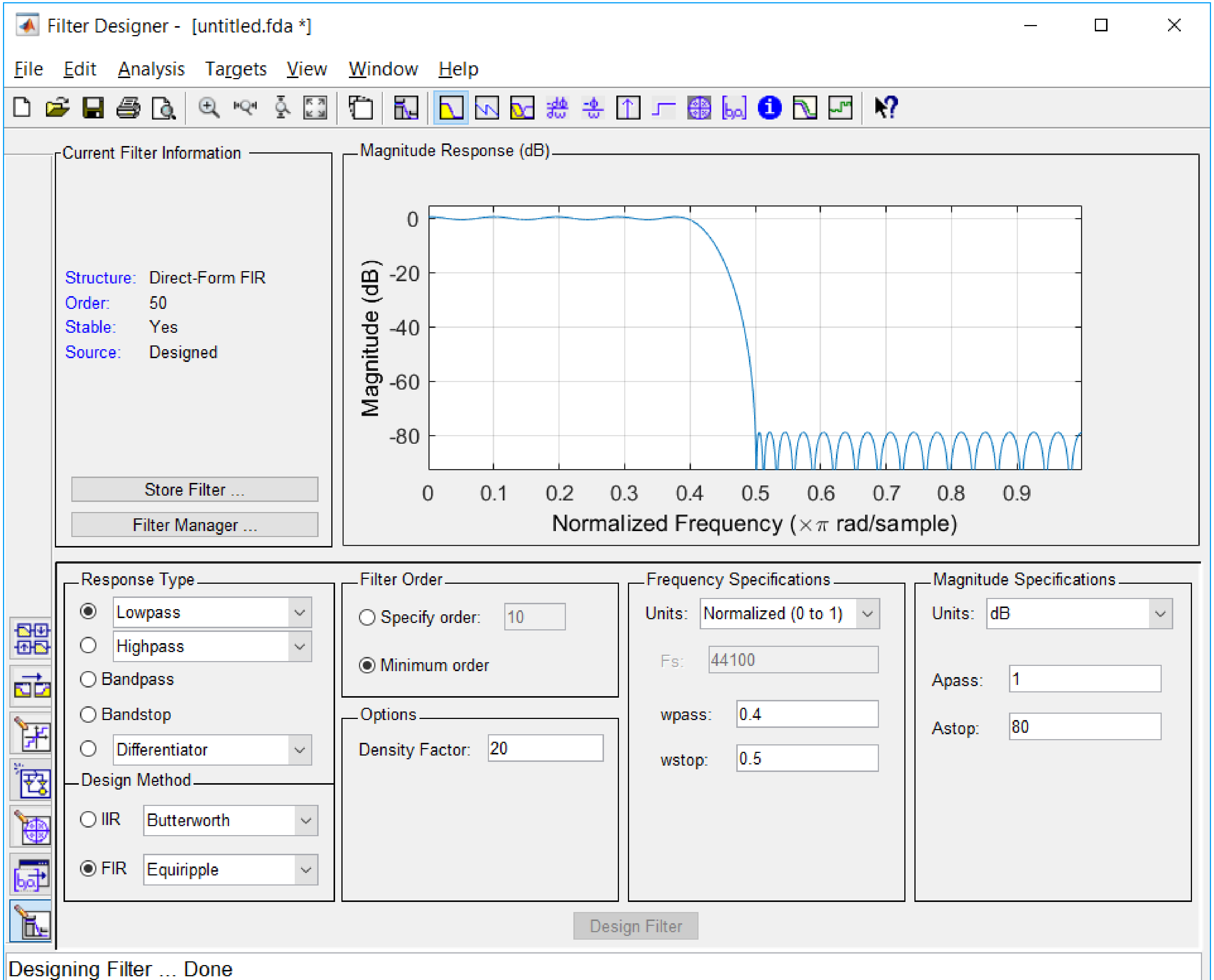
Simple Examples





Filter Design Tool

- A tool to play with
  - The *filterDesigner* (*fdatool* for old versions) in Matlab
- How to use?
  - Just type *filterDesigner* in the Command Window


Filter Designer - [untitled.fda *]
File Edit Analysis Targets View Window Help
Current Filter Information
Structure: Direct-Form FIR
Order: 50
Stable: Yes
Source: Designed
Store Filter ...
Filter Manager ...
Magnitude Response (dB)
Magnitude (dB)
0
-20
-40
-60
-80
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
Normalized Frequency (×π rad/sample)
Response Type
Lowpass
Highpass
Bandpass
Bandstop
Differentiator
Design Method
IIR Butterworth
FIR Equiripple
Filter Order
Specify order: 10
Minimum order
Options
Density Factor: 20
Frequency Specifications
Units: Normalized (0 to 1)
Fs: 44100
wpass: 0.4
wstop: 0.5
Magnitude Specifications
Units: dB
Apass: 1
Astop: 80
Design Filter
Designing Filter ... Done

Output of Filter Design

- A filter design process is to determine the filter coefficients

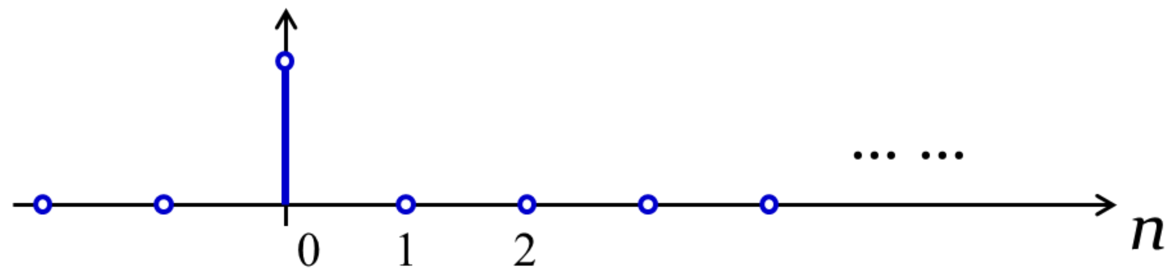
```
Num =

  Columns 1 through 14

   -0.0009   -0.0027   -0.0025    0.0037    0.0137    0.0174    0.0077   -0.0066   -0.0077    0.0061    0.0139    0.0004   -0.0169   -0.0089

  Columns 15 through 28

    0.0174    0.0207   -0.0123   -0.0342   -0.0010    0.0478    0.0274   -0.0594   -0.0823    0.0672    0.3100    0.4300    0.3100    0.0672

  Columns 29 through 42

   -0.0823   -0.0594    0.0274    0.0478   -0.0010   -0.0342   -0.0123    0.0207    0.0174   -0.0089   -0.0169    0.0004    0.0139    0.0061

  Columns 43 through 51

   -0.0077   -0.0066    0.0077    0.0174    0.0137    0.0037   -0.0025   -0.0027   -0.0009
```

Filter coefficients are also called the impulse response of a filter

The Unit Impulse and Impulse Response

- Unit impulse

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



- Impulse response: the response of a system to a unit impulse sequence

Why Impulse Response Matters

- It is the “DNA” of Linear Time-invariant systems

Linearity

- Linearity:

  If $y_1[n] = T\{x_1[n]\}$, and $y_2[n] = T\{x_2[n]\}$

  - Superposition:

  $$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$$

  - Homogeneity:

  $$T\{ax_1[n]\} = aT\{x_1[n]\} = ay_1[n]$$

  Overall: $T\{a_1x_1[n] + a_2x_2[n]\} = a_1y_1[n] + a_2y_2[n]$

Time Invariance

- Time invariance:

  If: $y[n] = T\{x[n]\}$

  Then: $y[n-n_0] = T\{x[n-n_0]\}$ for all integer n_0

- For a specified input, the output is independent of the time the input is being applied

Output of LTI Systems

- Compute the output of an LTI system using $h[n]$ for the input:

$$x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] - 0.75\delta[n-5]$$

Output of LTI Systems

- Since the system is time-invariant, we have

Input		Output
$\delta[n+2]$	$\rightarrow$	
$\delta[n-1]$	$\rightarrow$	
$\delta[n-2]$	$\rightarrow$	
$\delta[n-5]$	$\rightarrow$	

Output of LTI Systems

- Since the system is linear, we have

Input		Output
$0.5\delta[n+2]$	$\rightarrow$	
$1.5\delta[n-1]$	$\rightarrow$	
$\delta[n-2]$	$\rightarrow$	
$0.75\delta[n-5]$	$\rightarrow$	

- According to the superposition property, we get

$$y[n] = 0.5h[n+2] + 1.5h[n-1] - h[n-2] + 0.75h[n-5]$$

Output of LTI Systems

- The impulse response $h[n]$ completely characterizes an LTI system

"DNA of LTI"

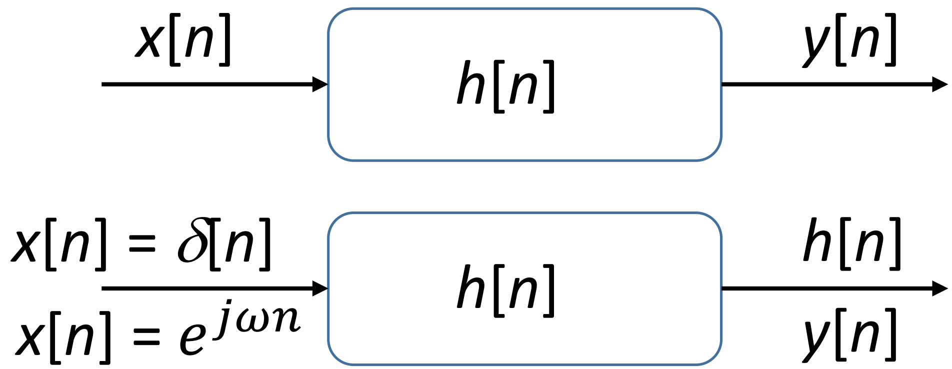
$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n-k]$$

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow x[k]h[n-k]$$

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution $\circledast$



$$y[n] = h[n] \circledast e^{j\omega n} = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)}$$

$$= \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \right) e^{j\omega n}$$

- Let

$$H\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

Eigenfunctions for LTI Systems

- Then, we can write

$$y[n] = H\left(e^{j\omega}\right)e^{j\omega n}$$

- So, $e^{j\omega n}$, is an eigenfunction of the system

Linear Combination

- If a signal can be represented as a linear combination of complex exponentials:

$$x[n] = \sum_k a_k e^{j\omega_k n}$$

- Knowing the response of an LTI system to a single complex exponential, we can determine its response to more complicated signals by making use of superposition property

The Concept of Filtering

- Any discrete-time signal may be expressed as

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- Any frequency component $e^{j\omega n}$ may be scaled by a frequency response $H(e^{j\omega})$ at frequency ω, such that the frequency component is passed or attenuated
- For example, if we have an ideal LTI system with magnitude response given by

$$|H(e^{j\omega})| = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

The Frequency Response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

is defined to be the **frequency response** of the LTI system with impulse response $h[n]$

- $H(e^{j\omega}) = |H(e^{j\omega})|e^{j\theta(\omega)}$, where, $\theta(\omega) = \arg\{H(e^{j\omega})\}$
- $|H(e^{j\omega})|$: **magnitude response**
- $\theta(\omega)$: **phase response**

A Simple Example

- For $x[n] = A\cos\omega_1 n + B\cos\omega_2 n,\ 0 < \omega_1 < \omega_2 < \pi$
- Because of linearity, the output of the system is

$$
\begin{aligned}
&y[n] \\
&= A\left|H\left(e^{j\omega_1}\right)\right|\cos(\omega_1 + \theta(\omega_1)) \\
&+ B\left|H\left(e^{j\omega_2}\right)\right|\cos(\omega_2 + \theta(\omega_2))
\end{aligned}
$$

- As $\left|H\left(e^{j\omega_1}\right)\right| = 1$, and $\left|H\left(e^{j\omega_2}\right)\right| = 0$, the output reduces to $y[n] = A\cos(\omega_1 + \theta(\omega_1))$
- The LTI system acts like a lowpass filter.

Frequency Response in Decibels

- Gain Function:

$$\mathcal{G}(\omega) = 20\log_{10}\left|H\left(e^{j\omega}\right)\right|$$

the unit is in dB

- Attenuation (or loss function):

$$\mathcal{A}(\omega) = -20\log_{10}\left|H\left(e^{j\omega}\right)\right|$$

is the negative of the gain function.

Design Example

- A signal, consisting of two sinusoids of angular frequencies of 0.1 rad/sample and 0.4 rad/sample, is to be filtered to keep the high-frequency component, but block the low-frequency component

- For simplicity, we assume a filter of length 3 with an impulse response: $h[0]=h[2]=\alpha$, and $h[1]=\beta$

- The input-output relation in time-domain is:

$$y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2]$$
$$= \alpha x[n] + \beta x[n-1] + \alpha x[n-2]$$

- **Design objective:** Choose suitable values of α and β, such that the output contains only the 0.4 rad/sample component
- The frequency response of the filter is given by

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} = \alpha\left(1 + e^{-j2\omega}\right) + \beta e^{-j\omega}$$
$$= 2\alpha\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)e^{-j\omega} + \beta e^{-j\omega} = (2\alpha\cos\omega + \beta)e^{-j\omega}$$

- To block the low-frequency component, let

$$H\left(e^{j0.1}\right) = (2\alpha\cos(0.1) + \beta) = 0$$

- To pass the high-frequency component, let

$$H\left(e^{j0.4}\right) = (2\alpha\cos(0.4) + \beta) = 1$$

- Result in:

$$\alpha = -6.76185, \qquad \beta = 13.456335$$

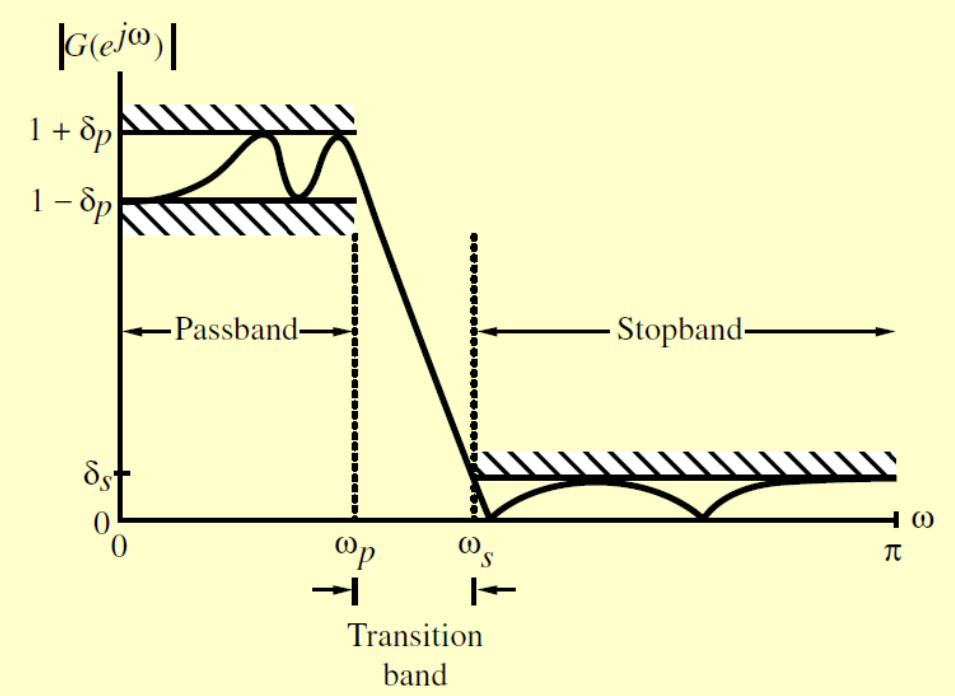
i.e., $h[n] = \{-6.76185, 13.456335, -6.76185\}$, for $n = 0, 1, 2$

- So the designed filter has the input-output relation in time-domain given by

$$y[n] = -6.76185(x[n] + x[n-2]) + 13.456335x[n-1]$$

and the input is $x[n] = (\cos(0.1n) + \cos(0.4n))\mu[n]$

Typical magnitude Specifications



- Passband edge: ω_p
- Stopband edge: ω_s
- Peak ripple value in passband: δ_p
- Peak ripple value in stopband: δ_s

- **Passband:** $\omega \leq \omega_p, \;\; 1-\delta_p \leq |G(e^{j\omega})| \leq 1+\delta_p$
- **Stopband:** $\omega_s \leq \omega \leq \pi, \;\; |G(e^{j\omega})| \leq \delta_s$
- **Transition band:** $\omega_p < \omega < \omega_s$, arbitrary response

Specifications Given as Loss function

- Loss Function

$$\mathcal{A}(\omega) = -20\log_{10}\left|G\left(e^{j\omega}\right)\right|$$

- Peak passband ripple:

$$\alpha_p = -20\log_{10}\left(1-\delta_p\right), \text{in dB}$$

- Minimum stopband attenuation

$$\alpha_s = -20\log_{10}(\delta_s), \qquad \text{in dB}$$

- **Example of ripples:** the peak passband ripple α_p and the minimum stopband attenuation α_s of a digital filter are, respectively, 0. 1 dB and 35dB. Determine their corresponding peak ripple values δ_p and δ_s.
- A: $\delta_p = 1 - 10^{-\frac{\alpha_p}{20}} = 1 - 10^{-0.005} = 0.0144690$

$$\delta_s = 10^{-\frac{\alpha_s}{20}} = 10^{-1.75} = 0.01778279$$

Obtain Band Edge Frequencies

- **Example** – For ECG signal, some studies are interested in low frequency range 0.03 Hz to 0.12 Hz and high frequency range 0.12 Hz to 0.488 Hz. If the ECG signal is sampled at 300 Hz, what are the passband edges for filters to extract the corresponding signal?
- A: Low frequency part:

$$\omega_{p1} = \frac{0.03\times 2\pi}{300} = 0.0002\pi, \omega_{p2} = \frac{0.12\times 2\pi}{300} = 0.0008\pi.$$

B: High frequency part:

$$\omega_{p1} = \frac{0.12\times 2\pi}{300} = 0.0008\pi, \omega_{p2} = \frac{0.488\times 2\pi}{300} = 0.00325\pi.$$

Does Phase Matters?

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

is defined to be the **frequency response** of the LTI system with impulse response $h[n]$

- $H(e^{j\omega}) = |H(e^{j\omega})|e^{j\theta(\omega)}, \theta(\omega) = \arg\{H(e^{j\omega})\}$
- $|H(e^{j\omega})|$: **magnitude response**
- $\theta(\omega)$: **phase response**

The Headphone: ANC

Engine Noise

Noise-Cancelling Headphone

Before

After

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The Headphone: ANC

Bose QuietComfort 25 review:

The best noise-canceling headphones get better