Review

1. Logical Equivalence

 $A \equiv B$ means A and B has the same truth value for every truth assignment.

Prove: $A \equiv B$

- By definition, $A^{-1}(\mathrm{T})=B^{-1}(\mathrm{T})$. i.e. $A^{-1}(\mathrm{T})\subseteq B^{-1}(\mathrm{T})$ and $B^{-1}(\mathrm{T})\subseteq A^{-1}(\mathrm{T})$
- $A \leftrightarrow B \equiv \mathrm{T}, \ A \to B \equiv B \to A \equiv \mathrm{T}, \ A \Rightarrow B \ \mathrm{and} \ B \Rightarrow A$
- Laws. e.g. $A \to B \equiv \neg A \lor B$

2. Tautological Implications

 $A \Rightarrow B$ means every truth assignment that causes A to be true causes B to be true.

Prove: $A \Rightarrow B$

- By definition, $A^{-1}(\mathrm{T}) \subseteq B^{-1}(\mathrm{T})$ or $B^{-1}(\mathrm{F}) \subseteq A^{-1}(\mathrm{F})$
- $A \to B \equiv \mathrm{T} \ \mathrm{or} \ \neg B \to \neg A \equiv \mathrm{T}$
- $A \wedge \neg B \equiv \mathbf{F}$
- Laws. e.g. $P \wedge Q \Rightarrow P$

3. Propositional function, Predicate logic

(i) Translation

 \forall uses \rightarrow , \exists uses \land . Why?

- On empty domain, $\forall x P(x) \equiv \mathrm{T}, \ \exists x P(x) \equiv \mathrm{F}$
- If $\forall x (P(x) \land \cdots)$, if x not in domain, making P(x) false, the whole formula is directly F.
- If $\exists x\, (P(x) \to \cdots)$, if x not in domain, making P(x) false, the whole formula is directly $\mathrm{T}.$

(ii) Proof of \equiv and \Rightarrow

4. Graph definitions

Homework 10

1.

(a)
$$\forall x (P(x) \rightarrow \exists y (P(y) \land \neg E(x,y) \land L(x,y)))$$

(b)
$$\exists x (P(x) \land \forall y (P(y) \land \neg E(x,y) \rightarrow L(y,x)))$$

5. Proof:
$$\exists x (P(x) \lor Q(x)) \equiv \exists x P(x) \lor \exists x Q(x)$$
.

(i) Show that
$$\exists x (P(x) \lor Q(x)) \Rightarrow \exists x P(x) \lor \exists x Q(x)$$
.

Suppose that $\exists x (P(x) \lor Q(x))$ is T in an interpretation I.

- There is an x_0 s.t. $P(x_0) \vee Q(x_0)$ is T in I.
- At least one of $P(x_0)$ and $Q(x_0)$ is T in I. WLOG, suppose it's $P(x_0)$.
- $\exists x P(x) \text{ is T in } I$.
- $\exists x P(x) \vee \exists x Q(x)$ is T in I.

(ii) Show that
$$\exists x (P(x) \lor Q(x)) \Leftarrow \exists x P(x) \lor \exists x Q(x)$$
.

Suppose that $\exists x P(x) \vee \exists x Q(x)$ is T in an interpretation *I*.

- At least one of $\exists x P(x)$ and $\exists x Q(x)$ is T in I. WLOG, suppose it's $\exists x P(x)$.
- There is an x_0 s.t. $P(x_0)$ is T in I
- $P(x_0) \vee Q(x_0)$ is T in I.
- $\exists x (P(x) \lor Q(x))$ is T in I.

6. Proof:
$$\forall x (P(x) \rightarrow Q(x)) \Rightarrow \forall x P(x) \rightarrow \forall x Q(x)$$

Suppose that $\forall x P(x) \rightarrow \forall x Q(x)$ is F in an interpretation I.

- $\forall x P(x)$ is T and $\forall x Q(x)$ is F in I.
- P(x) is T for every x in I and there is an x_0 s.t. $Q(x_0)$ is F in I.
- $P(x_0) \rightarrow Q(x_0)$ is F in I.
- $\forall x (P(x) \rightarrow Q(x))$ is F in I.