Discrete Mathematics
Lecture 12

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Summary of Lecture 11

THEOREM: There is a T-route from $A = (a, \alpha)$ to $B = (b, \beta)$ iff

(1) $b > a$; (2) $b - a \geq |\beta - \alpha|$; and (3) $2 | (b + \beta - a - \alpha)$.

THEOREM: If $A = (a, \alpha), B = (b, \beta)$ satisfy the T-condition.

- \# of T-routes from A to B is $\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta-\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta-\alpha}{2}\right)!}$
- $\alpha, \beta > 0$: # of T-Routes intersecting the x-axis is $\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta+\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta+\alpha}{2}\right)!}$

THEOREM: The number of solutions of the equation system

$$\begin{cases} x_1 + x_2 + \cdots + x_{2n+1} = n \\ x_1 + x_2 + \cdots + x_i < i/2, i = 1,2, \dots, 2n+1 \\ x_i \in \{0,1\}, i = 1,2, \dots, 2n+1 \end{cases}$$

is $C_n = \frac{(2n)!}{n!(n+1)!}$

Catalan Number: # of ways of parenthesizing
$a_1 * a_2 * \cdots * a_n * a_{n+1}$

Combinations of Sets

DEFINITION: Let $A = \{a_1, \dots, a_n\}$ and let $r \in \{0,1,\dots,n\}$.

- ***r*-combination of *A***: an r-subset of A.
  - Notation: $\{a_{i_1}, \dots, a_{i_r}\}$ with $1 \le i_1 < \cdots < i_r \le n$
  - $\binom{n}{r}$: the number of r-combinations of an n-element set $\binom{n}{r} r! = P(n,r)$

THEOREM: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ for all $n \in \mathbb{Z}^+$ and $r \in \{0,1,\dots,n\}$.

DEFINITION: Let $A = \{a_1, \dots, a_n\}$ and let $r \ge 0$.

- ***r*-combination of *A* with repetition**: a multiset $\{x_1 \cdot a_1, \dots, x_n \cdot a_n\}$ of r elements, where $x_1, \dots, x_n \ge 0$ are integers and $x_1 + \cdots + x_n = r$.
  - Notation: $\{a_{i_1}, \dots, a_{i_r}\}$ with $1 \le i_1 \le i_2 \le \cdots \le i_r \le n$

THEOREM: The number of r-combinations of an n element set with repetition is $\binom{n+r-1}{r}$

Combinations of Sets

$\forall u_i \in [n]$

- $\mathcal{U}$: the set of all r-combinations of A with repetition
- $\mathcal{V}$: the set of all r-combinations of $[n+r-1]$ without repetition $\to \{1,2,3,\ldots,n+r-1\}$
  - Let $U = \{u_1, u_2, \ldots, u_r\} \in \mathcal{U}$ and $1 \leq u_1 \leq u_2 \leq \cdots \leq u_r \leq n$.
    - $1 \leq u_1 < u_2 + 1 < u_3 + 2 < \cdots < u_r + r - 1 \leq n + r - 1$

      保证值不同
      - $\{u_1, u_2 + 1, \ldots, u_r + r - 1\} \in \mathcal{V}$
      - $f: \mathcal{U} \to \mathcal{V}\ \ \{u_1, u_2, \ldots, u_r\} \mapsto \{u_1, u_2 + 1, \ldots, u_r + r - 1\}$
    - f is bijective. Hence, $|\mathcal{U}| = |\mathcal{V}| = \binom{n+r-1}{r}$

THEOREM: The number of natural number solutions of the equation $x_1 + x_2 + \cdots + x_n = r$ is $\binom{n+r-1}{r}$.

- $\mathcal{X} = \{(x_1, \ldots, x_n): x_1, \ldots, x_n \in \mathbb{N} \text{ and } x_1 + \cdots + x_n = r\}$
- $\mathcal{Y}$: the set of all r-combinations of $[n]$ with repetition
- $f: \mathcal{X} \to \mathcal{Y}\quad (x_1, \ldots, x_n) \mapsto \{x_1 \cdot 1, x_2 \cdot 2, \ldots, x_n \cdot n\}$
  - f is bijective. Hence, $|\mathcal{X}| = |\mathcal{Y}| = \binom{n+r-1}{r}$.

Application

EXAMPLE: What is the value of k after the program execution?

- $k := 0;$
- for $i_1 := 1$ to n do
  - for $i_2 := 1$ to i_1 do
    - $\vdots$
      - for $i_r := 1$ to i_{r-1} do
        - $k := k + 1;$

Analysis:

- Loop variables: $1 \leq i_r \leq i_{r-1} \leq \cdots \leq i_1 \leq n$
- The number of iterations is equal to the number of r-combinations of the set $[n]$ with repetition
- In every iteration, k increases by 1.
  - After the program execution, $k = \binom{n+r-1}{r}$

Combinations of Multiset

DEFINITION: Let $A = \{n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k\}$ be an n-multiset. Let $r \in \{0,1,\dots,n\}$.

- r**-combination of** $\boldsymbol{A}$: an r-subset (multiset) of A
  - Notation: $\{x_1 \cdot a_1, x_2 \cdot a_2, \dots, x_k \cdot a_k\}$, where $0 \le x_i \le n_i$ for every $i \in [k]$ and $x_1 + x_2 + \cdots + x_k = r$.

EXAMPLE: $A = \{1 \cdot \mathrm{a}, 2 \cdot \mathrm{b}, 3 \cdot \mathrm{c}\}$

- $\{1 \cdot b, 2 \cdot c\}$ is a 3-combination of A; a 3-subset of A

REMARK:

- For every $r \in \{0,1,\dots,n\}$, an r-combination of $A = \{a_1, a_2, \dots, a_n\}$ without repetition is an r-combination of $\{1 \cdot a_1, 1 \cdot a_2, \dots, 1 \cdot a_n\}$.
- For every $r \ge 0$, an $r-$combination of $A = \{a_1, a_2, \dots, a_n\}$ with repetition is an $r-$combination of $\{\infty \cdot a_1, \infty \cdot a_2, \dots, \infty \cdot a_n\}$.

Inverse Binomial Transform

二项式变换

DEFINITION: The **binomial transform** of $\{a_n\}_{n\geq s}$ is a sequence $\{b_n\}_{n\geq s}$ such that

$$b_n = \sum_{k=s}^{n} \binom{n}{k} a_k \qquad (1)$$

逆二项式变换

DEFINITION: The **inverse binomial transform** of $\{a_n\}_{n\geq s}$ is a sequence $\{b_n\}_{n\geq s}$ such that

$$b_n = \sum_{k=s}^{n} \boxed{(-1)^{n-k}} \binom{n}{k} a_k \qquad (2)$$

QUESTION: Given (1), how to find the sequence $\{a_n\}$?

- Answer: $\{a_n\}$ is the inverse binomial transform of $\{b_n\}$
- Application: determine $\{a_n\}$ via $\{b_n\}$
- Proof?

Combinatorial Proofs

DEFINITION: A **combinatorial proof** of an identity $L = R$ is

- **a double counting proof,** which shows that L, R count the same set of objects but in different ways:
  - 相等原则 $L = |X| = R$ and L, R count $|X|$ in different ways.
- **a bijective proof,** which shows a bijection between the sets of objects counted by L and R:
  - $L = |X|, R = |Y|$ and there is a bijection $f: X \to Y$.

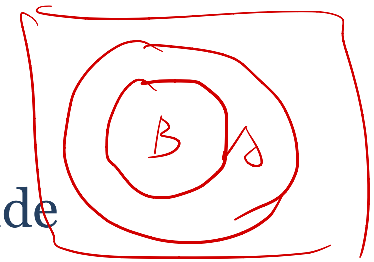
EXAMPLE: $\binom{n}{r} = \binom{n}{n-r}$

- $X = \{s \in \{0,1\}^n : s \text{ contains } \underline{r \text{ 0s}}\} = \{s \in \{0,1\}^n : s \text{ contains } \underline{n - r \text{ 1s}}\}$
  - $\binom{n}{r} = |X|$
  - $\binom{n}{n-r} = |X|$

Inverse Binomial Transform

引理　　先A再B　　先B再A

LEMMA: $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ for any $n, k, r \in \mathbb{N}$ such that $n \geq k \geq r$.



- Let $U = \{u_1, u_2, \ldots, u_n\}$ be a finite set of n elements
- $S = \{(A, B): A \subseteq U, |A| = k, B \subseteq A, |B| = r\}$
  - choose A then choose B: $|S| = \binom{n}{k}\binom{k}{r}$, the left-hand side
  - choose B then choose A: $|S| = \binom{n}{r}\binom{n-r}{k-r}$, the right-hand side

LEMMA: $\sum_{k=r}^{n}(-1)^{n-k}\binom{n}{k}\binom{k}{r} = \begin{cases} 1 & n = r \\ 0 & n > r \end{cases}$ when $n \geq r$.

- $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ as $n \geq k \geq r \geq 0$
- **left** $= \sum_{k=r}^{n}(-1)^{n-k}\binom{n}{r}\binom{n-r}{k-r} = \binom{n}{r}\sum_{k=r}^{n}(-1)^{(n-r)-(k-r)}\binom{n-r}{k-r}$

令 $k-r=i$

$$= \binom{n}{r}\sum_{i=0}^{n-r}(-1)^{(n-r)-i}\binom{n-r}{i}$$

?

$= \textbf{right}$

再看看

Inverse Binomial Transform

交换求和顺序

LEMMA: Let $n, s \in \mathbb{N}, s \leq n$. Then $\underbrace{\sum_{k=s}^{n} \sum_{i=s}^{k}}_{\alpha_k} a_{k,i} = \underbrace{\sum_{i=s}^{n} \sum_{k=i}^{n}}_{\beta_i} a_{k,i}$

k \ i	s	$s+1$	$s+2$	$\cdots$	n	row sum
s	$a_{s,s}$			$\cdots$		α_s
$s+1$	$a_{s+1,s}$	$a_{s+1,s+1}$		$\cdots$		α_{s+1}
$s+2$	$a_{s+2,s}$	$a_{s+2,s+1}$	$a_{s+2,s+2}$	$\cdots$		α_{s+2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
n	$a_{n,s}$	$a_{n,s+1}$	$a_{n,s+2}$	$\cdots$	$a_{n,n}$	α_n
col sum	β_s	β_{s+1}	β_{s+2}	$\cdots$	β_n	$\sum\sum$

THEOREM: Let $\{a_n\}, \{b_n\}$ be two sequences s.t. for all $n \geq s$,

$$a_n = \sum_{k=s}^{n} \binom{n}{k} b_k. \text{ Then } b_n = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} a_k \ (n \geq s).$$

- $\sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} a_k = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} \sum_{i=s}^{k} \binom{k}{i} b_i$

$$= \sum_{i=s}^{n} \sum_{k=i}^{n} (-1)^{n-k} \binom{n}{k} \binom{k}{i} b_i = b_n$$

$n=i$ 1

$n>i$ 0

(第二类斯特林数)

Distributing Objects into Boxes

The Problem Statement: distributing n objects into k boxes

不同的 标号

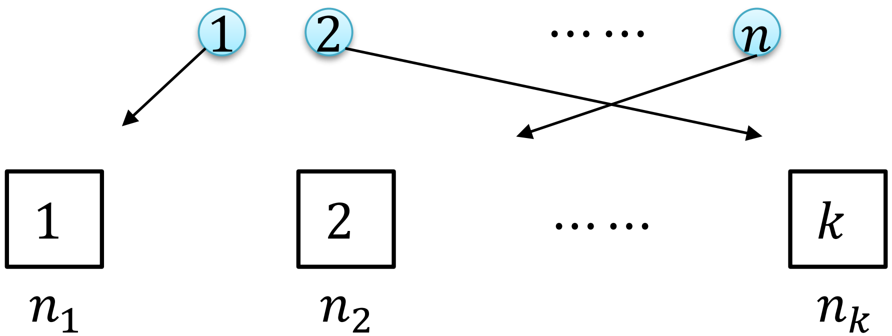
- Objects may be distinguishable (**labeled** with numbers $1,2,\ldots,n$) or 相同的 indistinguishable (**unlabeled**) 不标号
- Boxes may be distinguishable (**labeled** with numbers $1,2,\ldots,k$) or indistinguishable (**unlabeled**)
- ? What is the # of distributing n objects into k?

Problem Type	Objects	Boxes
1	labeled	labeled
2	unlabeled	labeled
3	labeled	unlabeled
4	unlabeled	unlabeled

Problem Classification

Type 1

Problem: distributing n labeled objects into k labled boxes



Classifications

$n_1 + n_2 + \cdots + n_k = n$

$n_1, n_2, \ldots, n_k \in \mathbb{N}$

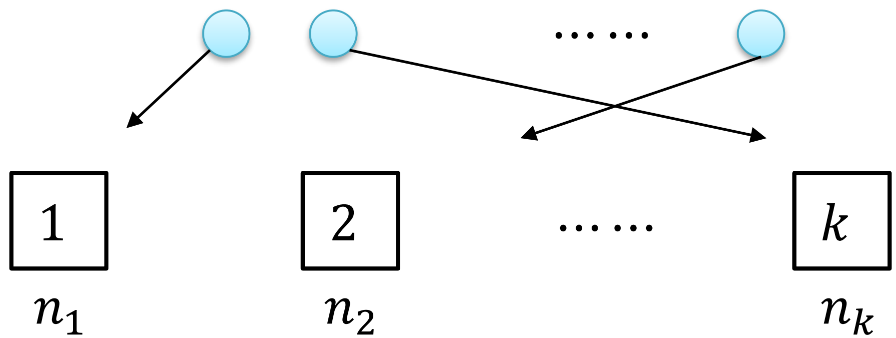
THEOREM: The number of ways of distributing n labeled objects into k labeled boxes such that n_i objects are placed into box i for every $i \in [k]$ is $N_1 = n!/(n_1!\, n_2! \cdots n_k!)$.

- S: the set of the expected distributing schemes
- $|S| = \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}$

REMARK: $N_1 = \#$ of permutations of $\{n_1 \cdot 1, \ldots, n_k \cdot k\}$.

Type 2

Problem: distributing n unlabeled objects into k labled boxes



Classifications

$n_1 + n_2 + \cdots + n_k = n$

$n_1, n_2, \dots, n_k \in \mathbb{N}$

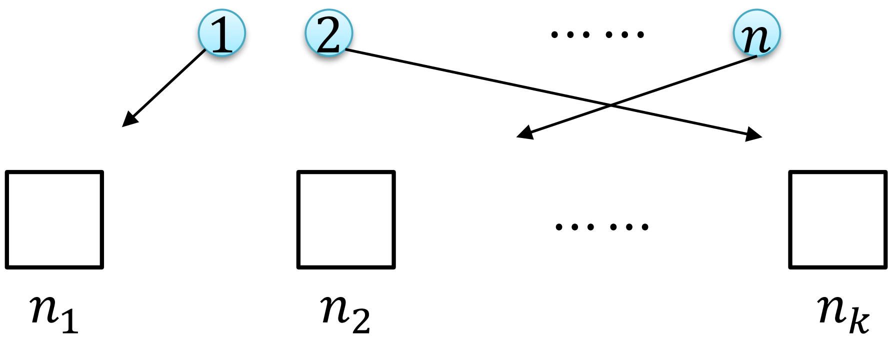
THEOREM: The number of ways of distributing n unlabeled objects into k labeled boxes is $N_2 = \binom{n+k-1}{n}$.

- S: the set of the expected distributing schemes
- $T = \{(n_1, n_2, \dots, n_k): n_1 + n_2 + \cdots + n_k = n; n_1, n_2, \dots, n_k \in \mathbb{N}\}$
- $f: T \to S \quad (n_1, n_2, \dots, n_k) \mapsto$ a scheme where n_i objects are put into box i
  - f is a bijection. Hence, $|S| = |T| = \binom{n+k-1}{n}$

REMARK: $N_2 = \#$ of n-combinations of $\{\infty \cdot 1, \dots, \infty \cdot k\}$

Type 3

Problem: distributing n labeled objects into k unlabled boxes



Classifications

$n_1 + n_2 + \cdots + n_k = n$

$n_1, n_2, \ldots, n_k \in \mathbb{N}$

$n_1 \geq n_2 \geq \cdots \geq n_k$

EXAMPLE: Assigning 4 employees {a, b, c, d} into 3 unlabeled offices. Each office can contain any number of employees.

- 4 0 0: [abcd − −]
- 3 1 0: [abc d −] [abd c −] [acd b −] [bcd a −]
- 2 2 0: [ab cd −] [ac bd −] [ad bc −]
- 2 1 1: [ab c d][ac b d] [ad b c] [bc a d] [bd a c] [cd a b]

REMARK: The schemes can be classified with $\{n_1, \ldots, n_k\}$

$S_2(n, j)$

第2类斯特林数

DEFINITION: $S_2(n, j)$, the **Stirling number of the second kind**, is defined as the number of different ways of distributing n labeled objects into j unlabeled boxes so that no box is empty.

n个标号物体

THEOREM: $S_2(n, j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i \binom{j}{i}(j-i)^n$ when $n \geq j \geq 1$.

THEOREM: The number of schemes of distributing n labeled objects into k unlabeled boxes is

$$\sum_{j=1}^{k} S_2(n, j) = \sum_{j=1}^{k} \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i \binom{j}{i}(j-i)^n$$

- $S_2(n, j)$: the number of schemes that use exactly j boxes, $j = 1,2, \ldots, k$

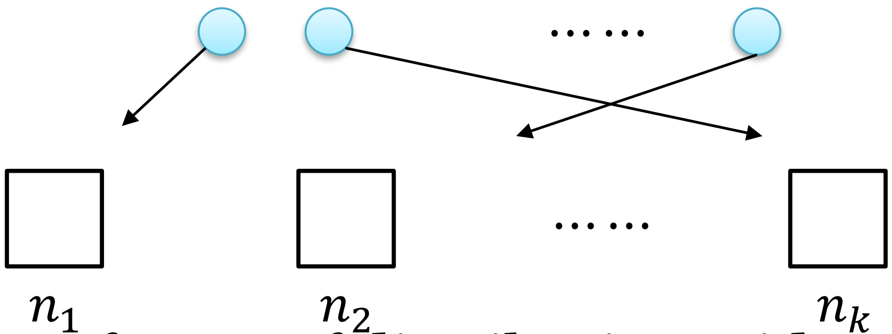
$S_2(n,j)$

THEOREM: $S_2(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n$ when $n \geq j \geq 1$.

- $T(n,j)$: the number of ways of distributing n labeled objects into j labeled boxes such that no box is empty
  - $T(n,j) = j! \cdot S_2(n,j)$
    - $T(n,j) = ?$
- X: the set of ways of distributing n labeled objects into j labeled boxes.
  - By the product rule, $|X| = j^n$
- $X_i \subseteq X$: the set of ways where exactly i boxes are used, $i = 1,2,\ldots,j$
  - $\{X_1, X_2, \ldots, X_j\}$ is a partition of X and $|X_i| = \binom{j}{i}T(n,i)$
  - $j^n = |X| = \sum_{i=1}^{j}|X_i| = \sum_{i=1}^{j}\binom{j}{i}T(n,i)$
  - $T(n,j) = \sum_{i=1}^{j}(-1)^{j-i}\binom{j}{i}i^n = \sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n$ //inversion
- $S_2(n,j) = \frac{1}{j!}\cdot T(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n$

Type 4

Problem: distributing n unlabeled objects into k unlabled boxes



Classifications

$$n_1 + n_2 + \cdots + n_k = n$$
$$n_1, n_2, \ldots, n_k \in \mathbb{N}$$
$$n_1 \geq n_2 \geq \cdots \geq n_k$$

EXAMPLE: # of ways of distributing 4 identical books into 3 identical boxes.

- 4 0 0
- 3 1 0
- 2 2 0
- 2 1 1

REMARK: The schemes are determined by $\{n_1, \ldots, n_k\}$

Partitions of Integers

DEFINITION: $n = a_1 + a_2 + \cdots + a_j$ is called an $\boldsymbol{n}$**-partition** with exactly j parts if $a_1 \geq a_2 \geq \cdots \geq a_j$ are all positive integers.

- $p_j(n) = |\{(a_1, \ldots, a_j): a_1 + \cdots + a_j = n, a_1 \geq a_2 \geq \cdots \geq a_j \geq 1 \text{ are integers}\}|$
  - $p_j(n)$: # of ways of writing n as the sum of j positive integers.

EXAMPLE: The integer 4 has four different partitions:

- $4 = 4$
- $4 = 3 + 1$
- $4 = 2 + 2$
- $4 = 2 + 1 + 1$

REMARK: solution to the type 4 problem=$\sum_{j=1}^{k} p_j(n)$

Partitions of Integers

THEOREM: For $n \in \mathbb{Z}^+, j \in [n]$, $p_j(n+j) = \sum_{k=1}^{j} p_k(n)$

- $p_1(n) = 1$, $p_n(n) = 1$
- Let $S_k = \{\text{partitions of } n \text{ into } k \text{ positive integers}\}, k \in [j]$
- Let $S = \bigcup_{k=1}^{j} S_k$.
  - $|S| = |S_1| + \cdots + |S_j| = p_1(n) + \cdots + p_j(n)$
- Let $T = \{\text{partitions of } n+j \text{ into } j \text{ positive integers}\}$
  - $|T| = p_j(n+j)$
- $f: S \to T \quad (n_1, \ldots, n_k) \mapsto (n_1 + 1, \ldots, n_k + 1, \underbrace{1, \ldots, 1}_{j-k})$
  - f is bijective
  - $|T| = |S|$

EXAMPLE: determine $p_3(6)$ and $p_4(6)$ with the above theorem

- $p_3(6) = p_3(3+3) = p_1(3) + p_2(3) + p_3(3) = 1 + 1 + 1 = 3$
- $p_4(6) = p_4(2+4) = p_1(2) + p_2(2) + p_3(2) + p_4(2) = 1 + 1 + 0 + 0 = 2$

Computing $p_j(n)$ Recursively

$$
\begin{array}{ccccccccc}
 & & & & p_1(1) & & & & \\
 & & & p_1(2) & & p_2(2) & & & \\
 & & p_1(3) & & p_2(3) & & p_3(3) & & \\
 & p_1(4) & & p_2(4) & & p_3(4) & & p_4(4) & \\
p_1(5) & & p_2(5) & & p_3(5) & & p_4(5) & & p_5(5)
\end{array}
$$

$$p_1(6) \quad p_2(6) \quad p_3(6) \quad p_4(6) \quad p_5(6) \quad p_6(6)$$

$$p_1(7) \quad p_2(7) \quad p_3(7) \quad p_4(7) \quad p_5(7) \quad p_6(7) \quad p_7(7)$$

$$p_1(8) \quad p_2(8) \quad p_3(8) \quad p_4(8) \quad p_5(8) \quad p_6(8) \quad p_7(8) \quad p_8(8)$$

$$p_1(9) \quad p_2(9) \quad p_3(9) \quad p_4(9) \quad p_5(9) \quad p_6(9) \quad p_7(9) \quad p_8(9) \quad p_9(9)$$

Principle of Inclusion–Exclusion

Problem: S is a finite set and $A_1, A_2, \dots, A_n \subseteq S$.

- $|\bigcup_{i=1}^{n} A_i| = ?$
- $|\bigcap_{i=1}^{n} A_i| = ?$

EXAMPLE: Let S be the set of permutations of $[n]$. Find $|A|$ for

$$A = \{x_1 x_2 \dots x_n : x_1 x_2 \dots x_n \in S;\; x_i \neq i \text{ for all } i \in [n]\}.$$

- $A_i = \{x_1 x_2 \cdots x_n : x_1 x_2 \cdots x_n \in S; x_i = i\}, i = 1,2, \dots, n$
  - $A = S - \bigcup_{i=1}^{n} A_i$
  - $|S| = n!$
  - $|\bigcup_{i=1}^{n} A_i| = ?$

Principle of IE (Two Sets)

THEOREM: Let S be a finite set. Let A_1, A_2 be subsets of S. Then

- $|S - A_1| = |S| - |A_1|$; $|A_1 - A_2| = |A_1| - |A_1 \cap A_2|$
  - $S = A_1 \cup (S - A_1\ ),\ A_1 \cap (S - A_1) = \emptyset;$
    - $\{A_1, S - A_1\}$ is a partition of S
      - $|S| = |A_1| + |S - A_1|$
        - $|S - A_1| = |S| - |A_1|$
    - $A_1 - A_2 = A_1 - A_1 \cap A_2$
      - $|A_1 - A_2| = |A_1| - |A_1 \cap A_2|$
- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
  - $A_1 \cup A_2 = (A_1 - A_2) \cup A_2, (A_1 - A_2) \cap A_2 = \emptyset;$
    - $\{A_1 - A_2, A_2\}$ is a partition of $A_1 \cup A_2$
      - $|A_1 \cup A_2| = |A_1 - A_2| + |A_2| = |A_1| - |A_1 \cap A_2| + |A_2|$
- $|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2|$

Principle of IE (Three Sets)

THEOREM: Let S be a finite set. Let A_1, A_2, A_3 be subsets of S.

Then $\left|\bigcup_{i=1}^{3} A_i\right| = \sum_{t=1}^{3}(-1)^{t-1}\sum_{1\le i_1<\cdots<i_t\le 3}|A_{i_1}\cap\cdots\cap A_{i_t}|$

- $\left|\bigcup_{i=1}^{3} A_i\right| = |(A_1 \cup A_2)\cup A_3| = |A_1\cup A_2| + |A_3| - |(A_1\cup A_2)\cap A_3|$
  - $|A_1\cup A_2| = |A_1| + |A_2| - |A_1\cap A_2|$
  - $|(A_1\cup A_2)\cap A_3| = |(A_1\cap A_3)\cup(A_2\cap A_3)|$

    $= |A_1\cap A_3| + |A_2\cap A_3| - |(A_1\cap A_3)\cap(A_2\cap A_3)|$

    $= |A_1\cap A_3| + |A_2\cap A_3| - |A_1\cap A_2\cap A_3|$
    - $\left|\bigcup_{i=1}^{3} A_i\right| = |A_1| + |A_2| - |A_1\cap A_2| + |A_3|$

      $-(|A_1\cap A_3| + |A_2\cap A_3| - |A_1\cap A_2\cap A_3|)$
- $\left|\bigcap_{i=1}^{3} A_i\right| = \sum_{t=1}^{3}(-1)^{t-1}\sum_{1\le i_1<\cdots<i_t\le 3}|A_{i_1}\cup\cdots\cup A_{i_t}|$

Principle of IE (n Sets)

THEOREM: Let S be a finite set. Let $A_1, A_2, \ldots, A_n$ be subsets of S.

Then $|\bigcup_{i=1}^{n} A_i| = \sum_{t=1}^{n}(-1)^{t-1} \sum_{1 \le i_1 < \cdots < i_t \le n} |A_{i_1} \cap \cdots \cap A_{i_t}|$

- $n = 1$: $|A_1| = |A_1|$
- $n = 2$: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- $n = 3$: $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3|$
  $-|A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$
- **Induction hypothesis**: the identity holds for $n \le k$ ($k \ge 3$)
- Need to show the identity for $n = k + 1$
- $|A_1 \cup \cdots \cup A_{k+1}| = |A_1 \cup \cdots \cup A_k| + |A_{k+1}| - |(A_1 \cup \cdots \cup A_k) \cap A_{k+1}|$
  $= \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\bigcup_{i=1}^{k}(A_i \cap A_{k+1})\right|$

Principle of IE (n Sets)

- $\left|\bigcup_{i=1}^{k} A_i\right| = \sum_{t=1}^{k}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le k} |A_{i_1} \cap \cdots \cap A_{i_t}|$
- $\left|\bigcup_{i=1}^{k}(A_i \cap A_{k+1})\right| = \sum_{t=1}^{k}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le k} |(A_{i_1} \cap A_{k+1}) \cap \cdots \cap (A_{i_t} \cap A_{k+1})|$
- $\left|\bigcup_{i=1}^{k+1} A_i\right| = \sum_{t=1}^{k}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le k} |A_{i_1} \cap \cdots \cap A_{i_t}| + |A_{k+1}| -$

$$\sum_{t=1}^{k}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le k} |(A_{i_1} \cap A_{k+1}) \cap \cdots \cap (A_{i_t} \cap A_{k+1})|$$

$$= \sum_{t=1}^{k+1}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le k+1} |A_{i_1} \cap \cdots \cap A_{i_t}|$$

THEOREM: Let S be a finite set. Let $A_1, A_2, \ldots, A_n$ be subsets of S.

Then $|\bigcap_{i=1}^{n} A_i| = \sum_{t=1}^{n}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le n} |A_{i_1} \cup \cdots \cup A_{i_t}|$

Principle of Inclusion-Exclusion

EXAMPLE: Let S be the set of permutations of $[n]$. Find $|A|$ for

$$A = \{x_1x_2 \dots x_n : x_1x_2 \dots x_n \in S;\ x_i \neq i \text{ for all } i \in [n]\}.$$

- $A_i = \{x_1x_2 \cdots x_n : x_1x_2 \cdots x_n \in S; x_i = i\}, i = 1,2, \dots, n$
  - $A = S - \bigcup_{i=1}^{n} A_i$
    - $|S| = n!$
    - $|\bigcup_{i=1}^{n} A_i| =?$
- $|\bigcup_{i=1}^{n} A_i| = \sum_{t=1}^{n}(-1)^{t-1} \sum_{1\leq i_1<\cdots<i_t\leq n} |A_{i_1} \cap \cdots \cap A_{i_t}|$
  - $\left|A_{i_1} \cap \cdots \cap A_{i_t}\right| = (n-t)!$ for $t = 1,2, \dots, n$
- $|A| = |S| - |\bigcup_{i=1}^{n} A_i|$

$$= n! - \left(\binom{n}{1} * (n-1)! - \binom{n}{2} * (n-2)! + \cdots + (-1)^{n-1} * \binom{n}{n} * 1\right)$$

$$= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^t \frac{1}{t!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

Pigeonhole Principle 鸽笼原理

EXAMPLE: There are 15 workstations $W_1, \dots, W_{15}$ and 10 servers $S_1, \dots, S_{10}$. A cable can connect a workstation to a server. Connect the workstations and servers such that any ≥ 10 workstations have access to all servers. How many cables are needed?

- **Solution 1**: Connecting every workstation directly to every server. 150
- **Solution 2**: S_i is connected to W_i for every $i \in [10]$; and each of $W_{11}, W_{12}, W_{13}, W_{14}, W_{15}$ is connected to all servers.
  - This solution requires 60 lines.
  - Is this solution optimal?

Cover

DEFINITION: A **cover**覆盖 of a finite set A is a family $\{A_1, A_2, \dots, A_n\}$ of subsets of A such that $\bigcup_{i=1}^{n} A_i = A$. //partition is disjoint cover

LEMMA: Let $\{A_1, A_2, \dots, A_n\}$ be a cover of a finite set A. Then $|A| \leq \sum_{i=1}^{n} |A_i|$.

- $n = 1$: $|A| = |A_1|$
- $n = 2$: $|A| = |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \leq |A_1| + |A_2|$
- Suppose true when $n \leq k$ $(k \geq 2)$.
- When $n = k + 1$, $|A| = \left|\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right|$

$$\leq \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}|$$

$$\leq \sum_{i=1}^{k} |A_i| + |A_{k+1}|$$

$$= \sum_{i=1}^{k+1} |A_i|$$

Pigeonhole Principle (simple form)

THEOREM: Let A be a set with $\geq n+1$ elements. Let $\{A_1, A_2, \ldots, A_n\}$ be a cover of A. Then $\exists k \in [n]$ such that $|A_k| \geq 2$.

- Suppose that $|A_i| \leq 1$ for every $i \in [n]$. Then $n + 1 \leq |A| \leq \sum_{i=1}^{n} |A_i| \leq n$.
  - If $\geq n + 1$ objects are distributed into n boxes, then there is at least one box containing ≥ 2 objects.

EXAMPLE: Given 367 people, there are two with the same birthday.

- $A = \{a_1, \ldots, a_{367}\}$
- $A_i = \{a \in A: \text{the birthday of } a \text{ is the } i\text{th day of a year}\}, i = 1,2, \ldots, 366$
- $\{A_1, A_2, \ldots, A_{366}\}$ is a cover of A
  - $\exists k \in [366]$ such that $|A_k| \geq 2$

Simple Form

EXAMPLE: Let $n \in \mathbb{Z}^+$. Let $A \subseteq \{1,2,\dots,2n\}$ have $n+1$ elements. Then there exist $x, y \in A$ such that $x|y$.

- Let $A = \{a_1, \dots, a_{n+1}\} \subseteq [2n]$ be any subset of $n+1$ elements.
- $a_j = 2^{u_j} \cdot v_j$, where $u_j \in \mathbb{N}$ and $v_j \in [2n]$ is odd for all $j = 1,2,\dots,n+1$
  - $\{v_1, v_2, \dots, v_{n+1}\} \subseteq \{1,3,\dots,2n-1\}$
- $A_i = \{a_j : v_j = i\}$ for $i = 1,3,\dots,2n-1$
- $\{A_1, A_3, \dots, A_{2n-1}\}$ is a cover of A
  - $\exists k \in \{1,3,\dots,2n-1\}$ such that $|A_k| \geq 2$
    - $a_s, a_t \in A_k \Rightarrow (a_s = 2^{u_s} \cdot v_s) \wedge (a_t = 2^{u_t} \cdot v_t) \wedge (v_s = v_t = k)$
      - $(x, y) = \begin{cases} (a_s, a_t), & \text{if } u_s \leq u_t \\ (a_t, a_s), & \text{if } u_s > u_t \end{cases}$

Pigeonhole Principle (general form)

THEOREM: Let A be a set with $\geq N$ elements. Let $\{A_1, A_2, \ldots, A_n\}$ be a cover of A. Then $\exists k \in [n]$ such that $|A_k| \geq \lceil N/n \rceil$.

- If $|A_i| < \lceil N/n \rceil$ for all $i \in [n]$, then $N \leq |A| \leq \sum_{i=1}^{n} |A_i| \overset{?}{<} n \cdot N/n = N$
  - If we distribute $\geq N$ objects into n boxes, then there is at least one box that contains $\geq \lceil N/n \rceil$ objects.

EXAMPLE: How many students are needed in a discrete math class to ensure that ≥ 6 will receive the same grade? The possible grades are A+, A, A-, B+, B, B-, C+, C, C-, and F.

- Let $A = \{a_1, a_2, \ldots, a_N\}$ be a set of students. Let s_j be the score of a_j.
- $A_1 = \{a_j \in A : s_j = \mathrm{A+}\}; A_2 = \{a_j \in A : s_j = \mathrm{A}\}; \ldots ; A_{10} = \{a_j \in A : s_j = \mathrm{F}\}$
  - $\{A_1, \ldots, A_{10}\}$ is a cover of A
    - $\exists k \in [10]$ such that $|A_k| \geq \lceil N/10 \rceil$
      - $\lceil N/10 \rceil \geq 6 \Rightarrow N \geq 51$

General Form

EXAMPLE: There are 15 workstations $W_1, \ldots, W_{15}$ and 10 servers $S_1, \ldots, S_{10}$. A cable can connect a workstation to a server. Connect the workstations and servers such that any ≥ 10 workstations have access to all servers. How many cables are needed?

- **Solution 2**: S_i is connected to W_i for every $i \in [10]$; and each of $W_{11}, W_{12}, W_{13}, W_{14}, W_{15}$ is connected to all servers. // 60 lines, optimal?
- Consider an optimal scheme Π.
  - Let $A = \{(W_i, S_j): i \in [15], j \in [10], W_i \text{ is not connected to } S_j\} \text{ in } \Pi$
  - $A_t = \{(W_i, S_j) \in A: j = t\} \text{ for } t = 1,2, \ldots, 10$
    - $\{A_1, A_2, \ldots, A_{10}\}$ is a cover of A
- If there are <60 lines in Π, then $|A| > 150 - 60 = 90$.
  - $\exists k \in [10]$ such that $|A_k| \geq \lceil 91/10 \rceil = 10$
    - There are 10 workstations not connected to S_k