SI 120 Discrete Mathematics (Spring 2021), Midterm Exam

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Instructions

- Time: 8:15–9:55am (100 minutes)
- This exam is closed-book, you may bring nothing but a pen. Put all the study materials and electronic devices into your bag and put your bag in the front, back, or sides of the classroom.
- You can write your answers in either English or Chinese.
- Two blank pieces of paper are attached, which you can use as scratch paper. Raise your hand if you need more paper.

1 Multiple choice (60 pt)

Each question has only one correct answer. *Write your answers in the table below.*

1	2	3	4	5	6	7	8	9	10

11	12	13	14	15	16	17	18	19	20

1. Which of the following is not a proposition? A

   A. There are no integers between x and $x+1$.

   B. The moon is made of green cheese.

   C. $7 \cdot 11 \cdot 13 = 1001$.

   D. $e^{i\pi} + 1 = 0$.

2. Let $|$ be the binary logical connective defined by $p|q = \neg(p \wedge q)$. Which of the following is not true?

   A

   A. $(p|q)|r \equiv p|(q|r)$

   B. $p|(q \vee r) \equiv (p|q) \wedge (p|r)$

   C. $\neg(p|q) \equiv p \wedge q$

   D. $p|q \equiv q|p$

3. Which of the following is not a tautology? A

   A. $(\neg p \vee q) \wedge (q \to \neg r \wedge \neg p) \wedge (p \vee r)$

B. $(p \leftrightarrow q) \leftrightarrow (\neg p \vee q) \wedge (p \vee \neg q)$

C. $(q \to r) \wedge (p \to q) \to (p \to r)$

D. $(p \vee q) \wedge (\neg p \vee r) \to (q \vee r)$

4. Let $a =$“A comes to the party.”, $b =$“B comes to the party.”, $c =$“C comes to the party.”, $d =$“D comes to the party.”. Which of the following is the correct translation of “A sufficient condition for A coming to the party is that,if B does not come, then at least one of C and D must come.”? A

   A. $(\neg b \to (c \vee d)) \to a$

   B. $(\neg b \to \neg(c \wedge d)) \to a$.

   C. $a \to (\neg b \to (c \vee d))$

   D. $a \leftrightarrow (\neg b \to \neg(c \wedge d))$.

5. Let $C(x)$ be the statement “x has a cat,” let $D(x)$ be the statement “x has a dog,” and let $T(x)$ be the statement “x is a student.” Which of the following best describes $\exists x\,(T(x) \wedge \neg C(x) \wedge \neg D(x))$? A

   A. Not every student has either a cat or a dog.

   B. There exists a student that he has a cat and a dog.

   C. Not every student has both a cat and a dog.

   D. There exists a student that he has either a cat or a dog.

6. The predicate formula $\exists x P(x) \to P(0)$ is A

   A. satisfiable

   B. unsatisfiable

   C. logically valid

   D. none of the above

7. Which of the following is logically valid? A

   A. $\forall x(P(x) \wedge Q(x)) \leftrightarrow \forall x P(x) \wedge \forall x Q(x)$

   B. $\forall x(P(x) \vee Q(x)) \leftrightarrow (\forall x P(x) \vee \forall x Q(x))$

   C. $\forall x(P(x) \to Q(x)) \leftrightarrow (\exists x P(x) \to \exists x Q(x))$

   D. $\forall x(P(x) \to Q(x)) \leftrightarrow (\forall x P(x) \to \forall x Q(x))$

8. Which of the following is not true?

   A. The sets A, B have the same cardinality if and only if there is a bijection $f : A \to B$.

   B. A set is uncountable if its power set is uncountable.

   C. If A, B are countably infinite, then so is $A \cup B$.

   D. If A, B are countably infinite, then so is $A \times B$.

9. Which of the following is not true?

   A. Let A be any set of sets. Then $\cup \mathcal{P}(A) = A$.

   B. Let A be any set of sets. Then $\mathcal{P}(\cup A) = A$.

   C. Let $A = \{1 \cdot a, 2 \cdot b, 3 \cdot c, 100 \cdot z\}, T = \{1 \cdot b, 98 \cdot z\}$. Then T is a 99-subset of A.

   D. Let $A = \{1 \cdot a, 2 \cdot b, 3 \cdot c\}$. Then the 3-permutations of A is 19.

10. Which of the following sets has different cardinality comparing to others?

    A. The set $\mathbb{R}^+$ of positive real numbers.

    B. The set $\{(x, y) : x, y \in \mathbb{R}, x^2 + y^2 = 1\}$.

    C. The set $\{(x, y) : x, y \in \mathbb{R}, x^2 + y^2 < 1\}$.

D. The set $\{S : S \subseteq \mathbb{Z}^+, |S| < \infty\}$.

11. According to the basic rules of counting, which of the following statement is not correct?

    A. The number of four-digit decimal odd number that each digit is different from each other is 2240. $5\times(A_9^3 - A_8^3)$

    B. The number of the composite divisors of $N = 3^4 \times 5^6 \times 7^8$ is 188. $5\times7\times9-1-3$

    C. There are 12 0's at the end of decimal representation of 50!.

    D. If there is an injection from set A to set B, then we can say that $|A| \leqslant |B|$

12. Which of the following choices allows a T path from A to B?

    A. $A = (2, 1), B = (49, 51)$

    B. $A = (2, 1), B = (49, 52)$

    C. $A = (1, 2), B = (48, 48)$

    D. $A = (1, 2), B = (51, 48)$

13. Let $\{a_n\}_{n=3}^{\infty}$ be a sequence such that $n(n-1) = \sum_{k=3}^{n} \binom{n}{k} a_k$. Then $a_6 =$ ____________.

    A. -10

    B. -20

    C. -30

    D. -40

14. The number of surjections from $\{1, 2, 3, 4, 5, 6\}$ to $\{A, B, C\}$ is ____________.

    A. 510

    B. 520

    C. 530

    D. 540

15. The number of ways of distributing 6 different books into 4 identical boxes such that at most one box is empty is ____________.

    A. 122

    B. 133

    C. 144

    D. 155

16. There are 21 identical (相同的) seats in an meeting room. The number of ways of arranging them in three (different) rows, such that any two rows are majority of them (i.e. greater or equal than 11) is ____________.

    ≤ 10

    A. 36

    B. 45

    C. 55

    D. 66

例 3 会议室中有 $2n+1$ 个座位,现摆成 3 排,要求任何两排的座位数都要占大多数.问有多少种摆法?

10
11
0

解 这个问题相当于把 $2n+1$ 个完全相同的球分配到 3 个不同的盒子里,如果没有附加限制,应该有

$$\binom{(2n+1)+3-1}{2n+1} = \binom{2n+3}{2}$$

种方案.不符合题意的摆法的特征是有一排至少有 $n+1$ 个座位,这相当于将 $n+1$ 个座位先放到 3 排中的某一排,再将剩下的 $(2n+1)-(n+1)=n$ 个座位任意分到 3 排中,这种摆法共有

$$3\cdot\binom{(2n+1)-(n+1)+3-1}{(2n+1)-(n+1)} = 3\cdot\binom{n+2}{2}$$

种方案.因此,符合题意的摆法有

$$\binom{2n+3}{2} - 3\cdot\binom{n+2}{2} = \binom{n+1}{2}$$

种方案.

17. Let $A = \{a : 1 \leqslant a \leqslant 1000, a$ is divisible by 3 and 5, but not divisible by 7$\}$. Then $|A| =$ ____________.

    A. 56

    B. 57

    C. 58

    D. 59

例 2 任意 n^2+1 个实数

$$a_1, a_2, a_3, \cdots, a_{n^2+1} \tag{1.2.1}$$

组成的序列中，必有一个长为 $n+1$ 的非降子序列，或必有一个长为 $n+1$ 的非升子序列.

在证明本例之前先看一个具体的例子，对于序列 $(n=3)$

5, 3, 16, 10, 15, 14, 9, 11, 6, 7,

从中可以选出如下几个递增子序列：

{5,16}, {5,10,15}, {3,9,11}, {3,6,7}, …;

也可以选出如下几个递减子序列：

{5,3}, {16,10,9,6}, {16,15,14,11,7}, ….

证明 **方法 1** 假设长为 n^2+1 的实数序列(1.2.1)中没有长度为 $n+1$ 的非降子序列，下面证明其必有一长度为 $n+1$ 的非升子序列.

令 m_k 表示从 a_k 开始的最长非降子序列的长度，因为实数序列(1.2.1)中没有长度为 $n+1$ 的非降子序列，所以有

$$1 \leqslant m_k \leqslant n \ (k = 1, 2, \cdots, n^2+1).$$

这相当于把 n^2+1 个物品 $m_1, m_2, \cdots, m_{n^2+1}$ 放入 n 个盒子 $1, 2, \cdots, n$ 中，由鸽巢原理知，必有一盒子 i 里面至少有 $n+1$ 个物品，即存在

$$k_1 < k_2 < \cdots < k_{n+1}$$

及

$$1 \leqslant i \leqslant n,$$

使得

$$m_{k_1} = m_{k_2} = \cdots = m_{k_{n+1}} = i. \tag{1.2.2}$$

对应于这些下标的实数序列必满足

$$a_{k_1} \geqslant a_{k_2} \geqslant \cdots \geqslant a_{k_{n+1}}, \tag{1.2.3}$$

它们构成一长为 $n+1$ 的非增子序列. 否则，若有某个 j $(1 \leqslant j \leqslant n)$ 使得 $a_{k_j} < a_{k_{j+1}}$，那么由从 $a_{k_{j+1}}$ 开始的最长非降子序列加上 a_{k_j}，就得到一个从 a_{k_j} 开始的长度为 $m_{k_{j+1}}+1$ 的非降子序列. 由 m_{k_j} 的定义知

$$m_{k_j} \geqslant m_{k_{j+1}} + 1,$$

这与(1.2.2)式矛盾. 因此(1.2.3)式成立，从而定理的结论成立.

方法 2 对应于实数序列(1.2.1)中的每个 a_i，定义一个有序偶

$$(l_i, m_i),$$

其中，l_i 为从 a_i 开始的最长非降子序列的长度，m_i 为从 a_i 开始的最长非升子序列的长度. 则对应于序列(1.2.1)，有以下的有序偶序列

$$(l_1, m_1), (l_2, m_2), \cdots, (l_{n^2+1}, m_{n^2+1}). \tag{1.2.4}$$

若实数序列(1.2.1)中既没有长为 $n+1$ 的非升子序列，也没有长为 $n+1$ 的非降子序列，则有

$$1 \leqslant l_i \leqslant n, \ 1 \leqslant m_i \leqslant n \ (i = 1, 2, \cdots, n^2+1). \tag{1.2.5}$$

满足条件(1.2.5)的有序偶最多只有 n^2 个，由鸽巢原理知，序列(1.2.4)中至少有两个有序偶相同. 即存在 $1 \leqslant i \neq j \leqslant n^2+1$，使得

$$(l_i, m_i) = (l_j, m_j),$$

即

$$l_i = l_j, \ m_i = m_j.$$

不妨设 $i<j$，由方法 1 的分析知，若 $a_i \leqslant a_j$，则 $l_i > l_j$，与 $l_i = l_j$ 矛盾；若 $a_i > a_j$，则 $m_i > m_j$，与 $m_i = m_j$ 矛盾. 所以，实数序列(1.2.1)中必有一长为 $n+1$ 的非降子序列，或有一长为 $n+1$ 的非升子序列.

18. Which of the following statement is not correct?

A. Given 367 persons, at least two of them have same birthday. (数的個数, 列的個数) → 不可能相同

B. Given a sequence of distinct real numbers $\{a_1, a_2, \cdots, a_{50}\}$, the length of the longest strictly monotonous (strictly increasing or decreasing) subsequence is at least ~~9~~ 8. (Example: $\{1, 3\}$ is a strictly increasing subsequence of $\{1, 4, 3, 2\}$)

C. During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Then there must be a period of some number of consecutive days during which the team must play exactly 14 games.

D. In order to connect 50 computers to 15 printers, such that any 15 computers are able to connect to all 15 printers, at least 540 cables are needed. $15+35\times15=540$

19. Which of the following could be a solution to the linear recurrence relation $a_n = 7a_{n-1} - 15a_{n-2} + 9a_{n-3} \quad (n \geq 3)$?

A. $a_n = 2 \cdot 3^n + 5$

B. $a_n = (n^2+3) \cdot 3^n$

C. $a_n = 2 \cdot 3^n - n + 1$

D. $a_n = 5^n$

20. Let $a_n = 2n+2$ for every integer $n \geq 0$. The generating function of $\{a_n\}_{n=0}^{\infty}$ is ________.

A. $\frac{1}{1-x}$

B. $\frac{1}{(1-x)^2}$

C. $\frac{2}{(1-x)^2}$

D. $\frac{1}{2(1-x)^2}$

Solution:

1	2	3	4	5	6	7	8	9	10
A	A	A	A	A	A	A	B	B	D

11	12	13	14	15	16	17	18	19	20
B	D	C	D	D	C	B	B	A	C

2 Logic (16 pt)

Let $P(x) =$"x is a person", $H(x, y) =$"x hates y" and $E(x, y) =$"$x = y$". Translate the following statements into formulas: *Only the final formula for each statements is needed.*

(a) "Every person hates some person."

(b) "Every person hates some other person."

(c) "There is a person who is hated by every person."

(d) "There is a person who is not hated by any other person."

Solution:

(a) $\forall x(P(x) \to \exists y(P(y) \land H(x, y)))$ or $\forall x\exists y(P(x) \to (P(y) \land H(x, y)))$

(b) $\forall x(P(x) \to \exists y(P(y) \land \neg E(x, y) \land H(x, y)))$ or $\forall x\exists y(P(x) \to (P(y) \land \neg E(x, y) \land H(x, y)))$

(c) $\exists x(P(x) \land \forall y(P(y) \to H(y, x)))$ or $\exists x\forall y(P(x) \land (P(y) \to H(y, x)))$

(d) $\exists x(P(x) \land \forall y(P(y) \boxed{\land \neg E(x, y)} \to \neg H(y, x)))$ or $\exists x\forall y(P(x) \land (P(y) \land \neg E(x, y) \to \neg H(y, x)))$ or $\exists x(P(x) \land \neg\exists y(P(y) \land \neg E(x, y) \land H(y, x)))$.

3 All Disordered Permutation (10 pt)

Let $A_n = \{x_1x_2\cdots x_n : x_1x_2\cdots x_n$ is a permutation of $[n]$ and $x_i \neq i, \forall i \in [n]\}$. Determine $|A_n|$.

Solution:

Two major ways of solution are listed here:

Inverse Binomial Transform

Let $S_n = \{\text{All the permutations of } [n]\}$, and $S_{n,i} = \{x_1x_2...x_n : x_1x_2...x_n \in S_n, \text{for} P \subseteq [n], |P| = i, x_j = j \text{if and only if} i \in P, \}$. Then we have $S_{n,i} = \binom{n}{i}|A_{n-i}$. Note that $\{S_{n,0}, S_{n,1}, ..., S_{n,n}\}$ is a partition of S_n, so

$$n! = \sum_{k=0}^{n} \binom{n}{k} |A_k|$$

According to inverse binomial transform, we have:

$$\begin{aligned}
|A_n| &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k! \\
&= \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!(n-k)!} k! \\
&= n! \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} \\
&= n! \sum_{k=0}^{n} \frac{(-1)^{k}}{k!}
\end{aligned}$$

Principle of IE

Let Let $S_n = \{\text{All the permutations of } [n]\}$, so $|S_n| = n!$. Define $S_i = \{x_1x_2...x_n : x_1x_2...x_n \in S_n, x_i = i\}$, it is obvious that $|S_i| = (n-1)!$.
According to the definition, we have that

$$|A_n| = |S_n| - |\cup_{t=1}^{n} S_t|$$

Based on principle of IE, we have

$$|\cup_{t=1}^{n} S_t| = \sum_{t=1}^{n} (-1)^{t-1} \sum_{1 \leq i_1 < ... < i_t \leq n} |S_{i_1} \cap ... \cap S_{i_t}|$$

Note that for $S_{i_1} \cap ... \cap S_{i_t}$, it is the permutation with t numbers on their original position. So

$$|S_{i_1} \cap ... \cap S_{i_t}| = (n-t)!$$

So finally, we have

$$\begin{aligned}
|A_n| &= n! - \sum_{t=1}^{n} (-1)^{t-1} \binom{n}{t} (n-t)! \\
&= \sum_{t=0}^{n} (-1)^{t} \binom{n}{t} (n-t)! \\
&= n! \sum_{t=0}^{n} \frac{(-1)^{t}}{t!}
\end{aligned}$$

4 Generating Function Application (14 pt)

For every integer $r \geq 1$, let a_r be the number of elements in $A_r = \{s : s \in \{0, 1, 2\}^r$, s has even number (including 0) of 1s, odd number of 2s and no more than two 0s.$\}$. Calculate a_{14} with the generating function of $\{a_n\}_{n=0}^{\infty}$.

Solution:

Here we define $R_0 = \{0, 1, 2\}$, $R_1 = \{0, 2, 4, \cdots\}$, $R_2 = \{1, 3, 5, \cdots\}$. To count permutations with generating function, according to the theorem, we have

$$
\begin{aligned}
\sum_{r=0}^{\infty} \frac{a_r}{r!} x^r =& \prod_{j=1}^{3} \sum_{i \in R_j} \frac{x^i}{i!} \\
=& (1 + x + \frac{x^2}{2!})(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots)(\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots) \\
=& (1 + x + \frac{x^2}{2}) \cdot \frac{e^x + e^{-x}}{2} \cdot \frac{e^x - e^{-x}}{2} \\
=& (1 + x + \frac{x^2}{2}) \cdot \frac{e^{2x} - e^{-2x}}{4} \\
=& \frac{1}{4}(1 + x + \frac{x^2}{2})[\sum_{r=0}^{\infty} \frac{(2x)^r}{r!} - \sum_{r=0}^{\infty} \frac{(-2x)^r}{r!}] \\
=& x + \frac{1}{4} \sum_{r=2}^{\infty} \left\{ \frac{2^r - (-2)^r}{r!} + \frac{2^{r-1} - (-2)^{r-1}}{(r-1)!} + \frac{2^{r-2} - (-2)^{r-2}}{2(r-2)!} \right\} x^r
\end{aligned}
$$

a_{14} is the coefficient of x^{14}, which could be computed as follow:

$$
\begin{aligned}
a_{14} =& \frac{1}{4} \left\{ \frac{2^{14} - (-2)^{14}}{14!} + \frac{2^{13} - (-2)^{13}}{13!} + \frac{2^{12} - (-2)^{12}}{2 \times 12!} \right\} \times 14! \\
=& \frac{2^{14}}{4 \times 13!} 14! \\
=& 2^{12} \times 14 \\
=& 57344
\end{aligned}
$$