Discrete Mathematics
Lecture 15

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Summary of Lecture 14

LHRR of degree k with constant coefficients:

$$a_n = \sum_{i=1}^{k} c_i a_{n-i}$$

- Existence and uniqueness: given k initial terms
- characteristic equation: $r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$
  - k distinct roots $r_1, r_2, \dots, r_k$: $x_n = \sum_{j=1}^{k} \alpha_j r_j^n$
  - Roots $\{m_1 \cdot r_1, \dots, m_t \cdot r_t\}$: $x_n = \sum_{j=1}^{t} \left(\sum_{\ell=0}^{m_j-1} \alpha_{j,\ell}\, n^{\ell} \right) r_j^n$

LNRR of degree k with constant coefficients:

$$a_n = \sum_{i=1}^{k} c_i a_{n-i} + F(n)$$

- Existence and uniqueness: given k initial terms
- General solutions: $z_n = x_n + y_n$
- Particular solutions: if $F(n) = (f_l n^l + \cdots + f_1 n + f_0) s^n$ and s is a root of multiplicity m, $x_n = (p_l n^l + \cdots + p_1 n + p_0) s^n n^m$

Generating Functions

DEFINITION: The **generating function** of a sequence $\{a_r\}_{r=0}^{\infty}$ is defined as $G(x) = \sum_{r=0}^{\infty} a_r x^r$.

形式

- Generating functions are **formal power series**. 幂级数
- We do not discuss their convergence.

EXAMPLE: generating functions of sequences

- $a_r = 3, G(x) = 3(1 + x + \cdots + x^r + \cdots)$ ∞
- $a_r = 2^r, G(x) = 1 + 2x + \cdots + (2x)^r + \cdots$ ∞
- $a_r = \binom{n}{r}, G(x) = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n$ 共(n+1)项

DEFINITION: Let $A(x) = \sum_{r=0}^{\infty} a_r x^r$, $B(x) = \sum_{r=0}^{\infty} b_r x^r$

- $A(x) = B(x)$ if $a_r = b_r$ for all $r = 0,1,2, \ldots$

Operations

DEFINITION: Let $A(x) = \sum_{r=0}^{\infty} a_r x^r$, $B(x) = \sum_{r=0}^{\infty} b_r x^r$

- $A(x) + B(x) = \sum_{r=0}^{\infty}(a_r + b_r)\, x^r$
- $A(x) - B(x) = \sum_{r=0}^{\infty}(a_r - b_r)\, x^r$
- $A(x) \cdot B(x) = \sum_{r=0}^{\infty}(\sum_{j=0}^{r} a_j b_{r-j})\, x^r$
- $\lambda \cdot A(x) = \sum_{r=0}^{\infty} \lambda a_r x^r$ for any constant $\lambda \in \mathbb{R}$
- We say that $B(x)$ is an **inverse** of $A(x)$ if $A(x)B(x) = 1$.
    - The inverse of $A(x)$: $A^{-1}(x)$
    - When $A(x)$ has an inverse, define $\frac{C(x)}{A(x)} = A^{-1}(x) \cdot C(x)$

Operations

THEOREM: $A(x) = \sum_{r=0}^{\infty} a_r x^r$ has an inverse iff $a_0 \neq 0$.

EXAMPLE: Let $A(x) = \sum_{r=0}^{\infty} x^r$. Find $A^{-1}(x)$.

- $a_0 = 1 \neq 0$: $A^{-1}(x)$ exists
- Denote $A^{-1}(x) = \sum_{r=0}^{\infty} b_r x^r$; $b_0, b_1, \ldots$ are undetermined coefficients
- $A(x)A^{-1}(x) = 1$:
  - $(1 + x + x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) = 1 + 0 \cdot x + 0 \cdot x^2 + \cdots$
    - Coefficient of x^0: $b_0 = 1$
    - Coefficient of x^1: $b_1 + b_0 = 0$
    - Coefficient of x^2: $b_2 + b_1 + b_0 = 0$
    - Coefficient of x^r: $b_r + b_{r-1} + \cdots + b_0 = 0$
      - $b_1 = -1, b_2 = 0, \ldots, b_r = 0$
      - $A^{-1}(x) = 1 - x$

Operations

DEFINITION: $A(x) = \sum_{r=0}^{\infty} a_r x^r$

- $A'(x) = \sum_{r=1}^{\infty} r a_r x^{r-1}$
  - $A^{(0)}(x) = A(x)$
  - $A^{(k)}(x) = (A^{(k-1)}(x))'$ for all integers $k \geq 1$
- $\int A(x)\mathrm{d}x = \sum_{r=0}^{\infty} \frac{1}{r+1} a_r x^{r+1} + C$, where C is a constant

THEOREM: Let $A(x) = \sum_{r=0}^{\infty} a_r x^r$ and $B(x) = \sum_{r=0}^{\infty} b_r x^r$.

- $\left(\alpha A(x) + \beta B(x)\right)' = \alpha A'(x) + \beta B'(x)$
- $\left(A(x)B(x)\right)' = A'(x)B(x) + A(x)B'(x)$
- $\left(A^k(x)\right)' = kA^{k-1}(x)\, A'(x)$

$$(1+\alpha x)^u$$

R

DEFINITION: Let $u \in \mathbb{R}$ and $r \in \mathbb{N}$. The **extended binomial**

扩展二项式系数

coefficient $\binom{u}{r} = \begin{cases} u(u-1)\cdots(u-r+1)/r! & r>0 \\ 1 & r=0 \end{cases}$

THEOREM: Let x be a real number with $|x|<1$ and let u be a real number. Then $(1+x)^u = \sum_{r=0}^{\infty} \binom{u}{r} x^r$.

EXAMPLE:

- $(1-\alpha x)^{-1} = \sum_{r=0}^{\infty} \alpha^r x^r$
- $(1-\alpha x)^{-n} = \sum_{r=0}^{\infty} \binom{r+n-1}{r} \alpha^r x^r$

Counting Combinations with GFs

1. 连续 ⇒ 用 $(1+\alpha x)^n$ 展开

QUESTION: Let $n > 0, R_1, \dots, R_n \subseteq \mathbb{N}$. For every $r \geq 0$, let a_r be the number of r-combinations of $[n]$ with repetition where every $i \in [n]$ appears R_i times.

- $a_r = |\{(r_1, \dots, r_n): r_1 \in R_1, \dots, r_n \in R_n, r_1 + \cdots + r_n = r\}|$
  - This is also the number of ways of distributing r unlabeled objects into n labeled boxes such that R_i objects are sent to box i

THEOREM: $\sum_{r=0}^{\infty} a_r\, x^r = \prod_{i=1}^{n} \sum_{r_i \in R_i} x^{r_i}$.

- $\prod_{i=1}^{n} \sum_{r_i \in R_i} x^{r_i} = \sum_{r_1 \in R_1} x^{r_1} \cdot \sum_{r_2 \in R_2} x^{r_2} \cdots \sum_{r_n \in R_n} x^{r_n}$

$$= \sum_{r=0}^{\infty} \Big(\sum_{r_1 \in R_1, \dots, r_n \in R_n, r_1 + \cdots + r_n = r} 1 \Big) x^r$$

$$= \sum_{r=0}^{\infty} a_r x^r$$

Counting Combinations with GFs

EXAMPLE: Let a_r be the number of ***r*-combinations of [*n*] without repetition**. Determine a_r.

- $a_r =$ the number of r-combinations of the multiset $\{1 \cdot 1, 1 \cdot 2, \dots, 1 \cdot n\}$
- $a_r =$ the number of r-subsets of the multiset $\{1 \cdot 1, 1 \cdot 2, \dots, 1 \cdot n\}$
- $a_r = |\{(r_1, \dots, r_n) : r_1, \dots, r_n \in \{0,1\}, r_1 + \cdots + r_n = r\}|$
  - $R_1 = \{0,1\}, R_2 = \{0,1\}, \dots, R_n = \{0,1\}$
    - $\sum_{r=0}^{\infty} a_r x^r = \prod_{i=1}^{n} \sum_{r_i \in R_i} x^{r_i}$
      $= (1+x)^n$
      - $a_r =$ the coefficient of x^r in the expansion of $(1+x)^n$
        - We know that $(1+\alpha x)^u = \sum_{r=0}^{\infty} \binom{u}{r} \alpha^r x^r$
        - $(1+x)^n = \sum_{r=0}^{\infty} \binom{n}{r} x^r$
          - $a_r = \binom{n}{r}$

Counting Combinations with GFs

EXAMPLE: Let a_r be the number of ***r*-combinations of** $[\boldsymbol{n}]$ **with repetition**. Determine a_r.

- $a_r =$ the number of r-combinations of the multiset $\{\infty \cdot 1, \infty \cdot 2, \dots, \infty \cdot n\}$
- $a_r =$ the number of r-subsets of the multiset $\{\infty \cdot 1, \infty \cdot 2, \dots, \infty \cdot n\}$
- $a_r = |\{(r_1, \dots, r_n): r_1, \dots, r_n \geq 0, r_1 + \cdots + r_n = r\}|$
  - $R_1 = \{0,1, \dots\}, R_2 = \{0,1, \dots\}, \dots, R_n = \{0,1, \dots\}$
    - $\sum_{r=0}^{\infty} a_r x^r = \prod_{i=1}^{n} \sum_{r_i \in R_i} x^{r_i}$
      $= (1 + x + \cdots)^n$
      $= (1 - x)^{-n}$
      - $a_r =$ the coefficient of x^r in the expansion of $(1 - x)^{-n}$
        - We know that $(1 + \alpha x)^u = \sum_{r=0}^{\infty} \binom{u}{r} \alpha^r x^r$
        - $(1 - x)^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} (-1)^r x^r$
          - $a_r = \binom{-n}{r} (-1)^r = \binom{n + r - 1}{r}$

Counting Combinations with GFs

EXAMPLE: Let a_r be the number of ways of distributing r identical books to 5 persons such that person 1, 2, 3, and 4 receive $\geq 3, \geq 2, \geq 4, \geq 6$ books, respectively. Calculate a_{20}.

- $a_r = |\{(r_1, \dots, r_5): r_1 \geq 3, r_2 \geq 2, r_3 \geq 4, r_4 \geq 6, r_5 \geq 0, r_1 + \cdots + r_5 = r\}|$
  - $R_1 = \{3,4,\dots\}; R_2 = \{2,3,\dots\}; R_3 = \{4,5,\dots\};$
    $R_4 = \{6,7,\dots\}; R_5 = \{0,1,2\dots\}$
  - $\sum_{r=0}^{\infty} a_r x^r = \prod_{i=1}^{5} \sum_{r_i \in R_i} x^{r_i}$

$$= (x^3 + \cdots)(x^2 + \cdots)(x^4 + \cdots)(x^6 + \cdots)(1 + x + \cdots)$$

$$= \frac{x^3}{1-x}\frac{x^2}{1-x}\frac{x^4}{1-x}\frac{x^6}{1-x}\frac{1}{1-x} = \frac{x^{15}}{(1-x)^5}$$

$$= x^{15} \sum_{r=0}^{\infty} \binom{-5}{r} (-1)^r x^r$$

- $a_{20} = \binom{-5}{5} (-1)^5 = 126$

Counting Permutations with GFs

2. 跳跃(奇、偶) ⇒ e^x 展开

QUESTION: Let $n > 0, R_1, \dots, R_n \subseteq \mathbb{N}$. For every $r \geq 0$, let a_r be the number of r-permutations of $[n]$ with repetition where every $i \in [n]$ appears R_i times.

- $a_r = \sum_{r_1 \in R_1, r_2 \in R_2, \dots, r_n \in R_n,\ r_1 + r_2 + \cdots + r_n = r} \frac{r!}{r_1! r_2! \cdots r_n!}$
  - This is the number of ways of distributing r labeled objects into n labeled boxes such that R_i objects are sent to box i for all $i \in [n]$

THEOREM: $\sum_{r=0}^{\infty} \frac{a_r}{r!} x^r = \prod_{i=1}^{n} \sum_{r_i \in R_i} \frac{x^{r_i}}{r_i!}$.

- $\prod_{i=1}^{n} \sum_{r_i \in R_i} \frac{x^{r_i}}{r_i!} = \sum_{r_1 \in R_1} \frac{x^{r_1}}{r_1!} \cdot \sum_{r_2 \in R_2} \frac{x^{r_2}}{r_2!} \cdots \sum_{r_n \in R_n} \frac{x^{r_n}}{r_n!}$

$$= \sum_{r=0}^{\infty} \left(\sum_{r_1 \in R_1, r_2 \in R_2, \dots, r_n \in R_n,\ r_1 + r_2 + \cdots + r_n = r} \frac{r!}{r_1! r_2! \cdots r_n!} \right) \frac{x^r}{r!}$$

$$= \sum_{r=0}^{\infty} \frac{a_r}{r!} x^r$$

Counting Permutations with GFs

EXAMPLE: Let a_r be the number of ***r*-permutations of** **[*n*] without repetition**. Determine a_r.

- a_r = the number of r-permutations of $[n]$ with repetition where every $i \in [n]$ appears 0 or 1 time.
  - $R_1 = \{0,1\}, R_2 = \{0,1\}, \dots, R_n = \{0,1\}$
  - $\sum_{r=0}^{\infty} \frac{a_r}{r!} x^r = \prod_{i=1}^{n} \sum_{r_i \in \{0,1\}} \frac{x^{r_i}}{r_i!}$
    $= \prod_{i=1}^{n} (1 + x)$
    $= (1 + x)^n$
    $= \sum_{r=0}^{\infty} \binom{n}{r} x^r$
    - $\frac{a_r}{r!} = \binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}$
      - $a_r = n(n-1)\cdots(n-r+1)$

Counting Permutations with GFs

EXAMPLE: Let a_r be the number of $\boldsymbol{r}$**-permutations of** $[\boldsymbol{n}]$ **with repetition**. Determine a_r.

- $a_r =$ the number of r-permutations of $[n]$ with repetition where every $i \in [n]$ appears ≥ 0 times.
  - $R_1 = \{0,1,\ldots\}, R_2 = \{0,1,\ldots\}, \ldots, R_n = \{0,1,\ldots\}$
  - $\sum_{r=0}^{\infty} \frac{a_r}{r!} x^r = \prod_{i=1}^{n} \sum_{r_i \in \{0,1,\ldots\}} \frac{x^{r_i}}{r_i!}$

    $= \prod_{i=1}^{n} \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)$

    $= e^{nx}$

    $= \sum_{r=0}^{\infty} \frac{1}{r!} (nx)^r$

    - $\frac{a_r}{r!} = \frac{1}{r!} n^r$
      - $a_r = n^r$

Counting Permutations with GFs

EXAMPLE: Show that $S_2(r,n) = \frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^r$

- We proved that $S_2(n,j) = \frac{1}{j!}\sum_{i=0}^{j}(-1)^i\binom{j}{i}(j-i)^n$ in lecture 9.
- $S_2(r,n) =$ the number of ways of distributing r different objects into n identical boxes such that ≥ 1 objects are sent to box i for all $i \in [n]$.
- Let a_r be the number of ways of distributing r different objects into n different boxes such that ≥ 1 objects are sent to box i for all $i \in [n]$.
- $a_r = S_2(r,n) \cdot n!$
- $R_1 = R_2 = \cdots = R_n = \{1,2,\ldots\}$
- $\sum_{r=0}^{\infty}\frac{a_r}{r!}x^r = \prod_{i=1}^{n}\sum_{r_i\in R_i}\frac{x^{r_i}}{r_i!}$

$$= \prod_{i=1}^{n}\left(\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)$$
$$= \prod_{i=1}^{n}(e^x-1)$$
$$= (e^x-1)^n$$
$$= \sum_{i=0}^{n}(-1)^i\binom{n}{i}e^{(n-i)x}$$
$$= \sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{r=0}^{\infty}\frac{(n-i)^r}{r!}x^r$$
$$= \sum_{r=0}^{\infty}\frac{1}{r!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^r x^r$$

Counting Permutations with GFs

EXAMPLE: Find $a_r = \{s \in \{1,2,3,4\}^r : s \text{ has an even number of } 1s\}$

- a_r = the number of r-permutations of $\{1,2,3,4\}$ with repetition where 1 appears an even number of times
- $R_1 = \{0,2,4,\dots\}, R_2 = R_3 = R_4 = \{0,1,2,\dots\}$
  - $$\sum_{r=0}^{\infty} \frac{a_r}{r!} x^r = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)^3$$
    $$= \frac{e^x + e^{-x}}{2} \cdot e^{3x}$$
    $$= \frac{e^{4x} + e^{2x}}{2}$$
    $$= \frac{1}{2} \sum_{r=0}^{\infty} \left(\frac{(4x)^r}{r!} + \frac{(2x)^r}{r!}\right)$$
    - $$\frac{a_r}{r!} = \frac{1}{2} \cdot \left(\frac{4^r}{r!} + \frac{2^r}{r!}\right)$$
      - $$a_r = \frac{4^r + 2^r}{2}$$

Counting Integer Partitions with GFs

THEOREM: Let $p_k(r)$ be the number of **partitions of *r*** that has k parts. Then $\sum_{r=0}^{\infty} p_k(r)\, x^r = x^k \prod_{j=1}^{k} \frac{1}{1-x^j}$.

- $p_k(r) = p_{k-1}(r-1) + p_k(r-k) \quad (r \geq k)$ // contain 1; do not contain 1
- $$\begin{aligned} Q_k(x) = \sum_{r=0}^{\infty} p_k(r)\, x^r &= \sum_{r=0}^{\infty} \big(p_{k-1}(r-1) + p_k(r-k)\big)\, x^r \\ &= \sum_{r=k}^{\infty} \big(p_{k-1}(r-1) + p_k(r-k)\big)\, x^r \\ &= \sum_{r=k}^{\infty} p_{k-1}(r-1)\, x^r + \sum_{r=k}^{\infty} p_k(r-k)\, x^r \\ &= xQ_{k-1}(x) + x^k Q_k(x) \end{aligned}$$
- $$\begin{aligned} Q_k(x) = \frac{x}{1-x^k} Q_{k-1}(x) = \cdots &= \frac{x^{k-1}}{(1-x^k)\cdots(1-x^2)} Q_1(x) \\ &= \frac{x^{k-1}}{(1-x^k)\cdots(1-x^2)} \frac{x}{1-x} \\ &= x^k \prod_{j=1}^{k} \frac{1}{1-x^j} \end{aligned}$$

部分分式分解

Partial Fraction Decomposition

LEMMA: Let $Q(x), P(x)$ be two polynomials s.t. $\deg(Q) > \deg(P)$. If $Q(x) = (1 - r_1 x)^{m_1} \cdots (1 - r_t x)^{m_t}$ for distinct non-zero numbers $r_1, \dots, r_t$ and integers $m_1, \dots, m_t \geq 1$, then there exist unique coefficients $\{\alpha_{j,u}: j \in [t], u \in [m_j]\}$ such that

$$\frac{P(x)}{Q(x)} = \sum_{j=1}^{t} \sum_{u=1}^{m_j} \frac{\alpha_{j,u}}{(1-r_j x)^u} .$$

EXAMPLE: Find the partial fraction decomposition of $\frac{1-9x}{1-18x+80x^2}$

- $P(x) = 1 - 9x$; $Q(x) = 1 - 18x + 80x^2$
  - $\deg(P) < \deg(Q)$
- $Q(x) = (1 - 8x)(1 - 10x)$
  - $t = 2$; $r_1 = 8, m_1 = 1$; $r_2 = 10, m_2 = 1$
- By the lemma, there exist constants $\alpha_{1,1}, \alpha_{2,1}$ such that $\frac{P(x)}{Q(x)} = \frac{\alpha_{1,1}}{1-8x} + \frac{\alpha_{2,1}}{1-10x}$
  - $\frac{P(x)}{Q(x)} = \frac{1}{2(1-8x)} + \frac{1}{2(1-10x)}$

Solving LNRR with GFs

EXAMPLE: Solve the LNRR $a_n = 8a_{n-1} + 10^{n-1}$ with the initial condition $a_0 = 1$ using generating function.

- $A(x) = \sum_{n=0}^{\infty} a_n x^n$

  $= 1 + \sum_{n=1}^{\infty}(8a_{n-1} + 10^{n-1})x^n$

  $= 1 + 8xA(x) + \frac{x}{1-10x}$

- $A(x) = \frac{1-9x}{(1-8x)(1-10x)} = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$

  $= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n$

- $a_n = \frac{1}{2}(8^n + 10^n) \quad (n \geq 0)$