Discrete Mathematics
Lecture 14

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Summary of Lecture 13

THEOREM: $S_2(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i \binom{j}{i}(j-i)^n$ when $n \geq j \geq 1$.

Partition of Integers $p_j(n)$: the number of ways of writing n as the sum of j positive integers

- solution to the type 4 problem=$\sum_{j=1}^{k} p_j(n)$

THEOREM: For $n \in \mathbb{Z}^+, j \in [n]$, $p_j(n+j) = \sum_{k=1}^{j} p_k(n)$

Principle of Inclusion–Exclusion:

- $|\bigcup_{i=1}^{n} A_i| = \sum_{t=1}^{n}(-1)^{t-1} \sum_{1 \leq i_1 < \cdots < i_t \leq n} |A_{i_1} \cap \cdots \cap A_{i_t}|$
- $|\bigcap_{i=1}^{n} A_i| = \sum_{t=1}^{n}(-1)^{t-1} \sum_{1 \leq i_1 < \cdots < i_t \leq n} |A_{i_1} \cup \cdots \cup A_{i_t}|$

Pigeonhole Principle (general form): $\{A_1, A_2, \ldots, A_n\}$ is a cover of A and $|A| \geq N \Rightarrow \exists k \in [n], |A_k| \geq \lceil N/n \rceil$.

- $N = n + 1$: $|A_k| \geq 2$ (simple form)

Pigeonhole Principle

EXAMPLE: Connect 15 workstations $W_1, \dots, W_{15}$ to 10 servers $S_1, \dots, S_{10}$ such that any ≥ 10 workstations have access to all servers. How many cables are needed?

- **Solution 2**: S_i is connected to W_i for every $i \in [10]$; and each of $W_{11}, W_{12}, W_{13}, W_{14}, W_{15}$ is connected to all servers. // 60 lines, optimal?
- Consider an optimal scheme Π.
  - Let $A = \{(W_i, S_j): i \in [15], j \in [10], W_i \text{ is not connected to } S_j\}$ in Π
  - $A_t = \{(W_i, S_j) \in A: j = t\}$ for $t = 1, 2, \dots, 10$
    - $\{A_1, A_2, \dots, A_{10}\}$ is a cover of A

$A_i \cap A_j = \emptyset$

$\bigcup_{i=1}^{10} A_i = A$

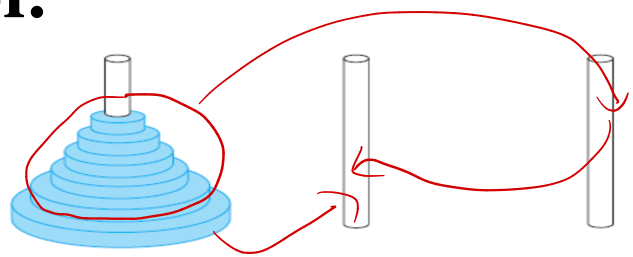
- If there are < 60 lines in Π, then $|A| > 150 - 60 = 90$.
  - $\exists k \in [10]$ such that $|A_k| \geq \lceil 91/10 \rceil = 10$
    - There are 10 workstations not connected to S_k

Recurrence Relation (RR)

递推关系

Fibonacci Sequence: The solution is a sequence $\{f_n\}_{n\geq 0}$ such that $f_0 = 1, f_1 = 1,\ \ f_n = f_{n-1} + f_{n-2}$ for every $n \geq 2$

The Tower of Hanoi:



- Every time move only 1 disk from one peg to another peg
- Always place a smaller disk on top of a larger disk
- Move all the disks from peg 1 to peg 2.
  - H_n: the smallest number of moves (n disks).
    - $H_1 = 1, H_2 = 3, H_n = 2H_{n-1} + 1$ for $n \geq 2$

$H_n = H_{n-1} + 1 + H_{n-1}$

QUESTION: f_n =? H_n =? Find explicit formulas.

Linear Homogeneous RR

k阶常系数线性递推关系。

DEFINITION: A **linear homogeneous RR (LHRR) of degree k with constant coefficients** is an RR of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$, where $n \geq k$, $\{c_i\}_{i=1}^{k}$ are constant real numbers, and $c_k \neq 0$.

- **degree k**: every term depends on k terms preceding it
- **constant coefficients:** $c_1, \ldots, c_k$ are independent of n
- **linear:** the right-hand side is a linear combination of $a_1, a_2, \ldots, a_{n-1}$.
- **homogeneous:** every term is a multiple of some a_j.
  - $f_n = f_{n-1} + f_{n-2}, n \geq 2$ LHRR of degree 2 with constant coefficients
  - $H_n = 2H_{n-1} + 1, n \geq 2$ not homomogenous
- $\{x_n\}_{n\geq 0}$ is a **solution** if $x_n = \sum_{i=1}^{k} c_i x_{n-i}$ for all $n \geq k$

Existence and Uniqueness

THEOREM: For any $a_0, a_1, \ldots, a_{k-1}$, $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ has a unique solution $\{x_n\}_{n\geq 0}$ such that $x_i = a_i$ for every $0 \leq i < k$.

- **Existence**:
  - $x_n = a_n$ for all $0 \leq n < k$
  - $x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k}$ for all $n \geq k$
- **Uniqueness**:
  a) $x'_n = a_n$ for all $0 \leq n < k$
  b) $x'_n = c_1\, x'_{n-1} + c_2 x'_{n-2} + \cdots + c_k x'_{n-k}$ $(n \geq k)$
  c) $x_n = a_n$ for all $0 \leq n < k$
  d) $x_n = c_1\, x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k}$ $(n \geq k)$
    - a) + c) $\Rightarrow x'_n = x_n$ for all $0 \leq n < k$
    - b) + d) $\Rightarrow x'_n = x_n$ for all $n \geq k$

Characteristic Roots

THEOREM: $\{r^n\}_{n\geq 0}$ is a solution of the LHRR $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ if and only if $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k\, r^{n-k}$.

特征方程

- **characteristic equation**: $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$

特征根

- **characteristic roots**: solutions of the characteristic equation.

EXAMPLE: Solve the LHRR $f_n = f_{n-1} + f_{n-2}, n \geq 2$.

- characteristic equation: $r^2 - r - 1 = 0$ 无限制初始条件时.
- characteristic roots: $r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$ 有无数组解
  - $\{r_1^n\}_{n\geq 0}$, $\{r_2^n\}_{n\geq 0}$ are solutions

LHRR (no multiple roots)

无重根

THEOREM: If $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ has k distinct characteristic roots $r_1, r_2, \ldots, r_k$, then $\{x_n\}_{n\geq 0}$ is a solution of the LHRR iff $x_n = \sum_{j=1}^{k} \alpha_j r_j^n$ for some constants $\alpha_1, \ldots, \alpha_k$.

EXAMPLE: Solve $f_n = f_{n-1} + f_{n-2}$ with $f_0 = f_1 = 1$.

- Characteristic equation: $r^2 - r - 1 = 0$
- Characteristic roots: $r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$
- $f_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
  - $f_0 = 1 \Rightarrow \alpha_1 * r_1^0 + \alpha_2 * r_2^0 = 1$
  - $f_1 = 1 \Rightarrow \alpha_1 * r_1^1 + \alpha_2 * r_2^1 = 1$
    - $\alpha_1 = \frac{1}{\sqrt{5}} \cdot \frac{1+\sqrt{5}}{2}, \alpha_2 = -\frac{1}{\sqrt{5}} \cdot \frac{1-\sqrt{5}}{2}$
- $f_n = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \quad (n \geq 0)$

LHRR (multiple roots)

THEOREM: If $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ has distinct characteristic roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, then $\{x_n\}_{n\geq 0}$ is a solution of the LHRR iff $x_n = \sum_{j=1}^{t} \left(\sum_{\ell=0}^{m_j-1} \alpha_{j,\ell}\, n^{\ell} \right) r_j^n$ for some constants $\{\alpha_{j,\ell} : j \in [t], 0 \leq \ell < m_j\}$.

$m_1 + \cdots + m_t = k$

重数

EXAMPLE: Solve $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1, a_1 = 6$.

- Characteristic equation: $r^2 - 6r + 9 = 0$
- Characteristic roots: $r_1 = 3$, $m_1 = 2$
- $a_n = \alpha_{1,0} 3^n + \alpha_{1,1} n 3^n$
  - $a_0 = 1 \Rightarrow \alpha_{1,0} * 3^0 + \alpha_{1,1} * 0 * 3^0 = 1$
  - $a_1 = 6 \Rightarrow \alpha_{1,0} * 3^1 + \alpha_{1,1} * 1 * 3^1 = 6$
- $\alpha_{1,0} = 1, \alpha_{1,1} = 1$
- $a_n = 3^n + n3^n = 3^n(n+1)$

Linear Nonhomogeneous RR

k阶常系数非齐次线性递归关系

DEFINITION: A **linear nonhomogeneous RR (LNRR) of degree *k* with constant coefficients** is an RR of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$, where $c_1, c_2, \ldots, c_k$ are constants, $c_k \neq 0$, and $F(n) \neq 0$.

相伴的

- **Associated LHRR**: $a_n = \sum_{i=1}^{k} c_i a_{n-i}$
- $\{x_n\}_{n\geq 0}$ is a **solution** if $x_n = \sum_{i=1}^{k} c_i x_{n-i} + F(n)$ for all $n \geq k$.

EXAMPLE: $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$

- $c_1 = 1, c_2 = 1, F(n) = n^2 + n + 1$
- LNRR of degree 2 with constant coefficients
- associated LHRR: $a_n = a_{n-1} + a_{n-2}$

Existence and Uniqueness

THEOREM: For any $a_0, a_1, \dots, a_{k-1}$, $a_n = \sum_{i=1}^{k} c_i a_{n-i} + F(n)$ has a unique solution $\{x_n\}_{n\geq 0}$ such that $x_n = a_n$ for all $0 \leq n < k$.

- **Existence**:
  - $x_n = a_n$ for all $0 \leq n < k$
  - $x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k} + F(n)$ for all $n \geq k$
- **Uniqueness**:

  a) $x'_n = a_n$ for all $0 \leq n < k$

  b) $x'_n = c_1\, x'_{n-1} + c_2 x'_{n-2} + \cdots + c_k x'_{n-k} + F(n) \;\; (n \geq k)$

  c) $x_n = a_n$ for all $0 \leq n < k$

  d) $x_n = c_1\, x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k} + F(n) \;\; (n \geq k)$

  - a) + c) $\Rightarrow x'_n = x_n$ for all $0 \leq n < k$
  - b) + d) $\Rightarrow x'_n = x_n$ for all $n \geq k$

General Solutions

THEOREM: If $\{x_n\}$ is a solution of $a_n = \sum_{i=1}^{k} c_i a_{n-i} + F(n)$, then $\{z_n\}$ is a solution iff $z_n = x_n + y_n$ for some solution $\{y_n\}$ of the associated LHRR $a_n = \sum_{i=1}^{k} c_i a_{n-i}$.

- $\Leftarrow$: we prove that $z_n = x_n + y_n$ is a solution of the LNRR
  - $x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} + F(n)$
  - $y_n = c_1 y_{n-1} + \cdots + c_k y_{n-k}$
  - $x_n + y_n = c_1(x_{n-1} + y_{n-1}) + \cdots + c_k\,(x_{n-k} + y_{n-k}) + F(n)$
  - $\{x_n + y_n\}$ is a solution of the LNRR
- $\Rightarrow$: we prove that a solution $\{z_n\}$ of the LNRR has the form $z_n = x_n + y_n$
  - $x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} + F(n)$
  - $z_n = c_1 z_{n-1} + \cdots + c_k z_{n-k} + F(n)$
  - Let $y_n = z_n - x_n$. Then $y_n = c_1 y_{n-1} + \cdots + c_k y_{n-k}$
    - $\{y_n\}$ is a solution of the associated LHRR

Particular Solutions

F(n) 为这种形式时，可解得 x_n

THEOREM: Let $a_n = \sum_{i=1}^{k} c_i a_{n-i} + F(n)$ be an LNRR with $F(n) = \left(f_l n^l + \cdots + f_1 n + f_0\right) s^n = f(n) s^n$, where $c_i, f_j \in \mathbb{R}$. Suppose that s is a root of $(r^k - c_1 r^{k-1} - \cdots - c_k)$ with multiplicity m, then the LNRR has a particular solution of the form $x_n = \left(p_l n^l + \cdots + p_1 n + p_0\right) s^n n^m$, where $\{p_j\}$ are undetermined coefficients.

EXAMPLE: Particular solution for $a_n = 4a_{n-1} - 4a_{n-2} + n2^n$.

- Characteristic equation of the associated LHRR: $r^2 - 4r + 4 = 0$
- Characteristic roots: $r_1 = 2$ (with multiplicity $m_1 = 2$)
  - Particular solution: $x_n = (p_1 n + p_0)2^n n^2$

Solving LNRR

EXAMPLE: Solve $a_n = 4a_{n-1} - 4a_{n-2} + n2^n$ with $a_0 = 1, a_1 = 4$.

- Particular solution of the LNRR: $x_n = (p_1 n + p_0)2^n n^2$
- General solution of the associated LHRR: $y_n = \left(\alpha_{1,0} + \alpha_{1,1} n\right)2^n$
- General solution of the LNRR:
  - $z_n = x_n + y_n = \left(\alpha_{1,0} + \alpha_{1,1} n + p_0 n^2 + p_1 n^3\right)2^n$
  - Initial conditions give an equation system:
    - $a_0 = 1$: $\left(\alpha_{1,0} + \alpha_{1,1} \cdot 0 + p_0 \cdot 0^2 + p_1 \cdot 0^3\right)2^0 = 1$
    - $a_1 = 4$: $\left(\alpha_{1,0} + \alpha_{1,1} \cdot 1 + p_0 \cdot 1^2 + p_1 \cdot 1^3\right)2^1 = 4$
    - $a_2 = 20$: $\left(\alpha_{1,0} + \alpha_{1,1} \cdot 2 + p_0 \cdot 2^2 + p_1 \cdot 2^3\right)2^2 = 20$
    - $a_3 = 88$: $\left(\alpha_{1,0} + \alpha_{1,1} \cdot 3 + p_0 \cdot 3^2 + p_1 \cdot 3^3\right)2^3 = 88$

$$\begin{cases} \alpha_{1,0} & & & & = 1 \\ \alpha_{1,0} + & \alpha_{1,1} + & p_1 + & p_0 & = 2 \\ \alpha_{1,0} + & 2\alpha_{1,1} + & 4p_1 + & 8p_0 & = 5 \\ \alpha_{1,0} + & 3\alpha_{1,1} + & 9p_1 + & 27p_0 & = 11 \end{cases}$$

$$\begin{cases} \alpha_{1,0} & & & & = 1 \\ & \alpha_{1,1} + & p_1 + & p_0 & = 1 \\ & 2\alpha_{1,1} + & 4p_1 + & 8p_0 & = 4 \\ & 3\alpha_{1,1} + & 9p_1 + & 27p_0 & = 10 \end{cases}$$

$$\begin{cases} \alpha_{1,0} & & & & = 1 \\ & \alpha_{1,1} + & p_1 + & p_0 & = 1 \\ & & 2p_1 + & 6p_0 & = 2 \\ & & 6p_1 + & 24p_0 & = 7 \end{cases}$$

$$\begin{cases} \alpha_{1,0} & & & & = 1 \\ & \alpha_{1,1} + & p_1 + & p_0 & = 1 \\ & & 2p_1 + & 6p_0 & = 2 \\ & & & 6p_0 & = 1 \end{cases}$$

$$\begin{cases} \alpha_{1,0} & & & & = 1 \\ & \alpha_{1,1} + & p_1 + & p_0 & = 1 \\ & & 2p_1 + & 6p_0 & = 2 \\ & & & p_0 & = \frac{1}{6} \end{cases}$$

$$\begin{cases} \alpha_{1,0} & & & & = 1 \\ & \alpha_{1,1} + & p_1 & & = \frac{5}{6} \\ & & 2p_1 & & = 1 \\ & & & p_0 & = \frac{1}{6} \end{cases}$$

$$\begin{cases} \alpha_{1,0} & & & & = 1 \\ & \alpha_{1,1} + & p_1 & & = \frac{5}{6} \\ & & p_1 & & = \frac{1}{2} \\ & & & p_0 & = \frac{1}{6} \end{cases}$$

$$\begin{cases} \alpha_{1,0} & & & & = 1 \\ & \alpha_{1,1} & & & = \frac{1}{3} \\ & & p_1 & & = \frac{1}{2} \\ & & & p_0 & = \frac{1}{6} \end{cases}$$

- The solution $(\alpha_{1,0}, \alpha_{1,1}, p_0, p_1) = \left(1, \frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)$ gives

$$a_n = \left(1 + \frac{1}{3}n + \frac{1}{2}n^2 + \frac{1}{6}n^3\right)2^n$$

Generating Functions

生成函数

DEFINITION: The **generating function** of a sequence $\{a_r\}_{r=0}^{\infty}$ is defined as $G(x) = \sum_{r=0}^{\infty} a_r x^r$.

- Generating functions are **formal power series**.
- We do not discuss their convergence.

EXAMPLE: generating functions of sequences

- $a_r = 3, G(x) = 3(1 + x + \cdots + x^r + \cdots)$
- $a_r = 2^r, G(x) = 1 + 2x + \cdots + (2x)^r + \cdots$
- $a_r = \binom{n}{r}, G(x) = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n$

DEFINITION: Let $A(x) = \sum_{r=0}^{\infty} a_r x^r$, $B(x) = \sum_{r=0}^{\infty} b_r x^r$

- $A(x) = B(x)$ if $a_r = b_r$ for all $r = 0,1,2, \ldots$

Operations

DEFINITION: Let $A(x) = \sum_{r=0}^{\infty} a_r x^r$, $B(x) = \sum_{r=0}^{\infty} b_r x^r$

- $A(x) + B(x) = \sum_{r=0}^{\infty}(a_r + b_r)\, x^r$
- $A(x) - B(x) = \sum_{r=0}^{\infty}(a_r - b_r)\, x^r$
- $A(x) \cdot B(x) = \sum_{r=0}^{\infty}(\sum_{j=0}^{r} a_j b_{r-j})\, x^r$
- $\lambda \cdot A(x) = \sum_{r=0}^{\infty} \lambda a_r x^r$ for any constant $\lambda \in \mathbb{R}$
- We say that $B(x)$ is an **inverse** of $A(x)$ if $A(x)B(x) = 1$.
  - The inverse of $A(x)$: $A^{-1}(x)$
  - When $A(x)$ has an inverse, define $\frac{C(x)}{A(x)} = A^{-1}(x) \cdot C(x)$

Operations

THEOREM: $A(x) = \sum_{r=0}^{\infty} a_r x^r$ has an inverse iff $a_0 \neq 0$.

EXAMPLE: Let $A(x) = \sum_{r=0}^{\infty} x^r$. Find $A^{-1}(x)$.

- $a_0 = 1 \neq 0$: $A^{-1}(x)$ exists
- Denote $A^{-1}(x) = \sum_{r=0}^{\infty} b_r x^r$; $b_0, b_1, \ldots$ are undetermined coefficients
- $A(x)A^{-1}(x) = 1$:
  - $(1 + x + x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) = 1 + 0 \cdot x + 0 \cdot x^2 + \cdots$
    - Coefficient of x^0: $b_0 = 1$
    - Coefficient of x^1: $b_1 + b_0 = 0$
    - Coefficient of x^2: $b_2 + b_1 + b_0 = 0$
    - Coefficient of x^r: $b_r + b_{r-1} + \cdots + b_0 = 0$
      - $b_1 = -1, b_2 = 0, \ldots, b_r = 0$
      - $A^{-1}(x) = 1 - x$

Operations

DEFINITION: $A(x) = \sum_{r=0}^{\infty} a_r x^r$

- $A'(x) = \sum_{r=1}^{\infty} r a_r x^{r-1}$
  - $A^{(0)}(x) = A(x)$
  - $A^{(k)}(x) = (A^{(k-1)}(x))'$ for all integers $k \geq 1$
- $\int A(x)\mathrm{d}x = \sum_{r=0}^{\infty} \frac{1}{r+1} a_r x^{r+1} + C$, where C is a constant

THEOREM: Let $A(x) = \sum_{r=0}^{\infty} a_r x^r$ and $B(x) = \sum_{r=0}^{\infty} b_r x^r$.

- $\left(\alpha A(x) + \beta B(x)\right)' = \alpha A'(x) + \beta B'(x)$
- $\left(A(x)B(x)\right)' = A'(x)B(x) + A(x)B'(x)$
- $\left(A^k(x)\right)' = kA^{k-1}(x)\, A'(x)$

Extended Binomial Coefficients

DEFINITION: Let $u \in \mathbb{R}$ and $r \in \mathbb{N}$. The **extended binomial coefficient** $\binom{u}{r} = \begin{cases} u(u-1)\cdots(u-r+1)/r! & r > 0 \\ 1 & r = 0 \end{cases}$

THEOREM: Let x be a real number with $|x| < 1$ and let u be a real number. Then $(1+x)^u = \sum_{r=0}^{\infty} \binom{u}{r} x^r$.

EXAMPLE:

- $(1-\alpha x)^{-1} = \sum_{r=0}^{\infty} \alpha^r x^r$
- $(1-\alpha x)^{-n} = \sum_{r=0}^{\infty} \binom{r+n-1}{r} \alpha^r x^r$