Discrete Mathematics
Lecture 13

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Summary of Lecture 12

***r*-Combination of Set** $A = \{a_1, a_2, \dots, a_n\}$

- Without repetition: an r-subset of A;
- With repetition: an r-multiset of the form $\{x_1 \cdot a_1, \dots, x_n \cdot a_n\}$

***r*-Combination of Multiset** $A = \{n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k\}$

- an r-subset of A

Binomial Transform: $b_n = \sum_{k=s}^{n} \binom{n}{k} a_k$

Inverse Binomial Transform: $b_n = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} a_k$

- $a_n = \sum_{k=s}^{n} \binom{n}{k} b_k \Rightarrow b_n = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} a_k \ (n \geq s)$

Distributing Objects into Boxes:

- Labeled/unlabeled objects + labeled/unlabeled box

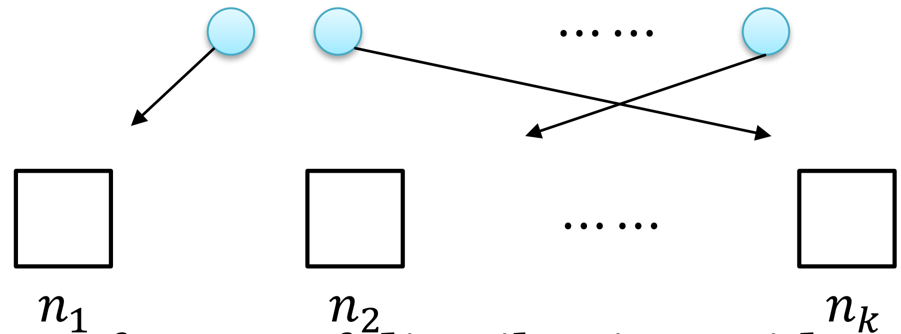
$S_2(n,j)$

THEOREM: $S_2(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i \binom{j}{i}(j-i)^n$ when $n \geq j \geq 1$.

- $T(n,j)$: the number of ways of distributing n labeled objects into j labeled boxes such that no box is empty
  - $T(n,j) = j! \cdot S_2(n,j)$
    - $T(n,j) =?$
- X: the set of ways of distributing n labeled objects into j labeled boxes.
  - By the product rule, $|X| = j^n$
- $X_i \subseteq X$: the set of ways where exactly i boxes are used, $i = 1,2,\ldots,j$
  - $\{X_1, X_2, \ldots, X_j\}$ is a partition of X and $|X_i| = \binom{j}{i} T(n,i)$
  - $j^n = |X| = \sum_{i=1}^{j}|X_i| = \sum_{i=1}^{j}\binom{j}{i} T(n,i)$
  - $T(n,j) = \sum_{i=1}^{j}(-1)^{j-i}\binom{j}{i} i^n = \sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n$ //inversion
- $S_2(n,j) = \frac{1}{j!}\cdot T(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n$

Type 4

Problem: distributing n unlabeled objects into k unlabled boxes



Classifications

$$n_1 + n_2 + \cdots + n_k = n$$
$$n_1, n_2, \ldots, n_k \in \mathbb{N}$$
$$n_1 \geq n_2 \geq \cdots \geq n_k$$

EXAMPLE: # of ways of distributing 4 identical books into 3 identical boxes.

- 4 0 0
- 3 1 0
- 2 2 0
- 2 1 1

REMARK: The schemes are determined by $\{n_1, \ldots, n_k\}$

Partitions of Integers

整数的分拆

DEFINITION: $n = a_1 + a_2 + \cdots + a_j$ is called an ***n*-partition** with exactly j parts if $a_1 \geq a_2 \geq \cdots \geq a_j$ are all positive integers.

- $p_j(n) = |\{(a_1, \ldots, a_j): a_1 + \cdots + a_j = n, a_1 \geq a_2 \geq \cdots \geq a_j \geq 1 \text{ are integers}\}|$
  - $p_j(n)$: # of ways of writing n as the sum of j positive integers.

EXAMPLE: The integer 4 has four different partitions:

- $4 = 4$
- $4 = 3 + 1$
- $4 = 2 + 2$
- $4 = 2 + 1 + 1$

REMARK: solution to the type 4 problem=$\sum_{j=1}^{k} p_j(n)$

Partitions of Integers

THEOREM: For $n \in \mathbb{Z}^+, j \in [n]$, $p_j(n+j) = \sum_{k=1}^{j} p_k(n)$

- $p_1(n) = 1, p_n(n) = 1$
- Let $S_k = \{\text{partitions of } n \text{ into } k \text{ positive integers}\}, k \in [j]$
- Let $S = \bigcup_{k=1}^{j} S_k$.
  - $|S| = |S_1| + \cdots + |S_j| = p_1(n) + \cdots + p_j(n)$
- Let $T = \{\text{partitions of } n+j \text{ into } j \text{ positive integers}\}$
  - $|T| = p_j(n+j)$
- $f: S \to T \quad (n_1, \ldots, n_k) \mapsto (n_1+1, \ldots, n_k+1, \underbrace{1, \ldots, 1}_{j-k})$
  - f is bijective
  - $|T| = |S|$

EXAMPLE: determine $p_3(6)$ and $p_4(6)$ with the above theorem

- $p_3(6) = p_3(3+3) = p_1(3) + p_2(3) + p_3(3) = 1 + 1 + 1 = 3$
- $p_4(6) = p_4(2+4) = p_1(2) + p_2(2) + p_3(2) + p_4(2) = 1 + 1 + 0 + 0 = 2$

Computing $p_j(n)$ Recursively

$$
\begin{array}{ccccccccc}
 & & & & p_1(1) & & & & \\
 & & & p_1(2) & & p_2(2) & & & \\
 & & p_1(3) & & p_2(3) & & p_3(3) & & \\
 & p_1(4) & & p_2(4) & & p_3(4) & & p_4(4) & \\
p_1(5) & & p_2(5) & & p_3(5) & & p_4(5) & & p_5(5)
\end{array}
$$

$p_1(6)$ $p_2(6)$ $p_3(6)$ $p_4(6)$ $p_5(6)$ $p_6(6)$

$p_1(7)$ $p_2(7)$ $p_3(7)$ $p_4(7)$ $p_5(7)$ $p_6(7)$ $p_7(7)$

$p_1(8)$ $p_2(8)$ $p_3(8)$ $p_4(8)$ $p_5(8)$ $p_6(8)$ $p_7(8)$ $p_8(8)$

$p_1(9)$ $p_2(9)$ $p_3(9)$ $p_4(9)$ $p_5(9)$ $p_6(9)$ $p_7(9)$ $p_8(9)$ $p_9(9)$

Principle of Inclusion–Exclusion

容斥原理

Problem: S is a finite set and $A_1, A_2, \ldots, A_n \subseteq S$.

- $|\bigcup_{i=1}^{n} A_i| =?$ $\to \cap$
- $|\bigcap_{i=1}^{n} A_i| =?$ $\to \cup$

EXAMPLE: Let S be the set of permutations of $[n]$. Find $|A|$ for

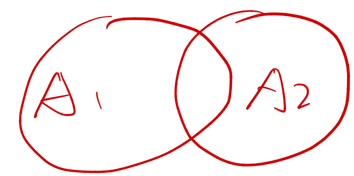
$$A = \{x_1x_2 \ldots x_n \colon x_1x_2 \ldots x_n \in S;\ x_i \neq i \text{ for all } i \in [n]\}.$$

- $A_i = \{x_1x_2 \cdots x_n \colon x_1x_2 \cdots x_n \in S; x_i = i\}, i = 1,2, \ldots, n$ 错排问题
  - $A = S - \bigcup_{i=1}^{n} A_i$
  - $|S| = n!$
  - $|\bigcup_{i=1}^{n} A_i| =?$

Principle of IE (Two Sets)

THEOREM: Let S be a finite set. Let A_1, A_2 be subsets of S. Then

- $|S - A_1| = |S| - |A_1|$; $|A_1 - A_2| = |A_1| - |A_1 \cap A_2|$
  - $S = A_1 \cup (S - A_1\ ),\ A_1 \cap (S - A_1) = \emptyset$;
    - $\{A_1, S - A_1\}$ is a partition of S
      - $|S| = |A_1| + |S - A_1|$
        - $|S - A_1| = |S| - |A_1|$
    - $A_1 - A_2 = A_1 - A_1 \cap A_2$
      - $|A_1 - A_2| = |A_1| - |A_1 \cap A_2|$
- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$



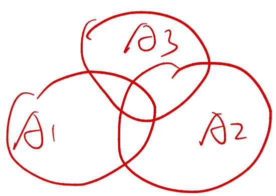
  - $A_1 \cup A_2 = (A_1 - A_2) \cup A_2, (A_1 - A_2) \cap A_2 = \emptyset$;
    - $\{A_1 - A_2, A_2\}$ is a partition of $A_1 \cup A_2$
      - $|A_1 \cup A_2| = |A_1 - A_2| + |A_2| = |A_1| - |A_1 \cap A_2| + |A_2|$
- $|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2|$

Principle of IE (Three Sets)

THEOREM: Let S be a finite set. Let A_1, A_2, A_3 be subsets of S.

Then $\left|\bigcup_{i=1}^{3} A_i\right| = \sum_{t=1}^{3}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le 3} |A_{i_1} \cap \cdots \cap A_{i_t}|$

- $\left|\bigcup_{i=1}^{3} A_i\right| = |(A_1 \cup A_2) \cup A_3| = |A_1 \cup A_2| + |A_3| - |(A_1 \cup A_2) \cap A_3|$
  - $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
  - $|(A_1 \cup A_2) \cap A_3| = |(A_1 \cap A_3) \cup (A_2 \cap A_3)|$



$$= |A_1 \cap A_3| + |A_2 \cap A_3| - |(A_1 \cap A_3) \cap (A_2 \cap A_3)|$$

$$= |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

  - $\left|\bigcup_{i=1}^{3} A_i\right| = |A_1| + |A_2| - |A_1 \cap A_2| + |A_3|$

$= |(A_1 \cup A_2) \cup A_3|$ $-(|A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|)$

- $\left|\bigcap_{i=1}^{3} A_i\right| = \sum_{t=1}^{3}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le 3} |A_{i_1} \cup \cdots \cup A_{i_t}|$

Principle of IE (n Sets)

THEOREM: Let S be a finite set. Let $A_1, A_2, \dots, A_n$ be subsets of S.

Then $|\bigcup_{i=1}^{n} A_i| = \sum_{t=1}^{n}(-1)^{t-1} \sum_{1 \le i_1 < \cdots < i_t \le n} |A_{i_1} \cap \cdots \cap A_{i_t}|$

t组合

- $n = 1$: $|A_1| = |A_1|$
- $n = 2$: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- $n = 3$: $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3|$

  $-|A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

数学归纳法

- **Induction hypothesis**: the identity holds for $n \le k$ ($k \ge 3$)
- Need to show the identity for $n = k + 1$
- $|A_1 \cup \cdots \cup A_{k+1}| = |A_1 \cup \cdots \cup A_k| + |A_{k+1}| - |(A_1 \cup \cdots \cup A_k) \cap A_{k+1}|$

  $= \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\bigcup_{i=1}^{k}(A_i \cap A_{k+1})\right|$

Principle of IE (n Sets)

- $\left|\bigcup_{i=1}^{k} A_i\right| = \sum_{t=1}^{k}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le k} |A_{i_1} \cap \cdots \cap A_{i_t}|$
- $\left|\bigcup_{i=1}^{k}(A_i \cap A_{k+1})\right| = \sum_{t=1}^{k}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le k} |(A_{i_1} \cap A_{k+1}) \cap \cdots \cap (A_{i_t} \cap A_{k+1})|$
- $\left|\bigcup_{i=1}^{k+1} A_i\right| = \sum_{t=1}^{k}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le k} |A_{i_1} \cap \cdots \cap A_{i_t}| + |A_{k+1}| -$

$$\sum_{t=1}^{k}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le k} |(A_{i_1} \cap A_{k+1}) \cap \cdots \cap (A_{i_t} \cap A_{k+1})|$$

$$= \sum_{t=1}^{k+1}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le k+1} |A_{i_1} \cap \cdots \cap A_{i_t}|$$

THEOREM: Let S be a finite set. Let $A_1, A_2, \dots, A_n$ be subsets of S. Then $|\bigcap_{i=1}^{n} A_i| = \sum_{t=1}^{n}(-1)^{t-1} \sum_{1\le i_1<\cdots<i_t\le n} |A_{i_1} \cup \cdots \cup A_{i_t}|$

Principle of Inclusion-Exclusion

错排列

EXAMPLE: Let S be the set of permutations of $[n]$. Find $|A|$ for

$$A = \{x_1x_2 \ldots x_n : x_1x_2 \ldots x_n \in S;\ x_i \neq i \text{ for all } i \in [n]\}.$$

- $A_i = \{x_1x_2 \cdots x_n : x_1x_2 \cdots x_n \in S; x_i = i\}, i = 1,2, \ldots, n$ Ai 第i个位置为i
  - $A = S - \bigcup_{i=1}^{n} A_i$
    - $|S| = n!$
    - $|\bigcup_{i=1}^{n} A_i| = ?$
- $|\bigcup_{i=1}^{n} A_i| = \sum_{t=1}^{n} (-1)^{t-1} \sum_{1 \le i_1 < \cdots < i_t \le n} |A_{i_1} \cap \cdots \cap A_{i_t}|$
  - $\left|A_{i_1} \cap \cdots \cap A_{i_t}\right| = (n-t)!$ for $t = 1,2, \ldots, n$
- $|A| = |S| - |\bigcup_{i=1}^{n} A_i|$

$$= n! - \left(\binom{n}{1} * (n-1)! - \binom{n}{2} * (n-2)! + \cdots + (-1)^{n-1} * \binom{n}{n} * 1\right)$$

$$= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^t \frac{1}{t!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

Pigeonhole Principle

鸽巢原理

工作站 服务器

EXAMPLE: Connect 15 workstations $W_1, \dots, W_{15}$ to 10 servers $S_1, \dots, S_{10}$ such that any ≥ 10 workstations have access to all servers. How many cables are needed?

- **Solution 1**: Connecting every workstation directly to every server. 150
- **Solution 2**: S_i is connected to W_i for every $i \in [10]$; and each of $W_{11}, W_{12}, W_{13}, W_{14}, W_{15}$ is connected to all servers.
  - This solution requires 60 lines.
  - Is this solution optimal?

Cover

覆盖

DEFINITION: A **cover** of a finite set A is a family $\{A_1, A_2, \dots, A_n\}$ of subsets of A such that $\bigcup_{i=1}^{n} A_i = A$.

LEMMA: Let $\{A_1, A_2, \dots, A_n\}$ be a cover of a finite set A. Then $|A| \le \sum_{i=1}^{n} |A_i|$.

- $n = 1$: $|A| = |A_1|$
- $n = 2$: $|A| = |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \le |A_1| + |A_2|$
- Suppose true when $n \le k$ ($k \ge 2$).
- When $n = k + 1$, $|A| = \left|\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right|$

$$\le \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}|$$

$$\le \sum_{i=1}^{k} |A_i| + |A_{k+1}|$$

$$= \sum_{i=1}^{k+1} |A_i|$$

Pigeonhole Principle

THEOREM: (simple form) Let A be a set with $\geq n+1$ elements. Let $\{A_1, A_2, \ldots, A_n\}$ be a cover of A. Then $\exists k \in [n], |A_k| \geq 2$.

- Suppose that $|A_i| \leq 1$ for every $i \in [n]$. Then $n+1 \leq |A| \leq \sum_{i=1}^{n} |A_i| \leq n$.
  - If $\geq n + 1$ objects are distributed into n boxes, then there is at least one box containing ≥ 2 objects.

THEOREM: (general form) Let A be a set with $\geq N$ elements. Let $\{A_1, A_2, \ldots, A_n\}$ be a cover of A. Then $\exists k \in [n], |A_k| \geq \lceil N/n \rceil$.

- If $|A_i| < \lceil N/n \rceil$ for all $i \in [n]$, then $N \leq |A| \leq \sum_{i=1}^{n} |A_i| < n \cdot N/n = N$
  - If we distribute $\geq N$ objects into n boxes, then there is at least one box that contains $\geq \lceil N/n \rceil$ objects.

Pigeonhole Principle

EXAMPLE: Connect 15 workstations $W_1, \ldots, W_{15}$ to 10 servers $S_1, \ldots, S_{10}$ such that any ≥ 10 workstations have access to all servers. How many cables are needed?

- **Solution 2**: S_i is connected to W_i for every $i \in [10]$; and each of $W_{11}, W_{12}, W_{13}, W_{14}, W_{15}$ is connected to all servers. // 60 lines, optimal?
- Consider an optimal scheme Π. 优化方案
  - Let $A = \{(W_i, S_j): i \in [15], j \in [10], W_i \text{ is not connected to } S_j\}$ in Π
  - $A_t = \{(W_i, S_j) \in A: j = t\}$ for $t = 1,2, \ldots, 10$
    - $\{A_1, A_2, \ldots, A_{10}\}$ is a cover of A
- If there are < 60 lines in Π, then $|A| > 150 - 60 = 90$.
  - $\exists k \in [10]$ such that $|A_k| \geq \lceil 91/10 \rceil = 10$
    - There are 10 workstations not connected to S_k