Discrete Mathematics: Lecture 21

predicate logic, WFFs, from NL to WFFs, logic equivalence, tautological implication

Xuming He
Associate Professor

School of Information Science and Technology
ShanghaiTech University

Spring Semester, 2022

Notes by Prof. Liangfeng Zhang

Well-Formed Formulas

DEFINIITON: well-formed formulas合式公式/formulas

1) propositional constants, propositional variables, and propositional functions without connectives are WFFs
2) If A is a WFF, then $\neg A$ is also a WFF
3) If A, B are WFFs and there is no individual variable x which is bound in one of A, B but free in the other, then $(A \wedge B), (A \vee B), (A \rightarrow B), (A \leftrightarrow B)$ are WFFs.
4) If A is a WFF with a free individual variable x, then $\forall x\ A, \exists x\ A$ are WFFs.
5) WFFs can be constructed with 1)-4).
   - Example: $\forall x\ F(x) \vee G(x), \forall x P(y)$ are not WFFs
   - Example: $\exists x\ (A(x) \rightarrow \forall y\ B(x, y))$ is a WFF

Precedence: $\forall, \exists$ have higher precedence than $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

- $\forall x\ P(x) \rightarrow Q(y)$ means $\big(\forall x P(x)\big) \rightarrow Q(y)$, not $\forall x(P(x) \rightarrow Q(y))$

From Natural Language to WFFs

The Method of Translation:

- Introduce symbols to represent propositional constants, propositional variables, individual constants, individual variables, predicate constants, predicate variables, functions of individuals
- Construct WFFs with 1)-4) such that WFFs reflect the real meaning of the natural language

EXAMPLE: All irrational numbers are real numbers.

- Every irrational number is a real number.
- For every x, if x is an irrational number, then x is a real number.
  - $I(x) =$"x is an irrational number"
  - $R(x) =$"x is a real number"
    - Translation: $\forall x\ (I(x) \rightarrow R(x))$

From Natural Language to WFFs

EXAMPLE: Some real numbers are irrational numbers.

- There is a real number which is also an irrational number.
- There is an x such that x is a real number and also an irrational number.
  - $I(x) =$"x is an irrational number"
  - $R(x) =$"x is a real number"
    - Translation: $\exists x\ (R(x) \land I(x))$

EXAMPLE: There is a symbol that can not be understood by any person's brain.

- There is a symbol such that any person's brain can not understand it.
- There is an x such that x is a symbol and any person's brain can not understand x.
  - $S(x)$: "x is a symbol"
  - Translation: $\exists x\ (S(x) \land (\cdots))$

From Natural Language to WFFs

EXAMPLE: There is a symbol that can not be understood by any person's brain.

- Any person's brain can not understand x.
- For any y, if y is a person, then y's brain cannot understand x.
  - $P(y)$: "y is a person"
  - Translation: $\forall y\ (P(y) \rightarrow (\cdots))$
- y's brain cannot understand x
  - $U(z, x)$: "z can understand x"
  - $b(y)$ =the brain of y
    - Translation: $\neg U(b(y), x)$
- Translation: $\exists x\ \left(S(x) \wedge \forall y\ \left(P(y) \rightarrow \neg U(b(y), x)\right)\right)$

Interpretation

DEFINITION: an **interpretation**解释 requires one to (remove all uncertainty)

- assign a concrete proposition to every **proposition variable**
- assign a concrete predicate to every **predicate variable**
- restrict the domain of every **bound individual variable**
- assign a concrete individual to every **free individual variable**
- choose a concrete **function**, if there is any

EXAMPLE: $\exists x P(x) \to q$

- Domain of x={Alice, Bob, Eve}
- $P(x) =$"x gets A+"
- $q =$"I get A+"
- If at least one of Alice, Bob, and Eve gets A+, then I get A+.

Types of WFFs

DEFINITION: A WFF is **logically valid**普遍有效 if it is **T** in every interpretation

- $\forall x\,(P(x) \vee \neg P(x))$ is logically valid

DEFINITION: A WFF is **unsatisfiable**不可满足 if it is **F** in every interpretation

- $\exists x\,(P(x) \wedge \neg P(x))$ is unsatisfiable

DEFINITION: A WFF is **satisfiable**可满足 if it is **T** in some interpretation

- $\forall x\,(x^2 > 0)$
  - true when domain= nonzero real numbers

THEOREM: Let A be any WFF. A is logically valid iff $\neg A$ is unsatisfiable.

<u>**Rule of Substitution**</u>: Let A be a tautology in propositional logic. If we substitute any propositional variable in A with an arbitrary任意的 WFF from predicate logic, then we get a logically valid WFF.

- $p \vee \neg p$ is a tautology; hence, $P(x) \vee \neg P(x)$ is logically valid

Logical Equivalence

DEFINITION: Two WFFs A,B are **logically equivalent**等值 if they always have the same truth value in every interpretation.

- notation: $A \equiv B$; example: $\boxed{\forall x\, P(x) \wedge \forall x\, Q(x) \equiv \forall x\, (P(x) \wedge Q(x)}$)

THEOREM: $A \equiv B$ iff $A \leftrightarrow B$ is logically valid.

- $A \equiv B$
- iff A, B have the same truth value in every interpretation I
- iff $A \leftrightarrow B$ is true in every interpretation I
- iff $A \leftrightarrow B$ is logically valid

THEOREM: $A \equiv B$ iff $A \rightarrow B$ and $B \rightarrow A$ are both logically valid.

- $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$

Rule of Substitution

METHOD*:* Applying the rule of substitution to the logical equivalences in propositional logic, we get logical equivalences in predicate logic.

$P \vee Q \equiv Q \vee P$	$A(x) \vee B(y) \equiv B(y) \vee A(x)$
$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	$\big(A(x) \wedge B(y)\big) \wedge c \equiv A(x) \wedge (B(y) \wedge c)$
$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	$A(x) \wedge (B(y) \vee c) \equiv \big(A(x) \wedge B(y)\big) \vee (A(x) \wedge c)$
$P \wedge (P \vee Q) \equiv P$	$A(x) \wedge \big(A(x) \vee B(y)\big) \equiv A(x)$
$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	$\neg\big(A(x) \wedge B(y)\big) \equiv \neg A(x) \vee \neg B(y)$
$P \rightarrow Q \equiv \neg P \vee Q$	$A(x) \rightarrow (\forall y\, B(y)) \equiv \neg A(x) \vee \big(\forall y\, B(y)\big)$
$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$	$A(x) \leftrightarrow c \equiv (A(x) \rightarrow c) \wedge (c \rightarrow A(x))$

De Morgan's Laws for Quantifiers

THEOREM: $\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$

- <u>Show that $\neg\forall x\ P(x) \rightarrow \exists x\ \neg P(x)$ is logically valid</u>
    - Suppose that $\neg\forall x\ P(x)$ is $\mathbf{T}$ in an interpretation I
    - $\forall x\ P(x)$ is $\mathbf{F}$ in I
    - There is an x_0 such that $P(x_0)$ is $\mathbf{F}$ in I
    - There is an x_0 such that $\neg P(x_0)$ is $\mathbf{T}$ in I
    - $\exists x\ \neg P(x)$ is $\mathbf{T}$ in I
- <u>Show that $\exists x\ \neg P(x) \rightarrow \neg\forall x\ P(x)$ is logically valid</u>
    - Suppose that $\exists x \neg P(x)$ is $\mathbf{T}$ in an interpretation I
    - There is an x_0 such that $\neg P(x_0)$ is $\mathbf{T}$ in I
    - There is an x_0 such that $P(x_0)$ is $\mathbf{F}$ in I
    - $\forall x P(x)$ is $\mathbf{F}$ in I
    - $\neg\forall x P(x)$ is $\mathbf{T}$ in I

THEOREM: $\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$.

De Morgan’s Laws for Quantifiers

EXAMPLE: $R(x)$:“x is a real number”; $Q(x)$:“x is a rational number”

- $\neg\forall x\ (R(x) \rightarrow Q(x))$
  - Not all real numbers are rational numbers
- Negation: $\exists x\ \neg\big(R(x) \rightarrow Q(x)\big) \equiv \exists x\ (R(x) \wedge \neg Q(x))$
  - There is a real number which is not rational

EXAMPLE: Let the domain be the set of all real numbers. Let $Q(x)$:“x is a rational number” and $I(x)$:“x is an irrational number”

- $\neg\exists x\ (Q(x) \wedge I\ (x))$
  - No real number is both rational and irrational.
- Negation: $\forall x\ \neg\big(Q(x) \wedge I(x)\big) \equiv \forall x\ (\neg Q(x) \vee \neg I(x))$
  - Any real number is either not rational or not irrational.

Distributive Laws for Quantifiers

THEOREM: $\forall x\left(P(x) \wedge Q(x)\right) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$

- Show that $\forall x\left(P(x) \wedge Q(x)\right) \rightarrow \forall x\, P(x) \wedge \forall x\, Q(x)$ is logically valid
  - Suppose that $\forall x\left(P(x) \wedge Q(x)\right)$ is $\mathbf{T}$ in an interpretation I
  - $P(x) \wedge Q(x)$ is $\mathbf{T}$ for every x in I
  - $P(x)$ is $\mathbf{T}$ for every x in I and $Q(x)$ is $\mathbf{T}$ for every x in I
  - $\forall x\, P(x)$ is $\mathbf{T}$ in I and $\forall x\, Q(x)$ is $\mathbf{T}$ in I
  - $\forall x\, P(x) \wedge \forall x\, Q(x)$ is $\mathbf{T}$ in I
- Show that $\forall x\, P(x) \wedge \forall x\, Q(x) \rightarrow \forall x\left(P(x) \wedge Q(x)\right)$ is logically valid.
  - Suppose that $\forall x\, P(x) \wedge \forall x\, Q(x)$ is $\mathbf{T}$ in an interpretation I
  - $\forall x\, P(x)$ is $\mathbf{T}$ in I and $\forall x\, Q(x)$ is $\mathbf{T}$ in I
  - $P(x)$ is $\mathbf{T}$ for every x in I and $Q(x)$ is $\mathbf{T}$ for every x in I
  - $P(x) \wedge Q(x)$ is $\mathbf{T}$ for every x in I
  - $\forall x\left(P(x) \wedge Q(x)\right)$ is $\mathbf{T}$ in I

THEOREM: $\exists x\left(P(x) \vee Q(x)\right) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$.

Tautological Implication

DEFINITION: Let A and B be WFFs in predicate logic. A **tautologically implies**(重言蕴涵) B if every interpretation that causes A to be true causes B to be true.

- notation: $A \Rightarrow B$, called a **tautological implication**(重言蕴涵)

THEOREM: $A \Rightarrow B$ iff $A \rightarrow B$ is logically valid.

- $A \Rightarrow B$
- iff every interpretation that causes A to be true causes B to be true
- iff there is no interpretation such that $(A, B) = (\mathbf{T}, \mathbf{F})$
- Iff $A \rightarrow B$ is true in every interpretation
- iff $A \rightarrow B$ is logically valid

THEOREM: $A \Rightarrow B$ iff $A \land \neg B$ is unsatisfiable.

- $A \rightarrow B \equiv \neg A \lor B \equiv \neg(A \land \neg B)$

Rule of Substitution

Name	Tautological Implication	NO.
Conjunction(合取)	$(P) \land (Q) \Rightarrow P \land Q$	1
Simplification(化简)	$P \land Q \Rightarrow P$	2
Addition(附加)	$P \Rightarrow P \lor Q$	3
Modus ponens(假言推理)	$P \land (P \to Q) \Rightarrow Q$	4
Modus tollens(拒取)	$\neg Q \land (P \to Q) \Rightarrow \neg P$	5
Disjunctive syllogism(析取三段论)	$\neg P \land (P \lor Q) \Rightarrow Q$	6
Hypothetical syllogism(假言三段论)	$(P \to Q) \land (Q \to R) \Rightarrow (P \to R)$	7
Resolution (归结)	$(P \lor Q) \land (\neg P \lor R) \Rightarrow Q \lor R$	8

EXAMPLE: $P \land (P \to Q) \Rightarrow Q$ is a TI in propositional logic.

- $A(x) \land \big(A(x) \to B(y)\big) \Rightarrow B(y)$ must be a TI in predicate logic.
  - Rule of substitution: let $P = A(x)$ and $Q = B(y)$

Tautological Implications

- $\forall x P(x) \vee \forall x\, Q(x) \Rightarrow \forall x\, (P(x) \vee Q(x))$
- $\exists x\big(P(x) \wedge Q(x)\big) \Rightarrow \exists x\, P(x) \wedge \exists x Q(x)$
- $\forall x\, \big(P(x) \rightarrow Q(x)\big) \Rightarrow \forall x P(x) \rightarrow \forall x\, Q(x)$
- $\forall x\, \big(P(x) \rightarrow Q(x)\big) \Rightarrow \exists x\, P(x) \rightarrow \exists x\, Q(x)$
- $\forall x\, \big(P(x) \leftrightarrow Q(x)\big) \Rightarrow \forall x\, P(x) \leftrightarrow \forall x\, Q(x)$
- $\forall x\, \big(P(x) \leftrightarrow Q(x)\big) \Rightarrow \exists x\, P(x) \leftrightarrow \exists x\, Q(x)$
- $\forall x\, \big(P(x) \rightarrow Q(x)\big) \wedge \forall x\, \big(Q(x) \rightarrow R(x)\big) \Rightarrow \forall x\, (P(x) \rightarrow R(x))$
- $\forall x\, \big(P(x) \rightarrow Q(x)\big) \wedge P(a) \Rightarrow Q(a)$

Examples

EXAMPLE: $\underline{\forall x \left(P(x) \to Q(x)\right) \land P(a)} \Rightarrow \underline{Q(a)}$

- Suppose that the left hand side is true in an interpretation I (domain=D)
  - $\forall x \left(P(x) \to Q(x)\right)$ is $\mathbf{T}$ and $P(a)$ is $\mathbf{T}$
    - $P(a) \to Q(a)$ is $\mathbf{T}$ and $P(a)$ is $\mathbf{T}$
      - $Q(a)$ is $\mathbf{T}$ in I.

EXAMPLE: Tautological implication in the following proof?

- All rational numbers are real numbers $\boxed{\forall x\left(P(x) \to Q(x)\right)}$
- 1/3 is a rational number $\boxed{P(1/3)}$
- 1/3 is a real number $\boxed{Q(1/3)}$
  - $P(x)$ =“x is a rational number”
  - $Q(x)$ =“x is a real number”
  - rule of inference: $\forall x \left(P(x) \to Q(x)\right) \land P(1/3) \Rightarrow Q(1/3)$

Examples

EXAMPLE: $\underline{\forall x\left(P(x) \to Q(x)\right) \wedge \forall x\left(Q(x) \to R(x)\right)} \Rightarrow \underline{\forall x\,(P(x) \to R(x))}$

- Suppose that the left hand side is $\mathbf{T}$ in an interpretation I (domain=D)
  - $\forall x\left(P(x) \to Q(x)\right)$ is $\mathbf{T}$ and $\forall x\left(Q(x) \to R(x)\right)$ is $\mathbf{T}$
  - $P(x) \to Q(x)$ is $\mathbf{T}$ for all $x \in D$ and $Q(x) \to R(x)$ is $\mathbf{T}$ for all $x \in D$
    - $P(x) \to R(x)$ is $\mathbf{T}$ for all $x \in D$
    - $\forall x\,(P(x) \to R(x))$ is $\mathbf{T}$ in I.

EXAMPLE: Tautological implication in the following proof?

- All integers are rational numbers. $\boxed{\forall x\left(P(x) \to Q(x)\right)}$
- All rational numbers are real numbers. $\boxed{\forall x\left(Q(x) \to R(x)\right)}$
- All integers are real numbers. $\boxed{\forall x\left(P(x) \to R(x)\right)}$
  - $P(x) =$"x is an integer"
  - $Q(x) =$"x is a rational number"
  - $R(x) =$"x is a real number"
  - rule of inference:$\forall x\left(P(x) \to Q(x)\right) \wedge \forall x\left(Q(x) \to R(x)\right) \Rightarrow \forall x\,(P(x) \to R(x))$