SI 120 Discrete Mathematics (Spring 2022), Final Exam

Instructions

- Time: 10:30am–12:30pm (110 minutes 答卷+ 10 minutes 交卷)
- This exam is open-book with the following restriction: you can only bring course ppt slides (printed or on screen) and one A4 cheat sheet.
- You can write your answers in either English or Chinese.
- See the online exam guideline ppt for detailed instructions.

Note: Question Sheet has 4 pages and 10 questions in total.

1 From natural language to formulas (10 pt)

a) Propositional Logic (7 pt):

Denote by A the proposition: "I am hungry", in B, "it is three o'clock now", and in C, "it is time to have dinner". Translate the following statements into formulas in propositional logic (命题逻辑), including A, B, and C:

(1) If it is three o'clock now or I am hungry, then it is time to have dinner.

(2) If I am hungry, then it is time to have dinner. But I am not hungry. So,

either it is not three o'clock now, or it is not time to dinner

Answer:

(1) (3 pt) $B \vee A \ \rightarrow C$.

(2) (4 pt) $\big((A \rightarrow C) \wedge (\neg A)\big) \rightarrow \big((\neg B) \vee (\neg C)\big)$.

b) Predicate Logic (3 pt):

The number A is a limit (极限) of the function $f(x)$ at the point x_0 if and only if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \ \ (|x - x_0| < \delta \rightarrow |f(x) - A| < \varepsilon)$$

With the help of quantifiers (量词) and De Morgan's Laws, write the statement: "the limit of the function $f(x)$ at the point x_0 is not equal to A".

Answer:

$$\exists \epsilon > 0 \forall \delta > 0 \exists x \big((|x - x_0| < \delta) \wedge (|f(x) - A| \geq \varepsilon)\big)$$

2 Propositional formulas (10 pt)

Let p, q and r be propositional (命题) variables. Determine the types of the followingformulas (tautology (重言式), contradiction (矛盾式) or contingency (可能式)). Explain your answers.

(1) $(p \leftrightarrow \neg r) \to (q \leftrightarrow r)$

(2) $\left(\neg(p \leftrightarrow q) \to \left((p \wedge \neg q) \vee (\neg p \wedge q)\right)\right) \vee r$

Answer:

(1) contingency

p	q	r	C
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

(2)tautology

p	q	r	A
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

3 Logic Equivalence (10 pt)

a) Propositional Logic (5 pt):

Let p, q and r be propositional variables, and

$$A = (p \wedge q) \vee (\neg p \wedge q \wedge r)$$

$$B = \left(p \vee (q \wedge r)\right) \wedge \left(q \vee (\neg p \wedge r)\right)$$

Show that $A \equiv B$ using the rule of replacement. (You can use any laws in lec2.)

Answer: 以下为例，答案不唯一

$$
\begin{aligned}
A &= (p \wedge q) \vee (\neg p \wedge q \wedge r) \\
&\Leftrightarrow (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r) \vee (\neg p \wedge q \wedge r) \\
&\Leftrightarrow m_3 \vee m_6 \vee m_7 \qquad \text{（重排序）} \\
B &= (p \vee (q \wedge r)) \wedge (q \vee (\neg p \wedge r)) \\
&\Leftrightarrow (p \wedge q) \vee (p \wedge \neg p \wedge r) \vee (q \wedge r) \vee (\neg p \wedge q \wedge r) \\
&\Leftrightarrow (p \wedge q) \vee (q \wedge r) \vee (\neg p \wedge q \wedge r) \\
&\Leftrightarrow (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r) \vee (\neg p \wedge q \wedge r) \\
&\Leftrightarrow m_3 \vee m_6 \vee m_7 \qquad \text{（重排序）}
\end{aligned}
$$

b) Predicate Logic (5 pt):

Identify two mistakes in the following logic equivalence:

$$
\begin{aligned}
\neg\exists x \forall y \Big(F(x) \wedge \big(G(y) \rightarrow H(x,y)\big)\Big) &\equiv \forall x\, \exists y\, (F(x)\ \wedge (G(y) \rightarrow H(x,y)) \\
&\equiv \forall x \exists y \Big(\big(F(x) \wedge G(y)\big) \rightarrow H(x,y)\Big)
\end{aligned}
$$

Answer:

7. 在演算的第一步中，应用量词否定等值式时丢掉了否定联结词“$\neg$”. 在演算的第二步中，在原错的基础上又用错了等值式，即

$$F(x) \wedge (G(y) \rightarrow H(x,y)) \Leftrightarrow (F(x) \wedge G(y)) \rightarrow H(x,y)$$

4 Rule of Inference in Propositional Logic (10 pt)

Show the following inference is valid using the tautological implications (and the resulting valid argument forms)

$$(\neg p \vee r) \wedge (\neg q \vee s) \wedge (p \wedge q) \Rightarrow (t \rightarrow (r \wedge s))$$

Answer: （答案可能不唯一）

① $p \wedge q$	前提引入
② p	①化简
③ q	①化简
④ $\neg p \vee r$	前提引入
⑤ r	②④析取三段论
⑥ $\neg q \vee s$	前提引入
⑦ s	③⑥析取三段论
⑧ $r \wedge s$	⑤⑦合取
⑨ $\neg t \vee (r \wedge s)$	⑧附加
⑩ $t \rightarrow (r \wedge s)$	⑨置换

5 Predicate Logic (10 pt)

a) Show that $\Big(\big(\forall x F(x) \rightarrow \exists y G(y)\big) \wedge \forall x F(x)\Big) \rightarrow \exists y G(y)$ is logically valid (普遍有效).

Answer:

（2）方法一　用（1）中使用的方法证明.设 I 为任意的解释，D 为 I 的个体域，若 $\forall xF(x)$ 为假，则显然该式的解释为真.若 $\forall xF(x)$ 为真，当 $\exists yG(y)$ 为真时，该式解释的前件、后件均为真，故为真；当 $\exists yG(y)$ 为假时，该式解释的前件、后件均为假，故也为真.因此，该式在任何解释下均为真，故为永真式.

方法二　取 $A=\forall xF(x)$，$B=\exists yG(y)$，则该式是假言推理定律 $(A\to B)\land A\Rightarrow B$ 的代换实例.由主教材中的定理 4.1 可知，该式为永真式.

b) Show that $\neg(\forall x\, F(x) \to \exists y G(y)) \land \exists y G(y)$ is unsatisfiable (不可满足).

Answer:

$\neg(\forall xF(x)\to\exists yG(y))\land\exists yG(y)$ 是 $\neg(A\to B)\land B$ 的代换实例，而

$$\neg(A\to B)\land B \Leftrightarrow A\land\neg B\land B\Leftrightarrow 0$$

6 Graph Basics (10 pt)

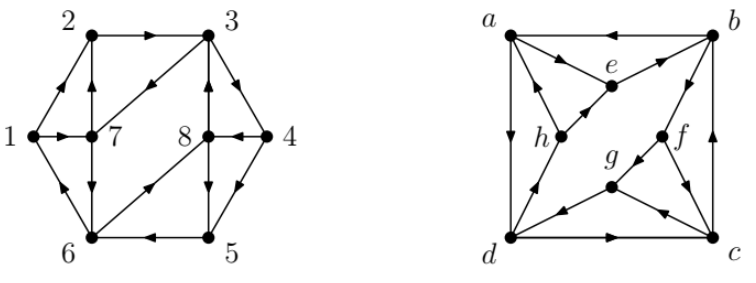
a) Let *G(V,E)* be an undirected graph in which the number of vertices equals the number of edges, i.e., $|V|=|E|$. Assume that it has 2 vertices of degree 2, 2 vertices of degree 3, and the rest of vertices have degree 1. Determine the number of vertices.

Answer:

$$2m = \sum_{i=1}^{n} d(v_i) = 2\times 2 + 3\times 2 + 1\times(n-4) = n + 6 = m + 6$$

解得 $m=6$.

b) Find whether the following digraphs D1 and D2 are isomorphic, and why?

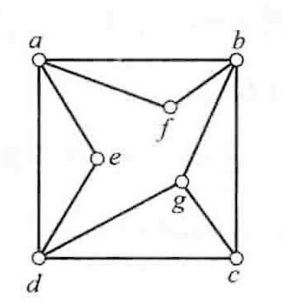


D_1　　D_2

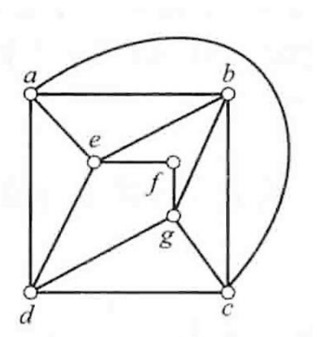
Answer: The presented digraphs are isomorphic. The respective bijection H : V1 → V2 is as follows: H = {(1, h), (2, e), (3, b), (4, f), (5, g), (6, d), (7, a), (8, c)}.

7 Euler Circuit & Connectivity (10 pt)

a) Do the following graphs below admit any Euler path or Euler circuit? If yes, draw one, otherwise, explain why there is no Euler path nor Euler circuit.



G1



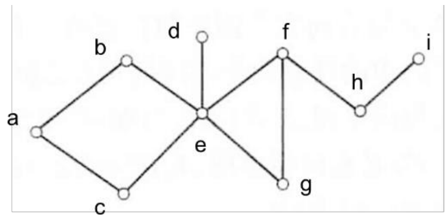
G2

Answer:

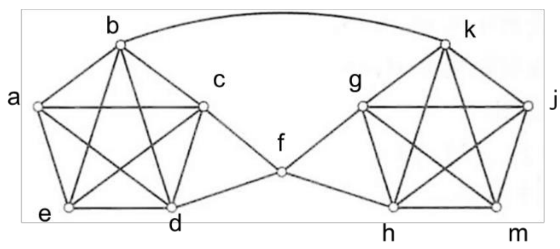
G1: Euler Path: gbfaedgcbadc

G2: Euler Circuit: abcdacgbefgdea

b) Find all the cut vertices and bridges in the following graphs, and compute their κ, λ.



G3



G4

Answer:

G3: $\kappa(G)=\lambda(G)=1.$ Cut vertices: e, f, h; Bridges: de, fh, hi

G4: $\kappa(G)=2, \lambda(G)=3$, Cut vertices and bridges: none

8 Planar Graph & Coloring (10 pt)

a) Let G be a planar simple graph with n=10 vertices, m=8 edges and r=3 regions. Compute the number of connected components p in G.

Answer:

由欧拉公式的推广 $n-m+r=k+1$，解得 $k=n+r-m-1=10+3-8-1=4.$

b) In a school, seven lessons are scheduled for Saturday: mathematics, literature, rhetoric, nature, physical education, drawing, and solfeggio. Each lesson takes exactly one hour. Some lessons can be conducted in parallel, and others, for example, those conducted by the same teacher, should be conducted in series. In Table below with “×” are marked the lessons that cannot be conducted simultaneously.

Find the minimal time required to conduct all the seven lessons on

Saturday.

Subject	Mathematics	Literature	Rhetoric	Nature	Physical education	Drawing	Solfeggio
Mathematics				×	×		
Literature				×			×
Rhetoric					×	×	×
Nature	×	×					
Physical education	×		×			×	×
Drawing			×		×		×
Solfeggio		×	×		×	×	

Answer:

Denote the Saturday disciplines by Latin letters: mathematics—a, literature—b, rhetoric—c, nature—d, physical education—e, drawing—f, solfeggio—g. The set of disciplines $V = \{a, b, c, d, e, f, g\}$.

Construct the graph $G(V, E)$, whose vertices correspond to the lessons, where two vertices of G are adjacent if and only if these lessons cannot be conducted simultaneously (Fig. 5.20).

Then any regular coloring specifies some schedule in the sense that the lessons that have obtained the same colors can be conducted simultaneously. We obtain that the minimal number of hours required to conduct all the lessons is $\chi(G)$.

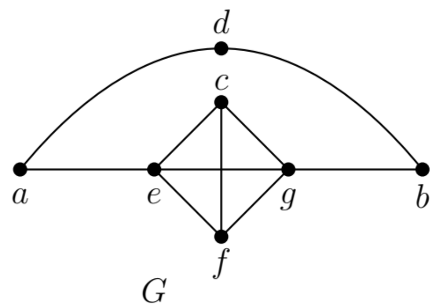


Fig. 5.20 The graph G to Problem 5.29

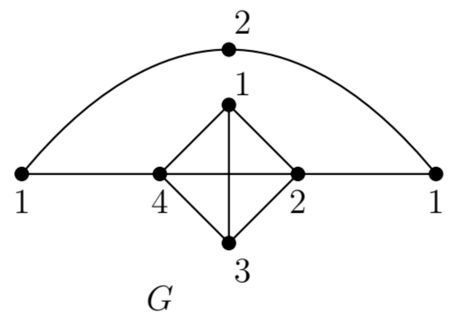


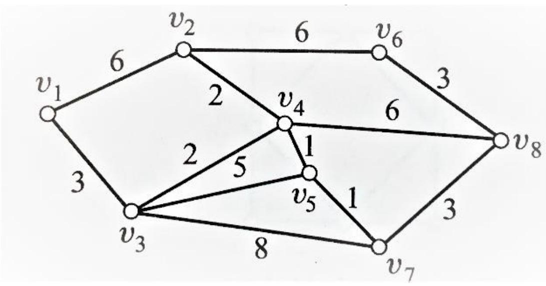
Fig. 5.21 Regular coloring of the graph G (to Problem 5.29)

Since the graph G contains the complete subgraph K_4, then $\chi(G) \geqslant 4$.

Figure 5.21 shows one of the permissible regular colorings. So, the minimal number of hours required to conduct seven lessons on Saturday is four.

9 Shortest Path (10 pt)

For the weighted graph shown in the figure below, use Dijkstra's algorithm to compute the distance $d(v_1, v)$ for every $v \in V$. For each step k of the algorithm write down explicitly the set S_k and the labels $L_k(v)$ for every $v \in V$.



Answer:

顶点	步骤							
	1	2	3	4	5	6	7	8
v_1	$(v_1,0)^*$							
v_2	(v_1,∞)	$(v_1,6)$	$(v_1,6)$	$(v_1,6)^*$				
v_3	(v_1,∞)	$(v_1,3)^*$						
v_4	(v_1,∞)	(v_1,∞)	$(v_3,5)^*$					
v_5	(v_1,∞)	(v_1,∞)	$(v_3,8)$	$(v_4,6)$	$(v_4,6)^*$			
v_6	(v_1,∞)	(v_1,∞)	(v_1,∞)	(v_1,∞)	$(v_2,12)$	$(v_2,12)$	$(v_2,12)$	$(v_2,12)^*$
v_7	(v_1,∞)	(v_1,∞)	$(v_3,11)$	$(v_3,11)$	$(v_3,11)$	$(v_5,7)^*$		
v_8	(v_1,∞)	(v_1,∞)	(v_1,∞)	$(v_4,11)$	$(v_4,11)$	$(v_4,11)$	$(v_7,10)^*$	

v_1 到 v_2 的最短路径：v_1v_2，距离：6.

v_1 到 v_3 的最短路径：v_1v_3，距离：3.

v_1 到 v_4 的最短路径：$v_1v_3v_4$，距离：5.

v_1 到 v_5 的最短路径：$v_1v_3v_4v_5$，距离：6.

v_1 到 v_6 的最短路径：$v_1v_2v_6$，距离：12.

v_1 到 v_7 的最短路径：$v_1v_3v_4v_5v_7$，距离：7.

v_1 到 v_8 的最短路径：$v_1v_3v_4v_5v_7v_8$，距离：10.

10 Tree (10 pt)

a) It is known that the tree T has one vertex of degree 3, six vertices of degree 2, and seven vertices of degree 1. The rest of the vertices are of degree 4. How many vertices of degree 4 does the tree T have?

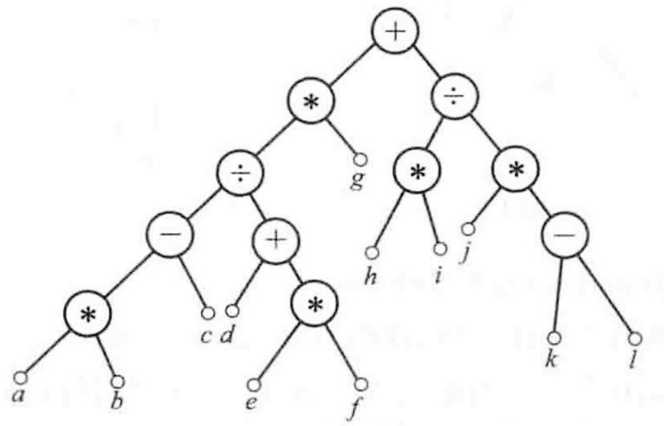
Answer:

Let n be the total number of vertices of the given tree. Then the sum of their degrees is equal to

$$4(n-14)+3\cdot 1+2\cdot 6+1\cdot 7=4n-34.$$

On the other hand, the number of edges of the tree T must be one less than the number of vertices, i. e., the number of edges is $n-1$. Relying upon the handshaking lemma, we obtain the relation: $4n-34=2(n-1)$. Hence, $n=16$, whence we obtain, that the number of vertices of the tree T of degree 4 is two.

b) The tree shown in the figure below represent an arithmetic expression. Use inorder traversal to write out the expression (with parentheses) and also write out its postfix form (reverse Polish notation).



Answer:

$(((a*b-c)\div(d+e*f))*g)+((h*i)\div(j*(k-l)))$

$ab*c-def*+\div g*hi*jkl-*\div+$