

# SI120 Discussion 10

## homework 10

1. (10 points) Let  $P(x)$  = “ $x$  is a person”,  $L(x, y)$  = “ $x$  likes  $y$ ” and  $E(x, y)$  = “ $x = y$ ”. Translate the following statements into formulas:

(a) “Every person likes some other person.”

(b) “There is a person who is liked by every other person.”

$$1. (a) \quad \forall x (P(x) \rightarrow \exists y (P(y) \wedge \neg E(x, y) \wedge L(x, y)))$$

$$(b) \quad \exists x (P(x) \wedge \forall y (P(y) \wedge \neg E(x, y) \rightarrow L(y, x)))$$

一般存在 $\exists$ 后面跟 $\wedge$ ，任意 $\forall$ 后面跟 $\rightarrow$ 。

2. (10 points) Let  $A$  be the formula  $\forall x \left( \forall y \left( (x \neq y) \rightarrow \forall z \left( (z = x) \vee (z = y) \right) \right) \right)$

(a) Find a domain  $D_1 \neq \emptyset$  such that  $A$  is true when  $x, y, z$  are taken over  $D_1$ .

(b) Find a domain  $D_2$  such that  $A$  is false when  $x, y, z$  are taken over  $D_2$ .

$$(a) \quad D_1 = \{0, 1\}$$

$$(b) \quad D_2 = \{0, 1, 2\}$$

3. (10 points) Determine if the following formulas are logically valid, satisfiable or unsatisfiable.

(a)  $(\exists x P(x) \leftrightarrow \exists x Q(x)) \rightarrow \exists x (P(x) \leftrightarrow Q(x))$

(b)  $\exists x (\mathbf{T} \vee P(x) \rightarrow \mathbf{F})$

(c)  $\forall x (P(x) \vee \neg \exists y (Q(y) \wedge \neg Q(y)))$

3. (a) satisfiable

$$D: x \in \mathbb{N}$$

$$T: P(x): x \text{ is an even number}$$

$$Q(x): 2|x$$

$$F: P(x): 2|x \quad Q(x): 2 \nmid x$$

$$\begin{aligned} \exists x P(x) = T & \Rightarrow \exists x P(x) \leftrightarrow \exists x Q(x) \\ \exists x Q(x) = T & \end{aligned}$$

$$\exists x (P(x) \leftrightarrow Q(x)) = F$$

(b) unsatisfiable

$$\exists x (T \vee P(x) \rightarrow F) = \exists x (T \rightarrow F) = \exists x (\neg T \vee F) = \exists x F$$

(c) logically valid

$$\forall x (P(x) \vee \neg \exists y (Q(y) \wedge \neg Q(y))) = \forall x (P(x) \vee \neg \exists y F)$$

$$= \forall x (P(x) \vee \forall y T) = \forall x (\forall y T)$$

4. (20 points) Show the following statements with interpretations of the formulas.

(a)  $\forall x(P(x) \vee Q(x))$  and  $\forall xP(x) \vee \forall xQ(x)$  are not logically equivalent.

(b)  $\exists x(P(x) \wedge Q(x))$  and  $\exists xP(x) \wedge \exists xQ(x)$  are not logically equivalent.

$$4. (a) \quad x \in \mathbb{R} \quad P(x): x \geq 0 \quad Q(x): x < 0$$

$$\forall x (P(x) \vee Q(x)) = \text{True}, \quad \forall x P(x) \vee \forall x Q(x) = \text{False}$$

$$(b) \quad x \in \mathbb{R} \quad P(x): x \geq 0 \quad Q(x): x < 0$$

$$\exists x P(x) \wedge \exists x Q(x) = \text{True}, \quad \exists x (P(x) \wedge Q(x)) = \text{False}$$

5. (10 points) Show that  $\exists x(P(x) \vee Q(x)) \equiv \exists xP(x) \vee \exists xQ(x)$ .

5,  $\Rightarrow$ : Let  $x_0$  satisfy  $P(x_0) \vee Q(x_0)$ ,

Then  $P(x_0)$  is right, or  $Q(x_0)$  is right

When  $P(x_0) = T$ ,  $\exists x P(x) = T$

When  $Q(x_0) = T$ ,  $\exists x Q(x) = T$

So  $\exists x P(x) \vee \exists x Q(x) = T$

$\Leftarrow$ : when  $\exists x P(x)$ , Let  $x_1$  satisfy  $P(x_1)$ ,

so  $x_1$  satisfies  $P(x_1) \vee Q(x_1)$ ,

so  $\exists x (P(x) \vee Q(x)) = T$

when  $\exists x Q(x)$ , same

6. (20 points) Show that  $\forall x (P(x) \rightarrow Q(x)) \Rightarrow \forall x P(x) \rightarrow \forall x Q(x)$ .

$$\forall x P(x) \rightarrow Q(x) = \forall x \neg P(x) \vee Q(x)$$

$$\text{if } \forall x P(x) \rightarrow \forall x Q(x) = F.$$

$$\text{i.e. } \neg(\forall x P(x)) \vee (\forall x Q(x)) = F.$$

$$\text{i.e. } \forall x P(x) = T \text{ and } \forall x Q(x) = F$$

$$\text{for } x_0, P(x_0) = T, Q(x_0) = F$$

$$\Rightarrow \neg P(x_0) \vee Q(x_0) = F. \text{ contradict}$$

7. (20 points) Show that  $\exists x P(x) \wedge \forall x Q(x) \Rightarrow \exists x (P(x) \wedge Q(x))$ .

suppose  $x_0$  satisfy  $P(x_0)$   
as  $\forall x Q(x)$ ,  $Q(x_0) = \top$   
so  $P(x_0) \wedge Q(x_0) = \top$   
so  $\exists x (P(x) \wedge Q(x))$