Discrete Mathematics
Lecture 11

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Summary of Lecture 10

Countable: A is **countable** if $|A| < \infty$ or $|A| = |\mathbb{Z}^+|$

- A is countably infinite $\Leftrightarrow A = \{a_1, a_2, \ldots\}$
- A is countably infinite $\Rightarrow$ so is any infinite subset of A
- A is uncountable $\Rightarrow$ so is any super set of A
- A, B are countably infinite $\Rightarrow$ so are $A \cup B$ and $A \times B$

Schröder-Bernstein: $|A| \leq |B|$ and $|B| \leq |A| \Rightarrow |A| = |B|$

- $|\mathcal{P}(\mathbb{Z}^+)| = |[0,1)|$
- $|\mathbb{Z}^+| < |\mathcal{P}(\mathbb{Z}^+)| = |[0,1)| = |(0,1)| = |\mathbb{R}|$

Basic Rules of Counting: Sum, Product, Bijection

Permutation of Set: r-permutation (w/o repetition)

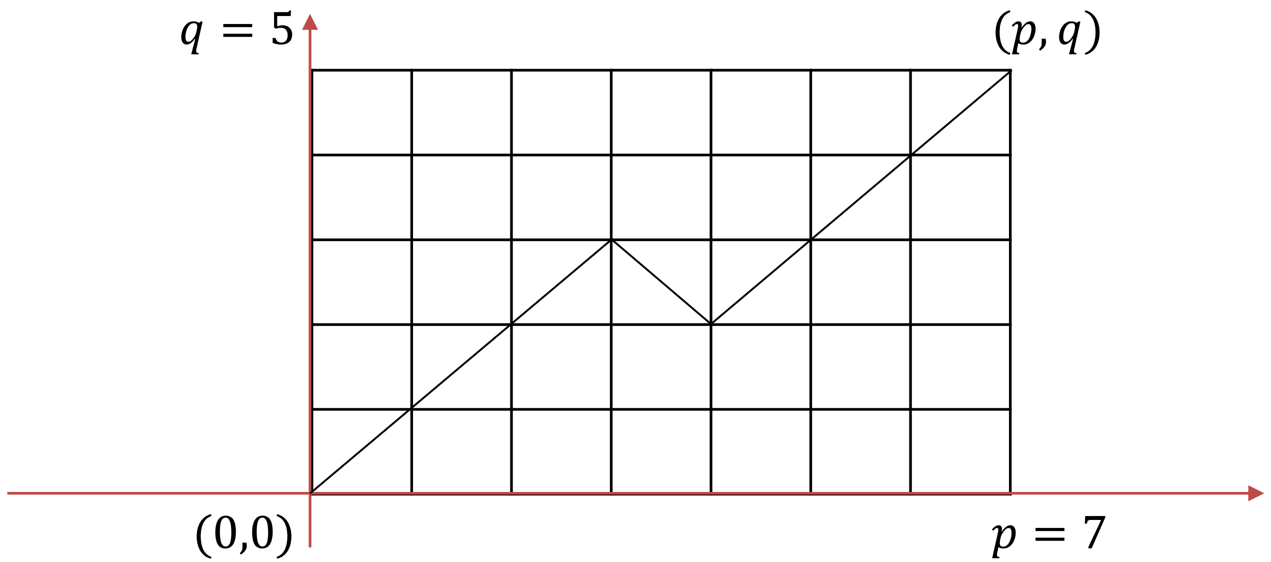
Permutation of Multiset: r-permutation

- $A = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$ has $\frac{(n_1+n_2+\cdots+n_k)!}{n_1!n_2!\cdots n_k!}$ permutations.

T-Route

DEFINITION: Let $A = (x, y), B \in \mathbb{Z}^2$. //**integral points**整点

- A **T-Step** at A is a segment from A to $(x+1, y+1)$ or $(x+1, y-1)$.
- A **T-Route** from A to B is a route where each step is a T-step.



T-Route

THEOREM: There is a T-route from $A = (a, \alpha)$ to $B = (b, \beta)$ only if (1) $b > a$; (2) $b - a \geq |\beta - \alpha|$; and (3) $2|\ (b + \beta - a - \alpha)$.

- Let $A = P_0, P_1, \ldots, P_k = B$ be a T-route from A to B, where $P_i = (x_i, y_i)$.
  - $x_0 = a, y_0 = \alpha; x_k = b, y_k = \beta$; $\overrightarrow{P_{i-1}P_i} = (1,1)/(1,-1)$
  - $x_i - x_{i-1} = 1; y_i - y_{i-1} \in \{\pm 1\}$ for every $i = 1, 2, \ldots, k$
- $b - a = x_k - x_0 = (x_k - x_{k-1}) + (x_{k-1} - x_{k-2}) + \cdots + (x_1 - x_0) = k > 0$
- $\beta - \alpha = y_k - y_0 = (y_k - y_{k-1}) + (y_{k-1} - y_{k-2}) + \cdots + (y_1 - y_0)$
  - $|\beta - \alpha| \leq |y_k - y_{k-1}| + |y_{k-1} - y_{k-2}| + \cdots + |y_1 - y_0| = k = b - a$
- $b + \beta - a - \alpha = \sum_{i=1}^{k} (y_i - y_{i-1} + x_i - x_{i-1})$
  - $y_i - y_{i-1} + x_i - x_{i-1} \in \{0, 2\}$
  - $2|(b + \beta - a - \alpha)$

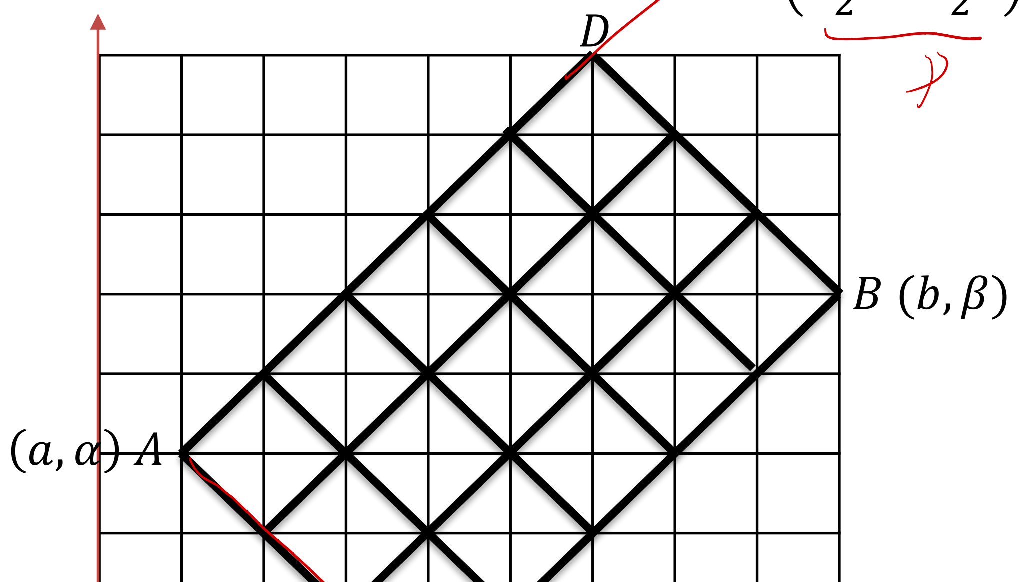
REMARK: The T-condition (1)+(2)+(3) is also sufficient for the existence of a T-route.

Number of T-Routes

THEOREM: If $A = (a, \alpha), B = (b, \beta)$ satisfy the T-condition. Then the number of T-routes from A to B is

$$\frac{\overbrace{(b-a)!}^{p+q}}{\underbrace{\left(\frac{b-a}{2}+\frac{\beta-\alpha}{2}\right)}_{p}!\underbrace{\left(\frac{b-a}{2}-\frac{\beta-\alpha}{2}\right)}_{q}!}.$$

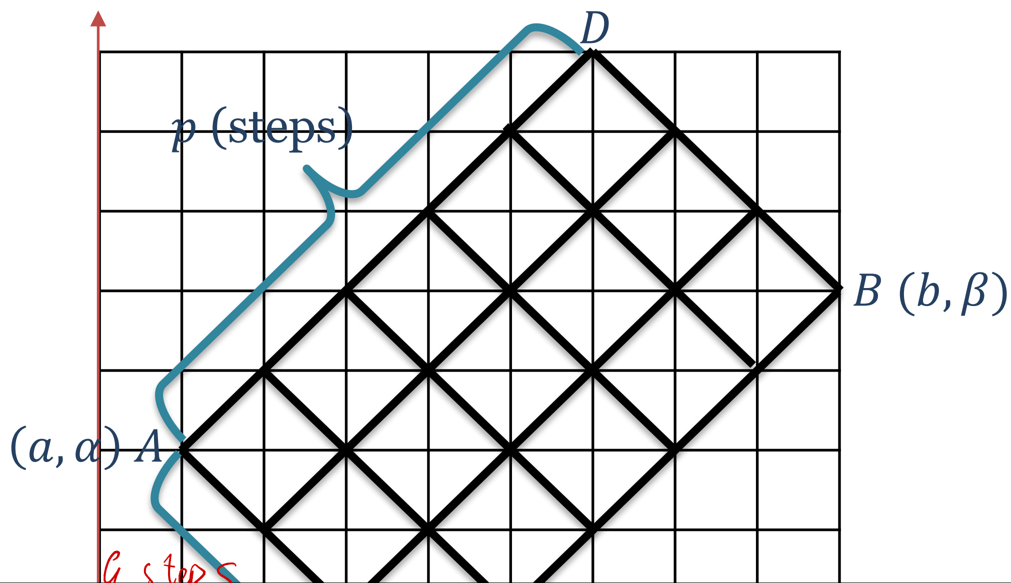
$$= \frac{(p+q)!}{p!\,q!}$$



The number of T routes from A to B = the number of shortest paths from A to B on the $p \times q$-grid.

- $AC: y-\alpha = -(x-a)$; $AD: y-\alpha = x-a$;
- $BC: y-\beta = x-b$; $BD: y-\beta = -(x-b)$.
- $p = \frac{1}{2}\cdot(a+b-\alpha+\beta) - a = \frac{1}{2}\cdot(b-a) + \frac{1}{2}\cdot(\beta-\alpha)$
- $q = \frac{1}{2}\cdot(\alpha-\beta+a+b) - a = \frac{1}{2}\cdot(b-a) - \frac{1}{2}\cdot(\beta-\alpha)$

$1/2\cdot(a+b-\alpha+\beta, \alpha+\beta-a+b)$

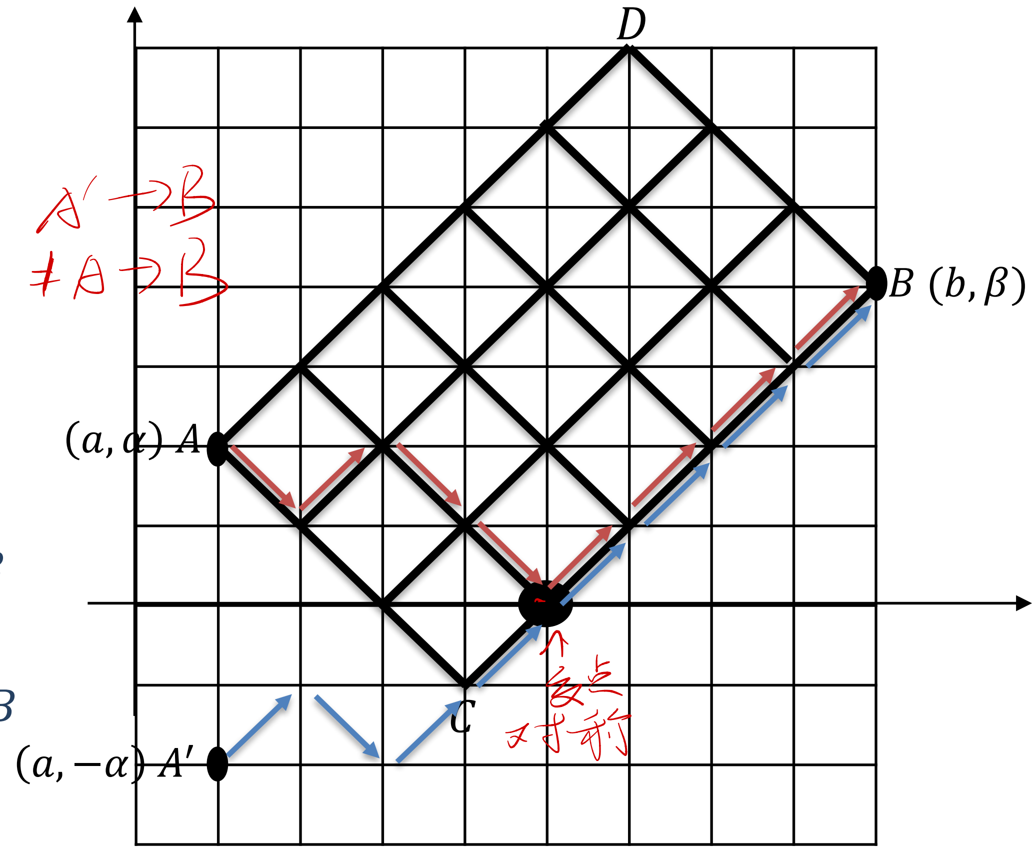


The number of T routes from A to B= the number of shortest paths from A to B on the $p\times q$-grid. This number is $\frac{(p+q)!}{p!q!} = \frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta-\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta-\alpha}{2}\right)!}$

Number of T-Routes

THEOREM: Let $A=(a,\alpha), B=(b,\beta)$ satisfy the T-condition, where $\alpha,\beta>0$. Then # of T-routes from A to B that intersect the x-axis=# of T routes from $A'(a,-\alpha)$ to B. And this number is $\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta+\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta+\alpha}{2}\right)!}$.

- Ω: the set of T-routes from A to B
- $U=\{\omega\in\Omega: \omega \text{ intersects } y=0\}$
- V: the set of T-routes from A' to B
- $f: U\to V \quad u\mapsto f(u)$
  - u: the brown T route
  - $f(u)$: the blue T route
  - f is a bijection



$$|U|=|V|=\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta+\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta+\alpha}{2}\right)!}$$

Number of T Routes

THEOREM: Let $A = (a, \alpha), B = (b, \beta) \in \mathbb{Z}^2$ satisfy the T-condition, where $\alpha, \beta > 0$. Then # of T routes from A to B that do not intersect the x-axis is

$$\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta-\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta-\alpha}{2}\right)!} -$$

A → B

$$\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta+\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta+\alpha}{2}\right)!}$$

A → B 지나고 x

扩号问题

Parenthesization

PROBLEM: Let $a_1, a_2, \dots, a_n, a_{n+1}$ be $n+1$ numbers. Let $*$ be any binary operator. Let C_n be the number of different ways of parenthesizing $a_1 * a_2 * \cdots * a_n * a_{n+1}$ such that the calculation is not ambiguous. What is C_n?

bijection rule ☆

- $n = 3$: there are 5 different ways of parenthesizing the expression

波兰表达式

a_1 a_2 a_3 a_4

$((a_1 * a_2) * a_3) * a_4$	a_1 ↲ a_2 ↲$*$ a_3 ↲$*$ a_4 ↲$*$	↲↲$*$↲$*$↲$*$	0010101
$(a_1 * a_2) * (a_3 * a_4)$	a_1 ↲ a_2 ↲$*$ a_3 ↲ a_4 ↲$**$	↲↲$*$↲↲$**$	0010011
$(a_1 * (a_2 * a_3)) * a_4$	a_1 ↲ a_2 ↲ a_3 ↲$**$ a_4 ↲$*$	↲↲↲$**$↲$*$	0001101
$a_1 * ((a_2 * a_3) * a_4)$	a_1 ↲ a_2 ↲ a_3 ↲$*$ a_4 ↲$**$	↲↲↲$*$↲$**$	0001011
$a_1 * (a_2 * (a_3 * a_4))$	a_1 ↲ a_2 ↲ a_3 ↲ a_4 ↲$***$	↲↲↲↲$***$	0000111

- $\mathcal{A}_3$: the set of all different parenthesizations of $a_1 * a_2 * a_3 * a_4$
- $\mathcal{C}_3$: the set of all $x = x_1x_2x_3x_4x_5x_6x_7 \in \{0,1\}^7$ such that
  - There are exactly three 1's in x
  - In any prefix of x, the number of 1's < the number of 0's

4数3符

逆波兰表达式 (后缀表达式)

前缀

Parenthesization

THEOREM: C_n is the number of solutions of the equation system

$$\begin{cases} x_1 + x_2 + \cdots + x_{2n+1} = n \\ x_1 + x_2 + \cdots + x_i < i/2, i = 1,2, \dots, 2n+1 \\ x_i \in \{0,1\}, i = 1,2, \dots, 2n+1 \end{cases}$$

In particular, $C_n = \frac{(2n)!}{n!(n+1)!}$ 卡特兰数

- $\mathcal{A}_n$: the set of all different parenthesizations of $a_1 * a_2 * \cdots * a_n * a_{n+1}$
- $\mathcal{C}_n$: the set of all $x = x_1 x_2 \cdots x_{2n+1} \in \{0,1\}^{2n+1}$ such that
  - The number of 1's in x is exactly equal to n
  - In any prefix of x, the number of 1's < the number of 0's
- There is a bijection f: $\mathcal{A}_n \to \mathcal{C}_n$
- $C_n = |\mathcal{A}_n| = |\mathcal{C}_n|$
- $\mathcal{C}_n$ is the set of all solutions of the equation system 不恰x轴
- $\mathcal{T}_n$: the set of all T-routes from (1,1) to $(2n+1, 1)$ above the x-axis

Parenthesization

- From $\mathcal{C}_n$ to $\mathcal{T}_n$: Given a solution $(x_1, x_2, \dots, x_{2n+1})$ of the equation system
  - Let $P_i = (i, 1 - 2x_1 + \cdots + 1 - 2x_i)$ for all $i = 1,2, \dots., 2n + 1$
  - $1 - 2x_1 + \cdots + 1 - 2x_i > 0$ for $i = 1,2, \dots 2n + 1$
  - $P_1 = (1,1 - 2x_1) = (1,1)$; $P_{2n+1} = (2n + 1, 1)$
  - $P_1, P_2, \dots, P_{2n+1}$ is a T-route above the x-axis

$P_1, P_2, \cdots P_{2n+1}$ 均在 x 轴上方

- From $\mathcal{T}_n$ to $\mathcal{C}_n$: Let $\{P_i = (u_i, v_i) : 1 \le i \le 2n + 1\}$ be the points on a T-Route from $P_1 = (1,1)$ to $P_{2n+1} = (2n + 1,1)$, where the T-Route is above the x-axis
  - $x_1 = (1 - v_1)/2 = 0$
  - $x_i = (1 - (v_i - v_{i-1}))/2 \in \{0,1\}, i = 2, \dots, 2n + 1$
  - $x_1 + x_2 + \cdots + x_{2n+1} = (2n + 1 - v_{2n+1})/2 = n$
  - $x_1 + x_2 + \cdots + x_i = (i - v_i)/2 < i/2, i = 1,2, \dots, 2n + 1$
- $A = P_1 = (1,1)$: $a = 1, \alpha = 1$; $B = P_{2n-1} = (2n + 1,1)$: $b = 2n + 1, \beta = 1$
  - $|\mathcal{C}_n| = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} = \frac{(2n)!}{n!(n+1)!}$

Combinations of Sets

DEFINITION: Let $A = \{a_1, \dots, a_n\}$ and let $r \in \{0,1,\dots,n\}$.

- ***r*-combination of *A***: an r-subset of A.
  - Notation: $\{a_{i_1}, \dots, a_{i_r}\}$ with $1 \le i_1 < \cdots < i_r \le n$
  - $\binom{n}{r}$: the number of r-combinations of an n-element set

THEOREM: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ for all $n \in \mathbb{Z}^+$ and $r \in \{0,1,\dots,n\}$.

DEFINITION: Let $A = \{a_1, \dots, a_n\}$ and let $r \ge 0$.

- ***r*-combination of *A* with repetition**: a multiset $\{x_1 \cdot a_1, \dots, x_n \cdot a_n\}$ of r elements, where $x_1, \dots, x_n \ge 0$ are integers and $x_1 + \cdots + x_n = r$.
  - Notation: $\{a_{i_1}, \dots, a_{i_r}\}$ with $1 \le i_1 \le i_2 \le \cdots \le i_r \le n$

THEOREM: The number of r-combinations of an n element set with repetition is $\binom{n+r-1}{r}$

Combinations of Sets

- $\mathcal{U}$: the set of all r-combinations of A with repetition
- $\mathcal{V}$: the set of all r-combinations of $[n+r-1]$ without repetition
  - Let $U = \{u_1, u_2, \dots, u_r\} \in \mathcal{U}$ and $1 \le u_1 \le u_2 \le \cdots \le u_r \le n$.
    - $1 \le u_1 < u_2 + 1 < u_3 + 2 < \cdots < u_r + r - 1 \le n + r - 1$
      - $\{u_1, u_2 + 1, \dots, u_r + r - 1\} \in \mathcal{V}$
      - $f\colon \mathcal{U} \to \mathcal{V}\ \ \{u_1, u_2, \dots, u_r\} \mapsto \{u_1, u_2 + 1, \dots, u_r + r - 1\}$
    - f is bijective. Hence, $|\mathcal{U}| = |\mathcal{V}| = \binom{n+r-1}{r}$

THEOREM: The number of natural number solutions of the equation $x_1 + x_2 + \cdots + x_n = r$ is $\binom{n+r-1}{r}$.

- $\mathcal{X} = \{(x_1, \dots, x_n) \colon x_1, \dots, x_n \in \mathbb{N} \text{ and } x_1 + \cdots + x_n = r\}$
- $\mathcal{Y}$: the set of all r-combinations of $[n]$ with repetition
- $f\colon \mathcal{X} \to \mathcal{Y}\quad (x_1, \dots, x_n) \mapsto \{x_1 \cdot 1, x_2 \cdot 2, \dots, x_n \cdot n\}$
  - f is bijective. Hence, $|\mathcal{X}| = |\mathcal{Y}| = \binom{n+r-1}{r}$.

Application

EXAMPLE: What is the value of k after the program execution?

- $k := 0;$
- for $i_1 := 1$ to n do
  - for $i_2 := 1$ to i_1 do
    - $\vdots$
      - for $i_r := 1$ to i_{r-1} do
        - $k := k + 1;$

Analysis:

- Loop variables: $1 \leq i_r \leq i_{r-1} \leq \cdots \leq i_1 \leq n$
- The number of iterations is equal to the number of r-combinations of the set $[n]$ with repetition
- In every iteration, k increases by 1.
  - After the program execution, $k = \binom{n+r-1}{r}$

Combinations of Multiset

DEFINITION: Let $A = \{n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k\}$ be an n-multiset. Let $r \in \{0,1,\dots,n\}$.

- r**-combination of** $\boldsymbol{A}$: an r-subset (multiset) of A
  - Notation: $\{x_1 \cdot a_1, x_2 \cdot a_2, \dots, x_k \cdot a_k\}$, where $0 \le x_i \le n_i$ for every $i \in [k]$ and $x_1 + x_2 + \cdots + x_k = r$.

EXAMPLE: $A = \{1 \cdot \mathrm{a}, 2 \cdot \mathrm{b}, 3 \cdot \mathrm{c}\}$

- $\{1 \cdot b, 2 \cdot c\}$ is a 3-combination of A; a 3-subset of A

REMARK:

- For every $r \in \{0,1,\dots,n\}$, an r-combination of $A = \{a_1, a_2, \dots, a_n\}$ without repetition is an r-combination of $\{1 \cdot a_1, 1 \cdot a_2, \dots, 1 \cdot a_n\}$.
- For every $r \ge 0$, an $r-$combination of $A = \{a_1, a_2, \dots, a_n\}$ with repetition is an $r-$combination of $\{\infty \cdot a_1, \infty \cdot a_2, \dots, \infty \cdot a_n\}$.

Inverse Binomial Transform

DEFINITION: The **binomial transform** of $\{a_n\}_{n\geq s}$ is a sequence $\{b_n\}_{n\geq s}$ such that

$$b_n = \sum_{k=s}^{n} \binom{n}{k} a_k \qquad (1)$$

DEFINITION: The **inverse binomial transform** of $\{a_n\}_{n\geq s}$ is a sequence $\{b_n\}_{n\geq s}$ such that

$$b_n = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} a_k \quad (2)$$

QUESTION: Given (1), how to find the sequence $\{a_n\}$?

- Answer: $\{a_n\}$ is the inverse binomial transform of $\{b_n\}$
- Application: determine $\{a_n\}$ via $\{b_n\}$
- Proof?

Combinatorial Proofs

DEFINITION: A **combinatorial proof** of an identity $L = R$ is

- **a double counting proof,** which shows that L, R count the same set of objects but in different ways:
  - $L = |X| = R$ and L, R count $|X|$ in different ways.
- **a bijective proof,** which shows a bijection between the sets of objects counted by L and R:
  - $L = |X|, R = |Y|$ and there is a bijection $f: X \to Y$.

EXAMPLE: $\binom{n}{r} = \binom{n}{n-r}$

- $X = \{s \in \{0,1\}^n : s \text{ contains } r \text{ 0s}\} = \{s \in \{0,1\}^n : s \text{ contains } n - r \text{ 1s}\}$
  - $\binom{n}{r} = |X|$
  - $\binom{n}{n-r} = |X|$

Inverse Binomial Transform

LEMMA: $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ for any $n, k, r \in \mathbb{N}$ such that $n \geq k \geq r$.

- Let $U = \{u_1, u_2, \dots, u_n\}$ be a finite set of n elements
- $S = \{(A, B): A \subseteq U, |A| = k, B \subseteq A, |B| = r\}$
  - choose A then choose B: $|S| = \binom{n}{k}\binom{k}{r}$, the left-hand side
  - choose B then choose A: $|S| = \binom{n}{r}\binom{n-r}{k-r}$, the right-hand side

LEMMA: $\sum_{k=r}^{n}(-1)^{n-k}\binom{n}{k}\binom{k}{r} = \begin{cases} 1 & n = r \\ 0 & n > r \end{cases}$ when $n \geq r$.

- $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ as $n \geq k \geq r \geq 0$
- **left** $= \sum_{k=r}^{n}(-1)^{n-k}\binom{n}{r}\binom{n-r}{k-r} = \binom{n}{r}\sum_{k=r}^{n}(-1)^{(n-r)-(k-r)}\binom{n-r}{k-r}$

$$= \binom{n}{r}\sum_{i=0}^{n-r}(-1)^{(n-r)-i}\binom{n-r}{i}$$

$=$ **right**

Inverse Binomial Transform

LEMMA: Let $n, s \in \mathbb{N}, s \leq n$. Then $\underbrace{\sum_{k=s}^{n} \sum_{i=s}^{k}}_{\alpha_k} a_{k,i} = \underbrace{\sum_{i=s}^{n} \sum_{k=i}^{n}}_{\beta_i} a_{k,i}$

$k \backslash i$	s	$s+1$	$s+2$	$\cdots$	n	row sum
s	$a_{s,s}$			$\cdots$		α_s
$s+1$	$a_{s+1,s}$	$a_{s+1,s+1}$		$\cdots$		α_{s+1}
$s+2$	$a_{s+2,s}$	$a_{s+2,s+1}$	$a_{s+2,s+2}$	$\cdots$		α_{s+2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
n	$a_{n,s}$	$a_{n,s+1}$	$a_{n,s+2}$	$\cdots$	$a_{n,n}$	α_n
col sum	β_s	β_{s+1}	β_{s+2}	$\cdots$	β_n	$\sum\sum$

THEOREM: Let $\{a_n\}, \{b_n\}$ be two sequences s.t. for all $n \geq s$,

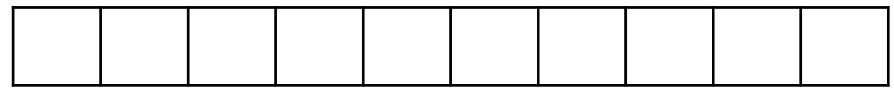
$$a_n = \sum_{k=s}^{n} \binom{n}{k} b_k. \text{ Then } b_n = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} a_k \;\; (n \geq s).$$

- $\sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} a_k = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} \sum_{i=s}^{k} \binom{k}{i} b_i$

$$= \sum_{i=s}^{n} \sum_{k=i}^{n} (-1)^{n-k} \binom{n}{k} \binom{k}{i} b_i = b_n$$

Inverse Binomial Transform

EXAMPLE: Let $h(n, m)$ be the # of ways of coloring a $m \times 1$ grid with n colors such that:



$m = 10$

(1) all colors are used; and

(2) the adjacent squares receive different colors.

Find a formula for $h(n, m)$. // $n \geq 2$

- B_n: the set of ways of coloring a $m \times 1$ grid with n colors s.t. (1), (2) hold
  - Let $b_n = |B_n|$. Then $h(n, m) = b_n$.
- A_n: the set of ways of coloring a $m \times 1$ grid with n colors s.t. (2) holds
  - Let $a_n = |A_n|$. Then $a_n = n(n-1)^{m-1}$.
  - $a_n = \sum_{k=2}^{n} \binom{n}{k} b_k$ // the sum rule
- $b_n = \sum_{k=2}^{n} (-1)^{n-k} \binom{n}{k} a_k = \sum_{k=2}^{n} (-1)^{n-k} \binom{n}{k} k(k-1)^{m-1}$ // inversion