Alternating Direction Method of Multipliers

Prof S. Boyd

EE364b, Stanford University

source:

Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers (Boyd, Parikh, Chu, Peleato, Eckstein)

Goals

robust methods for

- arbitrary-scale optimization
  - machine learning/statistics with huge data-sets
  - dynamic optimization on large-scale network
- decentralized optimization
  - devices/processors/agents coordinate to solve large problem, by passing relatively small messages

Outline

Dual problem

- convex equality constrained optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array} \rightarrow p^*$$

theory

- Lagrangian: $L(x, y) = f(x) + y^T(Ax - b)$
- dual function: $g(y) = \inf_x L(x, y)$
- dual problem: $\underset{y}{\text{maximize}}\ g(y) \rightarrow y^*$
- recover $x^\star = \operatorname{argmin}_x L(x, y^\star)$

algorithm

key problem:
how to solve them?

$\underset{x}{\text{minimize}} \quad f(x)$

Dual ascent

gradient method: $x^{k+1} = x^k - \alpha^k \nabla f(x^k)$

- gradient method for dual problem: $y^{k+1} = y^k + \alpha^k \nabla g(y^k)$
- $\nabla g(y^k) = A\tilde{x} - b$, where $\tilde{x} = \operatorname{argmin}_x L(x, y^k)$
- dual ascent method is

$$\Delta \left\{ \begin{array}{lcll} x^{k+1} & := & \operatorname{argmin}_x L(x, y^k) & \text{// } x\text{-minimization} \\ y^{k+1} & := & y^k + \alpha^k \underbrace{(Ax^{k+1} - b)}_{\nabla g(y^k)} & \text{// dual update} \end{array} \right.$$

- works, with lots of strong assumptions

Dual decomposition

Lagrangian : $L(x,y) = f(x) + \underbrace{y^T A x}_{\sum_{i=1}^{n} y^T A_i x_i} - y^T b$

- suppose f is separable:

$$f(x) = f_1(x_1) + \cdots + f_N(x_N), \quad x = (x_1, \ldots, x_N)$$

- then L is separable in x: $L(x,y) = L_1(x_1,y) + \cdots + L_N(x_N,y) - y^T b$,

$$L_i(x_i,y) = f_i(x_i) + y^T A_i x_i$$

- x-minimization in dual ascent splits into N separate minimizations

$$x_i^{k+1} \quad := \quad \underset{x_i}{\operatorname{argmin}}\, L_i(x_i, y^k)$$

which can be carried out in parallel

Dual decomposition

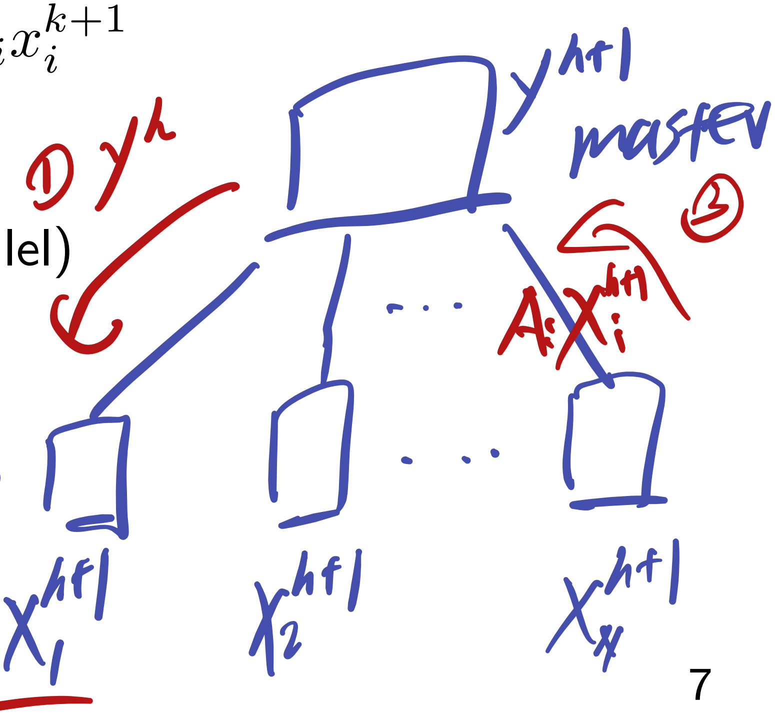
- dual decomposition (Everett, Dantzig, Wolfe, Benders 1960–65)

$$
\begin{aligned}
x_i^{k+1} &:= \operatorname{argmin}_{x_i} L_i(x_i, y^k), \quad i = 1, \ldots, N \\
y^{k+1} &:= y^k + \alpha^k \left(\textstyle\sum_{i=1}^N A_i x_i^{k+1} - b\right)
\end{aligned}
$$

- scatter y^k; update x_i in parallel; gather $A_i x_i^{k+1}$

① ②

- solve a large problem
  - by iteratively solving subproblems (in parallel)
  - dual variable update provides coordination
- works, with lots of assumptions; often slow



Outline

Method of multipliers

- a method to robustify dual ascent
- use **augmented Lagrangian** (Hestenes, Powell 1969), $\rho > 0$

$$L_\rho(x, y) = f(x) + y^T(Ax - b) + (\rho/2)\|Ax - b\|_2^2$$

- method of multipliers (Hestenes, Powell; analysis in Bertsekas 1982)

$$\begin{aligned} x^{k+1} &:= \operatorname*{argmin}_x L_\rho(x, y^k) \\ y^{k+1} &:= y^k + \rho(Ax^{k+1} - b) \end{aligned}$$

(note specific dual update step length ρ)

Method of multipliers dual update step

(KKT condition) original problem

- optimality conditions (for differentiable f):

$$\text{①}\quad Ax^\star - b = 0, \qquad \text{②}\quad \nabla f(x^\star) + A^T y^\star = 0$$

(primal and dual feasibility)

- since x^{k+1} minimizes $L_\rho(x, y^k)$

y^{k+1}

$$\begin{aligned}
\nabla_x L_\rho(x, y) = 0 \qquad 0 &= \nabla_x L_\rho(x^{k+1}, y^k) \\
&= \nabla_x f(x^{k+1}) + A^T \left(y^k + \rho(Ax^{k+1} - b) \right) \\
&= \nabla_x f(x^{k+1}) + A^T y^{k+1}
\end{aligned}$$

②
- dual update $y^{k+1} = y^k + \rho(Ax^{k+1} - b)$ makes (x^{k+1}, y^{k+1}) *dual feasible*
- *primal feasibility* achieved in limit: $Ax^{k+1} - b \to 0$

①

Method of multipliers

(compared to dual decomposition)

- *good news*: converges under much more relaxed conditions (f can be nondifferentiable, take on value $+\infty$, ...)

  sub-gradient

- *bad news*: quadratic penalty destroys splitting of the x-update, so can't do decomposition

  A (?)

Outline

Dual decomposition

Method of multipliers

Alternating direction method of multipliers

Common patterns

Examples

Consensus and exchange

Conclusions

Alternating direction method of multipliers

- a method ①
  - with good robustness of method of multipliers
  - which can support decomposition

  ②

- “robust dual decomposition” or “decomposable method of multipliers”
- proposed by Gabay, Mercier, Glowinski, Marrocco in 1976

Alternating direction method of multipliers

- ADMM problem form (with f, g convex)

$$
\begin{array}{ll}
\underset{x,z}{\text{minimize}} & f(x) + g(z) \\
\text{subject to} & Ax + Bz = c
\end{array}
$$

  - two sets of variables, with separable objective

- $L_\rho(x, z, y) = f(x) + g(z) + y^T(Ax + Bz - c) + (\rho/2)\|Ax + Bz - c\|_2^2$

$$(x^{k+1}, z^{k+1}) = \underset{x,z}{\operatorname{argmin}}\ \overline{L_\rho(x, z; y^k)}\ \Delta$$

- ADMM:

$$
\Delta \left\{
\begin{array}{lll}
x^{k+1} & := & \operatorname{argmin}_x L_\rho(x, z^k, y^k) \\
z^{k+1} & := & \operatorname{argmin}_z L_\rho(x^{k+1}, z, y^k)
\end{array}
\right.
$$

$$
\begin{array}{lll}
y^{k+1} & := & y^k + \rho(Ax^{k+1} + Bz^{k+1} - c)
\end{array}
$$

// x-minimization
// z-minimization
// dual update

Alternating direction method of multipliers

- if we minimized over x and z jointly, $\Rightarrow$ reduces to method of multipliers
- instead, we do one pass of a Gauss-Seidel method
- we get splitting since we minimize over x with z fixed, and vice versa

ADMM and optimality conditions

$L(x, z, y) = f(x) + g(z) + y^T(Ax + Bz - c)$

- optimality conditions (for differentiable case): (KKT condition)
  - primal feasibility: $Ax + Bz - c = 0$
  - dual feasibility: $\nabla f(x) + A^T y = 0, \quad \nabla g(z) + B^T y = 0$ ②

$\nabla_x L(x, z, y) = 0$

- since z^{k+1} minimizes $L_\rho(x^{k+1}, z, y^k)$ we have $\nabla_z L(x, z, y)$

$$
\begin{aligned}
0 &= \nabla g(z^{k+1}) + B^T y^k + \rho B^T (Ax^{k+1} + Bz^{k+1} - c) \\
&= \nabla g(z^{k+1}) + B^T y^{k+1}
\end{aligned}
$$

②

$y^{k+1} = y^k + \rho(Ax^{k+1} + Bz^{k+1} - c)$

- so with ADMM dual variable update, $(x^{k+1}, z^{k+1}, y^{k+1})$ satisfies second dual feasibility condition

- primal and first dual feasibility are achieved as $k \to \infty$

ADMM with scaled dual variables

- combine linear and quadratic terms in augmented Lagrangian

$$
\begin{aligned}
L_\rho(x, z, y^k) &= f(x) + g(z) + y^T(Ax + Bz - c) + (\rho/2)\|Ax + Bz - c\|_2^2 \\
&= f(x) + g(z) + (\rho/2)\|Ax + Bz - c + u^k\|_2^2 + \text{const.}
\end{aligned}
$$

with $u^k = (1/\rho)y^k$ $\|Ax+Bz-c+u\|_2^2 = \|Ax+Bz-c\|_2^2 + 2u^T(Ax+Bz-c) + \|u\|_2^2$

- ADMM (scaled dual form):

$$
\Delta \left\{
\begin{aligned}
x^{k+1} &:= \operatorname*{argmin}_x \left(f(x) + (\rho/2)\|Ax + \underline{Bz^k - c + u^k}\|_2^2\right) \\
z^{k+1} &:= \operatorname*{argmin}_z \left(g(z) + (\rho/2)\|Ax^{k+1} + Bz - c + u^k\|_2^2\right)
\end{aligned}
\right.
$$

$$
u^{k+1} := u^k + (Ax^{k+1} + Bz^{k+1} - c)
$$

Convergence

- assume (very little!)
  - f, g convex, closed, proper
  - L_0 has a saddle point
- then ADMM converges:
  - iterates approach feasibility: $Ax^k + Bz^k - c \to 0$
  - objective approaches optimal value: $f(x^k) + g(z^k) \to p^\star$

Related algorithms

- operator splitting methods
  (Douglas, Peaceman, Rachford, Lions, Mercier, . . . 1950s, 1979)
- proximal point algorithm (Rockafellar 1976)
- Dykstra's alternating projections algorithm (1983)
- Spingarn's method of partial inverses (1985)
- Rockafellar-Wets progressive hedging (1991)
- proximal methods (Rockafellar, many others, 1976–present)
- Bregman iterative methods (2008–present)
- most of these are special cases of the proximal point algorithm

Outline

Common patterns

- x-update step requires minimizing $f(x) + (\rho/2)\|Ax - v\|_2^2$ (with $v = Bz^k - c + u^k$, which is constant during x-update)
- similar for z-update
- several special cases come up often
- can simplify update by exploit ing structure in these cases

"proximal algorithms", by N. Parikh and S. Boyd

Foundations and Trends in optimizations

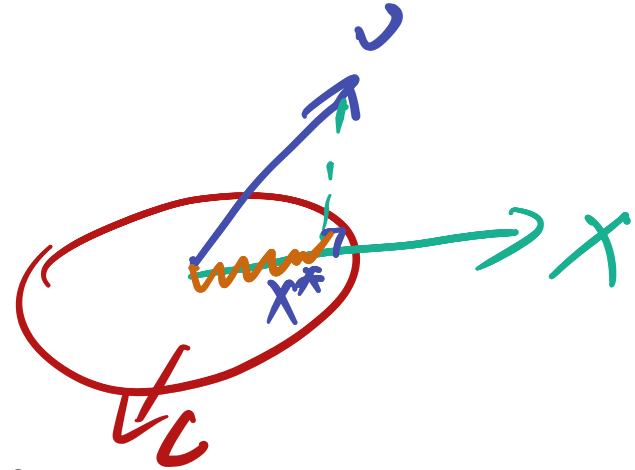
Decomposition

- suppose f is block-separable,

$$f(x) = f_1(x_1) + \cdots + f_N(x_N), \qquad x = (x_1, \ldots, x_N)$$

- A is conformably block separable: $A^T A$ is block diagonal
- then x-update splits into N parallel updates of x_i

Proximal operator



- consider x-update when $A = I$

$$x^+ = \underset{x}{\operatorname{argmin}} \left(f(x) + (\rho/2)\|x - v\|_2^2 \right) = \mathbf{prox}_{f,\rho}(v)$$

- some special cases:

$$I_C(z) = \begin{cases} 0, & z \in C \\ +\infty, & z \notin C \end{cases}$$

$f = I_C$ (indicator fct. of set C) $\quad x^+ := \Pi_C(v)$ (projection onto C)

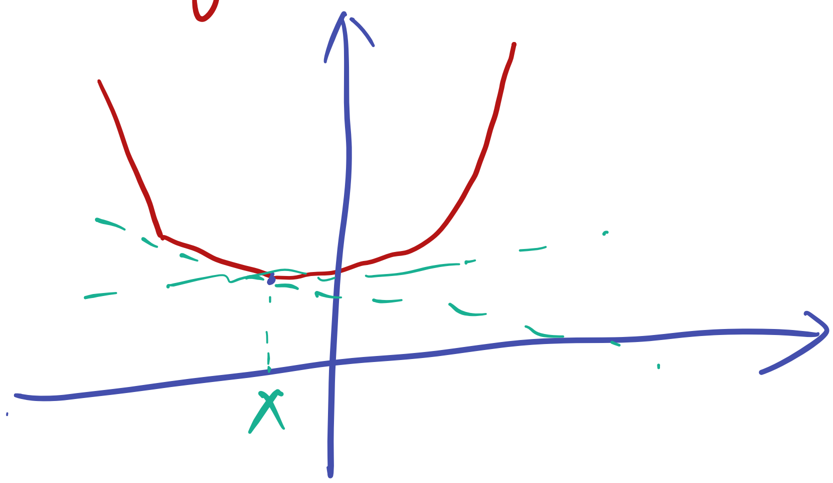
Δ $f = \lambda \| \cdot \|_1$ (ℓ_1 norm) (sub-gradient) $\quad x_i^+ := S_{\lambda/\rho}(v_i)$ (soft thresholding)

$(S_a(v) = (v - a)_+ - (-v - a)_+)$

subgradients :

We say g is a subgradient of f at the point x if

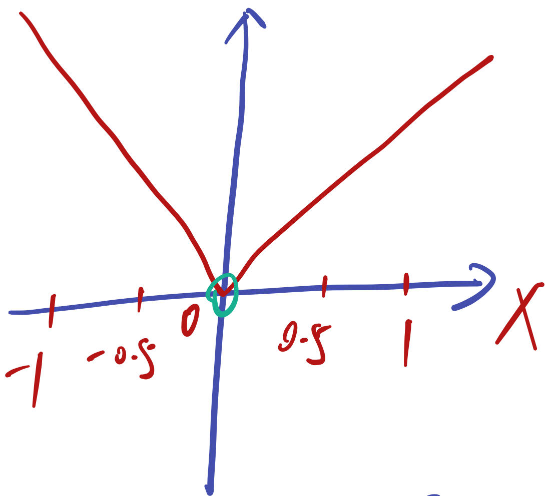
$$f(z) \geq \underbrace{f(x) + g^T(z-x)}_{\text{a linear global under-estimate of } f}, \quad \forall z$$



- the set of all subgradients of f at x is called the subdifferential of f at x, denoted by $\partial f(x)$

is a set, not unique

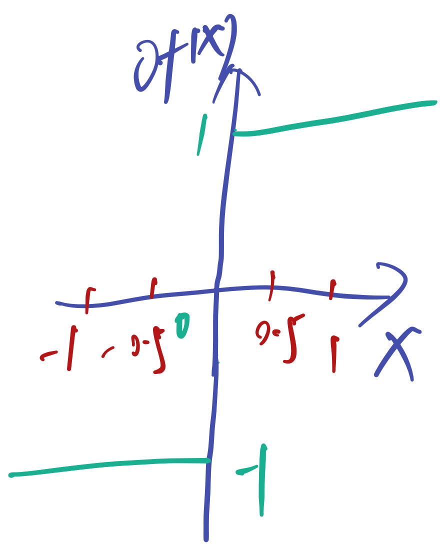
example: $f(x) = |x|$



$$f(z) \geq f(x) + g^T(z - x), \forall z$$

$$|z| \geq 0 + g^T(z - 0), \forall z$$

$$\Rightarrow g \in [-1, 1]$$



$$\partial f(x) = \begin{cases} -1, & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

example : l_1-norm

$$f(x) = \|x\|_1 = \sum_{i=1}^{n} \underbrace{|x_i|}_{: f_i(x)} \qquad e_i = \begin{array}{cc} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array} \end{array}$$

since

$$\partial f_i(x) = \begin{cases} \mathrm{sgn}(x_i)\, e_i & \text{if } x_i \neq 0 \\ [-1, 1]\, e_i, & \text{if } x_i = 0 \end{cases}$$

we have

$$\sum_{i,\, x_i \neq 0} \mathrm{sgn}(x_i)\, e_i \in \partial f(x)$$

$$x^{+} = \arg\min_{x} \left\{ f(x) + \frac{\rho}{2}\|x - v\|_2^2 \right\} = \mathrm{prox}_{f,\rho}(v)$$

is the proximal operator f any convex function f

In this case, if $f(x) = \lambda\|x\|_1$, l_1-norm

to minimize $\lambda\|x\|_1 + \frac{\rho}{2}\|x - v\|_2^2 = g(x)$

It is obvious that x^* is the minimizer $\Leftrightarrow$

$0 \in \partial g(x^*)$ — ① (subdifferential)

Proof: $\Rightarrow \forall y$,

$$g(y) \geq g(y^*) \Rightarrow g(y) \geq g(x^*) + 0(y - x^*)$$

$$\Rightarrow 0 \in \partial g(x^*)$$

$\Leftarrow$ if $0 \in \partial g(x^*)$, $g(y) \geq g(x^*) + 0 \cdot (y - x^*)$

$$\Rightarrow g(y) \geq g(x^*), \ \forall y$$

Based on ①, and

$$\partial g(x) = \rho(x - v) + \partial(\lambda \|x\|_1)$$

$$\Rightarrow 0 \in \rho(x^* - v) + \partial(\lambda \|x^*\|_1) \quad \Delta$$

① if $x_i^* > 0$, $\lambda |x_i^*| = \lambda x_i^* \Rightarrow \partial(\lambda |x_i^*|) = \lambda \Rightarrow$

$$x_i^* = v_i - \frac{\lambda}{\rho} > 0 \Rightarrow v_i > \frac{\lambda}{\rho} \quad \Delta$$

else if

② $x_i^* < 0$, $\lambda |x_i^*| = -\lambda x_i^* \Rightarrow \partial |\lambda | x_i^*|) = -\lambda \Rightarrow$

$$x_i^* = v_i + \frac{\lambda}{\rho} < 0 \Rightarrow v_i < -\frac{\lambda}{\rho}$$

else

③ $x_i^* = 0$, $\partial(\lambda |x_i^*|) \in [-\lambda, \lambda] \Rightarrow$

$$v_i \in \left[-\frac{\lambda}{\rho}, \frac{\lambda}{\rho}\right]$$

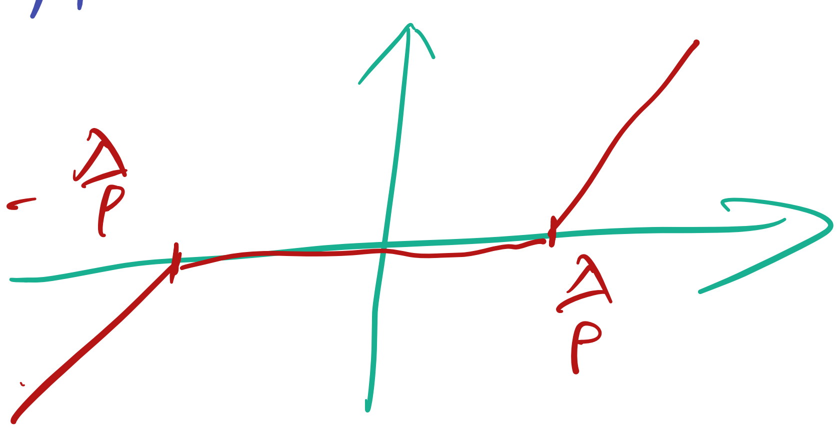
To conclude,

$$x_i^* = x_i^+ = \begin{cases} v_i - \frac{\lambda}{\rho}, & v_i > \frac{\lambda}{\rho} \\ v_i + \frac{\lambda}{\rho}, & v_i < -\frac{\lambda}{\rho} \\ 0, & \text{else} \end{cases} \Rightarrow S_{\frac{\lambda}{\rho}}(v_i)$$

equivalent form:

$$S_\delta(v_i) = (v_i - \delta)_+ - (-v_i - \delta)_+$$

$$(\cdot)_+ = \max\{\cdot, 0\}$$



Quadratic objective

- $f(x) = (1/2)x^T Px + q^T x + r$
- $x^+ := (P + \rho A^T A)^{-1}(\rho A^T v - q)$
- use matrix inversion lemma when computationally advantageous

$$(P + \rho A^T A)^{-1} = P^{-1} - \rho P^{-1} A^T (I + \rho A P^{-1} A^T)^{-1} A P^{-1}$$

- (direct method) cache factorization of $P + \rho A^T A$ (or $I + \rho A P^{-1} A^T$)
- (iterative method) warm start, early stopping, reducing tolerances

Smooth objective

- f smooth
- can use standard methods for smooth minimization
  - gradient, Newton, or quasi-Newton
  - preconditionned CG, limited-memory BFGS (scale to very large problems)
- can exploit
  - warm start
  - early stopping, with tolerances decreasing as ADMM proceeds

Outline

Constrained convex optimization

- consider ADMM for generic problem

$$
\begin{array}{ll}
\text{minimize} & f(x) \\
\text{subject to} & x \in \mathcal{C}
\end{array}
$$

$$\Updownarrow \qquad I_C(z) = \begin{cases} 0, & z \in C \\ +\infty, & z \notin C \end{cases}$$

- ADMM form: take g to be indicator of $\mathcal{C}$

$$
\begin{array}{ll}
\text{minimize} & f(x) + g(z) \\
\text{subject to} & x - z = 0 \qquad A
\end{array}
$$

- algorithm:

$$
\begin{array}{lll}
x^{k+1} & := & \underset{x}{\operatorname{argmin}} \left(f(x) + (\rho/2)\|x - z^k + u^k\|_2^2 \right) \\
z^{k+1} & := & \Pi_{\mathcal{C}}(x^{k+1} + u^k) \rightarrow a^k \\
u^{k+1} & := & u^k + x^{k+1} - z^{k+1}
\end{array}
$$

$$z^{k+1} = \underset{z \in C}{\operatorname{argmin}} \|z - a^k\|_2^2$$

Lasso

- lasso problem:

$$\text{minimize} \quad (1/2)\|Ax - b\|_2^2 + \lambda\|x\|_1$$

- ADMM form:

$$\begin{array}{ll} \text{minimize} & (1/2)\|Ax - b\|_2^2 + \lambda\|z\|_1 \\ \text{subject to} & x - z = 0 \end{array}$$

- ADMM:

$$\begin{aligned} x^{k+1} &:= (A^TA + \rho I)^{-1}(A^Tb + \rho z^k - y^k) \\ z^{k+1} &:= S_{\lambda/\rho}(x^{k+1} + y^k/\rho) \\ y^{k+1} &:= y^k + \rho(x^{k+1} - z^{k+1}) \end{aligned}$$

soft-thresholding

Homogeous self-dual embedding system:

① $\boxed{\begin{array}{ll} \text{find} & (u, v) \\ \text{subject to} & v = Qu \\ & (u, v) \in C \times C^* \end{array}}$

ADMM form:

$$\begin{array}{ll} \underset{x, z}{\text{minimize}} & f(x) + g(z) \\ \text{subject to} & x = z \end{array}$$

$$\Downarrow ①$$

$$\begin{array}{ll} \text{minimize} & I_{C \times C^*}(u, v) + I_{Qu=v}(\tilde{u}, \tilde{v}) \\ \text{subject to} & (u, v) = (\tilde{u}, \tilde{v}) \end{array}$$

$\Downarrow$ ADMM algorithm

$$△ \begin{cases} \tilde{u}^{k+1} = (I + Q)^{-1}(u^k + v^k) & \text{— subspace projection} \\ u^{k+1} = \Pi_C(\tilde{u}^{k+1} - v^k) & \text{— parallel cone projection} \\ v^{k+1} = v^k - \tilde{u}^{k+1} + u^{k+1} & (C = C_1 \times C_2 \times \cdots \times C_n) \end{cases}$$

"Conic optimization via operator splitting and homogenous self-dual embedding" by Brendan.

Lasso example

- example with dense $A \in \mathbf{R}^{1500 \times 5000}$
  $(1500$ measurements; 5000 regressors)

- computation times

factorization (same as ridge regression)	1.3s
subsequent ADMM iterations	0.03s
lasso solve (about 50 ADMM iterations)	2.9s
full regularization path (30 λ's)	4.4s

- not bad for a *very short* Matlab script

Sparse inverse covariance selection

- S: empirical covariance of samples from $\mathcal{N}(0, \Sigma)$, with Σ^{-1} sparse (*i.e.*, Gaussian Markov random field)
- estimate Σ^{-1} via ℓ_1 regularized maximum likelihood

$$\text{minimize} \quad \mathbf{Tr}(SX) - \log\det X + \lambda\|X\|_1$$

- methods: COVSEL (Banerjee et al 2008), graphical lasso (FHT 2008)

Sparse inverse covariance selection via ADMM

- ADMM form:

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}(SX) - \log\det X + \lambda \|Z\|_1 \\ \text{subject to} & X - Z = 0 \end{array}$$

- ADMM:

$$\begin{aligned} X^{k+1} &:= \operatorname*{argmin}_X \left(\mathbf{Tr}(SX) - \log\det X + (\rho/2)\|X - Z^k + U^k\|_F^2 \right) \\ Z^{k+1} &:= S_{\lambda/\rho}(X^{k+1} + U^k) \\ U^{k+1} &:= U^k + (X^{k+1} - Z^{k+1}) \end{aligned}$$

Analytical solution for X-update

- compute eigendecomposition $\rho(Z^k - U^k) - S = Q\Lambda Q^T$
- form diagonal matrix $\tilde{X}$ with

$$\tilde{X}_{ii} = \frac{\lambda_i + \sqrt{\lambda_i^2 + 4\rho}}{2\rho}$$

- let $X^{k+1} := Q\tilde{X}Q^T$
- cost of X-update is an eigendecomposition

Sparse inverse covariance selection example

- Σ^{-1} is 1000×1000 with 10^4 nonzeros
  - graphical lasso (Fortran): 20 seconds – 3 minutes
  - ADMM (Matlab): 3 – 10 minutes
  - (depends on choice of λ)

- very rough experiment, but with no special tuning, ADMM is in ballpark of recent specialized methods

- (for comparison, COVSEL takes 25+ min when Σ^{-1} is a 400×400 tridiagonal matrix)

Outline

Consensus optimization

- want to solve problem with N objective terms

$$\text{minimize} \quad \sum_{i=1}^N f_i(x)$$

  - *e.g.*, f_i is the loss function for ith block of training data

- ADMM form:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^N f_i(x_i) \\ \text{subject to} & x_i - z = 0 \quad i=1, \cdots, N \end{array}$$

  - x_i are *local variables*
  - z is the *global variable*
  - $x_i - z = 0$ are *consistency* or *consensus* constraints
  - can add regularization using a $g(z)$ term

Consensus optimization via ADMM

- $L_\rho(x,z,y) = \sum_{i=1}^N \left(f_i(x_i) + y_i^T(x_i - z) + (\rho/2)\|x_i - z\|_2^2\right)$

- ADMM:

$$
\begin{array}{llcl}
1 & x_i^{k+1} & := & \underset{x_i}{\operatorname{argmin}} \left(f_i(x_i) + y_i^{kT}(x_i - z^k) + (\rho/2)\|x_i - z^k\|_2^2\right) \\
2 & z^{k+1} & := & \dfrac{1}{N}\displaystyle\sum_{i=1}^N \left(x_i^{k+1} + (1/\rho)y_i^k\right) \\
 & y_i^{k+1} & := & y_i^k + \rho(x_i^{k+1} - z^{k+1})
\end{array}
$$

- with regularization, averaging in z update is followed by $\mathbf{prox}_{g,\rho}$

Consensus optimization via ADMM

- using $\sum_{i=1}^N y_i^k = 0$, algorithm simplifies to

$$
\begin{aligned}
x_i^{k+1} &:= \operatorname*{argmin}_{x_i} \left(f_i(x_i) + y_i^{kT}(x_i - \overline{x}^k) + (\rho/2)\|x_i - \overline{x}^k\|_2^2 \right) \\
y_i^{k+1} &:= y_i^k + \rho(x_i^{k+1} - \overline{x}^{k+1})
\end{aligned}
$$

  where $\overline{x}^k = (1/N)\sum_{i=1}^N x_i^k$

- in each iteration
  - gather x_i^k and average to get $\overline{x}^k$
  - scatter the average $\overline{x}^k$ to processors
  - update y_i^k locally (in each processor, in parallel)
  - update x_i locally

Statistical interpretation

- f_i is negative log-likelihood for parameter x given ith data block
- x_i^{k+1} is MAP estimate under prior $\mathcal{N}(\overline{x}^k + (1/\rho)y_i^k, \rho I)$
- prior mean is previous iteration's consensus shifted by 'price' of processor i disagreeing with previous consensus
- processors only need to support a Gaussian MAP method
  - type or number of data in each block not relevant
  - consensus protocol yields global maximum-likelihood estimate

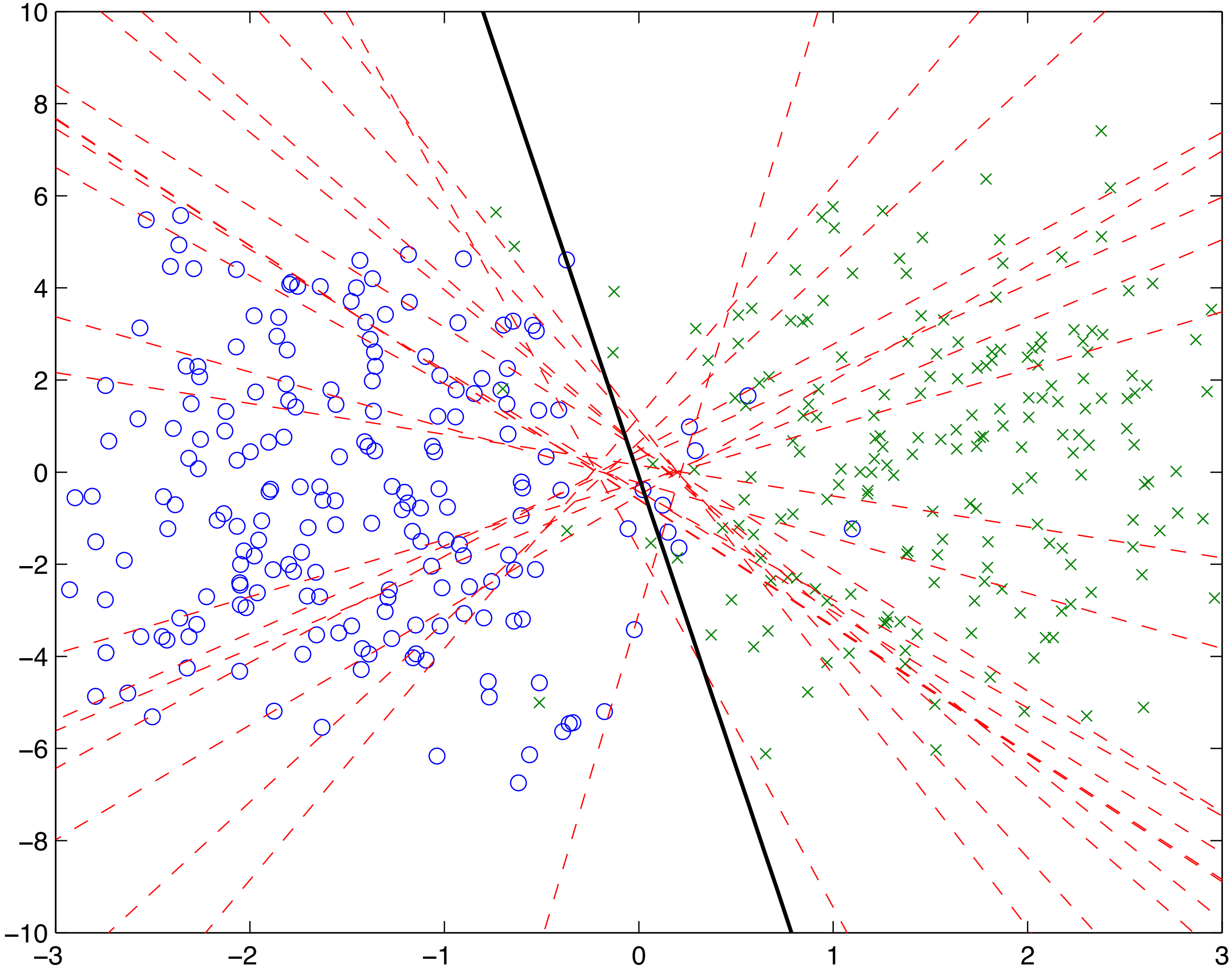
Consensus classification

- data (examples) (a_i, b_i), $i = 1, \ldots, N$, $a_i \in \mathbf{R}^n$, $b_i \in \{-1, +1\}$
- linear classifier $\mathbf{sign}(a^T w + v)$, with weight w, offset v
- margin for ith example is $b_i(a_i^T w + v)$; want margin to be positive
- loss for ith example is $l(b_i(a_i^T w + v))$
  - l is loss function (hinge, logistic, probit, exponential, ...)
- choose w, v to minimize $\frac{1}{N} \sum_{i=1}^{N} l(b_i(a_i^T w + v)) + r(w)$
  - $r(w)$ is regularization term (ℓ_2, ℓ_1, ...)
- split data and use ADMM consensus to solve

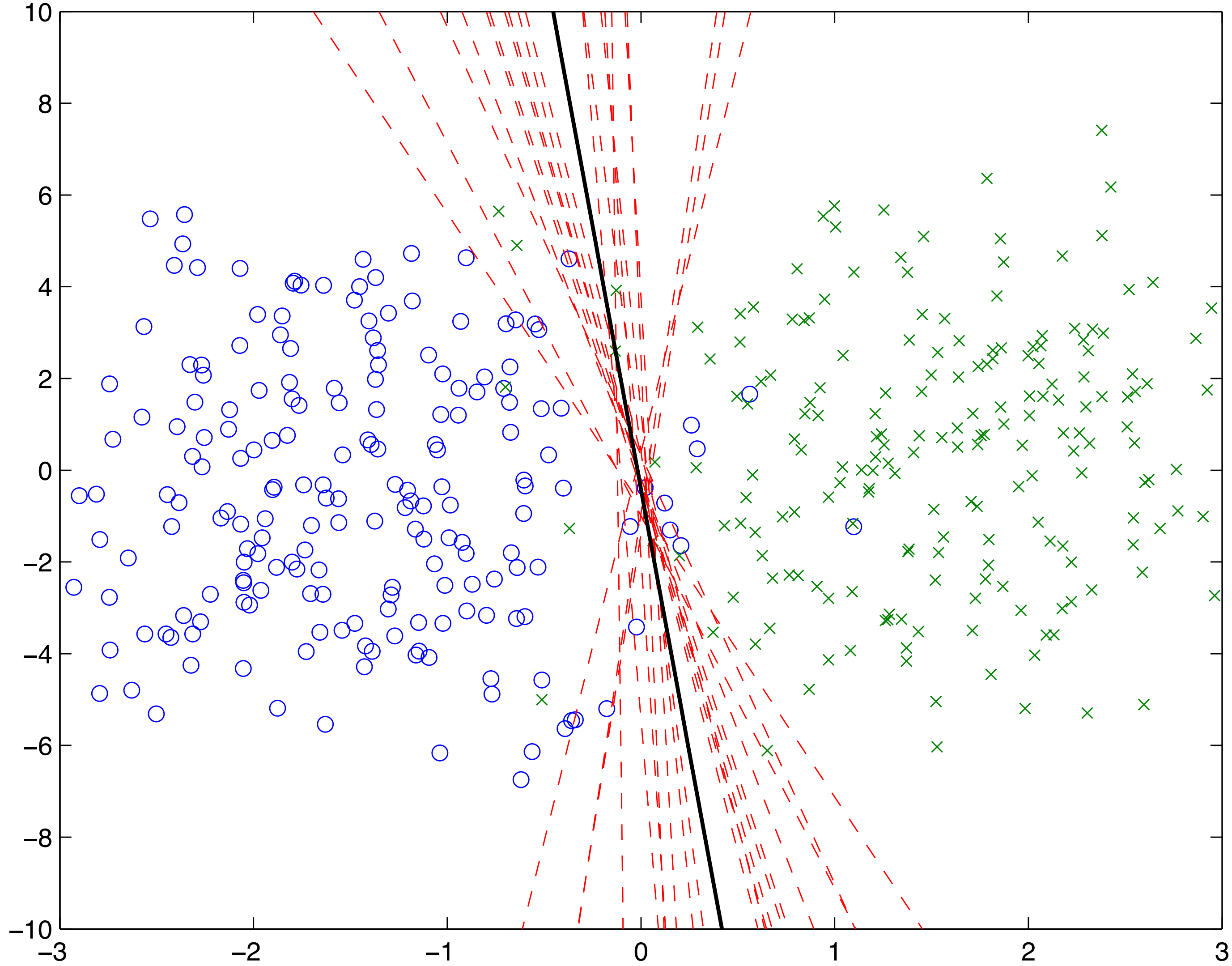
Consensus SVM example

- hinge loss $l(u) = (1-u)_+$ with ℓ_2 regularization
- baby problem with $n = 2$, $N = 400$ to illustrate
- examples split into 20 groups, in worst possible way: each group contains only positive or negative examples

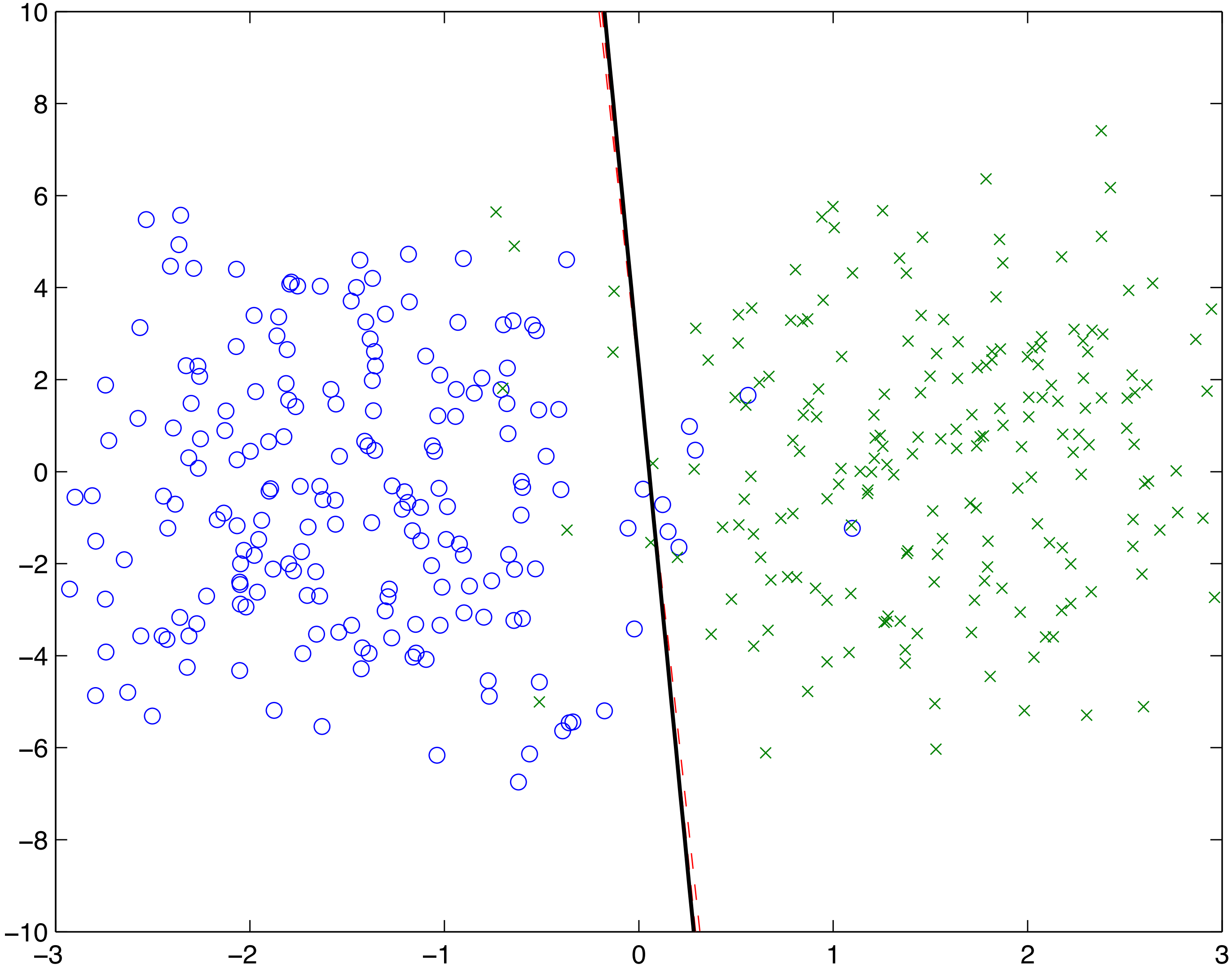
Iteration 1



Iteration 5



Iteration 40



Distributed lasso example

- example with **dense** $A \in \mathbf{R}^{400000 \times 8000}$ (roughly 30 GB of data)
  - distributed solver written in C using MPI and GSL
  - no optimization or tuned libraries (like ATLAS, MKL)
  - split into 80 subsystems across 10 (8-core) machines on Amazon EC2

- computation times

loading data	30s
factorization	5m
subsequent ADMM iterations	0.5–2s
lasso solve (about 15 ADMM iterations)	5–6m

Exchange problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^N f_i(x_i) \\ \text{subject to} & \sum_{i=1}^N x_i = 0 \end{array}$$

- another canonical problem, like consensus
- in fact, it's the dual of consensus
- can interpret as N agents exchanging n goods to minimize a total cost
- $(x_i)_j \geq 0$ means agent i *receives* $(x_i)_j$ of good j from exchange
- $(x_i)_j < 0$ means agent i *contributes* $|(x_i)_j|$ of good j to exchange
- constraint $\sum_{i=1}^N x_i = 0$ is *equilibrium* or *market clearing* constraint
- optimal dual variable $y^\star$ is a set of valid prices for the goods
- suggests real or virtual cash payment $(y^\star)^T x_i$ by agent i

Exchange ADMM

- solve as a generic constrained convex problem with constraint set

$$\mathcal{C} = \{x \in \mathbf{R}^{nN} \mid x_1 + x_2 + \cdots + x_N = 0\}$$

- scaled form:

$$\begin{aligned} x_i^{k+1} &:= \underset{x_i}{\operatorname{argmin}} \left(f_i(x_i) + (\rho/2)\|x_i - x_i^k + \overline{x}^k + u^k\|_2^2 \right) \\ u^{k+1} &:= u^k + \overline{x}^{k+1} \end{aligned}$$

- unscaled form:

$$\begin{aligned} x_i^{k+1} &:= \underset{x_i}{\operatorname{argmin}} \left(f_i(x_i) + y^{kT} x_i + (\rho/2)\|x_i - (x_i^k - \overline{x}^k)\|_2^2 \right) \\ y^{k+1} &:= y^k + \rho \overline{x}^{k+1} \end{aligned}$$

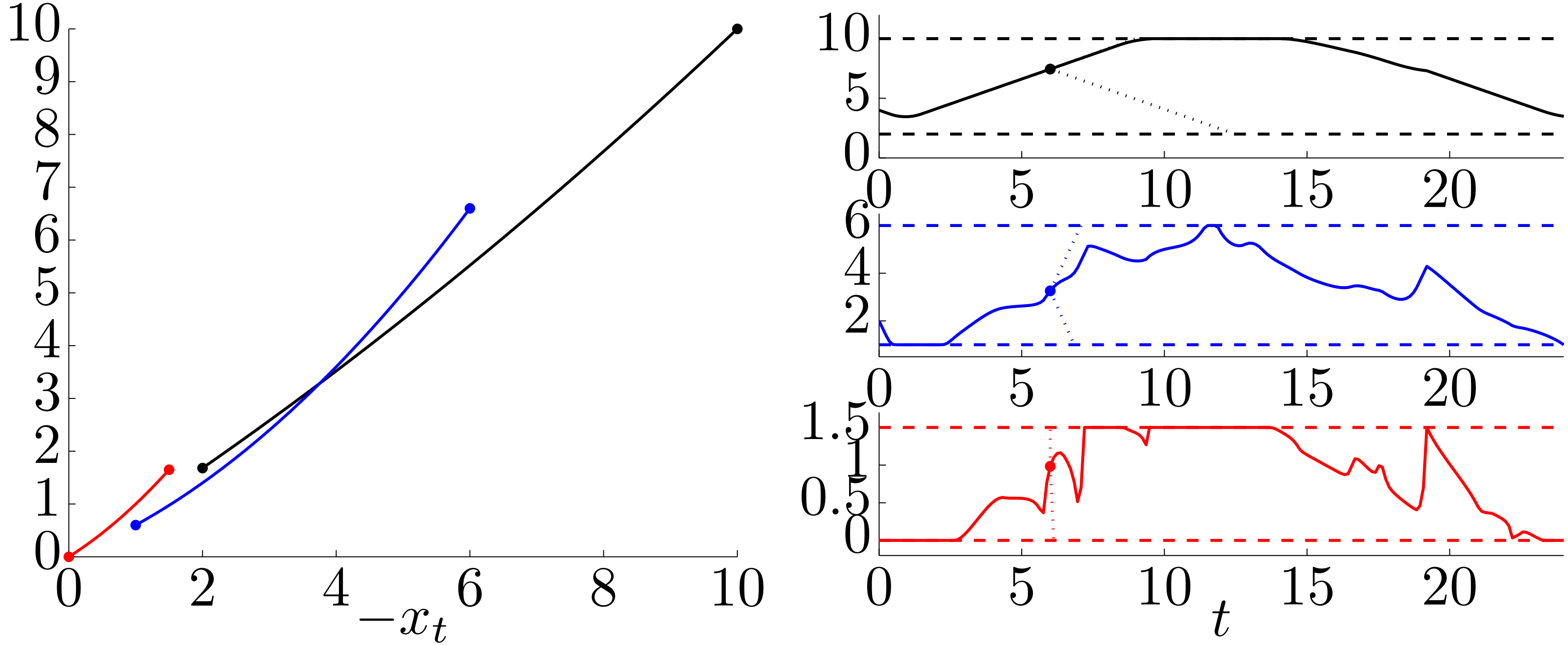
Interpretation as tâtonnement process

- *tâtonnement process*: iteratively update prices to clear market
- work towards equilibrium by increasing/decreasing prices of goods based on excess demand/supply
- dual decomposition is the simplest tâtonnement algorithm
- ADMM adds proximal regularization
  - incorporate agents' prior commitment to help clear market
  - convergence far more robust convergence than dual decomposition

Distributed dynamic energy management

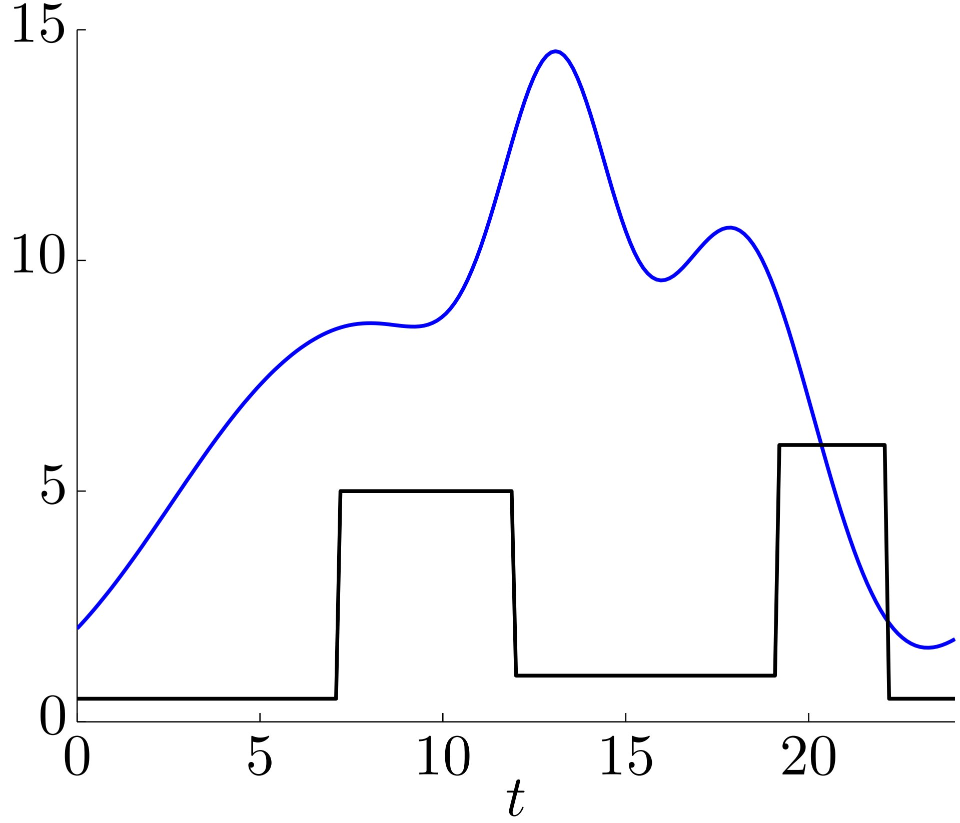
- N devices exchange power in time periods $t = 1, \dots, T$
- $x_i \in \mathbf{R}^T$ is power flow *profile* for device i
- $f_i(x_i)$ is cost of profile x_i (and encodes constraints)
- $x_1 + \cdots + x_N = 0$ is energy balance (in each time period)
- dynamic energy management problem is exchange problem
- exchange ADMM gives distributed method for dynamic energy management
- each device optimizes its own profile, with quadratic regularization for coordination
- residual (energy imbalance) is driven to zero

Generators



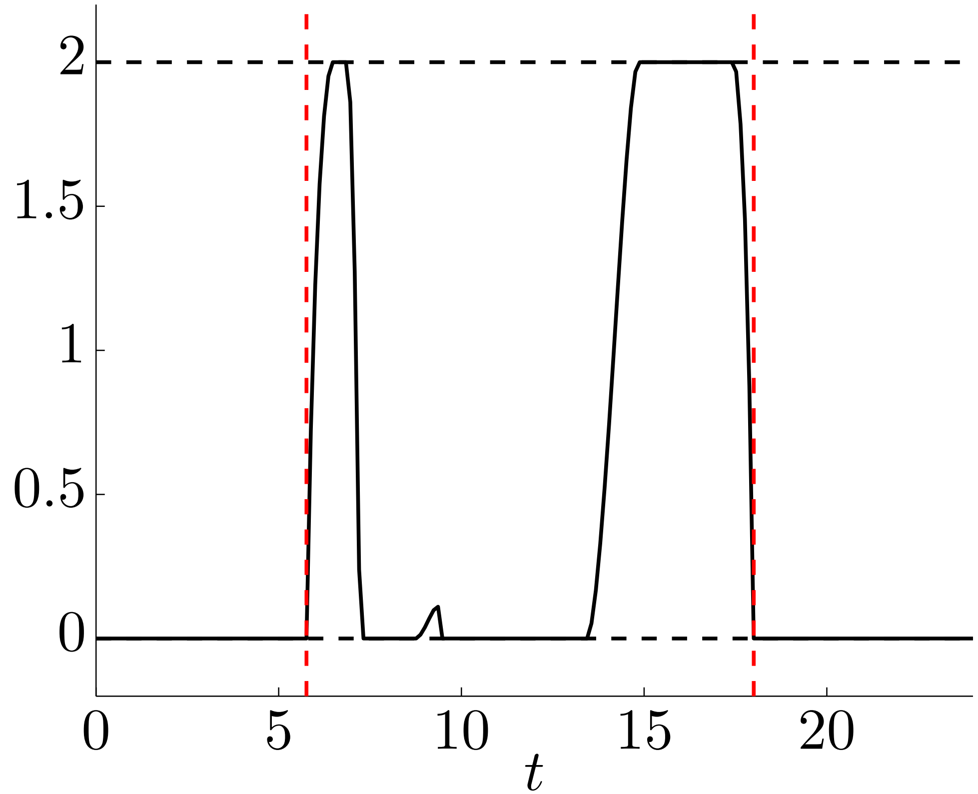
- 3 example generators
- left: generator costs/limits; right: ramp constraints
- can add cost for power changes

Fixed loads



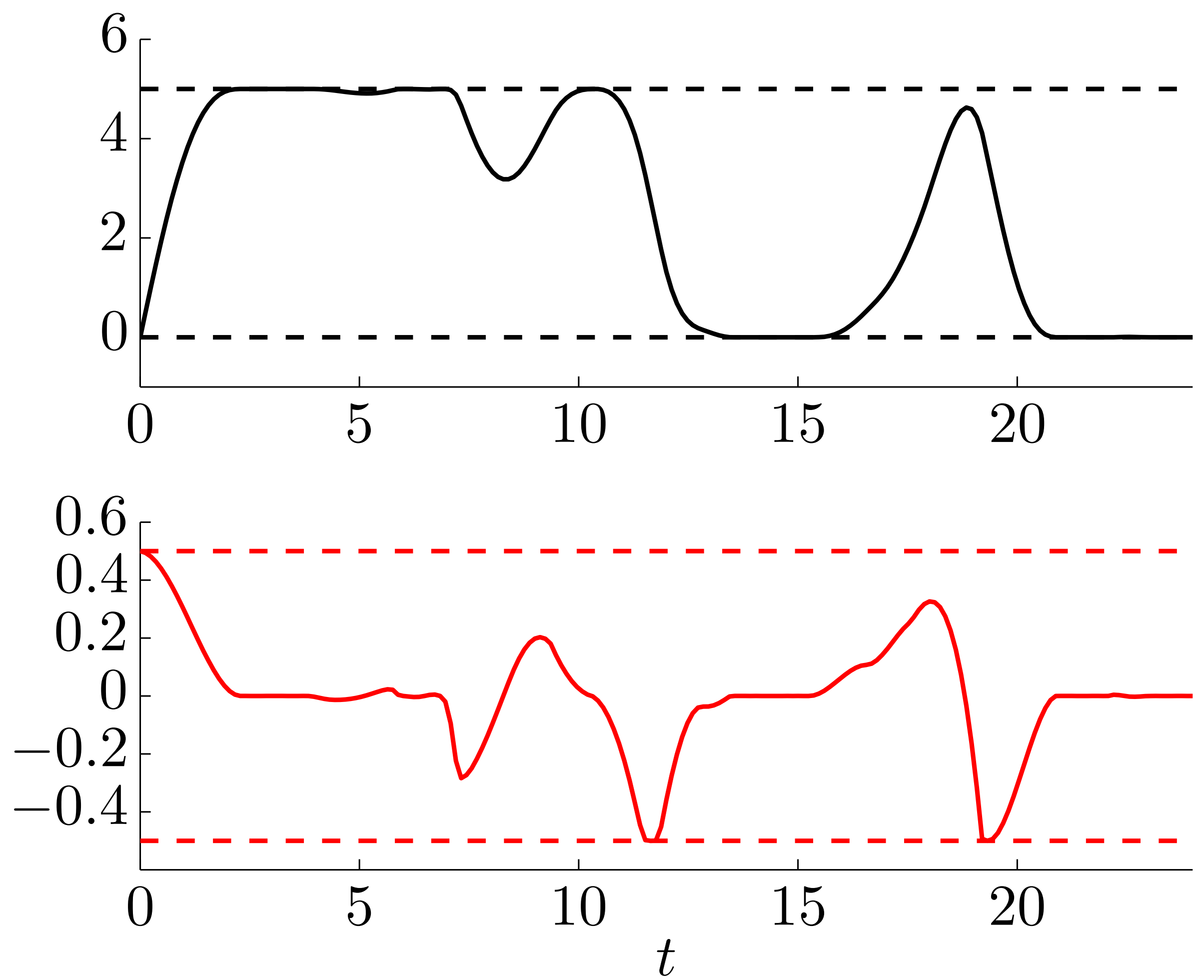
- 2 example fixed loads
- cost is $+\infty$ for not supplying load; zero otherwise

Shiftable load



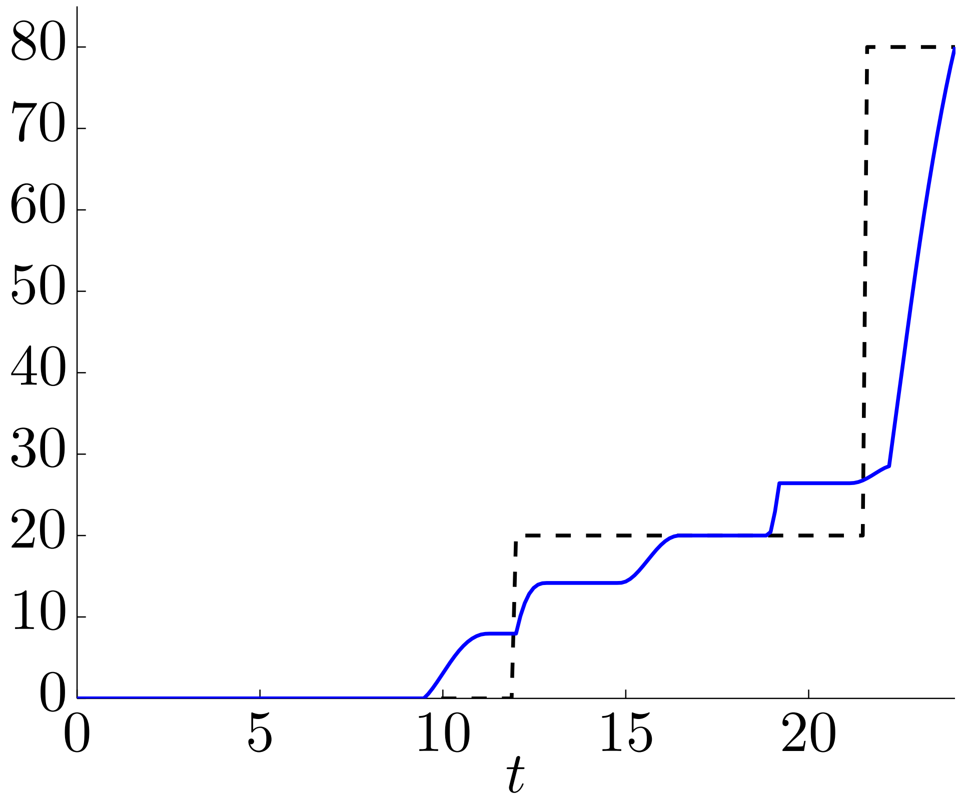
- total energy consumed over an interval must exceed given minimum level
- limits on energy consumed in each period
- cost is $+\infty$ for violating constraints; zero otherwise

Battery energy storage system



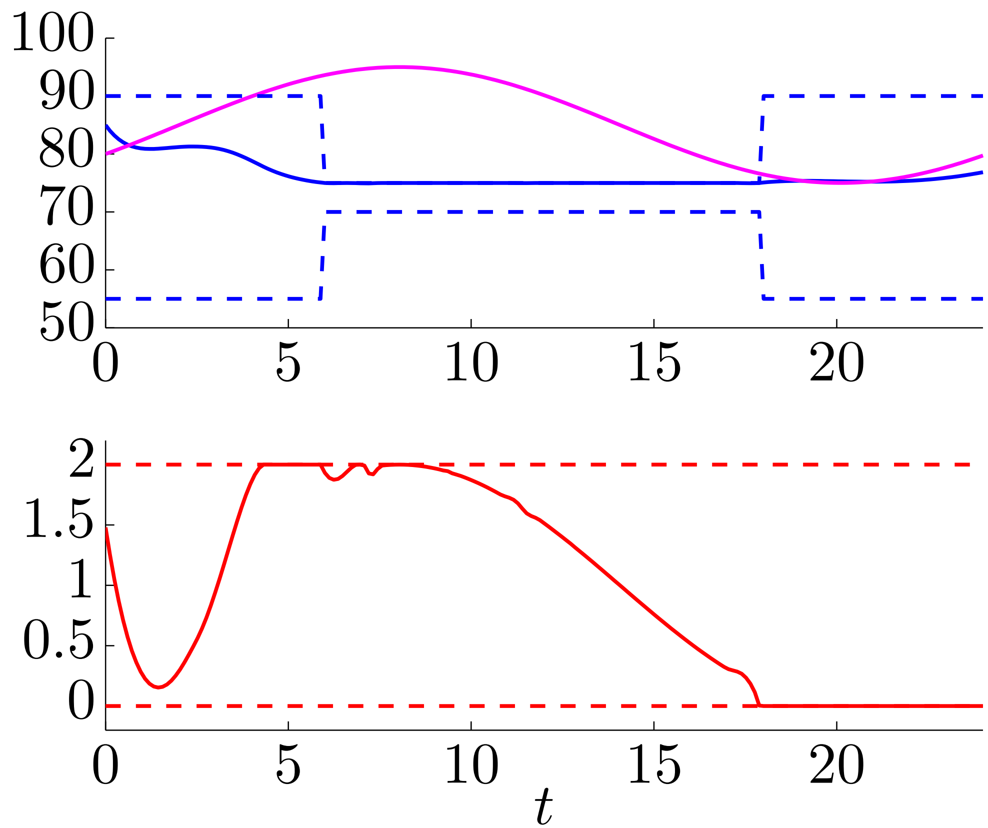
- energy store with maximum capacity, charge/discharge limits
- black: battery charge, red: charge/discharge profile
- cost is $+\infty$ for violating constraints; zero otherwise

Electric vehicle charging system



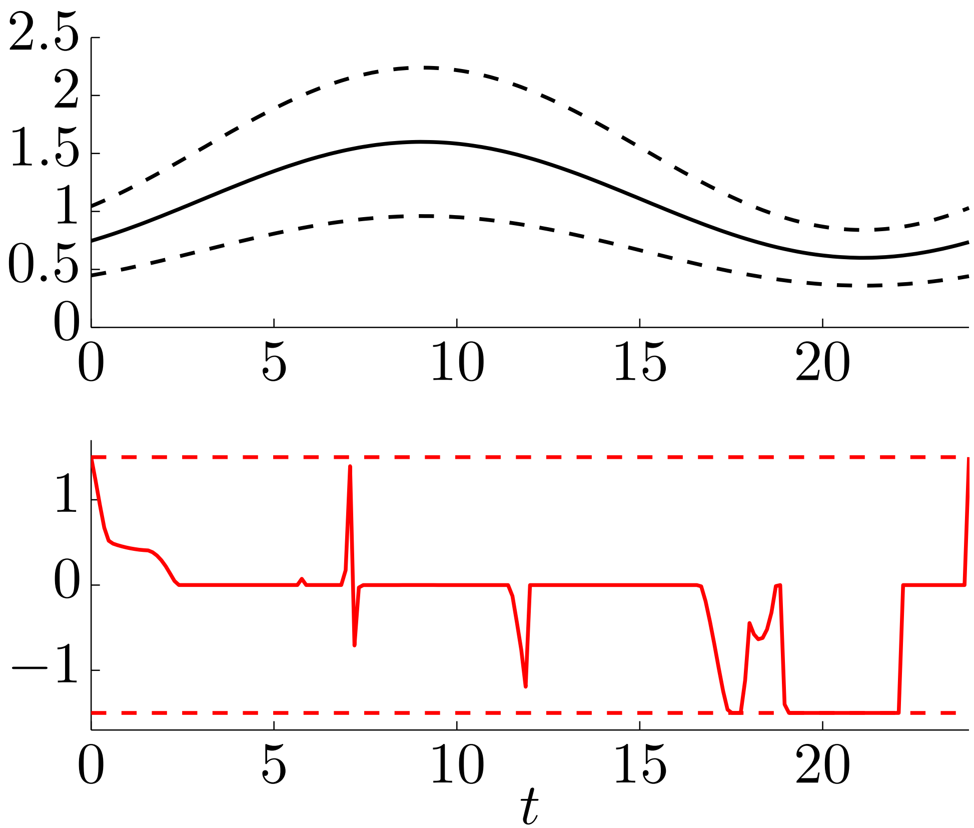
- black: desired charge profile; blue: charge profile
- shortfall cost for not meeting desired charge

HVAC



- thermal load (*e.g.*, room, refrigerator) with temperature limits
- magenta: ambient temperature; blue: load temperature
- red: cooling energy profile
- cost is $+\infty$ for violating constraints; zero otherwise

External tie



- buy/sell energy from/to external grid at price $p^{\mathrm{ext}}(t) \pm \gamma(t)$
- solid: $p^{\mathrm{ext}}(t)$; dashed: $p^{\mathrm{ext}}(t) \pm \gamma(t)$

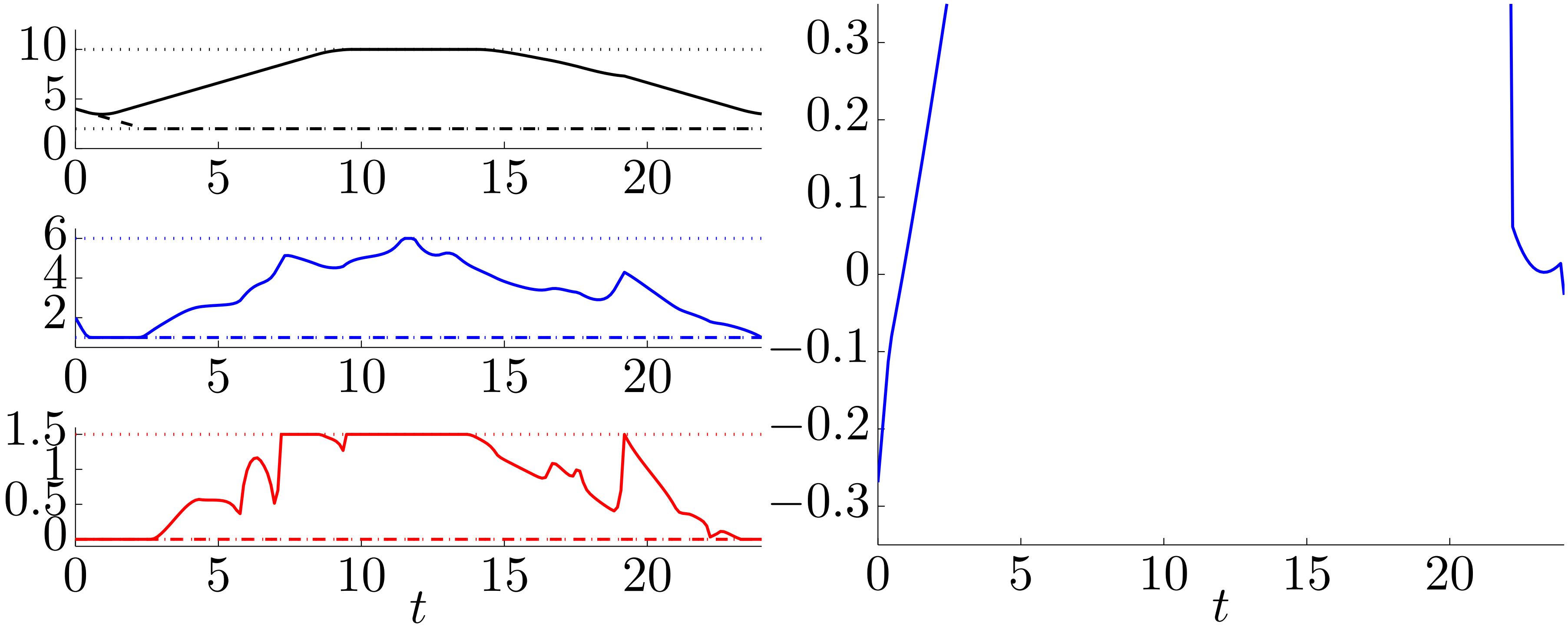
Smart grid example

10 devices (already described above)

- 3 generators
- 2 fixed loads
- 1 shiftable load
- 1 EV charging systems
- 1 battery
- 1 HVAC system
- 1 external tie

Convergence

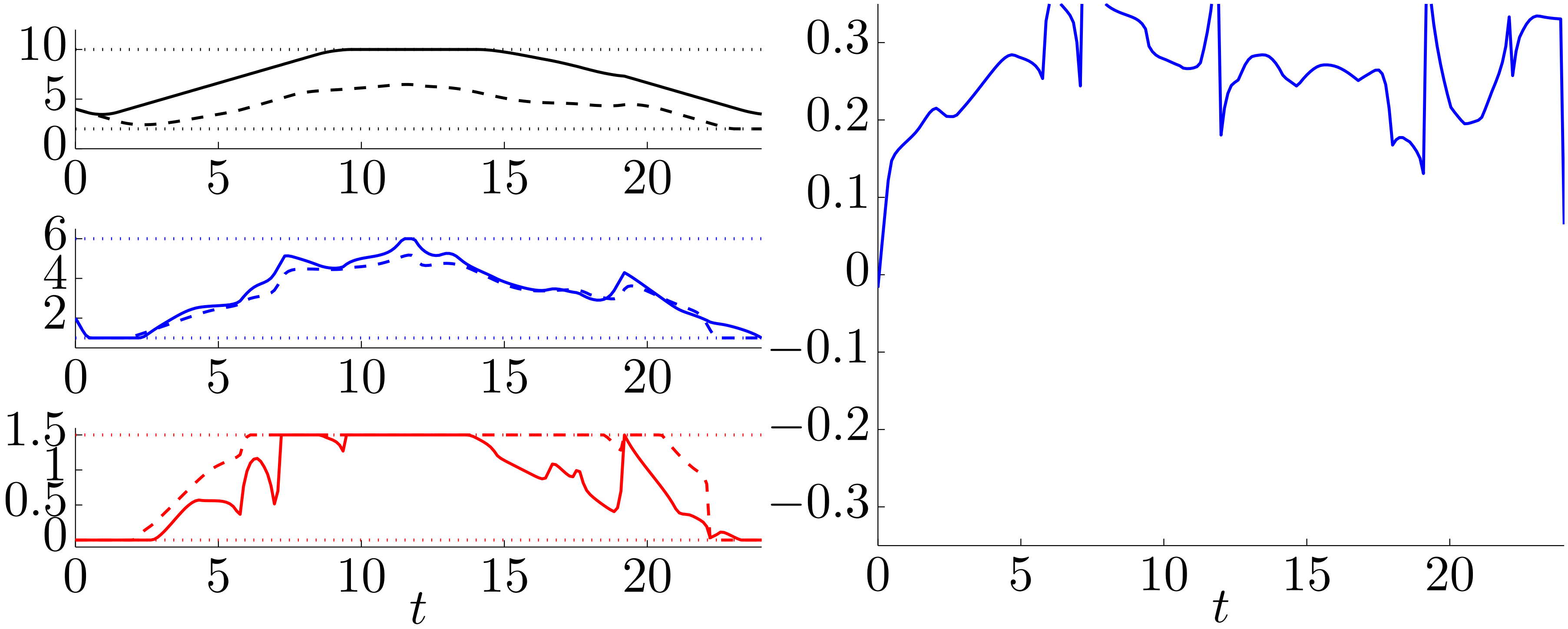
iteration: $k = 1$



- left: solid: optimal generator profile, dashed: profile at kth iteration
- right: residual vector $\bar{x}^k$

Convergence

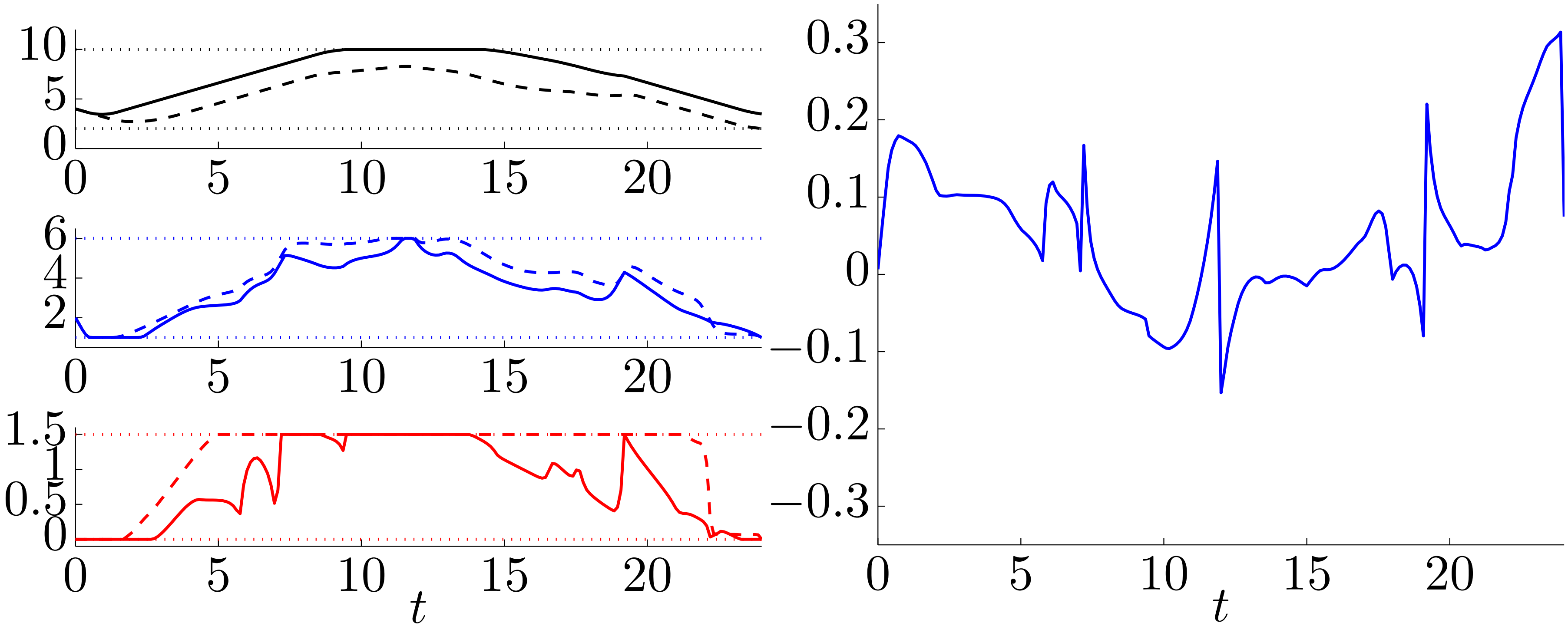
iteration: $k = 3$



- left: solid: optimal generator profile, dashed: profile at kth iteration
- right: residual vector $\bar{x}^k$

Convergence

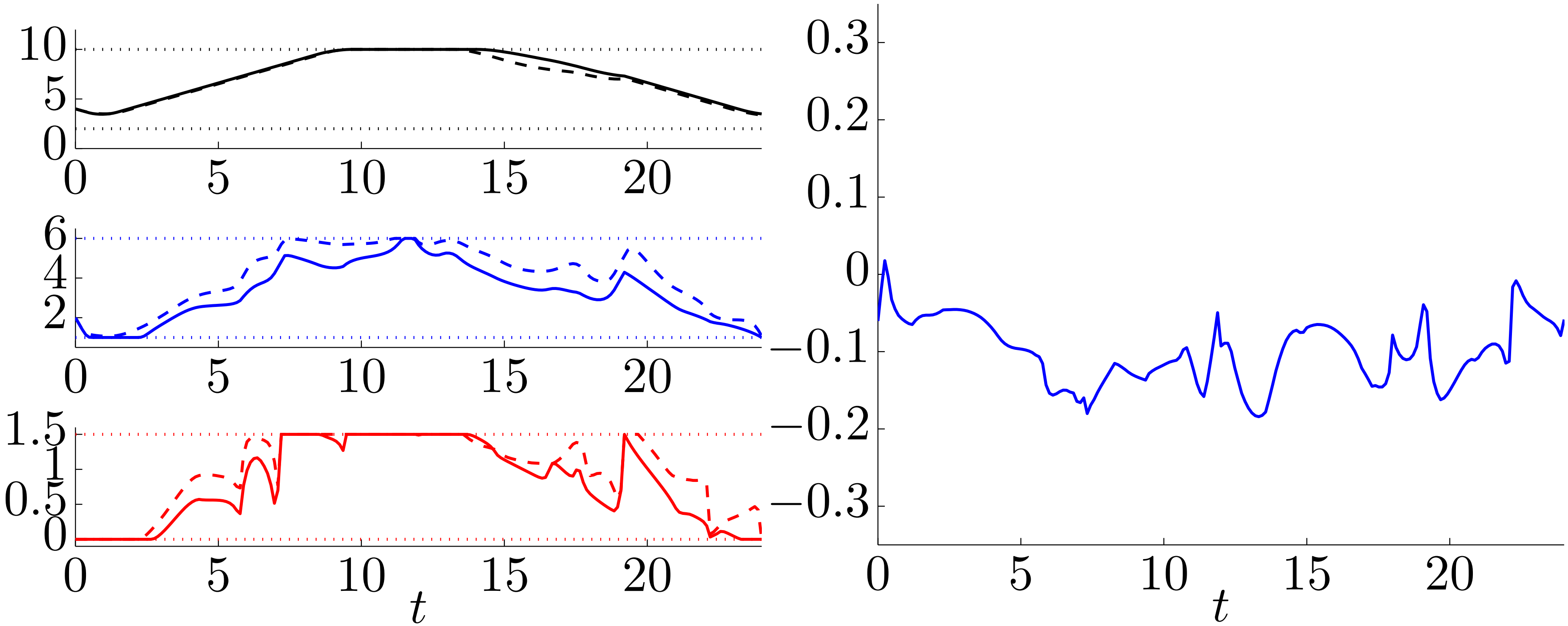
iteration: $k = 5$



- left: solid: optimal generator profile, dashed: profile at kth iteration
- right: residual vector $\bar{x}^k$

Convergence

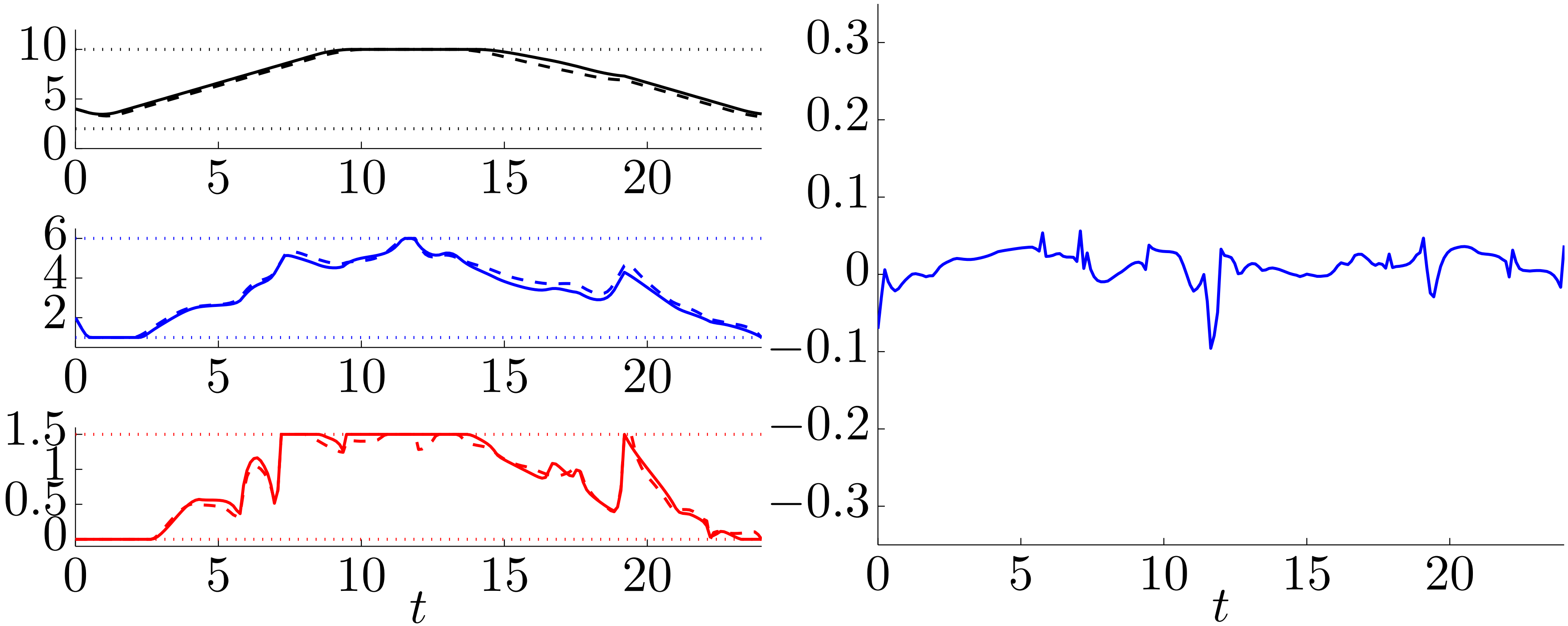
iteration: $k = 10$



- left: solid: optimal generator profile, dashed: profile at kth iteration
- right: residual vector $\bar{x}^k$

Convergence

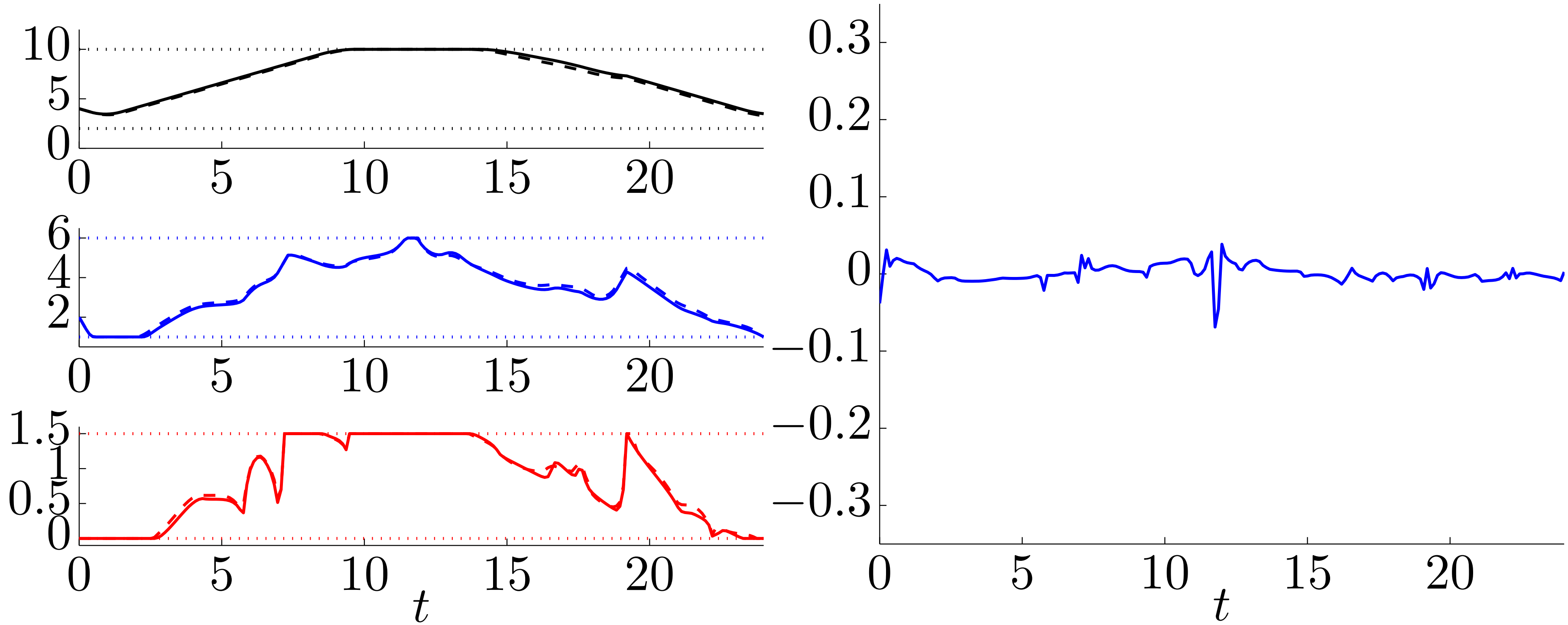
iteration: $k = 15$



- left: solid: optimal generator profile, dashed: profile at kth iteration
- right: residual vector $\bar{x}^k$

Convergence

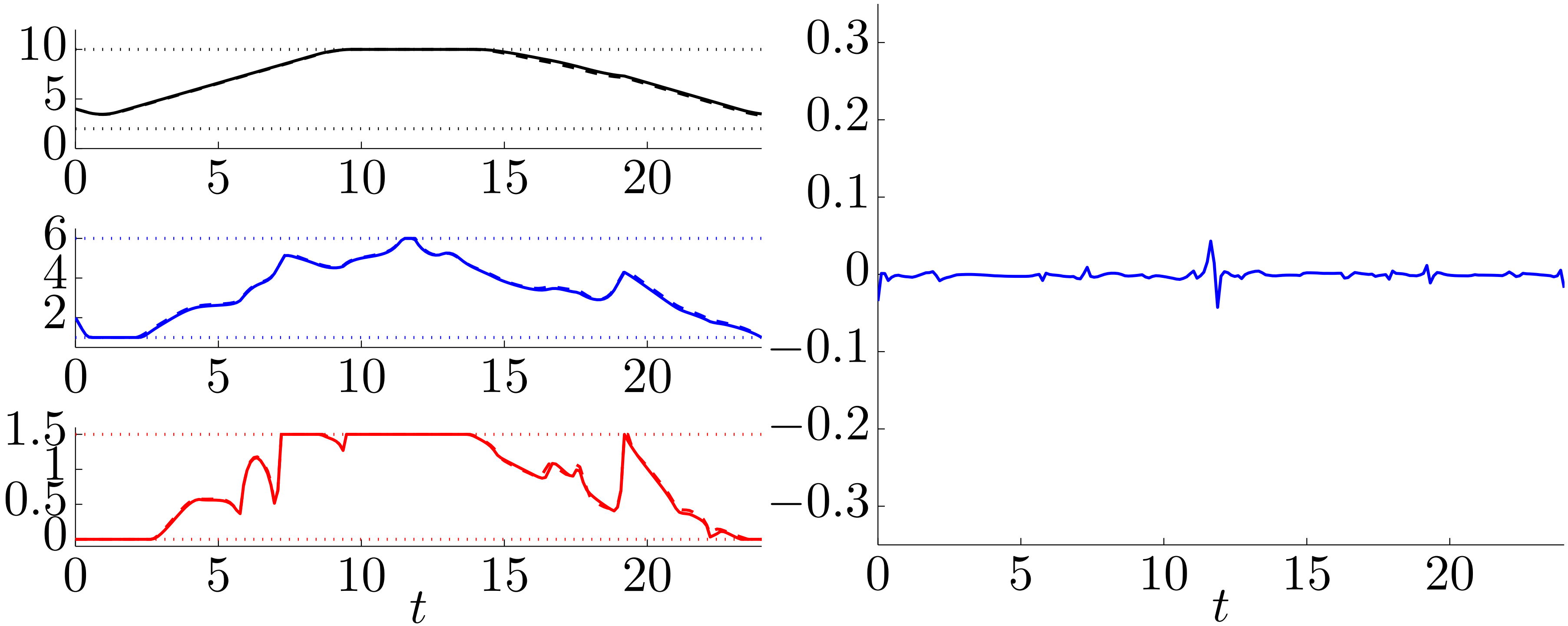
iteration: $k = 20$



- left: solid: optimal generator profile, dashed: profile at kth iteration
- right: residual vector $\bar{x}^k$

Convergence

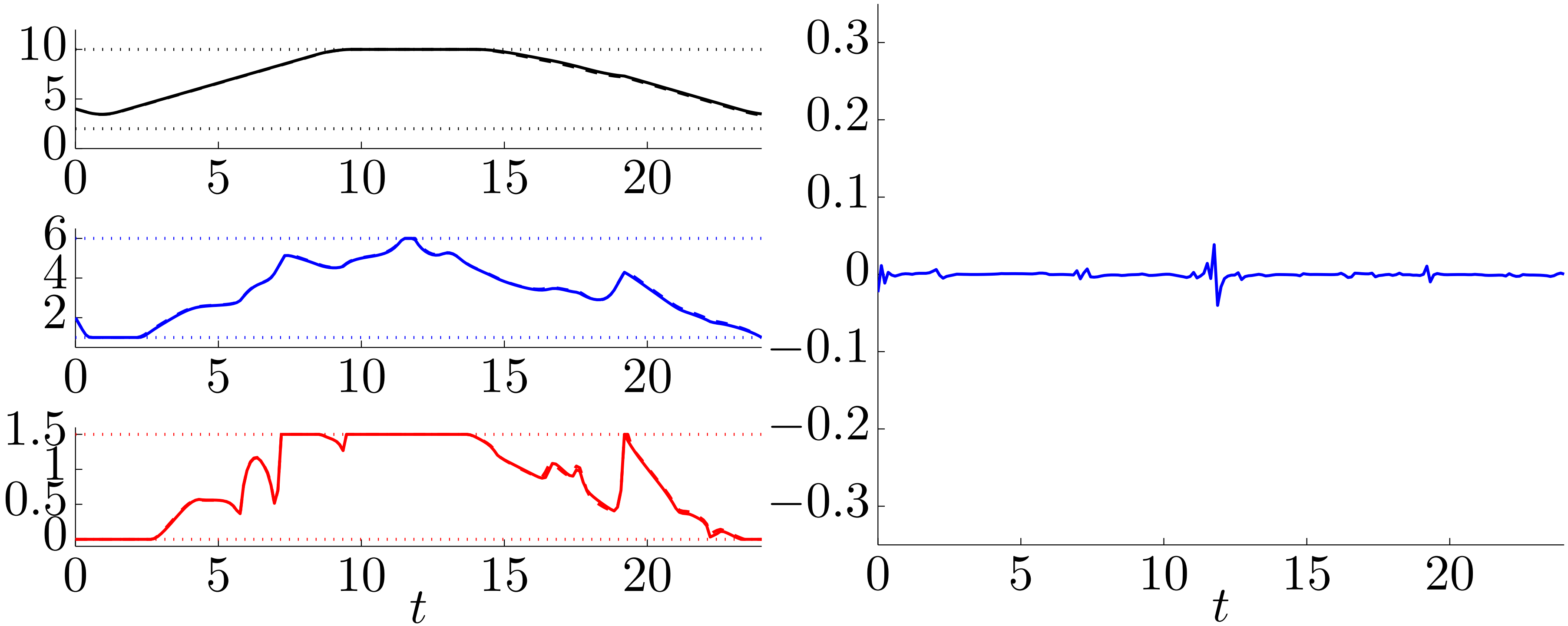
iteration: $k = 25$



- left: solid: optimal generator profile, dashed: profile at kth iteration
- right: residual vector $\bar{x}^k$

Convergence

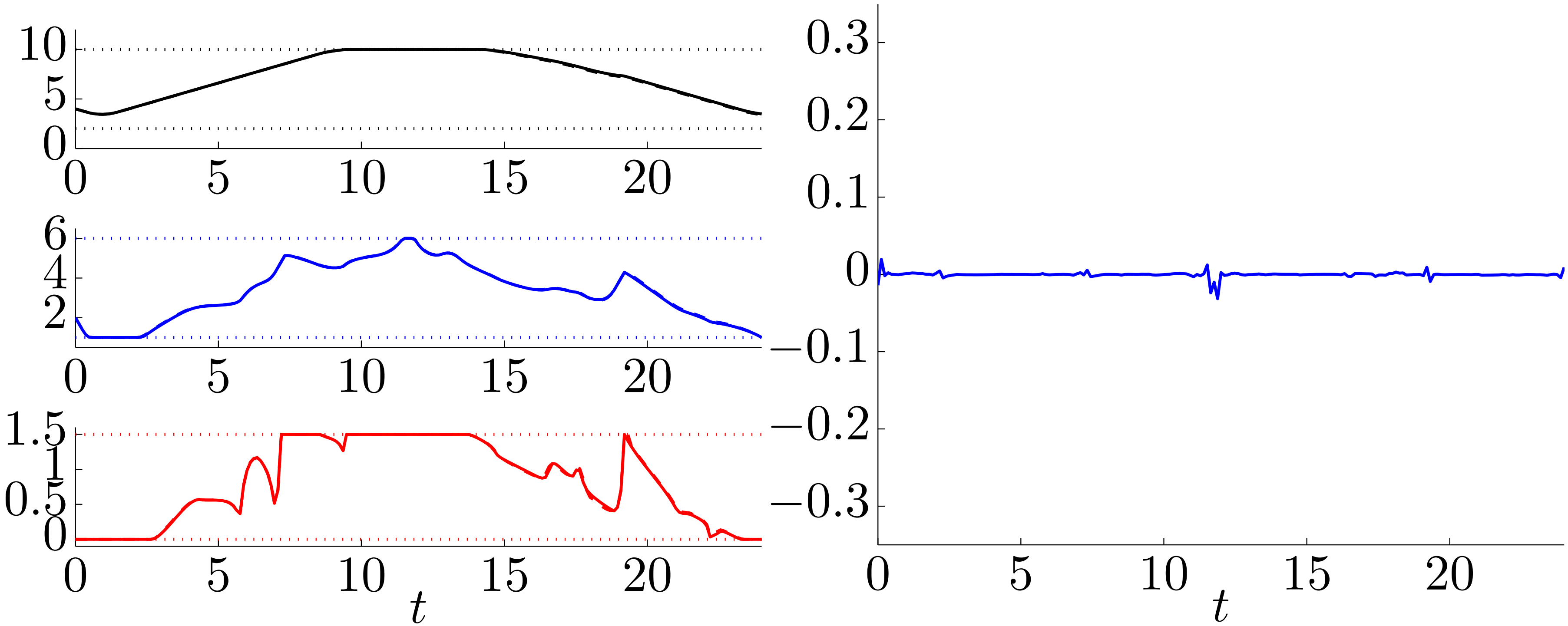
iteration: $k = 30$



- left: solid: optimal generator profile, dashed: profile at kth iteration
- right: residual vector $\bar{x}^k$

Convergence

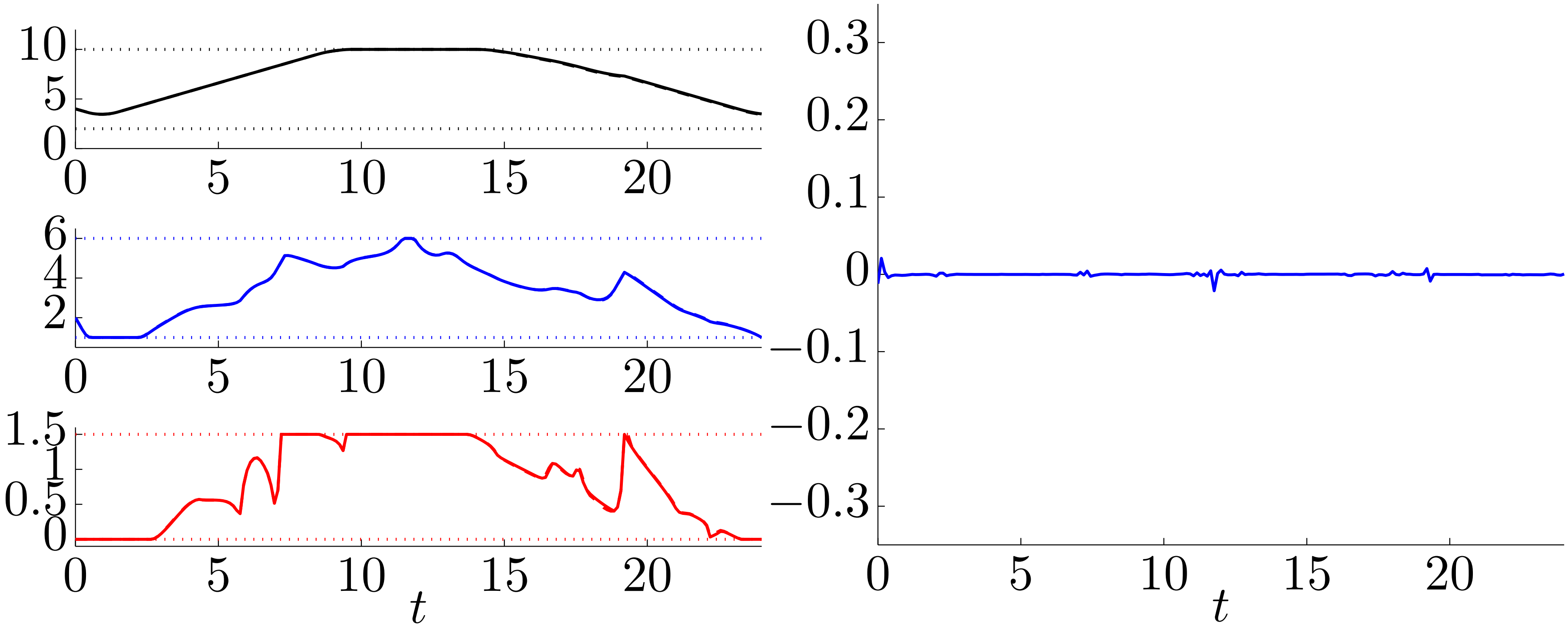
iteration: $k = 35$



- left: solid: optimal generator profile, dashed: profile at kth iteration
- right: residual vector $\bar{x}^k$

Convergence

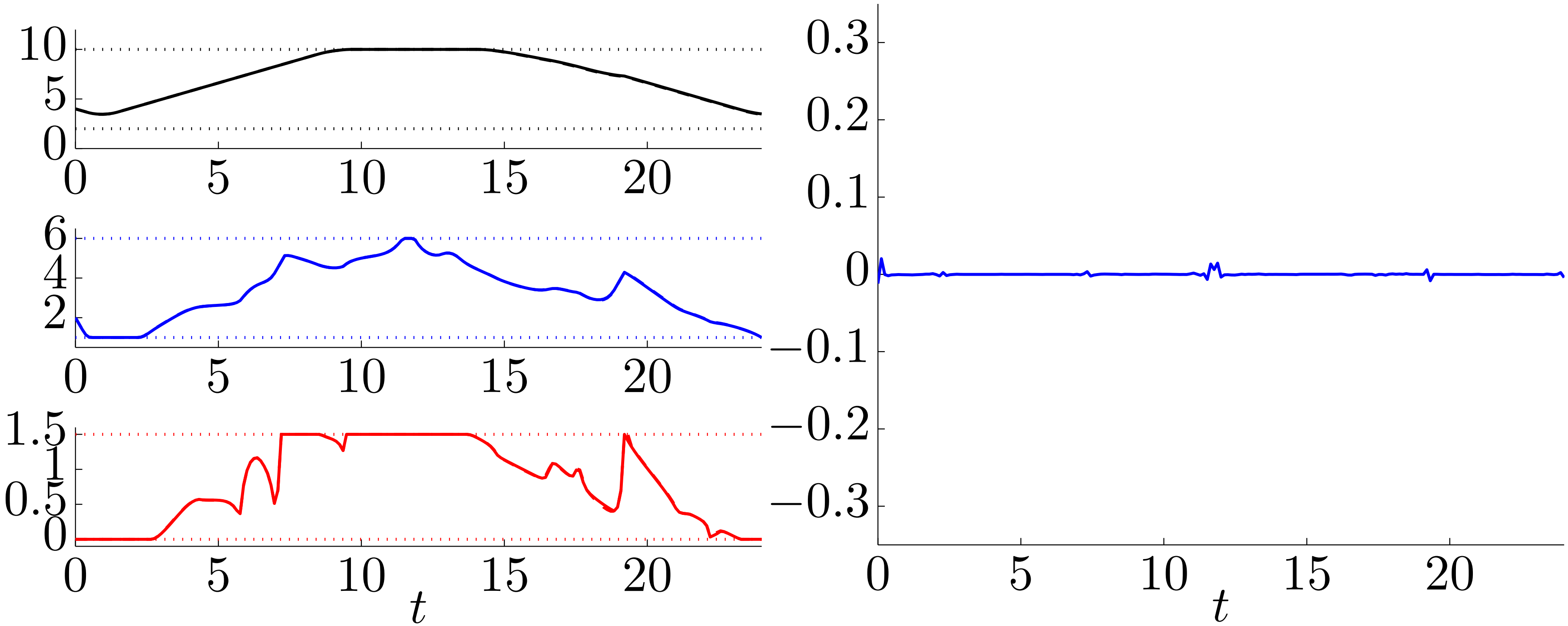
iteration: $k = 40$



- left: solid: optimal generator profile, dashed: profile at kth iteration
- right: residual vector $\bar{x}^k$

Convergence

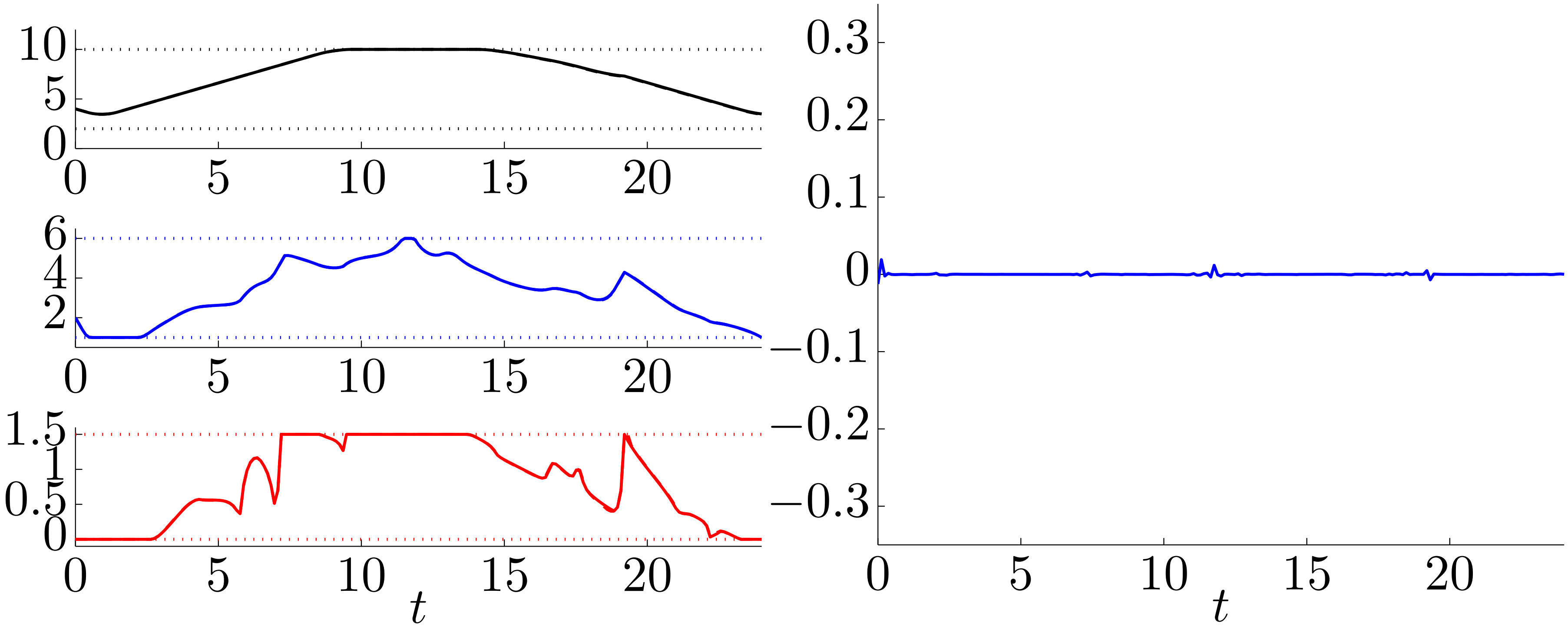
iteration: $k = 45$



- left: solid: optimal generator profile, dashed: profile at kth iteration
- right: residual vector $\bar{x}^k$

Convergence

iteration: $k = 50$



- left: solid: optimal generator profile, dashed: profile at kth iteration
- right: residual vector $\bar{x}^k$

Outline

Summary and conclusions

ADMM

- is the same as, or closely related to, many methods with other names
- has been around since the 1970s
- gives simple single-processor algorithms that can be competitive with state-of-the-art
- can be used to coordinate many processors, each solving a substantial problem, to solve a very large problem