Numerical Optimization, 2023 Fall
Homework 2

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1 Standard Form

Convert the following problem to a linear program in standard form. [20pts]

$$\begin{aligned} \max_{\boldsymbol{x}\in\mathbb{R}^4} \quad & 2x_1 - x_3 + x_4 \\ \text{s.t.} \quad & x_1 + x_2 \geq 5 \\ & x_1 - x_3 \leq 2 \\ & 4x_2 + 3x_3 - x_4 \leq 10 \\ & x_1 \geq 0 \end{aligned} \tag{1}$$

The standard form should have the objective function as a minimization problem.
So the objective function can be written as:

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^4} \quad & -(2x_1 - x_3 + x_4) \\ i.e. \min_{\boldsymbol{x}\in\mathbb{R}^4} \quad & -2x_1 + x_3 - x_4 \end{aligned} \tag{2}$$

Let s_1, s_2, s_3 be the slack variables for the first, second and third constraints, respectively.
And $s_1, s_2, s_3 \geq 0$.
So the inequality constraints can be written as:

$$\begin{aligned} & x_1 + x_2 = 5 + s_1 \\ & x_1 - x_3 = 2 - s_2 \\ & 4x_2 + 3x_3 - x_4 = 10 - s_3 \end{aligned} \tag{3}$$

Since there are no constraints on the boundary of x_2, x_3 and x_4 separately.
So let $x_2 = u_2 - v_2$, $x_3 = u_3 - v_3$, $x_4 = u_4 - v_4$, where $u_2, u_3, u_4, v_2, v_3, v_4 \geq 0$.
And put them into the origin problem, we can get the standard form of the origin problem:

So the standard form of the origin problem is:

$$\begin{aligned} \min_{x_1,u_2,u_3,u_4,v_2,v_3,v_4,s_1,s_2,s_3} \quad & -2x_1 + u_3 - v_3 - u_4 + v_4 \\ \text{s.t.} \quad & x_1 + u_2 - v_2 - s_1 = 5 \\ & x_1 - u_3 + v_3 + s_2 = 2 \\ & 4u_2 - 4v_2 + 3u_3 - 3v_3 - u_4 + v_4 + s_3 = 10 \\ & x_1, u_2, u_3, u_4, v_2, v_3, v_4, s_1, s_2, s_3 \geq 0 \end{aligned} \tag{4}$$

2 Two-Phase Simplex

Use the two-phase simplex procedure to solve the following problem. [40pts]

$$\begin{aligned}\min_{\boldsymbol{x}\in\mathbb{R}^4} \quad & -3x_1 + x_2 + 3x_3 - x_4 \\ \text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 = 0 \\ & 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9 \\ & x_1 - x_2 + 2x_3 - x_4 = 6 \\ & x_1, x_2, x_3, x_4 \geq 0\end{aligned} \tag{5}$$

Since the origin problem is already the standard form, we can directly use the two-phase simplex procedure to solve it.

1. Phase one:

The supporting problem is:

$$\begin{aligned}\min_{\boldsymbol{x}\in\mathbb{R}^7} \quad & x_5 + x_6 + x_7 \\ \text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 + x_5 = 0 \\ & 2x_1 - 2x_2 + 3x_3 + 3x_4 + x_6 = 9 \\ & x_1 - x_2 + 2x_3 - x_4 + x_7 = 6 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0\end{aligned} \tag{6}$$

And the supporting problem's simplex tableau is:

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \boldsymbol{b} \\ \hline x_5 & 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\ x_6 & 2 & -2 & 3 & 3 & 0 & 1 & 0 & 9 \\ x_7 & 1 & -1 & 2 & -1 & 0 & 0 & 1 & 6 \\ \hline \boldsymbol{c}^T/\boldsymbol{r}^T & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array} \tag{7}$$

The basic is $B = (x_5, x_6, x_7)$, and $\boldsymbol{x} = (0, 0, 0, 0, 0, 9, 6)^T$.

Then add the row 1,2,3 to the row 4, to let the base variables' reduced cost become 0, we can get:

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \hline x_5 & \boxed{1} & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\ x_6 & 2 & -2 & 3 & 3 & 0 & 1 & 0 & 9 \\ x_7 & 1 & -1 & 2 & -1 & 0 & 0 & 1 & 6 \\ \hline \boldsymbol{r}^T & -4 & 1 & -4 & -3 & 0 & 0 & 0 & -15 \end{array} \tag{8}$$

The basic is $B = (x_5, x_6, x_7)$.

We choose the leftmost column with negative reduced cost, which is x_1.

And we choose the row with the minimum ratio, which is row 1, and pivot, let x_1 in base and x_5 out base.

$$
\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \boldsymbol{B}^{-1}\boldsymbol{b} \\
\hline
x_1 & 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\
x_6 & 0 & -6 & \boxed{5} & 1 & -2 & 1 & 0 & 9 \\
x_7 & 0 & -3 & 3 & -2 & -1 & 0 & 1 & 6 \\
\hline
\boldsymbol{r}^T & 0 & 9 & -8 & 1 & 4 & 0 & 0 & -15
\end{array} \tag{9}
$$

The basic is $B = (x_1, x_6, x_7)$.
We choose the leftmost column with negative reduced cost, which is x_3.
And we choose the row with the minimum ratio, which is row 2, and pivot, let x_3 in base and x_6 out base.

$$
\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \boldsymbol{B}^{-1}\boldsymbol{b} \\
\hline
x_1 & 1 & \frac{4}{5} & 0 & \frac{6}{5} & \frac{3}{5} & \frac{1}{5} & 0 & \frac{9}{5} \\
x_3 & 0 & -\frac{6}{5} & 1 & \frac{1}{5} & -\frac{2}{5} & \frac{1}{5} & 0 & \frac{9}{5} \\
x_7 & 0 & \boxed{\frac{3}{5}} & 0 & -\frac{13}{5} & -\frac{1}{5} & -\frac{3}{5} & 1 & \frac{3}{5} \\
\hline
\boldsymbol{r}^T & 0 & -\frac{3}{5} & 0 & \frac{13}{5} & \frac{4}{5} & \frac{8}{5} & 0 & -\frac{3}{5}
\end{array} \tag{10}
$$

The basic is $B = (x_1, x_3, x_7)$.
We choose the leftmost column with negative reduced cost, which is x_2.
And we choose the row with the minimum ratio, which is row 3, and pivot, let x_2 in base and x_7 out base.

$$
\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \boldsymbol{B}^{-1}\boldsymbol{b} \\
\hline
x_1 & 1 & 0 & 0 & \frac{14}{3} & \frac{1}{3} & 1 & -\frac{4}{3} & 1 \\
x_3 & 0 & 0 & 1 & -5 & 0 & -1 & 2 & 3 \\
x_2 & 0 & 1 & 0 & -\frac{13}{3} & \frac{1}{3} & -1 & \frac{5}{3} & 1 \\
\hline
\boldsymbol{r}^T & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array} \tag{11}
$$

The basic is $B = (x_1, x_2, x_3)$.
And all the reduced cost are non-negative, so the supporting problem is feasible.
So the phase one is finished.
And the basic feasible solution is $\boldsymbol{x} = (1, 1, 3, 0, 0, 0, 0)^T$.

2. Phase two:

The tableau of the origin problem is:

$$
\begin{array}{c|cccc|c}
 & x_1 & x_2 & x_3 & x_4 & \boldsymbol{b} \\
\hline
x_1 & 1 & 0 & 0 & \frac{14}{3} & 1 \\
x_3 & 0 & 0 & 1 & -5 & 3 \\
x_2 & 0 & 1 & 0 & -\frac{13}{3} & 1 \\
\hline
\boldsymbol{c}^T/\boldsymbol{r}^T & -3 & 1 & 3 & -1 & 0
\end{array} \tag{12}
$$

Then let the base variables' reduced cost become 0, we can get:

$$
\begin{array}{c|cccc|c}
 & x_1 & x_2 & x_3 & x_4 & \boldsymbol{B}^{-1}\boldsymbol{b} \\
\hline
x_1 & 1 & 0 & 0 & \frac{14}{3} & 1 \\
x_3 & 0 & 0 & 1 & -5 & 3 \\
x_2 & 0 & 1 & 0 & -\frac{13}{3} & 1 \\
\hline
\boldsymbol{r}^T & 0 & 0 & 0 & \frac{97}{3} & -7
\end{array}
\tag{13}
$$

So above all, the basic feasible solution of the origin problem is $\boldsymbol{x} = (1, 1, 3, 0)^T$. And the optimal value is 7.

3 Extreme Point

3.1 Q1

Prove that the extreme points of the following two sets are in one-to-one correspondence. [20pts]

$$\begin{aligned} S_1 &= \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq 0\} \\ S_2 &= \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}, \boldsymbol{x} \geq 0, \boldsymbol{y} \geq 0\} \end{aligned} \tag{14}$$

, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$.

Suppose that the extreme points of S_1 compose the set P_1.
And the extreme points of S_2 compose the set P_2.
So we can construct the mapping from P_1 to P_2.
$\forall \boldsymbol{x} \in P_1$, we can get that $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \geq 0$, and $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x}$ is the extreme point of S_1.
so there must $\exists \boldsymbol{y} \in \mathbb{R}^m$, such that $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}$, where $\boldsymbol{y}_i = (\boldsymbol{A}\boldsymbol{x})_i - \boldsymbol{b}_i$.
Where $(\boldsymbol{A}\boldsymbol{x})_i$ be the ith element of the vector $(\boldsymbol{A}\boldsymbol{x})$, and $\boldsymbol{y}_i$ be the ith element of the vector $\boldsymbol{y}$, $\boldsymbol{b}_i$ be the ith element of the vector $\boldsymbol{b}$, $i = 1, \cdots, m$.
Then we say that $(\boldsymbol{x}, \boldsymbol{y})$ is the mapping result.
So we have constructed the mapping from P_1 to P_2 that $\boldsymbol{x} \to (\boldsymbol{x}, \boldsymbol{y})$.

Then we just need to prove that the mapping is bijective.
$\forall \boldsymbol{x} \in P_1$, we get $(\boldsymbol{x}, \boldsymbol{y})$ throw the mapping.
1. First, we need to prove that the mapping result is in S_2:
Since $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$, i.e. $(\boldsymbol{A}\boldsymbol{x})_i \leq \boldsymbol{b}_i$,
so $\boldsymbol{y}_i = \boldsymbol{b}_i - (\boldsymbol{A}\boldsymbol{x})_i \geq 0$, $i = 1, \cdots, m$.
i.e. $\boldsymbol{y} \geq 0$.
So $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{y} \in \mathbb{R}^m, \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}, \boldsymbol{x} \geq 0, \boldsymbol{y} \geq 0$.
So we have proved that $(\boldsymbol{x}, \boldsymbol{y}) \in S_2$.

2. Then we need to prove that its also an extreme point of S_2, i.e. $(\boldsymbol{x}, \boldsymbol{y}) \in P_2$.
We can prove this by contradiction.
Suppose that $(\boldsymbol{x}, \boldsymbol{y})$ is not an extreme point of S_2.
Then there must $\exists \lambda \in (0, 1)$, and $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2) \in S_2, (\boldsymbol{x}_1, \boldsymbol{y}_1) \neq (\boldsymbol{x}_2, \boldsymbol{y}_2)$.
i.e. $\boldsymbol{x} = \lambda\boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2$.
And since $\boldsymbol{x}$ is generated throw the mapping, so $\boldsymbol{x} \in P_1$.
i.e. $\boldsymbol{x}$ is the extreme point in S_1.
But since $\boldsymbol{x} = \lambda\boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2$, so it is contradict with $\boldsymbol{x}$ is the extreme point of S_1.
So the assumption not valid.
So above all, we have proved that $(\boldsymbol{x}, \boldsymbol{y}) \in P_2$.

Combine 1. and 2., we have proved that the mapping from P_1 to P_2 is surjective.

3. Then we need to prove that the mapping from P_1 to P_2 is injective.
If $\boldsymbol{x}_1 = \boldsymbol{x}_2 \in P_1$,
then $\boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{A}\boldsymbol{x}_2$,
$\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_2$,
Since $\boldsymbol{y}_1 = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_1, \boldsymbol{y}_2 = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_2$
So $\boldsymbol{y}_1 = \boldsymbol{y}_2$.
So above all, if $\boldsymbol{x}_1 = \boldsymbol{x}_2$, then $(\boldsymbol{x}_1, \boldsymbol{y}_1) = (\boldsymbol{x}_2, \boldsymbol{y}_2) \in P_2$.
So the mapping from P_1 to P_2 is injective.

From 1., 2., we have get that the mapping is surjective, and from 3., we get that the mapping is injective. So the mapping is bijective.

So above all, the mapping is bijective. i.e. We have proved that the extreme points of S_1 and S_2 are one-to-one correspondence.

3.2 Q2

Does the set $P = \{\boldsymbol{x} \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$ have extreme points? What is its standard form? Does it have extreme points in its standard form? If so, give a extreme point and explain why it is a extreme point. [20pts]

Since the set P is the polyhedron consists of two parallel lines, the polyhedron contain a line, so it has no extreme points.

Since x_2 is not bounded, so we can let $x_2 = u - v$, where $u, v \geq 0$.
And since $x_1 \leq 1$, so we can add slack variable s_1 to the inequality constraint,
i.e. $x_1 + s_1 = 1$.

so the standard form of P is:

$$\begin{aligned} \min_{x_1, s_1, u, v} \quad & \textit{constant} \\ \text{s.t.} \quad & x_1 + s_1 = 1 \\ & x_1, s_1, u, v \geq 0 \end{aligned} \tag{15}$$

And its variables are x_1, s_1, u, v

And the standard form has extreme points, and $(x_1, s_1, u, v) = (1, 0, 0, 0)$ is one of the extreme points. This is because:
$x_1 + s_1 = 1, s_1 = 0, u = 0, v = 0$
This makes the constraints

$$\begin{aligned} & x_1 + s_1 = 1 \\ & s_1 \geq 0 \\ & v \geq 0 \\ & u \geq 0 \end{aligned} \tag{16}$$

activate, and these 4 constrains are independent.
Since the number of variables of the standard form is 4, and the number of independent activate constraints is 4, so $(x_1, s_1, u, v) = (1, 0, 0, 0)$ is one of the extreme points.

So above all, P has no extreme points.
The standard form of P is:

$$\begin{aligned} \min_{x_1, s_1, u, v} \quad & \textit{constant} \\ \text{s.t.} \quad & x_1 + s_1 = 1 \\ & x_1, s_1, u, v \geq 0 \end{aligned} \tag{17}$$

The standard form has extreme points, and $(x_1, s_1, u, v) = (1, 0, 0, 0)$ is one of the extreme points. The reasons are above.