Numerical Optimization, 2023 Fall

Homework 3

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Problem 1. Prove the dual of the dual of a linear programming (standard form) is itself.[25pts]

Consider a linear programming that is in standard form:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned} \tag{1}$$

The Lagrangian of the above linear programming is:

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{\lambda}^T (A\boldsymbol{x} - \boldsymbol{b}) \tag{2}$$

The dual problem is that:

$$\begin{aligned} \max_{\boldsymbol{\lambda}} \quad & \boldsymbol{\lambda}^T \boldsymbol{b} \\ \text{s.t.} \quad & A^T \boldsymbol{\lambda} \leq \boldsymbol{c} \\ & \boldsymbol{\lambda} \geq 0 \end{aligned} \tag{3}$$

The Lagrangian of the dual problem is:

$$L(\boldsymbol{\lambda}, \boldsymbol{x}) = \boldsymbol{\lambda}^T \boldsymbol{b} + \boldsymbol{x}^T (A^T \boldsymbol{\lambda} - \boldsymbol{c}) \tag{4}$$

The dual of the dual problem is that:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned} \tag{5}$$

So above all, the dual of the dual of a linear programming (standard form) is itself.

Problem 2. Prove the dual objective increases after a pivot of the dual simplex method.[25pts]

Problem 3. Let $L(\boldsymbol{x}, \boldsymbol{\lambda})$ be the Lagrangian of a linear programming problem, and $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ be the optimal primal-dual solution. Prove that

$$L(\boldsymbol{x}, \boldsymbol{\lambda}^*) \geq L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \geq L(\boldsymbol{x}^*, \boldsymbol{\lambda}),$$

for any primal feasible $\boldsymbol{x}$ and dual feasible $\boldsymbol{\lambda}$.[25pts]

Problem 4. Construct a linear programming problem for which both the primal and the dual problem has no feasible solution.[25pts]

Construct a linear programming problem that is:

$$\begin{aligned} \min_{x_1, x_2} \quad & x_1 - 2x_2 \\ \text{s.t.} \quad & x_1 - x_2 \le 1 \\ & x_1 - x_2 \ge 2 \\ & x_1, x_2 \le 0 \end{aligned} \tag{6}$$

Since it is impossible to satisfy $x_1 - x_2 \le 1$ and $x_1 - x_2 \ge 2$ at the same time, so the primal problem has no feasible solution.
And the dual problem is that:

$$\begin{aligned} \max_{\lambda_1, \lambda_2} \quad & \lambda_1 + 2\lambda_2 \\ \text{s.t.} \quad & \lambda_1 + \lambda_2 \le 1 \\ & -\lambda_1 - \lambda_2 \le -2 \\ & \lambda_1 \le 0, \lambda_2 \ge 0 \end{aligned} \tag{7}$$

The second constrain $-\lambda_1 - \lambda_2 \le -2$ can be written as $\lambda_1 + \lambda_2 \ge 2$.
Since it is impossible to satisfy $\lambda_1 + \lambda_2 \le 1$ and $\lambda_1 + \lambda_2 \ge 2$ at the same time, so the dual problem has no feasible solution.

So above all, the above construction's primal and dual problem has no feasible solution.