Numerical Optimization, 2023 Fall

Homework 3

Name: Zhou Shouchen
Student ID: 2021533042

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Problem 1. Prove the dual of the dual of a linear programming (standard form) is itself.[25pts]

We can prove this with the help of the Duality Scheme.
Consider a linear programming that is in standard form:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{1}$$

The Lagrangian of the above linear programming is $L(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{c}^T \boldsymbol{x} - \boldsymbol{\lambda}^T (A\boldsymbol{x} - \boldsymbol{b})$.

Since for the primal question, the variables are $\boldsymbol{x} \geq 0$, so for $x_i \geq 0$, the dual constrain is that:

$$a_i^T \boldsymbol{\lambda} \leq c_i$$

where a_i is the i-th column of A.

So for the dual problem, the constrains are $A^T \boldsymbol{\lambda} \leq \boldsymbol{c}$.

And since for the primal question, the constrain it that $A\boldsymbol{x} = \boldsymbol{b}$, i.e. $\sum_{j=1}^{n} a_{ij} x_j = b_i$.
So for the dual question, the variables $\boldsymbol{\lambda}$ is free.

So the dual problem is that:

$$\begin{aligned} \max_{\boldsymbol{\lambda}} \quad & \boldsymbol{\lambda}^T \boldsymbol{b} \\ \text{s.t.} \quad & A^T \boldsymbol{\lambda} \leq \boldsymbol{c} \\ & \boldsymbol{\lambda} \text{ is free} \end{aligned} \tag{2}$$

To easily get the dual of the dual problem, we can first convert the objective function of the dual problem to a minimization problem. And take $M = A^T$, the dual problem becomes:

$$\begin{aligned} \min_{\boldsymbol{\lambda}} \quad & \boldsymbol{\lambda}^T (-\boldsymbol{b}) \\ \text{s.t.} \quad & M \boldsymbol{\lambda} \leq \boldsymbol{c} \\ & \boldsymbol{\lambda} \text{ is free} \end{aligned} \tag{3}$$

The Lagrangian of the dual problem is:

$$L(\boldsymbol{\lambda}, \boldsymbol{y}) = \boldsymbol{\lambda}^T (-\boldsymbol{b}) - \boldsymbol{y}^T (M\boldsymbol{\lambda} - \boldsymbol{c})$$

Since for the dual question, the variables are $\boldsymbol{\lambda}$ is free, so for λ_i is free, the dual of the dual constrain is that:

$$m_i^T \boldsymbol{\lambda} = (-\boldsymbol{b})_i = -b_i$$

where m_i is the i-th column of M.

So for the dual problem, the constrains are

$$M^T\boldsymbol{y} = -\boldsymbol{b}$$

And since for the dual question, the constrain it that $M\boldsymbol{\lambda} \leq \boldsymbol{c}$, i.e.$\sum_{j=1}^{n} m_{ij}\lambda_j \leq c_i$. so for the dual question, the variables are $\boldsymbol{y} \leq \boldsymbol{0}$.

So the dual of the dual problem is that:

$$\begin{aligned} \max_{\boldsymbol{y}} \quad & \boldsymbol{c}^T\boldsymbol{y} \\ \text{s.t.} \quad & M^T\boldsymbol{y} = -\boldsymbol{b} \\ & \boldsymbol{y} \leq \boldsymbol{0} \end{aligned} \tag{4}$$

We can take $\boldsymbol{x} = -\boldsymbol{y}$. And since $M = A^T$, so $M^T = (A^T)^T = A$.

Consider the objective function is

$$\max_{\boldsymbol{y}} \quad \boldsymbol{c}^T\boldsymbol{y}$$

We can convert it into a minimization problem by taking $-\boldsymbol{y}$ as the variable.
i.e.

$$\min_{\boldsymbol{y}} \quad \boldsymbol{c}^T(-\boldsymbol{y})$$

And since we have $\boldsymbol{x} = -\boldsymbol{y}$, so we can convert the above minimization problem into:

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T\boldsymbol{x}$$

And for the first constrain, we have:

$$M^T\boldsymbol{y} = -\boldsymbol{b}$$

Since $M^T = A$, and move the "-" from the right to the left, we have:

$$A(-\boldsymbol{y}) = \boldsymbol{b}$$

i.e.

$$A\boldsymbol{x} = \boldsymbol{b}$$

For the second constrain, we have:

$$\boldsymbol{y} \leq \boldsymbol{0}$$

We can convert it into:

$$-\boldsymbol{y} \geq \boldsymbol{0}$$

i.e.

$$\boldsymbol{x} \geq \boldsymbol{0}$$

So with the convertions above, we can get that the dual of the dual problem is that:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{5}$$

Which is exactly same with the primal problem.
So above all, the dual of the dual of a linear programming (standard form) is itself.

Problem 2. Prove the dual objective increases after a pivot of the dual simplex method.[25pts]

Consider the dual simplex method.

Suppose that the current state is feasible, and after the pivot, it is also feasible.

Then for the pivot process, we have:

$r^T \geq 0$ always satisfies, this is to make sure the dual problem is feasible.

As for choosing the pivot element, we choose the p-th row s.t. the current b_p in the tableau is nagetive.

i.e. $b_p < 0$.

Suppose that the p-th row is $y_{p1}, y_{p2}, \cdots, y_{pn}, b_p$, and we choose the pivot element a_{pq}.

s.t. $\hat{\epsilon} = \dfrac{r_q}{-y_{pq}} = \min\{\dfrac{r_i}{-y_{pi}} \mid a_{pi} < 0, i = 1, \cdots, n\}$.

Then we pivot with y_{pq}, and we need to update the tableau to make the r_q become 0 by:

Let the last line of the simplex tableau adds $\hat{\epsilon}$ times the p-th row.

So we have

$$r'_q = r_q + \hat{\epsilon} y_{pq} = r_q + \frac{r_q}{-y_{pq}} y_{pq} = 0$$

$$-f' = -f + \hat{\epsilon} b_p = -f + \frac{r_q}{-y_{pq}} b_p$$

We know that the lower-right corner if $-f$, where f is the dual objective value of the dual problem. So after the pivot, the lower-right corner becomes:

$$-f' = -f - \frac{r_q}{y_{pq}} b_p$$

Since $r_q > 0, y_{pq} < 0, b_p < 0$, so $\frac{r_q}{y_{pq}} b_p > 0$ So

$$-f = -f - \frac{r_q}{y_{pq}} b_p < -f'$$

i.e.

$$f' > f$$

So above all, the dual objective increases after a pivot of the dual simplex method.

Problem 3. Let $L(\boldsymbol{x}, \boldsymbol{\lambda})$ be the Lagrangian of a linear programming problem, and $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ be the optimal primal-dual solution. Prove that

$$L(\boldsymbol{x}, \boldsymbol{\lambda}^*) \geq L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \geq L(\boldsymbol{x}^*, \boldsymbol{\lambda}),$$

for any primal feasible $\boldsymbol{x}$ and dual feasible $\boldsymbol{\lambda}$.[25pts]

Suppose that the primal problem is that:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned} \tag{6}$$

And the dual problem is that:

$$\begin{aligned} \max_{\boldsymbol{\lambda}} \quad & \boldsymbol{\lambda}^T \boldsymbol{b} \\ \text{s.t.} \quad & A^T \boldsymbol{\lambda} \leq \boldsymbol{c} \end{aligned} \tag{7}$$

And the Lagrangian of the primal problem is that:

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{\lambda}^T (A\boldsymbol{x} - \boldsymbol{b})$$

Since we can considering the primal feasible $\boldsymbol{x}$ and dual feasible $\boldsymbol{\lambda}$, so we have:

$$A\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{c}^T \boldsymbol{x} \geq \boldsymbol{c}^T \boldsymbol{x}^*$$

If we put $A\boldsymbol{x} - \boldsymbol{b} = \boldsymbol{0}$ into the Lagrangian, we have:

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{\lambda}^T (A\boldsymbol{x} - \boldsymbol{b}) = \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{\lambda}^T \boldsymbol{0} = \boldsymbol{c}^T \boldsymbol{x}$$

And since $\boldsymbol{c}^T \boldsymbol{x} \geq \boldsymbol{c}^T \boldsymbol{x}^*$, and $L(\boldsymbol{x}, \boldsymbol{\lambda})$ do not contain $\boldsymbol{\lambda}$, so we have:

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) \geq L(\boldsymbol{x}^*, \boldsymbol{\lambda}) = L(\boldsymbol{x}^*, \boldsymbol{\lambda})$$

So above all, we have prove that

$$L(\boldsymbol{x}, \boldsymbol{\lambda}^*) \geq L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \geq L(\boldsymbol{x}^*, \boldsymbol{\lambda})$$

Problem 4. Construct a linear programming problem for which both the primal and the dual problem has no feasible solution.[25pts]

Construct a linear programming problem that is:

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1 - 2x_2 \\ \text{s.t.} \quad & x_1 - x_2 \leq 1 \\ & x_1 - x_2 \geq 2 \\ & x_1, x_2 \leq 0 \end{aligned} \tag{8}$$

Since it is impossible to satisfy $x_1 - x_2 \leq 1$ and $x_1 - x_2 \geq 2$ at the same time, so the primal problem has no feasible solution.
And the dual problem is that:

$$\begin{aligned} \max_{\lambda_1,\lambda_2} \quad & \lambda_1 + 2\lambda_2 \\ \text{s.t.} \quad & \lambda_1 + \lambda_2 \leq 1 \\ & -\lambda_1 - \lambda_2 \leq -2 \\ & \lambda_1 \leq 0, \lambda_2 \geq 0 \end{aligned} \tag{9}$$

The second constrain $-\lambda_1 - \lambda_2 \leq -2$ can be written as $\lambda_1 + \lambda_2 \geq 2$.
Since it is impossible to satisfy $\lambda_1 + \lambda_2 \leq 1$ and $\lambda_1 + \lambda_2 \geq 2$ at the same time, so the dual problem has no feasible solution.

So above all, the above construction's primal and dual problem has no feasible solution.