Homework 1

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Problem i. Write the gradient and Heissan matrix of the following formula. [10pts]

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} + \mathrm{c} \quad (\mathbf{A} \in \mathbf{R}^{\mathbf{n*n}}, \mathbf{b} \in \mathbf{R}^{\mathbf{n}}, \mathrm{c} \in \mathbf{R})$$

Let $f(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} + \mathrm{c}$

1. the gradient of the formula is
$\nabla f = \dfrac{\partial f}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}^{\mathbf{T}})\mathbf{x} + \mathbf{b}$

2. the Heissan matrix of the formula is
$\nabla^2 f = \nabla(\nabla f) = \dfrac{\partial(\nabla f)}{\partial \mathbf{x}} = \mathbf{A} + \mathbf{A}^{\mathbf{T}}$

Problem ii. Write the gradient and Heissan matrix of the following formula. [10pts]

$$\|\mathbf{Ax}-\mathbf{b}\|_2^2 \quad (\mathbf{A} \in \mathbf{R}^{\mathbf{m*n}}, \mathbf{b} \in \mathbf{R}^{\mathbf{m}})$$

Let $f(\mathbf{x}) = \|\mathbf{Ax}-\mathbf{b}\|_2^2 = (\mathbf{Ax}-\mathbf{b})^T(\mathbf{Ax}-\mathbf{b}) = \mathbf{x^TA^TAx} - \mathbf{2x^TA^Tb} + \mathbf{b^Tb}$

1. the gradient of the formula is

$\nabla f = \dfrac{\partial f}{\partial \mathbf{x}} = \mathbf{2A^TAx} - \mathbf{2A^Tb}$

2. the Heissan matrix of the formula is

$\nabla^2 f = \nabla(\nabla f) = \dfrac{\partial(\nabla f)}{\partial \mathbf{x}} = \mathbf{2A^TA}$

Problem iii. Convert the following problem to linear programming. [10pts]

$$\min_{\mathbf{x}\in\mathbf{R^n}} \|\mathbf{Ax}-\mathbf{b}\|_1 + \|\mathbf{x}\|_\infty \quad (\mathbf{A}\in\mathbf{R^{m*n}}, \mathbf{b}\in\mathbf{R^m})$$

1. for the first part,

let $z_i = |\mathbf{c_i}^T\mathbf{x} - b_i|$, where $\mathbf{c_i}$ is the i-th row of the matrix $\mathbf{A}$, and b_i is the i-th element in the column vector $\mathbf{b}$.
So $\|\mathbf{Ax}-\mathbf{b}\|_1 = \sum\limits_{i=1}^{m} z_i$
And $-z_i \leq \mathbf{c_i}^T\mathbf{x} - b_i \leq z_i$ and $z_i \geq 0$

2. for the second part,

let $w = \max\limits_{i=1,2,\cdots,n} x_i$, where x_i is the i-th element in the vector $\mathbf{x}$
So $\|\mathbf{x}\|_\infty = w$
And $w \geq x_i$

So combine the two parts, we convert the problem into the linear programming:

$$\begin{aligned}
\min_{\mathbf{x}\in\mathbf{R^n}} \quad & (\sum_{i=1}^{m} z_i) + w \\
\textit{subject to} \quad & -z_i \leq \mathbf{c_i}^T\mathbf{x} - b_i \leq z_i, \quad i = 1, 2, \cdots, m \\
& z_i \geq 0, \quad i = 1, 2, \cdots, m \\
& w \geq x_i, \quad i = 1, 2, \cdots, n
\end{aligned}$$

Problem vi. Proof the convergence rates of the following point sequences. [30pts]

$$\mathbf{x}^{\mathrm{k}} = \frac{1}{\mathrm{k}}$$
$$\mathbf{x}^{\mathrm{k}} = \frac{1}{\mathrm{k}!}$$
$$\mathbf{x}^{\mathrm{k}} = \frac{1}{2^{2^{\mathrm{k}}}}$$

(Hint: Given two iterates $\mathbf{x}^{\mathrm{k}+1}$ and $\mathbf{x}^{\mathrm{k}}$, and its limit point $\mathbf{x}^*$, there exists real number $\mathrm{q} > 0$, satisfies

$$\lim_{\mathrm{k}\to\infty} \frac{\left\|\mathbf{x}^{\mathrm{k}+1} - \mathbf{x}^*\right\|}{\left\|\mathbf{x}^{\mathrm{k}} - \mathbf{x}^*\right\|} = \mathrm{q}$$

if $0 < \mathrm{q} < 1$, then the point sequence Q-linear convergence; if $\mathrm{q} = 1$, then the point sequence Q-sublinear convergence; if $\mathrm{q} = 0$, then the point sequence Q-superlinear convergence)

1. for $\mathbf{x}^k = \dfrac{1}{k}$:

since $\lim\limits_{k\to\infty} \mathbf{x}^k = 0$, so $\mathbf{x}^* = 0$

$$\text{so } \lim_{k\to\infty} \frac{\left\|\mathbf{x}^{k+1} - \mathbf{x}^*\right\|}{\left\|\mathbf{x}^k - \mathbf{x}^*\right\|} = \lim_{k\to\infty} \frac{\left\|\frac{1}{k+1}\right\|}{\left\|\frac{1}{k}\right\|} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

i.e. $q = 1$. so the point sequence is Q-sublinear convergence.

2. for $\mathbf{x}^k = \dfrac{1}{k!}$:

since $\lim\limits_{k\to\infty} \mathbf{x}^k = 0$, so $\mathbf{x}^* = 0$

$$\text{so } \lim_{k\to\infty} \frac{\left\|\mathbf{x}^{k+1} - \mathbf{x}^*\right\|}{\left\|\mathbf{x}^k - \mathbf{x}^*\right\|} = \lim_{k\to\infty} \frac{\left\|\frac{1}{(k+1)!}\right\|}{\left\|\frac{1}{k!}\right\|} = \lim_{k\to\infty} \frac{k!}{(k+1)!} = \lim_{k\to\infty} \frac{1}{k+1} = 0$$

i.e. $q = 0$. so the point sequence is Q-superlinear convergence.

3. for $\mathbf{x}^k = \dfrac{1}{2^{2^k}}$:

since $\lim\limits_{k\to\infty} \mathbf{x}^k = 0$, so $\mathbf{x}^* = 0$

$$\text{so } \lim_{k\to\infty} \frac{\left\|\mathbf{x}^{k+1} - \mathbf{x}^*\right\|}{\left\|\mathbf{x}^k - \mathbf{x}^*\right\|} = \lim_{k\to\infty} \frac{\left\|\frac{1}{2^{2^{k+1}}}\right\|}{\left\|\frac{1}{2^{2^k}}\right\|} = \lim_{k\to\infty} \frac{2^{2^k}}{2^{2^{k+1}}} = \lim_{k\to\infty} 2^{-2^k} = 0$$

i.e. $q = 0$. so the point sequence is Q-superlinear convergence.

So above all, the point sequence $\mathbf{x}^k = \dfrac{1}{k}$ is Q-sublinear convergence, the point sequence $\mathbf{x}^k = \dfrac{1}{k!}$ and $\mathbf{x}^k = \dfrac{1}{2^{2^k}}$ are Q-superlinear convergence.

Problem v. Select the Haverly Pool Problem or the Horse Racing Problem in the courseware, compile the program using AMPL model language and submit it to https://neos-server.org/neos/solvers/index.html.(Hint: both AMPL solver and NEOS solver can be used, please indicate the type of solver used in the submitted job, show the solution results (eg: screenshots attached to the PDF file), and submit the source code together with the submitted job, please package as .zip file, including your PDF and source code.) [40pts]

Select the Haverly Pooling Problem.
The codes are in the additional files.
And since the problem is a bilinear problem, i.e. a non linear problem, so I used the the LOQO solver in the nonlinearly constrianed optimization to solve the problem.
The results are as follows:

```
You are using the solver loqo.
Checking ampl.mod for loqo_options...
Executing AMPL.
processing data.
processing commands.
Executing on prod-exec-5.neos-server.org

9 variables:
3 nonlinear variables
6 linear variables
6 constraints; 23 nonzeros
3 nonlinear constraints
3 linear constraints
4 equality constraints
2 inequality constraints
1 linear objective; 6 nonzeros.

LOQO 7.00: LOQO 7.00: optimal solution (25 iterations, 25 evaluations)
primal objective 1799.999995
 dual objective 1800.000021
A = 100
B = 300
Cx = 3.59196e-07
Cy = 2.10653e-07
Px = 200
Py = 200
```

Figure 1: The results of the Haverly Pooling Problem