Numerical Optimization, 2023 Fall Homework 3

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Problem 1. Prove the dual of the dual of a linear programming (standard form) is itself. [25pts]

Consider a linear programming that is in standard form:

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^{T} \boldsymbol{x}$$
s.t. $A\boldsymbol{x} = \boldsymbol{b}$ (1)
$$\boldsymbol{x} \ge 0$$

The Lagrangian of the above linear programming is:

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{\lambda}^T (A\boldsymbol{x} - \boldsymbol{b})$$
 (2)

The dual problem is that:

$$\max_{\lambda} \quad \lambda^{T} \mathbf{b}$$
s.t. $A^{T} \lambda \leq \mathbf{c}$ (3)
$$\lambda \geq 0$$

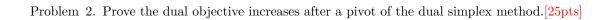
The Lagrangian of the dual problem is:

$$L(\lambda, x) = \lambda^T b + x^T (A^T \lambda - c)$$
(4)

The dual of the dual problem is that:

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^{T} \boldsymbol{x}$$
s.t. $A\boldsymbol{x} = \boldsymbol{b}$ (5)
$$\boldsymbol{x} \ge 0$$

So above all, the dual of the dual of a linear programming (standard form) is itself.



Problem 3. Let $L(x, \lambda)$ be the Lagrangian of a linear programming problem, and (x^*, λ^*) be the optimal primal-dual solution. Prove that

$$L(\boldsymbol{x}, \boldsymbol{\lambda}^*) \ge L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \ge L(\boldsymbol{x}^*, \boldsymbol{\lambda}),$$

for any primal feasible \boldsymbol{x} and dual feasible $\boldsymbol{\lambda}.[25 \mathrm{pts}]$

Problem 4. Construct a linear programming problem for which both the primal and the dual problem has no feasible solution. [25pts]

Construct a linear programming problem that is:

$$\min_{x_1, x_2} \quad x_1 - 2x_2$$
s.t.
$$x_1 - x_2 \le 1$$

$$x_1 - x_2 \ge 2$$

$$x_1, x_2 \le 0$$
(6)

Since it is impossible to satisfy $x_1 - x_2 \le 1$ and $x_1 - x_2 \ge 2$ at the same time, so the primal problem has no feasible solution.

And the dual problem is that:

$$\max_{\lambda_1, \lambda_2} \quad \lambda_1 + 2\lambda_2$$
s.t.
$$\lambda_1 + \lambda_2 \le 1$$

$$-\lambda_1 - \lambda_2 \le -2$$

$$\lambda_1 \le 0, \lambda_2 \ge 0$$

$$(7)$$

The second constrain $-\lambda_1 - \lambda_2 \le -2$ can be written as $\lambda_1 + \lambda_2 \ge 2$.

Since it is impossible to satisfy $\lambda_1 + \lambda_2 \leq 1$ and $\lambda_1 + \lambda_2 \geq 2$ at the same time, so the dual problem has no feasible solution.

So above all, the above construction's primal and dual problem has no feasible solution.