Numerical Optimization, 2023 Fall

Homework 2

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1 Standard Form

Convert the following problem to a linear program in standard form. [20pts]

$$\begin{aligned} \max_{\boldsymbol{x}\in\mathbb{R}^4} \quad & 2x_1 - x_3 + x_4 \\ \text{s.t.} \quad & x_1 + x_2 \geq 5 \\ & x_1 - x_3 \leq 2 \\ & 4x_2 + 3x_3 - x_4 \leq 10 \\ & x_1 \geq 0 \end{aligned} \tag{1}$$

Let s_1, s_2, s_3 be the slack variables for the first, second and third constraints, respectively. And $s_1, s_2, s_3 \geq 0$.
So the inequality constraints can be written as:

$$\begin{aligned} & x_1 + x_2 = 5 + s_1 \\ & x_1 - x_3 = 2 - s_2 \\ & 4x_2 + 3x_3 - x_4 = 10 - s_3 \end{aligned} \tag{2}$$

Also, the standard form should have the objective function as a minimization problem.
So the objective function can be written as:

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^4} \quad & -(2x_1 - x_3 + x_4) \\ i.e. \min_{\boldsymbol{x}\in\mathbb{R}^4} \quad & -2x_1 + x_3 - x_4 \end{aligned} \tag{3}$$

Since there are no constraints on the boundary of x_2, x_3 and x_4 separately.
So let $x_2 = u_2 - v_2$, $x_3 = u_3 - v_3$, $x_4 = u_4 - v_4$, where $u_2, u_3, u_4, v_2, v_3, v_4 \geq 0$.
And put them into the origin problem, we can get the standard form of the origin problem:

So the standard form of the origin problem is:

$$\begin{aligned} \max_{x_1,u_2,u_3,u_4,v_2,v_3,v_4,s_1,s_2,s_3} \quad & 2x_1 - u_3 + v_3 + u_4 - v_4 \\ \text{s.t.} \quad & x_1 + u_2 - v_2 - s_1 = 5 \\ & x_1 - u_3 + v_3 + s_2 = 2 \\ & 4u_2 - 4v_2 + 3u_3 - 3v_3 - u_4 + v_4 + s_3 = 10 \\ & x_1, u_2, u_3, u_4, v_2, v_3, v_4, s_1, s_2, s_3 \geq 0 \end{aligned} \tag{4}$$

2 Two-Phase Simplex

Use the two-phase simplex procedure to solve the following problem. [40pts]

$$
\begin{aligned}
\min_{\boldsymbol{x}\in\mathbb{R}^4} \quad & -3x_1 + x_2 + 3x_3 - x_4 \\
\text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 = 0 \\
& 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9 \\
& x_1 - x_2 + 2x_3 - x_4 = 6 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned} \tag{5}
$$

Since the origin problem is already the standard form, we can directly use the two-phase simplex procedure to solve it.

1. Phase one:

The supporting problem is:

$$
\begin{aligned}
\min \boldsymbol{x} \in \mathbb{R}^7 \quad & x_5 + x_6 + x_7 \\
\text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 + x_5 = 0 \\
& 2x_1 - 2x_2 + 3x_3 + 3x_4 + x_6 = 9 \\
& x_1 - x_2 + 2x_3 - x_4 + x_7 = 6 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0
\end{aligned} \tag{6}
$$

And the supporting problem's simplex tableau is:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	1	2	-1	1	1	0	0	0
	2	-2	3	3	0	1	0	9
	1	-1	2	-1	0	0	1	6
c^T/r^T	0	0	0	0	1	1	1	0

(7)

The basic is $B = (x_5, x_6, x_7)$, and $\boldsymbol{x} = (0,0,0,0,0,9,6)^T$.

Then add the row 1,2,3 to the row 4, to let the base variables' reduced cost become 0, we can get:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	$\boxed{1}$	2	-1	1	1	0	0	0
	2	-2	3	3	0	1	0	9
	1	-1	2	-1	0	0	1	6
r^T	-4	1	-4	-3	0	0	0	-15

(8)

The basic is $B = (x_5, x_6, x_7)$.

We choose the leftmost column with negative reduced cost, which is x_1.

And we choose the row with the minimum ratio, which is row 1, and pivot, let x_1 in base and x_5 out base.

$$
\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{b} \\
\hline
 & 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\
 & 0 & -6 & \boxed{5} & 1 & -2 & 1 & 0 & 9 \\
 & 0 & -3 & 3 & -2 & -1 & 0 & 1 & 6 \\
\hline
r^T & 0 & 9 & -8 & 1 & 4 & 0 & 0 & -15
\end{array} \tag{9}
$$

The basic is $B = (x_1, x_6, x_7)$.
We choose the leftmost column with negative reduced cost, which is x_3.
And we choose the row with the minimum ratio, which is row 2, and pivot, let x_3 in base and x_6 out base.

$$
\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{b} \\
\hline
 & 1 & \frac{4}{5} & 0 & \frac{6}{5} & \frac{3}{5} & \frac{1}{5} & 0 & \frac{9}{5} \\
 & 0 & -\frac{6}{5} & 1 & \frac{1}{5} & -\frac{2}{5} & \frac{1}{5} & 0 & \frac{9}{5} \\
 & 0 & \boxed{\dfrac{3}{5}} & 0 & -\frac{13}{5} & -\frac{1}{5} & -\frac{3}{5} & 1 & \frac{3}{5} \\
\hline
r^T & 0 & -\frac{3}{5} & 0 & \frac{13}{5} & \frac{4}{5} & \frac{8}{5} & 0 & -\frac{3}{5}
\end{array} \tag{10}
$$

The basic is $B = (x_1, x_3, x_7)$.
We choose the leftmost column with negative reduced cost, which is x_2.
And we choose the row with the minimum ratio, which is row 3, and pivot, let x_2 in base and x_7 out base.

$$
\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{b} \\
\hline
 & 1 & 0 & 0 & \frac{14}{3} & \frac{1}{3} & 1 & -\frac{4}{3} & 1 \\
 & 0 & 0 & 1 & -5 & 0 & -1 & 2 & 3 \\
 & 0 & 1 & 0 & -\frac{13}{3} & \frac{1}{3} & -1 & \frac{5}{3} & 1 \\
\hline
r^T & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array} \tag{11}
$$

The basic is $B = (x_1, x_2, x_3)$.
And all the reduced cost are non-negative, so the supporting problem is feasible.
So the phase one is finished.
And the basic feasible solution is $\boldsymbol{x} = (1, 1, 3, 0, 0, 0, 0)^T$.

2. Phase two:

The tableau of the origin problem is:

$$
\begin{array}{c|ccccc}
 & x_1 & x_2 & x_3 & x_4 & \text{b} \\
\hline
 & 1 & 0 & 0 & \frac{14}{3} & 1 \\
 & 0 & 0 & 1 & -5 & 3 \\
 & 0 & 1 & 0 & -\frac{13}{3} & 1 \\
\hline
c^T/r^T & -3 & 1 & 3 & -1 & 0
\end{array} \tag{12}
$$

Then let the base variables' reduced cost become 0, we can get:

$$
\begin{array}{c|ccccc}
 & x_1 & x_2 & x_3 & x_4 & \mathrm{b} \\
\hline
 & 1 & 0 & 0 & \frac{14}{3} & 1 \\
 & 0 & 0 & 1 & -5 & 3 \\
 & 0 & 1 & 0 & -\frac{13}{3} & 1 \\
\hline
r^T & 0 & 0 & 0 & \frac{97}{3} & -7
\end{array}
\tag{13}
$$

So above all, the basic feasible solution of the origin problem is $\boldsymbol{x} = (1, 1, 3, 0)^T$. And the optimal value is 7.

3 Extreme Point

3.1 Q1

Prove that the extreme points of the following two sets are in one-to-one correspondence. [20pts]

$$\begin{aligned} S_1 &= \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq 0\} \\ S_2 &= \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}, \boldsymbol{x} \geq 0, \boldsymbol{y} \geq 0\} \end{aligned} \tag{14}$$

, where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$.

Suppose that the extreme points of S_1 compose the set P_1.
And the extreme points of S_2 compose the set P_2.

$\forall \boldsymbol{x} \in P_1$
So $P_1 \subseteq P_2$.

Similarly, $\forall (\boldsymbol{x}, \boldsymbol{y}) \in P_2$
So $P_2 \subseteq P_1$.

Since $P_1 \subseteq P_2$ and $P_2 \subseteq P_1$ So the extreme points of the sets S_1, S_2 are one-to-one correspondence.

3.2 Q2

Does the set $P = \{\boldsymbol{x} \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$ have extreme points? What is its standard form? Does it have extreme points in its standard form? If so, give a extreme point and explain why it is a extreme point. [20pts]

So above all, P has no extreme points.
The standard form of P is:

The standard form has extreme points, and ... is one of the extreme points. The reasons are above.