Numerical Optimization, 2023 Fall

Homework 2

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1 Standard Form

Convert the following problem to a linear program in standard form. [20pts]

$$
\begin{aligned}
\max_{\boldsymbol{x}\in\mathbb{R}^4} \quad & 2x_1 - x_3 + x_4 \\
\text{s.t.} \quad & x_1 + x_2 \geq 5 \\
& x_1 - x_3 \leq 2 \\
& 4x_2 + 3x_3 - x_4 \leq 10 \\
& x_1 \geq 0
\end{aligned} \tag{1}
$$

Let s_1, s_2, s_3 be the slack variables for the first, second and third constraints, respectively.
And $s_1, s_2, s_3 \geq 0$.
So the inequality constraints can be written as:

$$
\begin{aligned}
& x_1 + x_2 = 5 + s_1 \\
& x_1 - x_3 = 2 - s_2 \\
& 4x_2 + 3x_3 - x_4 = 10 - s_3
\end{aligned} \tag{2}
$$

Also, the standard form should have the objective function as a minimization problem.
So the objective function can be written as:

$$
\begin{aligned}
\min_{\boldsymbol{x}\in\mathbb{R}^4} \quad & -(2x_1 - x_3 + x_4) \\
i.e. \min_{\boldsymbol{x}\in\mathbb{R}^4} \quad & -2x_1 + x_3 - x_4
\end{aligned} \tag{3}
$$

Since there are no constraints on the boundary of x_2, x_3 and x_4 separately.
So let $x_2 = u_2 - v_2$, $x_3 = u_3 - v_3$, $x_4 = u_4 - v_4$, where $u_2, u_3, u_4, v_2, v_3, v_4 \geq 0$.
And put them into the origin problem, we can get the standard form of the origin problem:

So the standard form of the origin problem is:

$$
\begin{aligned}
\max_{x_1,u_2,u_3,u_4,v_2,v_3,v_4,s_1,s_2,s_3} \quad & 2x_1 - u_3 + v_3 + u_4 - v_4 \\
\text{s.t.} \quad & x_1 + u_2 - v_2 - s_1 = 5 \\
& x_1 - u_3 + v_3 + s_2 = 2 \\
& 4u_2 - 4v_2 + 3u_3 - 3v_3 - u_4 + v_4 + s_3 = 10 \\
& x_1, u_2, u_3, u_4, v_2, v_3, v_4, s_1, s_2, s_3 \geq 0
\end{aligned} \tag{4}
$$

2 Two-Phase Simplex

Use the two-phase simplex procedure to solve the following problem. [40pts]

$$\begin{aligned}
\min_{\boldsymbol{x}\in\mathbb{R}^4} \quad & -3x_1 + x_2 + 3x_3 - x_4 \\
\text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 = 0 \\
& 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9 \\
& x_1 - x_2 + 2x_3 - x_4 = 6 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned} \tag{5}$$

Since the origin problem is already the standard form, we can directly use the two-phase simplex procedure to solve it.

1. Phase one:

The supporting problem is:

$$\begin{aligned}
\min \boldsymbol{x} \in \mathbb{R}^7 \quad & x_5 + x_6 + x_7 \\
\text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 + x_5 = 0 \\
& 2x_1 - 2x_2 + 3x_3 + 3x_4 + x_6 = 9 \\
& x_1 - x_2 + 2x_3 - x_4 + x_7 = 6 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0
\end{aligned} \tag{6}$$

And the supporting problem's simplex tableau is:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	1	2	-1	1	1	0	0	0
	2	-2	3	3	0	1	0	9
	1	-1	2	-1	0	0	1	6
c^T	0	0	0	0	1	1	1	0

(7)

The basic is $B = (x_5, x_6, x_7)$, and $\boldsymbol{x} = (0, 0, 0, 0, 0, 9, 6)^T$.

3 Extreme Point

3.1 Q1

Prove that the extreme points of the following two sets are in one-to-one correspondence. [20pts]

$$\begin{aligned} S_1 &= \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq 0\} \\ S_2 &= \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}, \boldsymbol{x} \geq 0, \boldsymbol{y} \geq 0\} \end{aligned} \tag{8}$$

, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$.

3.2 Q2

Does the set $P = \{\boldsymbol{x} \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$ have extreme points? What is its standard form? Does it have extreme points in its standard form? If so, give a extreme point and explain why it is a extreme point. [20pts]