姓名：

学号：

学院：

年级：

# 上海科技大学

## 2023-2024 学年第1 学期期末考试卷

开课单位：信息学院

授课教师：王浩

考试科目：数值最优化

课程序号：SI152

**考生须知：**

1．请严格遵守考场纪律，禁止任何形式的作弊行为。

2．参加闭卷考试的考生，除携带必要考试用具外，书籍、笔记、掌上电脑和其他电子设备等物品一律按要求放在指定位置。

3．参加开卷考试的考生，可以携带教师指定的材料独立完成考试，但不准相互讨论，不准交换材料。

**考试成绩录入表：**

| 题目 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 总分 |
|---|---|---|---|---|---|---|---|---|---|
| 计分 | | | | | | | | | |
| 复核 | | | | | | | | | |

**评卷人签名：** **复核人签名：**

**日期：** **日期：**

# Numerical Optimization Final Exam, 2023 Fall

Jan. 11, 2024

**Problem 1.** Suppose you were restoring a signal from a linear observation operator, i.e., $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\theta}$, where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{y} \in \mathbb{R}^m$, and $\boldsymbol{\theta} \in \mathbb{R}^m$ is the random noise.

(1) Construct a linear programming model to find the signal that can best fit the linear model $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$. [5pts]

(2) Suppose you know that the signal does not have large differences between adjacent elements. Therefore, you also want to add the "total difference of the signal" $\sum_{i=1}^{n-1} |x_{i+1} - x_i|$ to the objective. Transform this problem to a linear programming model. [5pts]

**Problem 2.** Prove the primal objective decreases after a non-degenerate pivot of the simplex method. [10pts]

**Problem 3.** For the standard form

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0. \end{aligned}$$

Consider the following 5 conditions.

$Ax = b$ $\qquad\qquad$ $x \geq 0$

$x^T s = 0$

$A^T y + s = c$ $\qquad\qquad$ $s \geq 0$

(1) Draw circles on the conditions that the iterates of the primal simplex must satisfy. [4pts]

(2) Draw squares on the conditions that the iterates of the dual simplex must satisfy. [4pts]

(3) Draw lines under the conditions that the iterates of the interior point method must satisfy. [4pts]

**Problem 4.** Fill out the following form. Put a "✓" to indicate that this case could happen. Put a "×" to indicate that this case cannot happen. (see the example in the bottom right cell). [4pts]

| Primal \ Dual | Infeasible | Unbounded (from above) | Optimal solution exists |
|---|---|---|---|
| Infeasible | | | |
| Unbounded (from below) | | | |
| Optimal solution exists | | | ✓ |

**Problem 5.** The key of Ouasi-Newton Methods is to update Hessian $H^{k+1}$ from one iteration to the next. To compute $H^{k+1}$, we have the new model:

$$m_{k+1}(d) := f(x^{k+1}) + \nabla f(x^{k+1})^T d + \frac{1}{2} d^T H^{k+1} d.$$

(1) Derive the secant equation with $s^k = x^{k+1} - x^k = \alpha_k d^k$ and $y_k = \nabla f(x^{k+1}) - \nabla f(x^k)$. [5pts]

(2) Show the secant equation is equal to the Secant Methods with stepsize $\alpha_k = 1$ if $x$ is one-dimensional. [5pts]
Hint: In Secant Methods, we have

$$x^{k+2} = x^{k+1} - \frac{x^{k+1} - x^k}{f'(x^{k+1}) - f'(x^k)} f'(x^{k+1}).$$

(3) The BFGS update of $H^{k+1}$ is

$$H^{k+1} = (I - \rho_k y^k (s^k)^T)^T H^k (I - \rho_k y^k (s^k)^T) + \rho_k s^k (s^k)^T,$$

where $\rho_k = \frac{1}{(s^k)^T y^k}$. Prove that if $H^k \succ 0$ and the curvature condition $(s^k)^T y^k > 0$ holds, then $H^{k+1} \succ 0$. [5pts]

**Problem 6.** Suppose in the optimality condition for standard form you relax the complementarity condition to have $x_i s_i = \tau > 0$. Suppose your interior point method has the iteration:

$$(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}, \boldsymbol{s}^{k+1}) \longleftarrow (\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{s}^k) + (\Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{s}).$$

(1) For any $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{s}^k)$ with $\boldsymbol{x}^k \geq 0, \boldsymbol{s}^k \geq 0$, derive the system of linear equations that is used for computing $(\Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{s})$. [5pts]

(2) For any $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{s}^k)$ with $\boldsymbol{x}^k \geq 0, \boldsymbol{s}^k \geq 0$, show the linear system matrix you have in (1) is nonsingular. [5pts]

(3) Suppose for any symmetric matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$, you have a very efficient subroutine to compute its Jordan Canonical form decomposition: $\boldsymbol{Q} = \boldsymbol{P}\Lambda\boldsymbol{P}^T$, where $\boldsymbol{P}^T = \boldsymbol{P}^{-1}$ and $\Lambda$ is a diagonal matrix consisting of the eigenvalues of $\boldsymbol{Q}$. Now use this decomposition subroutine to design an efficient algorithm for solving the linear equations you have in (1). [5pts]

**Problem 7.** Derive the optimal solution to the following problem: [10pts]

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbf{R}^n} \quad & \frac{1}{2}\|\boldsymbol{x}-\boldsymbol{y}\|_2^2 \\ \text{s.t.} \quad & \|\boldsymbol{x}\|_2^2 \le 1. \end{aligned}$$

**Problem 8.** A large manufacturing company produces liquid nitrogen in 3 plants spread out in Jiangsu Province. Each plant has monthly production capacity.

| Plant $i$ | 1 | 2 | 3 |
|---|---|---|---|
| Capacity $p_i$ | 100 | 90 | 80 |

It has 4 retailers in the same area. Each retailer has a monthly demand to be satisfied.

| Retailer $j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Demand $d_j$ | 72 | 64 | 55 | 83 |

Transportation between any plant $i$ and any retailer $j$ has a cost of $c_{ij}$ dollars per volume unit of nitrogen.

(1) Build a model to decide each retailer should be served by which plant. Your objective is to minimize the total transportation cost. [5pts]

(2) Since the monthly rental fee is rising up for each plant. The company is considering to shut down some of the plant while still keep all the retailers' demands satisfied. Suppose the rental fee for each plant is $r_i$, build a model to minimize the total cost. [5pts]

(3) Suppose you want to use the Lagrangian relaxation method. Which constraint(s) do you want to relax? Give your reason(s) for your choice. [5pts]

(4) Show that for any multipliers in your relaxation, the optimal value of your relaxed problem is always a lower bound for the optimal value of the original problem. [5pts]

**Problem 9.** The following are the outputs of solving a nonlinear problem by two different solvers. Please figure out the local convergence rate for each run and state the reason. (Infeas.: constraint violation, Pen.Par.: penalty parameter, KKT Opt.: the L2 norm of KKT residual) [4pts]

| Iter. | Objective | Infeas. | Pen. Par. | KKT opt. |
|---|---|---|---|---|
| 0 | +1.8846e+00 | 0.0000e+00 | 1.0000e+00 | 5.3640e+00 |
| 1 | +3.5674e-01 | 0.0000e+00 | 5.0000e-01 | 5.3640e+00 |
| 2 | +5.1695e-02 | 0.0000e+00 | 5.0000e-01 | 8.0823e-01 |
| 3 | +1.1837e-03 | 0.0000e+00 | 5.0000e-01 | 2.0933e-01 |
| 4 | +1.4589e-06 | 0.0000e+00 | 5.0000e-01 | 2.8125e-02 |
| 5 | +3.6207e-12 | 0.0000e+00 | 5.0000e-01 | 8.8401e-04 |

| Iter. | Objective | Infeas. | Pen. Par. | KKT Opt. |
|---|---|---|---|---|
| 0 | +7.1400e+02 | 0.0000e+00 | 1.0000e+00 | 7.1315e-01 |
| 1 | +7.1097e+02 | 0.0000e+00 | 1.0000e+00 | 4.0118e-01 |
| 2 | +6.7284e+02 | 1.7554e+01 | 5.0000e-01 | 1.5689e-01 |
| 3 | +6.7850e+02 | 3.0670e+00 | 5.0000e-01 | 5.8659e-02 |
| 4 | +6.7951e+02 | 9.7923e-01 | 5.0000e-01 | 3.9763e-03 |
| 5 | +6.8063e+02 | 2.9867e-03 | 5.0000e-01 | 1.2208e-05 |
| 6 | +6.8063e+02 | 2.1996e-08 | 5.0000e-01 | 2.8831e-01 |
| 7 | +6.8063e+02 | 2.1996e-08 | 1.0000e-01 | 5.7662e-02 |
| 8 | +6.8063e+02 | 2.1996e-08 | 2.0000e-02 | 1.1532e-02 |
| 9 | +6.8063e+02 | 2.1996e-08 | 4.0000e-03 | 2.3065e-03 |
| 10 | +6.8063e+02 | 2.1996e-08 | 8.0000e-04 | 4.6130e-04 |
| 11 | +6.8063e+02 | 2.1996e-08 | 1.6000e-04 | 9.2259e-05 |
| 12 | +6.8063e+02 | 2.1996e-08 | 3.2000e-05 | 1.8452e-05 |
| 13 | +6.8063e+02 | 2.1996e-08 | 6.4000e-06 | 3.6904e-06 |
| 14 | +6.8063e+02 | 2.1996e-08 | 1.2800e-06 | 7.3807e-07 |