

Numerical Optimization, 2023 Fall

Homework 2

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Due 23:59 (CST), Nov. 2, 2023

1 Standard Form

Convert the following problem to a linear program in standard form. [20pts]

$$\begin{aligned}
 \max_{\mathbf{x} \in \mathbb{R}^4} \quad & 2x_1 - x_3 + x_4 \\
 \text{s.t.} \quad & x_1 + x_2 \geq 5 \\
 & x_1 - x_3 \leq 2 \\
 & 4x_2 + 3x_3 - x_4 \leq 10 \\
 & x_1 \geq 0
 \end{aligned} \tag{1}$$

Let s_1, s_2, s_3 be the slack variables for the first, second and third constraints, respectively. And $s_1, s_2, s_3 \geq 0$.

So the inequality constraints can be written as:

$$\begin{aligned}
 x_1 + x_2 &= 5 + s_1 \\
 x_1 - x_3 &= 2 - s_2 \\
 4x_2 + 3x_3 - x_4 &= 10 - s_3
 \end{aligned} \tag{2}$$

Also, the standard form should have the objective function as a minimization problem. So the objective function can be written as:

$$\begin{aligned}
 \min_{\mathbf{x} \in \mathbb{R}^4} \quad & -(2x_1 - x_3 + x_4) \\
 \text{i.e.} \min_{\mathbf{x} \in \mathbb{R}^4} \quad & -2x_1 + x_3 - x_4
 \end{aligned} \tag{3}$$

Since there are no constraints on the boundary of x_2, x_3 and x_4 separately. So let $x_2 = u_2 - v_2, x_3 = u_3 - v_3, x_4 = u_4 - v_4$, where $u_2, u_3, u_4, v_2, v_3, v_4 \geq 0$. And put them into the origin problem, we can get the standard form of the origin problem:

So the standard form of the origin problem is:

$$\begin{aligned}
 \max_{x_1, u_2, u_3, u_4, v_2, v_3, v_4, s_1, s_2, s_3} \quad & 2x_1 - u_3 + v_3 + u_4 - v_4 \\
 \text{s.t.} \quad & x_1 + u_2 - v_2 - s_1 = 5 \\
 & x_1 - u_3 + v_3 + s_2 = 2 \\
 & 4u_2 - 4v_2 + 3u_3 - 3v_3 - u_4 + v_4 + s_3 = 10 \\
 & x_1, u_2, u_3, u_4, v_2, v_3, v_4, s_1, s_2, s_3 \geq 0
 \end{aligned} \tag{4}$$

2 Two-Phase Simplex

Use the two-phase simplex procedure to solve the following problem. [40pts]

$$\begin{aligned}
 \min_{\mathbf{x} \in \mathbb{R}^4} \quad & -3x_1 + x_2 + 3x_3 - x_4 \\
 \text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 = 0 \\
 & 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9 \\
 & x_1 - x_2 + 2x_3 - x_4 = 6 \\
 & x_1, x_2, x_3, x_4 \geq 0
 \end{aligned} \tag{5}$$

Since the origin problem is already the standard form, we can directly use the two-phase simplex procedure to solve it.

1. Phase one:

The supporting problem is:

$$\begin{aligned}
 \min_{\mathbf{x} \in \mathbb{R}^7} \quad & x_5 + x_6 + x_7 \\
 \text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 + x_5 = 0 \\
 & 2x_1 - 2x_2 + 3x_3 + 3x_4 + x_6 = 9 \\
 & x_1 - x_2 + 2x_3 - x_4 + x_7 = 6 \\
 & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0
 \end{aligned} \tag{6}$$

And the supporting problem's simplex tableau is:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	1	2	-1	1	1	0	0	0
	2	-2	3	3	0	1	0	9
	1	-1	2	-1	0	0	1	6
c^T/r^T	0	0	0	0	1	1	1	0

(7)

The basic is $B = (x_5, x_6, x_7)$, and $\mathbf{x} = (0, 0, 0, 0, 9, 6)^T$.

Then add the row 1,2,3 to the row 4, to let the base variables' reduced cost become 0, we can get:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	1	2	-1	1	1	0	0	0
	2	-2	3	3	0	1	0	9
	1	-1	2	-1	0	0	1	6
r^T	-4	1	-4	-3	0	0	0	-15

(8)

The basic is $B = (x_5, x_6, x_7)$.

We choose the leftmost column with negative reduced cost, which is x_1 .

And we choose the row with the minimum ratio, which is row 1, and pivot, let x_1 in base and x_5 out base.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	1	2	-1	1	1	0	0	0
	0	-6	5	1	-2	1	0	9
	0	-3	3	-2	-1	0	1	6
r^T	0	9	-8	1	4	0	0	-15

(9)

The basic is $B = (x_1, x_6, x_7)$.

We choose the leftmost column with negative reduced cost, which is x_3 .

And we choose the row with the minimum ratio, which is row 2, and pivot, let x_3 in base and x_6 out base.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	1	$\frac{4}{5}$	0	$\frac{6}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{9}{5}$
	0	$-\frac{6}{5}$	1	$\frac{1}{5}$	$-\frac{2}{5}$	$\frac{1}{5}$	0	$\frac{9}{5}$
	0	$\frac{3}{5}$	0	$-\frac{13}{5}$	$-\frac{1}{5}$	$-\frac{3}{5}$	1	$\frac{3}{5}$
r^T	0	$-\frac{3}{5}$	0	$\frac{13}{5}$	$\frac{4}{5}$	$\frac{8}{5}$	0	$-\frac{3}{5}$

(10)

The basic is $B = (x_1, x_3, x_7)$.

We choose the leftmost column with negative reduced cost, which is x_2 .

And we choose the row with the minimum ratio, which is row 3, and pivot, let x_2 in base and x_7 out base.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	1	0	0	$\frac{14}{3}$	$\frac{1}{3}$	1	$-\frac{4}{3}$	1
	0	0	1	-5	0	-1	2	3
	0	1	0	$-\frac{13}{3}$	$\frac{1}{3}$	-1	$\frac{5}{3}$	1
r^T	0	0	0	0	1	1	1	0

(11)

The basic is $B = (x_1, x_2, x_3)$.

And all the reduced cost are non-negative, so the supporting problem is feasible.

So the phase one is finished.

And the basic feasible solution is $\mathbf{x} = (1, 1, 3, 0, 0, 0, 0)^T$.

2. Phase two:

The tableau of the origin problem is:

	x_1	x_2	x_3	x_4	b
	1	0	0	$\frac{14}{3}$	1
	0	0	1	-5	3
	0	1	0	$-\frac{13}{3}$	1
c^T/r^T	-3	1	3	-1	0

(12)

Then let the base variables' reduced cost become 0, we can get:

	x_1	x_2	x_3	x_4	b
	1	0	0	$\frac{14}{3}$	1
	0	0	1	-5	3
	0	1	0	$-\frac{13}{3}$	1
r^T	0	0	0	$\frac{97}{3}$	-7

(13)

So above all, the basic feasible solution of the origin problem is $\boldsymbol{x} = (1, 1, 3, 0)^T$.
And the optimal value is 7.

3 Extreme Point

3.1 Q1

Prove that the extreme points of the following two sets are in one-to-one correspondence. [20pts]

$$\begin{aligned} S_1 &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq 0\} \\ S_2 &= \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \mathbf{Ax} + \mathbf{y} = \mathbf{b}, \mathbf{x} \geq 0, \mathbf{y} \geq 0\} \end{aligned} \tag{14}$$

, where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$.

Suppose that the extreme points of S_1 compose the set P_1 .

And the extreme points of S_2 compose the set P_2 .

1. $\forall \mathbf{x} \in P_1$, we can get that $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \geq 0$, and $\mathbf{Ax} \leq \mathbf{b}$,
so there must $\exists \mathbf{y} \in \mathbb{R}^m$, such that $\mathbf{Ax} + \mathbf{y} = \mathbf{b}$, and $\mathbf{y}_i = (\mathbf{Ax})_i - \mathbf{b}_i$.
Where $(\mathbf{Ax})_i$ be the i th element of the vector (\mathbf{Ax}) , and \mathbf{y}_i be the i th element of the vector \mathbf{y} , \mathbf{b}_i be the i th element of the vector \mathbf{b} , $i = 1, \dots, m$.
And since $\mathbf{Ax} \leq \mathbf{b}$, i.e. $(\mathbf{Ax})_i \leq \mathbf{b}_i$,
so $\mathbf{y}_i = \mathbf{b}_i - (\mathbf{Ax})_i \geq 0$, $i = 1, \dots, m$.
i.e. $\mathbf{y} \geq 0$.
So we have proved that $(\mathbf{x}, \mathbf{y}) \in S_2$.

2. And we need to prove that its also an extreme point of S_2 .
We can prove this by contradiction.
Suppose that (\mathbf{x}, \mathbf{y}) is not an extreme point of S_2 .

TODO!!!!

Combine 1. and 2., we have proved that the mapping from P_1 to P_2 is surjective.

3. Then we need to prove that the mapping from P_1 to P_2 is injective.

TODO!!!!

So above all, since the mapping from P_1 to P_2 is injective and surjective, so its bijective.
i.e. We have proved that the extreme points of S_1 and S_2 are one-to-one correspondence.

3.2 Q2

Does the set $P = \{\mathbf{x} \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$ have extreme points? What is its standard form? Does it have extreme points in its standard form? If so, give a extreme point and explain why it is a extreme point. [20pts]

Since the set P is the intersection of two parallel lines, so it has no extreme points.

Since x_2 is not bounded, so we can let $x_2 = u - v$, where $u, v \geq 0$.

And since $x_1 \leq 1$, so we can add slack variable s_1 to the inequality constraint, i.e. $x_1 + s_1 = 1$.

so the standard form of P is:

$$\begin{aligned} \min_{x_1, s_1, u, v} \quad & \text{constant} \\ \text{s.t.} \quad & x_1 + s_1 = 1 \\ & x_1, s_1, u, v \geq 0 \end{aligned} \tag{15}$$

And its variables are x_1, s_1, u, v

And the standard form has extreme points, and $(x_1, s_1, u, v) = (1, 0, 0, 0)$ is one of the extreme points.

This is because:

$$x_1 + s_1 = 1, s_1 = 0, u = 0, v = 0$$

This makes the constraints

$$\begin{aligned} \min_{x_1, s_1, u, v} \quad & \text{constant} \\ \text{s.t.} \quad & x_1 + s_1 = 1 \\ & s_1 \geq 0 \\ & v \geq 0 \\ & u \geq 0 \end{aligned} \tag{16}$$

activate, and these 4 constraints are independent.

Since the number of variables of the standard form is 4, and the number of independent activate constraints is 4, so $(x_1, s_1, u, v) = (1, 0, 0, 0)$ is one of the extreme points.

So above all, P has no extreme points.

The standard form of P is:

$$\begin{aligned} \min_{x_1, s_1, u, v} \quad & \text{constant} \\ \text{s.t.} \quad & x_1 + s_1 = 1 \\ & x_1, s_1, u, v \geq 0 \end{aligned} \tag{17}$$

The standard form has extreme points, and $(x_1, s_1, u, v) = (1, 0, 0, 0)$ is one of the extreme points. The reasons are above.