

Homework 1

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Monday 16th October, 2023

Problem i. Write the gradient and Hessian matrix of the following formula. [10pts]

$$\mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x} + c \quad (\mathbf{A} \in \mathbf{R}^{n \times n}, \mathbf{b} \in \mathbf{R}^n, c \in \mathbf{R})$$

Let $f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x} + c$

1. the gradient of the formula is

$$\nabla f = \frac{\partial f}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}^T) \mathbf{x} + \mathbf{b}$$

2. the Hessian matrix of the formula is

$$\nabla^2 f = \nabla(\nabla f) = \frac{\partial(\nabla f)}{\partial \mathbf{x}} = \mathbf{A} + \mathbf{A}^T$$

Problem ii. Write the gradient and Hessian matrix of the following formula. [10pts]

$$\|\mathbf{Ax} - \mathbf{b}\|_2^2 \quad (\mathbf{A} \in \mathbf{R}^{m \times n}, \mathbf{b} \in \mathbf{R}^m)$$

Let $f(\mathbf{x}) = \|\mathbf{Ax} - \mathbf{b}\|_2^2 = (\mathbf{Ax} - \mathbf{b})^T(\mathbf{Ax} - \mathbf{b}) = \mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x} - 2\mathbf{x}^T \mathbf{A}^T \mathbf{b} + \mathbf{b}^T \mathbf{b}$

1. the gradient of the formula is

$$\nabla f = \frac{\partial f}{\partial \mathbf{x}} = 2\mathbf{A}^T \mathbf{A} \mathbf{x} - 2\mathbf{A}^T \mathbf{b}$$

2. the Hessian matrix of the formula is

$$\nabla^2 f = \nabla(\nabla f) = \frac{\partial(\nabla f)}{\partial \mathbf{x}} = 2\mathbf{A}^T \mathbf{A}$$

Problem iii. Convert the following problem to linear programming. [10pts]

$$\min_{\mathbf{x} \in \mathbf{R}^n} \|\mathbf{Ax} - \mathbf{b}\|_1 + \|\mathbf{x}\|_\infty \quad (\mathbf{A} \in \mathbf{R}^{m \times n}, \mathbf{b} \in \mathbf{R}^m)$$

1. for the first part,

let $z_i = |\mathbf{c}_i^T \mathbf{x} - b_i|$, where \mathbf{c}_i is the i -th row of the matrix \mathbf{A} , and b_i is the i -th element in the column vector \mathbf{b} .

$$\text{So } \|\mathbf{Ax} - \mathbf{b}\|_1 = \sum_{i=1}^m z_i$$

$$\text{And } -z_i \leq \mathbf{c}_i^T \mathbf{x} - b_i \leq z_i \text{ and } z_i \geq 0$$

2. for the second part,

let $w = \max_{i=1,2,\dots,n} x_i$, where x_i is the i -th element in the vector \mathbf{x}

$$\text{So } \|\mathbf{x}\|_\infty = w$$

$$\text{And } w \geq x_i$$

So combine the two parts, we convert the problem into the linear programming:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbf{R}^n} \quad & \left(\sum_{i=1}^m z_i \right) + w \\ \text{subject to} \quad & -z_i \leq \mathbf{c}_i^T \mathbf{x} - b_i \leq z_i, \quad i = 1, 2, \dots, m \\ & z_i \geq 0, \quad i = 1, 2, \dots, m \\ & w \geq x_i, \quad i = 1, 2, \dots, n \end{aligned}$$

Problem vi. Proof the convergence rates of the following point sequences. [30pts]

$$\mathbf{x}^k = \frac{1}{k}$$

$$\mathbf{x}^k = \frac{1}{k!}$$

$$\mathbf{x}^k = \frac{1}{2^{2^k}}$$

(Hint: Given two iterates \mathbf{x}^{k+1} and \mathbf{x}^k , and its limit point \mathbf{x}^* , there exists real number $q > 0$, satisfies

$$\lim_{k \rightarrow \infty} \frac{\|\mathbf{x}^{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}^k - \mathbf{x}^*\|} = q$$

if $0 < q < 1$, then the point sequence Q-linear convergence; if $q = 1$, then the point sequence Q-sublinear convergence; if $q = 0$, then the point sequence Q-superlinear convergence)

1. for $\mathbf{x}^k = \frac{1}{k}$:

since $\lim_{k \rightarrow \infty} \mathbf{x}^k = 0$, so $\mathbf{x}^* = 0$

$$\text{so } \lim_{k \rightarrow \infty} \frac{\|\mathbf{x}^{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}^k - \mathbf{x}^*\|} = \lim_{k \rightarrow \infty} \frac{\left\| \frac{1}{k+1} \right\|}{\left\| \frac{1}{k} \right\|} = \lim_{k \rightarrow \infty} \frac{k}{k+1} = 1$$

i.e. $q = 1$. so the point sequence is Q-sublinear convergence.

2. for $\mathbf{x}^k = \frac{1}{k!}$:

since $\lim_{k \rightarrow \infty} \mathbf{x}^k = 0$, so $\mathbf{x}^* = 0$

$$\text{so } \lim_{k \rightarrow \infty} \frac{\|\mathbf{x}^{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}^k - \mathbf{x}^*\|} = \lim_{k \rightarrow \infty} \frac{\left\| \frac{1}{(k+1)!} \right\|}{\left\| \frac{1}{k!} \right\|} = \lim_{k \rightarrow \infty} \frac{k!}{(k+1)!} = \lim_{k \rightarrow \infty} \frac{1}{k+1} = 0$$

i.e. $q = 0$. so the point sequence is Q-superlinear convergence.

3. for $\mathbf{x}^k = \frac{1}{2^{2^k}}$:

since $\lim_{k \rightarrow \infty} \mathbf{x}^k = 0$, so $\mathbf{x}^* = 0$

$$\text{so } \lim_{k \rightarrow \infty} \frac{\|\mathbf{x}^{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}^k - \mathbf{x}^*\|} = \lim_{k \rightarrow \infty} \frac{\left\| \frac{1}{2^{2^{k+1}}} \right\|}{\left\| \frac{1}{2^{2^k}} \right\|} = \lim_{k \rightarrow \infty} \frac{2^{2^k}}{2^{2^{k+1}}} = \lim_{k \rightarrow \infty} 2^{-2^k} = 0$$

i.e. $q = 0$. so the point sequence is Q-superlinear convergence.

So above all, the point sequence $\mathbf{x}^k = \frac{1}{k}$ is Q-sublinear convergence, the point sequence $\mathbf{x}^k = \frac{1}{k!}$ and $\mathbf{x}^k = \frac{1}{2^{2^k}}$ are Q-superlinear convergence.

Problem v. Select the Haverly Pool Problem or the Horse Racing Problem in the courseware, compile the program using AMPL model language and submit it to <https://neos-server.org/neos/solvers/index.html>. (Hint: both AMPL solver and NEOS solver can be used, please indicate the type of solver used in the submitted job, show the solution results (eg: screenshots attached to the PDF file), and submit the source code together with the submitted job, please package as .zip file, including your PDF and source code.) [40pts]

Select the Haverly Pooling Problem.

The codes are in the additional files.

And since the problem is a bilinear problem, i.e. a non linear problem, so I used the the LOQO solver in the nonlinearly constrained optimization to solve the problem.

The results are as follows:

```
You are using the solver loqo.
Checking ampl.mod for loqo_options...
Executing AMPL.
processing data.
processing commands.
Executing on prod-exec-5.neos-server.org

9 variables:
3 nonlinear variables
6 linear variables
6 constraints; 23 nonzeros
3 nonlinear constraints
3 linear constraints
4 equality constraints
2 inequality constraints
1 linear objective; 6 nonzeros.

LOQO 7.00: LOQO 7.00: optimal solution (25 iterations, 25 evaluations)
primal objective 1799.999995
dual objective 1800.000021
A = 100
B = 300
Cx = 3.59196e-07
Cy = 2.10653e-07
Px = 200
Py = 200
```

Figure 1: The results of the Haverly Pooling Problem