Numerical Optimization, 2023 Fall
Homework 2

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1 Standard Form

Convert the following problem to a linear program in standard form. [20pts]

$$
\begin{aligned}
\max_{\boldsymbol{x} \in \mathbb{R}^4} \quad & 2x_1 - x_3 + x_4 \\
\text{s.t.} \quad & x_1 + x_2 \geq 5 \\
& x_1 - x_3 \leq 2 \\
& 4x_2 + 3x_3 - x_4 \leq 10 \\
& x_1 \geq 0
\end{aligned} \tag{1}
$$

Let s_1, s_2, s_3 be the slack variables for the first, second and third constraints, respectively. And $s_1, s_2, s_3 \geq 0$.
So the inequality constraints can be written as:

$$
\begin{aligned}
& x_1 + x_2 = 5 + s_1 \\
& x_1 - x_3 = 2 - s_2 \\
& 4x_2 + 3x_3 - x_4 = 10 - s_3
\end{aligned} \tag{2}
$$

Also, the standard form should have the objective function as a minimization problem.
So the objective function can be written as:

$$
\begin{aligned}
\min_{\boldsymbol{x} \in \mathbb{R}^4} \quad & -(2x_1 - x_3 + x_4) \\
i.e. \min_{\boldsymbol{x} \in \mathbb{R}^4} \quad & -2x_1 + x_3 - x_4
\end{aligned} \tag{3}
$$

Since there are no constraints on the boundary of x_2, x_3 and x_4 separately.
So let $x_2 = u_2 - v_2$, $x_3 = u_3 - v_3$, $x_4 = u_4 - v_4$, where $u_2, u_3, u_4, v_2, v_3, v_4 \geq 0$.
And put them into the origin problem, we can get the standard form of the origin problem:

So the standard form of the origin problem is:

$$
\begin{aligned}
\max_{x_1, u_2, u_3, u_4, v_2, v_3, v_4, s_1, s_2, s_3} \quad & 2x_1 - u_3 + v_3 + u_4 - v_4 \\
\text{s.t.} \quad & x_1 + u_2 - v_2 - s_1 = 5 \\
& x_1 - u_3 + v_3 + s_2 = 2 \\
& 4u_2 - 4v_2 + 3u_3 - 3v_3 - u_4 + v_4 + s_3 = 10 \\
& x_1, u_2, u_3, u_4, v_2, v_3, v_4, s_1, s_2, s_3 \geq 0
\end{aligned} \tag{4}
$$

2 Two-Phase Simplex

Use the two-phase simplex procedure to solve the following problem. [40pts]

$$\begin{aligned}
\min_{\boldsymbol{x}\in\mathbb{R}^4} \quad & -3x_1 + x_2 + 3x_3 - x_4 \\
\text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 = 0 \\
& 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9 \\
& x_1 - x_2 + 2x_3 - x_4 = 6 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned} \tag{5}$$

Since the origin problem is already the standard form, we can directly use the two-phase simplex procedure to solve it.

1. Phase one:

The supporting problem is:

$$\begin{aligned}
\min \boldsymbol{x} \in \mathbb{R}^7 \quad & x_5 + x_6 + x_7 \\
\text{s.t.} \quad & x_1 + 2x_2 - x_3 + x_4 + x_5 = 0 \\
& 2x_1 - 2x_2 + 3x_3 + 3x_4 + x_6 = 9 \\
& x_1 - x_2 + 2x_3 - x_4 + x_7 = 6 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0
\end{aligned} \tag{6}$$

And the supporting problem's simplex tableau is:

(7)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	1	2	-1	1	1	0	0	0
	2	-2	3	3	0	1	0	9
	1	-1	2	-1	0	0	1	6
c^T/r^T	0	0	0	0	1	1	1	0

The basic is $B = (x_5, x_6, x_7)$, and $\boldsymbol{x} = (0, 0, 0, 0, 0, 9, 6)^T$.

Then add the row 1,2,3 to the row 4, to let the base variables' reduced cost become 0, we can get:

(8)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	$\boxed{1}$	2	-1	1	1	0	0	0
	2	-2	3	3	0	1	0	9
	1	-1	2	-1	0	0	1	6
r^T	-4	1	-4	-3	0	0	0	-15

The basic is $B = (x_5, x_6, x_7)$.

We choose the leftmost column with negative reduced cost, which is x_1.

And we choose the row with the minimum ratio, which is row 1, and pivot, let x_1 in base and x_5 out base.

$$
\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{b} \\
\hline
 & 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\
 & 0 & -6 & \boxed{5} & 1 & -2 & 1 & 0 & 9 \\
 & 0 & -3 & 3 & -2 & -1 & 0 & 1 & 6 \\
\hline
r^T & 0 & 9 & -8 & 1 & 4 & 0 & 0 & -15
\end{array}
\tag{9}
$$

The basic is $B = (x_1, x_6, x_7)$.
We choose the leftmost column with negative reduced cost, which is x_3.
And we choose the row with the minimum ratio, which is row 2, and pivot, let x_3 in base and x_6 out base.

$$
\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{b} \\
\hline
 & 1 & \frac{4}{5} & 0 & \frac{6}{5} & \frac{3}{5} & \frac{1}{5} & 0 & \frac{9}{5} \\
 & 0 & -\frac{6}{5} & 1 & \frac{1}{5} & -\frac{2}{5} & \frac{1}{5} & 0 & \frac{9}{5} \\
 & 0 & \boxed{\frac{3}{5}} & 0 & -\frac{13}{5} & -\frac{1}{5} & -\frac{3}{5} & 1 & \frac{3}{5} \\
\hline
r^T & 0 & -\frac{3}{5} & 0 & \frac{13}{5} & \frac{4}{5} & \frac{8}{5} & 0 & -\frac{3}{5}
\end{array}
\tag{10}
$$

The basic is $B = (x_1, x_3, x_7)$.
We choose the leftmost column with negative reduced cost, which is x_2.
And we choose the row with the minimum ratio, which is row 3, and pivot, let x_2 in base and x_7 out base.

$$
\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{b} \\
\hline
 & 1 & 0 & 0 & \frac{14}{3} & \frac{1}{3} & 1 & -\frac{4}{3} & 1 \\
 & 0 & 0 & 1 & -5 & 0 & -1 & 2 & 3 \\
 & 0 & 1 & 0 & -\frac{13}{3} & \frac{1}{3} & -1 & \frac{5}{3} & 1 \\
\hline
r^T & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}
\tag{11}
$$

The basic is $B = (x_1, x_2, x_3)$.
And all the reduced cost are non-negative, so the supporting problem is feasible.
So the phase one is finished.
And the basic feasible solution is $\boldsymbol{x} = (1, 1, 3, 0, 0, 0, 0)^T$.

2. Phase two:

The tableau of the origin problem is:

$$
\begin{array}{c|ccccc}
 & x_1 & x_2 & x_3 & x_4 & \text{b} \\
\hline
 & 1 & 0 & 0 & \frac{14}{3} & 1 \\
 & 0 & 0 & 1 & -5 & 3 \\
 & 0 & 1 & 0 & -\frac{13}{3} & 1 \\
\hline
c^T/r^T & -3 & 1 & 3 & -1 & 0
\end{array}
\tag{12}
$$

Then let the base variables' reduced cost become 0, we can get:

$$
\begin{array}{c|ccccc}
 & x_1 & x_2 & x_3 & x_4 & \text{b} \\
\hline
 & 1 & 0 & 0 & \frac{14}{3} & 1 \\
 & 0 & 0 & 1 & -5 & 3 \\
 & 0 & 1 & 0 & -\frac{13}{3} & 1 \\
\hline
r^T & 0 & 0 & 0 & \frac{97}{3} & -7
\end{array}
\tag{13}
$$

So above all, the basic feasible solution of the origin problem is $\boldsymbol{x} = (1, 1, 3, 0)^T$. And the optimal value is 7.

3 Extreme Point

3.1 Q1

Prove that the extreme points of the following two sets are in one-to-one correspondence. [20pts]

$$\begin{aligned} S_1 &= \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \ge 0\} \\ S_2 &= \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}, \boldsymbol{x} \ge 0, \boldsymbol{y} \ge 0\} \end{aligned} \tag{14}$$

, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$.

Suppose that the extreme points of S_1 compose the set P_1.
And the extreme points of S_2 compose the set P_2.

1. $\forall \boldsymbol{x} \in P_1$, we can get that $\boldsymbol{x} \in \boldsymbol{R}^n, \boldsymbol{x} \ge 0$, and $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$,
so there must $\exists \boldsymbol{y} \in \mathbb{R}^m$, such that $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}$, and $\boldsymbol{y}_i = (\boldsymbol{A}\boldsymbol{x})_i - \boldsymbol{b}_i$.
Where $(\boldsymbol{A}\boldsymbol{x})_i$ be the ith element of the vector $(\boldsymbol{A}\boldsymbol{x})$, and $\boldsymbol{y}_i$ be the ith element of the vector $\boldsymbol{y}$, $\boldsymbol{b}_i$ be the ith element of the vector $\boldsymbol{b}$, $i = 1, \cdots, m$.
And since $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$, i.e. $(\boldsymbol{A}\boldsymbol{x})_i \le \boldsymbol{b}_i$,
so $\boldsymbol{y}_i = \boldsymbol{b}_i - (\boldsymbol{A}\boldsymbol{x})_i \ge 0$, $i = 1, \cdots, m$.
i.e. $\boldsymbol{y} \ge 0$.
So we have proved that $(\boldsymbol{x}, \boldsymbol{y}) \in S_2$.

2. And we need to prove that its also an extreme point of S_2.
We can prove this by contradiction.
Suppose that $(\boldsymbol{x}, \boldsymbol{y})$ is not an extreme point of S_2.
Then there must $\exists \lambda \in (0, 1)$, and $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2) \in P_2, (\boldsymbol{x}_1, \boldsymbol{y}_1) \ne (\boldsymbol{x}_2, \boldsymbol{y}_2)$.
i.e. $\boldsymbol{x} = \lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2$.

Since $\boldsymbol{y}_1 = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_1, \boldsymbol{y}_2 = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_2$, so $(\boldsymbol{x}_1, \boldsymbol{y}_1) \ne (\boldsymbol{x}_2, \boldsymbol{y}_2) \Rightarrow \boldsymbol{x}_1 \ne \boldsymbol{x}_2$ and $\boldsymbol{y}_1 \ne \boldsymbol{y}_2$.
So we mutiply matrix $\boldsymbol{A}$ to both side of the equation $\boldsymbol{x} = \lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2$.
$\Rightarrow \boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{A}\boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{A}\boldsymbol{x}_2$.
Put $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}, \boldsymbol{A}\boldsymbol{x}_1 + \boldsymbol{y}_1 = \boldsymbol{b}, \boldsymbol{A}\boldsymbol{x}_2 + \boldsymbol{y}_2 = \boldsymbol{b}$ into the equation, we can get that

WRONG!!!!!! TODO!!!!!!! $\boldsymbol{b} - \boldsymbol{y} =$

Combine 1. and 2., we have proved that the mapping from P_1 to P_2 is surjective.

3. Then we need to prove that the mapping from P_1 to P_2 is injective. If $\boldsymbol{x}_1 = \boldsymbol{x}_2 \in P_1$,
then $\boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{A}\boldsymbol{x}_2$,
$\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_2$,
Since $\boldsymbol{y}_1 = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_1, \boldsymbol{y}_2 = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_2$
So $\boldsymbol{y}_1 = \boldsymbol{y}_2$.
So above all, if $\boldsymbol{x}_1 = \boldsymbol{x}_2$, then $(\boldsymbol{x}_1, \boldsymbol{y}_1) = (\boldsymbol{x}_2, \boldsymbol{y}_2) \in P_2$.
So the mapping from P_1 to P_2 is injective.

So above all, since the mapping from P_1 to P_2 is injective and surjective, so its bijective. i.e. We have proved that the extreme points of S_1 and S_2 are one-to-one correspondence.

3.2 Q2

Does the set $P = \{\boldsymbol{x} \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$ have extreme points? What is its standard form? Does it have extreme points in its standard form? If so, give a extreme point and explain why it is a extreme point. [20pts]

Since the set P is the intersection of two parallel lines, so it has no extreme points.

Since x_2 is not bounded, so we can let $x_2 = u - v$, where $u, v \geq 0$.
And since $x_1 \leq 1$, so we can add slack variable s_1 to the inequality constraint,
i.e. $x_1 + s_1 = 1$.

so the standard form of P is:

$$\begin{aligned} \min_{x_1,s_1,u,v} \quad & \mathit{constant} \\ \text{s.t.} \quad & x_1 + s_1 = 1 \\ & x_1, s_1, u, v \geq 0 \end{aligned} \tag{15}$$

And its variables are x_1, s_1, u, v

And the standard form has extreme points, and $(x_1, s_1, u, v) = (1, 0, 0, 0)$ is one of the extreme points.
This is because:
$x_1 + s_1 = 1, s_1 = 0, u = 0, v = 0$
This makes the constraints

$$\begin{aligned} \min_{x_1,s_1,u,v} \quad & \mathit{constant} \\ \text{s.t.} \quad & x_1 + s_1 = 1 \\ & s_1 \geq 0 \\ & v \geq 0 \\ & u \geq 0 \end{aligned} \tag{16}$$

activate, and these 4 constrains are independent.
Since the number of variables of the standard form is 4, and the number of independent activate constraints is 4, so $(x_1, s_1, u, v) = (1, 0, 0, 0)$ is one of the extreme points.

So above all, P has no extreme points.
The standard form of P is:

$$\begin{aligned} \min_{x_1,s_1,u,v} \quad & \mathit{constant} \\ \text{s.t.} \quad & x_1 + s_1 = 1 \\ & x_1, s_1, u, v \geq 0 \end{aligned} \tag{17}$$

The standard form has extreme points, and $(x_1, s_1, u, v) = (1, 0, 0, 0)$ is one of the extreme points. The reasons are above.