Monotone Operators and Base Splitting Schemes

Ernest K. Ryu and Wotao Yin

Large-Scale Convex Optimization via Monotone Operators

Main idea

Use monotone operators and base splitting schemes to derive and analyze a wide variety of classical and modern algorithms in a unified and streamlined manner:

(i) pose the problem at hand as a monotone inclusion problem
(ii) use one of the base splitting schemes to encode the solution as a fixed point of a related operator
(iii) find the solution with a fixed-point iteration.

Outline

Set-valued operator

$\mathbb{T}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued operator on $\mathbb{R}^n$ if $\mathbb{T}$ maps a point in $\mathbb{R}^n$ to a (possibly empty) subset of $\mathbb{R}^n$.

Other names: point-to-set mapping, set-valued mapping, multi-valued function, correspondence. For simplicity, write $\mathbb{T}x = \mathbb{T}(x)$.

If $\mathbb{T}x$ is a singleton or empty for all x, then $\mathbb{T}$ is a function or is single-valued with domain $\{x \mid \mathbb{T}(x) \neq \emptyset\}$ and write $\mathbb{T}x = y$ (although $\mathbb{T}x = \{y\}$ would be strictly correct).

Graph of an operator:

$$\operatorname{Gra}\mathbb{T} = \{(x, u) \,|\, u \in \mathbb{T}x\} \subseteq \mathbb{R}^n \times \mathbb{R}^n.$$

We will often not distinguish $\mathbb{T}$ and $\operatorname{Gra}\mathbb{T}$ and write $\mathbb{T}$ when we really mean $\operatorname{Gra}\mathbb{T}$.

Operator definitions

Domain and range of $\mathbb{T}$:

$$\operatorname{dom} \mathbb{T} = \{x \mid \mathbb{T}x \neq \emptyset\}, \qquad \operatorname{range} \mathbb{T} = \{y \mid y \in \mathbb{T}x,\, x \in \mathbb{R}^n\}$$

Image of $C \subseteq \mathbb{R}^n$ under $\mathbb{T}$: $\mathbb{T}(C) = \cup_{c \in C} \mathbb{T}(c)$

Composition of operators:

$$\mathbb{T} \circ \mathbb{S}x = \mathbb{T}\mathbb{S}x = \mathbb{T}(\mathbb{S}(x))$$

Sum of operators:

$$(\mathbb{T} + \mathbb{S})x = \mathbb{T}(x) + \mathbb{S}(x)$$

Equivalent definitions that use the graph:

$$\mathbb{T}\mathbb{S} = \{(x, z) \mid \exists\, y \ (x, y) \in \mathbb{S},\ (y, z) \in \mathbb{T}\}$$
$$\mathbb{T} + \mathbb{S} = \{(x, y + z) \mid (x, y) \in \mathbb{T},\ (x, z) \in \mathbb{S}\}$$

Operator definitions

Identity and zero operators:

$$\mathbb{I} = \{(x, x) \mid x \in \mathbb{R}^n\} \qquad \mathbb{0} = \{(x, 0) \mid x \in \mathbb{R}^n\}$$

So $\mathbb{T} + \mathbb{0} = \mathbb{T}$, $\mathbb{T}\mathbb{I} = \mathbb{T}$, and $\mathbb{I}\mathbb{T} = \mathbb{T}$.

$\mathbb{T}$ is L-Lipschitz if $\nabla f(x)$

$$\|\mathbb{T}x - \mathbb{T}y\| \le L\|x - y\| \qquad \forall\, x, y \in \operatorname{dom} \mathbb{T}.$$

(This definition generalizes the Lipschitz continuity to functions with domain $\mathbb{R}^n$ to operators without full domain.)

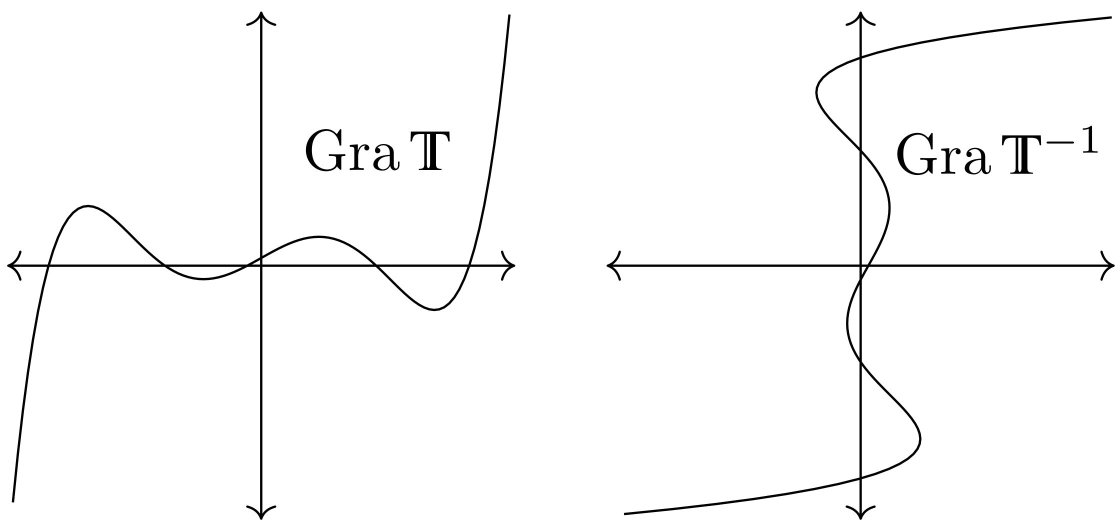
If $\mathbb{T}$ is L-Lipschitz, it is single-valued; if $\mathbb{T}x$ is not a singleton, then we have a contradiction by setting $y = x$.

Operator Inverse

The *inverse operator* of $\mathbb{T}$:

$$\mathbb{T}^{-1} = \{(y, x) \mid (x, y) \in \mathbb{T}\}$$

$\mathbb{T}^{-1}$ is always well defined ($\mathbb{T}^{-1}$ need not be single-valued).



$(\mathbb{T}^{-1})^{-1} = \mathbb{T}$ and $\operatorname{dom} \mathbb{T}^{-1} = \operatorname{range} \mathbb{T}$

$\mathbb{T}^{-1}$ is not an inverse in the usual sense since $\mathbb{T}^{-1}\mathbb{T} \neq \mathbb{I}$ possible.

Zero

If $0 \in \mathbb{T}(x)$, x is a zero of $\mathbb{T}$.

Zero set of an operator $\mathbb{T}$:

$$\operatorname{Zer} \mathbb{T} = \{x \mid 0 \in \mathbb{T}x\} = \mathbb{T}^{-1}(0)$$

Many interesting problems can be posed as finding zeros of an operator.

Subdifferential

When f is convex:

- ∂f is a set-valued operator
- $\operatorname{argmin} f = \operatorname{Zer} \partial f$ $\Rightarrow$ $0 \in \partial f(x)$
- when f is differentiable, write ∇f instead of ∂f

Subdifferential of conjugate

When f is CCP,

convex, closed, proper

$$(\partial f)^{-1} = \partial f^*$$

Proof.

$\max_z \langle u^T z - f(z) \rangle$

$$\begin{aligned}
u \in \partial f(x) \quad &\Leftrightarrow \quad 0 \in \partial f(x) - u \\
&\Leftrightarrow \quad x \in \operatorname*{argmin}_z \{f(z) - u^\intercal z\} \\
&\Leftrightarrow \quad -f(x) + u^\intercal x = f^*(u) \\
&\Leftrightarrow \quad f(x) + f^*(u) = u^\intercal x \\
&\Leftrightarrow \quad f^{**}(x) + f^*(u) = u^\intercal x \\
&\Leftrightarrow \quad x \in \partial f^*(u)
\end{aligned}$$

The last step takes the whole argument backwards. $\square$

Subdifferential of conjugate

$g(y) = f^*(A^\intercal y)$, where f is CCP and $\mathcal{R}(A^\intercal) \cap \operatorname{ri} \operatorname{dom} f^* \neq \emptyset$,

$$
\begin{aligned}
u \in \partial g(y) \quad &\Leftrightarrow \quad u \in A \partial f^*(A^\intercal y) \\
&\Leftrightarrow \quad u = Ax,\ x \in \partial f^*(A^\intercal y) \\
&\Leftrightarrow \quad u = Ax,\ \partial f(x) \ni A^\intercal y \\
&\Leftrightarrow \quad u = Ax,\ 0 \in \partial f(x) - A^\intercal y \\
&\Leftrightarrow \quad u = Ax,\ x \in \underset{z}{\operatorname{argmin}} \{ f(z) - \langle y, Az \rangle \}
\end{aligned}
$$

Find element of ∂g by solving a minimization problem.

Outline

Monotone operators

$\mathbb{T}$ is monotone if

$$\langle u - v, x - y \rangle \geq 0 \qquad \forall\, (x, u),\ (y, v) \in \mathbb{T}.$$

Equivalently and more concisely, $\mathbb{T}$ is monotone if

$$\langle \mathbb{T}x - \mathbb{T}y, x - y \rangle \geq 0 \qquad \forall\, x, y \in \mathbb{R}^n.$$

$T(x) \to \nabla f(x)$

Maximal monotone operators

$\mathbb{T}$ is maximal monotone if $\nexists$ monotone $\mathbb{S}$ such that $\operatorname{Gra}\mathbb{T} \subset \operatorname{Gra}\mathbb{S}$ properly.

I.e., if $\mathbb{T}$ is monotone but not maximal, then $\exists\ (x, u) \notin \mathbb{T}$ such that $\mathbb{T} \cup \{(x, u)\}$ is monotone.

Maximality is a technical but fundamental detail.

Monotone operator example

Heaviside step function

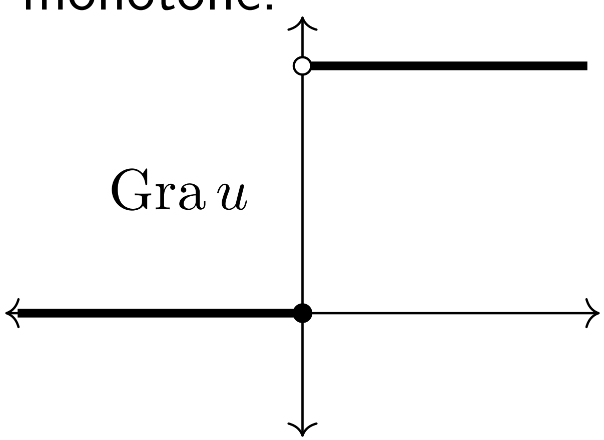
$$u(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 & \text{for } x > 0 \end{cases}$$

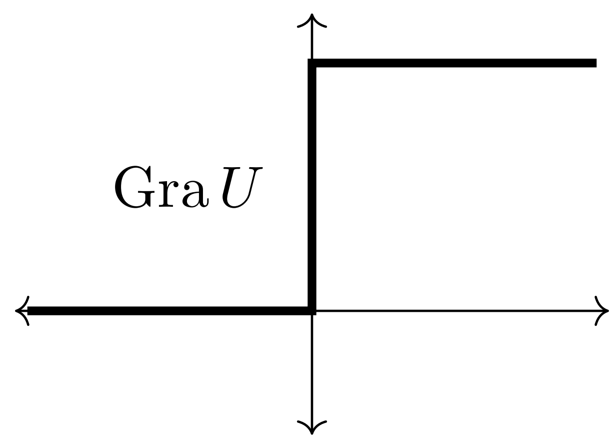
is monotone but not maximal. Operator

$|x|$

$$U(x) = \begin{cases} \{0\} & \text{for } x < 0 \\ [0,1] & \text{for } x = 0 \\ \{1\} & \text{for } x > 0 \end{cases}$$

is maximal monotone.





Monotonicity of subdifferentials

If f is convex and proper, then ∂f is monotone.
If f is CCP, then ∂f is maximal monotone.

Proof of monotonicity. Add

$$f(y) \geq f(x) + \langle \partial f(x), y - x \rangle, \quad (1) \qquad f(x) \geq f(y) + \langle \partial f(y), x - y \rangle \quad (2)$$

to get $\quad (1) + (2) \Rightarrow$

$$\langle \partial f(x) - \partial f(y), x - y \rangle \geq 0.$$

$\square$

Maximality proved later in §10.

[subdiff. CCP] $\subset$ [maximal monotone]

strict inclusion

Stronger monotonicity properties

$\mathbb{A} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is μ-strongly monotone or μ-coercive if $\mu > 0$ and

$$\langle u - v, x - y \rangle \geq \mu \|x - y\|^2 \qquad \forall\, (x, u), (y, v) \in \mathbb{A}.$$

$\mathbb{A}$ is strongly monotone if it is μ-strongly monotone for some $\mu \in (0, \infty)$.

$\mathbb{A}$ is β-cocoercive or β-inverse strongly monotone if $\beta > 0$

$$\langle u - v, x - y \rangle \geq \beta \|u - v\|^2 \qquad \forall\, (x, u), (y, v) \in \mathbb{A}.$$

We say $\mathbb{A}$ is cocoercive if it is β-cocoercive for some $\beta \in (0, \infty)$.

Cocoercivity and strong monotonicity are dual:
[$\mathbb{A}$ is β-cocoercive] $\Leftrightarrow$ [$\mathbb{A}^{-1}$ is β-strongly monotone]

Strongly monotone and cocoercive operators are monotone.

Stronger monotonicity properties

When $\mathbb{A}$ is β-cocoercive, Cauchy-Schwartz tells us

$$(1/\beta)\|x-y\| \geq \|\mathbb{A}x - \mathbb{A}y\| \qquad \forall\, x, y \in \mathbb{R}^n.$$

i.e., $\mathbb{A}$ is $(1/\beta)$-Lipschitz. Cocoercive operators are single-valued.

Converse is not true.

$$\mathbb{A}(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$$

is maximal monotone and Lipschitz, but not cocoercive since $\langle \mathbb{A}x - \mathbb{A}y, x - y\rangle = 0$.

More concisely express μ-strong monotonicity as

$$\langle \mathbb{A}x - \mathbb{A}y, x - y\rangle \geq \mu\|x-y\|^2 \qquad \forall\, x, y \in \mathbb{R}^n,$$

and, when $\mathbb{A}$ is a priori known or assumed to be single-valued, express β-cocoercivity as

$$\langle \mathbb{A}x - \mathbb{A}y, x - y\rangle \geq \beta\|\mathbb{A}x - \mathbb{A}y\|^2 \qquad \forall\, x, y \in \mathbb{R}^n.$$

Stronger monotonicity properties: Maximality

$\mathbb{A}$ is maximal μ-s.m. if $\nexists$ μ-s.m. $\mathbb{B}$ such that $\operatorname{Gra} \mathbb{A} \subset \operatorname{Gra} \mathbb{B}$ properly.
$\mathbb{A}$ is maximal β-coco. if $\nexists$ β-coco. $\mathbb{B}$ such that $\operatorname{Gra} \mathbb{A} \subset \operatorname{Gra} \mathbb{B}$ properly.

[Maximal coco.] and [maximal s.m.] are dual:
[$\mathbb{A}$ is maximal β-coco.] $\Leftrightarrow$ [$\mathbb{A}^{-1}$ is maximal β-s.m.].

If $\mathbb{A}$ cocoercive, [$\mathbb{A}$ maximal] $\Leftrightarrow$ [$\operatorname{dom} \mathbb{A} = \mathbb{R}^n$]. (We prove this in §10.)
Therefore, "$\mathbb{A} \colon \mathbb{R}^n \to \mathbb{R}^n$ is β-cocoercive" implicitly asserts
$\operatorname{dom} \mathbb{A} = \mathbb{R}^n$ and maximality of $\mathbb{A}$.

Stronger monotonicity properties: CCP functions

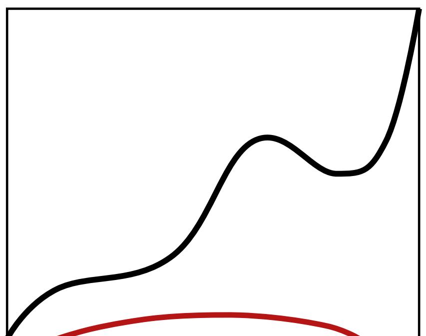
Assume f is CCP. Then

- [f is μ-strongly convex] $\Leftrightarrow$ [∂f is μ-strongly monotone]
- [f is L-smooth] $\Leftrightarrow$ [∂f is L-Lipschitz] $\Leftrightarrow$ [∂f is $(1/L)$-cocoercive]
- [f is μ-strongly convex] $\Leftrightarrow$ [f^* is $(1/\mu)$-smooth]

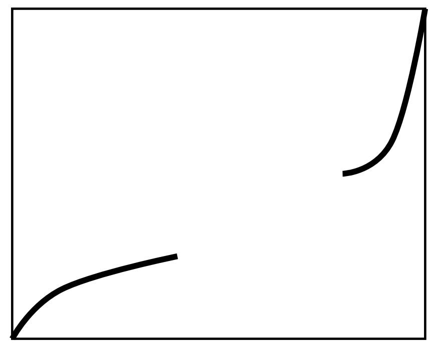
For ∂f, Lipschitz $=$ cocoercive.
For monotone operators, Lipschitz $\neq$ cocoercive.

Stronger monotonicity properties examples

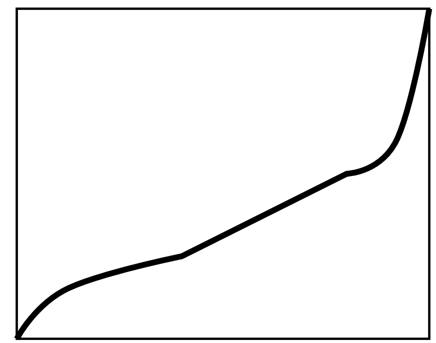
Operator on $\mathbb{R}$ is monotone if its graph is a nondecreasing curve in $\mathbb{R}^2$. Vertical portions, then multi-valued. Continuous with no end points, then maximal. Slope $\geq \mu$, then μ-strongly monotone. Slope $\leq L$, then L-Lipschitz. Lipschitz and cocoercivity coincide.



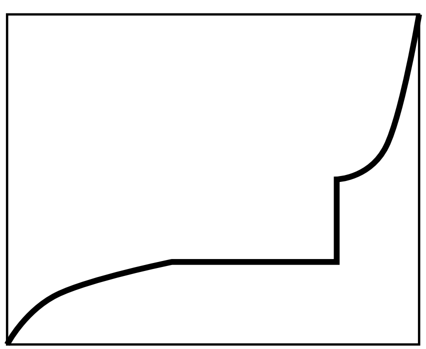
Not monotone



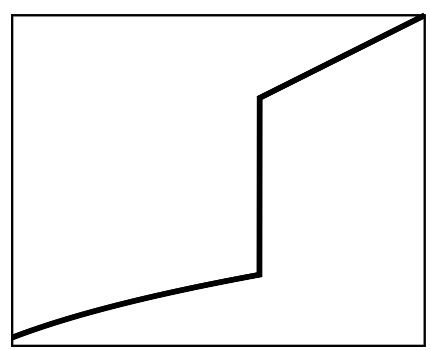
Monotone but not maximal



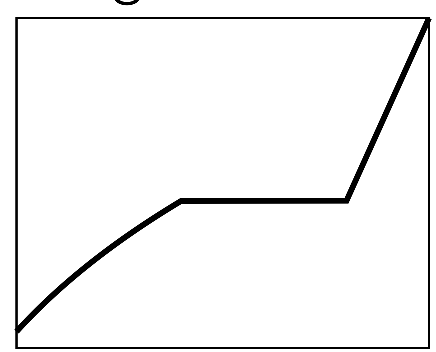
Maximal monotone and single-valued



Maximal monotone and multi-valued



Strongly monotone but not Lipschitz



Lipschitz but not strongly monotone

Operations preserving monotonicity

- $\mathbb{T}$ (maximal) monotone, then $\mathbb{S}(x) = y + \alpha\mathbb{T}(x+z)$ (maximal) monotone for any $\alpha > 0$ and $y, z \in \mathbb{R}^n$.
- $\mathbb{T}$ (maximal) monotone, then $\mathbb{T}^{-1}$ (maximal) monotone.
- $\mathbb{T}$ and $\mathbb{S}$ monotone, $\mathbb{T} + \mathbb{S}$ monotone.
- $\mathbb{T}$ and $\mathbb{S}$ maximal monotone and $\mathrm{dom}\,\mathbb{T} \cap \mathrm{int}\,\mathrm{dom}\,\mathbb{S} \neq \emptyset$, then $\mathbb{T} + \mathbb{S}$ maximal monotone.
- $\mathbb{T}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ monotone and $M \in \mathbb{R}^{n\times m}$, then $M^\intercal \mathbb{T} M$ monotone.
- $\mathbb{T}$ maximal and $\mathcal{R}(M) \cap \mathrm{int}\,\mathrm{dom}\,\mathbb{T} \neq \emptyset$, then $M^\intercal \mathbb{T} M$ maximal.

Proofs of maximality in §10.

Operations preserving monotonicity: Concatenation

If $\mathbb{R}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $\mathbb{S}\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, then $\mathbb{T}\colon \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{n+m}$

$$\mathbb{T}(x,y) = \{(u,v) \,|\, u \in \mathbb{R}x,\; v \in \mathbb{S}y\}$$

the concatenation of $\mathbb{R}$ and $\mathbb{S}$, is (maximal) monotone if $\mathbb{R}$ and $\mathbb{S}$ are.
Use notation

$$\mathbb{T} = \begin{bmatrix} \mathbb{R} \\ \mathbb{S} \end{bmatrix}, \qquad \mathbb{T}(x,y) = \begin{bmatrix} \mathbb{R}x \\ \mathbb{S}y \end{bmatrix}.$$

Operations preserving stronger monotonicity properties

$\mathbb{T}$ is μ-s.m., then $\alpha\mathbb{T}$ is $(\alpha\mu)$-s.m. for $\alpha > 0$.

$\mathbb{T}$ is μ-s.m. and $\mathbb{S}$ monotone, then $\mathbb{T} + \mathbb{S}$ is μ-s.m.

$\mathbb{T}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is μ-s.m., and $M \in \mathbb{R}^{n\times m}$ has rank m, then $M^\intercal \mathbb{T} M$ is $(\mu\sigma^2_{\min}(M))$-s.m.

$\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is L-Lipschitz and $M \in \mathbb{R}^{n\times m}$, then $M^\intercal \mathbb{T} M$ is $(L\sigma^2_{\max}(M))$-Lipschitz.

Example: Affine operators

Affine operator $\mathbb{T}(x) = Ax + b$:

- $[\mathbb{T}$ maximal monotone$] \Leftrightarrow [A + A^\intercal \succeq 0]$
- $[\mathbb{T} = \nabla f$ for CCP $f] \Leftrightarrow [A = A^\intercal$ and $A \succeq 0]$
- $\mathbb{T}$ is $\lambda_{\min}(A + A^\intercal)/2$-strongly monotone if $\lambda_{\min}(A + A^\intercal) > 0$ and $\sigma_{\max}(A)$-Lipschitz.

Example: Continuous operators

$\mathbb{T}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is continuous if
$\operatorname{dom} \mathbb{T} = \mathbb{R}^n$, $\mathbb{T}$ is single-valued, and $\mathbb{T}$ is continuous as a function.

A continuous monotone operator $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is maximal.

Maximality is only in question with discontinuous or set-valued operators.

Example: Differentiable operators

We say an operator is differentiable if it is continuous and differentiable.

For differentiable $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$,

- $[\mathbb{T}$ monotone$] \Leftrightarrow [D\mathbb{T}(x) + D\mathbb{T}(x)^\intercal \succeq 0,\ \forall x]$
- $[\mathbb{T}$ μ-s.m.$] \Leftrightarrow [D\mathbb{T}(x) + D\mathbb{T}(x)^\intercal \succeq 2\mu I,\ \forall x]$
- $[\mathbb{T}$ L-Lipschitz$] \Leftrightarrow [\sigma_{\max}(D\mathbb{T}(x)) \le L,\ \forall x]$

Continuously differentiable monotone $\mathbb{T}$,
$[\mathbb{T} = \nabla f$ for CCP $f] \Leftrightarrow [D\mathbb{T}(x)$ symmetric $\forall x]$
(When $n = 3$, this is $\nabla \times \mathbb{T} = 0$ condition of electromagnetic potentials.)

Example: Saddle subdifferential

For convex-concave $\mathbf{L} \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$,
saddle subdifferential operator $\partial \mathbf{L} \colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$:

$$\partial \mathbf{L}(x, u) = \left[\begin{array}{c} \partial_x \mathbf{L}(x, u) \\ \partial_u(-\mathbf{L}(x, u)) \end{array} \right]$$

$\operatorname{Zer} \partial \mathbf{L}$ is the set of saddle points of $\mathbf{L}$, i.e.,
$[0 \in \partial \mathbf{L}(x^\star, u^\star)] \Leftrightarrow [(x^\star, u^\star)$ is a saddle point of $\mathbf{L}]$

For most well-behaved ("closed proper") convex-concave saddle functions, their saddle subdifferentials are maximal monotone. We avoid this notion and instead verify the maximality of saddle subdifferentials on a case-by-case basis.

Example: KKT operator

Consider

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p, \end{array}$$

$f_0, \dots, f_m$ are CCP and $h_1, \dots, h_p$ are affine. Lagrangian

$$\mathbf{L}(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) - \delta_{\mathbb{R}_+^m}(\lambda)$$

is convex-concave. Consider the Karush–Kuhn–Tucker (KKT) operator

$$\mathbb{T}(x, \lambda, \nu) = \begin{bmatrix} \partial_x \mathbf{L}(x, \lambda, \nu) \\ -\mathbb{F}(x) + \mathbf{N}_{\mathbb{R}_+^m}(\lambda) \\ -\mathbb{H}(x) \end{bmatrix},$$

where

$$\mathbb{F}(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}, \qquad \mathbb{H}(x) = \begin{bmatrix} h_1(x) \\ \vdots \\ h_p(x) \end{bmatrix}.$$

Example: KKT operator

$$\mathbb{T}(x,\lambda,\nu) = \begin{bmatrix} \partial_x \mathbf{L}(x,\lambda,\nu) \\ -\mathbb{F}(x) + \mathbf{N}_{\mathbb{R}^m_+}(\lambda) \\ -\mathbb{H}(x) \end{bmatrix} = \begin{bmatrix} \partial_x \mathbf{L}(x,\lambda,\nu) \\ \partial_\lambda(-\mathbf{L}(x,\lambda,\nu)) \\ \partial_\nu(-\mathbf{L}(x,\lambda,\nu)) \end{bmatrix}$$

$\mathbb{T}$ is a special case of the saddle subdifferential, so monotone.

Arguments based on total duality tell us:
$[0 \in \mathbb{T}(x^\star, \lambda^\star, \nu^\star)] \Leftrightarrow [x^\star$ primal sol., $(\lambda^\star, \nu^\star)$ dual sol., strong duality$]$

Monotone inclusion problem

Monotone inclusion problem:

$$\underset{x \in \mathbb{R}^n}{\text{find}} \quad 0 \in \mathbb{A}x,$$

$\mathbb{A}$.

where $\mathbb{A}$ is monotone.

Many interesting problems can be formulated this way.

Outline

Nonexpansive and contractive operators

$\mathbb{T}$ is nonexpansive if

$$\|\mathbb{T}x - \mathbb{T}y\| \le \|x - y\| \qquad \forall\, x, y \in \operatorname{dom} \mathbb{T},$$

i.e., 1-Lipschitz. $\mathbb{T}$ is a contraction if L-Lipschitz with $L < 1$.

Mapping a pair of points by a contraction reduces their distance; mapping by a nonexpansive operator does not increase their distance.

Properties:

- If $\mathbb{T}$ and $\mathbb{S}$ nonexpansive, then $\mathbb{T}\mathbb{S}$ nonexpansive.
- If $\mathbb{T}$ or $\mathbb{S}$ furthermore contractive, then $\mathbb{T}\mathbb{S}$ contractive.
- If $\mathbb{T}$ and $\mathbb{S}$ nonexpansive, then $\theta\mathbb{T} + (1-\theta)\mathbb{S}$ with $\theta \in [0, 1]$ nonexpansive.
- If $\mathbb{T}$ is furthermore contractive and $\theta \in (0, 1]$, then $\theta\mathbb{T} + (1-\theta)\mathbb{S}$ contractive.

Averaged operators

For $\theta \in (0,1)$, $\mathbb{T}$ is θ-averaged if $\mathbb{T} = (1-\theta)\mathbb{I} + \theta\mathbb{S}$ for nonexpansive $\mathbb{S}$. Operator is averaged if θ-averaged for some $\theta \in (0,1)$. Operator is firmly nonexpansive if $(1/2)$-averaged.

$\mathbb{T}$ and $\mathbb{S}$ are averaged, composition $\mathbb{T}\mathbb{S}$ is averaged. (Proof later in §13.)

Averagedness is the basis for convergence of many splitting methods.

Averaged operators

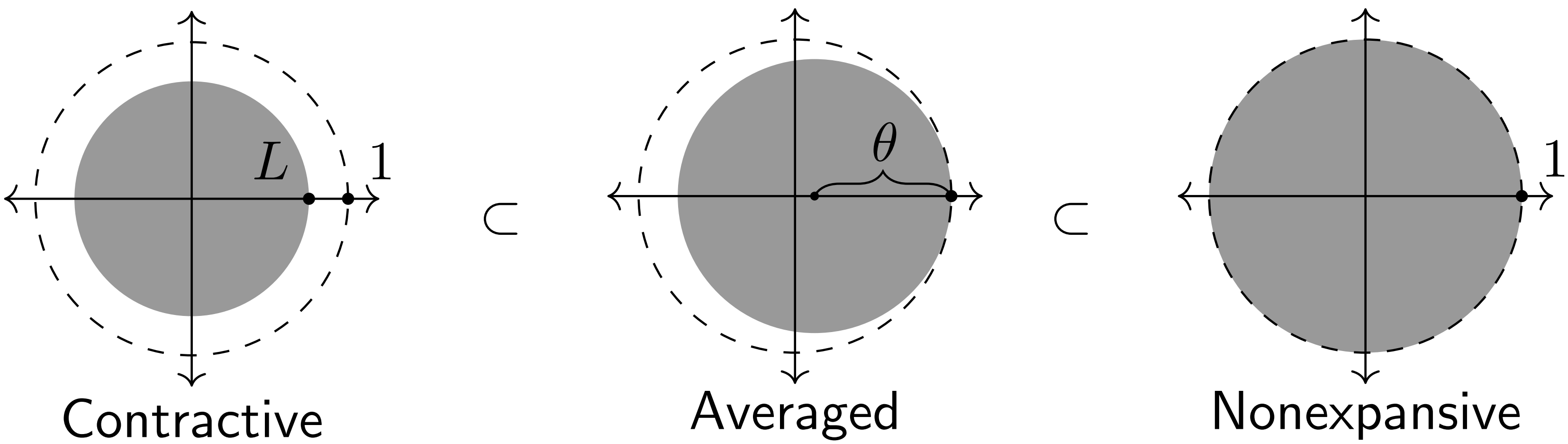


Illustration of classes of contractive, averaged, and nonexpansive operators. The figure illustrates the relationship contractive $\subset$ averaged $\subset$ nonexpansive. The precise meaning of these figures will be defined in §13.

Outline

Fixed points

x is a fixed point of $\mathbb{T}$ if $x = \mathbb{T}x$.

$$\operatorname{Fix}\mathbb{T} = \{x \mid x = \mathbb{T}x\} = (\mathbb{I} - \mathbb{T})^{-1}(0)$$

$\operatorname{Fix}\mathbb{T}$ can contain nothing (e.g. $\mathbb{T}x = x + 1$) or many points (e.g. $\mathbb{T}x = |x|$).

Fixed points

When $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive, $\operatorname{Fix}\mathbb{T}$ is closed and convex.

Proof. $\operatorname{Fix}\mathbb{T}$ is closed since $\mathbb{T}-\mathbb{I}$ is continuous.
Suppose $x,\ y \in \operatorname{Fix}\mathbb{T}$, $\theta \in [0,1]$, and $z = \theta x + (1-\theta)y$. Since $\mathbb{T}$ is nonexpansive,

$$\|\mathbb{T}z - x\| \le \|z - x\| = (1-\theta)\|y - x\|,$$

Similarly,

$$\|\mathbb{T}z - y\| \le \theta\|y - x\|.$$

So the triangle inequality

$$\|x - y\| \le \|\mathbb{T}z - x\| + \|\mathbb{T}z - y\|$$

holds with equality and $\mathbb{T}z$ is on the line segment between x and y.
From $\|\mathbb{T}z - y\| = \theta\|y - x\|$, we conclude $\mathbb{T}z = \theta x + (1-\theta)y = z$. $\square$

Fixed-point iteration

The fixed-point iteration (FPI) is

$$x^{k+1} = \mathbb{T}x^k$$

for $k = 0, 1, \ldots,$ where $x^0 \in \mathbb{R}^n$ is some starting point and $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$.

The FPI is used to find a fixed point of $\mathbb{T}$. Clearly, the algorithm stays at a fixed point if it starts at a fixed point.

Two steps of using FPI: (i) find a suitable operator whose fixed points are solutions to a monotone inclusion problem of interest. (ii) show that the iteration converges to a fixed point.

In general, FPI need not converge. We provide two guarantees.

FPI with contractive operators

If $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a contraction with $L < 1$, then FPI is a contraction mapping algorithm. For $x^\star \in \operatorname{Fix}\mathbb{T}$,

$$\|x^k - x^\star\| \le L\|x^{k-1} - x^\star\| \le \cdots \le L^k\|x^0 - x^\star\|.$$

Basis of classic Banach fixed-point theorem. When $\mathbb{T}$ is a contraction, convergence is simple.

In many optimization setups, however, a contraction is too much to ask for. We need convergence under weaker assumptions.

FPI with averaged operators

If $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is averaged, FPI is called an averaged or the Krasnosel'skiĭ–Mann iteration.

Theorem 1.

Assume $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is θ-averaged with $\theta \in (0,1)$ and $\operatorname{Fix}\mathbb{T} \neq \emptyset$. Then $x^{k+1} = \mathbb{T}x^k$ with any starting point $x^0 \in \mathbb{R}^n$ converges to one fixed point, i.e.,

$$x^k \to x^\star$$

for some $x^\star \in \operatorname{Fix}\mathbb{T}$. The quantities $\operatorname{dist}(x^k, \operatorname{Fix}\mathbb{T})$, $\|x^{k+1} - x^k\|$, and $\|x^k - x^\star\|$ for any $x^\star \in \operatorname{Fix}\mathbb{T}$ are monotonically nonincreasing with k. Finally, we have

$$\operatorname{dist}(x^k, \operatorname{Fix}\mathbb{T}) \to 0$$

and

$$\|x^{k+1} - x^k\|^2 \leq \frac{\theta}{(k+1)(1-\theta)} \operatorname{dist}^2(x^0, \operatorname{Fix}\mathbb{T}).$$

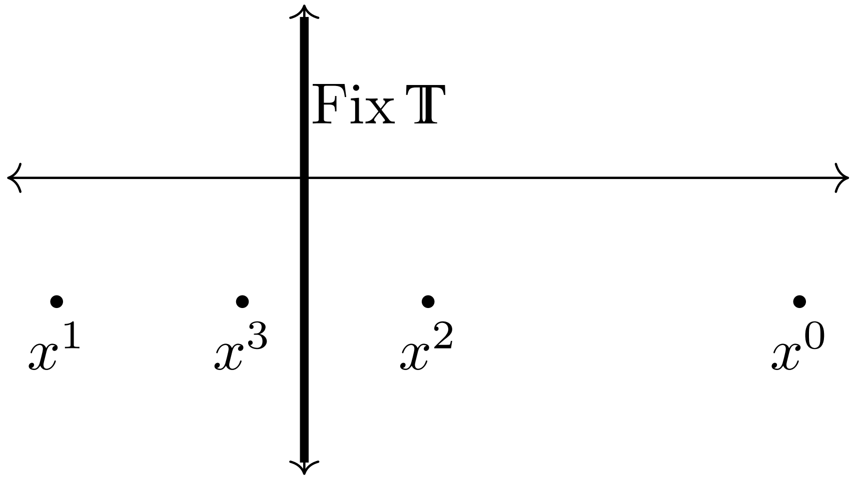
Discussion of Theorem 1

When $\mathbb{T}$ is nonexpansive but not averaged, we can use $(1-\theta)\mathbb{I}+\theta\mathbb{T}$ with $\theta \in (0,1)$ since $\operatorname{Fix}\mathbb{T} = \operatorname{Fix}((1-\theta)\mathbb{I}+\theta\mathbb{T})$.

For example, $\mathbb{T}\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbb{T}x = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix} x$$

is $(3/4)$-averaged with $\operatorname{Fix}\mathbb{T} = \{(0,z) \mid z \in \mathbb{R}\}$.



FPI with respect to $\mathbb{T}$ converges to one fixed point, which depends on the staring put x^0.

Proof outline of Theorem 1

Assume *nonnegative* sequences $V^0, V^1, \ldots$ and $S^0, S^1, \ldots$ satisfy

$$V^{k+1} \le V^k - S^k.$$

Consequences: (i) V^k is monotonically nonincreasing (although $V^k \nrightarrow 0$ possible) (ii) $S^k \to 0$. To see why, sum both sides from 0 to k to get

$$\sum_{i=0}^{k} S^i \le V^0 - V^{k+1} \le V^0.$$

Taking $k \to \infty$ gives us

$$\sum_{i=0}^{\infty} S^i \le V^0 < \infty.$$

$S^0, S^1, \ldots$ is summable. By summability, $S^k \to 0$. V^k is a the Lyapunov function and S^k the summable term. This is the summability argument.

Proof of Theorem 1

Stage 1. Note the identity

$$\|(1-\theta)x+\theta y\|^2 = (1-\theta)\|x\|^2 + \theta\|y\|^2 - \theta(1-\theta)\|x-y\|^2.$$

$\mathbb{T} = (1-\theta)\mathbb{I} + \theta\mathbb{S}$, where $\mathbb{S}$ is N.E. Then

$$x^{k+1} = \mathbb{T}x^k = (1-\theta)x^k + \theta\mathbb{S}x^k.$$

For any $x^\star \in \operatorname{Fix}\mathbb{T}$,

$$\begin{aligned}
&\|x^{k+1}-x^\star\|^2 \\
&\qquad = (1-\theta)\|x^k-x^\star\|^2 + \theta\|\mathbb{S}(x^k)-x^\star\|^2 - \theta(1-\theta)\|\mathbb{S}(x^k)-x^k\|^2 \\
&\qquad \le (1-\theta)\|x^k-x^\star\|^2 + \theta\|x^k-x^\star\|^2 - \theta(1-\theta)\|\mathbb{S}(x^k)-x^k\|^2 \\
&\qquad = \underbrace{\|x^k-x^\star\|^2}_{=V^k} - \underbrace{\theta(1-\theta)\|\mathbb{S}(x^k)-x^k\|^2}_{=S^k}. \qquad (1)
\end{aligned}$$

Proof of Theorem 1

Now establish the monotonic decreases. Core inequality (1) tells us

$$\|x^{k+1} - x^\star\| \le \|x^k - x^\star\|.$$

Minimize both sides with respect to $x^\star \in \operatorname{Fix} \mathbb{T}$:

$$\operatorname{dist}(x^{k+1}, \operatorname{Fix} \mathbb{T}) \le \operatorname{dist}(x^k, \operatorname{Fix} \mathbb{T}).$$

This is called Fejér monotonicity.

Fixed-point residual: $\mathbb{T}(x^k) - x^k = x^{k+1} - x^k$. We view $\|\mathbb{T}(x^k) - x^k\|$ as a measure of optimality for FPI. Since $\mathbb{T}$ is nonexpansive,

$$\|x^{k+1} - x^k\| = \|\mathbb{T}x^k - \mathbb{T}x^{k-1}\| \le \|x^k - x^{k-1}\|.$$

Proof of Theorem 1

Sum (1) from 0 to k

$$\|x^{k+1} - x^\star\|^2 \le \|x^0 - x^\star\|^2 - \frac{1-\theta}{\theta} \sum_{j=0}^{k} \|\mathbb{T}x^j - x^j\|^2$$

Reorganize

$$\sum_{j=0}^{k} \|\mathbb{T}x^j - x^j\|^2 \le \frac{\theta}{1-\theta}\|x^0 - x^\star\|^2 - \frac{\theta}{1-\theta}\|x^{k+1} - x^\star\|^2$$

Monotonicity of $\|x^{k+1} - x^k\|$

$$(k+1)\|x^{k+1} - x^k\|^2 \le \sum_{j=0}^{k} \|x^{j+1} - x^j\|^2 \le \frac{\theta}{1-\theta}\|x^0 - x^\star\|^2$$

Conclude

$$\|x^{k+1} - x^k\|^2 \le \frac{\theta}{(k+1)(1-\theta)}\|x^0 - x^\star\|^2.$$

Minimizing the right-hand side with respect to $x^\star \in \operatorname{Fix}\mathbb{T}$

$$\|x^{k+1} - x^k\|^2 \le \frac{\theta}{(k+1)(1-\theta)}\operatorname{dist}^2(x^0, \operatorname{Fix}\mathbb{T}).$$

Convergence proof of Theorem 1

Stage 2. Now show $x^k \to x^\star$ for some $x^\star \in \operatorname{Fix}\mathbb{T}$ with the steps: (i) x^k has an accumulation point (ii) this accumulation point is a solution (iii) this is the only accumulation point.

(i) Consider any $\tilde{x}^\star \in \operatorname{Fix}\mathbb{T}$. Then (1) tells us that $x^0, x^1, \dots$ lie within the compact set $\{x \mid \|x - \tilde{x}^\star\| \le \|x^0 - \tilde{x}^\star\|\}$, and $x^0, x^1, \dots$ has an accumulation point $x^\star$.

(ii) Accumulation point $x^\star$ satisfies $\mathbb{T}(x^\star) - x^\star = 0$, as $\mathbb{T}(x^k) - x^k \to 0$ and $\mathbb{T} - \mathbb{I}$ is continuous, i.e., $x^\star \in \operatorname{Fix}\mathbb{T}$.

(iii) Apply (1) to this accumulation point $x^\star \in \operatorname{Fix}\mathbb{T}$ to conclude $\|x^k - x^\star\|$ monotonically decreases to 0, i.e., the entire sequence converges to $x^\star$.

$\square$

Termination criterion

$\|x^{k+1} - x^k\| < \varepsilon$ can be used as a termination criterion.
Specific setups may have specific and better termination criteria.

We avoid the discussion of termination criterion for simplicity.

Methods: Gradient descent

Consider

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; f(x),$$

where f is CCP and differentiable.

$[x \in \operatorname{argmin} f] \Leftrightarrow [x = (\mathbb{I} - \alpha \nabla f)(x)$ for any nonzero $\alpha \in \mathbb{R}]$

The FPI

$$x^{k+1} = x^k - \alpha \nabla f(x^k)$$

is gradient method or gradient descent, and α is the step size.

Methods: Gradient descent

Assume f is L-smooth. By cocoercivity,

$$
\begin{aligned}
&\|(\mathbb{I}-(2/L)\nabla f)x-(\mathbb{I}-(2/L)\nabla f)y\|^2\\
&=\|x-y\|^2-\frac{4}{L}\left(\langle x-y,\nabla f(x)-\nabla f(y)\rangle-\frac{1}{L}\|\nabla f(x)-\nabla f(y)\|^2\right)\\
&\le\|x-y\|^2.
\end{aligned}
$$

Therefore, $\mathbb{I}-\alpha\nabla f$ is averaged for $\alpha\in(0,2/L)$ since

$$\mathbb{I}-\alpha\nabla f=(1-\theta)\mathbb{I}+\theta(\mathbb{I}-(2/L)\nabla f),$$

where $\theta=\alpha L/2<1$.

$x^k\to x^\star$ if a solution exists with rate

$$\|\nabla f(x^k)\|^2=O(1/k),$$

for any $\alpha\in(0,2/L)$.

If f is strongly convex, FPI is a contraction.

Methods: Forward step method

Consider

$$\underset{x \in \mathbb{R}^n}{\text{find}} \quad 0 = \mathbb{F}(x),$$

where $\mathbb{F}\colon \mathbb{R}^n \to \mathbb{R}^n$.

$[x \in \operatorname{Zer} \mathbb{F}] \Leftrightarrow [x \in \operatorname{Fix}(\mathbb{I} - \alpha\mathbb{F})$ for any nonzero $\alpha \in \mathbb{R}]$

The FPI

$$x^{k+1} = x^k - \alpha\mathbb{F}x^k,$$

is the forward step method.

$x^k \to x^\star$ if $\mathbb{F}$ is β-cocoercive, $\alpha \in (0, 2\beta)$, and $\operatorname{Fix}\mathbb{F} \neq \emptyset$.
Contraction for small $\alpha > 0$ if $\mathbb{F}$ is strongly monotone and Lipschitz.

Methods: Dual ascent

Consider primal-dual problem pair

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & Ax = b, \end{array} \qquad \underset{u \in \mathbb{R}^m}{\text{maximize}} \quad -f^*(-A^\intercal u) - b^\intercal u$$

and its associated Lagrangian

$$\mathbf{L}(x, u) = f(x) + \langle u, Ax - b \rangle.$$

Gradient method on $g(u) = f^*(-A^\intercal u) + b^\intercal u$, the FPI on $\mathbb{I} - \alpha \nabla g$

$$\begin{aligned} x^{k+1} &= \underset{x}{\operatorname{argmin}}\, \mathbf{L}(x, u^k) \\ u^{k+1} &= u^k + \alpha(Ax^{k+1} - b) \end{aligned}$$

Uzawa method or dual ascent. (∇g characterized in page 11.)

Methods: Dual ascent

If f is μ-strongly convex, then

$$\nabla g(u) = -A\nabla f^*(-A^\intercal u) + b$$

is Lipschitz with parameter $\sigma_{\max}^2(A)/\mu$.

If f is μ-strongly convex, total duality holds, and $0 < \alpha < 2\mu/\sigma_{\max}^2(A)$, then $x^k \to x^\star$ and $u^k \to u^\star$.

Outline

Resolvent and reflected resolvent

Resolvent $\mathbb{A}$:

$$\mathbb{J}_{\mathbb{A}} = (\mathbb{I} + \mathbb{A})^{-1}$$

Reflected resolvent of $\mathbb{A}$:

$$\mathbb{R}_{\mathbb{A}} = 2\mathbb{J}_{\mathbb{A}} - \mathbb{I}$$

also called Cayley operator or reflection operator.
Often use $\mathbb{J}_{\alpha\mathbb{A}}$ and $\mathbb{R}_{\alpha\mathbb{A}}$ with $\alpha > 0$.

If $\mathbb{A}$ is maximal monotone, $\mathbb{R}_{\mathbb{A}}$ is a nonexpansive (single-valued) with $\operatorname{dom}\mathbb{R}_{\mathbb{A}} = \mathbb{R}^n$, and $\mathbb{J}_{\mathbb{A}}$ is a $(1/2)$-averaged with $\operatorname{dom}\mathbb{J}_{\mathbb{A}} = \mathbb{R}^n$.

Nonexpansiveness of $\mathbb{R}_\mathbb{A}$ and $\mathbb{J}_\mathbb{A}$

Proof of nonexpansiveness and averagedness.

Proof. Assume $(x,u),(y,v) \in \mathbb{J}_\mathbb{A}$. Then

$$x \in u + \mathbb{A}u, \qquad y \in v + \mathbb{A}v.$$

By monotonicity of $\mathbb{A}$,

$$\langle (x-u)-(y-v), u-v\rangle \geq 0$$

and

$$\begin{aligned}\|(2u-x)-(2v-y)\|^2 &= \|x-y\|^2 - 4\langle (x-u)-(y-v), u-v\rangle \\ &\leq \|x-y\|^2.\end{aligned}$$

So $\mathbb{R}_\mathbb{A}$ is NE and $\mathbb{J}_\mathbb{A} = (1/2)\mathbb{I} + (1/2)\mathbb{R}_\mathbb{A}$ is $(1/2)$-averaged. $\square$

Domain of $\mathbb{R}_{\mathbb{A}}$ and $\mathbb{J}_{\mathbb{A}}$

Minty surjectivity theorem: $\operatorname{dom} \mathbb{J}_{\mathbb{A}} = \mathbb{R}^n$ when $\mathbb{A}$ is maximal monotone.

This result is easy to intuitively see in 1D but is non-trivial in higher dimensions. We prove this in §10.

Zero set of a maximal monotone operator

$\operatorname{Zer}\mathbb{A}$ is a closed convex set when $\mathbb{A}$ is maximal monotone.

Proof. $\operatorname{Zer}\mathbb{A} = \operatorname{Fix}\mathbb{J}_{\mathbb{A}}$ since

$$0 \in \mathbb{A}x \quad \Leftrightarrow \quad x \in x + \mathbb{A}x \quad \Leftrightarrow \quad \mathbb{J}_{\mathbb{A}}x = x.$$

Since $\mathbb{J}_{\mathbb{A}}$ is nonexpansive, $\operatorname{Fix}\mathbb{J}_{\mathbb{A}} = \operatorname{Zer}\mathbb{A}$ is a closed convex set. $\square$

Note that proof relies on maximality through $\operatorname{dom}\mathbb{J}_{\mathbb{A}} = \mathbb{R}^n$.

Example: Monotone linear operator

Let $\mathbb{A}$ be a monotone linear operator represented by a symmetric matrix.

Then, $\mathbb{A}$ has eigenvalues in $[0, \infty)$ and $\mathbb{J}_{\mathbb{A}} = (\mathbb{I} + \mathbb{A})^{-1}$ has eigenvalues in $(0, 1]$.

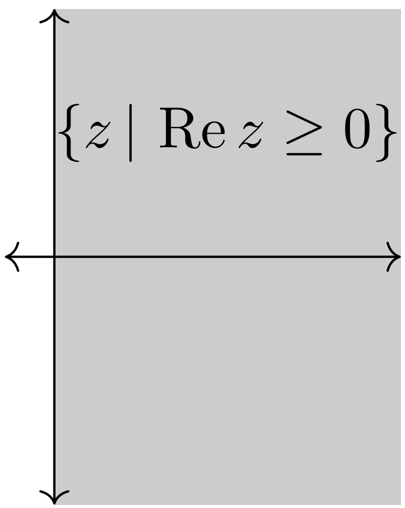
$$\mathbb{R}_{\mathbb{A}} = 2\mathbb{J}_{\mathbb{A}} - \mathbb{I} = (\mathbb{I} - \mathbb{A})(\mathbb{I} + \mathbb{A})^{-1} = (\mathbb{I} + \mathbb{A})^{-1}(\mathbb{I} - \mathbb{A}),$$

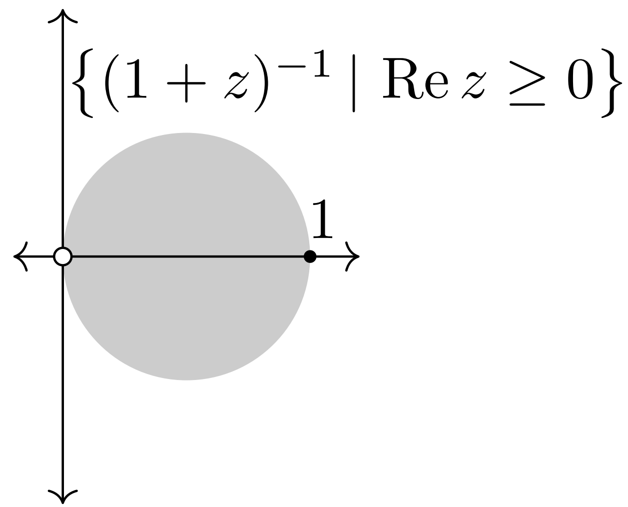
is the Cayley transform of $\mathbb{A}$ and has eigenvalues in $(-1, 1]$.

Example: Complex number as operator on $\mathbb{R}^2 \cong \mathbb{C}$

Identify $z \in \mathbb{C}$ with a linear operator from $\mathbb{C}$ to $\mathbb{C}$ defined by multiplication, i.e., $z\colon x \mapsto zx$.

Equip complex numbers with inner product $\langle x, y\rangle = \operatorname{Re} x\overline{y}$.





$z \in \mathbb{C}$ is monotone if and only if $\operatorname{Re} z \geq 0$. Resolvent $(1+z)^{-1}$ for monotone z is in disk with center $1/2$ and radius $1/2$ excluding origin.

Resolvent of subdifferential

For CCP f and $\alpha > 0$,

$$\mathbb{J}_{\alpha\partial f} = \mathrm{Prox}_{\alpha f}.$$

Proof.

$$\begin{aligned}
z = (I + \alpha\partial f)^{-1}(x) \quad &\Leftrightarrow \quad z + \alpha\partial f(z) \ni x \\
&\Leftrightarrow \quad 0 \in \partial_z \left(\alpha f(z) + \frac{1}{2}\|z - x\|^2\right) \\
&\Leftrightarrow \quad z = \underset{z}{\mathrm{argmin}} \left\{\alpha f(z) + \frac{1}{2}\|z - x\|^2\right\} \\
&\Leftrightarrow \quad z = \mathrm{Prox}_{\alpha f}(x)
\end{aligned}$$

$\square$

Resolvent of subdifferential of conjugate

If $g(u) = f^*(A^\intercal u)$, f CCP, and $\mathrm{ri}\,\mathrm{dom}\, f^* \cap \mathcal{R}(A^\intercal) \neq \emptyset$, then

$$v = \mathrm{Prox}_{\alpha g}(u) \quad \Leftrightarrow \quad \begin{array}{l} x \in \mathrm{argmin}_x \left\{ f(x) - \langle u, Ax \rangle + \frac{\alpha}{2} \|Ax\|^2 \right\} \\ v = u - \alpha Ax. \end{array}$$

Proof.

$$\begin{aligned}
v = (I + \alpha \partial g)^{-1}(u) \\
&\Leftrightarrow \quad v + \alpha A \partial f^*(A^\intercal v) \ni u \\
&\Leftrightarrow \quad v + \alpha Ax = u,\; x \in \partial f^*(A^\intercal v) \\
&\Leftrightarrow \quad v = u - \alpha Ax,\; \partial f(x) \ni A^\intercal v \\
&\Leftrightarrow \quad v = u - \alpha Ax,\; \partial f(x) \ni A^\intercal (u - \alpha Ax) \\
&\Leftrightarrow \quad v = u - \alpha Ax,\; x \in \operatorname*{argmin}_x \left\{ f(x) - \langle u, Ax \rangle + \frac{\alpha}{2} \|Ax\|^2 \right\}.
\end{aligned}$$

Projection is a resolvent

If $C \subset \mathbb{R}^n$ is nonempty closed convex, then

$$\mathbb{J}_{\mathbb{N}_C} = \mathrm{Prox}_{\delta_C} = \Pi_C.$$

The resolvent generalizes the projection operator in this sense.

KKT operator for linearly constrained problems

Consider the Lagrangian

$$\mathbf{L}(x,u) = f(x) + \langle u, Ax - b\rangle$$

which generates the primal problem

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & Ax = b. \end{array}$$

We can compute its resolvent with

$$\mathbb{J}_{\alpha\partial\mathbf{L}}(x,u) = (y,v) \quad \Leftrightarrow \quad \begin{array}{l} y = \operatorname{argmin}_z \left\{\mathbf{L}_\alpha(z,u) + \frac{1}{2\alpha}\|z-x\|^2\right\} \\ v = u + \alpha(Ay - b), \end{array}$$

where $\mathbf{L}_\alpha$ is the augmented Lagrangian

$$\mathbf{L}_\alpha(x,u) = f(x) + \langle u, Ax - b\rangle + \frac{\alpha}{2}\|Ax - b\|^2.$$

KKT operator for linearly constrained problems

Proof. For any $\alpha > 0$,

$$(y, v) = \mathbb{J}_{\alpha\partial\mathbf{L}}(x, u) \quad \Leftrightarrow \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} y \\ v \end{bmatrix} + \alpha \begin{bmatrix} \partial f(y) + A^\intercal v \\ b - Ay \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \alpha \begin{bmatrix} \partial f(y) \\ b \end{bmatrix} + \begin{bmatrix} I & \alpha A^\intercal \\ -\alpha A & I \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}.$$

Left-multiply invertible matrix

$$\begin{bmatrix} I & -\alpha A^\intercal \\ 0 & I \end{bmatrix}$$

to get

$$\Leftrightarrow \quad \begin{bmatrix} x - \alpha A^\intercal u \\ u \end{bmatrix} \in \alpha \begin{bmatrix} \partial f(y) - \alpha A^\intercal b \\ b \end{bmatrix} + \begin{bmatrix} I + \alpha^2 A^\intercal A & 0 \\ -\alpha A & I \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}.$$

First line is independent of v, so we can compute y first and then v.
(This is the Gaussian elimination technique of §3.4.)

KKT operator for linearly constrained problems

Reorganize to get

$$\begin{gathered}0 \in \partial f(y) + A^\intercal u - \alpha A^\intercal (Ay - b) + (1/\alpha)(y - x)\\ v = u + \alpha(Ay - b),\end{gathered}$$

and conclude

$$\begin{aligned}y &= \underset{z}{\operatorname{argmin}} \left\{ f(z) + \langle u, Az - b\rangle + \frac{\alpha}{2}\|Az - b\|^2 + \frac{1}{2\alpha}\|z - x\|^2 \right\}\\ v &= u + \alpha(Ay - b).\end{aligned}$$

$\square$

Resolvent identities

Let $\mathbb{A}$ maximal monotone and $\alpha > 0$.

If $\mathbb{B}(x) = \mathbb{A}(x) + t$,

$$\mathbb{J}_{\alpha\mathbb{B}}(u) = \mathbb{J}_{\alpha\mathbb{A}}(u - \alpha t).$$

If $\mathbb{B}(x) = \mathbb{A}(x - t)$,

$$\mathbb{J}_{\alpha\mathbb{B}}(u) = \mathbb{J}_{\alpha\mathbb{A}}(u - t) + t.$$

If $\mathbb{B}(x) = -\mathbb{A}(t - x)$,

$$\mathbb{J}_{\alpha\mathbb{B}}(u) = t - \mathbb{J}_{\alpha\mathbb{A}}(t - u).$$

Inverse resolvent identity

Inverse resolvent identity:

$$\mathbb{J}_{\alpha^{-1}\mathbb{A}}(x) + \alpha^{-1}\mathbb{J}_{\alpha\mathbb{A}^{-1}}(\alpha x) = x,$$

for maximal monotone $\mathbb{A}$ and $\alpha > 0$.

When $\alpha = 1$,

$$\mathbb{J}_{\mathbb{A}} + \mathbb{J}_{\mathbb{A}^{-1}} = \mathbb{I}.$$

Moreau identity: a special case, for CCP f

$$\mathrm{Prox}_{\alpha^{-1}f}(x) + \alpha^{-1}\mathrm{Prox}_{\alpha f^*}(\alpha x) = x.$$

Consequence: $\mathrm{Prox}_{\alpha f}$ and $\mathrm{Prox}_{\alpha f^*}$ require same computational cost.

Reflected resolvent identities

If $\mathbb{A}$ is maximal monotone and single-valued and $\alpha > 0$,

$$\mathbb{R}_{\alpha\mathbb{A}} = (\mathbb{I} - \alpha\mathbb{A})(\mathbb{I} + \alpha\mathbb{A})^{-1}.$$

Proof.

$$\begin{aligned}
\mathbb{R}_{\alpha\mathbb{A}} &= 2(\mathbb{I} + \alpha\mathbb{A})^{-1} - \mathbb{I} \\
&= 2(\mathbb{I} + \alpha\mathbb{A})^{-1} - (\mathbb{I} + \alpha\mathbb{A})(\mathbb{I} + \alpha\mathbb{A})^{-1} \\
&= (\mathbb{I} - \alpha\mathbb{A})(\mathbb{I} + \alpha\mathbb{A})^{-1}.
\end{aligned}$$

2nd line by Exercise 2.1. $\square$

Reflected resolvent identities

If $\mathbb{A}$ is maximal monotone (but not necessarily single-valued) and $\alpha > 0$,

$$\mathbb{R}_{\alpha\mathbb{A}}(\mathbb{I} + \alpha\mathbb{A}) = \mathbb{I} - \alpha\mathbb{A}.$$

Proof. For $x \in \operatorname{dom} \mathbb{A}$,

$$\begin{aligned}
\mathbb{R}_{\alpha\mathbb{A}}(\mathbb{I} + \alpha\mathbb{A})(x) &= 2(\mathbb{I} + \alpha\mathbb{A})^{-1}(\mathbb{I} + \alpha\mathbb{A})(x) - (\mathbb{I} + \alpha\mathbb{A})(x) \\
&= 2\mathbb{I}(x) - (\mathbb{I} + \alpha\mathbb{A})(x) \\
&= (\mathbb{I} - \alpha\mathbb{A})(x)
\end{aligned}$$

2nd line by Exercise 2.1. For $x \notin \operatorname{dom} \mathbb{A}$, both sides are empty. $\square$

Outline

Proximal point method

Consider

$$\underset{x \in \mathbb{R}^n}{\text{find}} \quad 0 \in \mathbb{A}x$$

where $\mathbb{A}$ is maximal monotone. Equivalent to finding $x \in \operatorname{Fix} \mathbb{J}_{\alpha\mathbb{A}}$.

The FPI

$$x^{k+1} = \mathbb{J}_{\alpha\mathbb{A}}(x^k)$$

is the proximal point method (PPM) or proximal minimization.

PPM converges to a solution if one exists, since $\mathbb{J}_{\alpha\mathbb{A}}$ is averaged.

Methods of multipliers

Consider the primal-dual problem pair

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & Ax = b, \end{array} \qquad \underset{u\in\mathbb{R}^m}{\text{maximize}} \quad -f^*(-A^\intercal u) - b^\intercal u$$

generated by the Lagrangian $\mathbf{L}(x,u) = f(x) + \langle u, Ax - b\rangle$.

Augmented Lagrangian:

$$\mathbf{L}_\alpha(x,u) = f(x) + \langle u, Ax - b\rangle + \frac{\alpha}{2}\|Ax - b\|^2.$$

Method of multipliers

Assume $\mathcal{R}(A^\intercal) \cap \operatorname{ri}\operatorname{dom} f^* \neq \emptyset$. Write $g(u) = f^*(-A^\intercal u) + b^\intercal u$.

The FPI $u^{k+1} = \mathbb{J}_{\alpha\partial g}(u^k)$

$$\begin{aligned} x^{k+1} &\in \operatorname*{argmin}_x \mathbf{L}_\alpha(x, u^k) \\ u^{k+1} &= u^k + \alpha(Ax^{k+1} - b) \end{aligned}$$

is the method of multipliers. ($\mathrm{Prox}_{\alpha g}$ calculation in pages 62 and 67.)

If a dual solution exists and $\alpha > 0$, then $u^k \to u^\star$.

Proximal method of multipliers

The FPI $(x^{k+1}, u^{k+1}) = \mathbb{J}_{\alpha\partial\mathbf{L}}(x^k, u^k)$

$$\begin{aligned}
x^{k+1} &= \underset{x}{\operatorname{argmin}} \left\{ \mathbf{L}_\alpha(x, u^k) + \frac{1}{2\alpha}\|x - x^k\|^2 \right\} \\
u^{k+1} &= u^k + \alpha(Ax^{k+1} - b)
\end{aligned}$$

is the proximal method of multipliers. ($\mathbb{J}_{\alpha\partial\mathbf{L}}$ calculation in page 64.)

If total duality holds and $\alpha > 0$, then $x^k \to x^\star$ and $u^k \to u^\star$.

Outline

Operator splitting

Operator splitting: split a monotone inclusion problem into smaller simpler components.

Specifically, transform monotone inclusion problems $x \in \mathrm{Zer}\,(\mathbb{A} + \mathbb{B})$ or $x \in \mathrm{Zer}\,(\mathbb{A} + \mathbb{B} + \mathbb{C})$ into fixed-point equations constructed from $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, and their resolvents.

Unified approach: formulate optimization problem as monotone inclusion problem, apply a splitting scheme, and use the FPI.

Forward-backward splitting

Consider
$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 \in (\mathbb{A}+\mathbb{B}),$$
where $\mathbb{A}$ and $\mathbb{B}$ maximal monotone, $\mathbb{A}$ single-valued.

For $\alpha > 0$,

$$\begin{aligned} 0 \in (\mathbb{A}+\mathbb{B})x \quad &\Leftrightarrow \quad 0 \in (\mathbb{I}+\alpha\mathbb{B})x - (\mathbb{I}-\alpha\mathbb{A})x \\ &\Leftrightarrow \quad (\mathbb{I}+\alpha\mathbb{B})x \ni (\mathbb{I}-\alpha\mathbb{A})x \\ &\Leftrightarrow \quad x = \mathbb{J}_{\alpha\mathbb{B}}(\mathbb{I}-\alpha\mathbb{A})x. \end{aligned}$$

So $[x \in \mathrm{Zer}\,(\mathbb{A}+\mathbb{B})] \Leftrightarrow [x \in \mathrm{Fix}\,\mathbb{J}_{\alpha\mathbb{B}}(\mathbb{I}-\alpha\mathbb{A})]$.

$\mathbb{J}_{\alpha\mathbb{B}}(\mathbb{I}-\alpha\mathbb{A})$ is forward-backward splitting (FBS).

Forward-backward splitting

Assume $\mathbb{A}$ is β-cocoercive and $\alpha \in (0, 2\beta)$.

Forward step $\mathbb{I} - \alpha\mathbb{A}$ and backward step $(\mathbb{I} + \alpha\mathbb{B})^{-1}$ are averaged.
So the composition $\mathbb{J}_{\alpha\mathbb{B}}(\mathbb{I} - \alpha\mathbb{A})$ is averaged.

FPI with FBS

$$x^{k+1} = \mathbb{J}_{\alpha\mathbb{B}}(x^k - \alpha\mathbb{A}x^k)$$

converges if $\alpha \in (0, 2\beta)$ and $\mathrm{Zer}\,(\mathbb{A} + \mathbb{B}) \neq \emptyset$.

Backward-forward splitting

Similar splitting with permuted order:

$$\begin{aligned} 0 \in (\mathbb{A}+\mathbb{B})x \quad &\Leftrightarrow \quad (\mathbb{I}+\alpha\mathbb{B})x \ni (\mathbb{I}-\alpha\mathbb{A})x \\ &\Leftrightarrow \quad z = (\mathbb{I}-\alpha\mathbb{A})x,\ z \in (\mathbb{I}+\alpha\mathbb{B})x \\ &\Leftrightarrow \quad z = (\mathbb{I}-\alpha\mathbb{A})x,\ \mathbb{J}_{\alpha\mathbb{B}}z = x \\ &\Leftrightarrow \quad z = (\mathbb{I}-\alpha\mathbb{A})\mathbb{J}_{\alpha\mathbb{B}}z,\ \mathbb{J}_{\alpha\mathbb{B}}z = x \end{aligned}$$

So $[x \in \operatorname{Zer}(\mathbb{A}+\mathbb{B})] \Leftrightarrow [z \in \operatorname{Fix}(\mathbb{I}-\alpha\mathbb{A})\mathbb{J}_{\alpha\mathbb{B}},\ x = \mathbb{J}_{\alpha\mathbb{B}}z]$.

$(\mathbb{I}-\alpha\mathbb{A})\mathbb{J}_{\alpha\mathbb{B}}$ is backward-forward splitting (BFS).

Backward-forward splitting

FPI with BFS

$$
\begin{aligned}
x^{k+1} &= \mathbb{J}_{\alpha \mathbb{B}} z^k \\
z^{k+1} &= x^{k+1} - \alpha \mathbb{A} x^{k+1}
\end{aligned}
$$

converges if $\alpha \in (0, 2\beta)$ and $\operatorname{Zer}(\mathbb{A} + \mathbb{B}) \neq \emptyset$.

BFS is FBS with the order permuted. BFS is more natural to work with in some setups considered in §5 and §6.

Peaceman–Rachford splitting

Consider

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 \in (\mathbb{A}+\mathbb{B})x,$$

where $\mathbb{A}$ and $\mathbb{B}$ maximal monotone.

For $\alpha > 0$, (2nd step uses identity of page 70)

$$\begin{aligned}
0 \in (\mathbb{A}+\mathbb{B})x \quad &\Leftrightarrow \quad 0 \in (\mathbb{I}+\alpha\mathbb{A})x - (\mathbb{I}-\alpha\mathbb{B})x \\
&\Leftrightarrow \quad 0 \in (\mathbb{I}+\alpha\mathbb{A})x - \mathbb{R}_{\alpha\mathbb{B}}(\mathbb{I}+\alpha\mathbb{B})x \\
&\Leftrightarrow \quad 0 \in (\mathbb{I}+\alpha\mathbb{A})x - \mathbb{R}_{\alpha\mathbb{B}}z, \;\; z \in (\mathbb{I}+\alpha\mathbb{B})x \\
&\Leftrightarrow \quad \mathbb{R}_{\alpha\mathbb{B}}z \in (\mathbb{I}+\alpha\mathbb{A})\mathbb{J}_{\alpha\mathbb{B}}z, \;\; x = \mathbb{J}_{\alpha\mathbb{B}}z \\
&\Leftrightarrow \quad \mathbb{J}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}z = \mathbb{J}_{\alpha\mathbb{B}}z, \;\; x = \mathbb{J}_{\alpha\mathbb{B}}z \\
&\Leftrightarrow \quad 2\mathbb{J}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}z - z = \mathbb{R}_{\alpha\mathbb{B}}z, \;\; x = \mathbb{J}_{\alpha\mathbb{B}}z \\
&\Leftrightarrow \quad 2\mathbb{J}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}z - \mathbb{R}_{\alpha\mathbb{B}}z = z, \;\; x = \mathbb{J}_{\alpha\mathbb{B}}z \\
&\Leftrightarrow \quad \mathbb{R}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}z = z, \;\; x = \mathbb{J}_{\alpha\mathbb{B}}z.
\end{aligned}$$

So $[x \in \operatorname{Zer}(\mathbb{A}+\mathbb{B})] \Leftrightarrow [z \in \operatorname{Fix}\mathbb{R}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}},\, x = \mathbb{J}_{\alpha\mathbb{B}}z]$.

$\mathbb{R}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}$ is Peaceman–Rachford splitting (PRS).

Peaceman–Rachford splitting

$\mathbb{R}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}$ merely nonexpansive. FPI with PRS

$$z^{k+1} = \mathbb{R}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}(z^k)$$

may not converge.

Douglas–Rachford splitting

Average to ensure convergence.

FPI with $\frac{1}{2}\mathbb{I} + \frac{1}{2}\mathbb{R}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}$, Douglas–Rachford splitting (DRS), is

$$\begin{aligned} x^{k+1/2} &= \mathbb{J}_{\alpha\mathbb{B}}(z^k) \\ x^{k+1} &= \mathbb{J}_{\alpha\mathbb{A}}(2x^{k+1/2} - z^k) \\ z^{k+1} &= z^k + x^{k+1} - x^{k+1/2} \end{aligned}$$

converges for any $\alpha > 0$ if $\mathrm{Zer}\,(\mathbb{A} + \mathbb{B}) \neq \emptyset$.

Davis–Yin splitting

Consider

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 \in (\mathbb{A}+\mathbb{B}+\mathbb{C})x,$$

where $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ maximal monotone, $\mathbb{C}$ single-valued.

For $\alpha > 0$,

$$\begin{aligned} 0 \in (\mathbb{A}+\mathbb{B}+\mathbb{C})x \quad \Leftrightarrow \quad & (1/2)\mathbb{I} + (1/2)\mathbb{T}z = z, \;\; x = \mathbb{J}_{\alpha\mathbb{B}}z, \\ & \mathbb{T} = \mathbb{R}_{\alpha\mathbb{A}}(\mathbb{R}_{\alpha\mathbb{B}} - \alpha\mathbb{C}\mathbb{J}_{\alpha\mathbb{B}}) - \alpha\mathbb{C}\mathbb{J}_{\alpha\mathbb{B}}. \end{aligned}$$

Davis–Yin splitting (DYS)

$$\frac{1}{2}\mathbb{I} + \frac{1}{2}\mathbb{T} = \mathbb{I} - \mathbb{J}_{\alpha\mathbb{B}} + \mathbb{J}_{\alpha\mathbb{A}}(\mathbb{R}_{\alpha\mathbb{B}} - \alpha\mathbb{C}\mathbb{J}_{\alpha\mathbb{B}})$$

Davis–Yin splitting

If $\mathbb{C}$ is β-cocoercive and $\alpha \in (0, 2\beta)$, then $(1/2)\mathbb{I} + (1/2)\mathbb{T}$ is averaged. We prove this in §13.

FPI with DYS

$$\begin{aligned} x^{k+1/2} &= \mathbb{J}_{\alpha\mathbb{B}}(z^k) \\ x^{k+1} &= \mathbb{J}_{\alpha\mathbb{A}}(2x^{k+1/2} - z^k - \alpha\mathbb{C}x^{k+1/2}) \\ z^{k+1} &= z^k + x^{k+1} - x^{k+1/2} \end{aligned}$$

converges for $\alpha \in (0, 2\beta)$ if $\mathrm{Zer}\,(\mathbb{A} + \mathbb{B} + \mathbb{C}) \neq \emptyset$.

DYS reduces to:

- BFS when $\mathbb{A} = 0$
- FBS when $\mathbb{B} = 0$
- DRS when $\mathbb{C} = 0$
- PPM when $\mathbb{A} = \mathbb{C} = 0$

Splitting for convex optimization

In §3, we combine the base splittings (FBS, DRS, DYS) with various techniques to derive many methods.

For now, we directly apply the base splittings to convex optimization.

Proximal gradient method

Consider

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x)+g(x),$$

where f, g CCP and f differentiable.
$[x \in \operatorname{argmin}(f+g)] \Leftrightarrow [x \in \operatorname{Zer}(\nabla f + \partial g)]$

FPI with FBS

$$x^{k+1} = \operatorname{Prox}_{\alpha g}(x^k - \alpha \nabla f(x^k))$$

is the proximal gradient method. If solution exists, f is L-smooth, and $\alpha \in (0, 2/L)$, then $x^k \to x^\star$.

Proximal gradient method

Equivalent to

$$x^{k+1} = \underset{x}{\operatorname{argmin}} \left\{ f(x^k) + \langle \nabla f(x^k), x - x^k \rangle + g(x) + \frac{1}{2\alpha} \|x - x^k\|_2^2 \right\},$$

which uses a first-order approximation of f about x^k.

When $g = \delta_C$

$$x^{k+1} = \Pi_C(x^k - \alpha \nabla f(x^k))$$

is the projected gradient method.

DRS for convex optimization

Primal-dual problem pair

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) + g(x) \qquad\qquad \underset{u \in \mathbb{R}^n}{\text{maximize}} \quad -f^*(-u) - g^*(u) \tag{2}$$

generated by

$$\mathbf{L}(x,u) = f(x) + \langle x, u \rangle - g^*(u),$$

where f, g CCP.

Primal problem equivalent to

$$\underset{x \in \mathbb{R}^n}{\text{find}} \quad 0 \in (\partial f + \partial g)x$$

when total duality holds. (Proof a few slides later.)

DRS for convex optimization

FPI with DRS:

$$
\begin{aligned}
x^{k+1/2} &= \mathrm{Prox}_{\alpha g}(z^k) \\
x^{k+1} &= \mathrm{Prox}_{\alpha f}(2x^{k+1/2} - z^k) \\
z^{k+1} &= z^k + x^{k+1} - x^{k+1/2}
\end{aligned}
$$

If total duality holds and $\alpha > 0$, then $x^k \to x^\star$ and $x^{k+1/2} \to x^\star$.
In §9, we furthermore show $z^k \to z^\star = x^\star + \alpha u^\star$.

DRS requires f and g to be CCP and $\alpha \in (0, \infty)$.
Prox-grad requires f to be L-smooth and $\alpha \in (0, 2/L)$.

DRS useful when evaluating $\mathrm{Prox}_{\alpha f}$ and $\mathrm{Prox}_{\alpha g}$ are easy.
Prox-grad useful when evaluating ∇f and $\mathrm{Prox}_{\alpha g}$ are easy.
PPM useful when evaluating $\mathrm{Prox}_{\alpha(f+g)}$ is easy.

Example: LASSO and ISTA

Consider $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $\lambda > 0$ and the LASSO problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2}\|Ax - b\|^2 + \lambda\|x\|_1.$$

FPI with DRS

$$\begin{aligned} x^{k+1/2} &= (I + \alpha A^\intercal A)^{-1}(z^k + \alpha A^\intercal b) \\ x^{k+1} &= S(2x^{k+1/2} - z^k; \alpha\lambda) \\ z^{k+1} &= z^k + x^{k+1} - x^{k+1/2}, \end{aligned}$$

converges for any $\alpha > 0$. FPI with FBS (prox-grad)

$$x^{k+1} = S(x^k - \alpha A^\intercal(Ax^k - b); \alpha\lambda)$$

is the Iterative Shrinkage-Thresholding Algorithm (ISTA). This converges for $0 < \alpha < 2/\lambda_{\max}(A^\intercal A)$. ($S(x;\kappa) = \mathrm{Prox}_{\kappa\|\cdot\|_1}(x)$c.f. §1.)

DRS uses the matrix inverse $(I + \alpha A^\intercal A)^{-1}$, which can be prohibitively expensive to compute when m and n are large. FBS is the more computationally effective splitting for large-scale LASSO problems.

DYS for convex optimization

Primal-dual problem pair

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x)+g(x)+h(x) \qquad\qquad \underset{u\in\mathbb{R}^n}{\text{maximize}} \quad -(f+h)^*(-u)-g^*(u)$$

generated by the Lagrangian

$$\mathbf{L}(x,u)=f(x)+h(x)+\langle x,u\rangle-g^*(u).$$

FPI with DYS:

$$\begin{aligned} x^{k+1/2} &= \mathrm{Prox}_{\alpha g}(z^k) \\ x^{k+1} &= \mathrm{Prox}_{\alpha f}(2x^{k+1/2}-z^k-\alpha\nabla h(x^{k+1/2})) \\ z^{k+1} &= z^k+x^{k+1}-x^{k+1/2} \end{aligned}$$

If total duality holds, h is L-smooth, and $\alpha\in(0,2/L)$, then $x^k\to x^\star$ and $x^{k+1/2}\to x^\star$. In §9, we furthermore show $z^k\to z^\star=x^\star+\alpha u^\star$.

Necessity and sufficiency of total duality

Role of total duality in splitting methods:

$$\operatorname{argmin}(f+g) = \operatorname{Zer}(\partial f+\partial g) \neq \emptyset \quad \Leftrightarrow \quad \text{total duality holds between (2)}$$

Therefore,

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x)+g(x) \qquad \Leftrightarrow \qquad \underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 \in (\partial f+\partial g)(x)$$

when total duality holds.

Necessity and sufficiency of total duality

Proof. First, assume that total duality holds. Then $x^\star \in \operatorname{argmin}(f+g)$ if and only if $(x^\star, u^\star)$ is a saddle point of

$$\mathbf{L}(x,u) = f(x) + \langle x, u\rangle - g^*(u)$$

for some $u^\star \in \mathbb{R}^n$, and

$$\begin{aligned}
(x^\star, u^\star) \text{ is a saddle point of } \mathbf{L} \quad &\Leftrightarrow \quad 0 \in \partial \mathbf{L}(x^\star, u^\star) \\
&\Leftrightarrow \quad 0 \in \partial_x \mathbf{L}(x^\star, u^\star),\ 0 \in \partial_u(-\mathbf{L})(x^\star, u^\star) \\
&\Leftrightarrow \quad -u^\star \in \partial f(x^\star),\ u^\star \in \partial g(x^\star) \\
&\Leftrightarrow \quad 0 \in (\partial f + \partial g)(x^\star).
\end{aligned}$$

We conclude $\operatorname{argmin}(f+g) = \operatorname{Zer}(\partial f + \partial g) \neq \emptyset$.

Next, assume $\operatorname{argmin}(f+g) = \operatorname{Zer}(\partial f + \partial g) \neq \emptyset$. Then any $x^\star \in \operatorname{argmin}(f+g)$ satisfies $0 \in (\partial f + \partial g)(x^\star)$. By a similar chain of arguments, $(x^\star, u^\star)$ is a saddle point of $\mathbf{L}$ for some $u^\star \in \mathbb{R}^n$, and we conclude total duality holds. $\square$

Discussion: Fixed-point encoding

Fixed-point encoding establishes a correspondence between solutions of a monotone inclusion problem and fixed points of a related operator.

PPM, FBS, BFS, DRS, DYS are fixed-point encodings.

Discussion: Why resolvent?

Splittings use resolvents or direct evaluations of single-valued operators. Why not use other operators such as $(\mathbb{I} - \alpha\mathbb{A})^{-1}$?

- Computational convenience; evaluating something like $(\mathbb{I} - \alpha\partial f)^{-1}$ is often difficult.
- Single-valued operators are algorithmically actionable; we can compute and store a vector but not a set in $\mathbb{R}^n$ on a computer.
  Multi-valued operators are useful mathematically.
  Single-valued operators are useful algorithmically.

Discussion: Role of maximality

$x^{k+1} = \mathbb{T}x^k$ becomes undefined if $x^k \notin \operatorname{dom} \mathbb{T}$. In Theorem 1, we implicitly assumed $\operatorname{dom} \mathbb{T} = \mathbb{R}^n$, satisfied with resolvents of maximal monotone operators. (Theoretical necessity.)

In practice, for non-maximal monotone operators (e.g. subgradient operator of a nonconvex function) we cannot efficiently compute the resolvent. (Practical necessity.)

Discussion: Computational efficiency

Base splitting methods are useful when the subroutines are efficient to compute. The DRS iteration

$$z^{k+1} = \left(\frac{1}{2}\mathbb{I} + \frac{1}{2}\mathbb{R}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}\right) z^k$$

always converges, but it is most useful when $\mathbb{R}_{\alpha\mathbb{A}}$ and $\mathbb{R}_{\alpha\mathbb{B}}$ are efficient.

For a given an optimization problem, there is more than one method.
Trick: find a method using computationally efficient split components.

Example: Consensus technique

Consider

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \sum_{i=1}^m g_i(x)$$

where $g_1, \dots, g_m$ are CCP. Equivalent to

$$\begin{array}{ll} \underset{\mathbf{x}\in\mathbb{R}^{nm}}{\text{minimize}} & \sum_{i=1}^m g_i(x_i) \\ \text{subject to} & \mathbf{x} \in C, \end{array}$$

where $\mathbf{x} = (x_1, \dots, x_m)$ and

$$C = \{(x_1, \dots, x_m) \,|\, x_1 = \cdots = x_m\}$$

is the consensus set. Equivalent to

$$\underset{\mathbf{x}\in\mathbb{R}^{nm}}{\text{find}} \quad 0 \in \begin{bmatrix} \partial g_1(x_1) \\ \vdots \\ \partial g_m(x_m) \end{bmatrix} + \mathbb{N}_C(\mathbf{x}),$$

assuming $\bigcap_{i=1}^m \operatorname{int}\operatorname{dom} g_i \neq \emptyset$.

Example: Consensus technique

Projection onto the consensus set is simple averaging:

$$\Pi_C \mathbf{x} = \overline{\mathbf{x}} = (\overline{x}, \overline{x}, \ldots, \overline{x}), \qquad \overline{x} = \frac{1}{m} \sum_{i=1}^{m} x_i.$$

DRS

$$\begin{aligned} x_i^{k+1} &= \mathrm{Prox}_{\alpha g_i}(2\overline{z}^k - z_i^k) \qquad \text{for } i = 1, \ldots, m, \\ \mathbf{z}^{k+1} &= \mathbf{z}^k + \mathbf{x}^{k+1} - \overline{\mathbf{z}}^k \end{aligned}$$

converges for any $\alpha > 0$, if $\bigcap_{i=1}^{m} \operatorname{int} \operatorname{dom} g_i \neq \emptyset$ and a solution exists. This method is well-suited for parallel distributed computing.

Example: Forward-Douglas–Rachford

Consider

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \sum_{i=1}^m (f_i(x)+g_i(x)),$$

where $g_1,\dots,g_m$ are CCP and $f_1,\dots,f_m$ are L-smooth.
With the consensus technique, equivalent to

$$\begin{array}{ll}\underset{\mathbf{x}\in\mathbb{R}^{nm}}{\text{minimize}} & \sum_{i=1}^m f_i(x_i)+\sum_{i=1}^m g_i(x_i)\\ \text{subject to} & \mathbf{x}\in C.\end{array}$$

DYS

$$\begin{aligned} x_i^{k+1} &= \mathrm{Prox}_{\alpha g_i}(2\overline{z}^k - z_i^k - \alpha\nabla f_i(\overline{z}^k)) \qquad \text{for } i=1,\dots,m,\\ \mathbf{z}^{k+1} &= \mathbf{z}^k + \mathbf{x}^{k+1} - \overline{\mathbf{z}}^k, \end{aligned}$$

is generalized forward-backward or forward-Douglas–Rachford. Converges if total duality holds, $\bigcap_{i=1}^m \mathrm{int\,dom}\, g_i \neq \emptyset$, and $\alpha\in(0,2/L)$.

Outline

Variable metric methods

The Euclidean norm played a special role:

$$\mathrm{Prox}_f(x_0) = \underset{x}{\mathrm{argmin}} \left\{ f(x) + \frac{1}{2}\|x - x_0\|^2 \right\},$$

is defined with $\|\cdot\|$ and Theorem 1 is stated in terms of $\|\cdot\|$.

Variable metric methods generalize with the M-norm, defined as $\|x\|_M^2 = x^\intercal M x$ for $M \succ 0$.

Why? (i) A good choice of M can act as a preconditioner and reduce the number of iterations needed. (ii) Sometimes $\mathbb{A}$ has structure and a well chosen M cancels terms to make $(M + \mathbb{A})^{-1}$ easy to evaluate. (c.f. §3)

Disclaimer: despite the name, the generalization only works with M-norms, which are induced by the inner product $\langle x, y\rangle_M = x^\intercal M y$, but not other metrics, such as the ℓ^1-norm.

Variable metric PPM

If $\mathbb{A}$ maximal monotone and $M \succ 0$, then $M^{-1/2}\mathbb{A}M^{-1/2}$ maximal monotone and the PPM

$$y^{k+1} = (\mathbb{I} + M^{-1/2}\mathbb{A}M^{-1/2})^{-1}y^k$$

converges.

Change of variables $x^k = M^{-1/2}y^k$ give

$$\begin{aligned}(\mathbb{I} + M^{-1/2}\mathbb{A}M^{-1/2})y^{k+1} &\ni y^k \\ (\mathbb{I} + M^{-1}\mathbb{A})x^{k+1} &\ni x^k\end{aligned}$$

and

$$\begin{aligned}x^{k+1} &= \mathbb{J}_{M^{-1}\mathbb{A}}x^k \\ &= (M + \mathbb{A})^{-1}Mx^k,\end{aligned}$$

variable metric PPM. x^k inherit convergence from y^k.

Variable metric FBS

Let $\mathbb{A}$ and $\mathbb{B}$ be maximal monotone and let $\mathbb{A}$ be single-valued.

FBS with $M^{-1/2}\mathbb{A}M^{-1/2}$ and $M^{-1/2}\mathbb{B}M^{-1/2}$, after change of variables,

$$\begin{aligned} x^{k+1} &= (M+\mathbb{B})^{-1}(M-\mathbb{A})x^k \\ &= \mathbb{J}_{M^{-1}\mathbb{B}}(\mathbb{I}-M^{-1}\mathbb{A})x^k. \end{aligned}$$

is variable metric FBS.

Converges if $\mathbb{I}-M^{-1/2}\mathbb{A}M^{-1/2}$ is averaged.

Proximal interpretation

When $\mathbb{A} = \nabla f$ and $\mathbb{B} = \partial g$, then

$$\mathbb{J}_{M^{-1}\partial g}(\mathbb{I} - M^{-1}\nabla f)x = \underset{z \in \mathbb{R}^d}{\operatorname{argmin}} \left\{ g(z) + \langle \nabla f(x), z \rangle + \frac{1}{2}\|z - x\|_M^2 \right\}.$$

Interpretation: Variable metric FBS is prox-grad with the norm $\|\cdot\|_M$.

If $\mathbb{A}$ is β-cocoercive, then $M^{-1/2}\mathbb{A}M^{-1/2}$ is $(\beta/\|M^{-1}\|)$-cocoercive.
So variable metric FBS converges if $\|M^{-1}\| < 2\beta$.

Averagedness with respect to $\|\cdot\|_M$

Assume $M \succ 0$. $\mathbb{T}$ is nonexpansive in $\|\cdot\|_M$ if

$$\|\mathbb{T}x - \mathbb{T}y\|_M \le \|x - y\|_M \qquad \forall\, x, y \in \operatorname{dom} \mathbb{T}.$$

For $\theta \in (0,1)$, $\mathbb{T}$ is θ-averaged in $\|\cdot\|_M$ if $\mathbb{T} = (1-\theta)\mathbb{I} + \theta\mathbb{S}$ for some $\mathbb{S}$ that is nonexpansive in $\|\cdot\|_M$.

$[M^{-1/2}\mathbb{T}M^{-1/2}$ nonexp. (in $\|\cdot\|$)$]$ $\Leftrightarrow$ $[M^{-1}\mathbb{T}$ nonexp. in $\|\cdot\|_M]$
Because

$$\|M^{-1/2}\mathbb{T}M^{-1/2}x - M^{-1/2}\mathbb{T}M^{-1/2}y\|^2 \le \|x - y\|^2$$

is equivalent to

$$\|M^{-1}\mathbb{T}\tilde{x} - M^{-1}\mathbb{T}\tilde{y}\|_M^2 \le \|\tilde{x} - \tilde{y}\|_M^2$$

with the change of variables $M^{-1/2}x = \tilde{x}$ and $M^{-1/2}y = \tilde{y}$.