

# SI251 - Convex Optimization, 2024 Spring

## Homework 1

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# 1 Convex sets

1. Please prove that the following sets are convex:

- 1)  $S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$ , where  $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$ . (5 pts)
- 2) (Ellipsoids)  $\left\{x \mid \sqrt{(x - x_c)^T P (x - x_c)} \leq r\right\}$  ( $x_c \in \mathbb{R}^n, r \in \mathbb{R}, P \succeq 0$ ). (5 pts)
- 3) (Symmetric positive semidefinite matrices)  $S_+^{n \times n} = \{P \in S^{n \times n} \mid P \succeq 0\}$ . (5 pts)
- 4) The set of points closer to a given point than a given set, i.e.,

$$\left\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\right\},$$

where  $S \in \mathbf{R}^n$ . (5 pts)

(1) For a fixed  $t \in [-\frac{\pi}{3}, \frac{\pi}{3}]$ , we could know that  $\cos t, \cos 2t, \dots, \cos mt$  are certain constants, so  $p(t)$  is a linear function of  $\mathbf{x}$ .

Since  $|p(t)| \leq 1 \Leftrightarrow -1 \leq p(t) \leq 1$ .

So let  $S_t = \{\mathbf{x} \mid -1 \leq x_1 \cos t + \cdots + x_n \cos nt \leq 1\}$ .

Since  $p(t)$  is linear function of  $\mathbf{x}$ , so  $S_t$  the intersection of two half spaces, which is a convex set.

And we could know that

$$S = \bigcap_{-\frac{\pi}{3} \leq t \leq \frac{\pi}{3}} S_t$$

From the theorem, we could know that the intersection of convex sets is also a convex set, so  $S$  is a convex set.

So above all, we have proved that  $S$  is a convex set.

(2) Let  $S$  be the Ellipsoids set, and we could know that  $\forall \mathbf{x}_1, \mathbf{x}_2 \in S$ , we have  $(\mathbf{x}_1 - \mathbf{x}_c)^T P (\mathbf{x}_1 - \mathbf{x}_c) \leq r^2$  and  $(\mathbf{x}_2 - \mathbf{x}_c)^T P (\mathbf{x}_2 - \mathbf{x}_c) \leq r^2$ .

And  $\forall \theta \in [0, 1]$ , since  $P \succeq 0$ , so  $P$  is symmetric, so we have

$$\begin{aligned} & [(\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2) - \mathbf{x}_c]^T P [(\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2) - \mathbf{x}_c] \\ &= \theta^2 (\mathbf{x}_1 - \mathbf{x}_c)^T P (\mathbf{x}_1 - \mathbf{x}_c) + 2\theta(1 - \theta) (\mathbf{x}_1 - \mathbf{x}_c)^T P (\mathbf{x}_2 - \mathbf{x}_c) + (1 - \theta)^2 (\mathbf{x}_2 - \mathbf{x}_c)^T P (\mathbf{x}_2 - \mathbf{x}_c) \\ &\leq \theta^2 r^2 + (1 - \theta)^2 r^2 + 2\theta(1 - \theta) (\mathbf{x}_1 - \mathbf{x}_c)^T P (\mathbf{x}_2 - \mathbf{x}_c) \end{aligned}$$

And since  $P \succeq 0$ , so  $P$  could be decomposition as  $P = Q^T \Lambda Q$ , so  $P^{\frac{1}{2}} = Q^T \Lambda^{\frac{1}{2}} Q$ , i.e.  $P^{\frac{1}{2}} \succeq 0$ .

So

$$\begin{aligned} (\mathbf{x}_1 - \mathbf{x}_c)^T P (\mathbf{x}_2 - \mathbf{x}_c) &= (\mathbf{x}_1 - \mathbf{x}_c)^T (P^{\frac{1}{2}})^T (P^{\frac{1}{2}}) (\mathbf{x}_2 - \mathbf{x}_c) \\ &= [P^{\frac{1}{2}} (\mathbf{x}_1 - \mathbf{x}_c)]^T [P^{\frac{1}{2}} (\mathbf{x}_2 - \mathbf{x}_c)] \\ &\leq \| [P^{\frac{1}{2}} (\mathbf{x}_1 - \mathbf{x}_c)] \|_2 \cdot \| P^{\frac{1}{2}} (\mathbf{x}_2 - \mathbf{x}_c) \|_2 \end{aligned}$$

Since  $\mathbf{x}_1 \in S$ , so

$$(\mathbf{x}_1 - \mathbf{x}_c)^T P (\mathbf{x}_1 - \mathbf{x}_c) = (\mathbf{x}_1 - \mathbf{x}_c)^T (P^{\frac{1}{2}})^T [P^{\frac{1}{2}} (\mathbf{x}_1 - \mathbf{x}_c)] = \| P^{\frac{1}{2}} (\mathbf{x}_1 - \mathbf{x}_c) \|_2^2 \leq r^2$$

i.e.

$$\|P^{\frac{1}{2}}(\mathbf{x}_1 - \mathbf{x}_c)\| \leq r$$

Similarly, we have

$$\|P^{\frac{1}{2}}(\mathbf{x}_2 - \mathbf{x}_c)\| \leq r$$

So

$$\begin{aligned} (\mathbf{x}_1 - \mathbf{x}_c)^T P(\mathbf{x}_2 - \mathbf{x}_c) &\leq \| [P^{\frac{1}{2}}(\mathbf{x}_1 - \mathbf{x}_c)] \|_2 \cdot \| P^{\frac{1}{2}}(\mathbf{x}_2 - \mathbf{x}_c) \|_2 \\ &\leq r \cdot r \\ &= r^2 \end{aligned}$$

So

$$\begin{aligned} &[(\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2) - \mathbf{x}_c]^T P[(\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2) - \mathbf{x}_c] \\ &\leq \theta^2 r^2 + (1 - \theta)^2 r^2 + 2\theta(1 - \theta)(\mathbf{x}_1 - \mathbf{x}_c)^T P(\mathbf{x}_2 - \mathbf{x}_c) \\ &\leq \theta^2 r^2 + (1 - \theta)^2 r^2 + 2\theta(1 - \theta)r^2 \\ &= r^2 \end{aligned}$$

So  $\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2 \in S$ .

So above all, we have proved that  $\forall \mathbf{x}_1, \mathbf{x}_2 \in S, \forall \theta \in [0, 1]$ , we have  $\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2 \in S$ . So  $S$  i.e. the Ellipsoids is a convex set.

(3)  $\forall A, B \in S_+^{n \times n}$ , we have  $A^T = A, B^T = B$ , and  $\forall \mathbf{y} \in \mathbb{R}^n, \mathbf{y}^T A \mathbf{y} \geq 0, \mathbf{y}^T B \mathbf{y} \geq 0$ .

So  $\forall \theta \in [0, 1]$ , we have

$$(\theta A + (1 - \theta) B)^T = \theta A^T + (1 - \theta) B^T = \theta A + (1 - \theta) B$$

And

$$\mathbf{y}^T (\theta A + (1 - \theta) B) \mathbf{y} = \theta \mathbf{y}^T A \mathbf{y} + (1 - \theta) \mathbf{y}^T B \mathbf{y} \geq 0$$

So  $\theta A + (1 - \theta) B$  is symmetric and semi-positive defined.

So  $\theta A + (1 - \theta) B \in S_+^{n \times n}$ .

So above all, we have proved that  $S_+^{n \times n}$  is a convex set.

(4) Let  $\mathcal{C} = \left\{ \mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2 \text{ for all } \mathbf{y} \in S \right\}$ .

$\forall \mathbf{x} \in \mathcal{C}$ , and for a fixed  $\mathbf{y}$ , we have

$$\begin{aligned} \|\mathbf{x} - \mathbf{x}_0\|_2 &\leq \|\mathbf{x} - \mathbf{y}\|_2 \\ \|\mathbf{x} - \mathbf{x}_0\|_2^2 &\leq \|\mathbf{x} - \mathbf{y}\|_2^2 \\ (\mathbf{x} - \mathbf{x}_0)^T (\mathbf{x} - \mathbf{x}_0) &\leq (\mathbf{x} - \mathbf{y})^T (\mathbf{x} - \mathbf{y}) \\ \mathbf{x}^T (\mathbf{x}_0 - \mathbf{y}) &\geq \frac{1}{2} (\mathbf{x}_0^T \mathbf{x}_0 - \mathbf{y}^T \mathbf{y}) \end{aligned}$$

From the definition, we know that for a fixed  $\mathbf{y}$ ,  $\mathbf{x}^T (\mathbf{x}_0 - \mathbf{y}) \geq \frac{1}{2} (\mathbf{x}_0^T \mathbf{x}_0 - \mathbf{y}^T \mathbf{y})$  is a half-space  $S_{\mathbf{y}}$ .

So  $\forall \mathbf{y} \in S$ , we could see that  $\mathcal{C} = \bigcap_{\mathbf{y} \in S} S_{\mathbf{y}}$ .

And since each  $S_{\mathbf{y}}$  is a half-space, which is a convex set. And from the theorem we have known, that the intersection of convex sets is also a convex set, so  $\mathcal{C}$  is a convex set.

2. (15 pts) For a given norm  $\|\cdot\|$  on  $\mathbf{R}^n$ , the dual norm, denoted  $\|\cdot\|_*$ , is defined as

$$\|y\|_* = \sup_{x \in \mathbf{R}^n} \{y^T x \mid \|x\| \leq 1\}.$$

Show that the dual of Euclidean norm is the Euclidean norm, i.e.,  $\sup_{x \in \mathbf{R}^n} \{z^T x \mid \|x\|_2 \leq 1\} = \|z\|_2$ .

$$\begin{aligned} & z^T x \\ & \leq \|z\|_2 \|x\|_2 \quad (\text{Cauchy-Schwarz inequality}) \\ & \leq \|z\|_2 \quad (\|x\|_2 \leq 1) \end{aligned}$$

If and only if  $\|x\|_2 = 1$  and  $\cos \langle z, x \rangle = 1$  will take the equation condition.

So above all, we have proved that  $\sup_{x \in \mathbf{R}^n} \{z^T x \mid \|x\|_2 \leq 1\} = \|z\|_2$ .

3. (15 pts) Define a norm cone as

$$\mathcal{C} \equiv \{(x, t) : x \in \mathbb{R}^d, t \geq 0, \|x\| \leq t\} \subseteq \mathbb{R}^{d+1}$$

Show that the norm cone is convex by using the definition of convex sets.

$\forall (\mathbf{x}_1, t_1), (\mathbf{x}_2, t_2) \in \mathcal{C}$ , we have  $\|\mathbf{x}_1\| \leq t_1, \|\mathbf{x}_2\| \leq t_2, t_1 \geq 0, t_2 \geq 0$ .

And  $\forall \theta \in [0, 1]$ , we have

$$\begin{aligned} & \|\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2\| \\ & \leq \|\theta \mathbf{x}_1\| + \|(1 - \theta) \mathbf{x}_2\| \\ & = \theta \|\mathbf{x}_1\| + (1 - \theta) \|\mathbf{x}_2\| \\ & = \theta t_1 + (1 - \theta) t_2 \end{aligned}$$

Also, since  $t_1, t_2 \geq 0$ , so  $\theta t_1 + (1 - \theta) t_2 \geq 0$ .

So  $\theta(\mathbf{x}_1, t_1) + (1 - \theta)(\mathbf{x}_2, t_2) = (\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2, \theta t_1 + (1 - \theta) t_2) \in \mathcal{C}$ .

So above all, we have proved that  $\mathcal{C}$  is a convex set.

## 2 Convex functions

4. (18 pts) Let  $C \subset \mathbb{R}^n$  be convex and  $f : C \rightarrow \mathbb{R}^*$ . Show that the following statements are equivalent:

(a)  $\text{epi}(f)$  is convex.

(b) For all points  $x_i \in C$  and  $\{\lambda_i | \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1, i = 1, 2, \dots, n\}$ , we have

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

(c) For  $\forall x, y \in C$  and  $\lambda \in [0, 1]$ ,

$$f\left((1-\lambda)x + \lambda y\right) \leq (1-\lambda)f(x) + \lambda f(y).$$

• (a)  $\Rightarrow$  (c)

$\forall \mathbf{x}, \mathbf{y} \in C$ , we have  $(\mathbf{x}, f(\mathbf{x})), (\mathbf{y}, f(\mathbf{y})) \in \text{epi}(f)$ .

From (a), we have known that  $\text{epi}(f)$  is convex, so  $\forall \lambda \in [0, 1]$ , we have

$$((1-\lambda)\mathbf{x} + \lambda\mathbf{y}, (1-\lambda)f(\mathbf{x}) + \lambda f(\mathbf{y})) \in \text{epi}(f)$$

which means that

$$f((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) \leq (1-\lambda)f(\mathbf{x}) + \lambda f(\mathbf{y})$$

So (a)  $\Rightarrow$  (c) has been proved.

• (c)  $\Rightarrow$  (a)

$\forall \mathbf{x}, \mathbf{y} \in C$ , and  $\forall (\mathbf{x}, t_1), (\mathbf{y}, t_2) \in \text{epi}(f)$ , we have  $t_1 \geq f(\mathbf{x}), t_2 \geq f(\mathbf{y})$ .

And  $\forall \lambda \in [0, 1]$ , we have

$$\begin{aligned} & f((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) \\ & \leq (1-\lambda)f(\mathbf{x}) + \lambda f(\mathbf{y}) \\ & \leq (1-\lambda)t_1 + \lambda t_2 \end{aligned}$$

So  $((1-\lambda)\mathbf{x} + \lambda\mathbf{y}, (1-\lambda)t_1 + \lambda t_2) \in \text{epi}(f)$ , so  $\text{epi}(f)$  is convex.

So (c)  $\Rightarrow$  (a) has been proved.

• (b)  $\Rightarrow$  (c)

Let  $n = 2, \lambda_1 = 1 - \lambda, \lambda_2 = \lambda$ , then we have

$$f((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) \leq (1-\lambda)f(\mathbf{x}) + \lambda f(\mathbf{y})$$

So (b)  $\Rightarrow$  (c) has been proved.

• (c)  $\Rightarrow$  (b)

We can use induction to prove this.

When  $n = 2$ , let  $\lambda_1 = 1 - \lambda, \lambda_2 = \lambda$ , then we have

$$f\left(\sum_{i=1}^n \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^n \lambda_i f(\mathbf{x}_i)$$

And since  $f$  is convex, so we have  $\sum_{i=1}^n \lambda_i \mathbf{x}_i \in C$ .

Suppose when  $n = k$ , (b) holds.

i.e.  $\forall \mathbf{y}_i \in C$  and  $\{\nu_i | \nu_i \geq 0, \sum_{i=1}^k \nu_i = 1, i = 1, 2, \dots, k\}$ , we have

$$f\left(\sum_{i=1}^k \nu_i \mathbf{y}_i\right) \leq \sum_{i=1}^k \nu_i f(\mathbf{y}_i)$$

And also suppose that  $\mathbf{z} = \sum_{i=1}^k \nu_i f(\mathbf{y}_i) \in C$ .

Then for  $n = k + 1$ ,  $\forall \mathbf{x} \in C$ , we have:

$\forall \lambda \in [0, 1]$ , since  $\mathbf{x}, \mathbf{z} \in C$ , so we can get that

$$\begin{aligned} f((1 - \lambda)\mathbf{x} + \lambda\mathbf{z}) &\leq (1 - \lambda)f(\mathbf{x}) + \lambda f(\mathbf{z}) && \text{(from (c))} \\ &= (1 - \lambda)f(\mathbf{x}) + \lambda f\left(\sum_{i=1}^k \nu_i f(\mathbf{y}_i)\right) \\ &\leq (1 - \lambda)f(\mathbf{x}) + \lambda \left(\sum_{i=1}^k \nu_i f(\mathbf{y}_i)\right) && \text{(the assumption when } n = k) \\ &= (1 - \lambda)f(\mathbf{x}) + \sum_{i=1}^k (\lambda \nu_i) f(\mathbf{y}_i) \end{aligned}$$

Let  $\lambda_{k+1} = 1 - \lambda$ ,  $\mathbf{x}_{k+1} = \mathbf{x}$ ,  $\lambda_i = \lambda \nu_i$ ,  $\mathbf{x}_i = \mathbf{y}_i$ , where  $i = 1, 2, \dots, k$ .

Since we have  $\sum_{i=1}^k \nu_i = 1$ ,

so  $\sum_{i=1}^n \lambda_i = (1 - \lambda) + \lambda \left(\sum_{i=1}^k \nu_i\right) = 1$ ,  $\lambda_i \geq 0$ .

So we have proved that when  $n = k + 1$ , (b) holds.

Also, since  $\mathbf{x}, \mathbf{z} \in C$ , and  $C$  is a convex set, so  $\forall \lambda \in [0, 1]$ ,  $(1 - \lambda)\mathbf{x} + \lambda\mathbf{z} \in C$ , i.e.  $\sum_{i=1}^{k+1} \lambda_i \mathbf{x}_i \in C$

So  $\forall n \geq 2$ , we have proved that (c)  $\Rightarrow$  (b).

Since we have proved that (a)  $\Leftrightarrow$  (c) and (b)  $\Leftrightarrow$  (c), so we could say that (a),(b),(c) three statements are equivalent.

5. (14 pts) Monotone Mappings. A function  $\psi : \mathbf{R}^n \rightarrow \mathbf{R}^n$  is called monotone if for all  $x, y \in \text{dom}\psi$ ,

$$(\psi(x) - \psi(y))^T(x - y) \geq 0$$

Suppose  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is a differentiable convex function. Show that its gradient  $\nabla f$  is monotone. Is the converse true, i.e., is every monotone mapping the gradient of a convex function?

1. If  $f$  is a differentiable convex function, we could know that  $\forall \mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ ,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x})$$

$$f(\mathbf{x}) \geq f(\mathbf{y}) + \nabla f(\mathbf{y})^T(\mathbf{x} - \mathbf{y})$$

Add these two inequality, we will get that

$$\nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) + \nabla f(\mathbf{y})^T(\mathbf{x} - \mathbf{y}) \leq 0$$

$$\nabla f(\mathbf{x})^T(\mathbf{x} - \mathbf{y}) - \nabla f(\mathbf{y})^T(\mathbf{x} - \mathbf{y}) \geq 0$$

$$(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y}))^T(\mathbf{x} - \mathbf{y}) \geq 0$$

So we have proved that a differentiable convex function's gradient  $\nabla f$  is monotone.

2. Suppose that  $\psi(x_1, x_2) = (-x_2, x_1)$ .

Let  $\mathbf{x} = (x_1, x_2)$  and  $\mathbf{y} = (y_1, y_2)$ , then

$$\begin{aligned} (\psi(\mathbf{x}) - \psi(\mathbf{y}))^T(\mathbf{x} - \mathbf{y}) &= [(-x_2, x_1) - (-y_2, y_1)]^T[(x_1, x_2) - (y_1, y_2)] \\ &= (-x_2 + y_2, x_1 - y_1)^T[(x_1 - y_1, x_2 - y_2)] \\ &= (x_1 - y_1) \cdot (x_2 - y_2) + (x_1 - y_1) \cdot [-(x_2 - y_2)] \\ &= 0 \\ &\geq 0 \end{aligned}$$

So our constructed  $\psi(\mathbf{x})$  is monotone.

Let  $f$  be the primitive function of  $\psi$ , then  $\frac{\partial^2 f}{\partial x_1 \partial x_2} = -1, \frac{\partial^2 f}{\partial x_2 \partial x_1} = 1$ .

But for a differentiable convex function  $f$ , it must have  $\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$ .

So for a monotone function, it may not a gradient of a differentiable convex function.

So above all, differentiable convex function's gradient  $\nabla f$  is monotone, but its converse is not true.



6. (18 pts) Please determine whether the following functions are convex, concave or none of those, and give a detailed explanation for your choice.

1)

$$f_1(x_1, x_2, \dots, x_n) = \begin{cases} -(x_1 x_2 \cdots x_n)^{\frac{1}{n}}, & \text{if } x_1, \dots, x_n > 0 \\ \infty & \text{otherwise;} \end{cases}$$

2)  $f_2(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ , where  $0 \leq \alpha \leq 1$ , on  $\mathbb{R}_{++}^2$ ;

3)  $f_3(x, u, v) = -\log(uv - x^T x)$  on  $\text{dom } f = \{(x, u, v) | uv > x^T x, u, v > 0\}$ .

(1) We could see that  $f_1$  is twice continuously differentiable over  $\text{dom } f_1$ , and  $\forall \mathbf{x} \in \text{dom } f_1$ , its Hessian matrix is that:

$$\nabla^2 f_1(\mathbf{x}) = \frac{-(x_1 x_2 \cdots x_n)^{\frac{1}{n}}}{n^2} \begin{bmatrix} \frac{1-n}{x_1^2} & \frac{1}{x_1 x_2} & \cdots & \frac{1}{x_1 x_n} \\ \frac{1}{x_2 x_1} & \frac{1-n}{x_2^2} & \cdots & \frac{1}{x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n x_1} & \frac{1}{x_1 x_2} & \cdots & \frac{1-n}{x_n^2} \end{bmatrix}$$

And  $\forall \mathbf{x} \in \text{dom } f_1, \mathbf{y} \in \mathbb{R}^n$ , we have

$$\begin{aligned} \mathbf{y}^T \nabla^2 f_1(\mathbf{x}) \mathbf{y} &= \frac{-(x_1 x_2 \cdots x_n)^{\frac{1}{n}}}{n^2} \cdot \left[ \sum_{i=1}^n \frac{y_i^2 (1-n)}{x_i^2} + \sum_{i \neq j} \frac{y_i y_j}{x_i x_j} \right] \\ &= \frac{-(x_1 x_2 \cdots x_n)^{\frac{1}{n}}}{n^2} \cdot \left[ \left( \sum_{i=1}^n \frac{y_i}{x_i} \right)^2 - n \sum_{i=1}^n \left( \frac{y_i}{x_i} \right)^2 \right] \end{aligned}$$

From the multivariate mean inequality, we could know that

$$\left( \frac{\sum_{i=1}^n a_i}{n} \right)^2 \leq \frac{1}{n} \sum_{i=1}^n a_i^2 \Rightarrow \left( \sum_{i=1}^n a_i \right)^2 \leq n \sum_{i=1}^n a_i^2$$

So we could know that

$$\left[ \left( \sum_{i=1}^n \frac{y_i}{x_i} \right)^2 - n \sum_{i=1}^n \left( \frac{y_i}{x_i} \right)^2 \right] \leq 0$$

And since  $\frac{-(x_1 x_2 \cdots x_n)^{\frac{1}{n}}}{n^2} < 0$ , so we could know that

$$\forall \mathbf{x} \in \text{dom } f_1, \mathbf{y} \in \mathbb{R}^n, \mathbf{y}^T \nabla^2 f_1(\mathbf{x}) \mathbf{y} \geq 0$$

So above all,  $f_1(x_1, \dots, x_n)$  is convex.

(2) The Hessian of  $f_2$  is that:

$$\begin{aligned} \nabla^2 f_2(\mathbf{x}) &= \begin{bmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} & \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} \\ \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} & (1-\alpha)(-\alpha)x_1^\alpha x_2^{-\alpha-1} \end{bmatrix} \\ &= \alpha(\alpha-1)x_1^\alpha x_2^{1-\alpha} \begin{bmatrix} \frac{1}{x_1^2} & -\frac{1}{x_1 x_2} \\ -\frac{1}{x_1 x_2} & \frac{1}{x_2^2} \end{bmatrix} \end{aligned}$$

So  $\forall \mathbf{y} = (y_1, y_2)$ , we have

$$\mathbf{y}^T \nabla^2 f_2(\mathbf{x}) \mathbf{y} = \alpha(\alpha - 1)x_1^\alpha x_2^{1-\alpha} \left( \frac{y_1}{x_1} - \frac{y_2}{x_2} \right)^2 \leq 0$$

So above all,  $f_2(x_1, x_2)$  is concave.

$$(3) f_3(\mathbf{x}, u, v) = -\log(uv - \mathbf{x}^T \mathbf{x}) = -\log \left( u \left( v - \frac{\mathbf{x}^T \mathbf{x}}{u} \right) \right) = -\log u - \log \left( v - \frac{\mathbf{x}^T \mathbf{x}}{u} \right).$$

From we have known about the perspective: if  $f(\mathbf{x})$  is convex, then its perspective  $g(\mathbf{x}, t) = tf\left(\frac{\mathbf{x}}{t}\right)$  is convex.

Since  $f(\mathbf{x}) = \mathbf{x}^T \mathbf{x}$  is convex, so  $g(\mathbf{x}, t) = \frac{\mathbf{x}^T \mathbf{x}}{t}$  is convex.

And since  $v$  is affine,  $-\frac{\mathbf{x}^T \mathbf{x}}{t}$  is concave, so  $\left( v - \frac{\mathbf{x}^T \mathbf{x}}{u} \right)$  is concave.

Since  $h(x) = -\log x$  is convex and non-increasing,  $\left( v - \frac{\mathbf{x}^T \mathbf{x}}{u} \right)$  is concave, so from the composition with scalar functions, we could know that  $-\log \left( v - \frac{\mathbf{x}^T \mathbf{x}}{u} \right)$  is convex.

Also, since  $-\log u$  is convex, so  $f_3(\mathbf{x}, u, v) = -\log u - \log \left( v - \frac{\mathbf{x}^T \mathbf{x}}{u} \right)$  is convex.

So above all,  $f_3(\mathbf{x}, u, v)$  is convex.