SI251 - Convex Optimization, 2024 Spring
Homework 3

Name: **Zhou Shouchen**
Student ID: 2021533042

Due 23:59 (CST), Apr. 24, 2024

1. (50 pts) **L-smooth functions**. Suppose the function $f : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable. Please prove that the following relations holds for all $x, y \in \mathbb{R}$ if f with an L-Lipschitz continuous conditions,

$$[1] \Rightarrow [2] \Rightarrow [3]$$

$$\begin{aligned}
&[1]\ \langle \nabla f(x) - \nabla f(y), x - y \rangle \le L\|x - y\|^2, \\
&[2]\ f(y) \le f(x) + \nabla f(x)^T (y - x) + \frac{L}{2}\|y - x\|^2, \\
&[3]\ f(y) \ge f(x) + \nabla f(x)^T (y - x) + \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|^2, \forall x, y.
\end{aligned}$$

Solution:

$[1] \Rightarrow [2]$:

So we have proved that $[1] \Rightarrow [2]$.

$[2] \Rightarrow [3]$:

So we have proved that $[2] \Rightarrow [3]$.

So above all, we have proved that $[1] \Rightarrow [2] \Rightarrow [3]$.

2. (50 pts) **Backtracking line search**. Please show the convergence of backtracking line search on a m-strongly convex and M-smooth objective function f as

$$f\left(x^{(k)}\right) - p^{\star} \leq c^{k}\left(f\left(x^{(0)}\right) - p^{\star}\right)$$

where $c = 1 - \min\{2m\alpha, 2\beta\alpha m/M\} < 1$.

Algorithm 9.2 *Backtracking line search.*

given a descent direction Δx for f at $x \in \mathbf{dom}\, f$, $\alpha \in (0, 0.5)$, $\beta \in (0, 1)$.
$t := 1$.
while $f(x + t\Delta x) > f(x) + \alpha t \nabla f(x)^{T} \Delta x, \quad t := \beta t$.

Solution: