# SI251 - Convex Optimization, 2024 Spring

# Homework 1

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## 1 Convex sets

1. Please prove that the following sets are convex:

   1) $S = \{x \in \boldsymbol{R}^m \mid \mid p(t) \mid \leq 1 \text{ for } |t| \leq \pi/3\}$, where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$. (5 pts)
   2) (Ellipsoids) $\left\{x | \sqrt{(x-x_c)^T P(x-x_c)} \leq r\right\} \quad (x_c \in \mathbb{R}^n, r \in \mathbb{R}, P \succeq 0)$. (5 pts)
   3) (Symmetric positive semidefinite matrices) $S_+^{n\times n} = \left\{P \in S^{n\times n} | P \succeq 0\right\}$. (5 pts)
   4) The set of points closer to a given point than a given set, i.e.,

   $$\left\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\right\},$$

   where $S \in R^n$. (5 pts)

(1)

(2)

(3)

(4)

2. (15 pts) For a given norm $\|\cdot\|$ on $\boldsymbol{R}^n$, the dual norm, denoted $\|\cdot\|_*$, is defined as

$$\|y\|_* = \sup_{x \in \boldsymbol{R}^n} \{y^T x \mid \|x\| \leq 1\}.$$

Show that the dual of Euclidean norm is the Euclidean mom, i.e., $\sup_{x \in \boldsymbol{R}^n} \{z^T x \mid \|x\|_2 \leq 1\} = ||z||_2$.

3. (15 pts) Define a norm cone as

$$\mathcal{C} \equiv \left\{(x, t) : x \in \mathbb{R}^d, t \geq 0, \|x\| \leq t\right\} \subseteq \mathbb{R}^{d+1}$$

Show that the norm cone is convex by using the definition of convex sets.

## 2 Convex functions

4. (18 pts) Let $C \subset \mathbb{R}^n$ be convex and $f : C \to R^{\star}$. Show that the following statements are equivalent:

   (a) epi$(f)$ is convex.

   (b) For all points $x_i \in C$ and $\{\lambda_i | \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1, i = 1, 2, \cdots, n\}$, we have

   $$f\Big(\sum_{i=1}^{n} \lambda_i x_i\Big) \leq \sum_{i=1}^{n} \lambda_i f(x_i).$$

   (c) For $\forall x, y \in C$ and $\lambda \in [0, 1]$,

   $$f\Big((1-\lambda)x + \lambda y\Big) \leq (1-\lambda) f(x) + \lambda f(y).$$

(a)

(b)

(c)

5. (14 pts) Monotone Mappings. A function $\psi : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is called monotone if for all $x, y \in \boldsymbol{dom}\psi$,

$$(\psi(x) - \psi(y))^T (x - y) >= 0$$

Suppose $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a differentiable convex function. Show that its gradient $\nabla f$ is monotone. Is the convex true, i.e., is every monotone mapping the gradient of a convex function?

6. (18 pts) Please determine whether the following functions are convex, concave or none of those, and give a detailed explanation for your choice.

1)

$$f_1(x_1, x_2, \cdots, x_n) = \begin{cases} -(x_1 x_2 \cdots x_n)^{\frac{1}{n}}, & \text{if } x_1, \cdots, x_n > 0 \\ \infty & \text{otherwise;} \end{cases}$$

2) $f_2(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbb{R}^2_{++}$;

3) $f_3(x, u, v) = -\log(uv - x^T x)$ on $dom f = \{(x, u, v) | uv > x^T x, \quad u, v > 0\}$.

(1)

(2)

(3)