Convex Sets

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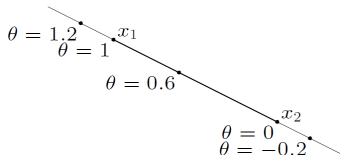
Outline

Definition of Affine Set

❧ **Line:** through x_1, x_2: all points

$$x = \theta x_1 + (1 - \theta) x_2 \quad (\theta \in \mathbb{R})$$

❧ **Affine set:** contains the line through any two distinct points in the set

❧ **Example:** solution set of linear equations $\{x | Ax = b\}$
(conversely, every affine set can be expressed as solution set of system of linear equations)

Definition of Convex Set

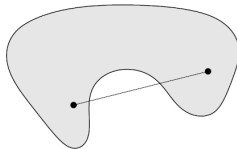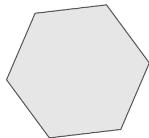❧ **Line segment:** between x_1 and x_2: all points

$$x = \theta x_1 + (1 - \theta) x_2$$

with $0 \le \theta \le 1$

❧ **Convex set:** contains line segment between any two points in the set C

$$x_1, x_2 \in C, 0 \le \theta \le 1 \implies \theta x_1 + (1 - \theta) x_2 \in C$$

❧ **Examples** (one convex, two nonconvex sets)

Outline

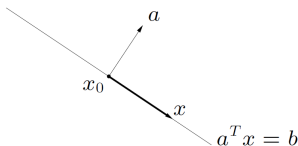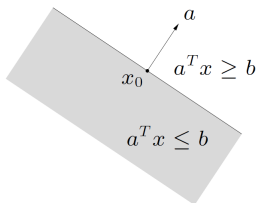# Examples: Hyperplanes and Halfspaces

❧ **Hyperplane:** set of the form $\{x | a^T x = b\}(a \neq 0)$

❧ **Halfspace:** set of the form $\{x | a^T x \leq b\}(a \neq 0)$
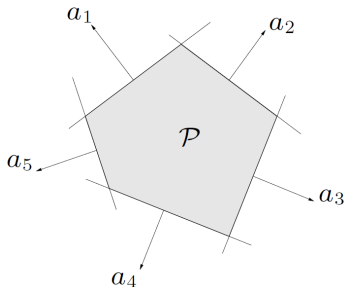

  ❧ a is the normal vector
  ❧ hyperplanes are affine and convex; halfspaces are convex

Example: Polyhedra

Solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \qquad Cx = d$$

($A \in \mathbb{R}^{m \times n}, C \in \mathbb{R}^{p \times n}, \preceq$ is componentwise inequality)

polyhedron is intersection of finite number of halfspaces and hyperplanes
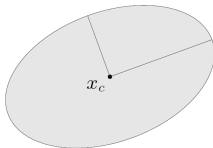
Examples: Euclidean Balls and Ellipsoids

❧ **(Euclidean) Ball** with center $\boldsymbol{x}_c$ and radius r:

$$B(\boldsymbol{x}_c, r) = \{\boldsymbol{x} | \|\boldsymbol{x} - \boldsymbol{x}_c\|_2 \le r\} = \{\boldsymbol{x}_c + r\boldsymbol{u} | \|\boldsymbol{u}\|_2 \le 1\}$$

❧ **Ellipsoid:** set of the form

$$
\begin{aligned}
E(\boldsymbol{x}_c, \boldsymbol{P}) &= \{\boldsymbol{x} | (\boldsymbol{x} - \boldsymbol{x}_c)^T \boldsymbol{P}^{-1}(\boldsymbol{x} - \boldsymbol{x}_c) \le 1\} \\
&= \{\boldsymbol{x}_c + \boldsymbol{A}\boldsymbol{u} | \|\boldsymbol{u}\|_2 \le 1\}
\end{aligned}
$$

with $\boldsymbol{P} \in \mathbb{S}_{++}^n$ (i.e., $\boldsymbol{P}$ symmetric positive definite), $\boldsymbol{A}$ square and nonsingular
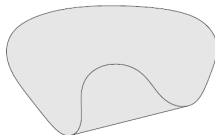
Convex Combination and Convex Hull

❧ **Convex combination** of $x_1, \cdots, x_k$: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1, \theta_i \geq 0$

❧ **Convex hull** conv S: set of all convex combinations of points in S
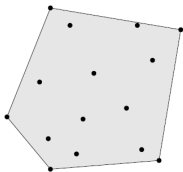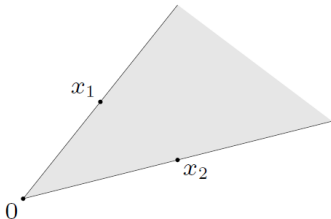
Conic Combination and Convex Cone

❧ **Conic (nonnegative) combination** of x_1 and x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0, \theta_2 \geq 0$

❧ **Convex cone:** set that contains all conic combinations of points in the set

Convex Cones: Norm Balls and Norm Cones

❧ **Norm:** a function $\| \cdot \|$ that satisfies

    ❧ $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
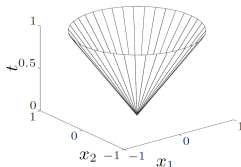   ❧ $\|tx\| = |t|\|x\|$ for $t \in \mathbb{R}$
    ❧ $\|x + y\| \leq \|x\| + \|y\|$

    notation: $\| \cdot \|$ general (unspecified) norm; $\| \cdot \|_{\text{symb}}$ a particular norm

❧ **Norm ball** with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$

❧ **Norm cone:** $\{(x, t) \in \mathbb{R}^{n+1} \mid \|x\| \leq t\}$

Euclidean norm cone or second-order cone (aka ice-cream cone)

Positive Semidefinite Cone

❧ **Notation**

  ❧ $\mathbb{S}^n$ is set of symmetric $n \times n$ matrices
  ❧ $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n | X \succeq 0\}$: positive semidefinite $n \times n$ matrices

  $$X \in \mathbb{S}^n_+ \quad \Longleftrightarrow \quad z^\top X z \geq 0 \text{ for all } z$$

  $\mathbb{S}^n_+$ is a convex cone
  ❧ $\mathbb{S}^n_{++} = \{X \in \mathbb{S}^n | X \succ 0\}$: positive definite $n \times n$ matrices

❧ **Example:** $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbb{S}^2_+$
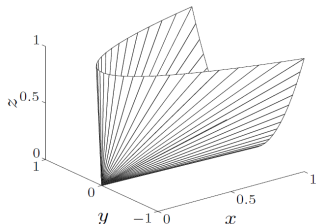
Outline

Operations that Preserve Convexity

How to establish the convexity of a given set C

   ✎ Apply the definition (can be cumbersome)

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in C, 0 \leq \theta \leq 1 \quad \implies \quad \theta \boldsymbol{x}_1 + (1 - \theta)\boldsymbol{x}_2 \in C$$

   ✎ Show that C is obtained from simple convex sets(hyperplanes, halfspaces, norm balls,$\cdots$) by operations that preserve convexity

      ✎ intersection
      ✎ affine functions
      ✎ perspective function
      ✎ linear-fractional functions

Intersection

❧ **Intersection:** if $S_1, S_2, \ldots, S_k$ are convex, then $S_1 \cap S_2 \cap \cdots \cap S_k$ is convex (k can be any positive integer)

❧ Example 1: a polyhedron is the intersection of halfspaces and hyper-planes

❧ Example 2:
$$S = \{\boldsymbol{x} \in \mathbb{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$
where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$
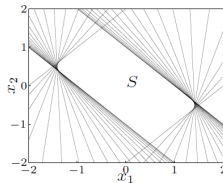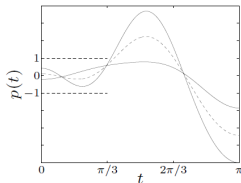
for $m = 2$

Affine Function

suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is affine ($f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ with $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$)

- the image of a convex set under f is convex
$$S \subseteq \mathbb{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(\boldsymbol{x}) | \boldsymbol{x} \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ a convex set under f is convex
$$C \subseteq \mathbb{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{\boldsymbol{x} \in \mathbb{R}^n | f(\boldsymbol{x}) \in C\} \text{ convex}$$

Examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{\boldsymbol{x} | x_1 \boldsymbol{A}_1 + \cdots + x_m \boldsymbol{A}_m \preceq \boldsymbol{B}\}$ (with $\boldsymbol{A}_i, \boldsymbol{B} \in \mathbb{S}^p$)
- $\{(\boldsymbol{x}, t) \in \mathbb{R}^{n+1} | \|\boldsymbol{x}\| \leq t\}$ is convex, so is

$$\{\boldsymbol{x} \in \mathbb{R}^n | \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\| \leq \boldsymbol{c}^T \boldsymbol{x} + d\}$$

Perspective and Linear-fractional Function I

❦ **Perspective function** $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$

$$P(\boldsymbol{x}, t) = \boldsymbol{x}/t, \quad \mathrm{dom}P = \{(\boldsymbol{x}, t) | t > 0\}$$

images and inverse images of convex sets under perspective are convex

❦ **Linear-fractional function** $f : \mathbb{R}^n \to \mathbb{R}^m$

$$f(\boldsymbol{x}) = \frac{\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}}{\boldsymbol{c}^T\boldsymbol{x} + d}, \quad \mathrm{dom}f = \{\boldsymbol{x} | \boldsymbol{c}^T\boldsymbol{x} + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

Perspective and Linear-fractional Function II

❧ **Examples** of a linear-fractional function

$$f(\boldsymbol{x}) = \frac{1}{x_1 + x_2 + 1} \boldsymbol{x}$$
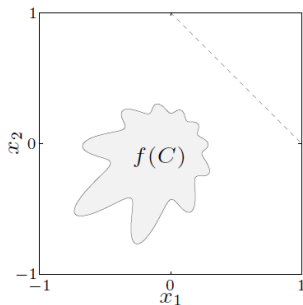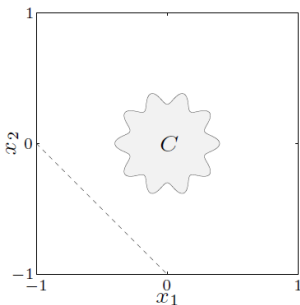
Outline

Generalized Inequalities I

- A convex cone $K \subseteq \mathbb{R}^n$ is a **proper cone** if

    - K is closed (contains its boundary)
    - K is solid (has nonempty interior)
    - K is pointed (contains no line)

- **Examples**

    - nonnegative orthant

    $$K = \mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n | x_i \geq 0, i = 1, \cdots, n\}$$

    - positive semidefinite cone

    $$K = \mathbb{S}^n_+ = \{\boldsymbol{X} \in \mathbb{R}^{n \times n} | \boldsymbol{X} = \boldsymbol{X}^T \succeq \boldsymbol{0}\}$$

    - nonnegative polynomials on $[0, 1]$:

    $$K = \{\boldsymbol{x} \in \mathbb{R}^n | x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

Generalized Inequalities II

Generalized inequality defined by a proper cone K:

$$y \succeq_K x \quad \Longleftrightarrow \quad y - x \succeq_K 0 \text{ or } y - x \in K$$

Examples

- Componentwise inequality ($K = \mathbb{R}^n_+$)

$$y \succeq_{\mathbb{R}^n_+} x \quad \Longleftrightarrow \quad y_i \geq x_i, \quad i = 1, \cdots, n$$

- Matrix inequality ($K = \mathbb{S}^n_+$)

$$Y \succeq_{\mathbb{S}^n_+} X \quad \Longleftrightarrow \quad Y - X \text{ positive semidefinite}$$

Outline

Separating Hyperplane Theorem

If C and D are nonempty disjoint convex sets, there exist $\boldsymbol{a} \neq \boldsymbol{0}$ and b, such that

$$\boldsymbol{a}^T \boldsymbol{x} \leq b \text{ for } \boldsymbol{x} \in C, \quad \boldsymbol{a}^T \boldsymbol{x} \geq b \text{ for } \boldsymbol{x} \in D$$
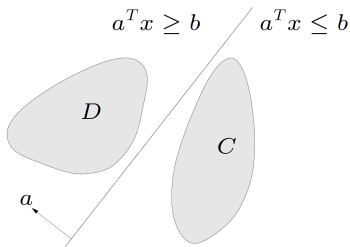
the hyperplane $\{\boldsymbol{x} | \boldsymbol{a}^T \boldsymbol{x} = b\}$ separates C and D

Supporting Hyperplane Theorem

Supporting hyperplane to set C at boundary point x_0:

$$\{x | a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$
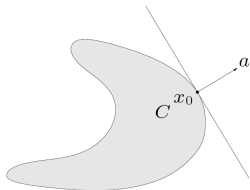
Supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

Dual Cones and Generalized Inequalities

❧ **Dual cone** of a cone K:

$$K^* = \{\boldsymbol{y} | \boldsymbol{y}^T \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \in K\}$$

❧ Examples

  ❧ $K = \mathbb{R}_+^n$: $K^* = \mathbb{R}_+^n$
  ❧ $K = \mathbb{S}_+^n$: $K^* = \mathbb{S}_+^n$
  ❧ $K = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_2 \leq t\}$: $K^* = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_2 \leq t\}$
  ❧ $K = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_1 \leq t\}$: $K^* = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_\infty \leq t\}$

  First three examples are **self-dual** cones

❧ Dual cones of proper cones are proper, hence define generalized inequalities:

$$\boldsymbol{y} \succeq_{K^*} \boldsymbol{0} \quad \Longleftrightarrow \quad \boldsymbol{y}^T \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \succeq_K \boldsymbol{0}$$

Reference

Chapter 2 of:

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.