SI251 - Convex Optimization Homework 4

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1. You can use Word, Latex or handwriting to complete this assignment. If you want to submit a handwritten version, scan it clearly.
2. The **report** has to be submitted as a PDF file to Gradescope, other formats are not accepted.
3. The submitted file name is **student_id+your_student_name.pdf**.
4. Late policy: You have 4 free late days for the quarter and may use up to 2 late days per assignment with no penalty. Once you have exhausted your free late days, we will deduct a late penalty of 25% per additional late day. Note: The timeout period is recorded in days, even if you delay for 1 minute, it will still be counted as a 1 late day.
5. You are required to follow ShanghaiTech's academic honesty policies. You are not allowed to copy materials from other students or from online or published resources. Violating academic honesty can result in serious sanctions.

Any plagiarism will get Zero point.

1 Proximal Operator

For each of the following convex functions, compute the proximal operator prox_f:

(1) (10 pts) $f(x) = \lambda \|x\|_1$, where $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}_+$ is the regularization parameter.

(2) (20 pts) $f(X) = \lambda \|X\|_*$, where $X \in \mathbb{R}^{d \times m}$ is a matrix, $\|X\|_*$ denotes the nuclear norm, and $\lambda \in \mathbb{R}_+$ is the regularization parameter.

Solution:

(1)

(2)

2 Alternating Direction Method of Multipliers

(35 pts) Consider the following problem.

$$\begin{aligned} \text{minimize} \quad & -\log\det X + \operatorname{Tr}(XC) + \rho\|X\|_1 \\ \text{subject to} \quad & X \succeq 0 \end{aligned} \tag{1}$$

In (1), $\|\cdot\|_1$ is the entrywise ℓ_1-norm. This problem arises in estimation of sparse undirected graphical models. C is the empirical covariance matrix of the observed data. The goal is to estimate a covariance matrix with sparse inverse for the observed data. In order to apply ADMM we rewrite (1) as

$$\begin{aligned} \text{minimize} \quad & -\log\det X + \operatorname{Tr}(XC) + \mathbb{I}_{X\succeq 0}(X) + \rho\|Y\|_1 \\ \text{subject to} \quad & X = Y \end{aligned} \tag{2}$$

where $\mathbb{I}_{X\succeq 0}(\cdot)$ is the indicator function associated with the set $X \succeq 0$. Please provide the ADMM update (the derivation process is required) for each variable at the t-th iteration.

Solution:

3 Monotone Operators and Base Splitting Schemes

(35 pts) Proof the theorem below:

Theorem 1. *Let z^* be a solution to the monotone operator splitting problem , and define J: $J = \mathbf{D}_{prox}(Wz^* + Ux + b)$ denotes the Clarke generalized Jacobian of the nonlinearity evaluated at the point $Wz^* + Ux + b$. . Then for $v \in \mathbb{R}^n$ the solution of the equation*

$$u^* = (I - JW)^{-T} v \tag{3}$$

is given by

$$u^* = v + W^T \tilde{u}^* \tag{4}$$

where $\tilde{u}^*$ is a zero of the operator splitting problem $0 \in (F + G)(\tilde{u}^*)$, with operators defined as

$$F(\tilde{u}) = (I - W^T)(\tilde{u}), \quad G(\tilde{u}) = D\tilde{u} - v \tag{5}$$

where D is a diagonal matrix defined by $J = (I + D)^{-1}$ (where we allow for the possibility of $D_{ii} = \infty$ for $J_{ii} = 0$).

Solution: