SI251 - Convex Optimization homework 2

Deadline: 2024-4-10 23:59:59

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2. The **report** has to be submitted as a PDF file to Gradescope, other formats are not accepted.
3. The submitted file name is **student_id+your_student_name.pdf**.
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1. (50 pts) **Robust quadratic programming**. In the lecture, we have learned about robust linear programming as an application of second-order cone programming. Now we will consider a similar robust variation of the convex quadratic program

$$\begin{array}{ll} \text{minimize} & (1/2)x^T Px + q^T x + r \\ \text{subject to} & Ax \preceq b. \end{array}$$

For simplicity, we assume that only the matrix P is subject to errors, and the other parameters (q, r, A, b) are exactly known. The robust quadratic program is defined as

$$\begin{array}{ll} \text{minimize} & \sup_{P\in\mathcal{E}} \left((1/2)x^T Px + q^T x + r\right) \\ \text{subject to} & Ax \prec b \end{array}$$

where $\mathcal{E}$ is the set of possible matrices P.

For each of the following sets $\mathcal{E}$, express the robust QP as a convex problem in a standard form (e.g., QP, QCQP, SOCP, SDP).

(a) A finite set of matrices: $\mathcal{E} = \{P_1, ..., P_K\}$, where $P_i \in S^n_+, i = 1, ..., K$.

(b) A set specified by a nominal value $P_0 \in S^n_+$ plus a bound on the eigenvalues of the deviation $P - P_0$:

$$\mathcal{E} = \{P \in \mathbf{S}^n \mid -\gamma I \preceq P - P_0 \preceq \gamma I\}$$

where $\gamma \in \mathbf{R}$ and $P_0 \in \mathbf{S}^n_+$.

(c) An ellipsoid of matrices:

$$\mathcal{E} = \left\{ P_0 + \sum_{i=1}^{K} P_i u_i \mid \|u\|_2 \le 1 \right\}.$$

You can assume $P_i \in \mathbf{S}^n_+, i = 0, \dots, K$.

Solution:

(a) The objective function is a maximum of convex function, hence convex.
We can write the problem as

$$\begin{array}{lll} \text{minimize} & t & \\ \text{subject to} & (1/2)x^T P_i x + q^T x + r \le t, & i = 1, \dots, K \\ & Ax \preceq b, & \end{array}$$

which is a QCQP in the variables x and t.

(b) For given x, the supremum of $x^T \Delta P x$ over $-\gamma I \le \Delta P \le \gamma I$ is given by

$$\sup_{-\gamma I \preceq \Delta P \preceq \gamma I} x^T \Delta P x = \gamma x^T x$$

Therefore we can express the robust QP as

$$\begin{array}{ll} \text{minimize} & (1/2)x^T (P_0 + \gamma I) x + q^T x + r \\ \text{subject to} & Ax \preceq b \end{array}$$

which is a QP.

(c) For given x, the quadratic objective function is

$$\begin{aligned}&(1/2)\left(x^T P_0 x + \sup_{\|u\|_2\le 1} \sum_{i=1}^K u_i \left(x^T P_i x\right)\right) + q^T x + r\\ &= (1/2)x^T P_0 x + (1/2)\left(\sum_{i=1}^K \left(x^T P_i x\right)^2\right)^{1/2} + q^T x + r.\end{aligned}$$

This is a convex function of x : each of the functions $x^T P_i x$ is convex since $P_i \succeq 0$. The second term is a composition $h\left(g_1(x), \ldots, g_K(x)\right)$ of $h(y) = \|y\|_2$ with $g_i(x) = x^T P_i x$. The functions g_i are convex and nonnegative. The function h is convex and, for $y \in \mathbf{R}_+^K$, nondecreasing in each of its arguments. Therefore the composition is convex. The resulting problem can be expressed as

$$\begin{array}{ll}\text{minimize} & (1/2)x^T P_0 x + \|y\|_2 + q^T x + r\\ \text{subject to} & (1/2)x^T P_i x \le y_i, \quad i = 1, \ldots, K\\ & Ax \preceq b\end{array}$$

which can be further reduced to a SOCP

$$\begin{array}{ll}\text{minimize} & u + t\\ \text{subject to} & \left\|\begin{bmatrix} P_0^{1/2} x \\ 2u - 2q^T x - 1/4 \end{bmatrix}\right\|_2 \le 2u - 2q^T x + 1/4\\ & \left\|\begin{bmatrix} P_i^{1/2} x \\ 2y_i - 1/4 \end{bmatrix}\right\|_2 \le 2y_i + 1/4, \quad i = 1, \ldots, K\\ & \|y\|_2 \le t\\ & Ax \preceq b.\end{array}$$

The variables are x, u, t , and $y \in \mathbf{R}^K$. Note that if we square both sides of the first inequality, we obtain

$$x^T P_0 x + (2u - 2q^T x - 1/4)^2 \le (2u - 2q^T x + 1/4)^2,$$

i.e., $x^T P_0 x + 2q^T x \le 2u$. Similarly, the other constraints are equivalent to $(1/2)x^T P_i x \le y_i$.

2. (50 pts) **Water-filling.** Please consider the convex optimization problem and calculate its solution

$$\begin{array}{ll}\text{minimize} & -\sum_{i=1}^n \log\left(\alpha_i + x_i\right)\\ \text{subject to} & x \succeq 0, \quad \mathbf{1}^T x = 1,\end{array}$$

Solution:

Introducing Lagrange multipliers $\lambda^\star \in \mathbf{R}^n$ for the inequality constraints $x^\star \succeq 0$, and a multiplier $\nu^\star \in \mathbf{R}$ for the equality constraint $\mathbf{1}^T x = 1$, we obtain the KKT conditions

$$\begin{gathered}x^\star \succeq 0, \quad \mathbf{1}^T x^\star = 1, \quad \lambda^\star \succeq 0, \quad \lambda_i^\star x_i^\star = 0, \quad i = 1, \ldots, n,\\ -1/\left(\alpha_i + x_i^\star\right) - \lambda_i^\star + \nu^\star = 0, \quad i = 1, \ldots, n.\end{gathered}$$

We can directly solve these equations to find $x^\star, \lambda^\star$, and $\nu^\star$. We start by noting that $\lambda^\star$ acts as a slack variable in the last equation, so it can be eliminated, leaving

$$\begin{gathered}x^\star \succeq 0, \quad \mathbf{1}^T x^\star = 1, \quad x_i^\star\left(\nu^\star - 1/\left(\alpha_i + x_i^\star\right)\right) = 0, \quad i = 1, \ldots, n\\ \nu^\star \ge 1/\left(\alpha_i + x_i^\star\right), \quad i = 1, \ldots, n.\end{gathered}$$

If $\nu^\star < 1/\alpha_i$, this last condition can only hold if $x_i^\star > 0$, which by the third condition implies that $\nu = 1/\left(\alpha_i + x_i^\star\right)$. Solving for $x_i^\star$, we conclude that $x_i^\star = 1/\nu^\star - \alpha_i$ if $\nu^\star < 1/\alpha_i$. If $\nu^\star \geq 1/\alpha_i$, then $x_i^\star > 0$ is impossible, because it would imply $\nu^\star \geq 1/\alpha_i > 1/\left(\alpha_i + x_i^\star\right)$, which violates the complementary slackness condition. Therefore, $x_i^\star = 0$ if $\nu^\star \geq 1/\alpha_i$. Thus we have

$$x_i^\star = \begin{cases} 1/\nu^\star - \alpha_i & \nu^\star < 1/\alpha_i \\ 0 & \nu^\star \geq 1/\alpha_i, \end{cases}$$

or, put more simply, $x_i^\star = \max\left\{0, 1/\nu^\star - \alpha_i\right\}$. Substituting this expression for $x_i^\star$ into the condition $\mathbf{1}^T x^\star = 1$ we obtain

$$\sum_{i=1}^{n} \max\left\{0, 1/\nu^\star - \alpha_i\right\} = 1.$$

The lefthand side is a piecewise-linear increasing function of $1/\nu^\star$, with breakpoints at α_i, so the equation has a unique solution which is readily determined.