SI251 - Convex Optimization homework 1

Deadline: 2024-03-27 23:59:59

1. You can use Word, Latex or handwriting to complete this assignment. If you want to submit a handwritten version, scan it clearly.
2. The **report** has to be submitted as a PDF file to Gradescope, other formats are not accepted.
3. The submitted file name is **student_id+your_student_name.pdf**.
4. Late policy: You have 4 free late days for the quarter and may use up to 2 late days per assignment with no penalty. Once you have exhausted your free late days, we will deduct a late penalty of 25% per additional late day. Note: The timeout period is recorded in days, even if you delay for 1 minute, it will still be counted as a 1 late day.
5. You are required to follow ShanghaiTech's academic honesty policies. You are not allowed to copy materials from other students or from online or published resources. Violating academic honesty can result in serious sanctions.

Any plagiarism will get Zero point.

1 Convex sets

1. Please prove that the following sets are convex:
    1) $S = \{x \in \mathbf{R}^m \mid \mid p(t) \mid \leq 1 \text{ for } |t| \leq \pi/3\}$, where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$. (5 pts)
    2) (**Ellipsoids**) $\left\{x | \sqrt{(x-x_c)^T P(x-x_c)} \leq r\right\} \quad (x_c \in \mathbb{R}^n, r \in \mathbb{R}, P \succeq 0)$. (5 pts)
    3) (**Symmetric positive semidefinite matrices**) $S_+^{n\times n} = \left\{P \in S^{n\times n} | P \succeq 0\right\}$. (5 pts)
    4) The set of points closer to a given point than a given set, i.e.,
    $$\left\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\right\},$$
    where $S \in R^n$. (5 pts)
2. (15 pts) For a given norm $\|\cdot\|$ on $\mathbf{R}^n$, the dual norm, denoted $\|\cdot\|_*$, is defined as
$$\|y\|_* = \sup_{x\in\mathbf{R}^n} \{y^T x \mid \|x\| \leq 1\}.$$
Show that the dual of Euclidean norm is the Euclidean mom, i.e., $\sup_{x\in\mathbf{R}^n}\{z^T x \mid \|x\|_2 \leq 1\} = ||z||_2$.
3. (15 pts) Define a norm cone as
$$\mathcal{C} \equiv \left\{(x,t) : x \in \mathbb{R}^d, t \geq 0, \|x\| \leq t\right\} \subseteq \mathbb{R}^{d+1}$$
Show that the norm cone is convex by using the definition of convex sets.

2 Convex functions

4. (18 pts) Let $C \subset \mathbb{R}^n$ be convex and $f : C \to R^\star$. Show that the following statements are equivalent:
    (a) epi(f) is convex.
    (b) For all points $x_i \in C$ and $\{\lambda_i | \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1, i = 1, 2, \cdots, n\}$, we have
    $$f\Big(\sum_{i=1}^n \lambda_i x_i\Big) \leq \sum_{i=1}^n \lambda_i f(x_i).$$
    (c) For $\forall x, y \in C$ and $\lambda \in [0,1]$,
    $$f\Big((1-\lambda)x + \lambda y\Big) \leq (1-\lambda) f(x) + \lambda f(y).$$
5. (14 pts) Monotone Mappings. A function $\psi : \mathbf{R}^n \to \mathbf{R}^n$ is called monotone if for all $x, y \in \mathbf{dom}\psi$,
$$(\psi(x) - \psi(y))^T (x - y) >= 0. \tag{1}$$
Suppose $f : \mathbf{R}^n \to \mathbf{R}^n$ is a differentiable convex function. Show that its gradient ∇f is monotone. Is the convex true, i.e., is every monotone mapping the gradient of a convex function?

6. (18 pts) Please determine whether the following functions are convex, concave or none of those, and give a detailed explanation for your choice.

   1)
   $$f_1(x_1, x_2, \cdots, x_n) = \begin{cases} -(x_1 x_2 \cdots x_n)^{\frac{1}{n}}, & \text{if } \ x_1, \cdots, x_n > 0 \\ \infty & \text{otherwise;} \end{cases}$$

   2) $f_2(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbb{R}^2_{++}$;

   3) $f_3(x, u, v) = -\log(uv - x^T x)$ on $\mathbf{dom} f = \{(x, u, v) | uv > x^T x, \quad u, v > 0\}$.