SI251 - Convex Optimization, 2024 Spring

Homework 3

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Due 23:59 (CST), Apr. 24, 2024

1. (50 pts) **L-smooth functions**. Suppose the function $f : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable. Please prove that the following relations holds for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ if f with an L-Lipschitz continuous conditions,

$$[1] \Rightarrow [2] \Rightarrow [3]$$

$$\begin{aligned}
&[1]\ \langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \leq L\|\mathbf{x} - \mathbf{y}\|^2, \\
&[2]\ f(\mathbf{y}) \leq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) + \frac{L}{2}\|\mathbf{y} - \mathbf{x}\|^2, \\
&[3]\ f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) + \frac{1}{2L}\|\nabla f(\mathbf{y}) - \nabla f(\mathbf{x})\|^2, \forall \mathbf{x}, \mathbf{y}.
\end{aligned}$$

Solution:

$[1] \Rightarrow [2]$:

Define $g(t) = f()$

So we have proved that $[1] \Rightarrow [2]$.

$[2] \Rightarrow [3]$:

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So above all, we have proved that $[1] \Rightarrow [2] \Rightarrow [3]$.

2. (50 pts) **Backtracking line search**. Please show the convergence of backtracking line search on a m-strongly convex and M-smooth objective function f as

$$f\left(x^{(k)}\right) - p^\star \leq c^k \left(f\left(x^{(0)}\right) - p^\star\right)$$

where $c = 1 - \min\{2m\alpha, 2\beta\alpha m/M\} < 1$.

Algorithm 9.2 *Backtracking line search.*

given a descent direction Δx for f at $x \in \textbf{dom}\, f$, $\alpha \in (0, 0.5)$, $\beta \in (0, 1)$.
$t := 1$.
while $f(x + t\Delta x) > f(x) + \alpha t \nabla f(x)^T \Delta x, \quad t := \beta t.$

Solution: