

SI251 - Convex Optimization, 2024 Spring

Homework 1

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1 Convex sets

1. Please prove that the following sets are convex:

- 1) $S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$, where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$. (5 pts)
- 2) (Ellipsoids) $\left\{x \mid \sqrt{(x - x_c)^T P (x - x_c)} \leq r\right\}$ ($x_c \in \mathbb{R}^n, r \in \mathbb{R}, P \succeq 0$). (5 pts)
- 3) (Symmetric positive semidefinite matrices) $S_+^{n \times n} = \{P \in S^{n \times n} \mid P \succeq 0\}$. (5 pts)
- 4) The set of points closer to a given point than a given set, i.e.,

$$\left\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\right\},$$

where $S \in \mathbf{R}^n$. (5 pts)

(1) For a fixed $t \in [-\frac{\pi}{3}, \frac{\pi}{3}]$, we could know that $\cos t, \cos 2t, \dots, \cos mt$ are certain constants, so $p(t)$ is a linear function of \mathbf{x} .

Since $|p(t)| \leq 1 \Leftrightarrow -1 \leq p(t) \leq 1$.

So let $S_t = \{\mathbf{x} \mid -1 \leq x_1 \cos t + \cdots + x_n \cos nt \leq 1\}$.

Since $p(t)$ is linear function of \mathbf{x} , so S_t is a convex set.

And we could know that

$$S = \bigcap_{-\frac{\pi}{3} \leq t \leq \frac{\pi}{3}} S_t$$

From the theorem, we could know that the intersection of convex sets is also a convex set, so S is a convex set.

So above all, we have proved that S is a convex set.

(2) Let S be the Ellipsoids set, and we could know that $\forall \mathbf{x}_1, \mathbf{x}_2 \in S$, we have $(\mathbf{x}_1 - \mathbf{x}_c)^T P (\mathbf{x}_1 - \mathbf{x}_c) \leq r^2$ and $(\mathbf{x}_2 - \mathbf{x}_c)^T P (\mathbf{x}_2 - \mathbf{x}_c) \leq r^2$.

And $\forall \theta \in [0, 1]$, since $P \succeq 0$, so P is symmetric, so we have

$$\begin{aligned} & [(\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2) - \mathbf{x}_c]^T P [(\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2) - \mathbf{x}_c] \\ &= \theta^2 (\mathbf{x}_1 - \mathbf{x}_c)^T P (\mathbf{x}_1 - \mathbf{x}_c) + 2\theta(1 - \theta) (\mathbf{x}_1 - \mathbf{x}_c)^T P (\mathbf{x}_2 - \mathbf{x}_c) + (1 - \theta)^2 (\mathbf{x}_2 - \mathbf{x}_c)^T P (\mathbf{x}_2 - \mathbf{x}_c) \\ &\leq \theta^2 r^2 + (1 - \theta)^2 r^2 + 2\theta(1 - \theta) (\mathbf{x}_1 - \mathbf{x}_c)^T P (\mathbf{x}_2 - \mathbf{x}_c) \end{aligned}$$

And since $P \succeq 0$, so P could be decomposition as $P = Q^T \Lambda Q$, so $P^{\frac{1}{2}} = Q^T \Lambda^{\frac{1}{2}} Q$, i.e. $P^{\frac{1}{2}} \succeq 0$, so

$$\begin{aligned} (\mathbf{x}_1 - \mathbf{x}_c)^T P (\mathbf{x}_2 - \mathbf{x}_c) &= (\mathbf{x}_1 - \mathbf{x}_c)^T (P^{\frac{1}{2}})^T (P^{\frac{1}{2}}) (\mathbf{x}_2 - \mathbf{x}_c) \\ &= [P^{\frac{1}{2}} (\mathbf{x}_1 - \mathbf{x}_c)]^T [P^{\frac{1}{2}} (\mathbf{x}_2 - \mathbf{x}_c)] \\ &\leq \| [P^{\frac{1}{2}} (\mathbf{x}_1 - \mathbf{x}_c)] \|_2 \cdot \| P^{\frac{1}{2}} (\mathbf{x}_2 - \mathbf{x}_c) \|_2 \end{aligned}$$

Since $\mathbf{x}_1 \in S$, so

$$(\mathbf{x}_1 - \mathbf{x}_c)^T P (\mathbf{x}_1 - \mathbf{x}_c) = (\mathbf{x}_1 - \mathbf{x}_c)^T (P^{\frac{1}{2}})^T [P^{\frac{1}{2}} (\mathbf{x}_1 - \mathbf{x}_c)] = \| P^{\frac{1}{2}} (\mathbf{x}_1 - \mathbf{x}_c) \|_2^2 \leq r^2$$

i.e.

$$\|P^{\frac{1}{2}}(\mathbf{x}_1 - \mathbf{x}_c)\| \leq r$$

Similarly, we have

$$\|P^{\frac{1}{2}}(\mathbf{x}_2 - \mathbf{x}_c)\| \leq r$$

So

$$\begin{aligned} (\mathbf{x}_1 - \mathbf{x}_c)^T P(\mathbf{x}_2 - \mathbf{x}_c) &\leq \| [P^{\frac{1}{2}}(\mathbf{x}_1 - \mathbf{x}_c)] \|_2 \cdot \| P^{\frac{1}{2}}(\mathbf{x}_2 - \mathbf{x}_c) \|_2 \\ &\leq r \cdot r \\ &= r^2 \end{aligned}$$

So

$$\begin{aligned} &[(\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2) - \mathbf{x}_c]^T P[(\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2) - \mathbf{x}_c] \\ &\leq \theta^2 r^2 + (1 - \theta)^2 r^2 + 2\theta(1 - \theta)(\mathbf{x}_1 - \mathbf{x}_c)^T P(\mathbf{x}_2 - \mathbf{x}_c) \\ &\leq \theta^2 r^2 + (1 - \theta)^2 r^2 + 2\theta(1 - \theta) r^2 \\ &= r^2 \end{aligned}$$

So $\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2 \in S$.

So above all, we have proved that $\forall \mathbf{x}_1, \mathbf{x}_2 \in S, \forall \theta \in [0, 1]$, we have $\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2 \in S$. So S i.e. the Ellipsoids is a convex set.

(3) $\forall A, B \in S_+^{n \times n}$, we have $A^T = A, B^T = B$, and $\forall \mathbf{y} \in \mathbb{R}^n, \mathbf{y}^T A \mathbf{y} \geq 0, \mathbf{y}^T B \mathbf{y} \geq 0$.

So $\forall \theta \in [0, 1]$, we have

$$(\theta A + (1 - \theta) B)^T = \theta A^T + (1 - \theta) B^T = \theta A + (1 - \theta) B$$

And

$$\mathbf{y}^T (\theta A + (1 - \theta) B) \mathbf{y} = \theta \mathbf{y}^T A \mathbf{y} + (1 - \theta) \mathbf{y}^T B \mathbf{y} \geq 0$$

So $\theta A + (1 - \theta) B$ is symmetric and semi-positive defined.

So $\theta A + (1 - \theta) B \in S_+^{n \times n}$.

So above all, we have proved that $S_+^{n \times n}$ is a convex set.

(4) Let $\mathcal{C} = \left\{ \mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2 \text{ for all } \mathbf{y} \in S \right\}$.

$\forall \mathbf{x} \in \mathcal{C}$, and for a fixed \mathbf{y} , we have

$$\begin{aligned} \|\mathbf{x} - \mathbf{x}_0\|_2 &\leq \|\mathbf{x} - \mathbf{y}\|_2 \\ \|\mathbf{x} - \mathbf{x}_0\|_2^2 &\leq \|\mathbf{x} - \mathbf{y}\|_2^2 \\ (\mathbf{x} - \mathbf{x}_0)^T (\mathbf{x} - \mathbf{x}_0) &\leq (\mathbf{x} - \mathbf{y})^T (\mathbf{x} - \mathbf{y}) \\ \mathbf{x}^T (\mathbf{x}_0 - \mathbf{y}) &\geq \frac{1}{2} (\mathbf{x}_0^T \mathbf{x}_0 - \mathbf{y}^T \mathbf{y}) \end{aligned}$$

From the definition, we know that for a fixed \mathbf{y} , $\mathbf{x}^T(\mathbf{x}_0 - \mathbf{y}) \geq \frac{1}{2}(\mathbf{x}_0^T \mathbf{x}_0 - \mathbf{y}^T \mathbf{y})$ is a half-space $S_{\mathbf{y}}$.

So $\forall \mathbf{y} \in S$, we could see that $\mathcal{C} = \bigcap_{\mathbf{y} \in S} S_{\mathbf{y}}$.

And since each $S_{\mathbf{y}}$ is a half-space, which is a convex set. And from the theorem we have known, that the intersection of convex sets is also a convex set, so \mathcal{C} is a convex set.

2. (15 pts) For a given norm $\|\cdot\|$ on \mathbf{R}^n , the dual norm, denoted $\|\cdot\|_*$, is defined as

$$\|y\|_* = \sup_{x \in \mathbf{R}^n} \{y^T x \mid \|x\| \leq 1\}.$$

Show that the dual of Euclidean norm is the Euclidean norm, i.e., $\sup_{x \in \mathbf{R}^n} \{z^T x \mid \|x\|_2 \leq 1\} = \|z\|_2$.

$$\begin{aligned} & z^T x \\ & \leq \|z\|_2 \|x\|_2 \\ & \leq \|z\|_2 \end{aligned}$$

If and only if $\|x\|_2 = 1$ and $\langle z, x \rangle = 1$ will take the equation condition.

So above all, we have proved that $\sup_{x \in \mathbf{R}^n} \{z^T x \mid \|x\|_2 \leq 1\} = \|z\|_2$.

3. (15 pts) Define a norm cone as

$$\mathcal{C} \equiv \{(x, t) : x \in \mathbb{R}^d, t \geq 0, \|x\| \leq t\} \subseteq \mathbb{R}^{d+1}$$

Show that the norm cone is convex by using the definition of convex sets.

$\forall (x_1, t_1), (x_2, t_2) \in \mathcal{C}$, we have $\|x_1\| \leq t_1, \|x_2\| \leq t_2, t_1 \geq 0, t_2 \geq 0$.

And $\forall \theta \in [0, 1]$, we have

$$\begin{aligned} & \|\theta x_1 + (1 - \theta)x_2\| \\ & \leq \|\theta x_1\| + \|(1 - \theta)x_2\| \\ & = \theta t_1 + (1 - \theta)t_2 \end{aligned}$$

So $\theta(x_1, t_1) + (1 - \theta)(x_2, t_2) = (\theta x_1 + (1 - \theta)x_2, \theta t_1 + (1 - \theta)t_2) \in \mathcal{C}$.

So above all, we have proved that \mathcal{C} is a convex set.

2 Convex functions

4. (18 pts) Let $C \subset \mathbb{R}^n$ be convex and $f : C \rightarrow \mathbb{R}^*$. Show that the following statements are equivalent:

(a) $\text{epi}(f)$ is convex.

(b) For all points $x_i \in C$ and $\{\lambda_i | \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1, i = 1, 2, \dots, n\}$, we have

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

(c) For $\forall x, y \in C$ and $\lambda \in [0, 1]$,

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y).$$

• (a) \Rightarrow (c)

$\forall x, y \in C$, we have $(x, f(x)), (y, f(y)) \in \text{epi}(f)$.

From (a), we have known that $\text{epi}(f)$ is convex, so $\forall \lambda \in [0, 1]$, we have

$$((1 - \lambda)x + \lambda y, (1 - \lambda)f(x) + \lambda f(y)) \in \text{epi}(f)$$

which means that

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)$$

So (a) \Rightarrow (c) has been proved.

• (c) \Rightarrow (a)

$\forall \mathbf{x}, \mathbf{y} \in C$, and $\forall (\mathbf{x}, t_1), (\mathbf{y}, t_2) \in \text{epi}(f)$, we have $t_1 \geq f(\mathbf{x}), t_2 \geq f(\mathbf{y})$.

And $\forall \lambda \in [0, 1]$, we have

$$\begin{aligned} & f((1 - \lambda)\mathbf{x} + \lambda\mathbf{y}) \\ & \leq (1 - \lambda)f(\mathbf{x}) + \lambda f(\mathbf{y}) \\ & \leq (1 - \lambda)t_1 + \lambda t_2 \end{aligned}$$

So $((1 - \lambda)\mathbf{x} + \lambda\mathbf{y}, (1 - \lambda)t_1 + \lambda t_2) \in \text{epi}(f)$, so $\text{epi}(f)$ is convex.

So (c) \Rightarrow (a) has been proved.

• (b) \Rightarrow (c)

Let $n = 2, \lambda_1 = 1 - \lambda, \lambda_2 = \lambda$, then we have

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)$$

So (b) \Rightarrow (c) has been proved.

• (c) \Rightarrow (b)

When $n = 2$, let $\lambda_1 = 1 - \lambda, \lambda_2 = \lambda$, then we have

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i)$$

When $n = 3$, $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in C, \theta_1, \theta_2 \in [0, 1]$, we have

$$f((1 - \theta_1)\mathbf{x} + \theta_1\mathbf{y}) \leq (1 - \theta_1)f(\mathbf{x}) + \theta_1f(\mathbf{y})$$

$$f((1 - \theta_2)\mathbf{x} + \theta_2\mathbf{z}) \leq (1 - \theta_2)f(\mathbf{x}) + \theta_2f(\mathbf{z})$$

Let $\mathbf{a} = (1 - \theta_1)\mathbf{x} + \theta_1\mathbf{y}, \mathbf{b} = (1 - \theta_2)\mathbf{x} + \theta_2\mathbf{z}$, from the definition of convex set, we could know that $\mathbf{a}, \mathbf{b} \in C$.

So $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} \in C$, so we have

$$f\left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right) \leq \frac{1}{2}f(\mathbf{a}) + \frac{1}{2}f(\mathbf{b})$$

$$f\left(\frac{1}{2}[(1 - \theta_1)\mathbf{x} + \theta_1\mathbf{y}] + \frac{1}{2}[(1 - \theta_2)\mathbf{x} + \theta_2\mathbf{z}]\right) \leq \frac{1}{2}[(1 - \theta_1)f(\mathbf{x}) + \theta_1f(\mathbf{y})] + \frac{1}{2}[(1 - \theta_2)f(\mathbf{x}) + \theta_2f(\mathbf{z})]$$

$$f\left(\left(1 - \frac{\theta_1}{2} - \frac{\theta_2}{2}\right)\mathbf{x} + \frac{\theta_1}{2}\mathbf{y} + \frac{\theta_2}{2}\mathbf{z}\right) \leq \left(1 - \frac{\theta_1}{2} - \frac{\theta_2}{2}\right)f(\mathbf{x}) + \frac{\theta_1}{2}f(\mathbf{y}) + \frac{\theta_2}{2}f(\mathbf{z})$$

Let $\lambda_1 = 1 - \frac{\theta_1}{2} - \frac{\theta_2}{2}, \lambda_2 = \frac{\theta_1}{2}, \lambda_3 = \frac{\theta_2}{2}$, then we have $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i)$$

Similarly, $\forall n > 3$, we could use the similar method to prove that (c) \Rightarrow (b).

Since (a) \Leftrightarrow (c) and (b) \Leftrightarrow (c), so we could say that (a),(b),(c) three statements are equivalent.

5. (14 pts) Monotone Mappings. A function $\psi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is called monotone if for all $x, y \in \text{dom}\psi$,

$$(\psi(x) - \psi(y))^T(x - y) \geq 0$$

Suppose $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is a differentiable convex function. Show that its gradient ∇f is monotone. Is the converse true, i.e., is every monotone mapping the gradient of a convex function?

1. If f is a differentiable convex function, we could know that $\forall \mathbf{x}, \mathbf{y} \in \mathbf{R}^n$,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x})$$

$$f(\mathbf{x}) \geq f(\mathbf{y}) + \nabla f(\mathbf{y})^T(\mathbf{x} - \mathbf{y})$$

Add these two inequality, we will get that

$$\nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) + \nabla f(\mathbf{y})^T(\mathbf{x} - \mathbf{y}) \leq 0$$

$$\nabla f(\mathbf{x})^T(\mathbf{x} - \mathbf{y}) - \nabla f(\mathbf{y})^T(\mathbf{x} - \mathbf{y}) \geq 0$$

$$(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y}))^T(\mathbf{x} - \mathbf{y}) \geq 0$$

So we have proved that a differentiable convex function's gradient ∇f is monotone.

2. Suppose that $\psi(x_1, x_2) = (-x_2, x_1)$.

Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, then

$$\begin{aligned} (\psi(\mathbf{x}) - \psi(\mathbf{y}))^T(\mathbf{x} - \mathbf{y}) &= [(-x_2, x_1) - (-y_2, y_1)]^T[(x_1, x_2) - (y_1, y_2)] \\ &= (-x_2 + y_2, x_1 - y_1)^T[(x_1 - y_1, x_2 - y_2)] \\ &= (x_1 - y_1) \cdot (x_2 - y_2) + (x_1 - y_1) \cdot [-(x_2 - y_2)] \\ &= 0 \\ &\geq 0 \end{aligned}$$

So our constructed $\psi(\mathbf{x})$ is monotone.

Let f be the primitive function of ψ , then $\frac{\partial^2 f}{\partial x_1 \partial x_2} = -1, \frac{\partial^2 f}{\partial x_2 \partial x_1} = 1$.

But for a differentiable convex function f , it must have $\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$.

So for a monotone function, it may not a gradient of a differentiable convex function.

So above all, differentiable convex function's gradient ∇f is monotone, but its converse is not true.

6. (18 pts) Please determine whether the following functions are convex, concave or none of those, and give a detailed explanation for your choice.

1)

$$f_1(x_1, x_2, \dots, x_n) = \begin{cases} -(x_1 x_2 \cdots x_n)^{\frac{1}{n}}, & \text{if } x_1, \dots, x_n > 0 \\ \infty & \text{otherwise;} \end{cases}$$

2) $f_2(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on \mathbb{R}_{++}^2 ;

3) $f_3(x, u, v) = -\log(uv - x^T x)$ on $\text{dom } f = \{(x, u, v) | uv > x^T x, u, v > 0\}$.

(1) We could see that f_1 is twice continuously differentiable over $\text{dom } f_1$ and its Hessian matrix is that:

$$\nabla^2 f_1(\mathbf{x}) = \frac{-(x_1 x_2 \cdots x_n)^{\frac{1}{n}}}{n^2} \begin{bmatrix} \frac{1-n}{x_1^2} & \frac{1}{x_1 x_2} & \cdots & \frac{1}{x_1 x_n} \\ \frac{1}{x_2 x_1} & \frac{1-n}{x_2^2} & \cdots & \frac{1}{x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n x_1} & \frac{1}{x_1 x_2} & \cdots & \frac{1-n}{x_n^2} \end{bmatrix}$$

And $\forall \mathbf{x} \in \text{dom } f_1, \mathbf{y} \in \mathbb{R}^n$, we have

$$\begin{aligned} \mathbf{y}^T \nabla^2 f_1(\mathbf{x}) \mathbf{y} &= \frac{-(x_1 x_2 \cdots x_n)^{\frac{1}{n}}}{n^2} \cdot \left[\sum_{i=1}^n \frac{y_i^2 (1-n)}{x_i^2} + \sum_{i \neq j} \frac{y_i y_j}{x_i x_j} \right] \\ &= \frac{-(x_1 x_2 \cdots x_n)^{\frac{1}{n}}}{n^2} \cdot \left[\left(\sum_{i=1}^n \frac{y_i}{x_i} \right)^2 - n \sum_{i=1}^n \left(\frac{y_i}{x_i} \right)^2 \right] \end{aligned}$$

From the multivariate mean inequality, we could know that

$$\left(\frac{\sum_{i=1}^n a_i}{n} \right)^2 \leq \frac{1}{n} \sum_{i=1}^n a_i^2 \Rightarrow \left(\sum_{i=1}^n a_i \right)^2 \leq n \sum_{i=1}^n a_i^2$$

So we could know that

$$\left[\left(\sum_{i=1}^n \frac{y_i}{x_i} \right)^2 - n \sum_{i=1}^n \left(\frac{y_i}{x_i} \right)^2 \right] \leq 0$$

And since $\frac{-(x_1 x_2 \cdots x_n)^{\frac{1}{n}}}{n^2} < 0$, so we could know that

$$\forall \mathbf{x} \in \text{dom } f_1, \mathbf{y} \in \mathbb{R}^n, \mathbf{y}^T \nabla^2 f_1(\mathbf{x}) \mathbf{y} \geq 0$$

So above all, $f_1(x_1, \dots, x_n)$ is convex.

(2) The Hessian of f_2 is that:

$$\begin{aligned} \nabla^2 f_2(\mathbf{x}) &= \begin{bmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} & \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} \\ \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} & (1-\alpha)(-\alpha)x_1^\alpha x_2^{-\alpha-1} \end{bmatrix} \\ &= \alpha(\alpha-1)x_1^\alpha x_2^{1-\alpha} \begin{bmatrix} \frac{1}{x_1^2} & -\frac{1}{x_1 x_2} \\ -\frac{1}{x_1 x_2} & \frac{1}{x_2^2} \end{bmatrix} \end{aligned}$$

So $\forall \mathbf{y} = (y_1, y_2)$, we have

$$\mathbf{y}^T \nabla^2 f_2(\mathbf{x}) \mathbf{y} = \alpha(\alpha - 1)x_1^\alpha x_2^{1-\alpha} \left(\frac{y_1}{x_1} - \frac{y_2}{x_2} \right)^2 \leq 0$$

So above all, $f_2(x_1, x_2)$ is concave.

$$(3) f_3(\mathbf{x}, u, v) = -\log(uv - \mathbf{x}^T \mathbf{x}) = -\log u - \log \left(v - \frac{\mathbf{x}^T \mathbf{x}}{u} \right).$$

From we have known about the perspective: if $f(\mathbf{x})$ is convex, then its perspective $g(\mathbf{x}, t) = tf\left(\frac{\mathbf{x}}{t}\right)$ is convex.

Since $f(\mathbf{x}) = \mathbf{x}^T \mathbf{x}$ is convex, so $g(\mathbf{x}, t) = \frac{\mathbf{x}^T \mathbf{x}}{t}$ is convex.

And since v is affine, $-\frac{\mathbf{x}^T \mathbf{x}}{t}$ is concave, so $\left(v - \frac{\mathbf{x}^T \mathbf{x}}{u}\right)$ is concave.

Since $h(x) = -\log x$ is convex and non-increasing, $\left(v - \frac{\mathbf{x}^T \mathbf{x}}{u}\right)$ is concave, so from the composition with scalar functions, we could know that $-\log \left(v - \frac{\mathbf{x}^T \mathbf{x}}{u}\right)$ is convex.

Also, since $-\log u$ is convex, so $f_3(\mathbf{x}, u, v) = -\log u - \log \left(v - \frac{\mathbf{x}^T \mathbf{x}}{u}\right)$ is convex.

So above all, $f_3(\mathbf{x}, u, v)$ is convex.