

Homework 1

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Due: 2025/03/09 11:59am

1. Let X, Y be jointly Gaussian random variables. Show the following equality:

$$E[Y|X] = L[Y|X] = E(Y) + \frac{\text{Cov}(X, Y)}{\text{Var}(X)}(X - E(X)).$$

2. We wish to estimate the probability of landing heads, denoted by θ , of a biased coin. We model θ as the value of a random variable Θ with a known prior PDF $f_{\Theta} \sim \text{Unif}(0, 1)$. We consider n independent tosses and let X be the number of heads observed. Find the MMSE $E[\Theta|X]$ and the LLSE $L[\Theta|X]$.
3. Sampling from the discrete distribution.
- (a) Given n positive real number a_1, \dots, a_n , where $\sum_{j=1}^n a_j = 1$; and n i.i.d. random variables $X_1, \dots, X_n \sim \text{Gumbel}(0, 1)$. show the following equality:

$$P\left(\log a_i + X_i = \max_{j \in \{1, \dots, n\}} (\log a_j + X_j)\right) = a_i.$$

- (b) Illustrate how the above result leads to new sampling methods for the discrete distribution.
4. Adopt the Acceptance-Rejection method to estimate the value of π , then evaluate the performance of Monte Carlo algorithms with finite number of samples.
5. Sampling from probability distributions. Show histograms and compare them to corresponding PDFs.
- (a) Sampling from the standard Normal distribution with both the Box-Muller method and the Acceptance-Rejection method. Discuss the pros and cons of both methods.
- (b) Sampling from the distribution with the following pdf:

$$f(x) \propto \exp\left(-\frac{1}{2}x^2\right) (\sin^2(6x) + 3\cos^2(x)\sin^2(4x) + 1).$$

6. Given a random variable $X \sim N(0, 1)$, evaluate the tail probability $c = P(X > 8)$ by Monte Carlo methods with & without importance sampling. Discuss the pros and cons of importance sampling.

7. Generate uniform distributions over the following geometric objects:

(a) Elliptic ($a = 2, b = 1$):

$$E_2(a, b) = \left\{ (x, y) \in \mathbb{R}^2 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1 \right\}.$$

(b) Sphere ($r = 1$):

$$S_2(r) = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2 \}.$$

(c) Ball ($r = 1$):

$$B_3(r) = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq r^2 \}.$$

(d) Torus ($r_0 = 2, r = 1$):

$$T_2(r_0, r) = \left\{ (x, y, z) \in \mathbb{R}^3 : \left(r_0 - \sqrt{x^2 + y^2}\right)^2 + z^2 = r^2 \right\}.$$

8. The Curse and Blessing of Dimensionality. Denote $\mathbf{x} = (x_1, \dots, x_d)$.

(a) The d -dimensional hyperball of radius r is denoted as

$$B_d(r) = \left\{ \mathbf{x} \in \mathbb{R}^d : \sum_{i=1}^d x_i^2 \leq r^2 \right\}$$

Find the volume of $B_d(r)$ and plot a figure to show how such volume changes with d when $r = 1$.

(b) The d -dimensional hypersphere of radius r is denoted as

$$S_{d-1}(r) = \left\{ \mathbf{x} \in \mathbb{R}^d : \sum_{i=1}^d x_i^2 = r^2 \right\}$$

When $d \gg 1$, adopt concentration inequalities to show almost all the volume of the high-dimensional hypersphere lies near its equator.

(c) The d -dimensional hypercube of radius r is denoted as

$$C_d(r) = \{ \mathbf{x} \in \mathbb{R}^d : -r \leq x_i \leq r \}$$

When $d \gg 1$, adopt concentration inequalities to show almost all the volume of the high-dimensional cube is located in its corners.