SI252 Reinforcement Learning: Homework #02

Due on March 23, 2025 at 11:59 p.m.(CST)

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Problem 1

[BH Chapter 11, Problem 2]. Let $X_0, X_1, X_2 \ldots$ be an irreducible Markov chain with state space $\{1, 2, \ldots, M\}$. $M \geq 3$, transition matrix $Q = (q_{i,j})$, and stationary distribution $\mathbf{s} = (s_1, \ldots, s_M)$. Let the initial state X_0 follow the stationary distribution, i.e., $P(X_0 = i) = s_i$.
(a) On average, how many of $X_0, X_1, \ldots, X_9$ equal 3? (In terms of $\mathbf{s}$; simplify.)
(b) Let $Y_n = (X_n - 1)(X_n - 2)$. For $M = 3$, find an example of Q (the transition matrix for the *original* chain $X_0, X_1, \ldots$) where $Y_0, Y_1, \ldots$ is Markov, and another example of Q where $Y_0, Y_1, \ldots$ is not Markov. In your examples, make $q_{i,i} > 0$ for at least one i and make sure it is possible to get from any state to any other state eventually.

Solution

(a) Let the indicator $\mathbb{I}_i$ donate wheher $X_i = 3$, and N be the number of $X_0, \ldots, X_9$ equal to 3. Since the initial state X_0 follows the stationary distribution, so $X_1, \ldots, X_9$ also follow the stationary distribution. i.e.

$$P(X_i = 3) = s_3, \forall i \in \{0, \ldots, 9\}$$

Then we can get that:

$$\mathbb{E}[N] = \mathbb{E}\left[\sum_{i=0}^{9} \mathbb{I}_i\right] = \sum_{i=0}^{9} \mathbb{E}[\mathbb{I}_i] = \sum_{i=0}^{9} P(X_i = 3) = \sum_{i=0}^{9} s_3 = 10s_3$$

(b) Since $M = 3$, and the relationship between X_i and Y_i is:

$$\begin{aligned}
X_i = 1 &\Rightarrow Y_i = 0 \\
X_i = 2 &\Rightarrow Y_i = 0 \\
X_i = 3 &\Rightarrow Y_i = 2
\end{aligned}$$

Define $g(y)$ be a set of values of x such that $Y = g(X)$. i.e. $g(0) = \{1, 2\}, g(2) = \{3\}$.
To let $Y_0, Y_1, \ldots$ be Markov, we need to make sure that

$$P(Y_{n+1} = y_{n+1}|Y_n = y_n, \ldots, Y_0 = y_0) = P(Y_{n+1} = y_{n+1}|Y_n = y_n)$$

Since Y_i have 2 possible values, so we can discuss them in 4 cases:

- $Y_{n+1} = 2$, $Y_n = 2$:

$$\begin{aligned}
&P(Y_{n+1} = 2|Y_n = 2, Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0) \\
&= P(X_{n+1} = 3|X_n = 3, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) \\
&= P(X_{n+1} = 3|X_n = 3) \\
&= P(Y_{n+1} = 2|Y_n = 2)
\end{aligned}$$

- $Y_{n+1} = 0$, $Y_n = 2$:

$$\begin{aligned}
&P(Y_{n+1} = 0|Y_n = 2, Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0) \\
&= P(X_{n+1} \in g(0)|X_n = 3, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) \\
&= P(X_{n+1} = 1|X_n = 3, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) + P(X_{n+1} = 2|X_n = 3, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) \\
&= P(X_{n+1} = 1|X_n = 3) + P(X_{n+1} = 2|X_n = 3) \\
&= P(X_{n+1} \in g(0)|X_n = 3) \\
&= P(Y_{n+1} = 0|Y_n = 2)
\end{aligned}$$

Problem 1 continued on next page...

- $Y_{n+1}=2$, $Y_n=0$, combined with LOTP:

$$\begin{aligned}&P(Y_{n+1}=2|Y_n=0,Y_{n-1}=y_{n-1},\dots,Y_0=y_0)\\=&\sum_{x_n=1,2}P(X_{n+1}=3|X_n=x_n,X_n\in g(0),\dots,X_0\in g(y_0))P(X_n=x_n|X_n\in g(0),\dots,X_0\in g(y_0))\\=&\sum_{x_n=1,2}P(X_{n+1}=3|X_n=x_n)P(X_n=x_n|X_n\in g(0),\dots,X_0\in g(y_0))\end{aligned}$$

Let $\alpha_1=P(X_n=1|X_n\in g(0),\dots,X_0\in g(y_0)),\alpha_2=P(X_n=2|X_n\in g(0),\dots,X_0\in g(y_0))$. So we have

$$\alpha_1+\alpha_2=1$$
$$P(Y_{n+1}=2|Y_n=0,Y_{n-1}=y_{n-1},\dots,Y_0=y_0)=\alpha_1q_{1,3}+\alpha_2q_{2,3}$$

Since $g(0)$ have 2 elements, so there exists many combinations to make α_1,α_2 take different values, but $\alpha_1+\alpha_2$ always holds. However, to make the Markov property holds, we need to make sure that $\alpha_1q_{1,3}+\alpha_2q_{2,3}=P(Y_{n+1}=2|Y_n=0)$, where $P(Y_{n+1}=2|Y_n=0),q_{2,3}$ are constants. Thus it must have

$$\alpha_1(q_{1,3}-q_{2,3})=P(Y_{n+1}=2|Y_n=0)-q_{2,3}\Rightarrow q_{1,3}=q_{2,3}$$

- $Y_{n+1}=0$, $Y_n=0$, combined with LOTP:

$$\begin{aligned}&P(Y_{n+1}=0|Y_n=0,Y_{n-1}=y_{n-1},\dots,Y_0=y_0)\\=&\sum_{x_n=1,2}P(X_{n+1}\in g(0)|X_n=x_n,X_n\in g(0),\dots,X_0\in g(y_0))P(X_n=x_n|X_n\in g(0),\dots,X_0\in g(y_0))\\=&\sum_{x_n=1,2}P(X_{n+1}\in g(0)|X_n=x_n)P(X_n=x_n|X_n\in g(0),\dots,X_0\in g(y_0))\end{aligned}$$

Let $\alpha_1'=P(X_n=1|X_n\in g(0),\dots,X_0\in g(y_0)),\alpha_2'=P(X_n=2|X_n\in g(0),\dots,X_0\in g(y_0))$. So we have

$$\alpha_1'+\alpha_2'=1$$
$$P(Y_{n+1}=0|Y_n=0,Y_{n-1}=y_{n-1},\dots,Y_0=y_0)=\alpha_1'(q_{1,1}+q_{1,2})+\alpha_2'(q_{2,1}+q_{2,2})$$

Similarly to the analysis above, to make the Markov property holds, it has

$$q_{1,1}+q_{1,2}=q_{2,1}+q_{2,2}$$

So above all, if $q_{1,3}=q_{2,3}$ and $q_{1,1}+q_{1,2}=q_{2,1}+q_{2,2}$, then $Y_0,Y_1,\dots$ is Markov.
And example of Q where $Y_0,Y_1,\dots$ is Markov is:

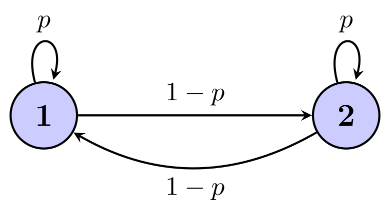
$$Q=\begin{pmatrix}\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{pmatrix}$$

Examplt of Q where $Y_0,Y_1,\dots$ is not Markov is:

$$Q=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{pmatrix}$$

Problem 2

[BH Chapter 11, Problem 4]. Consider the Markov chain shown below, where $0 < p < 1$ and the labels on the arrows indicate transition probabilities.



(a) Write down the transition matrix Q for this chain.
(b) Find the stationary distribution of the chain.
(c) What is the limit of Q^n as $n \to \infty$?

Solution

(a) The transition matrix for this chain is:

$$Q = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

(b) Suppose the stationary distribution is $\boldsymbol{\pi} = (\pi_1, \pi_2)$, then we have:

$$\boldsymbol{\pi} = \boldsymbol{\pi} Q \Rightarrow \begin{cases} \pi_1 = p\pi_1 + (1-p)\pi_2 \\ \pi_2 = (1-p)\pi_1 + p\pi_2 \end{cases}$$
$$\pi_1 + \pi_2 = 1$$

Solve the equations, we can get the stationary distribution:

$$\boldsymbol{\pi} = \left(\frac{1}{2}, \frac{1}{2}\right)$$

(c) It is obvious that the chain is irreducible, and since there exists self loops, the chain is aperiodic. So the chain is ergodic. Then from the Fundamental Limit Theorem for Ergodic Markov Chains, we have for the stationary distribution $\boldsymbol{\pi}$:

$$\pi_j = \lim_{n \to \infty} Q_{ij}^n$$

So the limit of Q^n as $n \to \infty$ is:

$$\lim_{n \to \infty} Q^n = \begin{pmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Problem 3

[BH Chapter 11, Problem 6]. Daenerys has three dragons: Drogon, Rhaegal, and Viserion. Each dragon independently explores the world in search of tasty morsels. Let X_n, Y_n, Z_n be the locations at time n of Drogon, Rhaegal, Viserion respectively, where time is assumed to be discrete and the number of possible locations is a finite number M. Their paths $X_0, X_1, X_2 \ldots$; $Y_0, Y_1, Y_2, \ldots$; and $Z_0, Z_1, Z_2, \ldots$ are independent Markov chains with the same stationary distribution $\mathbf{s}$. Each dragon starts out at a random location generated according to the stationary distribution.

(a) Let state 0 be home (so s_0 is the stationary probability of the home state). Find the expected number of times that Drogon is at home, up to time 24, i.e., the expected number of how many of $X_0, X_1, \ldots, X_{24}$ are state 0 (in terms of s_0).

(b) If we want to track all 3 dragons simultaneously, we need to consider the vector of positions, (X_n, Y_n, Z_n). There are M^3 possible values for this vector; assume that each is assigned a number from 1 to M^3, e.g., if $M = 2$ we could encode the statex $(0,0,0), (0,0,1), (0,1,0), \ldots, (1,1,1)$ as $1, 2, 3, \ldots, 8$ respectively. Let W_n be the number between 1 and M^3 representing (X_n, Y_n, Z_n). Determine whether $W_0, W_1, \ldots$ is a Markov chain.

(c) Given that all 3 dragons start at home at time 0, find the expected time it will take for all 3 to be at home again at the same time.

Solution

(a) Let the indicator $\mathbb{I}_i$ denote whether $X_i = 0$, and N be the number of $X_0, \ldots, X_{24}$ equal to 0. Since the initial state X_0 follows the stationary distribution, so $X_1, \ldots, X_{24}$ also follow the stationary distribution. i.e.

$$P(X_i = 0) = s_0, \forall i \in \{0, \ldots, 24\}$$

Then we can get that:

$$\mathbb{E}[N] = \mathbb{E}\left[\sum_{i=0}^{24} \mathbb{I}_i\right] = \sum_{i=0}^{24} \mathbb{E}\left[\mathbb{I}_i\right] = \sum_{i=0}^{24} P(X_i = 0) = \sum_{i=0}^{24} s_0 = 25 s_0$$

(b) $W_i = w_i$ and $(X_i, Y_i, Z_i) = (x_i, y_i, z_i)$ correspond one-to-one, so

$$\begin{aligned}
&P(W_{n+1} = w_{n+1} | W_0 = w_0, \ldots, W_n = w_n) \\
&= P(X_{n+1} = x_{n+1}, Y_{n+1} = y_{n+1}, Z_{n+1} = z_{n+1} | X_n = x_n, \ldots, X_0 = x_0, Y_n = y_n, \ldots, Y_0 = y_0, Z_n = z_n, \ldots, Z_0 = z_0) \\
&= P(X_{n+1} = x_{n+1} | X_n = x_n, \ldots, X_0 = x_0) P(Y_{n+1} = y_{n+1} | Y_n = y_n, \ldots, Y_0 = y_0) P(Z_{n+1} = z_{n+1} | Z_n = z_n, \ldots, Z_0 = z_0) \\
&= P(X_{n+1} = x_{n+1} | X_n = x_n) P(Y_{n+1} = y_{n+1} | Y_n = y_n) P(Z_{n+1} = z_{n+1} | Z_n = z_n) \\
&= P(X_{n+1} = x_{n+1}, Y_{n+1} = y_{n+1}, Z_{n+1} = z_{n+1} | X_n = x_n, Y_n = y_n, Z_n = z_n) \\
&= P(W_{n+1} = w_{n+1} | W_n = w_n)
\end{aligned}$$

The second equality and the second to last equality hold due to the the paths are independent markov chains. So $W_0, W_1, \ldots$ is a Markov chain.

(c) Let N_t be the number of times that all 3 dragons are at home from time 1 to time t. And let T_i be the time interval between the $(i-1)$-th time that all 3 dragons are at home and the i-th. Here we define that $X_0 = 0$ is the 0-th time that all 3 dragons are at home. Let T be the first time all 3 dragons are at home. From the Markov property, we can get that $T, T_1, \ldots$ are i.i.d. random variables. From the large number of law, we have

$$\mathbb{E}(T) = \lim_{n \to \infty} \frac{T_1 + \ldots + T_n}{n} = \lim_{t \to +\infty} \frac{T_1 + \ldots + T_{N_t}}{N_t}$$

Problem 3 continued on next page...

We can also get that

$$
\begin{aligned}
& T_1 + \ldots + T_{N_t} \le t \le T_1 + \ldots + T_{N_{t+1}} \\
\implies & \lim_{t \to +\infty} \frac{T_1 + \ldots + T_{N_t}}{N_t} \le \lim_{t \to +\infty} \frac{t}{N_t} \le \lim_{t \to +\infty} \frac{T_1 + \ldots + T_{N_{t+1}}}{N_{t+1}}
\end{aligned}
$$

Since $\lim_{t \to +\infty} \frac{T_1 + \ldots + T_{N_t}}{N_t} = \mathbb{E}(T)$, and $\lim_{t \to +\infty} \frac{T_1 + \ldots + T_{N_{t+1}}}{N_{t+1}} = \mathbb{E}(T)$, we have

$$
\lim_{t \to +\infty} \frac{t}{N_t} = \mathbb{E}(T)
$$

From the property as the stationary distribution of the Markov chain, we have the stationary distribution of $W_0, W_1, \ldots$ has

$$
P(W_i = 0) = P(X_i = 0)P(Y_i = 0)P(Z_i = 0) = s_0^3
$$

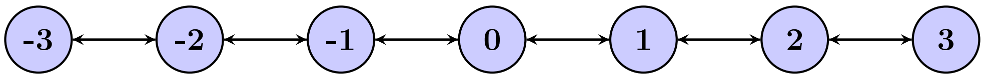
Thus we have

$$
\begin{aligned}
s_0^3 = \lim_{t \to +\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}(T)} \\
\implies \mathbb{E}(T) = \frac{1}{s_0^3}
\end{aligned}
$$

So above all, the expected time for all 3 dragons to be at home again at the same time is $\frac{1}{s_0^3}$.

Problem 4

[BH Chapter 11, Problem 7]. A Markov chain $X_0, X_1, \ldots$ with state space $\{-3,-2,-1,0,1,2,3\}$ proceeds as follows. The chain starts at $X_0 = 0$. If X_n is not an endpoint (-3 or 3), then X_{n+1} is $X_n - 1$ or $X_n + 1$, each with probability $\frac{1}{2}$. Otherwise, the chain gets reflected off the endpoint, i.e., from 3 it always goes to 2 and from -3 it always goes to -2. A diagram of the chain is shown below.



(a) Is $|X_0|, |X_1|, |X_2|, \ldots$ also a Markov chain? Explain.
Hint: For both (a) and (b), think about whether the past and future are conditionally independent given the present; don't do calculations with a 7 by 7 transition matrix!
(b) Let sgn be the sign function: $\operatorname{sgn}(x) = 1$ if $x > 0, \operatorname{sgn}(x) = -1$ if $x < 0$, and $\operatorname{sgn}(0) = 0$. Is $\operatorname{sgn}(X_0), \operatorname{sgn}(X_1), \operatorname{sgn}(X_2), \ldots$ a Markov chain? Explain.
(c) Find the stationary distribution of the chain $X_0, X_1, X_2, \ldots$.
(d) Find a simple way to modify some of the transition probabilities $q_{i,j}$ for $i \in \{-3, 3\}$ to make the stationary distribution of the modified chain uniform over the states.

Solution

(a) Let $Y_n = |X_n|$. Since the chain starts at $X_0 = 0$, and the chain is symmetric about state 0, we have

$$P(X_n = y_n) = P(X_n = -y_n)$$
$$P(X_n = y_n \mid |X_n| = y_n) = P(X_n = -y_n \mid |X_n| = y_n) = \frac{1}{2}$$
$$P(X_n = y_n \mid |X_n| = y_n, \ldots, |X_0| = y_0) = P(X_n = -y_n \mid |X_n| = y_n, \ldots, |X_0| = y_0) = \frac{1}{2}$$

Although when $y_n = 0$, $-y_n = y_n$, but for a better consistency when using LOTP, we can keep this format, then we have

$$\begin{aligned}
&P(Y_{n+1} = y_{n+1} | Y_n = y_n, \ldots, Y_0 = y_0) \\
&= P(|X_{n+1}| = y_{n+1} \mid |X_n| = y_n, \ldots, |X_0| = y_0) \\
&\overset{\text{LOTP}}{=} P(|X_{n+1}| = y_{n+1} | X_n = y_n, |X_n| = y_n, \ldots, |X_0| = y_0) P(X_n = y_n \mid |X_n| = y_n, \ldots, |X_0| = y_0) \\
&+ P(|X_{n+1}| = y_{n+1} | X_n = -y_n, |X_n| = y_n, \ldots, |X_0| = y_0) P(X_n = -y_n \mid |X_n| = y_n, \ldots, |X_0| = y_0) \\
&\overset{\text{symmetric}}{=} \frac{1}{2} [P(|X_{n+1}| = y_{n+1} | X_n = y_n, |X_n| = y_n, \ldots, |X_0| = y_0) + P(|X_{n+1}| = y_{n+1} | X_n = -y_n, |X_n| = y_n, \ldots, |X_0| = y_0)] \\
&\overset{\text{Markov property}}{=} \frac{1}{2} [P(|X_{n+1}| = y_{n+1} | X_n = y_n) + P(|X_{n+1}| = y_{n+1} | X_n = -y_n)] \\
&= P(|X_{n+1}| = y_{n+1} | X_n = y_n) P(X_n = y_n \mid |X_n| = y_n) + P(|X_{n+1}| = y_{n+1} | X_n = -y_n) P(X_n = -y_n \mid |X_n| = y_n) \\
&\overset{\text{LOTP}}{=} P(|X_{n+1}| = y_{n+1} \mid |X_n| = y_n) \\
&= P(Y_{n+1} = y_{n+1} | Y_n = y_n)
\end{aligned}$$

So above all, $|X_0|, |X_1|, |X_2|, \ldots$ is a Markov chain.
(b) Let $Q = (q_{i,j})$ be the transition matrix, then for example

$$P(\operatorname{sgn}(X_2) = 1 | \operatorname{sgn}(X_1) = 1) = P(X_2 \geq 1 | X_1 \geq 1, |X_2 - X_1| = 1) = q_{1,2} + q_{2,3} + q_{2,1} + q_{3,2}$$

However, if we have more information about the past, for example

$$P(\operatorname{sgn}(X_2) = 1 | \operatorname{sgn}(X_1) = 1, \operatorname{sgn}(X_0) = 0) = P(X_2 = 2 | X_1 = 1, X_0 = 0) = q_{1,2}$$

Problem 4 continued on next page...

So in the we have given an example that

$$P(\operatorname{sgn}(X_2) = 1 | \operatorname{sgn}(X_1) = 1, \operatorname{sgn}(X_0) = 0) \neq P(\operatorname{sgn}(X_2) = 1 | \operatorname{sgn}(X_1) = 1)$$

So $\operatorname{sgn}(X_0), \operatorname{sgn}(X_1), \operatorname{sgn}(X_2), \ldots$ is not a Markov chain.
(c) Let $Q = (q_{i,j})$ be the transition matrix, $\boldsymbol{\pi} = (\pi_i)$ be the stationary distribution, $i = -3, -2, -1, 0, 1, 2, 3$. Then we have the transition matrix is

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

For the stationary distribution, we have

$$\boldsymbol{\pi} = \boldsymbol{\pi} Q \Rightarrow \begin{cases} \pi_{-3} = \frac{1}{2}\pi_{-2} \\ \pi_{-2} = \pi_{-3} + \frac{1}{2}\pi_{-1} \\ \pi_{-1} = \frac{1}{2}\pi_{-2} + \frac{1}{2}\pi_0 \\ \pi_0 = \frac{1}{2}\pi_{-1} + \frac{1}{2}\pi_1 \\ \pi_1 = \frac{1}{2}\pi_0 + \frac{1}{2}\pi_2 \\ \pi_2 = \frac{1}{2}\pi_1 + \pi_3 \\ \pi_3 = \frac{1}{2}\pi_2 \\ \sum\limits_{i=-3}^{3} \pi_i = 1 \end{cases}$$

Solve above equations, we can get the stationary distribution is

$$\boldsymbol{\pi} = \frac{1}{12}(1, 2, 2, 2, 2, 2, 1) = \left(\frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}\right)$$

(d) From the theorem we have known, if Q is a double stochastic matrix, then the stationary distribution is uniform. So we can modify the transition matrix Q into a symmetric matrix easily. i.e. for the state -3, it has probability $\frac{1}{2}$ to go to -2 and stay at -3, and similarly for the state 3. So the modified transition matrix is

$$Q' = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Then the stationary distribution of the modified chain is uniform over the states.

Problem 5

[BH Chapter 11, Problem 8]. Let G be an undirected network with nodes labeled $1, 2, \ldots, M$ (edges from a node to itself are not allowed), where $M \geq 2$ and random walk on this network is irreducible. Let d_j be the degree of node j for each j. Create a Markow chain on the state space $1, 2, \ldots, M$, with transitions as follows. From state i, generate a proposal j by choosing a uniformly random j such that there is an edge between i and j in G; then go to j with probability $\min\left(\frac{d_i}{d_j}, 1\right)$, and stay at i otherwise.
(a) Find the transition probability $q_{i,j}$ from i to j for this chain, for all states i, j.
(b) Find the stationary distribution of this chain.

Solution

(a) Define $\mathcal{N}(i)$ be the set of neighbors of node i, i.e. there has edges between i and j. If $j \notin \mathcal{N}(i)$ and $j \neq i$, then $q_{i,j} = 0$.
If $j \in \mathcal{N}(i)$, then the probability to go from i to j is $\dfrac{1}{d_i} \cdot \min\left(\frac{d_i}{d_j}, 1\right) = \dfrac{1}{\max(d_i, d_j)}$.
Since it has a probability $1 - \min\left(\frac{d_i}{d_j}, 1\right)$ to stay at i, so $q_{i,i} = 1 - \sum\limits_{j \in \mathcal{N}(i)} \dfrac{1}{\max(d_i, d_j)}$.
So above all, the transition probability $q_{i,j}$ from i to j is:

$$q_{i,j} = P(X_{n+1} = j | X_n = i) = \begin{cases} \dfrac{1}{\max(d_i, d_j)} & \text{, if } j \in \mathcal{N}(i) \\ 1 - \sum\limits_{j \in \mathcal{N}(i)} \dfrac{1}{\max(d_i, d_j)} & \text{, if } i = j \\ 0 & \text{, otherwise} \end{cases}$$

(b) Since if $q_{i,j} > 0, i \neq j$, then $j \in \mathcal{N}(i) \Rightarrow i \in \mathcal{N}(j)$, and $q_{i,j} = \dfrac{1}{\max(d_i, d_j)} = q_{j,i}$, so the transition matrix Q is symmetric. Thus the sum of each row i of Q is:

$$\sum_{j=1}^{M} q_{j,i} = \sum_{j=1}^{M} q_{i,j} = 1$$

Which means that Q is a double stochastic matrix. Thus the stationary distribution is the uniform distribution:

$$(s_1, s_2, \ldots, s_M) = \left(\frac{1}{M}, \ldots, \frac{1}{M}\right)$$

Problem 6

[BH Chapter 11, Problem 14]. There are two urns with a total of $2N$ distinguishable balls. Initially, the first urn has N white balls and the second urn has N black balls. At each stage, we pick a ball at random from each urn and interchange them. Let X_n be the number of black balls in the first urn at time n. This is a Markov chain on the state space $\{0, 1, \dots, N\}$.
(a) Give the transition probabilities of the chain.
(b) Show that $(s_0, s_1, \dots, s_N)$ where

$$s_i = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}}$$

is the stationary distribution, by verifying the reversibility condition.

Solution

(a) Let $Q = (q_{i,j})$ be the transition matrix of the chain. Since each time one ball from each urn is picked and interchanged, the number of black balls in the first urn can only change by 1, which means $q_{i,j} = 0$ for $|i - j| > 1$.
And when $|i - j| \le 1$:
If there are i black balls in the first urn, then there are $N - i$ white balls in the first urn; i white balls, $N - i$ black balls in the second urn.
Let B_1 be the event that a black ball is picked from the first urn, B_2 be the event that a black ball is picked from the second urn. Thus the transition probabilities are:

$$\begin{aligned} q_{i,i+1} &= P(X_{n+1} = i+1 | X_n = i) = P(B_1^c)P(B_2) = \frac{N-i}{N}\frac{N-i}{N} = \frac{(N-i)^2}{N^2} \\ q_{i,i} &= P(X_{n+1} = i | X_n = i) = P(B_1)P(B_2) + P(B_1^c)P(B_2^c) = \frac{N-i}{N}\frac{i}{N} + \frac{i}{N}\frac{N-i}{N} = \frac{2i(N-i)}{N^2} \\ q_{i,i-1} &= P(X_{n+1} = i-1 | X_n = i) = P(B_1)P(B_2^c) = \frac{i}{N}\frac{i}{N} = \frac{i^2}{N^2} \\ q_{i,j} &= 0 \quad \text{, for} \quad |i-j| > 1 \end{aligned}$$

(b) To verify the reversibility condition, we need to verify that

$$s_i q_{i,j} = s_j q_{j,i} \Rightarrow \frac{s_i}{s_j} = \frac{q_{j,i}}{q_{i,j}}$$

And from the definition of s_i, we know that $s_i \neq 0$, and

$$\frac{s_i}{s_j} = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{N}{j}\binom{N}{N-j}} = \left(\frac{j!(N-j)!}{i!(N-i)!}\right)^2$$

It is obvious that $s_i q_{i,j} = s_j q_{j,i}$ holds when $|i - j| > 1$, as $q_{i,j} = q_{j,i} = 0$, so the reversibility condition holds. So we just consider the situations $q_{i,j} \neq 0$, i.e. $|i - j| \le 1$:

- When $j = i + 1$:

$$\begin{aligned} \frac{s_i}{s_j} &= \left(\frac{(i+1)!(N-i-1)!}{i!(N-i)!}\right)^2 = \left(\frac{i+1}{N-i}\right)^2 \\ \frac{q_{j,i}}{q_{i,j}} &= \frac{q_{i+1,i}}{q_{i,i+1}} = \frac{\frac{(i+1)^2}{N^2}}{\frac{(N-i)^2}{N^2}} = \left(\frac{i+1}{N-i}\right)^2 \\ \Rightarrow \frac{s_i}{s_j} &= \frac{q_{j,i}}{q_{i,j}} \end{aligned}$$

Problem 6 continued on next page...

- When $j = i$:

$$\frac{s_i}{s_j} = \frac{s_i}{s_i} = 1$$
$$\frac{q_{j,i}}{q_{i,j}} = \frac{q_{i,i}}{q_{i,i}} = 1$$
$$\Rightarrow \frac{s_i}{s_j} = \frac{q_{j,i}}{q_{i,j}}$$

- When $j = i - 1$:

$$\frac{s_i}{s_j} = \left(\frac{(i-1)!(N-i+1)!}{i!(N-i)!}\right)^2 = \left(\frac{N-i+1}{i}\right)^2$$
$$\frac{q_{j,i}}{q_{i,j}} = \frac{q_{i-1,i}}{q_{i,i-1}} = \frac{\frac{(N-i+1)^2}{N^2}}{\frac{i^2}{N^2}} = \left(\frac{N-i+1}{i}\right)^2$$
$$\Rightarrow \frac{s_i}{s_j} = \frac{q_{j,i}}{q_{i,j}}$$

So above all, we have proved that the reversibility condition

$$s_i q_{i,j} = s_j q_{j,i}$$

always holds.

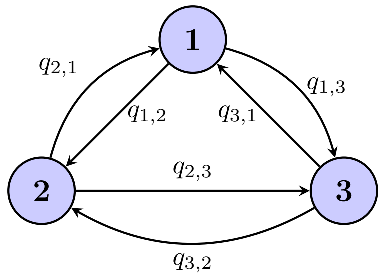
Also, we can use the story proof: To take out N ball from $2N$ balls without replacement and order, there are total $\binom{2N}{N}$ ways. Also, we can regard it as taking i balls from first N balls and taking $N-i$ balls from the rest N balls, where $i \in \{0, 1, \ldots, N\}$. So we use the story proof to prove that

$$\binom{2N}{N} = \sum_{i=0}^{N} \binom{N}{i}\binom{N}{N-i}$$
$$\Rightarrow \sum_{i=0}^{N} s_i = 1$$

So above all, it is obvious that $\forall i, s_i \geq 0$, and we have proved that $\sum\limits_{i=0}^{N} s_i = 1$, and $\forall i, j, s_i q_{i,j} = s_j q_{j,i}$, so $(s_0, s_1, \ldots, s_N)$ is the stationary distribution.

Problem 7

Given a three-state CTMC with transition rates between states shown in the following diagram:



Find holding time for each state, transition probability matrix of embedded chain, and the generator matrix.

Solution

Since for the generation matrix of CTMC, each row i has the property that $\sum\limits_{j=1}^{n} q_{i,j} = 0$, and from the diagram of the CTMC, we can get that $q_{i,i} = -\sum\limits_{j \neq i} q_{i,j}$. So we can get the generation matrix:

$$Q = \begin{pmatrix} -q_{1,2} - q_{1,3} & q_{1,2} & q_{1,3} \\ q_{2,1} & -q_{2,1} - q_{2,3} & q_{2,3} \\ q_{3,1} & q_{3,2} & -q_{3,1} - q_{3,2} \end{pmatrix}$$

The praamerter of the holding time for each state is

$$v_i = \sum_{j \neq i} q_{i,j}$$

And the holding time for each state is

$$T_i \sim \text{Expo}(v_i)$$

The transition probability matrix of the embedded chain is

$$p^e_{i,j} = \begin{cases} \dfrac{q_{i,j}}{v_i} & i \neq j \\ 0 & i = j \end{cases} \Rightarrow P^e = \begin{pmatrix} 0 & \frac{q_{1,2}}{q_{1,2}+q_{1,3}} & \frac{q_{1,3}}{q_{1,2}+q_{1,3}} \\ \frac{q_{2,1}}{q_{2,1}+q_{2,3}} & 0 & \frac{q_{2,3}}{q_{2,1}+q_{2,3}} \\ \frac{q_{3,1}}{q_{3,1}+q_{3,2}} & \frac{q_{3,2}}{q_{3,1}+q_{3,2}} & 0 \end{pmatrix}$$

So above all, the holding time for each state is

$$T_1 \sim \text{Expo}(q_{1,2} + q_{1,3}), T_2 \sim \text{Expo}(q_{2,1} + q_{2,3}), T_3 \sim \text{Expo}(q_{3,1} + q_{3,2})$$

The transition probability matrix of the embedded chain is

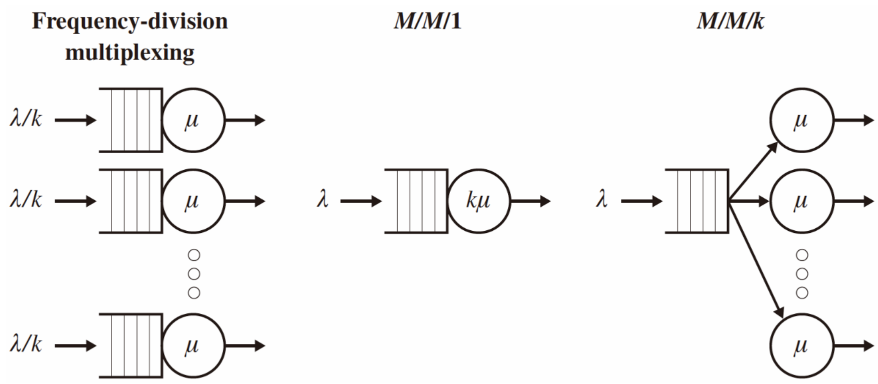
$$P^e = \begin{pmatrix} 0 & \frac{q_{1,2}}{q_{1,2}+q_{1,3}} & \frac{q_{1,3}}{q_{1,2}+q_{1,3}} \\ \frac{q_{2,1}}{q_{2,1}+q_{2,3}} & 0 & \frac{q_{2,3}}{q_{2,1}+q_{2,3}} \\ \frac{q_{3,1}}{q_{3,1}+q_{3,2}} & \frac{q_{3,2}}{q_{3,1}+q_{3,2}} & 0 \end{pmatrix}$$

And the generator matrix is

$$Q = \begin{pmatrix} -q_{1,2} - q_{1,3} & q_{1,2} & q_{1,3} \\ q_{2,1} & -q_{2,1} - q_{2,3} & q_{2,3} \\ q_{3,1} & q_{3,2} & -q_{3,1} - q_{3,2} \end{pmatrix}$$

Problem 8

Three Server Organizations: in the following figure, we show three data center systems with the same arriving rate λ and the same total service rate $k\mu$: FDM, M/M/1 and M/M/k. Please discuss the pros and cons of each system. Regarding the performance of average delay, which system is the best? Show your analysis (CTMC & Queueing Theory) and simulation results.

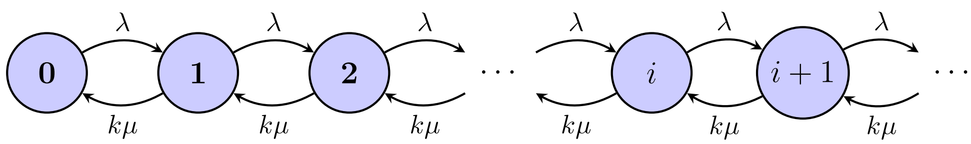


Solution

To ensure the total system is stable, the arrival rate should be less than the service rate, i.e.

$$\lambda < \mu$$

Intuitive understanding is that if the survice rate is less than the arrival rate, which means that the average survice time is less than the average arrival interval, then the system will continously queueing.
The numertical results for the systems' average delay:
1. M/M/1: the number of customers in the system could be modeled as the following CTMC:



Let $\boldsymbol{\pi}$ be the stationary distribution of the above CTMC, then according to the detailed balance equation, we have:

$$\pi_i \cdot \lambda = \pi_{i+1} \cdot (k\mu), \forall i \geq 0$$

i.e. $\pi_{i+1} = \dfrac{\lambda}{k\mu}\pi_i \Rightarrow \pi_i = \left(\dfrac{\lambda}{k\mu}\right)^i \pi_0$.
And since $\boldsymbol{\pi}$ is a probability distribution, we have:

$$\sum_{i=0}^{\infty} \pi_i = 1 \Rightarrow \pi_0 \sum_{i=0}^{\infty} \left(\frac{\lambda}{k\mu}\right)^i = 1 \Rightarrow \pi_0 = \frac{1}{\sum\limits_{i=0}^{\infty}\left(\dfrac{\lambda}{k\mu}\right)^i} = \frac{1}{\dfrac{1}{1-\dfrac{\lambda}{k\mu}}} = 1 - \frac{\lambda}{k\mu}$$

Thus, we have

$$\pi_i = \left(1 - \frac{\lambda}{k\mu}\right)\left(\frac{\lambda}{k\mu}\right)^i$$

The average number of customers L in the system is

$$L = \sum_{i=0}^{\infty} i \cdot \pi_i = \frac{\left(1 - \frac{\lambda}{k\mu}\right)\frac{\lambda}{k\mu}}{\left(1 - \frac{\lambda}{k\mu}\right)^2} = \frac{\lambda}{k\mu - \lambda}$$

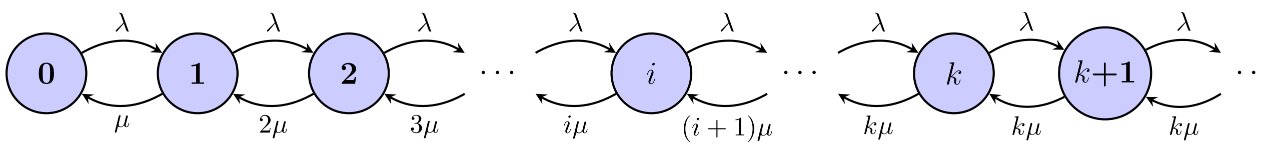
According to the Little's Law, the average delay W is

$$L = \lambda W \Rightarrow W = \frac{L}{\lambda} = \frac{1}{k\mu - \lambda}$$

2. Frequncy-division multiplexing (FDM):
Each server could be regard as a M/M/1 system, but the arriving rate is transformed to $\frac{\lambda}{k}$, and the service rate is transformed to $\frac{k\mu}{k} = \mu$. and the average delay is

$$W = \frac{k \cdot \frac{1}{\mu - \frac{\lambda}{k}}}{k} = \frac{1}{\mu - \frac{\lambda}{k}}$$

3. M/M/k: the number of customers in the system could be modeled as the following CTMC:



Let T_i be the survice time of the i-th server, i.e. $T_i \sim \text{Expo}(\mu)$. Since we have known that

$$\min\left(\text{Expo}(\lambda_1), \ldots, \text{Expo}(\lambda_n)\right) \sim \text{Expo}(\lambda_1 + \ldots + \lambda_n)$$

So when t servers are working, the time of the next server to be idle is

$$T = \min\left(T_1, \ldots, T_t\right) \sim \text{Expo}(t\mu)$$

Thus the death rate when t servers are working is $t\mu$. So the above CTMC is a birth-death process with parameter shown in the chain.
Let $\boldsymbol{\pi}$ be the stationary distribution of the above CTMC, then according to the detailed balance equation, we have:

$$\begin{aligned} \pi_i \cdot \lambda &= \pi_{i+1} \cdot ((i+1)\mu), \forall i \le k-1 \\ \pi_i \cdot \lambda &= \pi_{i+1} \cdot (k\mu) \qquad , \forall i \ge k \end{aligned}$$

Thus, we can express π_i with π_0 as:

$$\begin{aligned} \pi_i &= \frac{\lambda}{i\mu}\pi_{i-1} = \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i \pi_0 \qquad , \forall 1 \le i \le k-1 \\ \pi_i &= \frac{\lambda}{k\mu}\pi_{i-1} = \frac{k^k}{k!}\left(\frac{\lambda}{k\mu}\right)^i \pi_0, \forall i \ge k \end{aligned}$$

And since $\boldsymbol{\pi}$ is a probability distribution, we have:

$$\sum_{i=0}^{\infty} \pi_i = 1 \Rightarrow \pi_0 = \left[\left(\sum_{i=0}^{k-1} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i\right) + \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{1 - \frac{\lambda}{k\mu}}\right]^{-1}$$

Problem 8 continued on next page...

Since if there are less or equal to k customers, they would never be in the queue, but just use the survice for the survices, so the average number of customers L_q in the queue is

$$\begin{aligned}
L_q &= \sum_{i=k+1}^{\infty} (i-k)\cdot \pi_i \\
&= \sum_{i=k+1}^{\infty} (i-k)\cdot \frac{k^k}{k!}\left(\frac{\lambda}{k\mu}\right)^i \pi_0 \\
&\overset{n=i-k}{=} \left(\frac{\lambda}{k\mu}\right)^k \frac{k^k\pi_0}{k!}\sum_{n=1}^{\infty} n\left(\frac{\lambda}{k\mu}\right)^n \\
&= \left(\frac{\lambda}{\mu}\right)^k \frac{\pi_0}{k!}\frac{\frac{\lambda}{k\mu}}{(1-\frac{\lambda}{k\mu})^2}
\end{aligned}$$

According to the Little's Law, the average time the customers are in the queue W_q is

$$L_q = \lambda W_q \Rightarrow W_q = \frac{L_q}{\lambda} = \left(\frac{\lambda}{\mu}\right)^k \frac{\pi_0}{(k!)(k\mu)(1-\frac{\lambda}{k\mu})^2}$$

Where

$$\pi_0 = \left[\left(\sum_{i=0}^{k-1}\frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i\right) + \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{1-\frac{\lambda}{k\mu}}\right]^{-1}$$

Since the surver's service rate is μ, i.e. the survice time is Expo(μ), so the average time customers spend in the surver is $\frac{1}{\mu}$, so the average delay W is

$$W = W_q + \frac{1}{\mu} = \left(\frac{\lambda}{\mu}\right)^k \frac{\pi_0}{(k!)(k\mu)(1-\frac{\lambda}{k\mu})^2} + \frac{1}{\mu}$$

The performance comparison of the systems:

Define $\rho = \frac{\lambda}{\mu}$. As mentioned in the beginning, we have $\lambda < \mu \Rightarrow \rho \in (0,1)$.

1. Prove $W_{\mathrm{M/M/1}} \le W_{\mathrm{M/M/k}}$, the following steps are what we need to prove:

$$\begin{aligned}
\frac{1}{k\mu-\lambda} &\le \frac{1}{\mu} + \rho^k\frac{1}{(k!)(k\mu)\left(1-\frac{\rho}{k}\right)^2}\frac{1}{\pi_0} \\
\frac{\lambda-(k-1)\mu}{(k\mu-\lambda)\mu}\cdot\pi_0 &\le \frac{\rho^k}{(k!)(k\mu)\left(1-\frac{\rho}{k}\right)^2} \\
\left(\sum_{i=0}^{k-1}\frac{\rho^i}{i!}\right) + \frac{1}{k!}\frac{\rho^k}{1-\frac{\rho}{k}} &\le \frac{\rho^k\cdot(k\mu-\lambda)}{k(k!)\left(1-\frac{\rho}{k}\right)^2[\lambda-(k-1)\mu]} \\
\sum_{i=0}^{k-1}\frac{\rho^i}{i!} &\le \frac{\rho^k\cdot\left[k^2\mu+(2k-\rho)\lambda\right]}{k(k!)\left(1-\frac{\rho}{k}\right)^2[\lambda-(k-1)\mu]}
\end{aligned}$$

When $k=1$, the inequality obviously holds, and the equality holds.

Use the deduction, suppose the inequality holds $\forall n = 1,2,\dots,k$, i.e.

$$\sum_{i=0}^{n-1}\frac{\rho^i}{i!} \le \frac{\rho^n\cdot\left[n^2\mu+(2n-\rho)\lambda\right]}{n(n!)\left(1-\frac{\rho}{n}\right)^2[\lambda-(n-1)\mu]}$$

then for $n = k + 1$, we only need to prove that

$$\frac{\rho^n}{n!} \leq \frac{\rho^{n+1} \cdot \left[(n+1)^2\mu + (2(n+1) - \rho)\lambda\right]}{(n+1)((n+1)!)\left(1 - \frac{\rho}{n+1}\right)^2 [\lambda - ((n+1)-1)\mu]} - \frac{\rho^n \cdot \left[n^2\mu + (2n-\rho)\lambda\right]}{n(n!)\left(1 - \frac{\rho}{n}\right)^2 [\lambda - (n-1)\mu]}$$

$$1 \leq \frac{\rho \cdot \left[(n+1)^2\mu + (2(n+1) - \rho)\lambda\right]}{(n+1)^2\left(1 - \frac{\rho}{n+1}\right)^2 [\lambda - n\mu]} - \frac{n^2\mu + (2n-\rho)\lambda}{n\left(1 - \frac{\rho}{n}\right)^2 [\lambda - (n-1)\mu]}$$

Since $\rho \in (0,1)$, so the above inequality holds, thus

$$W_{\mathrm{M/M/1}} \leq W_{\mathrm{M/M/k}}$$

2. Prove $W_{\mathrm{M/M/k}} \leq W_{\mathrm{FDM}}$, we have a similar steps:

$$\frac{1}{\mu} + \rho^k \frac{1}{(k!)(k\mu)\left(1 - \frac{\rho}{k}\right)^2} \frac{1}{\pi_0} \leq \frac{1}{\mu - \frac{\lambda}{k}}$$

$$\sum_{i=0}^{k-1} \frac{\rho^i}{i!} \leq \frac{\rho^k \cdot \left[k^2\mu + (2k-\rho)\lambda\right]}{(k!)\left(1 - \frac{\rho}{k}\right)^2 [\lambda - (k-1)\mu]}$$

It has a similar process, instread that a k is times to the right hand side, thus

$$W_{\mathrm{M/M/k}} \leq W_{\mathrm{FDM}}$$

So above all, the average delay of the three systems are

$$W_{\mathrm{FDM}} = \frac{1}{\mu - \frac{\lambda}{k}}$$

$$W_{\mathrm{M/M/1}} = \frac{1}{k\mu - \lambda}$$

$$W_{\mathrm{M/M/k}} = \frac{1}{\mu} + \left(\frac{\lambda}{\mu}\right)^k \frac{1}{(k!)(k\mu)(1 - \frac{\lambda}{k\mu})^2} \left[\left(\sum_{i=0}^{k-1} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i\right) + \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{1 - \frac{\lambda}{k\mu}}\right]^{-1}$$

And the performance of average delay of the three systems are

$$W_{\mathrm{M/M/1}} \leq W_{\mathrm{M/M/k}} \leq W_{\mathrm{FDM}}$$

If and only if when $k = 1$, the equality holds.
Pros and cons:

- FDM:
  Pros: It is simply implemented and fault-isolated. It would not cause too much trouble if some of the queues or servers have faults.
  Cons: Its drawbacks include lower resource utilization and load imbalances that can lead to increased latency.

- M/M/1:
  Pros: Centralized queuing facilitates optimal resource sharing, which can significantly reduce the average latency. It concentrates efforts to accomplish great things.
  Cons: It does risk a single point of bottleneck, if the queue or the server has errors, the whole system will collapse.

- M/M/k:
  Pros: Parallel servers offer a certain degree of fault tolerance and distributed benefits
  Cons: Its scheduling efficiency is not as high as that of a centralized system, resulting in an average latency that falls between the other two.

Problem 8 continued on next page...

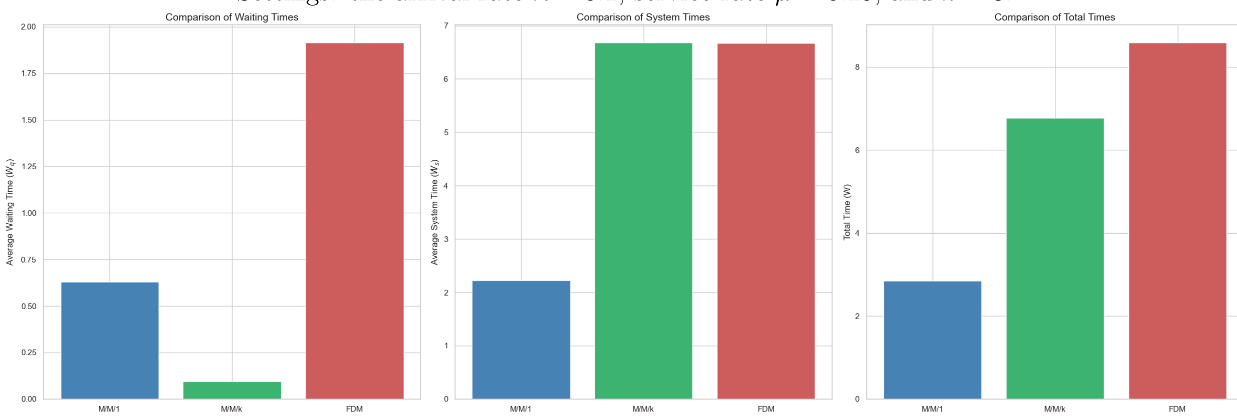
Therefore, as long as the system remains stable and the load is moderate, the M/M/1 system is the optimal choice in terms of average latency.

Simulation results:

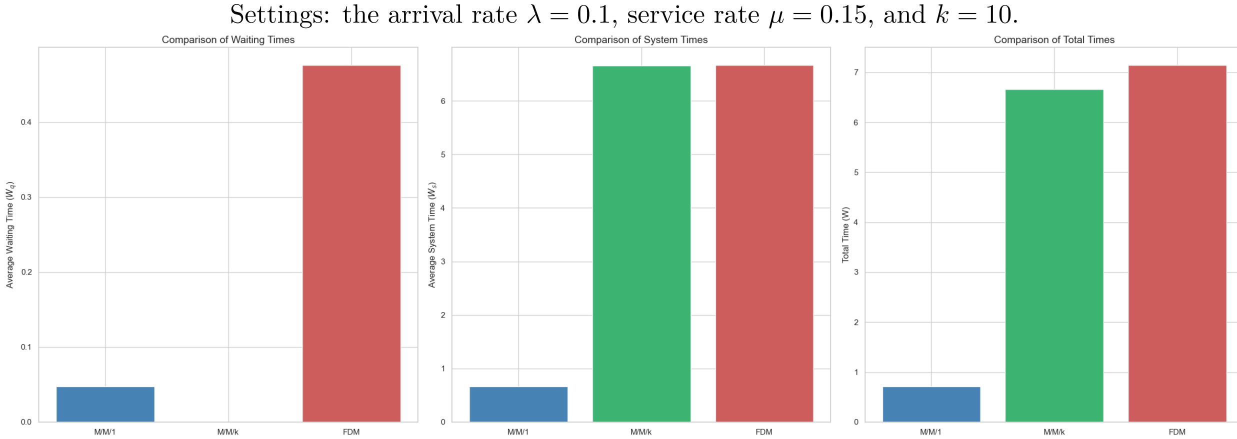
The simulation has total 1000000 customers, with different arrival rate and service rate, and k. The details could be seen in hw2_code.ipynb, theoretical results are also in it, which could be found that the simulated results are quite close to the theoretical results. And W_q represent the avreage waiting time, W_s represent the avreage service time, $W = W_q + W_s$ is the average delay.

The simulation with different settings are as followed.

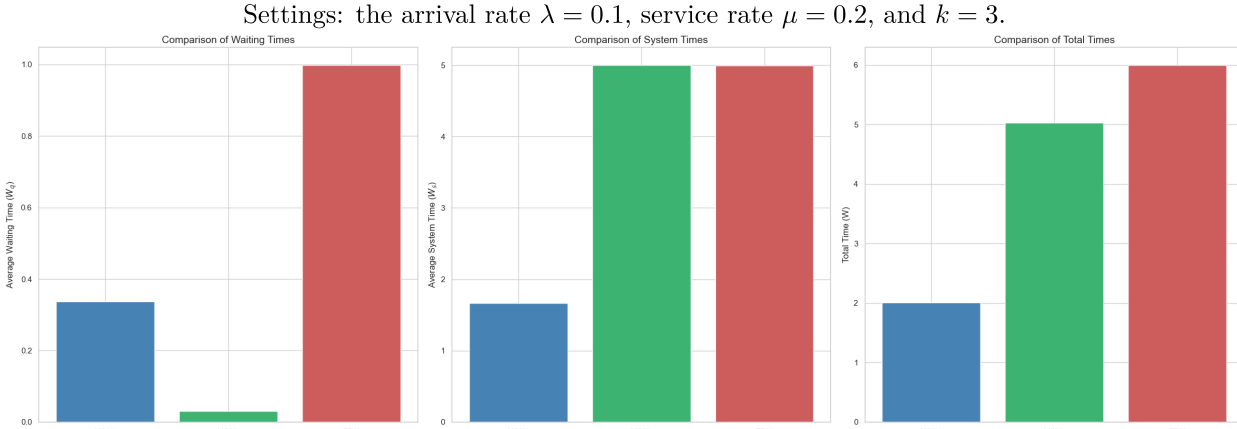
Settings: the arrival rate $\lambda = 0.1$, service rate $\mu = 0.15$, and $k = 3$.



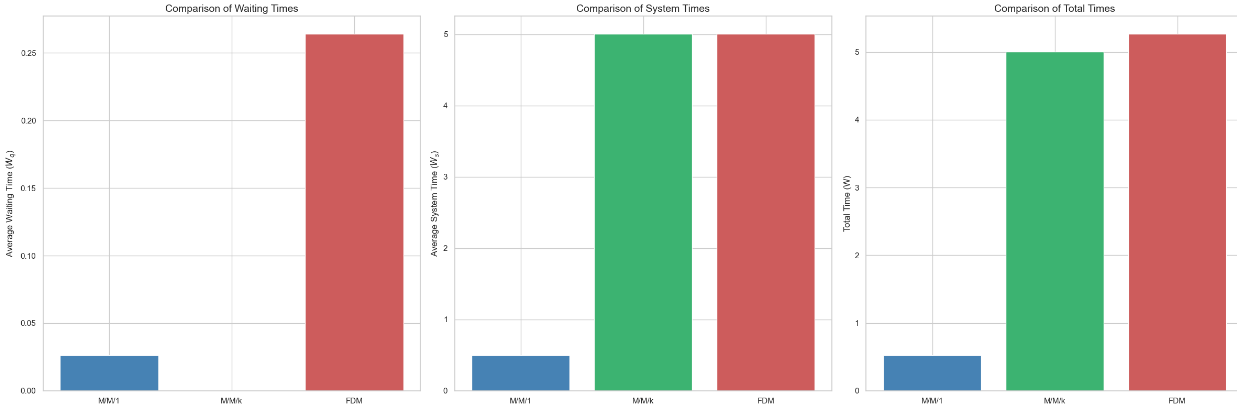
Settings: the arrival rate $\lambda = 0.1$, service rate $\mu = 0.15$, and $k = 10$.



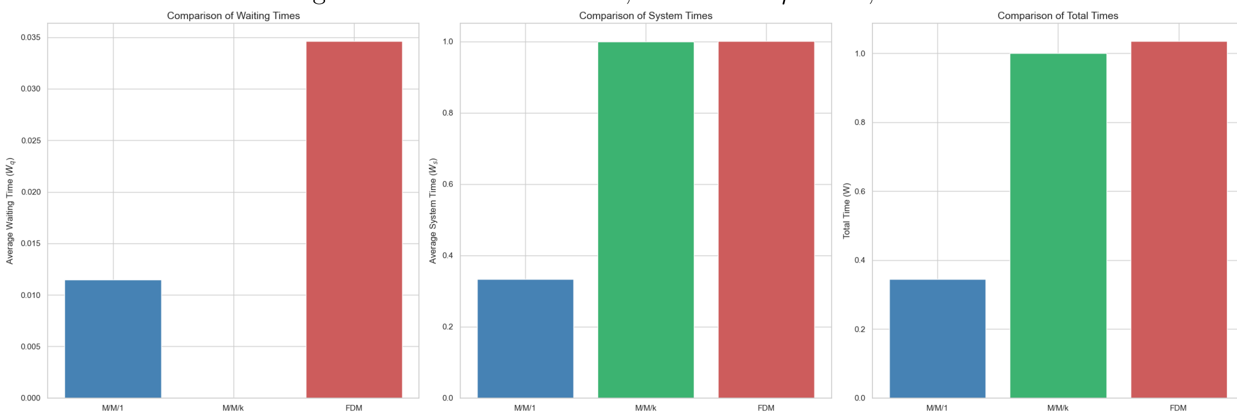
Settings: the arrival rate $\lambda = 0.1$, service rate $\mu = 0.2$, and $k = 3$.



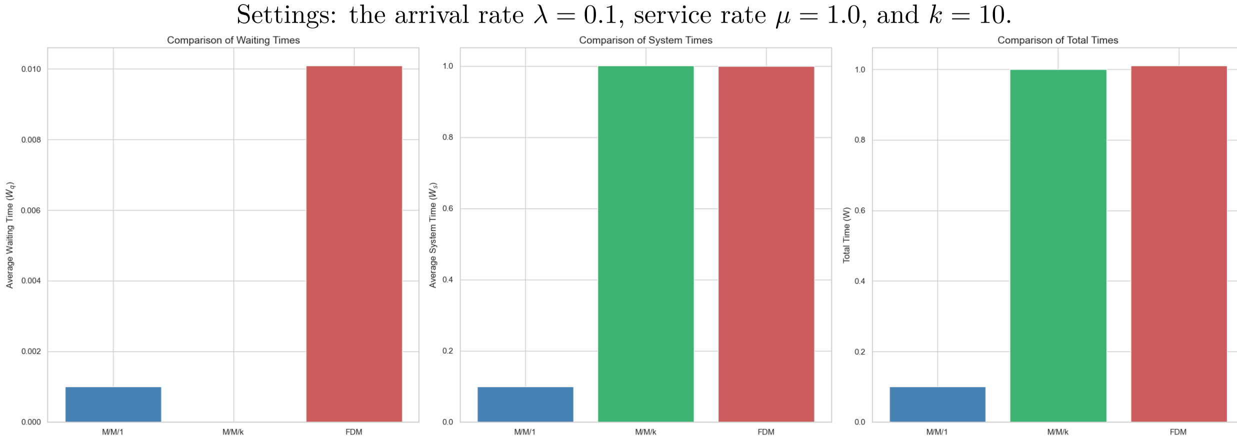
Settings: the arrival rate $\lambda = 0.1$, service rate $\mu = 0.2$, and $k = 10$.



Settings: the arrival rate $\lambda = 0.1$, service rate $\mu = 1.0$, and $k = 3$.



Settings: the arrival rate $\lambda = 0.1$, service rate $\mu = 1.0$, and $k = 10$.



From the results, we could see that the average waiting time W_q has M/M/k < M/M/1 < FDM, and the average survice time W_s has M/M/1 < M/M/k = FDM. The average delay W has M/M/1 < M/M/k < FDM. Which fits the theoretical results. The bigger k is, the advantage of M/M/1 on average time delay is more obvious. And the larger $\dfrac{\mu}{\lambda}$ is, the M/M/k is more similar to FDM, while M/M/1 seens to have a much better performance.

Problem 9

Consider a continuous-time Markov chain $\{X_t, t \geq 0\}$ with a finite discrete state space $\mathcal{X}$ and a transition rate matrix $Q = (q_{i,j}, i, j \in \mathcal{X})$. Let

$$p_{i,j}(t) = P\left(X_{t+s} = j \mid X_s = i\right) = P\left(X_t = j \mid X_0 = i\right)$$

Show the following results:
(a) $s, t \geq 0$

$$p_{i,j}(t+s) = \sum_{k \in \mathcal{X}} p_{i,k}(t) p_{k,j}(s)$$

(b) Kolmogorov's Backward Equation:

$$p'_{i,j}(t) = \sum_{k \neq i} q_{i,k} p_{k,j}(t) - v_i p_{i,j}(t)$$

where $v_i = -q_{i,i}$.
(c) Kolmogorov's Forward Equation:

$$p'_{i,j}(t) = \sum_{k \neq j} q_{k,j} p_{i,k}(t) - v_j p_{i,j}(t)$$

where $v_j = -q_{j,j}$.

Solution

(a) We can see that the Markov chain is time homogenous, and from the definition, we can get that:

$$\begin{aligned}
p_{i,j}(t+s) &= P(X_{t+s} = j | X_0 = i) \\
&= \sum_{k \in \mathcal{X}} P(X_{t+s} = j | X_t = k, X_0 = i) P(X_t = k | X_0 = i) \quad &\text{(LOTP)} \\
&= \sum_{k \in \mathcal{X}} P(X_{t+s} = j | X_t = k) P(X_t = k | X_0 = i) \quad &\text{(Markov property)} \\
&= \sum_{k \in \mathcal{X}} P(X_s = j | X_0 = k) P(X_t = k | X_0 = i) \quad &\text{(Time homogenous)} \\
&= \sum_{k \in \mathcal{X}} p_{k,j}(s) p_{i,k}(t) \\
&= \sum_{k \in \mathcal{X}} p_{i,k}(t) p_{k,j}(s)
\end{aligned}$$

So above all, we have proved that $\forall s, t \geq 0$:

$$p_{i,j}(t+s) = \sum_{k \in \mathcal{X}} p_{i,k}(t) p_{k,j}(s)$$

(b) We can use the definition of derivatives:

$$\begin{aligned}
p'_{i,j}(t) &= \lim_{\delta \to 0} \frac{p_{i,j}(t+\delta) - p_{i,j}(t)}{\delta} \\
&= \lim_{\delta \to 0} \frac{\sum\limits_{k \in \mathcal{X}} p_{i,k}(\delta) p_{k,j}(t) - p_{i,j}(t)}{\delta} \quad \text{(From (a))} \\
&= \lim_{\delta \to 0} \frac{\sum\limits_{k \neq i} p_{i,k}(\delta) p_{k,j}(t) - \left[p_{i,i}(\delta) - 1\right] p_{i,j}(t)}{\delta}
\end{aligned}$$

For the state i, the rate it transfer to other states is v_i, i.e. let T_i be the time it leaves state i, then we have $T_i \sim \text{Expo}(v_i)$. Since $\delta \to 0$, so

$$P(T_i \leq \delta) = 1 - e^{-v_i\delta} \approx 1 - (1 - v_i\delta) = v_i\delta$$

Let $P^e = \left(p^e_{i,j}\right)$ be the transition probability matrix of the embedded chain, then if $i \neq j$, we have $q_{i,j} = v_i \cdot p^e_{i,j}$. Thus:

$$p_{i,k}(\delta) = \begin{cases} P(T_i \leq \delta) \cdot p^e_{i,k} = (v_i\delta) \cdot p^e_{i,k} = q_{i,k} \cdot \delta & \text{if } k \neq i \\ P(T_i \geq \delta) = 1 - P(T_i \leq \delta) = 1 - v_i\delta & \text{if } k = i \end{cases}$$

So above all, we can get that

$$\begin{aligned} p'_{i,j}(t) &= \lim_{\delta\to 0} \frac{\sum\limits_{k\neq i} p_{i,k}(\delta)p_{k,j}(t) - \left[p_{i,i}(\delta) - 1\right] p_{i,j}(t)}{\delta} \\ &= \lim_{\delta\to 0} \frac{\sum\limits_{k\neq i} \left(q_{i,k} \cdot \delta\right) p_{k,j}(t) - \left[(1 - v_i\delta) - 1\right] p_{i,j}(t)}{\delta} \\ &= \sum_{k\neq i} q_{i,k}p_{k,j}(t) - v_i p_{i,j}(t) \end{aligned}$$

Which means that the Kolmogorov's Backward Equation holds:

$$p'_{i,j}(t) = \sum_{k\neq i} q_{i,k}p_{k,j}(t) - v_i p_{i,j}(t)$$

(c) Similarly with the backward equation, for the state k, the rate it transfer to other states is v_k, i.e. let T_k be the time it leaves state k, then we have $T_k \sim \text{Expo}(v_k)$. Since $\delta \to 0$, so

$$P(T_k \leq \delta) = 1 - e^{-v_k\delta} \approx 1 - (1 - v_k\delta) = v_k\delta$$

Let $P^e = \left(p^e_{i,j}\right)$ be the transition probability matrix of the embedded chain, then if $i \neq j$, we have $q_{i,j} = v_i \cdot p^e_{i,j}$. Thus:

$$p_{k,j}(\delta) = \begin{cases} P(T_k \leq \delta) \cdot p^e_{k,j} = (v_k\delta) \cdot p^e_{k,j} = q_{k,j} \cdot \delta & \text{if } k \neq j \\ P(T_k \geq \delta) = 1 - P(T_k \leq \delta) = 1 - v_k\delta = 1 - v_j\delta & \text{if } k = j \end{cases} \tag{1}$$

So above all, we can get that

$$\begin{aligned} p'_{i,j}(t) &= \lim_{\delta\to 0} \frac{p_{i,j}(t+\delta) - p_{i,j}(t)}{\delta} \\ &\overset{(a)}{=} \lim_{\delta\to 0} \frac{\sum\limits_{k\in\mathcal{X}} p_{i,k}(t)p_{k,j}(\delta) - p_{i,j}(\delta)}{\delta} \\ &= \lim_{\delta\to 0} \frac{\sum\limits_{k\neq j} p_{k,j}(\delta)p_{i,k}(t) - \left[p_{j,j}(\delta) - 1\right] p_{i,j}(t)}{\delta} \\ &\overset{(1)}{=} \lim_{\delta\to 0} \frac{\sum\limits_{k\neq j} \left(q_{k,j} \cdot \delta\right) p_{i,k}(t) - \left[(1 - v_j\delta) - 1\right] p_{i,j}(t)}{\delta} \\ &= \sum_{k\neq j} q_{k,j}p_{i,k}(t) - v_j p_{i,j}(t) \end{aligned}$$

Which means that the Kolmogorov's Forward Equation holds:

$$p'_{i,j}(t) = \sum_{k\neq j} q_{k,j}p_{i,k}(t) - v_j p_{i,j}(t)$$

Problem 10

Consider a continuous-time Markov chain $\{X_t, 0 \le t \le T\}$ with a finite discrete state space $\mathcal{X}$. It's transition rate matrix is **time-varying** and is denoted by $Q_t = \{Q_t(x, y), x, y \in \mathcal{X}\}$. The state distribution in time $t \in [0, T]$ is denoted by $\pi_t = \{\pi_t(x), x \in \mathcal{X}\}$. Now we consider the reverse time process of continuous-time Markov chain $\{X_t, 0 \le t \le T\}$, and denote it as $\left\{\overline{X}_t, 0 \le t \le T\right\}$. Then we have $\overline{X}_t = X_{T-t}$. Show the following results:
(a) The reverse time process $\left\{\overline{X}_t, 0 \le t \le T\right\}$ is a continuous-time Markov chain.
(b) Denote the **time-varying** transition rate matrix of such reverse time process as $R_t = \{R_t(x, y), x, y \in \mathcal{X}\}$, then

$$R_t(x, y) = \frac{\pi_t(y)}{\pi_t(x)} Q_t(y, x)$$

Solution

(a) $\forall t, s \in [0, T]$:

$$P\left(\overline{X}_{t+s} = j | \overline{X}_s = i, \overline{X}_u = x_u, 0 \le u \le s\right) = P\left(X_{T-(t+s)} = j | X_{T-s} = i, X_{t-u} = x_u, 0 \le u \le s\right)$$

Since $\{X_t, 0 \le t \le T\}$ is a CTMC, which means that given present state X_{T-s}, the past state $X_{T-(t+s)}$ and the future state X_{T-u} are conditional independent. Thus we have

$$\begin{aligned}
&P\left(X_{T-(t+s)} = j | X_{T-s} = i, X_{t-u} = x_u, 0 \le u \le s\right) \\
=& P\left(X_{T-(t+s)} = j | X_{T-s} = i\right) \\
=& P\left(\overline{X}_{t+s} = j | \overline{X}_s = i\right)
\end{aligned}$$

Which means that we have proved that $\forall t, s \in [0, T]$:

$$P\left(\overline{X}_{t+s} = j | \overline{X}_s = i, \overline{X}_u = x_u, 0 \le u \le s\right) = P\left(\overline{X}_{t+s} = j | \overline{X}_s = i\right)$$

So the reverse time process $\left\{\overline{X}_t, 0 \le t \le T\right\}$ is a CTMC.
(b) Let $p_{s,t}(x, y) = P(X_t = y | X_s = x)$. Then from the definition of conditional probability, we have:

$$\begin{aligned}
P(X_t = y, X_s = x) &= P(X_t = y | X_s = x) P(X_s = x) = \pi_s(x) p_{s,t}(x, y) \\
&= P(X_s = x | X_t = y) P(X_t = y) = \pi_t(y) p_{t,s}(y, x) \\
\Rightarrow \pi_s(x) p_{s,t}(x, y) &= \pi_t(y) p_{t,s}(y, x)
\end{aligned}$$

Similarly with the last problem(Problem 9), we can get that when $\delta \to 0$:

$$\begin{aligned}
\pi_{t+\delta}(x) &= \sum_y P(X_{t+\delta} = x | X_t = y) \pi_t(y) \\
&= \sum_{y \ne x} P(X_{t+\delta} = x | X_t = y) \pi_t(y) + P(X_{t+\delta} = x | X_t = x) \pi_t(x) \\
&\approx \sum_{y \ne x} \pi_t(y) \cdot Q_t(y, x)\delta + \pi_t(x) \cdot (1 + Q_t(x, x)\delta) \\
&= \pi_t(x) + \pi_t(x) Q_t(x, x)\delta + \sum_{y \ne x} \pi_t(y) \cdot Q(y, x)\delta
\end{aligned}$$

When $\delta \to 0$, all terms containing δ tend towards 0, therefore:

$$\lim_{\delta \to 0} \pi_{t+\delta}(x) = \pi_t(x) \tag{2}$$

Which means that we can regard $\pi_t(x)$ as continuous in time.

Let $s = t + \delta, \delta \to 0$, then we can get that:

$$\begin{aligned}
p_{t,s}(y,x) &= P(X_{t+\delta} = x | X_t = y) \\
&= \begin{cases} Q_t(y,x) \cdot \delta & \text{if } y \neq x \\ 1 + Q_t(y,x) \cdot \delta & \text{if } y = x \end{cases} \\
&= \mathbb{I}_{x=y} + Q_t(y,x) \cdot \delta
\end{aligned}$$

As for the reverse proceses, R_t is the transition rate for time t in the reverse process, which represent for the time $T - t$ in the original process. Thus,

$$\begin{aligned}
p_{s,t}(x,y) &= P(\overline{X}_{T-t} = y | \overline{X}_{T-(t+\delta)} = x) \\
&= \begin{cases} R_{t+\delta}(x,y) \cdot \delta & \text{if } y \neq x \\ 1 + R_{t+\delta}(x,y) \cdot \delta & \text{if } y = x \end{cases} \\
&= \mathbb{I}_{x=y} + R_{t+\delta}(x,y) \cdot \delta
\end{aligned}$$

Which means we have:

$$\pi_{t+\delta}(x) \left[\mathbb{I}_{x=y} + R_{t+\delta}(x,y) \cdot \delta \right] = \pi_t(y) \left[\mathbb{I}_{x=y} + Q_t(y,x) \cdot \delta \right]$$

And from equation 2, we know that $\pi_t(x)$ is continuous in time, also the transition rate matrix should be continuous in time, which means that as $\delta \to 0$:

$$\begin{gathered}
\pi_{t+\delta}(x) = \pi_t(x) \\
R_{t+\delta}(x,y) = R_t(x,y) \\
\Rightarrow \pi_t(x) \left[\mathbb{I}_{x=y} + R_t(x,y) \cdot \delta \right] = \pi_t(y) \left[\mathbb{I}_{x=y} + Q_t(y,x) \cdot \delta \right]
\end{gathered}$$

$$\Rightarrow \begin{cases} R_t(x,y) = Q_t(y,x) & \text{if } x = y \\ \pi_t(x) R_t(x,y) = \pi_t(y) Q_t(y,x) & \text{if } x \neq y \end{cases}$$

Thus merge all the cases, we can get that

$$R_t(x,y) = \frac{\pi_t(y)}{\pi_t(x)} Q_t(y,x)$$