Linear Algebra Tutorial 11

2023.12.19

homework

Find the standard matrix for the stated composition in $\mathbb{R}^3$.

1. (2 points) A rotation of $\frac{\pi}{6}$ about the positive x-axis, followed by a rotation of $\frac{\pi}{6}$ about the positive z-axis, followed by a contraction with factor $k = 1/4$.
2. (2 points) A reflection about the xy-plane, followed by a reflection about the xz-plane, followed by an orthogonal projection onto the yz-plane.
3. (2 points) A rotation of $\frac{3\pi}{2}$ about the positive x-axis, followed by a rotation of $\frac{\pi}{4}$ about the positive y-axis, followed by a rotation of π about the z-axis.

theorem

Theorem 4.51. 令$V = \mathbb{R}^n$，$W = \mathbb{R}^m$，$T : \mathbb{R}^n \to \mathbb{R}^m$为线性变换，$[T] \in M_{m\times n}$为$T$的标准矩阵。那么我们有

1. *$Ker(T) = Null([T])$;*
2. *$RAN(T) = Col([T])$;*
3. T是单射当且仅当$Ker(T) = Null([T]) = \{\mathbf{0}\}$，当且仅当$nullity([T]) = 0$;
4. T是满射当且仅当$Col([T]) = \mathbb{R}^m$，当且仅当$rank([T]) = m$;
5. 如果$m = n$，那么$T : \mathbb{R}^n \to \mathbb{R}^n$为双射当且仅当$[T]$可逆，当且仅当$T$是单射，当且仅当$T$是满射。

homework

Problem D(6 Points)

判断以下说法是否正确。如果正确写出证明，如果错误举出反例。

1. (2 points) 如果$n > m$，那么任何矩阵变换$T : \mathbb{R}^n \to \mathbb{R}^m$都不可能是一一映射(one-to-one)。
2. (2 points) 如果$n < m$，那么任何矩阵变换$T : \mathbb{R}^n \to \mathbb{R}^m$都不可能是满射(surjective, onto)，即不可能有$R(T) = \mathbb{R}^m$。
3. (2 points) 令$V = C^1(-\infty, \infty)$为所有连续可微函数的集合，令$W = C(-\infty, \infty)$为所有连续函数的集合，令$D : V \to W$为求导运算：$D(f(x)) = f'(x)$。那么D即是一一映射又是满射。

- 如果 $\dim(V) > \dim(W)$,那么T必然不可能是一一映射.
- 如果 $\dim(V) < \dim(W)$,那么T必然不可能是满射.

homework

E

2. (3 points) Let U, V, W be vector spaces and let $T : V \to W$, $S : W \to U$ be linear transformations. Prove that $\text{rank}(S \circ T) \leq \min(\text{rank}(S), \text{rank}(T))$, $\text{rank}(S \circ T) = \text{rank}(S)$ if T is surjective, $\text{rank}(S \circ T) = \text{rank}(T)$ if S is one-to-one.

- one-to-one / injective

  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

  > 只有一一映射才存在逆映射(反函数)
  > $T : V \to W$ is one-to-one $\Leftrightarrow T^{-1}$ exists
  > $T^{-1} : R(T) \to V$

- onto / surjective

  $T : V \to W$ is onto $\Leftrightarrow R(T) = W$

- bijective = injective + surjective

- 矩阵变换要从右往左看
  $T(\mathbf{x}) = (T_3(T_2(T_1(\mathbf{x})))) = (T_3 \circ T_2 \circ T_1)(\mathbf{x})$
  $[T] = [T_3][T_2][T_1]$
- $T : \mathbb{R}^n \to \mathbb{R}^m$
  $[T] \in M_{m\times n}$
  $\mathbf{y} = T(\mathbf{x}) = [T]\mathbf{x}, \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m$
  所以在使用rank-nullity theorem时, 注意是$rank([T]) + nullity([T]) = n$
- $T : V \to W$, W 为T的到达域(codomain),而不是值域(range)
  原因: T不一定是满射

  > T的值域(range)是W的子集

Linear transformation

$T : V \to W$ is linear transformation

- Kernel
  $Ker(T) : \{v \in V | T(v) = 0\}$
  $Ker(T) \subseteq V$ is a subspace of V
  $Ker(T) \Leftrightarrow Null([T])$

- range
  $RAN(T)/R(T) : \{w \in W | w = T(v), v \in V\}$
  $R(T) \subseteq W$ is a subspace of W
  $R(T) \Leftrightarrow Col([T])$

- $\dim(Ker(T)) + \dim(R(T)) = \dim(V)$

Isomorphism (同构)

- $T : V \to W$ is bijective linear transformation
  双射的线性变换
- 则称T是一个同构 (isomorphism)
- V 和 W 是同构的 (isomorphic)
- V 和 W 是同构的 $\Leftrightarrow \dim(V) = \dim(W)$

Isomorphism (同构)

prove $T: V \to W$ is isomorphism:

- T is linear transform
  $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
  $T(k\mathbf{u}) = kT(\mathbf{u})$
- T is bijective(双射)
  injective(单射/一一映射) + onto(满射)

$\forall V$, if $\dim(V) = n \Rightarrow V, \mathbb{R}^n$ are isomorphic

Isomorphism (同构)

$T : V \to W$ is a linear transformation, $\dim(V) = \dim(W) = n$

equivalent statements:

- T is injective
- T is onto
- T is isomorphism
- $Ker(T) = \{\mathbf{0}\}$
- $R(T) = W$

> $n = \dim(V) = rank(T) + nullity(T).$

Inverse Transformations(逆映射)

$T: V \to W$ is a linear transformation, and T is injective/one-to-one (单射/一一映射)
那么存在一个逆映射 $T^{-1}: R(T) \to V$ 使得

- $T(\mathbf{v}) = \mathbf{w} \Leftrightarrow T^{-1}(\mathbf{w}) = \mathbf{v}$
- $(T^{-1} \circ T)(\mathbf{v}) = \mathbf{v}, \mathbf{v} \in V$
- $(T \circ T^{-1})(\mathbf{w}) = \mathbf{w}, \mathbf{w} \in R(T)$
- T^{-1} is linear transformation, T^{-1} is onto and T^{-1} is one-to-one
  So $R(T), V$ are isomorphic

Inverse Transformations(逆映射)

$T_1 : U \to V$ and $T_2 : V \to W$ are injective

- $T_2 \circ T_1$ is injective
- $(T_2 \circ T_1)^{-1} : R(T_2 \circ T_1) \to U$ has inverse transformation
  $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$

Matrices for general linear transformations

(线性变换的矩阵表示)

$$\begin{tikzcd}
\boldsymbol{v} \in V \arrow[rr, "T"] \arrow[d, "f_B"] & & T(\boldsymbol{v}) \in W \arrow[d, "g_{B'}"] \\
([\boldsymbol{v}]_B \in \mathbb{R}^n) \arrow[rr, "T_A"] & & ([T(\boldsymbol{v})]_{B'} \in \mathbb{R}^m)
\end{tikzcd}$$

- f_B: 将$\mathbf{v}$转换成以B为基的坐标
- $g_{B'}$: 将$T(\mathbf{v})$转换成以B'为基的坐标
- T_A: $\mathbb{R}^n \to \mathbb{R}^m$的线性变换

$T_A(\mathbf{x}) = A\mathbf{x}$, s.t. $[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = [T]_{B',B}[\mathbf{v}]_B$

Matrices for general linear transformations

为T关于基底B与B'的矩阵表示

(The matrix for T relative to B and B')

$T_A \Rightarrow [T]_{B',B}$

$B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$, $B' = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}$

$[T_A(\mathbf{v})]_{B'} = [T]_{B',B}[\mathbf{v}]_B$

$[T]_{B',B} = [[T(\mathbf{v}_1)]_{B'} \ \cdots \ [T(\mathbf{v}_n)]_{B'}]$

Matrices for general linear transformations

Let V be the subspace in $F(-\infty, \infty)$ spanned by $B = \{\sin x, \cos x\}$. Let $T : V \to \mathbb{R}^2$ be defined by $T(f) = (f(0), f(\frac{\pi}{2}))$ for $f \in V$. Take $B' = \{(1,0), (0,1)\}$ as a basis of $\mathbb{R}^2$. Then the matrix for T relative to B and B' is

$$[T]_{B',B} = ?$$