Solutions to the Review Problems

A Reference Version

I. Multiple Choice Questions (By Yunfei Xu)

Policy for grading the Multiple choice questions:

For a multiple choice question, denote by C the set of all correct choices, and by A the set of your choices. If $A \nsubseteq C$, get zero points; If $A \subsetneq C$, get partial credits depending on the size of A.

- If $|C| = 4$, get one point for each correct choice when $|A| < 4$, and get full points when $|A| = 4$;
- If $|C| = 3$, get two points for each correct choice when $|A| < 3$, and get full points when $|A| = 3$;
- If $|C| = 2$, get three points when $|A| = 1$, and get full points when $|A| = 2$.

The unlisted remaining case for $|C| = 1$ should be self evident.

a). (5 *points*) Which of the following sets are subspaces of the given vector space? ( )

(A) $\{(x_1, x_2, x_3) \subseteq \mathbb{R}^3 : x_1 + 5x_2 + 3x_3 = 0, x_2 - 2x_3 = 0\} \subseteq \mathbb{R}^3$.

(B) $\{(x_1, x_2, x_3) \subseteq \mathbb{R}^3 : x_1 > x_2 > x_3\} \subseteq \mathbb{R}^3$.

(C) $\{(x^2, x, 1) \subseteq \mathbb{R}^3 : x \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

(D) $\{A \in \mathbb{M}_{3\times 3} : A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{M}_{3\times 3}$, where $\mathbf{x} = [1, 0, 1]^T$.

I. a) AD

(A) 1' for $\forall\ (x_1, x_2, x_3)\ (x_1', x_2', x_3') \subseteq V$

we have $\begin{cases} (x_1+x_1') + 5(x_2+x_2') + 3(x_3+x_3') = 0 \\ (x_2+x_2') - 2(x_3+x_3') = 0 \end{cases}$

$\therefore (x_1+x_1', x_2+x_2', x_3+x_3') \subseteq V$

2' for $\forall\ (x_1, x_2, x_3) \subseteq V$ and $k \subseteq R$

we have $\begin{cases} kx_1 + 5kx_2 + 3kx_3 = 0 \\ kx_2 - 2kx_3 = 0 \end{cases}$

$\therefore k(x_1, x_2, x_3) \subseteq V$

$\therefore V$ is a subspace of R^3

(B) $(0,0,0) \notin V$

$\therefore V$ is not a subspace of R^3

(C) $(0,0,0) \notin V$

$\therefore V$ is not a subspace of R^3

(D) 1' for $\forall\ A_1, A_2 \subseteq V$

we have $(A_1+A_2)x = A_1x + A_2x = 0$

2' for $\forall\ A_1 \subseteq V$ and $k \subseteq R$

we have $(kA_1)x = k(A_1x) = 0$

$\therefore V$ is a subspace of $M_{3\times 3}$

b). (5 *points*) Let $A \in \mathbb{M}_{n\times n}$ and $B \in \mathbb{M}_{n\times n}$. Determine which of the following statements are true. ( )

(A) If $\det(A-B)=0$, then $A=B$

(B) If $A^2=B^2$, then $A=B$ or $A=-B$

(C) If $\det(A-B)=1$ and there is an $\mathbf{x}\in\mathbb{R}^n$ such that $A\mathbf{x}=B\mathbf{x}$, then $\mathbf{x}=\mathbf{0}$

(D) If $\det(A-B)=1$, then $dim\,(row\,(A-B))=n$

b) CD

(A) $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $B=\begin{bmatrix}1&0\\0&0\end{bmatrix}$

$\det(A-B)=\det(\begin{bmatrix}0&0\\0&1\end{bmatrix})=0$

but $A\neq B$

(B) $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $B=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$A^2=\begin{bmatrix}0&0\\0&0\end{bmatrix}=B^2$

but $A\neq B$ and $A\neq -B$

(C) $Ax=Bx \Rightarrow (A-B)x=0$

$\because \det(A-B)=1\neq 0$

$\therefore$ the row vectors of $A-B$ are linearly independent

$\therefore x=0$

(D) $\because \det(A-B)=1\neq 0$

$\therefore$ the row vectors of $A-B$ are linearly independent

$\therefore \dim(row(A-B))=rank(A-B)=n$

c). (5 *points*) Let $U, W \subseteq V$ be 4-dimensional subspaces of a 6-dimensional vector space V, which of the following can not be the possible dimension of $U\cap W$? ( )

(A) 4 (B) 3 (C) 2 (D) 1

c) D

we have $\dim(U+W)=\dim(U)+\dim(W)-\dim(U\cap W)$,

which $\dim(U)=\dim(W)=4$ and $4\leq\dim(U+W)\leq 6$

$\therefore 2\leq \dim(U\cap W)\leq 4$

II. Fill in the blanks (By Yunfei Xu)

a.) (5 *points*) Suppose that $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (0,1,-2a)$, $\mathbf{v}_3 = (a,0,1)$ form a basis for $\mathbb{R}^3$.

If $(-2,-7,-12)$ has coordinates $(3a, -7-3a, a)$ relative to this basis, then a=________.

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2. a) $\begin{bmatrix} -2 \\ -7 \\ -12 \end{bmatrix} = 3a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - (7+3a)\begin{bmatrix} 0 \\ 1 \\ -2a \end{bmatrix} + a\begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a^2+3a \\ -7 \\ 6a^2+18a \end{bmatrix}$

$\Rightarrow a = -2$ or $a = -1$, but when $a=-1$, v_1, v_2, v_3 is not a basis. so $a = -2$

b). (5 *points*) Suppose that A is a 3×3 matrix with $\det(A) = -3$, then $\det(-2\,adj(A)) =$

________.

b) $\because \det(A) = -3 \neq 0 \quad \therefore A$ is invertible and $\det(A^{-1}) = -\frac{1}{3}$

$$\begin{aligned} \det(-2\,adj(A)) &= (-2)^3 \det(adj(A)) \\ &= -8 \det(\det(A)A^{-1}) \\ &= -8 \det^3(A) \cdot \det(A^{-1}) \\ &= -8 \times (-3)^3 \times (-\tfrac{1}{3}) \\ &= -72 \end{aligned}$$

c.) (5 *points*) Let $A = [a_{ij}]$ be a square matrix of size n with all its entries being zero except the $(i, i+1)$-th entries $a_{i,i+1}$ which equals 1 for $i = 1, \cdots, n-1$. Then $r(A^{n-1}) =$

________.

c) $A = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 1 & & \\ & 0 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix}$

...

$\Rightarrow A^{n-1} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ & 0 & \ddots & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{bmatrix}$

$\therefore r(A^{n-1}) = 1$

III. (By Yiyang Gong)

3. (10 *points*) Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -6 & 8 \\ 4 & 5 \\ 2 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$. Find a matrix X such that $A(X - B) = C$.

Sol. Noticed that $\det(A) = -1$, hence A is invertible.

Instead of computing A^{-1} directly, I choose to compute $\operatorname{adj}(A)$ and use fomula $A \cdot \operatorname{adj}(A) = \det(A) I$ such that $A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$

$$\operatorname{adj}(A) = \begin{pmatrix} \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 4 & 1 & -7 \\ 1 & 0 & -1 \\ -3 & -1 & 5 \end{pmatrix}$$ thus

$$A^{-1} = \begin{pmatrix} -4 & -1 & 7 \\ -1 & 0 & 1 \\ 3 & 1 & -5 \end{pmatrix}$$

$$X = A^{-1}C + B = \begin{pmatrix} -4 & -1 & 7 \\ -1 & 0 & 1 \\ 3 & 1 & -5 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} -6 & 8 \\ 4 & 5 \\ 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 15 & -7 \\ 2 & -2 \\ -10 & 6 \end{pmatrix} + \begin{pmatrix} -6 & 8 \\ 4 & 5 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 1 \\ 6 & 7 \\ -8 & 8 \end{pmatrix}$$

IV. (By Yiyang Gong)

4. (10 *points*) Let A be a square matrix of size n with cofactor matrix $C = [C_{ij}]_{1\le i,j\le n}$. Suppose that the sum of the entries of A in the ith row is equal to i and suppose that the determinant of A is 1. Compute the value of $C_{11} + 2C_{21} + 3C_{31} + \cdots + nC_{n1}$

Sol. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$, when we compute $\det A$,

we add i-th column to the first column $(2\le i\le n)$ in order.

which doesn't change the determinant. Denote by A', where

$$A' = \begin{bmatrix} \sum_{i=1}^n a_{1i} & a_{12} & a_{13} & \cdots & a_{1n} \\ \sum_{i=1}^n a_{2i} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^n a_{ni} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ 2 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

then $\det(A') = 1C'_{11} + 2C'_{21} + 3C'_{31} + \cdots + nC'_{n1}$

Notice that ① we only change the first column from A to A'

hence $C'_{j1} = C_{j1}$, $1\le j\le n$.

② $\det(A') = \det(A) = 1$.

Therefore, $1C_{11} + 2C_{21} + \cdots + nC_{n1} = 1C'_{11} + 2C'_{21} + \cdots + nC'_{n1} = \det(A')$

$= \det(A) = 1$.

V. (By Yiyang Gong)

5. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

a) (7 *points*) Compute a basis of W and the dimension of W.

b) (8 *points*) Let U be the set of all vectors in $\mathbb{R}^4$ that are orthogonal to all vectors of W (the orthogonal complement of W). Compute a basis and the dimension of U.

Sol. We write the five vectors as row vectors of a matrix A:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & 6 & -4 & 2 \\ -6 & 1 & 1 & 2 \\ 2 & -1 & 1 & 0 \\ 2 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) the RREF of A is as right above

then since elementary row operations don't change row space

A basis of W is $\{(1,0,0,0), (0,1,0,1)\ (0,0,1,1)\}$

the dimension of W is 3

b) We use the fact that $\text{row}(A)^{\perp} = \text{null}(A)$. by RREF we can directly solve $A\vec{x} = \vec{0}$ by $\begin{cases} x_1 = 0 \\ x_2 = -t \\ x_3 = -t \\ x_4 = t \end{cases}$, hence

the null space of $A = \text{span}\{(0,-1,-1,1)\}$, where $(0,-1,-1,1)$ is a basis with dimension 1.

VI. (By Yiyang Gong)

6. Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are three linearly independent vectors in $\mathbb{R}^3$. Let $\mathcal{S}$ and $\mathcal{S}'$ be two sets of vectors in $\mathbb{R}^3$ that are given respectively by

$$\mathcal{S} = \{\mathbf{v}_1 + \mathbf{v}_2,\ \mathbf{v}_2 + \mathbf{v}_3,\ \mathbf{v}_1 + \mathbf{v}_3\}; \quad \mathcal{S}' = \{2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3,\ \mathbf{v}_1 + \mathbf{v}_3,\ 2\mathbf{v}_1 + 2\mathbf{v}_2 + 2\mathbf{v}_3\}$$

a) (5 *points*) Verify that both $\mathcal{S}$ and $\mathcal{S}'$ are basis of $\mathbb{R}^3$.

b) (5 *points*) Find the transition matrix from $\mathcal{S}'$ to $\mathcal{S}$.

c) (5 *points*) Suppose that $\mathbf{u} \in \mathbb{R}^3$ has coordinates $[\mathbf{u}]_{\mathcal{S}} = [1, 2, 1]^T$ relative to $\mathcal{S}$, find its coordinates $[\mathbf{u}]_{\mathcal{S}'}$ relative to $\mathcal{S}'$.

Sol. Write S and S' as coordinates.

$S = \{(1,1,0), (0,1,1), (1,0,1)\}$. $S' = \{(2,1,1), (1,0,1), (2,2,2)\}$

a) $\det\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) = 1 \neq 0$ $\quad \det\left(\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}\right) = -2 \neq 0$.

then vectors are linearly independent separately in S and S'.

Since they are vectors of $\mathbb{R}^3$ and length of S, S' are both 3.

Both of them are bases.

b) $P_{S \leftarrow S'}$, let S' be the linear combination of S, where

$(2,1,1) = 1(1,1,0) + 1(1,0,1)$; $(1,0,1) = 1(1,0,1)$

$(2,2,2) = 1(1,1,0) + 1(0,1,1) + 1(1,0,1)$

so $P_{S \leftarrow S'} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

c) $[u]_{S'} = P_{S' \leftarrow S}[u]_S$ so we need to compute $P_{S \to S'}$

Notice that $(1,1,0) = 1(2,1,1) - 1(1,0,1)$

$(0,1,1) = -1(2,1,1) + 1(2,2,2)$

$(1,0,1) = 1(1,0,1)$

then $P_{S' \leftarrow S} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(or you can directly compute $(P_{S \leftarrow S'})^{-1} = P_{S' \leftarrow S}$).

Then $[u]_{S'} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

VII. (By Siyuan Huang)

7. (10 *points*) Let $\mathbf{v}_1 = (1, 0, 2, 3)$, $\mathbf{v}_2 = (-3, -2, 0, 1)$, $\mathbf{v}_3 = (0, 1, -3, 2)$, $\mathbf{u} = (1, 0, 1, 0)$, $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$ be vectors in $\mathbb{R}^4$. Suppose that the orthogonal projections of $\mathbf{w}$ along $\mathbf{v}_i$ are the same as that of $\mathbf{u}$ along $\mathbf{v}_i$ for $i = 1, 2, 3$, that is

$$proj_{\mathbf{v}_i}\mathbf{u} = proj_{\mathbf{v}_i}\mathbf{w} \quad i = 1, 2, 3,$$

find the length of $\mathbf{w}$.

Solution: v_1,v_2,v_3 is orthogonal to each other, since

$$\begin{aligned} v_1 \cdot v_2 &= 1 * (-3) + 0 * (-2) + 2 * 0 + 3 * 1 = 0 \\ v_1 \cdot v_3 &= 1 * 0 + 0 * 1 + 2 * (-3) + 3 * 2 = 0 \\ v_2 \cdot v_3 &= (-3) * 0 + (-2) * 1 + 0 * (-3) + 1 * 2 = 0 \end{aligned}$$

We can solve k_i by calculating $proj_{v_i} w$ and making use of the orthogonality between v_i

$$proj_{v_i} w = \frac{v_i \cdot w}{v_i \cdot v_i} v_i = \frac{v_i \cdot \sum_{j=1}^{3} k_j v_j}{v_i \cdot v_i} v_i = \frac{k_i (v_i \cdot v_i)}{v_i \cdot v_i} = k_i v_i$$

We can calculate $proj_{v_i} u$ directly,

$$\begin{aligned} proj_{v_1} u &= \frac{v_1 \cdot u}{v_1 \cdot v_1} v_1 = \frac{3}{14} v_1 \\ proj_{v_2} u &= \frac{v_2 \cdot u}{v_2 \cdot v_2} v_2 = -\frac{3}{14} v_2 \\ proj_{v_3} u &= \frac{v_3 \cdot u}{v_3 \cdot v_3} v_3 = -\frac{3}{14} v_3 \end{aligned}$$

Since $proj_{v_i} u = proj_{v_i} w$, by comparing the coefficients we know that $k_1 = \frac{3}{14}$, $k_2 = -\frac{3}{14}$, $k_3 = -\frac{3}{14}$.

And the length of w is,

$$\|w\| = (w \cdot w)^{\frac{1}{2}} = (k_1^2 \|v_1\|^2 + k_2^2 \|v_2\|^2 + k_3^2 \|v_3\|^2)^{\frac{1}{2}} = \frac{3\sqrt{42}}{14}$$

VIII. (By Siyuan Huang)

8. (10 *points*) Let $\mathbf{u}$ and $\mathbf{v}$ be two linearly independent vectors in $\mathbb{R}^3$. Viewing $\mathbf{u}$, $\mathbf{v}$ as matrices of size 3×1, and compute the rank of A and B that are given by

$$A = \mathbf{u}\mathbf{v}^T \qquad B = \begin{bmatrix} \mathbf{u}^T\mathbf{u} & \mathbf{u}^T\mathbf{v} \\ \mathbf{v}^T\mathbf{u} & \mathbf{v}^T\mathbf{v} \end{bmatrix}$$

Solution:

(1) rank(A)=1

$$\begin{aligned} A = uv^T &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \\ &= \begin{bmatrix} v_1 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} & v_2 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} & v_3 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \end{bmatrix} \end{aligned}$$

We can see that each column of A is a multiple of u, thus rank(A)=1

(2) rank(B)=2

Method 1:
To prove rank(B)=2, we just need to prove $det(B) \neq 0$

Suppose that detB $= 0$, the following equation has nontrivial solutions.

$$\begin{bmatrix} u^T u & u^T v \\ v^T u & v^T v \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Rewrite the equation in a more explicit way.

$$\begin{bmatrix} u^T u & u^T v \\ v^T u & v^T v \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 u^T u + k_2 u^T v \\ k_1 v^T u + k_2 v^T v \end{bmatrix} = \begin{bmatrix} u^T(k_1 u + k_2 v) \\ v^T(k_1 u + k_2 v) \end{bmatrix} = \begin{bmatrix} u \cdot (k_1 u + k_2 v) \\ v \cdot (k_1 u + k_2 v) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since u and v are linearly independent vectors, $k_1 u + k_2 v \neq 0$. However,

$$\|k_1 u + k_2 v\|^2 = (k_1 u + k_2 v) \cdot (k_1 u + k_2 v) = k_1 u \cdot (k_1 u + k_2 v) + k_2 u \cdot (k_1 u + k_2 v) = k_1 * 0 + k_2 * 0 = 0$$

That's a contradiction. So it must be that $detB \neq 0$ and rank(B)=2.

Method 2:

$$detB = \begin{vmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{vmatrix} = (u \cdot u)(v \cdot v) - (u \cdot v)^2 \geq 0$$

That's Cauchy-Schwartz inequality. Remind that the condition for equality is that u and v are colinear.
Since u and v are linear independent vectors here, there is always that $detB > 0$.