Linear Algebra Tutorial 12

2023.12.26

homework

Problem C(6 Points), 2021-2022年线性代数期末考试题

Let V be an n–dimensional vector space, and $T : V \to V$ a linear operator on V. Suppose that there is a $\boldsymbol{v} \in V$ such that $T^{n-1}(\boldsymbol{v}) \neq \mathbf{0}$ and $T^n(\boldsymbol{v}) = \mathbf{0}$. Here we use T^k $(k \geq 1)$ to denote the composition of T for k times, i.e.,

$$T^1 = T, \quad T^2 = T \circ T, \quad T^3 = T \circ T \circ T, \quad \ldots, \quad T^k = \underbrace{T \circ T \circ \ldots \circ T}_{k \text{ times}}.$$

1. (4 points) Prove that $B = \{\boldsymbol{v}, T(\boldsymbol{v}), T^2(\boldsymbol{v}), \ldots, T^{n-1}(\boldsymbol{v})\}$ is a basis of V.
2. (2 points) Find the matrix for T relative to B, i.e., $[T]_{B,B}$.

1. (3 points) Which of the following statements are true?

- A. Let V be an n–dimensional vector space, B is a basis of V, then the coordinate vector mapping $f_B : V \to \mathbb{R}^n$, $f_B(\boldsymbol{v}) = [\boldsymbol{v}]_B$ is an isomorphism.
- B. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $\ker(T) \neq \{\mathbf{0}\}$ and $\mathrm{RAN}(T) \neq \{\mathbf{0}\}$. Then there exists a non–zero vector $\boldsymbol{x} \in \mathbb{R}^2, \boldsymbol{x} \neq \mathbf{0}$ such that $\boldsymbol{x} \in \ker(T) \cap \mathrm{RAN}(T)$. (Here, $\ker(T)$ is the kernel of T, $\mathrm{RAN}(T)$ is the range of T.)
- C. Let V, W be vector space and $T : V \to W$ is an isomorphism. Let $B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a basis of V, then $B' = \{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)\}$ is a basis of W.
- D. Let V be an n–dimensional vector space, and $T : V \to V$ be the identity operator, i.e., $T(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in V$. Then for any basis B of V, we have $[T]_{B,B} = I_n$, where $I_n \in M_{n\times n}$ is the identity matrix.

2. (3 points) Which of the following statements are true?

- A. Let $T : V \to W$ be a linear transformation, and $W' \subset W$ is a subspace of W. Then the set

$$V' = \{\boldsymbol{v} \in V : T(\boldsymbol{v}) \in W'\}$$

  is always a subspace of V.

- B. If $T : U \to V$ is a surjective (满射) linear transformation, $S : V \to W$ is also a surjective linear transformation, then $S \circ T$ is always surjective.
- C. Let $A \in M_{n\times n}$. Consider the following partitioned matrix $C \in M_{2n\times 2n}$

$$C = \begin{bmatrix} A & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & A^\top A \end{bmatrix},$$

  here $\mathbf{0}_{n\times n}$ denotes the zero matrix in $M_{n\times n}$. Then we always have $\text{rank}(C) = 2\text{rank}(A)$.
- D. The function

$$T : M_{n\times n} \to \mathbb{R}, \quad T(A) = \text{tr}(A)$$

  is a surjective linear transformation. Here $\text{tr}(A)$ denotes the trace of A.

Bonus: 不计入分数

假设V为一个向量空间，令$W = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \subset V$为$V$的一个子空间，这里$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$线性无关。对于一个给定的$\boldsymbol{v} \in V$，考虑子空间

$$U = \mathrm{span}\{\boldsymbol{v}_1 + \boldsymbol{v}, \boldsymbol{v}_2 + \boldsymbol{v}, \boldsymbol{v}_3 + \boldsymbol{v}\}.$$

由期中考试不定项选择题的第三小题，我们知道$\dim(U)$可以等于2或者3。现在请对此提供一个严格的数学证明，特别地，证明$\dim(U)$不可能小于$3 - 1 = 2$。

线性变换的矩阵表示

Matrices for general linear transformations

$$\begin{tikzcd} \boldsymbol{v} \in V \arrow[r, "T"] \arrow[d, "f_B"] & T(\boldsymbol{v}) \in W \arrow[d, "g_{B'}"] \\ ([\boldsymbol{v}]_B \in \mathbb{R}^n) \arrow[r, "T_A"] & ([T(\boldsymbol{v})]_{B'} \in \mathbb{R}^m) \end{tikzcd}$$

为T关于基底B与B'的矩阵表示

(The matrix for T relative to B and B')

$$[T]_{B',B} = [[T(\mathbf{v}_1)]_{B'} \ \cdots \ [T(\mathbf{v}_n)]_{B'}]$$
$$[T(\mathbf{v})]_{B'} = [T]_{B',B}[\mathbf{v}]_B$$

Matrices for general linear transformations

为T关于基底B与B'的矩阵表示
(The matrix for T relative to B and B')

$T_A \Rightarrow [T]_{B',B}$
$B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$, $B' = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}$

$[T_A(\mathbf{v})]_{B'} = [T]_{B',B}[\mathbf{v}]_B$

$[T]_{B',B} = [[T(\mathbf{v}_1)]_{B'} \ \cdots \ [T(\mathbf{v}_n)]_{B'}]$

- $rank(T) = rank([T]_{B\prime,B}), nullity(T) = nullity([T]_{B\prime,B})$
- $[T^{-1}]_{B,B'} = ([T]_{B',B})^{-1}$

Matrices for general linear transformations

Let V be the subspace in $F(-\infty, \infty)$ spanned by $B = \{\sin x, \cos x\}$. Let $T : V \to \mathbb{R}^2$ be defined by $T(f) = (f(0), f(\frac{\pi}{2}))$ for $f \in V$. Take $B' = \{(1,0), (0,1)\}$ as a basis of $\mathbb{R}^2$. Then the matrix for T relative to B and B' is

$$[T]_{B',B} = ?$$

composition of linear transformations

$$\begin{tikzcd}
\boldsymbol{u} \in U \arrow[r, "T_1"] \arrow[d, "f_B"'] & T_1(\boldsymbol{u}) \in V \arrow[r, "T_2"] \arrow[d, "h_{\tilde{B}}"'] & T_2(T_1(\boldsymbol{u})) \in W \arrow[d, "g_{B'}"'] \\
([\boldsymbol{u}]_B \in \mathbb{R}^n) \arrow[r, "T_A"] & ([T_1(\boldsymbol{u})]_{\tilde{B}} = A[\boldsymbol{u}]_B \in \mathbb{R}^k) \arrow[r, "T_C"] & ([T_2(T_1(\boldsymbol{u}))]_{B'} = (CA)[\boldsymbol{u}]_B \in \mathbb{R}^m)
\end{tikzcd}$$

$$T_1 : U \to V, T_2 : V \to W$$
$$[T_2 \circ T_1]_{B',B} = [T_2]_{B',B''}[T_1]_{B'',B}$$

> cancel the nearest basis(B'')

Theorem 5.16. 设V，W为有限维向量空间，$T : V \to W$为一个线性变换。

1. 如果$S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\} \subset V$为$V$里的一个线性无关集合，且$T : V \to W$为**一一映射**，那么$S' = \{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_r)\} \subset W$也是一个线性无关集合。
2. 如果$B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\} \subset V$为$V$里的一组基底，且$T : V \to W$为**同构**，那么$B' = \{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)\} \subset W$是$W$的一组基底。

inverse of linear transformations

$$\begin{tikzcd}
\boldsymbol{v} \in V \arrow[r, "T"] \arrow[d, "f_B"] & T(\boldsymbol{v}) \in W \\
([\boldsymbol{v}]_B \in \mathbb{R}^n) \arrow[r, "T_A"'] & ([T(\boldsymbol{v})]_{B'} = A[\boldsymbol{v}]_B \in \mathbb{R}^m) \arrow[u, "g_{B'}^{-1}"]
\end{tikzcd}$$

$$[T^{-1}]_{B,B'} = ([T]_{B',B})^{-1}$$

property

Theorem 5.17. 令V为n维向量空间，令W为m维向量空间，$T: V \to V$为V上的线性算子，且令B为V的一组基底，B'为W的一组基底。令$[T]_{B',B} \in M_{m\times n}$为$T$关于基底$B$与$B'$的矩阵表示。那么我们有

1. $rank(T) = rank([T]_{B',B})$，$nullity(T) = nullity([T]_{B',B})$。
2. T是一个一一映射当且仅当$nullity([T]_{B',B}) = 0$。
3. T是一个满射当且仅当$rank([T]_{B',B}) = m = \dim(W)$。
4. 若$n = m$，那么T是一个同构当且仅当$[T]_{B',B} \in M_{n\times n}$可逆。此时$T^{-1}: W \to V$关于基底$B'$与$B$的矩阵表示为

$$[T^{-1}]_{B,B'} = ([T]_{B',B})^{-1}.$$

T是一个一一映射 $\Leftrightarrow [T]_{B',B}$ 可逆

theorem

Theorem 5.18. 令V为n维向量空间，$T : V \to V$为V上的线性算子，且令B为V上的一组基底。那么以下说法等价：

1. T是一个一一映射。
2. $[T]_{B,B}$可逆。

此时T^{-1}关于基底B与B的矩阵表示为

$$[T^{-1}]_{B,B} = ([T]_{B,B})^{-1}.$$

- $[T]_{B,B} = [T]_B$
- $[T]_{B',B'} = [T]_{B'}$

相似矩阵 Similarity

$T: V \to V$是一个线性变换,
B是V的一个基底, $A = [T]_{B,B}$: T关于B的矩阵表示
V的另一组基底B', $A' = [T]_{B',B'}$: T关于B'的矩阵表示

- $[T]_{B',B'} = (P_{B \leftarrow B'})^{-1}[T]_{B,B}P_{B \leftarrow B'}$

- $[T]_{B',B'} = (P_{B\leftarrow B'})^{-1}[T]_{B,B}P_{B\leftarrow B'}$

explaination

$$[T]_{B',B'} = [I \circ T \circ I]_{B',B'} = [I]_{B',B}[T]_{B,B}[I]_{B,B'}.$$

这个过程如下图所示：

$$\begin{tikzcd}
\boldsymbol{v} \in V \arrow[r, "I"] \arrow[d, "f_{B'}"'] & \boldsymbol{v} \in V \arrow[r, "T"] \arrow[d, "f_B"'] & T(\boldsymbol{v}) \in V \arrow[r, "I"] \arrow[d, "f_B"'] & T(\boldsymbol{v}) \in V \arrow[d, "f_{B'}"'] \\
([\boldsymbol{v}]_{B'}) \arrow[r, "T_P"] & ([I]_{B,B'}[\boldsymbol{v}]_{B'} = [\boldsymbol{v}]_B) \arrow[r, "T_A"] & (A[\boldsymbol{v}]_B = [T(\boldsymbol{v})]_B) \arrow[r, "T_Q"] & ([I]_{B',B}[T(\boldsymbol{v})]_B = [T(\boldsymbol{v})]_{B'})
\end{tikzcd}$$

> 1. 以B'为基的坐标转成以B为基的坐标
> 2. 以B为基的坐标进行线性变换
> 3. 以B为基的坐标转成以B'为基的坐标

Similarity

$B = P^{-1}AP$, then A and B are similar, A is similar to B

- 自反性 A is similar to A $(P = I)$
- 对称性 A is similar to $B \Leftrightarrow B$ is similar to A
- 传递性 A is similar to B, B is similar to C, then A is similar to C
- **矩阵多项式** if A is similar to B, then $p(A)$ is similar to $p(B)$
  $p(A) = a_0 + a_1 A + \cdots + a_n A^n$

> $[T]_{B',B'} = (P_{B \leftarrow B'})^{-1}[T]_{B,B}P_{B' \leftarrow B}$
> 所以说 $[T]_{B,B}$ 和 $[T]_{B',B'}$ 是相似的

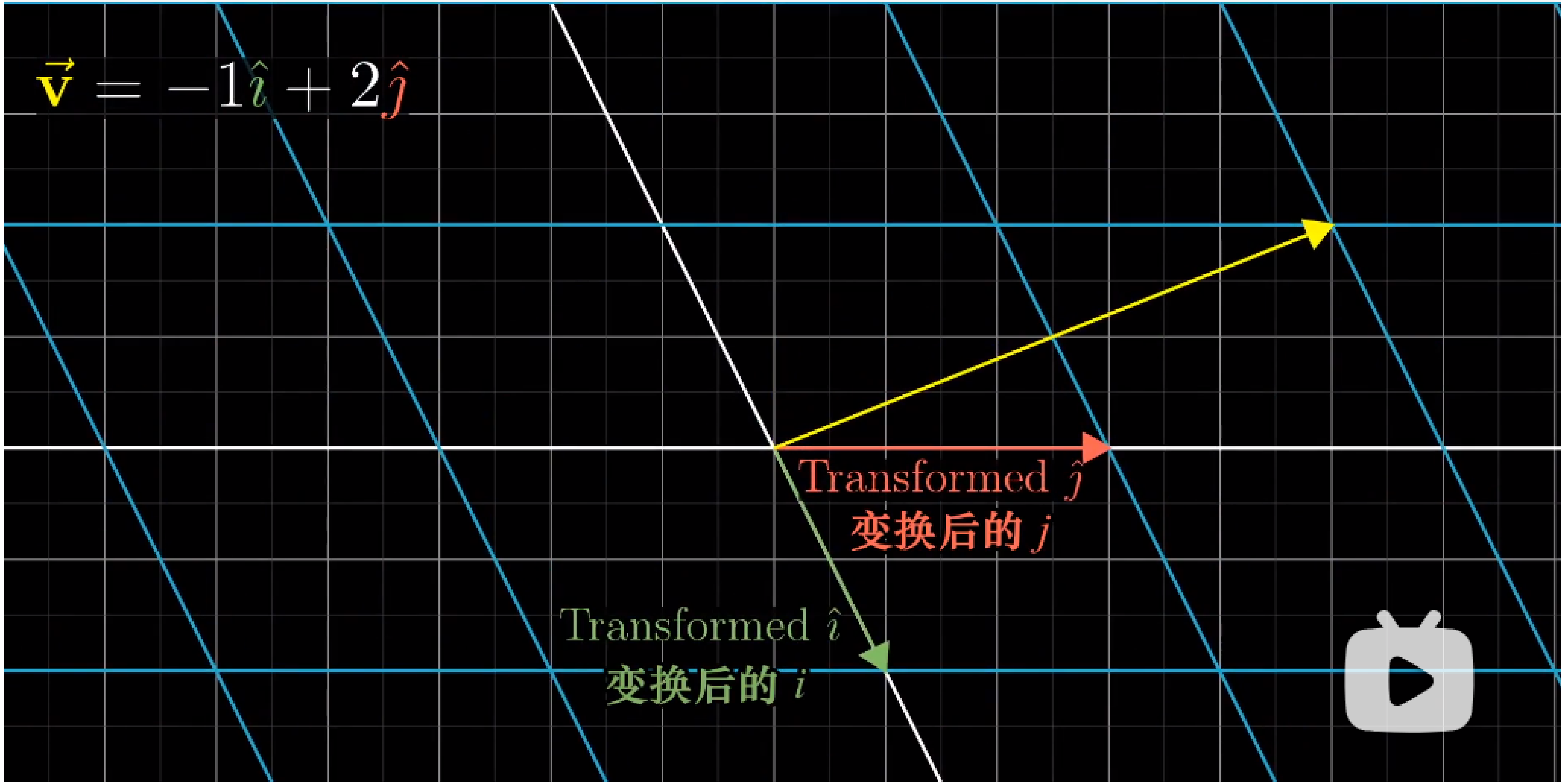
相似不变量 similarity invariants

$B = P^{-1}AP$

- $|A| = |B|$
- $tr(A) = tr(B)$
  (可以通过$tr(A) = \lambda_1 + \cdots + \lambda_n$证明)
- $rank(A) = rank(B)$
  $B = P^{-1}AP \Rightarrow rank(B) \leq rank(A)$
  $A = PBP^{-1} \Rightarrow rank(A) \leq rank(B)$
- $nullity(A) = nullity(B)$

特征值(eigenvalue)与特征向量(eigenvector)

> chapter 5 in the textbook



特征值(eigenvalue)

$$A\mathbf{x} = \lambda\mathbf{x}$$
$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$\mathbf{x} \neq \mathbf{0}$$
$$\left|\lambda I - A\right| = 0$$

- $p(\lambda) = \left|\lambda I - A\right|$: eigen polynomial 特征多项式
- $p(\lambda) = 0$: characteristic equation 特征方程

eigen polynomial 特征多项式

$p(\lambda) = \left|\lambda I - A\right|$

$p(\lambda)$为关于λ,最高次项为n的多项式

$p(\lambda) = (\lambda - \lambda_1)\cdots(\lambda - \lambda_n) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$

- let $\lambda = 0$, then $p(0) = \left|-A\right| = (-1)^n\lambda_1\cdots\lambda_n = a_0 \Rightarrow a_0 = (-1)^n|A|$
  $|A| = \lambda_1\cdots\lambda_n = (-1)^n a_0$
- 由行列式的另一种定义(不同行不同列的元素的积之和), λ_{n-1}的系数只能由 $(\lambda - a_{11})\cdots(\lambda - a_{nn})$产生, 即$a_{n-1} = -(a_{11} + \cdots + a_{nn})$
  $tr(A) = a_{11} + \cdots + a_{nn} = -a_{n-1}$

特征向量(eigenvector)

$A\mathbf{x} = \lambda\mathbf{x}$
$(A - \lambda I)\mathbf{x} = \mathbf{0}$
$\mathbf{x} \neq \mathbf{0}$

- the nontrivial solutions of $(A - \lambda I)\mathbf{x} = \mathbf{0}$
- $\mathbf{x} \in null(A - \lambda I)$
- $\mathbf{x}$: the eigenvectors(特征向量) of A corresponding to λ
- $null(A - \lambda I)$: the eigenspace(特征空间) of A corresponding to λ

The number of the eigenvectors of A corresponding to λ_i is same as the multiplicity of roots of λ_i of $p(\lambda)$

eigenvalue and eigenvector

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

find the eigenvalues and eigenvectors of A

property

Theorem 6.7. 以下说法都成立：

1. 对于任何n阶方阵$A \in M_{n\times n}$，它最多有n个不同的特征值。
2. 对于上三角矩阵$A = \begin{bmatrix} a_{11} & \star & \star & \dots & \star \\ 0 & a_{22} & \star & \dots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$它有特征值为对角线上的元素$a_{11}, \dots, a_{nn}$。该结论对下三角矩阵也成立。
3. 假设$\lambda \in \mathbb{R}$为A的一个特征值，$\boldsymbol{x} \in \mathbb{R}^n$为$A$关于$\lambda$的一个特征向量，那么对于任何多项式$f(x) = a_0 + a_1 x + \dots + a_m x^m$，$f(\lambda)$是多项式矩阵$f(A) = a_0 I_n + a_1 A + \dots + a_m A^m$的一个特征值，$\boldsymbol{x}$为$f(A)$关于$f(\lambda)$的一个特征向量。
4. A可逆当且仅当$\lambda = 0$不是A的特征值。
5. 如果$\lambda \in \mathbb{R}$为A的一个特征值，$\boldsymbol{x} \in \mathbb{R}^n$为$A$关于$\lambda$的一个特征向量，且$A$可逆，那么对于任何多项式$f(x) = a_0 + a_1 x + \dots + a_m x^m$，$f(\frac{1}{\lambda})$是多项式矩阵$f(A^{-1}) = a_0 I_n + a_1 A^{-1} + \dots + a_m A^{-m}$的一个特征值，$\boldsymbol{x}$为$f(A^{-1})$关于$f(\frac{1}{\lambda})$的一个特征向量。

example

A is a 2×2 matrix with eigenvalues $\lambda_1, \lambda_2 \in \mathbb{Z}$, $B = A^{-2} - 6A^{-1}$, the eigenvalues of B are -5 and 7, find λ_1, λ_2

example

A is a 2×2 matrix with eigenvalues $\lambda_1, \lambda_2 \in \mathbb{Z}$, $B = A^{-2} - 6A^{-1}$, the eigenvalues of B are -5 and 7, find λ_1, λ_2

$$\frac{1}{\lambda_1^2} - 6\frac{1}{\lambda_1} = -5$$
$$\frac{1}{\lambda_2^2} - 6\frac{1}{\lambda_2} = 7$$

$\lambda_1 = 1, \lambda_2 = -1$

similarity invariants 相似不变量

$B = P^{-1}AP$

- determinant, rank, nullity, trace
- A,B有相同的特征多项式(eigen polynomial)
  $|\lambda I - B| = |\lambda P^{-1}IP - P^{-1}AP| = |P^{-1}(\lambda I - A)P| = |\lambda I - A|$
- A,B有相同的特征值(eigenvalues)
- A,B 特征空间的**维度数**相同,特征空间不一定相同

similarity

$B = P^{-1}AP$

- A, B 有相同的eigenvalues λ
- 若$\mathbf{x}$是A的eigenvector, 则$P^{-1}\mathbf{x}$是B的eigenvector
- 若$\mathbf{y}$是B的eigenvector, 则$P\mathbf{y}$是A的eigenvector

(相似)对角化 Diagonalization

若一个矩阵A可写作$D = P^{-1}AP$,即$A = PDP^{-1}$, 则称A可对角化(diagonalizable)

usage:
$A^n = PDP^{-1}PDP^{-1}\cdots PDP^{-1} = PD^nP^{-1}$

Diagonalization

e.g. $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$, find A^n

思路: 将A对角化, $A = PDP^{-1}$, $A^n = PD^nP^{-1}$

1. 求A的特征值和特征向量
2. 将特征向量组成P
   原因: $A = PDP^{-1} \Leftrightarrow AP = PD$

D: 特征值
P: 特征空间的基拼成(对应特征值)

$P = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Diagonalization

可对角化: 对于一个有着k重根的特征值, 其特征空间的维度也为k

即能找到n个线性无关的特征向量

- 若 $A \in M_{n\times n}$拥有n个不同的特征值，那么A可对角化
  每个特征值至少有一个对应的特征向量
- 不同特征值的对应的特征向量线性无关
- A的任何一个特征值λ, 几何重数(geometric multiplicity)(特征空间的维度) $\leq$ 代数重数(algebraic multiplicity)(λ的重数)

property

- **对称矩阵**不同特征值对应的特征向量彼此正交
  设$\lambda_1 \neq \lambda_2$, 其对应的特征向量为$\mathbf{x}_1, \mathbf{x}_2$
  $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1, A\mathbf{x}_2 = \lambda_2\mathbf{x}_2$
  $\mathbf{x}_1^\top A\mathbf{x}_2 = \mathbf{x}_1^\top \lambda_2\mathbf{x}_2 = \lambda_2\mathbf{x}_1^\top\mathbf{x}_2$
  $(A\mathbf{x}_1)^\top\mathbf{x}_2 = (\lambda_1\mathbf{x}_1)^\top\mathbf{x}_2$
  $(\lambda_2 - \lambda_1)\mathbf{x}_1^\top\mathbf{x}_2 = x_1^\top A\mathbf{x}_2 - x_1^\top A^\top\mathbf{x}_2 = 0$
- 若将**对称矩阵**的特征向量的基**单位化**得到P,则P是正交矩阵
  $P^\top P = I$
- 实对称矩阵一定可以相似对角化
- $A^\top A$的特征值一定都是非负的
  $\forall x, x^\top A^\top A x = \|Ax\|^2 \geq 0 \Leftrightarrow A^\top A \succeq 0 \Leftrightarrow A^\top A$的特征值都是非负的
- 同理, $AA^\top$的特征值一定都是非负的

数列的特征值

e.g. $a_0 = a_1 = 1, a_{n+2} = a_{n+1} + a_n$

$x^2 = x + 1$

$x_1 = \frac{1+\sqrt{5}}{2}, x_2 = \frac{1-\sqrt{5}}{2}$

$a_n = c_1 x_1^n + c_2 x_2^n$

带入$a_0 = a_1 = 1$得到c_1, c_2的值

$c_1 = \frac{1}{\sqrt{5}}, c_2 = -\frac{1}{\sqrt{5}}$

所以$a_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

数列的特征值

e.g. $a_0 = a_1 = 1, a_{n+2} = a_{n+1} + a_n$

$$\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$$

$$\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$$

对$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$做特征值分解:

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$