Student Name: ________________

Student Number: ________________

School: ________________

Year of Entrance: ________________

ShanghaiTech University Final Examination Cover Sheet

Academic Year: 2021 to 2022 Term: 1

Course-offering School: IMS

Instructor: *Ye, ShuYang* ☐ / *Xue, BoQing* ☐ / *Zheng, Kai* ☐

Course Name: **Linear Algebra I**

Course Number: MATH1112

Exam Instructions for Students:

1. All examination rules must be strictly observed throughout the entire test, and any form of cheating is prohibited.

2. Other than allowable materials, students taking closed-book tests must place their books, notes, tablets and any other electronic devices in places designated by the examiners.

3. Students taking open-book tests may use allowable materials authorized by the examiners. They must complete the exam independently without discussion with each other or exchange of materials.

For Marker's Use:

Section	1	2	3	4	5	6	7	Total
Marks								
Recheck								

Marker's Signature:

Date:

Rechecker's Signature:

Date:

Instructions for Examiners:

1. The format of the exam papers and answer sheets shall be determined by the school and examiners according to actual needs. All pages should be marked by the page numbers in order (except the cover page). All text should be legible, visually comfortable and easy to bind on the left side. A4 double-sided printing is recommended for the convenience of archiving (There are all-in-one printers in the university).

2. The examiners should make sure that exam questions are correct and appropriate, If errors are found in exam questions during the exam, the examiners should be responsible to respond on site, which will be taking into account in the teaching evaluation.

Specific Instructions for students:

- Please check the box □ behind the name of your instructor on the cover page.
- The time duration for this exam is 120 **minutes**.
- Computers and calculators are prohibited in the exam.
- Answers can be written in **either Chinese or English**.

★ For problems 2-7, please show details of calculations or deductions. A correct answer with no details can not earn points.

Notations and conventions:

- $\mathbb{R}$ is the set of real numbers. All the scalars here are real numbers. We do NOT consider complex vector space, complex matrix, complex eigenvalues, etc.
- I denotes an identity matrix of suitable order.
- 0 or $\mathbf{0}$ may denote the number zero, a zero vector, a zero matrix, or a zero transformation.
- $\dim(V)$ is the dimension of a vector space V.
- $\mathsf{tr}(A)$ is the trace of a matrix A.
- $\mathsf{rank}(A)$ is the rank of a matrix A.
- M_n is the vector space of all $n \times n$ matrices.
- P_n is the vector space of all polynomials of degree $\leq n$.
- $\mathsf{rank}(T)$ is the rank of a linear transformation T, i.e., the dimension of the range space of T.
- $\mathsf{nullity}(T)$ is the nullity of a linear transformation T, i.e., the dimension of the kernel space of T.
- $p'(x)$ or $p''(x)$ are derivatives of $p(x)$ of order 1 or 2, respectively.

1. Fill in the blanks.

(1) (4 points) Let V be an inner product space, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal set in V. Then the norm

$$\|5\mathbf{v}_1 - 2\mathbf{v}_2 + 3\mathbf{v}_3\| = \underline{\hspace{4cm}}.$$

(2) (4 points) In $\mathbb{R}^3$, let

$$\mathbf{v}_1 = (3,1,2), \quad \mathbf{v}_2 = (5,3,4), \quad \mathbf{v}_3 = (1,1,1), \quad \mathbf{v}_4 = (4,2,3).$$

The dimension of the subspace spanned by the above vectors is

$$\dim\left(\mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}\right) = \underline{\hspace{4cm}}.$$

(3) (4 points) Let $A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$. All the singular values of A are $\underline{\hspace{4cm}}$.

(4) (4 points) The inner product on the vector space M_2 is given by $\langle A, B\rangle = \mathrm{tr}(B^T A)$. Let

$$C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}.$$

The distance between C and D is $d(C, D) = \underline{\hspace{4cm}}$.

(5) (6 points) The linear transformation $T: P_2 \to P_3$ is defined by

$$T(p(x)) = x\,p(2x+1) - 2x\,p(x)$$

for any $p(x) \in P_2$. Then

$$\mathsf{rank}(T) = \underline{\hspace{4cm}}, \quad \mathsf{nullity}(T) = \underline{\hspace{4cm}}.$$

2. Let $V = C(-\infty, +\infty)$, the vector space of all continuous functions on $(-\infty, +\infty)$. Define operators T_1 and T_2 on V by

$$T_1(f(x)) = e^x f(x), \quad T_2(f(x)) = f(5x - 1)$$

for any $f(x) \in V$.

(1) (6 points) Prove that T_1 and T_2 are both linear operators.

(2) (6 points) Take $g(x) = e^{-x}$. Find $(T_1 \circ T_2)(g(x))$ and $(T_2 \circ T_1)(g(x))$.

3. Let $V = P_2$. Define $\langle \cdot, \cdot \rangle$ on V by

$$\langle p(x), q(x) \rangle = p(1)q(1) + p'(1)q'(1) + p''(1)q''(1)$$

for any $p(x), q(x) \in V$.

(1) (7 points) Prove that $\langle \cdot, \cdot \rangle$ is an inner product on V.

(2) (8 points) Let

$$p_1(x) = 1, \quad p_2(x) = x, \quad p_3(x) = x^2.$$

Apply the Gram-Schmidt process to transform the standard basis $\{p_1(x), p_2(x), p_3(x)\}$ into an orthonormal basis $\{h_1(x), h_2(x), h_3(x)\}$.

(Do NOT change the order of $\{p_1(x), p_2(x), p_3(x)\}$ in the process.)

4. (12 points) Find A^{2022}, where

$$A = \begin{bmatrix} \frac{5}{4} & \frac{3\sqrt{3}}{4} \\ \frac{3\sqrt{3}}{4} & -\frac{1}{4} \end{bmatrix}.$$

5. (12 points) Suppose that $A \in M_3$ is symmetric and $\mathsf{rank}(A) = 2$. Suppose that

$$A \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$$

Find all the eigenvalues of A. For each eigenvalue, find an eigenvector corresponding to it.

(Please give your explanation.)

6. Let V be a vector space of dimension n, and T be a linear operator on V. Suppose that there is a vector $\mathbf{v} \in V$ such that $T^{n-1}(\mathbf{v}) \neq \mathbf{0}$ and $T^n(\mathbf{v}) = \mathbf{0}$.

(1) (8 points) Prove that $B = \{\mathbf{v}, T(\mathbf{v}), T^2(\mathbf{v}), \ldots, T^{n-1}(\mathbf{v})\}$ is a basis for V.

(2) (4 points) Find the matrix $[T]_B$ for T relative to the basis B.

7. Let $A \in M_n$. Suppose that ρ is the largest eigenvalue of $A^T A$.

(1) (6 points) Prove that $\|A\mathbf{x}\| \leq \sqrt{\rho}\|\mathbf{x}\|$ for any $\mathbf{x} \in \mathbb{R}^n$, here $\|\cdot\|$ is the ordinary Euclidean norm on $\mathbb{R}^n$.

(2) (4 points) Prove that if $\rho < 1$, then $I - A$ is invertible.

(3) (5 points) Suppose that A is invertible. Prove that the matrix A can be written as $A = RH$, where R, H are matrices in M_n satisfying
(i) R is orthogonal;
(ii) $H \in M_n$ is symmetric, and all its eigenvalues are positive.