MATH 1112.01 - Linear Algebra - Fall 2020

Midterm Exam

NAME & ID (please print legibly): ______________________________

- **NO CALCULATORS/BOOKS/NOTES are allowed on this exam.**
- **Please SHOW ALL YOUR WORK! You may not receive full credit for a correct answer if there is no work shown.**

QUESTION	VALUE	SCORE
1	10	
2	10	
3	15	
4	5	
5	12	
6	8	
7	12	
8	12	
9	10	
10	6	
TOTAL	100	

1. (10 points) Find the inverses of the following matrix.

$$A = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \end{bmatrix}$$

2. (10 points) Evaluate the determinant of the following matrix.

$$D_{n\times n} = \begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & & \vdots & & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}.$$

3. (15 points) Solve the following linear systems.

(a) (10 points)

$$\left\{\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 8 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 1 \\ 3x_1 & - & 7x_2 & + & 4x_3 & = & 10 \end{array}\right.$$

(b) (5 points) Only solve for x_2 and x_4 for the following system:

$$\left\{\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \\ x_1 & + & 3x_2 & - & 2x_3 & + & 4x_4 & = & -1 \\ x_1 & + & 9x_2 & + & 4x_3 & + & 16x_4 & = & 1 \\ x_1 & + & 27x_2 & - & 8x_3 & + & 64x_4 & = & -1 \end{array}\right.$$

4. (5 points) Using Cauchy-Schwarz inequality, prove that for all positive x, y, z one has

$$x + y + z \leq 2\left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right).$$

5. (12 points) Let $x = (1, 1, 1)$, $y = (1, 1, 0)$.

(a) (4 points) Normalize x.

(b) (4 points) Decompose $y = y_1 + y_0$ where y_1 is a scale of x and y_0 is orthogonal to x.

(c) (4 points) Find a vector z_0 such that $z_0 \cdot x = 0$ and $z_0 \cdot y_0 = 0$. (Cross Product)

6. (8 points) It is known that the set of all 2×2 matrices (with real entries, standard addition and scalar multiplication) is a vector space. Prove that the set of matrices in the form of

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

with $a_{ij} \in \mathbb{R}$ is a subspace.

7 (12 points) Let $v_1 = (2, 1, 0, 3)$, $v_2 = (3, -1, 5, 2)$, $v_3 = (-1, 0, 2, 1)$. Is $\{v_1, v_2, v_3\}$ a basis for $\mathbb{R}^4$? If not, enlarge the set by finding a vector v_4 so that $\{v_1, v_2, v_3, v_4\}$ forms a basis.

8. (12 points) Let

$$u_1 = (2, 1, 1), \quad u_2 = (2, -1, 1), \quad u_3 = (1, 2, 1)$$
$$v_1 = (3, 1, -5), \quad v_2 = (1, 1, -3), \quad v_3 = (-1, 0, 2).$$

It is known that $B_1 = \{u_1, u_2, u_3\}$ and $B_2\{v_1, v_2, v_3\}$ are bases of $\mathbb{R}^3$.

1. Find the transition matrix B_1 to B_2.

2. For $x = (-2, 4, 6)$, find the coordinate $(x)_{B_1}$ and then find $(x)_{B_2}$.

9. (10 points) Let

$$A = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Find bases for the null space and column space of A. What's the nullity of A?

10. (6 points) Let A be an $m \times n$ matrix. (A may or may not be a square.) Prove that $\operatorname{rank}(A) \leq \min\{m, n\}$.