线性代数(2023-2024)第十一次作业

1 复习知识点

- 坐标向量映射以及它的逆映射的表示：如果$B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$是$V$的一组基底，那么对于任何$\boldsymbol{v} \in V$，存在唯一确定的一组系数$k_1, \ldots, k_n$使得$\boldsymbol{v} = k_1\boldsymbol{v}_1 + \ldots + k_n\boldsymbol{v}_n$。我们定义坐标向量映射$f_B : V \to \mathbb{R}^n$为$f_B(\boldsymbol{v}) = (k_1, \ldots, k_n) \in \mathbb{R}^n$，它的逆映射为$f_B^{-1} : \mathbb{R}^n \to V$，$f_B^{-1}(x_1, \ldots, x_n) = x_1\boldsymbol{v}_1 + \ldots + x_n\boldsymbol{v}_n$。
- 熟练掌握如何计算线性变换$T : V \to W$关于V的基底B与W的基底B'的矩阵表示$[T]_{B',B}$。建议仔细阅读英文教材8.4节的所有例子。
- 相似矩阵的定义，牢记常见的相似不变量，尤其是讲义Theorem 5.21。
- 相似矩阵的一些基本性质，尤其是讲义Definition 5.19下方的内容。这里总结如下：
  1. 若A与B相似，则$A^\top$与$B^\top$相似。
  2. 若A与B相似，且A可逆，那么B必然可逆，且A^{-1}与B^{-1}也相似。
  3. 若A与B相似，那么对任何多项式$f(x) = a_0 + a_1x + \ldots + a_nx^n$，多项式矩阵$f(A)$与$f(B)$相似。
- 讲义Theorem 5.20：线性变换对于不同基底的矩阵表示与相似矩阵的关系。
- 特征值，特征向量，特征多项式，特征方程的定义与计算方法。

2 习题部分

Problem A(6 Points)

1. (2 points) Let V be a vector space spanned by its basis $B = \{1, \sin x, \cos x\}$. Let $D : V \to V$ be the differential operator on V such that for any $f(x) \in V$, $D(f(x)) = f'(x)$. Find the matrix $[D]_{B,B}$ of D relative to the basis B.
2. (2 points) Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a given matrix in $M_{2\times2}$. Prove that ad : $M_{2\times2} \to M_{2\times2}$, defined by
$$\text{ad}(A) = BA - AB$$
for $A \in M_{2\times2}$ is a linear operator on $M_{2\times2}$. Then find the matrix $[\text{ad}]_{B,B}$ of ad relative to the standard basis B of $M_{2\times2}$.
3. (2 points) Let V, W be two–dimensional vector spaces. Let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be a basis of V, let $B' = \{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ be a basis of W. Let $T : V \to W$ be a linear operator

such that

$$T(3\boldsymbol{v}_1 + 2\boldsymbol{v}_2) = \boldsymbol{w}_2, \quad T(\boldsymbol{v}_1 - 4\boldsymbol{v}_2) = \boldsymbol{w}_1 - \boldsymbol{w}_2.$$

Find the matrix $[T]_{B',B}$ relative to B and B'.

Problem B(6 Points) 仔细阅读英文教材8.4节的Example 2与Example 6，然后回答以下问题。

- Let $T : P_2 \to M_{2\times 2}$ be the linear transformation defined by

$$T(p(x)) = \begin{bmatrix} p(0) & p(1) \\ p(-1) & p(0) \end{bmatrix}.$$

  Let B be the standard basis for $M_{2\times 2}$, let $B' = \{1, x, x^2\}$ be the standard basis for $P_2 = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$, $B'' = \{1, 1+x, 1+x^2\}$ be another basis for P_2.

  1. (2 points) Find $[T]_{B,B'}$ and $[T]_{B,B''}$.
  2. (2 points) Let $q(x) = 2+2x+x^2$, compute $[T]_{B,B'}[q(x)]_{B'}$ and $[T]_{B,B''}[q(x)]_{B''}$, then use $[T]_{B,B'}[q(x)]_{B'}$ and $[T]_{B,B''}[q(x)]_{B''}$ to compute $T(q(x))$, i.e., compute $f_B^{-1}([T]_{B,B'}[q(x)]_{B'})$ and $f_B^{-1}([T]_{B,B''}[q(x)]_{B''})$, where $f_B : M_{2\times 2} \to \mathbb{R}^4$, $f_B(A) = [A]_B$, is the coordinate vector transformation (坐标向量映射) on $M_{2\times 2}$.

- (2 points) Let $T_1 : P_1 \to P_2$ be the linear transformation defined by

$$T_1(a_0 + a_1 x) = 2a_0 - 3a_1 x,$$

  and let $T_2 : P_2 \to P_3$ be the linear transformation defined by

$$T_2(a_0 + a_1 x + a_2 x^2) = 3a_0 x + 3a_1 x + 3a_2 x^3.$$

  Let $B = \{1, x\}$, $B'' = \{1, x, x^2\}$, $B' = \{1, x, x^2, x^3\}$. Compute $[T_2 \circ T_1]_{B',B}$, $[T_2]_{B',B''}$, and $[T_1]_{B'',B}$. What is the relation between $[T_2 \circ T_1]_{B',B}$, $[T_2]_{B',B''}$, and $[T_1]_{B'',B}$?

Problem C(6 Points)

1. (2 points) Let $T : P_2 \to P_2$ be defined by

$$T(p(x)) = p(2x + 1).$$

   Determine whether T is invertible and if it is invertible, find T^{-1}, and compute $[T^{-1}]_{B,B}$ for the standard basis $B = \{1, x, x^2\}$ of P_2.

2. (2 points) Let $J : P_3 \to \mathbb{R}$ be the integral transformation, $I(p(x)) = \int_{-1}^{1} p(x)dx$. Let $B' = \{1\}$ be the standard basis of $\mathbb{R}$, $B = \{1, x, x^2, x^3\}$ be the standard basis of P_3, what is the matrix $[J]_{B',B}$? What is the kernel of J?
3. (2 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3, -x_2, x_1 + 7x_3).$$

Let B be the standard basis of $\mathbb{R}^3$, $B' = \{(1,0,0), (1,1,0), (1,1,1)\}$ be another basis of $\mathbb{R}^3$. Find $[T]_{B,B}$, $[T]_{B',B'}$ and a invertible matrix P such that $[T]_{B',B'} = P^{-1}[T]_{B,B}P$.

Problem D(6 Points) 令V为一个向量空间，$T : V \to V$为一个线性变换。我们说$\lambda \in \mathbb{R}$是T的一个特征值(eigenvalue)，如果存在一个非零$\boldsymbol{v} \in V$，$\boldsymbol{v} \neq \boldsymbol{0}$，使得$T(\boldsymbol{v}) = \lambda\boldsymbol{v}$。此时我们称$\boldsymbol{v}$为$T$关于$\lambda$的特征向量(eigenvector)。所有$T$关于$\lambda$的特征向量的集合被称为$T$关于$\lambda$的特征空间(eigenspace)。

1. (2 points) 假设$B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$是$V$的一组基底，证明$\lambda$是$T$的一个特征值当且仅当$\lambda$是矩阵$[T]_{B,B} \in M_{n\times n}$的一个特征值。证明$\boldsymbol{v} \in V$是$T$关于$\lambda$的一个特征向量当且仅当$[\boldsymbol{v}]_B$是矩阵$[T]_{B,B} \in M_{n\times n}$关于$\lambda$的一个特征向量。
2. (4 points) Let $T : P_2 \to P_2$ be defined by

$$T(a_0 + a_1x + a_2x^2) = (5a_0 + 6a_1 + 2a_2) - (a_1 + 8a_2)x + (a_0 - 2a_2)x^2.$$

Find the eigenvalues of T and find a basis for the eigenspaces of T.

Problem E(6 Points) Let $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 0 \\ -5 & 5 & 10 \end{bmatrix}$.

1. (3 points) Compute the eigenvalues of A, and find a basis of the eigensapces of A.
2. (3 points) Let adj(A) be the adjoint matrix of A. Find the eigenvalues of $3I_3 +$ adj(A).

Bonus：不计入分数

假设V为一个n维向量空间，W为一个m维向量空间。

1. 对于$T : V \to W$和$S : V \to W$两个线性变换，定义它们的加法为 $S + T : V \to W$为$(S + T)(\boldsymbol{v}) = S(\boldsymbol{v}) + T(\boldsymbol{v})$。对于实数$k \in \mathbb{R}$，定义$k$与$T$的标量积$kT : V \to W$为$(kT)(\boldsymbol{v}) = kT(\boldsymbol{v})$。令$L(V, W)$为所有从$V$到$W$的线性变换

的集合。证明$L(V,W)$装配上以上加法和标量积之后是一个向量空间，尤其是$S+T$和kT依然是从V到W的线性变换。

2. 证明$L(V,W)$的维数$\dim(L(V,W)) = nm$ (n乘以m)，并找出它的一组基底。

Deadline: 22:00, December 31.

作业提交截止时间：12月31日晚上22：00。