Linear Algebra Tutorial4

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homework

- A, B are upper triangular matrix, then $AB, A^k ...$ are also upper triangular matrix
  It is theorem on book, we do not need to prove it but need to mention it
- the symbol [] and ||
  [] for matrix, || for determinant
- **iff** $\Leftrightarrow$ if and only if $\Leftrightarrow$ 当且仅当

determinant properties

> compare with the elementary row(column) operations

1. B is obtained from A by interchanging two rows(columns)
   $|B| = -|A|$
2. B is obtained from A by multiplying one row(column) by a nonzero scalar k
   $|B| = k|A|$
3. B is obtained from A by adding a multiple of one row(column) to another row(column)
   $|B| = |A|$

> we can mix row and column operations when calculating the determinant
> but we can only use row or column operations when calculating the inverse matrix!!!!

determinant properties

> If a matrix A have two same rows(columns), then $|A| = 0$

suppose B is A swapping the two same rows(columns), from property 1, $|B| = -|A|$

but $B = A$, so $|B| = |A|$

so $|A| = -|A|$, $|A| = 0$

determinant property

$$C = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1}+b_{i1} & a_{i2}+b_{i2} & \dots & a_{in}+b_{in}\,(\text{第}i\text{行}) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in}\,(\text{第}i\text{行}) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{i1} & b_{i2} & \dots & b_{in}\,(\text{第}i\text{行}) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

> $|C| = |A| + |B|$

determinant property

- similary

$$C = \begin{bmatrix} a_{11} & \dots & a_{1j} + b_{1j} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nj} + b_{nj} & \dots & a_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & \dots & b_{1j} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & b_{nj} & \dots & a_{nn} \end{bmatrix}$$

> $|C| = |A| + |B|$
> essentially: determinant expansion by column

generally, $|C| \neq |A| + |B|$!!!

determinant property

- $|AB| = |A||B|$
  $\Rightarrow R = ABCD\ldots, |R| = |A||B||C|\ldots$
- $|A^T| = |A|$
- $|A^{-1}| = \dfrac{1}{|A|}$
- $|A^k| = |A|^k$
- $|\lambda A| = \lambda^n |A|$

Cramer's rule

$A\mathbf{x} = \mathbf{b}$, $|A| \neq 0$ $(A_{n\times n})$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

$$A_i = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & & & & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{bmatrix}$$

then $x_i = \dfrac{|A_i|}{|A|}$

where A_i is the matrix obtained from A by replacing the ith column of A by $\mathbf{b}$

Cramer's rule

- e.g. when the coefficient is complicated

$$(e^u + \sin v)u_x \quad + u\cos v \quad v_x = 1$$

$$(e^u - \cos v)u_x \quad + u\sin v \quad v_x = 0$$

$$e^u \cdot u\sin v + u\sin^2 v - e^u \cdot u\cos v + u\cos^2 v$$

Cramer's rule

$$(e^u + \sin v)\frac{\partial u}{\partial x} + (u\cos v)\frac{\partial v}{\partial x} = 1$$
$$(e^u - \cos v)\frac{\partial u}{\partial x} + (u\sin v)\frac{\partial v}{\partial x} = 0$$

with Cramer's rule, we can get

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} 1 & u\cos v \\ 0 & u\sin v \end{vmatrix}}{\begin{vmatrix} e^u + \sin v & u\cos v \\ e^u - \cos v & u\sin v \end{vmatrix}} = \frac{\sin v}{e^u(\sin v - \cos v) + 1},$$

$$\frac{\partial v}{\partial x} = \frac{\begin{vmatrix} e^u + \sin v & 1 \\ e^u - \cos v & 0 \end{vmatrix}}{\begin{vmatrix} e^u + \sin v & u\cos v \\ e^u - \cos v & u\sin v \end{vmatrix}} = \frac{-(e^u - \cos v)}{(e^u(\sin v - \cos v) + 1)u}$$

determinant

5.

$$A = \begin{bmatrix} 1 & 1 & 5 & 4 \\ 2 & 3 & 2 & 4 \\ 1 & 6 & 0 & 3 \\ 4 & 2 & 5 & 1 \end{bmatrix}$$

find $C_{21} + C_{22} + 5C_{23} + 4C_{24}$

where C_{ij} is the cofactor of a_{ij}

determinant

5.

$$A = \begin{bmatrix} 1 & 1 & 5 & 4 \\ 2 & 3 & 2 & 4 \\ 1 & 6 & 0 & 3 \\ 4 & 2 & 5 & 1 \end{bmatrix}$$

construct:

$$A' = \begin{bmatrix} 1 & 1 & 5 & 4 \\ 1 & 1 & 5 & 4 \\ 1 & 6 & 0 & 3 \\ 4 & 2 & 5 & 1 \end{bmatrix}$$

so $C_{21} + C_{22} + 5C_{23} + 4C_{24} = |A'|$

and since A' has two same rows, $|A'| = 0$

so $C_{21} + C_{22} + 5C_{23} + 4C_{24} = 0$

determinant property

$\forall A \in \mathbb{R}^{n\times n}$

- if $i \neq k$, then
  $a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = 0$
- if $i = k$, then
  $a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = |A|$

  > so called **Laplace expansion**

adjoint matrix

- $C_{i,j}$: matrix of cofactors of $a_{i,j}$
  代数余子式
- C : matrix of cofactors from A
  A的代数余子式矩阵

i.e. $C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$

- $A^* = adj(A) = C^T$
- **no matter A is invertible or not,** $AA^* = A^*A = |A|I$
- If A is **invertible**, $A^{-1} = \dfrac{1}{|A|}A^*$

adjoint matrix

- $|A^*| = |A|^{n-1}$
  $AA^* = |A|I$
  $|AA^*| = |A|^n|I| = |A|^n$
  $|A^*||A| = |A|^n$
  $|A^*| = |A|^{n-1}$

> any constrains?

adjoint matrix

- but if A is not invertible??
  i.e. $|A| = 0$
  is $|A^*| = 0$?

- Proof by Contradiction
  suppose $|A^*| \neq 0$, then
  $(AA^*)(A^*)^{-1} = |A|I(A^*)^{-1} = \mathbf{0}$
  also, $(AA^*)(A^*)^{-1} = A(A^*(A^*)^{-1}) = A$

  so $A = 0 \Rightarrow A^* = 0$, which is a contradiction with $|A^*| \neq 0$

  so $|A^*| = 0$

> so $|A^*| = |A|^{n-1}$ is always true

determinant

1.

$$D = \begin{bmatrix} a_1 & c_2 & c_3 & \cdots & c_n \\ b_2 & a_2 & & & \\ b_3 & & a_3 & & \\ \vdots & & & \ddots & \\ b_n & & & & a_n \end{bmatrix}$$

> $$det(D) = \prod_{i=1}^{n} a_i - \sum_{i=2}^{n} \frac{Ab_n c_n}{a_n}$$
> where $A = \prod_{i=2}^{n} a_i$

determinant

2.

$$D = \begin{bmatrix} a & b & 0 & \cdots & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ b & 0 & 0 & \cdots & 0 & a \end{bmatrix}$$

> $|D| = a^n + (-1)^{1+n} b^n$

determinant

3.

$$D_n = \begin{bmatrix} b & -1 & 0 & \cdots & 0 & 0 \\ 0 & b & -1 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & b & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & b+a_1 \end{bmatrix}$$

determinant

4.

$$\text{Calculate } D_n = \begin{vmatrix} 1+x_1^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & 1+x_2^2 & \cdots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \cdots & 1+x_n^2 \end{vmatrix}.$$

some mostly used tricks

- row(column) operations, construct a row(column) with as many zeros as possible, then use the Laplace expansion
- try to calculate the determinant with lower dimension, the use induction
- **practice more!!**

linear space

inner product $< \cdot, \cdot >$

- $\mathbb{R}^n$ vector: dot product $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x}$
- $M_{m \times n}(\mathbb{R})$ matrix: $< A, B >= tr(B^T A)$
- $P_n(\mathbb{R})$ polynomial: $< p, q >= \int_0^1 p(x)q(x)dx$
- $C[a, b]$ continuous function: $< f, g >= \int_a^b f(x)g(x)dx$

  $\vdots$

linear space

angle θ between two vectors $\mathbf{x}, \mathbf{y}$

$$cos\theta = \frac{< \mathbf{x}, \mathbf{y} >}{\sqrt{< \mathbf{x}, \mathbf{x} >} \cdot \sqrt{< \mathbf{y}, \mathbf{y} >}} = \frac{\mathbf{x}^T \mathbf{y}}{|| \mathbf{x} || \, \|\mathbf{y}\|}$$

so $\mathbf{x} \cdot \mathbf{y} = \; || \mathbf{x} || \, \|\mathbf{y}\| cos\theta$

$\mathbf{x} \cdot \mathbf{y} = 0 \Leftrightarrow \mathbf{x} \perp \mathbf{y}$

cross product

$$\mathbf{x} \times \mathbf{y} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix}$$

- $\mathbf{x} \times \mathbf{y}$ is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$
- $\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\| \, \|\mathbf{y}\| sin\theta$
- $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$
- $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = \mathbf{x} \times \mathbf{y} + \mathbf{x} \times \mathbf{z}$
- $(k\mathbf{x}) \times \mathbf{y} = k(\mathbf{x} \times \mathbf{y})$

cross product

- the cross product of two vectors can be represented by the determinant

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

- $\mathbf{x} \times \mathbf{y} = \mathbf{0} \Leftrightarrow \mathbf{x} \parallel \mathbf{y}$
- we can also transfer $\mathbf{x}$ into a matrix $[\mathbf{x}]\mathbf{y}$

where $[\mathbf{x}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$

> notice that $[\mathbf{x}]$ is skew-symmetric, so its rank is odd
> actually, rank($[\mathbf{x}]$)=2