Linear Algebra Tutorial 7

2023.11.21

homework

- P_n 所有次数$\leq n$的多项式的集合
  $P_n = \{p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_1, \cdots, a_n \in \mathbb{R}\}$
- matrix has no Binomial Theorem
  e.g. $(A+B)^2 = A^2 + AB + BA + B^2$
  AB may not equal to BA!!!
- about $\vec{v}$ and $\mathbf{v}$
- a small tip for hw7
  $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
  $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
  $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$

homework

- for the problem E:

Problem E(6 Points). 对每个给定的$\boldsymbol{x} \in \mathbb{R}^n$与$\boldsymbol{b} \in \mathbb{R}^m$，考虑以下$M_{m\times n}$里的子集

$$W_{\boldsymbol{x},\boldsymbol{b}} = \{A \in M_{m\times n} : A\boldsymbol{x} = \boldsymbol{b}\}.$$

找出所有能够使得集合$W_{\boldsymbol{x},\boldsymbol{b}}$成为$M_{m\times n}$里的子空间的向量$\boldsymbol{x}$与$\boldsymbol{b}$。

$$A\mathbf{x} = \mathbf{0} \nRightarrow \mathbf{x} = \mathbf{0}$$

midterm review **期中复习**

- 考试时间: 2023.12.6 星期三 8:15~9:55
- 考试地点: 教学中心202
- 考试内容: 第一章到第四章的4.8节（包含）
- 期中考试占总成绩 30%!
- 试卷为全英文, 不涉及数学的问题可以找监考人员翻译
- 作答中英文均可

> 欢迎大家有问题随时在群里/私聊提问, 尽量不要拖延问题
> 尽早复习，不要等到考前一天才开始复习
> start early!!!

一些要强调的事情

- iff $\Leftrightarrow$ if and only if $\Leftrightarrow$ 当且仅当,
  $\Rightarrow$ 充分性, $\Leftarrow$ 必要性, 都要证
  或者全程使用$\Leftrightarrow$等价表述

- free variables
  eg. $x_3 = s, x_4 = r$
  $s, r \in \mathbf{R}$!

- consistent 有解的

- inconsistent 无解的

- trival solution 平凡解(无解)

- the symbol [] and ||
  [] for matrix, || for determinant

一些要强调的事情

- 注意公式不要记错了
  - $||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$ 不要忘记根号
  - $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}$ 分母不要忘记开方
  - $proj_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{v}||^2}\mathbf{v}$ 分母不要忘记平方,公式背不过的话可以自己考场推一下

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

- $A + I = ?, A - I = ?$ 注意I是单位矩阵,只有对角线上的元素为1!!!
- P_n 所有次数$\leq n$的多项式的集合
  $P_n = \{p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_1, \cdots, a_n \in \mathbb{R}\}$

一些要强调的事情

- 行列式交换两行后，记得要有一个负号
- 关于叉乘

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1)$$

  注意中间有个负号(原因:a_{12}的代数余子式的符号是$(-1)^{1+2}$)
- 行列式按行/列的展开时:

$$|A| = \sum_{i=1}^{n} a_{ij}C_{ij}$$

  注意C_{ij}是a_{ij}的**代数余子式**,有一个$(-1)^{i+j}$!!!
- 矩阵没有二项式定理

  e.g. $(A+B)^2 = A^2 + AB + BA + B^2$

review list 复习清单

- chapter1 线性方程组
  - 矩阵
  - 高斯消元
  - 矩阵求逆
- chapter2 行列式
- chapter3 欧氏空间
- chapter4 向量空间
  - 子空间
  - 线性相关、线性无关
  - 基、维数、基变换
  - 行空间、 列空间、 零空间
  - 矩阵的秩、零度、 矩阵基本空间

the dimension of a vector space

(defined in last tutorial)

V is a finite-dimensional vector space, $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ is a subset of V

If

- S is linearly independent
- $V = span(S)$

then we call S a basis of V

the number of vectors in S is the dimension of V, denoted as $dim(V)$

coordinate

$S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \subset \mathbb{R}^n$ is a basis of $\mathbb{R}^n$
$\forall \mathbf{v} \in V$, $\exists c_1, \cdots, c_n \in \mathbb{R}$ s.t.
$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$

coordinate vector of $\mathbf{v}$ relative to S
$(\mathbf{v})_S = (c_1, \cdots, c_n)$

> the coordinate is actually the coefficient of the linear combination

the dimension of a vector space

The dimension of V is the number of vectors in any basis of V

- $\dim(\mathbb{R}^n) = n$
- $\dim(P_n) = n + 1$
- $\dim(M_{m\times n}) = \dim(\mathbb{R}^{mn}) = mn$
- for zero space $\dim(\{\mathbf{0}\}) = 0$

dimension and basis

$S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$

- $W = span(S)$, then S is a basis of W
- if S is a linear independent set
  $\dim(W) = \dim(span(S)) = |S| = n$

dimension and basis

$S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be the basis of V

- notice that S is a independent set
- if $m > n$, then $M = \{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ must be a **linear dependent** set
- if $m < n$, then $M = \{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ must **not span** V

  $\Downarrow$

- If $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, $M = \{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ are basis of V, then $n = m$

Plus/Minus Theorem

V is a vector space, $S \subset V$

- If S is an independent set, and $\mathbf{v} \in V, \mathbf{v} \notin S$, then $S \cup \{\mathbf{v}\}$ is also an **independent set**

> proof by contradiction, suppose that $S \cup \{\mathbf{v}\}$ is linear dependent
> $\Rightarrow \mathbf{v} = span(S)$

- If $\mathbf{v} \in S$, and $\mathbf{v}$ can be written as a linear combination of other vectors in S, then $span(S - \mathbf{v}) = span(S)$

> $\mathbf{v} \in S$, WLOG, take $\mathbf{v} = v_1$, consider $\forall \mathbf{w} \in span(S)$, can be writen as linear combination of $\mathbf{v}_2, \cdots, \mathbf{v}_n$

coordinate

$n \geq 1, \dim(V) = n, S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis of V

- 1. for any vector set $M = \{\mathbf{w}_1, \cdots, \mathbf{w}_r\} \subset V$
  M is an independent set $\Leftrightarrow [\mathbf{v}_1]_S, \cdots, [\mathbf{v}_r]_S$ are independent set

> proof: set $[\mathbf{v}_i] = (a_{i1}, \cdots, a_{in})$, then $\mathbf{v}_i = a_{i1}\mathbf{v}_1 + \cdots + a_{in}\mathbf{v}_n$

- 2. for vector set $M = \{\mathbf{w}_1, \cdots, \mathbf{w}_n\} \subset V$
  M is the basis of $V \Leftrightarrow [\mathbf{v}_1]_S, \cdots, [\mathbf{v}_n]_S$ is the basis of $\mathbb{R}^n$
  $\Leftrightarrow [\mathbf{v}_1]_S, \cdots, [\mathbf{v}_n]_S$ is the standard basis of $\mathbb{R}^n$

> from 1., we know that M is independent $\Rightarrow [\mathbf{v}_1]_S, \cdots, [\mathbf{v}_n]_S$ is independent, so we just need to prove that $span\{[\mathbf{w}_1]_S, \cdots, [\mathbf{w}_n]_S\} = \mathbb{R}^n$

example

例子：(2022年线性代数考试题)

Let $p_1(x) = 1 + 3x$,$p_2(x) = 2 + 4x$,$p_3(x) = -4x^2$ be three polynomials in P_2.

1. Let $M = \{1, x, x^2\}$ be the standard basis of P_2. Let $A = \begin{bmatrix} [p_1(x)]_M & [p_2(x)]_M & [p_3(x)]_M \end{bmatrix}$ be the matrix such that its columns are $[p_1(x)]_M$, $[p_2(x)]_M$ and $[p_3(x)]_M$. Compute the adjoint matrix of A.
2. Prove that $S = \{p_1(x), p_2(x), p_3(x)\}$ is a basis of P_2.

basis' theorem

V is a vector space,$\dim(V) = n$, $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\} \subset V$

- $span(S) = V, m > n$, then we can delete some of the vectors in S to get a basis of V
- if S is a linear independent set, $m < n$, then we can add some vectors to S to get a basis of V

basis' theorem

V is a vector space, $\dim(V) = n$, $W \subset V$ is a subspace.

- let $m = \dim(W)$, then $m \leq n$
- $W = V$ iff $m = n$

Change of basis

V is the vector space, B, B' are two bases of V

- $B = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$
- $B' = \{\mathbf{v}'_1, \cdots, \mathbf{v}'_n\}$
  If we have $(\mathbf{v})_B = (c_1, \cdots, c_n)$
  how could we find $(\mathbf{v})_{B'} = (c'_1, \cdots, c'_n)$?

> as we known that $[\mathbf{v}]_B$, $[\mathbf{v}]_{B'}$ has the unique expression

Change of basis

- transition matrix(过渡矩阵) P
  P is invertible, P^{-1} is called the transition matrix from B to B'
- transition matrix from B to B'
  $P_{B\to B'}$ or $P_{B'\leftarrow B}$
- transition matrix from B' to B
  $P_{B'\to B}$ or $P_{B\leftarrow B'}$
- $P_{B\leftarrow B'}P_{B'\leftarrow B} = I$

> notice that the defination of the transition matrix may be different with some of the Chinese textbooks!!

Change of basis

- We can represent the transition matrix as $P_{B \to B'}$ or $P_{B' \leftarrow B}$
- $[v]_{B'} = P_{B' \leftarrow B}[v]_B$
- $[v]_B = P_{B \leftarrow B'}[v]_{B'}$
- method to get the transition matrix
  $[B'|B] \Rightarrow [I|P_{B' \leftarrow B}]$

example of transition matrix

例子： 考虑$\mathbb{R}^3$的两组基底$B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$与$B' = \{\boldsymbol{u}'_1, \boldsymbol{u}'_2, \boldsymbol{u}'_3\}$:

$$\boldsymbol{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix};$$

$$\boldsymbol{u}'_1 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \boldsymbol{u}'_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \boldsymbol{u}'_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

我们利用之前介绍的计算步骤求解转移矩阵$P_{B' \leftarrow B}$。