Student Name: ________________
Student Number: ________________
School: ________________
Year of Entrance: ________________

ShanghaiTech University Final Examination Cover Sheet

Academic Year: 2023 to 2024 Term: 1
Course-offering School: IMS
Instructor: *Mingliang Cai* ☐ / *Chong Liu* ☐ / *Qixiao Ma* ☐
Daniel Skodlerack ☐ / *Qiang Wang* ☐ / *Boqing Xue* ☐
Course Name: **Linear Algebra I**
Course Number: MATH1112

Exam Instructions for Students:

1. All examination rules must be strictly observed throughout the entire test, and any form of cheating is prohibited.

2. Other than allowable materials, students taking closed-book tests must place their books, notes, tablets and any other electronic devices in places designated by the examiners.

3. Students taking open-book tests may use allowable materials authorized by the examiners. They must complete the exam independently without discussion with each other or exchange of materials.

For Marker's Use:

Section	1	2	3	4	5	6	7	8	Total
Marks									
Recheck									

Marker's Signature: Rechecker's Signature:
Date: Date:

Instructions for Examiners:

1. The format of the exam papers and answer sheets shall be determined by the school and examiners according to actual needs. All pages should be marked by the page numbers in order (except the cover page). All text should be legible, visually comfortable and easy to bind on the left side. A4 double-sided printing is recommended for the convenience of archiving (There are all-in-one printers in the university).

2. The examiners should make sure that exam questions are correct and appropriate, If errors are found in exam questions during the exam, the examiners should be responsible to respond on site, which will be taking into account in the teaching evaluation.

Specific Instructions for students:

- Please check the box □ after the name of your instructor on the cover page.
- The time duration for this exam is 120 **minutes**.
- Computers and calculators are prohibited in the exam.
- Answers can be written in **either Chinese or English**.

★ For problems 3-8, please show details of calculations or deductions. A correct answer with no details can not earn points.

Policy for grading the Multiple choice questions:

For a multiple choice question, denote by C the set of all correct choices, and by A the set of your choices. If A is not a subset of C, get zero points; If A is a subset of C, get partial credits depending on the size of A.

- If $|C| = 4$, get one point for each correct choice when $|A| < 4$, and get full points when $|A| = 4$;
- If $|C| = 3$, get two points for each correct choice when $|A| < 3$, and get full points when $|A| = 3$;
- If $|C| = 2$, get three points when $|A| = 1$, and get full points when $|A| = 1$;
- If $|C| = 1$, get full points when $|A| = 1$.

Notations and conventions:

- m, n always denote positive integers.
- $\mathbb{R}$ is the set of real numbers. All the scalars here are real numbers. We do NOT consider complex vector space, complex matrix, complex eigenvalues, complex inner product, etc.
- I denotes an identity matrix of suitable order.
- 0 or $\mathbf{0}$ may denote the number zero, a zero vector, a zero matrix, or a zero transformation.
- A^{T} is the transpose of a matrix A.
- $\det(A)$ is the determinant of a matrix A.
- $\mathrm{tr}(A)$ is the trace of a matrix A.
- $\mathrm{rank}(A)$ is the rank of a matrix A.
- M_n is the vector space of all $n \times n$ real matrices.
- P_n is the vector space of all polynomials of degree $\leq n$ with real coefficients.
- $\mathrm{proj}_W(\mathbf{x})$ is the orthogonal projectioin of the vector $\mathbf{x}$ onto the subspace W in an inner product space.

1. Multiple choice questions (20 points)

(1) Which of the following map $T : V \to V$ is/are linear isomorphim (i.e., both one-to-one/injective and onto/surjective)? (________________)

A. $V = \mathbb{R}^2$, $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$.

B. $V = \mathbb{R}^{2024}$, $T(x_1, x_2, x_3, \ldots, x_{2023}, x_{2024}) = (0, x_1, x_2, \ldots, x_{2022}, x_{2023})$.

C. $V = P_{2024}$, $T(p(x)) = xp'(x)$.

D. $V = P_{2024}$, $T(p(x)) = p(3x - 2) + 2$.

(2) Let $A \in M_n$. The characteristic polynomial of A is

$$\lambda^4 - 6\lambda^3 + 11\lambda^2 - 6\lambda.$$

Which of the following is/are true? (________________)

A. The matrix A is invertible.

B. The matrix A is diagonalizable.

C. The number 4 is an eigenvalue of the matrix $A + I$.

D. The determinant of the matrix $A - I$ is 0.

(3) Let $Q \in M_3$ be an orthogonal matrix. Which of the following statements is/are true? (________________)

A. For every symmetric matrix $A \in M_3$, the matrix $Q^{-1}AQ$ is symmetric.

B. For every column vector $\mathbf{v} \in \mathbb{R}^3$, the vectors $Q\mathbf{v}$ and $\mathbf{v}$ have the same length, i.e., $\|Q\mathbf{v}\| = \|\mathbf{v}\|$ for the Euclidean norm $\|\cdot\|$.

C. There is a nonzero column vector $\mathbf{v} \in \mathbb{R}^3$ such that $Q\mathbf{v} = \mathbf{v}$ or $Q\mathbf{v} = -\mathbf{v}$.

D. Q is orthogonally diagonalizable.

(4) Suppose that the singular values of a 2×2 matrix A are $\sigma_1 = 2$ and $\sigma_2 = 1$. Which of the following is/are true? (________________)

A. The set of eigenvalues of A can not be $\{0, 4\}$.

B. The determinant of A can not be -2.

C. The trace of A can not be -1.

D. The set of eigenvalues of A can be $\{\sqrt{2}\}$.

2. Fill in the blanks (16 points)

(1) Let $x \in \mathbb{R}$. Suppose that A and B are similar, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & x & 1 \\ 0 & -5 & -2 \end{bmatrix}$$

and $\mathrm{tr}(B) = 1$. Then

$$x = \underline{\hspace{4cm}}.$$

(2) Let $V = \mathrm{span}\{1, \cos x, \sin x\}$, the inner product on V is given by $\langle f(x), g(x)\rangle = \int_0^\pi f(x)g(x)dx$. Let $W = \mathrm{span}\{\cos x, \sin x\}$. Then

$$\mathrm{proj}_W(1) = \underline{\hspace{4cm}}.$$

(3) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Suppose that the QR-decomposition of A is given by $A = QR$, where Q is orthogonal and R is upper triangular. Then

$$Q = \underline{\hspace{4cm}}, \qquad R = \underline{\hspace{4cm}}.$$

(4) Let $y \in \mathbb{R}$ and $A = \begin{bmatrix} -1 & y \\ 2 & 3 \end{bmatrix}$. Suppose that the quadratic form $\mathbf{x}^\top (A^\top A)\mathbf{x}$ is not positive definite. Then

$$y = \underline{\hspace{4cm}}.$$

3. (10 points) Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

(1) Find all eigenvalues of A.

(2) Find all eigenvalues of A^2. And find a basis for each eigenspace of A^2.

4. (12 points) Let P_2 be the vector space of polynomials with degree ≤ 2. Let $T_1 : P_2 \to P_2$ be given by

$$T_1(p(x)) = p(-x) - x^2p(1).$$

Let $T_2 : P_2 \to P_2$ be given by

$$T_2(p(x)) = p(3x - 1).$$

(1) Prove that T_1 is a linear operator.

(2) Find the rank of T_1.

(3) Find the matrix $[T_1 \circ T_2]_S$ of $T_1 \circ T_2$ relative to the standard basis $S = \{1, x, x^2\}$.

5. (10 points) Given four points on the plane:

$$(1,-1),\quad (0,-2),\quad (2,0),\quad (-1,-3).$$

Fit a polynomial $\vec{p}$ with degree ≤ 2 to these four points, that is, find a polynomial

$$\vec{p}(x) = a_0 + a_1 x + a_2 x^2$$

such that

$$\left\| \begin{bmatrix} \vec{p}(1) \\ \vec{p}(0) \\ \vec{p}(2) \\ \vec{p}(-1) \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \\ 0 \\ -3 \end{bmatrix} \right\|$$

is the smallest possible. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^4$.

6. (10 points) Let c be a parameter. Consider

$$A = \begin{bmatrix} 0 & c & 1 \\ c & 1 & 0 \\ c & 0 & 1 \end{bmatrix}$$

For which values of $c \in \mathbb{R}$ is A not diagonalisable? Prove your answer.

7. (12 points) Define $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = 6x_1y_1 + 4x_1y_2 + 4x_2y_1 + 3\overset{x_2}{\cancel{y_2}}y_2.$$

Let $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (-3, 2)$.

(1) Prove that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^2$.

(2) Find the angle θ $(0 \leq \theta \leq \pi)$ between $\mathbf{u}$ and $\mathbf{v}$ with respect to this inner product $\langle \cdot, \cdot \rangle$.

(3) Let $\phi(\mathbf{x}) = x_1$ for $\mathbf{x} = (x_1, x_2)$. Find a vector $\mathbf{z} = (z_1, z_2)$ such that $\phi(\mathbf{x}) = \langle \mathbf{x}, \mathbf{z} \rangle$ for all $\mathbf{x} \in \mathbb{R}^2$.

8. (10 points) Consider the quadratic form

$$Q(x_1, x_2, x_3) = (a_1x_1 + a_2x_2 + a_3x_3)^2 + 2(b_1x_1 + b_2x_2 + b_3x_3)^2.$$

Let

$$\vec{\alpha} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ and } \vec{\beta} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

(1) Prove that the matrix corresponding to the quadratic form is $\vec{\alpha}\vec{\alpha}^{\top} + 2\vec{\beta}\vec{\beta}^{\top}$.

(2) Suppose that the set $\{\vec{\alpha}, \vec{\beta}\}$ is orthonormal. Prove that there exists an orthogonal matrix $P \in M_3$ such that, under the orthogonal change of variables $[x_1\ x_2\ x_3]^{\top} = P[y_1\ y_2\ y_3]^{\top}$, the quadratic form Q has the standard form

$$Q(y_1, y_2, y_3) = y_1^2 + 2y_2^2.$$

Please also find one such matrix P (write P in terms of $a_1, a_2, a_3, b_1, b_2, b_3$).