Linear Algebra Tutorial8

2023.11.28

Plus/Minus Theorem

V is a vector space, $S \subset V$

- If S is an independent set, and $\mathbf{v} \in V, \mathbf{v} \notin S$, then $S \cup \{\mathbf{v}\}$ is also an **independent set**

> proof by contradiction, suppose that $S \cup \{\mathbf{v}\}$ is linear dependent
> $\Rightarrow \mathbf{v} = span(S)$

- If $\mathbf{v} \in S$, and $\mathbf{v}$ can be written as a linear combination of other vectors in S, then $span(S - \mathbf{v}) = span(S)$

> $\mathbf{v} \in S$, WLOG, take $\mathbf{v} = v_1$, consider $\forall \mathbf{w} \in span(S)$, can be writen as linear combination of $\mathbf{v}_2, \cdots, \mathbf{v}_n$

coordinate

$n \geq 1, \dim(V) = n, S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis of V

- 1. for any vector set $M = \{\mathbf{w}_1, \cdots, \mathbf{w}_r\} \subset V$
  M is an independent set $\Leftrightarrow [\mathbf{v}_1]_S, \cdots, [\mathbf{v}_r]_S$ are independent set

> proof: set $[\mathbf{v}_i] = (a_{i1}, \cdots, a_{in})$, then $\mathbf{v}_i = a_{i1}\mathbf{v}_1 + \cdots + a_{in}\mathbf{v}_n$

- 2. for vector set $M = \{\mathbf{w}_1, \cdots, \mathbf{w}_n\} \subset V$
  M is the basis of $V \Leftrightarrow [\mathbf{v}_1]_S, \cdots, [\mathbf{v}_n]_S$ is the basis of $\mathbb{R}^n$
  $\Leftrightarrow [\mathbf{v}_1]_S, \cdots, [\mathbf{v}_n]_S$ is the standard basis of $\mathbb{R}^n$

> from 1., we know that M is independent $\Rightarrow [\mathbf{v}_1]_S, \cdots, [\mathbf{v}_n]_S$ is independent, so we just need to prove that $span\{[\mathbf{w}_1]_S, \cdots, [\mathbf{w}_n]_S\} = \mathbb{R}^n$

example

例子：(2022年线性代数考试题)

Let $p_1(x) = 1 + 3x$,$p_2(x) = 2 + 4x$,$p_3(x) = -4x^2$ be three polynomials in P_2.

1. Let $M = \{1, x, x^2\}$ be the standard basis of P_2. Let $A = \begin{bmatrix} [p_1(x)]_M & [p_2(x)]_M & [p_3(x)]_M \end{bmatrix}$ be the matrix such that its columns are $[p_1(x)]_M$, $[p_2(x)]_M$ and $[p_3(x)]_M$. Compute the adjoint matrix of A.
2. Prove that $S = \{p_1(x), p_2(x), p_3(x)\}$ is a basis of P_2.

basis' theorem

V is a vector space,$\dim(V) = n$, $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\} \subset V$

- $span(S) = V, m > n$, then we can delete some of the vectors in S to get a basis of V
- if S is a linear independent set, $m < n$, then we can add some vectors to S to get a basis of V

basis' theorem

V is a vector space, $\dim(V) = n$, $W \subset V$ is a subspace.

- let $m = \dim(W)$, then $m \leq n$
- $W = V$ iff $m = n$

Change of basis

V is the vector space, B, B' are two bases of V

- $B = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$
- $B' = \{\mathbf{v}'_1, \cdots, \mathbf{v}'_n\}$
  If we have $(\mathbf{v})_B = (c_1, \cdots, c_n)$
  how could we find $(\mathbf{v})_{B'} = (c'_1, \cdots, c'_n)$?

> as we known that $[\mathbf{v}]_B$, $[\mathbf{v}]_{B'}$ has the unique expression

Change of basis

- transition matrix(过渡矩阵/转移矩阵) P
  P is invertible, P^{-1} is called the transition matrix from B to B'
- transition matrix from B to B'
  $P_{B\to B'}$ or $P_{B'\leftarrow B}$
- transition matrix from B' to B
  $P_{B'\to B}$ or $P_{B\leftarrow B'}$
- $P_{B\leftarrow B'}P_{B'\leftarrow B} = I$

> notice that the defination of the transition matrix may be different with some of the Chinese textbooks!!

Change of basis

- We can represent the transition matrix as $P_{B \to B'}$ or $P_{B' \leftarrow B}$
- $[v]_{B'} = P_{B' \leftarrow B}[v]_B$
- $[v]_B = P_{B \leftarrow B'}[v]_{B'}$
- method to get the transition matrix
  $[B'|B] \Rightarrow [I|P_{B' \leftarrow B}]$

example of transition matrix

例子： 考虑$\mathbb{R}^3$的两组基底$B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$与$B' = \{\boldsymbol{u}_1', \boldsymbol{u}_2', \boldsymbol{u}_3'\}$:

$$\boldsymbol{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix};$$

$$\boldsymbol{u}_1' = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \boldsymbol{u}_2' = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \boldsymbol{u}_3' = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

我们利用之前介绍的计算步骤求解转移矩阵$P_{B' \leftarrow B}$。

Row space, Column space and Null space

A is a $m \times n$ matrix

- row space 行空间
  $row(A) = span\{\mathbf{r}_1, \cdots, \mathbf{r}_m\}$
- column space 列空间
  $col(A) = span\{\mathbf{c}_1, \cdots, \mathbf{c}_n\}$
- null space 零空间
  $null(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$
- left null space 左零空间
  $null(A^T) = \{\mathbf{x} \in \mathbb{R}^m : A^T\mathbf{x} = \mathbf{0}\}$

Fundamental Matrix Spaces

Definition 4.31. 对$m \times n$-矩阵A，以下六个向量空间被称为A的基本空间(*the fundamental spaces of* A):

- A的行空间$Row(A)$，
- A的列空间$Col(A)$，
- $A^\top$的行空间$Row(A^\top)$，
- $A^\top$的列空间$Col(A^\top)$，
- A的零空间$Null(A)$，
- $A^\top$的零空间$Null(A^\top)$。

- 行空间和零空间互为正交补
- 列空间和左零空间互为正交补

正交补(Orthogonal Complements)

> 正交: $col(A) \perp null(A)$
> 补: $col(A) + null(A) = \mathbb{R}^n$

Row space, Column space and Null space

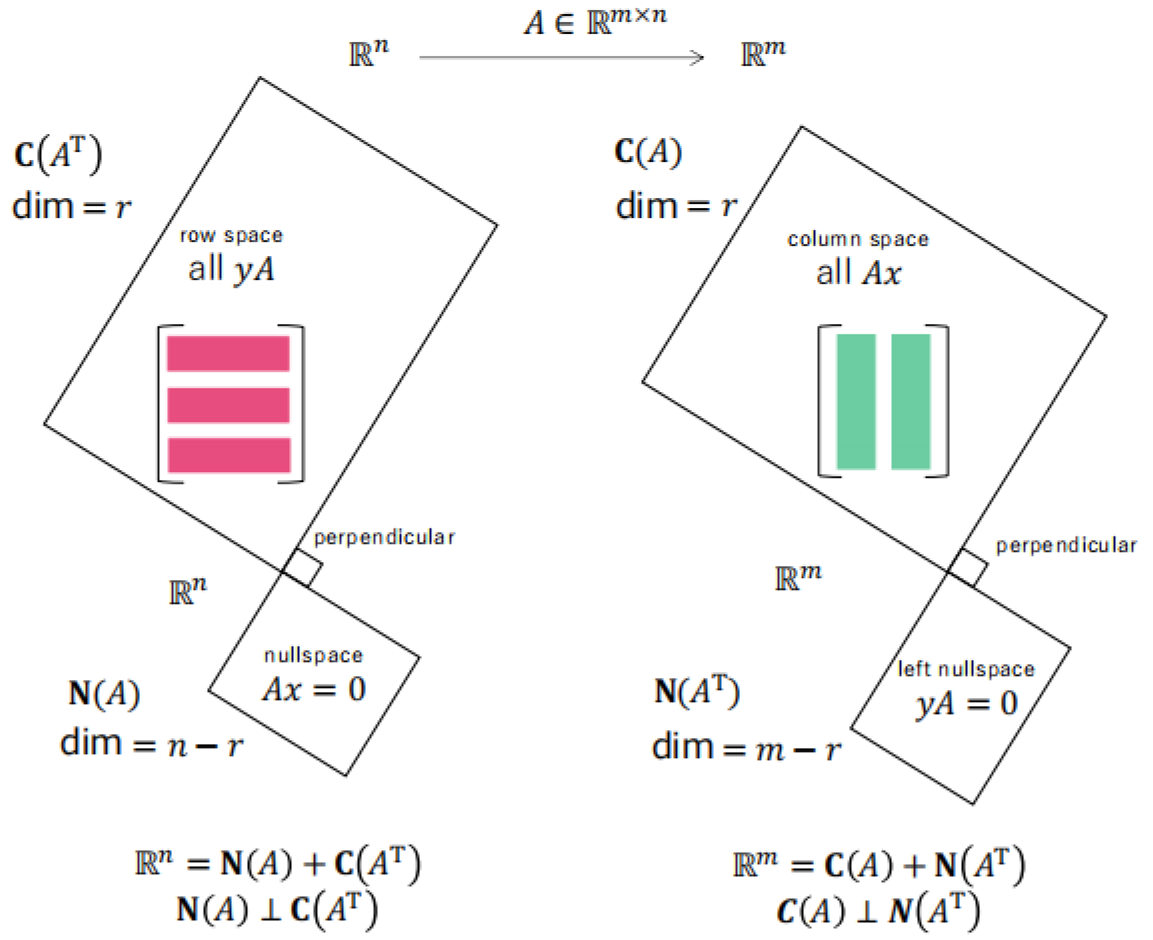
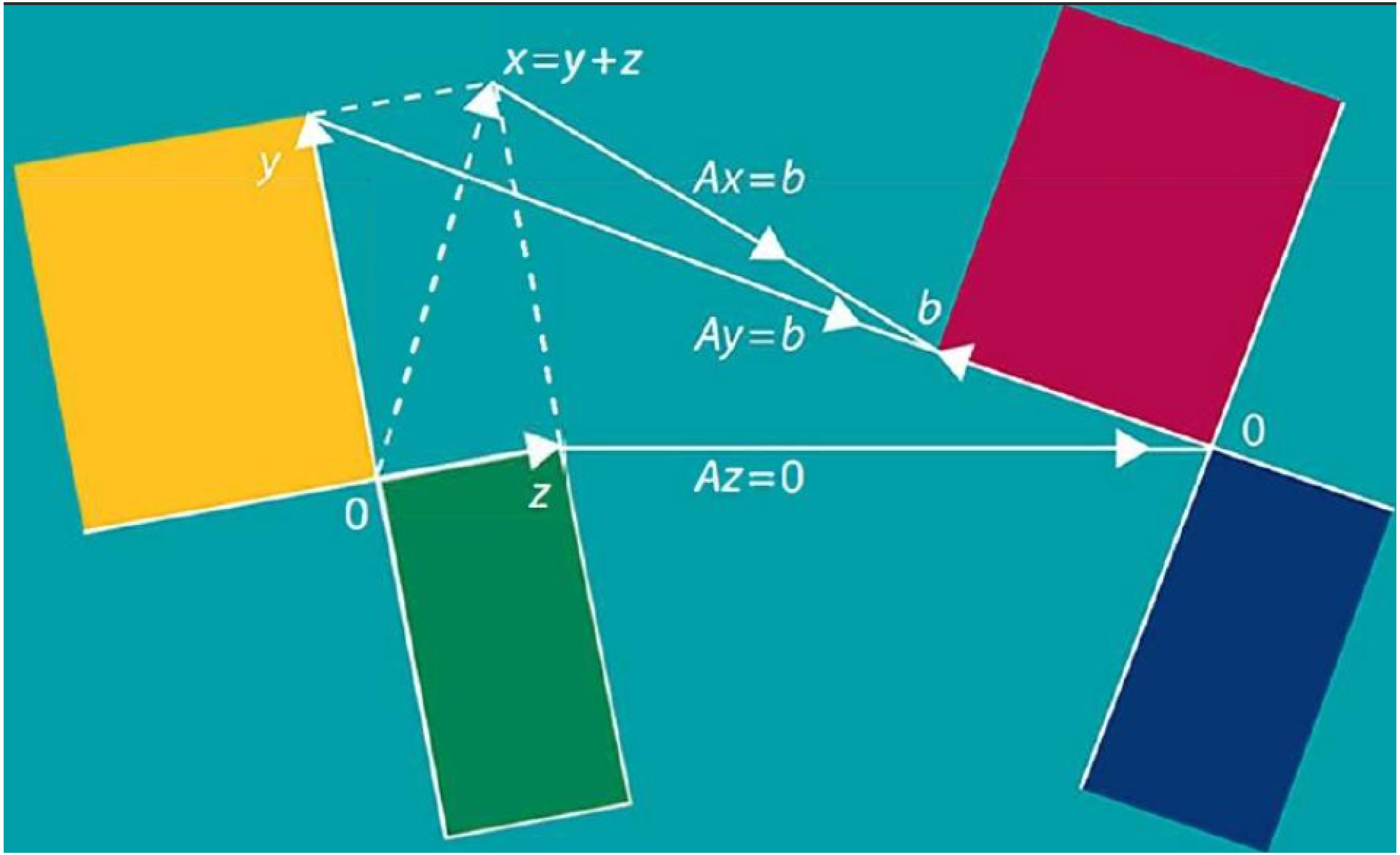


Figure 5: 四个子空间



x=y+z
y
Ax=b
b
Ay=b
0
z
Az=0
0

Null space

For a linear system $A\mathbf{x} = \mathbf{b}$

We can write the solutions as

$\mathbf{s} = \mathbf{s}_0 + c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$

where $\mathbf{s}_0$ is a particular solution, $\mathbf{v}_1, \cdots, \mathbf{v}_k$ is the basis of the null space of A

- $\mathbf{s}_0$: 特解
- $Null(A) = span\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$
  so $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are the basis of the null space of A

Rank, Nullity

- $rank(A) = dim(row(A)) = dim(col(A))$
  A 的秩 = A 的行阶梯矩阵的首一个数
  $\Rightarrow rank(A) \leq \min(n, m)$
- $rank(A)$ 可看作行阶梯矩阵的首一(非零行/主元) 个数
  $nullity(A) = dim(Null(A))$ 可看作自由元的个数
  $\Rightarrow rank(A) + nullity(A) = n$

rank property

- $A \in M_{m \times n}, W = \{A\mathbf{v} : \mathbf{v} \in \mathbb{R}^n\}$
  then $W = Col(A)$
- $rank(AB) \leq \min(rank(A), rank(B))$
  矩阵相乘秩不增
- $rank(A^T A) = rank(A)$

Equivalent expression

$A \in M_{n\times n}$

- A可逆；
- $A\boldsymbol{x}=\boldsymbol{0}$只有平凡解；
- A的简化阶梯型为单位矩阵；
- A是一组初等矩阵的乘积；
- $A\boldsymbol{x}=\boldsymbol{b}$对任何$n\times 1$的列向量$\boldsymbol{b}$都有解；
- $A\boldsymbol{x}=\boldsymbol{b}$对任何$n\times 1$的列向量$\boldsymbol{b}$都有且只有一个解；
- $\det(A)\neq 0$；
- A的所有n个行向量线性无关；
- A的所有n个列向量线性无关；
- $span(Row(A))=\mathbb{R}^n$；
- $span(Col(A))=\mathbb{R}^n$；
- A的所有n个行向量构成$\mathbb{R}^n$的一组基底；
- A的所有n个列向量构成$\mathbb{R}^n$的一组基底；
- $rank(A)=n$；
- $Null(A)=0$。

Geometry review

- detail in tutorial 5
- 平面的一般式
  $Ax + By + Cz + D = 0$
- 平面的法向量
  $\mathbf{n} = (A, B, C)$
- 空间中一点(x_0, y_0, z_0)到平面的距离
  $$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$
- 过空间内一点(x_0, y_0, z_0), 法向量为$\mathbf{v} = (a, b, c)$ 的平面的方程
  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$