线性代数(2023-2024)第十次作业

Problem A(6 Points), 部分取自2021-2022年线性代数期末考试题

Let $P_2 = \{a_0+a_1x+a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$, $P_3 = \{a_0+a_1x+a_2x^2+a_3x^3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$. The function $T : P_2 \to P_3$ is defined by

$$T(p(x)) = xp(2x+1) - 2xp(x)$$

for any $p(x) \in P_2$.

1. (2 points) Prove that $T : P_2 \to P_3$ is a linear transformation.

   答案： 直接验证对于任何$p(x) \in P_2, q(x) \in P_2$，$k \in \mathbb{R}$，

   $$\begin{aligned} T(p(x)+q(x)) &= x(p+q)(2x+1) - 2x(p+q)(x) \\ &= xp(2x+1) - 2xp(x) + xq(2x+1) - 2xq(x) \\ &= T(p(x)) + T(q(x)), \end{aligned}$$

   $T(kp(x)) = x(kp)(2x+1) - 2x(kp)(x) = k(xp(2x+1) - 2xp(x)) = kT(P(x))$,
   因此T是一个线性变换。

2. (2 points) Let $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3\}$ be the standard basis of P_2 and P_3, respectively. Compute the matrix for T relative to B and B', i.e., $[T]_{B',B}$.

   答案： 注意对于$p_1(x) = 1$，我们有

   $$T(p_1(x)) = xp_1(2x+1) - 2xp_1(x) = x - 2x = -x \Rightarrow [T(p_1(x))]_{B'} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

   对于$p_2(x) = x$，我们有

   $$T(p_2(x)) = xp_2(2x+1) - 2xp_2(x) = x(2x+1) - 2x(x) = x \Rightarrow [T(p_2(x))]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

   对于$p_3(x) = x^2$，我们有

   $$T(p_3(x)) = xp_3(2x+1) - 2xp_3(x) = x(2x+1)^2 - 2x(x^2) = 2x^3 + 4x^2 + x \Rightarrow [T(p_3(x))]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 2 \end{bmatrix},$$

因此由于$B=\{p_1(x),p_2(x),p_3(x)\}$，我们得到所求矩阵为

$$[T]_{B',B}=\Big[[T(p_1(x))]_{B'}\quad [T(p_2(x))]_{B'}\quad [T(p_3(x))]_{B'}\Big]=\begin{bmatrix}0&0&0\\-1&1&1\\0&0&4\\0&0&2\end{bmatrix}.$$

3. (2 points) Compute rank(T) and nullity(T).

答案： 容易计算出$[T]_{B',B}$的简化阶梯型为

$$R=\begin{bmatrix}1&-1&-1\\0&0&1\\0&0&1\\0&0&0\end{bmatrix},$$

因此可得rank$(T)=$ rank$([T]_{B',B})=2$，以及

$$\text{nullity}(T)=\text{nullity}([T]_{B',B})=\dim(P_2)-\text{rank}([T]_{B',B})=3-2=1.$$

Problem B(6 Points), 2021-2022年线性代数期末考试题

Let $V=C(-\infty,\infty)$ be the vector space of continuous functions on $\mathbb{R}$. Define functions $T_1:V\to V$ and $T_2:V\to V$ by

$$T_1(f(x))=e^xf(x),\quad T_2(f(x))=f(5x-1)$$

for any $f\in V$.

1. (3 points) Prove that T_1 and T_2 are linear operators on V.

答案： 直接通过定义验证即可。对于$f(x)\in V,g(x)\in V$， $k\in\mathbb{R}$，有

$$T_1(f(x))=e^x(f+g)(x)=e^x(f(x)+g(x))=e^xf(x)+e^xg(x)=T_1(f(x))+T_1(g(x)),$$

$$T_1(kf(x))=e^x(kf(x))=ke^xf(x)=kT_1(f(x)),$$

$$T_2(f(x)+g(x))=(f+g)(5x-1)=f(5x-1)+g(5x-1)=T_2(f(x))+T_2(g(x)),$$

$$T_2(kf(x))=kf(5x-1)=kT_2(f(x)).$$

2. (3 points) Take $g(x)=e^{-x}$. Find $(T_1\circ T_2)(g(x))$ and $(T_2\circ T_1)(g(x))$.

答案： $T_2(g(x))=g(5x-1)=e^{-(5x-1)}=e^{-5x+1}$，$T_1(e^{-5x+1})=e^xe^{-5x+1}=e^{-4x+1}$，因此$(T_1\circ T_2)(x)=e^{-4x+1}$。
$T_1(g(x))=e^xe^{-x}=1$， $T_2(1)=1$，因此$(T_2\circ T_1)(x)=1$。

Problem C(6 Points), 2021-2022年线性代数期末考试题

Let V be an n–dimensional vector space, and $T: V \to V$ a linear operator on V. Suppose that there is a $\boldsymbol{v} \in V$ such that $T^{n-1}(\boldsymbol{v}) \neq \mathbf{0}$ and $T^n(\boldsymbol{v}) = \mathbf{0}$. Here we use T^k ($k \geq 1$) to denote the composition of T for k times, i.e.,

$$T^1 = T, \quad T^2 = T \circ T, \quad T^3 = T \circ T \circ T, \quad \ldots, \quad T^k = \underbrace{T \circ T \circ \ldots \circ T}_{k \text{ times}}.$$

(We also define $T^0 = \mathrm{Id}_V$ to be the identity map on V, i.e., $T^0(\boldsymbol{u}) = \mathrm{Id}_V(\boldsymbol{u}) = \boldsymbol{u}$ for any $\boldsymbol{u} \in V$).

1. (4 points) Prove that $B = \{\boldsymbol{v}, T(\boldsymbol{v}), T^2(\boldsymbol{v}), \ldots, T^{n-1}(\boldsymbol{v})\}$ is a basis of V.

答案： 由于V为n维向量空间且$B = \{\boldsymbol{v}, T(\boldsymbol{v}), T^2(\boldsymbol{v}), \ldots, T^{n-1}(\boldsymbol{v})\}$包含了$n$个向量，因此要证明$B$是$V$的基底只需证明$B$为一个线性无关集合。

不妨假设$n \geq 2$($n = 1$的情况非常简单)。考虑等式$k_1\boldsymbol{v} + k_2T(\boldsymbol{v}) + \ldots + k_nT^{n-1}(\boldsymbol{v}) = \mathbf{0}$。若该等式对于系数$k_1, \ldots, k_n$成立，那么必然有

$$T^{n-1}(k_1\boldsymbol{v} + k_2T(\boldsymbol{v}) + \ldots + k_nT^{n-1}(\boldsymbol{v})) = T^{n-1}(\mathbf{0}) = \mathbf{0},$$

因为T^{n-1}作为一个线性变换必然满足$T^{n-1}(\mathbf{0}) = \mathbf{0}$。
现在我们利用T^{n-1}的线性性质将$T^{n-1}(k_1\boldsymbol{v} + k_2T(\boldsymbol{v}) + \ldots + k_nT^{n-1}(\boldsymbol{v}))$展开：

$$\begin{aligned} T^{n-1}(k_1\boldsymbol{v} + k_2T(\boldsymbol{v}) + \ldots + k_nT^{n-1}(\boldsymbol{v})) &= k_1T^{n-1}(\boldsymbol{v}) + k_2T^{n-1}(T(\boldsymbol{v})) + \ldots + k_nT^{n-1}(T^{n-1}(\boldsymbol{v})) \\ &= k_1T^{n-1}(\boldsymbol{v}) + k_2T^n(\boldsymbol{v}) + \ldots + k_nT^{2n-2}(\boldsymbol{v}). \end{aligned}$$

由假设，$T^n(\boldsymbol{v}) = \mathbf{0}$，因此对于任何$k \geq n$，都有$T^k(\boldsymbol{v}) = \mathbf{0}$；特别地，我们有

$$k_2T^n(\boldsymbol{v}) + \ldots + k_nT^{2n-2}(\boldsymbol{v}) = k_2\mathbf{0} + \ldots + k_n\mathbf{0} = \mathbf{0}.$$

由以上分析可得，

$$\mathbf{0} = T^{n-1}(k_1\boldsymbol{v} + k_2T(\boldsymbol{v}) + \ldots + k_nT^{n-1}(\boldsymbol{v})) = k_1T^{n-1}(\boldsymbol{v}).$$

又因为已经假设$T^{n-1}(\boldsymbol{v}) \neq \mathbf{0}$，等式$\mathbf{0} = k_1T^{n-1}(\boldsymbol{v})$成立必然要求$k_1 = 0$。因此原等式$k_1\boldsymbol{v} + k_2T(\boldsymbol{v}) + \ldots + k_nT^{n-1}(\boldsymbol{v}) = \mathbf{0}$现在变为

$$k_2T(\boldsymbol{v}) + \ldots + k_nT^{n-1}(\boldsymbol{v}) = \mathbf{0}.$$

类似的，将T^{n-2}作用在以上等式上，可得

$$T^{n-2}(k_2T(\boldsymbol{v}) + \ldots + k_nT^{n-1}(\boldsymbol{v})) = T^{n-2}(\mathbf{0}) = \mathbf{0}.$$

那么重复以上分析可得$T^{n-2}(k_2T(\boldsymbol{v})+\ldots+k_nT^{n-1}(\boldsymbol{v}))=k_2T^{n-1}(\boldsymbol{v})=\mathbf{0}$，因此$k_2=0$必须成立。继续该步骤，我们可以最终得到：等式$k_1\boldsymbol{v}+k_2T(\boldsymbol{v})+\ldots+k_nT^{n-1}(\boldsymbol{v})=\mathbf{0}$要成立只能让$k_1=k_2=\ldots=k_n=0$。证毕。

2. (2 points) Find the matrix for T relative to B, i.e., $[T]_{B,B}$.

答案： 注意，对于$B=\{\boldsymbol{v},T(\boldsymbol{v}),\ldots,T^{n-1}(\boldsymbol{v})\}$，有

$$[T(\boldsymbol{v})]_B=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}=\boldsymbol{e}_2\in\mathbb{R}^n,$$

$$[T(T(\boldsymbol{v}))]_B=[T^2(\boldsymbol{v})]_B=\begin{bmatrix}0\\0\\0\\1\\\vdots\\0\end{bmatrix}=\boldsymbol{e}_3\in\mathbb{R}^n,$$

依次类推，可得

$$[T(T^k(\boldsymbol{v}))]_B=\boldsymbol{e}_{k+2}$$

对于任何$k=0,\ldots,n-2$成立，以及$[T(T^{n-1}(\boldsymbol{v}))]_B=\mathbf{0}\in\mathbb{R}^n$。因此所求矩阵为

$$[T]_{B,B}=\begin{bmatrix}\boldsymbol{e}_2 & \boldsymbol{e}_3 & \ldots & \boldsymbol{e}_n & \mathbf{0}\end{bmatrix}=\begin{bmatrix}0&0&0&\ldots&0&0\\1&0&0&\ldots&0&0\\0&1&0&\ldots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&\ldots&1&0\end{bmatrix}.$$

Problem D(6 Points), 2022-2023年线性代数期末考试题

Suppose that $n\geq 2$ is a positive integer to be determined. Let $T:P_n\to P_n$ be a linear operator given by

$$T(p(x))=p'(x)+p(1)x^2$$

for any $p(x)\in P_n=\{a_0+a_1x+\ldots+a_nx^n:a_0,a_1,\ldots,a_n\in\mathbb{R}\}$. Suppose that for $T^2=T\circ T$ we have $\operatorname{rank}(T^2)=3$. Find all possible values of this integer n.

答案： 由$T(p(x)) = p'(x) + p(1)x^2$可得

$T^2(p(x)) = p''(x)+p'(1)x^2+p(1)(2x)+p(1)(1)^2x^2 = p''(x)+2p(1)x+(p'(1)+p(1))x^2.$

现在取P_n的标准基底$B = \{1, x, x^2, \ldots, x^n\}$，通过计算可得

$$T^2(1) = 2x + x^2,$$

$$T^2(x) = 2x + 2x^2,$$

$$T^2(x^2) = 2 + 2x + 3x^2,$$

$$T^2(x^3) = 8x + 4x^2,$$

$$T^2(x^4) = 2x + 17x^2,$$

以及

$$T^2(x^n) = 2x + (n+1)x^2 + n(n-1)x^{n-2}$$

对于$n \geq 5$。由此可得，当$n = 2$时，

$$[T^2]_{B,B} = \begin{bmatrix} 0 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix},$$

容易验证此时rank$(T^2) = $ rank$([T^2]_{B,B}) = 3$。

当$n = 3$时，

$$[T^2]_{B,B} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 2 & 2 & 8 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

容易验证此时rank$(T^2) = $ rank$([T^2]_{B,B}) = 3$。

当$n = 4$时，

$$[T^2]_{B,B} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 2 & 2 & 2 & 8 & 2 \\ 1 & 2 & 3 & 4 & 17 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

容易验证此时rank$(T^2) = $ rank$([T^2]_{B,B}) = 3$。

当$n \geq 5$时，

$$[T^2]_{B,B} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & \ldots \\ 2 & 2 & 2 & 8 & 2 & 2 & \ldots \\ 1 & 2 & 3 & 4 & 17 & 6 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 20 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \end{bmatrix}$$

显然此时rank$(T^2) = $ rank$([T^2]_{B,B}) \geq 4$。因此只有当$n = 2, 3, 4$时，有rank$(T^2) = 3$。

Problem E(6 Points), Multiple Choices

1. (3 points) Which of the following statements are true?

   - A. Let V be an n–dimensional vector space, B is a basis of V, then the coordinate vector mapping $f_B : V \to \mathbb{R}^n$, $f_B(\boldsymbol{v}) = [\boldsymbol{v}]_B$ is an isomorphism.

     正确。 见讲义Theorem 5.7.

   - B. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $\ker(T) \neq \{\mathbf{0}\}$ and RAN$(T) \neq \{\mathbf{0}\}$. Then there exists a non–zero vector $\boldsymbol{x} \in \mathbb{R}^2, \boldsymbol{x} \neq \mathbf{0}$ such that $\boldsymbol{x} \in \ker(T) \cap$ RAN(T). (Here, $\ker(T)$ is the kernel of T, RAN(T) is the range of T.)

     错误。 反例：令$T = T_A$， $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$，显然RAN$(T) = \{k \begin{bmatrix} 1 \\ 1 \end{bmatrix} : k \in \mathbb{R}\}$，$\ker(T) = \{k \begin{bmatrix} 0 \\ 1 \end{bmatrix} : k \in \mathbb{R}\}$，因此RAN$(T) \cap \ker(T) = \{\mathbf{0}\}$，即不存在非零的$\boldsymbol{v} \in$ RAN$(T) \cap \ker(T)$。

   - C. Let V, W be vector space and $T : V \to W$ is an isomorphism. Let $B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a basis of V, then $B' = \{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)\}$ is a basis of W.

     正确。 由于$T : V \to W$为一个同构，必然有$\dim(V) = \dim(W)$。我们现在证明$B' = \{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)\}$是一个线性无关集合(因此它是$W$的一组基底)。显然，如果存在$k_1, \ldots, k_n \in \mathbb{R}$使得$k_1 T(\boldsymbol{v}_1) + \ldots + k_n T(\boldsymbol{v}_n) = \mathbf{0}$，那么可以得到$k_1 T(\boldsymbol{v}_1) + \ldots + k_n T(\boldsymbol{v}_n) = T(k_1 \boldsymbol{v}_1 + \ldots + k_n \boldsymbol{v}_n) = \mathbf{0}$，从而有$k_1 \boldsymbol{v}_1 + \ldots + k_n \boldsymbol{v}_n \in \ker(T)$。由于$T$为同构，特别地，为一个一一映射，我们有$\ker(T) = \{\mathbf{0}\}$，因此$k_1 \boldsymbol{v}_1 + \ldots + k_n \boldsymbol{v}_n = \mathbf{0}$。因此，由于$B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$是$V$的基底，利用它们的线性无关性可得$k_1 = \ldots = k_n = 0$。

   - D. Let V be an n–dimensional vector space, and $T : V \to V$ be the identity operator, i.e., $T(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in V$. Then for any basis B of V, we have

$[T]_{B,B} = I_n$, where $I_n \in M_{n\times n}$ is the identity matrix.

正确。 对于$B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$，可以验证$[T(\boldsymbol{v}_i)]_B = [\boldsymbol{v}_i]_B = \boldsymbol{e}_i$对于任何$i = 1, \ldots, n$。因此$[T]_{B,B} = I_n$。

2. (3 points) Which of the following statements are true?

- A. Let $T : V \to W$ be a linear transformation, and $W' \subset W$ is a subspace of W. Then the set
$$V' = \{\boldsymbol{v} \in V : T(\boldsymbol{v}) \in W'\}$$
is always a subspace of V.

  正确。 直接验证关于加法与标量积的封闭性即可。

- B. If $T : U \to V$ is a surjective (满射) linear transformation, $S : V \to W$ is also a surjective linear transformation, then $S \circ T$ is always surjective.

  正确。 对任何$\boldsymbol{w} \in W$，由于S是满射，存在$\boldsymbol{v} \in V$使得$S(\boldsymbol{v}) = \boldsymbol{w}$。对于$\boldsymbol{v} \in V$，由于$T$为满射，存在$\boldsymbol{u} \in U$使得$T(\boldsymbol{u}) = \boldsymbol{v}$，因此$(S \circ T)(\boldsymbol{u}) = S(\boldsymbol{v}) = \boldsymbol{w}$，所以$S \circ T$是满射。

- C. Let $A \in M_{n\times n}$. Consider the following partitioned matrix $C \in M_{2n\times 2n}$
$$C = \begin{bmatrix} A & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & A^\top A \end{bmatrix},$$
here $\mathbf{0}_{n\times n}$ denotes the zero matrix in $M_{n\times n}$. Then we always have $\text{rank}(C) = 2\text{rank}(A)$.

  正确。 已知$\text{rank}(A) = \text{rank}(A^\top A)$，见讲义Theorem 4.38. 另外容易看出来C的阶梯型为
$$R = \begin{bmatrix} R_1 & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & R_2 \end{bmatrix}$$
R_1为A的阶梯型，R_2为$A^\top A$的阶梯型。显然R的主一数量等于R_1与R_2的主一之和。由于R_1与R_2的主一个数相同，可得可得$\text{rank}(C) = 2\text{rank}(A)$。

- D. The function
$$T : M_{n\times n} \to \mathbb{R}, \quad T(A) = \text{tr}(A)$$

is a surjective linear transformation. Here $\mathrm{tr}(A)$ denotes the trace of A.

正确。 只需要注意对于单位矩阵I_n，$\mathrm{tr}(I_n) = n \neq 0$，因此$\mathrm{rank}(T = \mathrm{tr}) = 1$。

Bonus: 不计入分数

假设V为一个向量空间，令$W = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \subset V$为$V$的一个子空间，这里$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$线性无关。对于一个给定的$\boldsymbol{v} \in V$，考虑子空间

$$U = \mathrm{span}\{\boldsymbol{v}_1 + \boldsymbol{v}, \boldsymbol{v}_2 + \boldsymbol{v}, \boldsymbol{v}_3 + \boldsymbol{v}\}.$$

由期中考试不定项选择题的第三小题，我们知道$\dim(U)$可以等于2或者3。现在请对此提供一个严格的数学证明，特别地，证明$\dim(U)$不可能小于$3 - 1 = 2$。

答案： 首先假设$\boldsymbol{v} \in V$但$\boldsymbol{v} \notin W = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$。此时$\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}$线性无关(见讲义Theorem 4.18)。我们可以证明$\boldsymbol{v}_1 + \boldsymbol{v}, \boldsymbol{v}_2 + \boldsymbol{v}, \boldsymbol{v}_3 + \boldsymbol{v}$线性无关：考虑等式$\mathbf{0} = k_1(\boldsymbol{v}_1 + \boldsymbol{v}) + k_2(\boldsymbol{v}_2 + \boldsymbol{v}) + k_3(\boldsymbol{v}_3 + \boldsymbol{v}) = k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + k_3\boldsymbol{v}_3 + (k_1 + k_2 + k_3)\boldsymbol{v}$。由于$\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}$线性无关，以上等式成立的必要条件是$k_1 = k_2 = k_3 = 0$。因此$\{\boldsymbol{v}_1 + \boldsymbol{v}, \boldsymbol{v}_2 + \boldsymbol{v}, \boldsymbol{v}_3 + \boldsymbol{v}\}$是$U$的一组基底，即此时$\dim(U) = 3$。

现在假设$\boldsymbol{v} \in W$，并且$\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3$。依据提示，我们知道

$$\dim U = \dim(\mathrm{span}\{\boldsymbol{v}_1 + \boldsymbol{v}, \boldsymbol{v}_2 + \boldsymbol{v}, \boldsymbol{v}_3 + \boldsymbol{v}\}) = \mathrm{rank}([T]_{B,B}).$$

由于$[\boldsymbol{v}_1 + \boldsymbol{v}]_B = \begin{bmatrix} 1 + c_1 \\ c_2 \\ c_3 \end{bmatrix}$，$[\boldsymbol{v}_2 + \boldsymbol{v}]_B = \begin{bmatrix} c_1 \\ 1 + c_2 \\ c_3 \end{bmatrix}$，$[\boldsymbol{v}_3 + \boldsymbol{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ 1 + c_3 \end{bmatrix}$，可知

$$[T]_{B,B} = \begin{bmatrix} 1 + c_1 & c_1 & c_1 \\ c_2 & 1 + c_2 & c_2 \\ c_3 & c_3 & 1 + c_3 \end{bmatrix}.$$

现在将$[T]_{B,B}$拆分为

$$[T]_{B,B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} c_1 & c_1 & c_1 \\ c_2 & c_2 & c_2 \\ c_3 & c_3 & c_3 \end{bmatrix}.$$

令$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$，$C = \begin{bmatrix} c_1 & c_1 & c_1 \\ c_2 & c_2 & c_2 \\ c_3 & c_3 & c_3 \end{bmatrix}$。以上等式$[T]_{B,B} = A + C$可以写为$A = [T]_{B,B} - C$。假设$\mathrm{rank}([T]_{B,B}) = \dim(\mathrm{RAN}([T]_{B,B})) = r$，且$\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$为$\mathrm{RAN}([T]_{B,B})$的

一组基底，那么由于RAN(C)的基底显然可以选取为$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$，我们容易看出RAN$([T]_{B,B} - C) \subset \text{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r, \boldsymbol{c}\}$，从而有

$$\dim(\text{RAN}([T]_{B,B} - C)) \leq r + 1.$$

现在我们可以得到

$$3 = \dim(\text{RAN}(A)) = \dim(\text{RAN}([T]_{B,B} - C)) \leq r + 1,$$

从而有$r = \text{rank}([T]_{B,B}) \geq 3 - 1 = 2$。因此我们最终得到

$$2 \leq r = \text{rank}([T]_{B,B}) \leq 3.$$

Deadline: 22:00, December 24.

作业提交截止时间：12月24日晚上22：00。