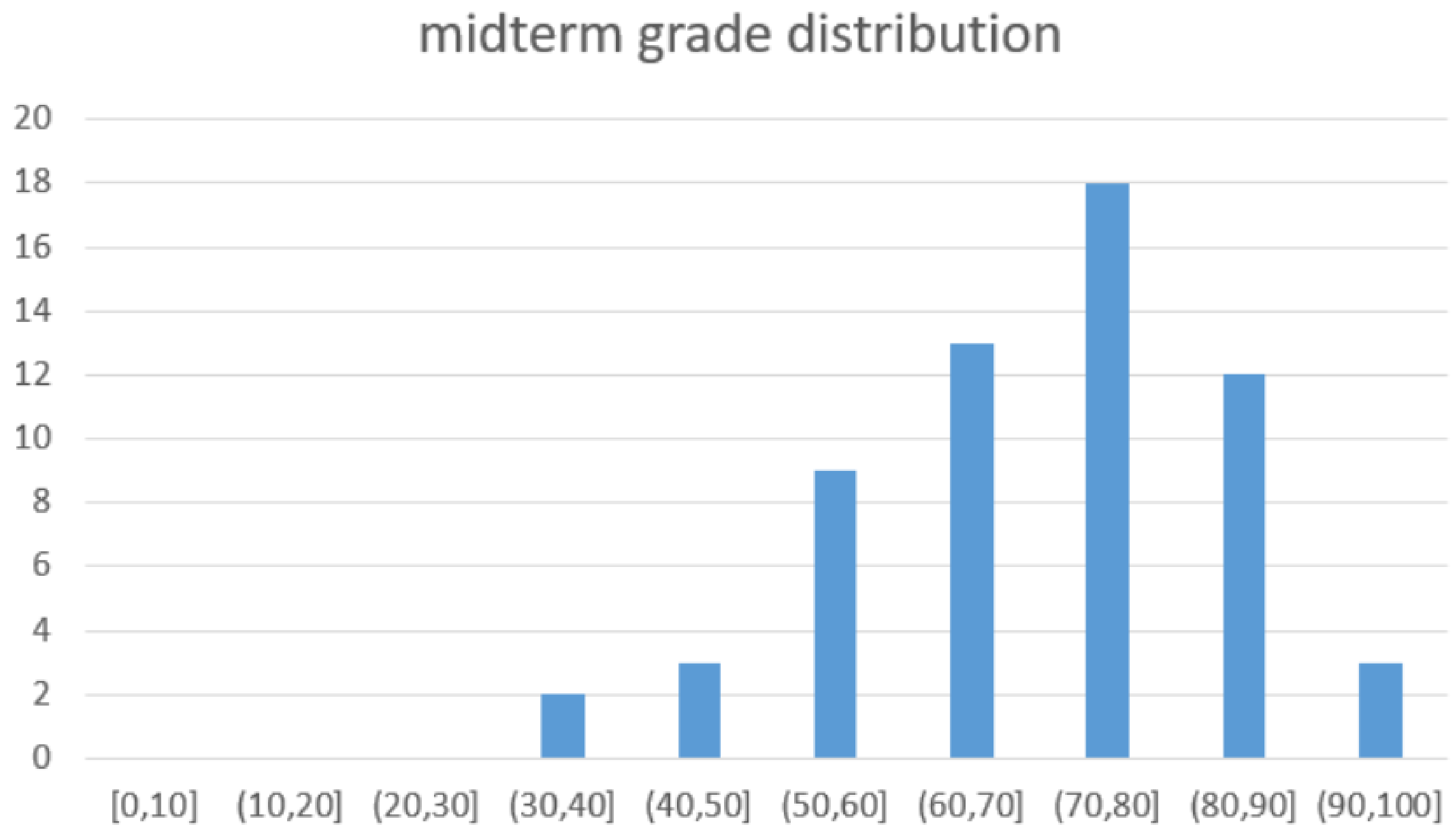
Linear Algebra Tutorial 10

2023.12.12

mid-term



mean	69.6353
median	71
std	14.3608

mid-term

a). (5 *points*) Which of the following sets are vector spaces? (   )

(A) $\{(a,b) \in \mathbb{R}^2 : \; b = 2a + 3\} \subseteq \mathbb{R}^2$, with the usual "$+$" and "$\cdot$" as in $\mathbb{R}^2$.

(B) $\{\mathbf{v} \in \mathbb{R}^3 : \; ||\mathbf{v}|| = 1\} \subseteq \mathbb{R}^3$, with the usual "$+$" and "$\cdot$" as in $\mathbb{R}^3$.

(C) {All polynomials in P_2 that are divisible by $x - 2$}, with the usual "$+$" and "$\cdot$" as in P_2.

(D) The set $\mathbb{R}^2$, with addition and scalar multiplication given by: for $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $k \in \mathbb{R}$, $\mathbf{x} + \mathbf{y} := (x_1 + 2y_1,\; x_2 + 3y_2)$, $\;k\mathbf{x} := (kx_1, kx_2)$.

mid-term

THEOREM 4.2.1 *If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.*

(*a*) *If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u}+\mathbf{v}$ is in W.*

(*b*) *If k is a scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.*

V is **vectoe space**

- $x+(-x)=0$
  $-x=(-x_1,-x_2)$
  $x+(-x)=(x_1-2x_1,x_2-3x_2)\neq 0$

linearty 线性

$T: U \rightarrow V$ linear transform: 线性变换

- additivity 可加性
  $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- homogeneity 齐次性
  $T(k\mathbf{x}) = kT(\mathbf{x})$

> T 可看作是一个函数f, 所做的操作是将$\mathbf{x}$映射到$T(\mathbf{x})$, $(\mathbf{x} \rightarrow A\mathbf{x})$

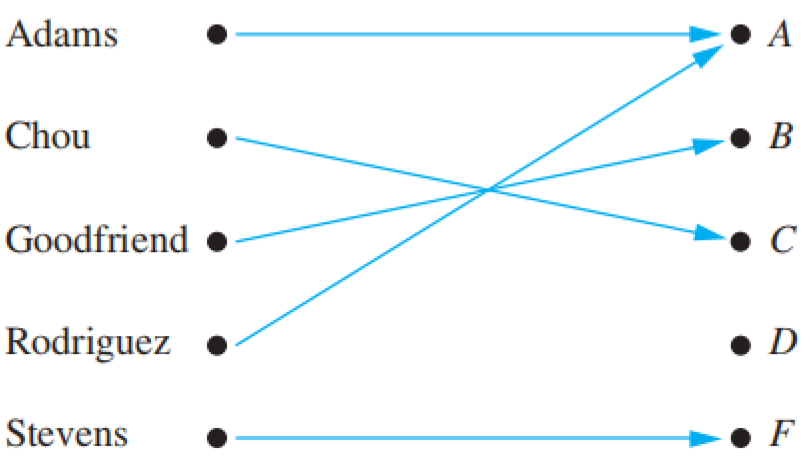
Matrix transformation

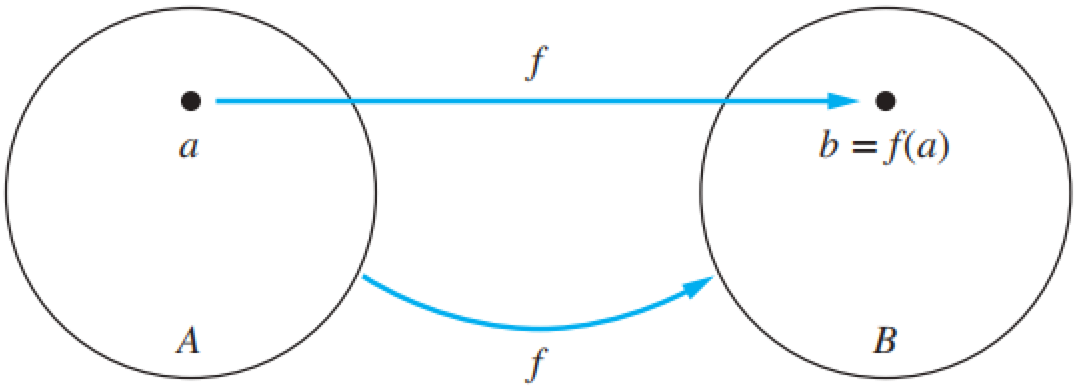
- mutiplication of matrix and vector is a linear transformation
- $A = [T] = [T(\mathbf{e}_1), T(\mathbf{e}_2), \cdots, T(\mathbf{e}_n)]$
  A: the standard matrix of T

> 复合函数(composition) $f(g(x))$ 也可写作 $(f \circ g)(x)$
> 同理: $(T_2 \circ T_1)(\mathbf{x}) = [T_2][T_1](\mathbf{x})$

concepts

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b. The *range*, or *image*, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f *maps* A to B.





- image 像 ⇔ range 值域
- mapping 映射

concepts

- domain 定义域
  {Adams, Chou, Goodfriend, Rodriguez, Stevens}
- codomain 陪域
  A, B, C, D, F
- range 值域
  A, B, C, F
- 定义域是映射的集合, 值域是**被**映射**到**的集合, 陪域是**可被**映射到的集合
  值域是陪域的子集

concepts

- kernel 核

  $Ker(T) = \{\mathbf{x} \in V | T(\mathbf{x}) = \mathbf{0}\}$

  $Ker(T) \Leftrightarrow Null(A)$

- range 值域

  $RAN(T) = \{\mathbf{y} \in W | \mathbf{y} = T(\mathbf{x}), \mathbf{x} \in V\}$

  $RAN(T) \Leftrightarrow Col(A)$

concepts

- injective(one-to-one) 单射(一一映射)

  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

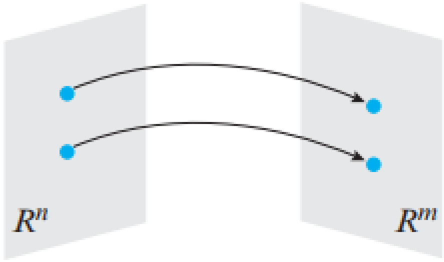
  > 只有一一映射才存在逆映射(反函数)

- surjective(onto) 满射

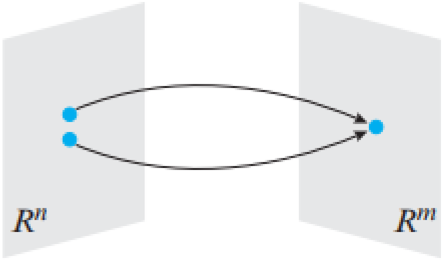
  $\forall y \in Y, \exists x \in X, s.t. f(x) = y$

- bijective 双射

  injective + surjective



One-to-one



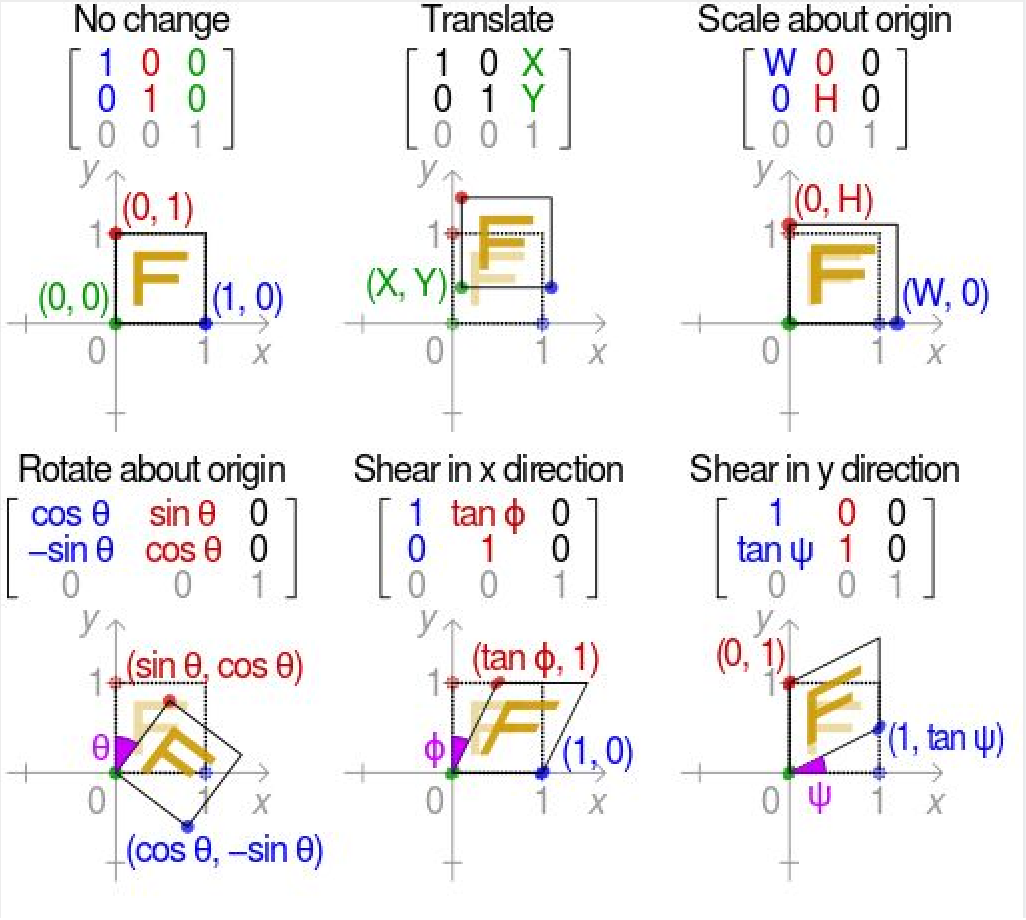
Not one-to-one

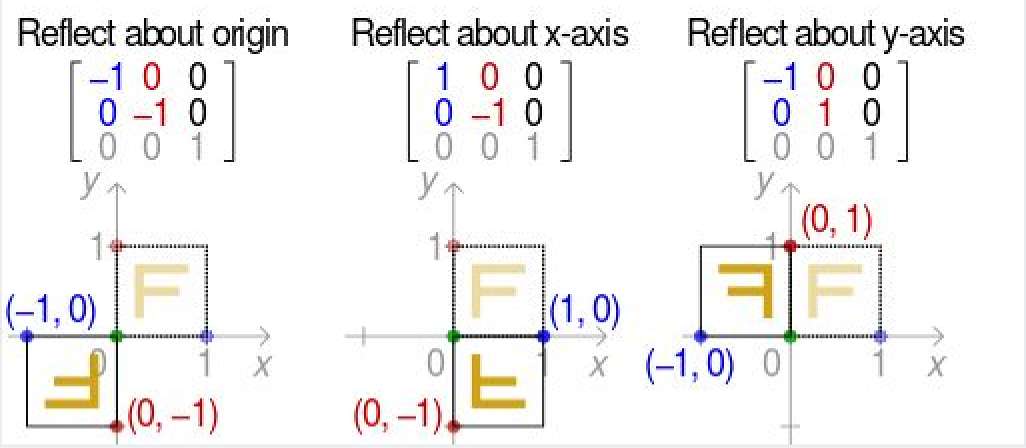
Matrix transformation

https://www.bilibili.com/video/BV1ys411472E?p=4&vd_source=6b1c6ae9b58bc4261b8429b79364410d

- method : 推出(x, y, z)变换后的坐标(x', y', z') 即可得到变换矩阵

affine transformation 仿射变换

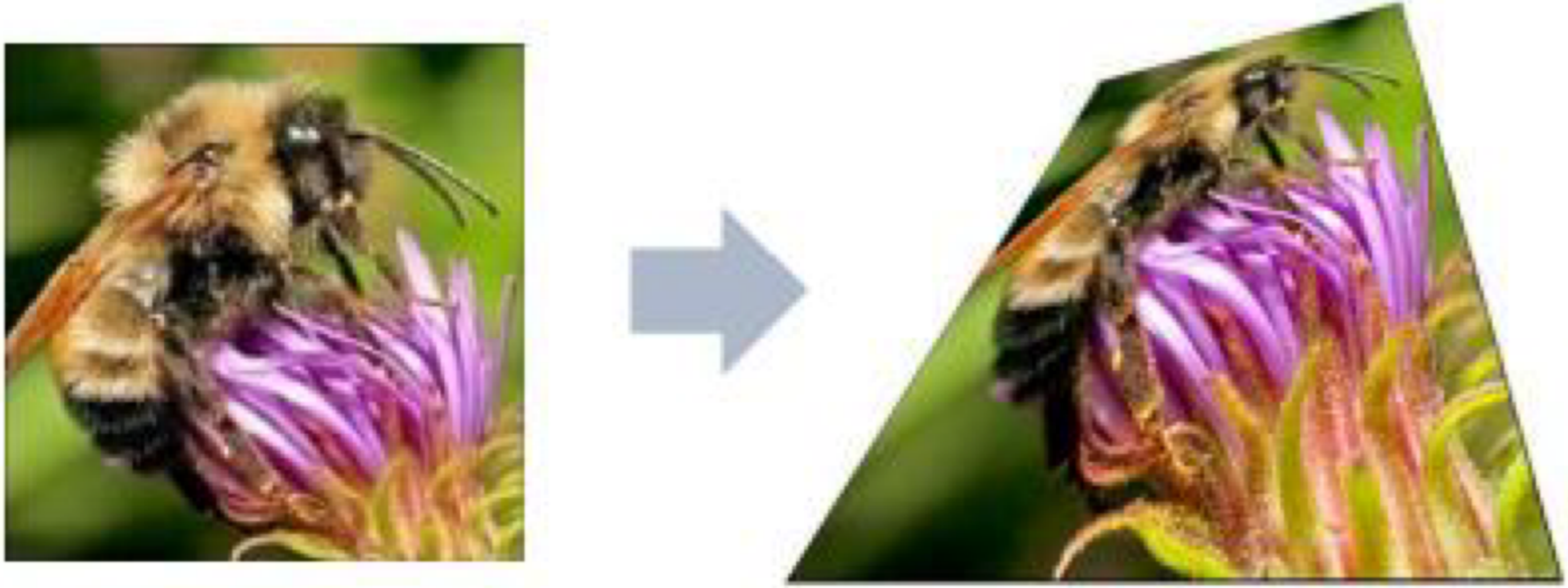




仿射变换: 平行不变性

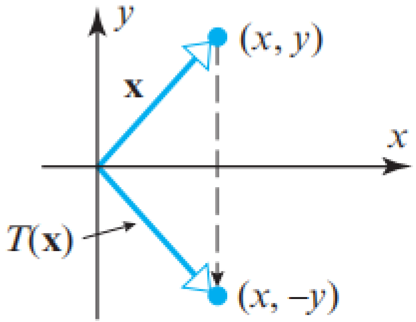
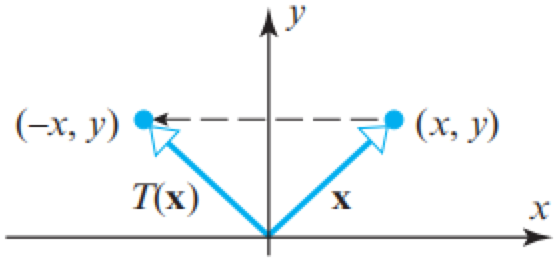
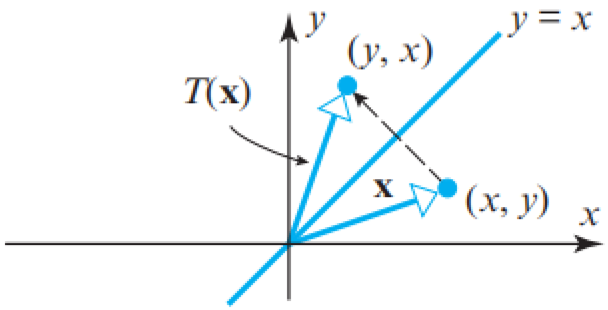
homography projective transformation(单应性变换)

- 想要任意变换(正方形变为任意四边形)



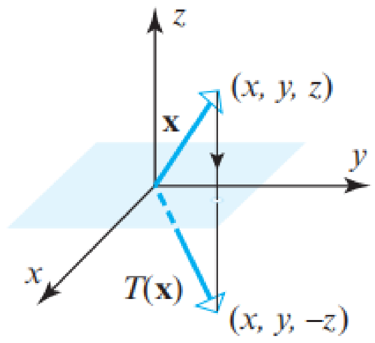
$$H = K_2 R_2 \left(I - \frac{1}{d}(-R_1^{-1} t_1 + R_2^{-1} t_2) n^T R_1\right) R_1^T K_1^{-1}$$

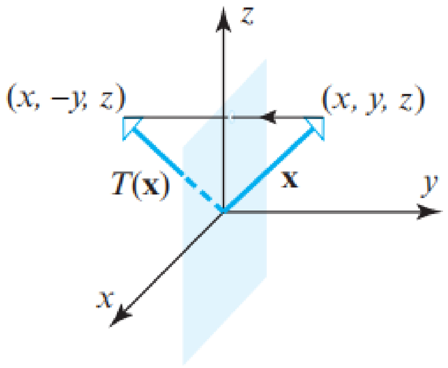
Reflection on $\mathbb{R}^2$

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Reflection about the x-axis $T(x, y) = (x, -y)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, -1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the y-axis $T(x, y) = (-x, y)$		$T(\mathbf{e}_1) = T(1, 0) = (-1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x, y) = (y, x)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 1)$ $T(\mathbf{e}_2) = T(0, 1) = (1, 0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Reflection on $\mathbb{R}^3$

<table>
<tr><th>Operator</th><th>Illustration</th><th>Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$</th><th>Standard Matrix</th></tr>
<tr><td>Reflection about the xy-plane

$T(x, y, z) = (x, y, -z)$</td><td>
</td><td>$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$
$T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$
$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, -1)$</td><td>$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$</td></tr>
<tr><td>Reflection about the xz-plane

$T(x, y, z) = (x, -y, z)$</td><td>
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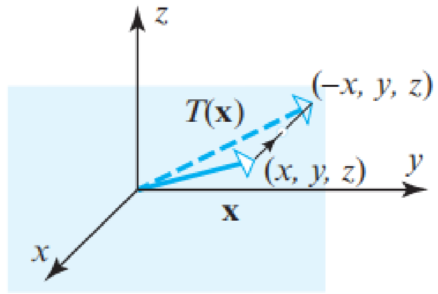
$T(x, y, z) = (-x, y, z)$</td><td>
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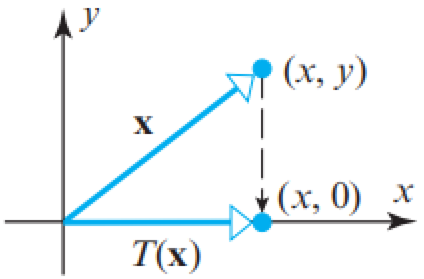
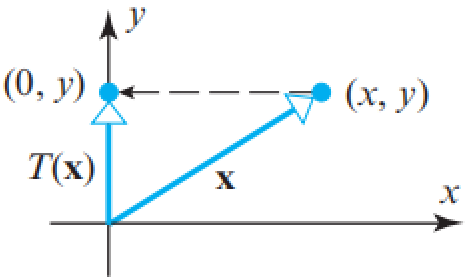
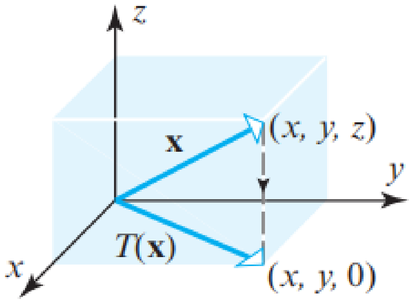
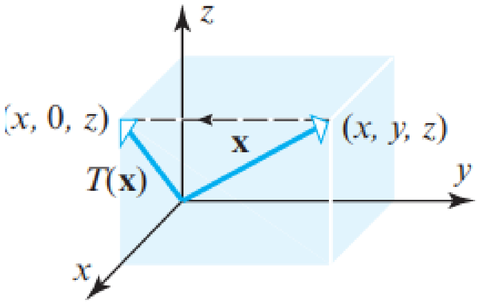
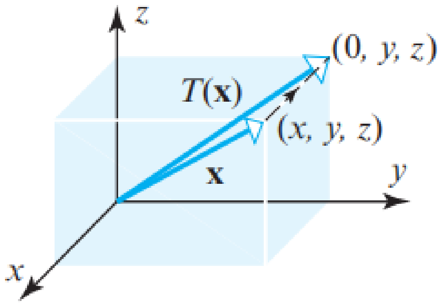
$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$</td><td>$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$</td></tr>
</table>

orthogonal projection on $\mathbb{R}^2$

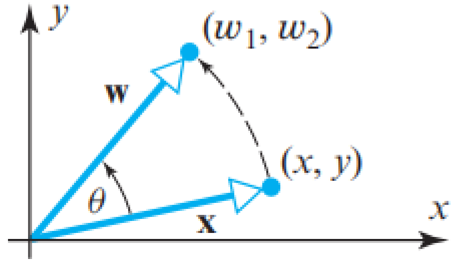
Operator	**Illustration**	**Images of $\mathbf{e}_1$ and $\mathbf{e}_2$**	**Standard Matrix**
Orthogonal projection onto the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection onto the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

> projection operation : the rank of the matrix is not full $\Rightarrow$ dimension reduction

orthogonal projection on $\mathbb{R}^3$

Operator	**Illustration**	**Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$**	**Standard Matrix**
Orthogonal projection onto the xy-plane $T(x, y, z) = (x, y, 0)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection onto the xz-plane $T(x, y, z) = (x, 0, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 0, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection onto the yz-plane $T(x, y, z) = (0, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (0, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

rotation on $\mathbb{R}^2$

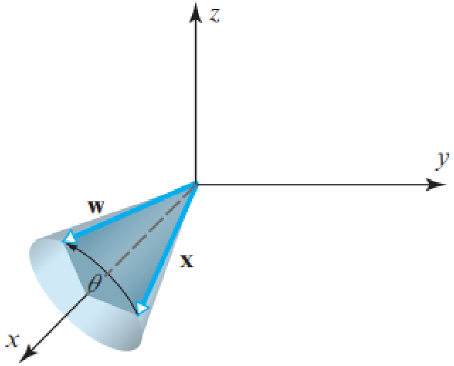
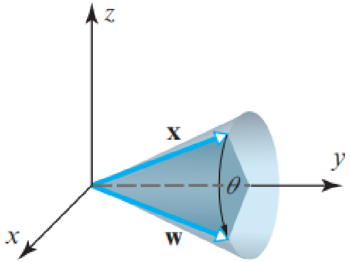
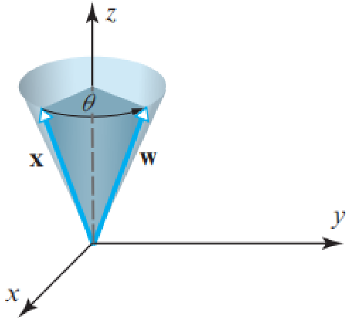
Operator	**Illustration**	**Rotation Equations**	**Standard Matrix**
Counterclockwise rotation about the origin through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

> for the rotation matrix R , $R^T = R^{-1}$, i.e. $RR^T = I$(orthogonal matrix)

> without scaling, i.e. $|R| = 1$

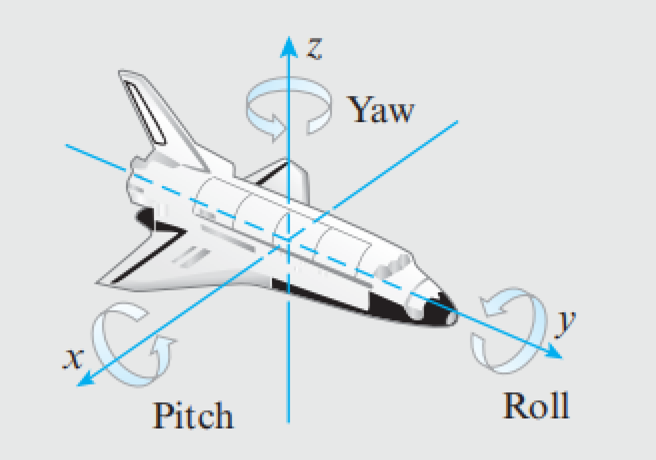
> the rotation is counterclockwise 默认是逆时针旋转了θ

rotation on $\mathbb{R}^3$

Operator	Illustration	Rotation Equations	Standard Matrix
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

> notice the difference of y axis

Euler angle

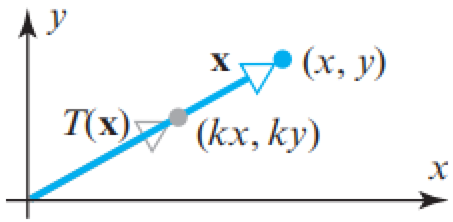
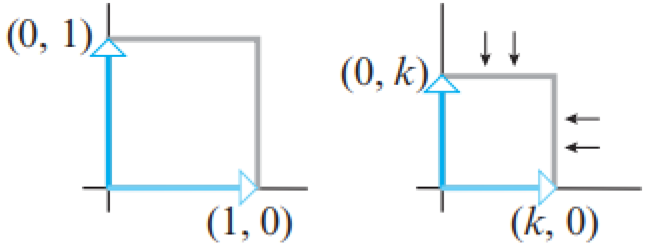
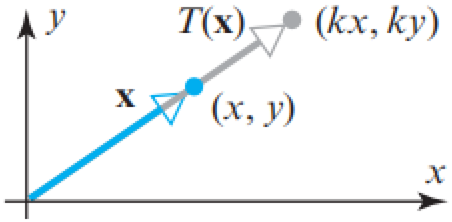
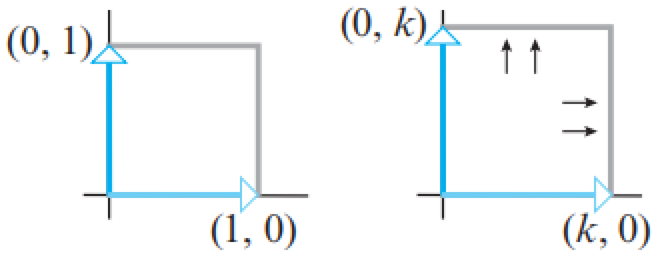


rotation clockwise around x axis **pitch** (α), around y axis **roll**(β), around z axis **yaw**(γ)

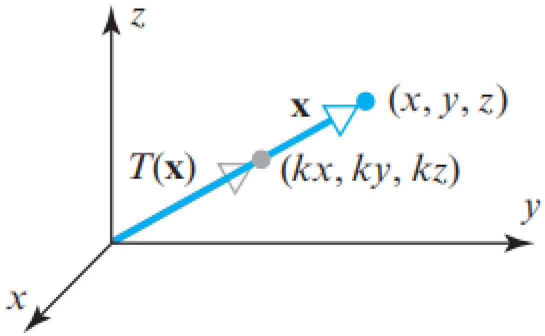
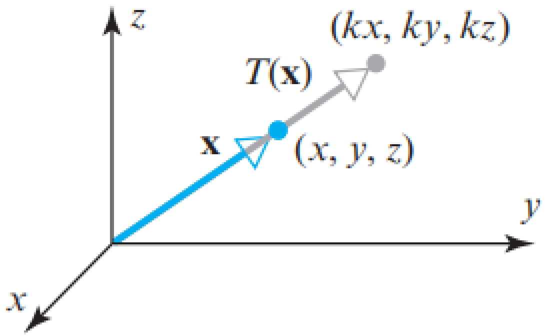
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

> 注意变换的顺序, 从右往左
> 若改变旋转的顺序, 则变换矩阵也要改变(结果不同)

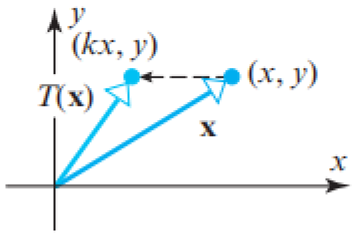
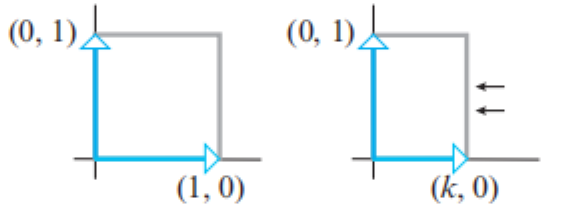
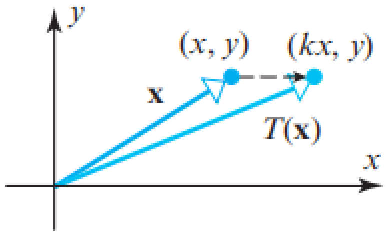
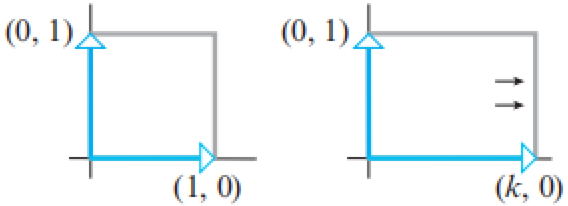
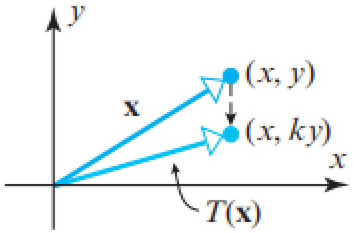
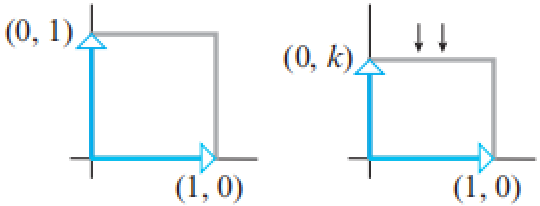
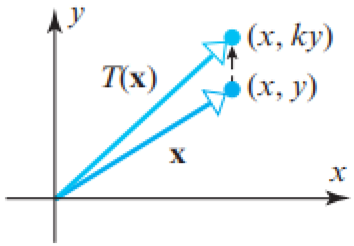
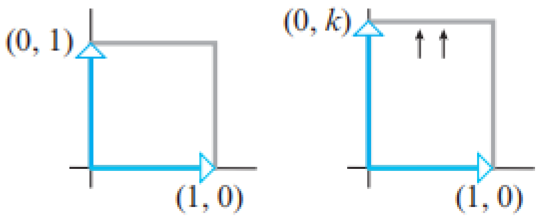
Dilations(拉伸) and Contractions(收缩) on $\mathbb{R}^2$

Operator	**Illustration** $T(x, y) = (kx, ky)$	**Effect on the Unit Square**	**Standard Matrix**
Contraction with factor k in R^2 $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Dilation with factor k in R^2 $(k > 1)$			

Dilations(拉伸) and Contractions(收缩) on $\mathbb{R}^3$

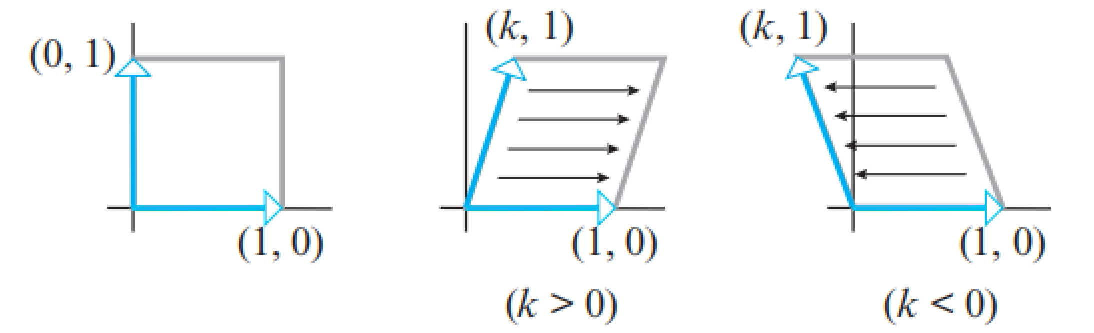
Operator	Illustration $T(x, y, z) = (kx, ky, kz)$	Standard Matrix
Contraction with factor k in R^3 $(0 \leq k < 1)$		$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$
Dilation with factor k in R^3 $(k > 1)$		

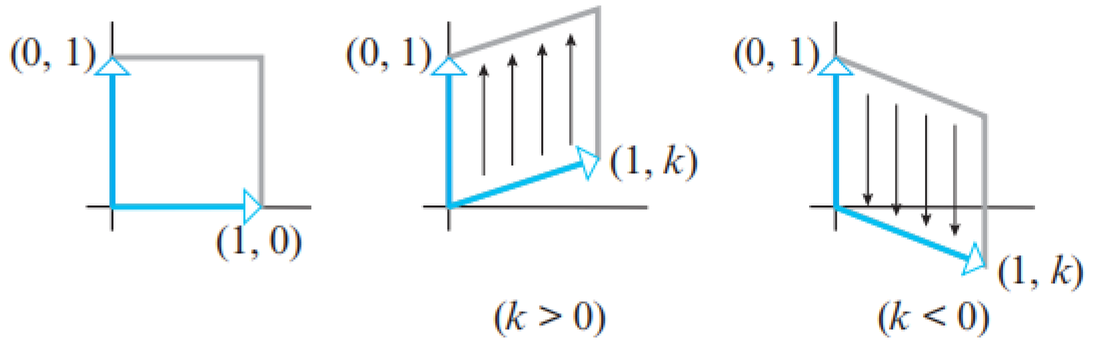
Expansions(拉伸) and Compressions(压缩)

Operator	Illustration $T(x, y) = (kx, y)$	Effect on the Unit Square	Standard Matrix
Compression in the x-direction with factor k in R^2 $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion in the x-direction with factor k in R^2 $(k > 1)$			
Compression in the y-direction with factor k in R^2 $(0 \le k < 1)$			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion in the y-direction with factor k in R^2 $(k > 1)$			

shear 推移

<table>
<tr><th>Operator</th><th>Effect on the Unit Square</th><th>Standard Matrix</th></tr>
<tr><td>Shear in the x-direction by a factor k in R^2

$T(x, y) = (x + ky, y)$</td><td>
</td><td>$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$</td></tr>
<tr><td>Shear in the y-direction by a factor k in R^2

$T(x, y) = (x, y + kx)$</td><td>
</td><td>$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$</td></tr>
</table>