

Linear Algebra Tutorial4

2023.10.31

homework

- A, B are upper triangular matrix, then $AB, A^k \dots$ are also upper triangular matrix
It is theorem on book, we do not need to prove it but need to mention it
- the symbol $[]$ and $||$
 $[]$ for matrix, $||$ for determinant
- iff \Leftrightarrow if and only if \Leftrightarrow 当且仅当

determinant properties

compare with the elementary row(column) operations

1. B is obtained from A by interchanging two rows(columns)

$$|B| = -|A|$$

2. B is obtained from A by multiplying one row(column) by a nonzero scalar k

$$|B| = k|A|$$

3. B is obtained from A by adding a multiple of one row(column) to another row(column)

$$|B| = |A|$$

we can mix row and column operations when calculating the determinant
but we can only use row or column operations when calculating the inverse matrix!!!!

determinant properties

| If a matrix A have two same rows(columns), then $|A| = 0$

suppose B is A swapping the two same rows(columns), from property 1, $|B| = -|A|$

but $B = A$, so $|B| = |A|$

so $|A| = -|A|$, $|A| = 0$

determinant property

$$C = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} + b_{i1} & a_{i2} + b_{i2} & \dots & a_{in} + b_{in} \text{ (第 } i \text{ 行)} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \text{ (第 } i \text{ 行)} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{i1} & b_{i2} & \dots & b_{in} \text{ (第 } i \text{ 行)} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$|C| = |A| + |B|$$

determinant property

- similiary

$$C = \begin{bmatrix} a_{11} & \dots & a_{1j} + b_{1j} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nj} + b_{nj} & \dots & a_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & \dots & b_{1j} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & b_{nj} & \dots & a_{nn} \end{bmatrix}$$

$$|C| = |A| + |B|$$

essentially: determinant expansion by column

generally, $|C| \neq |A| + |B|!!!$

determinant property

- $|AB| = |A||B|$
 $\Rightarrow R = ABCD\dots, |R| = |A||B||C|\dots$
- $|A^T| = |A|$
- $|A^{-1}| = \frac{1}{|A|}$
- $|A^k| = |A|^k$
- $|\lambda A| = \lambda^n |A|$

Cramer's rule

$$A\mathbf{x} = \mathbf{b}, |A| \neq 0 (A_{n \times n})$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

$$A_i = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{bmatrix}$$

$$\text{then } x_i = \frac{|A_i|}{|A|}$$

where A_i is the matrix obtained from A by replacing the i th column of A by \mathbf{b}

Cramer's rule

- e.g. when the coefficient is complicated

$$(e^u + \sin v) u_x + u \cos v v_x = 1$$

$$(e^u - \cos v) u_x + u \sin v v_x = 0$$

$$e^u \cdot u \sin v + u \sin^2 v - e^u \cdot u \cos v + u \cos^2 v$$

Cramer's rule

$$\begin{aligned}(e^u + \sin v) \frac{\partial u}{\partial x} + (u \cos v) \frac{\partial v}{\partial x} &= 1 \\ (e^u - \cos v) \frac{\partial u}{\partial x} + (u \sin v) \frac{\partial v}{\partial x} &= 0\end{aligned}$$

with Cramer's rule, we can get

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\begin{vmatrix} 1 & u \cos v \\ 0 & u \sin v \end{vmatrix}}{\begin{vmatrix} e^u + \sin v & u \cos v \\ e^u - \cos v & u \sin v \end{vmatrix}} = \frac{\sin v}{e^u(\sin v - \cos v) + 1}, \\ \frac{\partial v}{\partial x} &= \frac{\begin{vmatrix} e^u + \sin v & 1 \\ e^u - \cos v & 0 \end{vmatrix}}{\begin{vmatrix} e^u + \sin v & u \cos v \\ e^u - \cos v & u \sin v \end{vmatrix}} = \frac{-(e^u - \cos v)}{(e^u(\sin v - \cos v) + 1)u}\end{aligned}$$

determinant

5.

$$A = \begin{bmatrix} 1 & 1 & 5 & 4 \\ 2 & 3 & 2 & 4 \\ 1 & 6 & 0 & 3 \\ 4 & 2 & 5 & 1 \end{bmatrix}$$

find $C_{21} + C_{22} + 5C_{23} + 4C_{24}$

where C_{ij} is the cofactor of a_{ij}

determinant

5.

$$A = \begin{bmatrix} 1 & 1 & 5 & 4 \\ 2 & 3 & 2 & 4 \\ 1 & 6 & 0 & 3 \\ 4 & 2 & 5 & 1 \end{bmatrix}$$

construct:

$$A' = \begin{bmatrix} 1 & 1 & 5 & 4 \\ 1 & 1 & 5 & 4 \\ 1 & 6 & 0 & 3 \\ 4 & 2 & 5 & 1 \end{bmatrix}$$

so $C_{21} + C_{22} + 5C_{23} + 4C_{24} = |A'|$

and since A' has two same rows, $|A'| = 0$

so $C_{21} + C_{22} + 5C_{23} + 4C_{24} = 0$

determinant property

$$\forall A \in \mathbb{R}^{n \times n}$$

- if $i \neq k$, then

$$a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = 0$$

- if $i = k$, then

$$a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = |A|$$

so called Laplace expansion

adjoint matrix

- $C_{i,j}$: matrix of cofactors of $a_{i,j}$

代数余子式

- C : matrix of cofactors from A

A 的代数余子式矩阵

$$\text{i.e. } C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

- $A^* = \text{adj}(A) = C^T$
- no matter A is invertible or not, $AA^* = A^*A = |A|I$
- If A is invertible, $A^{-1} = \frac{1}{|A|} A^*$

adjoint matrix

- $|A^*| = |A|^{n-1}$

$$AA^* = |A|I$$

$$|AA^*| = |A|^n |I| = |A|^n$$

$$|A^*||A| = |A|^n$$

$$|A^*| = |A|^{n-1}$$

| any constraints?

adjoint matrix

- but if A is not invertible??

i.e. $|A| = 0$

is $|A^*| = 0$?

- Proof by Contradiction

suppose $|A^*| \neq 0$, then

$$(AA^*)(A^*)^{-1} = |A|I(A^*)^{-1} = \mathbf{0}$$

$$\text{also, } (AA^*)(A^*)^{-1} = A(A^*(A^*)^{-1}) = A$$

so $A = \mathbf{0} \Rightarrow A^* = \mathbf{0}$, which is a contradiction with $|A^*| \neq 0$

so $|A^*| = 0$

so $|A^*| = |A|^{n-1}$ is always true

determinant

1.

$$D = \begin{bmatrix} a_1 & c_2 & c_3 & \cdots & c_n \\ b_2 & a_2 & & & \\ b_3 & & a_3 & & \\ \vdots & & & \ddots & \\ b_n & & & & a_n \end{bmatrix}$$

$$\det(D) = \prod_{i=1}^n a_i - \sum_{i=2}^n \frac{A b_i c_i}{a_i}$$

$$\text{where } A = \prod_{i=2}^n a_i$$

determinant

2.

$$D = \begin{bmatrix} a & b & 0 & \cdots & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ b & 0 & 0 & \cdots & 0 & a \end{bmatrix}$$

$$|D| = a^n + (-1)^{1+n} b^n$$

determinant

3.

$$D_n = \begin{bmatrix} b & -1 & 0 & \cdots & 0 & 0 \\ 0 & b & -1 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & b & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & b + a_1 \end{bmatrix}$$

determinant

4.

$$\text{Calculate } D_n = \begin{vmatrix} 1 + x_1^2 & x_1 x_2 & \cdots & x_1 x_n \\ x_2 x_1 & 1 + x_2^2 & \cdots & x_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n x_1 & x_n x_2 & \cdots & 1 + x_n^2 \end{vmatrix}.$$

some mostly used tricks

- row(column) operations, construct a row(column) with as many zeros as possible, then use the Laplace expansion
- try to calculate the determinant with lower dimension, then use induction
- **practice more!!**

linear space

inner product $\langle \cdot, \cdot \rangle$

- \mathbb{R}^n vector: dot product $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x}$
- $M_{m \times n}(\mathbb{R})$ matrix: $\langle A, B \rangle = \text{tr}(B^T A)$
- $P_n(\mathbb{R})$ polynomial: $\langle p, q \rangle = \int_0^1 p(x)q(x)dx$
- $C[a, b]$ continuous function: $\langle f, g \rangle = \int_a^b f(x)g(x)dx$
- \vdots

linear space

angle θ between two vectors \mathbf{x}, \mathbf{y}

$$\cos\theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle \cdot \langle \mathbf{y}, \mathbf{y} \rangle} = \frac{\mathbf{x}^T \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}$$

$$\text{so } \mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos\theta$$

$$\mathbf{x} \cdot \mathbf{y} = 0 \Leftrightarrow \mathbf{x} \perp \mathbf{y}$$

cross product

$$\mathbf{x} \times \mathbf{y} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix}$$

- $\mathbf{x} \times \mathbf{y}$ is orthogonal to both \mathbf{x} and \mathbf{y}
- $\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\| \|\mathbf{y}\| \sin\theta$
- $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$
- $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = \mathbf{x} \times \mathbf{y} + \mathbf{x} \times \mathbf{z}$
- $(k\mathbf{x}) \times \mathbf{y} = k(\mathbf{x} \times \mathbf{y})$

cross product

- the cross product of two vectors can be represented by the determinant

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

- $\mathbf{x} \times \mathbf{y} = \mathbf{0} \Leftrightarrow \mathbf{x} \parallel \mathbf{y}$
- we can also transfer \mathbf{x} into a matrix $[\mathbf{x}]\mathbf{y}$

$$\text{where } [\mathbf{x}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

notice that $[\mathbf{x}]$ is skew-symmetric, so its rank is odd
actually, $\text{rank}([\mathbf{x}])=2$