Linear Algebra Tutorial6

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homework

- about cross product
  is applicable **only** to vectors in 3-space
- the norm of a vector
  $$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$
- the projection of a vector
  $$proj_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}$$

determinant practice

7. *(10 points)* Evaluate the following determinant of order n $(n \geq 2)$:

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 1 & 2 & \dots & n-2 & n-1 \\ 3 & 2 & 1 & \dots & n-3 & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n-2 & n-3 & \dots & 1 & 2 \\ n & n-1 & n-2 & \dots & 2 & 1 \end{vmatrix}$$

exercise

- $\mathbf{a}, \mathbf{b}, \mathbf{c}$ satisfy $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$ and $\|\mathbf{a}\|=3, \|\mathbf{b}\|=5, \|\mathbf{c}\|=7$.
  find the angle between $\mathbf{a}$ and $\mathbf{b}$
- $\|\mathbf{a}\|=2, \|\mathbf{b}\|=\sqrt{2}, \mathbf{a}\cdot\mathbf{b}=2$.
  find $\|\mathbf{a}\times\mathbf{b}\|$

representation of lines and planes

- line

  $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

  $$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

  $\mathbf{v} = (v_1, v_2, v_3)$ is the direction vector of the line

- plane

  $\mathbf{r} = \mathbf{r}_0 + s\mathbf{v} + t\mathbf{w}$

  $n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$

  $\mathbf{n} = (n_1, n_2, n_3)$ is the normal vector of the plane

  $\mathbf{n} = \mathbf{v} \times \mathbf{w}$

distance from a point to a line/plane

- line(2-dimensional):
  from $P_0(x_0, y_0)$ to $ax + by + c = 0$
  $$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$
  $\mathbf{n} = (a, b)$ is the normal vector of the line

- plane(3-dimensional):
  from $P_0(x_0, y_0, z_0)$ to $ax + by + cz + d = 0$
  $$d = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$
  $\mathbf{n} = (a, b, c)$ is the normal vector of the plane

homogeneous coordinates*

- In 2D space, a point $p(x, y)$
- its homogeneous coordinates
  $p(x, y, 1)$
- a vector $\mathbf{v} = (x, y)$
  its homogeneous coordinates is
  $\mathbf{v} = (x, y, 0)$
- a line $ax + by + c = 0$
  its homogeneous coordinates is
  $\mathbf{v} = (a, b, c)$
- benefit1:
  we can judge whether a point is on the line by $\mathbf{v} \cdot \mathbf{p} = 0$

homogeneous coordinates*

benefit2:

- we can easily get the line crossing two points:
  $\mathbf{l} = \mathbf{p} \times \mathbf{q}$
- we can easily get the intersection of two lines:
  $\mathbf{x} = \mathbf{l} \times \mathbf{m}$

homogeneous coordinates*

benefit3:

we can easily transform a point in Euclidean space

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = \begin{bmatrix} R & \mathbf{t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

where $\begin{bmatrix} R & \mathbf{t} \\ 0 & 1 \end{bmatrix}$ is the transformation matrix

> https://zhuanlan.zhihu.com/p/625678401

Eucledian space $\Rightarrow$ Vector space

$\mathbb{R}^n \Rightarrow V$

the Euclidean space $\mathbb{R}^n$ is a special kind of vector space V

> vector space is also called linear space

property of vector space

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a ***zero vector*** for V, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

- If and only if the 10 conditions are satisfied, then V is a vector space.
- we call every element in the vector space "vector"

> compare with the properties of Euclidean space

- $+$ is an abstract operation, it may not be the addition in Euclidean space
  eg. we can define that $a+b=ab$
- $-u$ is the inverse(negative) of u, it may have $-u \neq -1u$
  but in vector space, $-u=-1u$
- the most important thing is the defination of "+", "-", "0"

> the above defination may not be a vector space

example of vector space

- $\mathbb{R}^n$
- $V = M_{m\times n}$

  "+" is the addition of matrices, $\mathbf{0}$ is the zero matrix
- all function $f : \mathbb{R} \to \mathbb{R}$ is the vector in $V = F(-\infty, +\infty)$
- the set of all polynomials of degree $\leq n$

  "+" the addtion for every coefficient

theorem of vector space

Theorem 4.3. 令V为一个向量空间，$\mathbf{0} \in V$为V里的零向量，$\boldsymbol{u} \in V$，$c \in \mathbb{R}$。那么

1. $0\boldsymbol{u} = \mathbf{0}$。
2. $k\mathbf{0} = \mathbf{0}$。
3. $(-1)\boldsymbol{u} = -\boldsymbol{u}$。
4. $c\boldsymbol{u} = \mathbf{0} \Rightarrow c = 0$ 或者 $\boldsymbol{u} = \mathbf{0}$。

subspace

V is a vector space, W is a subset of V **iff**

- W is closed under addition
  $\forall \mathbf{u}, \mathbf{v} \in W, \mathbf{u} + \mathbf{v} \in W$
- W is closed under scalar multiplication
  $\forall \mathbf{u} \in W, \forall c \in \mathbb{R}, c\mathbf{u} \in W$

W is the subspace of V with the defination of "+" and "$\cdot$"
W **is also a vector space**

examples of subspace

- $\mathbf{u} = (1, 1, 1) \in \mathbb{R}^3, \mathbf{v} = (1, 1, 0) \in \mathbb{R}^3, \mathbf{0} = (0, 0, 0) \in \mathbb{R}^3$
  $W = \{k\mathbf{u} + c\mathbf{v}, k \in \mathbb{R}, c \in \mathbb{R}\}$
  W is a subspace of $\mathbb{R}^3$
- $W = \{A = M_{n\times n : A^T = A}\}$
  W is a subspace of $M_{n\times n}$
- $P_n \subset P_\infty \subset C^\infty(-\infty, \infty) \subset C^m(-\infty, \infty) \subset C^1(-\infty, \infty)$
  $\subset C(-\infty, \infty) \subset F(-\infty, \infty)$

P_n is the set of all polynomials of degree $\leq n$
All these are subspaces of $F(-\infty, \infty)$

examples of subspace

- the set of all invertible matrices is **not** a subspace of $M_{n\times n}$

subspace

- the intersection of two subspaces is also a subspace
  $W_1, \cdots, W_r$ are the subspaces of V, then $W_1 \cap \cdots \cap W_r$ is also a subspace of V
- the union of two subspaces is not necessarily a subspace(usually not)

zero space

for all vector space V, $\{\mathbf{0}\}$ is a subspace of V

- $\mathbf{0} + \mathbf{0} = \mathbf{0}$
- $c\mathbf{0} = \mathbf{0}$

linear combination

V is a vector space, $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_r \in V\}$
$W = \{c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r, c_1, \cdots, c_r \in \mathbb{R}\}$

- W is all the linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_r$
- W is a subspace of V
- **W is the smallest subspace of V containing $\mathbf{v}_1, \cdots, \mathbf{v}_r$**
- We call $W = span(\mathbf{v}_1, \cdots, \mathbf{v}_r)$ or $W = span(S)$
  the span of $\mathbf{v}_1, \cdots, \mathbf{v}_r$

span

$S = \{\mathbf{v}_1, \cdots, \mathbf{v}_r \in V\}$ is a subset of V
$W = span(S)$ is the span of S

- W is the smallest subspace of V containing S
- W is all the linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_r$

span

Let $e_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{R}^n$, where 1 is in the i-th position

$S = \{e_1, \cdots, e_n\}$ is the standard basis of $\mathbb{R}^n$

- $span(S) = \mathbb{R}^n$
- $\forall \mathbf{v} \in \mathbb{R}^n, \mathbf{v} = v_1 e_1 + \cdots + v_n e_n$

> what set of vectors can span $\mathbb{R}^n$?

- can $\mathbf{v}_1 = (1, 1, 2), \mathbf{v}_2 = (1, 0, 1), \mathbf{v}_3 = (2, 1, 3)$ span $\mathbb{R}^3$?
- can $\mathbf{v}_1 = (1, 0, 0), \mathbf{v}_2 = (0, 1, 0)$ span $\mathbb{R}^3$?
- can $\mathbf{v}_1 = (1, 0, 0), \mathbf{v}_2 = (0, 1, 0), \mathbf{v}_3 = (0, 0, 1), \mathbf{v}_4 = (1, 1, 1)$ span $\mathbb{R}^3$?

> when will it have a unique solution?

linear independence

V is a vector space, $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_r \in V\}$, $r \geq 2$

If we say S is linearly independent set, or say $\mathbf{v}_1, \cdots, \mathbf{v}_r$ are linearly independent vectors, we mean that

- $\forall i, \mathbf{v}_i$ can not be represented by the linear combination of the other vectors in S

linear independence

The way we usually use to prove that $\mathbf{v}_1, \cdots, \mathbf{v}_r$ are linearly independent

- We call S a linearly independent set **iff**
  $\exists c_1, \cdots, c_r \in \mathbb{R}$,
  $c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r = \mathbf{0}$
  only when $c_1 = \cdots = c_r = 0$

> proof by contradiction
> WLOG, we usually set $\mathbf{v}_1$ can be represented by the linear combination of the other vectors.

linear independence

- $S = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ in $\mathbb{R}^n$ is the standard basis of $\mathbb{R}^n$
- $p_0(x) = 1, p_1(x) = x, p_2(x) = x^2, \cdots, p_n(x) = x^n$ are linearly independent in P_n

linear independence

Let $c \in \mathbb{R}$, suppose that

$p_1(x) = 1 - 2x, p_2(x) = 3 + x - cx^2, p_3(x) = -1 + 3x^2,$

$p_4(x) = 1 + 2021x + 2021^2x^2 + 2021^3x^3$

Find c s.t. p_1, p_2, p_3, p_4 are linearly independent

linear independence's property

V is a vector space, $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_r \in V\}$ is a subset if V

S is a linearly independent set **iff**

every vector in $W = span(S)$ has an unique representation

> proof by contradiction

linear independence's property

- Let $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ be a subset of $\mathbb{R}^n$
  If $r > n$, then S is linearly dependent

proof:

Let $k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$, where $k_1, \cdots, k_r \in \mathbb{R}$

Then the coefficient matrix is $A = \begin{bmatrix} v_{11} & \cdots & v_{1r} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nr} \end{bmatrix}$

since $r > n$, which means the numbers of variables is more than the numbers of equations, then there must be a non-trivial solution, which means S is linearly dependent

Coordinates and basis

V is a finite-dimensional vector space, $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ is a subset of V
If

- S is linearly independent
- $V = span(S)$

then we call S a basis of V
the number of vectors in S is the dimension of V, denoted as $dim(V)$

Coordinates and basis

We have mentioned that

$S = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$

$span(S) = \mathbb{R}^n$

We can S the standard basis of $\mathbb{R}^n$

Coordinates and basis

Some equality relations

$$
\begin{aligned}
S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \subset \mathbb{R}^n \text{ is linearly independent} &\Leftrightarrow span(S) = \mathbb{R}^n \\
&\Leftrightarrow S \text{ is a basis of } \mathbb{R}^n \\
&\Leftrightarrow A = [\mathbf{v}_1 \cdots \mathbf{v}_n] \text{is invertible}
\end{aligned}
$$

Coordinates and basis

$S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \subset \mathbb{R}^n$ is a basis of $\mathbb{R}^n$
$\forall \mathbf{v} \in V$, $\exists c_1, \cdots, c_n \in \mathbb{R}$ s.t.
$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$

coordinate vector of $\mathbf{v}$ relative to S
$(\mathbf{v})_S = (c_1, \cdots, c_n)$