Linear Algebra Tutorial 14

2024.1.9

homework

inner product space

Theorem 7.14. 令V为一个内积空间，其内积记为$\langle\cdot,\cdot\rangle$。那么我们有以下结果：

1. 对于任何正交集合$S=\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r\}\subset V$，如果$\boldsymbol{v}_i\neq\mathbf{0}$对所有$i=1,\ldots,r$成立，那么$S$一定是线性无关集合。
2. 对于任何V的标准正交基底S，任何$\boldsymbol{u}\in V,\boldsymbol{v}\in V$，将其关于$S$的坐标向量记为

$$(\boldsymbol{u})_S=(u_1,\ldots,u_n),\quad(\boldsymbol{v})_S=(v_1,\ldots,v_n),$$

那么总是有

$$\|\boldsymbol{u}\|^2=\langle\boldsymbol{u},\boldsymbol{u}\rangle=u_1^2+\ldots+u_n^2,$$

$$d(\boldsymbol{u},\boldsymbol{v})=\|\boldsymbol{u}-\boldsymbol{v}\|=(\langle\boldsymbol{u}-\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v}\rangle)^{\frac{1}{2}}=\sqrt{(u_1-v_1)^2+\ldots+(u_n-v_n)^2},$$

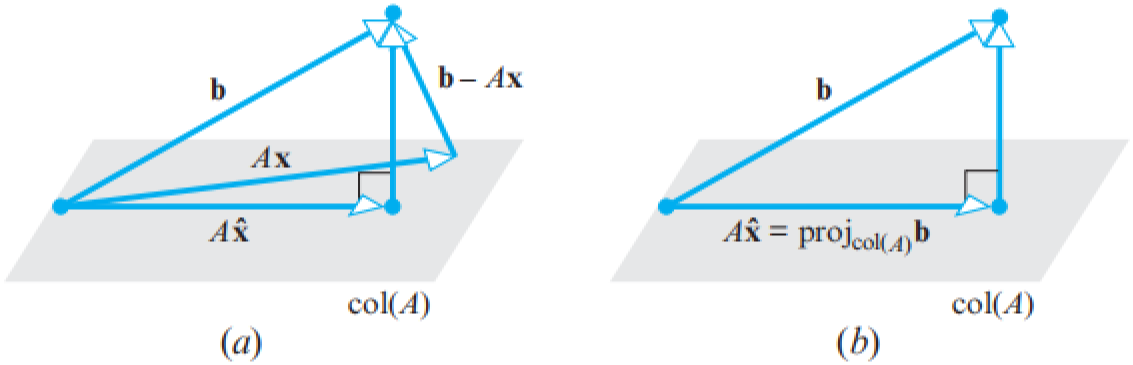
$$\langle\boldsymbol{u},\boldsymbol{v}\rangle=u_1v_1+\ldots+u_nv_n.$$

3. 若V为有限维向量空间，S与S'为V的两组标准正交基底。那么$P_{S\leftarrow S'}$与$P_{S'\leftarrow S}$都是正交矩阵。

- 与欧式空间类似, 只需把点乘替换成内积即可$<\cdot,\cdot>$

Projection Theorem for inner product space

least square approximation*



least square approximation*

Finding Least Squares Solutions

One way to find a least squares solution of $A\mathbf{x} = \mathbf{b}$ is to calculate the orthogonal projection $\text{proj}_W \mathbf{b}$ on the column space W of A and then solve the equation

$$A\mathbf{x} = \text{proj}_W \mathbf{b} \tag{2}$$

However, we can avoid calculating the projection by rewriting (2) as

$$\mathbf{b} - A\mathbf{x} = \mathbf{b} - \text{proj}_W \mathbf{b}$$

and then multiplying both sides of this equation by A^T to obtain

$$A^T(\mathbf{b} - A\mathbf{x}) = A^T(\mathbf{b} - \text{proj}_W \mathbf{b}) \tag{3}$$

Since $\mathbf{b} - \text{proj}_W \mathbf{b}$ is the component of $\mathbf{b}$ that is orthogonal to the column space of A, it follows from Theorem 4.8.7(b) that this vector lies in the null space of A^T, and hence that

$$A^T(\mathbf{b} - \text{proj}_W \mathbf{b}) = \mathbf{0}$$

Thus, (3) simplifies to

$$A^T(\mathbf{b} - A\mathbf{x}) = \mathbf{0}$$

which we can rewrite as

$$A^TA\mathbf{x} = A^T\mathbf{b} \tag{4}$$

least square approximation*

- more intuitive:

  $$\min_{\mathbf{x}\in\mathbb{R}^n} \mathcal{L} = \|\mathbf{b} - A\mathbf{x}\|_2^2$$

$$\mathcal{L} = \|\mathbf{b} - A\mathbf{x}\|_2^2 = (\mathbf{b} - A\mathbf{x})^T(\mathbf{b} - A\mathbf{x}) = \mathbf{b}^T\mathbf{b} - \mathbf{b}^T A\mathbf{x} - \mathbf{x}^T A^T\mathbf{b} + \mathbf{x}^T A^T A\mathbf{x}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = -2A^T\mathbf{b} + 2A^T A\mathbf{x} = 0$$

$$\Rightarrow A^T A\mathbf{x} = A^T\mathbf{b}$$

Gram–Schmidt 施密特正交化

Quadratic Forms 二次型

奇异值分解 Singular Value Decomposition (SVD)