Student Name: ____________________

Student Number: ____________________

School/Institute: ____________________

Year of Entrance: ____________________

ShanghaiTech University Midterm Examination

Academic Year: 2021 to 2022 Term: 1

Course-offering School: IMS

Instructor: *Ye, ShuYang* ☐ / *Xue, Boqing* ☐ / *Zheng, Kai* ☐

Course Name: **Linear Algebra I**

Course Number: MATH1112

Exam Instructions for Students:

1. All examination rules must be strictly observed throughout the entire test, and any form of cheating is prohibited.

2. Other than allowable materials, students taking closed-book tests must place their books, notes, tablets and any other electronic devices in places designated by the examiners.

3. Students taking open-book tests may use allowable materials authorized by the examiners. They must complete the exam independently without discussion with each other or exchange of materials.

For Marker's Use:

Problem	1	2	3	4	5	6	7	8	Total
Points	26	8	10	12	12	12	10	10	100
Scores									
Recheck									

Marker's Signature: Rechecker's Signature:

Date: Date:

Specific Instructions:

- Please check the box ☐ behind the name of your instructor on the cover page.
- The time duration for this exam is 120 **minutes**.
- Computers and calculators are prohibit in the exam.
- Answers can be written in **either Chinese or English**.
- You may ask for direct translation during the exam, if needed.

★ For problems 2-8, please show details of calculations or deductions. A correct answer with no details can not earn points.

Notations and conventions:

- $\mathbb{R}$ is the set of real numbers. All the scalars here are real numbers.
- I denotes an identity matrix of suitable order.
- 0 or $\mathbf{0}$ may denote the number zero, a zero vector, or a zero matrix.
- $\det(A)$ is the determinant of a matrix A.
- $\mathsf{tr}(A)$ is the trace of a matrix A.
- $\mathsf{adj}(A)$ is the adjoint (adjunct) matrix of A.
- A linear system is said to be consistent if it has at least one solution.

1. Fill in the blanks.

(1) *(8 points)* In $\mathbb{R}^3$, let $\mathbf{u} = (1, 0, -1)$ and $\mathbf{v} = (0, 1, 2)$.

(a) $\|\mathbf{u} - \mathbf{v}\| =$ $\sqrt{11}$;

(b) $\mathbf{u} \cdot \mathbf{v} =$ -2;

(c) $\mathbf{u} \times \mathbf{v} =$ $(1, -2, 1)$;

(d) The orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$ is $\text{proj}_{\mathbf{v}}(\mathbf{u}) =$ $(0, -\frac{2}{5}, -\frac{4}{5})$.

(2) *(4 points)* Let $A = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$ and $p(x) = (x+1)(x-1)$. Then

$$p(A) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}.$$

(3) *(4 points)* The inverse of the matrix $A = \begin{bmatrix} -2 & 1 \\ 5 & 3 \end{bmatrix}$ is $\begin{pmatrix} -\frac{3}{11} & \frac{1}{11} \\ \frac{5}{11} & \frac{2}{11} \end{pmatrix}$.

(4) *(6 points)* Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 6 \end{bmatrix}$. Compute the following trace and determinant:

$$\mathsf{tr}(A^T) = 9, \quad \det\left(A^{2021}\right) = -1.$$

(5) *(4 points)* In $\mathbb{R}^3$, suppose that Π is the plane given by $2x - y - 3z + 2 = 0$. The distance between the point $P_0(1, -1, 0)$ and the plane Π is $\frac{5\sqrt{14}}{14}$.

2. *(8 points)* Find an invertible matrix P such that $PA = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} - a_{21} & a_{32} - a_{22} & a_{33} - a_{23} \end{bmatrix}.$$

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} - a_{21} & a_{32} - a_{22} & a_{33} - a_{23} \end{bmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \cdot A$$

$$\therefore P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

3. *(10 points)* Let $c \in \mathbb{R}$ be a parameter. Suppose that

$$p_1(x) = 1 - 2x, \quad p_2(x) = 3 + x - cx^2, \; p_3(x) = -1 + 3x^2,$$

$$p_4(x) = 1 + 2021x + 2021^2x^2 + 2021^3x^3.$$

Find a value of c such that p_1, p_2, p_3, p_4 are *linearly dependent* in the vector space P_∞ of all polynomials.

系数阵 $A = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 3 & 1 & -c & 0 \\ -1 & 0 & 3 & 0 \\ 1 & 2021 & 2021^2 & 2021^3 \end{pmatrix}$

$\because$ 相关 $\therefore |A| = 0$

即: $|A| = \begin{vmatrix} 1 & -2 & 0 & 0 \\ 3 & 1 & -c & 0 \\ -1 & 0 & 3 & 0 \\ 1 & 2021 & 2021^2 & 2021^3 \end{vmatrix} = 2021^3 \begin{vmatrix} 1 & -2 & 0 \\ 3 & 1 & -c \\ 1 & 0 & 3 \end{vmatrix}$

$= 2021^3 \cdot (2c + 21) = 0$

$\therefore c = \frac{21}{2}.$

4. *A square matrix A is called idempotent if $A^2 = A$.*

A square matrix A is said to be involutory if $A^2 = I$.

(1) *(6 points)* Suppose that A, B are both idempotent. Prove that $A + B$ is idempotent if and only if $AB + BA = 0$.

(2) *(6 points)* Suppose that A, B are both involutory. Prove that AB is involutory if and only if $AB = BA$.

(1). $AB + BA = 0 \iff A^2 + B^2 = A^2 + B^2 + AB + BA$

①. 充分必要都需证. $\iff A^2 + B^2 = (A+B)^2$

$\because A^2 = A$. $B^2 = B$

$\therefore A^2 + B^2 = A + B = (A+B)^2$

② 若用 A^{-1}, B^{-1}, 需证明

③ $(A+B)^2$ 展开

(2). $A^2 = E$. $B^2 = E$

若 $AB = BA$.

$(AB)^2 = ABAB = ABBA = AB^2A = AEA = A^2 = E$

若 $AB^2 = E$

则 $ABAB = E$. 则 $A^2BAB^2 = AB$ $\therefore$ $BA = AB$

5. Let A be an $n \times n$ invertible matrix. Prove the following two statements:

(1) *(6 points)* $\mathsf{adj}(A^{-1}) = \big(\mathsf{adj}(A)\big)^{-1}$.

(2) *(6 points)* $\mathsf{adj}\big(\mathsf{adj}(A)\big) = \big(\det(A)\big)^{n-2}A$.

(1). 证. $(A^{-1})^* = (A^*)^{-1}$

$\because A \cdot A^* = |A| \cdot E$ ①

$\therefore (A^{-1}) \cdot (A^{-1})^* = |A^{-1}| \cdot E$

两边同乘 A

$(A^{-1})^* = \frac{A}{|A|} \cdot E$.

由①知. $\frac{A}{|A|} \cdot A^* = E$

$\therefore (A^*)^{-1} = \frac{A}{|A|} = (A^{-1})^*$

得证.

(2) 证. $(A^*)^* = |A|^{n-2} \cdot A$.

$\because A \cdot A^* = |A| \cdot E$

$\therefore (A^*)(A^*)^* = |A^*| \cdot E$

$\therefore (A^*) \cdot (A^*)^* = |A|^{n-1} \cdot E$

由(1)知. $(A^*)^{-1} = \frac{A}{|A|}$

$\therefore (A^*)^* = (A^*)^{-1} \cdot |A|^{n-1} \cdot E$

$= \frac{A}{|A|} \cdot |A|^{n-1}$

$= A \cdot |A|^{n-2}$

6. *(12 points)* Let $a, b \in \mathbb{R}$ be parameters. Consider the linear system of equations

$$\begin{cases} -2x & +y & +z & = & -2 \\ x & -2y & +z & = & a \\ x & +y & +(b-2)z & = & a^2+b \end{cases}$$

When is the above linear system consistent? When it is consistent, find the solutions.

增广: $\begin{pmatrix} -2 & 1 & 1 & -2 \\ 1 & -2 & 1 & a \\ 1 & 1 & b-2 & a^2+b \end{pmatrix}$, 行变换得:

$$\begin{pmatrix} 1 & -2 & 1 & a \\ 0 & 3 & b-3 & a^2-a+b \\ 0 & 0 & b & a^2+a+b-2 \end{pmatrix}$$

① $b \neq 0$ 时. 必有解: $\begin{cases} x = \dfrac{a^2-3a+2}{b} + \dfrac{7}{3} + a \\ y = \dfrac{a^2-a-2}{b} + \dfrac{5}{3} \\ z = \dfrac{a^2+a-2}{b} + 1 \end{cases}$

② $b=0$ 时 $a \neq 1$ 且 $a \neq 2$ 时 无解.

$a=2$ 时. 解为 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$a=1$ 时. 解为 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

7. *(10 points)* Evaluate the following determinant of order n ($n \geq 2$):

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 1 & 2 & \dots & n-2 & n-1 \\ 3 & 2 & 1 & \dots & n-3 & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n-2 & n-3 & \dots & 1 & 2 \\ n & n-1 & n-2 & \dots & 2 & 1 \end{vmatrix}.$$

① 逐行相减，化为:

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 1 & 1 & & -1 \end{vmatrix}$$

② 用第一列消.

$$= \begin{vmatrix} 1 & 1 & 2 & \cdots & n-1 \\ 1 & -2 & -2 & \cdots & -2 \\ 1 & 0 & -2 & \cdots & -2 \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 0 & 0 & & -2 \end{vmatrix}$$

③ 逐行相减，下减上.

$$= \begin{vmatrix} 1 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & -2 & & \cdots & -2 & -2 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & \cdots & 2 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 2 & 0 \end{vmatrix} = (-1)^{n+1} \cdot 2^{n-2}(n+1)$$

8. *A matrix A is said to be skew-symmetric if $A^T = -A$.*

Let A be an $n \times n$ skew-symmetric matrix with $n \geq 2$.

(1) *(2 points)* Suppose that n is odd (i.e., $n = 2k - 1$). Prove that $\det(A) = 0$.

(2) *(8 points)* Suppose that n is even (i.e., $n = 2k$) and $\det(A) = 0$. Prove that the linear system $A\mathbf{x} = \mathbf{0}$ has at least two linearly independent solutions.

(1) $\because$ $|-A| = (-1)^n \cdot |A|$. 且 $n=2k-1$ $(-1)^n = -1$

$\therefore$ $|-A| = -|A|$ 又$\because$ $|A^T| = |A|$. 且 $A^T = -A$

则 $|A^T| = |-A|$ 即 $|A| = -|A|$. $\therefore$ $|A| = 0$.

(2) 由 $A^T = -A$. 可写 A 的一般表达.

$$A = \begin{array}{c} \begin{array}{cccccc} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n & \end{array} \\ \begin{pmatrix} 0 & a_2 & a_3 & \cdots & a_n \\ -a_2 & 0 & b_3 & \cdots & \\ -a_3 & -b_3 & 0 & \ddots & \\ \vdots & \vdots & & \ddots & \\ -a_n & & & & 0 \end{pmatrix} \end{array} \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_n \end{array}$$

$\because$ $|A| = 0$, $\therefore$ $r(A) < n$.

即 A 得行(列)向量相关，至少有一个向量可由其余向量表示.

设 α 为列向量，β 为行向量.

对行向量. 设 β_k 可由其余表示，

即 $\beta_k = k_1\beta_1 + k_2\beta_2 + \cdots + k_{k-1}\beta_{k-1} + k_{k+1}\beta_{k+1} + \cdots + k_n\beta_n$

$\because$ 每个行向量与其对应列向量元素相同，只有正负差别.

即 $\alpha_k^T = -\beta_k$.

$\therefore$ 有 $\alpha_k = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_{k-1}\alpha_{k-1} + k_{k+1}\alpha_{k+1} + \cdots + k_n\alpha_n$

$\therefore$ 除去 α_k 与 β_k 的矩阵仍满足 $A'^T = -A'$

对于 ——————— A'，其为 $n-1$ 阶

由(1)知，$n-1$ 阶反对称行列式均为0. 即 $|A'| = 0$

$\therefore$ $r(A') < n-1$ $\therefore$ $r(A) < n-1$ $r(A) \leq n-2$.

当 $r(A) = n-2$ 时，其解有两个无关的自由变量. $\therefore$ 至少俩.

$r(A) < n-2$ 时，其解有多于两个无关的自由变量.