第六章，第七章复习自测题

1 不定项选择, Multiple Choices. 15 points

1-1 (5 points). Determine which of the following statements is/are true.

- (A) If $A \in M_{n\times n}$ is an orthogonal matrix, and $\lambda \in A$ is an eigenvalue of A, then $\lambda = 1$ or $\lambda = -1$.
- (B) Let $A, B \in M_{n\times n}$. Then AB is invertible if and only if A and B are invertible.
- (C) Every orthogonal matrix $A \in M_{n\times n}$ is diagonalizable.
- (D) If $A \in M_{n\times n}$ has n linearly independent eigenvectors, then A is a symmetric matrix.

1-2 (5 points). Determine which of the following functions $\langle\cdot,\cdot\rangle$ is/are inner product.

- (A) $V = \mathbb{R}^2$, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$, $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{x}^\top A\boldsymbol{y}$, where $A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$.
- (B) $V = \mathbb{R}^2$, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$, $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{x}^\top A\boldsymbol{y}$, where $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
- (C) $V = M_{n\times n}$, for $A, B \in V$, $\langle A, B\rangle = \mathrm{tr}(B^\top A)$.
- (D) $V = P_2 = \{a_0 + a_1 + a_2x^2; a_0, a_1, a_2 \in \mathbb{R}\}$, for $p(x), q(x) \in V$, $\langle p(x), q(x)\rangle = p(1)q(1) + p(2)q(2) + p(3)q(3)$.

1-3 (5 points). Determine which of the following properties is/are similar invariants.

- (A) rank.
- (B) The dimension of eigenspace.
- (C) Eigenvector.
- (D) Characteristic polynomial.

2 填空题, Fill in the blanks. 15 points

2-1 (5 points). Let $A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$, then the singular values of A are ______.

2-2 (5 points). If the quadratic form $f(x_1, x_2, x_3) = a(x_1^2 + x_2^2 + x_3^2) + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$ is changed to standard form $f = 6y_1^2$ by the orthogonal change of variables $\boldsymbol{x} = P\boldsymbol{y}$, then $a =$ _____.

2-3 (5 points).

Let $A \in M_{3\times 3}$ be a diagonalizable matrix: there are diagonal matrix D and invertible matrix P such that $D = P^{-1}AP$. Suppose that $\mathrm{tr}(A) = -5$, and $A^2 + 2A - 3I_3 = \mathbf{0}_{3\times 3}$ is the zero matrix. Then $D =$ _____.

3 10 points

Suppose that $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & a \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & b \end{bmatrix}$, and A is similar to B.

(a) (4 points) Find a and b.

(b) (6 points) Find an invertible matrix P such that $B = P^{-1}AP$.

4 10 points

Consider P_2 with the inner product

$$\langle p(x), q(x)\rangle = \int_{-1}^{1} p(x)q(x)dx$$

for $p(x), q(x) \in P_2$. Apply the Gram-Schmidt process to transform the standard basis $B = \{1, x, x^2\}$ to an orthonormal basis of P_2.

5 10 points

Consider the following quadratic form

$$f(x_1, x_2, x_3) = 2x_1x_2 + kx_3^2 + 2x_3x_4 + 2x_4^2, \quad k \in \mathbb{R}.$$

(a) (3 points) Find the symmetric matrix A such that $f(x_1, x_2, x_3, x_4) = \boldsymbol{x}^\top A\boldsymbol{x}$, where
$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

(b) (4 points) If 3 is an eigenvalue of A, find the value for k and find an orthogonal matrix P such that $D = P^{-1}AP$ where D is a diagonal matrix.

(c) (3 points) Decide whether A is positive definite.

6 10 points

Let $A \in M_{n\times n}$ be a matrix such that $\|A\boldsymbol{x}\|= 1$ for all unit vector $\boldsymbol{x} \in \mathbb{R}^n$ (i.e., $\|\boldsymbol{x}\|= 1$), where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. Denote the column vectors of A by $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$, i.e.,
$$A = \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \ldots & \boldsymbol{c}_n \end{bmatrix}.$$
Compute $\|A^{2023}(\boldsymbol{c}_1 + \ldots + \boldsymbol{c}_n)\|$.

7 15 points

Let V be a finite dimensional inner product space with inner product $\langle\cdot,\cdot\rangle$. Let U, W be two subspaces of V.

(a) (5 points) Suppose that $U \subset W$, prove that $W^\perp \subset U^\perp$.

(b) (5 points) Suppose that $\text{proj}_U = \text{proj}_U \circ \text{proj}_W$, prove that $U \subset W$.

(c) (5 points) Suppose that $U \subset W$, prove that $\text{proj}_U = \text{proj}_W \circ \text{proj}_U$.

8 15 points

Let $A \in M_{n\times n}$. Suppose that ρ is the largest eigenvalue of $A^\top A$.

(a) (3 points) Prove that $\|A\boldsymbol{x}\|\leq \sqrt{\rho}\|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^n$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

(b) (5 points) Prove that if $\rho < 1$, then $I_n - A$ is invertible.

(c) (7 points) Suppose that A is invertible. Prove that A can be written as $A = RH$, where $R \in M_{n\times n}$ is orthogonal, $H \in M_{n\times n}$ is symmetric and all of H's eigenvalues are positive.