线性代数(2023-2024)第十次作业

1 复习知识点

- 对于一般的线性变换$T : V \to W$，值域(range)，核(kernel)，单射(injective)/一一映射(one-to-one)，满射(surjective,onto)，双射(bijective)/同构(isomorphism)的定义。
- 对于一般线性变换$T : V \to W$的秩–零化度定理，即讲义Theorem 5.5。
- 线性变换的复合和逆。
- 熟练掌握如何计算线性变换$T : V \to W$关于V的基底B与W的基底B'的矩阵表示$[T]_{B',B}$。

2 习题部分

Problem A(6 Points), 部分取自2021-2022年线性代数期末考试题

Let $P_2 = \{a_0+a_1x+a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$, $P_3 = \{a_0+a_1x+a_2x^2+a_3x^3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$. The function $T : P_2 \to P_3$ is defined by

$$T(p(x)) = xp(2x + 1) - 2xp(x)$$

for any $p(x) \in P_2$.

1. (2 points) Prove that $T : P_2 \to P_3$ is a linear transformation.
2. (2 points) Let $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3\}$ be the standard basis of P_2 and P_3, respectively. Compute the matrix for T relative to B and B', i.e., $[T]_{B',B}$.
3. (2 points) Compute rank(T) and nullity(T).

Problem B(6 Points), 2021-2022年线性代数期末考试题

Let $V = C(-\infty, \infty)$ be the vector space of continuous functions on $\mathbb{R}$. Define functions $T_1 : V \to V$ and $T_2 : V \to V$ by

$$T_1(f(x)) = e^x f(x), \quad T_2(f(x)) = f(5x - 1)$$

for any $f \in V$.

1. (3 points) Prove that T_1 and T_2 are linear operators on V.
2. (3 points) Take $g(x) = e^{-x}$. Find $(T_1 \circ T_2)(f(x))$ and $(T_2 \circ T_1)(f(x))$.

Problem C(6 Points), 2021-2022年线性代数期末考试题

Let V be an n–dimensional vector space, and $T : V \to V$ a linear operator on V. Suppose that there is a $\boldsymbol{v} \in V$ such that $T^{n-1}(\boldsymbol{v}) \neq \mathbf{0}$ and $T^n(\boldsymbol{v}) = \mathbf{0}$. Here we use T^k ($k \geq 1$) to denote the composition of T for k times, i.e.,

$$T^1 = T, \quad T^2 = T \circ T, \quad T^3 = T \circ T \circ T, \quad \ldots, \quad T^k = \underbrace{T \circ T \circ \ldots \circ T}_{k \text{ times}}.$$

1. (4 points) Prove that $B = \{\boldsymbol{v}, T(\boldsymbol{v}), T^2(\boldsymbol{v}), \ldots, T^{n-1}(\boldsymbol{v})\}$ is a basis of V.
2. (2 points) Find the matrix for T relative to B, i.e., $[T]_{B,B}$.

Problem D(6 Points), 2022-2023年线性代数期末考试题

Suppose that $n \geq 2$ is a positive integer to be determined. Let $T : P_n \to P_n$ be a linear operator given by

$$T(p(x)) = p'(x) + p(1)x^2$$

for any $p(x) \in P_n = \{a_0 + a_1 x + \ldots + a_n x^n : a_0, a_1, \ldots, a_n \in \mathbb{R}\}$. Suppose that for $T^2 = T \circ T$ we have $\text{rank}(T^2) = 3$. Find all possible values of this integer n.

Problem E(6 Points), Multiple Choices

1. (3 points) Which of the following statements are true?
   - A. Let V be an n–dimensional vector space, B is a basis of V, then the coordinate vector mapping $f_B : V \to \mathbb{R}^n$, $f_B(\boldsymbol{v}) = [\boldsymbol{v}]_B$ is an isomorphism.
   - B. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $\ker(T) \neq \{\mathbf{0}\}$ and $\text{RAN}(T) \neq \{\mathbf{0}\}$. Then there exists a non–zero vector $\boldsymbol{x} \in \mathbb{R}^2, \boldsymbol{x} \neq \mathbf{0}$ such that $\boldsymbol{x} \in \ker(T) \cap \text{RAN}(T)$. (Here, $\ker(T)$ is the kernel of T, $\text{RAN}(T)$ is the range of T.)
   - C. Let V, W be vector space and $T : V \to W$ is an isomorphism. Let $B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a basis of V, then $B' = \{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)\}$ is a basis of W.
   - D. Let V be an n–dimensional vector space, and $T : V \to V$ be the identity operator, i.e., $T(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in V$. Then for any basis B of V, we have $[T]_{B,B} = I_n$, where $I_n \in M_{n \times n}$ is the identity matrix.
2. (3 points) Which of the following statements are true?
   - A. Let $T : V \to W$ be a linear transformation, and $W' \subset W$ is a subspace of W. Then the set

     $$V' = \{\boldsymbol{v} \in V : T(\boldsymbol{v}) \in W'\}$$

     is always a subspace of V.

- B. If $T: U \to V$ is a surjective (满射) linear transformation, $S: V \to W$ is also a surjective linear transformation, then $S \circ T$ is always surjective.
- C. Let $A \in M_{n\times n}$. Consider the following partitioned matrix $C \in M_{2n\times 2n}$

$$C = \begin{bmatrix} A & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & A^\top A \end{bmatrix},$$

here $\mathbf{0}_{n\times n}$ denotes the zero matrix in $M_{n\times n}$. Then we always have $\text{rank}(C) = 2\text{rank}(A)$.
- D. The function

$$T: M_{n\times n} \to \mathbb{R}, \quad T(A) = \text{tr}(A)$$

is a surjective linear transformation. Here $\text{tr}(A)$ denotes the trace of A.

Bonus: 不计入分数

假设V为一个向量空间，令$W = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \subset V$为$V$的一个子空间，这里$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$线性无关。对于一个给定的$\boldsymbol{v} \in V$，考虑子空间

$$U = \text{span}\{\boldsymbol{v}_1 + \boldsymbol{v}, \boldsymbol{v}_2 + \boldsymbol{v}, \boldsymbol{v}_3 + \boldsymbol{v}\}.$$

由期中考试不定项选择题的第三小题，我们知道$\dim(U)$可以等于2或者3。现在请对此提供一个严格的数学证明，特别地，证明$\dim(U)$不可能小于$3-1=2$。

提示： 首先证明如果$\boldsymbol{v} \notin W = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$，那么$\dim(U) = \dim(W) = 3$。再分析当$\boldsymbol{v} \in W = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$时可能发生的情况。这里可以尝试考虑引入一个线性变换$T: W \to W$，它满足$T(\boldsymbol{v}_1) = \boldsymbol{v}_1 + \boldsymbol{v}, T(\boldsymbol{v}_2) = \boldsymbol{v}_2 + \boldsymbol{v}, T(\boldsymbol{v}_3) = \boldsymbol{v}_3 + \boldsymbol{v}$；因此我们有$U = \text{span}\{\boldsymbol{v}_1 + \boldsymbol{v}, \boldsymbol{v}_2 + \boldsymbol{v}, \boldsymbol{v}_3 + \boldsymbol{v}\} = R(T)$，此时关于$\dim(U)$的问题转化为研究$\text{rank}(T)$的取值可能。由于$B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$为$W$的基底，我们现在可以得到矩阵$[T]_{B,B} \in M_{3\times 3}$，那么利用讲义Theorem 5.17，此时问题进一步转为研究$\text{rank}([T]_{B,B})$的取值可能。对矩阵$[T]_{B,B} \in M_{3\times 3}$进行相关分析即可得到结论。

Deadline: 22:00, December 24.

作业提交截止时间：12月24日晚上22：00。