Student Name: __________________
Student Number: __________________
School: __________________
Year of Entrance: __________________

ShanghaiTech University Midterm Examination Cover Sheet

Academic Year: 2022 to 2023 Term: 1
Course-offering School: IMS
Instructor: *Xue, Boqing*
Course Name: **Linear Algebra I**
Course Number: MATH1112.01/MATH1455.01

Exam Instructions for Students:

1. All examination rules must be strictly observed throughout the entire test, and any form of cheating is prohibited.
2. Other than allowable materials, students taking closed-book tests must place their books, notes, tablets and any other electronic devices in places designated by the examiners.
3. Students taking open-book tests may use allowable materials authorized by the examiners. They must complete the exam independently without discussion with each other or exchange of materials.

For Marker's Use:

Section	1	2	3	4	5	6	7	Total
Marks								
Recheck								

Marker's Signature: Rechecker's Signature:
Date: Date:

Instructions for Examiners:

1. The format of the exam papers and answer sheets shall be determined by the school and examiners according to actual needs. All pages should be marked by the page numbers in order (except the cover page). All text should be legible, visually comfortable and easy to bind on the left side. A4 double-sided printing is recommended for the convenience of archiving (There are all-in-one printers in the university).
2. The examiners should make sure that exam questions are correct and appropriate. If errors are found in exam questions during the exam, the examiners should be responsible to respond on site, which will be taking into account in the teaching evaluation.

Specific Instructions for students:

- The time duration for this exam is 100 **minutes**.
- Computers and calculators are prohibited in the exam.
- Answers can be written in **either Chinese or English**.

★ For problems 2-7, please show details of calculations or deductions. A correct answer with no details can not earn points.

Notations and conventions:

- $\mathbb{R}$ is the set of real numbers. All the scalars here are real numbers.
- m, n always denote positive integers.
- I denotes an identity matrix of suitable order.
- 0 or $\mathbf{0}$ may denote the number zero, a zero vector, a zero matrix, or a zero transformation.
- $\mathsf{tr}(A)$ is the trace of a matrix A.
- $\mathsf{rank}(A)$ is the rank of a matrix A.
- $M_{m\times n}$ is the vector space of all $m \times n$ matrices. And M_n is the vector space of all $n \times n$ matrices.
- P_n is the vector space of all polynomials of degree $\leq n$.
- For a square matrix $A = [a_{ij}]$, the number C_{ij} is the cofactor of entry a_{ij}.
- In $\mathbb{R}^3$, $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$.
- In $\mathbb{R}^n$, the vector $\text{proj}_{\mathbf{u}}\mathbf{v}$ is the orthogonal projection of $\mathbf{v}$ on $\mathbf{u}$.

1. Fill in the blanks.

(1) (5 points) For the matrix $A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 3 & 2 \\ 5 & 1 & -4 \end{bmatrix}$, the cofactor $C_{23} =$ ______________.

(2) (5 points) Let $A = \begin{bmatrix} -3 & 0 & -15 \\ -1 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$. Then $\mathsf{rank}(A) =$ ______________.

(3) (5 points) In $\mathbb{R}^3$, let $\mathbf{u} = (3, -2, 1)$ and $\mathbf{v} = (-1, k, -5)$. Suppose that $||\text{proj}_{\mathbf{u}}\mathbf{v}|| = \sqrt{14}$, then $k =$ ______________.

(4) (5 points) In $\mathbb{R}^3$, let

$$\mathbf{u} = -\mathbf{i}, \quad \mathbf{v} = 2\mathbf{j}, \quad \mathbf{w} = 3\mathbf{k}.$$

Then

$$||\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{w} + \mathbf{w} \times \mathbf{u}|| = \underline{\hspace{4cm}}.$$

(5) (5 points) In P_2, let

$$p_1(x) = 1, \quad p_2(x) = x - 2, \quad p_3(x) = (x-2)^2.$$

Write $q(x) = x^2$ as a linear combination of $p_1(x)$, $p_2(x)$ and $p_3(x)$:

$$q(x) = \underline{\hspace{8cm}}.$$

2. (12 points) In $\mathbb{R}^3$, let $\mathbf{u} = (1, 1, 2)$ and $\mathbf{v} = (0, 2, 3)$. Suppose that H is the plane passing through the point $P(1, 0, 1)$ and parallel to both $\mathbf{u}$ and $\mathbf{v}$. Find an equation of H in the form $Ax + By + Cz + D = 0$.

3. (12 points) Evaluate $\det\big((P^{-1}AP)^{2023}\big)$, where

$$A = \begin{bmatrix} 4 & 1 & 6 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} \sin 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos 1 \end{bmatrix}.$$

4. (8 points) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Suppose that $\|\mathbf{u}\| = \|\mathbf{v}\|$. Prove that the vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal.

5. (8 points) Let $A, B \in M_n$. Suppose that A is symmetric and B is skew-symmetric (also called anti-symmetric). Is $AB - BA$ symmetric? skew-symmetric? or not? Please prove your conclusion.

6. (15 points) Let A, B, C be 3×3 matrices such that

$$B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix},$$

and $(I - C^{-1}B)^T C^T A = I$. Find A.

7. Let V be the real vector space that consists of all 3×3 upper triangular matrices with identical diagonal entries, i.e.,

$$V = \left\{ \begin{bmatrix} k & a & b \\ 0 & k & c \\ 0 & 0 & k \end{bmatrix} : \; k, a, b, c \in \mathbb{R} \right\}.$$

Let

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Define

$$W = \{A \in V : \; \mathsf{tr}(B^T A) = \mathsf{tr}(C^T A) = 0\}.$$

(i) (8 points) Prove that W is a subspace of V.

(ii) (12 points) Find a basis for W.