线性代数(2023-2024)第八次作业

1 复习知识点

- 矩阵的秩(rank)的定义，如何计算一个矩阵的秩。
- Rank-Nullity Theorem的内容，秩与线性方程组主元之间的关系，零化度与线性方程组自由元之间的关系。
- $\mathbb{R}^2, \mathbb{R}^3$里由方程确定的直线与平面，仔细阅读目前讲义128页-131页内容，尤其是讲义Theorem 4.34里的两个计算距离的公式。

2 习题部分

Problem A(6 Points) (2022年线性代数期中考试题).

Consider two parallel (中文：平行的) planes in $\mathbb{R}^3$ whose equations are given by $x + 2y + 2z = 4$ and $x + 2y + 2z = -5$. Compute the distance between these two planes.

Problem B(6 Points) (2022年线性代数期中考试题).

In $\mathbb{R}^3$, let $\boldsymbol{u} = (1, 1, 2), \boldsymbol{v} = (0, 2, 3)$. Suppose that H is the plane passing through the point $P = (1, 0, 1)$ and parallel to $\boldsymbol{u}$ and $\boldsymbol{v}$. Find an equation of H in the form $Ax + By + Cz + D = 0$.

Problem C(6 Points) (2022年线性代数期中考试题).

Consider the matrix $A = \begin{bmatrix} 2x & -1 & 2 & 3 \\ x & -2 & 1 & 2 \\ 1 & x & x & x \end{bmatrix}$, where x is a real number.

1. (2 points) Prove that $\operatorname{rank}(A) \geq 2$ for all $x \in \mathbb{R}$.
2. (4 points) Find all $x \in \mathbb{R}$ such that $\operatorname{rank}(A) = 2$.

Problem D(6 Points)

假设$\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5$是$\mathbb{R}^5$里的五个列向量。假设矩阵 $A = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}$ (即以$\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$依次作为列向量)的秩为$\operatorname{rank}(A) = 3$，矩阵$B = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 \end{bmatrix}$的秩为$\operatorname{rank}(B) = 3$，矩阵$C = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_5 \end{bmatrix}$的秩为$\operatorname{rank}(C) = 4$。证明矩阵

$$D = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_5 - \boldsymbol{v}_4 \end{bmatrix}$$

的秩为rank(D) = 4。

Problem E(6 Points) (2022年线性代数期中考试题)

Determine whether the following statements are true or false, and explain why.

1. (2 points) For any matrix with two rows and three columns, the matrix $A^\top A$ is always singular. (注：一个方阵被称为singular，如果它不可逆。)
2. (2 points) For any matrix with three rows and two columns, the matrix $A^\top A$ is always singular.
3. (2 points) In any vecotr space of dimension 5, one can find 4 linearly independent vectors.

Deadline: 22:00, December 03.

作业提交截止时间：12月3日晚上22：00。