Student Name: ________________

Student Number: ________________

School: ________________

Year of Entrance: ________________

# ShanghaiTech University Midterm Examination Cover Sheet

Academic Year: 2023 to 2024 Term: 1

Course-offering School: IMS

Instructor: *Mingliang Cai, Chong Liu, Qixiao Ma, Daniel Skodlerack, Qiang Wang, Boqing Xue*

Course Name: **Linear Algebra I**

Course Number: MATH1112/MATH1455

**Exam Instructions for Students:**

1. All examination rules must be strictly observed throughout the entire test, and any form of cheating is prohibited.
2. Other than allowable materials, students taking closed-book tests must place their books, notes, tablets and any other electronic devices in places designated by the examiners.
3. Students taking open-book tests may use allowable materials authorized by the examiners. They must complete the exam independently without discussion with each other or exchange of materials.

**For Marker's Use:**

| Section | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Marks | | | | | | | | | |
| Recheck | | | | | | | | | |

Marker's Signature:

Date:

Rechecker's Signature:

Date:

**Specific Instructions for students:**

- The time duration for this exam is 100 minutes.
- Computers and calculators are prohibited in the exam.
- Answers can be written in either Chinese or English.

★ For problems 3-8, please show details of calculations or deductions. A correct answer with no details can not earn points.

**Policy for grading the Multiple choice questions:**

For a multiple choice question, denote by $C$ the set of all correct choices, and by $A$ the set of your choices. If $A \not\subseteq C$, get zero points; If $A \subsetneq C$, get partial credits depending on the size of $A$.

- If $|C| = 4$, get one point for each correct choice when $|A| < 4$, and get full points when $|A| = 4$;
- If $|C| = 3$, get two points for each correct choice when $|A| < 3$, and get full points when $|A| = 3$;
- If $|C| = 2$, get three points when $|A| = 1$, and get full points when $|A| = 1$.

The unlisted remaining case for $|C| = 1$ should be self evident.

**Notations and conventions:**

- $\mathbb{R}$ is the set of real numbers.
- $I$ denotes an identity matrix of suitable size.
- 0 or $\mathbf{0}$ may denote the number zero, a zero vector, or a zero matrix.
- $\mathbf{M}_{m\times n}$ is the vector space of all $m \times n$ matrices (with real entries).
- For a square matrix $A = [a_{ij}]$, $M_{ij}$ is the minor of entry $a_{ij}$; $C_{ij}$ is the cofactor of entry $a_{ij}$; $adj(A)$ is the adjoint (adjunct) matrix of $A$.
- For a square matrix $A$, both $det(A)$ and $|A|$ denote the determinant of $A$.
- Give a matrix $A$, we denote by $null(A)$, $row(A)$, $col(A)$ the null space, row space, column space of $A$ respectively. And $nullity(A)$ and $r(A)$ denotes the nullity and rank of $A$.
- For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, we denote by $\mathbf{u} \bullet \mathbf{v}$ the *dot product* of $\mathbf{u}$ and $\mathbf{v}$, and by $\mathbf{u} \times \mathbf{v}$ their *cross product.*
- $P_n$ is the vector space of all polynomials (with real coefficients) with degree no more than $n$.
- The transition matrix $P_{B'\leftarrow B}$ from basis $B = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ to basis $B' = \{\mathbf{v}'_1, \cdots, \mathbf{v}'_n\}$ is given through $B = B'P_{B'\leftarrow B}$, i.e, $[\mathbf{v}_1, \cdots, \mathbf{v}_n] = [\mathbf{v}'_1, \cdots, \mathbf{v}'_n]P_{B'\leftarrow B}$. Equivalently,

$$P_{B'\leftarrow B} := [[\mathbf{v}_1]_{B'} \ [\mathbf{v}_2]_{B'} \ \cdots \ [\mathbf{v}_n]_{B'}]$$

**1.** Multiple choice questions.

a). (5 *points*) Which of the following sets are vector spaces? ( C ) $(0,0)$

~~(A)~~ $\{(a,b) \in \mathbb{R}^2 : \ b = 2a+3\} \subseteq \mathbb{R}^2$, with the usual " + " and " · " as in $\mathbb{R}^2$.

~~(B)~~ $\{\mathbf{v} \in \mathbb{R}^3 : \ ||\mathbf{v}|| = 1\} \subseteq \mathbb{R}^3$, with the usual " + " and " · " as in $\mathbb{R}^3$. $\}\ \vec{0}$

(C) {All polynomials in $P_2$ that are divisible by $x-2$}, with the usual " + " and " · " as in $P_2$.

(D) The set $\mathbb{R}^2$, with addition and scalar multiplication given by: for $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $k \in \mathbb{R}$, $\mathbf{x}+\mathbf{y} := (x_1 + 2y_1,\ x_2 + 3y_2)$, $k\mathbf{x} := (kx_1, kx_2)$.

b). (5 *points*) Determine which of the following statements are true. ( AD )

(A) If $A \in \mathbf{M}_{n\times n}$ is invertible, then its adjoint $adj(A)$ is also invertible. ✓

~~(B)~~ Let $E \in \mathbf{M}_{3\times 3}$ be an elementary matrix such that $det(E) = 1$, then $E$ must be the identity matrix in $\mathbf{M}_{3\times 3}$.

(C) Let $V \subsetneq \mathbb{R}^5$ be a subspace, then any set of five vectors in $V$ is linearly dependent.

(D) If $A \in \mathbf{M}_{4\times 7}$ and $dim\,(null(A)) = 3$, then for all $\mathbf{b} \in \mathbb{R}^4$, the linear system $A\mathbf{x} = \mathbf{b}$ has at least one solution. rank = 4 3 free

c). (5 *points*) Consider a linearly independent set $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\} \subseteq V$ for some $m \geq 1$, and let $\mathbf{v} \in V$. Which possible values can $dim\,(span\{\mathbf{v}_1 + \mathbf{v}, \cdots, \mathbf{v}_m + \mathbf{v}\})$ take? ( A ) B

$v = -v_1$ $v = v_1$

(*A*) $m-1$ (*B*) $m$ ~~(*C*)~~ $m+1$ ~~(*D*)~~ $m+2$

**2.** Fill in the blanks. $|A| = 2$ $AA^* = |A|I$

a.) (5 *points*) Let $A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$. Then $(adj(A))^{-1} =$ $\begin{bmatrix} 1 & 1 \\ \frac{3}{2} & 2 \end{bmatrix}$. $(A^*)^{-1} = \frac{A}{|A|}$

b). (5 *points*) Let $B = \{1, x, x^2\}$ and $B' = \{1+x^2, x+x^2, 1+2x+x^2\}$ be two basis for $P_2$.

$[B' | B] \rightarrow [I \,|\, P_{B' \leftarrow B}]$

Then the transition matrix $P_{B' \leftarrow B}$ from $B$ to $B'$ is $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

c.) (5 *points*) Let $A = [a_{ij}] \in \mathbf{M}_{n\times n}$ be given such that $a_{ij} = ij$ for all $i, j = 1, \cdots, n$. Assuming that $n \geq 2$, then $det\,A =$ 0.

$$\begin{bmatrix} 1\cdot 1 & 1\cdot 2 & \cdots & 1\cdot n \\ 2\cdot 1 & 2\cdot 2 & \cdots & 2\cdot n \\ \vdots & \vdots & \ddots & \vdots \\ n\cdot 1 & n\cdot 2 & & n\cdot n \end{bmatrix} > \text{same}$$

**3.** (10 *points*) Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, and suppose that $A^2 - AB = I_3$. Find $B$.

$|A| = -1 \neq 0$

$AB = A^2 - I$

$B = A^{-1}(A^2 - I)$

$A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ , $A^2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^2 - I = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**4.** Let $A \in \mathbf{M}_{4\times 5}$ be the following matrix

$$\begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & 3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}$$

a). (10 *points*) Compute $r(A)$, $nullity(A)$, and find basis for $row\,(A)$, $col\,(A)$ and $null(A)$.

$$A \sim \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & -9 & 9 & -9 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(A) = 3$, $nullity(A) = n - r(A) = 2$

$col(A)$: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -3 \\ 0 \end{bmatrix} \right\}$

$Ax = 0$

$x_2 = s$, $x_4 = t$, $x_5 = 0$

$x_1 = -3s - 3t$, $x_3 = t$

$$x = s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$s, t \in \mathbb{R}$

basis: $null(A)$: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$A^T = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 3 & 6 & 9 & 9 \\ 4 & 6 & 3 & 0 \\ -1 & 0 & 6 & 9 \\ 2 & 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 9 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$col(A^T) = row(A)$ basis: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 6 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 3 \\ 6 \\ -3 \end{bmatrix} \right\}$

b). (5 *points*) Determine whether $\mathbf{u} = [2, 1, 7, -12]^T$ belongs to $col(A)$.

augmented
$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 2 & 6 & 3 & 1 \\ 3 & 3 & -3 & 7 \\ 3 & 0 & 0 & -12 \end{array}\right]$$

$$\sim \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 19 \\ 1 & 0 & 0 & -4 \end{bmatrix}$$

有解

$\therefore u \in col(A)$

c). (5 *points*) Find the space of all vectors in $\mathbb{R}^4$ that are orthogonal to $col(A)$, i.e. the *orthogonal complement* of $col(A)$ in $\mathbb{R}^4$.

$col(A)^{\perp} = null(A^T)$

$$a): A^T \sim \begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3 = 0, \; x_4 = s \quad x_2 = -6s, \; x_1 = 9s$

$$x = s\begin{bmatrix} 9 \\ -6 \\ 0 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R}$$

**5.** Let $M_{2\times 2}$ denote the vector space of all $2 \times 2$ matrices with real entries. Consider the following two subsets of $M_{2\times 2}$

$$U = \left\{ \begin{bmatrix} x & -x \\ y & z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}; \qquad W = \left\{ \begin{bmatrix} a & b \\ -a & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

a). (10 *points*) Verify that both $U$ and $W$ are vector subspaces of $M_{2\times 2}$. And find a basis and the dimension of $U$ and $W$.

$u_1, u_2 \in U$, $k_1 \in R$

$$u_1 + u_2 = \begin{bmatrix} x_1+x_2 & -(x_1+x_2) \\ y_1+y_2 & z_1+z_2 \end{bmatrix} \in U$$

$$k_1 u_1 = \begin{bmatrix} k_1x_1 & -(k_1x_1) \\ k_1y_1 & k_1z_1 \end{bmatrix} \in U$$

$w_1, w_2 \in W$, $k_2 \in R$

$$w_1 + w_2 = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ -(a_1+a_2) & c_1+c_2 \end{bmatrix} \in V$$

$$k w_1 = \begin{bmatrix} k_2a_1 & k_2b_1 \\ -(k_2a_1) & k_2c_1 \end{bmatrix} \in V$$

$\forall u \in U$ $\quad u = \begin{bmatrix} x & -x \\ y & z \end{bmatrix} = x\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\dim(U) = 3$, basis: $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\dim(W) = 3$ basis: $\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

b). (10 *points*) Find the dimensions and basis of the subspaces $U + W$ and $U \cap W$.

$$U+W = \begin{bmatrix} x+a & -x+b \\ y-a & z+c \end{bmatrix}$$

$$\dim(U+W) = 4$$

basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap W)$$

$$\Rightarrow \dim(U \cap W) = 2$$

$$\begin{bmatrix} x & -x \\ -x & z \end{bmatrix}$$

basis $\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

**7. a).** (5 *points*) Let $\mathbf{v}_1 = [1, 3, 0, 2]^T$, $\mathbf{v}_2 = [-1, 0, 1, 0]^T$, $\mathbf{v}_3 = [5, 9, -2, 6]^T$ be vectors in $\mathbb{R}^4$. Is it possible to find a set of numbers $\{a_{ij} \mid i, j = 1, 2, 3\}$, such that the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent? Here $\mathbf{w}_i$'s are given by

$$\mathbf{w}_1 = a_{11}\mathbf{v}_1 + a_{12}\mathbf{v}_2 + a_{13}\mathbf{v}_3$$
$$\mathbf{w}_2 = a_{21}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + a_{23}\mathbf{v}_3$$
$$\mathbf{w}_3 = a_{31}\mathbf{v}_1 + a_{32}\mathbf{v}_2 + a_{33}\mathbf{v}_3$$

Please give full explanation of your claim.

$$k_1 w_1 + k_2 w_2 + k_3 w_3 = 0$$

$$\Longrightarrow \quad c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$[v_1 | v_2 | v_3] = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 0 & 9 \\ 0 & 1 & -2 \\ 2 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Linear dependent

X

b). (5 *points*) You should have already known the fact (from the review problems) that a matrix of of the form $A = \mathbf{u}\mathbf{v}^T$ has rank 1, here $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are any $n$ dimensional non zero column vectors. What about the converse? That is, is it true that any rank 1 square matrix of size $n$ can be written as $\mathbf{u}\mathbf{v}^T$ for some $n$ dimensional non zero column vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$? Prove your claim.

$$uv^T = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

$$= \left[ v_1 \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \cdots, v_n \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \right]$$

$$= \begin{bmatrix} u_1 \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \\ \vdots \\ u_n \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \end{bmatrix}$$

$\forall A \in M_{n \times n}$, $\text{rank}(A) = 1$

$\Rightarrow \quad r_i = k_i r_1 \Rightarrow v_1 = 1,\ v_i = k_i$

or $c_i = k_i' c_1 \Rightarrow u_1 = 1,\ u_i = k_i'$

**8.** (10 *points*) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ be two linearly independent sets of vectors in $\mathbb{R}^n$ for some integer $n$ such that $\mathbf{v}_i \bullet \mathbf{w}_j = 0$ for all $i = 1, 2, 3$ and $j = 1, 2$. Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_1, \mathbf{w}_2\}$ still linearly independent? Verify your claim.

Let $v = a_1v_1 + a_2v_2 + a_3v_3 + b_1w_1 + b_2w_2$

$= 0$

then $v \cdot v = 0$

$(a_1v_1 + a_2v_2 + a_3v_3 + b_1w_1 + b_2w_2) \cdot (a_1v_1 + a_2v_2 + a_3v_3 + b_1w_1 + b_2w_2)$

$$= (a_1v_1 + a_2v_2 + a_3v_3) \cdot (a_1v_1 + a_2v_2 + a_3v_3) + \left.\begin{matrix} (a_1v_1 + a_2v_2 + a_3v_3) \cdot (b_1w_1 + b_2w_2) \\ + (b_1w_1 + b_2w_2) \cdot (a_1v_1 + a_2v_2 + a_3v_3) \end{matrix}\right\} = 0 + (b_1w_1 + b_2w_2) \cdot (b_1w_1 + b_2w_2)$$

$= \|(a_1v_1 + a_2v_2 + a_3v_3)\|^2 + \|(b_1w_1 + b_2w_2)\|^2$

$= 0$

$\therefore \underbrace{a_1v_1 + a_2v_2 + a_3v_3 = 0}, \underbrace{b_1w_1 + b_2w_2 = 0}$

linear independent

$\Rightarrow \underbrace{a_1 = a_2 = a_3 = 0 \quad b_1 = b_2 = 0}$

$\therefore$ $\{v_1, v_2, v_3, w_1, w_2\}$ linear independent