Linear Algebra Tutorial5

2023.11.7

homework

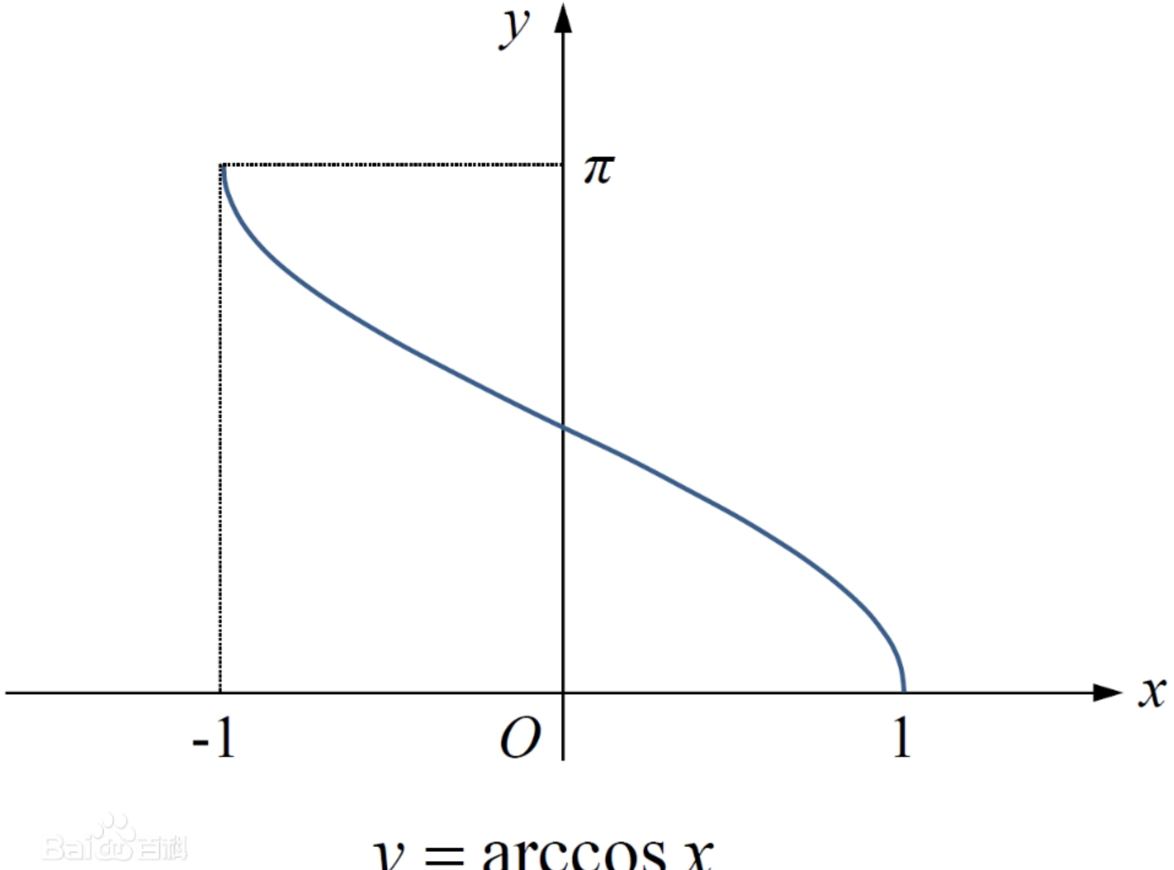
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

- $(A^2 - 1)B = A + I$
  $|B| =?$

$$\mathbf{v} = (2, -2, 2)$$

- $\|\mathbf{v}\| =?$

arccos



$y = \arccos x$

homework

- bonus

  $A, B \in M_{3\times 3}$

  $A^2 - AB - 2B^2 = A - 2BA - B$

  $$B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 3 & 2 \end{bmatrix}$$

  $det(A - B) \neq 0$

Cramer's rule

$A\mathbf{x} = \mathbf{b}$, $|A| \neq 0$ ($A_{n\times n}$)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

$$A_i = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & & & & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{bmatrix}$$

then $x_i = \dfrac{|A_i|}{|A|}$

where A_i is the matrix obtained from A by replacing the ith column of A by $\mathbf{b}$

Cramer's rule

- e.g. when the coefficient is complicated
- **directly solve one of the variables**

(b) (5 points) Only solve for x_2 and x_4 for the following system:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 3x_2 - 2x_3 + 4x_4 = -1 \\ x_1 + 9x_2 + 4x_3 + 16x_4 = 1 \\ x_1 + 27x_2 - 8x_3 + 64x_4 = -1 \end{cases}$$

determinant property

$$C = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} + b_{i1} & a_{i2} + b_{i2} & \dots & a_{in} + b_{in}\,(\text{第}i\text{行}) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in}\,(\text{第}i\text{行}) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{i1} & b_{i2} & \dots & b_{in}\,(\text{第}i\text{行}) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

> $|C| = |A| + |B|$

determinant property

calculate $\begin{vmatrix} ax+by & ay+bz & az+bx \\ ay+bz & az+bx & ax+by \\ az+bx & ax+by & ay+bz \end{vmatrix}$

determinant practice

7. *(10 points)* Evaluate the following determinant of order n ($n \geq 2$):

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 1 & 2 & \dots & n-2 & n-1 \\ 3 & 2 & 1 & \dots & n-3 & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n-2 & n-3 & \dots & 1 & 2 \\ n & n-1 & n-2 & \dots & 2 & 1 \end{vmatrix}$$

math review

- vector
- Cauchy inequality
- dot product
- cross product

$\vdots$

Euclidean space

- $\mathbb{R}^n$ n-dimensional Euclidean space(can be seen as ordered n-tuples)
  $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} = (v_1, \cdots, v_n)$
- **linear combination**
  $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$
  $\mathbf{w}$ is the linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_r$
  $k_1, \cdots, k_r$ are the coefficients of $\mathbf{v}_1, \cdots, \mathbf{v}_r$
- **span**
  the set of all linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_r$ is called the span of $\mathbf{v}_1, \cdots, \mathbf{v}_r$
  $span\{\mathbf{v}_1, \cdots, \mathbf{v}_r\} = \{\mathbf{w} | \mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r, k_1, \cdots, k_r \in \mathbb{R}\}$

linear independence

- standard unit vector 标准单位向量
  there are n standard unit vectors in $\mathbb{R}^n$
  $\mathbf{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)$
  $\mathbf{e}_i$ has 1 in the i-th position and 0 elsewhere
  eg. $\mathbf{w} = (1, 2)$, $\mathbf{w} = 1\mathbf{e}_1 + 2\mathbf{e}_2$
- is the representation unique?
  $\mathbf{w} = (1, 2)$
  $\mathbf{v_1} = (1, 0), \mathbf{v_2} = (0, 1), \mathbf{v_3} = (1, 1)$
  $\mathbf{w} = 1\mathbf{v_1} + 2\mathbf{v_2} + 0\mathbf{v_3}$
  $\mathbf{w} = -1\mathbf{v_1} + 0\mathbf{v_2} + 1\mathbf{v_3}$
  $\mathbf{w} = 0\mathbf{v_1} + 1\mathbf{v_2} + 1\mathbf{v_3}$

> unique represetation $\Leftrightarrow$ linearly independent

linear independence

- $\mathbf{v}_1, \cdots, \mathbf{v}_r$ are linearly independent if the only solution of $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$ is $k_1 = k_2 = \cdots = k_r = 0$
- $\mathbf{v}_1, \cdots, \mathbf{v}_r$ are linearly dependent if and only if one of them is a linear combination of the others

norm

- norm(Euclidean norm)

  $$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

  distance to the origin

  $$\|\mathbf{v}\| \geq 0$$

  $$\|\mathbf{v}\| = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$$

  $$\|k\mathbf{v}\| = |k| \, \|\mathbf{v}\|$$

- normalization

  $$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

  get the **unit vector** $\mathbf{u}$: normailize $\mathbf{v}$

other different norms*

p norm of a vector $\mathbf{v} = (v_1, \cdots, v_n)$

$$|| \mathbf{v} ||_p = \sqrt[p]{|v_1|^p + |v_2|^p + \cdots + |v_n|^p}$$

- 0 norm:

  $|| \mathbf{v} ||_0$ is the number of nonzero entries in $\mathbf{v}$

  not good due to it is **not convex**

- 1 norm:

  $$|| \mathbf{v} ||_1 = |v_1| + |v_2| + \cdots + |v_n|$$

- 2 norm(Euclidean norm):

  $$|| \mathbf{v} ||_2 = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- ∞ norm:

  $$|| \mathbf{v} ||_\infty = max\{|v_1|, |v_2|, \cdots, |v_n|\}$$

properties of norm*

p-norm : $\| \mathbf{v} \|_p = \sqrt[p]{|v_1|^p + |v_2|^p + \cdots + |v_n|^p}$, $p \geq 1$

- 0 norm is actually not a norm, but matric(度量)
- Cauchy inequality

  $$\sum_{i=1}^{n} |a_i b_i| \leq \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2}$$

- Hölder's inequality (赫尔德不等式)

  $$p, q \geq 1, \frac{1}{p} + \frac{1}{q} = 1$$

  $$\sum_{i=1}^{n} |a_i b_i| \leq \| x \|_p \|y\|_q$$

  $$\sum_{i=1}^{n} |a_i b_i| \leq \| x \|_1 \|y\|_\infty$$

  $$\sum_{i=1}^{n} |a_i b_i| \leq \| x \|_2 \|y\|_2$$ -> Cauchy inequality

properties of norm*

- Minkowski's inequality(闵可夫斯基不等式)
  $\|\mathbf{x}+\mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$
  when $p=2$: triangle inequality
- $\|\mathbf{x}\| \geq 0$, $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$
- $\forall a$, $\|a\mathbf{x}\| = |a| \|\mathbf{x}\|$
- all norm $f: \mathbb{R}^n \to \mathbb{R}$ are convex.
- to learn more in IML/numerical optimazation/convex optimazation/...

properties of norm*

Introduction to Machine Learning, Fall 2023

Homework 2

(Due Tuesday Nov. 14 at 11:59pm (CST))

October 25, 2023

1. [10 points] [Convex Optimization Basics]

   (a) Proof any norm $f : \mathbb{R}^n \to \mathbb{R}$ is convex. [2 points]

   (b) Determine the convexity (i.e., convex, concave or neither) of $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{>0}$. [2 points]

   (c) Determine the convexity of $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{>0}$. [2 points]

   (d) Recall Jensen's inequality $f(\mathbb{E}(X)) \leq \mathbb{E}(f(X))$ if f is convex for any random variable X. Proof the log sum inequality:

$$\sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i\right) \log \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

   where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are positive numbers. Hints: $f(x) = x \log x$ is strictly convex. [4 points]

Solution:

distance

the Euclidean distance between $\mathbf{u}$ and $\mathbf{v}$

$$d(u, v) = d(u, v) = \| \mathbf{u} - \mathbf{v} \| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

Cauchy-Schwarz inequality

- vector version

  $|\mathbf{u} \cdot \mathbf{v}| \leq \; ||\mathbf{u}|| \; \cdot \; ||\mathbf{v}||$

- calculus version

  $\int_a^b f(x)g(x)dx \leq \sqrt{\int_a^b f^2(x)dx}\sqrt{\int_a^b g^2(x)dx}$

- probability version

  $|E(XY)| \leq \sqrt{E(X^2)}\sqrt{E(Y^2)}$

triangle inequality

- $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$
- $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

dot(inner) product

- $\mathbf{u} \cdot \mathbf{w} = u_1 w_1 + u_2 w_2 + \cdots + u_n w_n$

So the Euclidean norm can be represented by the dot product

- $||\, \mathbf{v} \,|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

regard $\mathbf{v}, \mathbf{w}$ as a column vector
$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} = \mathbf{v}^T \mathbf{w} = \mathbf{w}^T \mathbf{v}$

if $A_{n \times n}$, $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$

- $\mathbf{Av} \cdot \mathbf{w} = (\mathbf{Av})^T \mathbf{w} = \mathbf{v}^T \mathbf{A}^T \mathbf{w} = \mathbf{v} \cdot \mathbf{A}^T \mathbf{w}$
- $\mathbf{v} \cdot \mathbf{Aw} = \mathbf{v}^T (\mathbf{Aw}) = (\mathbf{v}^T \mathbf{A}) \mathbf{w} = (\mathbf{Av}^T)^T \mathbf{w} = \mathbf{A}^T \mathbf{v} \cdot \mathbf{w}$

dot prodcut

THEOREM 3.2.2 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **[Symmetry property]**

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **[Distributive property]**

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ **[Homogeneity property]**

(*d*) $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$ **[Positivity property]**

THEOREM 3.2.3 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

(*b*) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(*c*) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(*d*) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

(*e*) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

- Theorem

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2+\|\boldsymbol{u}-\boldsymbol{v}\|^2=2(\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2),$$

$$\boldsymbol{u}\cdot\boldsymbol{v}=\frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2-\frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2.$$

angle

angle θ between two vectors $\mathbf{x}, \mathbf{y} \leftarrow$ **no zero vector**

- $cos\theta = \dfrac{< \mathbf{x}, \mathbf{y} >}{\sqrt{< \mathbf{x}, \mathbf{x} >} \cdot \sqrt{< \mathbf{y}, \mathbf{y} >}} = \dfrac{\mathbf{x}^T \mathbf{y}}{|| \mathbf{x} || \, \|\mathbf{y}\|}$

so $\mathbf{x} \cdot \mathbf{y} = \; || \mathbf{x} || \, \|\mathbf{y}\| cos\theta$

- $\mathbf{x} \cdot \mathbf{y} = 0 \Leftrightarrow \mathbf{x} \perp \mathbf{y}$
- $\theta = \arccos(\dfrac{\mathbf{x} \cdot \mathbf{y}}{|| \mathbf{x} || \, \|\mathbf{y}\|})$

  Cauchy inequality:
  $|\dfrac{\mathbf{x} \cdot \mathbf{y}}{|| \mathbf{x} || \, \|\mathbf{y}\|}| \leq 1$

Orthogonal

- $\mathbf{u}, \mathbf{v}$ are orthogonal **iff** $\mathbf{u} \cdot \mathbf{v} = 0$
- orthogonal set
  $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are orthogonal
  $\|\mathbf{v}_1 + \cdots + \mathbf{v}_n\| = \|\mathbf{v}_1\| + \cdots + \|\mathbf{v}_n\|$
  proof in homework

Projection Theorem

- orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$
  $\mathbf{w}_1 = proj_{\mathbf{v}}(\mathbf{u}) = \dfrac{\mathbf{u} \cdot \mathbf{v}}{||\,\mathbf{v}\,||^{\,2}}\mathbf{v}$
- the vector component of \mathbf{u} orthogonal to \mathbf{v}
  $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \mathbf{u} - proj_{\mathbf{v}}(\mathbf{u}) = \mathbf{u} - \dfrac{\mathbf{u} \cdot \mathbf{v}}{||\,v\,||^{\,2}}\mathbf{v}$

The geometry of linear systems

- For linear system, $A\mathbf{x} = \mathbf{b}$
  $\mathbf{w}$ is a solution of $A\mathbf{x} = \mathbf{0}$(homogeneous system)
  $\mathbf{s}$ is a specific solution of $A\mathbf{x} = \mathbf{b}$

Then every solution of $A\mathbf{x} = \mathbf{b}$ is of the form $\mathbf{v} = \mathbf{s} + \mathbf{w}$

important in the whole learning process of Linear Algebra

The geometry of linear systems

- for the homogeneous system $A\mathbf{x} = \mathbf{0}$
- consider the A as a sequence of row vectors,

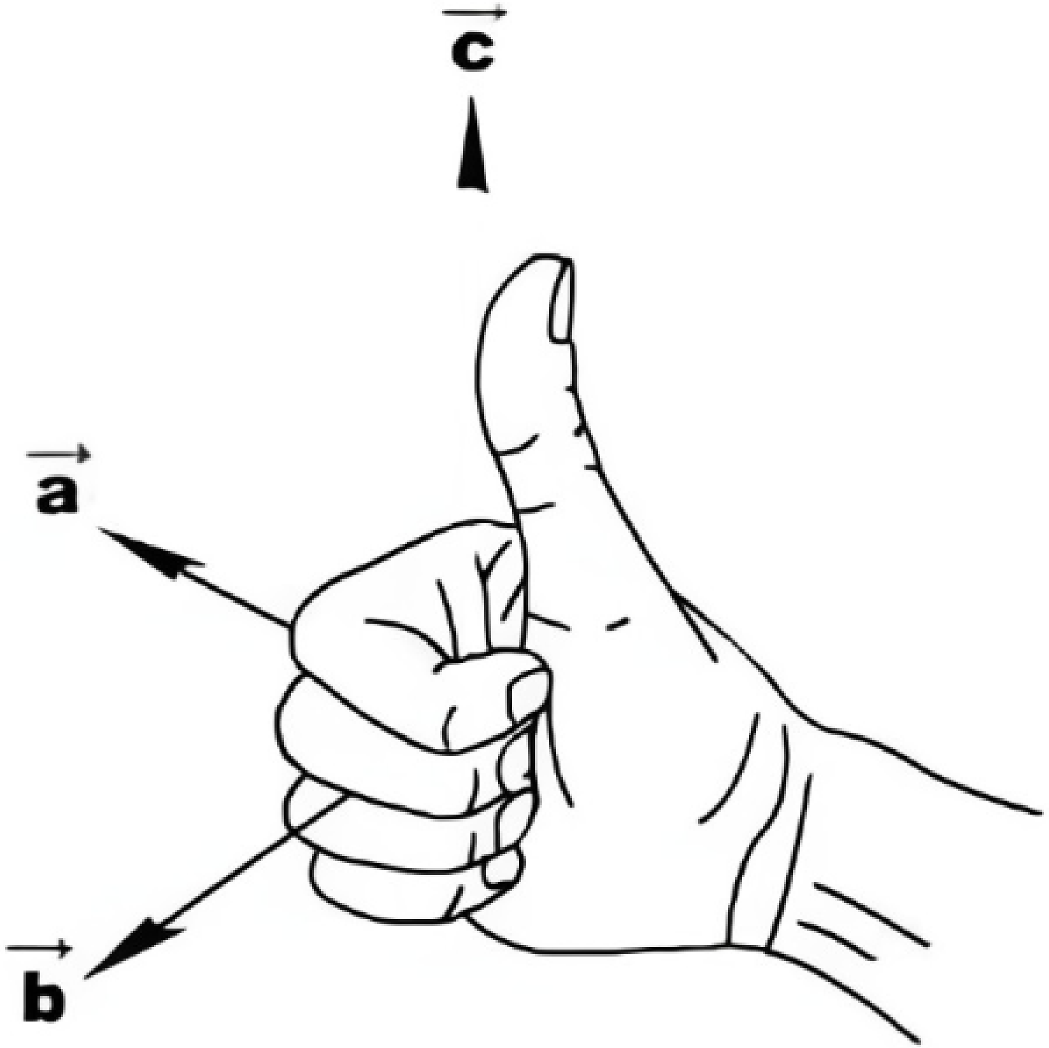
  i.e. $A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$
- and $\mathbf{s}$ is a solution to the homogenous system

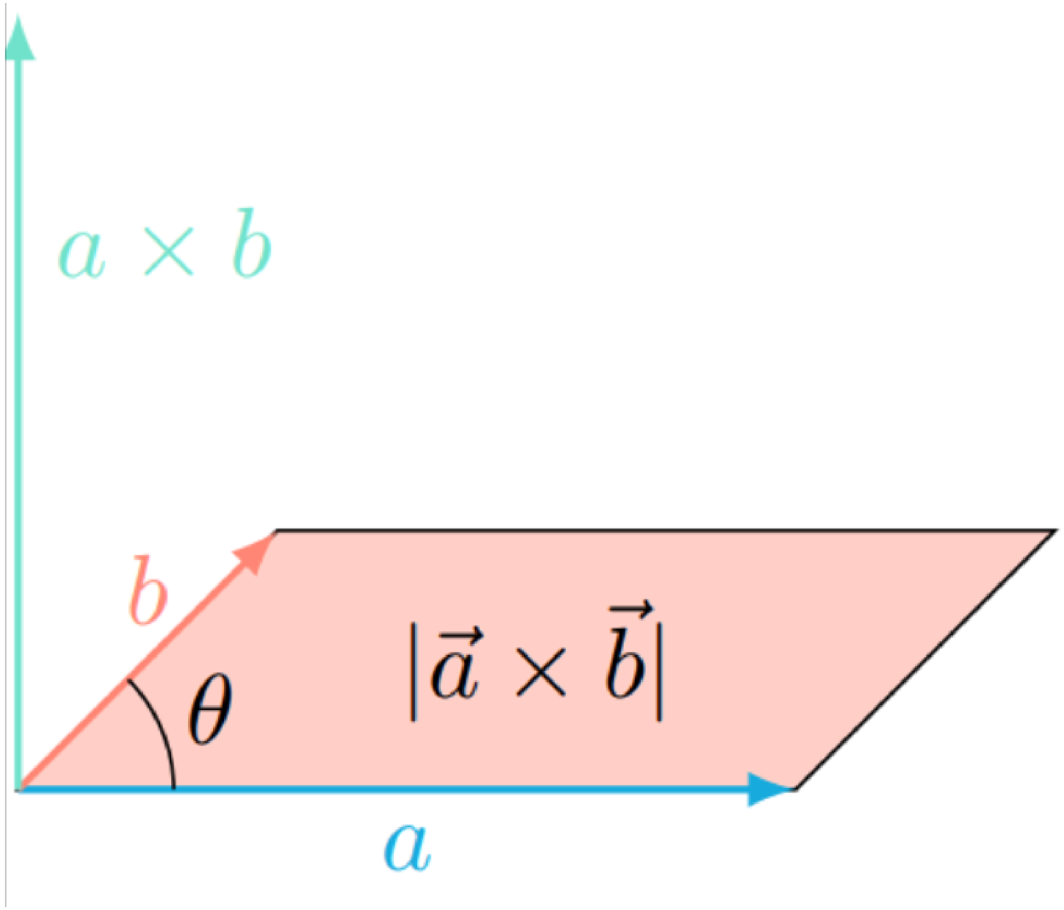
Then we have $\forall i, \mathbf{r}_i \cdot \mathbf{s} = 0$

and $\sum_{i=1}^{m} \mathbf{r}_i \cdot s_i = 0$

> linear independent

cross product





cross product

$$\mathbf{x} \times \mathbf{y} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix}$$

- $\mathbf{x} \times \mathbf{y}$ is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$
- $\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\| \, \|\mathbf{y}\| sin\theta$

cross product

- the cross product of two vectors can be represented by the determinant

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

- $\mathbf{x} \times \mathbf{y} = \mathbf{0} \Leftrightarrow \mathbf{x} \parallel \mathbf{y}$
- we can also transfer $\mathbf{x}$ into a matrix $[\mathbf{x}]\mathbf{y}$

where $[\mathbf{x}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$

> notice that $[\mathbf{x}]$ is skew-symmetric, so its rank is odd
> actually, rank($[\mathbf{x}]$)=2

cross product property

THEOREM 3.5.2 Properties of Cross Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in 3-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

- just prove it in homework by the definition of cross product

scalar triple product

- 标量三重积(混合积)

$$\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

- $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \mathbf{y} \cdot (\mathbf{z} \times \mathbf{x}) = \mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})$
- $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \mathbf{0} \Leftrightarrow \mathbf{x}, \mathbf{y}, \mathbf{z}$ are coplanar
- $\mathbf{x} \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \cdot (\mathbf{y} \times \mathbf{x}) = 0$
- Lagrange's identity

$$||\,\mathbf{u} \times \mathbf{v}\,||^2 = ||\,\mathbf{u}\,||^2\,||\,\mathbf{v}\,||^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

cross product geometric meaning

- parallelogram area
  平行四边形面积
  $S = \bar{AB} * \bar{AC} sin\theta = \|\mathbf{AB} \times \mathbf{AC}\|$
- volume of parallelepiped
  平行六面体体积
  $V = S_{base} h = \|\mathbf{AB} \times \mathbf{AC}\| \cdot \|\mathbf{AD}\| cos\alpha = \mathbf{AD} \cdot (\mathbf{AB} \times \mathbf{AC})$

exercise (21-mid)

1. Fill in the blanks.

(1) *(8 points)* In $\mathbb{R}^3$, let $\mathbf{u} = (1, 0, -1)$ and $\mathbf{v} = (0, 1, 2)$.

(a) $\|\mathbf{u} - \mathbf{v}\| =$ ________________;

(b) $\mathbf{u} \cdot \mathbf{v} =$ ________________;

(c) $\mathbf{u} \times \mathbf{v} =$ ________________;

(d) The orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$ is $\text{proj}_{\mathbf{v}}(\mathbf{u}) =$ ________________.

exercise (20-mid)

5. (12 points) Let $x = (1, 1, 1)$, $y = (1, 1, 0)$.

(a) (4 points) Normalize x.

(b) (4 points) Decompose $y = y_1 + y_0$ where y_1 is a scale of x and y_0 is orthogonal to x.

(c) (4 points) Find a vector z_0 such that $z_0 \cdot x = 0$ and $z_0 \cdot y_0 = 0$. (Cross Product)

exercise

- $\mathbf{a}, \mathbf{b}, \mathbf{c}$ satisfy $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ and $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 5, \|\mathbf{c}\| = 7$.
  find the angle between $\mathbf{a}$ and $\mathbf{b}$
- $\|\mathbf{a}\| = 2, \|\mathbf{b}\| = \sqrt{2}, \mathbf{a} \cdot \mathbf{b} = 2$.
  find $\|\mathbf{a} \times \mathbf{b}\|$

representation of lines and planes

- line

  $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

  $$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

  $\mathbf{v} = (v_1, v_2, v_3)$ is the direction vector of the line

- plane

  $\mathbf{r} = \mathbf{r}_0 + s\mathbf{v} + t\mathbf{w}$

  $n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$

  $\mathbf{n} = (n_1, n_2, n_3)$ is the normal vector of the plane

  $\mathbf{n} = \mathbf{v} \times \mathbf{w}$

distance from a point to a line/plane

- line(2-dimensional):
  from $P_0(x_0, y_0)$ to $ax + by + c = 0$
  $$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$
  $\mathbf{n} = (a, b)$ is the normal vector of the line

- plane(3-dimensional):
  from $P_0(x_0, y_0, z_0)$ to $ax + by + cz + d = 0$
  $$d = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$
  $\mathbf{n} = (a, b, c)$ is the normal vector of the plane

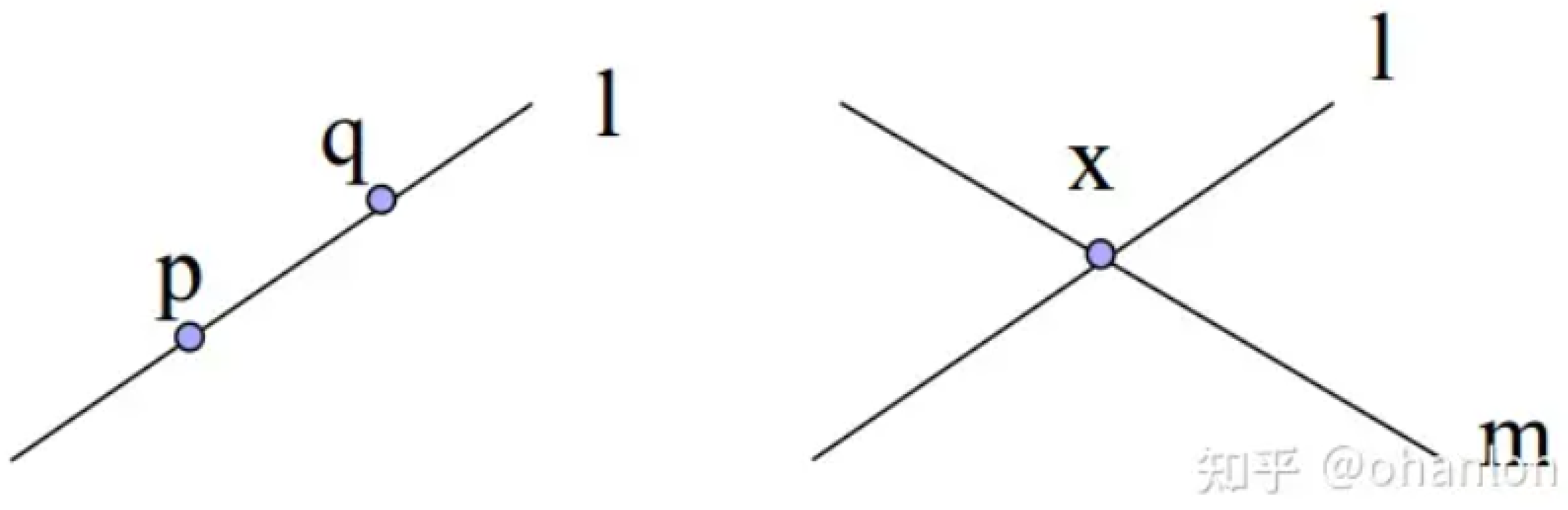
homogeneous coordinates*

- In 2D space, a point $p(x, y)$
- its homogeneous coordinates
  $p(x, y, 1)$
- a vector $\mathbf{v} = (x, y)$
  its homogeneous coordinates is
  $\mathbf{v} = (x, y, 0)$
- a line $ax + by + c = 0$
  its homogeneous coordinates is
  $\mathbf{v} = (a, b, c)$
- benefit1:
  we can judge whether a point is on the line by $\mathbf{v} \cdot \mathbf{p} = 0$

homogeneous coordinates*

benefit2:

- we can easily get the line crossing two points:
  $\mathbf{l} = \mathbf{p} \times \mathbf{q}$
- we can easily get the intersection of two lines:
  $\mathbf{x} = \mathbf{l} \times \mathbf{m}$



homogeneous coordinates*

benefit3:

we can easily transform a point in Euclidean space

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = \begin{bmatrix} R & \mathbf{t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

where $\begin{bmatrix} R & \mathbf{t} \\ 0 & 1 \end{bmatrix}$ is the transformation matrix

> https://zhuanlan.zhihu.com/p/625678401