Linear Algebra Tutorial 1

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self introduction

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About the tutorial

- time: Every Tuesday 18:00
- place: SIST 1A-106

1. ~~hand out homework~~

   grade distribution:

   平时成绩 30%； 期中考试30%；期末考试 40%
2. some discussion(please post your questions before the tutorial)
3. why in English?

线性代数1周二习题课

群号: 572318922

扫一扫二维码，加入群聊

QQ

About academic integrity

- No Plagiarism
- No Plagiarism
- No Plagiarism

着重强调学术诚信问题, 禁止抄袭!

鼓励讨论交流

feel free to contact TAs

Why Linear Algebra?

- Linear algebra is the study of vectors and linear transformations.
- It is a fundamental mathematical subject with applications in many fields.
- It is a prerequisite for many other courses.
- ...

for everyone

- mathmatic anylisis
- probability and statistics
- ...

for cs students

Review: Linear Algebra

- 为什么用到线性代数:
  - 线性代数是描述空间和变换的工具，让描述问题变得简单
  - 大量学习算法通过建模输入空间到输出空间的变换来解决问题
  - 线性代数的矩阵分解理论提供了寻找主成分的理论基础
- 用哪些线性代数:
  - 矩阵的基本运算和性质（回忆一下特殊矩阵：对称矩阵、对角矩阵、单位矩阵、正交矩阵、上三角矩阵）
  - 常用的两种矩阵分解：特征值分解、SVD分解
  - 最小二乘法
  - 矩阵求导*（由于将向量记作行向量还是列向量有分歧，因此有两套矩阵求导公式，请注意如果没有特殊说明，我们均默认列向量）

from *Introduction to Machine Learning*'s tutorial1

What is Linear

- linearity = additivity + homogeneity

1. additivity

   $f(x+y) = f(x) + f(y)$

2. homogeneity

   $f(ax) = af(x)$

- more properties will be introduced later(chapter 4 Linear Spaces)
- In this cource, we will mostly focus on linear spaces and linear transformations.

Linear equation(s)

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{aligned}
$$

supplement

(column) vector

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- also an $m \times 1$ matrix

coeffcient matrix & augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

coeffcient matrix A and augmented matrix $\bar{A} = (A|\mathbf{b})$

- specific: $\vec{b} = \vec{0}$:

  called **homogeneous linear system**

  i. at least a **trivial solution**

  ii. may have non-trivial solutions(linear independence)

  > one non-trivial solution $\Rightarrow$ infinite non-trivial solutions

elementary row operations

- multiply a row by a nonzero constant
- add a multiple of one row to another row
- interchange two rows

row echelon form & reduced row echelon form

row echelon form

- leading 1
- Any row in the matrix that is 0 must be below the row that is not 0
- upper row's leading 1 must be to the left of the lower row's leading 1

reduced row echelon form

- The leading 1 contained in any row of the matrix that is not 0 is the only term in the column that is not 0

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 7 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

row echelon form：

reduced row echelon form：

neither：

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

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(d) $\begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 7 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

row echelon form: (a) (c) (d) (g)

reduced row echelon form: (g)

neither: (b) (e) (f)

Gauss elimination

$$\begin{bmatrix} 1 & 0 & \dots & 0 & s_1 \\ 0 & 1 & \dots & 0 & s_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & s_n \end{bmatrix},$$

- if $m = n$
  i.e. unknowns = equations

  if the agumented matrix can be transformed into the form of above then the solution is unique

- if $m \neq n$?

some possible situations

1. no solution
2. fixed solution
3. infinite solutions
4. general solution
5. specific solution

- specific
  trivial solution: $\vec{x} = \vec{0}$
- usage
  eg. $Ax = b$
  general solution = specific solution + non-trivial solutions(homogeneous)

leading variable & free variable

B is an augmented matrix of a linear system, its row echelon form is $\tilde{B}$

The unknowns corresponding to the leading 1 in the rows that are not 0 in $\tilde{B}$ are called the **leading variables** of the system of equations, and all the unknowns other than the leading variables are called the **free variables** of the system of equations.

$$
\begin{aligned}
-2x_3 + 7x_5 &= 12 \\
2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 &= 28 \\
2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 &= -1
\end{aligned}
$$

$$
\begin{bmatrix}
0 & 0 & -2 & 0 & 7 & 12 \\
2 & 4 & -10 & 6 & 12 & 28 \\
2 & 4 & -5 & 6 & -5 & -1
\end{bmatrix}
$$

$$
\begin{bmatrix}
\color{red}{1} & \color{blue}{2} & 0 & \color{blue}{3} & 0 & 7 \\
0 & 0 & \color{red}{1} & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \color{red}{1} & 2
\end{bmatrix}
$$

general solutions:

$x_1 = 7 - 3r - 2s, x_2 = s, x_3 = 1, x_4 = r, x_5 = 2$

$r, s \in \mathbb{R}$

example for format

Problem B(6 Points). Consider the following linear system

$$3x_1 + x_2 + x_3 + x_4 = 0$$
$$5x_1 - x_2 + x_3 - x_4 = 10.$$

1. (1 point) Write down the augmented matrix of this linear system.
2. (3 points) Transform the augmented matrix to a row echelon form or to a reduced row echelon form, and indicate which elementary row operation is used in every step.
3. (1 point) Determine the leading variables and free variables of this linear system.
4. (1 point) Solve this linear system by using the row echelon form or reduced row echelon form obtained in 2., and give the general solution.

True or False

1) If a matrix is in reduced row echelon form, then it is also in row echel on form.
2) If an elementary row operation is applied to a matrix that is in row ec helon form, the resulting matrix will still be in row echelon form.
3) Every matrix has a unique row echelon form.
4) A homogeneous linear system in n unknowns whose corresponding a ugmented matrix has a reduced row echelon form with r leading 1's has $n - r$ free variables.
5) All leading 1's in a matrix in row echelon form must occur in different columns.
6) If a linear system has more unknowns than equations, then it must have infinitely many solutions.

True or False

1) If a matrix is in reduced row echelon form, then it is also in row echelon form.
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6) If a linear system has more unknowns than equations, then it must have infinitely many solutions.

TFFTTF

Some useful resources

- 3B1B Essence of Linear Algebra
  youtube or bilibili