Linear Algebra Tutorial2

2023.10.17

homework

free variables

eg. $x_3 = s, x_4 = r$

$s, r \in \mathbf{R}$!

- consistent 有解的
- inconsistent 无解的

Problem C(6 Points). Consider the following linear system

$$\begin{aligned} v + 3w - 2x &= 0 \\ 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0. \end{aligned}$$

1. (1 point) Write down the augmented matrix of this linear system.
2. (3 points) Transform the augmented matrix to a row echelon form or to a reduced row echelon form, and indicate which elementary row operation is used in every step.
3. (1 point) Determine the leading variables and free variables of this linear system.
4. (1 point) Solve this linear system by using the row echelon form or reduced row echelon form obtained in 2., and give the general solution.

Matrices

- representation form
  $M_{m\times n}$: a $m \times n$ -matrix
- diagonal
- row vector
- column vector

> A matrix can be view as a set of vectors(row/column)

Matrices

- square matrix
  diagonal
- diagonal matrix
- upper(lower) triangular matrix
- identity matrix
- zero matrix
- symmetric matrix
- skew-symmetric matrix
- orthogonal matrix
- ....

Matrix Operations

- equality
- addition(subtraction)
- scalar multiplication(product)
- matrix multiplication
- division?

matrix operation practice

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, find $A + B$, $A - B$,AB

some concepts

- linear combination of $A1, A2, \cdots, A_r$ with coefficients $c_1, c_2, \cdots, c_r$

> A matrix can be view as a set of vectors(row/column)

- matrix multiplication can be seen as
  $AB = A[\mathbf{c_1}\ \mathbf{c_2}\ \cdots\ \mathbf{c_n}] = [\mathbf{Ac_1}\ \mathbf{Ac_2}\ \cdots\ \mathbf{Ac_n}]$
- similarly
  $AB = [\mathbf{c_1}\ \mathbf{c_2}\ \cdots\ \mathbf{c_n}]^T B = [\mathbf{Ac_1}\ \mathbf{Ac_2}\ \cdots\ \mathbf{c_{nB}}]$

> proof? Try on homework

- the linear equation system can be seen as $\mathbf{Ax} = \mathbf{B}$

trace

> only for square matrix

$$trace(A) = \sum_{i=1}^{n} a_{ii}$$

- $trace(AB) = trace(BA)$
  proof:
  $$trace(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}$$
  $$= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} = \sum_{j=1}^{n} (BA)_{jj} = trace(BA)$$
- matrix inner product
  $< A, B >= trace(A^T B) = trace(B^T A)$

trace practice

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, find $trace(A), trace(B), trace(AB), trace(BA)$

transpose

$(A^\top)_{ij} = (A)_{ji}$

- $(AB)^T = B^T A^T$
- AA^T must be a symmetric matrix

matrix properties

- no commutative law!!

  $AB = BA$

  most common counterexample is A, B are not square matrices

- no cancellation law!!

  $AB = AC \nRightarrow B = C$

  $AB = 0 \nRightarrow A = 0$ or $B = 0$

  > 0 matrix is a matrix with all elements are 0. Not a number.

polynomial of a matrix

- the binomial theorem

  $$(A+B)^n = \sum_{k=0}^{n} C_n^k A^k B^{n-k}$$

> It does not work

eg.

$A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$, $p(x) = x^2 - 2x - 3$, find $p(A)$

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$p(A) = A^2 - 2A - 3I_2 =$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

partitioned matrix

$$A = \left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, A_{12} = \begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix}, A_{21} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}, A_{22} = \begin{bmatrix} a_{34} \end{bmatrix}$$

$A_{11}, A_{12}, A_{21}, A_{22}$ are called submatrices of A

partitioned matrix

$$A = \begin{bmatrix} A_{11} & | & A_{12} \\ - & - & - \\ A_{21} & | & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & | & B_{12} \\ - & - & - \\ B_{21} & | & B_{22} \end{bmatrix}$$

$$A \times B = C = \begin{bmatrix} C_{11} & | & C_{12} \\ - & - & - \\ C_{21} & | & C_{22} \end{bmatrix}$$

Inverses of matrices

- nonsingular matrix $\leftrightarrow$ invertible matrix
  $AB = BA = I_n$
  $B = A^{-1}$
  otherwise, A is singular and has no inverse
  $(|A| = 0)$

  > How about $A_{3\times 4}$?

properties of inverse matrices

- inverse matrix of A is unique
  B, C are A's inverse matrices
  $B = BI_n = B(AC) = (BA)C = I_nC = C$
- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$

> proof: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I_n$

- $(A^T)^{-1} = (A^{-1})^T \to A^{-T}$

> proof: $A^T(A^{-1})^T = (A^TA)^{-1} = I_n$

inverse of a matrix with 2×2

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| = ad - bc \neq 0$, find A^{-1}

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

> actually, $A^{-1} = \frac{1}{|A|} A^*$, where A^* is the adjugate matrix of A

elementary operations $\to$ elementary matrices' product

- every elementary row operation can be implemented by left multiplying an elementary matrix
- similarly, every elementary column operation can be implemented by right multiplying an elementary matrix

> 左行右列

例子(2021年线性代数期中考试题)：Find an invertible matrix P such that $PA = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} - a_{21} & a_{32} - a_{22} & a_{33} - a_{23} \end{bmatrix}.$$

use elementary row operations to find A^{-1}

important!

> will be proved tomorrow

simply proof:

$$E_k E_{k-1} \cdots E_1 A = I_n$$
$$A^{-1}(A|I_n) = (I_n|A^{-1})$$

- we can also use the foundamental column operations to find A^{-1}

$$\begin{bmatrix} A \\ B \end{bmatrix} A^{-1} = \begin{bmatrix} E \\ BA^{-1} \end{bmatrix}$$

but we cannot mix row and column operations!!!

- an example

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}$$

- verify

equivalent conditions

- A is a square matrix of order n

1. A is invertible
2. homogeneous equation system $Ax = 0$ has only trivial solution
3. A's reduced row echolon form is I_n
4. A can be expressed as a product of elementary matrices
5. for a $n \times 1$ vector b, the equation system $Ax = b$ has a (unique) solution