线性代数(2023-2024)第十一次作业参考答案

Problem A(6 Points)

1. (2 points) Let V be a vector space spanned by its basis $B = \{1, \sin x, \cos x\}$. Let $D : V \to V$ be the differential operator on V such that for any $f(x) \in V$, $D(f(x)) = f'(x)$. Find the matrix $[D]_{B,B}$ of D relative to the basis B.

   答案： 令$\boldsymbol{v}_1 = 1$，$\boldsymbol{v}_2 = \sin x$，$\boldsymbol{v}_3 = \cos x$，即$B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$。由于$D(\boldsymbol{v}_1) = D(1) = 0$(常值函数1的导数当然为常值函数0)，我们有$[D(\boldsymbol{v}_1)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$，由于$D(\boldsymbol{v}_2) = \sin'(x) = \cos x$，我们有 $[D(\boldsymbol{v}_2)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$，由于$D(\boldsymbol{v}_3) = \cos'(x) = -\sin x$，我们有$[D(\boldsymbol{v}_3)]_B = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.因此，将以上三个列向量拼成一个矩阵，我们得到

   $$[D]_{B,B} = \Big[[D(\boldsymbol{v}_1)]_B \quad [D(\boldsymbol{v}_2)]_B \quad [D(\boldsymbol{v}_3)]_B\Big] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

2. (2 points) Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a given matrix in $M_{2\times 2}$. Prove that $\mathrm{ad} : M_{2\times 2} \to M_{2\times 2}$, defined by

   $$\mathrm{ad}(A) = BA - AB$$

   for $A \in M_{2\times 2}$ is a linear operator on $M_{2\times 2}$. Then find the matrix $[\mathrm{ad}]_{B,B}$ of ad relative to the standard basis B of $M_{2\times 2}$.

   答案： 对于$A_1, A_2 \in M_{2\times 2}$，有

   $$\mathrm{ad}(A_1+A_2) = B(A_1+A_2)-(A_1+A_2)B = BA_1-A_1B+BA_2-A_2B = \mathrm{ad}(A_1)+\mathrm{ad}(A_2),$$

   类似可证$\mathrm{ad}(kA) = k\mathrm{ad}(A)$。因此ad是一个线性变换。
   将M_2的标准基底B记为$B = \{A_1, A_2, A_3, A_4\}$，$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_3 =$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$。经计算可得

$$\mathrm{ad}(A_1) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} - \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix},$$

即$\mathrm{ad}(A_1) = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix} = 0A_1 - bA_2 + cA_3 + 0A_4 \Rightarrow [\mathrm{ad}(A_1)]_B = \begin{bmatrix} 0 \\ -b \\ c \\ 0 \end{bmatrix}$。类似可得

$$\mathrm{ad}(A_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -c & a-d \\ 0 & c \end{bmatrix}$$

即$\mathrm{ad}(A_2) = -cA_1 + (a-d)A_2 + 0A_3 + cA_4 \Rightarrow [\mathrm{ad}(A_2)]_B = \begin{bmatrix} -c \\ a-d \\ 0 \\ c \end{bmatrix}$；

$$\mathrm{ad}(A_3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & 0 \\ d-a & -b \end{bmatrix}$$

即$\mathrm{ad}(A_3) = bA_1 + 0A_2 + (d-a)A_3 - bA_4 \Rightarrow [\mathrm{ad}(A_3)]_B = \begin{bmatrix} b \\ 0 \\ d-a \\ -b \end{bmatrix}$；

$$\mathrm{ad}(A_4) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix},$$

即$\mathrm{ad}(A_4) = 0A_1 + bA_2 - cA_3 + 0A_4 \Rightarrow [\mathrm{ad}(A_4)]_B = \begin{bmatrix} 0 \\ b \\ -c \\ 0 \end{bmatrix}$。综上，可得

$$[\mathrm{ad}]_{B,B} = \Big[[\mathrm{ad}(A_1)]_B \quad [\mathrm{ad}(A_2)]_B \quad [\mathrm{ad}(A_3)]_B \quad [\mathrm{ad}(A_4)]_B \Big] = \begin{bmatrix} 0 & -c & b & 0 \\ -b & a-d & 0 & b \\ c & 0 & d-a & -c \\ 0 & c & -b & 0 \end{bmatrix}.$$

3. (2 points) Let V, W be two–dimensional vector spaces. Let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ be a basis of V, let $B' = \{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ be a basis of W. Let $T : V \to W$ be a linear operator

such that

$$T(3\boldsymbol{v}_1 + 2\boldsymbol{v}_2) = \boldsymbol{w}_2, \quad T(\boldsymbol{v}_1 - 4\boldsymbol{v}_2) = \boldsymbol{w}_1 - \boldsymbol{w}_2.$$

Find the matrix $[T]_{B',B}$ relative to B and B'.

答案： 要写出矩阵$[T]_{B',B}$，我们需要知道如何用$\boldsymbol{w}_1, \boldsymbol{w}_2$的线性组合表示$T(\boldsymbol{v}_1), T(\boldsymbol{v}_2)$。因此首先我们利用$T$的线性性质以及已知等式$T(3\boldsymbol{v}_1 + 2\boldsymbol{v}_2) = \boldsymbol{w}_2, \quad T(\boldsymbol{v}_1 - 4\boldsymbol{v}_2) = \boldsymbol{w}_1 - \boldsymbol{w}_2$计算$T(\boldsymbol{v}_1)$与$T(\boldsymbol{v}_2)$。首先来计算$T(\boldsymbol{v}_1)$：

$$\begin{aligned} T(\underbrace{2(3\boldsymbol{v}_1 + 2\boldsymbol{v}_2) + (\boldsymbol{v}_1 - 4\boldsymbol{v}_2)}_{=7\boldsymbol{v}_1}) &= 2T(3\boldsymbol{v}_1 + 2\boldsymbol{v}_2) + T(\boldsymbol{v}_1 - 4\boldsymbol{v}_2) \\ &= 2\boldsymbol{w}_2 + (\boldsymbol{w}_1 - \boldsymbol{w}_2) \\ &= \boldsymbol{w}_1 + \boldsymbol{w}_2, \end{aligned}$$

即$T(7\boldsymbol{v}_1) = \boldsymbol{w}_1 + \boldsymbol{w}_2$，因此$T(\boldsymbol{v}_1) = \frac{1}{7}(\boldsymbol{w}_1 + \boldsymbol{w}_2)$。由此可得

$$T(3\boldsymbol{v}_1+2\boldsymbol{v}_2) = 3T(\boldsymbol{v}_1)+2T(\boldsymbol{v}_2) = \frac{3}{7}(\boldsymbol{w}_1+\boldsymbol{w}_2)+2T(\boldsymbol{v}_2) = \boldsymbol{w}_2 \Rightarrow T(\boldsymbol{v}_2) = -\frac{3}{14}\boldsymbol{w}_1+\frac{2}{7}\boldsymbol{w}_2.$$

显然，由于$[T(\boldsymbol{v}_1)]_{B'} = \begin{bmatrix} 1/7 \\ 1/7 \end{bmatrix}$，$[T(\boldsymbol{v}_2)]_{B'} = \begin{bmatrix} -3/14 \\ 2/7 \end{bmatrix}$，我们得到答案为$[T]_{B',B} = \begin{bmatrix} 1/7 & 1/7 \\ -3/14 & 2/7 \end{bmatrix}$。

Problem B(6 Points) 仔细阅读英文教材8.4节的Example 2与Example 6，然后回答以下问题。

- Let $T : P_2 \to M_{2\times 2}$ be the linear transformation defined by

$$T(p(x)) = \begin{bmatrix} p(0) & p(1) \\ p(-1) & p(0) \end{bmatrix}.$$

Let B be the standard basis for $M_{2\times 2}$, let $B' = \{1, x, x^2\}$ be the standard basis for $P_2 = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$, $B'' = \{1, 1 + x, 1 + x^2\}$ be another basis for P_2.

1. (2 points) Find $[T]_{B,B'}$ and $[T]_{B,B''}$.

答案： 将M_2的标准基底B记为$B = \{A_1, A_2, A_3, A_4\}$，$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$。将$B' = \{1, x, x^2\}$记为$B' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$，

即$bv_1=1$，$\boldsymbol{v}_2=x$，$\boldsymbol{v}_3=x^2$。那么我们有

$$T(\boldsymbol{v}_1)=T(1)=\begin{bmatrix}1&1\\1&1\end{bmatrix}$$

(因为作为常值函数，$\boldsymbol{v}_1=1$在$0,1,-1$这三个点的取值都等于1)，即$[T(\boldsymbol{v}_1)]_B=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$；类似可得$T(\boldsymbol{v}_2)=T(x)=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$(因为对于$p(x)=x$，$p(1)=1,p(-1)=-1,p(0)=0$)，即$[T(\boldsymbol{v}_2)]_B=\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$；$T(\boldsymbol{v}_3)=T(x^2)=\begin{bmatrix}0&1\\1&0\end{bmatrix}$，即$[T(\boldsymbol{v}_3)]_B=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$。因此$[T]_{B,B'}=\begin{bmatrix}[T(\boldsymbol{v}_1)]_B & [T(\boldsymbol{v}_2)]_B & [T(\boldsymbol{v}_3)]_B\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&1\\1&-1&1\\1&0&0\end{bmatrix}$。

对于$B''=\{\boldsymbol{w}_1,\boldsymbol{w}_2,\boldsymbol{w}_3\}$，$\boldsymbol{w}_1=1$，$\boldsymbol{w}_2=1+x$，$\boldsymbol{w}_3=1+x^2$，按照以上步骤进行计算，可得

$$T(\boldsymbol{w}_1)=\begin{bmatrix}1&1\\1&1\end{bmatrix}\Rightarrow[T(\boldsymbol{w}_1)]_B=\begin{bmatrix}1\\1\\1\\1\end{bmatrix},$$

$$T(\boldsymbol{w}_2)=\begin{bmatrix}1&2\\0&1\end{bmatrix}\Rightarrow[T(\boldsymbol{w}_2)]_B=\begin{bmatrix}1\\2\\0\\1\end{bmatrix},$$

$$T(\boldsymbol{w}_3)=\begin{bmatrix}1&2\\2&1\end{bmatrix}\Rightarrow[T(\boldsymbol{w}_3)]_B=\begin{bmatrix}1\\2\\2\\1\end{bmatrix}$$

因此$[T]_{B,B''} = \begin{bmatrix} [T(\boldsymbol{w}_1)]_B & [T(\boldsymbol{w}_2)]_B & [T(\boldsymbol{w}_3)]_B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$。

2. (2 points) Let $q(x) = 2+2x+x^2$, compute $[T]_{B,B'}[q(x)]_{B'}$ and $[T]_{B,B''}[q(x)]_{B''}$, then use $[T]_{B,B'}[q(x)]_{B'}$ and $[T]_{B,B''}[q(x)]_{B''}$ to compute $T(q(x))$, i.e., compute $f_B^{-1}([T]_{B,B'}[q(x)]_{B'})$ and $f_B^{-1}([T]_{B,B''}[q(x)]_{B''})$, where $f_B : M_{2\times 2} \to \mathbb{R}^4, f_B(A) = [A]_B$, is the coordinate vector transformation (坐标向量映射) on $M_{2\times 2}$.

答案： 对于$B' = \{1, x, x^2\}$，可知$[q(x)]_B = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$。因此$[T]_{B,B'}[q(x)]_{B'} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \\ 2 \end{bmatrix}$，从而得到

$$T(q(x)) = f_B^{-1}([T]_{B,B'}[q(x)]_{B'}) = f_B^{-1}(\begin{bmatrix} 2 \\ 5 \\ 1 \\ 2 \end{bmatrix}) = 2A_1+5A_2+1A_3+2A_4 = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}.$$

对于$B'' = \{1, 1+x, 1+x^2\}$，假设$[q(x)]_{B''} = (k_1, k_2, k_3)$，即$q(x) = 2+2x+x^2 = k_1+k_2(1+x)+k_3(1+x^2) = (k_1+k_2+k_3)+k_2x+k_3x^2$，那么容易确定$k_3 = 1, k_2 = 2, k_1 = -1$。因此$[T]_{B,B''}[q(x)]_{B''} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \\ 2 \end{bmatrix}$，因此$f_B^{-1}([T]_{B,B''}[q(x)]_{B''}) = f_B^{-1}(\begin{bmatrix} 2 \\ 5 \\ 1 \\ 2 \end{bmatrix}) = 2A_1 + 5A_2 + 1A_3 + 2A_4 = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$。

- (2 points) Let $T_1 : P_1 \to P_2$ be the linear transformation defined by

$$T_1(a_0 + a_1x) = 2a_0 - 3a_1x,$$

and let $T_2 : P_2 \to P_3$ be the linear transformation defined by

$$T_2(a_0 + a_1x + a_2x^2) = 3a_0x + 3a_1x + 3a_2x^3.$$

Let $B = \{1, x\}$, $B'' = \{1, x, x^2\}$, $B' = \{1, x, x^2, x^3\}$. Compute $[T_2 \circ T_1]_{B',B}$, $[T_2]_{B',B''}$, and $[T_1]_{B'',B}$. What is the relation between $[T_2 \circ T_1]_{B',B}$, $[T_2]_{B',B''}$, and $[T_1]_{B'',B}$?

答案： 由T_1的定义以及$B = \{1, x\}$，$B'' = \{1, x, x^2\}$，可得$T_1(1) = T_1(1 + 0x) = 2 \times 1 - (3 \times 0)x = 2$，$T_2(x) = T_2(0 + 1 \times x) = 2 \times 0 - (3 \times 1)x = -3x$，因此$[T_1(1)]_{B''} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $[T_1(x)]_{B''} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$，因此

$$[T_1]_{B'',B} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}.$$

对于$B'' = \{1, x, x^2\}$和$B' = \{1, x, x^2, x^3\}$，由T_2的定义，可知

$$T_2(1) = T_2(1 + 0x + 0x^2) = 3x \Rightarrow [T_2(1)]_{B'} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix},$$

$$T_2(x) = T_2(0 + 1x + 0x^2) = 3x \Rightarrow [T_2(x)]_{B'} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix},$$

$$T_3(x^2) = T_3(0 + 0x + 1x^2) = 3x^3 \Rightarrow [T_2(x)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}.$$

因此$[T_2]_{B',B''} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$。因此$[T_2 \circ T_1]_{B',B} = [T_2]_{B',B''}[T_1]_{B'',B} = \begin{bmatrix} 0 & 0 \\ 6 & -9 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$。

Problem C(6 Points)

1. (2 points) Let $T : P_2 \to P_2$ be defined by

$$T(p(x)) = p(2x+1).$$

Determine whether T is invertible and if it is invertible, find T^{-1}, and compute $[T^{-1}]_{B,B}$ for the standard basis $B = \{1, x, x^2\}$ of P_2.

答案： 由于$B = \{1, x, x^2\}$是P_2的标准基底，且

$$T(1) = 1, T(x) = 2x+1, T(x^2) = (2x+1)^2 = 4x^2 + 4x + 1,$$

可得

$$[T(1)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [T(x)]_B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, [T(x^2)]_B = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix},$$

因此$[T]_{B,B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$。显然$\det([T]_{B,B}) = 8 \neq 0$，因此$[T]_{B,B}$可逆，即$T$可逆，且此时$[T^{-1}]_{B,B} = [T]_{B,B}^{-1}$。经计算可得

$$[T^{-1}]_{B,B} = [T]_{B,B}^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

这意味着T^{-1}满足

$$[T^{-1}(1)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow T^{-1}(1) = 1;$$

$$[T^{-1}(x)]_B = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \Rightarrow T^{-1}(x) = \frac{1}{2}x - \frac{1}{2};$$

$$[T^{-1}(x^2)]_B = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \Rightarrow T^{-1}(x^2) = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4} = (\frac{1}{2}x - \frac{1}{2})^2.$$

综上，对任何$p(x) = a_0 + a_1 x + a_2 x^2 \in P_2$，我们有

$$T^{-1}(p(x)) = a_0 + a_1(\frac{1}{2}x - \frac{1}{2}) + a_2(\frac{1}{2}x - \frac{1}{2})^2 = p(\frac{1}{2}x - \frac{1}{2}).$$

2. (2 points) Let $J : P_3 \to \mathbb{R}$ be the integral transformation, $I(p(x)) = \int_{-1}^{1} p(x)dx$. Let $B' = \{1\}$ be the standard basis of $\mathbb{R}$, $B = \{1, x, x^2, x^3\}$ be the standard basis of P_3, what is the matrix $[J]_{B',B}$? What is the kernel of J?

答案： 由于

$$J(1) = \int_{-1}^{1} 1dx = 1 - (-1) = 2 \Rightarrow [J(1)]_{B'} = 2,$$

$$J(x) = \int_{-1}^{1} xdx = \frac{1}{2}x^2|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0 \Rightarrow [J(x)]_{B'} = 0,$$

$$J(x^2) = \int_{-1}^{1} x^2dx = \left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3} \Rightarrow [J(x^2)]_{B'} = \frac{2}{3},$$

$$J(x^3) = \int_{-1}^{1} x^3dx = \left[\frac{1}{4}x^4\right]_{-1}^{1} = \frac{1}{4} - (\frac{1}{4}) = 0 \Rightarrow [J(x^3)]_{B'} = 0,$$

我们得到$[J]_{B',B} = \begin{bmatrix} 2 & 0 & \frac{2}{3} & 0 \end{bmatrix}$。那么由$[J]_{B',B}$作为系数矩阵的方程组为

$$2x_1 + \frac{2}{3}x_3 = 0.$$

求解该方程组可得通解为$\boldsymbol{s} = (-\frac{1}{3}s, t, s, r) = (-\frac{1}{3}, 0, 1, 0)s + (0, 1, 0, 0)t + (0, 0, 0, 1)r$，因此

$$\ker([J]_{B',B}) = \text{span}\{\begin{bmatrix} -1/3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\}.$$

令$f_B : P_3 \to \mathbb{R}^4$作为$P_3$关于$B$的坐标向量映射，即$f_B(p(x)) = [p(x)]_B$，那么$f_B^{-1}(\begin{bmatrix} -1/3 \\ 0 \\ 1 \\ 0 \end{bmatrix}) = -\frac{1}{3} + x^2$，$f_B^{-1}(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}) = x$，$f_B^{-1}(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}) = x^3$，我们最终得到

$$\ker(J) = \text{span}\{-\frac{1}{3} + x^2, x, x^3\}.$$

3. (2 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3, -x_2, x_1 + 7x_3).$$

Let B be the standard basis of $\mathbb{R}^3$, $B' = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ be another basis of $\mathbb{R}^3$. Find $[T]_{B,B}$, $[T]_{B',B'}$ and a invertible matrix P such that $[T]_{B',B'} =$

$P^{-1}[T]_{B,B}P.$

答案： 对于$B=\{\boldsymbol{e}_1=(1,0,0),\boldsymbol{e}_2=(0,1,0),\boldsymbol{e}_3=(0,0,1)\}$，我们有

$$T(\boldsymbol{e}_1)=T(1,0,0)=(1,0,1)\Rightarrow[T(\boldsymbol{e}_1)]_B=\begin{bmatrix}1\\0\\1\end{bmatrix},$$

$$T(\boldsymbol{e}_2)=T(0,1,0)=(2,-1,0)\Rightarrow[T(\boldsymbol{e}_2)]_B=\begin{bmatrix}2\\-1\\0\end{bmatrix},$$

$$T(\boldsymbol{e}_3)=T(0,0,1)=(-1,0,7)\Rightarrow[T(\boldsymbol{e}_3)]_B=\begin{bmatrix}-1\\0\\7\end{bmatrix},$$

因此

$$[T]_{B,B}=\begin{bmatrix}1&2&-1\\0&-1&0\\1&0&7\end{bmatrix}.$$

对于$B'=\{\boldsymbol{u}_1'=(1,0,0),\boldsymbol{u}_2'=(1,1,0),\boldsymbol{u}_3'=(1,1,1)\}$，我们知道$[T]_{B'.B'}$与$[T]_{B,B}$相似，且有

$$[T]_{B'.B'}=P_{B'\leftarrow B}[T]_{B,B}P_{B\leftarrow B'}.$$

由转移矩阵$P_{B\leftarrow B'}$的定义(见讲义4.6节)可知，

$$P_{B\leftarrow B'}=\begin{bmatrix}[\boldsymbol{u}_1']_B & [\boldsymbol{u}_2']_B & [\boldsymbol{u}_3']_B\end{bmatrix}=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$$

从而有$P_{B'\leftarrow B}=P_{B\leftarrow B'}^{-1}=\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}$。综上，我们得到

$$[T]_{B'.B'}=P_{B'\leftarrow B}[T]_{B,B}P_{B\leftarrow B'}=\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&-1\\0&-1&0\\1&0&7\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&4&3\\-1&-2&-9\\1&1&8\end{bmatrix}.$$

Problem D(6 Points) 令V为一个向量空间，$T:V\to V$为一个线性变换。我们说$\lambda\in\mathbb{R}$是T的一个特征值(eigenvalue)，如果存在一个非零$\boldsymbol{v}\in V$，$\boldsymbol{v}\neq\boldsymbol{0}$，使得$T(\boldsymbol{v})=\lambda\boldsymbol{v}$。此时我们称$\boldsymbol{v}$为$T$关于$\lambda$的特征向量(eigenvector)。所有$T$关于$\lambda$的特征向量的集合被称为$T$关于$\lambda$的特征空间(eigenspace)。

1. (2 points) 假设$B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$是$V$的一组基底，证明$\lambda$是$T$的一个特征值当且仅当$\lambda$是矩阵$[T]_{B,B} \in M_{n\times n}$的一个特征值。证明$\boldsymbol{v} \in V$是$T$关于$\lambda$的一个特征向量当且仅当$[\boldsymbol{v}]_B$是矩阵$[T]_{B,B} \in M_{n\times n}$关于$\lambda$的一个特征向量。

   答案： 令$f_B : V \to \mathbb{R}^n$为V关于B的坐标向量映射，即$f_B(\boldsymbol{v}) = [\boldsymbol{v}]_B$。那么我们知道
$$T = f_B^{-1} \circ T_A \circ f_B,$$
   这里$A = [T]_{B,B}$。如果$\lambda \in \mathbb{R}$是$A = [T]_{B,B}$的一个特征值，$[\boldsymbol{v}]_B \in V$是A关于λ的一个特征向量，即$A[\boldsymbol{v}]_B = \lambda[\boldsymbol{v}]_B$，我们利用以上等式可得
$$T(\boldsymbol{v}) = f_B^{-1}\circ T_A \circ f_B(\boldsymbol{v}) = f_B^{-1}(T_A[\boldsymbol{v}]_B) = f_B^{-1}(A[\boldsymbol{v}]_B) = f_B^{-1}(\lambda[\boldsymbol{v}]_B) = \lambda f_B^{-1}([\boldsymbol{v}]_B) = \lambda\boldsymbol{v},$$
   即$\boldsymbol{v} \in V$是T关于λ的一个特征向量，λ是T的一个特征值。
   反之，由于$T_A = f_B \circ T \circ f_B^{-1}$ $(A = [T]_{B,B})$，如果$\boldsymbol{v} \in V$是T关于λ的一个特征向量，那么与之前推理类似，可得
$$A[\boldsymbol{v}]_B = T_A([\boldsymbol{v}]_B) = f_B\circ T\circ \underbrace{f_B^{-1}([\boldsymbol{v}]_B)}_{=\boldsymbol{v}} = f_B(T(\boldsymbol{v})) = f_B(\lambda\boldsymbol{v}) = \lambda f_B(\boldsymbol{v}) = \lambda[\boldsymbol{v}]_B,$$
   因此$[\boldsymbol{v}]_B$是矩阵$A = [T]_{B,B}$关于λ的一个特征向量。

2. (4 points) Let $T : P_2 \to P_2$ be defined by
$$T(a_0 + a_1x + a_2x^2) = (5a_0 + 6a_1 + 2a_2) - (a_1 + 8a_2)x + (a_0 - 2a_2)x^2.$$
   Find the eigenvalues of T and find a basis for the eigenspaces of T.

   答案： 令$B = \{1, x, x^2\}$为P_2的标准基底，那么容易求出
$$[T]_{B,B} = \begin{bmatrix} [T(1)]_B & [T(x)]_B & [T(x^2)]_B \end{bmatrix} = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}.$$
   利用第一问的结论，要求T的特征值与特征空间基底，**我们可以先求$[T]_{B,B}$的特征值与特征空间基底，再利用坐标向量映射f_B的逆映射f_B^{-1}得到T的特征值与特征空间基底。**
   令$A = [T]_{B,B}$，计算$\det(\lambda I_3 - A)$可得A的特征多项式为
$$p_A(\lambda) = \lambda^3 - 2\lambda^2 - 15\lambda + 36 = (\lambda + 4)(\lambda - 3)^2.$$
   令$p_A(\lambda) = 0$可求得它的特征值为$\lambda_1 = -4, \lambda_2 = 3$。
   现在我们找出A关于$\lambda_1 = -4$的特征空间的基底。考虑方程组
$$(\lambda_1 I_3 - A)\boldsymbol{x} = \boldsymbol{0} \Rightarrow \begin{bmatrix} -9 & -6 & -2 \\ 0 & -3 & 8 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

利用高斯消元法可求得通解为

$$\boldsymbol{s} = r\begin{bmatrix} -2 \\ 8/3 \\ 1 \end{bmatrix}, r \in \mathbb{R}.$$

因此A关于$\lambda_1 = -4$的特征空间基底为$\begin{bmatrix} -2 \\ 8/3 \\ 1 \end{bmatrix}$。

接下来求出A关于$\lambda_2 = 3$的特征空间的基底。考虑方程组

$$(\lambda_1 I_3 - A)\boldsymbol{x} = \boldsymbol{0} \Rightarrow \begin{bmatrix} -2 & -6 & -2 \\ 0 & -4 & -8 \\ -1 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

利用高斯消元法可求得通解为

$$\boldsymbol{s} = r\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, r \in \mathbb{R}.$$

因此A关于$\lambda_2 = 3$的特征空间基底为$\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$。

综上，T关于$\lambda_1 = -4$的特征空间的一组基底为$f_B^{-1}(\begin{bmatrix} -2 \\ 8/3 \\ 1 \end{bmatrix}) = -2 + \frac{8}{3}x + x^2$，

T关于$\lambda_2 = 3$的特征空间的一组基底为$f_B^{-1}(\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}) = 5 - 2x + x^2$。

Problem E(6 Points) Let $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 0 \\ -5 & 5 & 10 \end{bmatrix}$.

1. (3 points) Compute the eigenvalues of A, and find a basis of the eigensapces of A.

答案： $\det(\lambda I_3 - A) = \det(\begin{bmatrix} \lambda-1 & 2 & -2 \\ 2 & \lambda-1 & 0 \\ 5 & -5 & \lambda-10 \end{bmatrix}) = (\lambda+1)(\lambda-5)(\lambda-8)$。

因此特征值为$\lambda_1 = -1, \lambda_2 = 5, \lambda_3 = 8$。

对于$\lambda_1=-1$，有

$$\lambda_1 I_3 - A = \begin{bmatrix} -2 & 2 & -2 \\ 2 & -2 & 0 \\ 5 & -5 & -11 \end{bmatrix} \underbrace{\Rightarrow}_{\text{行变换}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{s} = r \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, r \in \mathbb{R}$$

因此$\lambda_1=-1$的特征空间的基底为$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$。

对于$\lambda_2=5$，有

$$\lambda_2 I_3 - A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 4 & 0 \\ 5 & -5 & -5 \end{bmatrix} \underbrace{\Rightarrow}_{\text{行变换}} \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{s} = r \begin{bmatrix} 2/3 \\ -1/3 \\ 1 \end{bmatrix}, r \in \mathbb{R}$$

因此$\lambda_2=5$的特征空间的基底为$\begin{bmatrix} 2/3 \\ -1/3 \\ 1 \end{bmatrix}$。

对于$\lambda_3=8$，有

$$\lambda_3 I_3 - A = \begin{bmatrix} 7 & 2 & -2 \\ 2 & 7 & 0 \\ 5 & -5 & -2 \end{bmatrix} \underbrace{\Rightarrow}_{\text{行变换}} \begin{bmatrix} 1 & 0 & -14/45 \\ 0 & 1 & 4/45 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{s} = r \begin{bmatrix} 14/45 \\ 4/45 \\ 1 \end{bmatrix}, r \in \mathbb{R}$$

因此$\lambda_3=8$的特征空间的基底为$\begin{bmatrix} 14/45 \\ 4/45 \\ 1 \end{bmatrix}$。

2. (3 points) Let adj(A) be the adjoint matrix of A. Find the eigenvalues of $3I_3 + \text{adj}(A)$.

答案： 由于$A \in M_{3\times 3}$有三个不同的特征值，A可被对角化且有$\det(A) = \lambda_1\lambda_2\lambda_3 = -40 \neq 0$，因此$A$可逆。那么由于$A\text{adj}(A) = \det(A)I_3$，我们得到

$$\text{adj}(A) = \det(A)A^{-1} = -40A^{-1}.$$

又因为A^{-1}的特征值为$1/\lambda_1 = -1, 1/\lambda_2 = 1/5, 1/\lambda_3 = 1/8$，adj$(A)$的特征值为$-40 \times 1/\lambda_1 = 40, -40 \times 1/\lambda_2 = -8, -40 \times 1/\lambda_3 = -5$，因此$3I_3 + \text{adj}(A)$的特征值为

$$3 + 40 = 43, 3 + (-8) = -5, 3 + (-5) = -2.$$

Bonus：不计入分数

假设V为一个n维向量空间，W为一个m维向量空间。

1. 对于$T: V \to W$和$S: V \to W$两个线性变换，定义它们的加法为$S+T: V \to W$为$(S+T)(\boldsymbol{v}) = S(\boldsymbol{v}) + T(\boldsymbol{v})$。对于实数$k \in \mathbb{R}$，定义$k$与$T$的标量积$kT: V \to W$为$(kT)(\boldsymbol{v}) = kT(\boldsymbol{v})$。令$L(V, W)$为所有从$V$到$W$的线性变换的集合。证明$L(V, W)$装配上以上加法和标量积之后是一个向量空间，尤其是$S+T$和$kT$依然是从$V$到$W$的线性变换。

   答案： 容易验证。略过具体步骤。

2. 证明$L(V, W)$的维数$\dim(L(V, W)) = nm$ (n乘以m)，并找出它的一组基底。

   答案： 令$B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$为$V$的一组基底，令$B' = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$是$W$的一组基底。令$f_B: V \to \mathbb{R}^n$为$V$关于$B$的坐标向量映射，即$f_B(\boldsymbol{v}) = [\boldsymbol{v}]_B$；令$g_{B'}: W \to \mathbb{R}^m$为$V$关于$B'$的坐标向量映射，即$f_B(\boldsymbol{w}) = [\boldsymbol{w}]_{B'}$。对于任何一个矩阵$A \in M_{m \times n}$，我们定义$T^A: V \to W$为

   $$T^A(\boldsymbol{v}) = g_{B'}^{-1} \circ T_A \circ f_B(\boldsymbol{v}).$$

   显然，作为三个线性变换的复合，T是一个线性变换，即$T \in L(V, W)$。容易验证，对$A, B \in M_{m \times n}, k \in \mathbb{R}$，有

   $$T^{A+B} = T^A + T^B, T^{kA} = kT_A.$$

   因此$S: M_{m \times n} \to L(V, W)$是一个线性变换。我们现在证明$S$是一个同构。假设$S(A) = T^A \in L(V, W)$是一个零变换，即$T^A(\boldsymbol{v}) = g_{B'}^{-1} \circ T_A \circ f_B(\boldsymbol{v}) = g_{B'}^{-1}(A[\boldsymbol{v}]_B) = \mathbf{0}$对任何$\boldsymbol{v} \in V$都成立，那么由于$g_{B'}^{-1}$是one-to-one的，这必然意味着$A[\boldsymbol{v}]_B = \mathbf{0}$对任何$\boldsymbol{v} \in V$成立，也即(利用$f_B: V \to \mathbb{R}^n$是onto的)$A\boldsymbol{x} = \mathbf{0}$对任何$\boldsymbol{x} \in \mathbb{R}^n$成立。显然满足这一要求的唯一矩阵是零矩阵，即$A = \mathbf{0}_{m \times n}$。这意味着$\ker(S) = \{\mathbf{0}_{m \times n}\}$，即$S: M_{m \times n} \to L(V, W)$是one-to-one的。另一方面，对任何线性变换$T \in L(V, W)$，容易看出$S([T]_{B', B}) = T$，因此$S: M_{m \times n} \to L(V, W)$是onto的。综上，$S: M_{m \times n} \to L(V, W)$是一个同构，特别地，我们有$\dim(L(V, W)) = \dim(M_{m \times n}) = mn$。此外，选取$M_{m \times n}$的标准基底$M = \{M^{ij} : i = 1, \ldots, m, j = 1, \ldots, n\}$，即$M^{ij}$除了第$i$行第$j$列上的项为1外，其余项均为0，那么由讲义Theorem 5.16可知，$\{S(M^{ij}) : i = 1, \ldots, m, j = 1, \ldots, n\}$是$L(V, W)$的一组基底。注意$S(M^{ij}) \in L(V, W)$将$\boldsymbol{v}_j$映射到$\boldsymbol{w}_i$，将其余的$\boldsymbol{v}_k, k \neq j$映射到$\mathbf{0} \in W$。