Linear Algebra Tutorial3

2023.10.24

homework

- please notice that the ddl is $22:00$!
- please notice which homework you are handing in

homework

- iff: if and only if

  which means you need to prove both **if** "$\Leftarrow$" and **only if** "$\Rightarrow$"

  Or always uses $\Leftrightarrow$

Problem E(6 Points). (2021年线性代数期中考试题)

A square matrix A is called idempotent if $A^2 = A$.

A square matrix A is called involutory if $A^2 = I$ (I is the identity matrix).

1. (2 points) Suppose that A, B are both idempotent. Prove that $A + B$ is idempotent if and only if $AB + BA = 0$. (if and only if: 当且仅当)
2. (3 Points) Suppose that A, B are both involutory. Prove that AB is involutory if and only if $AB = BA$.

- we are mainly talking about the determinant these times, so without announcement, all the matrixes are square matrixes

Diagonal matrices

- for diagonal matrix D, $D_{ij} = 0$ for $i \neq j$
  so it can be written as $D = diag(d_1, d_2, \cdots, d_n)$
- the power of diagonal matrix is easy to compute
  $D^k = diag(d_1^k, d_2^k, \cdots, d_n^k)$
  $\rightarrow$ similarity and diagonalizable(much later)
- the diagonal matrix D is invertible if and only if $\forall i, d_i \neq 0$

> $|D| = \Pi d_i$

triangular matrix

- upper triangular matrix

  the elements **below** the diagonal are all zero

  (the elements on the diagonal can be zero or not)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix},$$

then elements on the diagonal of A^k are $a_{11}^k, a_{22}^k, \cdots, a_{nn}^k$

> hint: try to prove it by induction

- similar to the lower triangular matrix

symmetric matrix

- $A^T = A$

all properties of symmetric matrix are based on this definition(this time)
(more propertires later such as similarity and diagonalizable)

- $\forall A_{m\times n}, AA^T$ or A^TA are symmetric matrix

determinant

- a function mapping a matrix A into a scalar $det(A)$ or $|A|$
  $A^{-1} = \frac{1}{|A|} A^*$
- the most simple usage: invertibility

minor and cofactor

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad \tilde{A}^{ij} = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

- minor: 余子式 $M_{ij} = det(\tilde{A}_{ij})$
- cofactor: 代数余子式 $C_{ij} = (-1)^{i+j} M_{ij}$
- cofactor expansion along the i-th row

  $$det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

- similarly, we can expand along the j-th column

determinant example

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

- expand along the row?
- expand along the column?

> we can pick any row or column to expand, so just make it as easy as possible

trangular matrix determinant

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

- $det(A) = \prod_{i=1}^{n} a_{ii}$
- and the lower triangular matrix is the same

determinant properties

> compare with the elementary row(column) operations

1. B is obtained from A by interchanging two rows(columns)
   $|B| = -|A|$
2. B is obtained from A by multiplying one row(column) by a nonzero scalar k
   $|B| = k|A|$
3. B is obtained from A by adding a multiple of one row(column) to another row(column)
   $|B| = |A|$

> we can mix row and column operations when calculating the determinant
> but we can only use row or column operations when calculating the inverse matrix!!!!

determinant properties

$$|A^T| = |A|$$
$$\lambda A = \lambda^n |A|$$
$$|AB| = |A||B|$$
$$|A^{-1}| = \frac{1}{|A|}$$

Vandermonde determinant

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

- $det(V) = \prod_{1 \le i < j \le n} (x_j - x_i)$

proof: induction

- verify $n = 3$

> https://blog.csdn.net/u011089523/article/details/72845136

$$\begin{vmatrix} bc & 1 & a \\ ac & 1 & b \\ ab & 1 & c \end{vmatrix}$$

$$\begin{vmatrix} bc & 1 & a \\ ac & 1 & b \\ ab & 1 & c \end{vmatrix}$$

> ans=(c-a)(c-b)(b-a)

determinant

1.

$$D = \begin{bmatrix} a_1 & c_2 & c_3 & \cdots & c_n \\ b_2 & a_2 & & & \\ b_3 & & a_3 & & \\ \vdots & & & \ddots & \\ b_n & & & & a_n \end{bmatrix}$$

> $$det(D) = \prod_{i=1}^{n} a_i - \sum_{i=2}^{n} \frac{Ab_nc_n}{a_n}$$
> where $A = \prod_{i=2}^{n} a_i$

determinant

2.

$$D = \begin{bmatrix} a & b & 0 & \cdots & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ b & 0 & 0 & \cdots & 0 & a \end{bmatrix}$$

> $|D| = a^n + (-1)^{1+n} b^n$

determinant

3.

$$D_n = \begin{bmatrix} b & -1 & 0 & \cdots & 0 & 0 \\ 0 & b & -1 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & b & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & b+a_1 \end{bmatrix}$$

> $$|D_n| = b \cdot |D_{n-1}| + a_n(-1)^{n+1}(-1)^{n-1}$$
> $$|D_n| = b \cdot |D_{n-1}| + a_n$$

Calculate $D_n = \begin{vmatrix} 1+x_1^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & 1+x_2^2 & \cdots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \cdots & 1+x_{\mathrm{n}}^2 \end{vmatrix}$.

Given $A=\begin{bmatrix} 1 & 1 & 5 & 4 \\ 2 & 3 & 2 & 4 \\ 1 & 6 & 0 & 3 \\ 4 & 2 & 5 & 1 \end{bmatrix}$, please calculate $A_{21}+A_{22}+5A_{23}+4A_{24}$