Cluster State Quantum Computing

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Abstract— **Any quantum computation can be performed via sequences of one-qubit measurements on a specific type of initially entangled state – the *cluster state*. Each computational step is a projective measurement that destroys a quantum state, leaving a final state that relies on the outcomes of earlier computations. The model of interest is the one-way quantum computer which is based on this measurement scheme. This paper will present background regarding computation using only measurements, a brief introduction into the preparation of cluster states, a discussion of one way quantum computers (1WQC), and the computational power of various configurations of a 1WQC.**

I. Background

Over the past few decades, advances in science and technology have greatly contributed to the development of modern computers. While these computers are efficient and convenient for everyday needs, they fail at certain computational tasks. Instead, quantum computers promise faster large scale factorization and database searches that are intractable for their classical counterparts. The first quantum computer designs were based off of classical models; sequences of one- and multi-qubit gate operations are performed on chosen quantum bits and a final measurement would convert quantum information into classical bits. A new model, proposed by Briegel and Raussendorf [1], demonstrates that quantum computation can be achieved by using single qubit measurements as computational steps. This so-called cluster model or *one-way quantum computer (1WQC)* relies on an entangled state of a large number of qubits or *cluster state* as the resource. The fascinating feature about 1WQC is that they have no classical analogues and probe into new territory in regards to entanglement and measurements.

A. Cluster States

Consider a set of qubits $\mathcal{C}$ labeled by an integer index, that are distributed in some lattice such that every qubit can be said to have adjacent neighbors. For these to collectively form a cluster state, their quantum mechanical state would be characterized by the set of eigenvalue equations [2],

$$K_a \left| \Phi \right\rangle_{\mathcal{C}} = \kappa \left| \Phi \right\rangle_{\mathcal{C}} \tag{1}$$

for a family of operators $K_a = X^{(a)} \bigotimes_{\gamma \in \Gamma(a)} Z^{(\gamma)}$, $a \in \mathcal{C}$, where $\Gamma(a)$ is the set of indices of all qubits in the "adjacent neighborhood" of a. The matrix $X^{(a)}$ is used to denote an X operation on qubit-a, and so on. The eigenvalue $\kappa = \pm 1$ is determined by the specific occupation pattern of the neighboring sites.

A.1 Preparation of Linear Cluster State

Intuitively, a cluster state can be thought of as a graph where every vertex represents a qubit, and every edge represents the application of a C_z gate to both adjacent vertices.
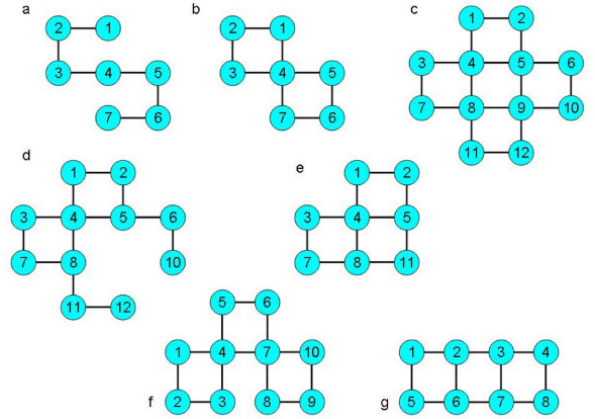

Fig. 1. Figure from [5], showing representative 2-D cluster shapes. The vertices are qubits with integer indices, and the edges indicate entanglement connectivity between select neighbors.
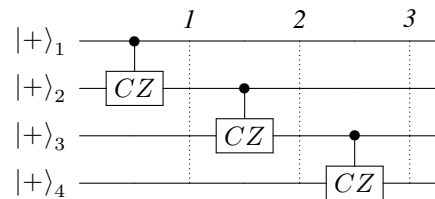
A cluster state can be represented as a graph $G = (N, E)$, where the $n \in N$ is a qubit and $e \in E$ is the application of a Controlled-Z (C_z) gate, where:

$$C_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

A *linear* cluster state is one where $degree(n) \leq 2 \forall n \in N$.

4-Node Linear Cluster State
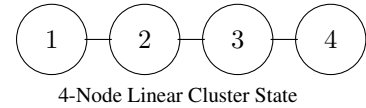
A method to prepare such a cluster state is given in [3], consisting of "cascading" C_z gates on n qubits as follows:

We can then analyze the state of the qubits at each of the dotted lines:

1:

$$\left(\frac{|0\rangle_1 |+\rangle_2 + |1\rangle_1 |-\rangle_2}{\sqrt{2}}\right) |+\rangle_3 |+\rangle_4$$
$$\equiv \left(\frac{|+\rangle_1 |0\rangle_2 + |-\rangle_1 |1\rangle_2}{\sqrt{2}}\right) |+\rangle_3 |+\rangle_4$$

2:

$$\left(\frac{|+\rangle_1 |0\rangle_2 |+\rangle_3 + |-\rangle_1 |1\rangle_2 |-\rangle_3}{\sqrt{2}}\right) |+\rangle_4$$
$$\equiv \left(\frac{|0\rangle_1 |+\rangle_2 + |1\rangle_1 |-\rangle_2}{2} |0\rangle_3 + \frac{|0\rangle_1 |-\rangle_2 + |1\rangle_1 |+\rangle_2}{2} |1\rangle_3\right) |+\rangle_4$$

3:

$$\left(\frac{|+\rangle_1 |0\rangle_2 |+\rangle_3 + |-\rangle_1 |1\rangle_2 |-\rangle_3}{2}\right) |0\rangle_4$$
$$+ \left(\frac{|+\rangle_1 |0\rangle_2 |-\rangle_3 + |-\rangle_1 |1\rangle_2 |+\rangle_3}{2}\right) |1\rangle_4$$
$$\equiv \frac{(|+\rangle_1 |0\rangle_2 + |-\rangle_1 |1\rangle_2) |0\rangle_3 |+\rangle_4 + (|+\rangle_1 |0\rangle_2 - |-\rangle_1 |1\rangle_2) |1\rangle_3 |-\rangle_4}{2}$$

The action of the C_z gate in the computational basis can be seen to be $|x, y\rangle \to (-1)^{xy} |x, y\rangle$. Cluster states of arbitrary shape and connectivity can similarly be prepared via the recursive use of the Hadamard gate and two-qubit fusion operations [4], [5].
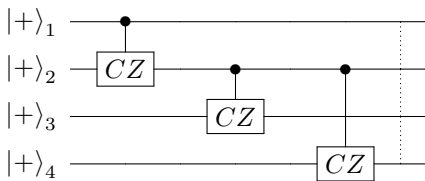
A.2 Preparation of T-Shaped Cluster State

A cluster state without the limitation on the degree of a node allows us to build *nonlinear* cluster states:
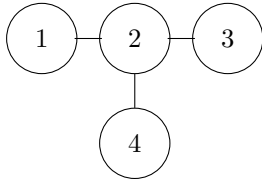
4-Node T-Shaped Cluster State

The circuit creating this cluster state will look as follows:

The state of the qubits after the application of the first two C_z gates is identical to the linear case, and the state after the last C_z (at the dotted line) is given by:

$$\frac{|+\rangle_1 |0\rangle_2 |+\rangle_3 |+\rangle_4 + |-\rangle_1 |1\rangle_2 |-\rangle_3 |-\rangle_4}{\sqrt{2}}$$
$$\equiv \frac{1}{\sqrt{2}} \left[\left(\frac{|+\rangle_1 |0\rangle_2 |+\rangle_3 + |-\rangle_1 |1\rangle_2 |-\rangle_3}{\sqrt{2}}\right) |0\rangle_4 \right.$$
$$\left. + \left(\frac{|+\rangle_1 |0\rangle_2 |+\rangle_3 - |-\rangle_1 |1\rangle_2 |-\rangle_3}{\sqrt{2}}\right) |1\rangle_4\right]$$

It is important to emphasize that the order in which the C_z gates are applied to grow the cluster state is irrelevant, as all of these pair-wise operations commute. This feature will be exploited later when discussing parallelizability.

B. The Effects of Measurement on a Cluster State

As is clear from the form of the expressions of all cluster states illustrated thus far, measuring any node in the computational basis severs it from the remaining graph by cutting all of it's edges with it's neighboring nodes. Should the outcome of said measurement be 1, then a Z gate/transform gets applied to all of it's erstwhile neighbors in the leftover cluster state. Thus, a large cluster state can be arbitrarily trimmed, split, and/or reshaped by removing qubits from the cluster. This is accomplished by measuring the target qubit in the computational basis, and performing appropriate unitary rotations on its former neighbors based on the measurement outcome.

The effect of an X-measurement (*i.e.,* a computational basis measurement following a Hadamard transformation) on any node of the cluster state is much more involved. This is best illustrated when demonstrating the use of a linear cluster state as a wire for quantum information. For this exercise, we start with a linear cluster state with three nodes (labelled 1, 2, and 3). A single qubit of quantum information $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ is stored in a physical qubit labelled 0 as illustrated below.
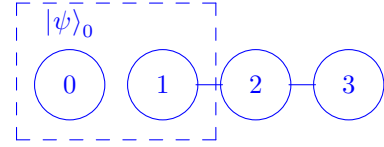
Gate $C_z^{(0,1)}$, followed by measurements $M_X^{(0)}$, $M_X^{(1)}$, & $M_X^{(2)}$.

To teleport the state $|\psi\rangle$ to physical qubit number 3, we must first supply the quantum information to the "wire." This is achieved by applying a C_z gate between physical qubits 0 and 1. Using $|\mathcal{LC}\rangle_{123}$ to denote the linear cluster state, we have

$$C_z^{(0,1)} |\psi\rangle_0 \otimes |\mathcal{LC}\rangle_{123}$$
$$= \frac{1}{\sqrt{2}} [\alpha |0\rangle_0 |+\rangle_1 |0\rangle_2 |+\rangle_3 + \beta |1\rangle_0 |-\rangle_1 |0\rangle_2 |+\rangle_3$$
$$\alpha |0\rangle_0 |-\rangle_1 |1\rangle_2 |-\rangle_3 + \beta |1\rangle_0 |+\rangle_1 |1\rangle_2 |-\rangle_3]$$

Following this, we perform X-measurements on physical qubits 0, 1, and 2 in that order. Let us denote an X-measurement operation on the jth-node with $M_X^{(j)}$, and let the outcome of any measurement on the same node be m_j. Then, the end result of these operations is the state

$$X^{m_2} Z^{m_1} X^{m_0} H |\psi\rangle_3, \qquad m_j \in \{0, 1\}.$$

The quantum information $|\psi\rangle$ has successfully been teleported to physical qubit 3, up to application of Pauli operators depending on the measurement outcomes. Since the leftover/extra Pauli operators do not commute, these measurements had to have been carried out in a specific order. However, the operation $C_z^{(2,3)}$, which was employed to grow the linear cluster state, commutes with $C_z^{(0,1)}$, as well as measurements $M_X^{(0/1)}$. Thus, further links to the chain can be grown as earlier links are being subjected to measurements. This aids in parallelizability, as well as physical implementation of cluster state quantum computing schemes.

Briegel and Raussendorf show that any quantum logic circuit can be implemented on a cluster state, which demonstrates universality of the proposed scheme [1]. Nielsen [6] extended this result to no longer require coherent dynamics, instead relying on a method to teleport quantum gates, and he provided a concise algorithm to accomplish this.

C. One-Way Quantum Computation

All quantum computation schemes may be characterized by some combination of state preparation, unitary transformation of said states, and measurements on the same. Human-usable computational tasks necessarily require both input and final output to be classical information. The classical input information can influence the quantum computation in choice of initial states, the choice of unitary transforms (*i.e.,* algorithm), and the choice of measurement bases. The output is always a classical function of the measurement outcomes. In typical models for quantum computation, entire algorithms are implemented as a sequence of unitary transformations on a prepared quantum state (stored in qubits) of size appropriate to the problem, with a round of measurements as the final step. In such models, the unitary transformation stage is completely reversible. The splitting of the effective unitary matrix into sequential steps can be arbitrary and entirely dependent on physical hardware limitations. There is no correspondence with "computational steps" or "clock cycles" in the classical sense, as the quantum state of the computer in the midst of the unitary stages is inaccessible for diagnosis or debugging purposes. Any leakage of information into computer memory or environment constitutes decoherence, and will introduce errors in the computation.

One-way quantum computation, on the other hand, revolves around single qubit measurements as a progression of computational steps. Measurements are a crucial component to quantum information processing because they irreversibly destroy a quantum state. Entanglement, on the other hand, will ensure that the state of the final qubit relies on the outcomes of preceding measurements. Given a cluster state, a series one-qubit measurements can be performed at each qubit to implement a quantum gate [3]. The unidirectionality of cluster state computation is inherent, due to the fact that quantum information cannot be accurately recovered once a measurement has been made. Consider a two-dimensional array of entangled qubits, information propagates horizontally through a row of qubits while vertical qubit neighbors are used for two-qubit gates. Similarly, three-dimensional clusters can be used to implement topologically protected gates [7], where the gate function only depends upon the way "connected defects" are wound around one another, but not on the details of their shape. This degree of freedom affords the design some fault tolerance.
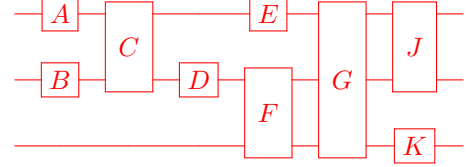

Need to draw 2-D cluster state graph that looks identical to picture above.

C.1 Gates through teleportation

Xiaodi mentioned that gate application via teleportation was the motivation for cluster states to begin with. This is basically Maira's segment of the talk. Can she turn it into motivational prose?

C.2 Applying a 2-qubit gate via 2-D cluster state

The ability to implement gates of the form HZ_α, and any 2-qubit controlled-phase gate, and to prepare arrays of $|+\rangle$ states as inputs, constitute a set of resources that is universal for quantum computation [9]. In this subsection, we will demonstrate the use of two-dimensional cluster states to implement a C_z gate between two input kets bearing quantum information $|\phi_A\rangle_A |\phi_B\rangle_B$ encoded in physical qubits labelled 'A' and 'B.' Here, $|\phi_j\rangle := \alpha_j |0\rangle + \beta_j |1\rangle$. For the scheme, we will use an I-shaped cluster states with six nodes, as shown in the figure.
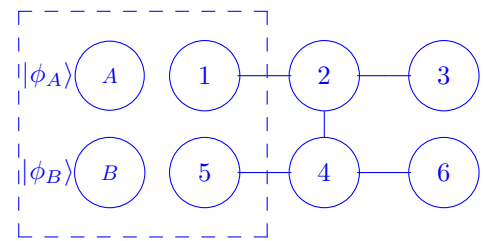
Apply $C_z^{(A,1)}$ and $C_z^{(B,5)}$ to input quantum information into cluster state.

The procedure would be to first entangle the input quantum information qubits into the cluster state via the application of C_z gates. Then we perform single qubit measurements on all qubits but numbers 3 and 6. Using the Z-measurement property of nodes on cluster states, we can pick node 4 as a representative anchor and write down the total state of the current system as

$$\frac{1}{\sqrt{2}} \left[|\mathcal{LC}\rangle_{123} |0\rangle_4 |+\rangle_5 |+\rangle_6 + Z_2 |\mathcal{LC}\rangle_{123} |0\rangle_4 |-\rangle_5 |-\rangle_6 \right]$$
$$\otimes |\phi_A\rangle_A |\phi_B\rangle_B \,,$$

where Z_2 is being applied on qubit 2.

The single qubit mesurements have to be performed in the order $(M_X^{(A)}, M_X^{(1)}, M_X^{(2)})$ and $(M_X^{(B)}, M_B^{(5)}, M_X^{(4)})$. These two subsets are parallelizable, and one could perform the measurements $\{M_X^{(A)}, M_X^{(B)}\}$ in parallel, and then $\{M_X^{(1)}, M_X^{(5)}\}$, and so on. But here, we will introduce the concept of layers of measurement, by performing all bottom row operations at once (including input entanglement), and then working on the top row.

Enacting the gate $C_z^{(B,5)}$ on the initial state, followed by measurements $M_X^{(B)}, M_B^{(5)}, M_X^{(4)}$ yields random outcomes $m_B, m_5, m_4 \in \{0,1\}$ and gives us the state

$$X_6^{m_4} Z_6^{m_5} X_6^{m_B} C_Z^{(2,6)} |\mathcal{LC}\rangle_{123} |\phi_B\rangle_6 \otimes |\phi_A\rangle_A \,.$$

Note that even though the original cluster state had an edge between nodes 2 and 4 in the graph representation, a measurement on qubit 4 did not sever the link between node 6 and the linear chain $|\mathcal{LC}\rangle_{123}$, due to the measurements being in the X basis. Now, if we follow the above operations by application of the $C_Z^{(A,1)}$ gate, and the measurements $M_X^{(A)}, M_X^{(1)}, M_X^{(2)}$ in that order, we get the random results $m_A, m_1, m_2 \in \{0,1\}$ and the final state

$$X_3^{m_2} Z_3^{m_1} X_3^{m_A} H_3 X_6^{m_4} Z_6^{m_5} X_6^{m_B} H_6 C_X^{(6,3)} |\phi_A\rangle_3 |\phi_B\rangle_6 \,,$$

where $C_X^{(j,k)}$ is the controlled-NOT gate. All single-qubit operations on qubit 3 commute with single-qubit operations on qubit 6. Using the fact that $H_j^2 = I_j$, as well as $C_X^{(6,3)} \equiv (H_3 \otimes I_6) C_Z^{(3,6)} (H_3 \otimes I_6)$, the final state is equivalently

$$X_3^{m_2} X_6^{m_4} Z_3^{m_1} Z_6^{m_5} X_3^{m_A} X_6^{m_4} H_6 C_Z^{(3,6)} H_3 |\phi_A\rangle_3 |\phi_B\rangle_6 \,,$$

which is the desired result, up to overall Pauli transformations on the individual qubits. Thus, we have proven that a universal set of quantum operations can be implemented using the cluster state model.

C.3 Commutations and parallelizability

The previous subsections demostrated that although measurements in a specific layer (an ordered set of sequentially connected nodes) of a cluster state need to occur in a specific order, operations on different layers commute, and can therefore be interspersed. This, combined with the ability to delay the growth of the cluster state to occur just before the measurements catch up with the remaining nodes, offers several

possibilities for process parallelization. Indeed, in the 2-qubit gate example, the quantum information in one of the qubits was put into the system first, followed by some measurement-based teleporation, and the other input qubit was brought in later. We effectively managed to entangle the two qubits using only single-qubit measurements. This seems to indicate that all multi-qubit interactions can be pre-computed by setting the topological layout of the graph of the cluster state before any of the quantum information has even been introduced into the system. Thus, outside of the physical implementation considerations, cluster state model isomorphisms offer a new analysis tool that disentangles the quintessential influence of quantum formalism on the complexity class of various algorithms.

D. Computational Power and Complexity

The spacial layout of the graph representation of the cluster state plays a role in the computational power of that state. If a cluster state can be prepared linearly via the cascading C_z technique mentioned above, it can be represented as a "one-dimensional" graph (i.e., some graph $G = (V, E)$, $\forall v \in V$, $\deg(v) \leq 2$). Operations on a(?) linearly prepared cluster state can be efficiently simulated on a classical computer in $O(n \log^c(1/n))$, where n is the initial number of qubits, and c is the cost of floating point multiplication [8]. Though the author consequently dismisses linearly prepared cluster states as a substrate for quantum computation, it would be interesting to know which class of problems they would be able to solve. Zach, there may be something worth digging deep for here. Let's talk about this.

In general, measurement based models can be polynomial time reduced to the gate array model, and thus have the same power, but they are more easily parallelizable [9].

The gate teleportation algorithm [6] has a time complexity of $O(\log(1/\epsilon))$, where ϵ is the failure probability.

To pad this out to ten pages, we can work on physical systems, limitations of linear optics, 3-D fault tolerant model, and complexity theory. If we REALLY have time, there may or may not be something to be done with qudits.

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