Assignment #4
CIS 427/527

Group 2

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1

Complete the proof of soundness of propositional logic (given in van Dalen, Lemma 1.5.1) with the case of $\rightarrow_E$.

Solution

Inductive Hypothesis: $\overset{D}{\varphi}$ and $\varphi \overset{D'}{\Rightarrow} \psi$ are derivations. For each γ, γ' containing the hypothesis of D,D', $\gamma \models \varphi$ and $\gamma' \models \varphi \Rightarrow \psi$. γ' contains the hypotheses of $(\overset{D}{\varphi}, \varphi \overset{D'}{\Rightarrow} \psi) \vdash \psi$

$$\gamma'' \supseteq \gamma \cup \gamma'$$

Then $\gamma'' \models \varphi$ and $\gamma'' \models \varphi \Rightarrow \psi$.

$$\gamma'' \models \varphi \wedge (\varphi \Rightarrow \psi)$$
$$\gamma'' \models \psi \text{ (Modus ponens)}$$

2

Prove the soundness of the $\vee$ rules ($\vee I$ and $\vee E$).

Solution

(V_I)
Inductive Hypothesis: $\overset{D}{\varphi}$ is a derivation, let γ contain the hypotheses of D. Then $\gamma \models \varphi$. γ' contains the hypotheses of $\varphi \overset{\overset{D}{\varphi}}{\vee} \psi$

$$\gamma' \supseteq \gamma$$
$$\gamma' \models \varphi$$

If $\varphi_1 = 1$, then $[\varphi \vee \psi]_v = 1$, therefore $\gamma' \models \varphi \vee \psi$.

(V_E)
Inductive Hypothesis: $\varphi \overset{D}{\vee} \psi$ is a derivation, let γ contain the hypotheses of D. Then $\gamma \models \varphi \vee \psi$. γ' contains the hypotheses of $(\varphi \overset{D}{\vee} \psi, \overset{[\varphi]}{\sigma}, \overset{[\psi]}{\sigma}) \vdash \sigma$

$$\gamma' \supseteq \gamma$$
$$\gamma' \models \varphi \vee \psi$$
$$\gamma' \models \varphi \rightarrow \sigma$$
$$\gamma' \models \psi \rightarrow \sigma$$

For all valuations v which make all formula in γ' true,

$$[\varphi \vee \psi]_v = 1$$
$$[\varphi \to \sigma]_v = 1$$
$$[\psi \to \sigma]_v = 1$$

$[\varphi \vee \psi]_v = 1$, so either $[\varphi]_v = 1$ or $[\psi]_v = 1$.

Case 1: $[\varphi]_v = 1$
Since $[\varphi]_v = 1$ and $[\varphi \to \sigma]_v = 1$, $[\sigma]_v = 1$.

Case 2: $[\psi]_v = 1$
Since $[\psi]_v = 1$ and $[\psi \to \sigma]_v = 1$, $[\sigma]_v = 1$.

In either case, $\gamma' \models \sigma$.

3

Do we have $\models (p \to q) \vee (q \to r)$?

Solution

p	q	r	$(p \to q)$	$(q \to r)$	$(p \to q) \vee (q \to r)$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	1	1	0	1
0	1	1	1	1	1
1	0	1	0	1	1
1	0	1	0	1	1
1	1	1	1	0	1
1	1	1	1	1	1

4

Do we have $(q \to (p \vee (q \to p))) \vee \neg(p \to q) \models p$?

Solution

If both p and q are false, we have $q \to (p \vee (q \to p)) \vee \neg(p \to q) = 1$ while $p = 0$, breaking the semantic entailment.

5

Assuming the soundness and completeness of natural deduction for propositional logic, suppose that you need to show that ϕ is not a semantic consequence of $\phi_1, \phi_2, ..., \phi_n$, but that you are only allowed to base your argument on the use of natural deduction rules. Which judgement would you need to prove in order to guarantee that $\phi_1, \phi_2, ..., \phi_n \not\models \phi$? Do you need completeness and soundness for this to work out?

Solution

You need RAA, completeness, and soundess for this to work.

$$\begin{aligned} \varphi_1, \varphi_2, ...\varphi_n &\models \varphi \\ &\models (\varphi_1 \wedge ... \wedge \varphi_n) \to \varphi \\ &\vdash (\varphi_1 \wedge ... \wedge \varphi_n) \to \varphi \text{ (completeness)} \end{aligned}$$

Then construct a derivation:

$$\dfrac{\dfrac{(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \varphi}{\dfrac{D}{\bot}}}{\neg[(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \varphi]}\,\text{RAA}$$

Therefore $\varphi_1, \varphi_2, \ldots\varphi_n \nvdash \varphi$, and by soundess, $\varphi_1, \varphi_2, \ldots\varphi_n \not\models \varphi$.

6

Consider the following axiom based system, called Hilbert system:

$$\begin{aligned}&(\phi \to (\psi \to \phi))\\ &((\phi \to (\psi \to \sigma)) \to ((\phi \to \psi) \to (\phi \to \sigma)))\\ &((\neg\phi \to \neg\psi) \to ((\neg\phi \to \psi) \to \phi))\end{aligned}$$

Combined with the Modus Ponens inference rule, which corresponds to the elimination rule of the implication connective.

Prove according to this system the judgement $\vdash \phi \to \phi$.

Solution

$$\dfrac{\dfrac{((\phi \to (\psi \to \sigma)) \to ((\phi \to \psi) \to (\phi \to \sigma))) \qquad (\phi \to (\psi \to \phi))}{(\phi \to (\psi \to \phi)) \to (\phi \to \phi)}\,\text{Let } \psi = \psi \to \phi \text{ and } \sigma = \phi \qquad (\phi \to (\psi \to \phi))}{\phi \to \phi}$$

7

Consider classical logic given in the handout "Natural deduction in sequent form" in Figure 5. Prove the following judgements:
$\vdash \phi \vee \neg\phi$ (This is called Law of Excluded Middle).
$((\phi \to \psi) \to \phi) \to \phi$ (This is called Peirce's Law).

Solution

- $\vdash \phi \vee \neg\phi$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg(\phi \vee \neg\phi)]^1 \qquad \dfrac{[\phi]^2}{\phi \vee \neg\phi}\,\vee\mathrm{I}}{\bot}\,\to\mathrm{E}}{\neg\phi}\,\to\mathrm{I}_2}{\phi \vee \neg\phi}\,\vee\mathrm{I} \qquad [\neg(\phi \vee \neg\phi)]^1}{\dfrac{\bot}{\phi \vee \neg\phi}\,\mathrm{RAA}_1}\,\to\mathrm{E}$$

- $((\phi \to \psi) \to \phi) \to \phi$

  Let LEM be the proof described above.

$$\dfrac{\dfrac{\dfrac{LEM}{(\phi \to \psi) \to \phi \vdash \phi \vee \neg\phi} \qquad \dfrac{}{(\phi \to \psi) \to \phi, \phi \vdash \phi} \qquad \dagger}{(\phi \to \psi) \to \phi \vdash \phi}}{\cdot \vdash ((\phi \to \psi) \to \phi) \to \phi}$$

$\dagger$

Let $\Gamma = (\phi \to \psi) \to \phi$

$$\dfrac{\dfrac{LEM}{\Gamma, \neg\phi \vdash (\phi \to \psi) \vee \neg(\phi \to \psi)} \qquad \dfrac{\dfrac{\ddagger}{\Gamma, \neg\phi, (\phi \to \psi) \vdash \phi \wedge \neg\psi}}{\Gamma, \neg\phi, \neg(\phi \to \psi) \vdash \phi} \qquad \dfrac{\Gamma, \neg\phi, (\phi \to \psi) \vdash \Gamma \qquad \Gamma, \neg\phi, \phi \to \psi \vdash \phi \to \psi}{\Gamma, \neg\phi, (\phi \to \psi) \vdash \phi}}{\Gamma, \neg\phi \vdash \phi}$$

‡

Let $\Gamma = \neg(\phi \to \psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \neg\phi, \phi \vdash \phi \qquad \Gamma, \neg\phi, \phi \vdash \neg\phi}{\bot}\,{\to}\mathrm{E}}{\Gamma, \neg\phi, \phi \vdash \psi}\,\mathrm{EFQ}}{\Gamma, \neg\phi \vdash \phi \to \psi}\,{\to}\mathrm{I}_2 \qquad \Gamma, \neg\phi \vdash \neg(\phi \to \psi)}{\Gamma, \neg\phi \vdash \bot}\,{\to}\mathrm{E}}{\Gamma \vdash \phi}\,\mathrm{RAA}_1 \qquad \dfrac{\dfrac{\dfrac{\Gamma, \psi, \phi \vdash \phi \qquad \Gamma, \psi, \phi \vdash \psi}{\Gamma, \psi \vdash \phi \to \psi}\,{\to}\mathrm{I} \qquad \Gamma, \psi \vdash \neg(\phi \to \psi)}{\Gamma, \psi \vdash \bot}\,{\to}\mathrm{E}}{\Gamma \vdash \neg\psi}\,{\to}\mathrm{I}}{\Gamma \vdash \phi \wedge \neg\psi}\,\wedge\mathrm{I}$$

8

Prove the following judgements:
$A \to B \to A$
$(A \to B \to C) \to (A \to B) \to (A \to C)$
$(A \wedge B \to C) \to (A \to B \to C)$
$(A \to B \to C) \to (A \wedge B \to C)$
Annotate each proof with lambda-terms.

Solution

$A \to B \to A$
$(\lambda x : A.\ \lambda y : B.\ x)$
$(A \to B \to C) \to (A \to B) \to (A \to C)$
$(\lambda x : A \to B \to C.\ \lambda y : A \to B.\ \lambda z : A.\ x\ z\ (y\ z))$
$(A \wedge B \to C) \to (A \to B \to C)$
$(\lambda x : A \wedge B \to C.\ \lambda y : A.\ \lambda z : B.\ x(y * z))$
$(A \to B \to C) \to (A \wedge B \to C)$
$(\lambda x : A \to B \to C.\ \lambda y : A \wedge B.\ x(\text{fst } y)(\text{snd } y))$