Assignment #4
CIS 427/527

Group 2

February 3, 2016

1

Complete the proof of soundness of propositional logic (given in val Dalen, Lemma 1.5.1) with the case of $\rightarrow_E$.

Solution

2

Prove the soundness of the $\vee$ rules ($\vee I$ and $\vee E$).

Solution

3

Do we have $\models (p \rightarrow q) \vee (q \rightarrow r)$?

Solution

p	q	r	$(p \rightarrow q)$	$(q \rightarrow r)$	$(p \rightarrow q) \vee (q \rightarrow r)$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	1	1	0	1
0	1	1	1	1	1
1	0	1	0	1	1
1	0	1	0	1	1
1	1	1	1	0	1
1	1	1	1	1	1

4

Do we have $(q \rightarrow (p \vee (q \rightarrow p))) \vee \neg(p \rightarrow q) \models p$?

Solution

If both p and q are false, we have $q \rightarrow (p \vee (q \rightarrow p)) \vee \neg(p \rightarrow q) = 1$ while $p = 0$, breaking the semantic entailment.

5

Assuming the soundness and completeness of natural deduction for propositional logic, suppose that you need to show that ϕ is not a semantic consequence of $\phi_1, \phi_2, ..., \phi_n$, but that you are only allowed to base your argument on the use of natural deduction rules. Which judgement would you need to prove in order to guarantee that $\phi_1, \phi_2, ..., \phi_n \not\models \phi$? Do you need completeness and soundness for this to work out?

Solution

6

Consider the following axiom based system, called Hilbert system:

$$
\begin{aligned}
&(\phi \to (\psi \to \phi))\\
&((\phi \to (\psi \to \sigma)) \to ((\phi \to \psi) \to (\phi \to \sigma)))\\
&((\neg\phi \to \neg\psi) \to ((\neg\phi \to \psi) \to \phi))
\end{aligned}
$$

Combined with the Modus Ponens inference rule, which corresponds to the elimination rule of the implication connective.

Prove according to this system the judgement $\vdash \phi \to \phi$.

Solution

$$
\dfrac{\dfrac{((\phi \to (\psi \to \sigma)) \to ((\phi \to \psi) \to (\phi \to \sigma))) \qquad (\phi \to (\psi \to \phi))}{(\phi \to (\psi \to \phi)) \to (\phi \to \phi)}\text{Let } \psi = \psi \to \phi \text{ and } \sigma = \phi \qquad (\phi \to (\psi \to \phi))}{\phi \to \phi}
$$

7

Consider classical logic given in the handout "Natural deduction in sequent form" in Figure 5. Prove the following judgements:
$\vdash \phi \vee \neg\phi$ (This is called Law of Excluded Middle).
$((\phi \to \psi) \to \phi) \to \phi$ (This is called Peirce's Law).

Solution

- $\vdash \phi \vee \neg\phi$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\neg(\phi \vee \neg\phi)]^1 \qquad \dfrac{[\phi]^2}{\phi \vee \neg\phi}\vee\mathrm{I}}{\bot}\to\mathrm{E}}{\neg\phi}\to\mathrm{I}_2}{\phi \vee \neg\phi}\vee\mathrm{I} \qquad [\neg(\phi \vee \neg\phi)]^1}{\dfrac{\bot}{\phi \vee \neg\phi}\mathrm{RAA}_1}\to\mathrm{E}
$$

- $((\phi \to \psi) \to \phi) \to \phi$

  Let LEM be the proof described above.

$$
\dfrac{\dfrac{\dfrac{LEM}{(\phi \to \psi) \to \phi \vdash \phi \vee \neg\phi} \qquad \dfrac{}{(\phi \to \psi) \to \phi, \phi \vdash \phi} \qquad \dagger}{(\phi \to \psi) \to \phi \vdash \phi}}{\cdot \vdash ((\phi \to \psi) \to \phi) \to \phi}
$$

$\dagger$

  Let $\Gamma = (\phi \to \psi) \to \phi$

$$
\dfrac{\dfrac{LEM}{\Gamma, \neg\phi \vdash (\phi \to \psi) \vee \neg(\phi \to \psi)} \qquad \dfrac{\dfrac{\ddagger}{\Gamma, \neg\phi, (\phi \to \psi) \vdash \phi \wedge \neg\psi}}{\Gamma, \neg\phi, \neg(\phi \to \psi) \vdash \phi} \qquad \dfrac{\Gamma, \neg\phi, (\phi \to \psi) \vdash \Gamma \qquad \Gamma, \neg\phi, \phi \to \psi \vdash \phi \to \psi}{\Gamma, \neg\phi, (\phi \to \psi) \vdash \phi}}{\Gamma, \neg\phi \vdash \phi}
$$

‡

Let $\Gamma = \neg(\phi \to \psi)$

$$
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\dfrac{\Gamma, \neg\phi, \phi \vdash \phi \qquad \Gamma, \neg\phi, \phi \vdash \neg\phi}{\bot}\,{\to}\mathrm{E}
}{\Gamma, \neg\phi, \phi \vdash \psi}\,\mathrm{EFQ}
}{\Gamma, \neg\phi \vdash \phi \to \psi}\,{\to}\mathrm{I}_2
\qquad \Gamma, \neg\phi \vdash \neg(\phi \to \psi)
}{\Gamma, \neg\phi \vdash \bot}\,{\to}\mathrm{E}
}{\Gamma \vdash \phi}\,\mathrm{RAA}_1
\qquad
\dfrac{
\dfrac{
\dfrac{\Gamma, \psi, \phi \vdash \phi \qquad \Gamma, \psi, \phi \vdash \psi}{\Gamma, \psi \vdash \phi \to \psi}\,{\to}\mathrm{I}
\qquad \Gamma, \psi \vdash \neg(\phi \to \psi)
}{\Gamma, \psi \vdash \bot}\,{\to}\mathrm{E}
}{\Gamma \vdash \neg\psi}\,{\to}\mathrm{I}
}{\Gamma \vdash \phi \wedge \neg\psi}\,\wedge\mathrm{I}
$$

8

Prove the following judgements:
$A \to B \to A$
$(A \to B \to C) \to (A \to B) \to (A \to C)$
$(A \wedge B \to C) \to (A \to B \to C)$
$(A \to B \to C) \to (A \wedge B \to C)$
Annotate each proof with lambda-terms.

Solution

$A \to B \to A$
    $(\lambda x : A.\ \lambda y : B.\ x)$
$(A \to B \to C) \to (A \to B) \to (A \to C)$
    $(\lambda x : A \to B \to C.\ \lambda y : A \to B.\ \lambda z : A.\ x\ z\ (y\ z))$
$(A \wedge B \to C) \to (A \to B \to C)$
    $(\lambda x : A \wedge B \to C.\ \lambda y : A.\ \lambda z : B.\ x(y * z))$
$(A \to B \to C) \to (A \wedge B \to C)$
    $(\lambda x : A \to B \to C.\ \lambda y : A \wedge B.\ x(\mathrm{fst}\ y)(\mathrm{snd}\ y))$