Assignment #5
CIS 427/527

Group 2

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2.2.1

Which of the following strings are formulas in predicate logic?

Solution

(a),(b),(f),(g) are formulas.

(c) isn't, as $f(m)$ is a term.
(d) isn't, as B is expecting two terms, yet $B(m,x)$ is a formula.
(e) isn't, as $B(m)$ doesn't have enough arguments.
(h) isn't, as $B(x)$ doesn't have enough arguments.

2.5.3

Provide proofs for the following sequents:

Solution

(a) $\forall x P(x) \vdash \forall y P(y)$

$$\dfrac{\dfrac{\forall x P(x)}{P(x)}}{\forall y P(y)}$$

(b) $\forall x(P(x) \to Q(x)) \vdash (\forall x \neg Q(x)) \to (\forall x \neg P(x))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x(P(x) \to Q(x))}{P(x_0) \to Q(x_0)}\,\forall\text{E} \qquad [P(x)]^1}{Q(x_0)}\,\to\text{E} \qquad \dfrac{[\forall x \neg Q(x)]^2}{\neg Q(x)}\,\forall\text{E}}{\bot}\,\to\text{E}}{\neg P(x)}\,\to\text{I}^1}{\forall x \neg P(x)}\,\forall\text{ I}}{(\forall x \neg Q(x)) \to (\forall x \neg P(x))}\,\to\text{I}^2$$

(c) $\forall x(P(x) \to \neg Q(x)) \vdash \neg(\exists x(P(x) \wedge Q(x)))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x(P(x) \to \neg Q(x))}{P(x_0) \to \neg Q(x_0)}\,\forall\text{E} \qquad \dfrac{[\exists x(P(x) \wedge Q(x))]^1 \qquad \dfrac{\dfrac{[P(x) \wedge Q(x)]^2}{P(x)}}{P(x_0)}}{P(x_0)}\,\exists\text{E}^2}{\neg Q(x_0)}\,\to\text{E} \qquad \dfrac{[\exists x(P(x) \wedge Q(x))]^1 \qquad \dfrac{\dfrac{[P(x) \wedge Q(x)]^3}{Q(x)}}{Q(x_0)}}{Q(x_0)}\,\exists\text{E}^3}{\bot}\,\to\text{E}}{\neg(\exists x(P(x) \wedge Q(x)))}\,\to\text{I}^1$$

2.5.11

Prove the following sequents in predicate logic:

Solution

(a) $\forall x \forall y \forall z (S(x,y) \wedge S(y,z) \to S(x,z)), \forall x \neg S(x,x) \vdash \forall x \forall y (S(x,y) \to \neg S(y,x))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x \neg S(x,x)}{\neg S(x_0,x_0)}\ \forall\mathrm{E} \qquad \dfrac{\dfrac{\forall x\forall y\forall z(S(x,y)\wedge S(y,z)\to S(x,z))}{S(x_0,y_0)\wedge S(y_0,x_0)\to S(x_0,x_0)}\ \forall\mathrm{E} \qquad \dfrac{\dfrac{\dfrac{[S(x,y)]^1}{\forall x\forall y S(x,y)}\ \forall\mathrm{I}}{S(x_0,y_0)}\ \forall\mathrm{E} \quad \dfrac{\dfrac{[S(y,x)]^2}{\forall y\forall x S(y,x)}\ \forall\mathrm{I}}{S(y_0,x_0)}\ \forall\mathrm{E}}{S(x_0,y_0)\wedge S(y_0,x_0)}\ {\to}\mathrm{E}}{S(x_0,x_0)}\ {\to}\mathrm{E}}{\bot}\ {\to}\mathrm{E}}{S(x,y)\to\neg S(y,x)}\ {\to}\mathrm{I}^1}{\forall x\forall y(S(x,y)\to\neg S(y,x))}\ \forall\mathrm{I}$$

(b) $\forall x (P(x) \vee Q(x)), \exists x \neg Q(x), \forall x (R(x) \to \neg P(x)) \vdash \exists x \neg R(x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x(R(x)\to\neg P(x))}{R(x_0)\to\neg P(x_0)}\ \forall\mathrm{E} \quad [R(x_0)]^1}{\neg P(x_0)}\ {\to}\mathrm{E} \qquad \dfrac{\dfrac{\exists x\neg Q(x) \quad \dfrac{\dfrac{[\neg Q(x)]^2}{\forall x\neg Q(x)}\ \forall\mathrm{I}}{\neg Q(x_0)}\ \forall\mathrm{E}}{\neg Q(x_0)}\ \exists\mathrm{E}^2 \quad \dfrac{\forall x(P(x)\vee Q(x))}{P(x_0)\vee Q(x_0)}\ \forall\mathrm{E}}{P(x_0)}\ \vee\mathrm{E}}{\bot}\ {\to}\mathrm{E}}{\neg R(x_0)}\ {\to}\mathrm{I}^1}{\exists x\neg R(x)}\ \exists\mathrm{I}$$

(c) $\forall x (P(x) \to (Q(x) \vee R(x))), \neg \exists x (P(x) \wedge R(x)) \vdash \forall x P(x) \to Q(x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg Q(x)]^2}{\forall x\neg Q(x)}\ \forall\mathrm{I}}{\neg Q(x_0)}\ \forall\mathrm{E} \qquad \dfrac{\dfrac{\forall x(P(x)\to(Q(x)\vee R(x)))}{P(x_0)\to(Q(x_0)\vee R(x_0))}\ \forall\mathrm{E} \quad \dfrac{\dfrac{[P(x)]^1}{\forall x P(x)}\ \forall\mathrm{I}}{P(x_0)}\ \forall\mathrm{E}}{Q(x_0)\vee R(x_0)}\ {\to}\mathrm{E}}{R(x_0)}\ \vee\mathrm{E} \qquad \dfrac{\dfrac{[P(x)]^1}{\forall x P(x)}\ \forall\mathrm{I}}{P(x_0)}\ \forall\mathrm{E}}{P(x_0)\wedge R(x_0)}\ \wedge\mathrm{I} \qquad \dfrac{\dfrac{\dfrac{\neg\exists x(P(x)\wedge R(x))}{\forall x\neg(P(x)\wedge R(x))}}{\neg(P(x_0)\wedge R(x_0))}\ \forall\mathrm{E}}{(P(x_0)\wedge R(x_0))\to\bot}}{\bot}\ {\to}\mathrm{E}}{Q(x)}\ \mathrm{RAA}^2}{P(x)\to Q(x)}\ {\to}\mathrm{I}^1}{\forall x P(x)\to Q(x)}\ \forall\mathrm{I}$$

(e) $(\exists x P(x)) \to \forall y P(y) \vdash \exists x \forall y ((P(x) \to P(y)) \wedge (P(y) \wedge P(x)))$

(f) $\exists x (P(x) \wedge Q(x)), \forall y (P(y) \to R(y)) \vdash \exists x R(x) \wedge Q(x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall y(P(y)\to R(y))}{P(x_0)\to R(x_0)}\ \forall\mathrm{E} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[P(x)\wedge Q(x)]^2}{P(x)}\ \wedge\mathrm{E}}{\forall x P(x)}\ \forall\mathrm{I}}{P(x_0)}\ \forall\mathrm{E} \quad \exists x(P(x)\wedge Q(x))}{P(x_0)}\ \exists\mathrm{E}^2}{R(x_0)}\ {\to}\mathrm{E} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[P(x)\wedge Q(x)]^1}{Q(x)}\ \wedge\mathrm{E}}{\forall x Q(x)}\ \forall\mathrm{I}}{Q(x_0)}\ \forall\mathrm{E} \quad \exists x(P(x)\wedge Q(x))}{Q(x_0)}\ \exists\mathrm{E}^1}{R(x_0)\wedge Q(x_0)}\ \wedge\mathrm{I}}{\exists x R(x)\wedge Q(x)}\ \exists\mathrm{I}$$

2.6.1

Consider the formula

$$\phi = \forall x \; \exists y \; Q(g(x,y), g(y,y), z)$$

Find two models M and M' with respective environments l and l' such that $M \models_l \phi$ but $M' \not\models_l \phi$

Solution

M
$A = \{a, b\}$
$Q^M = \{(a, a, b)\}$
$g^M(x, y) = a$ for all input
$z^M = b$

Then, $\forall x \; \forall y \; Q(g(x,y), g(y,y), z) = Q(a, a, b)$, which holds, so $M \models \phi$.

M'
$A = \{a, b\}$
$Q^{M'} = \{(a, a, c)\}$
$g^{M'}(x, y) = c$ for all input
$z^{M'} = a$

Then, $\forall x \; \forall y \; Q(g(x,y), g(y,y), z) = Q(c, c, a)$, but $(c, c, a) \notin Q^{M'}$, so $M' \not\models \phi$.

2.6.2

Consider the sentence

$$\phi = \forall x \; \exists y \; \exists z \; (P(x,y) \wedge P(z,y) \wedge (P(x,z) \rightarrow P(z,x)))$$

Which of the following models satisfies ϕ?

(a) $P^M = \{(m,n)|m < n\}$
(b) $P^{M'} = \{(m, 2*m)|m \text{ natural number}\}$
(c) $P^{M''} = \{(m,n)|m < n+1\}$

Solution

(a) This model does not satisfy ϕ, because we either need to force $P(x,z)$ to be false by requiring z to be smaller than x (in which case we can escape the natural numbers), or by having $x < z \wedge z < x$, which cannot happen.
(b) Yes, because the first two properties say $y = 2*x$ and $y = 2*z$, which means $x = z$ making $P(x,z)$ always false.
(c) Yes, let $y = z = x$, then all the properties hold.

2.6.3

Let P be a predicate with two arguments. Find a model M which satisfies $\forall x \; \neg P(x,x)$. Find also a model M' such that $M' \not\models \forall x \; \neg P(x,x)$.

Solution

- Let $P^M = \{(x,y)|x < y\}$, then we have $\forall x \; \neg P(x,x)$.
- Let $P^{M'} = \{(x,y)|x = y\}$, then we have $M' \not\models \forall x \; \neg P(x,x)$ as desired.

2.7.5

Show $\forall x(P(x) \bigvee Q(x)) \not\models \forall x P(x) \bigvee \forall x Q(x)$. Thus, find a model which satisfies $\forall x(P(x) \bigvee Q(x))$, but not $\forall x P(x) \bigvee \forall x Q(x)$

Solution

- Let $A =$ natural numbers, and let $P^M = \{x|x \leq 5\}$, $Q^M = \{x|x > 5\}$.

Then $\forall x.x \leq 5 \bigvee x > 5$ holds, but it is not true that $\forall x.x \leq 5 \bigvee \forall x.x > 5$.