Assignment #2
CIS 427/527

Group 2

January 20, 2016

1

Since two of the men are telling the truth, we have a proposition of the form $P = (A \wedge B) \vee (A \wedge C) \vee (B \wedge C)$.

- If Adams is telling the truth, then Brown is lying ($A \implies \neg B$).
- If Brown is telling the truth, then both Adams and Clark are lying ($B \implies (\neg A \wedge \neg C)$).
- If Clark is telling the truth, then Brown is lying ($C \implies \neg B$).

If Brown were telling the truth, then P could never be satisfied, therefore Brown is lying, which makes him the killer.

2

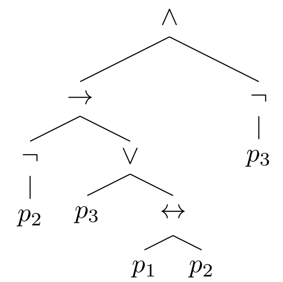
2. Show that $((\to \notin PROP$

Suppose $((\to \in X$ and X satisfies (i), (ii), (iii) of Definition 2.1.2. We claim that $Y = X \backslash \{((\to\}$ also satisfies (i), (ii), and (iii).

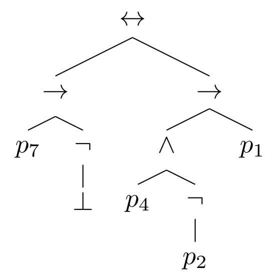
(i) $\perp, p_i \in Y$,
(ii) $\varphi, \psi \in Y$ and $(\varphi \square \psi) \neq ((\to$, it is clear that $(\varphi \square \psi) \in Y$
(iii) Similarly, $\varphi \in Y$ and $(\neg\varphi) \neq ((\to$, so $(\neg\varphi) \in Y$

Therefore X is not the smallest set satisfying (i), (ii), and (iii), so $((\to$ cannot belong to $PROP$.

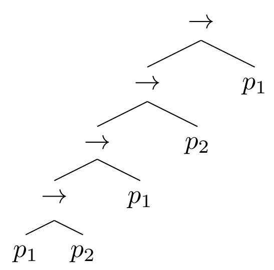
7. (a) Determine the trees of the proposition in Exercise 1
$(\neg p_2 \to (p_3 \vee (p_1 \leftrightarrow p_2))) \wedge \neg p_3$



$(p_7 \to \neg\perp) \leftrightarrow ((p_4 \wedge \neg p_2) \to p_1$



$(((p_1 \to p_2) \to p_1) \to p_2 \to p_1$



(b) Determine the propositions with the trees given

$\neg\neg\neg\bot$

$(p_0 \to \bot) \to ((p_0 \leftrightarrow p_1) \wedge p_5)$

$\neg((\neg p_1) \to (\neg p_1))$

9. Show that a proposition with n connectives has at most $2n+1$ subformulas

Base Case A proposition P with zero connectives has $\leq 2(0)+1 = 1$ subforumla(s).

Inductive Case Inductive Hypothesis: A proposition φ with n connectives has at most $2n+1$ subformulas, where $s(\varphi)$ gives the set of subformulas.

Assume propositions p, q with $n, m \leq n$ connectives resp., both of which have the property above. $|s(p)| \leq 2n+1$ and $|s(q)| \leq 2m+1$. Consider the proposition $r = (p \,\square\, q)$. r must have $n+m+1 = n'$ connectives (for the unary connective $\neg$, $|s(q)|$ is assumed to be 0). From the definition of s:

$$\begin{aligned} s(r) &= s(p) \cup s(q) \cup \{r\} \\ |s(r)| &\leq |s(p)| + |s(q)| + |\{r\}| \\ |s(r)| &\leq 2n+1+2m+1+1 \\ |s(r)| &\leq 2(n+m+1)+1 \\ |s(r)| &\leq 2n'+1 \end{aligned}$$

Since the desired property holds for r the inductive hypothesis holds $\forall n \in N$.

3

(a)

$(\neg\varphi \vee \psi) \iff (\psi \to \varphi)$	φ	ψ
1	0	0
0	0	1
1	1	0
1	1	1

Not a tautology.

(f)

$(\varphi \vee \neg\varphi)$	φ
1	0
1	1

Is a tautology.

(h)

$(\bot \to \varphi)$	φ
1	0
1	1

Is a tautology.

4

(a) $\varphi \models \varphi$

By definition of semantic entailment, $\varphi \models \varphi$ iff for all v : $[\![\ \varphi\ ]\!]_v = 1 \implies [\![\ \varphi\ ]\!]_v = 1$. Since $[\![\ \varphi\ ]\!]_v = 1 \implies [\![\ \varphi\ ]\!]_v = 1$ is a tautology, $\varphi \models \varphi$ holds.

(b) $\varphi \models \psi$ and $\psi \models \sigma \implies \varphi \models \sigma$

By definition of semantic entailment, we have $[\![\ \varphi\ ]\!]_v = 1\ \forall v \implies [\![\ \psi\ ]\!]_v = 1$ and $[\![\ \psi\ ]\!]_v = 1\ \forall v \implies [\![\ \sigma\ ]\!]_v = 1$, by the transitive property, $[\![\ \varphi\ ]\!]_v = 1\ \forall v \implies [\![\ \sigma\ ]\!]_v = 1$.

(c) $\models \varphi \to \psi \iff \varphi \models \psi$

Since $\models \varphi \to \psi$, we know that $[\![\ \psi\ ]\!]_v = 1\ \forall v$ (because if $[\![\ \psi\ ]\!]_v = 0$ for some v, then $\varphi \to \psi$ is not a tautology). Therefore, we can conclude that $[\![\ \varphi\ ]\!]_v = 1 \to [\![\ \psi\ ]\!]_v = 1$, since anytime $[\![\ \varphi\ ]\!]_v = 1$, $[\![\ \psi\ ]\!]_v = 1$.

5

1. Show by algebraic means:

(i) $\models (\varphi \implies \psi) \iff (\neg\psi \implies \neg\varphi)$ contraposition

$$\varphi \implies \psi \approx \neg\varphi \vee \psi \text{ (Thm 2.3.4.b)}$$
$$\varphi \implies \psi \approx \psi \vee \neg\varphi \text{ (Commutativity)}$$
$$\varphi \implies \psi \approx \neg\psi \implies \neg\varphi \text{ (Thm 2.3.4.b)}$$

(v) $\models \neg(\varphi \wedge \neg\varphi)$

$$\neg(\varphi \wedge \neg\varphi) \approx \neg\varphi \vee \neg(\neg\varphi) \text{ (deMorgan's)}$$
$$\neg(\varphi \wedge \neg\varphi) \approx \neg\varphi \vee \varphi \text{ (Double Negation law)}$$
$$\neg(\varphi \wedge \neg\varphi) \approx \varphi \implies \varphi \text{ (Thm 2.3.4.b)}$$
$$\neg(\varphi \wedge \neg\varphi) \approx \top \text{ (Absurdum)}$$

6

We can derive this using De Morgan's laws as follows:

$$\begin{aligned}\#(p,q) &= (p \vee q) \wedge \neg(p \wedge q)\\ &= (p \vee q) \wedge \neg(\neg(\neg p \vee \neg q))\\ &= (p \vee q) \wedge (\neg p \vee \neg q)\\ &= \neg(\neg(p \vee q) \vee \neg(\neg p \vee \neg q))\end{aligned}$$

7

A conjunctive normal form is a tautology iff every clause is also a tautology.

8

Let $\Gamma = \{\#\}$ (as defined in problem 6). Then it is impossible for $[\![\phi]\!]_v = [\![\psi]\!]_v = 1$ (due to the nature of XOR). It is therefore not the always the case that $[\![\phi]\!]_v = 1$ and it is not always the case that $[\![\psi]\!]_v = 1$, providing the required counterexample.

9

$P \to Q$**:** True when P is false; false when P is true and Q is false.
$P \vee Q \to P \wedge Q$**:** True when $P \iff Q$; false otherwise.
$\neg(P \vee Q \vee R)$**:** True when P, Q, R are all false; false when any P, Q, R are true.
$\neg(P \wedge Q) \wedge \neg(Q \vee R) \wedge (P \vee R)$**:** True when P is true, Q is false, R is false; false when P is false and R is false.