

# Assignment #4

## CIS 427/527

Group 2

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### 1

Complete the proof of soundness of propositional logic (given in val Dalen, Lemma 1.5.1) with the case of  $\rightarrow_E$ .

### Solution

### 2

Prove the soundness of the  $\vee$  rules ( $\vee I$  and  $\vee E$ ).

### Solution

### 3

Do we have  $\models (p \rightarrow q) \vee (q \rightarrow r)$ ?

### Solution

p	q	r	$(p \rightarrow q)$	$(q \rightarrow r)$	$(p \rightarrow q) \vee (q \rightarrow r)$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	1	1	0	1
0	1	1	1	1	1
1	0	1	0	1	1
1	0	1	0	1	1
1	1	1	1	0	1
1	1	1	1	1	1

### 4

Do we have  $(q \rightarrow (p \vee (q \rightarrow p))) \vee \neg(p \rightarrow q) \models p$ ?

### Solution

If both  $p$  and  $q$  are false, we have  $\neg(p \rightarrow q) = 0$  and  $q \rightarrow (p \vee (q \rightarrow p)) = 1$ , which results in  $q \rightarrow (p \vee (q \rightarrow p)) \vee \neg(p \rightarrow q) = 1$ , while  $p = 0$ , breaking the entailment.

### 5

Assuming the soundness and completeness of natural deduction for propositional logic, suppose that you need to show that  $\phi$  is not a semantic consequence of  $\phi_1, \phi_2, \dots, \phi_n$ , but that you are only allowed to base your argument on the use of natural deduction rules. Which judgement would you need to prove in order to guarantee that  $\phi_1, \phi_2, \dots, \phi_n \not\models \phi$ ? Do you need completeness and soundness for this to work out?

## Solution

### 6

Consider the following axiom based system, called Hilbert system:

$$\begin{aligned} &(\phi \rightarrow (\psi \rightarrow \phi)) \\ &((\phi \rightarrow (\psi \rightarrow \sigma)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \sigma))) \\ &((\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\phi \rightarrow \psi) \rightarrow \phi)) \end{aligned}$$

Combined with the Modus Ponens inference rule, which corresponds to the elimination rule of the implication connective.

Prove according to this system the judgement  $\vdash \phi \rightarrow \phi$ .

## Solution

$$\frac{\frac{((\phi \rightarrow (\psi \rightarrow \sigma)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \sigma))) \quad (\phi \rightarrow (\psi \rightarrow \phi))}{((\phi \rightarrow (\psi \rightarrow \varphi)) \rightarrow (\phi \rightarrow \varphi))} \text{ Let } \psi = \psi \rightarrow \varphi \text{ and } \sigma = \varphi}{\varphi \rightarrow \varphi} \quad (\phi \rightarrow (\psi \rightarrow \phi))$$

### 7

Consider classical logic given in the handout “Natural deduction in sequent form” in Figure 5. Prove the following judgements:

$\vdash \phi \vee \neg\phi$  (This is called Law of Excluded Middle).

$((\phi \rightarrow \psi) \rightarrow \phi) \rightarrow \phi$  (This is called Peirce’s Law).

## Solution

### 8

Prove the following judgements:

$A \rightarrow B \rightarrow A$

$(A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$

$(A \wedge B \rightarrow C) \rightarrow (A \rightarrow B \rightarrow C)$

$(A \rightarrow B \rightarrow C) \rightarrow (A \wedge B \rightarrow C)$

Annotate each proof with lambda-terms.

## Solution