Assignment #4
CIS 427/527

Group 2

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1

Complete the proof of soundness of propositional logic (given in val Dalen, Lemma 1.5.1) with the case of $\to_E$.

Solution

IH: $\overset{D}{\varphi}$ and $\overset{D'}{\varphi \Rightarrow \ \psi}$ are derivations. For each γ, γ' containing the hypothesis of D,D', $\gamma \models \varphi$ and $\gamma' \models \varphi \Rightarrow \ \psi$.

$$\gamma'' \supseteq \gamma \cup \gamma'$$

Then $\gamma'' \models \varphi$ and $\gamma'' \models \varphi \Rightarrow \ \psi$.

$$\gamma'' \models \varphi \wedge (\varphi \Rightarrow \ \psi)$$
$$\gamma'' \models \psi \text{ (Modus ponens)}$$

2

Prove the soundness of the $\vee$ rules ($\vee I$ and $\vee E$).

Solution

3

Do we have $\models (p \to q) \vee (q \to r)$?

Solution

p	q	r	$(p \to q)$	$(q \to r)$	$(p \to q) \vee (q \to r)$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	1	1	0	1
0	1	1	1	1	1
1	0	1	0	1	1
1	0	1	0	1	1
1	1	1	1	0	1
1	1	1	1	1	1

4

Do we have $(q \to (p \vee (q \to p))) \vee \neg(p \to q) \models p$?

Solution

If both p and q are false, we have $q \to (p \vee (q \to p)) \vee \neg(p \to q) = 1$ while $p = 0$, breaking the semantic entailment.

5

Assuming the soundness and completeness of natural deduction for propositional logic, suppose that you need to show that ϕ is not a semantic consequence of $\phi_1, \phi_2, ..., \phi_n$, but that you are only allowed to base your argument on the use of natural deduction rules. Which judgement would you need to prove in order to guarantee that $\phi_1, \phi_2, ..., \phi_n \not\models \phi$? Do you need completeness and soundness for this to work out?

Solution

6

Consider the following axiom based system, called Hilbert system:

$$
\begin{aligned}
&(\phi \to (\psi \to \phi)) \\
&((\phi \to (\psi \to \sigma)) \to ((\phi \to \psi) \to (\phi \to \sigma))) \\
&((\neg\phi \to \neg\psi) \to ((\neg\phi \to \psi) \to \phi))
\end{aligned}
$$

Combined with the Modus Ponens inference rule, which corresponds to the elimination rule of the implication connective.

Prove according to this system the judgement $\vdash \phi \to \phi$.

Solution

$$
\dfrac{\dfrac{((\phi \to (\psi \to \sigma)) \to ((\phi \to \psi) \to (\phi \to \sigma))) \qquad (\phi \to (\psi \to \phi))}{((\varphi \to (\psi \to \varphi)) \to (\varphi \to \varphi)))}\text{ Let } \psi = \psi \to \varphi \text{ and } \sigma = \varphi \qquad (\phi \to (\psi \to \phi))}{\varphi \to \varphi}
$$

7

Consider classical logic given in the handout "Natural deduction in sequent form" in Figure 5. Prove the following judgements:
$\vdash \phi \vee \neg\phi$ (This is called Law of Excluded Middle).
$((\phi \to \psi) \to \phi) \to \phi$ (This is called Peirce's Law).

Solution

- $\vdash \phi \vee \neg\phi$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg(\phi \vee \neg\phi)]^1 \qquad \dfrac{[\phi]^2}{\phi \vee \neg\phi}\vee\text{I}}{\bot}\to\text{E}}{\neg\phi}\to\text{I}_2}{\phi \vee \neg\phi}\vee\text{I}}{\dfrac{\bot}{\phi \vee \neg\phi}\text{RAA}_1}\vee\text{E}
$$

- $((\phi \to \psi) \to \phi) \to \phi$

8

Prove the following judgements:
$A \to B \to A$
$(A \to B \to C) \to (A \to B) \to (A \to C)$
$(A \wedge B \to C) \to (A \to B \to C)$
$(A \to B \to C) \to (A \wedge B \to C)$
Annotate each proof with lambda-terms.

Solution

$A \to B \to A$

$(\lambda x : A.\ \lambda y : B.\ x)$

$(A \to B \to C) \to (A \to B) \to (A \to C)$

$(\lambda x : A \to B \to C.\ \lambda y : A \to B.\ \lambda z : A.\ x\ z\ (y\ z))$

$(A \wedge B \to C) \to (A \to B \to C)$

$(\lambda x : A \wedge B \to C.\ \lambda y : A.\ \lambda z : B.\ x(y * z))$

$(A \to B \to C) \to (A \wedge B \to C)$

$(\lambda x : A \to B \to C.\ \lambda y : A \wedge B.\ x(\mathrm{fst}\ y)(\mathrm{snd}\ y))$