Assignment #5
CIS 427/527

Group 2

February 22, 2016

2.2.1

Which of the following strings are formulas in predicate logic?

Solution

(a),(b),(f),(g) are formulas.

(c) isn't, as $f(m)$ is a term.
(d) isn't, as B is expecting two terms, yet $B(m, x)$ is a formula.
(e) isn't, as $B(m)$ doesn't have enough arguments.
(h) isn't, as $B(x)$ doesn't have enough arguments.

2.5.3

Provide proofs for the following sequents:

Solution

(a) $\forall x P(x) \vdash \forall y P(y)$

$$
\begin{prooftree}
\AxiomC{$\forall x P(x)$}
\UnaryInfC{$P(x)$}
\UnaryInfC{$\forall y P(y)$}
\end{prooftree}
$$

(b) $\forall x(P(x) \to Q(x)) \vdash (\forall x \neg Q(x)) \to (\forall x \neg P(x))$

$$
\begin{prooftree}
\AxiomC{$\forall x(P(x) \to Q(x))$}
\RightLabel{$\forall$E}
\UnaryInfC{$P(x_0) \to Q(x_0)$}
\AxiomC{$[P(x)]^1$}
\RightLabel{$\to$E}
\BinaryInfC{$Q(x_0)$}
\AxiomC{$[\forall x \neg Q(x)]^2$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg Q(x)$}
\RightLabel{$\to$E}
\BinaryInfC{$\perp$}
\RightLabel{$\to$ I^1}
\UnaryInfC{$\neg P(x)$}
\RightLabel{$\forall$ I}
\UnaryInfC{$\forall x \neg P(x)$}
\RightLabel{$\to$ I^2}
\UnaryInfC{$(\forall x \neg Q(x)) \to (\forall x \neg P(x))$}
\end{prooftree}
$$

(c) $\forall x(P(x) \to \neg Q(x)) \vdash \neg(\exists x(P(x) \wedge Q(x)))$

$$
\begin{prooftree}
\AxiomC{$\forall x(P(x) \to \neg Q(x))$}
\RightLabel{$\forall$E}
\UnaryInfC{$P(x_0) \to \neg Q(x_0)$}
\AxiomC{$[\exists x(P(x) \wedge Q(x))]^1$}
\AxiomC{$[P(x) \wedge Q(x)]^2$}
\UnaryInfC{$P(x)$}
\UnaryInfC{$P(x_0)$}
\RightLabel{$\exists$E^2}
\BinaryInfC{$P(x_0)$}
\RightLabel{$\to$E}
\BinaryInfC{$\neg Q(x_0)$}
\AxiomC{$[\exists x(P(x) \wedge Q(x))]^1$}
\AxiomC{$[P(x) \wedge Q(x)]^3$}
\UnaryInfC{$Q(x)$}
\UnaryInfC{$Q(x_0)$}
\RightLabel{$\exists$E^3}
\BinaryInfC{$Q(x_0)$}
\RightLabel{$\to$E}
\BinaryInfC{$\perp$}
\RightLabel{$\to$I^1}
\UnaryInfC{$\neg(\exists x(P(x) \wedge Q(x)))$}
\end{prooftree}
$$

2.5.11

Prove the following sequents in predicate logic:

Solution

(a) $\forall x \forall y \forall z (S(x,y) \wedge S(y,z) \to S(x,z)), \forall x \neg S(x,x) \vdash \forall x \forall y (S(x,y) \to \neg S(y,x))$

$$
\begin{prooftree}
\AxiomC{$\forall x \neg S(x,x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg S(x_0,x_0)$}
\AxiomC{$\forall x \forall y \forall z (S(x,y) \wedge S(y,z) \to S(x,z))$}
\RightLabel{$\forall$E}
\UnaryInfC{$S(x_0,y_0) \wedge S(y_0,x_0) \to S(x_0,x_0)$}
\AxiomC{$[S(x,y)]^1$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x \forall y S(x,y)$}
\RightLabel{$\forall$E}
\UnaryInfC{$S(x_0,y_0)$}
\AxiomC{$[S(y,x)]^2$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall y \forall x S(y,x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$S(y_0,x_0)$}
\RightLabel{$\to$E}
\BinaryInfC{$S(x_0,y_0) \wedge S(y_0,x_0)$}
\RightLabel{$\to$E}
\BinaryInfC{$S(x_0,x_0)$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I^1}
\UnaryInfC{$S(x,y) \to \neg S(y,x)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x \forall y (S(x,y) \to \neg S(y,x))$}
\end{prooftree}
$$

(b) $\forall x (P(x) \vee Q(x)), \exists x \neg Q(x), \forall x (R(x) \to \neg P(x)) \vdash \exists x \neg R(x)$

$$
\begin{prooftree}
\AxiomC{$\forall x (R(x) \to \neg P(x))$}
\RightLabel{$\forall$E}
\UnaryInfC{$R(x_0) \to \neg P(x_0)$}
\AxiomC{$[R(x_0)]^1$}
\RightLabel{$\to$E}
\BinaryInfC{$\neg P(x_0)$}
\AxiomC{$\exists x \neg Q(x)$}
\AxiomC{$[\neg Q(x)]^2$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x \neg Q(x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg Q(x_0)$}
\RightLabel{$\exists$E^2}
\BinaryInfC{$\neg Q(x_0)$}
\AxiomC{$\forall x (P(x) \vee Q(x))$}
\RightLabel{$\forall$E}
\UnaryInfC{$P(x_0) \vee Q(x_0)$}
\RightLabel{$\vee$E}
\BinaryInfC{$P(x_0)$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I^1}
\UnaryInfC{$\neg R(x_0)$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists x \neg R(x)$}
\end{prooftree}
$$

(c) $\forall x (P(x) \to (Q(x) \vee R(x))), \neg \exists x (P(x) \wedge R(x)) \vdash \forall x P(x) \to Q(x)$

$$
\begin{prooftree}
\AxiomC{$[\neg Q(x)]^2$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x \neg Q(x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg Q(x_0)$}
\AxiomC{$\forall x (P(x) \to (Q(x) \vee R(x)))$}
\RightLabel{$\forall$E}
\UnaryInfC{$P(x_0) \to (Q(x_0) \vee R(x_0))$}
\AxiomC{$[P(x)]^1$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x P(x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$P(x_0)$}
\RightLabel{$\to$E}
\BinaryInfC{$Q(x_0) \vee R(x_0)$}
\RightLabel{$\vee$E}
\BinaryInfC{$R(x_0)$}
\AxiomC{$[P(x)]^1$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x P(x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$P(x_0)$}
\RightLabel{$\wedge$I}
\BinaryInfC{$P(x_0) \wedge R(x_0)$}
\AxiomC{$\neg \exists x (P(x) \wedge R(x))$}
\UnaryInfC{$\forall x \neg (P(x) \wedge R(x))$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg (P(x_0) \wedge R(x_0))$}
\UnaryInfC{$(P(x_0) \wedge R(x_0)) \to \bot$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{RAA^2}
\UnaryInfC{$Q(x)$}
\RightLabel{$\to$I^1}
\UnaryInfC{$P(x) \to Q(x)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x P(x) \to Q(x)$}
\end{prooftree}
$$

(e) $(\exists x P(x)) \to \forall y P(y) \vdash \exists x \forall y ((P(x) \to P(y)) \wedge (P(y) \wedge P(x)))$
(f) $\exists x (P(x) \wedge Q(x)), \forall y (P(y) \to R(y)) \vdash \exists x R(x) \wedge Q(x)$

$$
\begin{prooftree}
\AxiomC{$\forall y (P(y) \to R(y))$}
\RightLabel{$\forall$E}
\UnaryInfC{$P(x_0) \to R(x_0)$}
\AxiomC{$[P(x) \wedge Q(x)]^2$}
\RightLabel{$\wedge$E}
\UnaryInfC{$P(x)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x P(x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$P(x_0)$}
\AxiomC{$\exists x (P(x) \wedge Q(x))$}
\RightLabel{$\exists$E^2}
\BinaryInfC{$P(x_0)$}
\RightLabel{$\to$E}
\BinaryInfC{$R(x_0)$}
\AxiomC{$[P(x) \wedge Q(x)]^1$}
\RightLabel{$\wedge$E}
\UnaryInfC{$Q(x)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x Q(x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$Q(x_0)$}
\AxiomC{$\exists x (P(x) \wedge Q(x))$}
\RightLabel{$\exists$E^1}
\BinaryInfC{$Q(x_0)$}
\RightLabel{$\wedge$I}
\BinaryInfC{$R(x_0) \wedge Q(x_0)$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists x R(x) \wedge Q(x)$}
\end{prooftree}
$$

2.6.1

Consider the formula

$$\phi = \forall x \; \exists y \; Q(g(x,y), g(y,y), z)$$

Find two models M and M' with respective environments l and l' such that $M \models_l \phi$ but $M' \not\models_l \phi$

Solution

M
$A = \{a, b\}$
$Q^M = \{(a, a, b)\}$
$g^M(x, y) = a$ for all input
$z^M = b$

Then, $\forall x \; \forall y \; Q(g(x,y), g(y,y), z) = Q(a, a, b)$, which holds, so $M \models \phi$.

M'
$A = \{a, b\}$
$Q^{M'} = \{(a, a, c)\}$
$g^{M'}(x, y) = c$ for all input
$z^{M'} = a$

Then, $\forall x \; \forall y \; Q(g(x,y), g(y,y), z) = Q(c, c, a)$, but $(c, c, a) \notin Q^{M'}$, so $M' \not\models \phi$.

2.6.2

Consider the sentence

$$\phi = \forall x \; \exists y \; \exists z \; (P(x,y) \wedge P(z,y) \wedge (P(x,z) \to P(z,x)))$$

Which of the following models satisfies ϕ?

(a) $P^M = \{(m, n) | m < n\}$
(b) $P^{M'} = \{(m, 2 * m) | m \text{ natural number}\}$
(c) $P^{M''} = \{(m, n) | m < n + 1\}$

Solution

(a) This model does not satisfy ϕ, because we either need to force $P(x, z)$ to be false by requiring z to be smaller than x (in which case we can escape the natural numbers), or by having $x < z \wedge z < x$, which cannot happen.
(b) Yes, because the first two properties say $y = 2 * x$ and $y = 2 * z$, which means $x = z$ making $P(x, z)$ always false.
(c) Yes, let $y = z = x$, then all the properties hold.

2.6.3

Let P be a predicate with two arguments. Find a model M which satisfies $\forall x \; \neg P(x, x)$. Find also a model M' such that $M' \not\models \forall x \; \neg P(x, x)$.

Solution

- Let $P^M = \{(x, y) | x < y\}$, then we have $\forall x \; \neg P(x, x)$.
- Let $P^{M'} = \{(x, y) | x = y\}$, then we have $M' \not\models \forall x \; \neg P(x, x)$ as desired.

2.7.5

Show $\forall x (P(x) \bigvee Q(x)) \not\models \forall x P(x) \bigvee \forall x Q(x)$. Thus, find a model which satisfies $\forall x (P(x) \bigvee Q(x))$, but not $\forall x P(x) \bigvee \forall x Q(x)$

Solution

- Let $A =$ natural numbers, and let $P^M = \{x | x \leq 5\}$, $Q^M = \{x | x > 5\}$.

Then $\forall x . x \leq 5 \bigvee x > 5$ holds, but it is not true that $\forall x . x \leq 5 \bigvee \forall x . x > 5$.