Assignment #3
CIS 427/527

Group 2

January 26, 2016

1

Show that the following propositions are derivable:

(a) $\varphi \to \varphi$

$$\dfrac{[\varphi]^1}{\varphi \to \varphi} \to I^1$$

(b) $\bot \to \varphi$

$$\dfrac{\dfrac{[\bot]^1}{\varphi}\ \bot E}{\bot \to \varphi} \to I^1$$

(c) $\neg(\varphi \wedge \neg\varphi)$

$$\dfrac{\dfrac{\dfrac{[\varphi \wedge \neg\varphi]_1}{\varphi}\ \wedge E \qquad \dfrac{[\varphi \wedge \neg\varphi]_1}{\neg\varphi}\ \wedge E}{\bot} \to\mathrm{E}}{\neg(\varphi \wedge \neg\varphi)} \to\mathrm{I}^1$$

(d) $(\varphi \to \psi) \leftrightarrow \neg(\varphi \wedge \neg\psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \wedge \neg\psi]_1}{\varphi}\ \wedge\mathrm{E} \qquad [\varphi \to \psi]_2}{\psi} \qquad \dfrac{[\varphi \wedge \neg\psi]_1}{\neg\psi}\ \wedge\mathrm{E}}{\bot} \to\mathrm{E}}{\neg(\varphi \wedge \neg\psi)} \to\mathrm{I}^1}{(\varphi \to \psi) \to \neg(\varphi \wedge \neg\psi)} \to\mathrm{I}^2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg(\varphi \wedge \neg\psi)]_2 \qquad \dfrac{[\varphi]_3 \qquad [\neg\psi]_1}{\varphi \wedge \neg\psi}\ \wedge\mathrm{I}}{\bot} \to\mathrm{E}}{\psi}\ \mathrm{RAA}^1}{\varphi \to \psi} \to\mathrm{I}^3}{\neg(\varphi \wedge \neg\psi) \to (\varphi \to \psi)} \to\mathrm{I}^2}{(\varphi \to \psi) \leftrightarrow \neg(\varphi \wedge \neg\psi)}$$

(e) $(\varphi \wedge \psi) \leftrightarrow \neg(\varphi \to \neg\psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]_1}{\psi}\ \wedge\mathrm{E} \qquad \dfrac{\dfrac{[\varphi \wedge \psi]_1}{\varphi}\ \wedge\mathrm{E} \qquad [\varphi \to \neg\psi]_2}{\neg\psi} \to\mathrm{I}}{\bot}}{\neg(\varphi \to \neg\psi)} \to\mathrm{I}^2}{(\varphi \wedge \psi) \to \neg(\varphi \to \neg\psi)} \to\mathrm{I}^1 \qquad \dfrac{\dfrac{\dfrac{[\neg(\varphi \to \neg\psi)]_1 \qquad \dfrac{[\neg\psi]^2}{\varphi \to \neg\psi}}{\bot}\ \mathrm{RAA}^2}{\varphi \wedge \psi}\ \wedge I^1}{\neg(\varphi \to \neg\psi) \to (\varphi \wedge \psi)} \to I}{(\varphi \wedge \psi) \leftrightarrow \neg(\varphi \to \neg\psi)}$$

(f) $\varphi \to (\psi \to (\varphi \wedge \psi))$

$$\dfrac{\dfrac{\dfrac{[\varphi]^1 \qquad [\psi]^2}{\varphi \wedge \psi}\ \wedge I}{\psi \to (\varphi \wedge \psi)} \to I^2}{\varphi \to (\psi \to (\varphi \wedge \psi))} \to I^1$$

2

Show that the following propositions are derivable:

(a) $(\varphi \to \neg\varphi) \to \neg\varphi$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \to \neg\varphi]_1 \qquad [\varphi]_2}{\neg\varphi}\,{\to}\mathrm{E} \qquad [\varphi]_2}{\bot}\,{\to}\mathrm{E}}{\neg\varphi}\,{\to}\mathrm{I}^2}{(\varphi \to \neg\varphi) \to \neg\varphi}\,\to I_1$$

(b) $[\varphi \to (\psi \to \sigma)] \leftrightarrow [\psi \to (\varphi \to \sigma)]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\psi \to (\varphi \to \sigma)]_3 \qquad [\psi]_1}{\varphi \to \sigma}\,{\to}\mathrm{E} \qquad [\varphi]_2}{\sigma}\,{\to}\mathrm{E}}{\psi \to \sigma}\,{\to}\mathrm{I}^1}{\varphi \to (\psi \to \sigma)}\,{\to}\mathrm{I}^2}{[\psi \to (\varphi \to \sigma)] \to [\varphi \to (\psi \to \sigma)]}\,{\to}\mathrm{I}^3 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \to (\psi \to \sigma)]_3 \qquad [\varphi]_1}{\psi \to \sigma}\,{\to}\mathrm{E} \qquad [\psi]_2}{\sigma}\,{\to}\mathrm{E}}{\varphi \to \sigma}\,{\to}\mathrm{I}^1}{\psi \to (\varphi \to \sigma)}\,{\to}\mathrm{I}^2}{[\varphi \to (\psi \to \sigma)] \to [\psi \to (\varphi \to \sigma)]}\,{\to}\mathrm{I}^3}{[\varphi \to (\psi \to \sigma)] \leftrightarrow [\psi \to (\varphi \to \sigma)]}$$

(c) $(\varphi \to \psi) \wedge (\varphi \to \neg\psi) \to \neg\varphi$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(\varphi \to \psi) \wedge (\varphi \to \neg\psi)]_2}{\varphi \to \psi}\,\wedge\mathrm{E} \qquad [\varphi]_1}{\psi}\,{\to}\mathrm{E} \qquad \dfrac{\dfrac{[(\varphi \to \psi) \wedge (\varphi \to \neg\psi)]_2}{\varphi \to \neg\psi}\,\wedge\mathrm{E} \qquad [\varphi]_1}{\neg\psi}\,{\to}\mathrm{E}}{\bot}\,{\to}\mathrm{E}}{\neg\varphi}\,{\to}\mathrm{I}^1}{(\varphi \to \psi) \wedge (\varphi \to \neg\psi) \to \neg\varphi}\,{\to}\mathrm{I}^2$$

(d) $(\varphi \to \psi) \to [(\varphi \to (\psi \to \sigma)) \to (\varphi \to \sigma)]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \to \psi]_1 \qquad [\varphi]_2}{\psi}\,\to E \qquad \dfrac{[\varphi]_2 \qquad [(\varphi \to (\psi \to \sigma))]_3}{\psi \to \sigma}\,\to E}{\sigma}\,\to E}{\varphi \to \sigma}\,\to I_2}{(\varphi \to (\psi \to \sigma)) \to (\varphi \to \sigma)}\,\to I_3}{(\varphi \to \psi) \to [(\varphi \to (\psi \to \sigma)) \to (\varphi \to \sigma)]}\,\to I_1$$

3

Show:

(a) $\varphi \vdash \neg(\neg\varphi \wedge \psi)$

$$\dfrac{\dfrac{\dfrac{[\neg\varphi \wedge \psi]_1}{\neg\varphi}\,\wedge\mathrm{E} \qquad \varphi}{\bot}\,{\to}\mathrm{E}}{\neg(\neg\varphi \wedge \psi)}\,\to I_1$$

(b) $\neg(\varphi \wedge \neg\psi), \varphi \vdash \psi$

$$\dfrac{\dfrac{\dfrac{[\neg\psi]_1 \qquad \varphi}{\varphi \wedge \neg\psi}\,\wedge\mathrm{I} \qquad \neg(\varphi \wedge \neg\psi)}{\bot}\,{\to}\mathrm{E}}{\psi}\,\mathrm{RAA}_1$$

(c) $\neg\varphi \vdash (\varphi \to \psi) \leftrightarrow \neg\varphi$

$$
\begin{prooftree}
\AxiomC{$[\neg\varphi]_2$}
\AxiomC{$[\varphi]_1$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{EFQ}
\UnaryInfC{ψ}
\RightLabel{$\to$I^1}
\UnaryInfC{$\varphi \to \psi$}
\RightLabel{$\to$I^2}
\UnaryInfC{$\neg\varphi \to (\varphi \to \psi)$}
\AxiomC{$[\varphi \to \psi]_1$}
\AxiomC{$\neg\varphi$}
\RightLabel{$\wedge$I}
\BinaryInfC{$(\varphi \to \psi) \wedge (\neg\varphi)$}
\RightLabel{$\wedge$E}
\UnaryInfC{$\neg\varphi$}
\RightLabel{$\to$I^1}
\UnaryInfC{$(\varphi \to \psi) \to \neg\varphi$}
\RightLabel{$\wedge$I}
\BinaryInfC{$(\varphi \to \psi) \leftrightarrow \neg\varphi$}
\end{prooftree}
$$

(d) $\vdash \varphi \Rightarrow \vdash \psi \to \varphi$

Given $\vdash \varphi$, there exists a derivation $\begin{smallmatrix} D \\ \varphi \end{smallmatrix}$, with all hypotheses in D cancelled. Since $\begin{smallmatrix} D \\ \varphi \end{smallmatrix}, \psi$ are derivations, the definition of the set of derivations gives:

$$
\begin{prooftree}
\AxiomC{ψ}
\AxiomC{$\begin{smallmatrix} D \\ \varphi \end{smallmatrix}$}
\RightLabel{$\wedge$I}
\BinaryInfC{$\psi \wedge \varphi$}
\end{prooftree}
$$

From here we can apply derviation rules to prove $\psi \to \varphi$

$$
\begin{prooftree}
\AxiomC{$[\psi]_1$}
\AxiomC{$\begin{smallmatrix} D \\ \varphi \end{smallmatrix}$}
\RightLabel{$\wedge$I}
\BinaryInfC{$\psi \wedge \varphi$}
\RightLabel{$\wedge$E}
\UnaryInfC{φ}
\RightLabel{$\to$I$_1$}
\UnaryInfC{$\psi \to \varphi$}
\end{prooftree}
$$

(e) $\neg\varphi \vdash \varphi \to \psi$

$$
\begin{prooftree}
\AxiomC{$[\varphi]_1$}
\AxiomC{$\neg\varphi$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{EFQ}
\UnaryInfC{ψ}
\RightLabel{$\to I_1$}
\UnaryInfC{$\varphi \to \psi$}
\end{prooftree}
$$