Assignment #3
CIS 427/527

Group 2

January 27, 2016

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Show that the following propositions are derivable:

(a) $\varphi \to \varphi$

$$\dfrac{[\varphi]^1}{\varphi \to \varphi} \to I^1$$

(b) $\bot \to \varphi$

$$\dfrac{\dfrac{[\bot]^1}{\varphi}\,\bot E}{\bot \to \varphi} \to I^1$$

(c) $\neg(\varphi \wedge \neg\varphi)$

$$\dfrac{\dfrac{\dfrac{[\varphi \wedge \neg\varphi]_1}{\varphi}\,\wedge E \qquad \dfrac{[\varphi \wedge \neg\varphi]_1}{\neg\varphi}\,\wedge E}{\bot}\,\to\mathrm{E}}{\neg(\varphi \wedge \neg\varphi)}\,\to\mathrm{I}^1$$

(d) $(\varphi \to \psi) \leftrightarrow \neg(\varphi \wedge \neg\psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \wedge \neg\psi]_1}{\varphi}\,\wedge\mathrm{E} \qquad [\varphi \to \psi]_2}{\psi} \qquad \dfrac{[\varphi \wedge \neg\psi]_1}{\neg\psi}\,\wedge\mathrm{E}}{\bot}\,\to\mathrm{E}}{\neg(\varphi \wedge \neg\psi)}\,\to\mathrm{I}^1}{(\varphi \to \psi) \to \neg(\varphi \wedge \neg\psi)}\,\to\mathrm{I}^2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg(\varphi \wedge \neg\psi)]_2 \qquad \dfrac{[\varphi]_3 \qquad [\neg\psi]_1}{\varphi \wedge \neg\psi}\,\wedge\mathrm{I}}{\bot}\,\to\mathrm{E}}{\psi}\,\mathrm{RAA}^1}{\varphi \to \psi}\,\to\mathrm{I}^3}{\neg(\varphi \wedge \neg\psi) \to (\varphi \to \psi)}\,\to\mathrm{I}^2}{(\varphi \to \psi) \leftrightarrow \neg(\varphi \wedge \neg\psi)}$$

(e) $(\varphi \wedge \psi) \leftrightarrow \neg(\varphi \to \neg\psi)$
Note this proof got too wide – both implications are proved sequentially:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]_4 \qquad [\neg\varphi]_5}{\bot}\,\to\mathrm{E}}{\neg\psi}\,\mathrm{EFQ}}{\varphi \to \neg\psi}\,\to\mathrm{I}^4 \qquad [\neg(\varphi \to \neg\psi)]_3}{\bot}\,\to\mathrm{E}}{\varphi}\,\mathrm{RAA}^5 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg\psi]_1 \qquad [\varphi]_2}{\varphi \to \neg\psi}\,\to\mathrm{I}^2 \qquad [\neg(\varphi \to \neg\psi)]_3}{\bot}\,\to\mathrm{E}}{\psi}\,\mathrm{RAA}^1}{\varphi \wedge \psi}\,\wedge I}{\neg(\varphi \to \neg\psi) \to (\varphi \wedge \psi)}\,\to\mathrm{I}^3$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]_1}{\psi}\,\wedge\mathrm{E} \qquad \dfrac{\dfrac{[\varphi \wedge \psi]_1}{\varphi}\,\wedge\mathrm{E} \qquad [\varphi \to \neg\psi]_2}{\neg\psi}\,\to\mathrm{I}}{\bot}}{\neg(\varphi \to \neg\psi)}\,\to\mathrm{I}^2}{(\varphi \wedge \psi) \to \neg(\varphi \to \neg\psi)}\,\to\mathrm{I}^1$$

(f) $\varphi \to (\psi \to (\varphi \wedge \psi))$

$$\dfrac{\dfrac{\dfrac{[\varphi]^1 \qquad [\psi]^2}{\varphi \wedge \psi}\,\wedge I}{\psi \to (\varphi \wedge \psi)}\,\to I^2}{\varphi \to (\psi \to (\varphi \wedge \psi))}\,\to I^1$$

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Show that the following propositions are derivable:

(a) $(\varphi \to \neg\varphi) \to \neg\varphi$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \to \neg\varphi]_1 \qquad [\varphi]_2}{\neg\varphi}\,\to\mathrm{E} \qquad [\varphi]_2}{\bot}\,\to\mathrm{E}}{\neg\varphi}\,\to\mathrm{I}^2}{(\varphi \to \neg\varphi) \to \neg\varphi}\,\to I_1$$

(b) $[\varphi \to (\psi \to \sigma)] \leftrightarrow [\psi \to (\varphi \to \sigma)]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\psi \to (\varphi \to \sigma)]_3 \qquad [\psi]_1}{\varphi \to \sigma}\,\to\mathrm{E} \qquad [\varphi]_2}{\sigma}\,\to\mathrm{E}}{\psi \to \sigma}\,\to\mathrm{I}^1}{\varphi \to (\psi \to \sigma)}\,\to\mathrm{I}^2}{[\psi \to (\varphi \to \sigma)] \to [\varphi \to (\psi \to \sigma)]}\,\to\mathrm{I}^3 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \to (\psi \to \sigma)]_3 \qquad [\varphi]_1}{\psi \to \sigma}\,\to\mathrm{E} \qquad [\psi]_2}{\sigma}\,\to\mathrm{E}}{\varphi \to \sigma}\,\to\mathrm{I}^1}{\psi \to (\varphi \to \sigma)}\,\to\mathrm{I}^2}{[\varphi \to (\psi \to \sigma)] \to [\psi \to (\varphi \to \sigma)]}\,\to\mathrm{I}^3}{[\varphi \to (\psi \to \sigma)] \leftrightarrow [\psi \to (\varphi \to \sigma)]}$$

(c) $(\varphi \to \psi) \wedge (\varphi \to \neg\psi) \to \neg\varphi$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(\varphi \to \psi) \wedge (\varphi \to \neg\psi)]_2}{\varphi \to \psi}\,\wedge\mathrm{E} \qquad [\varphi]_1}{\psi}\,\to\mathrm{E} \qquad \dfrac{\dfrac{[(\varphi \to \psi) \wedge (\varphi \to \neg\psi)]_2}{\varphi \to \neg\psi}\,\wedge\mathrm{E} \qquad [\varphi]_1}{\neg\psi}\,\to\mathrm{E}}{\bot}\,\to\mathrm{E}}{\neg\varphi}\,\to\mathrm{I}^1}{(\varphi \to \psi) \wedge (\varphi \to \neg\psi) \to \neg\varphi}\,\to\mathrm{I}^2$$

(d) $(\varphi \to \psi) \to [(\varphi \to (\psi \to \sigma)) \to (\varphi \to \sigma)]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \to \psi]_1 \qquad [\varphi]_2}{\psi}\,\to E \qquad \dfrac{[\varphi]_2 \qquad [(\varphi \to (\psi \to \sigma))]_3}{\psi \to \sigma}\,\to E}{\sigma}\,\to E}{\varphi \to \sigma}\,\to I_2}{(\varphi \to (\psi \to \sigma)) \to (\varphi \to \sigma)}\,\to I_3}{(\varphi \to \psi) \to [(\varphi \to (\psi \to \sigma)) \to (\varphi \to \sigma)]}\,\to I_1$$

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Show:

(a) $\varphi \vdash \neg(\neg\varphi \wedge \psi)$

$$\dfrac{\dfrac{\dfrac{[\neg\varphi \wedge \psi]_1}{\neg\varphi}\,\wedge\mathrm{E} \qquad \varphi}{\bot}\,\to\mathrm{E}}{\neg(\neg\varphi \wedge \psi)}\,\to I_1$$

(b) $\neg(\varphi \wedge \neg\psi), \varphi \vdash \psi$

$$\dfrac{\dfrac{\dfrac{[\neg\psi]_1 \qquad \varphi}{\varphi \wedge \neg\psi}\,\wedge\mathrm{I} \qquad \neg(\varphi \wedge \neg\psi)}{\bot}\,\to\mathrm{E}}{\psi}\,\mathrm{RAA}_1$$

(c) $\neg\varphi \vdash (\varphi \to \psi) \leftrightarrow \neg\varphi$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\varphi]_2 \qquad [\varphi]_1}{\bot}\,\to\mathrm{E}}{\psi}\,\mathrm{EFQ}}{\varphi \to \psi}\,\to\mathrm{I}^1}{\neg\varphi \to (\varphi \to \psi)}\,\to\mathrm{I}^2 \qquad \dfrac{\dfrac{\dfrac{[\varphi \to \psi]_1 \qquad \neg\varphi}{(\varphi \to \psi) \wedge (\neg\varphi)}\,\wedge\mathrm{I}}{\neg\varphi}\,\wedge\mathrm{E}}{(\varphi \to \psi) \to \neg\varphi}\,\to\mathrm{I}^1}{(\varphi \to \psi) \leftrightarrow \neg\varphi}\,\wedge\mathrm{I}$$

(d) $\vdash \varphi \Rightarrow \vdash \psi \to \varphi$

Given $\vdash \varphi$, there exists a derivation $\genfrac{}{}{0pt}{}{D}{\varphi}$, with all hypotheses in D cancelled. Since $\genfrac{}{}{0pt}{}{D}{\varphi}, \psi$ are derivations, the definition of the set of derivations gives:

$$\dfrac{\psi \qquad \genfrac{}{}{0pt}{}{D}{\varphi}}{\psi \wedge \varphi}\,\wedge\mathrm{I}$$

From here we can apply derviation rules to prove $\psi \to \varphi$

$$\dfrac{\dfrac{\dfrac{[\psi]_1 \qquad \genfrac{}{}{0pt}{}{D}{\varphi}}{\psi \wedge \varphi}\,\wedge\mathrm{I}}{\varphi}\,\wedge\mathrm{E}}{\psi \to \varphi}\,\to\mathrm{I}_1$$

(e) $\neg\varphi \vdash \varphi \to \psi$

$$\dfrac{\dfrac{\dfrac{[\varphi]_1 \qquad \neg\varphi}{\bot}\,\to\mathrm{E}}{\psi}\,\mathrm{EFQ}}{\varphi \to \psi}\,\to I_1$$