Assignment #2
CIS 427/527

Group 2

January 18, 2016

1

There are three suspects for a murder: Adams, Brown, and Clark. Adams says: I didnt do it. The victim was an old acquaintance of Browns. But Clark hated him. Brown states I didnt do it. I didnt even know the guy. Besides I was out of town all that week. Clark says I didnt do it. I saw both Adams and Brown downtown with the victim that day; one of them must have done it. Assume that the two innocent men are telling the truth, but that the guilty man might not be. Who did it?

Solution

Since two of the men are telling the truth, we have a proposition of the form $P = (A \wedge B) \vee (A \wedge C) \vee (B \wedge C)$.

- If Adams is telling the truth, then Brown is lying ($A \implies \neg B$).
- If Brown is telling the truth, then both Adams and Clark are lying ($B \implies (\neg A \wedge \neg C)$).
- If Clark is telling the truth, then Brown is lying ($C \implies \neg B$).

If Brown were telling the truth, then P could never be satisfied, therefore Brown is lying, which makes him the killer.

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2. Show that $((\rightarrow \notin PROP$

Suppose $((\rightarrow \in X$ and X satisfies (i), (ii), (iii) of Definition 2.1.2. We claim that $Y = X \backslash \{((\rightarrow\}$ also satisfies (i), (ii), and (iii).

(i) $\perp, p_i \in Y$,
(ii) $\varphi, \psi \in Y$ and $(\varphi \Box \psi) \neq ((\rightarrow$, it is clear that $(\varphi \Box \psi) \in Y$
(iii)

Therefore X is not the smallest set satisfying (i), (ii), and (iii), so $((\rightarrow$ cannot belong to $PROP$.

7. **(a)** Determine the trees of the proposition in Exercise 1
(b) Determine the propositions with the following trees

9. Show that a proposition with n connectives has at most $2n + 1$ subformulas

Using Theorem 2.1.3 (Induction Principle for $PROP$)

Let $A(\varphi) := \varphi$ has n connectives and at most $2n + 1$ subformulas.

(i) $A(\perp)$ and $A(p_i)$ is true $\forall p_i \in PROP$

$\perp$ is atomic, and is itself a connective. $SUB(\perp) = \{\perp\}$, so $|\{\perp\}| = 1$. $1 \leq 2 \cdot 0 + 1$, thus $A(\perp)$.

Let $p_i \in PROP$ for some $i \in N$. Since p_i is atomic, p_i has 0 connectives, $SUB(p_i) = \{p_i\}$, and $|\{p_i\}| = 1$.

$$1 \leq 2 \cdot 0 + 1, \text{ thus } A(p_i).$$

(ii) $A(\varphi) \wedge A(\psi) \implies A((\varphi \Box \psi))$

Suppose $\varphi, \psi \in PROP$, with n, m connectives resp., and $A(\varphi), A(\psi)$
$|SUB(\varphi)| \leq 2n+1$ $\qquad$ $|SUB(\psi)| \leq 2M+1$
$SUB(\varphi \square \psi) = SUB(\varphi) \cup SUB(\psi) \cup \{\varphi\}$ (definition of SUB)

$$
\begin{aligned}
|SUB(\varphi \square \psi)| &\leq |SUB(\varphi)| + |SUB(\psi)| + 1 \\
|SUB(\varphi \square \psi)| &\leq 2n + 1 + 2m + 1 + 1 \\
|SUB(\varphi \square \psi)| &\leq 2(n + m + 1) + 1
\end{aligned}
$$

$(\varphi \square \psi)$ has $n+m+1$ connectives, and at most $2(n+m+1)$ subformulas, thus $A(\varphi), A(\psi) \implies A((\varphi \square \psi))$.

(iii) $A(\varphi) \implies A((\neg\varphi))$

Suppose $\varphi \in PROP$, with n connectives, and $A(\varphi)$, then $|SUB(\varphi)| \leq 2n+1$

$$
\begin{aligned}
SUB((\neg\varphi)) &= SUB(\varphi) \cup \{(\neg\varphi)\} \\
|SUB((\neg\varphi))| &\leq |SUB(\varphi)| + |\{(\neg\varphi)\} \\
|SUB((\neg\varphi))| &\leq 2n + 1 + 1 \\
|SUB((\neg\varphi))| &\leq 2(n+1) \\
|SUB((\neg\varphi))| &\leq 2(n+1) + 1
\end{aligned}
$$

$(\neg\varphi)$ has n+1 connectives, and at most $2(n+1)+1$ subformulas, thus $A(\varphi) \implies A((\neg\varphi))$.

From (i), (ii), (iii), and the induction principle for PROP, A proposition with n connectives has at most $2n+1$ subformulas.

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(a)

$(\neg\varphi \vee \psi) \iff (\psi \to \varphi)$	φ	ψ
1	0	0
0	0	1
1	1	0
1	1	1

Not a tautology.

(f)

$(\varphi \vee \neg\varphi)$	φ
1	0
1	1

Is a tautology.

(h)

$(\bot \to \varphi)$	φ
1	0
1	1

Is a tautology.

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