Assignment #2
CIS 427/527

Group 2

January 20, 2016

1

Since two of the men are telling the truth, we have a proposition of the form $P = (A \wedge B) \vee (A \wedge C) \vee (B \wedge C)$.

- If Adams is telling the truth, then Brown is lying ($A \implies \neg B$).
- If Brown is telling the truth, then both Adams and Clark are lying ($B \implies (\neg A \wedge \neg C)$).
- If Clark is telling the truth, then Brown is lying ($C \implies \neg B$).

If Brown were telling the truth, then P could never be satisfied, therefore Brown is lying, which makes him the killer.

2

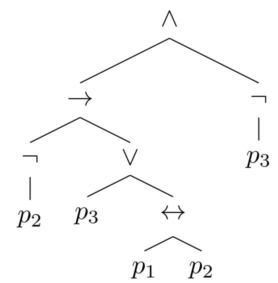
2. Show that $((\rightarrow \notin PROP$

Suppose $((\rightarrow \in X$ and X satisfies (i), (ii), (iii) of Definition 2.1.2. We claim that $Y = X \backslash \{((\rightarrow\}$ also satisfies (i), (ii), and (iii).

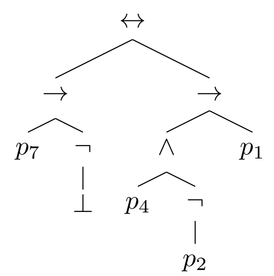
(i) $\perp, p_i \in Y$,
(ii) $\varphi, \psi \in Y$ and $(\varphi \Box \psi) \neq ((\rightarrow$, it is clear that $(\varphi \Box \psi) \in Y$
(iii) Similarly, $\varphi \in Y$ and $(\neg\varphi) \neq ((\rightarrow$, so $(\neg\varphi) \in Y$

Therefore X is not the smallest set satisfying (i), (ii), and (iii), so $((\rightarrow$ cannot belong to $PROP$.

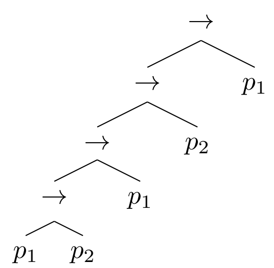
7. (a) Determine the trees of the proposition in Exercise 1
$(\neg p_2 \rightarrow (p_3 \vee (p_1 \leftrightarrow p_2))) \wedge \neg p_3$



$(p_7 \rightarrow \neg\perp) \leftrightarrow ((p_4 \wedge \neg p_2) \rightarrow p_1$



$(((p_1 \rightarrow p_2) \rightarrow p_1) \rightarrow p_2 \rightarrow p_1$



(b) Determine the propositions with the trees given

$\neg\neg\neg\bot$

$(p_0 \to \bot) \to ((p_0 \leftrightarrow p_1) \wedge p_5)$

$\neg((\neg p_1) \to (\neg p_1))$

9. Show that a proposition with n connectives has at most $2n + 1$ subformulas

Base Case A proposition P with zero connectives has $\leq 2(0) + 1 = 1$ subforumla(s).

Inductive Case Inductive Hypothesis: A proposition φ with n connectives has at most $2n + 1$ subformulas, where $s(\varphi)$ gives the number of subformulas (Def 2.1.3).

Assume propositions p, q with $n, m \leq n$ connectives resp., both of which have the property above. $s(p) \leq 2n + 1$ and $s(q) \leq 2m + 1$. Consider the proposition $r = (p \Box q)$. r must have $n + m + 1$ connectives. From the definition of s,

$$\begin{aligned} s(r) &= s(p) + s(q) + 1 \\ s(r) &= n + m + 1 \\ s(r) &\leq 2n + 1 \end{aligned}$$

$$\begin{aligned} s(r) &= s(p) \cup s(q) \cup \{r\} \\ |s(r)| &\leq |s(p)| + |s(q)| + |\{r\}| \\ |s(r)| &\leq 2n + 1 + 2m + 1 + 1 \\ |s(r)| &\leq 2(n + m + 1) + 1 \end{aligned}$$

Since the desired property holds for r the inductive hypothesis holds $\forall n \in N$.

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(a)

$(\neg\varphi \vee \psi) \iff (\psi \to \varphi)$	φ	ψ
1	0	0
0	0	1
1	1	0
1	1	1

Not a tautology.

(f)

$(\varphi \vee \neg\varphi)$	φ
1	0
1	1

Is a tautology.

(h)

$(\bot \to \varphi)$	φ
1	0
1	1

Is a tautology.

4

(a) $\varphi \models \varphi$

By definition of semantic entailment, $\varphi \models \varphi$ iff for all v : $[\![\varphi]\!]_v = 1 \implies [\![\varphi]\!]_v = 1$. Since $[\![\varphi]\!]_v = 1 \implies [\![\varphi]\!]_v = 1$ is a tautology, $\varphi \models \varphi$ holds.

(b) $\varphi \models \psi$ and $\psi \models \sigma \implies \varphi \models \sigma$

By definition of semantic entailment, we have $[\![\varphi]\!]_v = 1 \; \forall v \implies [\![\psi]\!]_v = 1$ and $[\![\psi]\!]_v = 1 \; \forall v \implies [\![\sigma]\!]_v = 1$, by the transitive property, $[\![\varphi]\!]_v = 1 \; \forall v \implies [\![\sigma]\!]_v = 1$.

(c) $\models \varphi \to \psi \iff \varphi \models \psi$

Since $\models \varphi \to \psi$, we know that $[\![\psi]\!]_v = 1 \; \forall v$ (because if $[\![\psi]\!]_v = 0$ for some v, then $\varphi \to \psi$ is not a tautology). Therefore, we can conclude that $[\![\varphi]\!]_v = 1 \to [\![\psi]\!]_v = 1$, since anytime $[\![\varphi]\!]_v = 1$, $[\![\psi]\!]_v = 1$.

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1. Show by algebraic means:

(i) $\models (\varphi \implies \psi) \iff (\neg\psi \implies \neg\varphi)$ contraposition

$$\varphi \implies \psi \approx \neg\varphi \lor \psi \text{Thm 2.3.4.b}$$
$$\varphi \implies \psi \approx \psi \lor \neg\varphi \text{commutativity}$$
$$\varphi \implies \psi \approx \neg\psi \implies \neg\varphi \text{Thm 2.3.4.b}$$

(v) $\models \neg(\varphi \land \neg\varphi)$

$$\neg(\varphi \land \neg\varphi) \approx \neg\varphi \lor \neg(\neg\varphi) deMorgan's$$
$$\neg(\varphi \land \neg\varphi) \approx \neg\varphi \lor \varphi DoubleNegationlaw$$
$$\neg(\varphi \land \neg\varphi) \approx \varphi \implies \varphi Thm2.3.4.b$$
$$\neg(\varphi \land \neg\varphi) \approx \top (Notsurewhichruleforthis)$$

6

$\# = \neg(\neg(\neg p \lor \neg q) \lor \neg(p \lor q))$

7

A conjunctive normal form is a tautology iff every clause is also a tautology.

8

9

$P \to Q$**:** True when P is false; false when P is true and Q is false.
$P \lor Q \to P \land Q$**:** True when $P \iff Q$; false otherwise.
$\neg(P \lor Q \lor R)$**:** True when P, Q, R are all false; false when any P, Q, R are true.
$\neg(P \land Q) \land \neg(Q \lor R) \land (P \lor R)$**:** True when P is true, Q is false, R is false; false when P is false and R is false.