Assignment #5
CIS 427/527

Group 2

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2.2.1

Which of the following strings are formulas in predicate logic?

Solution

(a),(b),(f),(g) are formulas.

(c) isn't, as $f(m)$ is a term.
(d) isn't, as B is expecting two terms, yet $B(m, x)$ is a formula.
(e) isn't, as $B(m)$ doesn't have enough arguments.
(h) isn't, as $B(x)$ doesn't have enough arguments.

2.5.3

Provide proofs for the following sequents:

Solution

(a) $\forall x P(x) \vdash \forall y P(y)$

$$\dfrac{\dfrac{\forall x P(x)}{P(x)}}{\forall y P(y)}$$

(b) $\forall x(P(x) \to Q(x)) \vdash (\forall x \neg Q(x)) \to (\forall x \neg P(x))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x(P(x) \to Q(x))}{P(x_0) \to Q(x_0)}\,\forall\mathrm{E} \qquad [P(x)]^1}{Q(x_0)}\,\to\mathrm{E} \qquad \dfrac{[\forall x \neg Q(x)]^2}{\neg Q(x)}\,\forall\mathrm{E}}{\bot}\,\to\mathrm{E}}{\neg P(x)}\,\to\mathrm{I}^1}{\forall x \neg P(x)}\,\forall\ \mathrm{I}}{(\forall x \neg Q(x)) \to (\forall x \neg P(x))}\,\to\mathrm{I}^2$$

(c) $\forall x(P(x) \to \neg Q(x)) \vdash \neg(\exists x(P(x) \wedge Q(x)))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x(P(x) \to \neg Q(x))}{P(x_0) \to \neg Q(x_0)}\,\forall\mathrm{E} \qquad \dfrac{[\exists x(P(x) \wedge Q(x))]^1 \qquad \dfrac{\dfrac{[P(x) \wedge Q(x)]^2}{P(x)}}{P(x_0)}}{P(x_0)}\,\exists\mathrm{E}^2}{\neg Q(x_0)}\,\to\mathrm{E} \qquad \dfrac{[\exists x(P(x) \wedge Q(x))]^1 \qquad \dfrac{\dfrac{[P(x) \wedge Q(x)]^3}{Q(x)}}{Q(x_0)}}{Q(x_0)}\,\exists\mathrm{E}^3}{\bot}\,\to\mathrm{E}}{\neg(\exists x(P(x) \wedge Q(x)))}\,\to\mathrm{I}^1$$

2.5.11

Prove the following sequents in predicate logic:

Solution

(a) $\forall x \forall y \forall z (S(x,y) \land S(y,z) \to S(x,z)), \forall x \neg S(x,x) \vdash \forall x \forall y (S(x,y) \to \neg S(y,x))$

$$
\begin{prooftree}
\AxiomC{$\forall x \neg S(x,x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg S(x_0,x_0)$}
\AxiomC{$\forall x \forall y \forall z (S(x,y) \land S(y,z) \to S(x,z))$}
\RightLabel{$\forall$E}
\UnaryInfC{$S(x_0,y_0) \land S(y_0,x_0) \to S(x_0,x_0)$}
\AxiomC{$[S(x,y)]^1$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x \forall y S(x,y)$}
\RightLabel{$\forall$E}
\UnaryInfC{$S(x_0,y_0)$}
\AxiomC{$[S(y,x)]^2$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall y \forall x S(y,x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$S(y_0,x_0)$}
\RightLabel{$\to$E}
\BinaryInfC{$S(x_0,y_0) \land S(y_0,x_0)$}
\RightLabel{$\to$E}
\BinaryInfC{$S(x_0,x_0)$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I^1}
\UnaryInfC{$S(x,y) \to \neg S(y,x)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x \forall y (S(x,y) \to \neg S(y,x))$}
\end{prooftree}
$$

(b) $\forall x (P(x) \lor Q(x)), \exists x \neg Q(x), \forall x (R(x) \to \neg P(x)) \vdash \exists x \neg R(x)$

$$
\begin{prooftree}
\AxiomC{$\forall x (R(x) \to \neg P(x))$}
\RightLabel{$\forall$E}
\UnaryInfC{$R(x_0) \to \neg P(x_0)$}
\AxiomC{$[R(x_0)]^1$}
\RightLabel{$\to$E}
\BinaryInfC{$\neg P(x_0)$}
\AxiomC{$\exists x \neg Q(x)$}
\AxiomC{$[\neg Q(x)]^2$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x \neg Q(x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg Q(x_0)$}
\RightLabel{$\exists$E^2}
\BinaryInfC{$\neg Q(x_0)$}
\AxiomC{$\forall x (P(x) \lor Q(x))$}
\RightLabel{$\forall$E}
\UnaryInfC{$P(x_0) \lor Q(x_0)$}
\RightLabel{$\lor$E}
\BinaryInfC{$P(x_0)$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{$\to$I^1}
\UnaryInfC{$\neg R(x_0)$}
\RightLabel{$\exists$I}
\UnaryInfC{$\exists x \neg R(x)$}
\end{prooftree}
$$

(c) $\forall x (P(x) \to (Q(x) \lor R(x))), \neg \exists x (P(x) \land R(x)) \vdash \forall x P(x) \to Q(x)$

$$
\begin{prooftree}
\AxiomC{$[\neg Q(x)]^2$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x \neg Q(x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg Q(x_0)$}
\AxiomC{$\forall x (P(x) \to (Q(x) \lor R(x)))$}
\RightLabel{$\forall$E}
\UnaryInfC{$P(x_0) \to (Q(x_0) \lor R(x_0))$}
\AxiomC{$[P(x)]^1$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x P(x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$P(x_0)$}
\RightLabel{$\to$E}
\BinaryInfC{$Q(x_0) \lor R(x_0)$}
\RightLabel{$\lor$E}
\BinaryInfC{$R(x_0)$}
\AxiomC{$[P(x)]^1$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x P(x)$}
\RightLabel{$\forall$E}
\UnaryInfC{$P(x)_0$}
\RightLabel{$\land$I}
\BinaryInfC{$P(x_0) \land R(x_0)$}
\AxiomC{$\neg \exists x (P(x) \land R(x))$}
\UnaryInfC{$\forall x \neg (P(x) \land R(x))$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg (P(x_0) \land R(x_0))$}
\UnaryInfC{$(P(x_0) \land R(x_0)) \to \bot$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{RAA^2}
\UnaryInfC{$Q(x)$}
\RightLabel{$\to$I^1}
\UnaryInfC{$P(x) \to Q(x)$}
\RightLabel{$\forall$I}
\UnaryInfC{$\forall x P(x) \to Q(x)$}
\end{prooftree}
$$

(e) $(\exists x P(x)) \to \forall y P(y) \vdash \exists x \forall y ((P(x) \to P(y)) \land (P(y) \land P(x)))$
(f) $\exists x (P(x) \land Q(x)), \forall y (P(y) \to R(y)) \vdash \exists x R(x) \land Q(x)$

2.6.1

Consider the formula

$$\phi = \forall x \ \exists y \ Q(g(x,y), g(y,y), z)$$

Find two models M and M' with respective environments l and l' such that $M \models_l \phi$ but $M' \not\models_l \phi$

Solution

M
$A = \{a, b\}$
$Q^M = \{(a, a, b)\}$
$g^M(x, y) = a$ for all input

$z^M = b$

Then, $\forall x\ \forall y\ Q(g(x,y), g(y,y), z) = Q(a,a,b)$, which holds, so $M \models \phi$.

M'
$A = \{a, b\}$
$Q^{M'} = \{(a,a,c)\}$
$g^{M'}(x,y) = c$ for all input
$z^{M'} = a$

Then, $\forall x\ \forall y\ Q(g(x,y), g(y,y), z) = Q(c,c,a)$, but $(c,c,a) \notin Q^{M'}$, so $M' \not\models \phi$.

2.6.2

Consider the sentence

$$\phi = \forall x\ \exists y\ \exists z\ (P(x,y) \wedge P(z,y) \wedge (P(x,z) \rightarrow P(z,x)))$$

Which of the following models satisfies ϕ?

(a) $P^M = \{(m,n) | m < n\}$
(b) $P^{M'} = \{(m, 2*m) | m$ natural number$\}$
(c) $P^{M''} = \{(m,n) | m < n+1\}$

Solution

(a) This model does not satisfy ϕ, because we either need to force $P(x,z)$ to be false by requiring z to be smaller than x (in which case we can escape the natural numbers), or by having $x < z \wedge z < x$, which cannot happen.
(b) Yes, because the first two properties say $y = 2*x$ and $y = 2*z$, which means $x = z$ making $P(x,z)$ always false.
(c) Yes, let $y = z = x$, then all the properties hold.

2.6.3

Let P be a predicate with two arguments. Find a model M which satisfies $\forall x\ \neg P(x,x)$. Find also a model M' such that $M' \not\models \forall x\ \neg P(x,x)$.

Solution

- Let $P^M = \{(x,y) | x < y\}$, then we have $\forall x\ \neg P(x,x)$.
- Let $P^{M'} = \{(x,y) | x = y\}$, then we have $M' \not\models \forall x\ \neg P(x,x)$ as desired.

2.7.5

Show $\forall x(P(x) \bigvee Q(x)) \not\models \forall x P(x) \bigvee \forall x Q(x)$. Thus, find a model which satisfies $\forall x(P(x) \bigvee Q(x))$, but not $\forall x P(x) \bigvee \forall x Q(x)$

Solution

- Let $A =$ natural numbers, and let $P^M = \{x | x \leq 5\}$, $Q^M = \{x | x > 5\}$.

Then $\forall x. x \leq 5 \bigvee x > 5$ holds, but it is not true that $\forall x. x \leq 5 \bigvee \forall x. x > 5$.