This document outlines how to use the ratios given by the Saha equation to calculate the degrees of ionization, where the i^th degree of ionization is denoted y^i. This is written under the assumption the reader has read Math_Capstone_Paper.pdf

Hydrogen

Initially considering the hydrogen, the Saha equation can be expressed as

$$\frac{n_H^1}{n_H^0} = C T^{\frac{3}{2}} \frac{2 B_H^1}{B_H^0} \frac{e^{-\frac{\chi_H^0}{kT}}}{N_e}$$

where C is the combination of constant terms.

Since we never need to consider the zeroth degree of ionization, we only need to consider the first degree of ionization, given by

$$y_H^1 = \frac{n_H^1}{n_H^0 + n_H^1} = \frac{1}{\frac{1}{\frac{n_H^1}{n_H^0}} + 1}$$

Although this notation may seem cumbersome, it will make more sense when the element being considered has many ionization levels.

Helium

Helium has 3 ionization levels (0^{th}, 1^{st}, and 2^{nd}), we need to express the Saha equation for $\frac{n_{He}^1}{n_{He}^0}$ and $\frac{n_{He}^2}{n_{He}^1}$

$$\frac{n_{He}^1}{n_{He}^0}=C T^{\frac{3}{2}} \frac{2 B_{He}^1}{B_{He}^0} \frac{e^{-\frac{\chi_{He}^0}{k T}}}{N_e} \quad \text{and} \quad \frac{n_{He}^2}{n_{He}^1}=C T^{\frac{3}{2}} \frac{2 B_{He}^2}{B_{He}^1} \frac{e^{-\frac{\chi_{He}^1}{k T}}}{N_e}$$

We can express the first and second degrees of ionization as

$$y_{He}^1=\frac{n_{He}^1}{n_{He}^0+n_{He}^1+n_{He}^2}=\frac{1}{\frac{1}{\frac{n_{He}^1}{n_{He}^0}}+1+\frac{n_{He}^2}{n_{He}^1}} \quad \text{and} \quad y_{He}^2=\frac{n_{He}^2}{n_{He}^0+n_{He}^1+n_{He}^2}=\frac{1}{\frac{1}{\frac{n_{He}^1}{n_{He}^0}\frac{n_{He}^2}{n_{He}^1}}+\frac{1}{\frac{n_{He}^2}{n_{He}^1}}+1}$$

There exists a pattern. Consider the terms of the sum being indexed 0, 1, and 2. For the first degree of ionization, at index 0 is the term $\frac{1}{\frac{n_{He}^1}{n_{He}^0}}$, at index 1 is term 1, and at index 2 is term $\frac{n_{He}^2}{n_{He}^1}$. For the first degree of ionization, at index 0 is the term $\frac{1}{\frac{n_{He}^1}{n_{He}^0}\frac{n_{He}^2}{n_{He}^1}}$, at index 1 is $\frac{1}{\frac{n_{He}^2}{n_{He}^1}}$, and at index 2 is 1.

The pattern can be seen by observing the following:
-At the index corresponding to degree of ionization, the term is simply a 1
-At the index/indexes before the index corresponding to the degree of ionization, the denominator of the term is expressed as the ratios obtained from the Saha equation. The first index less than the degree of ionization is 1 over the ratio whose numerator corresponding to the current degree being calculated. Using the example of y_{He}^2, the term at this index is $\frac{1}{\frac{n_{He}^2}{n_{He}^1}}$. The next index less than the degree of ionization being computed must include another ratio in its denominator. This ratio is the one with the numerator corresponding to the current degree being calculated minus one. Continuing with our example, it is the term $\frac{1}{\frac{n_{He}^1}{n_{He}^0}\frac{n_{He}^2}{n_{He}^1}}$.
-At the index/indexes after the index corresponding to the degree of ionization, the term is expressed as the ratios obtained from the Saha equation. The first index greater than the degree of ionizaion is the ratio whose numerator is the current degree being calculated plus one. Using the example of y_{He}^1, the term at this index is $\frac{n_{He}^2}{n_{He}^1}$. The index greater than the degree of ionization being computed must include another ratio whose numerator is the current degree being calculated plus two. These ratio does not exist for the ionization degrees of helium.

Carbon

To see this pattern progress, we will consider carbon which has 7 ionization levels (0^{th} to 6^{th}). The Saha equations progress in a straightforward manner, however it is the ionization degrees that needs to be looked at. Taking the first, second, and third degrees of ionization, we have

$$y_C^1=\frac{n_C^1}{n_C^0+n_C^1+n_C^2+n_C^3+n_C^4+n_C^5+n_C^6}=\frac{1}{\frac{n_C^0}{n_C^1}+\frac{n_C^1}{n_C^1}+\frac{n_C^2}{n_C^1}+\frac{n_C^3}{n_C^1}+\frac{n_C^4}{n_C^1}+\frac{n_C^5}{n_C^1}+\frac{n_C^6}{n_C^1}}=$$

$$=\frac{1}{\frac{1}{\frac{n_C^1}{n_C^0}}+1+\frac{n_C^2}{n_C^1}+\frac{n_C^2}{n_C^1}\frac{n_C^3}{n_C^2}+\frac{n_C^2}{n_C^1}\frac{n_C^3}{n_C^2}\frac{n_C^4}{n_C^3}+\frac{n_C^2}{n_C^1}\frac{n_C^3}{n_C^2}\frac{n_C^4}{n_C^3}\frac{n_C^5}{n_C^4}+\frac{n_C^2}{n_C^1}\frac{n_C^3}{n_C^2}\frac{n_C^4}{n_C^3}\frac{n_C^5}{n_C^4}\frac{n_C^6}{n_C^5}}$$

$$y_C^2=\frac{n_C^2}{n_C^0+n_C^1+n_C^2+n_C^3+n_C^4+n_C^5+n_C^6}=\frac{1}{\frac{n_C^0}{n_C^2}+\frac{n_C^1}{n_C^2}+\frac{n_C^2}{n_C^2}+\frac{n_C^3}{n_C^2}+\frac{n_C^4}{n_C^2}+\frac{n_C^5}{n_C^2}+\frac{n_C^6}{n_C^2}}=$$

$$=\frac{1}{\frac{1}{\frac{n_C^1}{n_C^0}\frac{n_C^2}{n_C^1}}+\frac{1}{\frac{n_C^2}{n_C^1}}+1+\frac{n_C^3}{n_C^2}+\frac{n_C^3}{n_C^2}\frac{n_C^4}{n_C^3}+\frac{n_C^3}{n_C^2}\frac{n_C^4}{n_C^3}\frac{n_C^5}{n_C^4}+\frac{n_C^3}{n_C^2}\frac{n_C^4}{n_C^3}\frac{n_C^5}{n_C^4}\frac{n_C^6}{n_C^5}}$$

$$y_C^3=\frac{n_C^3}{n_C^0+n_C^1+n_C^2+n_C^3+n_C^4+n_C^5+n_C^6}=\frac{1}{\frac{n_C^0}{n_C^3}+\frac{n_C^1}{n_C^3}+\frac{n_C^2}{n_C^3}+\frac{n_C^3}{n_C^3}+\frac{n_C^4}{n_C^3}+\frac{n_C^5}{n_C^3}+\frac{n_C^6}{n_C^3}}=$$

$$=\frac{1}{\frac{1}{\frac{n_C^1}{n_C^0}\frac{n_C^2}{n_C^1}\frac{n_C^3}{n_C^2}}+\frac{1}{\frac{n_C^2}{n_C^1}\frac{n_C^3}{n_C^2}}+\frac{1}{\frac{n_C^3}{n_C^2}}+1+\frac{n_C^4}{n_C^3}+\frac{n_C^4}{n_C^3}\frac{n_C^5}{n_C^4}+\frac{n_C^4}{n_C^3}\frac{n_C^5}{n_C^4}\frac{n_C^6}{n_C^5}}$$

We can see how these terms change with each degree. The fourth, fifth, and sixth degrees of ionization proceed in a similar fashion. Using the knowledge of the current degree of ionization being calculated, and how the terms change at the indexes before and after the index of the current degree, the degree of ionization can solved in software.

Since this pattern remains the same regardless of the number of ionization levels an element has, many elements can be included in the iterative method using Saha's equations.