

SAHA EQUATION OF STATE
SAHA EQUATION REFORMULATION DOCUMENT
ZACHARIAH SCHRECENGOST

1 Saha Equation

The general Saha equation is given by

$$\frac{n_i^{r+1}}{n_i^r} = \left(\frac{2\pi m_e}{h^2} \right)^{\frac{3}{2}} (kT)^{\frac{3}{2}} \frac{2B_i^{r+1} \exp\left(\frac{-\chi_i^r}{kT}\right)}{B_i^r N_e} \quad (1)$$

where for element i and ionization r . For ease of expression, we will take

$$M_{i^{r+1}} = \left(\frac{2\pi m_e}{h^2} \right)^{\frac{3}{2}} \frac{2B_i^{r+1}}{B_i^r} \quad (2)$$

So the Saha equation is re-expressed as

$$\frac{n_i^{r+1}}{n_i^r} = M_{i^{r+1}} (kT)^{\frac{3}{2}} \exp\left(\frac{-\chi_i^r}{kT}\right) \frac{1}{N_e} \quad (3)$$

This is the form we will manipulate throughout this document.

2 Pressure

The general formula for pressure is given by

$$P = \left(\sum_i n_i + N_e \right) kT \quad (4)$$

where n_i is the number of atoms of element i . We can rearrange this to yield

$$kT = \frac{P}{\left(\sum_i n_i + N_e \right)} \quad (5)$$

as well as

$$\frac{1}{kT} = \frac{\left(\sum_i n_i + N_e \right)}{P} \quad (6)$$

We can substitute these into equation 3 which gives us

$$\frac{n_i^{r+1}}{n_i^r} = M_{i^{r+1}} \left(\frac{P}{\left(\sum_i n_i + N_e \right)} \right)^{\frac{3}{2}} \exp\left(\frac{-\chi_i^r}{P} \left(\sum_i n_i + N_e \right)\right) \frac{1}{N_e} \quad (7)$$

and our new "Saha constant" is

$$A_{i^{r+1}} = M_{i^{r+1}} P^{\frac{3}{2}} \quad (8)$$

The Saha equation using pressure, instead of temperature, is given by

$$\frac{n_i^{r+1}}{n_i^r} = A_{i^{r+1}} \left(\sum_i n_i + N_e \right)^{-\frac{3}{2}} \exp\left(\frac{-\chi_i^r}{P} \left(\sum_i n_i + N_e \right)\right) N_e^{-1} \quad (9)$$

2.1 Hydrogen

Initially we will consider a mixture of pure hydrogen. The Saha equation becomes

$$\frac{n_H^1}{n_H^0} = A_{H^1} (n_H + N_e)^{-\frac{3}{2}} \exp\left(\frac{-\chi_H^0}{P} (n_H + N_e)\right) N_e^{-1} \quad (10)$$

We will use the following to simplify our expression

$$s_{H^1} = \frac{n_H^1}{n_H^0} \quad (11)$$

$$B_H(N_e) = (n_H + N_e)^{-\frac{3}{2}} \quad (12)$$

$$C_{H^1}(N_e) = \exp\left(\frac{-\chi_H^0}{P} (n_H + N_e)\right) \quad (13)$$

$$D_H(N_e) = N_e^{-1} \quad (14)$$

Now, omitting that B_H , C_{H^1} , and D_H are explicitly functions of N_e

$$s_{H^1} = A_{H^1} B_H C_{H^1} D_H \quad (15)$$

Recall

$$N_e = n_h * \nu_e(H) \quad (16)$$

where

$$\nu_e(H) = 1.0 * y_H^1 \quad (17)$$

and

$$y_H^1 = \frac{1}{\frac{1}{\frac{n_H^1}{n_H^0}} + 1} = (s_{H^1}^{-1} + 1)^{-1} \quad (18)$$

Therefore

$$N_e = n_h * (s_{H^1}^{-1} + 1)^{-1} \rightarrow n_h * (s_{H^1}^{-1} + 1)^{-1} - N_e = 0 \quad (19)$$

We will consider the function, $G(N_e)$ given by

$$G(N_e) = n_h * (s_{H^1}^{-1} + 1)^{-1} - N_e = 0 \quad (20)$$

To solve for N_{e1} we will use Newton's Method, which is

$$N_{e1} = N_e - \frac{G(N_e)}{G'(N_e)} \quad (21)$$

We already have an expression for $G(N_e)$, so now we must find its derivative. We have

$$G'(N_e) = -n_h * (s_{H^1}^{-1} + 1)^{-2} [s_{H^1}^{-1} + 1]' - 1 \quad (22)$$

where

$$[s_{H^1}^{-1} + 1]' = -s_{H^1}^{-2} [s_{H^1}]' \quad (23)$$

We can therefore simplify equation 22 as

$$G'(N_e) = n_h * (s_{H^1} + 1)^{-2} * [s_{H^1}]' - 1 \quad (24)$$

Using equation 15, we have

$$[s_{H^1}]' = A_{H^1} (B_H' C_{H^1} D_H + B_H (C_{H^1}' D_H + C_{H^1} D_H')) \quad (25)$$

All we need now is to differentiate our ‘saha constants’ and substitute them into equation 25. Therefore

$$B'_H = \left[(n_H + N_e)^{-\frac{3}{2}} \right]' = -\frac{3}{2} (n_H + N_e)^{-\frac{5}{2}} \quad (26)$$

$$C'_{H^1} = \left[\exp \left(\frac{-\chi_H^0}{P} (n_H + N_e) \right) \right]' = C_{H^1} \left[\frac{-\chi_H^0}{P} (n_H + N_e) \right]' = -\frac{\chi_H^0}{P} C_{H^1} \quad (27)$$

$$D'_H = [N_e^{-1}]' = -N_e^{-2} \quad (28)$$

We now have all of the pieces that we need to compute Newton’s Method. That is

$$N_{e1} = N_e - \frac{n_h * (s_{H^1}^{-1} + 1)^{-1} - N_e}{n_h * (s_{H^1} + 1)^{-2} * A_{H^1} (B'_H C_{H^1} D_H + B_H (C'_{H^1} D_H + C_{H^1} D'_H)) - 1} \quad (29)$$

2.2 Hydrogen and Helium

Now we will take a mixture of hydrogen and helium. Considering equation 9, we have the following three equations

$$\frac{n_H^1}{n_H^0} = A_{H^1} (n_H + n_{He} + N_e)^{-\frac{3}{2}} \exp \left(\frac{-\chi_H^0}{P} (n_H + n_{He} + N_e) \right) N_e^{-1} \quad (30)$$

$$\frac{n_{He}^1}{n_{He}^0} = A_{He^1} (n_H + n_{He} + N_e)^{-\frac{3}{2}} \exp \left(\frac{-\chi_{He}^0}{P} (n_H + n_{He} + N_e) \right) N_e^{-1} \quad (31)$$

$$\frac{n_{He}^2}{n_{He}^1} = A_{He^2} (n_H + n_{He} + N_e)^{-\frac{3}{2}} \exp \left(\frac{-\chi_{He}^1}{P} (n_H + n_{He} + N_e) \right) N_e^{-1} \quad (32)$$