This document outlines how to use the ionization fraction given by the convergence of the iterative method using the Saha equation. The intention is to direct the reader toward incorporating more elements into these calculations. This is written under the assumption the reader has read Math_Capstone_Paper.pdf

Hydrogen

The current implementation only considers hydrogen. Our calculations of pressure, internal energy, and specific heat follow the methods outlined in http://www.astro.princeton.edu/~gk/A403/ioniz.pdf

Helium

The main purpose of this document is to share the current thoughts on how to move forward and include the ionization of more elements. The method described here is not complete, however hopefully it can serve as a starting place.

Initially, we will consider pressure. The formula becomes

$$P_g = (n_H + n_{He} + n_e) kT$$

where the number of electrons is due to singly ionized hydrogen, singly ionized helium, and doubly ionized helium. Pressure can thus be rewritten as

$$P_g = kT\left[(n_H + n_H^1) + (n_{He} + n_{He}^1 + 2n_{He}^2)\right] = kT\left[n_H\left(1 + \frac{n_H^1}{n_H}\right) + n_{He}\left(1 + \frac{n_{He}^1}{n_{He}} + 2\frac{n_{He}^2}{n_{He}}\right)\right]$$

Notice that the fractional terms are the degrees of ionization. Denoting the k^{th} degree of ionization of the i^{th} element as y_i^k we have

$$P_g = kT\left[n_H(1 + y_H^1) + n_{He}(1 + y_{He}^1 + 2y_{He}^2)\right]$$

Finally, the number density of the i^{th} element, denoted n_i, is given by

$$n_i = \frac{\rho x_i}{A_i} N_o$$

where ρ is the density, x_i is the grams of element *i* per grams of mixture, A_i is the molecular weight of element *i*, and N_o is Avogadro's number. Therefore, the pressure is given by

$$P_g = kT\rho N_o\left[\frac{x_H}{A_H}(1 + y_H^1) + \frac{x_{He}}{A_{He}}(1 + y_{He}^1 + 2y_{He}^2)\right]$$

Now considering internal energy, we have

$$U_g = 1.5\,P_g + n_e \chi$$

where the second term of the sum is given by

$$n_e \chi = n_H^1 \chi_H^0 + n_{He}^1 \chi_{He}^0 + 2 n_{He}^2 \chi_{He}^1$$

Notice

$$y_i^k = \frac{n_i^k}{n_i} \rightarrow n_i^k = y_i^k n_i$$

Therefore

$$n_e \chi = y_H^1 n_H \chi_H^0 + y_{He}^1 n_{He} \chi_{He}^0 + 2\, y_{He}^2 n_{He} \chi_{He}^1 = n_H\, y_H^1 \chi_H^0 + n_{He} \left(y_{He}^1 \chi_{He}^0 + 2\, y_{He}^2 \chi_{He}^1 \right)$$

Using our expressions for number density we have

$$n_e \chi = \frac{\rho\, x_H}{A_H} N_o\, y_H^1 \chi_H^0 + \frac{\rho\, x_{He}}{A_{He}} N_o \left(y_{He}^1 \chi_{He}^0 + 2\, y_{He}^2 \chi_{He}^1 \right) = \rho N_o \left(\frac{x_H}{A_H} y_H^1 \chi_H^0 + \frac{x_{He}}{A_{He}} \left(y_{He}^1 \chi_{He}^0 + 2\, y_{He}^2 \chi_{He}^1 \right) \right)$$

Therefore, the internal energy is given by

$$U_g = 1.5\,P_g + \rho N_o \left(\frac{x_H}{A_H} y_H^1 \chi_H^0 + \frac{x_{He}}{A_{He}} \left(y_{He}^1 \chi_{He}^0 + 2\, y_{He}^2 \chi_{He}^1 \right) \right)$$

The specific heat including helium still must be worked out.