Saha Equation of State

Saha Equation Reformulation Document

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1 Saha Equation

The general Saha equation is given by

$$\frac{n_i^{r+1}}{n_i^r} = \left(\frac{2\pi m_e k}{h^2}\right)^{\frac{3}{2}} T^{\frac{3}{2}} \frac{2B_i^{r+1} exp\left(\frac{-\chi_i^r}{kT}\right)}{B_i^r N_e} \tag{1}$$

where for element i and ionization r. For ease of expression, we will take

$$M_{i^{r+1}} = \left(\frac{2\pi m_e}{h^2}\right)^{\frac{3}{2}} \frac{2B_i^{r+1}}{B_i^r} \tag{2}$$

So the Saha equation is re-expressed by

$$\frac{n_i^{r+1}}{n_i^r} = M_{i^{r+1}} (kT)^{\frac{3}{2}} exp\left(\frac{-\chi_i^r}{kT}\right) \frac{1}{N_e} \tag{3}$$

This is the form we will manipulate throughout this document.

2 Pressure

The general formula for pressure is given by

$$P = \left(\sum_i n_i + N_e\right) kT \tag{4}$$

where n_i is the number of atoms of element i. We can rearrange this to yield

$$kT = \frac{P}{\left(\sum_i n_i + N_e\right)} \tag{5}$$

as well as

$$\frac{1}{kT} = \frac{\left(\sum_i n_i + N_e\right)}{P} \tag{6}$$

We can substitute these into equation 3 which gives us

$$\frac{n_i^{r+1}}{n_i^r} = M_{i^{r+1}} \left(\frac{P}{\left(\sum_i n_i + N_e\right)}\right)^{\frac{3}{2}} exp\left(\frac{-\chi_i^r}{P}\left(\sum_i n_i + N_e\right)\right) \frac{1}{N_e} \tag{7}$$

and our new "Saha constant" is

$$A_{i^{r+1}} = M_{i^{r+1}} P^{\frac{3}{2}} \tag{8}$$

The Saha equation using pressure, instead of temperature, is given by

$$\frac{n_i^{r+1}}{n_i^r} = A_{i^{r+1}} \left(\sum_i n_i + N_e\right)^{-\frac{3}{2}} exp\left(\frac{-\chi_i^r}{P}\left(\sum_i n_i + N_e\right)\right) N_e^{-1} \tag{9}$$

2.1 Hydrogen

Initially we will consider a mixture of pure hydrogen. The Saha equation becomes

$$\frac{n_H^1}{n_H^0} = A_{H^1}\left(n_H + N_e\right)^{-\frac{3}{2}} exp\left(\frac{-\chi_H^0}{P}\left(n_H + N_e\right)\right) N_e^{-1} \tag{10}$$

We will use the following to simplify our expression

$$s_{H^1} = \frac{n_H^1}{n_H^0} \tag{11}$$

$$B_H(N_e) = \left(n_H + N_e\right)^{-\frac{3}{2}} \tag{12}$$

$$C_{H^1}(N_e) = exp\left(\frac{-\chi_H^0}{P}\left(n_H + N_e\right)\right) \tag{13}$$

$$D_H(N_e) = N_e^{-1} \tag{14}$$

Now

$$s_{H^1} = A_{H^1} B_H(N_e) C_{H^1}(N_e) D_H(N_e) \tag{15}$$

Recall

$$N_e = n_h * \nu_e(H) \tag{16}$$

where

$$\nu_e(H) = 1.0 * y_H^1 \tag{17}$$

and

$$y_H^1 = \frac{1}{\frac{1}{\frac{n_H^1}{n_H^0}} + 1} = \left(s_{H^1}^{-1} + 1\right)^{-1} \tag{18}$$

Therefore

$$N_e = n_h * \left(s_{H^1}^{-1} + 1\right)^{-1} \rightarrow n_h * \left(s_{H^1}^{-1} + 1\right)^{-1} - N_e = 0 \tag{19}$$

We will consider the function, $G(N_e)$ given by

$$G(N_e) = n_h * \left(s_{H^1}^{-1} + 1\right)^{-1} - N_e = 0 \tag{20}$$

To solve for N_{e1} we will use Newton's Method, which is

$$N_{e1} = N_e - \frac{G(N_e)}{G'(N_e)} \tag{21}$$

We already have an expression for $G(N_e)$, so now we must find its derivative. We have

$$G'(N_e) = -n_h * \left(s_{H^1}^{-1} + 1\right)^{-2}\left[s_{H^1}^{-1} + 1\right]' - 1 \tag{22}$$

where

$$\left[s_{H^1}^{-1} + 1\right]' = -s_{H^1}^{-2}\left[s_{H^1}\right]' \tag{23}$$

Using equation 15, we have

$$\left[s_{H^1}\right]' = A_{H^1}(B'_H(N_e) C_{H^1}(N_e) D_H(N_e) + B_H(N_e)(C'_{H^1}(N_e) D_H(N_e) + C_{H^1}(N_e) D'_H(N_e))) \tag{24}$$

All we need now is to differentiate our 'saha constants' and substitute them into equation 24. Therefore

$$B'_H(N_e) = \left(n_H + N_e\right)^{-\frac{3}{2}} \tag{25}$$

$$C'_{H^1}(N_e) = exp\left(\frac{-\chi_H^0}{P}\left(n_H + N_e\right)\right) \tag{26}$$

$$D'_H(N_e) = N_e^{-1} \tag{27}$$