# Order-Reversal of Compactness Scores under Map Projections [DRAFT]

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**Abstract**

In political redistricting, the *compactness* of a district is used as a quantitative proxy for its fairness. Several well-established, yet competing, notions of geometric compactness are commonly used to evaluate the shapes of regions, including the Polsby-Popper score, the convex hull score, and the Reock score, and these scores are used to compare two or more districts or plans. In this paper, we prove mathematically that, given any *map projection* from the sphere to the plane, that there is some pair of regions whose score is reversed after the projection, for all three of these measures. Finally, we demonstrate empirically the existence of legislative districts whose scores are permuted under the choice of map projection. Our proofs use elementary techniques from geometry and analysis and should be accessible to an interested undergraduate.

## 1 Introduction

Striving for the *geometric compactness* of legislative districts is a traditional principle of redistricting, and, to that end, many jurisdictions have included the criterion of compactness in their legal code for drawing districts. However, there is no agreed-upon definition for what makes a district compact or not. Several competing mathematical definitions have emerged over the past two centuries, including the *Polsby-Popper score*, [cut: which measures the ratio of a district's area to the square of its perimeter,] the *convex hull score*, [cut: which measures the ratio of the area of a district to the smallest convex region containing it], and the *Reock score*, [cut: which measures the ratio of the area of a district to the smallest circle containing it]. Each of these measures is appealing at an intuitive level, since they each assign to a district a single scalar value between zero and one, allowing easy comparisons between proposed redistricting plans. Additionally, the mathematics underpinning each is widely understandable by the relevant parties, including lawmakers, judges, advocacy groups, and the general public. None of these measures is perfect, however[citation needed] [cut: For each, it is not difficult to construct a mathematical counterexample for which a human's intuition and the score's evaluation of a shape's compactness differ, such as a circle with slightly perturbed boundary for the Polsby-Popper measure and a very long, thin rectangle for the convex hull measure. ] Additionally, these scores often do not agree. The long, thin rectangle has a very good convex hull score, but a very poor Polsby-Popper score. These issues are well-studied by political scientists and mathematicians alike [2, 3, 5].

In this paper, we propose a further critique of these measures, namely *sensitivity under the choice of map projection*. Each of the compactness scores named above is defined as a tool to evaluate geometric shapes in the plane, but in reality we are interested in analyzing shapes which sit on the surface of the planet Earth, which is (roughly) spherical[citation needed] When a shape is assigned a compactness score, it is implicitly done with respect to some choice of map projection. We show, both mathematically and empirically, that this may have serious consequences for the evaluation of compactness. In particular, we define the analogue of the Polsby-Popper, convex hull, and Reock scores on the sphere, and demonstrate that for any choice of map projection, there are two regions, $A$ and $B$, such that $A$ is more compact than $B$ on the sphere but $B$ is more compact than $A$ when projected to the plane.

## 2 Organization of Paper

This paper contains multiple results about compactness scores, all of which require similar mathematical backgrounds. An overview of the required preliminaries is found in section 3.

For each of the compactness scores we analyze, our proof that no map projection can preserve their order follows a similar recipe. We first use the fact that any map projection which preserves an ordering must preserve the *maximizers* of that ordering[cut: , meaning that if $\Omega$ is a region for which $\mathcal{C}(\Omega) \geq \mathcal{C}(\Sigma)$ for all regions $\Sigma$, then it must at the very least be the case that $\mathcal{C}(\varphi(\Omega)) \geq \mathcal{C}(\varphi(\Sigma))$ if $\varphi$ preserves $\mathcal{C}$'s ordering.]

Using this fact, we can restrict our attention to those map projections which preserve the maximizers in the induced ordering, then argue that any projection in this restricted set must permute the order of scores of some pair of regions.

We spend some time developing background related to the gnomonic and stereographic projections, and showing that, in fact, any map preserving maximizers of convex-hull scores must be the gnomonic projection, and any map preserving maximizers of Reock must be stereographic. This analysis can be found in section 4.

Finally, we show that the gnomonic and stereographic projections do not preserve the ordering induced by convex hull or Reock scores respectively. We do this in sections 5 (for convex hull) and 6 (for Reock).

Being a slightly different argument, we dedicate section 7 to the analysis of Polsby-Popper scores. [Assaf: Still need to summarize sections 7 and 8]

## 3 Preliminaries, Definitions

We begin by introducing the necessary definitions and terminology, as well as a few observations about the mathematical objects of interest which will be of use later. We carefully lay out these definitions so as to align with an intuitive understanding of the concepts and to appease the astute reader who may be concerned with edge cases, geometric weirdness, and nonmeasurability.

**Definition 1.** A **region** $\Omega$ in $\mathbb{S}^2$ or in $\mathbb{R}^2$ is an non-empty open set together with its boundary such that the region is measurable, its boundary is measurable, and it is connected.

We choose this definition so that concepts of the 'area' and 'perimeter' of a region are well-defined concepts. Throughout, we restrict our attention to the plane $\mathbb{R}^2$ [cut: (or $\mathbb{C}$ if one prefers) ]and the surface of the unit sphere $\mathbb{S}^2$ equipped with the standard measures and metrics. We leave the consideration of other surfaces, measures, and metrics to future work.

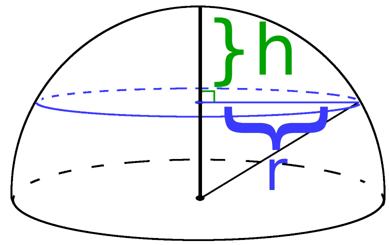


Figure 1: The height and radius of a spherical cap.

**Definition 2.** A **compactness score function** $\mathcal{C}$ is a function from the set of all regions to the positive real numbers. We adopt the convention that a region with a *higher* compactness score is *more* compact, and this naturally induces a partial order over the set of all regions, where $A$ is at least as compact as $B$ if and only if $\mathcal{C}(A) \geq \mathcal{C}(B)$.

The final major definition we need is that of a *map projection*. In reality, the regions we are interested in comparing sit on the surface of the Earth (i.e. a sphere), but these regions are often examined as being projected onto a flat sheet of paper or computer screen, and which means that the regions drawn in any flat map object are subject to such a projection.

**Definition 3.** A **map projection** $\varphi$ is a locally defined diffeomorphism from a region on the sphere to a region on the plane. [cut: This means that $\varphi$ is continuous, $\varphi^{-1}$ exists and is also continuous.]

[Assaf: condensed these definitions]

**Definition 4.** We use the word **transformation** [of the plane/sphere] to mean to a diffeomorphism from the plane or sphere to itself.

Since the image of a region under a map projection $\varphi$ is also a region, we can examine the compactness score of that region both before and after applying $\varphi$, and this is the heart of the problem we address in this paper. We demonstrate, for several standard choices of compactness scores $\mathcal{C}$, that the order induced by $\mathcal{C}$ is different than the order induced by $\mathcal{C} \circ \varphi$ for *any* choice of map projection $\varphi$.

**Definition 5.** We say that a map projection $\varphi$ **preserves** a compactness score $\mathcal{C}$ if for any regions $\Omega, \Omega'$, $\mathcal{C}(\Omega) \geq \mathcal{C}(\Omega')$ if and only if $\mathcal{C}(\varphi(\Omega)) \geq \mathcal{C}(\varphi(\Omega'))$.

*Remark* 1. $\varphi$ preserves a compactness score $\mathcal{C}$ if and only if $\varphi^{-1}$ does.

**Definition 6.** Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere. A **cap** on the sphere is a region of the form $\kappa(h) = \{(x, y, z) \in \mathbb{S}^2 : z \geq h\}$.

[Zach: new section who dis?]

## 4 The Gnomonic and Stereographic Projections

In this section, we'll present some results about two common projections from the sphere to the plane: the *gnomonic* and the *stereographic* projections. We build these results to later show that the convex hull score and the Reock score are not preserved by any map projection.

### 4.1 The Gnomonic Projection

[Zach: old defn 6]

**Definition 7.** A set $X$ in a metric space is **convex** if for any pair of points $x_1$ and $x_2$ in $X$, the shortest geodesic segment connecting $x_1$ and $x_2$ is entirely contained in $X$.

In the plane, these geodesic segments are ordinary line segments. On the sphere, the geodesics are segments of great circles. In particular, this means that on the sphere, caps are convex, and in the plane, circular regions are convex.

[Zach: old defn 8, lem 3, thm 2,3,4]

**Definition 8.** The **gnomonic projection** is the map projection from the lower half-sphere to the plane defined by placing the south pole of the sphere tangent to the plane at the origin and sending a point $p$ on the lower half-sphere to the unique point $q$ in the plane which lies on the line in $\mathbb{R}^3$ which passes through the center of the sphere, $p$ and $q$.

Intuitively, the image of a region on the sphere under the gnomonic projection is the shadow in the plane of that region which is created when we put the bottom of the sphere at the origin and turn on a light source at the center of the sphere.

[Zach: do we want to write the explicit parametrization of the gnomonic projection here? the xy version of it is pretty gross. the polar version is less messy. I don't know if it provides any extra insight...] [Assaf: A figure might be better]

We can observe that the gnomonic projection sends caps centered at the south pole to circles in the plane centered at the origin, and sends any segment of a great circle passing through the south pole to a line segment in the plane passing through the origin. What may be somewhat surprising is that it sends *every* convex set on the sphere to a convex set in the plane.

**Lemma 1.** *Let $\varphi$ be the gnomonic projection, and $\Omega \subset S^2$ a convex set. Then $\varphi(\Omega)$ is convex.*

*Proof.* To prove this, it suffices to show that the one-dimensional convex sets are preserved. In other words, we must show that any segment of a great circle on the sphere is sent to a line segment in the plane.

Let $p$ be the center of the sphere $S^2$, and let $C \subset S^2$ be a geogesic segment in the lower hemisphere. Note that $C$ is uniquely identified by the intersection of $S^2$ with a portion of a plane $\mathcal{P}$ passing through $p$ which is not parallel to the $xy$-plane, $\mathcal{P}_{xy}$. Since $\mathcal{P}$ contains all of the lines passing through $p$ and a point in $C$, by construction, $\varphi(C) \subset \mathcal{P} \cap \mathcal{P}_{xy}$, and two planes intersect along a line.

Since the gnomonic projection is injective, it follows that geodesic segments in $S^2$ cannot be sent to points, and hence must be sent to lines. [Zach: do we need this statement?] $\square$

[Assaf: I'm still hunting down the reference for this theorem. Some say it's due to Hartshorne, but I can't find the reference. Someone says it's due to von Staudt from the 1850's...]

**Theorem 1.** *Let $\Omega$ be some subset of $\mathbb{R}^n$ for $n \geq 2$. If $f : \Omega \to \mathbb{R}^n$ is bijective and sends line segments to line segments, then $f$ is an affine linear transformation.[Zach: should try to make work for injective]*

*Proof.* After possibly translating, we can assume without loss of generality that $0 \in \Omega$ and $f(0) = 0$, in which case we must show that $f$ is linear. We write $V = f(\Omega)$. Since $f$ is a bijection from $\Omega$ to $V$ sending straight lines to straight lines, it follows that if $\mathcal{P}$ is an affine plane in $\mathbb{R}^n$, then $f(\mathcal{P} \cap \Omega)$ must be of the form $\mathcal{Q} \cap V$, where $\mathcal{Q}$ is an affine plane in $\mathbb{R}^n$. Additionally, $f$ must preserve intersections of lines, and hence sends coplanar parallel lines to coplanar parallel lines, and hence nondegenerate parallelograms to nondegenerate parallelograms. [Assaf: Lorenzo says that this is enough, and I believe him, but I've never seen it...] Thus, $f$ is linear. $\square$

[Assaf: removed general result]

Using this, we can complete the construction of the main tool of this section – that any map projection which preserves convex sets can be written as the composition of the gnomonic projection and an affine linear transformation of the plane.

**Theorem 2.** *Let $\psi : \Omega \to \mathbb{R}^2$ be a map projection defined on a region contained in the lower-hemisphere, and let $\varphi$ be the gnomonic projection. If $\psi^{-1}$ sends convex sets to convex sets, then there exists an affine linear map $L$ such that $\psi = L \circ \varphi$.*

[Zach: this field automorphism business feels a little too abstract for the audience. In order to show $f$ is linear, you need to show $f(u + v) = f(u) + f(v)$ and $f(kx) = kf(x)$ and preserving parallelograms is strong enough to imply both of these. You showed additivity, and scaling is even easier. If a parallelogram is preserved, it sends parallel segments of equal length to parallel segments of equal length, and this extends to the statement that any pair of parallel segments has the ratio of their lengths preserved, so any pair of collinear segments have the ratio of their length preserved, which implies linear.]

*Proof.* By Lemma 1, the map $\varphi \circ \psi^{-1} : \psi(\Omega) \to \mathbb{R}^2$ sends convex sets to convex sets. It is bijective on its range, since $\psi$ and $\varphi$ are bijective, and hence satisfies the conditions of Theorem 1, meaning that there exists an affine linear isomorphism $L^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi = \psi \circ L^{-1}$. Thus, $\psi = \varphi \circ L$, as desired. $\square$

[Zach: we don't use this anywhere. it's cute, but I don't think it's relevant]

### 4.2 Stereographic Projection

[Zach: defn 9, lem 5 lem 6 lem 7 lem 8]

**Definition 9.** The **stereograpic projection** from the north pole $N$ is defined by placing a copy $\mathbb{R}^2$ tangent to the sphere at the south pole, and sending any $p \in \mathbb{S}^2 \setminus \{N\}$ to the unique point $q$ in $\mathbb{R}^2$ such that $q \in \overline{Np}$. [cut: If $(x, y, z)$ is a point on the sphere, then the action of the stereographic projection is

$$(x, y, z) \mapsto \left( \frac{x}{1 - z}, \frac{y}{1 - z} \right)$$

and its inverse, for $(u, v)$ in the plane:

$$(u, v) \mapsto \left( \frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{u^2 + v^2 - 1}{1 + u^2 + v^2} \right)$$

]

*Remark* 2. It is a classical result that this projection sends every cap on the sphere which doesn't pass through the north pole to a circle in the plane.

[Assaf: This is a fact that is usually taught at a second year complex analysis course - perhaps we can leave this out if we want to conserve space?] [Assaf: okay, Assaf, I'll do it.]

**Lemma 2.** *The stereographic projection sends every cap which does not pass through the north pole to a circle in the plane.*

We have in-hand a circle-preserving map projection, and we now turn to demonstrating that the stereographic projection is essentially the unique map projection with this property. We can observe that if we perform stereographic projection and compose it with a transformation of the plane which sends every circle to a circle, then this composition is a circle-preserving map projection. The next lemma demonstrates that the class of transformations of the plane which preserve circles is actually quite narrow. [Assaf: Localized the lemma to concern mobius transformations]

**Lemma 3.** *Let $f : \Omega \to V$ be a bijection between two planar regions sending generalised circles to generalised circles. Then $f$ is a Möbius transformation composed with a reflection.*

This is a slight rewording of a celebrated result by Carathéodory[1]:

**Theorem 3.** *Every arbitrary bijection of a disc $D$ to a bounded set $D'$ by which circles lying completely in $D$ are transformed into circles lying in $D'$ must be of the form $M$ or $\bar{M}$, where $M$ is a Möbius transformation.*

The proof of the lemma then follows:

*Proof.* By Carathéodory's theorem, we know that on any open ball in $\Omega$, $f = M$ or $f = \bar{M}$, where $M$ is a Möbius transformaion. Thus, $f$ or $\bar{f}$ are holomorphic, so by applying the identity theorem, it follows that $f = M$ or $f = \bar{M}$ on the entirety of $\Omega$. □

This gives us the desired characterization.

**Theorem 4.** *The map projections from the sphere to the plane which send every cap to a circle are exactly those which can be written as the composition of the stereographic projection followed by a Möbius transoformation and a reflection.*

*Proof.* Let $\varphi$ and $\psi$ be two map projections which preserve circles, and without loss of generality let $\varphi$ be the standard stereographic projection. Then the composition $M^{-1} = \psi \circ \varphi^{-1}$ is a transformation of the plane which preserves circles, so by the previous lemma, $M^{-1}$ is a Möbius transformation composed with a reflection. Then, we can write $\psi = M\varphi$, which is the composition of the stereographic projection and a scaled isometry. □

[Zach: not convinced we need this next bit anymore] [Assaf: I muted it]

# 5 Convex Hull

A commonly used compactness score is the *convex hull score*. We briefly recall the definition of a convex set and then define this score function.

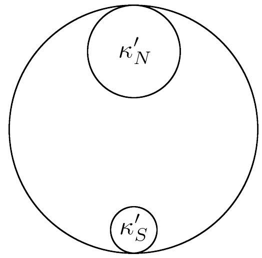


Figure 2: The construction used in Theorem 5.

**Definition 10.** Let conv($\Omega$) denote the *convex hull* of a region $\Omega$ in either the sphere or the plane, which is the smallest convex region containing $\Omega$. Then we define the *convex hull score* of $\Omega$ as

$$\mathrm{CH}(\Omega) = \frac{\mathrm{area}(\Omega)}{\mathrm{area}(\mathrm{conv}(\Omega))}$$

[cut: The convex hull score of a region $\Omega$ will be equal to one if and only if $\Omega$ is itself convex. We note that the convex hull and Polsby-Popper scores agree when $\Omega$ is a circle, provide a similar score when $\Omega$ is roughly circluar, and differ greatly if $\Omega$ is a highly non-circular but convex region, such as a long, thin rectangle.]

To demonstrate that any map projection cannot preserve the orders of convex hull scores we note that order-preserving map must, in particular, preserve the maximizers in the ordering. This means that convex regions on the sphere are mapped to convex regions in the plane.

By Theorem 2, such a map must be the gnomonic projection up to a linear transformaion. We first show that the gnomonic projection does not preserve convex-hull score orderings on any region of the sphere.

[Assaf: mute this and write it in later: This lemma lets us prove our theorem in two steps. We can first show that the gnomonic projection does not preserve the ordering of convex hull scores, then argue that this misordering cannot be corrected by any affine linear transformation of the plane. This first part we will prove by explicitly constructing two regions whose convex hull scores are permuted under the gnomonic projection, and to facilitate this construction, we make the following observation:]

We finally have the tools to prove the main result of this section, which is a direct consequence of the four previous lemmas.

**Theorem 5.** *Let $\Omega$ be a region on the sphere. Then there exist two regions $\kappa'_N, \kappa'_S \subset \Omega$ such that the convex hull scores of $\kappa'_N$ and $\kappa'_S$ are equal on the sphere, but under the gnomonic projection $\varphi$, the convex hull score of $\kappa'_S$ is strictly greater than that of $\kappa'_N$.*

*Proof.* First, since $\Omega$ is a region, we can find some cap $\kappa$ inside of $\Omega$ such that the center of $\kappa$ is not the south pole of the sphere, and the north pole is exterior to $\kappa$. This cap is bisected by a line of longitude, in particular the great circle which passes through the two poles and the center of $\kappa$. This line of longitude meets $\kappa$ at exactly two points, and we call $p_N$ the point closer to the north pole and $p_S$ the point closer to the south pole.

Choose some $r$ strictly less the radius of $\kappa$ and let $\kappa'_S$ be the region constructed by deleting a cap of radius $r$ from the interior of $\kappa$ tangent at $p_S$. Construct $\kappa'_N$ analogously by deleting a cap of radius $r$ tangent at $p_n$. Observe that the convex hull scores of $\kappa'_N$ and $\kappa'_S$ are identical, since each has the boundary of $\kappa$ as its smallest bounding cap and both convex hulls have the same area.

We now consider the images of $\kappa'_N$ and $\kappa'_S$ under the gnomonic projection $\varphi$. Since $\varphi$ preserves points of tangency, containment, and sends every cap to an ellipse, the images $\varphi(\kappa'_N)$ and $\varphi(\kappa'_S)$ are both regions which are ellipses with a smaller ellipse, tangent to a point on the circumference, deleted. Furthermore, since the boundary of $\kappa$ was the smallest bounding cap of both regions on the sphere, the image of the boundary of $\kappa$ under $\varphi$ is the smallest bounding circle of the images of these regions in the plane.

We can now observe that these two regions in the plane do not have the same convex hull score. Both have the same bounding ellipse, which is their convex hulls, but $\varphi(\kappa'_N)$ is strictly smaller than $\varphi(\kappa'_S)$. This is because the gnomonic projection distorts areas in a way such that regions further from the south pole have their areas magnified more than the same region closer to the south pole. If the regions in question are sufficiently small, letting $\theta$ denote the polar angle at which we consider a small cap on the sphere, a cap of radius $r$ will be sent to an ellipse with a major axis (along the line through the center of the ellipse and the origin) of length roughly $r/\sin^2\theta$[Zach: check this] , and a minor axis of length $r/\sin\theta$, and a straightforward examination of this as a function of $\theta$ shows that it grows faster as $\theta$ increases.

Since $\varphi(\kappa'_N)$ fills a smaller fraction of the bounding ellipse than $\varphi(\kappa'_N)$ does, its convex hull score is strictly worse, and the gnomonic projection does not preserve convex hull scores. □

Now that we've shown that the gnomonic projection alone cannot preserve convex hull scores, to complete the argument, we must now show that there is no affine linear transformation of the plane which can correct for this.

**Lemma 4.** *Let $L$ be an affine linear transformation of the plane. Then $L$ preserves the convex hull scores of all figures in the plane.*

*Proof.* Since affine linear transformations map lines to lines, they preserve convexity, and since they also preserve the containment of regions in other regions, if $\Omega$ is a region in the plane, then it must be the case that the convex hull of $\Omega$ is mapped to the convex hull of $L(\Omega)$ by $L$. In other words, the image of the convex hull of $\Omega$ is the convex hull of the image of $\Omega$. Finally, since affine linear transformations preserve the ratio of the areas of any two regions, $L$ must, in particular, preserve the ratio of the areas of $\Omega$ and its convex hull, so the convex hull score is preserved. □

# 6 Reock

Let $\mathrm{circ}(\Omega)$ denote the *smallest bounding circle* (smallest bounding *cap* on the sphere) of a region $\Omega$. Then the *Reock score* of $\Omega$ is

$$\mathrm{Reock}(\Omega) = \frac{\mathrm{area}(\Omega)}{\mathrm{area}(\mathrm{circ}(\Omega))}.$$

In words, it is the ratio of the area of the region to the area of the smallest circle containing that region, and it is equal to one if and only if $\Omega$ is a circle. We can observe that $\mathrm{CH}(\Omega) \geq \mathrm{Reock}(\Omega)$, since the convex hull of a region is never larger than the smallest bounding circle. This relation holds with equality if and only if the convex hull of $\Omega$ is a circle.

Since $\text{Reock}(\Omega) = 1$ if and only if $\Omega$ is a circle, we can observe that any candidate $\varphi$ for a map projection which preserves the ordering of Reock scores must, at the very least, preserve the maximizeres in the ordering, meaning it sends caps on the sphere to circles in the plane.

By Theorem 4, it follows that we can assume that $\varphi = R \circ M \circ \psi$, where $R$ is a reflection, $M$ is a Möbius transformation, and $\psi$ is the standard stereographic projection. However, we note that reflections do not change any compactness scores, so we only need to deal with $M \circ \psi$.

We now treat the Möbius transformation composition in the following manner:

**Claim 1.** *There exists a rigid transformation of the sphere, $T_M$, such that $M\psi = T_M\psi$.*

This follows by writing $T_M$

In other words, to show that $M\psi$ does not preserve Reock score rankings in some region $U$, it suffices to show that $\psi$ does not preserve Reock score rankings in the transformed region $T(U)$.

**Theorem 6.** *Let $A$ be a region on the sphere. Then there exist two regions $\kappa'_N, \kappa'_S \subset A$ such that the Reock scores of $\kappa'_N$ and $\kappa'_S$ are equal on the sphere, but under the stereographic projection $\varphi$, the Reock score of $\kappa'_S$ is strictly greater than that of $\kappa'_N$.*

*Proof.* The structure of this proof is similar to that of Theorem 5. We set $\kappa, \kappa'_S, \kappa'_N$ as before. [Assaf: Condensed construction]

We can now observe that these two regions in the plane do not have the same Reock score. Both have the same bounding circle, but $\varphi(\kappa'_N)$ is strictly smaller than $\varphi(\kappa'_S)$. This is because the stereographic projection distorts areas in a way such that figures further from the south pole have their areas magnified more than the same region closer to the south pole. If the regions in question are sufficiently small, letting $\theta$ denote the polar angle at which we consider a small cap on the sphere, a cap of radius $r$ will be sent to a circle with radius roughly $r/\cos^2\theta$[Zach: check this] , and a straightforward examination of this as a function of $\theta$ shows that it grows faster as $\theta$ increases.

Since $\varphi(\kappa'_N)$ fills a smaller fraction of the bounding circle than $\varphi(\kappa'_N)$ does, its Reock score is strictly worse, and the stereographic projection does not preserve Reock scores. □

Piecing this together yields:

**Theorem 7.** *There is no map projetion from the sphere to the plane which preserves the ordering of Reock scores.*

## 7 Polsby-Popper

The first compactness score we analyze is the *Polsby-Popper score*, which takes the form of an *isoperimetric quotient*, meaning it measures how much area a region's perimeter encloses, relative to all other regions with the same perimeter.

**Definition 11.** The Polsby-Popper score of a region $\Omega$ is defined to be

$$\text{PP}_X(\Omega) = \frac{4\pi \cdot \text{area}_X(\Omega)}{\text{perim}_X(\Omega)^2}$$

Where $X$ is either the sphere $S$ or the plane $\mathbb{R}^2$, and $\text{area}_X$ and $\text{perim}_X$ are the area and perimeter respectively of $A$ in $X$.

The ancient Greeks were first to observed that if $\Omega$ is a region in the plane, then $4\pi \cdot \text{area}(\Omega) \leq \text{perim}(\Omega)^2$, with equality if and only if $\Omega$ is a circle. This seemingly obvious fact took a long time to prove[Assaf: TODO: find a good reference on the history of the isoperimetric ratio] , and became known as the *isoperimetric inequality* in the plane. This means that $0 \leq \text{PP}_{\mathbb{R}^2}(\Omega) \leq 1$, where the Polsby-Popper score is equal to 1 only in the case of a circle. In particular, the Polsby-Popper score is scale-invariant in the plane. An isoperimetric inequality for the sphere exists, and we state it as the following lemma. For a more detailed treatment of isoperimetry in general, see [4], and for a proof of this inequality for the sphere, see [6].

**Lemma 5.** *If $\Omega$ is a region on the sphere with area $A$ and perimeter $P$, then $P^2 \geq A^2 - 4\pi A$ with equality if and only if $\Omega$ is a spherical cap.*

A consequence of this is that among all regions on the sphere with a fixed area $A$, a spherical cap with area $A$ has the shortest perimeter. However, the key difference between the isoperimetric quotient in the plane and on the sphere is that on the sphere, there is no scale invariance. [Lorenzo: It feels weird to me to call this theorem a "lemma"] [Zach: I stated it as a lemma because I don't think we need to or even should be proving isoperimetric inequalities in here. We can talk a little more about the audience, but it's one of those things that's intuitively true, but in order to get a rigorous proof, you need a few pages of symbol chasing and worrying about edge cases. Maybe we can put a proof in an appendix if we feel like we need to include it?]

**Lemma 6.** *Let $S$ be the unit sphere, and let $\kappa(h)$ be the cap at height $h$ (i.e. the region comprised of all points whose $z$-coordinate is greater than $1-h$, where we imagine the sphere as being embedded in $\mathbb{R}^3$, centered at the origin) Then $\text{PP}(\kappa(h))$ is a monotonically increasing function of $h$.*

[Zach: $A$ was a little overloaded, so I changed it to $\kappa$ ☺]

*Proof.* Let $r(h)$ be the radius of the circle bounding $\kappa(h)$. We compute:

$$\begin{aligned} 1 &= r(h)^2 + (1-h)^2, \text{ by right triangle trigonometry} \\ &= r(h)^2 + 1 - 2h + h^2 \end{aligned}$$

Rearranging, we get that $r(h)^2 = 2h - h^2$, which we can plug in to the standard formula for perimeter:

$$\text{perim}_S(\kappa(h)) = 2\pi r(h) = 2\pi\sqrt{2h - h^2}$$

We can now use the Archimedian equal-area projection defined by $(x, y, z) \to \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z\right)$ to compute $\text{area}_S(\kappa(h)) = 2\pi h$ and plug it in to get: [Zach: i think there's a better explanation than just asserting the existence of a projection that does this. What about 'Archimedes showed that the lateral surface area of a cylinder with radius 1 and height 2 is the same as the surface area of the spher, and that the lateral projection (do figure) from the sphere onto the cylinder preserves area. Using this, we have that the area of a spherical cap at height $h$ is'] [Assaf: I wrote something shorter and possibly more useful]

$$\text{PP}_S(\kappa(h)) = \frac{4\pi(2\pi h)}{4\pi^2(2h - h^2)} = \frac{2}{2-h}$$

Which is a monotonically increasing function of $h$. □

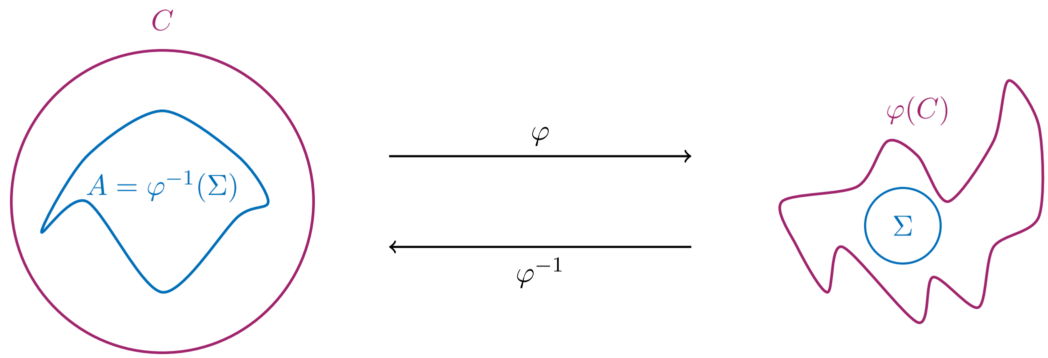


Figure 3: The construction of regions $A$, $C$, and $\hat{A}$ in the proof of Theorem 8.

**Corollary 1.** *On the sphere, Polsby-Popper scores of caps are monotonically increasing with area*

Using this, we can show the main theorem of this section, that no map projection from the half-sphere to the plane can preserve the ordering of Polsby-Popper scores for all regions.

**Theorem 8.** *If $\varphi$ is a map projection from the half-sphere to the plane, then there are two regions $A$ and $C$ in the half sphere such that the Polsby-Popper score of $C$ is greater than that of $A$ in the sphere, but the Polsby-Popper score of $\varphi(A)$ is greater than that of $\varphi(C)$ in the plane.*

*Proof.* Let $\varphi$ be a map projection, and take $C$ to be a cap in the half-sphere. Let $\Sigma$ be a circle in the plane such that $\Sigma \subsetneq \varphi(C)$ and let $A = \varphi^{-1}(\Sigma)$ (see Figure 7).

We now use the isoperimetric inequality for the sphere and Corollary 1 to claim that $A$ does not maximilze Polsby-Popper score in the sphere.

To see this, take $\hat{A}$ to be a cap in the sphere with area equal to that of $A$. By the isoperimetric inequality of the sphere, $\mathrm{PP}_S(\hat{A}) \geq \mathrm{PP}_S(A)$. Since map projections preserve containment, $\Sigma \subsetneq \varphi(C)$ implies that $A \subsetneq C$, meaning that $\text{area}(\hat{A}) = \text{area}(A) \lneq \text{area}(C)$. By Corollary 1, we know that $\mathrm{PP}_\mathrm{S}(\hat{A}) < \mathrm{PP}_\mathrm{S}(C)$, and combining this with the earlier inequality, we get

$$\mathrm{PP}_\mathrm{S}(A) \leq \mathrm{PP}_\mathrm{S}(\hat{A}) < \mathrm{PP}_\mathrm{S}(C)$$

Since $\Sigma = \varphi(A)$ maximizes the Polsby-Popper score in the plane, but $A$ does not do so in the sphere, we have shown that $\varphi$ does not preserve the maximal elements in the score ordering, and therefore it cannot preserve the ordering itself. □

The reason why every map projection fails to preserve the ordering of Polsby-Popper scores is because the score itself is constructed from the *planar* notion of isoperimetry, and there is no reason to expect this formula to move nicely back and forth between the sphere and the plane. This proof crucially exploits a scale invariance present in the plane but not the sphere. If we consider any circle in the plane, it's Polsby-Popper score is definitionally equal to one, but that is not true of every cap in the sphere. This naturally raises the question of whether being more careful, and defining a compactness score which uses the isoperimetric quotient of the surface the region is actually in will evade this problem. We show later that it does resolve the issue of scale-noninvariance, but it is still induces an ordering which is not preserved by any map projection. We address this in Section **??**, as it uses on machinery we develop in the interceding sections. [Assaf: muted newer sections]

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