ORDER REVERAL OF POLSBY-POPPER SCORES UNDER PROJECTIONS

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Definition 1. Let $U \subset \mathbb{R}^2$ be an open set with smooth boundary. We define:

$$PP_{\mathbb{R}^2}(U) = \frac{\text{Area}(U)}{\text{Perimeter}(U)^2}$$

The isoperimetric inequality states that $PP_{\mathbb{R}^2}(U) \leq \frac{1}{4\pi}$, with equality if and only if U is a disk. TODO:reference

On a spherical region, we similarly define:

Definition 2. Let U be an open set in S^2 with smooth boundary. We similarly define:

$$PP_S(U) = \frac{\text{Area}(U)}{\text{Perimeter}(U)^2}$$

Where area and perimeter are measured on the sphere.

[TODO - explain why this is the right notion.]

Remark 3. The Polsby-Popper score induces on order on the open regions with smooth boundary of a surface X, by assigning $U > V$ iff $PP_X(U) > PP_X(V)$. (To do: make notation consistent... change PP to $PP_{\mathbb{R}^2}$)

Example 4. Consider S^2 the unit sphere centered at $0 \in \mathbb{R}^3$, and define $U_h = \{(x, y, z) \in S^2 : z \geq h\}$. A straightforward integration by shells shows that $\text{Area}(U_h) = 2\pi(1 - h)$, and that $\text{Perimeter}(U_h) = 2\pi(\sqrt{1 - h^2})$. Overall, we get:

$$PP_S(U_h) = \frac{2\pi(1 - h)}{4\pi^2(1 - h^2)} = \frac{1}{2\pi}\frac{1}{1 + h}$$

Note that $PP_S(U_h)$ is monotonically strictly decreasing with h. This means that as the geodesic radius (TODO: vague?) of the "cap" U_h gets larger, so does the Polsby-Popper score.

The goal of this paper is to prove the following theorem:

Theorem 5. *Let $U \subset S^2$ be some open region. Then there does not exist a diffeomorphism $\varphi : U \to V \subset \mathbb{R}^2$ preserving orders of Polsby-Popper scores. In other words, for any such φ, there exist regions $A, B \subset U$ such that $PP_S(A) > PP_S(B)$, but $PP_{\mathbb{R}^2}(\varphi(A)) \leq PP_{\mathbb{R}^2}(\varphi(B))$.*

We will use the isoperimetric inequality on spheres [1] [2]

Theorem 6. *If L is the length of a simple curve γ on S^2, and A the area it encloses, then:*

$$L^2 \geq A^2 - 4\pi A$$

with equality if and only if γ is the boundary of a ball.

Corollary 7. *Balls minimize perimeters in their volume class, and hence maximize Polsby-Popper ratios.*

Among all regions with a fixed area a, spherical caps minimize perimeters, and hence Polsby-Popper ratios.

Proof of 5. We first observe that the theorem is equivalent to showing that φ does not preserve maximal elements in the ordering. Let $C \subset\subset U$ be a cap of S^2, and choose $A' \subset\subset \varphi(C) \subset \mathbb{R}^2$ be some disk, and $A = \varphi^{-1}(A')$. Since φ is a diffeomorphism, it follows that $\varphi(B), A$ are open sets with smooth boundary.

Claim 8. *A' maximizes Polsby-Popper score in $\varphi(C)$, but A does not maximize Polsby-Popper score in C.*

Proof. A' maximizes Polsby-Popper score in $\varphi(C)$, because it is a disk in the plane. On the sphere, let $\hat{A}$ be the cap with equal area to A. By corollary 7, it follows that $PP_S(\hat{A}) \geq PP_S(A)$. Since $A \subset\subset C$ it follows by construction that $\text{Area}(\hat{A}) \leq \text{Area}(C)$, meaning that $\hat{A}$ is a cap with geodesic radius smaller than C's.

By the computation in example 4, it follows that:

$$PP_S(A) \leq PP_S(\hat{A}) < PP_S(C)$$

as desired. $\square$

Notice that the claim proves the theorem, as any diffeomorphism preserving orders of Polsby-Popper scores would preserve maximal regions in the ordering. $\square$

Proof of 5. Let $C \subset\subset U$ be a cap of S^2, and $B \subset C$ some other cap. Let $A' \subset \varphi(B) \subset \varphi(C) \subset \mathbb{R}^2$ be some disk, and $A = \varphi^{-1}(A')$. Since φ is a diffeomorphism, it follows that $\varphi(B), \varphi(C), A$ are open sets with smooth boundary.

Claim 9. *A' maximizes Polsby-Popper score in $\varphi(C)$, but A does not maximize Polsby-Popper score in C.*

Proof. A' maximizes Polsby-Popper score in $\varphi(C)$, because it is a disk in the plane. On the sphere, let $\hat{A}$ be the cap with equal area to A. By corollary 7, it follows that $PP_S(\hat{A}) \geq PP_S(A)$. Since $A \subset B$ it follows by construction that $\text{Area}(\hat{A}) \leq \text{Area}(B)$, meaning that $\hat{A}$ is a cap with geodesic radius smaller than B's.

By the computation in example 4, it follows that:

$$PP_S(A) \leq PP_S(\hat{A}) \leq PP_S(B) < PP_S(C)$$

as desired. $\square$

Notice that the claim proves the theorem, as any diffeomorphism preserving orders of Polsby-Popper scores would preserve maximal regions in the ordering. $\square$

Proof. TODO: proof 2: delete?

Claim 10.

Proof. □

Let $C \subset U$ be an open ball in S^2. Let $A \subset C$ be some region, and B be a ball such that $\mu_S(A) = \mu_S(B)$. We note that since $Area(B) < Area(C)$, we can position B such that $B \subset C$. Moreover, by By corollary 7, it follows that $PP_S(B) \geq PP_S(A)$.

Note that $PP_S(B(r))$ is monotonically strictly increasing with the radius r, as shown by example 4. Note that $\sup_{A \subset C} PP_S(A) = PP_S(C)$, simply by taking an increasing sequence of balls whose radii converge to that of C. Since $PP_S(A) \leq PP_S(B) < PP_S(C)$, there does not exist a maximizer of PP_S within C.

However, on the plane, every open region has a maximizer of PP. In particular, these maximizers are the open discs. Suppose that A' is one such open disc in $\phi(C)$. Then $PP_{\mathbb{R}^2}(\varphi(A')) < PP_{\mathbb{R}^2}(C)$, so there is an open cap $B \subset C$ so that $PP_{\mathbb{R}^2}(\varphi(A')) < PP_{\mathbb{R}^2}(B)$.

Let $A = \varphi(A')$. Then $PP_{\mathbb{R}^2}(A) < PP_{\mathbb{R}^2}(B)$, but $PP_{\mathbb{R}^2}(\phi(A)) = PP_{\mathbb{R}^2}(A') \geq PP_{\mathbb{R}^2}(\phi(B))$, which is the conclusion of the theorem.

□

Remark 11. Note that the proof above actually tells us something stronger - namely, that for any projection, Polsby-Popper score orderings are not preserved in *any* cap.

TODO: comments on generalizing to rectifiable curves, TODO: further questions: Reock/Convex hull optimization, drop the assumption that the Earth is a sphere (it's flat)

Question. *Some further questions:*

(1) *PP_S goes to infinity, so what does it mean?*
(2) *PP_S optimizes for larger shapes, as shown by Example example 4. What does this mean?*
(3) *How does the choice of projection precisely game the Polsby-Popper scores?*
(4) *In law, is there a requirement to use a particular projection?*
(5) *How much of an "edge case" are the non-order preserved shapes? Probably not that far off, since anything close to a cap would run into these issues.*

References

[1] Tibo Radö, The Isoperimetric Inequality on the Sphere, American Journal of Mathematics Vol. 57, No. 4 (Oct., 1935), pp. 765-770

[2] Robert Osserman, Bonnesen-Style Isoperimetric Inequalities, The American Mathematical Monthly Vol. 86, No. 1 (Jan., 1979), pp. 1-29