Shape Analysis for Redistricting

modern geometry meets modern politics

Zachary Schutzman

University of Pennsylvania & Metric Geometry and Gerrymandering Group

Hyperbolic Lunch, U. of Toronto Mathematics

February 21st, 2019

This is joint work with...

Total Variation Isoperimetric Profiles
https://arxiv.org/abs/1809.07943

Daryl DeFord

Hugo Lavenant

Justin Solomon

Graph Laplacians

Emilia Alvarez

Daryl DeFord

James Murphy

Justin Solomon

Discrete Compactness

Assaf Bar-Natan

Moon Duchin

Adriana Rogers

Curve-Shortening Flow
https://zachschutzman.com/distflow

Emilia Alvarez

Daryl DeFord

Michelle Feng

Patrick Girardet

Natalia Hajlasz

Eduardo Chavez Heredia

Lorenzo Najt

Sloan Nietert

Aidan Perreault

Justin Solomon

WHAT IS COMPACTNESS?

Compactness is ...

Vaguely, it’s supposed to describe the niceness of the shape of a district.

Compactness is in the discourse

America’s most gerrymandered congressional districts

By Christopher Ingraham
May 15, 2014

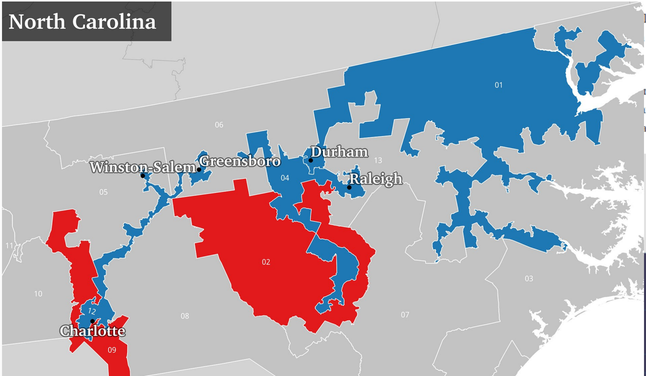
Michigan has districts in shapes that even a flexible salamander would be incapable of contorting itself into. To be sure, there are perfectly legitimate rea

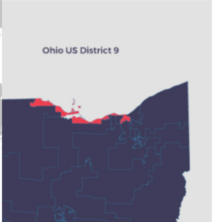
Democratic-le

Ohio’s wonky districts

l research company, ranked Ohio 11th on its
Congressional District, below, was named

re redrawn in 2011, the new 9th forced two
ry election. By eating up little pieces of
nother.





Compactness is in the law

Legislative Apportionment Commission shall attempt to form functionally contiguous and compact territories. For purposes of this section, a

Iowa law provides that congressional and legislative districts should be reasonably compact in form. As noted previously, the requirement to establish

(i) Each congressional district shall consist of areas of convenient territory contiguous by land. Areas that meet only at points of adjoining corners are not contiguous.

Compactness is poorly defined

contiguous and compact territories. For purposes of this section, a "functionally contiguous and compact territory" is one that facilitates representation by minimizing impediments to travel within the district.

In order to compare the relative compactness of two or more districts or of two or more alternative redistricting plans, the Code provides that two measures of compactness, length-width compactness[74] and perimeter compactness,[75] shall be used.

(vii) Compactness shall be determined by circumscribing each district within a circle of minimum radius and measuring the area, not part of the Great Lakes and not part of another state, inside the circle but not inside the district.

The measures are basic

Polsby-Popper

$$0 < \mathrm{PP}(\Omega) = \frac{4\pi \cdot \mathrm{Area}(\Omega)}{\mathrm{Perim}^2(\Omega)} \leq 1$$

The measures are basic

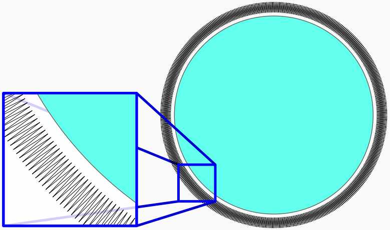
Polsby-Popper

scale-free

loves circles

isoperimetricky

sensitive



The measures are basic

Bounding regions

$$f(\Omega) = \frac{\mathrm{Area}(\Omega)}{\mathrm{Area}(\mathcal{B}(\Omega))}$$

The measures are basic

Bounding regions

$\mathcal{B}$ can be

- Circle [Reock]
- Square [Square Reock]
- Convex hull [Convex hull]
- Ellipse, rectangle
- Axis-aligned ellipse, rectangle
- ...

The measures are basic

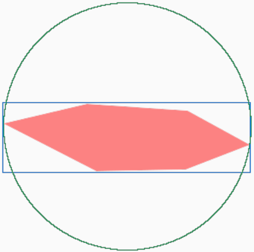
Bounding regions

scale-free

not sensitive at boundary

inconsistent

good interpretation?



The measures are basic

Miscellany

Largest inscribed circle

Just the perimeter

Longest axis by greatest orthogonal width

Population-weighted versions

Reciprocal of Polsby-Popper

What's the Takeaway?

The geometry is important, and a lot of geometry has been done in the last 2000 years. So, let’s use it. But, maybe we should care a little less.

This talk:

- The case for multiscale methods
- ‘Continuous’ definitions
    - Isoperimetric profiles/total variation
    - Curve-shortening flow
- ‘Discrete’ definitions
    - Constructing a dual graph
    - Discrete analogues
    - Graph spectrum
    - Discrete curvature?
- You should ask me questions

What's the dream?

Computable: we should have a good algorithm to find the measure

Informative: the score should say something about the geometry

Stable: similar shapes should have similar scores

Explainable: it should be easy to tell someone what’s going on

CONTINUOUS METHODS

Isoperimetric profiles

"Total Variation Isoperimetric Profiles" (2018), DeFord, Lavenant, Schutzman, & Solomon

"For all times $t \in (0, 1]$, what is the smallest perimeter of any inscribed subregion of Ω which fills a t-fraction of the area?"

Gives you a function or a curve or a vector from your shape.

What's so cool about it?

$t = 1$ recovers the Polsby-Popper score

Some basic algebra lets you get the largest inscribed circle

Stable under perturbations

The function and its derivative tell you some stuff about the shape at different resolutions

Formalization

$$\mathrm{TV}[f] = \int_{\mathbb{R}^n} \|\nabla f\|_2 \, dx.$$

$$\mathrm{area}(\partial\Sigma) = \mathrm{TV}[\mathbb{1}_\Sigma].$$

$$I_\Omega(t) = \begin{cases} \inf_{f \in L^1(\mathbb{R}^n)} & \mathrm{TV}[f] \\ \text{subject to} & \int_{\mathbb{R}^n} f(x)\mathrm{d}x = t \\ & 0 \le f \le \mathbb{1}_\Omega \\ & f(x) \in \{\, 0, 1 \,\} \; \forall x \in \mathbb{R}^n. \end{cases}$$

Convexify!

$$I_\Omega(t) = \begin{cases} \inf_{f \in L^1(\mathbb{R}^n)} & \mathrm{TV}[f] \\ \text{subject to} & \int_{\mathbb{R}^n} f(x)\mathrm{d}x = t \\ & 0 \le f \le \mathbb{1}_\Omega \\ & f(x) \in \{ 0, 1 \}\ \forall x \in \mathbb{R}^n. \end{cases}$$

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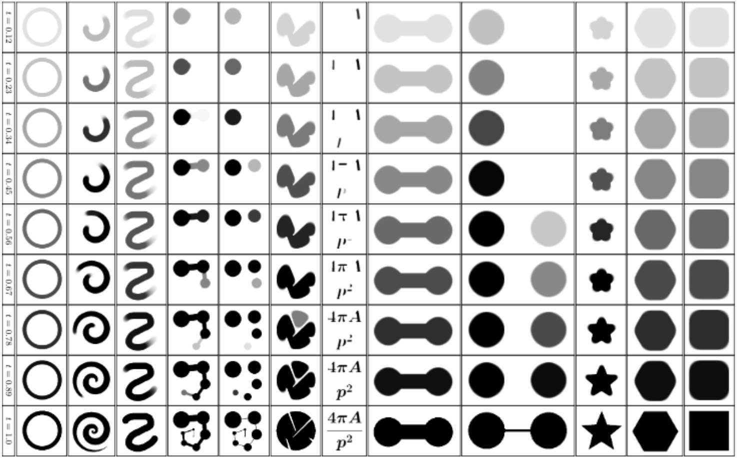
$$

Convexify!

$$I_\Omega(t) = \begin{cases} \inf_{f \in L^1(\mathbb{R}^n)} & \mathrm{TV}[f] \\ \text{subject to} & \int_{\mathbb{R}^n} f(x)\mathrm{d}x = t \\ & 0 \le f \le \mathbb{1}_\Omega \end{cases}$$

Using some duality arguments, we show that this is the *lower convex envelope* of the Isoperimetric profile. (See the paper)

See it in action!



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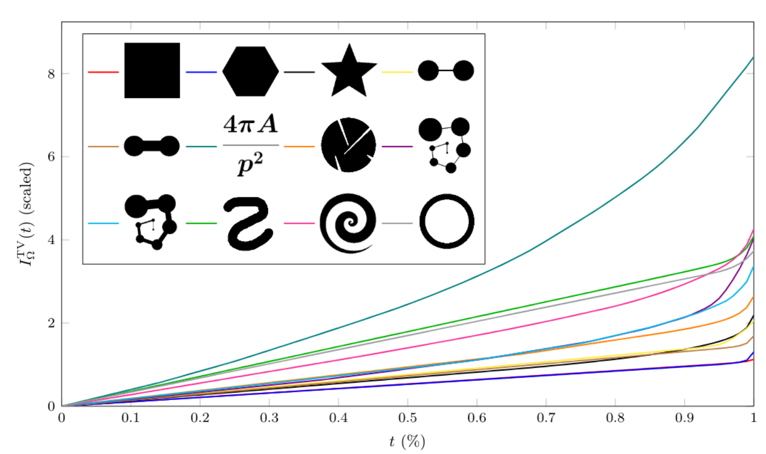

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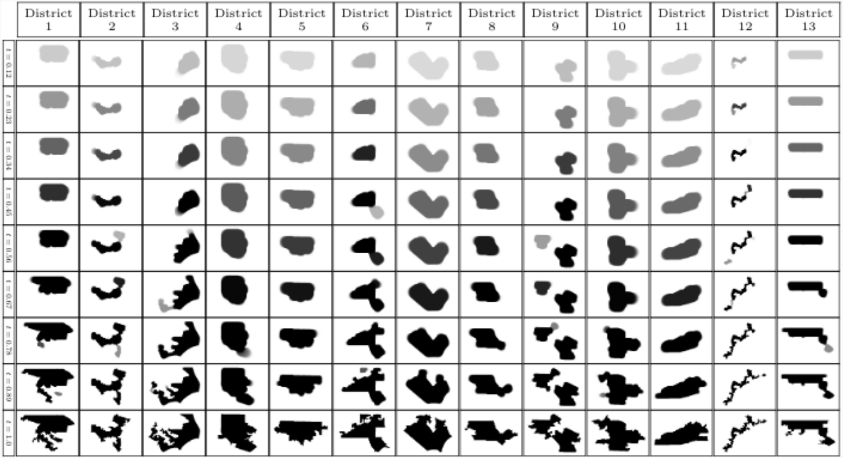

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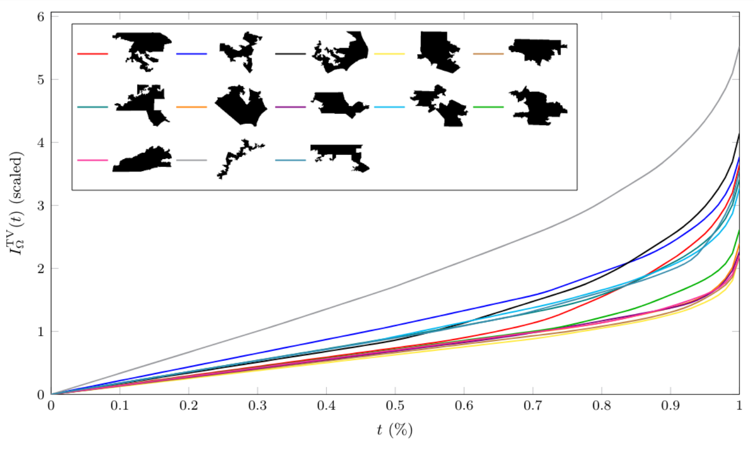


Nice properties

satisfies three of our desiderata

good algorithms to compute

we can make it measure-aware

isoperimetricky

An Open Problem

Open Problem

The TV relaxation works in $\mathbb{R}^n$ (examples of $\mathbb{R}^3$ in the paper) and should work over any metric space where all the calculus stuff makes sense.

Is there a good algorithm to compute the isoperimetric profile in $\mathbb{R}^2$?

Curve-shortening flow

take a (closed) smooth curve in the plane

at each time step, at each point:

(1) find the curvature κ

(2) move a distance proportional to κ ...

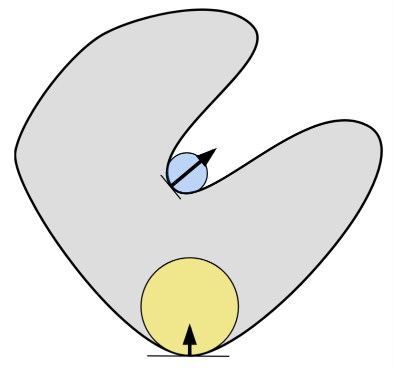
... in the direction normal to the tangent

(3) rescale the area

Curve-shortening flow

the perimeter shrinks

becomes a circle in finite time



Record the PP score at each time
This assigns a function to a shape

What's so cool about it?

$t = 0$ recovers the Polsby-Popper score

monotonically decreasing in t

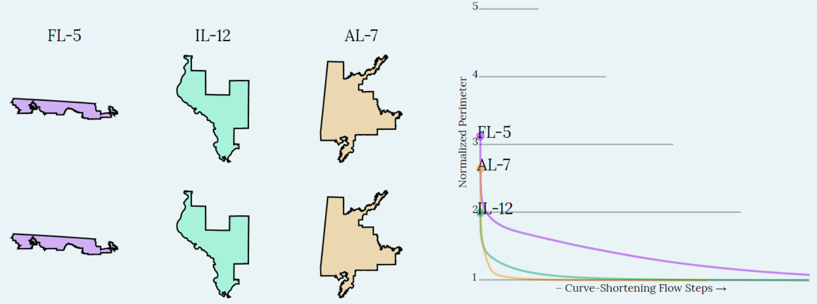
discretizes nicely

satisfies all four desiderata

The function and its derivative tell you some stuff about the shape at different resolutions

See it in action!

http://zachschutzman.com/distflow



Nice properties

satisfies our desiderata

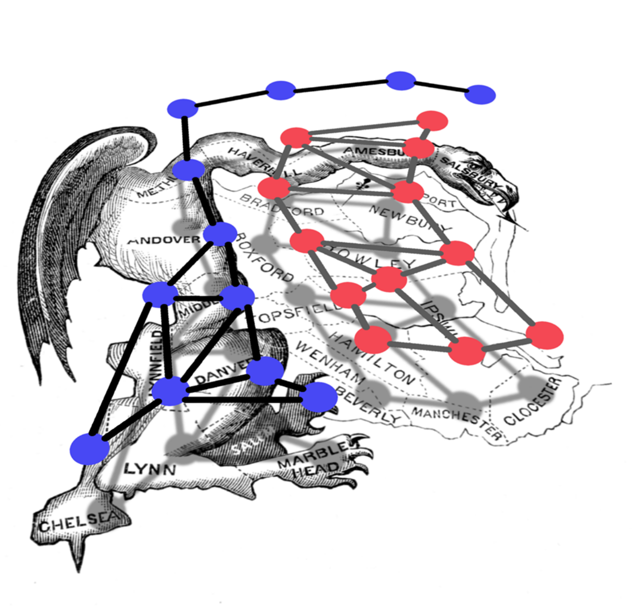
easy to compute

discretizes nicely

isoperimetricky

DISCRETE METHODS

Constructing the dual graph



Discrete geometry + classical scores

the districts are subgraphs

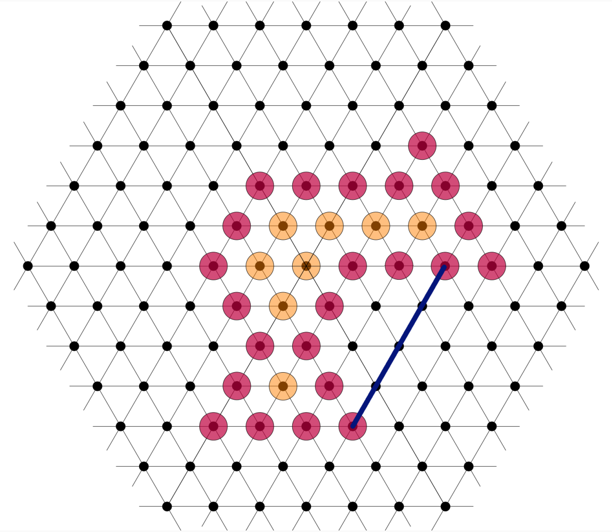
we can talk about ‘boundary’ and ‘interior’ nodes

there’s a natural metric to use

discrete polsby-popper

discrete convex hull

A quick illustration



Graphic adapted from Duchin & Tenner

What's the up side?

dual graphs have structure!

the sensitivity issue largely goes away

no longer depends on the $\mathbb{R}^2$ embedding

But, we know how to do more with graphs than just count vertices!

Graph Laplacian

Take a graph. Define the Laplacian $\mathcal{L}$ as the matrix with -1 in entry ij if edge ij is in the graph and $\deg(i)$ in entry ii. Zeros elsewhere.

This matrix is real and symmetric, so it's positive semi-definite

Let's consider its eigenvalues.

Laplacians

$$\mathcal{L}_S = \begin{bmatrix} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \end{bmatrix} \qquad \mathcal{L}_P = \begin{bmatrix} [\mathcal{L}_{d_1}] & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & [\mathcal{L}_{d_n}] \end{bmatrix}$$

$\mathcal{L}_P$ is $\mathcal{L}_S$ with some edges deleted.

These two matrices ‘know’ almost all of the discrete geometry of a districting plan.

Laplace spectrum: small eigenvalues

There's a zero eigenvalue for each connected component

The second eigenvalue is no more than the vertex connectivity

Moral truth: the kth eigenvalue says something about how easy it is to cut the graph into k pieces.

Laplace spectrum: large eigenvalues

The largest eigenvalue is less than the max degree

Summing in reverse, the degree sequence majorizes the eigenvalues

Kirchoff’s Matrix-Tree Theorem

Laplacians - Current work

help us do our research!

Summing eigenvalues correlates very strongly with geometric compactness measures. Why?

Do these have any meaning as operators?

Is there meaning to the Laplace eigenvectors?

THANK YOU!