Research Paper: Pricing and Hedging of American options

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This research project is part of the requirements imposed by the double-diploma that I am currently finalizing between ENSEIRB-MATMECA and Illinois Institute of Technology. I genuinely want to thank Prof. Bielecki for supervising me on this trading project as well as for teaching me key notions in Stochastic Processes and Continuous-time Financial models.

Contents

1 Motivation

The main purpose of this trading project is to hedge different American options with different positions. To do so, a paper account on Interactive Brokers (IB) has been used in order to access to real market data and to simulate the different trades realized.
Let's recall that an American-option is an option whose owner can decide to exercise at any time up to the expiration date (included).
Let's also provide a recap of the payoff diagrams that can be encountered for call and put options:

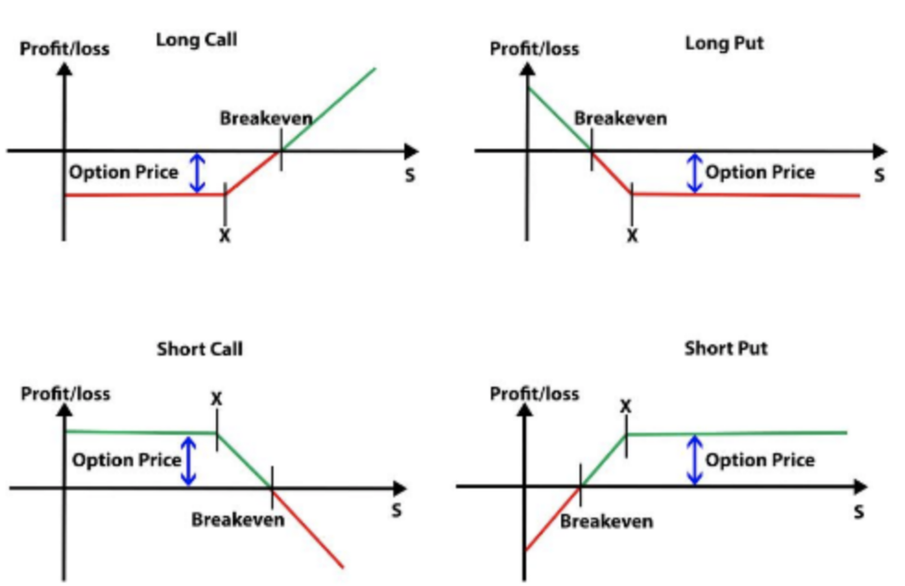

Figure 1: Source – Medium – Option payoff

First, we will start by giving more sense to what has been done by clarifying the context and the stakes of this project.

1.1 Pricing and Hedging

No matter how sophisticated a financial model is, it will always be an approximation of the reality. Each model has its own assumptions that eventually lead to different limitations in terms of capturing the reality.
A model can be used for different purposes such as pricing or hedging for example.
Pricing is the basis of Finance as every financial instrument needs to be continuously evaluated in order for an investor to approach the market with the best position possible. It is closely linked with the notion of **hedging** as a hedge is a move that is made with the intention of reducing the risk of adverse price movements in an asset.

There are different ways to hedge. In this project, one condition was to trade only the underlying stocks and borrow only from your IB paper account. No other assets were allowed to be traded. Therefore, a genuine focus has been made on The Greeks.

According to Investopedia, the variables that are used to assess risk in the options market are commonly referred to as **"The Greeks"**. The most common Greeks used include the delta, gamma and vega, which are the first partial derivatives of the options pricing model. Let's recall their expression:

- $\Delta = \frac{\partial V}{\partial S}$, represents he change in the value of an option in relation to the movement in the market price of the underlying asset.
- $\Gamma = \frac{\partial \Delta}{\partial S}$, represents the rate of change between an option's delta and the underlying asset's price.
- $\vartheta = \frac{\partial V}{\partial \sigma}$, represents the rate of change between an option's value and the underlying asset's implied volatility.

Greeks are used by options traders and portfolio managers to understand how their options investments will behave as prices move, and to hedge their positions accordingly.

During this project, I tried as much as possible to be Delta-Neutral. Obviously, even though I did not use them, I still computed gamma and vega as they are good risk indicators. A discussion will be carried later on ways to improve certain aspects of the hedging by considering those two metrics.

1.2 What needs to be modeled ?

A model provides dynamics about an underlying asset which means that it will follow certain properties and distributions according to assumptions imposed by the model. Usually, the main differences between models involve **implied volatility (constant or not), interest rates (constant or not) and the underlying/asset** (as the price evolves differently across asset classes).
Models used for simulating interest rates such as Vasicek or Hull-White have not been considered for this project as it has been realized during an internship that I was doing simultaneously. Another reason is the access for quotes that can be difficult for interest rates. The different GitHub repositories will be provided in the end of the paper.

1.3 Financial Models tested during the project

Two models have been considered. First of all, the main focus has been made on Black-Scholes-Merton model. **Black-Scholes-Merton model** is the pillar of model pricing in Finance. It is a popular model especially because it offers simple closed-form expressions for different financial instruments. It makes sense to study this model in the beginning as it lays the foundation to understand more complex models. This model has been used from the 5th of July to 20th of July.

Then, from the 20th of July until the end, **Heston model** has been considered and used. Heston model is a stochastic volatility model. Obviously, more details will be provided later in this paper.

The implementation has been realized on Python. In order to check if the functions were well coded, I also used Matlab that provided a Financial Toolbox containing functions related to those two models. For Black-Scholes-Merton, I can safely say that there is no different between my Python code and the Matlab function. For Heston, there is sometimes a slight difference as it is quite sensitive to the parameters used.

1.4 Presentation of the assets

Let us use the following notations Call(Ticker, T, K) for call option on underlying Ticker, maturity T, and Strike K. Similarly, we will use Put(Ticker, T, K) for the put options. All options are American style options.
The portfolios of the options to be hedged were defined as followed:

- Long 4000 Put(BAC, Jan 20 2023, 27.00)
- Short 6000 Call(CSCO, Jan 20 2023, 55.00)
- Short 2000 Put(GE, Jan 20 2023, 35.00)

Cisco develops, manufactures, and sells networking hardware, software, telecommunications equipment and other high-technology services and products. Here is the evolution of the stock over the last five years: What can be noticed is that the option is out of the money when trading started.

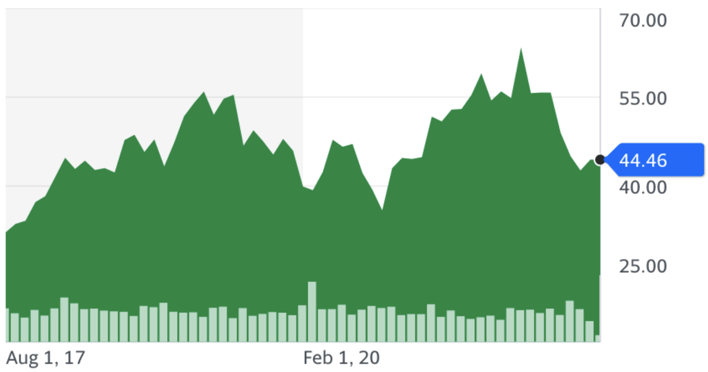

Figure 2: Evolution of CSCO Stock - Source: Yahoo Finance

Then, General Electric, that has been divided in three public companies in November 2021, focuses on different fields such as aerospace, healthcare, and energy (renewable energy, power and digital). Here is the evolution of the stock over the last five years:

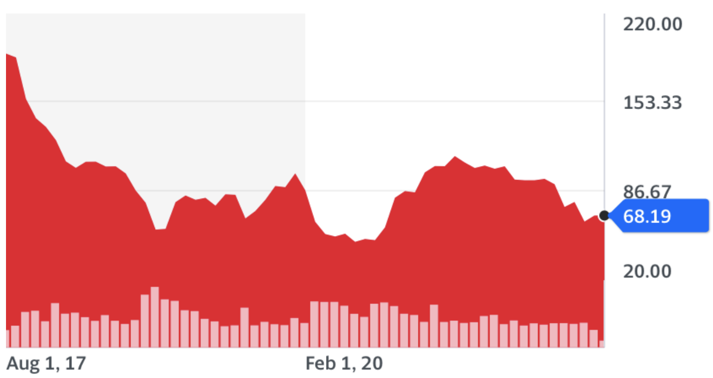

Figure 3: Evolution of GE Stock - Source: Yahoo Finance

What can be seen is that the option is far out of the money. It is an interesting phenomenon to study. Indeed, an option far out of the money usually displays a high implied volatility and a low volume traded.

At last, Bank of America is an American multinational investment bank and financial services holding company. Here is the evolution of the stock over the last five years:

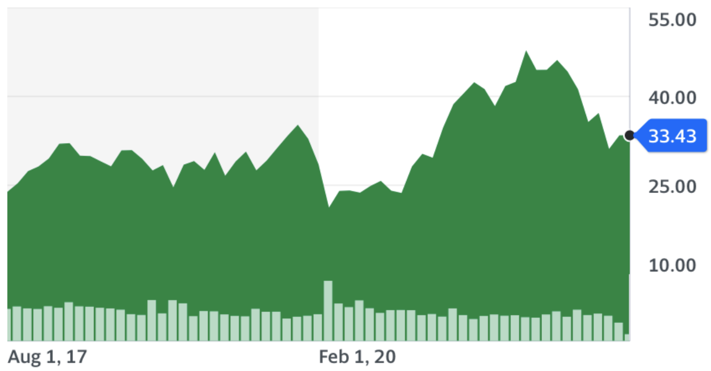

Figure 4: Evolution of BAC Stock - Source: Yahoo Finance

Therefore, **the first main challenge of the project is to be able to perfectly hedge the short positions**.
Ideally, the hedging portfolios will replicate through time the options sold. The value of the combined portfolio (hedging portfolio + short position in the option) has to be close to zero at every point in time, so that the hedging error (absolute value of the difference between the value of your hedging portfolio and the option price) is close to zero at every point in time. The final goal is to have the total hedging error as close to zero as possible.
Obviously, the goal is not to cheat but rather to trust the models and methods used in order to emphasize certain properties as well as certain limitations.

The second main challenge is to identify a way to derive the optimal time for the long position.

At last, some conditions were imposed for trading and rebalancing. First of all, the same number of the options has been kept at all the times till the end of the project. The hedging portfolio could be re-balanced as often as needed even though losses can be encountered because of bid-ask spread. I decided to re-balance the hedging portfolio every time the price of underlying asset changed by 3 % or more.

The first day of trading was the 5th of July 2022 and the last day was the 2nd of August.

At last, considering the choices made for implementing, I chose to stick to the analytical expressions (closed-form expressions) that can be encountered with the models picked. Simultaneously, I have been doing an internship using Monte-Carlo methods, so I wanted to differ from those methods in this project.

2 Black-Scholes-Merton model

Let's focus on the first model: Black-Scholes-Merton model. The goal is to provide an explanation of the different expressions that exist under that model. The well-known expressions have been used in the project in order to hedge and price. Thus, let's see how the option prices can be obtained as well as the Greeks.

2.1 Key assumptions

Black-Scholes-Merton model is not only famous for providing famous expressions to price derivatives but also for making genuine simple assumptions.

First and foremost, Black-Scholes-Merton model assumes that there are **no commissions nor transactions costs**. This means that there are no fees for buying and selling options and stocks. Moreover, the market is **perfectly liquid**: it is possible to purchase or sell any amount of stocks or options at any given time.
Afterwards, the Black-Scholes-Merton model only considers **European-style options** which can only be exercised on the expiration date whereas American-style options can be exercised at any time during the life of the option. This makes American options more valuable as they display a greater flexibility.
Volatility is a statistical measure of the dispersion of returns for a given security. In the securities markets, volatility is often associated with big swings in either direction. The volatility is assumed to be **constant**.
Similarly, the Black-Scholes-Merton model uses the **risk-free rate** to represent constant interest rates.

Finally, the model assumes that returns on the underlying stock are **log-normally distributed**. This means the stock is modeled by a geometric Brownian motion. The asset price St has instantaneous mean rate of return α and volatility σ. Here are the dynamics of the stock under Black-Scholes-Merton model:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t \tag{1}$$

where dt represents the variation of time and it is often referred as the drift term and Wt is a Brownian motion (or Wiener process) which means that it is a stochastic (random) process with normally independent distributed increments with a normal distribution N(0,$\sqrt{\Delta t}$). In other words, the volatility combined with a Brownian motion represents the uncertainty or the unexpected change that might occur in the variation of the stock, dS_t.
In the case of constant volatility, we can write:

$$S_t = S_0 \exp(\sigma W_t + (\alpha - \frac{1}{2}\sigma^2)t) \tag{2}$$

2.2 Presentation of the Partial Differential Equation - PDE

In order to present Black-Scholes-Merton Partial Differential Equation, there are two notions that are needed. First of all, the value of one dollar today is not the same as the value of one dollar dollar tomorrow: this is **discounting**. This is why it makes more sense to analyze the differential of the discounted portfolio value. Continuous compounding, with e^{-rt}, will be considered.

Secondly, another essential notion of Finance is needed to carry the explanation: **replication**. Let's denote X, an arbitrary contingent claim (an arbitrary derivative). Replication means that one can form a trading strategy such that the terminal value of this portfolio matches the payoff of X in all states.

Let's also recall **Itô-Doeblin formulas for an Itô process**:

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2}f_{xx}(t, X(t))dX(t)dX(t) \tag{3}$$

where f_t is the first-order partial derivative with respect to the time variable (t), f_x is the first-order partial derivative with respect to the stock price x and f_{xx} is the second-order partial derivative with respect to x.

Let's denote V_t, the value of the portfolio at each time t.
Let's define the investing context by considering an investor holding $\Delta(t)$ shares of stock. The remainder of the portfolio value is invested in the money market account with rate r.
Thus, the differential dV_t, of the investor's portfolio value comes from the capital gain on the stock position and the interest rate earnings. Therefore, we have:

$$\begin{aligned}
dV_t &= \Delta(t)dS_t + r(V_t - \Delta(t)S_t)dt \\
&= \Delta(t)(\alpha S_t dt + \sigma S_t dW_t) + r(V_t - \Delta(t)S_t)dt \\
&= rV_t dt + \Delta(t)(\alpha - r)S_t dt + \Delta(t)\sigma S_t dW_t
\end{aligned}$$

The goal is to use the replication property. To do so, we need to understand the variations of the discounted portfolio value as well as the variations of the discounted call price which is our contingent claim here.
First, let's apply this formula to the discounted portfolio value using: f(t,x)=$e^{-rt}x$. This leads to:

$$\begin{aligned}
d(e^{-rt}V_t) &= df(t, V_t) \\
&= f_t(t, V(t))dt + f_x(t, V(t))dV(t) + \frac{1}{2}f_{xx}(t, V(t))dV(t)dV(t) \\
&= -re^{-rt}V_t dt + e^{-rt}dV_t \\
&= \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW_t
\end{aligned}$$

Then, let's denote c(t,x), the value of the call at time t if the stock price at that time is S(t)=x. Let's compute the differential of c(t,S(t)):

$$\begin{aligned}
dc(t, S(t)) &= c_t(t, S(t))dt + c_x(t, S(t))dS(t) + \frac{1}{2}c_{xx}(t, S(t))dS(t)dS(t) \\
&= [c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t))]dt + \sigma S(t)c_x(t, S(t))dW_t
\end{aligned}$$

Let's apply Itô-Doeblin formula to discounted option price:

$$\begin{aligned} d(e^{-rt}c(t,S(t))) &= -re^{-rt}c(t,S(t))dt + e^{-rt}dc(t,S(t)) \\ &= e^{-rt}[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))]dt + e^{-rt}\sigma S(t)c_x(t,S(t))dW_t \end{aligned}$$

Now, the only thing that is required is to use the replication principle. A hedging portfolio starts with some initial capital V0 and invests in the stock and the banking account in order to match Vt at each time t with c(t,S(t)). To ensure this condition, the previous equations need to be equated such that for all t between [0,T]:

$$d(e^{-rt}V_t) = d(e^{-rt}c(t,S(t))$$

Subsequently, two key equations can be obtained. The first one is the delta-hedging rule. Indeed, if the dW_t terms are equated, we get:
For all t $\in [0,T)$,

$$\Delta_t = c_x(t,S(t)) \tag{4}$$

Finally, by equating the dt terms, the Black-Scholes-Merton partial differential equation appears as followed: For all $t \in [0,T)$, $x \geq 0$,

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x) \tag{5}$$

With the terminal condition,

$$c(T,x) = (x-K)^+ \tag{6}$$

2.3 Key Results

Now that the Black-Scholes-Merton PDE has been presented, it is interesting to study and to find the solution of this PDE. Here, the focus has been made on a call option. In the end of this section, a recap about all the expressions needed for trading is provided.

2.3.1 Risk-Neutral Pricing Formula

The goal is not to prove the continuous-time risk-neutral pricing formula but rather to use it. Still, let's recall the important steps that enable to obtain this important formula.
First of all, a discount process, D(t), must be introduced such that:

$$dD(t) = -R(t)D(t)dt$$

Then, the differential of the discounted stock price process can be written as:

$$\begin{aligned} d(D(t)S(t)) &= (\alpha(t) - R(t))D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) \\ &= \sigma(t)D(t)S(t)[\Theta(t)dt + dW(t)] \end{aligned}$$

Thanks to **Girsanov's Theorem**, the probability measure $\tilde{\mathbb{P}}$ can be introduced such that we may rewrite:

$$d(D(t)S(t)) = \sigma(t)D(t)S(t)d\tilde{W}(t)$$

The measure define in Girsanov's theorem is the risk-neutral measure because it is equivalent to the original measure $\mathbb{P}$ and it renders the discounted stock price D(t)S(t) into a martingale.
Therefore, we have:

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\tilde{W}(t)$$

In continuous-time, the change from the actual measure $\mathbb{P}$ to the risk-neutral measure $\tilde{\mathbb{P}}$ changes the mean rate of return of the stock but not the volatility.
Afterwards, the value of the portfolio process under the Risk-Neutral Measure must be investigated.

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt \\ &= R(t)X(t)dt + \Delta(t)\sigma(t)S(t)[\Theta(t)dt + dW(t)] \end{aligned}$$

Using Itô's product rule, we get:

$$d(D(t)X(t)) = \Delta(t)d(D(t)S(t)) = \Delta(t)\sigma(t)S(t)D(t)d\tilde{W}(t)$$

Hence, the discounted value of the portfolio is a martingale under the risk-neutral probability measure. This means that:

$$D(t)X(t) = \tilde{\mathbb{E}}[D(T)X(T)|\mathcal{F}_t] = \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}_t]$$

where V is the payoff of the derivative security. This is the the continuous-time risk-neutral pricing formula:

$$D(t)V(t) = \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}_t], 0 \leq t \leq T \tag{7}$$

2.3.2 Call expression under Black-Scholes-Merton Model

Let's now use this formula. However, before that, let's just adjust the dynamics in order to take dividends into account.
Consider a stock, modeled as a generalized Brownian motion, that pays dividends over time at a rate A(t) per unit time. Here A(t) is a non-negative process. Dividends paid by a stock reduce its value. Thus, the model for the stock price can be written as followed:

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) - A(t)S(t)dt \tag{8}$$

With constant coefficients σ, r, and a, this leads to:

$$S(t) = S(0)exp[\sigma\tilde{W}(t) + (r - a - \frac{1}{2}\sigma^2)t] \tag{9}$$

According to the risk-neutral pricing formula, the price at time t of a European call expiring at time T with strike K is:

$$V(t) = \tilde{\mathbb{E}}[e^{-r(T-t)}(S(T) - K)^+|\mathcal{F}_t] \tag{10}$$

We can compute c(t,x) using **Independence Lemma**.
This eventually leads to:

$$\begin{aligned} c(t,x) &= \tilde{\mathbb{E}}[e^{-r(T-t)}(xexp[\sigma(\tilde{W}(T) - \tilde{W}(t)) + (r - a - \frac{1}{2}\sigma^2)(T-t)] - K)^+)] \\ &= \tilde{\mathbb{E}}[e^{-r\tau}(xexp(-\sigma\sqrt{\tau}Y + (r - a - \frac{1}{2}\sigma^2)\tau) - K)^+] \end{aligned}$$

We denote $Y = -\frac{\tilde{W}(T)-\tilde{W}(t)}{\sqrt{T-t}}$, which is a standard normal random variable under $\tilde{\mathbb{P}}$. Let's define $d_{\pm}(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}[log\frac{x}{K} + (r - a \pm \frac{1}{2}\sigma^2)\tau]$.
We require $Y < d_-(\tau,x)$ so that we have:

$$\begin{aligned} c(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x} e^{-r\tau}(xexp(-\sigma\sqrt{\tau}y + (r - a - \frac{1}{2}\sigma^2)\tau) - K)e^{-\frac{1}{2}y^2}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x} xexp(-\sigma\sqrt{\tau}y - (a + \frac{1}{2}\sigma^2)\tau - \frac{1}{2}y^2)dy - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x} e^{-r\tau}Ke^{-\frac{1}{2}y^2}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x} xe^{-a\tau}exp(-\frac{1}{2}(y + \sigma\sqrt{\tau})^2)dy - e^{-r\tau}KN(d_-(\tau,x)) \end{aligned}$$

At last, we make the change of variable $z = y + \sigma\sqrt{\tau}$ in the integral, which leads us to the following formula:

$$c(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_+(\tau,x} xe^{-a\tau}e^{\frac{z^2}{2}}dz - e^{-r\tau}KN(d_-(\tau,x)) \tag{11}$$

$$= xe^{-a\tau}N(d_+(\tau,x)) - e^{-r\tau}KN(d_-(\tau,x)) \tag{12}$$

The expression for a put option can be obtained by applying the same method or by using Put-call parity.

The derivations for The Greeks are quite straightforward for Black-Scholes-Merton model. Here is a recap of all the expressions that have been used during the project:

	Call	**Put**
Option Price	$c(t,x) = xe^{-a\tau}N(d_+(\tau,x)) - e^{-r\tau}KN(d_-(\tau,x))$	$p(t,x) = Ke^{-r\tau}N(-d_-(\tau,x)) - xe^{-a\tau}N(-d_+(\tau,x))$
Delta	$\Delta_C = e^{-a\tau}N(d_+(T-t,x))$	$\Delta_P = e^{-a\tau}[N(d_+(T-t,x) - 1]$
Gamma	$\Gamma = \frac{e^{-a\tau}}{x\sigma\sqrt{T-t}}N'(d_+(T-t,x))$	$\Gamma = \frac{e^{-a\tau}}{x\sigma\sqrt{T-t}}N'(d_+(T-t,x))$
Vega	$\frac{1}{100}xe^{-a\tau}\sqrt{T-t}N'(d_+(T-t,x))$	$\Gamma = \frac{e^{-a\tau}}{x\sigma\sqrt{T-t}}N'(d_+(T-t,x))$

Figure 5: Expressions obtained and used under Black-Scholes-Merton Model

2.4 Some serious limitations

As it has been explained earlier, Black-Scholes-Merton model is based on several assumptions. Among them, some are reasonable. Indeed, assuming the market perfectly liquid or modeling the stock with a log-normal distribution (Geometric Brownian Motion) are far from being bad approximations for a financial model.

Nevertheless, assuming no fees nor taxes on transactions is a first drawback for this model. This will be discussed later in the section describing the different ways to improve the trading.
But more importantly, the biggest limitations remain the ones made on interest rates and volatility. During the life of the option, rates and volatility are assumed to be constant. Therefore, when I was trading using Black-Scholes-Merton model, I kept the same interest rate (1.68%) and volatility that I got from the first day, when I entered the position.

Supposing a risk-free rate (constant) is not a good way to model rates. It can be assimilated with the 10Y Treasury bond but sill it is a poor approximation. However, during this project, I did not try to model them using the dynamics of Vasicek or Hull-White for instance (I did this during my internship).
I chose to focus on volatility which is the other serious limitation of this model. The next section is dedicated to volatility to genuinely understand the stakes spawn by this parameter.

The performances under Black-Scholes-Merton model will be analyzed during Part VI. It remains a good model in such that it gives a good first approximation and an idea of the price of an option.

3 Focus on volatility

In this section, the goal is to understand why volatility needs to be modeled or approximated. **It is a key parameter that influences significantly the accuracy of pricing**.

3.1 Historical volatility vs Implied volatility

First of all, there are two types of volatility that exist: **historical volatility and implied volatility**. There are numerous ways to compute historical volatility but the easiest one is to the standard deviation of the annualized returns. There are other methods that exist using entropy for example and that can be found here.
According to Investopedia, the term implied volatility refers to a metric that captures the market's view of the likelihood of changes in a given security's price. Investors can use implied volatility to project future moves and supply and demand, and often employ it to price options contracts.

What needs to be understood is that implied volatility is not known through formulas but it can be derived when the real-market quotes are known. Indeed, when the they are known, volatility is the only unknown and as such it can be computed. For Black-Scholes-Merton model, it is quite straightforward as volatility can be isolated and computed. However, for other models, it can get more difficult and then optimization algorithms such as the gradient descent can be used in order to find a local minimum and thus the implied volatility.

3.2 Volatility smile $ Volatility skew

So, why does volatility need to be modeled ? Well, there is a phenomenon called **"volatility skew"** that needs to be studied which to some extent is linked with volatility smile. Volatility smiles have first been observed after the 1987 stock market crash.
According to Investopedia, when options with the same expiration date and the same underlying asset, but with different strike prices, are graphed for implied volatility, the tendency is for that graph to show a smile. The smile shows that the options that are furthest in the money (ITM) or out of the money (OTM) have the highest implied volatility. On the other hand, options with the lowest implied volatility have strike prices at the money (ATM) or near the money. Thus, if the implied volatility is plotted according to strike prices, a U-Shaped curve would be observed.

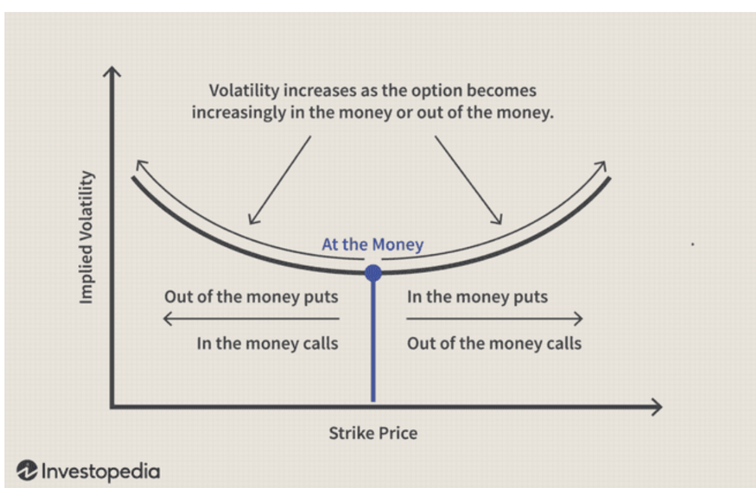

Image by Julie Bang © Investopedia 2019

Figure 6: U-Shaped curve

If the curve is unbalanced on one-side, it is called a volatility smirk. I emphasized this phenomenon using Yahoo Finance data on BAC. The second phenomenon that can be noticed is the "volatility skew".

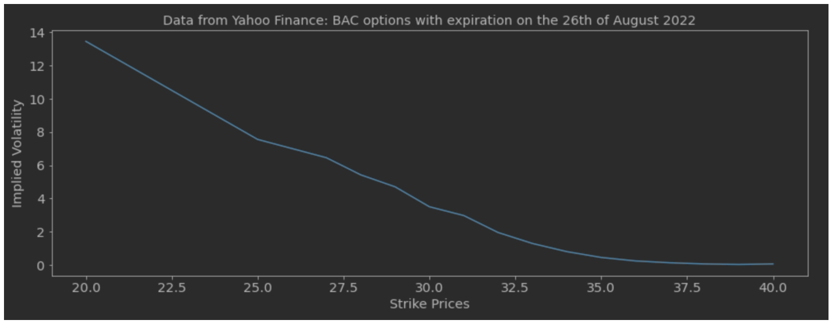

Figure 7: Volatility Smirk noted on BAC options

Options contracts for the same underlying asset with the same expiration date but different strike prices have a range of implied volatility. In other words, it's a graph plot of implied volatility points representing different strike prices or expiration dates for options contracts. For some underlying assets, there is a convex volatility "smile" that shows that demand for options is greater when they are in-the-money or out-of-the-money, versus at-the-money.

3.3 Volatility & Black-Scholes-Merton model

There is a real debate among traders about which volatility should be considered for Black-Scholes-Merton model. Indeed, the volatility smile is not predicted by the Black-Scholes-Merton model as the model predicts that the implied volatility curve is flat when plotted against varying strike prices. Based on the model, it would be expected that the implied volatility would be the same for all options expiring on the same date with the same underlying asset, regardless of the strike price. Yet, in the real world, it is not the case.
Therefore, some would say that the historical volatility should be considered as it is unique. Nonetheless, it gives poor result for options with a long expiration.
As a result, when I was trading using Black-Scholes-Merton model, on the fist day, I decided to estimate the implied volatility on the strike imposed and I kept this volatility until the end of the project.

This why more complex models need to be studied. There are are different ways to gain in accuracy by handling the volatility properties.

3.4 SABR model

First, volatility can be estimated thanks to **SABR model**. Indeed, the SABR model is used to model a forward LIBOR rate, a forward swap rate, a forward index price, or any other forward rate. It is an extension of Blackís model and of the CEV (Constant Elasticity Variance) model. But unlike other stochastic volatility models such as the Heston model, the model does not produce option prices directly. Rather, it produces an estimate of the implied volatility curve, which is subsequently used as an input in Blackís model to price swaptions, caps, and other interest rate derivatives.

The model has gained widespread use due to its tractable ability to capture both the correct shape of the smile, as well as the correct dynamics of the volatility smile.
SABR model assumes the volatility of the forward price is a stochastic variable. Thus, the forward price and the volatility can be defined as followed:

$$dF_t = \sigma_t (F_t)^{\beta} dW_1(t) \tag{13}$$

$$d\sigma_t = \alpha \sigma_t dW_2(t) \tag{14}$$

The two Wiener processes are correlated so that: $dW_1(t)dW_2(t) = \rho dt$.
The coefficient α is positive and controls the height of the implied volatility level. The correlation ρ controls the slope of the implied skew and β controls its curvature.
The prices of European call options in the SABR model are given by Black's model. Nonetheless, the volatility parameter is provided by the SABR model such that:

$$\sigma_B(K,f) = \frac{\alpha\left(1 + \left(\frac{(1-\beta)^2\alpha^2}{24(fK)^{1-\beta}} + \frac{\rho\beta\alpha v}{4(fK)^{0.5(1-\beta)}} + \frac{(2-3\rho^2)v^2}{24}\right)T\right)}{(fK)^{0.5(1-\beta)}\left(1 + \frac{(1-\beta)^2 ln^2 \frac{f}{K}}{24} + \frac{(1-\beta)^4 ln^4 \frac{f}{K}}{1920}\right)} \tag{15}$$

3.5 Stochastic Volatility model

A natural extension of considering fixed volatilities is to instead consider probabilistic volatilities. Rather than assume the volatility is a fixed quantity, the volatility is instead modeled by its own stochastic process (and hence, these are **"stochastic volatility models"**). The motivation behind these models is to try to better model the dynamics of the stock behavior, by more realistically reflecting the dynamics seen in real returns. One popular feature is to model volatility clustering through an autoregressive model, wherein the value of the volatility today is some function of the recent history of the volatility, plus some random noise. These models often also include some level of mean-reversion, to model the observed return of volatility to a particular level over time.

By adding a source of a second source of randomness, the equations governing the dynamics become more computationally complex.
The second model studied is Heston model which is one of the popular stochastic shifted volatility models.

4 Heston model

The second model that has been tested is Heston model. It is a stochastic volatility model which means that the volatility of the asset is not constant but it follows a random process. Actually, the instantaneous variance, is given by a Feller square-root or CIR process.

4.1 Closed-form expressions

The stochastic differential equation is:

$$dS_t = (r-q)S_t dt + \sqrt{\nu_t} S_t dW_t \tag{16}$$
$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma_\nu \sqrt{\nu_t} dW_t^\nu \tag{17}$$
$$\mathbb{E}[dW_t dW_t^\nu] = \rho dt \tag{18}$$

Let's explain the different parameters:

- r, is the continuous risk-free rate.
- q, is the continuous dividend yield.
- S_t, is the asset price at time t.
- ν_t, is the asset price variance at time t
- v_0, is the initial variance of the asset price at t = 0 for ($v_0 > 0$).
- θ, is the long-term variance level for (> 0).
- κ, is the mean reversion speed for the variance for ($\kappa > 0$).
- σ_ν, is the volatility of the variance for (v > 0).
- ρ, is the correlation between the Wiener processes W_t and W_t^ν for (-1 $\leq \rho \leq 1$).

Based on the dynamics, Heston model can be easily generated and simulated with Monte-Carlo methods especially to get the prices and the Greeks. Nonetheless, this is why I did during my internship so I decided for the project to study the analytical expressions that exist under Heston model. As Black-Scholes-Merton model, It is a strength of this model: it provides closed-form expressions.

Then, let's focus on the call price expression and especially on the numerical integration method under the Heston (1993) framework. This method is based on the following expressions:

$$Call(K) = S_t e^{-q\tau} P_1 - K e^{-r\tau} P_2 \tag{19}$$
$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re[\frac{e^{-i\phi ln(K)} f_j(\phi)}{i\phi}] d\phi \tag{20}$$

What can be noticed is that the call expression is really similar to the one obtained under Black-Scholes-Merton model. The only thing that differs is the probabilities P_1 and P_2 that were respectively $N(d_1)$ and $N(d_2)$. Generally, this equation applies to many models (CEV or jump diffusion from Merton and Kou for example) where only the probabilities are different according to the model. It is a result of the change of numéraire technique.

Let's give more details about the the characteristic function, $f_j(\phi)$ for P_j(j = 1,2).
First of all, P_1 is the probability of St > K under the asset price measure for the model and P_2 is the probability of St > K under the risk-neutral measure for the model.
The characteristic function $f_j(\phi)$ for j = 1 (asset price measure) and j = 2 (risk-neutral measure) is:

$$f_j(\phi) = exp(C_j + D_j\nu_0 + i\phi lnS_t) \tag{21}$$

$$C_j = (r-q)i\phi\tau + \frac{\kappa\theta}{\sigma_\nu^2}[(b_j - \rho\sigma_\nu i\phi + d_j)\tau - 2ln(\frac{1-g_je^{d_j\tau}}{1-g_j})] \tag{22}$$

$$D_j = \frac{b_j - \rho\sigma_\nu i\phi + d_j}{\sigma_\nu^2}(\frac{1-e^{d_j\tau}}{1-g_je^{d_j\tau}}) \tag{23}$$

$$g_j = \frac{b_j - \rho\sigma_\nu i\phi - d_j}{b_j - \rho\sigma_\nu i\phi + d_j} \tag{24}$$

$$d_j = \sqrt{(b_j - \rho\sigma_\nu i\phi)^2 - \sigma_\nu^2(2u_ji\phi - \phi^2)} \tag{25}$$

$$u_1 = \frac{1}{2}, u_2 = -\frac{1}{2} \tag{26}$$

$$b_1 = \kappa + \lambda_{VolRisk} - \rho\sigma_\nu, b_2 = \kappa + \lambda_{VolRisk} \tag{27}$$

where ϕ, is the characteristic function variable and $\lambda_{VolRisk}$, is the volatility risk premium.

There are different methods that exist that enable to derive the call price under Heston assumptions. They differ by the way the integral is computed (Lewis (2001), Car-Madan (1999) with the use of the inverse Fourier transform for example).
I chose to stick to the original framework presented above.

4.2 Calibration

One drawback of Heston model is calibration, even though it can be realized by minimizing the squared difference between the prices observed in the market and those calculated from the model. That is exactly what I have done. I got the real market quotes from Yahoo Finance and I minimized those quotes with prices from my model using the squared root error.
It worked pretty well, but it was really sensitive especially because Yahoo Finance quotes are not continuously updated. Therefore, after having a good first approximation, I finished the calibration on Matlab using the pre-existing functions.
Here is a recap of the parameters obtained during the project under Heston model:

	CSCO				
Date of Re-calibration	**V0**	**Kappa**	**Theta**	**Volvol**	**Rho**
05/07/2022	0,1125	1,6181	0,0268	0,3121	-0,6
20/07/2022	0,0825	1,66	0,0568	0,3158	-0,58
27/07/2022	0,0775	1,66	0,0568	0,3158	-0,58
	GE				
05/07/2022	0,3022	0,1211	0,0108	0,0412	0,2
27/07/2022	0,247	0,1211	0,0108	0,0412	0,2

Figure 8: Heston Calibration

5 American Put options

Obviously, the approach on the long position is genuinely different as the key is to know whether to exercise the option or not. Things would have been clear and straightforward with a call option, as no dividends were paid during the period of trading. Indeed, if the underlying stock does not pay dividends, then the price of an American Call option coincides with the price of the corresponding European call option, and it is optimal to exercise the option only at maturity.

On average, we expect that the stock price will go up, and thus longer we wait, deeper in the money the Call option gets. This argument also hints that if the stock pays dividends, then it may be optimal to exercise the Call option right before the dividend is paid; if the dividend is large, the stock may have not enough time till maturity to recover to desired level that makes the Call in the money.

However, for this project, the long position was on a Put option. We need to investigate a finite maturity option, so, in principle we should compute a time dependent exercise boundary. But, the maturity for all your options is January 2023, so we can pretend that these are perpetual options.
The simplest American option is the **perpetual American put**. it is interesting because the optimal exercise policy is not obvious, and it is simple because this policy can be determined explicitly.
The underlying asset has the price process S(t) given by:

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t)$$

where the interest rate r and the volatility σ are strictly positive constants and $d\tilde{W}(t)$ is a Brownian motion under the risk-neutral probability measure $\tilde{\mathbb{P}}$.
The perpetual American put pays K-S(t) if it is exercised at time t. This is its **intrinsic value**. The price of the perpetual American put is defined to be:
Let $\mathcal{T}$, be the set of all stopping times. Then,

$$v_*(x) = max\tilde{\mathbb{E}}[e^{-r\tau}(K - S(\tau))] \tag{28}$$

This means that the owner of the perpetual American put can choose an exercise time τ, subject only to the condition that she may not look ahead to determine when to exercise. The price of the option at time zero is the risk-neutral expected payoff of the option, discounted from the exercise time back to time zero.

The owner of the option should choose the exercise strategy that maximizes this expected payoff. The owner of the perpetual American put can exercise at any time. There is no expiration date after which the put can no longer be exercised. This makes every date like any other date: the time remaining to expiration is always the same (infinity). Therefore, it is reasonable to expect that the optimal exercise policy depends only on the value of S(t) and not on the time variable t.

The owner of the put should exercise as soon as S(t) falls "far enough" below K. In other words, we expect the optimal exercise policy to be of the form:

"Exercise the put as soon as S(t) falls to the level L_*."

Obviously, it is essential to find the value of L_* as well as the value of the put.
Suppose the owner of the perpetual American put sets a positive level L<K and resolves to exercise the put the first time the stock falls to L. If the initial stock is at or below L, then he exercises at time zero. The value of the put in this case is $v_L(S(0)) = K - S(0)$. If the initial stock price is above L, then he exercises at the stopping time:

$$\tau_L = mint \geq 0; S(t) = L$$

where τ_L is set to ∞ if the stock price never reaches the level L. At the time of exercise, the put pays $K - S(\tau_L) = K - L$. Discounting this back to time zero and taking the risk-neutral expected value, we compute the value of the put under this exercise strategy to be:

$$v_L(S(0)) = (K - L)\tilde{\mathbb{E}}e^{-r\tau_L} \tag{29}$$

for all S(0) $\geq L$.

The function $v_L(x)$ is given by the formula:

$$v_L(x) = \begin{cases} K - x, 0 \leq x \leq L \\ (K-L)(\frac{x}{L})^{\frac{-2r}{\sigma^2}}, x \geq L \end{cases}$$

This can be proved especially thanks to the Laplace transform used for first passage time of drifted Brownian motion.
Let's denote L_*, the optimal value of L that maximizes the value of the put when x is fixed.
Thus, we have:

$$v_L(x) = (K-L)L^{\frac{2r}{\sigma^2}}x^{\frac{-2r}{\sigma^2}}$$

and we define:

$$g(L) = (K-L)L^{\frac{2r}{\sigma^2}}$$

We seek the value of that maximizes this function over L ≥ 0 and this can be done by deriving the first order derivative that we solve equal to zero.

This eventually leads to:

$$L_* = \frac{2r}{2r+\sigma^2}K \tag{30}$$

$$g(L_*) = \frac{\sigma^2}{2r+\sigma^2}(\frac{2r}{2r+\sigma^2})^{\frac{2r}{\sigma^2}}K^{\frac{2r+\sigma^2}{\sigma^2}} \tag{31}$$

At last, what needs to be done in practice, is to check daily if the level L_* computed right above is ever reached during the Trading period.

6 Results & Performances

6.1 How did the stocks evolve during the Trading period ?

Obviously, before looking at the performances in terms of pricing and hedging, it is important to know how the stocks evolve during the trading period.
First, CSCO stock gradually increased. Our position was a short one on a call. So, obviously, the fact that the stock increases means that the owner is getting closer to the strike and therefore, as a seller, we need to buy more stocks to hedge the option position.

Figure 9: Evolution of CSCO stock during the Trading period

On the other hand, GE stock increased quite significantly. This is also linked with the fact that the company was announcing the earnings on 26th of July and they were quite astonishing. A short position on a put option was hold over this stock. As a seller, this means that we gradually sold stocks as the buyer was less likely to exercise his right. Indeed, the stock was going even further from the strike that was already deep out of the money.

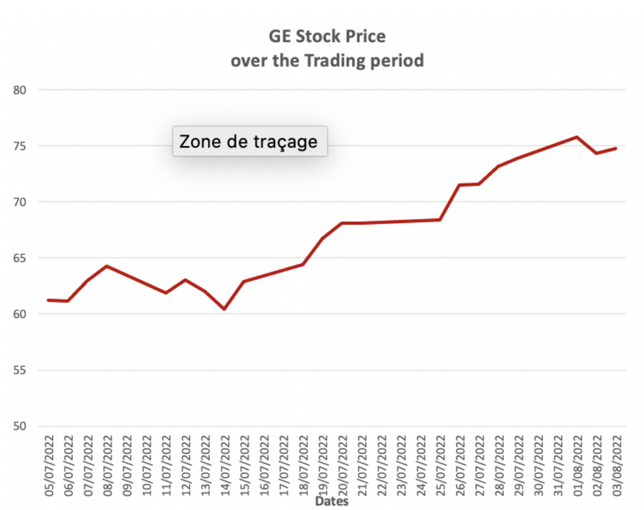

Figure 10: Evolution of GE stock during the Trading period

6.2 Call & Put options: Model Pricing vs Real Market Quotes

Then, what is essential to analyze is how well the models were to estimate the prices of the different options.
First, let's start with the call option on CSCO.

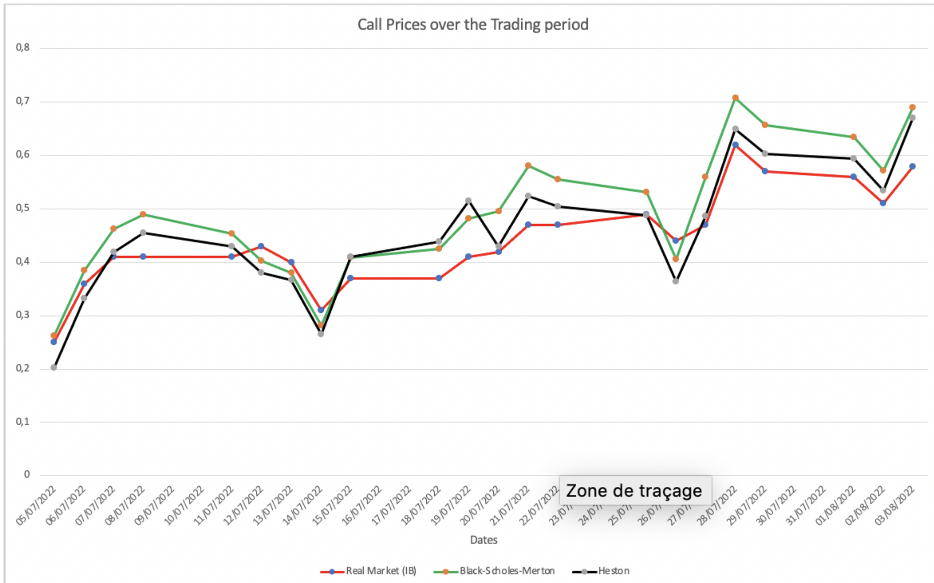

Figure 11: CSCO call prices: real market quotes vs Black-Scholes vs Heston

What can be observed is that Heston model does a better job at capturing the real-market data than Black-Scholes-Merton model. Nonetheless, it is also essential to highlight that Black-Scholes-Merton model gives a really good first approximation. This could be expected as we deal with a call option with no dividends, so Black-Scholes model that is usually only for European-style options can be used here for an American-style option.
Let's confirm those good results with Root-Mean Squared Error (RMSE) between real-market data and the estimations obtained from the models:

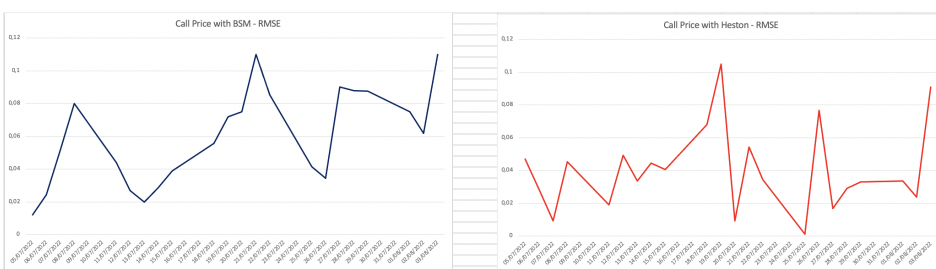

Figure 12: RMSE Black-Scholes & Real-Market vs RMSE Heston & Real-Market

Then, the same work has been made on the GE put option. Here the efficiency of Heston model over Black-Scholes-Merton model is even more relevant.

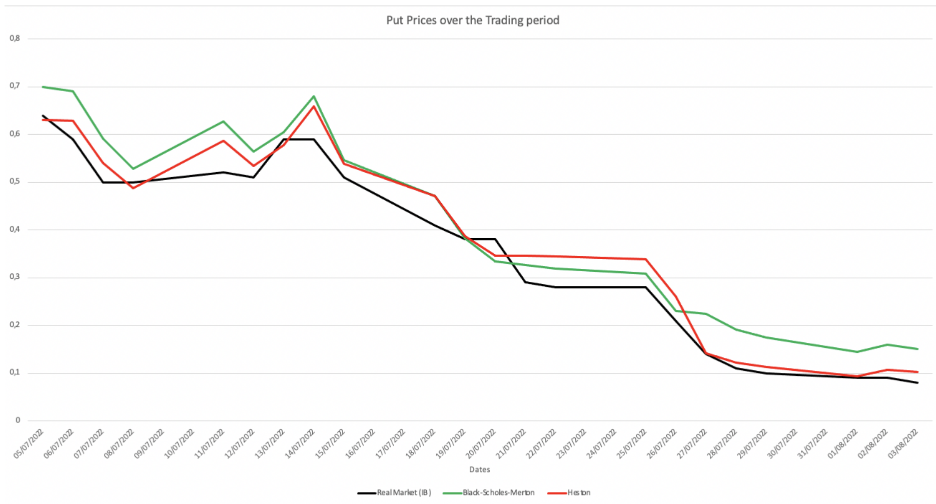

Figure 13: GE put prices: real market quotes vs Black-Scholes vs Heston

Once again, this can be verified by plotting the RMSE of both models:

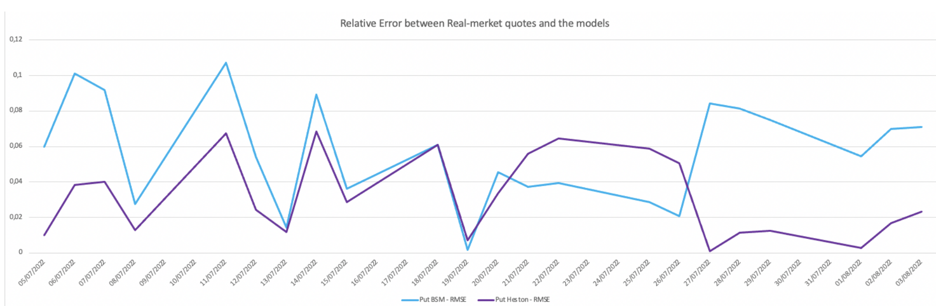

Figure 14: RMSE Black-Scholes & Real-Market vs RMSE Heston & Real-Market

6.3 Call & Put options: Hedging

One of the main goals of the project was also to be able to hedge the short positions on options (using the underlying stock exclusively). To do so, I used a Delta-neutral strategy with a rebalancing realized every time the underlying stock price changed by more than 3%.
Here is the hedging portfolio realized for the call option on CSCO:

Figure 15: CSCO Hedged portfolio

Likewise, here is the hedging portfolio realized on the put position on GE:

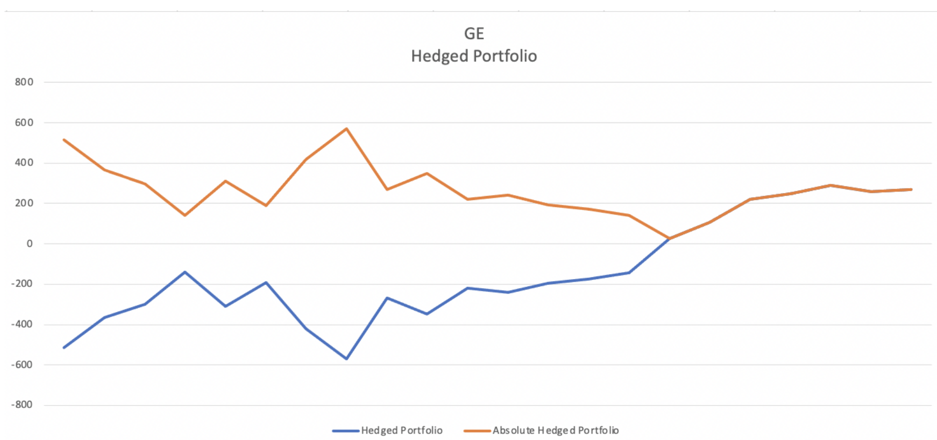

Figure 16: GE Hedged portfolio

Obviously, the plots are not flat as they are supposed to be. Many factors can explain that. First of all, fees and taxes are not taken into account with the models used. Moreover, maybe I should have re-balanced more often. But, in a way, the goal was also not to cheat but rather to trust the models in order to highlight their limitations.

At last, here is the total hedged portfolio which is a combination of the two presented above:

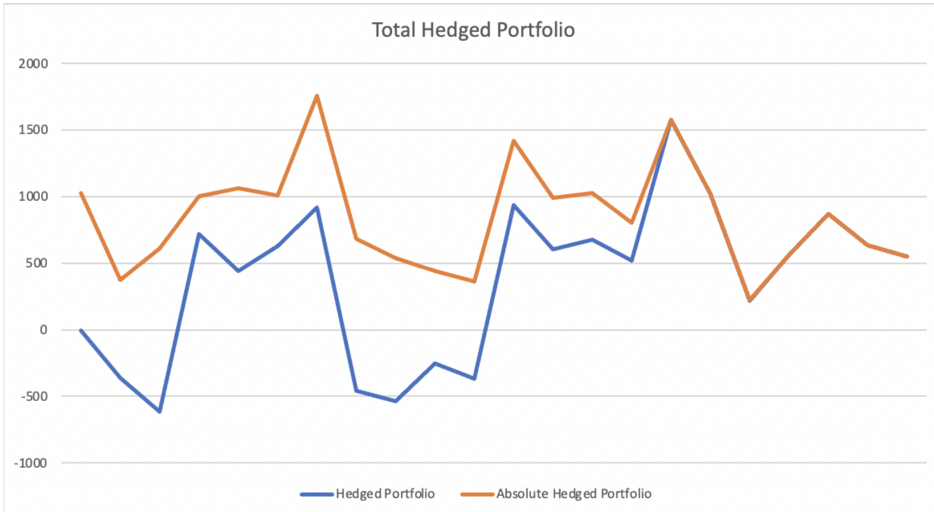

Figure 17: Total Hedged portfolio

It is not perfectly flat, but there are no trends (no monotonic plots observed) which is good point.

6.4 Long position

The approach on the long position was genuinely dependent on the evolution of stock price. In the beginning, it was quite close from the strike. And this trend got reinforced when Bank of America did underperform on annual health check realized by the Federal Reserve: `https://finance.yahoo.com/news/bank-sector-rallies-passing-stress-145418447.html`. Yet, then the earnings presented on the 18th of July were really convincing and the stock price went up quite significantly. Therefore, any chances of exercising the option disappeared as the stock was further from the strike.

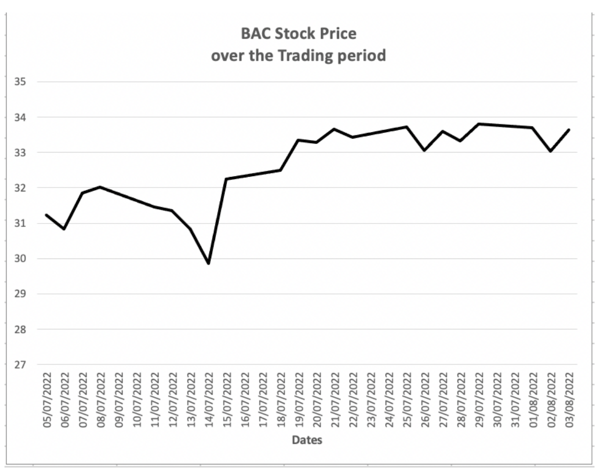

Figure 18: Evolution of BAC stock during the Trading period

7 Ways to improve the performances

There are different ways to improve the hedging realized during this project.
First of all, one of the restrictions was to only hedge with the underlying stock and the banking account. This is mainly why I have decided to be Delta-neutral. However, other hedging strategies exist but they may involve other assets such as options.

7.1 Delta-Gamma neutral

Among the famous strategies that go beyond the Delta-neutral position, there is the Delta-Gamma strategy. It consists in being neutral on Delta as well as on Gamma. The goal is to benefit from the **convexity** that can be observed with Vanilla options.
As an example, let's prove the convexity for a European put option such that:

$$\alpha p(K_1) + (1-\alpha)p(K_2) > p(\alpha K_1 + (1-\alpha)K_2) \tag{32}$$

for any $\alpha \in (0,1)$ and any $K_1, K_2 > 0$.
First, assuming contradiction:

$$\alpha p(K_1) + (1-\alpha)p(K_2) < p(\alpha K_1 + (1-\alpha)K_2)$$

Then, we can construct a portfolio with following strategy π:

- Long α put at K_1
- Long $(1-\alpha)$ put at K_2
- Short 1 put at $\alpha K_1 + (1-\alpha)K_2$
- Long $p(\alpha K_1 + (1-\alpha)K_2) - \alpha p(K_1) + (1-\alpha)p(K_2)$

and the value of this portfolio at time 0 is:

$$V^{\pi}(0) = \alpha p(K_1) + (1-\alpha)p(K_2) - p(\alpha K_1 + (1-\alpha)K_2) + (p(\alpha K_1 + (1-\alpha)K_2) - \alpha p(K_1) + (1-\alpha)p(K_2)) = 0$$

By assuming the bond value at time T is $FV(p(\alpha K_1 + (1-\alpha)K_2) - \alpha p(K_1) + (1-\alpha)p(K_2))$, the value of this portfolio becomes:

$$V^{\pi}(T) = \alpha(S_T - K_1)^+ + (1-\alpha)(S_T - K_2)^+ - (S_T - (\alpha K_1 + (1-\alpha)K_2))^+ + FV(p(\alpha K_1 + (1-\alpha)K_2) - \alpha p(K_1) + (1-\alpha)p(K_2))$$

Then:

$$V^{\pi}(T) = \begin{cases} FV(p(\alpha K_1 + (1-\alpha)K_2) - \alpha p(K_1) + (1-\alpha)p(K_2)) \geq 0 & S_T \in [0, K_1) \\ \alpha(S_T - K_1) + FV(p(\alpha K_1 + (1-\alpha)K_2) - \alpha p(K_1) + (1-\alpha)p(K_2)) \geq 0 & S_T \in [K_1, \alpha K_1 + (1-\alpha)K_2) \\ (1-\alpha)(K_2 - S_T) + FV(p(\alpha K_1 + (1-\alpha)K_2) - \alpha p(K_1) + (1-\alpha)p(K_2)) \geq 0 & S_T \in [\alpha K_1 + (1-\alpha)K_2, K_2) \\ FV(p(\alpha K_1 + (1-\alpha)K_2) - \alpha p(K_1) + (1-\alpha)p(K_2)) \geq 0 & S_T \in [K_2, \infty) \end{cases}$$

Obviously, $V^{\pi}(T) \geq 0$ indicates there is an arbitrage opportunity in this strategy, and the contradiction violate the arbitrage free assumption, which means $\alpha p(K_1) + (1-\alpha)p(K_2) > p(\alpha K_1 + (1-\alpha)K_2)$ stands.

Yet, would it have been relevant to be Delta-Gamma neutral during this project ? Well, let's have a look at the evolution of The Greeks during the trading period:

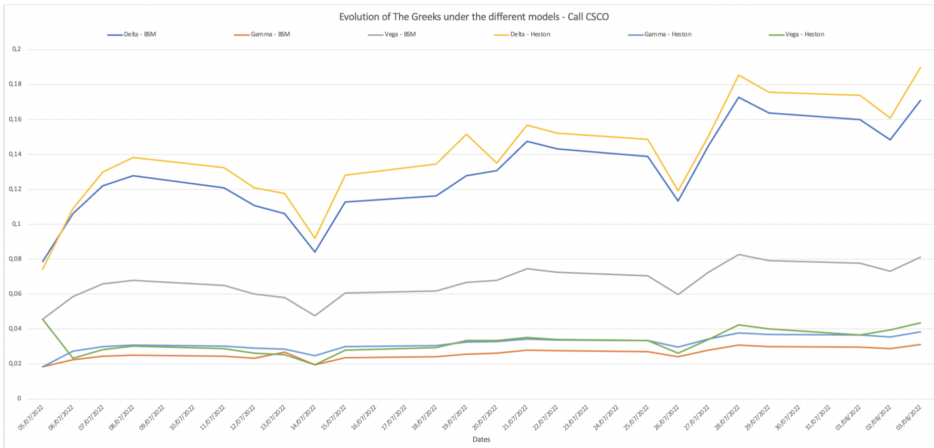

Figure 19: Evolution of the Greeks for the CSCO call

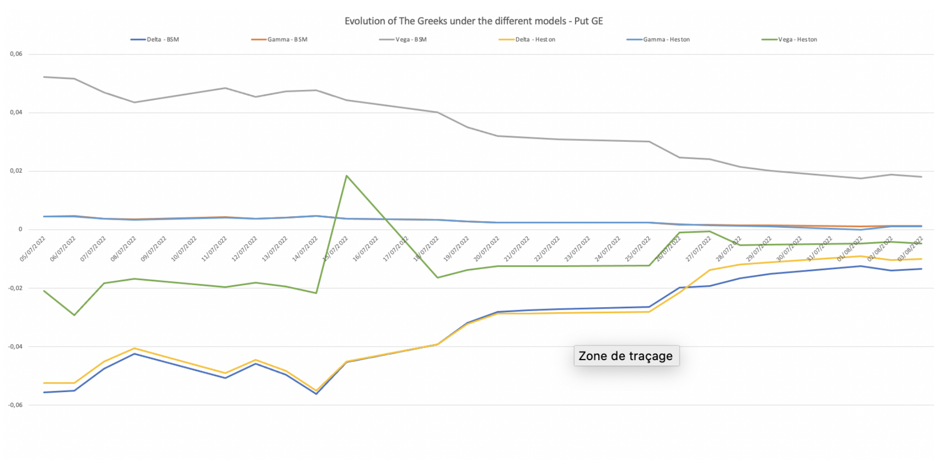

Figure 20: Evolution of the Greeks for the GE put

What can be noticed is that whether it is for the call or the put option, clearly a Gamma-neutral strategy wouldn't have changed the hedging that much. However, Vega seems to be at a much higher level and therefore it would maybe need Vega-neutral strategy. Let's dive into it.

7.2 Delta-Vega neutral

Another strategy that exists is the Vega hedging. Vega measures the sensitivity of the price of an option to the implied volatility of the underlying asset.

$$Vega = \frac{\partial V}{\partial \sigma} \tag{33}$$

Options at long positions come with positive vega and the ones at short positions come with negative vega. In a vega-neutral portfolio, the total vega of all the positions sums up to zero. A vega-neutral strategy makes profits from the bid-ask spread of implied volatility or the skew between the volatilities of the calls and puts.

Obviously, the value of a contract depends on the volatility of the underlying asset. This parameter is tough to estimate, therefore, to ensure that a portfolio value is insensitive to this parameter, vega hedging strategies are considered. This means that an option is hedged with both the underlying and another option in such a way that both the delta and the vega (or long gamma and vega) are zero.

7.3 Other Techniques

In the Black-Scholes-Merton model, the market is **complete**. An option can be perfectly hedged by dynamically trading the underlying stock. As a result, a portfolio consisting of a short position in a call option and Δ units of stocks is locally riskless.

However, in stochastic volatility models, the model is **incomplete**. A perfect hedging is impossible by trading only the underlying stock.
By dynamically trading the underlying asset and another option, both the price risk and the volatility risk can be hedged. Yet, in practice, continuous trading is impossible. For instance, traders can re-balance the portfolio daily but if the frequency is too high, the transaction cost can be a problem. The transaction cost is much higher for trading options than for trading stocks. Thus, it may be more relevant to use stocks to hedge options.
Researchers have proposed some hedging strategies using only stocks in incomplete markets. Superhedging, mean-variance hedging and shortfall hedging are three frequently discussed strategies in financial literature.

According to Investopedia, Super-hedging is a strategy that hedges positions with a self-financing trading plan. It utilizes the lowest price that can be paid for a hedged portfolio such that its worth will be greater or equal to the initial portfolio at a set future time.

7.4 Interest Rates

Obviously, modeling interest rates and not assume them constant would improve hedging in a significant way. For Black-Scholes-Merton model, I assume them constant whereas I directly used market quotes for Heston model.
I worked on interest rates during my internship that I was doing simultaneously, this is why I wanted to focus on volatility for this project.
Different models exist to model interest rates such as Vasicek and Hull-White models that enable rates to become negative, but also more complex models such HJM model. Let's recall the dynamics for Vasicek model:

$$dr_t = a(b - r_t)dt + \sigma dW_t \tag{34}$$

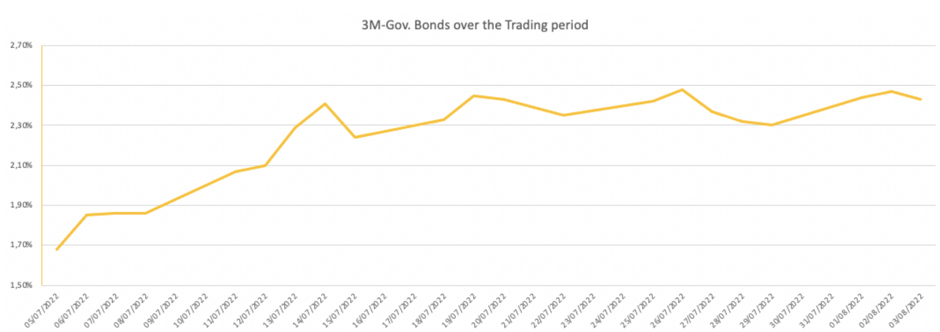

Figure 21: Evolution of 3Mo Government Bond over the Trading period

8 Conclusion

This trading project aimed to present the notions of Pricing and Hedging applied to American options. To do so, two models have been tested: Black-Scholes-Merton and Heston. Obviously, both models display strengths and weaknesses but it also enabled to emphasize the importance of volatility as a key parameter for hedging purposes.
A real focus has been realized on analytical expressions that those two models offer. As it has been explained during this paper, those models have been studied with Monte-Carlo methods , but during my internship. It is important to combine both methods in order to check if they have been well-implemented.

You will find the GitHub repository of this trading project here: Research Project: GitHub Repository containing Python files, Matlab files and an Excel follow-up.

Here is also the repository of the open-source library that I have developed during my internship aiming to price different derivatives using Monte-Carlo methods: Internship: GitHub Repository and the different articles that I wrote:

- Introduction about my library
- Focus on BSM PDE
- Negative Interest Rates

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