Project report

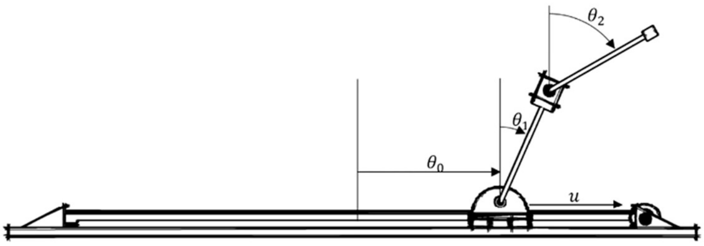
The goal of my project is to balance a double inverted pendulum using LQR.

The work was done in the project:

1. Compute the dynamic model for the double inverted pendulum
2. Establish a linear state space equation near the equilibrium point
3. Choose reasonable matrices Q and constant R for the cost function
4. Plot the results and make an animation of the double inverted pendulum.

1. dynamic model

Compute the dynamic model $\mathbf{D}(\mathbf{q})\mathbf{q}' + \mathbf{C}(\mathbf{q},\mathbf{q}') + \mathbf{g}(\mathbf{q}) = \mathbf{u}$ for the double inverted pendulum. In my project, I simplify the model by assuming there is no friction.



M	Cart weight
m_1	First pendulum weight
m_2	Second pendulum weight
L_1	First pendulum length
L_2	Second pendulum length
l_1	Distance between pivot point and center of mass of first pendulum
l_2	Distance between pivot point and center of mass of second pendulum
I_1	The moment of inertia of first pendulum
I_2	The moment of inertia of second pendulum
g	Gravitational acceleration
x, θ_1, θ_2	Cart position, angle of first and second pendulum
$\dot{x}, \dot{\theta}_1, \dot{\theta}_2$	Cart velocity, angular velocity of first and second pendulum

$$\mathbf{q} = \left[x, \dot{x}, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2\right]^{\boldsymbol{T}}$$

$$\mathbf{D}(\mathbf{q})\mathbf{q}' + \mathbf{C}(\mathbf{q},\mathbf{q}') + \mathbf{g}(\mathbf{q}) = \mathbf{u}$$

where

$$\boldsymbol{D}(\boldsymbol{q}) = \begin{bmatrix} d_1 & d_2 cos\theta_1 & d_3 cos\theta_2 \\ d_2 cos\theta_1 & d_4 & d_5\cos(\theta_1 - \theta_2) \\ d_3 cos\theta_2 & d_5\cos(\theta_1 - \theta_2) & d_6 \end{bmatrix}$$

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{bmatrix} 0 & -d_2\, sin(\theta_1)\dot{\theta}_1 & -d_3\, sin(\theta_2)\dot{\theta}_2 \\ 0 & 0 & d_5\, sin(\theta_1-\theta_2)\dot{\theta}_2 \\ 0 & -d_5\, sin(\theta_1-\theta_2)\dot{\theta}_1 & 0 \end{bmatrix}$$

$$\mathbf{f} = \boldsymbol{u} - \boldsymbol{g}(\boldsymbol{q}) = \begin{bmatrix} u \\ f_1 \sin\theta_1 \\ f_2 \sin\theta_2 \end{bmatrix}$$

For the elements for matrices $\boldsymbol{D}(\boldsymbol{q})$, $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\boldsymbol{u}-\boldsymbol{g}(\boldsymbol{q})$:

$$d_1 = M + m_1 + m_2$$

$$d_2 = m_1 l_1 + m_2 \mathrm{L}$$

$$d_3 = m_2 l_2$$

$$d_4 = m_1 l_1^2 + m_2 \mathrm{L}_1^2 + I_1$$

$$d_5 = m_2 L_1 l_2$$

$$d_6 = m_2 l_2^2 + I_2$$

Then I compute the exact $\dot{\boldsymbol{q}}$ use equations:

$$\begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & C_{12} & C_{13} \\ 0 & 0 & C_{23} \\ 0 & C_{32} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Firstly,

$$\ddot{\theta}_2 = \frac{a\dot{\theta}_1 + b\dot{\theta}_2 + c}{d}$$

where

$$a = -[(D_{31}D_{22} - D_{21}D_{32})(D_{31}C_{12} - D_{11}C_{32}) + (-D_{11}D_{32} + D_{31}D_{12})D_{21}C_{32}]$$

$$b = -[(D_{31}D_{22} - D_{21}D_{32})D_{31}C_{13} - (-D_{11}D_{32} + D_{31}D_{12})D_{21}C_{32}]$$

$$\mathrm{c} = -(\mathrm{D}_{11}\mathrm{D}_{32} + \mathrm{D}_{31}\mathrm{D}_{12})(\mathrm{D}_{31}\mathrm{f}_2 - \mathrm{D}_{21}\mathrm{f}_3) + (\mathrm{D}_{31}\mathrm{f}_1 - \mathrm{D}_{11}\mathrm{f}_3)(\mathrm{D}_{31}\mathrm{D}_{22} - \mathrm{D}_{21}\mathrm{D}_{22})$$

$$d = [(-D_{11}D_{32} + D_{31}D_{12})(D_{33}D_{21} - D_{31}D_{23}) + (D_{31}D_{22} - D_{21}D_{32})(D_{31}D_{13} - D_{11}D_{33})]$$

Secondly,

$$\ddot{\theta}_1 = \frac{(D_{33}D_{21} - D_{31}D_{23})\ddot{\theta}_2 + D_{21}C_{32}\dot{\theta}_1 - D_{31}C_{23}\dot{\theta}_2 + (\mathrm{D}_{31}\mathrm{f}_2 - \mathrm{D}_{21}\mathrm{f}_3)}{d}$$

Finally,

$$\ddot{x} = \frac{f_3 - D_{32}\ddot{\theta}_1 - D_{33}\ddot{\theta}_2 - C_{32}\dot{\theta}_1}{D_{31}}$$

Then discretize using a time step Δt:

$$x_{n+1} = x_n + \Delta t \cdot v_n$$

$$v_{n+1} = v_n + \Delta t \cdot \dot{v}_n$$

$$\theta_{1_{n+1}} = \theta_{1_n} + \Delta t \cdot \omega_{1_n}$$

$$\omega_{1_{n+1}} = \omega_{1_n} + \Delta t \cdot \dot{\omega}_{1_n}$$

$$\theta_{2_{n+1}} = \theta_{2_n} + \Delta t \cdot \omega_{2_n}$$

$$\omega_{2_{n+1}} = \omega_{2_n} + \Delta t \cdot \dot{\omega}_{2_n}$$

2. state space model

To use LQR, we should linearize the discretized equations of the double inverted pendulum near equilibrium position. Approximate some nonlinear variables such as trigonometric functions, then we can get:

$$\dot{\mathbf{q}}(\mathrm{t}) = \mathbf{A}\mathbf{q}(\mathrm{t}) + \mathbf{B}\mathbf{u}(\mathrm{t})$$

$$\mathbf{y}(\mathrm{t}) = \mathbf{C}\mathbf{q}(\mathrm{t}) + \mathbf{D}\mathbf{u}(\mathrm{t})$$

$$w_1 = m_0 + m_1 + m_2$$

$$d = I_1 I_2 w_1 + I_2 L_1^2 m_0 m_2 + I_2 L_1^2 m_1 m_2 + I_2 l_1^2 m_0 m_1 + I_1 l_2^2 m_0 m_2 + I_1 l_2^2 m_1 m_2 + I_2 l_1^2 m_1 m_2 - 2 I_2 L_1 l_1 m_1 m_2 + l_1^2 l_2^2 m_0 m_1 m_2$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & -\mu \dfrac{I_2 m_2 L_1^2 + m_1 m_2 l_1^2 l_2^2 + I_2 m_1 l_1^2 + I_1 m_2 l_2^2 + I_1 I_2}{d} & -g \dfrac{(L_1 m_2 + l_1 m_1)(l_1 m_1 m_2 l_2^2 + I_2 L_1 m_2 + I_2 l_1 m_1)}{d} & \\ 0 & 0 & 0 & \\ 0 & \mu \dfrac{l_1 m_1 m_2 l_2^2 + I_2 L_1 m_2 + I_2 l_1 m_1}{d} & g \dfrac{(L_1 m_2 + l_1 m_1)(I_2 w_1 + l_2^2 m_0 m_2 + l_2^2 m_1 m_2)}{d} & \\ 0 & 0 & 0 & \\ 0 & \mu l_2 m_2 \dfrac{m_1 l_1^2 - L_1 m_1 l_1 + I_1}{d} & -g l_2 m_2 \dfrac{(L_1 m_2 + l_1 m_1)(L_1 m_0 + L_1 m_1 - l_1 m_1)}{d} & \end{bmatrix}$$

$$\begin{matrix} 0 & 0 & 0 \\ 0 & -g l_2^2 m_2^2 \dfrac{m_1 l_1^2 - L_1 m_1 l_1 + I_1}{d} & 0 \\ 1 & 0 & 0 \\ 0 & -g l_2^2 m_2^2 \dfrac{L_1 m_0 + L_1 m_1 - l_1 m_1}{d} & 0 \\ 0 & 0 & 0 \\ 0 & g l_2 m_2 \dfrac{I_1 w_1 + l_1^2 m_0 m_1 + l_1^2 m_1 m_2 + L_1^2 m_0 m_2 + L_1^2 m_1 m_2 - 2 L_1 l_1 m_1 m_2}{d} & 0 \end{matrix}\Bigg]$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ k \dfrac{I_2 m_2 L_1^2 + m_1 m_2 l_1^2 l_2^2 + I_2 m_1 l_1^2 + I_1 m_2 l_2^2 + I_1 I_2}{d} \\ 0 \\ -k \dfrac{l_1 m_1 m_2 l_2^2 + I_2 L_1 m_2 + I_2 l_1 m_1}{d} \\ 0 \\ -k l_2 m_2 \dfrac{m_1 l_1^2 - L_1 m_1 l_1 + I_1}{d} \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{D} = [0]$$

3. Q and R for the cost function

$$Q = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 50 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1000 \end{bmatrix} \qquad R = 0.01$$