ECE-GY 9243 / ME-GY 7973
Optimal and learning control for robotics

Exercise series 2

For questions requesting a written answer, please provide a detailed explanation. Typesetted answers (e.g. using LaTeX[1]) are strongly preferred. Include plots where requested, either in a Jupyter Notebook or in the typesetted answers. For questions requesting a software implementation, please provide your code in a python file or in a Jupyter Notebook such that it can be run directly. Include comments explaining how the functions work and how the code should be run if necessary. Any piece of code that does not run out of the box or does not contain instructions to execute it will be considered invalid.

Exercise 1 [Dynamic Programming]

a) Consider the following dynamical system

$$x_{n+1} = x_n + u_n$$

where x_n is an integer between 0 and 5 and u_n is an integer such that $0 \leq x_{n+1} \leq 5$ (i.e. the control is any integer that drives the system to an admissible state). And consider the cost function

$$J = x_N^2 + \sum_{k=0}^{N-1} (x_k^2 + u_k^2)$$

Use the dynamic programming algorithm to solve the finite horizon optimal control problem that minimizes J, for $N = 3$ and $x_0 = 5$. In your solution, give the optimal sequence of x_k and u_k as well as the optimal cost J^*. Show the different steps of the algorithm as well.

b) Consider the same problem as a) with the additional constraint $x_3 = 5$ on the final state. *Hint:* For this problem you need to define a state space for x_3 that consists of just the value $x_3 = 5$.

c) Consider the dynamical system of question a) with a stochastic disturbance w_n added.

$$x_{n+1} = x_n + u_n + w_n$$

The disturbance takes the values -1 and 1 with equal probabilities for all x_k and u_k, except if $x_k + u_k$ is equal to 0 or 5, in which case $w_k = 0$ with probability 1. Apply the dynamic programming algorithm to solve the finite horizon optimal control problem, with $N = 3$ and $x_0 = 5$. In this case minimize the expectation of the cost J. In your solution, give the optimal sequence of x_k and the policy μ_k as well as the optimal expected cost J^*.

[1] `https://en.wikibooks.org/wiki/LaTeX`, NYU provides access to Overleaf to all the community `https://www.overleaf.com/edu/nyu`

Exercise 2 [Controllability]

Consider the following dynamical systems

a)

$$x_{n+1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1.5 & 0 \\ 1 & 0 & 0 \end{bmatrix} x_n + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_n \tag{1}$$

b)

$$x_{n+1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1.5 & 0 \\ 1 & 0 & 0 \end{bmatrix} x_n + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u_n \tag{2}$$

c)

$$x_{n+1} = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & -0.5 & -1 \\ 0 & 0 & 0.5 \end{bmatrix} x_n + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_n \tag{3}$$

d)

$$x_{n+1} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & -0.5 & -1 \\ -0.1 & 0 & 0.5 \end{bmatrix} x_n + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_n \tag{4}$$

Which of these systems stable when $u_n = 0$? (Justify)
Which of these systems is controllable? (Justify)
For which of these systems can you find a control law u_n to stabilize the system? (Justify)

Exercise 3 [Linear Quadratic Regulators]

Answer the 2 questions asked in the file *Linear Quadratic Regulators.ipynb*

Exercise 4 [Cart-Pole Model]

Answer the 3 questions asked in the file *Cart-Pole Model.ipynb*

Exercise 5 [Direct Transcription]

Answer the 3 questions asked in the file *Direct Transcription Methods.ipynb*