CS1231S
AY22/23 Sem 1 Midterm
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01. PROOFS

Sets of Numbers

$\mathbb{N}$: natural numbers ($\mathbb{Z}_{\geq 0}$)
$\mathbb{Z}$: integers
$\mathbb{Q}$: rational numbers
$\mathbb{R}$: real numbers
$\mathbb{C}$: complex numbers

Basic Properties of Integers

Closure (under $+$ and x)
$x + y \in \mathbb{Z} \wedge xy \in \mathbb{Z}$

Commutativity
$a + b = b + a \wedge ab = ba$

Associativity
$a + b + c = a + (b + c) = (a + b) + c$
$abc = a(bc) = (ab)c$

Distributivity
$a(b + c) = ab + ac$

Trichotomy
$(a < b) \vee (a > b) \vee (a = b)$

Number Definitions

Even/Odd
n is even $\leftrightarrow \exists k \in \mathbb{Z} \mid n = 2k$
n is odd $\leftrightarrow \exists k \in \mathbb{Z} \mid n = 2k + 1$

Prime/Composite
n is prime $\leftrightarrow n > 1$ and $\forall r, s \in \mathbb{Z}^+, n = rs \rightarrow (r = n) \vee (r = s)$
n is composite $\leftrightarrow n > 1$ and $\exists r, s \in \mathbb{Z}^+ s.t. n = rs$ and $1 < r < n$ and $1 < s < n$

Divisibility ("d divides n": $n, d \in \mathbb{Z}$ and $d \neq 0$)
$d \mid n \leftrightarrow \exists k \in \mathbb{Z} \mid n = kd$

Rational
r is rational $\leftrightarrow \exists a, b \in \mathbb{Z} \mid r = \frac{a}{b}$ and $b \neq 0$

Fraction in lowest term
$\frac{a}{b}$ where $b \neq 0$ is said to be in *lowest terms* if the largest integer that divides both a and b is 1.

Congruence
Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. a is **congruent** to b modulo n,
$a \equiv b \pmod{n} \leftrightarrow n \mid (a - b)$ or $\exists k \in \mathbb{Z}(a - b = nk)$

04. METHODS OF PROOF

Proof by Exhaustion/Cases

1. list out possible cases
   1.1. Case 1: n is odd OR If $n = 9$, ...
   1.2. Case 2: n is even OR If $n = 16$, ...
2. therefore ...

Proof by Contradiction

1. Suppose not, i.e. . . . (note: use De Morgan's Law if required)
   1.1. ¡proof¿
   1.2. . . . but this contradicts . . .
2. Hence the supposition that . . . is false. Therefore . . .

Proof by Contraposition

1. Contrapositive statement: $\sim q \rightarrow \sim p$
2. let $\sim q$
   2.1. ¡proof¿
   2.2. hence $\sim p$
3. $\therefore p \rightarrow q$

Proof by Construction

1. Let $x = 3, y = 4, z = 5$.
2. Then $x, y, z \in \mathbb{Z}_{\geq 1}$ and $x^2 + y^2 = 3^2 + 4^2 = 9 + 16 = 25 = 5^2$.
3. Thus $\exists x, y, z \in \mathbb{Z}_{\geq 1}$ such that $x^2 + y^2 = z^2$.

Proof by Induction

1. For each $n \in \mathbb{Z}_{\geq 1}$, let $P(n)$ be the proposition ". . . "
2. (base step) $P(1)$ is true because ¡manual method¿
3. (induction step)
   3.1. let $k \in \mathbb{Z}_{\geq 1}$ s.t. $P(k)$ is true
   3.2. Then . . .
   3.3. proof that $P(k + 1)$ is true - e.g. $P(k + 1) = P(k) + term_{k+1}$
   3.4. So $P(k + 1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geq 1} P(n)$ is true by MI.

Proofs for Sets

Equality of Sets (A=B)

1. $(\subseteq)$ Take any $z \in A$.
   1.1. . . .
   1.2. $\therefore z \in B$.
2. $(\supseteq)$ Take any $z \in B$.
   2.1. . . .
   2.2. $\therefore z \in A$.
3. Therefore, $A = B$ (by definition of set equality).

Element Method

1. $A \cap (B \setminus C) = \{x : x \in A \wedge x \in (B \setminus C)\}$ (by def. of $\cap$)
2. $= \{x : x \in A \wedge (x \in B \wedge x \notin C)\}$ (by def. of $\setminus$)
3. . . .
4. $= (A \cap B) \setminus C$ (by def. of $\setminus$)

Proofs for Relations

Equivalence Relation

1. ("Reflexivity") Take any $a \in A$
   1.1. . . .
   1.2. Thus, $a\ R\ a$ and R is reflexive.
2. ("Symmetry") Take any $a, b \in A$.
   2.1. Suppose $a\ R\ b$.
   2.2. . . .
   2.3. Thus, $b\ R\ a$ and R is symmetric.
3. ("Transitivity") Take any $a, b, c \in A$.
   3.1. Suppose $a\ R\ b$ and $b\ R\ c$.
   3.2. . . .
   3.3. Thus, $a\ R\ c$ and R is transitive.
4. Therefore, R is an equivalence relation.

Partial Order

1. ("Reflexivity") Take any $a \in A$
   1.1. . . .
   1.2. Thus, $a\ R\ a$ and R is reflexive.
2. ("Antisymmetry") Take any $a, b \in A$.
   2.1. Suppose $a\ R\ b$ and $b\ R\ a$.
   2.2. . . .
   2.3. Thus, $a = b$ and R is antisymmetric.
3. ("Transitivity") Take any $a, b, c \in A$.
   3.1. Suppose $a\ R\ b$ and $b\ R\ c$.
   3.2. . . .
   3.3. Thus, $a\ R\ c$ and R is transitive.
4. Therefore, R is a partial order.

Other Proofs

iff ($A \leftrightarrow B$)

1. ($\Rightarrow$) Suppose A.
   1.1. . . . ¡proof¿ . . .
   1.2. Hence $A \rightarrow B$
2. ($\Leftarrow$) Suppose B.
   2.1. . . . ¡proof¿ . . .
   2.2. Hence $B \rightarrow A$

Logical equivalence of multiple statements

1. ((i) $\rightarrow$ (ii))
   1.1. . . .
   1.2. Hence . . .
2. ((ii) $\rightarrow$ (iii))
   2.1. . . .
   2.2. Hence . . .
3. ((iii) $\rightarrow$ (i))
   3.1. . . .
   3.2. Hence . . .
4. Therefore, (i), (ii) and (iii) are logically equivalent.

02. COMPOUND STATEMENTS

Operations

1 $\sim$: negation (not)
2 $\wedge$: conjunction (and)
2 $\vee$: disjunction (or) - coequal to $\wedge$
3 $\rightarrow$: if-then/conditional

Logical Equivalence

- identical truth values in truth table
- definitions
- to show non-equivalence:
  - truth table method (only needs 1 row)
  - counter-example method

Conditional Statements

hypothesis/antecedent $\rightarrow$ conclusion/consequent

- **vacuously true** : hypothesis is false
- **implication law** : $p \rightarrow q \equiv \sim p \vee q$
- common if/then statements:
  - if p then q: $p \rightarrow q$
  - p if q: $q \rightarrow p$
  - p only if q: $p \rightarrow q$
  - p iff q: $p \leftrightarrow q$
- **contrapositive** : $\sim q \rightarrow \sim p$
- **inverse** : $\sim p \rightarrow \sim q$
- **converse** : $q \rightarrow p$

converse $\equiv$ inverse
statement $\equiv$ contrapositive

- r is a **necessary** condition for s: $\sim r \rightarrow \sim s$ and $s \rightarrow r$
- r is a **sufficient** condition for s: $r \rightarrow s$
- biconditional / **necessary** & **sufficient** : $\leftrightarrow$

Valid Arguments

- determining validity: construct truth table
  - valid $\leftrightarrow$ conclusion is true when premises are true
- **syllogism** : (argument form) 2 premises, 1 conclusion
- **sound argument** : is valid & all premises are true

Rules of Inference

Modus Ponens
$p \rightarrow q$
p
$\therefore q$

Modus Tollens
$p \rightarrow q$
$\sim q$
$\therefore \sim p$

Proof by Division into Cases
$p \vee q$
$p \rightarrow r$
$q \rightarrow r$
$\therefore r$

Generalisation
p
$\therefore p \vee q$

Specialisation
$p \wedge q$
$\therefore p$

Elimination
$p \vee q$
$\sim q$
$\therefore p$

Transitivity
$p \rightarrow q$
$q \rightarrow r$
$\therefore p \rightarrow r$

Conjunction
p
q
$p \wedge q$

Contradiction Rule
$\sim p \rightarrow$ **false**
$\therefore p$

Fallacies

Converse Error
$p \rightarrow q$
q
$\therefore p$

Inverse Error
$p \rightarrow q$
$\sim p$
$\therefore \sim q$

03. QUANTIFIED STATEMENTS

- **truth set** of $P(x) = \{x \in D \mid P(x)\}$
- $P(x) \Rightarrow Q(x) : \forall x(P(x) \rightarrow Q(x))$
- $P(x) \Leftrightarrow Q(x) : \forall x(P(x) \leftrightarrow Q(x))$

relation between $\forall, \exists, \wedge, \vee$

- $\forall x \in D, Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge \cdots \wedge Q(x_n)$
- $\exists x \in D \mid Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \cdots \vee Q(x_n)$

Similar to compound statements:
Contrapositive, converse, inverse, necessary and sufficient, only if, rules of inference

05. SETS

Notation

- Set Roster Notation [1]: $\{x_1, x_2, \ldots, x_n\}$
- Set Roster Notation [2]: $\{x_1, x_2, x_3, \ldots\}$
- Set-Builder Notation: $\{x \in \mathbb{U} : P(x)\}$ or $\{x \in \mathbb{U} \mid P(x)\}$
- Replacement Notation: $\{t(x) : x \in A\}$ or $\{t(x) \mid x \in A\}$
- Intervals of real numbers: $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- Unions/Intersection of Indexed Collection of Sets:
  $\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\} = A_0 \cup A_1 \cup \cdots \cup A_n$
  $\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\} = A_0 \cap A_1 \cap \cdots \cap A_n$

Set Definitions

- **Cardinality** of a set, $\mid A \mid$: number of elements
- **Singleton** : sets of size 1
  - $A = B \leftrightarrow (A \subseteq B) \wedge (A \supseteq B)$

- **Empty Set**, $\varnothing$: $\varnothing \subseteq$ all sets (T6.2.4)
- **Subset** : $A \subseteq B \leftrightarrow \forall x(x \in A \rightarrow x \in B)$
- **Proper Subset** : $A \subsetneq B \leftrightarrow (A \subseteq B) \wedge (A \neq B)$
- **Set Equality** :
  $A = B \leftrightarrow \forall x(x \in A \leftrightarrow x \in B) \leftrightarrow A \subseteq B \wedge B \subseteq A$
- **Disjoint** : $A \cap B = \varnothing$
- **Mutually/pairwise disjoint**: $A_i \cap A_j = \varnothing$ whenever $i \neq j$
- **Power Set** of A : $\mathcal{P}(A) = \{X \mid X \subseteq A\}$
  - $|\mathcal{P}(A)| = 2^{|A|}$, given that A is a finite set (T6.3.1)

methods of proof for sets

- direct proof
- element method
- truth table

Set Operations

- **Union:** $A \cup B = \{x : x \in A \vee x \in B\}$
- **Intersection:** $A \cap B = \{x : x \in A \wedge x \in B\}$
- **Difference** (of A minus B) or **Relative Complement** (of B in A): $A \setminus B = A - B = \{x : x \in A \wedge x \notin B\}$
- **Complement** (of B): $\bar{B}$ or $B^c = U \setminus B$
  - set difference law: $A \setminus B = A \cap \bar{B}$

Ordered Pairs and Cartesian Products

- **Ordered pair** : (x, y)
  - $(a, b) = (c, d) \leftrightarrow (a = c) \wedge (b = d)$
- **Cartesian product** : $A \times B = \{(x, y) : x \in A \wedge y \in B\}$
  - $|A \times B| = |A| \times |B|$
- **Ordered n-tuple** : expression of the form $(x_1, x_2, \ldots, x_n)$

Subset Relations (T6.2.1)

- **Inclusion of Intersection:** $A \cap B \subseteq A$ and $A \cap B \subseteq B$
- **Inclusion in Union:** $A \subseteq A \cup B$ and $B \subseteq A \cup B$
- **Transitive Property of Subsets:**
  $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$

06. RELATIONS

Relations Definitions

Let A and B be sets. A (binary) relation from A to B is a subset of $A \times B$. Given an ordered pair $(x, y) \in A \times B$, **x is related to y by R** or **x is R-related to y**

$x\, R\, y \leftrightarrow (x, y) \in R$

- **Domain**: $Dom(R) = \{a \in A : a\, R\, b$ for some $b \in B\}$
- **Co-domain**: $coDom(R) = B$
- **Range**: $Range(R) = \{b \in B : a\, R\, b$ for some $a \in A\}$
- A **relation on a set** A is a relation from A to A ($\subseteq$ of A^2).
- **Inverse Relation**:
  $R^{-1} = \{(y, x) \in B \times A : (x, y) \in R\}$
- **Composition** of R with S:
  $\forall x \in A, \forall z \in C\Big(x\, S \circ R\, z \leftrightarrow \big(\exists y \in B(x\, R\, y \wedge y\, S\, z)\big)\Big)$
- **n-ary Relation** R on $A_1 \times A_2 \times \cdots \times A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$ (2-ary = binary, 3-ary = ternary)

Properties of Relations

Let A be a set and R be a relation on A.

- **Reflexive**: $\forall x \in A\ (x\, R\, x)$
- **Symmetric**: $\forall x, y \in A\ (x\, R\, y \rightarrow y\, R\, x)$
- **Transitive**: $\forall x, y, z \in A\ (x\, R\, y \wedge y\, R\, z \rightarrow x\, R\, z)$
- Note: Relations are reflexive, elements are **related** to themselves.
- **Transitive Closure** of R is relation R^t on A such that:
  - R^t is transitive.
  - $R \subseteq R^t$.
  - If S is any other transitive relation that contains R, then $R^t \subseteq S$.
- **Antisymmetry**: $\forall x, y \in A(x\, R\, y \wedge y\, R\, x \rightarrow x = y)$

Equivalence Relations

Let A be a set and $\sim$ be a relation on A.

- $\sim$ is an **equivalence relation** on A iff $\sim$ is reflexive, symmetric and transitive.
- **Equivalence class** of a: $[a]_\sim = \{x \in A : a \sim x\}$
- Lemma Rel.1 Equivalence Classes:
  $x \sim y \leftrightarrow [x] = [y] \leftrightarrow [x] \cap [y] \neq \varnothing$
- T8.3.4: Distinct equivalence classes of R form a **partition** of A, i.e. the union of equivalence classes is all of A, and the intersection of any 2 distinct classes is empty.
- **Set of equivalence classes**: $A/R = \{[x]_\sim : x \in A\}$
- Theorem Rel.2 Equivalence classes form a partition:
  $A/\sim$ is a partition of A.

Partitions

- $\mathscr{C}$ is a **partition** of a set A if the following hold:
  - $\varnothing \neq S \subseteq A$ for all $S \in \mathscr{C}$ (all elements are non-empty subsets of A)
  - $\forall x \in A\ \exists S \in \mathscr{C}(x \in S)$ and
    $\forall x \in A\ \forall S_1, S_2 \in \mathscr{C}\big(x \in S_1 \wedge x \in S_2 \rightarrow S_1 = S_2\big)$
    OR $\forall x \in A\ \exists! S \in \mathscr{C}(x \in S)$
- **Components** : elements of a partition
- **Relation induced by a Partition**: $\forall x, y \in A$,
  $x\, R\, y \leftrightarrow \exists$ a component S of $\mathscr{C}$ s.t. $x, y \in S$
- Relation induced by a Partition is an **reflexive, symmetric and transitive**, i.e. an equivalence relation. (T8.3.1)

Partial Order

Let A be a set and R be a relation on A.

- R is a **partial order** if R is **reflexive, antisymmetric and transitive**.
- A is called a **partially ordered set/poset** w.r.t partial order relation R on A, denoted by (A, R).
- L6, Slide 68: Example of Partial order relations
  - $\leq$ relation on a set of real numbers
  - $\subseteq$ relation on a set of sets

Comparability

Let $\preccurlyeq$ be a partial order on a set A.

- **Hasse Diagram** of $\preccurlyeq$ satisfies for all distinct $x, y, m \in A$: If $x \preccurlyeq y$ and no $m \in A$ such that $x \preccurlyeq m \preccurlyeq y$, then x is placed below y with a line joining them, else no line joins x and y.
- **Comparable**: $\forall x, y \in A\ (x \preccurlyeq y \vee y \preccurlyeq x)$

Let $\preccurlyeq$ be a partial order on a set A and $c \in A$.

- **Minimal element** ("nothing below"):
  $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$
- **Maximal element** ("nothing above"):
  $\forall x \in A\ (c \preccurlyeq x \Rightarrow c = x)$
- **Smallest/Minimum/Least element** ("everything above"):
  $\forall x \in a\ (c \preccurlyeq x)$.
- **Largest/Maximum/Greatest** ("everything below"):
  $\forall x \in a\ (x \preccurlyeq c)$.
- L6, Slide 83: A smallest element is minimal.
- A is **well-ordered** iff
  $\forall S \in \mathcal{P}(A), S \neq \varnothing \rightarrow \big(\exists x \in S\ \forall y \in S(x \preccurlyeq y)\big)$
  - $(\mathbb{N}, \leq)$ is well-ordered.
  - $(\mathbb{Z}, \leq)$ is not well-ordered.

Linearization

- R is a **total order relation** on A iff
  R is a partial order and $\forall x, y \in A(x\, R\, y \vee y\, R\, x)$
- Linearization of $\preccurlyeq$ is a total order $\preccurlyeq^*$ on A such that
  $\forall x, y \in A(x \preccurlyeq y \rightarrow x \preccurlyeq^* y)$

LOGICAL EQUIVALENCES (T2.1.1)

Commutative Laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative Laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive Laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity Laws	$p \wedge$ **true** $\equiv p$	$p \vee$ **false** $\equiv p$
Idempotent Laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal Bound Laws	$p \vee$ **true** $\equiv$ **true**	$p \wedge$ **false** $\equiv$ **false**
Negation Laws	$p \vee {\sim}p \equiv$ **true**	$p \wedge {\sim}p \equiv$ **false**
Double Negation Law	${\sim}({\sim}p) \equiv p$	—
Absorption Laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
De Morgan's Laws	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$
Negation of **true** and **false**	${\sim}$**true** $\equiv$ **false**	${\sim}$**false** $\equiv$ **true**
Variant Absorption Laws (A1, Q1a)	$p \vee ({\sim}p \wedge q) \equiv p \vee q$	$p \wedge ({\sim}p \vee q) \equiv p \wedge q$

SET IDENTITIES (T6.2.2)

Commuative Laws	$A \cap B = B \cap A$	$A \cup B = B \cup A$
Associative Laws	$(A \cap B) \cap C = A \cap (B \cap C)$	$(A \cup B) \cup C = A \cup (B \cup C)$
Distributive Laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Identity Laws	$A \cap U = A$	$A \cup \emptyset = A$
Idempotent Laws	$A \cap A = A$	$A \cup A = A$
Universal Bound Laws	$A \cap \emptyset = \emptyset$	$A \cup U = U$
Complement Laws	$A \cap \bar{A} = \emptyset$	$A \cup \bar{A} = U$
Double **Complement** Law	$\overline{(\bar{A})} = A$	—
Absorption Laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
De Morgan's Laws	$\overline{A \cup B} = \bar{A} \cap \bar{B}$	$\overline{A \cap B} = \bar{A} \cup \bar{B}$
Complements of U and $\varnothing$	$\bar{U} = \varnothing$	$\bar{\varnothing} = U$
Set Difference Law	$A \setminus B = A \cap \bar{B}$	—

Proven:

Numbers

- T4.2.1 - Evey integer is a rational number
- T4.2.2 - The sum of any 2 rational numbers is rational
- C4.2.3 - The double of a rational number is rational
- T4.6.1 - There is no greatest integer
- P4.6.4 - For all integers n, if n^2 is even then n is even
- E1.1 - The product of 2 consecutive odd numbers is always odd
- E1.5 - The difference between 2 consecutive squares is always odd
- E1.7 - There exist irrational numbers p and q such that p^q is rational
- T4.7.1 - $\sqrt{2}$ is irrational.
- Tut 1, Q9 - Product of 2 odd integers is an odd integer
- Tut 1, Q10 - n^2 is odd iff n is odd
- Tut 2, Q3b - Rational numbers are closed under addition

Divisibility

- T4.3.1 - For all positive integers a and b, if $a|b$, then $a \leq b$.
- T4.3.2 - The only divisors of 1 are 1 and -1
- T4.3.3 - Transitivity of divisibility: $\forall a, b, c \in \mathbb{Z}(a \mid b \wedge b \mid c \rightarrow a \mid c)$
- T4.4.1 Quotient-Remainder Theorem: Given any integer n and positive integer d, there exist unique integers q and r such that $n = dq + r$ and $0 \leq r < d$.

Logic

- T3.2.1 - Negation of a universal statement:
  - $\sim(\forall x \in D, P(x)) \equiv \exists x \in D \mid \sim P(x)$
- T3.2.2 - Negation of an existential statement:
  - $\sim(\exists x \in D \mid P(x)) \equiv \forall x \in D, \sim P(x)$
- A1, Q1b - $\big((p \rightarrow q) \wedge (q \rightarrow r)\big) \rightarrow (p \rightarrow r)$

Sets

- T6.2.1 Subset Relations (see Sets section)
- T6.2.4 An empty subset is a subset of every set, i.e. $\varnothing \in A$ for all sets A
- T6.3.1 Suppose A is a finite set with n elements, then $\mathcal{P}(A)$ has 2^n elements
- Tut 3 Q5 - $A \cap (B \setminus C) = (A \cap B) \setminus C$
- Tut 3 Q6 - $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$
- Tut 3 Q7 - Symmetric difference ($\oplus$):
  $A \oplus B = (A \setminus B) \cup (B \setminus A)$ (given) $= (A \cup B) \setminus (A \cap B)$ (proven)
- Tut 3 Q8 - $A \subseteq B \Leftrightarrow A \cup B = B$
- Tut 3 Q12: Let $B_1, B_2, B_3, \dots, B_k$ and $C_1, C_2, C_3, \dots, C_l$ such that $\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{l} C_j$ - $B_i \subseteq C_j$ for any $i \in \{1, 2, \dots, k\}$ and any $j \in \{1, 2, \dots, l\}$

Relations

- T8.3.1 - Relation Induced by a Partition (see Partitions section)
- L Rel.1 - Equivalence Classes (see Equivalence Relations section)
- T8.3.4 - Partition Induced by an Equivalence Relation (see Equivalence Relations section)
- T Rel.2 - Equivalence Classes form a Partition (see Equivalence Relations section)
- Tut 4, Q2 - The following statements are logically equivalent:
  - $\forall x, y \in A(x \mathrel{R} y \rightarrow y \mathrel{R} x)$ (R is symmetric)
  - $\forall x, y \in A(x \mathrel{R} y \leftrightarrow y \mathrel{R} x)$
  - $R = R^-1$
- Tut 4, Q6 - Composition of relations is Associative
  $T \circ (S \circ R) = (T \circ S) \circ R$
- Tut 4, Q9 - Let $\mathcal{C}$ be a partition of a set A. Denote by $\sim$ the same-component relation with respect to $\mathcal{C}$, i.e. for all $x, y \in A$,
  - $x \sim y \Leftrightarrow x$ is in the same component of $\mathcal{C}$ as $y \Leftrightarrow x, y \in S$ for some $S \in \mathcal{C}$
  - Proven: $x \in S \in \mathcal{C} \rightarrow [x] = S$
  - Proven: $A/\sim = \mathcal{C}$
- Tut 5, Q3 - $\subseteq$ on $\mathcal{P}(A)$ is a partial order
- Tut 5, Q5 - $xSy \Leftrightarrow x = y \vee x \mathrel{R} y$ for all $x, y \in A$ is called the **reflexive closure** of R
- Tut 5, Q6 - Asymmetry: $\forall x, y \in A(x \mathrel{R} y \Rightarrow y \not\mathrel{R} x)$
  Every asymmetric relation is antisymmetric
- Tut 5, Q7 - In a total order $\preccurlyeq$ on A, all minimal elements are smallest
- Tut 5, Q8 - a, b are compatible iff there exists $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$
- Tut 5, Q10 - In all partially ordered sets, any two **comparable** elements are **compatible**.

Appendix A

Field Axioms

- F1 - Commutative Laws: $\forall a, b \in \mathbb{R}, a + b = b + a$ and $ab = ba$
- F2 - Associative Laws:
  $\forall a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$
- F3 - Distributive Laws:
  $\forall a, c, c \in \mathbb{R}, a(b + c) = ab + ac$ and $(b + c)a = ba + ca$
- F4 - Existence of Identity Elements: There exist two distinct real numbers, denoted 0 and 1, such that
  $\forall a \in \mathbb{R}, 0 + a = a + 0 = a$ and $1 \cdot a = a \cdot 1 = a$
- F5 - Existence of Additive Inverses: $\forall a \in \mathbb{R}$, there is a real number, denoted $-a$ and called the **additive inverse** of a, such that
  $a + (-a) = (-a) + a = 0$
- F6 - Existence of Reciprocals: $\forall a \in \mathbf{R}$, there is a real number, denoted $1/a$ or a^{-1}, called the **reciprocal** of a, such that
  $a \cdot (\frac{1}{a}) = (\frac{1}{a}) \cdot a = 1$

Algebra

Let a, b, c, d represent arbitrary real numbers.

- T1 - Cancellation Law for Addition: If $a + b = a + c$, then $b = c$
- T2 - Possibility of Subtraction: Given a and b, there is exactly one x such that $a + x = b$. This x is denoted by $b - a$. In particular, $0 - a$ is the additive inverse of a, $-a$.
- T3 - $b - a = b + (-a)$
- T4 - $-(-a) = a$
- T5 - $a(b - c) = ab - ac$
- T6 - $0 \cdot a = a \cdot 0 = 0$
- T7 - Cancellation Law for Multiplication: If $ab = ac$ and $a \neq 0$, then $b = c$
- T8 - Possibility of Division: Given a and b with $a \neq 0$, there is exactly one x such that $ax = b$. This x is denoted by b/a and is called the **quotient** of b and a. In particular, $\frac{1}{a}$ is the reciprocal of a
- T9 - If $a \neq 0$, then $b/a = b \cdot a^{-1}$
- T10 - If $a \neq 0$, then $(a^{-1})^{-1} = a$
- T11 - Zero Product Property: If $ab = 0$, then $a = 0$ or $b = 0$
- T12 - Rule for Multiplication with Negative Signs
  $(-a)b = a(-b) = -(ab)$, $(-a)(-b) = ab$
  and $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$
- T13 - Equivalent Fractions Property: $\frac{a}{b} = \frac{ac}{bc}$ if $b \neq 0$ and $c \neq 0$
- T14 - Rule for Addition of Fractions: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ if $b \neq 0$ and $d \neq 0$
- T15 - Rule for Multiplication of Fractions: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ if $b \neq 0$ and $d \neq 0$
- T16 - Rule for Division of Fractions: $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$ if $b \neq 0, c \neq 0, d \neq 0$

Order Axioms

- Ord1 - For any real nubmers a and b, if a and b are positive, so are $a + b$ and ab.
- Ord2 - For every real number $a \neq 0$, either a is positive or $-a$ is positive but not both.
- Ord3 - The number 0 is not positive.

Inequality

- T17 - Trichotomy Law: For arbitrary real numbers a and b, exactly one of three relations $a < b$, $b < a$ or $a = b$ holds
- T18 - Transitive Law: If $a < b$ and $b < c$, then $a < c$
- T19 - If $a < b$, then $a + c < b + c$
- T20 - If $a < b$ and $c > 0$, then $ac < bc$
- T21 - If $a \neq 0$, then $a^2 > 0$
- T22 - $1 > 0$
- T23 - If $a < b$ and $c < 0$, then $ac > bc$
- T24 - If $a < b$, then $-a > -b$. In particular, if $a < 0$, then $-a > 0$
- T25 - If $ab > 0$, then both a and b are positive or both are negative
- T26 - If $a < c$ and $b < d$, then $a + b < c + d$
- T27 - if $0 < a < c$ and $0 < b < d$, then $0 < ab < cd$

Abbreviations

- L - Lemma
- Ex.y - Example (Lecture X, Example Y)
- P - Proposition
- T - Theorem
- Tut - Tutorial
- A - Assignment