Lecture 3:
Factor Pricing Models

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FINM 25000: Portfolio Management

Notation

notation	description
$\tilde{r}$	excess return rate over the period
$\tilde{r}^i$	arbitrary asset i
$\tilde{r}^p$	arbitrary portfolio p
$\tilde{r}^t$	tangency portfolio
$\tilde{r}^m$	market portfolio
$\beta^{i,j}$	regression beta of $\tilde{r}^i$ on $\tilde{r}^j$
$\mathbb{E}\left[\tilde{r}^i\right]$	expected excess return
$\tilde{\boldsymbol{\mu}}$	in-sample mean excess return

Outline

Fundamental Theorem

The CAPM

Testing

Multi-Factor Models

Factor pricing

By no-arbitrage arguments,

$$\mathbb{E}\left[\tilde{r}^{i}\right] = \beta^{i,\mathtt{t}}\, \mathbb{E}\left[\tilde{r}^{\mathtt{t}}\right] \tag{1}$$

$$\beta^{i,\mathtt{t}} \equiv \frac{\operatorname{cov}\left(\tilde{r}^{i}, \tilde{r}^{\mathtt{t}}\right)}{\operatorname{var}\left(\tilde{r}^{\mathtt{t}}\right)}$$

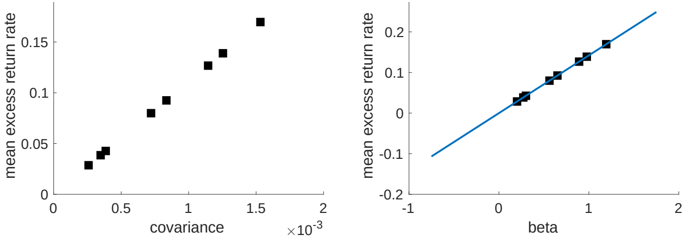
This says the expected (excess) return of an asset is proportional to its tangency-portfolio beta (or covariance).

Optional: Proof sketch

Mean excess returns are a linear function of the covariance with the tangency portfolio.

$$
\begin{aligned}
\mathrm{cov}\left(\tilde{\boldsymbol{r}}, \tilde{r}^{\mathrm{t}}\right) =& \boldsymbol{\Sigma} \boldsymbol{w}^{\mathrm{t}} \\
=& \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}\left(\frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}}\right) \\
=& \tilde{\boldsymbol{\mu}}\underbrace{\left(\frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}}\right)}_{\text{scaling}}
\end{aligned}
$$

Thus, covariance of any return to the tangency portfolio return is linear to mean excess returns.



- Left panel: Mean excess returns of asset classes, affine in covariance with tangency portfolio.
- Right panel: Mean excess returns of asset class indexes, linear in beta with tangency portfolio.

Optimization Conditions

The First Fundamental Theorem of Asset Pricing holds if—and only if—the mean-variance efficient portfolios are well defined. In our MV optimization language, the theorem relies on...

- Convexity of the control space.
- Convexity of the constraint and objective.
- Well-defined solution.

Payoff conditions

Portfolio Formation

For any two security payoffs Γ^i and Γ^j, the payoff $a\Gamma^i + b\Gamma^j$ is also an available security.

Law of One Price (LOOP)

The pricing function, $\mathcal{P}(\cdot)$ is linear:

$$\mathcal{P}(w^i\,\Gamma^i + w^j\,\Gamma^j) = w^i\,\mathcal{P}(\Gamma^i) + w^j\,\mathcal{P}(\Gamma^j)$$

No Arbitrage

For every payoff Γ, if

$$Pr(\Gamma \geq 0) = 1 \text{ and } Pr(\Gamma > 0) > 0 \Longrightarrow \mathcal{P}(\Gamma) > 0.$$

Portfolio implications

Portfolio Formation

- if $\boldsymbol{w}^i, \boldsymbol{w}^j$ are permissible,
- $\boldsymbol{w}^k = \tilde{\delta}\boldsymbol{w}^i + (1-\tilde{\delta})\boldsymbol{w}^j$ is permissible for any $\tilde{\delta} \in (-\infty, \infty)$.

LOOP

- $\tilde{r}^k = \tilde{\delta}\tilde{r}^i + (1-\tilde{\delta})\tilde{r}^j$
- More generally, *Suppose* $\boldsymbol{w}$ *is a vector of portfolio weights. If* $\boldsymbol{w}'\boldsymbol{\Gamma} = 0$ *for every state, then* $\boldsymbol{w}'\mathcal{P}(\boldsymbol{\Gamma}) = 0$.

No Arbitrage

- Non-trivial, limited-liability, portfolios have well-defined returns.

Meaning

Portfolio Formation

- Short-selling and leverage are allowed.

LOOP

- Prices and returns equal the sum of their parts.

No arbitrage

- If a portfolio has cash-flow in any contingency, (without incurring liabilities in any contingencies,) then the portfolio must have a positive price.

Modeling vs Estimation

This factor-beta notation seems to give us a model for all mean returns. But it depends on knowing $\tilde{r}^{\mathrm{t}}$ via

- a theoretical model for $\tilde{r}^{\mathrm{t}}$.
- direct empirical estimation of $\tilde{r}^{\mathrm{t}}$.

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Circularity in direct estimation

Suppose we want to use the factor pricing model to estimate $\tilde{\mu}^i$.

- The estimation of $\tilde{r}^{\mathrm{t}}$ requires $\tilde{\boldsymbol{\mu}}$.

$$\boldsymbol{w}^{\mathrm{t}} = \boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}\frac{1}{\phi}$$

- But $\boldsymbol{w}^{\mathrm{t}}$ should consider all available assets, meaning $\tilde{\boldsymbol{\mu}}$ should includes $\tilde{\mu}^i$ itself!
- So this cannot be a way to estimate $\tilde{\mu}^i$.

Imprecision in direct estimaiton

Suppose we instead estimate $\tilde{r}^{t}$ from one set of assets and then use it to estimate the mean return for some other asset, i.

- Still, direct estimation does not work well.
- We will be ignoring the weight of $\boldsymbol{w}^{t}$ that should be in i.
- Worse, the poor conditioning of $\boldsymbol{\Sigma}$ means that inverting it will greatly magnify the (substantial!) estimation errors in $\boldsymbol{\mu}$.

Thus, it is not practical to statistically extract an $\tilde{\mathrm{MV}}$ portfolio to use in the linear pricing formula above.

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Linear Factor Pricing Models

Linear factor pricing models (LFPM) are assertions about the identity of the tangency portfolio.

- This avoids the problems of direct estimation.
- But it relies on the assumption about the identity of the tangency portfolio (or some other mean-variance portfolio.)

Allocation vs Pricing

The theory does not assume investors allocate to this assumed MV portfolio.

- It assumes the portfolio is MV for the purposes of pricing expected returns.
- If we additionally assume investors prefer MV portfolios, then this portfolio will both price securities and be the equilibrium allocation.
- But note that even if we do not want to hold an MV portfolio, all expected returns are calculated as covariances to the MV portfolio!

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Outline

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The CAPM

The most famous Linear Factor Model is the Capital Asset Pricing Model (CAPM).

$$\mathbb{E}\left[\tilde{r}^i\right] = \beta^{i,m}\,\mathbb{E}\left[\tilde{r}^m\right] \tag{2}$$

$$\beta^{i,m} \equiv \frac{\operatorname{cov}\left(\tilde{r}^i, \tilde{r}^m\right)}{\operatorname{var}\left(\tilde{r}^m\right)}$$

where $\tilde{r}^m$ denotes the return on the entire market portfolio, meaning a portfolio that is value-weighted to every asset in the market.

The market portfolio

The CAPM identifies the **market portfolio** as the tangency portfolio.

- The market portfolio is the value-weighted portfolio of all available assets.
- It should include every type of asset, including non-traded assets.
- In practice, a broad equity index is typically used.

Explaining expected returns

The CAPM is about expected returns:

- The expected return of any asset is given as a function of two market statistics: the risk-free rate and the market risk premium.
- The coefficient is determined by a regression. If β were a free parameter, then this theory would be vacuous.
- In this form, the theory does not say anything about how the risk-free rate or market risk premium are given.
- Thus, it is a relative pricing formula.

Deriving the CAPM

If returns have a joint normal distribution...

1. The mean and variance of returns are sufficient statistics for the return distribution.
2. Thus, every investor holds a portfolio on the $\tilde{\text{MV}}$ frontier.
3. Everyone holds a combination of the tangency portfolio and the risk-free rate.
4. Then aggregating across investors, the market portfolio of all investments is equal to the tangency portfolio.

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Deriving CAPM by investor preferences

Even if returns are not normally distributed, the CAPM would hold if investors only care about mean and variance of return.

- This is another way of assuming all investors choose MV portfolios.
- But now it is not because mean and variance are sufficient statistics of the return distribution, but rather that they are sufficient statistics of investor objectives.
- So one derivation of the CAPM is about return distribution, while the other is about investor behavior.

CAPM assumptions and asset classes

But if we assume normally distributed and iid. returns...

- Application is almost exclusively for equities.
- The CAPM is often not even tried on derivative securities, or even debt securities.

The CAPM decomposition of risk premium

The CAPM says that the risk premium of any asset is proportional to the market risk premium.

$$\mathbb{E}\left[\tilde{r}^i\right] = \beta^{i,m}\,\mathbb{E}\left[\tilde{r}^m\right] \tag{3}$$

The **risk premium** of an asset is defined as the **expected excess return** of that asset.

- The scale of proportionality is given by a measure of risk—the market beta of asset i.
- What would a negative beta indicate?

Beta as the only priced risk

Equation (3) says that market beta is the **only** risk associated with higher average returns.

- No other characteristics of asset returns command a higher risk premium from investors.
- Beyond how it affects market beta, CAPM says volatility, skewness, other covariances do not matter for determining risk premia.

Return variance decomposition

The CAPM implies a clear relation between volatility of returns and risk premia.

$$\tilde{r}^i_t = \beta^{i,m}\tilde{r}^m_t + \epsilon_t$$

Take the variance of both sides of the equation to get

$$\sigma_i^2 = \underbrace{\left(\beta^{i,m}\right)^2 \left(\sigma^m\right)^2}_{\text{systematic}} + \underbrace{\sigma_\epsilon^2}_{\text{idiosyncratic}}$$

So CAPM implies...

- The variance of an asset's return is made up of a systematic (or market) portion and an idiosyncratic portion.
- Only the former risk is priced.

Proportional risk premium

To appreciate how idiosyncratic risk does not increase return, consider the following calculations for expected returns.

$$\mathbb{E}\left[\tilde{r}^i\right] = \beta^{i,m}\,\mathbb{E}\left[\tilde{r}^m\right]$$

- Using the definition of $\beta^{i,m}$,

$$\frac{\mathbb{E}\left[\tilde{r}^i\right]}{\sigma^i} = \left(\rho^{i,m}\right)\frac{\mathbb{E}\left[\tilde{r}^m\right]}{\sigma^m} \tag{4}$$

where $\rho^{i,m}$ denotes $\operatorname{corr}\left(\tilde{r}^m, \tilde{r}^i\right)$.

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The CAPM and Sharpe-Ratios

Using the definition of the Sharpe ratio in (4), we have

$$\mathrm{SR}^i = \left(\rho^{i,m}\right) \mathrm{SR}^m$$

- The Sharpe ratio earned on an asset depends only on the correlation between the asset return and the market.
- A security with large idiosyncratic risk, σ_ϵ^2, will have lower $\rho^{i,m}$ which implies a lower Sharpe Ratio.
- Thus, risk premia are determined only by systematic risk.

Treynor's Ratio

Treynor's measure is an alternative measure of the risk-reward tradeoff. For the return of asset, i,

$$\text{Treynor Ratio} = \frac{\mathbb{E}\left[\tilde{r}^i\right]}{\beta^{i,m}}$$

If CAPM does not hold, then Treynor's Measure does not appropriately capture systemic risk.

Outline

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CAPM and realized returns

The CAPM implies that expected returns for any security are

$$\mathbb{E}\left[\tilde{r}^i\right] = \beta^{i,m}\, \mathbb{E}\left[\tilde{r}^m\right]$$

This implies that realized returns can be written as

$$\tilde{r}^i_t = \beta^{i,m}\ \tilde{r}^m_t + \epsilon_t \qquad (5)$$

where ϵ_t is **not** assumed to be normal, but

$$\mathbb{E}\left[\epsilon\right] = 0$$

Of course, taking expectations of both sides we arrive back at the expected-return formulation.

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Testing the CAPM on an asset

Using any asset return i, we can test the CAPM.

- Run a time-series regression of excess returns i on the excess market return.

- Regression for asset i, across multiple data points t:

$$\tilde{r}^i_t = \alpha^i + \beta^{i,m}\ \tilde{r}^m_t + \epsilon^i_t$$

  Estimate α and β.

- The CAPM implies $\alpha^i = 0$.

Testing the CAPM on a group of assets

Can run a CAPM regression on various assets, to get various estimates α^i.

- CAPM claims every single α^i should be zero.
- A joint-test on the α^i should not be able to reject that all α^i are jointly zero.

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CAPM and realized returns

CAPM explains variation in $\mathbb{E}\left[\tilde{r}^i\right]$ across assets—NOT variation in $\tilde{r}^i$ across time!

$$\tilde{r}^i_t = \alpha^i + \beta^{i,m}\ \tilde{r}^m_t + \epsilon_t$$

- The CAPM does not say anything about the size of ϵ_t.
- Even if the CAPM were exactly true, it would not imply anything about the R^2 of the above regression, because σ_ϵ may be large.

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CAPM as practical model

For many years, the CAPM was the primary model in finance.

- In many early tests, it performed quite well.
- Some statistical error could be attributed to difficulties in testing.
- For instance, the market return in the CAPM refers to the return on all assets—not just an equity index. (Roll critique.)
- Further, working with short series of volatile returns leads to considerable statistical uncertainty.

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Industry portfolios

A famous test for the CAPM is a collection of industry portfolios.

- Stocks are sorted into portfolios such as manufacturing, telecom, healthcare, etc.
- Again, variation in mean returns is fine if it is accompanied by variation in market beta.

Industry portfolios: beta and returns

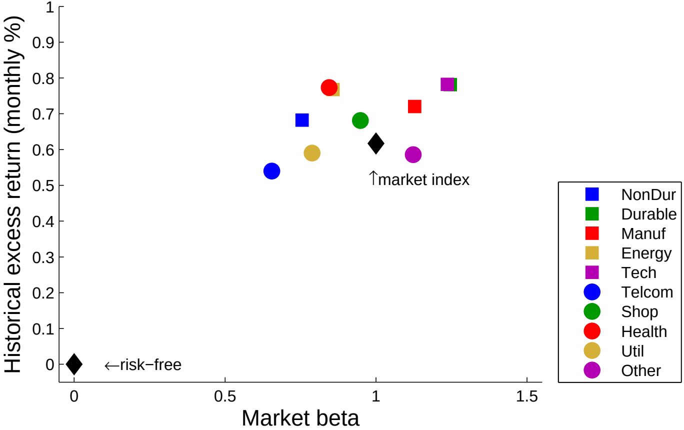


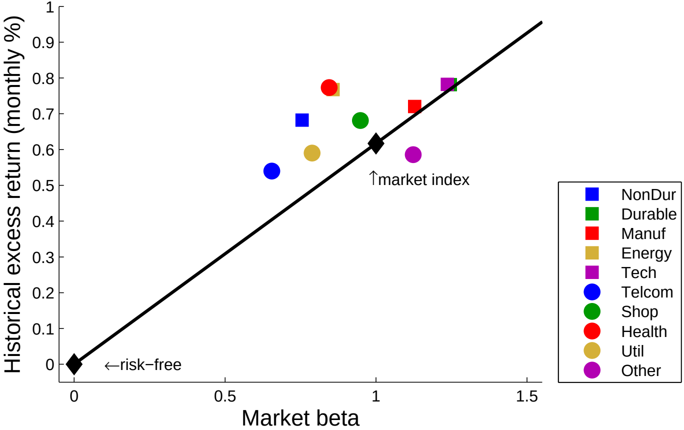
Figure: Data Source: Ken French. Monthly 1926-2011.

Evidence for CAPM?

The plot of industry portfolios shows monthly risk premia from about 0.5% to 0.8%.

- Still, there is substantial spread in betas, and the correlation seems to be positive.
- Note that the risk-free rate and market index are both plotted (black diamonds.)
- Note that the markers for the “Health” and “Tech” portfolio cover up most of the markers for “Energy” and “Durables”.

CAPM-implied relation between beta and returns



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Unrestricted SML for industry portfolios;

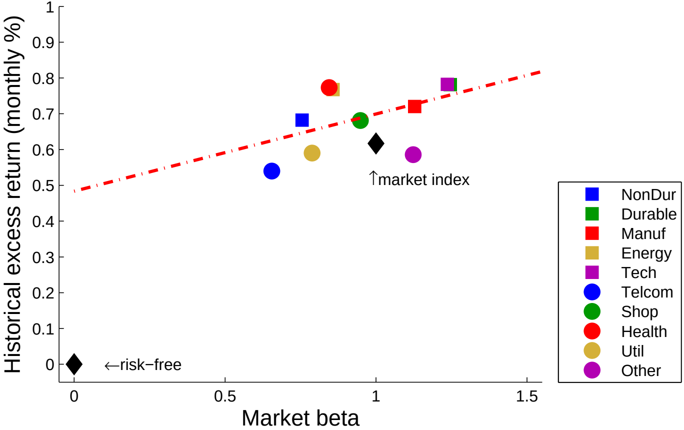
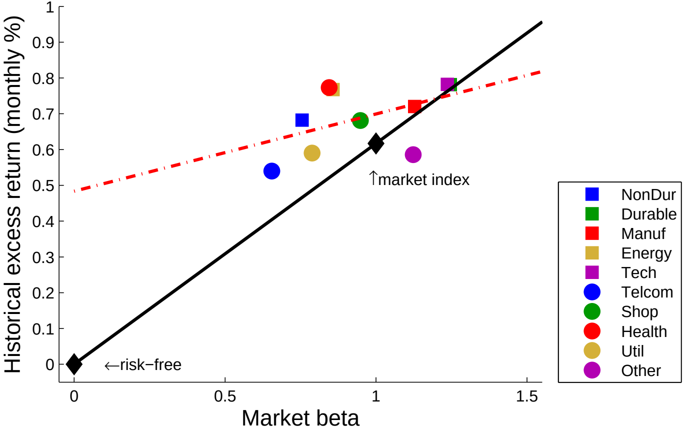


Figure: Data Source: Ken French. Monthly 1926-2011.

Risk-reward tradeoff is too flat relative to CAPM



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Multiple factors

Suppose we have a set of (excess) factor returns, $\tilde{\boldsymbol{r}}^{z}$, such that the tangency portfolio is a linear combination of them: (no need for linear combination to sum to one.)

$$\tilde{r}^{\mathrm{t}} = \boldsymbol{w}' \tilde{\boldsymbol{r}}^{z}$$

Then,

$$\mathbb{E}\left[\tilde{r}^{p}\right] = \left(\boldsymbol{\beta}^{p,z}\right)' \mathbb{E}\left[\tilde{\boldsymbol{r}}^{z}\right]$$

where $\boldsymbol{\beta}^{p,z}$ is the vector of betas from a multivariate regression of $\tilde{r}^{p}$ on $\tilde{\boldsymbol{r}}^{z}$.

Fama-French model

The Fama-French 3-factor model is one of the most well-known multifactor models.

$$\mathbb{E}\left[\tilde{r}^i\right] = \beta^{i,m}\,\mathbb{E}\left[\tilde{r}^m\right] + \beta^{i,s}\,\mathbb{E}\left[\tilde{r}^s\right] + \beta^{i,v}\,\mathbb{E}\left[\tilde{r}^v\right]$$

- $\tilde{r}^m$ is the excess market return as in the CAPM.
- $\tilde{r}^s$ is a portfolio that goes long small stocks and shorts large stocks.
- $\tilde{r}^v$ is a portfolio that goes long value stocks and shorts growth stocks.

Use of growth and value

The labels “growth” and “value” are widely used.

- Historically, value stocks have delivered higher average returns.
- So-called “value” investors try to take advantage of this by looking for stocks with low market price per fundamental or per cash-flow.
- Much research has been done to try to explain this difference of returns and whether it is reflective of risk.
- Many funds (ETF, mutual funds, hedge funds) orient themselves around being “value” or “growth”.

FF Measure of Value

The **book-to-market** (B/M) ratio is the market value of equity divided by the book (balance sheet) value of equity.

- High B/M means strong (accounting) fundamentals per market-value-dollar.
- High B/M are value stocks.
- Low B/M are growth stocks.

For portfolio value factor, this is the most common measure.

Other value measures

Many other measures of value based on some cash-flow or accounting value per market price.

- Earnings-price is a popular metric beyond value portfolios. Like B/M, the E/P ratio is accounting value per market valuation.
- EBITDA-price is similar, but uses accounting measure of profit that ignores taxes, financing, and depreciation.
- Dividend-price uses common dividends, but less useful for individual firms as many have no dividends.

Many other measures, and many competing claims to special/better measure of 'value'.

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Other Popular Factors

Sort portfolios of equities based on...

- Price movement. Momentum, mean reversion, etc.
- Volatility. Realized return volatility, market beta, etc.
- Profitability.*
- Investment.*

*As measured in financial statements.

Characteristics or Betas?

LFPM says security's beta matters, not its measure of the characteristic.

- So what does FF model expect of a stock with high B/M yet low correlation to other high B/M stocks?
- Beta earns premium—not the stock's characteristic.
- This is one difference between FF "value" investing and Buffett-Graham "value" investing.

Finding the right factors

Hundreds of tests and papers written about LFMs!
Does z^j help the model given the other $\boldsymbol{z}$?

- Really asking whether z^j adds to the MV frontier generated by $\boldsymbol{z}$.
- Calculate factor MV:

$$\boldsymbol{w} = \boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1} \boldsymbol{\lambda}_{\boldsymbol{z}} \frac{1}{\gamma}$$

- Any significant weight on factor z^j?
- Easy to formally test this using t-stat, chi-squared test, etc.