Calculus: Single Variable

Ichiro Obara

UCLA

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Single Variable Function

We start with a brief coverage of single variable calculus.

- It forms a basis for the study of multivariable calculus.
- We are often interested in the effect of one variable over another, which can be expressed by a function with single variable (keeping the other variables constant).

Example of single variable functions in Economics

- Demand function: $D(p)$ & Inverse demand function $P(q)$
- Profit function: $P(q)q - C(q)$

A Few Definitions

A **single variable function** maps a point on the real line $\Re$ to another point on the real line. We usually use the notation $f : X \to \Re$ to represent function f.

- $X \subset \Re$ is the set of numbers on which f is defined ($X \subset \Re$ means that X is a subset of the real line) and is called the **domain** of f. X is usually an **interval** and often $\Re$ or $\Re_+$ or $\Re_{++}$.
- The set of values that f can take (y for which there exists $x \in X$ such that $y = f(x)$) is called the **range** of f.

The set of (x, y) that f passes through (i.e. $y = f(x)$) is the **graph** of f.

Linear function

The simplest function is **linear function** such as $2x$, $3x+5$. Linear functions are functions whose graph is a line on a plane. It can be expressed as $f(x) = ax + b$, where a, b are some numbers.

- a is the **slope** of function f.
- b is the **y-intercept** of f, which is the value of f at $x = 0$.

The slope of f measures how much f increases as x increases. For any two points $(x', y') \neq (x'', y'')$ in the graph of $f(x) = ax + b$, $\frac{y''-y'}{x''-x'}$ is exactly a.

A linear function that has slope a and passes through (x_0, y_0) can be easily obtained by $a = \frac{f(x)-y_0}{x-x_0}$, hence $f(x) = ax - ax_0 + y_0$.

Examples of Some Common Functions

- **Polynomial**: $f(x) = a_k x^k + a_{k-1}x^{k-1} + ... + a_1 x + a_0$ is a polynomial function with **degree** k (assuming $a_k \neq 0$), where $a_k, ..., a_0$ are **coefficients**. A linear function is a special polynomial with degree 1 or 0.
- **Exponential Function**: $f(x) = a^x$. Constant $e = 2.718...$ is often used as parameter a.
- **Logarithmic Function**: $f(x) = \ln x$. This function is the **inverse** of $f(x) = e^x$. That is, for any $x > 0$, $f(x) = \ln x$ is defined as the unique number that satisfies $x = e^{f(x)}$.

Exercises

1. What would you take as the domain of $f(x) = e^x$? What is the range of this function for the domain?
2. What would you take as the domain of $f(x) = \log x$? What is the range of this function for the domain?
3. What is the linear function which has slope 5 and passes through $(2, 5)$?
4. What is the linear function that passes through $(-1, -1)$ and $(2, 11)$?

Continuity

We are often interested in the effect of one variable over another. In many cases, it is reasonable to assume that a small change of one variable leads to a small change of another. We would like to express this idea mathematically.

Sequence, Convergence, and Limit

We introduce sequence and convergence etc. to define continuity formally.

- A **sequence** of numbers $x_1, x_2, ... \in \Re$ is denoted by $\{x_n\}_n$.
- A sequence $\{x_n\}_n$ is **bounded** if there exists a number $K \in \Re$ such that $|x_n| \leq K$ for every n.
- A sequence $\{x_n\}_n$ **converges to** $x^* \in \Re$ if, for any $\epsilon > 0$, there exists an integer N such that $|x_n - x^*| < \epsilon$ for every $n \geq N$. We write this as $\lim_{n \to \infty} x_n = x^*$ or $x_n \to x^*$. x^* is called the **limit** of $\{x_n\}_n$.

Properties of Convergent Sequences

Here are some useful facts about convergent sequences. They follow almost immediately from the definition.

- A convergent sequence has only one limit.
- A convergent sequence is bounded.
- If $x_n \leq K$ for every n, then $x^* \leq K$.
- Every subsequence of a convergent sequence has the same limit (a **subsequence** is a subset of the sequence).

Continuous Function

Continuous Function

Function $f : X \to \Re$ is continuous at $x \in X$ if, for any sequence $\{x_n\}_n$ in X that converges to x, $f(x_n)$ converges to $f(x)$. f is continuous if it is continuous at every x in its domain X.

Note: An equivalent definition: Function $f : X \to \Re$ is continuous at $x \in X$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x') - f(x)| < \epsilon$ for any $x' \in X$ such that $|x' - x| < \delta$.

Polynomials, exponential functions, and logarithmic functions are all continuous.

An example of **discontinuous** function: $f(x) = 2x$ for $x < 0$ and $f(x) = 2x + 1$ for $x \geq 0$.

Another example: suppose that there is a **fixed cost** 500 to start a production of some product and a **variable cost** $0.5x^2$ to produce x units of the product. The producer solves the problem $\max\left\{px - 0.5x^2 - 500, 0\right\}$. Plot the optimal production $x^*(p)$ as a function of price p.

Monotonic Function

Monotonic function is a function for which the effect of x over $y = f(x)$ is always the same in sign. It often naturally arises in Economics, is particularly useful when continuity is not guaranteed.

Monotonic Function

Function $f : X \to \Re$ is **increasing (strictly increasing)** if $f(x') \geq (>) f(x)$ for any $x' \geq (>) x$, **decreasing (strictly decreasing)** if $f(x') \leq (<) f(x)$ for any $x' \geq (>) x$.

For example, $f(x) = e^x$ is a strictly increasing function.

Exercises

1. Let $\{x_n\}_n$ and $\{y_n\}_n$ be two converging sequences. Prove the following rules.
   - $\lim (x_n + y_n) = \lim x_n + \lim y_n$.
   - $\lim (x_n y_n) = \lim x_n \lim y_n$.
   - $\lim \left(\frac{x_n}{y_n}\right) = \frac{\lim x_n}{\lim y_n}$, when $\lim y_n \neq 0$.
2. Is a bounded sequence always a convergent sequence?
3. Let $f : \Re \to \Re$ and $g : \Re \to \Re$ be continuous functions. Show that $h(x) = g(f(x))$ is a continuous function.

Slope of Nonlinear Function

Consider a change from x to $x + \Delta x$ and the associated change of value from $f(x)$ to $f(x + \Delta x)$. For a linear function, the ratio of these changes $\frac{f(x+\Delta x)-f(x)}{\Delta}$, which is its slope, is independent of $\Delta x(\neq 0)$ and x. For general nonlinear function, this value depends on Δx and x.

It is useful to evaluate this ratio for a small Δx at x. The **derivative/slope** of function f at x is this ratio at the limit as Δx goes to 0.

Derivative

The formal definition of differentiability and derivative:

Differentiable Function and Derivative

A function f on (a, b) is **differentiable** at $x \in (a, b)$ if $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ converges to the same number for any sequence Δx such that $|\Delta x| \neq 0$ and $|\Delta x| \to 0$. This number is the **derivative** of f at x and is denoted by $\frac{df(x)}{dx}$ or $f'(x)$. f is differentiable on (a, b) if it is differentiable at every $x \in (a, b)$.

Note: A differentiable function is continuous by definition.

Examples

- The derivative of $f(x) = x^2$ is $2x$. To see this, note that $\frac{(x+\Delta x)^2 - x^2}{\Delta x} = \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \to 2x$ as $\Delta x \to 0$.
- More generally, $(x^k)' = kx^{k-1}$ for any $k \in \Re$.
- $(a^x)' = a^x \log a$.
- $(\ln x)' = \frac{1}{x}$.

Some Useful Rules

For differentiable f and g, the derivative of $f+g$ is clearly the sum of derivatives of f and g (i.e. $(f+g)' = f' + g'$). Here are some other useful rules about derivative.

- **Product Rule:** $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- **Quotient Rule:** $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
- **Chain Rule:** Consider a **composite function** $h(x) = g(f(x))$. If f is differentiable at x and g is differentiable at $f(x)$, then h is differentiable at x and its derivative is given by $h'(x) = g'(f(x)) \cdot f'(x)$.

Inverse and Its Derivative

We have already seen an **inverse** of an exponential function. We can define the inverse for a class of general functions. Suppose that, for any x' in some set X', there exists unique $x'' \in \Re$ such that $x' = f(x'')$. Then we can define the **inverse** f^{-1} of f by $f^{-1}(x') = x''$ if and only if $f(x'') = x'$.

We often use **inverse demand function** $P(q)$, which provides price at which q can be sold. As the name suggests, it is the inverse of demand function $D(p)$.

The derivative of inverse is easy to obtain. When f is differentiable at x and $f'(x) \neq 0$, f^{-1} is differentiable at $x' = f(x)$ and $f^{-1'}(x') = \frac{1}{f'(x)}$. For example, the inverse of $f(x) = x^3$ can be obtained by solving $x = (f_{-1}(x))^3$. The derivative of f at $x = 2$ is 12. The derivative of f^{-1} at $x = 8$ is indeed $\frac{1}{12}$ (check this).

Linear Approximation

The derivative of f at certain point x_0 can be used to construct a linear approximation of f around x_0. Graphically, it is a straight line that is tangent to f at x_0.

- What is the linear function $L(x)$ that has the slope $f'(x_0)$ and passes through $(x_0, f(x_0))$? It is $L(x) = f'(x_0)(x - x_0) + f(x_0)$.
- Compare the order of $|f(x) - L(x)|$ and $|x - x_0|$ at $x = x_0 + \Delta x$ around x_0.
  - Note that $\frac{f(x_0+\Delta x)-L(x_0+\Delta x)}{\Delta x} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} - f'(x_0)$, which converges to 0 as $\Delta x \to 0$ by definition. This means that the "error" $f(x_0 + \Delta x) - L(x_0 + \Delta x)$ goes to 0 much faster than Δx goes to 0. We can write this as $f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + o(\Delta x)$ ($o(\cdot)$ is called "little o" and it represents a term that satisfies $\frac{o(h)}{h} \to 0$ for $h \to 0$).

Mean Value Theorem

Let $M = \frac{f(x+\Delta x)-f(x)}{\Delta x}$ be the rate of change from x to $x + \Delta x > x$ for differentiable function f. Intuitively, this number is larger than the smallest derivative and smaller than the largest derivative of f between x and $x + \Delta x$.

Mean Value Theorem

Suppose that $f : (a, b) \to \Re$ is differentiable. Pick any point $x \in (a, b)$ and $\Delta x > 0$ such that $x + \Delta x \in (a, b)$. Then there exists $t \in (0, \Delta x)$ such that

$$f(x + \Delta x) - f(x) = f'(t)\Delta x$$

Higher Order Derivatives

The derivative is itself a function. So it is possible to consider things like the second derivative, the third derivative and so on.

A function is **continuously differentiable** if its derivative is continuous. A function is k-**times continuously differentiable** if it's k-times differentiable and its kth derivative is continuous. The set of k-times continuously differentiable functions is denoted by $\mathcal{C}^k$.

Approximation by Higher Order Polynomials

Higher order polynomials provide a better approximation than linear functions.

For any $f \in \mathcal{C}^n$, let $L^n(x : x_0) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}}{k!}(x - x_0)^k$ be the ***n*th order Taylor polynomial** of f around x_0, where $f^{(k)}$ is the kth derivative of f.

Fix any $\Delta x > 0$ and let M be such that $f(x_0 + \Delta x) - L^1(x_0 + \Delta x : x_0) = M(\Delta x)^2$. The RHS evaluates the residual of the linear approximation in terms of $(\Delta x)^2$.

Define a function $g(h) = f(x_0 + h) - L^1(x_0 + h) - Mh^2$. By definition $g(0) = g(\Delta x) = 0$. Hence $g'(s) = 0$ for some $x\,(0, \Delta x)$ by the mean value theorem. Note that $g'(0) = f'(x_0) - f'(x_0) = 0$, hence $g''(t) = 0$ for some $t\,(0, s)$ by the mean value theorem again. As $g''(t) = f''(x_0 + t) - 2M$, we have $M = \frac{f''(x_0+t)}{2}$.

Similarly, define $M = \frac{f(x_0+\Delta x) - L^2(x_0+\Delta x)}{(\Delta x)^3}$ so that $M(\Delta x)^3$ is the residual with respect to the second order Taylor polynomial. Then we can obtain $M = \frac{f^{(3)}(x_0+t)}{3!}$ for some $t \in (0, \Delta x)$ by applying the MVT three times. This implies that the residual goes to 0 as $\Delta x \to 0$ in the order of $(\Delta x)^3$ if the third derivative of f is bounded.

Taylor's Theorem

This observation generalizes to higher orders. Hence we have the following result.

Taylor's Theorem

Suppose that $f : (a, b) \to \Re$ is a $\mathcal{C}^n$ function and f is $(n+1)$-times differentiable. Pick any point $x \in (a, b)$ and $\Delta x > 0$ such that $x + \Delta x \in (a, b)$. Then there exists $t \in (0, \Delta x)$ such that

$$f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 + \cdots + \frac{f^{(n)}(x)}{n!}(\Delta x)^n + \frac{f^{(n+1)}(x+t)}{(n+1)!}(\Delta x)^{n+1}$$

So $f(x + \Delta x) - L_n(x + \Delta x : x) = \frac{f^{(n+1)}(x+t)}{(n+1)!}(\Delta x)^{n+1}$. Note that the mean value theorem corresponds to $n = 0$.

Exercises

1. What is the derivative of the following functions?
   - $f(x) = e^{2x}$
   - $f(x) = \frac{3x^2-2}{2x+1}$
   - $f(x) = \ln\left(\frac{1}{x}\right)$
2. Suppose that $f'(x) > 0$. Show that there exists $\epsilon > 0$ such that f is strictly increasing in $(x-\epsilon, x+\epsilon)$. Is the converse true?
3. Let $f(x) = x^3 + 4x^2 + 4x$ on $\Re_+$. What is the derivative of the inverse $f^{-1}(x)$ of f of at $x = 32$?
4. **L'Hopital's rule:** Suppose that f and g are differentiable at some point x' and $f(x') = g(x') = 0$. Show that $\lim_{x\to x'} \frac{f(x')}{g(x')} = \frac{f'(x')}{g'(x')}$.

Elasticity

The slope of a function depends on the unit of variables. For example, if $D(p) = -ap + b$ is a demand function in dollar, the demand function in terms of cents would be $-\frac{a}{100}p + b$.

We may want to define the degree of relative change independent of the choice of units. **Elasticity** measures the percentage change of a variable with respect to a percentage change of another variable.

Let x, y be two variables such that $y = f(x)$ for some differentiable function f. The **elasticity** of y with respect to x at x_0 is the limit of $\frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{\frac{f(x+\Delta x)}{f(x)}}{\frac{\Delta x}{x}}$ as Δx goes to 0. So it is $f'(x)\frac{x}{f(x)}$, which we denote by $\mathcal{E}(y, x)$.

We often measure quantity by its log amount. This is useful in terms of deriving the elasticity.

For example, let $q = D(p)$ be some demand function. Let $P = \ln p$ be the log of price and $Q = \ln q$ be the log of quantity. Then $Q = \ln(D(e^P))$. Applying the product rule, we get $\frac{dQ}{dP} = \frac{1}{D(e^P)} D'(e^P)e^P = \frac{p}{q} D'(p)$, which is exactly the elasticity. So if a demand function is given by a linear function $\ln q = a \ln p + b$, then a is the elasticity of the demand function (independent of p).

Newton's Method

- Suppose that you want to find x^* to solve $f(x) = 0$ for some complicated nonlinear function f. What to do?
- Pick some point x_0. Get a linear approximation of f at x_0, which is $L(x) = f(x_0) + f'(x_0)(x - x_0)$. Then find a solution for $L(x) = 0$ instead. This gives us $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. x_1 is not a solution for the original problem, but may be a better guess than x_0.
- We can repeat this and obtain $x_1, x_2, \ldots$ by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. If it happens that x_n converges to some number, then that is a solution.
- Convergence is not guaranteed in general and may depend on the choice of the initial point x_0. If f is $\mathcal{C}_1$ and $f'(x^*) \neq 0$, we can get a convergence to x^* if the starting point x_0 is close enough to x^*.

Exercises

1. Derive the elasticity of linear inverse demand function $p = -3q + 24$ as a function of $q \in [0, 8)$.
2. Derive the elasticity of $f(x) = 3x^2$ and show that it does not depend on x. More generally, discuss why any function with constant elasticity can be expressed as Ax^B with some $A > 0$ and $B \in \Re$.
3. Show $\mathcal{E}(f(x)g(x), x) = \mathcal{E}(f(x), x) + \mathcal{E}(g(x), x)$.