Optimization with Single Variable

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Many problems in Economics can be expressed as some type of optimization problem. Here we study the most elementary optimization problem: maximizing a single variable function. In this slide, we assume that every function is a $\mathcal{C}^1$ function defined on some interval.

Maximum and Minimum

We usually look for x^* that maximizes or minimizes f. Such $(x^*, f(x^*))$ is called **maximum** or **minimum** of f respectively.

Definition: Maximum and Minimum

For function $f : X \to \Re$,

- $(x^*, f(x^*))$ is a **maximum** of f if $f(x^*) \geq f(x)$ for all $x \in X$.
- $(x^*, f(x^*))$ is a **local maximum** of f if $f(x^*) \geq f(x)$ for all $x \in X$ such that $|x - x^*| < \epsilon$ for some $\epsilon > 0$.
- $(x^*, f(x^*))$ is a **minimum** of f if $f(x^*) \leq f(x)$ for all $x \in X$.
- $(x^*, f(x^*))$ is a **local minimum** of f if $f(x^*) \leq f(x)$ for all $x \in X$ such that $|x - x^*| < \epsilon$ for some $\epsilon > 0$.

First Order Condition

We first consider the case where the domain X of f is an open interval (a, b).

Remember the following fact: f is strictly increasing (resp. decreasing) at x if $f'(x) > 0$ (resp. $f'(x) < 0$). This means that the derivative of f must be 0 at any maximum (or minimum) point.

Theorem: First Order Condition

If f has a maximum (or a minimum) at $x^* \in (a, b)$, then $f'(x^*) = 0$.

$f'(x^*) = 0$ is called the **first order condition (FOC)**.

Second Order Condition

FOC is a necessary condition for any maximum, but not a sufficient condition. A point satisfying FOC can correspond to a (local) maximum, a (local) minimum, or neither.

If f is twice differentiable, then we can say a bit more about f's behavior around the point satisfying FOC.

Second Order Condition

Theorem: Second Order Condition

Suppose that $f : (a, b) \to \Re$ is twice differentiable and $f'(x^*) = 0$ at $x^* \in (a, b)$.

- If f has a local maximum at x^*, then $f''(x^*) \leq 0$.
- If $f''(x^*) < 0$, then f has a local maximum at x^*.
- If f has a local minimum at x^*, then $f''(x^*) \geq 0$.
- If $f''(x^*) > 0$, then f has a local minimum at x^*.

Proof

- The 2nd statement: If $f''(x^*) < 0$, then f' is strictly decreasing in $(x^* - \epsilon, x^* + \epsilon)$ for some $\epsilon > 0$. So $f' < 0$ below x^* and $f' > 0$ above x^*. Now suppose that, say, there exists $x' \in (x^*, a + \epsilon)$ such that $f(x^*) \leq f(x')$. Then there must be $\hat{x} \in (x^*, x')$ such that $f(x') = f(x^*) + f'(\hat{x})(x' - x^*)$ by the mean value theorem. But this contradicts $f'(\hat{x}) < 0$. Hence x^* is in fact the unique local maximum around x^*.
- Then the 3rd statement follows from x^* being the unique local maximum.
- The proof of the 4th statement, then the 1st statement is exactly the same.

Optimization on Closed Interval

Next we consider an optimization over closed intervals: $X = [a, b]$.

One important difference between this case and the previous case is that the existence of a maximum is guaranteed. To show this, we take for granted the following not-so-obvious fact about sequences.

Weierstrass Theorem

A bounded sequence in $\Re$ has a convergent subsequence.

Extreme Value Theorem

We can apply WT to prove the existence of maximum and minimum.

Extreme Value Theorem

A continuous function $f : [a, b] \to \Re$ has a maximum and a minimum in $[a, b]$

We use the following property of real number. For any $X \subset\in \Re$, there exists $\sup X$ (**least upper bound**) and $\inf X$ (**largest lower bound**). $\sup X$ is a number such that $x \leq \sup X$ for any $x \in X$ and for any ϵ, there exists $x \in X$ such that $x > \sup X - \epsilon$ (similar definition for $\inf X$). For example, for $X = \left\{1 - \frac{1}{n}\right\}_{n=1,2,..}$, $\sup X = 1$ and $\inf X = 0$.

Proof (for maximum only)

- Let $M = \sup_{[a,b]} f(x)$. Then there exists a sequence $\{x_n\}_n$ in $[a, b]$ such that $f(x_n) \to M$.
- There is a convergent subsequence by Weierstrass theorem, which we still denote by $\{x_n\}_n$, that converges to some x^*. Note that $x^* \in [a, b]$.
- Since f is continuous, $M = \lim_{n\to\infty} f(x_n) = f(x^*)$. Clearly f has a maximum at x^*.

Necessary Condition for Maximum

When f is maximized over $[a, b]$, $f'(x^*) = 0$ does not necessarily hold if f achieves a maximum at $x^* = a$ or $x^* = b$. It is possible that $f(x^*) \leq 0$ if $x^* = a$ and $f(x^*) \geq 0$ if $x^* = b$. The next theorem summarizes this observation and describes how FOC needs to be modified in this case.

Theorem: Necessary Condition on Closed Interval

Suppose that $f : [a, b] \to \Re$ has a maximum at $x^* \in [a, b]$. Then one of the following conditions must hold:

- $x^* \in (a, b)$ and $f'(x^*) = 0$
- $x^* = a$ and $f(x^*) \leq 0$
- $x^* = b$ and $f(x^*) \geq 0$

Average Cost and Marginal Cost

Consider a firm with cost function $F + C(q)$, where F is the fixed cost to start production. Let $AC(q) = \frac{F+C(q)}{q}$ be the **average cost** and $MC(q) = C'(q)$ be the **marginal cost**. Assume $C'' > 0$, so the marginal cost is increasing.

What is the production level $\underline{q}$ that minimizes the average cost? Solve $\min_{q\in(0,\infty)} AC(q)$. The first order condition is

$$AC'(q) = \frac{MC(q)q - F - C(q)}{q^2} = \frac{MC(q) - AC(q)}{q}$$

Hence $MC(q) = AC(q)$ at $\underline{q}$. It is easy to see that $AC'(q) < 0$ for q below $\underline{q}$ and $AC'(q) > 0$ for q above $\underline{q}$.

Monopoly Price

A monopoly firm of some product sets price p to maximize its profit $pD(p) - cD(p)$, where $D(\cdot)$ is the demand function and $c > 0$ is the marginal cost.

The firm solves $\max_{p \in [0,\infty)}$. The first order condition is given by

$$D(p) + pD'(p) - cD'(p) = 0$$

Hence $-\mathcal{E}(q,p)|_{p=p^*} = \frac{p^* - c}{p^*}$ holds at profit-maximizing p^* (assuming that it exists), which means that the demand elasticity at the optimal price is equal to the profit margin adjusting its sign.

Exercises

1. Find an example of $f : \Re_+ \to \Re$ such that f does not have any maximum in $\Re_+$.

2. Solve the following optimization problems.
   - $\max_{x \in \Re_{++}} f(x) = \ln x - 3x$
   - $\max_{x \in \Re_+} f(x) = -x^3 + 4x^2 - 5x - 2$

3. A consumer can use a part of her wealth $w = 10$ to buy some product at price $p > 0$. Suppose that her **utility** when purchasing $x \in \left[0, \frac{10}{p}\right]$ units of the product is given by $\frac{2x}{x+1} + 10 - px$. Find her utility-maximizing consumption $x(p)$ as a function of price.

Concave and Convex Function

Let $f : X \to \Re$ be a function on some interval X.

Concave and Convex Function

- f is **concave** if $f((1-a)x + ay) \geq (1-a)f(x) + af(y)$ for every $x, y \in X$ and $a \in [0, 1]$.
- f is **strictly concave** if $f((1-a)x + ay) > (1-a)f(x) + af(y)$ for every $x \neq y \in X$ and $a \in (0, 1)$.
- f is **convex** $f((1-a)x + ay) \leq (1-a)f(x) + af(y)$ for every $x, y \in X$ and $a \in [0, 1]$.
- f is **strictly convex** if $f((1-a)x + ay) < (1-a)f(x) + af(y)$ for every $x \neq y \in X$ and $a \in (0, 1)$.

A few useful facts: (we postpone the proof to the part on multivariate calculus).

- If f is twice differentiable and $f''(x) \leq 0$ for all $x \in X$, then f is concave.
- f is a concave function if and only if $f(y) \leq f(x) + f'(x)(y - x)$ for any $x, y \in X$

They are intuitive. $f''(x) \leq 0$ means that the slope is always decreasing. The condition in the second statement means that the linear approximation of a concave function always lies above the function.

Note: Of course similar results hold for convex functions. Note that f is convex if and only if $-f$ is concave.

Sufficiency

We've learned the first order condition (and its modified version for the case with the closed interval) is necessary for a maximum. So we may not be sure which of the points satisfying these conditions is a maximum.

They turn out to be sufficient if the objective function f is concave. So if we find a solution for the first order condition, then we know that it is a maximum.

FOC is Sufficient with Concavity

Theorem: Sufficiency

Suppose that f is a concave function.

- If $f'(x^*) = 0$ at $x^* \in (a, b)$, then f takes a maximum at x^* on (a, b).
- If (a) $f'(x^*) \leq 0$ and $x^* = a$ or (b) $f'(x^*) \geq 0$ and $x^* = b$, then f takes a maximum at x^* on $[a, b]$

Proof

Just for the first one. Since f is concave, $f(x) \leq f(x^*) + f'(x^*)(x - x^*) = f(x^*)$ for every $x \in (a, b)$. Hence f takes a maximum at x^* on (a, b).

Another Sufficiency Condition

Here is another condition that guarantees the sufficiency of FOC.

Theorem 2: Sufficiency

Suppose that $f : (a, b) \to \Re$ satisfies the FOC $f'(x) = 0$ only at $x = x^*$ and $f''(x^*) < 0$. Then f has a maximum at x^*.

Proof

- Suppose that f does not take a maximum at x^*. Then there exists a point such as, say, $x' \in (x^*, b)$ with $f(x') > f(x^*)$.
- Since $f''(x^*) \leq 0$, $f'(x) < 0$ for nearby x above x^*. Then f must take a minimum on $[x^*, x']$ at some point $\hat{x} \neq x^*, x'$.
- Then $f'(\hat{x}) = 0$ must hold, which is a contradiction.

Exercises

1. Prove the following statements:
   - If f is a concave function, then $-f$ is a convex function.
   - If f and g are concave, then $f+g$ is a concave.
   - If f is concave and convex, then f must be a linear function.
2. Solve $\max_{x \in \Re} 2xe^{-x}$.