Integral Calculus

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Integral

Consider the problem of measuring the area "below" some function f on $[a, b]$.

This area can be approximated as follows. Divide $[a, b]$ to n intervals of equal length $\Delta = \frac{b-a}{n}$. Let $x_0 = a, x_1 = a + \Delta, ..., x_k = a + k\Delta, ...$. The area is bounded below by $\sum_{k=1}^{n} \min_{x\in[x_{k-1},x_k]} f(x)\Delta$ and bounded above by $\sum_{k=1}^{n} \max_{x\in[x_{k-1},x_k]} f(x)\Delta$.

For a certain class of functions called **integrable functions** (we skip the precise definition, but they include continuous functions and monotonic functions), these bounds converge to the same number as $n \to \infty$. We call this number the **(definite) integral** of f on $[a, b]$, which we denote by $\int_a^b f(x)dx$.

Properties of Integral

For any integrable function f and g, the following properties are satisfied.

- $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
- $\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$ for $\alpha \in \Re$.
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for $c \in (a, b)$.
- If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

Derivative of Integral

Differentiation and integration can be regarded as inverse operations in a certain sense.

Let $F_a(x) = \int_a^x f(t)dt$ and regard this as a function on $[a, b]$. Then the following result can be easily shown (the proofs are left as exercises).

Theorem

- $F_a(x)$ is continuous in x.
- If f is continuous at x, then $F_a'(x)$ is differentiable at x and $F_a'(x) = f(x)$.

Antiderivative

We know how to take the derivative of typical functions. It is also useful to know how to derive the original function from its derivative.

Let f be an integrable function on some interval. A function F that satisfies $F' = f$ is an **antiderivative** of f. For example, if f is continuous, then $F_a(x)$ is an anti-derivative of f on $[a, b]$. There are many antiderivatives. If F is an antiderivative of f, then $F + C$ is an antiderivative of f as well for any constant C. In fact, any antiderivative of f can be expressed this way.

Examples:

For example...

- For $f(x) = x^k$, $F(x) = \frac{1}{k+1}x^{k+1} + C$.
- For $f(x) = e^x$, $F(x) = e^x + C$.
- For $f(x) = x^k, k \neq 1$, $F(x) = \frac{1}{k+1}x^{k+1} + C$.
- For $f(x) = x^{-1}$, $F(x) = \ln x + C$.

where C is any constant.

Fundamental Theorem of Calculus

Antiderivative is useful when computing the definite integral.

Theorem: Fundamental Theorem of Calculus

Suppose that f is integrable. Then the following formula holds for any antiderivative F of f.

$$\int_a^b f(x)dx = F(b) - F(a)$$

Here is why this formula holds:

Proof

- By the mean value theorem,

$$F(b) - F(a) = \sum_{k=1}^{n} F(x_k) - F(x_{k-1}) = \sum_{k=1}^{n} f(t_k)\Delta$$

  for some $t_k \in [x_{k-1}, x_k]$ for $k = 1, \ldots, n$.

- This number is between the upper bound and the lower bound, which converges to $\int_a^b f(x)dx$ as $\Delta \to 0$ by definition. Hence it must be $\int_a^b f(x)dx$.

Integration by Parts

Let U and V be some $\mathcal{C}^1$ functions with $u = U'$ and $v = V'$. We know from the product rule that $(UV)' = uV + Uv$. By taking the integral and applying the fundamental theorem of calculus, we get

$$\int_a^b u(x)V(x)dx = U(b)V(b) - U(a)V(a) - \int_a^b U(x)v(x)dx$$

This is called the **integration by parts** formula.

This formula is useful when U and v are simpler to work with than u and V respectively.

Exercises

1. Show that $|F_a(x') - F_a(x)| \leq K\,|x' - x|$ for some K for any $x', x \in [a, b]$. This implies that $F_a(x)$ is actually **uniformly continuous** in x.

2. Compute the following integrals:
    - $\int_a^b 2x \ln x dx$
    - $\int_a^b \ln x dx$
    - $\int_a^b 3xe^{-2x} dx$

Consumer Surplus

Consider two consumers who need 1 unit of some product. The first consumer is willing to pay up to \$200 and the second consumer is willing to pay up to \$100.

If the price is \$150, then the only first consumer buys it. The difference between \$200, which the first consumer is willing to pay, and the actual payment \$150 is called **consumer surplus**. If the price is \$80, then both consumers buy the product. Consumer surplus is $\$140 = \$200 + \$100 - 2 \times \$80$ in this case.

Draw the demand curve of these two consumers with p in y-axis. Note that the consumer surplus given price p' corresponds to the area below the inverse demand function $P(q)$ from 0 to $D(p')$. For any inverse demand function $P(q)$, we call $\int_{q=0}^{Q} P(q)dq$ consumer surplus when Q units of products are consumed.

Nonlinear Pricing

Suppose that consumer of type $\theta \in [0, \overline{\theta}]$ gets payoff $\theta q - t$ by consuming q units of some product and paying t.

If the price of the product is p, then $t = pq$ when the consumer buys q. But more generally, a seller can use any nonlinear pricing scheme $t(q)$ (ex. quantity discounts).

Given any pricing scheme, different type would choose different combination of (q, t). So let $(q(\theta), t(\theta))$ be the choice of type θ consumer.

Which $(q(\cdot), t(\cdot))$ could arise? Since θ is private information and no one is excluded from any deal, the following conditions must be satisfied for $(q(\cdot), t(\cdot))$:

- **IC condition:** $\theta q(\theta) - t(\theta) \geq \theta q(\theta') - t(\theta')$ for any θ, θ'.
- **IR condition:** $\theta q(\theta) - t(\theta) \geq 0$ for all θ.

We say that $(q(\cdot), t(\cdot))$ is **implementable** if IC and IR is satisfied.

Implementability

Let $U(\theta) = \theta q(\theta) - t(\theta)$. The we can work with $(q(\cdot), U(\cdot))$ instead of $(q(\cdot), t(\cdot))$. The following theorem characterizes implementability.

Theorem: Implementability

$(q(\cdot), U(\cdot))$ is implementable if and only if the following two conditions are satisfied.

1. $q(\cdot)$ is increasing.
2. $U(\cdot)$ satisfies $U(\theta) = U(0) + \int_{t=0}^{\theta} q(x)dx$ where $U(0) \geq 0$.

In the following proof, we assume that q is continuous.

Proof

- IC holds if and only if $(\theta' - \theta)\, q(\theta) \leq U(\theta') - U(\theta) \leq (\theta' - \theta)\, q(\theta')$ for any $\theta \leq \theta'$. This implies that $q(\cdot)$ must be increasing.
- The above formula implies that U satisfies $U'(\theta) = q(\theta)$ at every θ as q is continuous (hence U is $\mathcal{C}^1$).
- By the FTC, $U(\theta) - U(0) = \int_{t=0}^{\theta} q(x)dx$. Then IR is satisfied if and only if $U(0) \geq 0$.
- To see that the converse holds, just note that $U(\theta') - U(\theta) = \int_{x=\theta}^{\theta'} q(x)dx$.

Remark:

- U is (uniformly) continuous, increasing, and convex.
- This result holds without restriction to continuous q: q can be any increasing function (remember that monotone function is integrable). When q is not continuous, $U'(\theta) = q(\theta)$ except for (at most finite) discontinuous points of q.

Exercises

1. Let $D(p) = 10 - 2p$ be the demand function for some product. Suppose that the price of the product is 1.5. Compute the consumer surplus given this price.
2. Suppose that a tax $t > 0$ is imposed for each consumption of this product and consumers face price $1.5 + t$. What is the loss of consumer surplus due to the introduction of this tax? What is the marginal loss at $t = 0$?
3. Suppose that the payoff of consumer of type $\theta \in [0, 1]$ is given by $\sqrt{\theta}q - t$. Consider nonlinear pricing scheme $t(q) = 0.5q^2$.
    1. Compute how many units of product does each type of consumer purchases.
    2. Compute the payoff of each type of consumer given the optimal