**Introduction to Machine Learning**
Fall 2018
University of Science and Technology of China


Lecturer: Jie Wang | Homework 2
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**Notice,** to get the full credits, please present your solutions step by step.

**Exercise 1: Projection Operator** 4pts

For a nonempty, closed, and convex set $S \subseteq \mathbb{R}^n$, the projection of an arbitrary point $x \in \mathbb{R}^n$ onto $S$ is defined by

$$P_S(x) = \mathop{\mathbf{argmin}}_{z\in S} \|x - z\|_2. \tag{1}$$

Show that

1. (1pt) $P_S(x)$ always exists and is unique;
2. (2pt) $y = P_S(x)$ if and only if $y \in S$ and
$$\langle z - y, x - y\rangle \le 0, \forall\, z \in S.$$
3. (1pt) for all $x, y \in \mathbb{R}^n$,
$$\|P_S(x) - P_S(y)\|_2 \le \|x - y\|_2.$$

**Solution:** (1) 1. if $s \in S$,the conclusion is obviously ok.

2. if $s \notin S$ .we consider a set that $B = \{z \in R^n|||z|| < 1\}$,we can find a $\beta$ big enough to make $D = S\bigcap(x + \beta B) \neq \emptyset$ ,and D is a bounded closed set,and $||x - z||_2$ is a continuous function,so that the function must have a minima at $D \in S$,so we prove $P_s(x)$ always exists.

To prove it is unique,wo assume $x_1$ can make $||x - z||_2$ get the minima which equal r, if there is another $x_2$ can make can make $||x-z||_2$get the minima too,so the $x_3 = \frac{x_1+x_2}{2}$,and $x_3$ must belong to S for S is a convex set. and $x_1$ , $x_2$, $z$ can construct a isosceles triangle,and we must have $||x_3 - z|| < r$,which is contradictory with r is the minima. So the it is unique.

(2) we first prove if $y = P_S(x)$,there must have $< z - y, x - y >\le 0, \forall\, z \in S.$ ,we proof by contradiction,if there is a $z \in S$,can make $< z - y, x - y >> 0$,we can find a $z_1 = y + t(z - y)$,and t is a constant,we now prove $||z_1 - x||_2 < ||y - x||_2$when t is small enough.

$$||z_1-x||^2 = ||z_1-y||^2+||y-x||^2+2 < z_1-y, y-x >= t^2||z-y||^2+||y-x||^2-2t < z-y, y-x >$$

,when t is small enough ,we can get

$$||z_1 - x||^2 < ||y - x||^2$$

which is contradiction with $y = P_S(x)$. then we prove if $< z - y, x - y >\le 0, \forall\, z \in S.$,we have $y = P_S(x)$.we can easy prove it by the proof above,for any $z_{n-1} \in S$, we always can find a ponit $z_n = y + t(z_{n-1} - y)$ can make $||z_n - y||$smaller than $||z_{n-1} - y||$,and the limit of $Z_n$ is $y$ ,so we prove it.

(3) ■

**Exercise 2: Separation Theorems** 6pts

Let $S_1, S_2 \subseteq \mathbb{R}^n$ be nonempty convex sets. Show that

1. (2pt) if $S_1$ is closed, then for $x \notin S_1$, there exists a nonzero vector $v \in \mathbb{R}^n$ and $\epsilon > 0$ such that
$$\langle v, y\rangle \leq \langle v, x\rangle - \epsilon, \forall\, y \in S_1;$$
2. (1pt) if $S_1$ and $S_2$ are closed, $S_1$ is bounded and $S_1 \bigcap S_2 = \emptyset$, then there exists a nonzero vector $v \in \mathbb{R}^n$ and $\epsilon > 0$ such that
$$\langle v, x_1\rangle \leq \langle v, x_2\rangle - \epsilon, \forall\, x_1 \in S_1, x_2 \in S_2;$$
3. (2pt) for $x \notin S_1$, there exists a nonzero vector $v \in \mathbb{R}^n$ such that
$$\langle v, y\rangle \leq \langle v, x\rangle, \forall\, y \in S_1;$$
4. (1pt) if $S_1 \bigcap S_2 = \emptyset$, then there exists a nonzero vector $v \in \mathbb{R}^n$ such that
$$\langle v, x_1\rangle \leq \langle v, x_2\rangle, \forall\, x_1 \in S_1, x_2 \in S_2.$$

**Solution:** 1.we assume that $x_1 = P_s(x)$,and make $v = x - x_1$,we have prove the $\langle v, y - x_1\rangle \leq 0$ ,so $\langle v, y\rangle >< \langle v, x_1\rangle$ ,and $\langle v, x - x_1\rangle > 0$,so $\langle v, x\rangle > \langle v, x_1\rangle$, so$\langle v, y\rangle \leq \langle v, x\rangle - \epsilon$, $\forall y \in S_1$.

2. we construct a new set
$$S_0 = S_1 - S_2 = \{x_1 - x_2 | x_1 \in S_1,\ S_2 \in S_2\}$$
and we know S is convex,closed and $S \neq \emptyset$,and $S_1 \bigcap S_2 = \emptyset$ ,so $O \notin S$,so there must exist v can make
$$\langle v, x - 0\rangle > 0, \forall x \in S \text{the conclusion of question 1}$$

we prove it.

3.when $S_1$ is closed ,we have prove it,and when $S_1$is't closed,what we can define $P_s(x)$ as a limit point of $z_k \in S_1$ which can make $||z_k - x|| < ||z_{k-1} - x||$,,and we easy to know $P_s(x) \in S_1^c$because $S_1$is't closed,so the proof of 1 the x can be $x_1$,so $\langle v, x - x_1\rangle$ 0 should change to$\langle v, x - x_1\rangle \geq 0$,the = will be ok when x in the border of set $S_1$.

so we get $\langle v, y\rangle \leq \langle v, x\rangle, \forall\, y \in S_1$

4. we can use the conclusion of 3 and do the same thing as 2 and we can prove it easily. ■