136-final-notes

1 Week 1

1.1 Topic 2: Dot Product

Lemma 1.1 (Properties of Dot Product)**.** If $v, w, x \in R^n \to R^n$, then

1. $v \cdot w = w \cdot v$ (called symmetry)
2. $(v + w) \cdot x = v \cdot x + w \cdot x$ (first property of linearity)
3. $(aw) \cdot v = a(w \cdot v)$ (second property of linearity)
4. $v \cdot v \geq 0 \longleftrightarrow 0$

Definition 1. The **length** or **norm** of a vector v is given by $\|\mathbf{v}\| = \sqrt{v \cdot v}$

Definition 2. The **unit vector** of a vector v is given by $\|\mathbf{v}\| = 0$. So it's length is 0

Definition 3. The **normalization** of a vector v is given by $\hat{v} = \frac{v}{\|\mathbf{v}\|}$. This produces a unit vector

Definition 4. Two vectors are **orthogonal, perpendicular** if their dot-product is zero.

1. $(-b, a)^T$ is orthogonal to $(a, b)^T$ in R^2
2. $(-b, a, 0)^T$ is orthogonal to (a, b, c) in R^3

Definition 5. Angle between two vectors v, w is given by the formula

$$v \cdot w = \|\mathbf{v}\|\|\mathbf{w}\| \cos(\theta) \tag{1}$$

$$\theta = \arccos\left(\frac{v \cdot w}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) \tag{2}$$

Definition 6. Projection of v along w, or projection of v in the w direction, is defined by

$$proj_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}$$

Definition 7. Component/Scalar Component of v along w is: $\|\mathbf{v}\| \cos(\theta)$

Definition 8. Remainder/perp is $\mathbf{r} = \mathbf{v} - proj_{\mathbf{w}}\mathbf{v} = perp_{\mathbf{w}}\mathbf{v}$ where $w \neq 0$. Read it as **perp** of **v** onto **w**

Lemma 1.2. The projection of **v** along **w** and the remainder are orthogonal to each other.

1.2 Topic 3: Inner Product on $\mathbf{C}^n$

Definition 9. Standard Inner Product or just simply dot product for complex numbers is exact same as regular dot product except: $w_1\overline{z_1} + w_2\overline{z_2} + \cdots + w_n\overline{z_n}$. Just remember that $\overline{z}$ basically means the imaginary part of the complex number will have its sign flipped:

So if $\mathbf{w} = 2 + 3i$, then $\overline{\mathbf{w}} = 2 - 3i$

Lemma 1.3 (Properties of Inner Product)**.** Suppose $\mathbf{v}, \mathbf{w}, \mathbf{z}$ are complex vectors and (x, x) is a inner(dot) product function

1. $(\mathbf{v}, \mathbf{z}) = \overline{(\mathbf{z}, \mathbf{v})}$ (called conjugate symmetry)
2. $(\mathbf{v} + \mathbf{w})\mathbf{z} = (\mathbf{v}, \mathbf{z}) + (\mathbf{w}, \mathbf{z})$ first property of linearity in first argument
3. $(a\mathbf{v}, \mathbf{w}) = a(\mathbf{v}, \mathbf{w})$ second property of linearity in first argument
4. $(\mathbf{v}, \mathbf{v}) \geq 0 \iff \mathbf{v} = 0$ (non-negativity)

Definition 10 (Length of complex vectors)**.** if v is a complex vector, then its **norm, length** is given by $\|\mathbf{v}\| = \sqrt{(\mathbf{v}, \mathbf{v})}$, where (x, x) is a inner(dot) product function.

note: every single properties listed in topic 2 applies to complex numbers, so I will not be writing them out

2 Week 2

2.1 Topic 4: Cross Product

Definition 11 (Cross Product)**.** If $\mathbf{u}, \mathbf{v} \in R^3$, and $u = (u_1, u_2, u_3)^T$ and $v = (v_1, v_2, v_3)^T$, then their cross product, defined as $\mathbf{u} \times \mathbf{v}$ is given by:

$$\begin{pmatrix} u_2v_3 - u_3v_2 \\ -[u_1v_3 - u_3v_1] \\ u_1v_2 - u_2v_1 \end{pmatrix} \tag{3}$$

Always remember that cross product is only defined for R^3

Lemma 2.1 (Properties of cross product)**.** Let $\mathbf{u}, \mathbf{v}, \in R^3, \mathbf{z} = \mathbf{u} \times \mathbf{v}$. Then:

1. $\mathbf{z}$ is orthogonal to both vectors $\mathbf{u}, \mathbf{v}$.
2. We say that the cross product is skew-symmetric: Which means $\mathbf{v} \times \mathbf{u} = -\mathbf{z} = -(\mathbf{u} \times \mathbf{v})$
3. If θ is the angle between $\mathbf{u}, \mathbf{v}$, then the length of $\mathbf{z}$ is given by $\|\mathbf{z}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$, which is the area of parallelogram formed by these two vectors.

note: see topic 4, cross product for picture

Lemma 2.2 (Linearity of Cross Product)**.** The cross product is linear in both arguments. That is, it satisfies the $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$

2.2 Topic 5: Linear Combination and Span

Definition 12 (Linear Combination)**.** Let $\mathbf{u}, \mathbf{v}$ be vectors from F^n, and a, b be scalars from F. Then the linear combination of vector $\mathbf{u}, \mathbf{v}$, means, $a\mathbf{u} + b\mathbf{v}$ and this concept can be extended up to any amount of vectors

Definition 13 (Span)**.** Let $\mathbf{v_1}, \mathbf{v_2} \cdots \mathbf{v_n}$ be vectors in F^n. Then we define **Span** of $\mathbf{v_1}, \mathbf{v_2} \cdots \mathbf{v_n}$ as $Span(\{\mathbf{v_1}, \mathbf{v_2} \cdots \mathbf{v_n}\})$, to mean the set of all linear combinations of $\mathbf{v_1}, \mathbf{v_2} \cdots \mathbf{v_n}$: that is,

$$Span(\{\mathbf{v_1}, \mathbf{v_2} \cdots \mathbf{v_n}\}) = \{a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n} : a_1, a_2, \cdots, a_n \in F\}$$

Note: If one or more vectors exists in $Span, S$ where those two vectors can be written as linear combination of other vectors from S, then that vector cannot belong to the **"spanning set"** of the span S

To determine if a vector $\mathbf{v}$,lies in a span of some spanning set S, we simply try to find some coefficients, $a_1, a_2 \cdots a_n$, such that the vector $\mathbf{v}$, can be written as linear combination of $\mathbf{z} \in S$. So to do this, we simply create an augmented matrix and reduce it to row-echelon form

Definition 14 (Spanning Set)**.** We say that set S is a **spanning set** of some vector space V if every vector in V can be written as linear combination of vectors from S. So what this is saying is that all vectors in V can be formed from just the vectors from the spanning set.

Important: For this, we say set S **spans** vector space V.

2.3 Topic 6A: Lines

Note: for this topic, it is best to read official pdf.

Definition 15 (Parametric equation)**.** A parametric equation of a line in R^2 is just an equation of the form $x = x_1 + pt$ or $y = y_1 + qt, t \in R$, where t is called the parameter and goes through the point (x_1, y_1) with the slope $\frac{q}{p}, p \neq 0$.

Definition 16 (Vector Equations)**.** When we say the vector equations in R^2, we mean the expression,

$$x = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t \begin{bmatrix} p \\ q \end{bmatrix}, t \in R$$

This all simplifies down to the vector equation in line in R^2 which is written as $\mathbf{x} = \mathbf{v} + t\mathbf{w} : t \in R$.

Definition 17 (Lines in R^2)**.** Let $\mathbf{u}, \mathbf{v}$, be vectors in R^2, where $\mathbf{w}$, tangent vector to the line with $\mathbf{w} \neq 0$, we refer to the set of vectors: $L = \{\mathbf{v} + t\mathbf{w} : t \in R\}$ as the line in R^2 passing through $\mathbf{v}$, in the direction $\mathbf{w}$, or just the line L.

Example 1 Give the vector equation and the parametric equations, of the line through the point V with coordinates $(2, -3, 4)$ and pointing in the direction of the $\mathbf{w} = (-2, 4, 1)^T)$

The vector equation of the line is given by

$$x = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + t \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}, t \in R$$

Note that the terminal point W (with coordinates $(-2, 4, 1)$) is not a point in this line
The parametric equation of the line is

$$\begin{aligned} x &= 2 - 2t \\ y &= -3 + 4t \\ z &= 5 + t \end{aligned}$$

2.4 Topic 7A: System of Linear Equations

Definition 18 (Solution Set)**.** Solution set is a set S such that it contains all the solutions to the system of some linear equations. The solution set is called **consistent** if there consists of at least one solution. Otherwise, it is called **inconsistent**.

Two solution set is called **equivalent** if all the solutions set in both sets are the same.

Definition 19 (Elementary Row Operations)**.** Elementary row operations are given by:

1. interchange two equations
2. multiply one equation by non-zero constant
3. add to one equation a multiple of another equation

Definition 20 (Trivial equation)**.** Equation $0 = 0$ is called a **trivial equation**. Similarly, any equation that is not $0 = 0$ is called **non-trivial equation**

Definition 21 (Homogeneous equation)**. Homogeneous equations** are of the form $Ax = 0$ whereas **Non-homogeneous equations** are of the form $Ax = b$ where $b \neq 0$. Homogeneous system of equation is always consistent since they always have the trivial solution. Non-homogeneous system may or may not be consistent.

We refer to linear system $Cx = 0$ as the **associated homogeneous linear system**.

Definition 22 (Trivial Solution)**.** It is the zero solution in the homogeneous equation. If a system of equation produces a trivial solution, then this means it has **no free variable**.

Definition 23 (Free and Basic Variable)**. Free variable** is a kind of variable where we can assign any value of our choice to it. **Basic variable** on the other hand, is not a free variable. So it has fixed values.

2.5 Topic 8B: Gauss Jordan (REF)

Definition 24 (Pivot)**.** We say that column 1 of the coefficient/augmented matrix is a **pivot column**.
We say that the $(1, 1)$ position in the coefficient/augmented matrix is a **pivot position**.
We call any **non-zero entry** in the $(1, 1)$ position of the coefficient/augmented matrix,a **pivot**

Definition 25 (Row echelon form)**.** We say that matrix is in row-echelon form to mean that

1. all zero rows occur at the last rows of the matrix
2. the first entry, from the LHS, in any non-zero row appears to the right of the first entry in any rows above it

This is also called the **"Gauss"** part of the algorithm

Definition 26 (Leading entry/variable)**.** The first non-zero entry from the left-hand-side of any row is called the **leading entry**. If a leading entry lies in some column k, then x_k is called the **leading variable**

Definition 27 (Reduced row echelon form)**.** We say that matrix is in reduced row-echelon form to mean that

1. It is in echelon form

2. All the pivots are one

3. the only non-zero element in pivot-column is the pivot itself

Lemma 2.3. If A is a matrix, then there is a **unique** matrix R such that $R = RREF(A)$.

For some remaining pivot column, row, element properties, see the last page in topic 8B. Not that necessary tho.

3 Week 4

3.1 Topic 9: Some language and counting

Definition 28 (Rank)**.** We say that some matrix has **rank** r to mean that there are exactly r pivot elements in the matrix after it has been reduced to reduced row-echelon form.

If A is an $m \times n$ order matrix and if $rank(A) = m$, then this means all the rows in matrix A has a pivot element. And if we are trying to find some solution to some other vector b in the matrix A, since $rank(A) = m$, then the augmented matrix $(A|b)$ is also consistent, i.e. will always produce a solution.

Lemma 3.1. The system of linear equations is consistent if and only if $rank(A) = rank(A|b)$ for some vector b. To see its proof, see the official pdf on topic 9, week 4.

Definition 29 (Nullity)**.** Let A be a matrix of $m \times n$ order and have $r = rank(A)$. Then, we define **nullity** of A, as $nullity(A) = n - r$.

Lemma 3.2 (Rank-Nullity-Theorem)**.** If the system of equation is consistent, where $A \in M_{m \times n}$ and $rank(A) = r$, then the solution set of this equation will contain $n - r$ parameters, which is the nullity of A.

3.2 Topic 10A: Real Examples

Definition 30 (Null space)**.** Let A be the augmented matrix of a homogeneous system of linear equations and let S be its solution set. Then the **nullspace** of A, or $N(A)$ is just that solution set. To see an example on how to calculate this, see the pdf from topic 10a, near end of pdf. 10b contains how to solve complex equations. But basically, you just solve the augmented matrix, then write the solution. The set of coefficient vectors (after solving the solution) will be the **spanning set** of the nullspace.

3.3 Topic 11B: Solution Sets

Lemma 3.3. Linear combinations of solutions to the homogeneous system of linear equations are also solutions to the homogeneous system. That is: $x_1, x_2 \in S, a_1 \in F$, then $(x_1+x_2) \in S, a_1x_1 \in S$.

Lemma 3.4. If we have solutions of an homogeneous system of linear equations, then their difference is also a solution to the associated homogeneous system of these linear equations: $y_1, y_2 \in S, (y_1 - y_2) \in S$

Lemma 3.5. The solution set S can be constructed from the solution set of S and a single particular solution, that is: $y_p \in S, S = \{y_p + x : x \in S\}$

4 Week 5

4.1 Topic 11C: Matrix Multiplication II

Definition 31 (Column Space)**.** The **column space** of matrix A is the span of the columns of matrix A. So Column space is subspace of R^n. To find the column space of matrix A, convert it to rref, determine the pivot columns. Then the original columns, with the corresponding pivot columns are the vectors in the spanning set of columns space of A.

Lemma 4.1 (Solutions of linear system)**.** The system of linear equation $Ax = b$ has a solution if and only if $b \in Col(A)$. So in other words, just from the columns of matrix A, we can write b as some linear combination of it. Because $Col(A) spans(R^n)$. (To see full proof, topic 11C, last page)

4.2 Topic 12A: Properties of Matrices

Definition 32 (Equality)**.** Two matrix can only be equal if they are of the same size and the i^{th}, h^{th}, in both matrix correspond to the same element

Definition 33 (Addition)**.** If both A, B are some matrices of same size, then we can define their addition as

1. $D = A + B$, D is also the same size
2. $d_{ij} = a_{ij} + b_{ij}$

Note: Addition is **only** defined for two matrices with same size

Lemma 4.2 (Properties of Addition of Matrices)**.** If A, B, C are $m \times n$ order matrices then:

1. $A + B = B + A$
2. $A + B + C = (A + B) + C = A + (B + C)$
3. $\exists Z, Z + A = A + Z = Z$ where Z is a zero matrix (called the identity matrix of addition)

Lemma 4.3 (Properties of Multiplication of Matrix by Scalar)**.** If A, B, C are $m \times n$ order matrices and $c, d \in F$ then:

1. $cA = Ac$
2. $c(A + B) = cA + cB$

3. $(c+d)A = cA + dA$

4. $c(dA) = (cd)A$

5. $c(AC) = (cA)C = A(cC) = cAC$

If b has m rows and x has n rows, then $Ax = b$, the order of A is $m \times n$, and $b \in R^m$ and $x \in R^n$

Definition 34 (Transpose of Matrix)**.** If A is an $m \times n$ order matrix, then its **transpose**, defined by A^T is now an $n \times m$ order matrix and, $(A^T)_{pq} = A_{qp}$. You can sorta rotate in any direction, except you need to keep in mind to follow the order you first started. Examples will help clarify this:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \text{ then } A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Lemma 4.4 (Properties of Transpose)**.** If A, B are both $m \times n$ order matrix and $c \in F$, then:

1. $(A+B)^T = A^T + B^T$

2. $(cA)^T = cA^T$

3. $(A^T)^T = A$

Lemma 4.5 (Properties of Matrix Multiplication)**.** If $A, G \in M_{m\times n}, B, D \in M_{n\times p}, C \in M_{p\times q}$, then:

1. $(A+G)B = AB + GB$

2. $A(B+D) = AB + AD$

3. $(AB)C = A(BC) = ABC$

4. $(AB)^T = B^T A^T$

Warning: If A is $m \times n$ order and B is $n \times q$ order. If $m \neq q$, then the multiplication BA is not defined but AB is defined. If $m = q$, then there is also no reason why AB will be same as BA. In fact, on most cases, this will never be same

4.3 Special Square Matrices

Definition 35 (Symmetric)**.** Some properties of square matrices of order $n \times n$:

1. B is **symmetric** to mean that $B = B^T$.

2. C is **skew-symmetric** to mean that $C = -C^T$

Definition 36 (Upper and Lower Triangle)**.** Example of upper triangle $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$.

Example of lower triangle $\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$.

Transpose of lower triangle is upper triangle and vise versa. The product of lower triangle is lower triangle and vise versa.

Definition 37 (Diagonal Matrix)**.** Diagonal matrix simply means a kind of matrix that has both upper and lower triangle. It will only have non-zero elements in the **main diagonal**, at $a_{11}, a_{22}, \cdots, a_{nn}$. We call it $diag(A) = (1, 2, 3)$, to mean $a_{11} = 1, a_{22} = 2, a_{33} = 3$

Definition 38 (Identity Matrix)**.** Identity matrix is just a matrix whose main diagonal, $diag(A)$ is all 1 only. Identity matrix is also always elementary matrix.

Definition 39 (Elementary Matrix)**.** Elementary matrix just means that after you do one row operation, you will end up with identity matrix.

Lemma 4.6 (See notes to understand better)**.** Suppose that C is an $m \times n$ order matrix, then if B is an elementary matrix of C and E is another elementary formed from the I, (which I was formed from B), then $B = EC$. For a good understanding of this lemma, see topic 12B at the end of page.

4.4 Topic 13A: Linear Transformation

Definition 40 (Function determined by matrix A)**.** Let A be an $m \times n$ order matrix. Then the **function determined by matrix** A, is given to us by $T_A = F^n \longrightarrow F^m, T_A(x) = Ax$, where T_A is the function.

Definition 41 (Linear Transformation)**.** We mean that the function determined by matrix is a **linear transformation** to mean that the function satisfies the 2 properties of linearity, the linearity over addition and linearity over scalar multiplication.

Lemma 4.7. The zero vector is mapped to the zero vector. To see its proof, see the official pdf. It also consists of great examples

Definition 42 (Range)**.** Let T be a function from $F^n \longrightarrow F^m$, then we define the **range** of T, as $R(T)$ to mean the set of image points of T: $R(T) = \{T(x) : x \in F^n\}$. The range of T is a subset of $F^m, not F^n$

Lemma 4.8. Let A be an $m \times n$ order matrix and T_A be a function from $F^n \longrightarrow F^m$, then $R(T_A) = Col(A)$. See proof on this near the end of pdf.

Definition 43 (onto (surjective))**.** We say that the function $T : F^n \longrightarrow F^m$, is **onto** to mean that $R(T) = F^m$. Onto will always have at least 1 pre-image.

$T(x)$ is called onto if all the rows are pivot rows. This basically means $Col(A) = F^m$ if A is $m \times n$ order matrix. As well as if $rank(A) = m$.

Definition 44 (one-to-one (injective))**.** We say that the function $T : F^n \longrightarrow F^m$ is a **one-to-one** to mean that if $x \neq y, x, y \in F^n$, then $T(x) \neq T(y)$. Basically what this is saying is that it has at most only 1 pre-image.

$T(x)$ is called one-to-one if all the columns are pivot columns. In other words if the Null space of $T(x) = Ax$ is 0. This is also the case when $rank(A) = n$.

Definition 45 (Nullspace/Kernal of A)**.** The **Nullspace** of A is the set of all vectors x such that $T(x) = Ax = 0$ (image under T is 0). . The nullspace of an $m \times n$ order matrix is a subspace of R^n. Note, $Null(A)$ can never be empty since it always consists of the zero vector. Also, both addition and scalar multiplication property of linearity applies.

5 Week 6

note: Notes are ommited for now since it only contains examples: However, it has some good examples to chcek out. 1) Projection onto a line through the origin, Projection onto a plane through the origin, Rotating about the origin by an angle theta, Reflection about a line through origin.

5.1 13C: Linear Transformation III

Definition 46 (Compositions of a function)**.** It is defined as $T(x) = (T_2 o T_1)(x) = T_2(T_1(x))$ where T is called the composite function of T_2 and T_1. This is also linear in both arguments. See the pdf for the proof, in the first page.

Important property: $[T]_S = [T_2]_S[T_1]_S$

Useful examples here: Standard matrix of their composite function, standard matrix of transformation with rotation and projection.

Definition 47 (Invertibility of Matrices)**.** Let A be a square matrix of $n \times n$ order. We say that A is invertible to mean that there exist another matrix B of the same order, such that $AB = BA = I$ where I is the identity matrix. Then B is called the **inverse** of A. B is also a **unique** matrix.

Definition 48 (Singular)**.** We say a matrix is **singular** to mean that the matrix is not invertible. If it is **non-singular**, then the matrix is invertible.

Lemma 5.1. If A is a square matrix of n order, then $Ax = b$ has a **unique** solution, $z = A^{-1}b, \forall b \in R$

Lemma 5.2 (Properties of the inverse)**.** Let A, B be $n \times n$ order matrix which are invertible, and $c \in F$, non-zero scalar. Then:

1. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$
2. cA is invertible and $(cA)^{-1} = c^{-1}A^{-1}$
3. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$ order really matters
4. If $C, D \in M_{n\times p}(F)$ and $AC = AD$, then $C = D$

Lemma 5.3 (Inverse of Elementary Matrices)**.** Any elementary matrix has an inverse and inverse of elementary matrix is also elementary matrix and it has the same type as the original elementary matrix.

Lemma 5.4. If T is a linear transformation from $F^n \longrightarrow F^n$, then if T is invertible, then its inverse is unique and linear.

Also, T is invertible if and only if $[T]_S$ is invertible, which we can write as $[T^{-1}]_S = ([T]_S)^{-1}$

Lemma 5.5. If A is an $n \times n$ order matrix and $Ax = b$ has a unique solution for any vector b, then A is invertible.

Definition 49 (Isomorphism)**.** An invertible linear transformation is called **isomorphism**. To check if a matrix is isomorphism, we simply need to check if it is either one to one or onto, since if it is either one-to-one or onto, then it is **both** one-to-one and onto (one-to-one onto). Also called **bijective**.

5.2 Topic 14: Matrix Inverse

Definition 50 (Invertiblitiy of matrix)**.** Suppose that A is some order square matrix, then A is invertible if and only if $rank(A) = n$. In other words, A is invertible if and only if $rref(A) = I$ where I is the idenitity matrix. So to convert it to its inverse, create an augmented matrix $(A|I)$ then row reduce it to convert it to $(I|B)$. If we can form I, then it is invertible and B is its inverse. If we cannot form I, then A is not invertible.

Lemma 5.6 (Inverse of 2x2 matrix)**.** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A^{-1} \iff ad - bc \neq 0$ and

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(A)} Adj_(A)$$

where $d(A)$ is the **determinant** of matrix A and $Adj(A)$ is the adjoint (adjugate) of matrix A. **Adjoint** of A is the transpose of the matrix of all the cofactors of A.

6 Week 7

6.1 Topic 15A: The Determinant I

Definition 51 (Submatrix)**.** This is basically the "walk" concept we learned. If we are going to "walk" from a_{ij}, then we will ignore the i^{th} row and j^{th} column, and include everything else

Definition 52 (Determinant)**.** Determinant is defined as a function $\det : M_{n\times n} \longrightarrow F$ by

$$\det(A) = \sum_{i=1}^{j=n} a_{1j}(-1)^{j+1} \det(M_{1j}(A))$$

Definition 53 (Cofactor)**.** We define the i^{th}, j^{th} **cofactor** of A, as $C_{ij}(A)$, which means:

$$C_{ij}(A) = (-1)^{j+i} \det(M_{ij}(A))$$

Lemma 6.1. If A is a square matrix of n order, then $\det(A^T) = \det(A)$. See the proof near end of 15A

Lemma 6.2. If A is a square matrix of n order and A is either upper or lower triangle or both, then $\det(A)$ can be calculated by simply multiplying all the elements in $diag(A)$.

Corollary 6.2.1. Let A be a square matrix of n order. If A contains any two identical rows or columsn, then immediately we can say that $\det(A) = 0$.

Corollary 6.2.2. To calculate the determinant of some higher order matrix, we can simply convert it to echelon form (not rref), and simply multiply all the elements in the diagonal of that echelon form matrix.

Corollary 6.2.3. Let A be a square matrix of n order. Then A is invertible if and only if $\det(A) \neq 0$.

Corollary 6.2.4. $\det(AB) = \det(A)\det(B)$ if both A, B are square matrix of n order.

Corollary 6.2.5. If A is a square matrix of n order and it is invertible, then $\det(A^{-1}) = (\det(A))^{-1}$

6.2 Topic 15C: The Determinant III

Lemma 6.3. Let A be a square matrix with order of n. Then $A \cdot adj(A) = adj(A) \cdot A = \det(A)I_n$, where I_n is the identity matrix

This section contains some useful examples such as: Find area of parallelogram of the sides.

Lemma 6.4 (Area of Parallelogram)**.** In R^2, the area of the parallelogram can be calculated via $\left|\det\left(\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}\right)\right|$, where $v = (v_1, v_2)^T$ and $w = (w_1, w_2)^T$

Lemma 6.5 (LEMMA 6)**.** I don't understand wtf this is saying so just read the course notes

7 Week 8

7.1 Topic 16A: Diagonalization and Eigenvalue I

Definition 54 (Eigenvector, eigenvalue, eigenspace)**.** Suppose that A is a square matrix with n order. We then say that the **non-zero vector x** is an **eigenvector** if there exists some scalar λ such that $Ax = \lambda x$. Such λ is called the **eigenvalue** and together, $(\lambda, \mathbf{x}$ is called the **eigenpair**.

Definition 55 (Eigenvalue equation)**.** The eigenvalue equation is simply

$$Ax = x \implies (A - \lambda I_n)\, x = 0$$

This equation is used to determine x which is the eigenvector.

From the above equation, we want to find the non-trivial solution, (since trivial solution will always work). Then this is only possible **if and only if** the matrix M is not invertible, which means $\det(M) = 0$. So then, we need to find λ first before we can find the x.

Definition 56 (Characteristic Polynomial)**.** The characteristic polynomial of matrix A means that $\Delta_A(t) = \det(A - \lambda I)$. When we set this polynomial to zero, then we are finding the solutions to the **characteristic equation**, which basically means we are solving $\Delta_A(t) = \det(A - \lambda I) = 0$. We can find eigenvalues from this.

Note: When writing eigenvalues, we tend to write them in decreasing order.

Definition 57 (Eigenspace)**. Eigenspace** is basically a set that contains all the eigenvectors of the matrix M, **including zero vector**, since eigenvalue equation always has a trivial solution. (since homogeneous equation always have trivial solution).

7.2 Topic 16B: Diagonalization and Eigenvalue II

Definition 58 (Similar)**.** Let A, B be any two square matrix with same order n. Then A is **similar** to B if there exists some invertible matrix P such that $P^{-1}AP = B$ or $PBP^{-1} = A$.

The same concept can be applied to the linear transformation $T(x)$.

Definition 59 (Trace)**. Trace** of square matrix is the sum of the diagonal entries in the matrix M.

$$tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} (A)_{ii}$$

Lemma 7.1 (Properties of Similar Matrix)**.** If A and B are similar matrices then:

1. $\det(A) = \det(B)$
2. $tr(A) = tr(B)$

Definition 60 (Diagonalizable)**.** Suppose that A is a square matrix of n order. Then A is **diagonalizable** to mean that there exists invertible matrix P of the same order such that $P^{-1}AP = D$, where D is a diagonal matrix. This also means that A is similar to the diagonal matrix D.

To determine such P and D, we simply determine the eigenvalues and eigenvectors of the matrix A. Then $D = diag($ eigenvalues $)$ and P is the eigenvectors.

Lemma 7.2 (Properties of Characteristics Polynomial)**.** Suppose that the matrix A has a characteristic polynomial, $\Delta_A(t) = \det(A - \lambda I)$. Then:

1. $\Delta_A(t)$ is called the n^{th} order polynomial in t; basically the highest degree
2. The highest degree's leading coefficient's sign is given by $(-1)^n$.
3. The second highest $= (-1)^{n-1}(a_{11} + \cdots + a_{nn}) = (-1)^{n-1}tr(A)$
4. .The constant term of the polynomial is simply the $\det(A)$

Note: We use the above properties to "check" our ans only, most of the time.

Corollary 7.2.1. Matrix A is invertible if and only if the eigenvalue of the matrix A given by λ is not equal to 0.

Lemma 7.3. If two matrix A and B are similar, then they will have same characteristic polynomials and therefore have same eigenvalues.

7.3 Subspaces and Bases

Definition 61 (Subspace)**.** A subset S of vector space V is called the **subspace** if it satisfies the 3 conditions:

1. S contains the zero vector
2. Satisfies the closure property under addition, i.e. if $\mathbf{u}, \mathbf{v} \in S, (\mathbf{u} + \mathbf{v}) \in S$
3. Satisfies the closure property under multiplication, i.e. if $\mathbf{u} \in S, c \in F, c\mathbf{u} \in S$

The above criteria should work for all vectors in S and scalars in F.

Definition 62 (Linear Independence/Dependence)**.** We say that the vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots$ are linearly dependent to mean that there exists some scalars such that $c_1\mathbf{v_1} + c_2\mathbf{v_2} \cdots + c_n\mathbf{v_n} = 0$ and **at least one c is not zero**. What this means is that one of the vectors can be written as a linear combination of other vectors, so it is **linearly dependent on those other vectors**. The set containing such vectors is called **Linearly Dependent Set**.

Opposite of that is **linearly independent**. The set of vectors is called linearly independent if the only solution to their homogenous system of equation is the trivial solution. What this means is that all the coefficients c are 0. So no vectors are depending on the other vectors. Set containing these kinds of vectors is called **Linearly Independent Set**.

Definition 63 (Basis)**.** If the set S is a subset of vector space V and S is both **linearly independent** and **spans the vector space V**, then S is a **basis**.

8 Week 9

8.1 Topic 17B: Linear Independence

Lemma 8.1. Simply lemmas condensed below

1. The empty set is linearly independent
2. If a set S contains the zero vector, then S is linearly dependent.
3. if a set S contains a single vector, then S is linearly dependent **iff** that vector is zero vector.
4. If one of the vectors in S is a multiple of another in S, then S is linearly dependent.

Lemma 8.2. Suppose that set S is a subset of F^n and A be the matrix of those vectors in S. Then S is **linearly independent if and only if** $rank(A) = n$ if A is $m \times n$ order matrix. In other words, when we row reduce the matrix A, there are no **free variables, only trivial solution**. This means there should be exactly n pivots.

We can "extract" out all the linearly independent vectors by choosing the original vectors whose rref's pivots are non-zero. Call that set U. Then U will be linearly independent.

Now if set V contains U and vectors from S, then V will be **linearly dependent** since it might contain vectors which can be written as linear combination from other vectors, in which case, it makes the set linearly dependent.

$Span(U) = Span(S)$, since U is the spanning set, which means every other vector can be created from vectors from U. And since we already know that S is subset of F^n, then it will span F^n as well.

Corollary 8.2.1. If a set S which is a subset of the dimension F^n and it contains more vectors than n, then it will be **linearly dependent**.

Note: Example 8/9 is EXTREMELY useful. Make sure to know how to do that full example!!!

Lemma 8.3. Suppose that the set S is a subset of F^n and that it is linearly independent. Now if vector $\mathbf{w}$ is from F^n, then the set S including the vector $\mathbf{w}$ is linearly independent **if and only if** $\mathbf{w} \in Span(S)$.

Lemma 8.4. Suppose that the set S is a subset of F^n and that it is linearly independent. Now suppose we take some vector $\mathbf{v_k}$ from the set S. Then S is still linearly independent.

8.2 Topic 17C: Spanning and Bases

Lemma 8.5. Let V be a subspace of F^n, and let $S = \{v_1, v_2, \cdots, v_p\} \subset V$. Then $Span(S)$ is a subspace of V.

Lemma 8.6. Let $S = \{v_1, v_2, \cdots, v_p\}$, for some vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n} \in F^n$. Then $Span(S) = F^n$ **if and only if** $rank((\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_p})) = n$. In other words, if you take the rref of the vectors in the set S, then there should be exactly n pivots rows, the last column cannot be pivot column, and it has exactly n rows. So we would need at least n vectors in S for it to span in R^n.

Lemma 8.7. Let $S = \{v_1, v_2, \cdots, v_p\}$, for some vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n} \in F^n$. Then if S is a basis for F^n, then S has exactly n vectors, that is, $p = n$.

Lemma 8.8. Let $S = \{v_1, v_2, \cdots, v_p\}$, for n distinct vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n} \in F^n$. Then S is linearly independent **if and only if** $Span(S) = F^n$. Which means, there are three ways to check whether a subset S is a basis for some F^n.

1. Check S for linear independence and check S for Spanning
2. Count the vectors in S and check for Spanning
3. Count the vectors in S and check for linear independence (the fastest way)

Definition 64 (Dimension)**.** We refer to the number n, which is equal to the number of elements in the basis for F^n, as the dimension of F^n, and write it as $dim(F^n = n$

The dimension of a space is equal to the number of elements in a basis for that space.

Theorem 8.9 (Unique Representation Theorem)**.** Let $B = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ be the basis for the F^n. Then for every vector v in F^n, there exists unique scalars $c_1, c_2, \cdots, c_n$ from the field F, such that $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$

9 Week 10

9.1 Topics 17D: Bases, Coordinates, and Components

Definition 65 (Coordinates and Components)**.** Recall from the **unique representation theorem** that for any vector v from F^n, there exists unique scalars from F such that $v = c_1v_1 + c_2v_2 \cdots + c_nv_n$ where $v_1, v_2, \cdots, v_n$ belongs to the basis for F^n. We call the $c_1, c_2, \cdots, c_n$ the **components or coordinates**.

We write it out as

$$[\mathbf{v}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{pmatrix}$$

. The vector $(c_1, c_2, \cdots, c_n)^T$ is called the **coordinate vector** and we read $[\mathbf{v}]_B$ as **The coordinate of vector v relative to the basis,** B

Lemma 9.1 (Change of Basis)**.** Let $B_1 = \{v_1, v_2 \cdots v_n\}$ and $B_2 = \{w_1, w_2 \cdots w_n\}$ both be the basis for F^n. Let x be a vector from F^n and $[\mathbf{x}]_{B_1} = \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{pmatrix}$ and $[\mathbf{x}]_{B_2} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}$.

Then, $[\mathbf{x}]_{B_2} = B_2\,[I]\,B_1 \cdot [\mathbf{x}]_{B_1}$ and $[\mathbf{x}]_{B_1} = B_1\,[I]\,B_2 \cdot [\mathbf{x}]_{B_2}$, where

$$B_2\,[I]\,B_1 = \left([\mathbf{v_1}]_{B_2}, [\mathbf{v_2}]_{B_2}, \cdots, [\mathbf{v_n}]_{B_2}\right) \tag{4}$$
$$B_1\,[I]\,B_2 = \left([\mathbf{w_1}]_{B_1}, [\mathbf{w_2}]_{B_1}, \cdots, [\mathbf{w_n}]_{B_1}\right) \tag{5}$$

Lemma 9.2 (Inverse of Basis)**.** The two change-of-basis matrices, $B_2\,[I]\,B_1$ and $B_1\,[I]\,B_2$ are inverses of each other, so

$$(B_2\,[I]\,B_1)(B_1\,[I]\,B_2) = (B_1\,[I]\,B_2)(B_2\,[I]\,B_1) = I$$

.

What this means is that if we need to change the basis of a matrix to another basis, we can simply just take its inverse

9.2 Topic 18: Matrix Representation of Linear Operators

Definition 66 (Linear Operator)**.** Linear Operator basically means a linear transformation except we usually use it only for square matrices.

Let B be a basis for F^n. Then the **Matrix representation of T relative to the basis B** to be $[T]_B = ([T[(v_1)]_B T[(v_3)]_B \cdots T[(v_n)]_B)$

Definition 67. Suppose that T is a linear operator and let B be a basis for F^n. If v belongs to F^n, then $[T(v)]_B = [T]_B[v]_B$

Lemma 9.3. Suppose that T is a linear operator in R^n. If B_1 and B_2 are bases for F^n, then $[T]_{B_1}, [T]_{B_2}$ are similar to each other and:

$$[T]_{B_2} = (B_1[I]B_2)^{-1}[T]_{B_1}(B_1[I]B_2)$$
$$[T]_{B_1} = (B_2[I]B_1)^{-1}[T]_{B_2}(B_2[I]B_1)$$

9.3 Topic 19A: Diagonalization of Linear Operators I

Lemma 9.4. Let T be a linear operator and B be a basis, both in F^n. Then $(\lambda, \mathbf{x})$ is an **eigenpair iff** $(\lambda, [\mathbf{x}]_B)$ is an eigenpair of the matrix $[T]_B$

Definition 68 (Diagonalizable)**.** Let T be a linear operator in F^n. We say that T is a diagonalizable to mean that there exists a basis B of F^n such that $[T]_B$ is a diagonal matrix where B consists of eigenvectors of T.

Lemma 9.5. Let T be a linear operator and B_1 be basis, both for F^n. Then T is diagonalizable **iff** the matrix $[T]_{B_1}$ is diagonalizable.

Definition 69. To determine whether a linear operator T is diagonalizable or not, we can obtain the matrix representation of T in any basis B_1, $[T]B_1$, and test whether this matrix is diagonalizable.

1. If $[T]_{B_1}$ is diagonalizable and $P^{-1}[T]_{B_1}P = D$ is a diagonal matrix, then
2. the entries of D are the eigenvalues
3. The columns of P are the eigenvectors of T in basis B_1.

Let A be a square matrix of order n. Then A is diagonalizable **iff** there exists a basis of F^n of eigenvectors of A.

Lemma 9.6. If $v_1, v_2, \cdots, v_n$ are the eigenvectors that correspond to a distinct eigenvalues $\lambda_{12}, \cdots, \lambda_n$ of an $n \times n$ order matrix A, then the set $\{v_1, v_2, \cdots v_n\}$ is **linearly independent**.

10 Week 11

10.1 Topic 19B: Diagonalization of Linear Operators II

There are some problems we would need to address:

1. If the matrix has no (real) eigenvalues and eigenvectors, then that matrix is not diagonalizable.

2. If the matrix has only 1 real eigenvalue and 1 linearly independent eigenvector, then that matrix is not diagonalizable

Definition 70 (Algebraic Multiplicity)**.** Algebraic multiplicity simply means the eigenvalue with the **highest power** that divides the characteristic polynomial. So essentially, the number of times, the eigenvalue repeats.

The algebraic multiplicity of an eigenvalue tells us how many linearly independent eigenvectors we would like to obtain from that eigenvalue. So if an eigenvalue has multiplicity of 2, then we **"expect" (NOT "get")** 2 linearly independent eigenvectors.

Definition 71 (Geometric Multiplicity)**.** Geometric multiplicity simply means the number of eigenvectors we get for the corresponding eigenvalue. This is also the dimension of the eigenspace. So the geometric multiplicity of an eigenvalue tells us how many linearly independent vectors we **can obtain** from that eigenvalue.

Lemma 10.1. *Let λ be an eigenvalue of a matrix A. If the geometric multiplicity of λ is $_\lambda$ and the algebraic multiplicity of it is a_λ, then $1 \leq g_\lambda \leq a_\lambda$*

To determine quickly if a matrix is diagonalizable, we find the algebraic and geometric multiplicity. If they do not equal each other, then the matrix will not be diagonalizable.

11 Formula

11.1 Projection onto plane

Example Question: Let $refl(p)$ denote the reflection through the plane P, where P is defined by the scalar equation $-3x_1 - 2x_2 - 3x_3 = 0$. Find the standard matrix:

Proof. $\|n\|^2 = 22$. So now, just remember that standard matrix has e_1, e_2, e_3, so now just substitute in to this formula:

$$\mathbf{e_i} - proj_{\mathbf{n}}(\mathbf{e_i})$$

□