DCT-DHT Sample Exam Problems with Solutions

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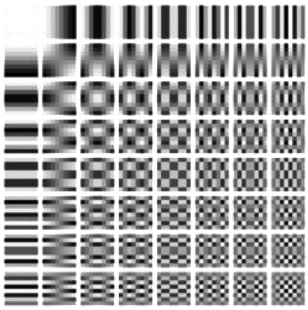

1. Let $f(x,y)$ denote a digital image of size 256×256. In order to compress this image, we take its Discrete Cosine Transform (DCT) $C(u,v), u,v = 0,\dots,255$ and keep only the Discrete Cosine Transform coefficients for $u,v = 0,\dots,n$ with $0 \le n \le 255$. The percentage of total energy of the original image that is preserved in that case is given by the formula $an + b + 85$ with a,b constants. Furthermore, the energy that is preserved if $n = 0$ is 85%. Find the constants a,b.

Solution

For $n = 0$, it is given that the preserved energy is 85%. This is the case where only the (0,0) frequency pair is kept. Therefore,

$$a \cdot 0 + b + 85 = 85 \Rightarrow b = 0$$

In case where the entire DCT is kept we have $n = 255$ and the preserved energy should be 100%. In that case: $a \cdot 255 + b + 85 = 100 \Rightarrow 255a = 15 \Rightarrow a = \frac{1}{17}$.

2. Let $f(x,y)$ denote a digital image of size $N \times N$ pixels that is zero outside $0 \le x \le N-1$, $0 \le y \le N-1$, where N is an integer and power of 2. In implementing the standard Discrete Hadamard Transform of $f(x,y)$, we relate $f(x,y)$ to a new $N \times N$ point sequence $H(u,v)$.
(i) State the main disadvantage(s) and advantages of the Discrete Hadamard Transform.
(ii) In the case of $N = 2$ and $f(x,y) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ calculate the Hadamard transform coefficients of $f(x,y)$.

Solution

(i) **<u>Disadvantages</u>**

The main disadvantage of DHT is that it is not ordered, which means that we must order it after calculating it.

<u>Advantages</u>

Energy compaction
A large fraction of the signal energy is packed within very few transform coefficients, namely, the ones near the origin but this is valid for the ordered Hadamard and not the standard Hadamard. By keeping the low index transform coefficients in the ordered Hadamard and replacing the rest with zero we can achieve image compression.

Resistant to errors
Basis functions consist of 1s and -1s and therefore, the transform is more resistant to errors.

Recursive relationship

The Hadamard matrix of large order can be computed from the Hadamard matrix of lower orders through the well-known recursive relationship.

(ii) We know that $N = 2^n$ and therefore, in case of $N = 2$ we have $n = 1$, x, y 0 or 1 and $b_0(0) = 0$, $b_0(1) = 1$. For $f(x,y) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ we calculate the Hadamard transform coefficients as follows.

$$H(u,v) = \frac{1}{N}\sum_{x=0}^{2-1}\sum_{y=0}^{2-1} f(x,y)\left[\prod_{i=0}^{n-1}(-1)^{(b_i(x)b_i(u)+b_i(y)b_i(v))}\right]$$

$$H(u,v) = \frac{1}{N}\sum_{x=0}^{1}\sum_{y=0}^{1} f(x,y)\left[\prod_{i=0}^{1-1}(-1)^{(b_i(x)b_i(u)+b_i(y)b_i(v))}\right]$$

$$H(u,v) = \frac{1}{N}\sum_{x=0}^{1}\sum_{y=0}^{1} f(x,y)(-1)^{(b_0(x)b_0(u)+b_0(y)b_0(v))}$$

$$= \frac{1}{2}f(0,0)(-1)^{(b_0(0)b_0(u)+b_0(0)b_0(v))} + \frac{1}{2}f(0,1)(-1)^{(b_0(0)b_0(u)+b_0(1)b_0(v))}$$

$$+\frac{1}{2}f(1,0)(-1)^{(b_0(1)b_0(u)+b_0(0)b_0(v))} + \frac{1}{2}f(1,1)(-1)^{(b_0(1)b_0(u)+b_0(1)b_0(v))}$$

$$= \frac{1}{2}f(0,0)(-1)^{(0\cdot b_0(u)+0\cdot b_0(v))} + \frac{1}{2}f(0,1)(-1)^{(0\cdot b_0(u)+1\cdot b_0(v))}$$

$$+\frac{1}{2}f(1,0)(-1)^{(1\cdot b_0(u)+0\cdot b_0(v))} + \frac{1}{2}f(1,1)(-1)^{(1\cdot b_0(u)+1\cdot b_0(v))}$$

$$= \frac{1}{2}f(0,0)(-1)^0 + \frac{1}{2}f(0,1)(-1)^{b_0(v)} + \frac{1}{2}f(1,0)(-1)^{b_0(u)} + \frac{1}{2}f(1,1)(-1)^{b_0(u)+b_0(v)}$$

$$= \frac{1}{2}(-1)^0 + \frac{1}{2}2(-1)^{b_0(v)} + \frac{1}{2}2(-1)^{b_0(u)} + \frac{1}{2}3(-1)^{b_0(u)+b_0(v)}$$

$$= \frac{1}{2} + (-1)^{b_0(v)} + (-1)^{b_0(u)} + \frac{3}{2}(-1)^{b_0(u)+b_0(v)}$$

$$H(0,0) = \frac{1}{2} + (-1)^{b_0(0)} + (-1)^{b_0(0)} + \frac{3}{2}(-1)^{b_0(0)+b_0(0)}$$

$$= \frac{1}{2} + (-1)^0 + (-1)^0 + \frac{3}{2}(-1)^0 = \frac{1}{2} + 1 + 1 + \frac{3}{2} = 4$$

$$H(0,1) = \frac{1}{2} + (-1)^{b_0(0)} + (-1)^{b_0(1)} + \frac{3}{2}(-1)^{b_0(0)+b_0(1)}$$

$$= \frac{1}{2} + (-1)^0 + (-1)^1 + \frac{3}{2}(-1)^{0+1} = \frac{1}{2} + 1 - 1 - \frac{3}{2} = -1$$

$$H(1,0) = \frac{1}{2} + (-1)^{b_0(1)} + (-1)^{b_0(0)} + \frac{3}{2}(-1)^{b_0(1)+b_0(0)}$$

$$= \frac{1}{2} + (-1)^1 + (-1)^0 + \frac{3}{2}(-1)^{1+0} = \frac{1}{2} - 1 + 1 - \frac{3}{2} = -1$$

$$H(1,1) = \frac{1}{2} + (-1)^{b_0(1)} + (-1)^{b_0(1)} + \frac{3}{2}(-1)^{b_0(1)+b_0(1)}$$

$$= \frac{1}{2} + (-1)^1 + (-1)^1 + \frac{3}{2}(-1)^{1+1} = \frac{1}{2} - 1 - 1 + \frac{3}{2} = 0$$

Therefore, $H(u,v) = \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix}$.

3. Let $f(x,y)$ denote the following constant 4×4 digital image that is zero outside $0 \le x \le 3$, $0 \le y \le 3$, with r a constant value.

$$\begin{bmatrix} r & r & r & r \\ r & r & r & r \\ r & r & r & r \\ r & r & r & r \end{bmatrix}$$

(i) Give the standard Hadamard Transform of $f(x, y)$. Comment on whether the result obtained could have been deducted without carrying out any mathematical manipulations.
(ii) Comment on the energy compaction property of the standard Hadamard Transform.

Solution

(i) The 4×4 Hadamard transform matrix is given below:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

This is applied to each row of the image.

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} r \\ r \\ r \\ r \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The following "intermediate" image is obtained by putting the result above as the "new" rows.

$$\begin{bmatrix} r & 0 & 0 & 0 \\ r & 0 & 0 & 0 \\ r & 0 & 0 & 0 \\ r & 0 & 0 & 0 \end{bmatrix}$$

Next, we apply the Hadamard transform in a column-by-column fashion. The final image is

$$\begin{bmatrix} 4r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the original image is constant, we can guess the form of the Hadamard transformed image. There is only one nonzero component corresponding to the (0,0) frequency pair and its value is the mean of the image.

The 4×4 Hadamard transform matrix can be obtained from the 2×2 Hadamard transform matrix using the well-known recursive relationship.

$H(x, u) = \prod_{i=0}^{n-1}(-1)^{b_i(x)b_i(u)}$. For signals of size 2 samples the Hadamard matrix is 2×2 and the Hadamard kernel has 4 samples as follows:

$$H(x, u) = \prod_{i=0}^{1-1}(-1)^{b_i(x)b_i(u)} = (-1)^{b_0(x)b_0(u)}$$

$H(0,0) = (-1)^{0\cdot 0} = 1$

$H(0,1) = (-1)^{0\cdot 1} = 1$

$H(1,0) = (-1)^{1\cdot 0} = 1$

$H(1,1) = (-1)^{1\cdot 1} = -1$

$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Using the recursive relationship of the Hadamard matrix we get:

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

(ii) The standard Hadamard transform does not possess the property of energy compaction since the basis functions in the transformation matrix are not sorted to have increasing sequency.

4. Let $f(x, y)$ denote the following constant 4×4 digital image that is zero outside $0 \leq x \leq 3$, $0 \leq y \leq 3$, with r a constant value.

$$\begin{bmatrix} r & r & r & r \\ r & r & r & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Give the standard Hadamard Transform of $f(x, y)$.

Hint: Use the recursive relationship of the Hadamard matrix and the separability property of the Hadamard Transform.

Solution

$$\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} r \\ r \\ r \\ r \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The following "intermediate" image is obtained by putting the result above as the "new" rows:

$$\begin{bmatrix} r & 0 & 0 & 0 \\ r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Next, we apply the Hadamard transform in a column-by-column fashion.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} r \\ r \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2r \\ 0 \\ 2r \\ 0 \end{bmatrix}$$

The final image is

$$\begin{bmatrix} 2r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$