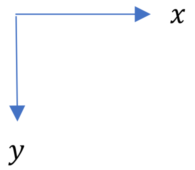
KLT Sample Exam Problems with Solutions

1. Consider the population of vectors $\underline{f}$ of the form

$$\underline{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$$



Each component $f_i(x,y), i = 1,2$ represents an image of size $M \times M$, where M is even. The population arises from their formation across the entire collection of pixels.
The two images are defined as follows:

$$f_1(x,y) = \begin{cases} r_1 & 1 \le x \le M, 1 \le y \le \frac{M}{2} \\ s_1 & 1 \le x \le M, \frac{M}{2} < y \le M \end{cases}, \quad f_2(x,y) = \begin{cases} r_2 & 1 \le y \le M, 1 \le x \le \frac{M}{2} \\ s_2 & 1 \le y \le M, \frac{M}{2} < x \le M \end{cases}$$

Consider now a population of random vectors of the form

$$\underline{g} = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}$$

where the vectors $\underline{g}$ are the Karhunen-Loeve (KL) transforms of the vectors $\underline{f}$.

(i) Find the images $g_1(x,y)$ and $g_2(x,y)$ using the Karhunen-Loeve (KL) transform.

(ii) Comment on whether you could obtain the result of (i) above using intuition.

Solution

(i)

The first image $f_1(x,y)$ has a solid horizontal edge. Its mean is $\frac{r_1+s_1}{2}$. The zero-mean version of it is $\bar{f}_1(x,y) = \begin{cases} \frac{r_1-s_1}{2} & 1 \le x \le M, 1 \le y \le \frac{M}{2} \\ \frac{s_1-r_1}{2} & 1 \le x \le M, \frac{M}{2} < y \le M \end{cases}$.

The second image $f_2(x,y)$ has a solid vertical edge. Its mean is $\frac{r_2+s_2}{2}$. The zero-mean version of it is $\bar{f}_2(x,y) = \begin{cases} \frac{r_2-s_2}{2} & 1 \le y \le M, 1 \le x \le \frac{M}{2} \\ \frac{s_2-r_2}{2} & 1 \le y \le M, \frac{M}{2} < x \le M \end{cases}$.

The variance of $f_1(x,y)$ is $\frac{(r_1-s_1)^2}{4}$ and the variance of $f_2(x,y)$ is $\frac{(r_2-s_2)^2}{4}$.

The covariance between the two images is zero. Note that the covariance is the mean of the product of the zero-mean versions of the two images. This is because $\bar{f}_1(x,y)$ is of the form $\begin{bmatrix} a & \vdots & a \\ \cdots & \vdots & \cdots \\ -a & \vdots & -a \end{bmatrix}$ and $\bar{f}_2(x,y)$ is of the form $\begin{bmatrix} b & \vdots & -b \\ \cdots & \vdots & \cdots \\ b & \vdots & -b \end{bmatrix}$ and, `therefore, $f_1(x,y)f_2(x,y) = \begin{bmatrix} ab & \vdots & -ab \\ \cdots & \vdots & \cdots \\ -ab & \vdots & ab \end{bmatrix}$.
Hence, the mean of $f_1(x,y)f_2(x,y)$ is zero.

Based on the above, the covariance matrix of the population is $C_{f_1 f_2} = \begin{bmatrix} \frac{(r_1-s_1)^2}{4} & 0 \\ 0 & \frac{(r_2-s_2)^2}{4} \end{bmatrix}$.

The eigenvalues of the covariance matrix are $\frac{(r_1-s_1)^2}{4}$ and $\frac{(r_2-s_2)^2}{4}$. The images $g_1(x,y)$ and $g_2(x,y)$ are simply the zero mean versions of the original images.

(ii)

There is no point of using the KL transform since it is obvious visually that the original images are already uncorrelated.

2. Consider the population of vectors $\underline{f}$ of the form

$$\underline{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{bmatrix}.$$

Each component $f_i(x,y),\ i = 1,2,3$ represents an image of size $M \times M$ where M is even. The population arises from the formation of the vectors $\underline{f}$ across the entire collection of pixels (x,y). The three images are defined as follows:

$$f_1(x,y) = \begin{cases} r_1 & 1 \le x \le \frac{M}{2}, 1 \le y \le M \\ r_2 & \frac{M}{2} < x \le M, 1 \le y \le M \end{cases}$$

$$f_2(x,y) = r_3,\ 1 \le x \le M,\ 1 \le y \le M$$

$$f_3(x,y) = r_4,\ 1 \le x \le M,\ 1 \le y \le M$$

The parameters r_1, r_2, r_3, r_4 are constants.
Consider now a population of random vectors of the form

$$\underline{g}(x,y) = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \\ g_3(x,y) \end{bmatrix}$$

where the vectors $\underline{g}$ are the Karhunen-Loeve (KL) transforms of the vectors $\underline{f}$.

(i) Find the images $g_1(x,y)$, $g_2(x,y)$ and $g_3(x,y)$ using the Karhunen-Loeve (KL) transform.
(ii) Comment on whether you could obtain the result of (i) above using intuition rather than by explicit calculation.

Solution

(i)

The mean value of $f_1(x,y)$ is $m_1 = \frac{r_1}{2} + \frac{r_2}{2}$. The zero-mean version of $f_1(x,y)$ is

$$f_1(x,y) - m_1 = \begin{cases} \frac{r_1}{2} - \frac{r_2}{2} & 1 \le x \le \frac{M}{2}, 1 \le y \le M \\ \frac{r_2}{2} - \frac{r_1}{2} & \frac{M}{2} < x \le M, 1 \le y \le M \end{cases}$$

The mean value of $f_2(x,y)$ is r_3. The zero-mean version of $f_2(x,y)$ is $f_2(x,y) - m_2 = 0$.
The mean value of $f_3(x,y)$ is r_4. The zero-mean version of $f_3(x,y)$ is $f_3(x,y) - m_3 = 0$.
The variance of $f_1(x,y)$ is $\frac{1}{2}\frac{1}{4}(r_1 - r_2)^2 + \frac{1}{2}\frac{1}{4}(r_1 - r_2)^2 = \frac{1}{4}(r_1 - r_2)^2$.
The variance of $f_2(x,y) - m_2$ is 0.
The variance of $f_3(x,y) - m_3$ is 0.
The covariance between $\bar{f}_1(x,y)$ and the other two images is 0. Therefore, the covariance matrix of the three images is $\begin{bmatrix} \frac{1}{4}(r_1 - r_2)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with eigenvalues $\frac{1}{4}(r_1 - r_2)^2$, 0, 0. Hence, by using

the Karhunen Loeve transform we produce three new images, with two of them being 0 and the other being $\bar{f}_1(x,y)$.

(ii)

As with Problem 1, there is no point of using the KL transform since it is obvious visually that the original images are already uncorrelated.

3. Consider the population of random vectors $\underline{f}$ of the form

$$\underline{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{bmatrix}$$

Each component $f_i(x,y)$ represents an image. The population arises from their formation across the entire collection of pixels. Suppose that $n > 2$, i.e. you have at least three images.
Consider now a population of random vectors of the form

$$\underline{g} = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \\ g_3(x,y) \end{bmatrix}$$

where the vectors $\underline{g}$ are the Karhunen-Loeve transforms of the vectors $\underline{f}$.
The covariance matrix of the population $\underline{f}$ calculated as part of the transform is

$$\underline{C_f} = \begin{bmatrix} a & 0 & b^2 \\ 0 & a & b^2 \\ b^2 & b^2 & a \end{bmatrix}$$

Suppose that a credible job could be done of reconstructing approximations to the three original images by using one or two principal component images. What would be the mean square error incurred in doing so in each case?

Solution

Values of a must be positive since it represents the variance of the three images. All three images have the same variance.
The eigenvalues of the covariance matrix $\underline{C_f}$ are obtained by solving the equation $det\left|\underline{C_f} - \lambda I\right| = 0 \Rightarrow (a-\lambda)^3 - 2b^4(a-\lambda) = 0 \Rightarrow \lambda = a, \lambda = a \pm \sqrt{2}b^2$.
The eigenvalues are sorted as follows: $a + \sqrt{2}b^2 \geq a \geq a - \sqrt{2}b^2$.
If we keep one image the error will be $2a - \sqrt{2}b^2$ and if we keep two images the error will be $a - \sqrt{2}b^2$.

4. Consider the population of vectors $\underline{f}$ of the form

$$\underline{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{bmatrix}.$$

Each component $f_i(x,y), i = 1,2,3$ represents an image. The population arises from the formation of the vectors across the entire collection of pixels.
Consider now a population of vectors $\underline{g}$ of the form

$$\underline{g} = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \\ g_3(x,y) \end{bmatrix}$$

where the vectors $\underline{g}$ are the Karhunen-Loeve transforms of the vectors $\underline{f}$.

The covariance matrix of the population $\underline{f}$ calculated as part of the transform is

$$\underline{C_f} = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$$

with $a, b, c > 0$.

(i) Suppose that a credible job could be done of reconstructing approximations to the three original images by using one principal component image. What would be the mean square error incurred in doing so, if it is known that $c < a - b$.

(ii) Suppose that a credible job could be done of reconstructing approximations to the three original images by using two principal component images. What would be the mean square error incurred in doing so, if it is known that $c > a + b$.

Solution

(i)

The eigenvalues of the covariance matrix $\underline{C_f} = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$ are found by the following relationship:

$$\det \begin{bmatrix} a-\lambda & b & 0 \\ b & a-\lambda & 0 \\ 0 & 0 & c-\lambda \end{bmatrix} = (c-\lambda)[(a-\lambda)^2 - b^2]$$

$$= (c-\lambda)[(a-\lambda) - b][(a-\lambda) + b] = 0 \Rightarrow \lambda_1 = c, \lambda_2 = a - b, \lambda_3 = a + b$$

If $c < a - b$ then because $c > 0$ (this condition is given but furthermore, c must be positive since it represents the variance of an image), the eigenvalues will be sorted according to their magnitude as $a + b \geq a - b > c$ and therefore, by using only one principal component the error of reconstruction will be $a - b + c$.

(ii)

If $c > a + b$ the eigenvalues will be sorted according to magnitude as $c > a + b \geq a - b$ and therefore, by using only two principal components the error of reconstruction will be $a - b$.

5. Consider the population of vectors $\underline{f}$ of the form

$$\underline{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{bmatrix}.$$

Each component $f_i(x,y), i = 1,2,3$ represents a real image. The population arises from the formation of the vectors across the entire collection of pixels. Consider now a population of vectors $\underline{g}$ which are the Karhunen-Loeve transforms of the vectors $\underline{f}$.

The covariance matrix of the population $\underline{f}$ calculated as part of the transform is

$$\underline{C}_{\underline{f}} = \begin{bmatrix} a & b & c \\ b & a & 0 \\ c & 0 & a \end{bmatrix}$$

(i) Find the covariance matrix of the population $\underline{g}$. Provide conditions for a,b,c so that both covariance matrices are valid.

(ii) Suppose that a credible job could be done of reconstructing approximations to the three original images by using one or two principal component images. What would be the mean square error incurred in each case?

Solution

(i)

The value of a must be positive.

(ii)

The eigenvalues of the covariance matrix $\underline{C_f}$ are obtained by solving the equation $det\left|\underline{C_f} - \lambda I\right| = 0 \Rightarrow (a-\lambda)^3 - b^2(a-\lambda) - c^2(a-\lambda) = 0 \Rightarrow \lambda = a,\ \lambda = a \pm \sqrt{b^2+c^2}$. The eigenvalues are sorted as follows: $a + \sqrt{b^2+c^2} \geq a \geq a - \sqrt{b^2+c^2}$.

If we keep one image the error will be $2a - \sqrt{b^2+c^2}$ and if we keep two images the error will be $a - \sqrt{b^2+c^2}$.