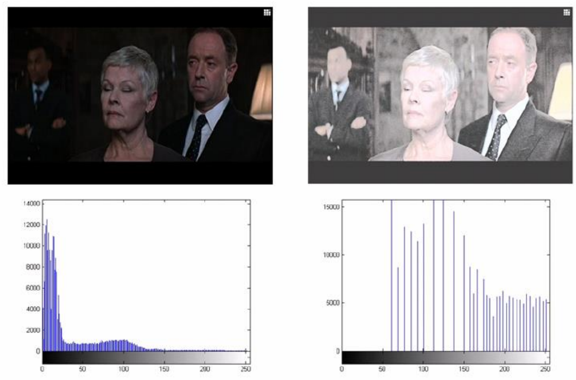
Histogram Processing Sample Exam Problems with Solutions

Imperial College UG4, MSc COMSP, MSc AML



1. (i) Knowing that adding uncorrelated images convolves their histograms, how would you expect the contrast of the sum of two uncorrelated images to compare with the contrast of its component images? Justify your answer.

   (ii) Consider an $N \times N$ image $f(x, y)$. From $f(x, y)$ create an image $g(x, y) = -2f(x, y) + f(x, y-1) + f(x, y+1)$. Comment on the histogram of $g(x, y)$ in relation to the histogram of $f(x, y)$.

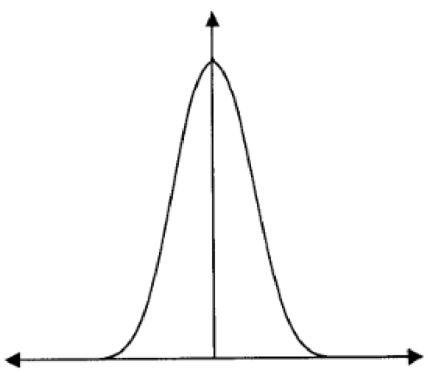
   Solution

   (i)

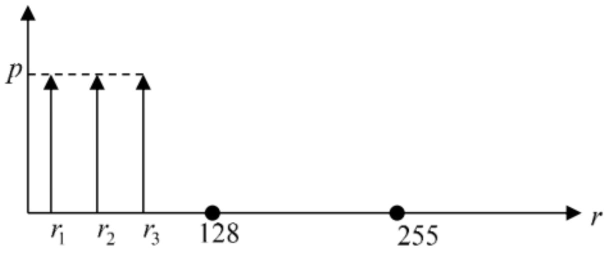
   The new histogram will be the convolution of the 2 original histograms and, therefore, it will occupy a wider range of values. Therefore, the new image will be an image of higher contrast compared to the original.

   (ii)

   By carrying out the proposed manipulation we see that most pixel values of pixels which belong to a constant or slowly varying area will turn into zeros. Furthermore, the resulting intensities will be both positive and negative. The resulting histogram will look as follows:



2. Consider an $8-$bit grey-level image $f(x, y)$ with histogram sketched below.



(i) What can we say about $f(x, y)$?
(ii) Propose an intensity transformation function which will improve the contrast of the image when it is used to modify the intensity of the image.
(iii) Sketch the histogram of the transformed intensity.
(iv) Calculate the mean and the variance of the two images.

Solution

(i)

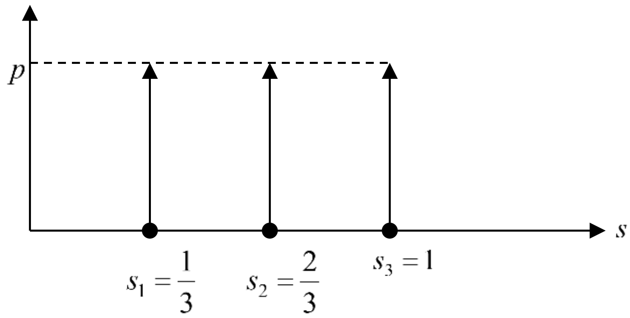
$f(x, y)$ will be a dark image since the intensities are concentrated in the lower half of the intensity range. Moreover, it will consist of three intensities only with equal probabilities, and therefore, $p(r_i) = p = \frac{1}{3}, i = 1,2,3$.

(ii)

We can use histogram equalization. By doing so, the three original intensities are mapped to the following new intensities: $s_1 = T(r_1) = \frac{1}{3}$, $s_2 = T(r_2) = \frac{2}{3}$, $s_3 = T(r_3) = 1$.
s_1 is mapped to $\frac{1}{3} \times 255 = 85$
s_2 is mapped to $\frac{2}{3} \times 255 = 170$
s_3 is mapped to $1 \times 255 = 255$

(iii)

The histogram of the new normalized intensities is shown below:

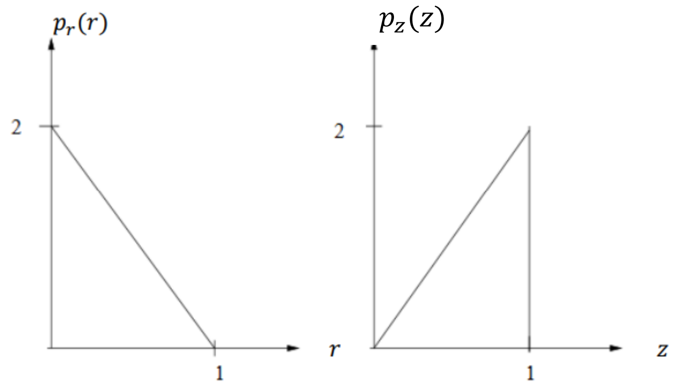


(iv)

For the original image we have:
Mean: $m_1 = \frac{1}{3}(r_1 + r_2 + r_3)$
Variance: $\sigma_1^2 = \frac{1}{3}(r_1^2 + r_2^2 + r_3^2) - \frac{1}{9}(r_1 + r_2 + r_3)^2$
For the equalised image we have:
Mean: $m_2 = \frac{1}{3}(s_1 + s_2 + s_3) = \frac{1}{3}(\frac{1}{3} + \frac{2}{3} + 1) = \frac{2}{3}$

Variance: $\sigma_2^2 = \frac{1}{3}(s_1^2 + s_2^2 + s_3^2) - \frac{4}{9} = \frac{1}{3}\left(\frac{1}{9} + \frac{4}{9} + 1\right) - \frac{4}{9} = \frac{1}{3}\frac{14}{9} - \frac{12}{27} = \frac{2}{27}$

3. An image has the grey level probability density function (or histogram normalized by the number of pixels) $p_r(r)$ shown below left.



(i) Find the pixel transformation $y = g(r)$ such that, after transformation the image has a flat PDF, i.e., which accomplishes histogram equalisation. Assume continuous variables r, y.

(ii) It is desired to find a transformation $z = f(r)$ such that the transformed image will have the PDF of $p_z(z)$ shown above right. Assume continuous quantities and determine the transformation function $z = f(r)$.

Solution

(i)

$p_r(r) = 2 - 2r$

$y = g(r) = (L-1)\int_0^r p_r(w)dw = (L-1)\int_0^r (2-2w)dw = (L-1)(2r - r^2)$

(ii)

$p_z(z) = 2z$

By equalizing z we obtain

$s = T(z) = (L-1)\int_0^z p_z(w)dw = (L-1)\int_0^z 2wdw = (L-1)\,z^2$

$y = s \Rightarrow\; g(r) = \; T(z) \Rightarrow 2r - r^2 = z^2$

$z = \sqrt{2r - r^2}$

4. Two images have the same histogram. Which of the following properties must they have in common? Justify your answer.

(i) Same total power.

(ii) Same entropy.

(iii) Same degree of pixel-to-pixel correlation.

Solution

Let $H(r)$ be the histogram of the image and r be the gray level distribution. The first two properties, namely, the total power (sum of the squares of pixel values) and the entropy must be the same.

Both depend only on the pixel values not the order they are arranged in the image and can be expressed as:

$$\text{Power} = \sum_r H(r) r^2$$
$$\text{Entropy} = \sum_r -H(r) \ln_2 H(r)$$

The inter-pixel covariance function, however, is not necessarily the same. One could take the pixels in an image and move them randomly around. In that case, the image histogram would be the same but the covariance between pixels would be different.

5. Consider a grey level image $f(x, y)$ of size 256×256 with $1 \leq x, y \leq 256$, which has the following intensities:

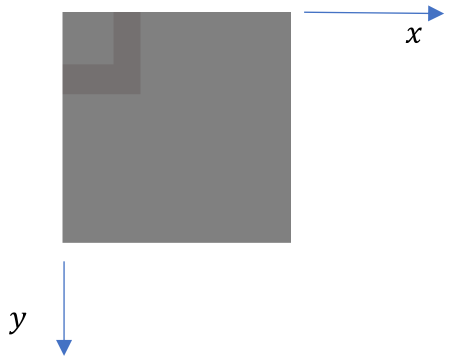
$$f(x,y) = \begin{cases} r+1 & 1 \leq x \leq 12 \text{ and } 1 \leq y \leq 12 \\ r & 13 \leq x \leq 16 \text{ and } 1 \leq y \leq 16 \text{ and } 1 \leq x \leq 12 \text{ and } 13 \leq y \leq 16 \\ r+2 & \text{elsewhere} \end{cases}$$

(i) Sketch the image $f(x, y)$ and comment on its visual appearance. Justify your answer.
(ii) Apply global histogram equalisation on the above image. Comment on the visual appearance of the resulting equalised image.
(iii) Apply local histogram equalisation on the above image using non-overlapping image patches of size 16×16. Comment on the visual appearance of the resulting locally equalised image.
(iv) Based on the above observations, which of the two types of equalisation processes would you choose for the visual improvement of the particular image? Justify your answer.

Solution

(i)

The image looks as the one below. The figure is not true to scale.



The three intensities are very close to each other, so their differences are not large enough to be perceived by the human eye. Therefore, the above image should appear constant with a grey level around r.

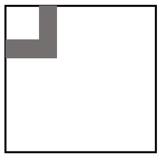
(ii)

$$p(r) = \frac{256 - 144}{256 \cdot 256} = \frac{112}{256 \cdot 256} = \frac{7}{16 \cdot 256}$$

$$p(r+1) = \frac{144}{256 \cdot 256} = \frac{9}{16 \cdot 256}$$
$$p(r+2) = \frac{256 \cdot 256 - 256}{256 \cdot 256} = \frac{255}{256}$$
Therefore, we get:
$$T(r) = p(r) = \frac{7}{16 \cdot 256}$$
$$T(r+1) = \frac{7+9}{16 \cdot 256} = \frac{1}{256}$$
$$T(r+2) = 1$$
By multiplying with 255 we get the new image intensities as follows:
$$r \to \frac{7}{16}\frac{255}{256} = 0$$
$$(r+1) \to \frac{255}{256} = 1$$
$$(r+2) \to 255$$
In the resulting image we will still not be able to distinguish the two new intensities which arise from the mappings of r and $r+1$.

(iii)

For the top left part of the image of size 16×16 we will get:
$$p(r) = \frac{256-144}{256} = \frac{112}{256} = \frac{7}{16}$$
$$p(r+1) = 1$$
By multiplying with 255 we get the new image intensities as follows:
$$r \to \frac{7}{16} 255 = 112$$
$$(r+1) \to 255$$
The rest of the image will turn white, i.e., it will be of intensity 255.
Therefore, the locally equalised image will look as below:



(iv)

Obviously local histogram equalisation is able to extract the local pattern on the top left part and therefore, it is preferable.

6. Consider the $3-$grey-level digital image $f(x,y)$ of size 256×256 shown below in **Figure 1** where $0 \le x, y \le 255$. The intensity of this image is constant and equal to r_1 for most of the pixel locations. Inside the image there is a pattern of two small rectangular areas. The smallest area is of intensity r_2 and occupies the pixels at locations $144 \le x, y \le 175$. This is placed within a slightly larger square area which occupies the pixels at locations $128 \le x, y \le 191$. The intensities of this area are r_3 everywhere apart from the locations that form the smallest square placed in the middle.

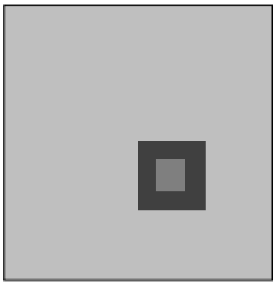


Figure 1

Intensities r_1, r_2, r_3 lie within 0 and 255, such that $r_3 < r_2 < r_1$ and $r_2 = r_3 + 1$.

(i) Apply global histogram equalization to the image $f(x,y)$. Let $h_{\text{out}}(s)$ denote the resulting (equalized) histogram of pixel values s, taking values in [0,255]. Sketch the plot of $h_{\text{out}}(s)$, and label the plot axes. Sketch the resulting histogram-equalized image.

(ii) Apply local histogram equalization to the image $f(x,y)$ by diving the image in non-overlapping patches of size 64×64. Sketch the resulting histogram equalized image.

(iii) Discuss which of the two transforms, i.e., the global or the local histogram equalization is more beneficial for the given image. [Note: the dark line that appears around the image is used to signify the boundary of the image but is not part of it.]

Solution

(i)

The intensities of the two inner squares are very similar and therefore, the inner pattern is not visible by the human eye. It basically looks like a single square instead of the following pattern:

The probabilities of the three intensities are as follows:

$$p(r_2) = \frac{32 \times 32}{256 \times 256} = \frac{1}{64}$$

$$p(r_3) = \frac{64 \times 64 - 32 \times 32}{256 \times 256} = \frac{3}{64}$$

$$p(r_1) = \frac{60}{64} = \frac{15}{16}$$

After histogram equalisation we obtain the following mapping:

$s_3 = T(r_3) = p(r_3) = \frac{3}{64}$ and with normalisation we get $\frac{3}{64} \times 255 \approx 12$.

$s_2 = T(r_2) = p(r_2) + p(r_3) = \frac{4}{64}$ and with normalisation we get $\frac{4}{64} \times 255 \approx 16$.

$s_1 = T(r_1) = 1$ and with normalization we get $1 \times 255 \approx 255$.

The intensities 12 and 16 are quite close and therefore, the inner pattern will still not be clearly visible to the human eye.

(ii)

In case we opt for local histogram equalisation the inner patch with the pattern will perfectly fit in a scanning patch. For that patch we have the following intensity transformations:

$$p(r_2) = \frac{32 \times 32}{64 \times 64} = \frac{1}{4}$$

$$p(r_3) = \frac{64 \times 64 - 32 \times 32}{64 \times 64} = \frac{3}{4}$$

After histogram equalisation we obtain the following mapping:

$s_3 = T(r_3) = p(r_3) = \frac{3}{4}$ and with normalisation we get $\frac{3}{4} \times 255 \approx 191$.
$s_2 = T(r_2) = p(r_3) + p(r_2) = 1$ and with normalization we get $1 \times 255 \approx 255$.
The difference between the new intensities is quite substantial and therefore, the inner pattern is now clearly visible.
The rest of the image will turn white after local histogram equalisation.

(iii)

Based on the above analysis local histogram equalisation is more beneficial.

7. The two images shown below in Figure 2 are quite different, but their histograms are identical. Both images have size 8×8, with black and white pixels. Suppose that both images are blurred with a 3×3 smoothing mask. Would the resultant histograms still be the same? Draw approximately the two histograms and explain your answer. [Note: the dark lines that appear around the two images are used to signify the boundaries of the images but are not part of them.]

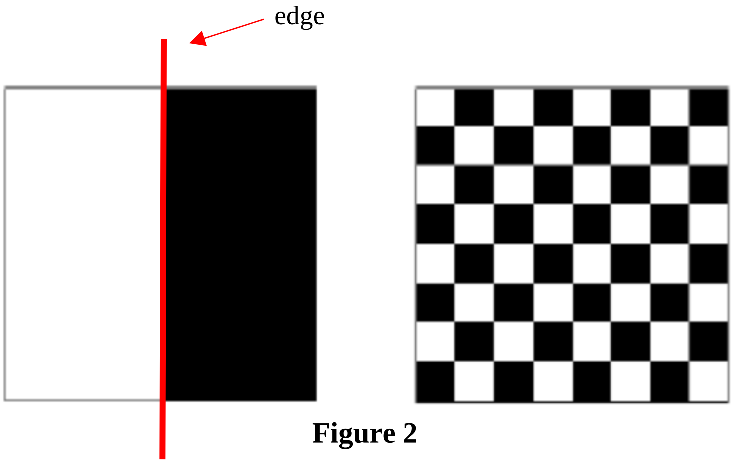


Figure 2

Solution

We assume that the images are extended by zeros.
The responses of the various pixels to a smoothing mask are as follows.

For the left image we have:

Response of black corners (2 on total): 0
Response of white corners (2 on total): $\frac{4}{9}$
White non-border pixels next to the image's edge (6 on total): $\frac{6}{9} = \frac{2}{3}$
Black non-border pixels next to the image's edge (6 on total): $\frac{3}{9} = \frac{1}{3}$
White border pixels next to the image's edge (2 on total): $\frac{4}{9}$
Black border pixels next to the image's edge (2 on total): $\frac{2}{9}$
Rest of white border pixels (10 on total): $\frac{6}{9} = \frac{2}{3}$
Rest of black border pixels (10 on total): 0
Rest of white (inner) pixels (12 on total): 1
Rest of black (inner) pixels (12 on total): 0

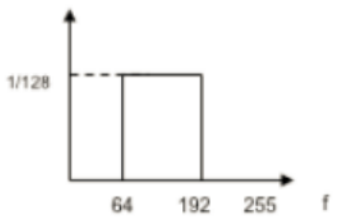
For the right image we have:

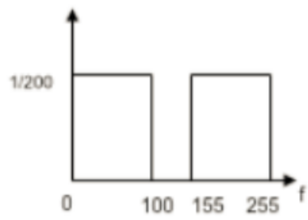
Response of black corners (2 on total): $\frac{2}{9}$

Response of white corners (2 on total): $\frac{2}{9}$
Rest of white border pixels (12 on total): $\frac{3}{9} = \frac{1}{3}$
Rest of black border pixels (12 on total): $\frac{3}{9} = \frac{1}{3}$
Rest of white (inner) pixels (18 on total): $\frac{5}{9}$
Rest of black (inner) pixels (18 on total): $\frac{4}{9}$

It is straightforward to see that the two histograms of the filtered images are different.

8. The histograms of two images are illustrated below. Sketch a transformation function for each image that will make the image have a better contrast. Use the axis provided below to sketch your transformation functions.





Solution

The transformation used for histogram equalisation is $s = T(r) = \int_0^r p_r(w)dw$. Based on that we get (do some work-out yourselves):