DSP Exam 2021

1. (a) Let $f(x,y)$ denote an $N \times N$ −point two-dimensional sequence, that has zero value outside $0 \leq x \leq N-1$, $0 \leq y \leq N-1$, where N is an integer power of 2. In implementing the standard Walsh Transform of $f(x,y)$, we relate $f(x,y)$ to a new $N \times N$ −point sequence $W(u,v)$.

   (i) Define the sequence $W(u,v)$ in terms of $f(x,y)$. Explain all the symbols used. **[1]**

   (ii) For the case $N = 2$ and $f(x,y) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ calculate the forward Walsh transform coefficients. **[4]**

   (b) Consider the population of vectors $\underline{f}$ of the form

$$\underline{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{bmatrix}.$$

   Each component $f_i(x,y)$, $i = 1,2,3$ represents an image of size $M \times M$ where M is even. The population arises from the formation of the vectors $\underline{f}$ across the entire collection of pixels (x,y).

   Consider now a population of vectors of the form

$$\underline{g}(x,y) = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \\ g_3(x,y) \end{bmatrix}$$

   where the vectors $\underline{g}$ are the Karhunen-Loeve (KL) transforms of the vectors $\underline{f}$.

   (i) Prove and explain the relationship between the covariance matrix of $\underline{g}(x,y)$ and the covariance matrix of $\underline{f}(x,y)$. What is the structure of the covariance matrix of $\underline{g}(x,y)$? **[3]**

   (ii) Can the elements of the covariance matrix of $\underline{g}(x,y)$ be negative? Justify your answer. **[2]**

   (iii) Suppose that $N = 8$ and that the eigenvalues of the covariance matrix of $\underline{f}(x,y)$ are:

$$[168 \quad 6.1 \quad 0.08 \quad 13 \quad 64 \quad 214 \quad 1.2 \quad 0.2]$$

   What will be the mean square error if we use principal component images associated with the largest eigenvalues for 2: 1 and 4: 1 data compression? **[4]**

   (c) Consider again Question 1(b) in the case of 2 images ($N = 2$). The covariance matrix of the population is $\underline{C_f}$ with elements defined as $C_{m,n} = \rho^{|m-n|}, 1 \leq m,n \leq 2, 0 < \rho < 1$.

   (i) Provide expressions for the variances of the images $g_1(x,y)$ and $g_2(x,y)$. **[4]**

   (ii) What will be the mean square error as a function of ρ if we use the principal component image from the set of images $g_1(x,y)$ and $g_2(x,y)$ to reconstruct the original images? **[2]**

2. (a) Consider a 64×64 image with 4 grey levels. The normalized grey levels are denoted by 0, $\frac{1}{3}, \frac{2}{3}$ and 1. The number of pixels with the corresponding grey levels, are shown in the following table.

Grey level	**Number of pixels**
0	1813
$\frac{1}{3}$	1506
$\frac{2}{3}$	574
1	203

(i) Draw the histogram of the image. **[1]**
(ii) Determine the equalized histogram. **[2]**
(iii) Draw the equalized histogram. **[1]**

(b) What happens if you apply histogram equalisation twice to the same image? Justify your answer. **[4]**

(c) After histogram equalisation will an image have more, the same or fewer distinct grey levels? Justify your answers. **[4]**

(d) A mean filter is a linear filter but a median filter is not. Why? Justify your answer. **[4]**

(e) Let $f(x, y)$ denote an $M \times N$ image. Suppose that the pixel intensities r are represented by 8 bits. Moreover, the histogram $h(r)$ of the image is available. Find the value of $\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |F(u, v)|^2$, with $F(u, v)$ the Discrete Fourier Transform of the image. **[4]**

3. We are given the degraded version $g(x,y)$ of an image $f(x,y)$ such that in lexicographic ordering

$$g = Hf + n$$

where H is the degradation matrix which is assumed to be block-circulant and n is the noise term which is assumed to be zero-mean, white and independent of the image f. The symbols f, n, g denote the lexicographically ordered original, noise and degraded image respectively. All images involved have size $N \times N$ after extension and zero-padding.

(a) (i) Under what assumptions can we perfectly restore the image using the technique of Inverse Filtering? **[3]**

(ii) How can we avoid erratic behaviour when the assumption is not met? **[3]**

(iii) Discuss the relation between Wiener filtering and both inverse and pseudo-inverse filtering. **[4]**

(b) Propose a technique to restore an image using a spatially adaptive Constrained Least Squares (CLS) filter. A full mathematical analysis is required. **[4]**

(c) One class of filters considered for reducing background noise in images has frequency response $W(u,v)$ given by:

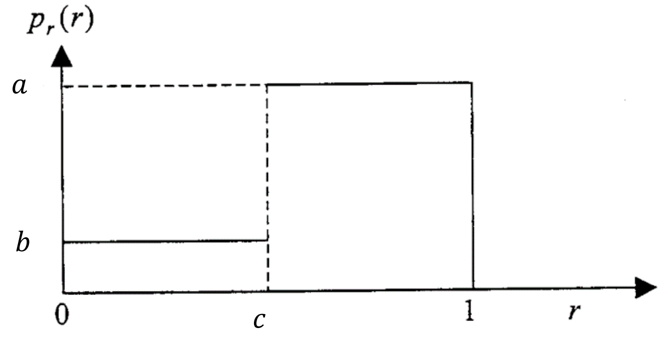
$$W(u,v) = \left[\frac{S_{ff}(u,v)}{S_{ff}(u,v) + S_{nn}(u,v)}\right]^{\beta}$$

where $S_{ff}(u,v)$ is the original image power spectrum and $S_{nn}(u,v)$ is the noise power spectrum. If $\beta = 1$, the filter is a Wiener filter. If $\beta = 1/2$, the filter is called a power spectrum filter. Suppose $S_{ff}(u,v)$ has a lowpass character and its amplitude decreases as u and v increase., while $S_{nn}(u,v)$ is approximately constant independent of u and v.

(i) For a given $S_{ff}(u,v)$ and $S_{nn}(u,v)$, which filter among the Wiener filter and the power spectrum filter performs better in term of noise reduction? Justify your answer. **[3]**

(ii) For a given $S_{ff}(u,v)$ and $S_{nn}(u,v)$, which filter among the Wiener filter and the power spectrum filter performs worse in term of image blurring? Justify your answer. **[3]**

4. (a) Consider an image with intensity $f(x, y)$ that can be modeled as a sample obtained from the probability density function sketched below:



(i) Suppose that four reconstruction levels are assigned to quantize the intensity $r = f(x, y)$. Determine these reconstruction levels using a uniform quantizer. In the above figure, $a = 1.6, b = 0.4$. **[4]**

(ii) Suppose that the four intensity levels found in (i) are to be transmitted using Huffman coding. Find the Huffman codewords. For your codeword assignment, determine the average number of bits required to represent r. **[4]**

(iii) Determine the entropy, the redundancy and the coding efficiency of the Huffman code for this example. **[4]**

(b) A set of images is to be compressed. Each pixel of each image has a value in the range $[0, 3]$, i.e., we require 2 bits/pixel. An image taken from this set is given below. The frequency of occurrence of pixels of different values in this image, is typical of the set of images as a whole.

$$\begin{bmatrix} 3 & 3 & 3 & 2 \\ 2 & 3 & 3 & 3 \\ 3 & 2 & 2 & 2 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

What is the degree of compression achievable using the following methods? **[4]**

(i) Huffman coding of the pixel values.

(ii) Forming differences between adjacent pixels (assuming zig-zag scan) and then Huffman coding these differences. For the top-left pixel use the value of 0 in this calculation. **[4]**