1. (a) The image of the square shown on the left of **Figure 1** below is represented by a function $f(x,y)$. The gray level $f(x,y)$ shows the value at a given point (x,y), with black being 1 and white being 0. Next to the square image $f(x,y)$ is a plot of the magnitude of its Fourier transform $F(u,v)$.

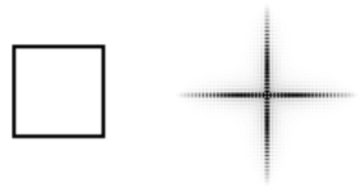


Figure 1

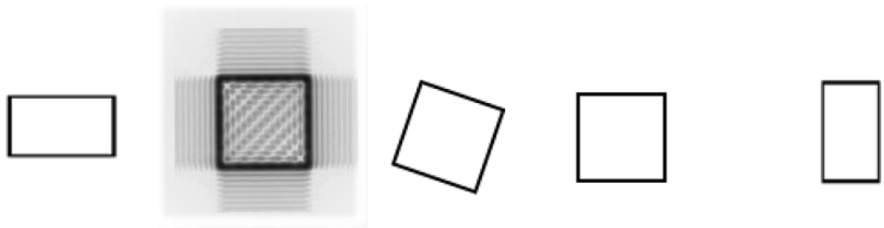
Let $a > 1$ be a fixed constant, and let A be the matrix

$$A = \begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix}$$

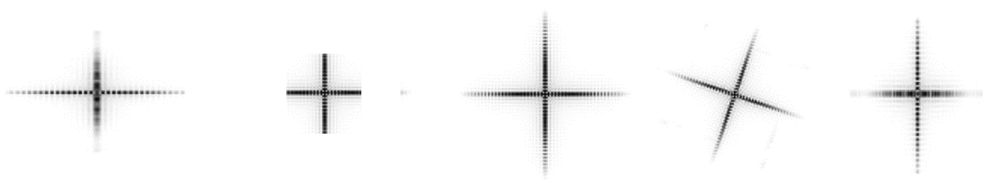
Consider the following five transformations of $f(x,y)$.
1. $f(ax,y)$
2. $f(x,ay)$
3. $f(x+a,y)$
4. $f(x_1,y_1)$ with $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$
5. $f(x,y) * [\mathrm{sinc}(ax)\cdot\mathrm{sinc}(ay)]$

The sign "$*$" indicates convolution. Match the transformation 1 – 5 of $f(x,y)$ with the corresponding plot (i) – (v) shown below and with the plot of the corresponding Fourier transform (A) – (E) shown below. Give explanations. Each correct matching counts 2 marks (10 marks in total).

(i) (ii) (iii) (iv) (v)



(A) (B) (C) (D) (E)



(b) Consider the population of vectors $\underline{f}$ of the form

$$\underline{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{bmatrix}.$$

Each component $f_i(x,y),\ i = 1,2,3$ represents an image of size $M \times M$ where M is even. The population arises from the formation of the vectors $\underline{f}$ across the entire collection of pixels (x,y). The three images are defined as follows:

$$f_1(x,y) = \begin{cases} r_1 & 1 \le x \le \frac{M}{2}, 1 \le y \le M \\ r_2 & \frac{M}{2} < x \le M, 1 \le y \le M \end{cases}$$

$$f_2(x,y) = r_3,\ 1 \le x \le M,\ 1 \le y \le M$$

$$f_3(x,y) = r_4,\ 1 \le x \le M,\ 1 \le y \le M$$

Consider now a population of random vectors of the form

$$\underline{g}(x,y) = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \\ g_3(x,y) \end{bmatrix}$$

where the vectors $\underline{g}$ are the Karhunen-Loeve (KL) transforms of the vectors $\underline{f}$.

(i) Find the images $g_1(x,y)$, $g_2(x,y)$ and $g_3(x,y)$ using the Karhunen-Loeve (KL) transform. **[7]**

(ii) Comment on whether you could obtain the result of (b)-(i) above using intuition rather than by explicit calculation.

[3]

2. (a) A two-dimensional filter $h(m,n)$ of size 3×3 is given below, where the centre position corresponds to $m = n = 0$:

$$h(m,n) = \begin{bmatrix} -1 & 2 & -1 \\ -2 & 4 & -2 \\ -1 & 2 & -1 \end{bmatrix}$$

(i) Is the filter separable? If so, give the one-dimensional filters in horizontal and vertical directions. **[4]**

(ii) Determine the two-dimensional Discrete Time Fourier Transform

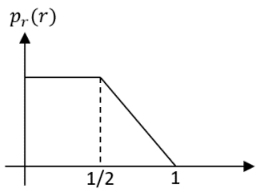
$$H(u,v) = \sum_{m=-1}^{m=1} \sum_{n=-1}^{n=1} h(m,n) e^{j\frac{(um+vn)}{3}}$$

of the filter and sketch the one-dimensional profiles $H(u,0)$, $H(u,1/2)$, $H(0,v)$, and $H(1/2,v)$. **[4]**

(iii) What is the functionality of the filter? Explain your reasoning. **[4]**

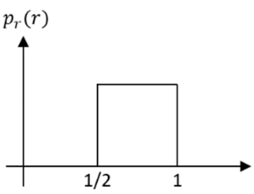
(b) Two images have grey level probability density function (or histogram normalized by the number of pixels) $p_r(r)$ illustrated in **Figure 2** below. For each image, find the transformation function that will help to equalize the histogram.

(i)



[6]

(ii)



[2]

Figure 2

3. We are given the degraded version g of an image f such that in lexicographic ordering

$$g = Hf + n$$

where H is the degradation matrix which is assumed to be block-circulant and n is the noise term which is assumed to be zero-mean, white and independent of the image f. All images involved have size $N \times N$ after extension and zero-padding.

(a) (i) Describe how inverse filtering can be used to restore the degraded image above. **[3]**

(i) Given knowledge of the exact degradation function, under what assumption can we perfectly restore the image? (2 points in total) How can we avoid erratic behaviour when the assumption is not met? (2 points in total) **[4]**

(ii) In a particular scenario the degradation process can be modelled as a linear filter with the two-dimensional impulse response given below:

$$h(x,y) = \begin{cases} 0.25 & x \pm 1, y = 0 \\ 0.5 & x = 0, y = 0 \\ 0 & \text{elsewhere} \end{cases}$$

Estimate the frequency pairs for which Inverse Filtering cannot be applied. **[4]**

(b) (i) Give the equation of Wiener filter in frequency domain. It is often referred to as the least square error filter. Explain the statement. **[3]**

(ii) Define the terms in the equation. **[3]**

(iii) Explain under which conditions the Wiener filter is equal to the inverse filtering procedure. **[3]**

4. (a) (i) Draw the Huffman tree corresponding to the encoding table in **Figure 4.1** below. The first column indicates the list of symbols, the second column their frequency and the third column their Huffman codeword. **[5]**

symbol	frequency	encoding
B	2	01111
F	1	01110
H	3	0110
I	?	00
L	5	010
M	15	10
S	15	11

Figure 4.1

(ii) State the range of frequency values that are possible for character I.
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 **[5]**

(b) A set of images is to be compressed by a lossless method. Each pixel of each image has a value in the range (0-3), i.e., 2 bits/pixel. An image from this set is given in **Figure 4.2** below; in this image the occurrence of pixels of different values is typical of the set of images as a whole.

3	3	3	2
2	3	3	3
3	2	2	2
2	1	1	0

Figure 4.2

What is the degree of compression achievable using the following methods?

(i) Using Huffman coding of the pixel values. **[5]**

(ii) Using differences between pixels in the following fashion: a pixel value is replaced by the difference between itself and the value of the pixel in the same row and the adjacent to the left column (horizontal scan). The top left pixel is ignored. Then, applying Huffman code to these differences. **[5]**