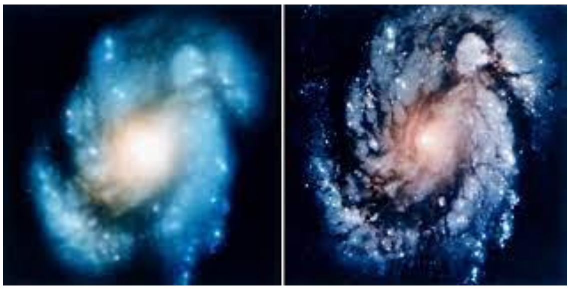
Image Restoration Sample Exam Problems

Imperial College UG4, MSc COMSP, MSc AML



1. We are given the noisy version $g(x,y)$ of an image $f(x,y)$. We wish to denoise $g(x,y)$ using a spatially adaptive image denoising mask with coefficients $h(m,n)$. The mask is estimated at each pixel (x,y) based on the local signal variance $\sigma_g^2(x,y)$. The local signal variance is estimated from the degraded image $g(x,y)$ using a local neighborhood around each pixel. The mask coefficients are obtained using the following equation:

$$h_{(x,y)}(m,n) = k_1(x,y)e^{-k_2(x,y)(m^2+n^2)}w(m,n)$$

where $w(m,n)$ is a 5×5-point rectangular window placed symmetrically around the origin, i.e.,

$$w(m,n) = \begin{cases} 1, & -2 \le m \le 2, -2 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}$$

The parameters $k_1(x,y)$ and $k_2(x,y)$ are constants, but they depend on the pixel location. We require that

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} h_{(x,y)}(m,n) = 1$$

(i) Give the form of $h_{(x,y)}(m,n)$ for a very small $k_2(x,y)$ (close to 0).
(ii) Give the form of $h_{(x,y)}(m,n)$ for a very large $k_2(x,y)$ (close to ∞).
(iii) Explain why random noise is typically less visible to human viewers in image regions of high detail, such as edge regions, than in image regions of low detail, such as uniform background regions.
(iv) Give one reasonable choice of $k_2(x,y)$ as a function of $\sigma_g^2(x,y)$. Justify your answer. For your choice of $k_2(x,y)$ determine $k_1(x,y)$.
(v) The image restoration system discussed here can exploit the observation stated in (iii). The system, however, cannot exploit the observation that random noise is typically less visible to human viewers in bright areas than in dark areas. How would you modify the image restoration system so that this additional piece of information can be exploited?

Solution

(i)

For a very small $k_2(x,y)$ the factor $e^{-k_2(x,y)(m^2+n^2)}$ tends to 1 and therefore, we can write $h_{(x,y)}(m,n) = k_1(x,y)w(m,n)$.

(ii)

For a very large $k_2(x,y)$ we can write

$$e^{-k_2(x,y)(m^2+n^2)} = \begin{cases} 1 & m = n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, we can write

$$h_{(x,y)}(m,n) = k_1(x,y)\delta(m,n)$$

(iii)

Noise is more visible within background areas (areas with slowly varying intensity) because the **local** Signal-to-Noise Ratio within these areas is lower. In image regions of high detail, such as edge regions, we don't want to do any lowpass filtering or we want to do mild filtering. This is because we don't want to ruin the sharp edges and fine details of the image. We can afford to do this, since noise is less visible withing regions of high detail.

(iv)

Based on the comment in (iii) above, we want large $k_2(x,y)$ in regions of high detail and small $k_2(x,y)$ in regions of low detail. Hence, we can choose $k_2(x,y) = \sigma_g^2(x,y)$

We wish the sum of the filter coefficients to be 1, so that we don't distort the mean intensity of the image, and therefore,

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} h_{(x,y)}(m,n) = 1 \Rightarrow \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} k_1(x,y)e^{-k_2(x,y)(m^2+n^2)}w(m,n) = 1$$

$$k_1(x,y) = \frac{1}{\sum_{m=-2}^{2}\sum_{n=-2}^{2} e^{-\sigma_g^2(x,y)(m^2+n^2)}}$$

(v)

The image restoration system discussed here can exploit the observation stated in (iii). The system, however, cannot exploit the observation that random noise is typically less visible to human viewers in bright areas than in dark areas. How would you modify the image restoration system so that this additional piece of information can be exploited?

A possible scenario is to implement weaker filtering in both high detail and bright areas. Bright areas can be identified from the local mean $m_g(x,y)$. More specifically, a large $m_g(x,y)$ indicates a bright area. Therefore, we can normalize $\sigma_g^2(x,y)$ and $m_g(x,y)$ so that they occupy the same range of values (for example from 0 to 1) to obtain $\tilde{\sigma}_g^2(x,y)$ and $\widetilde{m}_g(x,y)$ and then we can consider $k_2(x,y) = \tilde{\sigma}_g^2(x,y) + \widetilde{m}_g(x,y)$.

2. We are given the degraded version g of an image f such that in lexicographic ordering

$$g = Hf + n$$

where H is the degradation matrix which is assumed to be block-circulant and n is the noise term which is assumed to be zero-mean, white and independent of the image f. All images involved have size $N \times N$ after extension and zero-padding.

(i) Consider the Inverse Filtering image restoration technique. Give the general expressions for both the Inverse Filtering estimator and the restored image in both spatial and frequency domains and explain all symbols used.

(ii) In a particular scenario the degradation process can be modelled as a linear filter with the two-dimensional impulse response given below:

$$h(x,y) = \begin{cases} 1 & -1 \le x \le 1, y = 0 \\ 0 & \text{elsewhere} \end{cases}$$

Estimate the frequency pairs for which Inverse Filtering cannot be applied.

Solution

(i)

Bookwork.

(ii)

Inverse Filtering cannot be applied if $H(u,v) = 0$.
To be able to use the Discrete Fourier Transform, $h(x,y)$ is assumed to be periodic with period N in both dimensions. Therefore, $h(-1,0) = h(N-1,0) = 1$.

$$H(u,v) = \frac{1}{NN}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} h(x,y)e^{-j2\pi\left(\frac{ux}{N}+\frac{vy}{N}\right)} = \frac{1}{NN}\left(e^{\frac{j2\pi u(N-1)}{N}} + e^{\frac{j2\pi u}{N}} + 1\right)$$
$$= \frac{1}{NN}(e^{-j2\pi u/N} + e^{j2\pi u/N} + 1) = \frac{1}{NN}[2\cos(2\pi u/N) + 1]$$
$$H(u,v) = 0 \Rightarrow 2\cos(2\pi u/N) + 1 = 0 \Rightarrow \cos(2\pi u/N) = -1/2 \Rightarrow 2\pi u/N = 2k\pi \pm 2\pi/3$$
$$\Rightarrow u = \frac{N(2k\pi \pm 2\pi/3)}{2\pi} = N(k \pm 1/3), k = 0,1,2,\ldots$$
Therefore, the valid values are $u = N/3$ and $u = 2N/3$.

3. (i) Consider the Constrained Least Squares (CLS) filtering image restoration technique. Give the general expressions for both the CLS filter estimator and the restored image in both spatial and frequency domains and explain all symbols used.
   (ii) In a particular scenario, the degradation process can be modelled as a linear filter with the transfer function given below:
$$H(u,v) = \sqrt{2\pi}\sigma(u^2 + v^2)e^{-j2\pi^2\sigma^2(u^2+v^2)}$$
   In the above formulation σ is a constant parameter. Generate the expression of the CLS filter in frequency domain by assuming that the high pass filter used in CLS is a Laplacian filter.

Solution

(i)

Bookwork

(ii)

$$H^*(u,v) = \sqrt{2\pi}\sigma(u^2 + v^2)e^{j2\pi^2\sigma^2(u^2+v^2)}$$
$$|H(u,v)|^2 = H(u,v)H^*(u,v) = 2\pi\sigma^2(u^2 + v^2)^2$$

The CLS filter in frequency domain is $\frac{\sqrt{2\pi}\sigma(u^2+v^2)e^{j2\pi^2\sigma^2(u^2+v^2)}}{2\pi\sigma^2(u^2+v^2)^2+\alpha|C(u,v)|^2}$ where $C(u,v)$ is the Laplacian filter in frequency domain.

4. Let $f(x,y)$ be an image corrupted by noise $n(x,y)$, with its noisy version expressed as
$$g(x,y) = f(x,y) + n(x,y)$$
In Wiener filtering we assume that the Discrete Space Fourier Transform $S_{ff}(u,v)$ of the autocorrelation function of the original image is available.
One method of estimating $S_{ff}(u,v)$ is to model the autocorrelation function as follows

$$R_{ff}(k,l) = \underset{(x,y)}{E}\{f(x,y)f(x+k,y+l)\} = \rho^{|k|+|l|}$$

with ρ an unknown parameter with $0 < \rho < 1$.

(i) Assuming that $n(x,y)$ is a zero mean, white noise with unknown variance σ_n^2 and independent of $f(x,y)$, write down without proof the expressions for the Wiener filter estimator and the restored image in the frequency domain as functions of ρ and σ_n^2.

(ii) Develop a method of estimating ρ using the autocorrelation function samples of $g(x,y)$.

Hints:

1. The Discrete Space Fourier Transform $S_{ff}(u,v)$ of the autocorrelation function is defined as $S_{ff}(u,v) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} R_{ff}(k,l)e^{-j(uk+vl)}$.
2. The following result holds: $\sum_{k=0}^{\infty} a^x = \frac{1}{1-a}, |a| < 1$.

Solution

(i)

The noise is white with variance σ_n^2 and therefore, the autocorrelation function of the noise is

$$R_{nn}(k,l) = \underset{(x,y)}{E}\{n(x,y)n(x+k,y+l)\} = \sigma_n^2\delta(k,l)$$

and the power spectrum, $S_{nn}(u,v)$, of the noise is σ_n^2.

$$S_{ff}(u,v) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} R_{ff}(k,l)e^{-j(uk+vl)} = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \rho^{|k|+|l|}\, e^{-j(uk+vl)}$$

$$= \sum_{k=-\infty}^{\infty} \rho^{|k|}\, e^{-juk} \sum_{l=-\infty}^{\infty} \rho^{|l|}\, e^{-jvl}$$

We find that

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{-j\omega n} = \sum_{n=-\infty}^{-1} r^{-n}e^{-j\omega n} + \sum_{n=0}^{\infty} r^{n}e^{-j\omega n} = \sum_{n=1}^{\infty} r^{n}e^{j\omega n} + \sum_{n=0}^{\infty} r^{n}e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} r^{n}e^{j\omega n} + \sum_{n=0}^{\infty} r^{n}e^{-j\omega n} - 1$$

$$= \sum_{n=0}^{\infty}(re^{j\omega})^n + \sum_{n=0}^{\infty}(re^{-j\omega})^n - 1 = \frac{1}{1-re^{j\omega}} + \frac{1}{1-re^{-j\omega}} - 1 = \frac{2-2r\cos\omega}{1+r^2-2r\cos\omega} - 1$$

$$= \frac{1-r^2}{1+r^2-2r\cos\omega}$$

Therefore,

$$S_{ff}(u,v) = \frac{(1-\rho^2)^2}{(1+\rho^2-2\rho\cos u)(1+\rho^2-2\rho\cos v)}$$

and

$$W(u,v) = \frac{S_{ff}(u,v)}{S_{ff}(u,v)+S_{nn}(u,v)} = \frac{(1-\rho^2)^2}{(1-\rho^2)^2 + (1+\rho^2-2\rho\cos u)(1+\rho^2-2\rho\cos v)\sigma_n^2}$$

(ii)

We know that, since the original image and the noise are uncorrelated, we can write:

$$R_{gg}(k,l) = R_{ff}(k,l) + R_{nn}(k,l) = \rho^{|k|+|l|} + \sigma_n^2\delta(k,l)$$

If we choose a pair of values for the parameters k, l as $(k, l) = (1,0)$, the following relationship holds:
$R_{gg}(1,0) = \rho^{|1|+|0|} + \sigma_n^2\delta(1,0) = \rho$, where $R_{gg}(1,0)$ is estimated from the available data.

5. We are given the degraded version g of an image f such that in lexicographic ordering

$$g = Hf + n$$

where H is the degradation matrix which is assumed to be block-circulant, and n is the noise term which is assumed to be zero mean, independent and white. All images involved have size $N \times N$ after extension and zero-padding. In a particular scenario, the image under consideration is blurred to relative motion between the image and the camera. The pixel of the image g at location (x, y) is related to the corresponding pixel of the image f through the following relationship:

$$g(x,y) = f(x,y) + 2f(x,y-1) + f(x,y-2) + n(x,y)$$

(i) Consider the Inverse Filtering image restoration technique. Find the expressions for both the inverse filter estimator and the restored image in the frequency domain. Find the specific frequencies for which the restored image cannot be estimated.

(ii) Consider the Constrained Least Squares (CLS) image restoration technique. The two-dimensional high pass filtering operator used in the regularization term is given by the function below:

$$c(x,y) = \begin{cases} 1 & x = 0, y = 0 \\ -2 & x = 0, y = 1 \\ 1 & x = 0, y = 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the expressions for both the CLS filter estimator and the restored image in the frequency domain. Find whether there are any specific frequencies for which the restored image cannot be estimated.

Solution

(i)

$$h(x,y) = \begin{cases} 1 & x = 0, y = 0 \\ 2 & x = 0, y = 1 \\ 1 & x = 0, y = 2 \end{cases}$$

$$H(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} h(x,y)e^{-j\frac{2\pi}{N}(ux+vy)} = \frac{1}{N^2}(e^{-j\frac{2\pi}{N}v0} + 2e^{-j\frac{2\pi}{N}v} + e^{-j\frac{2\pi}{N}2v})$$
$$= \frac{1}{N^2}(1 + 2e^{-j\frac{2\pi}{N}v} + e^{-j\frac{2\pi}{N}2v}) = \frac{1}{N^2}e^{-j\frac{2\pi}{N}v}(e^{j\frac{2\pi}{N}v} + 2 + e^{-j\frac{2\pi}{N}v})$$
$$= \frac{2}{N^2}e^{-j\frac{2\pi}{N}v}[\cos(\frac{2\pi v}{N}) + 1]$$

The expressions for Inverse Filtering are book work.
Inverse Filtering cannot be implemented when $H(u,v) = 0$. To find the frequency points for which $H(u,v) = 0$, we set
$\cos(\frac{2\pi v}{N}) = -1 \Rightarrow \frac{2\pi v}{N} = k\pi, k$ odd.
Therefore, for $k = 1 \Rightarrow v = \frac{N}{2}$.
For $k = 3 \Rightarrow v = \frac{3N}{2}$. This value is invalid since $v \in [0, N-1]$.

(ii)

$$C(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} c(x,y)e^{-j\frac{2\pi}{N}(ux+vy)} = \frac{1}{N^2}(e^{-j\frac{2\pi}{N}v0} - 2e^{-j\frac{2\pi}{N}v} + e^{-j\frac{2\pi}{N}2v})$$

$$= \frac{1}{N^2}(1 - 2e^{-j\frac{2\pi}{N}v} + e^{-j\frac{2\pi}{N}2v}) = \frac{1}{N^2}e^{-j\frac{2\pi}{N}v}(e^{j\frac{2\pi}{N}v} - 2 + e^{-j\frac{2\pi}{N}v})$$
$$= \frac{2}{N^2}e^{-j\frac{2\pi}{N}v}[\cos(\frac{2\pi v}{N}) - 1]$$
The expression for the CLS estimator is book work.
In order for CLS to not be realisable, both $H(u,v)$ and $C(u,v)$ have to be 0.
$H(u,v) = 0$ for $v = \frac{N}{2}$.
For $C(u,v)$ we have:
$C(u,v) = 0 \Rightarrow \cos(\frac{2\pi v}{N}) = 1 \Rightarrow \frac{2\pi v}{N} = 2k\pi, k$ any integer.
Therefore, for $k = 0 \Rightarrow v = 0$
For $k = 1 \Rightarrow v = N$. This value is invalid since $v \in [0, N-1]$.
By comparing the frequency points for which both $H(u,v)$ and $C(u,v)$ are zero, we see that there are not any frequency points for which both functions are zero and therefore, the CLS estimator can be obtained for all frequencies.