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Planned Syllabus

Week	Topic	HW
1 (9/3)	Introduction, Tensor algebra & analysis	
2 (9/10)	Description of deformation	
3 (9/17)	Description of deformation / **BME Sport Day (educational break)**	
4 (9/24)	**Canceled**	
5 (10/1)	Description of deformation	
6 (10/8)	Stress measures	
7 (10/15)	Hyperelasticity	**HW 1**
8 (10/22)	**1st Mid-term exam/test**	
9 (10/29)	Hyperelasticity	
10 (11/5)	Hyperelasticity	
11 (11/12)	Velocity, acceleration, time derivatives	
12 (11/19)	Objectivity	
13 (11/26)	Fundamental principles	
14 (12/3)	**2nd Mid-term exam/test**	**HW 2**
Ratake week	**Retake exams/tests 1 & 2**	

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4 Hyperelasticity

4.1 General form for the stress

- Small-strain (linearized deformation) formulation:

$$\boldsymbol{\sigma} = \frac{\partial u}{\partial \boldsymbol{\varepsilon}} \qquad \boldsymbol{stress} = \frac{\partial(\text{strain energy})}{\partial(\boldsymbol{strain})}$$

$$u(strain\ measure, mat.parameters) = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{D}^e : \boldsymbol{\varepsilon}$$

- Finite strain formulation:

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{E}}$$

strain energy per unit reference volume

2nd Piola-Kirchhoff stress

Green-Lagrange strain

Main question: How to construct W ?

$$W = W(strain\ measure, mat.parameters)$$

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$$W = W(strain\ measure, mat.\ parameters)$$

Strain/deformation tensors:

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$$
$$\boldsymbol{B} = \boldsymbol{F}^{-1}\boldsymbol{F}^{-T}$$
$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$$
$$\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$$
$$\boldsymbol{c} = \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}$$
$$\boldsymbol{e} = \frac{1}{2}(\boldsymbol{i} - \boldsymbol{c})$$

Stress tensors:

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{P}\boldsymbol{F}^T = \frac{1}{J}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T = \frac{1}{J}\boldsymbol{\tau}$$
$$\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T} = \boldsymbol{F}\boldsymbol{S} = \boldsymbol{\tau}\boldsymbol{F}^{-T}$$
$$\boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T} = \boldsymbol{F}^{-1}\boldsymbol{P} = \boldsymbol{F}^{-1}\boldsymbol{\tau}\boldsymbol{F}^{-T}$$
$$\boldsymbol{\tau} = J\boldsymbol{\sigma} = \boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T = \boldsymbol{\tau}$$

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Relation between $\boldsymbol{E}$ and $\boldsymbol{C}$ implies:

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$$

$$\boldsymbol{C} = 2\boldsymbol{E} + \boldsymbol{I}$$

$$\boxed{\boldsymbol{S} = 2\frac{\partial W}{\partial \boldsymbol{C}}}$$

Using the transformation rules between the stress measures, we can obtain other forms for the stress tensors:

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P} \quad \Rightarrow \quad \boxed{\boldsymbol{P} = \boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{E}}} \qquad \boxed{\boldsymbol{P} = 2\boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{C}}}$$

$$\boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T} \quad \Rightarrow \quad \boxed{\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{E}}\boldsymbol{F}^T} \qquad \boxed{\boldsymbol{\sigma} = \frac{2}{J}\boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{C}}\boldsymbol{F}^T}$$

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} \quad \Rightarrow \quad \boxed{\boldsymbol{\tau} = \boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{E}}\boldsymbol{F}^T} \qquad \boxed{\boldsymbol{\tau} = \boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{C}}\boldsymbol{F}^T}$$

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If W is expressed with $\boldsymbol{b}$, then:

$$\boxed{\boldsymbol{\sigma} = \frac{2}{J}\frac{\partial W}{\partial \boldsymbol{b}}\boldsymbol{b}}$$

Furthermore, it can be shown, that

$$\boxed{\boldsymbol{P} = \frac{\partial W}{\partial \boldsymbol{F}}}$$

Conclusion: Once W is expressed as the function of $\boldsymbol{F}$ or $\boldsymbol{C}$ or $\boldsymbol{E}$ or $\boldsymbol{b}$ or **any deformation/strain tensor**, then all the stress tensors can be derived using transformation rules. But how should we define W?

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{E}} = 2\frac{\partial W}{\partial \boldsymbol{C}}$$

$$\boldsymbol{P} = \frac{\partial W}{\partial \boldsymbol{F}} = \boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{E}} = 2\boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{C}}$$

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{E}}\boldsymbol{F}^T = \frac{2}{J}\boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{C}}\boldsymbol{F}^T = \frac{2}{J}\boldsymbol{b}\frac{\partial W}{\partial \boldsymbol{b}} = \frac{2}{J}\frac{\partial W}{\partial \boldsymbol{b}}\boldsymbol{b}$$

$$\boldsymbol{\tau} = \boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{E}}\boldsymbol{F}^T = 2\boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{C}}\boldsymbol{F}^T = 2\boldsymbol{b}\frac{\partial W}{\partial \boldsymbol{b}} = 2\frac{\partial W}{\partial \boldsymbol{b}}\boldsymbol{b}$$

There is no “best” representation. They are equivalent but using different quantities in the representation!

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Examples for stress formulations:

Linearized theory (small-strains): $$\boxed{\boldsymbol{\sigma} = \frac{\partial u}{\partial \boldsymbol{\varepsilon}}}$$

Lagrangian form: $$\boxed{\boldsymbol{S} = 2\frac{\partial W}{\partial \boldsymbol{C}}}$$

Eulerian form: $$\boxed{\boldsymbol{\sigma} = \frac{2}{J}\frac{\partial W}{\partial \boldsymbol{b}}\boldsymbol{b}}$$

Mixed form: $$\boxed{\boldsymbol{P} = \frac{\partial W}{\partial \boldsymbol{F}}}$$

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4.2 Isotropic case

In case of isotropic behavior, W can be written using the principal invariants of the particular deformation and strain tensors:

$$W = W(I_1, I_2, I_3)$$

Denote the principal scalar invariants of $\boldsymbol{C}$ and $\boldsymbol{b}$:

$$I_1 = \mathrm{tr}\boldsymbol{C} = \mathrm{tr}\boldsymbol{b}$$

$$I_2 = \frac{1}{2}\left(I_1^2 - \mathrm{tr}(\boldsymbol{C}^2)\right) = \frac{1}{2}\left(I_1^2 - \mathrm{tr}(\boldsymbol{b}^2)\right)$$

$$I_3 = \det\boldsymbol{C} = \det\boldsymbol{b}$$

Then, using the chain rule we find:

$$\boldsymbol{S} = 2\frac{\partial W}{\partial \boldsymbol{C}} = 2\left(\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{C}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \boldsymbol{C}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \boldsymbol{C}}\right) = 2\sum_{i=1}^{3}\frac{\partial W}{\partial I_i}\frac{\partial I_i}{\partial \boldsymbol{C}}$$

Note, that

$$\frac{\partial I_1}{\partial \boldsymbol{C}} = \boldsymbol{I} \qquad \frac{\partial I_2}{\partial \boldsymbol{C}} = I_1\boldsymbol{I} - \boldsymbol{C} \qquad \frac{\partial I_3}{\partial \boldsymbol{C}} = I_3\boldsymbol{C}^{-1}$$

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To simplify the expressions, we introduce the following notations:

$$\frac{\partial W}{\partial I_1} = W_{,1} \qquad \frac{\partial W}{\partial I_2} = W_{,2} \qquad \frac{\partial W}{\partial I_3} = W_{,3}$$

Thus:

$$\boldsymbol{S} = 2\left[W_{,1}\frac{\partial I_1}{\partial \boldsymbol{C}} + W_{,2}\frac{\partial I_2}{\partial \boldsymbol{C}} + W_{,3}\frac{\partial I_3}{\partial \boldsymbol{C}}\right]$$

$$\boldsymbol{S} = 2\left[W_{,1}\boldsymbol{I} + W_{,2}(I_1\boldsymbol{I} - \boldsymbol{C}) + W_{,3}I_3\boldsymbol{C}^{-1}\right]$$

$$\boldsymbol{S} = 2\left[(W_{,1} + I_1W_{,2})\boldsymbol{I} - W_{,2}\boldsymbol{C} + I_3W_{,3}\boldsymbol{C}^{-1}\right]$$

Cauchy stress tensor:

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{FSF}^T = \frac{2}{J}\left[(W_{,1} + I_1W_{,2})\overbrace{\boldsymbol{FF}^T}^{\boldsymbol{b}} - W_{,2}\overbrace{\boldsymbol{FCF}^T}^{\boldsymbol{b}^2} + I_3W_{,3}\overbrace{\boldsymbol{FC}^{-1}\boldsymbol{F}^T}^{\boldsymbol{I}}\right]$$

$$\boldsymbol{FCF}^T = \boldsymbol{FF}^T\boldsymbol{FF}^T = \boldsymbol{b}^2 \qquad \boldsymbol{FC}^{-1}\boldsymbol{F}^T = \boldsymbol{FF}^{-1}\boldsymbol{F}^{-T}\boldsymbol{F}^T = \boldsymbol{I}$$

$$\boldsymbol{\sigma} = \frac{2}{J}\left[(W_{,1} + I_1W_{,2})\boldsymbol{b} - W_{,2}\boldsymbol{b}^2 + I_3W_{,3}\boldsymbol{I}\right]$$

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Therefore, the 2nd P-K stress tensor can be written as

$$\boxed{\mathbf{S} = 2\left[\left(W_{,1} + I_1 W_{,2}\right)\mathbf{I} - W_{,2}\mathbf{C} + I_3 W_{,3}\mathbf{C}^{-1}\right]}$$

Lagrangian form

$$W_{,1} = \frac{\partial W}{\partial I_1}$$

$$W_{,2} = \frac{\partial W}{\partial I_2}$$

$$W_{,3} = \frac{\partial W}{\partial I_3}$$

$$I_1 = \mathrm{tr}\mathbf{C} = \mathrm{tr}\mathbf{b}$$

$$I_2 = \frac{1}{2}\left(I_1^2 - \mathrm{tr}(\mathbf{C}^2)\right) = \frac{1}{2}\left(I_1^2 - \mathrm{tr}(\mathbf{b}^2)\right)$$

$$I_3 = \det\mathbf{C} = \det\mathbf{b}$$

Cauchy stress can be expressed as:

$$\boxed{\boldsymbol{\sigma} = \frac{2}{J}\left[\left(W_{,1} + I_1 W_{,2}\right)\mathbf{b} - W_{,2}\mathbf{b}^2 + I_3 W_{,3}\mathbf{I}\right]}$$

Eulerian form

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• Constitutive relation using the principal stretches

In this case, W is formulated as

$$W = W(\lambda_1, \lambda_3, \lambda_3)$$

The principal invariants I_1, I_3, I_3 can be expressed using the eigenvalues μ_1, μ_3, μ_3 of $\boldsymbol{C}$ (or $\boldsymbol{b}$):

$$I_1 = \mu_1 + \mu_2 + \mu_3$$

$$I_2 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3$$

$$I_3 = \mu_1\mu_2\mu_3$$

Furthermore, we can use the identity $\mu_i = \lambda_i^2$, where λ_i with $i = 1,2,3$ are the principal stretches (eigenvalues of $\boldsymbol{U}$ and $\boldsymbol{V}$).

$$I_1 = \mu_1 + \mu_2 + \mu_3 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$I_2 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 = (\lambda_1\lambda_2)^2 + (\lambda_1\lambda_3)^2 + (\lambda_2\lambda_3)^2$$

$$I_3 = \mu_1\mu_2\mu_3 = (\lambda_1\lambda_2\lambda_3)^2 = J^2$$

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Using the chain rule, we find

$$\boldsymbol{S} = 2\frac{\partial W}{\partial \boldsymbol{C}} = 2\left(\frac{\partial W}{\partial \mu_1}\frac{\partial \mu_1}{\partial \boldsymbol{C}} + \frac{\partial W}{\partial \mu_2}\frac{\partial \mu_2}{\partial \boldsymbol{C}} + \frac{\partial W}{\partial \mu_3}\frac{\partial \mu_3}{\partial \boldsymbol{C}}\right) = 2\sum_{i=1}^{3}\frac{\partial W}{\partial \mu_i}\frac{\partial \mu_i}{\partial \boldsymbol{C}}$$

Note that

$$\frac{\partial W}{\partial \lambda_i} = \frac{\partial W}{\partial \mu_i}\frac{\partial \mu_i}{\partial \lambda_i} = \frac{\partial W}{\partial \mu_i}(2\lambda_i) \quad \Longrightarrow \quad \frac{\partial W}{\partial \mu_i} = \frac{1}{2\lambda_i}\frac{\partial W}{\partial \lambda_i}$$

$$\frac{\partial \mu_i}{\partial \boldsymbol{C}} = \boldsymbol{N}_i \otimes \boldsymbol{N}_i$$

$\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3$ are the normalized eigenvectors of $\boldsymbol{C}$

Substituting back, we get

$$\boldsymbol{S} = 2\sum_{i=1}^{3}\frac{\partial W}{\partial \mu_i}\frac{\partial \mu_i}{\partial \boldsymbol{C}} = 2\sum_{i=1}^{3}\frac{1}{2\lambda_i}\frac{\partial W}{\partial \lambda_i}\boldsymbol{N}_i \otimes \boldsymbol{N}_i$$

$$\boxed{\boldsymbol{S} = \sum_{i=1}^{3}\frac{1}{\lambda_i}\frac{\partial W}{\partial \lambda_i}\boldsymbol{N}_i \otimes \boldsymbol{N}_i}$$

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The normalized eigenvectors of $\boldsymbol{C}$ and $\boldsymbol{b}$ are related through the expression

$$\boldsymbol{n}_i = \frac{1}{\lambda_i}(\boldsymbol{F}\boldsymbol{N}_i) \qquad \boldsymbol{N}_i = \lambda_i(\boldsymbol{F}^{-1}\boldsymbol{n}_i)$$

$$\boldsymbol{N}_i \otimes \boldsymbol{N}_i = \lambda_i(\boldsymbol{F}^{-1}\boldsymbol{n}_i) \otimes \lambda_i(\boldsymbol{F}^{-1}\boldsymbol{n}_i) = \lambda_i^2(\boldsymbol{F}^{-1}\boldsymbol{n}_i) \otimes \lambda_i(\boldsymbol{n}_i\boldsymbol{F}^{-T})$$

$$\boldsymbol{N}_i \otimes \boldsymbol{N}_i = \lambda_i^2\boldsymbol{F}^{-1}(\boldsymbol{n}_i \otimes \boldsymbol{n}_i)\boldsymbol{F}^{-T}$$

Thus, the stress expression can be formulated

$$\boldsymbol{S} = \sum_{i=1}^{3}\frac{1}{\lambda_i}\frac{\partial W}{\partial \lambda_i}\boldsymbol{N}_i \otimes \boldsymbol{N}_i = \sum_{i=1}^{3}\lambda_i\frac{\partial W}{\partial \lambda_i}\boldsymbol{F}^{-1}(\boldsymbol{n}_i \otimes \boldsymbol{n}_i)\boldsymbol{F}^{-T}$$

Comparing the result with the transformation formula $\boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}$, we find that

$$\boxed{\boldsymbol{\sigma} = \sum_{i=1}^{3}\frac{1}{J}\lambda_i\frac{\partial W}{\partial \lambda_i}\boldsymbol{n}_i \otimes \boldsymbol{n}_i}$$

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The spectral decomposition of the symmetric stress tensors $\boldsymbol{S}$ and $\boldsymbol{\sigma}$:

$$\boldsymbol{S} = \sum_{i=1}^{3} S_i \cdot \boldsymbol{N}_i \otimes \boldsymbol{N}_i \qquad \boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \cdot \boldsymbol{n}_i \otimes \boldsymbol{n}_i$$

Therefore, the principal stresses can be expressed as

$$\boxed{S_i = \frac{1}{\lambda_i} \frac{\partial W}{\partial \lambda_i}}$$ Principal 2nd Piola-Kirchhoff stresses

$$\boxed{\sigma_i = \frac{1}{J} \lambda_i \frac{\partial W}{\partial \lambda_i}}$$ Principal Cauchy stresses

$$\boxed{P_i = \frac{\partial W}{\partial \lambda_i}}$$ Principal 1st Piola-Kirchhoff stresses

$$\boxed{\tau_i = \lambda_i \frac{\partial W}{\partial \lambda_i}}$$ Principal Kirchhoff stresses

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Summary:

$$\boldsymbol{S} = 2\left[\left(W_{,1} + I_1 W_{,2}\right)\boldsymbol{I} - W_{,2}\boldsymbol{C} + I_3 W_{,3}\boldsymbol{C}^{-1}\right] = \sum_{i=1}^{3} \frac{1}{\lambda_i} \cdot \frac{\partial W}{\partial \lambda_i} \boldsymbol{N}_i \otimes \boldsymbol{N}_i$$

$$\boldsymbol{\sigma} = \frac{2}{J}\left[\left(W_{,1} + I_1 W_{,2}\right)\boldsymbol{b} - W_{,2}\boldsymbol{b}^2 + I_3 W_{,3}\boldsymbol{I}\right] = \frac{1}{J}\sum_{i=1}^{3} \lambda_i \cdot \frac{\partial W}{\partial \lambda_i} \boldsymbol{n}_i \otimes \boldsymbol{n}_i$$

$$\boldsymbol{S} = \sum_{i=1}^{3} S_i \cdot \boldsymbol{N}_i \otimes \boldsymbol{N}_i \qquad S_i = \frac{1}{\lambda_i} \cdot \frac{\partial W}{\partial \lambda_i}$$

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \cdot \boldsymbol{n}_i \otimes \boldsymbol{n}_i \qquad \sigma_i = \frac{1}{J}\lambda_i \cdot \frac{\partial W}{\partial \lambda_i}$$

Principal stresses

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4.3 Incompressible isotropic case

- Incompressible Hooke's law:

$$\boldsymbol{\sigma} = \boldsymbol{s} + \boldsymbol{p} = 2G\boldsymbol{e} + 3K\boldsymbol{\varepsilon}_V$$

bulk modulus: $K = \dfrac{E}{3(1-2\nu)} = \infty \rightarrow \nu = 0.5$

no volume change: $\boldsymbol{\varepsilon}_V = \mathbf{0}$

$$\boldsymbol{e} = \boldsymbol{\varepsilon} - \frac{1}{3}\mathrm{tr}[\boldsymbol{\varepsilon}]\boldsymbol{I} = \boldsymbol{\varepsilon}$$

$$\boldsymbol{\sigma} = 2G\boldsymbol{e} + \boldsymbol{p}$$

Cannot be determined from the constitutive equation

$$\boldsymbol{p} \neq \mathbf{0}$$

$$\boldsymbol{\sigma} = 2G\boldsymbol{\varepsilon} + \boldsymbol{p}$$

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Then, in this case:

$$\boldsymbol{\sigma} = \operatorname{dev}[\boldsymbol{\sigma}] + \operatorname{sph}[\boldsymbol{\sigma}] = \boldsymbol{s} + \boldsymbol{p}$$

$$\boldsymbol{\varepsilon} = \operatorname{dev}[\boldsymbol{\varepsilon}] + \operatorname{sph}[\boldsymbol{\varepsilon}] = \boldsymbol{e} + \boldsymbol{\varepsilon}_V$$

$$u = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{s} : \boldsymbol{e} + \frac{1}{2}\boldsymbol{p} : \cancel{\boldsymbol{\varepsilon}_V} = \frac{1}{2}\boldsymbol{s} : \boldsymbol{e} = \frac{1}{2}(2G\boldsymbol{e}) : \boldsymbol{e} = G\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}$$

$$u = u_d + \cancel{u_h}$$

In the coordinate system of the principal directions:

$$u_d = G\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} = G(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) \qquad \varepsilon_i = \lambda_i - 1$$

$$u_d = G(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2(\lambda_1 + \lambda_2 + \lambda_3) + 3)$$

incompressible case: $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0 \rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 3$

$$u_d = G(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$$

$$u_d = u_d(G, \lambda_1, \lambda_2, \lambda_3)$$

$$\boxed{u = G(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)}$$

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- Incompressible Hyperelasticity:

No volume change: $J = 1$

It implies that: $I_3 = (\lambda_1 \lambda_2 \lambda_3)^2 = 1$

No dependence on J:

$$W = W(I_1, I_2, I_3) \longrightarrow W = W(I_1, I_2)$$

Only the deviatoric stress can be obtained from the constitutive equation. $\boldsymbol{p}$ must be determined from the boundary conditions:

$$\boldsymbol{\sigma} = \mathrm{dev}[\boldsymbol{\sigma}] + \mathrm{sph}[\boldsymbol{\sigma}] = \boldsymbol{s} + \boldsymbol{p}$$

$$\boldsymbol{\sigma} = \mathrm{dev}\left[2\left(W_{,1} + I_1 W_{,2}\right)\boldsymbol{b} - 2W_{,2}\boldsymbol{b}^2\right] + \boldsymbol{p}$$

$$\boldsymbol{\sigma} = \mathrm{dev}\left[\sum_{i=1}^{3} \lambda_i \frac{\partial W}{\partial \lambda_i} \boldsymbol{n}_i \otimes \boldsymbol{n}_i\right] + \boldsymbol{p}$$

$$\boldsymbol{p} = p\boldsymbol{I} = ?$$

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Alternative representation:

$$\boldsymbol{p} = \mathrm{sph}[\boldsymbol{\sigma}]$$

$$\boldsymbol{\sigma} = \underbrace{\mathrm{dev}\left[2\left(W_{,1} + I_1 W_{,2}\right)\boldsymbol{b} - 2W_{,2}\boldsymbol{b}^2\right]} + \boldsymbol{p}$$

$$[\boldsymbol{p}] = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

It is deviatoric. It does not contain hydrostatic component.

$$\boldsymbol{\sigma} = \underbrace{2\left(W_{,1} + I_1 W_{,2}\right)\boldsymbol{b} - 2W_{,2}\boldsymbol{b}^2} + \boldsymbol{\eta}$$

Unknown hydrostatic component

It is not deviatoric.
It contains hydrostatic component too.

$$\boldsymbol{\eta} \neq \mathrm{sph}[\boldsymbol{\sigma}]$$

$$[\boldsymbol{\eta}] = \begin{bmatrix} \eta & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta \end{bmatrix}$$

$$\boldsymbol{p} = \mathrm{sph}[\boldsymbol{\sigma}] = \mathrm{sph}\left[2\left(W_{,1} + I_1 W_{,2}\right)\boldsymbol{b} - 2W_{,2}\boldsymbol{b}^2\right] + \mathrm{sph}[\boldsymbol{\eta}]$$

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4.4 Standard incompressible hyperelastic models

- **NEO-HOOKEAN**
  1 parameter

$$\boxed{W(I_1) = C_{10}(I_1 - 3)} = C_{10}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$$

incompressible Hooke's law: $u = G(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$

$C_{10} \neq G$!!!

- **MOONEY-RIVLIN**
  2 parameters

$$\boxed{W(I_1, I_2) = C_{10}(I_1 - 3) + C_{01}(I_2 - 3)}$$

- **YEOH**
  3 parameters

$$\boxed{W(I_1) = C_{10}(I_1 - 3) + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3}$$

- **OGDEN**
  $2K$ parameters

$$\boxed{W(\lambda_1, \lambda_2, \lambda_3) = \sum_{k=1}^{K} \frac{2\mu_k}{\alpha_k^2}\left(\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k} - 3\right)}$$

For $K = 1$ and $\alpha_1 = 2$ it degenerates to the NH model:

$$W = \underbrace{\frac{\mu_k}{2}}_{C_{10}}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$$

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There exist many more hyperelastic models !!!! For example:

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Nano Materials Science
Volume 4, Issue 2, June 2022, Pages 64-82

A comparative study of 85 hyperelastic constitutive models for both unfilled rubber and highly filled rubber nanocomposite material

Hong He[a,b], Qiang Zhang[a,b], Yaru Zhang[a], Jianfeng Chen[a], Liqun Zhang[a,c], Fanzhu Li[a,c]

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Model Name	Year	Strain Energy Function
Mooney-Rivlin [49]	1940	$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3)$
Neo-Hookean [48]	1943	$W = C_{10}(I_1 - 3)$
Isihara [51]	1951	$W = C_{10}(I_1 - 3) + C_{20}(I_1 - 3)^2 C_{01}(I_2 - 3)$
Biderman [52]	1958	$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3$
James-Green-Simpson [53]	1975	$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3$
Haines-Wilson [54]	1979	$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + C_{02}(I_2 - 3)^2 + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3$
Yeoh [55]	1990	$W = \sum_{i=1}^{3} C_{i0}(I_1 - 3)^i$
Lion [56]	1997	$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{50}(I_1 - 3)^5$
Haupt-Sedlan [57]	2001	$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + C_{02}(I_2 - 3)^2 + C_{30}(I_1 - 3)^3$
Hartmann-Neff [58]	2003	$W = \alpha(I_1^3 - 3^3) + \sum_{i=1}^{N} C_{i0}(I_1 - 3)^i + \sum_{j=1}^{N} C_{0j}(I_2^{3/2} - 3\sqrt{3})^j, N = 1, 2, 3\ldots$
Carroll [59]	2011	$W = AI_1 + BI_1^4 + CI_2^{1/2}$
Nunes [60]	2011	$W = C_1(I_1 - 3) + \frac{4}{3}C_2(I_2 - 3)^{3/4}$
Bahreman-Darijani [61]	2014	$W = A_2(I_1 - 3) + B_2(I_2 - 3) + A_4(I_1^2 - 2I_2 - 3) + A_6(I_1^3 - 3I_1 I_2)$
Zhao [62]	2019	$W = C^1_{-1}(I_2 - 3) + C^1_1(I_1 - 3) + C^1_2(I_1^2 - 2I_2 - 3) + C^2_2(I_1^2 - 2I_2 - 3)^2$

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Model Name	Year	Strain Energy Function
Knowles [65]	1977	$W=\frac{\mu}{2b}\left(\left(1+\frac{b(I_1-3)}{n}\right)^n-1\right)$
Swanson [66]	1985	$W=\frac{3}{2}\sum_{i=1}^{N}\frac{A_i}{1+\alpha_i}\left(\frac{I_1}{3}\right)^{1+\alpha_i}+\frac{3}{2}\sum_{i=1}^{N}\frac{B_i}{1+\beta_i}\left(\frac{I_2}{3}\right)^{1+\beta_i}, N=1,2,3\ldots$
Yamashita-Kawabata [67]	1992	$W=C_1(I_1-3)+C_2(I_2-3)+\frac{C_3}{N+1}(I_1-3)^{N+1}$
Davis-De-Thomas [68]	1994	$W=\frac{A}{2(1-n/2)}(I_1-3+C^2)^{(1-n/2)}+k(I_1-3)^2$
Gregory [69]	1997	$W=\frac{A}{2(1-n/2)}(I_1-3+C^2)^{(1-n/2)}+\frac{B}{2(1+m/2)}(I_1-3+C^2)^{(1+m/2)}$
modified Gregory	2021	$W=\frac{A}{1+\alpha}(I_1-3+M^2)^{1+\alpha}+\frac{B}{1+\beta}(I_1-3+N^2)^{1+\beta}$
Beda [70]	2005	$W=\frac{C_1}{\alpha}(I_1-3)^{\alpha}+C_2(I_1-3)+\frac{C_3}{\varsigma}(I_1-3)^{\varsigma}+\frac{K_0}{\beta}(I_2-3)^{\beta}$
Amin [71]	2006	$W=C_1(I_1-3)+\frac{C_2}{N+1}(I_1-3)^{N+1}+\frac{C_3}{M+1}(I_1-3)^{M+1}+C_4(I_2-3)$
Lopez-Pamies [72]	2010	$W=\sum_{i=1}^{N}\frac{3^{1-\alpha_i}}{\alpha_i}\mu_i(I_1^{\alpha_i}-3^{\alpha_i}), N=1,2,3\ldots$
gen-Yeoh [73]	2019	$W=K_1(I_1-3)^m+K_2(I_1-3)^p+K_3(I_1-3)^q$
Hart-Smith [74]	1966	$\frac{\partial W}{\partial I_1}=G\exp[k_1(I_1-3)^2], \frac{\partial W}{\partial I_2}=G\frac{k_2}{I_2}$
Veronda-Westmann [75]	1970	$W=C_1(e^{\alpha(I_1-3)}-1)+C_2(I_2-3)$
Fung-Demiray [76,77]	1972	$W=\frac{\mu}{2b}(e^{b(I_1-3)}-1)$
Vito [78]	1973	$W=\frac{\mu}{2b}(e^{b(\alpha(I_1-3)+(1-\alpha)(I_2-3))}-1)$

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Humphrey-Yin [79]	1987	$W = C_1(e^{C_2(I_1-3)}-1)$
modified Yeoh [80]	1993	$W = C_{10}(I_1-3)+C_{20}(I_1-3)^2+C_{30}(I_1-3)^3+\frac{\alpha}{\beta}(1-e^{-\beta(I_1-3)})$
Martins [81]	1998	$W = C_1(e^{C_2(I_1-3)}-1)+C_3(e^{C_4(\lambda_f-1)^2}-1)$
Chevalier-Marco [82]	2002	$\frac{\partial W}{\partial I_1} = \exp\{\sum_{i=0}^{N} a_i(I_1-3)^i\}, \frac{\partial W}{\partial I_2} = \sum_{i=0}^{N}\frac{b_i}{I_2^i}, N = 1,2,3...$
Gornet-Desmorat [83]	2012	$W = h_1\int e^{h_3(I_1-3)^2}dI_1+3h_2\int\frac{1}{\sqrt{I_2}}dI_2$
Mansouri-Darijani [84]	2014	$W = A_1[e^{m_1(I_1-3)}-1]+B_1[e^{n_1(I_2-3)}-1]$
Gent-Thomas [85]	1958	$W = C_1(I_1-3)+C_2\ln\left(\frac{I_2}{3}\right)$
Alexander [86]	1968	$W = C_1\int\exp k(I_1-3)^2dI_1+C_2\ln\left(\frac{(I_2-3)+\gamma}{\gamma}\right)+C_3(I_2-3)$
Lambert Diani-Rey [87]	1999	$W = \int\exp\{\sum_{i=0}^{2}a_i(I_1-3)^i\}dI_1+\int\exp\{\sum_{i=0}^{1}b_i(\ln I_2)^i\}dI_2$
Hoss-Marczak-I [13]	2010	$W = \frac{\alpha}{\beta}(1-e^{-\beta(I_1-3)})+\frac{\mu}{2b}\left(\left(1+\frac{b(I_1-3)}{n}\right)^n-1\right)$
Hoss-Marczak-II [13]	2010	$W = \frac{\alpha}{\beta}(1-e^{-\beta(I_1-3)})+\frac{\mu}{2b}\left(\left(1+\frac{b(I_1-3)}{n}\right)^n-1\right)+C_2\ln\left(\frac{I_2}{3}\right)$
Exp-Ln [88]	2013	$W = A\left(\frac{1}{a}e^{a(I_1-3)}+b(I_1-2)(1-\ln(I_1-2))-\frac{1}{a}-b\right)$

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Model Name	Year	Strain Energy Function
Warner [90]	1972	$W = -\frac{1}{2}\mu I_m \ln\left(1-\frac{I_1-3}{I_m-3}\right)$
Kilian [91]	1981	$W = -\mu J_L\left[\ln\left(1-\sqrt{\frac{I_1-3}{J_L}}\right)+\sqrt{\frac{I_1-3}{J_L}}\right]$
Van der Waals [92–94]	1986	$W = \mu\left\{-(\lambda_m^2-3)[\ln(1-\Theta)+\Theta]-\frac{2}{3}a\left(\frac{I-3}{2}\right)^{\frac{3}{2}}\right\}$
Gent [89]	1996	$W = -\frac{E}{6}(I_m-3)\ln\left(1-\frac{I_1-3}{I_m-3}\right)$
Takamizawa-Hayashi [95]	1987	$W = -c\ln\left[1-\left(\frac{I_1-2}{J_m}\right)^2\right]$
Yeoh-Fleming [96]	1997	$W = \frac{A}{B}(I_m-3)\left(1-e^{-B\frac{I_1-3}{I_m-3}}\right)-C_{10}(I_m-3)\ln\left(1-\frac{I_1-3}{I_m-3}\right)$
3 Parameters Gent [97]	1999	$W = \frac{\mu}{2}\left[-\alpha(I_m-3)\ln\left(1-\frac{I_1-3}{I_m-3}\right)+(1-\alpha)(I_2-3)\right]$
Pucci-Saccomandi [98]	2002	$W = K\ln\left(\frac{I_2}{3}\right)-\frac{\mu}{2}J_m\ln\left(1-\frac{I_1-3}{J_m}\right)$
Horgan-Saccomandi [99]	2004	$W = -\frac{\mu}{2}J\ln\left[\frac{J^2-J^2I_1+JI_2-1}{(J-1)^3}\right]$
Beatty [100]	2007	$W = -c\frac{I_m(I_m-3)}{2I_m-3}\ln\left(\frac{1-(I_1-3)/(I_m-3)}{1+(I_1-3)/I_m}\right)$
Horgan-Murphy [101]	2007	$W = -\frac{2\mu(I_m-3)}{c^2}\ln\left(1-\frac{\lambda_1^c+\lambda_2^c+\lambda_3^c-3}{I_m-3}\right)$

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Model Name	Year	Strain Energy Function
Valanis-Landel [102]	1967	$W = 2\mu \sum_{i=1}^{3} [\lambda_i(\ln \lambda_i - 1)]$
Peng-Landel [103]	1972	$W = E \sum_{i=1}^{3} \left[\lambda_i - 1 - \ln \lambda_i - \frac{1}{6}(\ln \lambda_i)^2 + \frac{1}{18}(\ln \lambda_i)^3 - \frac{1}{216}(\ln \lambda_i)^4\right]$
Ogden [104]	1972	$W = \sum_{i=1}^{N} \frac{2\mu_i}{\alpha_i^2}(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3)$
Attard [11]	2004	$W = \sum_{i=1}^{N} \frac{A_i}{2i}[(\lambda_1)^{2i} + (\lambda_2)^{2i} + (\lambda_3)^{2i} - 3] + \frac{B_i}{2i}[(\lambda_1)^{-2i} + (\lambda_2)^{-2i} + (\lambda_3)^{-2i} - 3],\ N = 1, 2, 3\ldots$
Shariff [106]	2000	$W = \sum_{i=1}^{3} \omega(\lambda_i),\ \omega(\lambda_i) = E \sum_{j=0}^{n} \alpha_j \varphi_j(\lambda_i)$
Arman-Narooei [107]	2014	$W = \sum_{i=1}^{N} A_i[e^{m_i(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3)} - 1] + \sum_{j=1}^{N} B_j[e^{n_j(\lambda_1^{-\beta_j} + \lambda_2^{-\beta_j} + \lambda_3^{-\beta_j} - 3)} - 1],\ N = 1, 2, 3\ldots$

Model Name	Year	Strain Energy Function
Continuum Hybrid [108]	2003	$W = K_1(I_1 - 3) + K_2 \ln \frac{I_2}{3} + \frac{\mu}{\alpha}(\lambda_1^{\alpha} + \lambda_2^{\alpha} + \lambda_3^{\alpha} - 3)$
Bechir-4term [109]	2006	$W = C_1^1(I_1 - 3) + \sum_{n=1}^{2} \sum_{r=1}^{2} C_n^r(\lambda_1^{2n} + \lambda_2^{2n} + \lambda_3^{2n} - 3)^r$
WFB [110]	2017	$W = \int_1^{L_f} \{F(\lambda_1)A(\lambda_1 e^{-BI_1}) + C(\lambda_1 I_1^{-D})\} \left(\lambda_1^2 - \frac{1}{\lambda_1}\right) d\lambda_1$

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Model Name	Year	Strain Energy Function
Gaussian [113]	1943	$W = \frac{1}{2}NkT(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$
Affine [114–116]	1946	$W = \frac{G}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3),\ G = \nu kT$
Phantom [117,118]	1947	$W = \frac{\nu kT}{2}\left(1 - \frac{2}{f}\right)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3),\ G_c = \nu kT(1 - 2/f)$
Edwards-Tube [119]	1967	$W = G_e(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$
Slip-Link [120]	1981	$W = \frac{1}{2}G_e\sum_{i=1}^{3}\left(\frac{(1+\eta)(1-\alpha^2)\lambda_i^2}{(1+\eta\lambda_i^2)(1-\alpha^2\sum_{i=1}^{3}\lambda_i^2)} + \ln(1 + \eta\sum_{i=1}^{3}\lambda_i^2)\right)$
Constrained Junctions [121,122]	1982	$W = W_{ph} + \frac{\nu kT}{2}\sum_t\left[\kappa\frac{\lambda_t^2 - 1}{\lambda_t^2 + \kappa} + \ln\left(\frac{\lambda_t^2 + \kappa}{1 + \kappa}\right) - \ln\lambda_t^2\right]$
Edwards-Vilgis [123]	1986	$W = \frac{G_c}{2}\left[\frac{(1-\alpha^2)I_1}{1-\alpha^2 I_1} + \ln(1 - \alpha^2 I_1)\right] + \frac{1}{2}G_e\sum_{i=1}^{3}\left(\frac{(1+\eta)(1-\alpha^2)\lambda_i^2}{(1+\eta\lambda_i^2)(1-\alpha^2\sum_{i=1}^{3}\lambda_i^2)} + \ln(1 + \eta\sum_{i=1}^{3}\lambda_i^2)\right)$
MCC [124]	1989	$W = \frac{1}{2}\xi kT\sum_i(\lambda_i^2 - 1) + \frac{1}{2}\mu kT\sum_i[B_t + D_t - \ln(1 + B_t) - \ln(1 + D_t)]$ $B_t = \kappa^2(\lambda_t^2 - 1)(\lambda_t^2 + \kappa)^{-2},\ D_t = \lambda_t^2 B_t/\kappa$
Tube [125]	1997	$W = \sum_{i=1}^{3}\frac{G_c}{2}(\lambda_i^2 - 1) + \frac{2G_e}{\beta^2}(\lambda_i^{-\beta} - 1)$
Nonaffine-Tube [126]	1997	$W = W_{ph} + W_{ent} = G_c\sum_{i=1}^{3}\frac{\lambda_i^2}{2} + G_e\sum_{i=1}^{3}\left(\lambda_i + \frac{1}{\lambda_i}\right)$

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Model Name	Year	Strain Energy Function
Three-Chain [117]	1943	$W=\frac{\mu N}{3}\sum_{i=1}^{3}\left(\sqrt{N^{-1}}\,\lambda_i\beta_i+\ln\frac{\beta_i}{\sinh\beta_i}\right)$
Four-Chain [128]	1943	–
Arruda-Boyce [129]	1993	$W=\mu N\left(\beta_{chain}\lambda_{chain}+\ln\frac{\beta_{chain}}{\sinh\beta_{chain}}\right)$
modified Flory-Erman [9,119]	1993	$W=W_{8ch}+\sum_{i=1}^{3}\frac{\mu}{2}\left[B_i+D_i-\ln(B_i+1)-\ln(D_i+1)\right]$ $B_t=\kappa^2(\lambda_t^2-1)(\lambda_t^2+\kappa)^{-2}\quad D_t=\lambda_t^2 B_t/\kappa$
Extended-Tube [130]	1999	$W=\frac{G_c}{2}\left[\frac{(1-\delta^2)(I_1-3)}{1-\delta^2(I_1-3)}+\ln(1-\delta^2(I_1-3))\right]+\frac{2G_e}{\beta^2}\sum_{i=1}^{3}(\lambda_i^{-\beta}-1)$
Meissner-Matějka (ABGI) [131,132]	2004	$W=\mu N\left(\gamma\lambda_{c,r}+\ln\left(\frac{\gamma}{\sinh\gamma}\right)\right)+\sum_{i=1}^{3}\frac{2\mu_e}{\beta^2}(\lambda_i^{-\beta}-1)$
Micro-Sphere [133]	2004	$W=\mu\left(\lambda_r L^{-1}(\lambda_r)+\ln\frac{L^{-1}(\lambda_r)}{\sinh L^{-1}(\lambda_r)}\right)$
Bootstrapped-8-Chain [134,135]	2009	$W=W_{8ch}\left(\frac{\lambda_1+\lambda_2+\lambda_3}{\sqrt{3N}}-\frac{\lambda_c}{\sqrt{N}}\right)+W_{8ch}\left(\frac{\lambda_c}{\sqrt{N}}\right)$
Davidson-Goulbourne [136]	2013	$W=\frac{1}{6}G_c I_1-G_c\lambda_{\max}^2\ln(3\lambda_{\max}^2-I_1)+G_e\sum_i\left(\lambda_i+\frac{1}{\lambda_i}\right)$
Network Averaging Tube [137]	2016	$W=\mu_c\kappa n\ln\frac{\sin\left(\frac{\pi}{\sqrt{n}}\right)\left(\frac{I_1}{3}\right)^{\frac{q}{2}}}{\sin\left(\frac{\pi}{\sqrt{n}}\left(\frac{I_1}{3}\right)^{\frac{q}{2}}\right)}+\mu_t\left[\left(\frac{I_2}{3}\right)^{\frac{1}{2}}-1\right]$
SpT [138]	2018	$W=G_c N\ln\left(\frac{3N+\frac{1}{2}I_1}{3N-I_1}\right)+G_e\sum_i\frac{1}{\lambda_i}$

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Model Name	Year	Strain Energy Function
Wu-Giessen (Full Network) [139–141]	1992	$W=\frac{\mu}{3}\sqrt{N}\sum_{i=1}^{3}\left(\lambda_i\beta_i+\sqrt{N}\ln\frac{\beta_i}{\sinh\beta_i}\right)(1-\rho)+\mu N\left(\lambda_{chain}\beta_{chain}+\ln\frac{\beta_{chain}}{\sinh\beta_{chain}}\right)\rho$
Zuniga-Beatty [142]	2002	$W=W_{3ch}\left(1-\frac{\lambda_1}{\sqrt{N_3}}\right)+W_{8ch}\sqrt{\frac{\lambda_1^2+\lambda_2^2+\lambda_3^2}{3N_8}}$
Lim [143]	2005	$W=W_{Gaussian}(1-f)+fW_{8ch}$
Bechir-Chevalier [144]	2010	$W=\mu_f N_8\left[\lambda_r\beta+\ln\left(\frac{\beta}{\sinh\beta}\right)\right]+\frac{\mu_c}{3}N_3\sum_{j=1}^{j=3}\left[\bar{\beta}_j\bar{\lambda}_{jr}+\ln\left(\frac{\bar{\beta}_j}{\sinh\bar{\beta}_j}\right)\right]$

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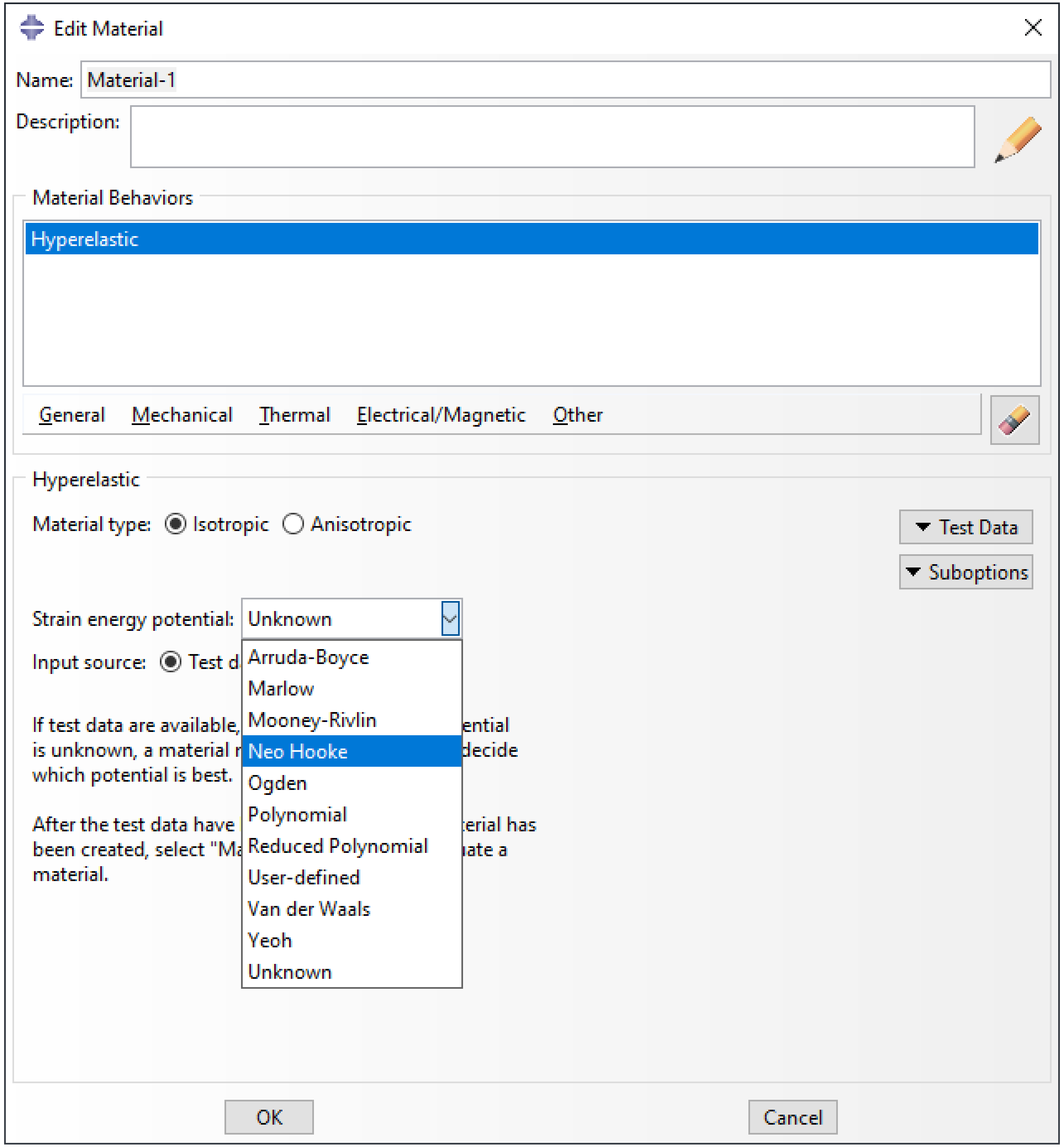
ABAQUS:

In FE solvers, only the basic models are available as built-in hyperealastic models



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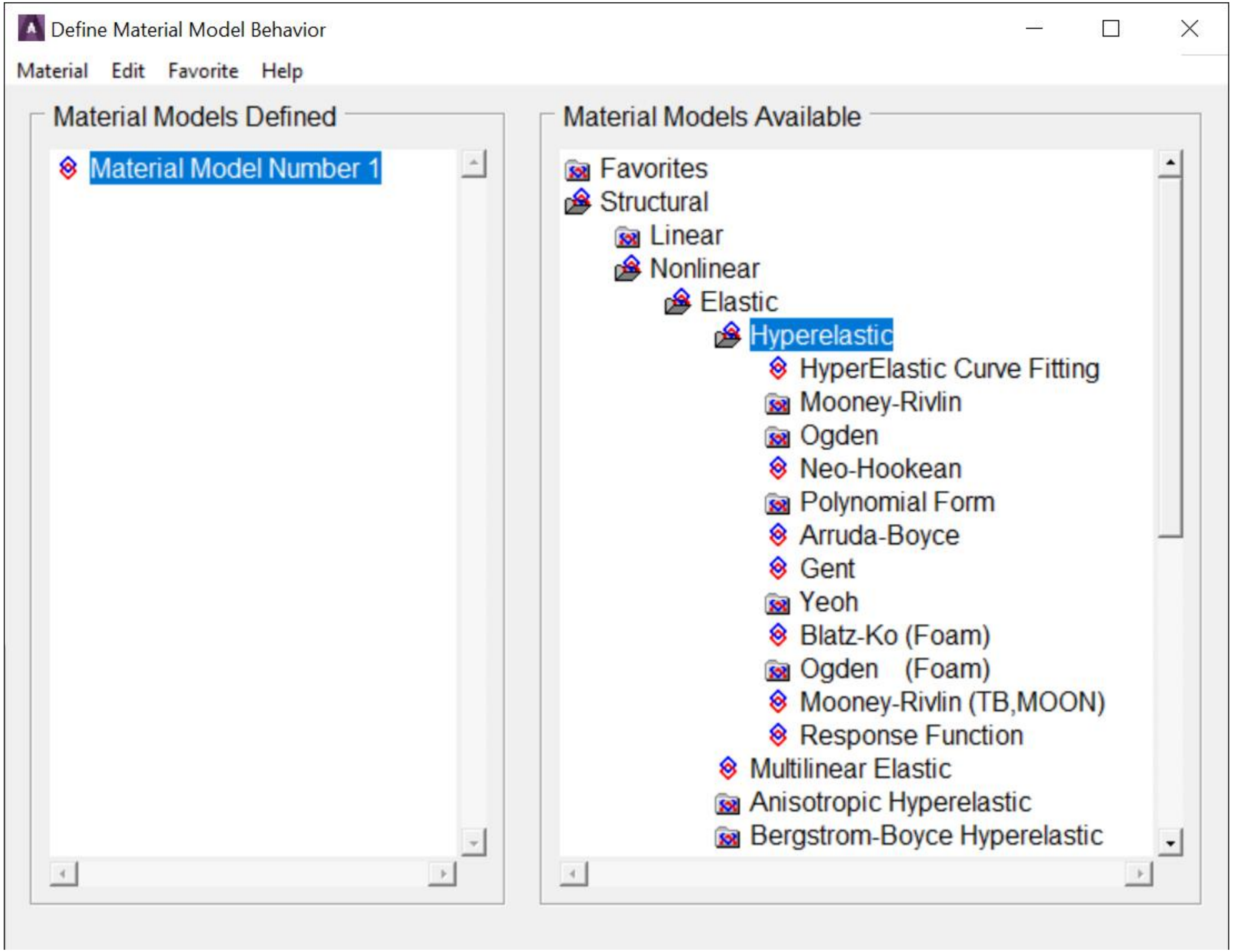

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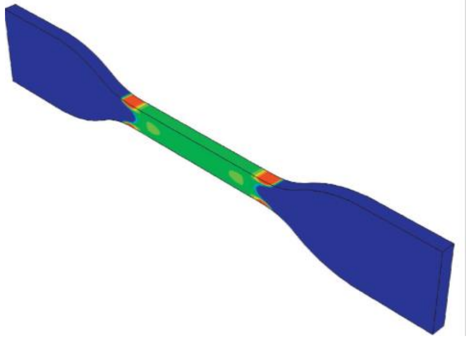
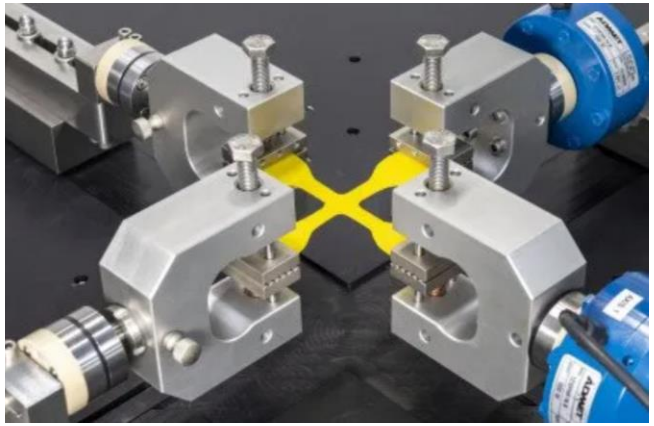
4.5 Standard homogeneous deformations

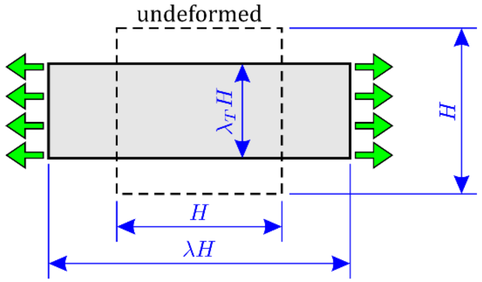
Tests



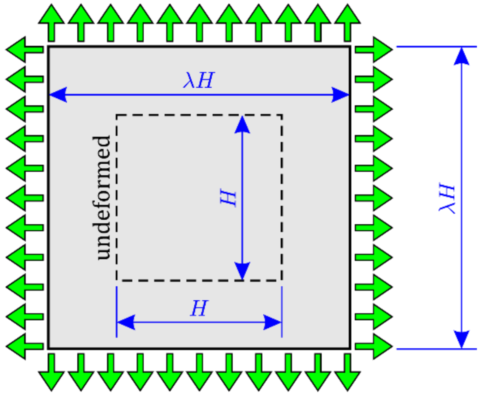


Idealized mechanical models

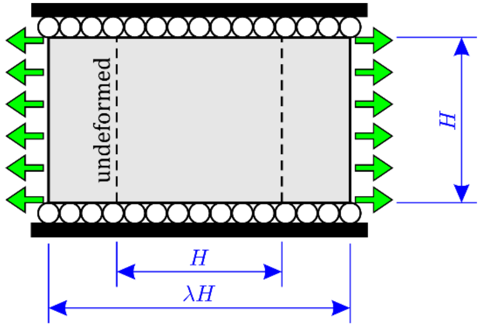
UNIAXIAL



EQUIBIAXIAL



PLANAR



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UNIAXIAL loading

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \lambda \cdot X_1$$
$$x_2 = \lambda_T \cdot X_2$$
$$x_3 = \lambda_T \cdot X_3$$

$$[\boldsymbol{F}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda_T & 0 \\ 0 & 0 & \lambda_T \end{bmatrix}$$

$$[\boldsymbol{C}] = [\boldsymbol{b}] = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda_T^2 & 0 \\ 0 & 0 & \lambda_T^2 \end{bmatrix}$$

$$I_1 = \lambda^2 + 2\lambda_T^2$$
$$I_2 = 2\lambda^2\lambda_T^2 + \lambda_T^4$$
$$I_3 = \lambda^2\lambda_T^4 \qquad J = \lambda\lambda_T^2$$

$$\lambda_1 = \lambda \qquad \boldsymbol{N}_1 = \boldsymbol{n}_1 = \boldsymbol{e}_1$$
$$\lambda_2 = \lambda_T \qquad \boldsymbol{N}_2 = \boldsymbol{n}_2 = \boldsymbol{e}_2$$
$$\lambda_3 = \lambda_T \qquad \boldsymbol{N}_3 = \boldsymbol{n}_3 = \boldsymbol{e}_3$$

INCOMPRESSIBLE CASE

$$x_1 = \lambda \cdot X_1$$
$$x_2 = \lambda^{-1/2} \cdot X_2$$
$$x_3 = \lambda^{-1/2} \cdot X_3$$

$$[\boldsymbol{F}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix}$$

$$[\boldsymbol{C}] = [\boldsymbol{b}] = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}$$

$$I_1 = \lambda^2 + 2/\lambda$$
$$I_2 = 2\lambda + \lambda^{-2}$$
$$I_3 = 1 \qquad J = 1$$

$$\lambda_1 = \lambda \qquad \boldsymbol{N}_1 = \boldsymbol{n}_1 = \boldsymbol{e}_1$$
$$\lambda_2 = \lambda^{-1/2} \qquad \boldsymbol{N}_2 = \boldsymbol{n}_2 = \boldsymbol{e}_2$$
$$\lambda_3 = \lambda^{-1/2} \qquad \boldsymbol{N}_3 = \boldsymbol{n}_3 = \boldsymbol{e}_3$$

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EQUIBIAXIAL loading

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \lambda \cdot X_1$$
$$x_2 = \lambda \cdot X_2$$
$$x_3 = \lambda_T \cdot X_3$$

$$[\boldsymbol{F}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_T \end{bmatrix}$$

$$[\boldsymbol{C}] = [\boldsymbol{b}] = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda_T^2 \end{bmatrix}$$

$$I_1 = 2\lambda^2 + \lambda_T^2$$
$$I_2 = 2\lambda^2\lambda_T^2 + \lambda^4$$
$$I_3 = \lambda^4\lambda_T^2 \qquad J = \lambda^2\lambda_T$$

$$\lambda_1 = \lambda \qquad \boldsymbol{N}_1 = \boldsymbol{n}_1 = \boldsymbol{e}_1$$
$$\lambda_2 = \lambda \qquad \boldsymbol{N}_2 = \boldsymbol{n}_2 = \boldsymbol{e}_2$$
$$\lambda_3 = \lambda_T \qquad \boldsymbol{N}_3 = \boldsymbol{n}_3 = \boldsymbol{e}_3$$

INCOMPRESSIBLE CASE

$$x_1 = \lambda \cdot X_1$$
$$x_2 = \lambda \cdot X_2$$
$$x_3 = \lambda^{-2} \cdot X_3$$

$$[\boldsymbol{F}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}$$

$$[\boldsymbol{C}] = [\boldsymbol{b}] = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^{-4} \end{bmatrix}$$

$$I_1 = 2\lambda^2 + \lambda^{-4}$$
$$I_2 = 2\lambda^{-2} + \lambda^4$$
$$I_3 = 1 \qquad J = 1$$

$$\lambda_1 = \lambda \qquad \boldsymbol{N}_1 = \boldsymbol{n}_1 = \boldsymbol{e}_1$$
$$\lambda_2 = \lambda \qquad \boldsymbol{N}_2 = \boldsymbol{n}_2 = \boldsymbol{e}_2$$
$$\lambda_3 = \lambda^{-2} \qquad \boldsymbol{N}_3 = \boldsymbol{n}_3 = \boldsymbol{e}_3$$

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PLANAR loading

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \lambda \cdot X_1$$
$$x_2 = 1$$
$$x_3 = \lambda_T \cdot X_1$$

$$[\boldsymbol{F}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda_T \end{bmatrix}$$

$$[\boldsymbol{C}] = [\boldsymbol{b}] = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda_T^2 \end{bmatrix}$$

$$I_1 = \lambda^2 + \lambda_T^2 + 1$$
$$I_2 = \lambda_T^2 + \lambda^2 + \lambda^2\lambda_T^2$$
$$I_3 = \lambda^2\lambda_T^2 \qquad J = \lambda\lambda_T$$

$$\lambda_1 = \lambda \qquad \boldsymbol{N}_1 = \boldsymbol{n}_1 = \boldsymbol{e}_1$$
$$\lambda_2 = 1 \qquad \boldsymbol{N}_2 = \boldsymbol{n}_2 = \boldsymbol{e}_2$$
$$\lambda_3 = \lambda_T \qquad \boldsymbol{N}_3 = \boldsymbol{n}_3 = \boldsymbol{e}_3$$

INCOMPRESSIBLE CASE

$$x_1 = \lambda \cdot X_1$$
$$x_2 = 1$$
$$x_3 = \lambda^{-1} \cdot X_3$$

$$[\boldsymbol{F}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}$$

$$[\boldsymbol{C}] = [\boldsymbol{b}] = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}$$

$$I_1 = \lambda^2 + \lambda^{-2} + 1$$
$$I_2 = I_1$$
$$I_3 = 1 \qquad J = 1$$

$$\lambda_1 = \lambda \qquad \boldsymbol{N}_1 = \boldsymbol{n}_1 = \boldsymbol{e}_1$$
$$\lambda_2 = 1 \qquad \boldsymbol{N}_2 = \boldsymbol{n}_2 = \boldsymbol{e}_2$$
$$\lambda_3 = \lambda^{-1} \qquad \boldsymbol{N}_3 = \boldsymbol{n}_3 = \boldsymbol{e}_3$$

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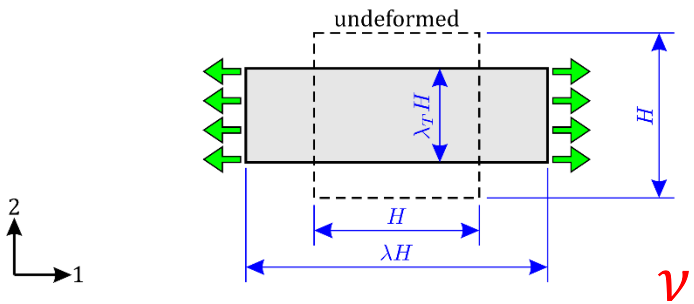
- The Poisson's ratio in **small-strain** theory:



$$\varepsilon = \lambda - 1$$

$$\varepsilon_T = \lambda_T - 1$$

$$\boxed{\varepsilon_T = -\nu \cdot \varepsilon}$$

ν is independent of strain $\qquad \nu = -\dfrac{\varepsilon_T}{\varepsilon}$

Incompressible case: $\varepsilon_T = -0.5 \cdot \varepsilon$

- Apparent Poisson's ratio or Poisson's function in **finite strain**:

$$\boxed{\lambda_T = \lambda^{-\nu}}$$

ν is dependent of strain: $\nu = \nu(\lambda)$

$$\ln\lambda_T = \ln\lambda^{-\nu}$$

$$\ln\lambda_T = -\nu \cdot \ln\lambda$$

$$\varepsilon_T^{\text{true}} = -\nu \cdot \varepsilon^{\text{true}}$$

Incompressible case: $\lambda_T = \lambda^{-0.5} = \dfrac{1}{\sqrt{\lambda}}$

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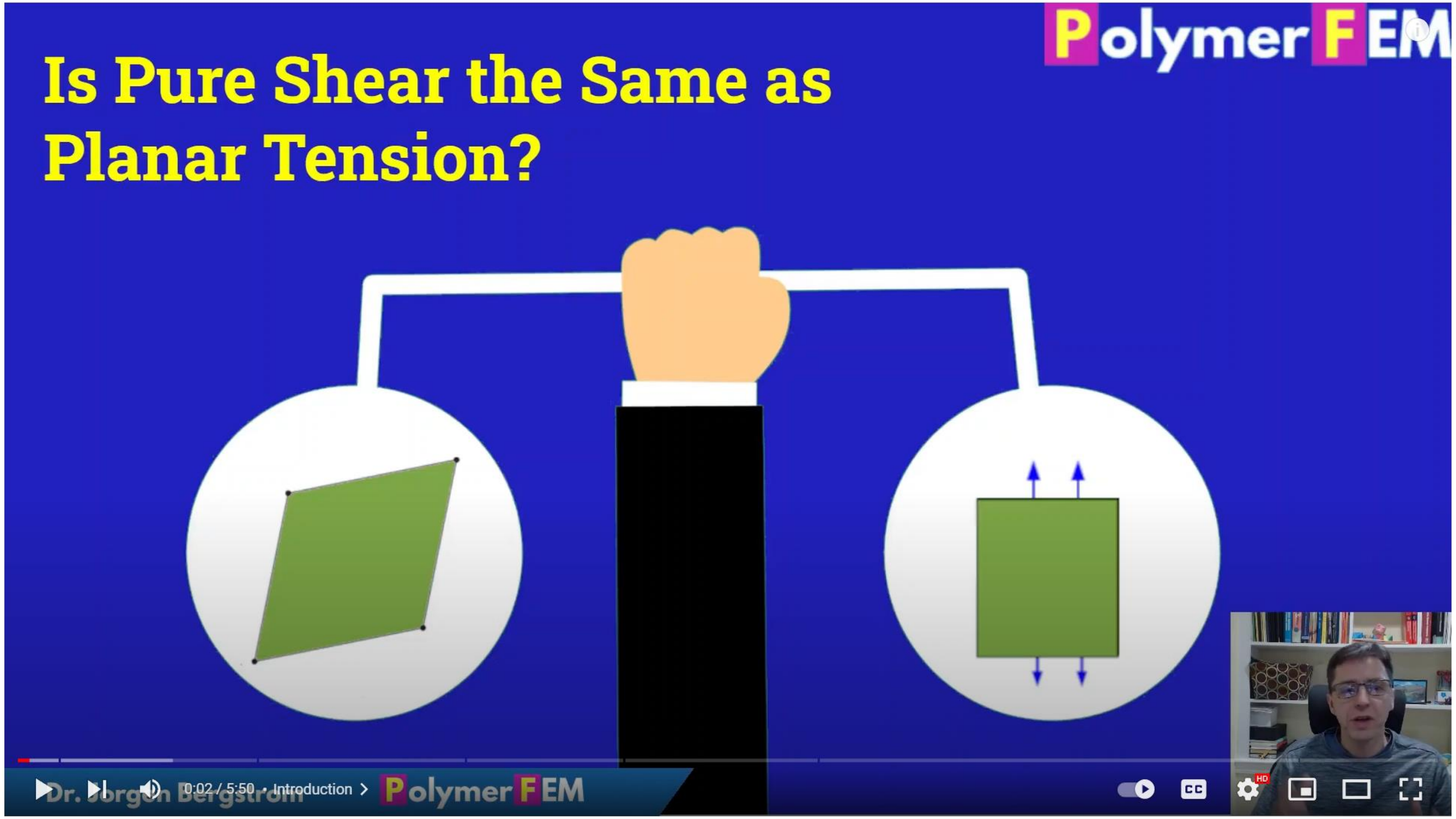
For incompressible materials, the pure shear and planar tension behaviors are nearly the same at moderate strains.

https://www.youtube.com/watch?v=vovwCovZeGA



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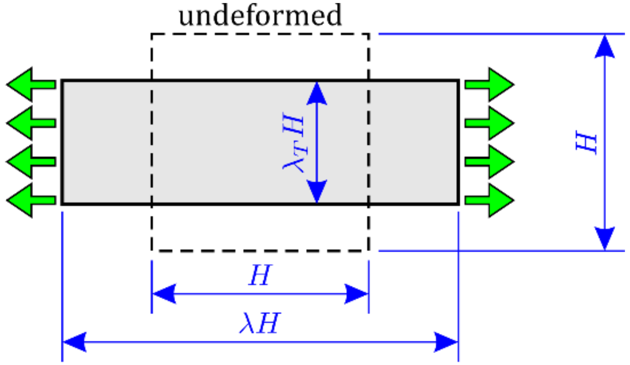
Example: Stress solution for the incompressible **neo-Hookean** model in uniaxial loading:



$$[\boldsymbol{F}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix} \qquad J = \det\boldsymbol{F} = 1$$

$$[\boldsymbol{b}] = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix} \qquad I_1 = \lambda^2 + 2\lambda^{-1}$$

Strain energy: $\boxed{W = C_{10}(I_1 - 3)}$ $\qquad W_{,1} = C_{10}$

General solution for compressible case:

$$\boldsymbol{\sigma} = \frac{2}{J}\left[\left(W_{,1} + I_1 W_{,2}\right)\boldsymbol{b} - W_{,2}\boldsymbol{b}^2 + I_3 W_{,3}\boldsymbol{I}\right]$$

General solution for incompressible case reduces to:

$$\boldsymbol{\sigma} = \operatorname{dev}\left[2\left(W_{,1} + I_1 W_{,2}\right)\boldsymbol{b} - 2W_{,2}\boldsymbol{b}^2\right] + \boldsymbol{p} = 2C_{10}\operatorname{dev}[\boldsymbol{b}\,] + \boldsymbol{p}$$

$$[\operatorname{dev}[\boldsymbol{b}\,]] = \frac{1}{3\lambda}\begin{bmatrix} 2(\lambda^3 - 1) & 0 & 0 \\ 0 & 1 - \lambda^3 & 0 \\ 0 & 0 & 1 - \lambda^3 \end{bmatrix}$$

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$$\boldsymbol{\sigma} = \operatorname{dev}\left[2\left(W_{,1} + I_1 W_{,2}\right)\boldsymbol{b} - 2W_{,2}\boldsymbol{b}^2\right] + \boldsymbol{p} = 2C_{10}\operatorname{dev}[\boldsymbol{b}\,] + \boldsymbol{p}$$

$$[\boldsymbol{\sigma}] = \frac{2C_{10}}{3\lambda}\begin{bmatrix} 2(\lambda^3 - 1) & 0 & 0 \\ 0 & 1 - \lambda^3 & 0 \\ 0 & 0 & 1 - \lambda^3 \end{bmatrix} + \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_{22} = 0 \implies p = \frac{2C_{10}}{3\lambda}(\lambda^3 - 1)$$

$$[\boldsymbol{\sigma}] = \frac{2C_{10}}{3\lambda}\begin{bmatrix} 2(\lambda^3 - 1) & 0 & 0 \\ 0 & 1 - \lambda^3 & 0 \\ 0 & 0 & 1 - \lambda^3 \end{bmatrix} + \frac{2C_{10}}{3\lambda}(\lambda^3 - 1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Cauchy stress: $\sigma = 2C_{10}(\lambda^2 - 1/\lambda)$

Nominal (engineering, 1st Piola-Kirchhoff) stress:

$$\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}$$

$$P = \sigma/\lambda = 2C_{10}(\lambda - 1/\lambda^2)$$

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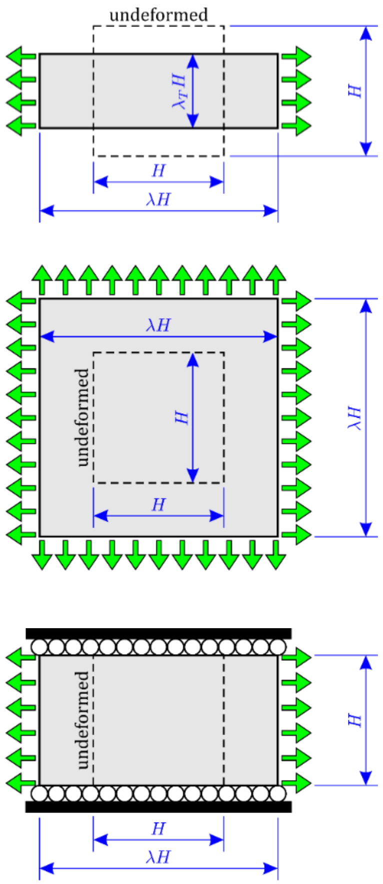

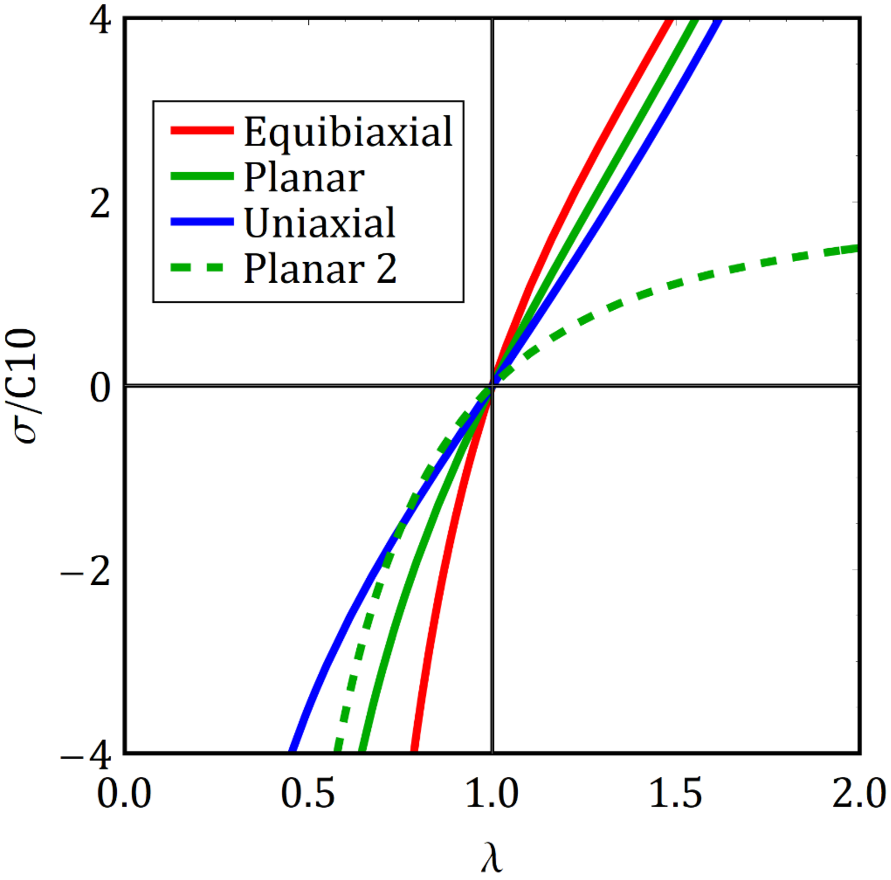

neo-Hookean model
True stress solutions

$$\sigma^U = 2C_{10}(\lambda^2 - \lambda^{-1})$$

$$\sigma^B = 2C_{10}(\lambda^2 - \lambda^{-4})$$

$$\sigma^P = 2C_{10}(\lambda^2 - \lambda^{-2})$$

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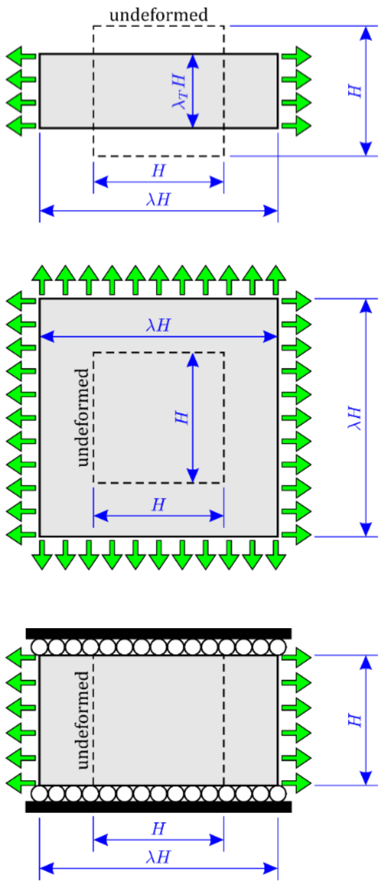

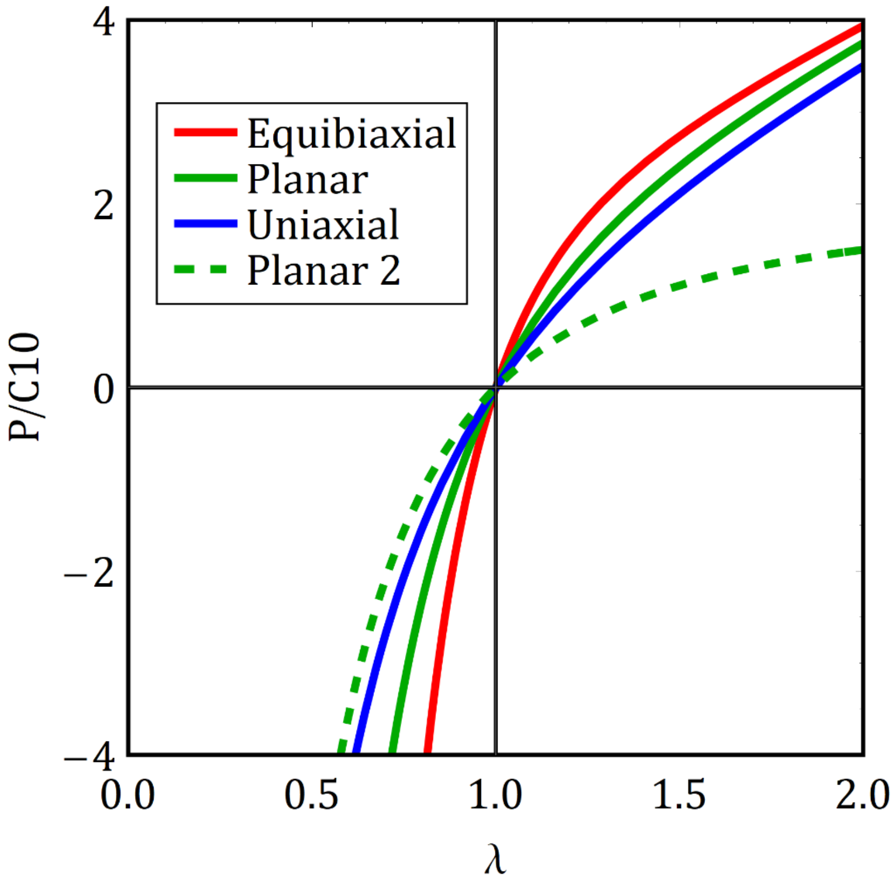

neo-Hookean model
Nominal stress solutions

$$P^{U} = 2C_{10}(\lambda - \lambda^{-2})$$

$$P^{B} = 2C_{10}(\lambda - \lambda^{-5})$$

$$P^{P} = 2C_{10}(\lambda - \lambda^{-3})$$

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• Stress solutions for some hyperelastic models

Notations:

U: Uniaxial
B: Equibiaxial
P: Planar

NH: neo-Hookean
MR: Mooney-Rivlin
YE: Yeoh
OG: Kth-order Ogden

$$P_U^{NH} = 2C_{10}(\lambda - \lambda^{-2}) \qquad P_U^{MR} = 2C_{10}(\lambda - \lambda^{-2}) + 2C_{01}(1 - \lambda^{-3})$$

$$P_B^{NH} = 2C_{10}(\lambda - \lambda^{-5}) \qquad P_B^{MR} = 2C_{10}(\lambda - \lambda^{-5}) + 2C_{01}(\lambda^3 - \lambda^{-3})$$

$$P_P^{NH} = 2C_{10}(\lambda - \lambda^{-3}) \qquad P_P^{MR} = 2C_{10}(\lambda - \lambda^{-3}) + 2C_{01}(\lambda - \lambda^{-3})$$

$$P_U^{YE} = 2(\lambda - \lambda^{-2})(C_{10} + 2C_{20}(\lambda^2 + 2\lambda^{-1} - 3) + 3C_{30}(\lambda^2 + 2\lambda^{-1} - 3)^2)$$

$$P_B^{YE} = 2(\lambda - \lambda^{-5})(C_{10} + 2C_{20}(2\lambda^2 + 2\lambda^{-4} - 3) + 3C_{30}(2\lambda^2 + 2\lambda^{-4} - 3)^2)$$

$$P_P^{YE} = 2(\lambda - \lambda^{-3})(C_{10} + 2C_{20}(\lambda^2 + \lambda^{-2} - 2) + 3C_{30}(\lambda^2 + \lambda^{-2} - 2)^2)$$

$$P_U^{OG} = \sum_{k=1}^{K} \frac{2\mu_k}{\alpha_k}\left(\lambda^{\alpha_k - 1} - \lambda^{-\alpha_k/2 - 1}\right) \qquad P_P^{OG} = \sum_{k=1}^{K} \frac{2\mu_k}{\alpha_k}\left(\lambda^{\alpha_k - 1} - \lambda^{-\alpha_k - 1}\right)$$

$$P_B^{OG} = \sum_{k=1}^{K} \frac{2\mu_k}{\alpha_k}\left(\lambda^{\alpha_k - 1} - \lambda^{-2\alpha_k - 1}\right)$$

Hooke's law:

$$\sigma = P = E\varepsilon = E(\lambda - 1)$$

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