Contents

Chapter 1

Theory of Linear Viscoelasticity

1.1 Introduction

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1.2.1 Linear Elastic Element

The one-dimensional linear elastic element can be represented with a *spring* as shown in Figure 1.1.

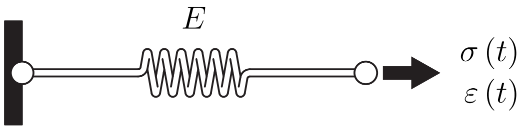


Figure 1.1: Illustration of the linear elastic element.

The behavior of the elastic element is described with the constant elastic modulus E. The relation between the stress and the strain is given by the relation

$$\sigma(t) = E\varepsilon(t), \qquad \varepsilon(t) = \frac{1}{E}\sigma(t). \tag{1.1}$$

The rate-forms of the expressions can be also formulated:

$$\dot{\sigma}(t) = E\dot{\varepsilon}(t), \qquad \dot{\varepsilon}(t) = \frac{1}{E}\dot{\sigma}(t), \tag{1.2}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = E\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}, \qquad \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{1}{E}\frac{\mathrm{d}\sigma}{\mathrm{d}t}, \tag{1.3}$$

$$\mathrm{d}\sigma = E\mathrm{d}\varepsilon, \qquad \mathrm{d}\varepsilon = \frac{1}{E}\mathrm{d}\sigma. \tag{1.4}$$

The model is rate-independent, the effect of the strain-rate can be eliminated. The unit for the elastic modulus is [Pa].

1.2.2 Linear Viscous Element

The linear viscous element is usually represented with a *dashpot* as shown in Figure 1.2.

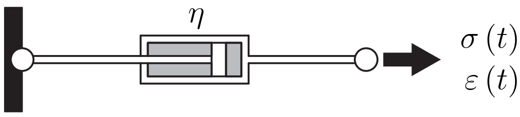


Figure 1.2: Illustration of the linear viscous element.

The mechanical behavior of the dashpot is characterized by the coefficient of viscosity, η. The constitutive relation for this element reads as

$$\sigma(t) = \eta\dot{\varepsilon}(t), \qquad \dot{\varepsilon}(t) = \frac{1}{\eta}\sigma(t). \tag{1.5}$$

The unit for the coefficient of viscosity is $[\mathrm{Pa}\cdot\mathrm{s}]$.

1.2.3 The Maxwell Model

The Maxwell model is constructed by connecting a linear spring and a dashpot in series, as shown in Figure 1.3. The stress and strain associated with the elastic element are denoted with quantities using subscript e, whereas the quantities corresponding to the viscous element are labelled with v. The total stress and strain are denoted with σ and ε.

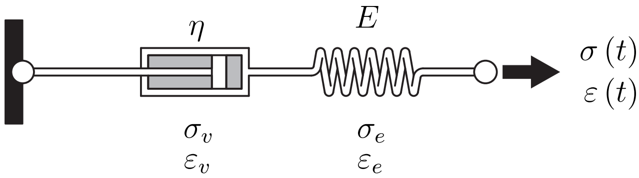


Figure 1.3: Illustration of the Maxwell element.

The total stress and strain of the Maxwell model can be determined as:

$$\sigma(t) = \sigma_e(t) = \sigma_v(t), \qquad \varepsilon(t) = \varepsilon_e(t) + \varepsilon_v(t). \tag{1.6}$$

These expression can be also written in rate-from:

$$\dot{\sigma}(t) = \dot{\sigma}_e(t) = \dot{\sigma}_v(t), \qquad \dot{\varepsilon}(t) = \dot{\varepsilon}_e(t) + \dot{\varepsilon}_v(t). \tag{1.7}$$

The constitutive relations for the elements:

$$\sigma_e(t) = E\varepsilon_e(t), \qquad \sigma_v(t) = \eta\dot{\varepsilon}_v(t), \tag{1.8}$$

$$\varepsilon_e(t) = \frac{1}{E}\sigma_e(t), \qquad \dot{\varepsilon}_v(t) = \frac{1}{\eta}\sigma_v(t). \tag{1.9}$$

Combining the expressions for the stresses and strains we can express the total strain rate and total stress rate as

$$\dot{\varepsilon}(t) = \dot{\varepsilon}_e(t) + \dot{\varepsilon}_v(t) = \frac{1}{E}\dot{\sigma}_e(t) + \frac{1}{\eta}\sigma_v(t), \tag{1.10}$$

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}, \qquad \dot{\sigma} = E\dot{\varepsilon} - \frac{E}{\eta}\sigma, \tag{1.11}$$

where the dependence on t is removed in the last expression to simplify the representation. By introducing the relaxation time, $\tau = \eta/E$, we can express the last equation as

$$\dot{\sigma} = E\dot{\varepsilon} - \frac{1}{\tau}\sigma. \tag{1.12}$$

1.2.4 The Kelvin-Voigt Model

The model is constructed by a parallel connection of a linear spring and a linear dashpot, as shown in Figure 1.4.

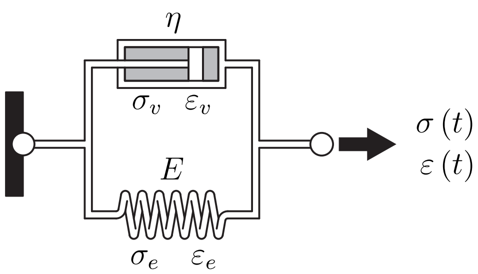


Figure 1.4: Illustration of the Kelvin-Voigt element.

The total stress and strain of the Kelvin-Voigt model can be evaluated as

$$\sigma(t) = \sigma_e(t) + \sigma_v(t), \qquad \varepsilon(t) = \varepsilon_e(t) = \varepsilon_v(t), \tag{1.13}$$

$$\dot{\sigma}(t) = \dot{\sigma}_e(t) + \dot{\sigma}_v(t), \qquad \dot{\varepsilon}(t) = \dot{\varepsilon}_e(t) = \dot{\varepsilon}_v(t). \tag{1.14}$$

Combining the expressions, we can formulate the definition for the total stress as

$$\sigma(t) = E\varepsilon_e(t) + \eta\dot{\varepsilon}_v(t) = E\varepsilon(t) + \eta\dot{\varepsilon}(t), \tag{1.15}$$

$$\sigma = E\varepsilon + \eta\dot{\varepsilon}, \qquad \dot{\varepsilon} = \frac{1}{\eta}\sigma - \frac{E}{\eta}\varepsilon. \tag{1.16}$$

1.2.5 Standard Linear Solid

A three-parameter model constructed from one spring and one Maxwell unit in parallel, and known as the Standard Linear Solid (SLS) model, is shown in Figure 1.5.

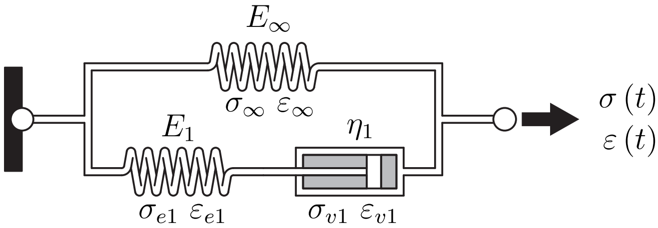


Figure 1.5: Illustration of the Standard Linear Solid model.

The relations between the stress and strain components are

$$\sigma_{ve1} = \sigma_{e1} = \sigma_{v1}, \qquad \sigma = \sigma_\infty + \sigma_{ve1} = \sigma_\infty + \sigma_{e1} = \sigma_\infty + \sigma_{v1}, \tag{1.17}$$

$$\varepsilon_{ve1} = \varepsilon_{e1} + \varepsilon_{v1}, \qquad \varepsilon = \varepsilon_\infty = \varepsilon_{ve1}. \tag{1.18}$$

The constitutive relations for the components:

$$\sigma_\infty = E_\infty\varepsilon_\infty = E_\infty\varepsilon, \qquad \dot{\sigma}_\infty = E_\infty\dot{\varepsilon}, \tag{1.19}$$

$$\sigma_{e1} = E_1\varepsilon_{e1}, \qquad \dot{\sigma}_{e1} = E_1\dot{\varepsilon}_{e1}, \tag{1.20}$$

$$\sigma_{v1} = \eta_1\dot{\varepsilon}_{v1}. \tag{1.21}$$

By combining the expression above, we arrive at the following constitutive relation connecting the total stress and total strain:

$$\dot{\varepsilon}_{ve1} = \dot{\varepsilon}_{e1} + \dot{\varepsilon}_{v1} = \frac{\dot{\sigma}_{ve1}}{E_1} + \frac{\sigma_{ve1}}{\eta_1} = \frac{\dot{\sigma} - \dot{\sigma}_\infty}{E_1} + \frac{\sigma - \sigma_\infty}{\eta_1} = \frac{\dot{\sigma} - E_\infty \dot{\varepsilon}}{E_1} + \frac{\sigma - E_\infty \varepsilon_\infty}{\eta_1}, \tag{1.22}$$

$$= \frac{\dot{\sigma} - E_\infty \dot{\varepsilon}}{E_1} + \frac{\sigma - E_\infty \varepsilon}{\eta_1} = \frac{\dot{\sigma}}{E_1} - \frac{E_\infty}{E_1}\dot{\varepsilon} + \frac{\sigma}{\eta_1} - \frac{E_\infty}{\eta_1}\varepsilon \tag{1.23}$$

$$= \frac{\dot{\sigma}}{E_1} - \frac{E_\infty}{E_1}\dot{\varepsilon} + \frac{\sigma}{\eta_1} - \frac{E_\infty}{\eta_1}\varepsilon, \tag{1.24}$$

$$\dot{\varepsilon} + \frac{E_\infty}{E_1}\dot{\varepsilon} + \frac{E_\infty}{\eta_1}\varepsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta_1}, \tag{1.25}$$

$$\left(1 + \frac{E_\infty}{E_1}\right)\dot{\varepsilon} + \frac{E_\infty}{\eta_1}\varepsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta_1}, \tag{1.26}$$

$$\left(\frac{E_1 + E_\infty}{E_1}\right)\dot{\varepsilon} + \frac{E_\infty}{\eta_1}\varepsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta_1}, \tag{1.27}$$

$$\frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta_1} = \left(\frac{E_1 + E_\infty}{E_1}\right)\dot{\varepsilon} + \frac{E_\infty}{\eta_1}\varepsilon, \tag{1.28}$$

$$\frac{\eta_1}{E_1}\dot{\sigma} + \sigma = \frac{\eta_1}{E_1}\left(E_1 + E_\infty\right)\dot{\varepsilon} + E_\infty \varepsilon. \tag{1.29}$$

By introducing the relaxation time, $\tau_1 = \eta_1/E_1$, and the so-called instantaneous modulus, $E_0 = E_1 + E_\infty$, we can convert the last expression to

$$\tau_1 \dot{\sigma} + \sigma = \tau_1 E_0 \dot{\varepsilon} + E_\infty \varepsilon. \tag{1.30}$$

Note that the unit of the relaxation time is [s].

1.2.6 Stress solution for strain loading with constant strain-rate

Consider a stress- and strain-free configuration for a one dimensional model. Consequently $\varepsilon = 0$ and $\sigma = 0$ Pa at $t = 0$. The loading is prescribed by an applied constant strain-rate as

$$\varepsilon\left(t\right) = \dot{\varepsilon} \cdot t. \tag{1.31}$$

The linear strain paths are illustrated in Figure 1.6 for various values of the applied strain rate.

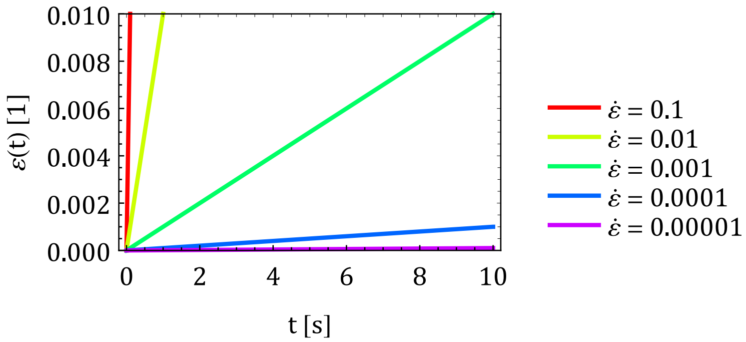


Figure 1.6: Applied linear strain paths for various values of $\dot{\varepsilon}$.

The stress solutions of the particular models can be obtained by solving the corresponding differential equations. The stress solutions are summarized in Table 1.1.

Table 1.1: Stress solutions of the basic models for strain loading with constant strain rate.

Model	Stress solution
Linear elastic element	$\sigma(t) = E\dot{\varepsilon}t = E\varepsilon$
Linear viscous element	$\sigma(t) = \eta\dot{\varepsilon}$
Maxwell element	$\sigma(t) = \eta\dot{\varepsilon}\left(1 - e^{-\frac{E}{\eta}t}\right) = \eta\dot{\varepsilon}\left(1 - e^{-\frac{t}{\tau}}\right)$
Kelvin-Voigt element	$\sigma(t) = E\dot{\varepsilon}t + \eta\dot{\varepsilon} = \left(\frac{t}{\tau} + 1\right)\eta\dot{\varepsilon} = (Et + \eta)\dot{\varepsilon}$
Standard Linear Solid model	$\sigma(t) = E_\infty\dot{\varepsilon}t + \eta_1\dot{\varepsilon}\left(1 - e^{-\frac{E_1}{\eta_1}t}\right) = E_\infty\dot{\varepsilon}t + \eta_1\dot{\varepsilon}\left(1 - e^{-\frac{t}{\tau_1}}\right)$

Note that the stress solution for the Kelvin-Voigt model is the sum of the solutions corresponding to the linear elastic and viscous elements. The stress solutions are plotted for each model using various set for the parameters in Figure 1.7-1.15.

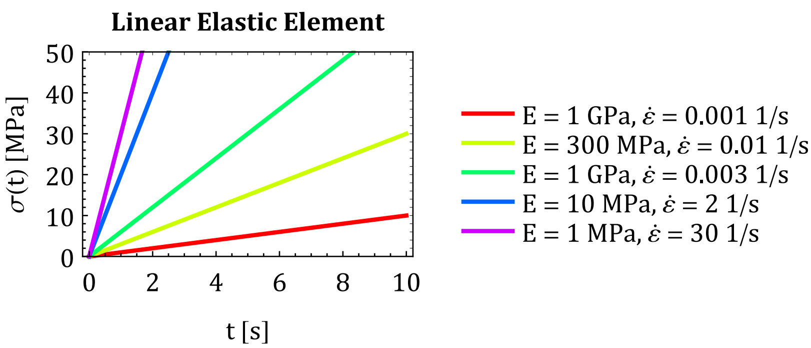


Figure 1.7: Stress solutions for different parameters of the linear elastic element.

The solution for the elastic element can be visualized in a different way by plotting the dimensionless ratio $\sigma(t)/E$ versus the applied strain $\varepsilon(t)$ as shown in Figure 1.8.

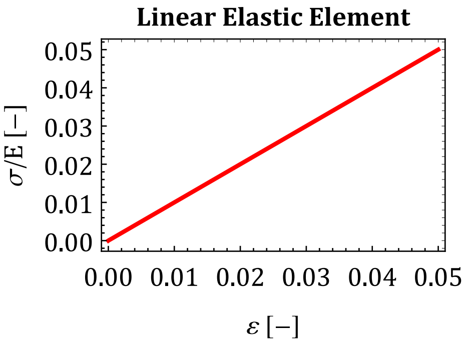


Figure 1.8: Stress solution of the linear elastic element.

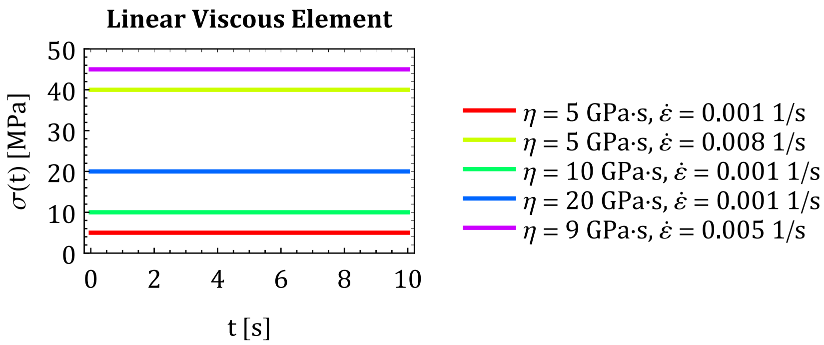


Figure 1.9: Stress solutions for different parameters of the linear viscous element.

The solution for the linear viscous element can be visualized in a different way by plotting the dimensionless ratio $\sigma\left(t\right) / \left(\eta\dot{\varepsilon}\right)$ versus the applied strain $\varepsilon\left(t\right)$ as shown in Figure 1.10.

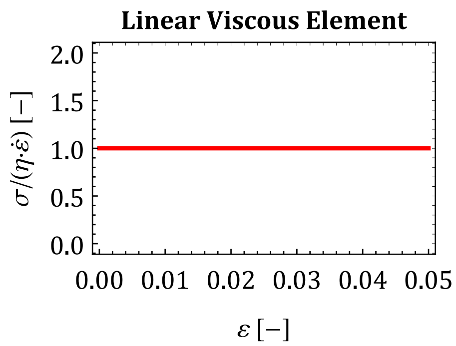


Figure 1.10: Stress solution of the linear viscous element.

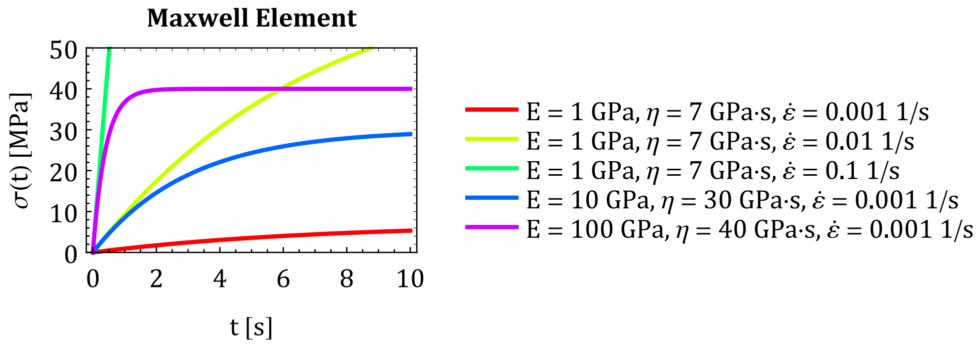


Figure 1.11: Stress solutions for different parameters of the Maxwell element.

The solution for the Maxwell element can be visualized in a different way by plotting the dimensionless ratio $\sigma(t) / (\eta\dot{\varepsilon})$ versus time t for different values of the relaxation time τ as shown in Figure 1.12.

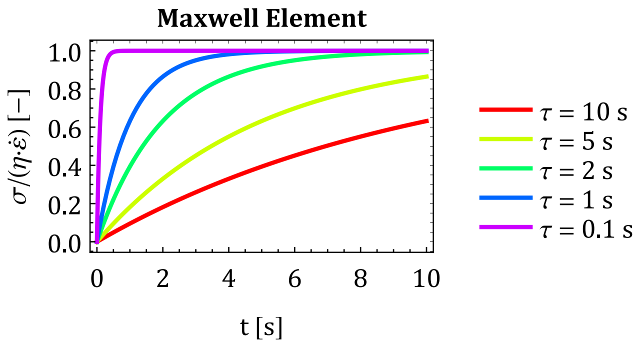


Figure 1.12: Stress solutions for different parameters of the Maxwell element.

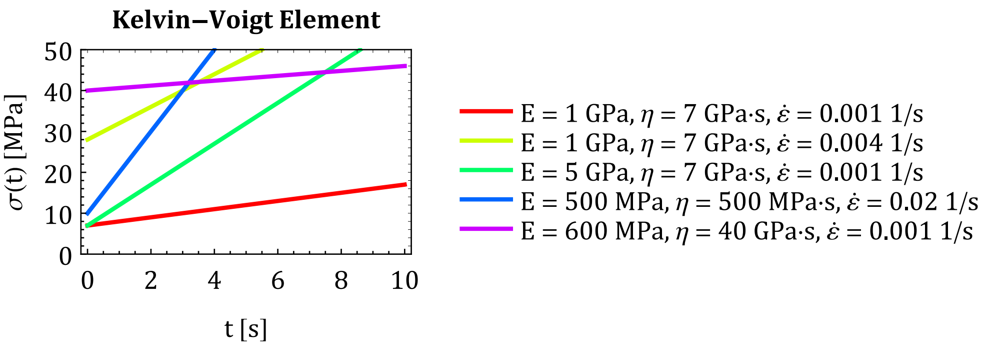


Figure 1.13: Stress solutions for different parameters of the Kelvin-Voigt element.

The solution for the Kelvin-Voigt element can be visualized in a different way by plotting the dimensionless ratio $\sigma(t) / (\eta\dot{\varepsilon})$ versus time t for different values of the relaxation time τ as shown in Figure 1.14.

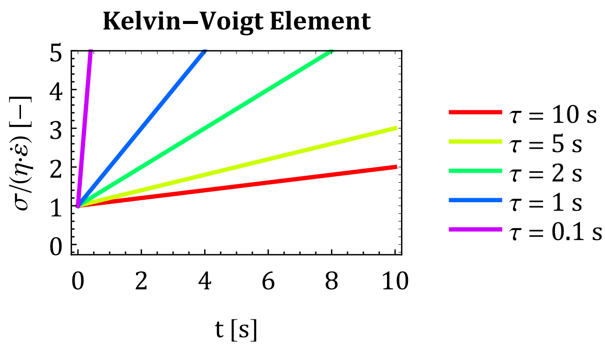


Figure 1.14: Stress solutions for different parameters of the Kelvin-Voigt element.

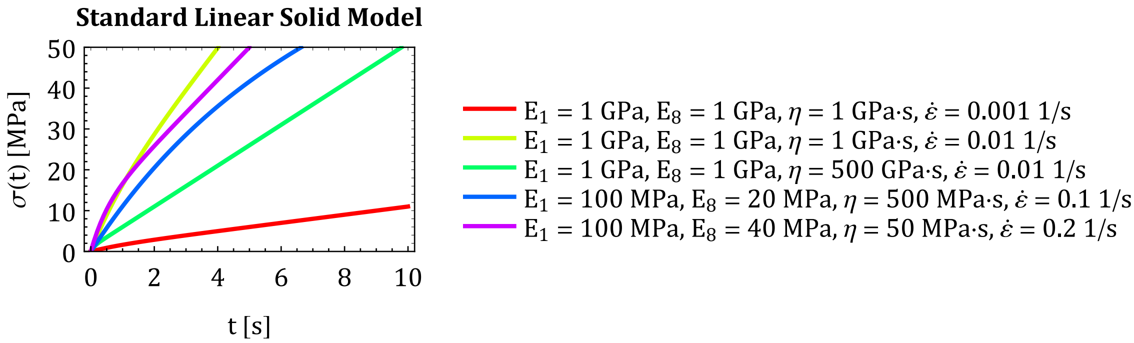


Figure 1.15: Stress solutions for different parameters of the Standard Linear Solid model.

The solution for the Standard Linear Solid model can be visualized in a different way by plotting the dimensionless ratio $\sigma(t) / (\eta_1 \dot{\varepsilon})$ versus time t for different values of the relaxation times τ_1 and η_1 / E_∞ as shown in Figure 1.16.

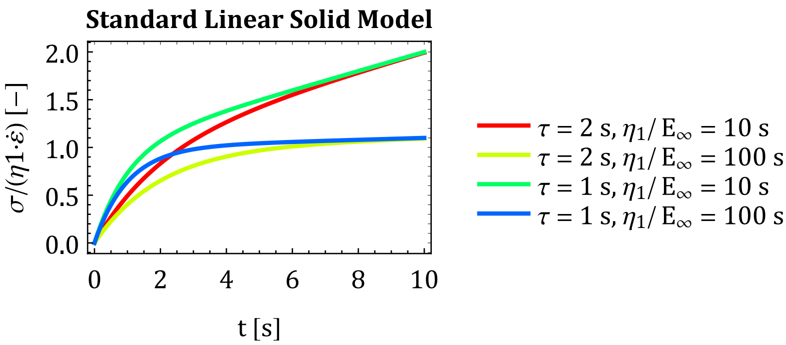


Figure 1.16: Stress solutions for different parameters of the Standard Linear Solid model.

The stress solutions plotted above illustrate the nature of the solutions of the particular models. Figure 1.17 shows one particular solution for each model for comparison purposes. The strain rate is identical for each model: $\dot{\varepsilon} = 0.001\ 1/\mathrm{s}$. The following abbreviations are used: LE: linear elastic element; LV: linear viscous element; MA: Maxwell element; KV: Kelvin-Voigt element; SLS: Standard Linear Solid Model.

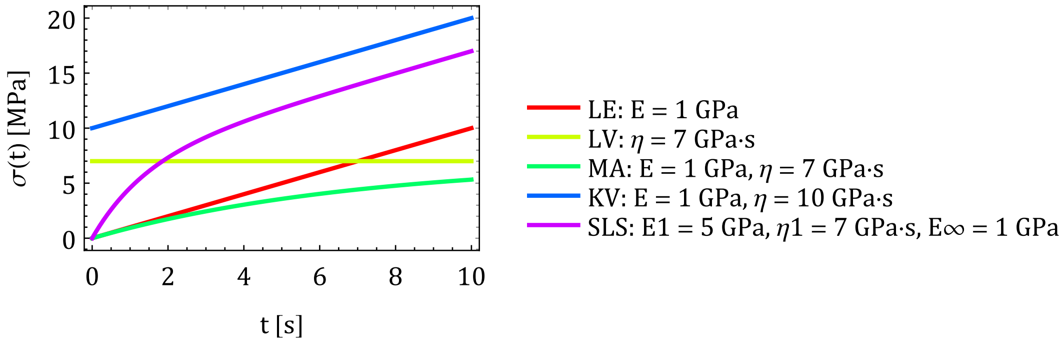


Figure 1.17: Comparison of the nature of the stress solutions.

1.2.7 Strain solution for stress loading with constant stress-rate

Consider a stress and strain-free configuration for a one dimensional model. Consequently $\varepsilon = 0$ and $\sigma = 0$ Pa at $t = 0$. The loading is prescribed with an applied constant stress-rate as

$$\sigma(t) = \dot{\sigma} \cdot t. \tag{1.32}$$

The linear stress paths are illustrated in Figure 1.18 for various values of the applied stress rate.

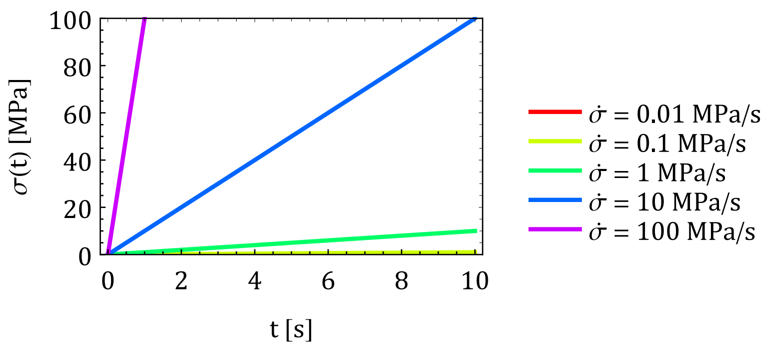


Figure 1.18: Applied linear stress paths for various values of $\dot{\sigma}$.

The strain solutions of the particular models can be obtained by solving the corresponding differential equations. The strain solutions are summarized in Table 1.2.

Table 1.2: Stress solutions of the basic models for strain loading with constant strain rate.

Model	Strain solution
Linear elastic element	$\varepsilon(t) = \frac{1}{E}\dot{\sigma}t = \frac{1}{E}\sigma$
Linear viscous element	$\varepsilon(t) = \frac{1}{2\eta}\dot{\sigma}t^2 = \frac{\sigma t}{2\eta}$
Maxwell element	$\varepsilon(t) = \dot{\sigma}t\left(\frac{1}{E} + \frac{t}{2\eta}\right) = \frac{\sigma}{E}\left(1 + \frac{E}{\eta}\frac{t}{2}\right) = \frac{\sigma}{E}\left(1 + \frac{t}{2\tau}\right)$
Kelvin-Voigt element	$\varepsilon(t) = \frac{\dot{\sigma}}{E}\left(t - \frac{\eta}{E}\left(1 - e^{-\frac{E}{\eta}t}\right)\right) = \frac{\dot{\sigma}}{E}\left(t - \tau\left(1 - e^{-t/\tau}\right)\right)$
Standard Linear Solid model	$\varepsilon(t) = \frac{\dot{\sigma}}{E_\infty}\left(t - \frac{\eta_1}{E_\infty}\left(1 - e^{-\frac{E_1 E_\infty}{(E_1+E_\infty)\eta}t}\right)\right)$

Note that the stress solution for the Maxwell model is the sum of the solutions corresponding to the linear elastic and viscous elements. The strain solutions are plotted for each model using various set for the parameters in Figure 1.19-1.27.

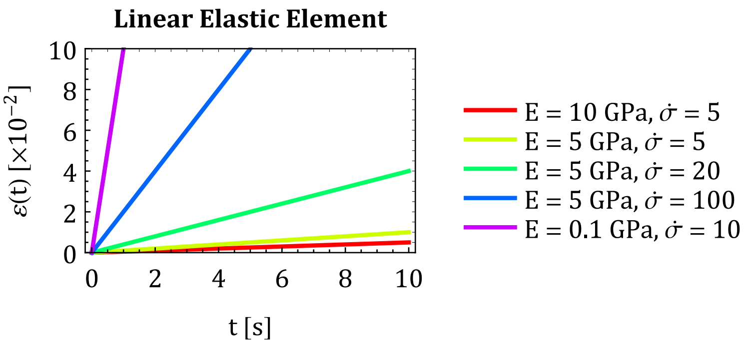


Figure 1.19: Strain solutions for different parameters of the linear elastic element.

The solution for the elastic element can be visualized in a different way by plotting $E\varepsilon$ versus the applied stress $\sigma(t)$ as shown in Figure 1.20.

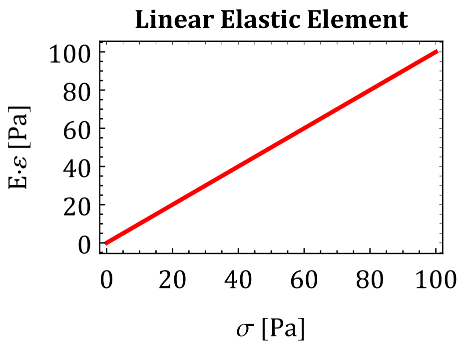


Figure 1.20: Strain solution the linear elastic element.

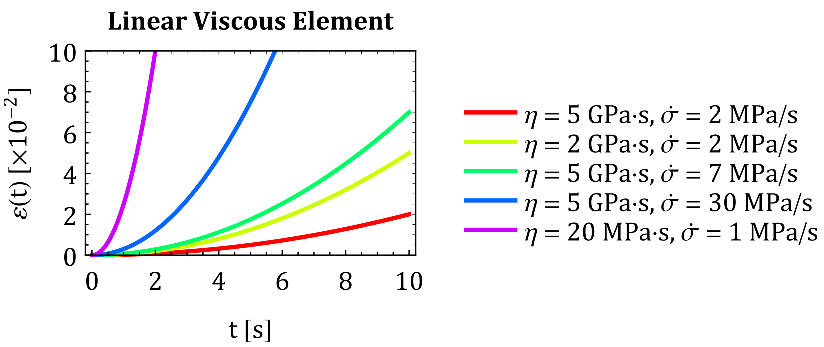


Figure 1.21: Strain solutions for different parameters of the linear viscous element.

The solution for the linear viscous element can be visualized in a different way by plotting $\eta\varepsilon/\dot{\sigma}$ versus time t as shown in Figure 1.22.

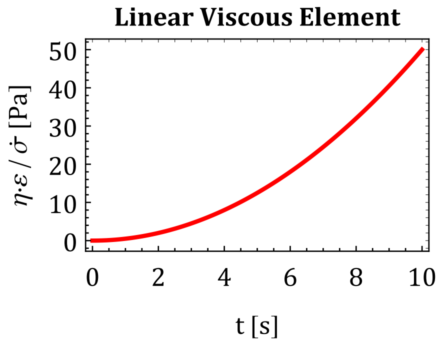


Figure 1.22: Strain solutions for the linear viscous element.

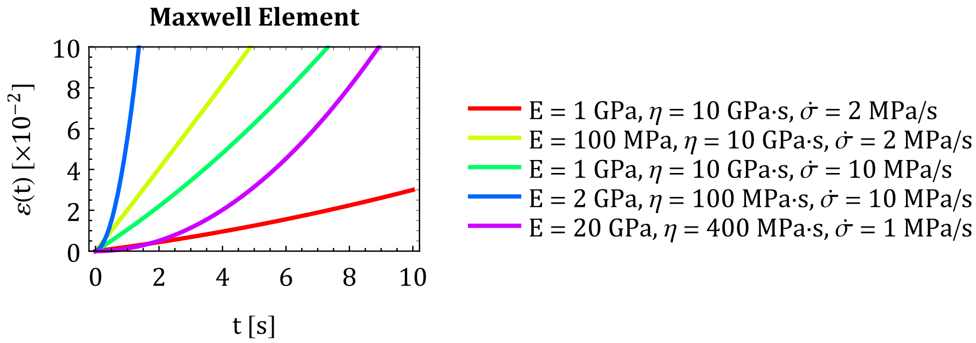


Figure 1.23: Strain solutions for different parameters of the Maxwell element.

The solution for the Maxwell element can be visualized in a different way by plotting $E\varepsilon/\sigma$ versus time t as shown in Figure 1.24.

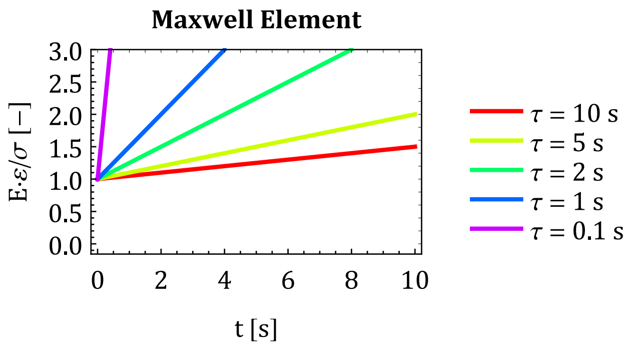


Figure 1.24: Strain solutions for different parameters of the Maxwell element.

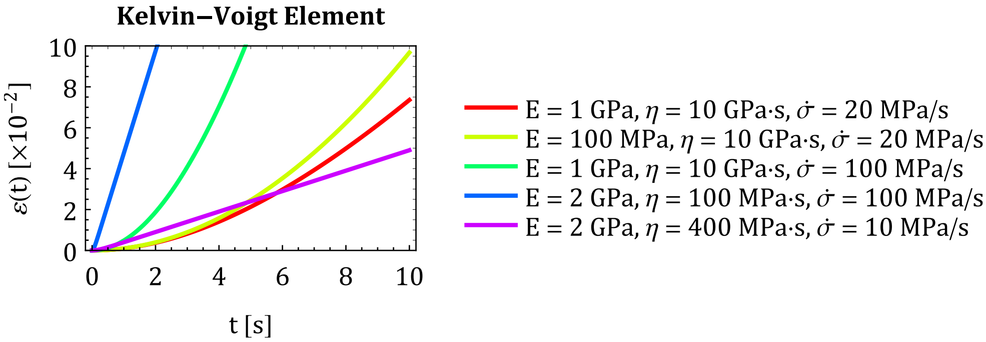


Figure 1.25: Strain solutions for different parameters of the Kelvin-Voigt element.

The solution for the Kelvin-Voigt element can be visualized in a different way by plotting $E\varepsilon/\dot{\sigma}$ versus time t as shown in Figure 1.26.

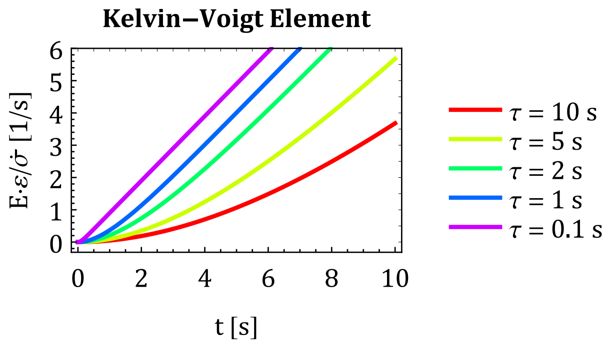


Figure 1.26: Strain solutions for different parameters of the Kelvin-Voigt element.

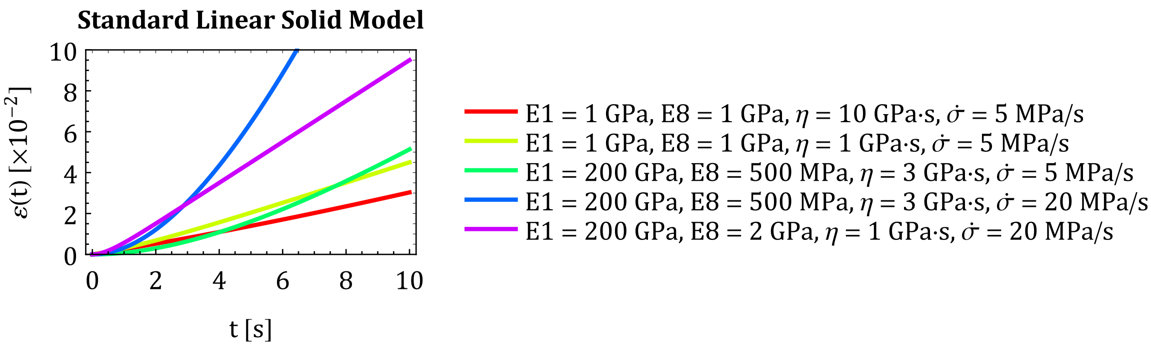


Figure 1.27: Strain solutions for different parameters of the Standard Linear Solid model.

The solution for the Standard Linear Solid model can be visualized in a different way by plotting $E_\infty\varepsilon/\dot{\sigma}$ versus time t as shown in Figure 1.26.

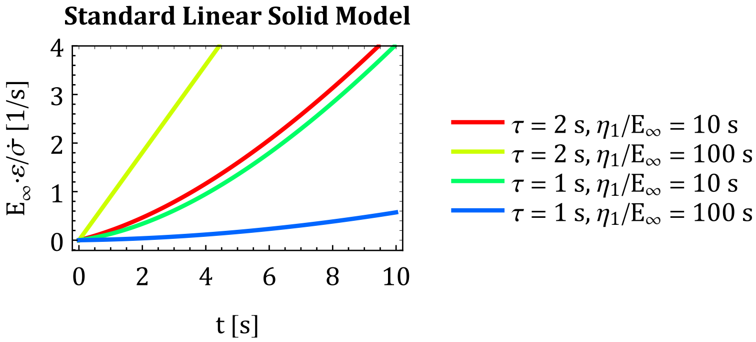


Figure 1.28: Strain solutions for different parameters of the Standard Linear Solid model.

The strain solutions plotted above illustrate the nature of the solutions of the particular models. Figure 1.29 shows one particular solution for each model for comparison purposes. The stress rate is identical for each model: $\dot{\sigma} = 10$ MPa/s. The following abbreviations are used: LE: linear elastic element; LV: linear viscous element; MA: Maxwell element; KV: Kelvin-Voigt element; SLS: Standard Linear Solid Model.

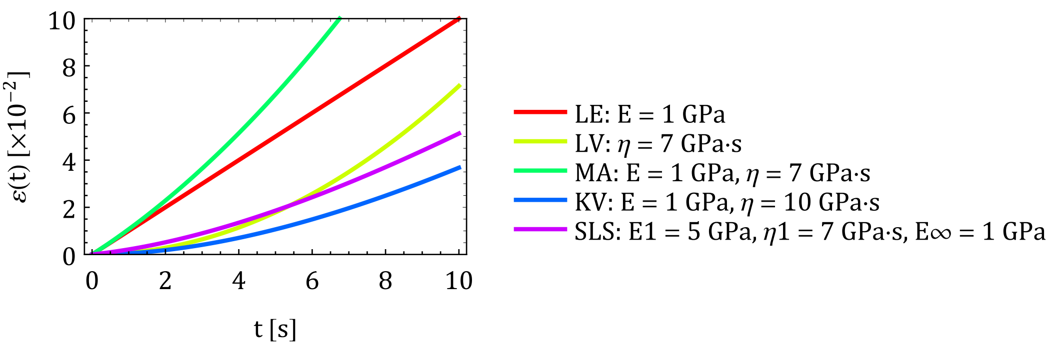


Figure 1.29: Comparison of the nature of the strain solutions.

1.2.8 Stress solution for idealized relaxation test

In the idealized stress relaxation test, an instantaneous strain ε of magnitude ε_0 is imposed on the material and maintained at that value while the resulting stress $\sigma(t)$ is recorded as a function of time t. We introduce the unit step function:

$$U(t - t_1) = \begin{cases} 1 & t > t_1 \\ 0 & t \leq t_1 \end{cases} . \tag{1.33}$$

The function is illustrated in Figure 1.30 for $t_1 = 2$.

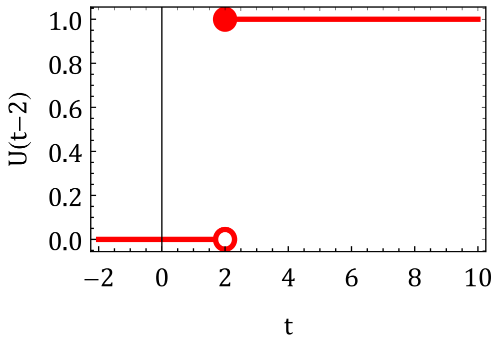


Figure 1.30: Illustration of the unit step function $U\,(t-2)$.

The derivative of $U\,(t-t_1)$ is the delta function, which is defined by the relations

$$\delta\,(t-t_1) = \frac{\mathrm{d}U\,(t-t_1)}{\mathrm{d}t}, \tag{1.34}$$

$$\delta\,(t-t_1) = 0, \qquad t \neq t_1 \qquad \text{and} \qquad \int_{t_1^-}^{t_1^+} \delta\,(t-t_1)\,\mathrm{d}t = 1. \tag{1.35}$$

The strain loading for the idealized relaxation test is

$$\varepsilon\,(t) = \varepsilon_0 U\,(t) \qquad \rightarrow \qquad \dot{\varepsilon}\,(t) = \varepsilon_0 \delta\,(t)\,. \tag{1.36}$$

The resulting stress solutions of the basic elements are listed in Table 1.3.

Table 1.3: Stress solutions of the basic models for the idealized relaxation test.

Model	Stress solution
Linear elastic element	$\sigma\,(t) = E\varepsilon_0 U\,(t)$
Linear viscous element	$\sigma\,(t) = \eta\varepsilon_0\delta\,(t)$
Maxwell element	$\sigma\,(t) = E\varepsilon_0 U\,(t)\cdot e^{-t/\tau}$
Kelvin-Voigt element	$\sigma\,(t) = E\varepsilon_0\,(U\,(t) + \tau\delta\,(t))$
Standard Linear Solid model	$\sigma\,(t) = \varepsilon_0 U\,(t)\left(E_\infty + E_1\cdot e^{-t/\tau_1}\right)$

The stress solution for the linear viscous element indicates that infinite stress is required to impose the prescribed strain ε_0. In other words, the viscous element behaves as a rigid link for instantaneous strain jumps. This conclusion is valid for the Kelvin-Voigt element too.
The initial slope of the stress solution for the Maxwell element is $-E\varepsilon_0/\tau$. Since the starting stress value is $E\varepsilon_0$ it follows that if the stress would decrease with the same rate then it would become zero at $t = \tau$.
The stress solutions for the basic models are illustrated in Figure 1.31-1.35.

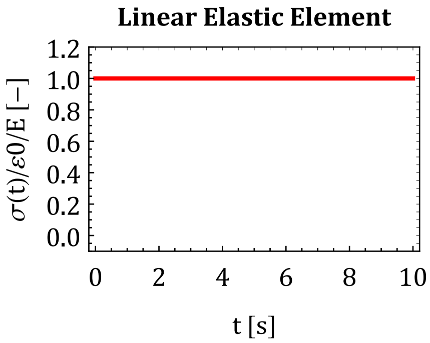


Figure 1.31: Stress solution for the linear elastic element in the idealized relaxation test.

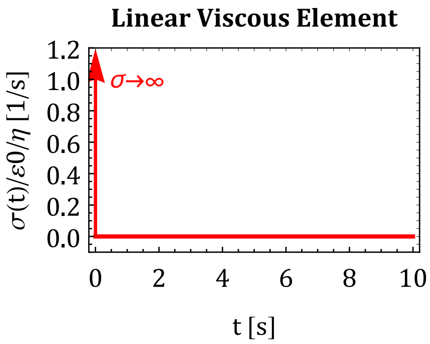


Figure 1.32: Stress solution for the linear viscous element in the idealized relaxation test.

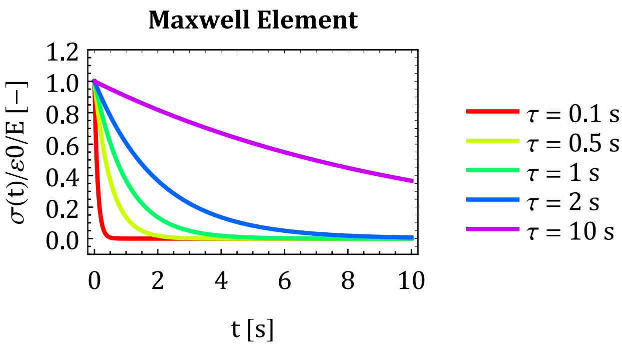


Figure 1.33: Stress solution for the Maxwell element in the idealized relaxation test.

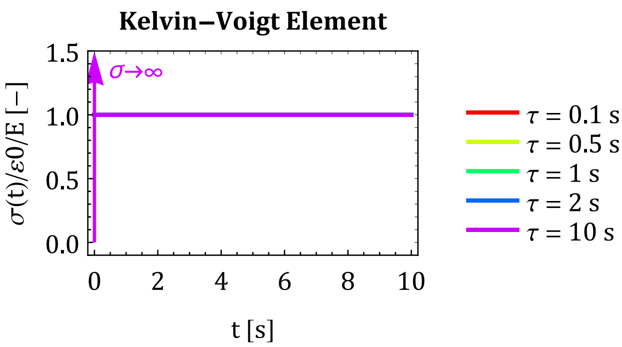


Figure 1.34: Stress solution for the Kelvin-Voigt element in the idealized relaxation test.

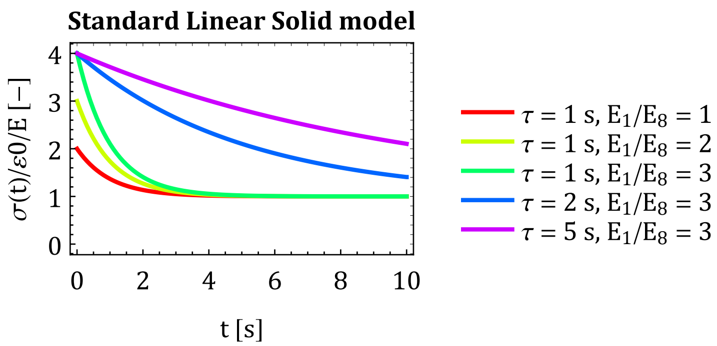


Figure 1.35: Stress solution for the Standard Linear Solid Model in the idealized relaxation test.

We can introduce the relaxation function $E(t)$ as

$$E(t) = \frac{\sigma(t)}{\varepsilon_0 U(t)}. \tag{1.37}$$

The relaxation function for the standard linear solid is

$$E(t) = E_\infty + E_1 \cdot e^{-t/\tau_1}. \tag{1.38}$$

Its particular value at $t = 0$ is the so-called instantaneous modulus $E_0 = E_\infty + E_1$. The asymptotic value of the relaxation modulus is termed as long-term elastic modulus:

$$E_\infty = \lim_{t\to\infty} E(t). \tag{1.39}$$

The relaxation function $E(t)$ is illustrated in Figure 1.36 for the following (dimensionless) parameters:

$$E_\infty = 1, \qquad E_1 = 3, \qquad \tau_1 = 2. \tag{1.40}$$

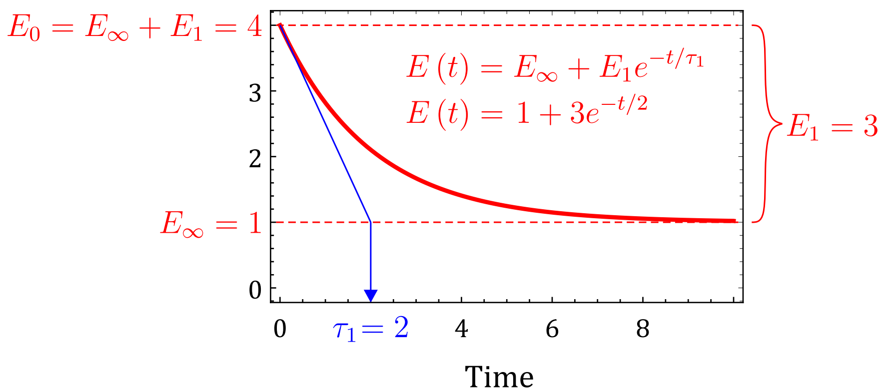


Figure 1.36: Illustration of the relaxation function of the Standard Linear Solid model.

1.2.9 Strain solution for the idealized creep test

In the idealized creep test, the material is instantaneously subjected to a stress of magnitude σ_0 and this stress value is kept constant. The resulting strain $\varepsilon(t)$, is called the creep strain. Thus, the stress loading can be specified as

$$\sigma(t) = \sigma_0 U(t) \qquad \rightarrow \qquad \dot{\sigma}(t) = \sigma_0 \delta(t). \tag{1.41}$$

The resulting strain solutions of the basic elements are listed in Table 1.4.

Table 1.4: Strain solutions of the basic models for the idealized creep test.

Model	Stress solution
Linear elastic element	$\varepsilon(t) = \sigma_0 U(t)/E$
Linear viscous element	$\varepsilon(t) = \sigma_0 t/\eta$
Maxwell element	$\varepsilon(t) = \sigma_0 U(t)\left(\frac{1}{E} + \frac{t}{\eta}\right) = \sigma_0 U(t)\frac{1}{E}\left(1 + \frac{t}{\tau}\right)$
Kelvin-Voigt element	$\varepsilon(t) = \sigma_0 \frac{1}{E}\left(1 - e^{-t/\tau}\right)$
Standard Linear Solid model	$\varepsilon(t) = \sigma_0 \frac{1}{E_\infty}\left(1 - \frac{E_1}{E_1+E_\infty} e^{-\frac{E_\infty}{(E_\infty+E_1)}\cdot\frac{t}{\tau_1}}\right)$

The strain solutions are plotted in Figure 1.37-1.41.

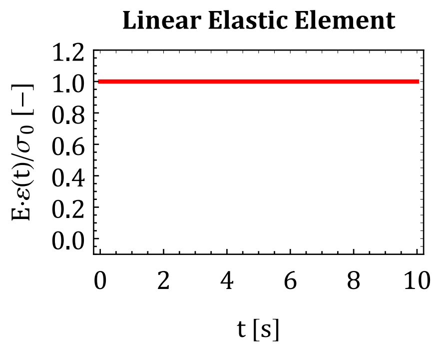


Figure 1.37: Strain solution for the linear elastic element in the idealized creep test.

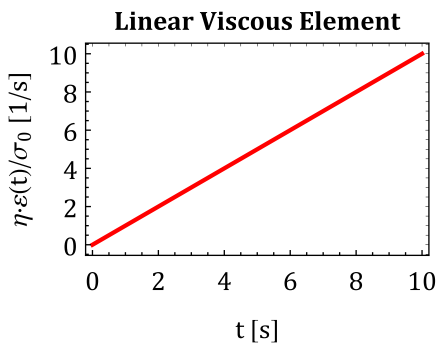


Figure 1.38: Strain solution for the linear viscous element in the idealized creep test.

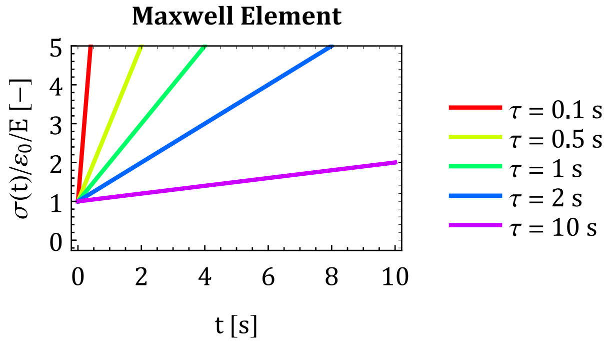


Figure 1.39: Strain solution for the Maxwell element in the idealized creep test.

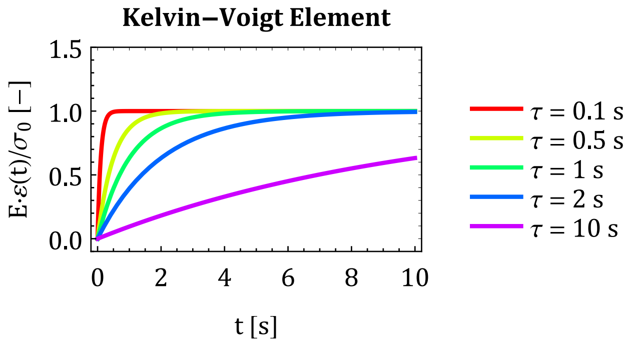


Figure 1.40: Strain solution for the Kelvin-Voigt element in the idealized creep test.

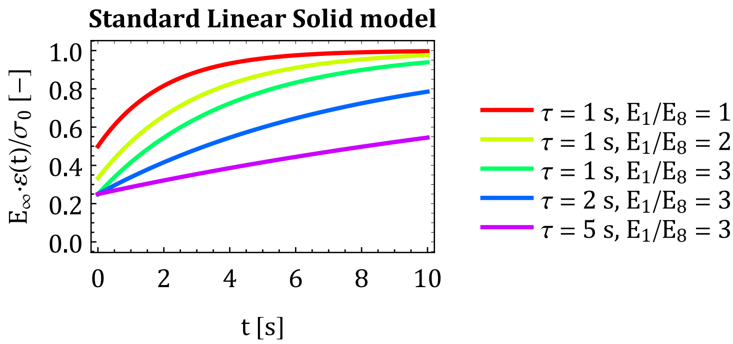


Figure 1.41: Strain solution for the Standard Linear Solid Model in the idealized creep test.

The creep function in general is defined as

$$J\left(t\right) = \frac{\varepsilon\left(t\right)}{\sigma_0 U\left(t\right)}. \tag{1.42}$$

The creep function of the *Standard Linear Solid* model is given by

$$J\left(t\right) = \frac{1}{E_\infty} - \frac{E_1}{E_\infty\left(E_1 + E_\infty\right)} e^{-\frac{E_\infty}{\left(E_\infty + E_1\right)} \cdot \frac{t}{\tau_1}} = \frac{1}{E_\infty} - \frac{E_1}{E_\infty\left(E_1 + E_\infty\right)} e^{-\frac{t}{\tilde{\tau}_1}}, \tag{1.43}$$

where

$$\tilde{\tau}_1 = \frac{E_1 + E_\infty}{E_\infty}\tau_1. \tag{1.44}$$

The creep function $J(t)$ for the Standard Linear Solid model is illustrated in Figure 1.42 for the following (dimensionless) parameters:

$$E_\infty = 0.5, \qquad E_1 = 1.5, \qquad \tau_1 = 2. \tag{1.45}$$

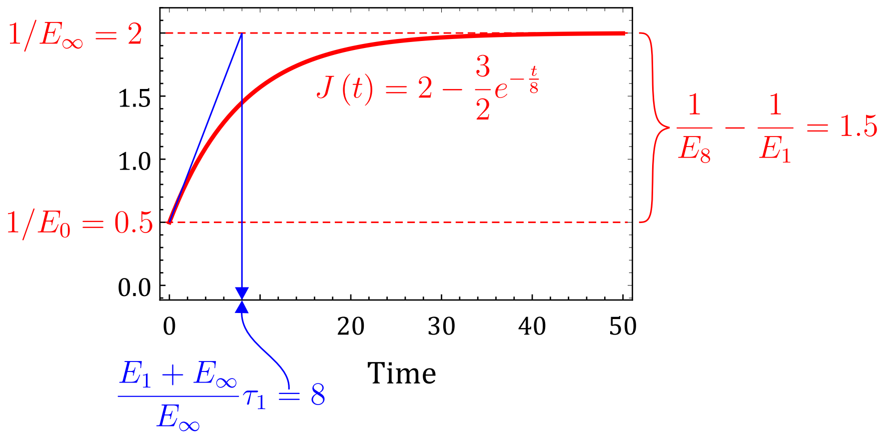


Figure 1.42: Illustration of the creep function of the Standard Linear Solid model.

1.2.10 Stress solution of the SLS model for ramp loading

The instantaneous application of a given strain loading cannot be realized in reality. Therefore the idealized relaxation experiment cannot be performed in a laboratory test. Based on the characteristics of the testing machine, a certain time (t_0) is needed to reach the desired strain (ε_0) value.

The strain history for a relaxation test in reality is given by the ramp loading shown in Figure 1.43, where $t_0 = 2$ s and $\varepsilon_0 = 5\%$.

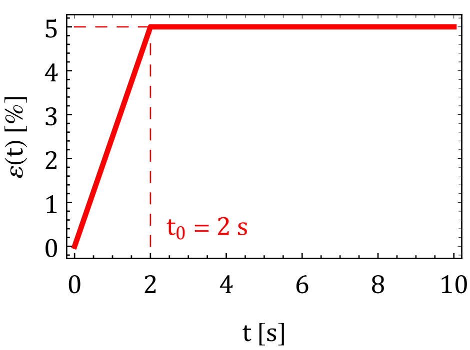


Figure 1.43: Illustration of the ramp loading in a relaxation test.

Note that the strain rate $\dot{\varepsilon} = \varepsilon_0/t_0$ is constant for domain $t = 0 ... t_0$. Consequently, the solution in Table 1.1 can be used to determine the stress values in this region:

$$\sigma\left(t\right)=\frac{\varepsilon_0}{t_0}\left(E_\infty t+\eta_1\left(1-e^{-\frac{t}{\tau_1}}\right)\right)\qquad\text{for}\qquad 0\leq t\leq t_0. \tag{1.46}$$

The stress value at t_0 is

$$\sigma_0=\sigma\left(t_0\right)=\frac{\varepsilon_0}{t_0}\left(E_\infty t_0+\eta_1\left(1-e^{-\frac{t_0}{\tau_1}}\right)\right)=\varepsilon_0\left(E_\infty t_0+E_1\frac{\tau_1}{t_0}\left(1-e^{-\frac{t_0}{\tau_1}}\right)\right). \tag{1.47}$$

The strain rate for $t>t_0$ is 0. Consequently, the stress solution for the relaxation part can be obtained by solving the differential equation (1.29) with $\dot{\varepsilon}=0$ and with the initial condition $\sigma\left(t_0\right)=\sigma_0$:

$$\frac{\eta_1}{E_1}\dot{\sigma}+\sigma=E_\infty\varepsilon_0\qquad\text{with}\qquad\sigma\left(t_0\right)=\sigma_0. \tag{1.48}$$

The solution is expressed as

$$\sigma\left(t\right)=\varepsilon_0\left(E_\infty-\frac{\eta_1}{t_0}e^{-\frac{E_1 t}{\eta_1}}\left(1-e^{-\frac{E_1 t_0}{\eta_1}}\right)\right)=\varepsilon_0\left(E_\infty-\frac{\eta_1}{t_0}e^{-\frac{t}{\tau_1}}\left(1-e^{-\frac{t_0}{\tau_1}}\right)\right). \tag{1.49}$$

$$\sigma\left(t\right)=\varepsilon_0\left(E_\infty-E_1\frac{\tau_1}{t_0}e^{-\frac{t}{\tau_1}}\left(1-e^{-\frac{t_0}{\tau_1}}\right)\right). \tag{1.50}$$

Therefore:

$$\sigma\left(t\right)=\left\{\begin{array}{ll}\varepsilon_0\left(E_\infty t_0+E_1\frac{\tau_1}{t_0}\left(1-e^{-\frac{t_0}{\tau_1}}\right)\right) & 0\leq t\leq t_0\\ \varepsilon_0\left(E_\infty-E_1\frac{\tau_1}{t_0}e^{-\frac{t}{\tau_1}}\left(1-e^{-\frac{t_0}{\tau_1}}\right)\right) & t_0<t\end{array}\right.. \tag{1.51}$$

The stress solution for the ramp loading is illustrated in Figure 1.44 for different values of parameter t_0. The material parameters are: $E_1=1$, $E_\infty=0.5$, $\eta_1=1$ ($\tau_1=1$). The applied strain is $\varepsilon_0=0.05$. The black dashed curve shows the idealized stress solution corresponding to the instantaneous application of strain ε_0. The gray dashed curve represents the limit curve of the peak stress value corresponding to $t=t_0$.

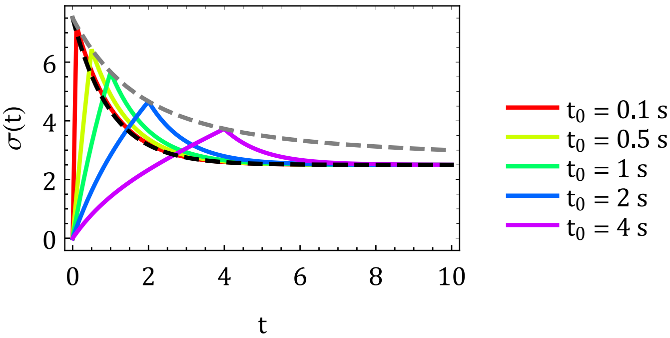


Figure 1.44: Illustration of the stress solution in a ramp loading for the Standard Linear Solid model.

Observe the following important phenomenon: the stress value at the beginning of the relaxation part of a ramp loading can be significantly higher than the stress value at the same time instant t resulting from the solution corresponding to the idealized relaxation test. The ratio of the two limit curves (gray and black) is shown in Figure 1.45 and given by the following expression:

$$\frac{e^t\,(2+t)-2}{(2+e^t)\,t}. \tag{1.52}$$

The maximum value for the current parameter set is located at $t = 2.64324$, which value was obtained using a numerical scheme. The corresponding value for the ratio is 1.49075.

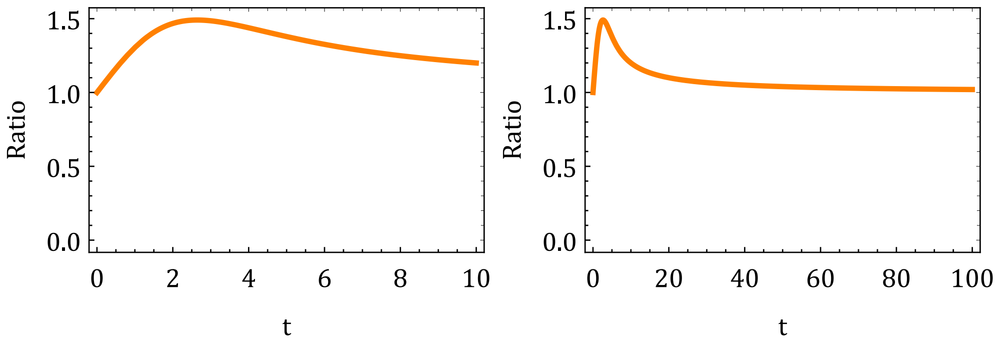


Figure 1.45: Illustration of the ratio of the limit curves.

The stress solution $\sigma(t)$ as the function of time t_0 is illustrated in Figure 1.46 using surface plots. The solutions for particular values of t_0 are highlighted.

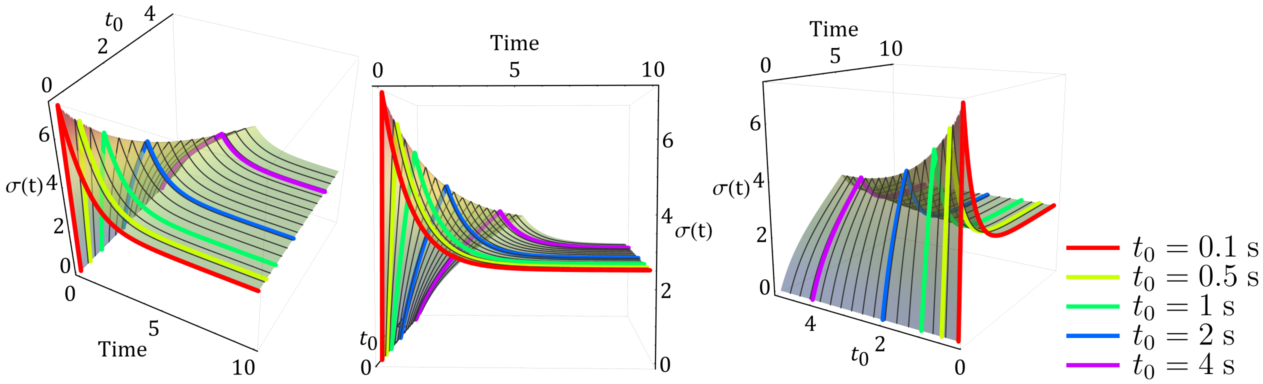


Figure 1.46: Illustration of the stress solution in a ramp loading for the Standard Linear Solid model using 3D surface plots.

If $t_0 \to 0$ then we arrive at the case, when the strain is suddenly applied. The solution for this case is obtained from (1.51) as

$$\sigma(t) = \varepsilon_0 \left(E_\infty + E_1 e^{-\frac{t}{\tau_1}}\right) = E(t)\,\varepsilon_0 \tag{1.53}$$

with

$$E(t) = E_\infty + E_1 e^{-\frac{t}{\tau_1}}. \tag{1.54}$$

1.2.11 Integral formulation for the Standard Linear Solid model

The stress solution can be expressed by the convolution integral

$$\sigma(t) = \int_0^t E(t-s)\,\frac{\mathrm{d}\varepsilon(s)}{\mathrm{d}s}\mathrm{d}s, \tag{1.55}$$

where the material is stress-free (and strain-free) at $t = 0$. The loading is prescribed by the applied strain history $\varepsilon(t)$. The relaxation function $E(t)$ is defined by

$$E(t) = E_\infty + E_1 \mathrm{exp}\left[-\frac{t}{\tau_1}\right] = E_0\left(e_\infty + e_1 \cdot \mathrm{exp}\left[-\frac{t}{\tau_1}\right]\right), \tag{1.56}$$

$$\dot{E}(t) = -E_0 \frac{e_1}{\tau_1} \mathrm{exp}\left[-\frac{t}{\tau_1}\right], \tag{1.57}$$

where $e_1 = E_1/E_0$ is a relative modulus. The instantaneous ($E_0 = E(0)$) and the long-term ($E_\infty = E(\infty)$) moduli are related as

$$E_0 = E_\infty + E_1 = \frac{E_\infty}{e_\infty}. \tag{1.58}$$

Therefore, it is evident that

$$e_\infty = 1 - e_1. \tag{1.59}$$

Another representation can be formulated by performing integration by parts. Equation (1.55) can be thus written in an alternative form as

$$\begin{aligned}
\sigma(t) &= \left[E(t-s)\,\varepsilon(s)\right]_0^t - \int_0^t \frac{\mathrm{d}E(t-s)}{\mathrm{d}s}\varepsilon(s)\,\mathrm{d}s && (1.60)\\
&= E(0)\,\varepsilon(t) - E(t)\,\varepsilon(0) - \int_0^t \frac{\mathrm{d}E(t-s)}{\mathrm{d}(t-s)}\frac{\mathrm{d}(t-s)}{\mathrm{d}s}\varepsilon(s)\,\mathrm{d}s && (1.61)\\
&= E(0)\,\varepsilon(t) + \int_0^t \frac{\mathrm{d}E(t-s)}{\mathrm{d}(t-s)}\varepsilon(s)\,\mathrm{d}s && (1.62)\\
&= E(0)\,\varepsilon(t) + \int_0^t \frac{\mathrm{d}E(s)}{\mathrm{d}(s)}\varepsilon(t-s)\,\mathrm{d}s && (1.63)\\
&= E_0\varepsilon(t) + \int_0^t \dot{E}(s)\,\varepsilon(t-s)\,\mathrm{d}s. && (1.64)
\end{aligned}$$

Denote $\sigma_0(t) = E_0\varepsilon(t)$ the instantaneous stress response. Then the stress solution becomes

$$\sigma(t) = \sigma_0(t) - \frac{e_1}{\tau_1}\int_0^t \sigma_0(t-s)\cdot\exp\left[-\frac{s}{\tau_1}\right]\mathrm{d}s. \tag{1.65}$$

We have two solutions: one for the region $t \le t_0$ and the other one for $t > t_0$. These regions are illustrated in Figure 1.47.

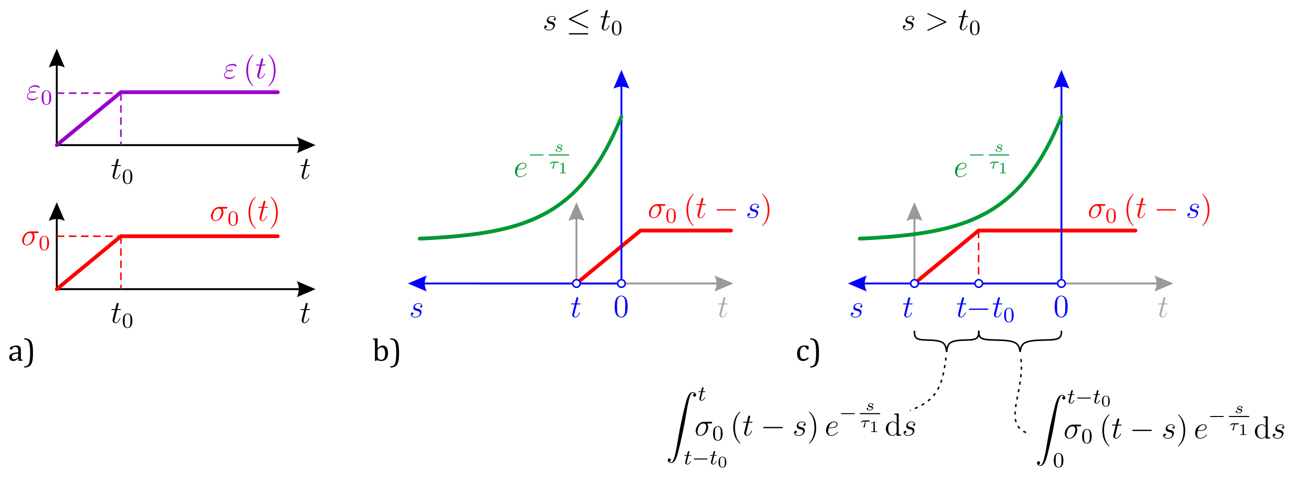


Figure 1.47: a) Strain loading and the $\sigma_0(t)$ stress response b) Illustration of the integral for $t \le t_0$ c) Illustration of the integral for $t > t_0$.

In the first domain ($t \le t_0$) the instantaneous stress solution is given by $\sigma_0(t) = E_0\varepsilon_0 t/t_0$. Therefore, the overall stress solution in this domain is given by:

$$\sigma(t) = E_0\frac{\varepsilon_0}{t_0}t - E_0\frac{\varepsilon_0}{t_0}\frac{e_1}{\tau_1}\int_0^t (t-s)\cdot\exp\left[-\frac{s}{\tau_1}\right]\mathrm{d}s, \tag{1.66}$$

$$\sigma(t) = \varepsilon_0 \left(E_\infty t_0 + E_1 \frac{\tau_1}{t_0} \left(1 - e^{-\frac{t_0}{\tau_1}} \right) \right), \tag{1.67}$$

which is identical to the solution (1.51).

In the second domain ($t > t_0$), the integral can be divided into two parts as follows:

$$\sigma(t) = \sigma_0(t) - \frac{e_1}{\tau_1} \int_0^{t-t_0} \sigma_0(t-s) \cdot \exp\left[-\frac{s}{\tau_1}\right] \mathrm{d}s - \frac{e_1}{\tau_1} \int_{t-t_0}^{t} \sigma_0(t-s) \cdot \exp\left[-\frac{s}{\tau_1}\right] \mathrm{d}s, \tag{1.68}$$

$$\sigma(t) = \sigma_0(t) - \frac{e_1}{\tau_1} \int_0^{t-t_0} (\varepsilon_0 E_0) \cdot \exp\left[-\frac{s}{\tau_1}\right] \mathrm{d}s - \frac{e_1}{\tau_1} \int_{t-t_0}^{t} \left(\varepsilon_0 E_0 \frac{t-s}{t_0} \right) \cdot \exp\left[-\frac{s}{\tau_1}\right] \mathrm{d}s, \tag{1.69}$$

$$\sigma(t) = \sigma_0(t) - \varepsilon_0 E_0 \frac{e_1}{\tau_1} \left(\int_0^{t-t_0} \exp\left[-\frac{s}{\tau_1}\right] \mathrm{d}s + \int_{t-t_0}^{t} \left(\frac{t-s}{t_0} \right) \cdot \exp\left[-\frac{s}{\tau_1}\right] \mathrm{d}s \right), \tag{1.70}$$

where

$$\int_0^{t-t_0} \exp\left[-\frac{s}{\tau_1}\right] \mathrm{d}s = \tau_1 \left(1 - \exp\left[-\frac{t-t_0}{\tau_1}\right] \right), \tag{1.71}$$

$$\int_{t-t_0}^{t} \left(\frac{t-s}{t_0} \right) \cdot \exp\left[-\frac{s}{\tau_1}\right] \mathrm{d}s = \frac{\tau_1}{t_0} \exp\left[-\frac{t}{\tau_1}\right] \left(\exp\left[\frac{t_0}{\tau_1}\right] (t_0 - \tau_1) + \tau_1 \right). \tag{1.72}$$

After simplification we get

$$\sigma(t) = \varepsilon_0 \left(E_\infty - E_1 \frac{\tau_1}{t_0} e^{-\frac{t}{\tau_1}} \left(1 - e^{-\frac{t_0}{\tau_1}} \right) \right), \tag{1.73}$$

which is identical to the solution (1.51).

1.3 Generalized Maxwell model

1.3.1 1D model

In general, the *one-dimensional* small-strain viscoelastic model refers to the generalized Maxwell model, where one spring element is arranged in parallel with arbitrary number N of Maxwell elements as depicted in Figure 1.48.

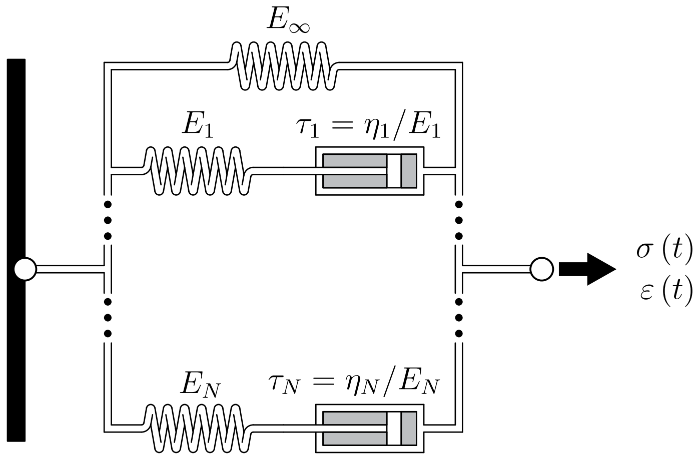


Figure 1.48: Schematic illustration of the generalized Maxwell model.

The total stress for this rheological model can be expressed by the convolution integral

$$\sigma(t) = \int_0^t E(t-s)\,\frac{\mathrm{d}\varepsilon(s)}{\mathrm{d}s}\mathrm{d}s, \tag{1.74}$$

where the material is stress-free (and strain-free) at $t = 0$. The loading is prescribed by the applied strain history $\varepsilon(t)$. The relaxation function $E(t)$ is defined by the Prony series

$$E(t) = E_\infty + \sum_{i=1}^{N} E_i \cdot \exp\left[-\frac{t}{\tau_i}\right] = E_0\left(e_\infty + \sum_{i=1}^{N} e_i \cdot \exp\left[-\frac{t}{\tau_i}\right]\right), \tag{1.75}$$

where

$$e_i = E_i/E_0 \tag{1.76}$$

are the relative moduli. The instantaneous ($E_0 = E(0)$) and the long-term ($E_\infty = E(\infty)$) moduli are related as

$$E_0 = E_\infty + \sum_{i=1}^{N} E_i = \frac{E_\infty}{e_\infty}. \tag{1.77}$$

Therefore, it is evident that

$$e_\infty = 1 - \sum_{i=1}^{N} e_i \tag{1.78}$$

and

$$E(t) = E_0\left(1 - \sum_{i=1}^{N} e_i\left(1 - \exp\left[-\frac{t}{\tau_i}\right]\right)\right). \tag{1.79}$$

An example for the Prony series representation is shown in Figure 1.49. Here, a 3-term Prony series is plotted for the following parameters:

$$e_1 = 0.1, \quad e_2 = 0.2, \quad e_3 = 0.4, \quad \tau_1 = 0.5 \text{ s}, \quad \tau_2 = 3 \text{ s}, \quad \tau_3 = 5 \text{ s}. \tag{1.80}$$

The long-term relative modulus is therefore $e_\infty = 0.3$.

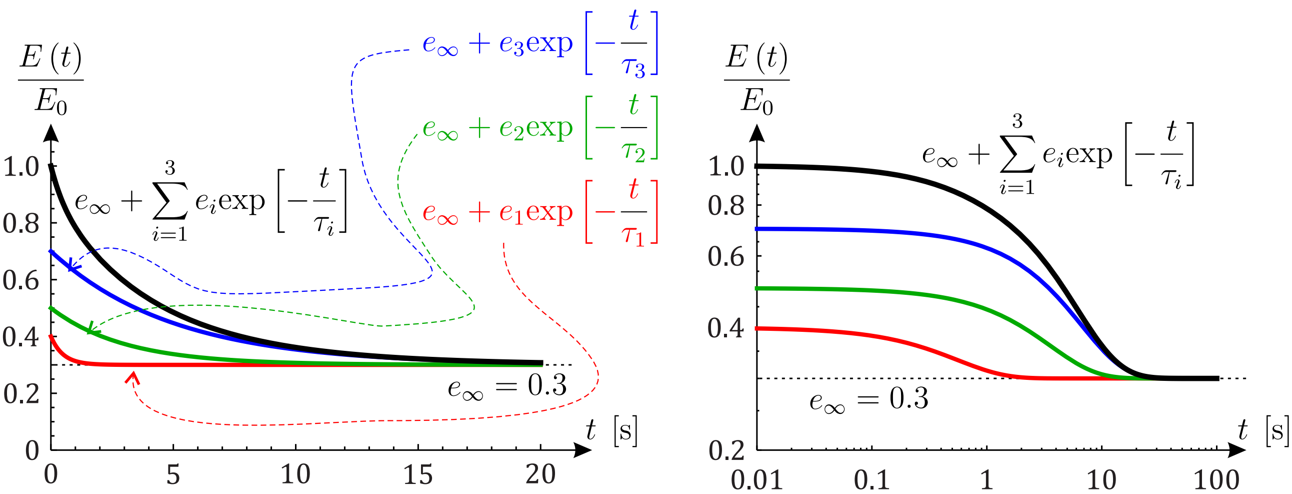


Figure 1.49: An example for a particular Prony series.

Another representation can be formulated by performing integration by parts. Equation (1.74) can be thus written in an alternative form as

$$\begin{aligned}
\sigma(t) &= \left[E(t-s)\,\varepsilon(s)\right]_0^t - \int_0^t \frac{\mathrm{d}E(t-s)}{\mathrm{d}s}\varepsilon(s)\,\mathrm{d}s & (1.81)\\
&= E(0)\,\varepsilon(t) - E(t)\,\varepsilon(0) - \int_0^t \frac{\mathrm{d}E(t-s)}{\mathrm{d}(t-s)}\frac{\mathrm{d}(t-s)}{\mathrm{d}s}\varepsilon(s)\,\mathrm{d}s & (1.82)\\
&= E(0)\,\varepsilon(t) + \int_0^t \frac{\mathrm{d}E(t-s)}{\mathrm{d}(t-s)}\varepsilon(s)\,\mathrm{d}s & (1.83)\\
&= E(0)\,\varepsilon(t) + \int_0^t \frac{\mathrm{d}E(s)}{\mathrm{d}(s)}\varepsilon(t-s)\,\mathrm{d}s & (1.84)\\
&= E_0\varepsilon(t) + \int_0^t \dot{E}(s)\,\varepsilon(t-s)\,\mathrm{d}s. & (1.85)
\end{aligned}$$

Denote $\sigma_0(t) = E_0\varepsilon(t)$ the instantaneous stress response. Then the stress solution becomes

$$\sigma(t) = \sigma_0(t) - \sum_{i=1}^{N}\frac{e_i}{\tau_i}\int_0^t \sigma_0(t-s)\cdot\exp\left[-\frac{s}{\tau_i}\right]\mathrm{d}s. \tag{1.86}$$

The stress solution for sudden strain input can be written by generalize the solution (1.53) for multiple terms:

$$\sigma(t) = \varepsilon_0\left(E_\infty + \sum_{i=1}^{N} E_i e^{-\frac{t}{\tau_i}}\right) = E(t)\,\varepsilon_0 \tag{1.87}$$

with

$$E\left(t\right) = E_{\infty} + \sum_{i=1}^{N} E_i e^{-\frac{t}{\tau_i}}. \tag{1.88}$$

1.3.2 3D compressible form

The three-dimensional small-strain viscoelastic model can be easily formulated by replacing the one-dimensional stress components in equation (1.86) with the tensorial stress components. Consequently, we have the form

$$\boldsymbol{\sigma}\left(t\right) = \boldsymbol{\sigma}_0\left(t\right) - \sum_{i=1}^{N} \frac{e_i}{\tau_i} \int_0^t \boldsymbol{\sigma}_0\left(t-s\right) \cdot \exp\left[-\frac{s}{\tau_i}\right] \mathrm{d}s, \tag{1.89}$$

where the instantaneous Hooke's law $\boldsymbol{\sigma}_0(t)$ employs the instantaneous shear (G_0) and bulk (K_0) moduli. Expression (1.89) assumes that the distortional and volumetric behaviors have the same relaxation characteristics, thus the shear relaxation modulus and the bulk relaxation modulus are expressed as

$$G\left(t\right) = G_0\left(e_{\infty} + \sum_{i=1}^{N} e_i \cdot \exp\left[-\frac{t}{\tau_i}\right]\right), \tag{1.90}$$

$$K\left(t\right) = K_0\left(e_{\infty} + \sum_{i=1}^{N} e_i \cdot \exp\left[-\frac{t}{\tau_i}\right]\right). \tag{1.91}$$

However, in general, we can allow different viscoelastic effects for the distortional and for the volumetric deformations. Consequently, the shear and bulk relaxation functions are expressed as

$$G\left(t\right) = G_0\left(g_{\infty} + \sum_{i=1}^{N_G} g_i \cdot \exp\left[-\frac{t}{\tau_i}\right]\right), \tag{1.92}$$

$$K\left(t\right) = K_0\left(k_{\infty} + \sum_{i=1}^{N_G} k_i \cdot \exp\left[-\frac{t}{\tau_i}\right]\right), \tag{1.93}$$

where $g_i = G_i/G_0$ and $k_i = K_i/K_0$ are the relative moduli of the ith term. N_G and N_K are the number of terms in the relaxation functions. Similar to (1.78) we have the identities

$$g_{\infty} + \sum_{i=1}^{N_G} g_i = 1, \qquad k_{\infty} + \sum_{i=1}^{N_K} k_i = 1. \tag{1.94}$$

The deviatoric ($\boldsymbol{s}$) and hydrostatic ($\boldsymbol{p}$) parts of the total stress tensor are expressed as

$$\boldsymbol{s}(t) = \boldsymbol{s}_0(t) - \sum_{i=1}^{N_G} \frac{g_i}{\tau_i} \int_0^t \boldsymbol{s}_0(t-s) \cdot \exp\left[-\frac{s}{\tau_i}\right] \mathrm{d}s, \tag{1.95}$$

$$\boldsymbol{p}(t) = \boldsymbol{p}_0(t) - \sum_{i=1}^{N_K} \frac{k_i}{\tau_i} \int_0^t \boldsymbol{p}_0(t-s) \cdot \exp\left[-\frac{s}{\tau_i}\right] \mathrm{d}s, \tag{1.96}$$

with the instantaneous stress responses

$$\boldsymbol{s}_0(t) = 2G_0\boldsymbol{e}(t), \qquad \boldsymbol{p}_0(t) = K_0 \mathrm{tr}\boldsymbol{\varepsilon}(t)\boldsymbol{I}, \tag{1.97}$$

where $\boldsymbol{e} = \mathrm{dev}[\boldsymbol{\varepsilon}]$ is the deviatoric strain tensor. The total stress is then calculated as $\boldsymbol{\sigma}(t) = \boldsymbol{s}(t) + \boldsymbol{p}(t)$.

1.3.3 3D incompressible form

If the material is *incompressible*, then only the deviatoric stress can be calculated from the Hooke's law and the hydrostatic stress can be obtained from the boundary conditions. Thus the total stress, in this case, is given by

$$\boldsymbol{\sigma}(t) = \boldsymbol{s}_0(t) - \sum_{i=1}^{N} \frac{g_i}{\tau_i} \int_0^t \boldsymbol{s}_0(t-s) \cdot \exp\left[-\frac{s}{\tau_i}\right] \mathrm{d}s + \boldsymbol{p}(t), \tag{1.98}$$

where $N = N_G$ is used for simplicity. Equation (1.98) can be expressed in a more convenient form by introducing a stress tensor $\boldsymbol{s}_i(t-s)$ associated to the ith term in the model:

$$\boldsymbol{s}_i(t) = \frac{g_i}{\tau_i} \int_0^t \boldsymbol{s}_0(t-s) \cdot \exp\left[-\frac{s}{\tau_i}\right] \mathrm{d}s. \tag{1.99}$$

Finally, we have the form

$$\boldsymbol{\sigma}(t) = \boldsymbol{s}(t) + \boldsymbol{p}(t) = \boldsymbol{s}_0(t) - \sum_{i=1}^{N} \boldsymbol{s}_i(t) + \boldsymbol{p}(t) \tag{1.100}$$

1.3.4 Solution for the 3D incompressible case in uniaxial extension with constant strain-rate

Hooke's law in 3D can be written as

$$\boldsymbol{\sigma} = \boldsymbol{s} + \boldsymbol{p} = 2G\boldsymbol{e} + K\mathrm{tr}\boldsymbol{\varepsilon}\boldsymbol{I}, \tag{1.101}$$

where $\boldsymbol{s} = \mathrm{dev}\,[\boldsymbol{\sigma}]$ is the deviatoric stress, whereas $\boldsymbol{p} = \mathrm{hyd}\,[\boldsymbol{\sigma}]$ is the hydrostatic part of the stress tensor. In the expression above, G and K represent the shear and bulk moduli, whereas $\boldsymbol{e} = \mathrm{dev}\,[\boldsymbol{\varepsilon}]$ is the deviatoric strain tensor and the quantity $\mathrm{tr}\boldsymbol{\varepsilon}$ denotes the volumetric strain. G and K are related to the Young's modulus (E) and the Poisson's ratio (ν) according to the definitions

$$G = \frac{E}{2\,(1+\nu)}, \qquad K = \frac{E}{3\,(1-2\nu)}. \tag{1.102}$$

For incompressible solids, the Hooke's law reduces to the form

$$\boldsymbol{\sigma} = \boldsymbol{s} + \boldsymbol{p} = \boldsymbol{s} + p\boldsymbol{I} = 2G\boldsymbol{e} + p\boldsymbol{I}, \tag{1.103}$$

where the hydrostatic component p is an undetermined parameter, which can be obtained from the boundary conditions and equilibrium equations but not from the constitutive equation. The strain tensor in uniaxial extension for an incompressible ($\nu = 0.5$) solid has the simple diagonal form

$$\boldsymbol{\varepsilon} = \varepsilon \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}, \tag{1.104}$$

where ε is the strain applied along the principal loading direction. Observe that, in this case, we have the identity $\boldsymbol{e} = \boldsymbol{\varepsilon}$. Consequently, the total stress can be expressed as

$$\boldsymbol{\sigma} = 2G\varepsilon \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.5 \end{bmatrix} + p\boldsymbol{I}, \tag{1.105}$$

where the undetermined hydrostatic component p can be obtained from the zero transverse stress constraint $\sigma_{22} = \sigma_{33} = 0$, resulting $p = G\varepsilon$. Consequently, the total stress becomes

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{with} \qquad \sigma = 3p = 3G\varepsilon. \tag{1.106}$$

The uniaxial extension of an incompressible solid with constant strain-rate is characterized by the applied strain loading $\varepsilon\,(t) = \dot{\varepsilon} \cdot t$ in (1.104), where $\dot{\varepsilon}$ is constant. Combining (1.104) with

(1.99) and (1.97) gives the expression for $\boldsymbol{s}_i\left(t\right)$ as

$$\boldsymbol{s}_i\left(t\right)=\frac{g_i}{\tau_i}\int_0^t 2G_0\dot{\varepsilon}\cdot\left(t-s\right)\cdot\exp\left[-\frac{s}{\tau_i}\right]\mathrm{d}s\cdot\begin{bmatrix}1&0&0\\0&-0.5&0\\0&0&-0.5\end{bmatrix}. \tag{1.107}$$

The convolution integral can be evaluated. Consequently, we have the solution

$$\boldsymbol{s}_i\left(t\right)=2G_0\dot{\varepsilon}g_i\tau_i\left(\frac{t}{\tau_i}+\exp\left[-\frac{t}{\tau_i}\right]-1\right)\cdot\begin{bmatrix}1&0&0\\0&-0.5&0\\0&0&-0.5\end{bmatrix}. \tag{1.108}$$

The undetermined pressure in (1.100) can be obtained from the zero transverse stress constraint $\sigma_{22}\left(t\right)=\sigma_{33}\left(t\right)=0$ as

$$p\left(t\right)=G_0\dot{\varepsilon}\left(t-\sum_{i=1}^{N}g_i\tau_i\left(\frac{t}{\tau_i}+\exp\left[-\frac{t}{\tau_i}\right]-1\right)\right). \tag{1.109}$$

Substituting the expression $p\left(t\right)$ into (1.100) gives the solution for the total stress as

$$\boldsymbol{\sigma}\left(t\right)=\begin{bmatrix}\sigma\left(t\right)&0&0\\0&0&0\\0&0&0\end{bmatrix} \tag{1.110}$$

with

$$\sigma\left(t\right)=3p\left(t\right)=\sigma_0\left(t\right)\cdot\left(1-\sum_{i=1}^{\dot{N}}g_i\left(1+\frac{\tau_i}{t}\left(\exp\left[-\frac{t}{\tau_i}\right]-1\right)\right)\right), \tag{1.111}$$

where $\sigma_0\left(t\right)=3G_0\dot{\varepsilon}t$ is the pure elastic solution for the incompressible material (see Eq. (1.106)).

Chapter 2

Finite-strain incompressible viscoelasticity

2.1 Incompressible hyperelasticity

The hyperelastic constitutive equation for a compressible material can be formulated as

$$\boldsymbol{P} = \frac{\partial W}{\partial \boldsymbol{F}}, \tag{2.1}$$

where $\boldsymbol{P}$ denotes the first Piola-Kirchhoff stress tensor, W is the strain energy per unit reference volume and $\boldsymbol{F}$ denotes the deformation gradient. The second Piola-Kirchhoff ($\boldsymbol{S}$), the Cauchy ($\boldsymbol{\sigma}$) and the Kirchhoff ($\boldsymbol{\tau}$) stress tensors are obtained as

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P} = \boldsymbol{F}^{-1}\boldsymbol{\tau}\boldsymbol{F}^{-T}, \qquad \boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{P}\boldsymbol{F}^{T}, \qquad \boldsymbol{\tau} = J\boldsymbol{\sigma} = \boldsymbol{P}\boldsymbol{F}^{T} = \boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^{T}, \tag{2.2}$$

where $J = \det\boldsymbol{F}$ denotes the volume ratio. For isotropic hyperelastic material, the constitutive relation can be expressed as

$$\boldsymbol{\sigma} = \frac{2}{J}\left(I_3\frac{\partial W}{\partial I_3}\boldsymbol{I} + \left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\boldsymbol{b} - \frac{\partial W}{\partial I_2}\boldsymbol{b}^2\right), \tag{2.3}$$

where $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^{T}$ is the left Cauchy-Green deformation tensor, whereas I_K $(K = 1, 2, 3)$ are the principal invariants of $\boldsymbol{b}$. The principal invariants of $\boldsymbol{b}$ are defined as

$$I_1 = \mathrm{tr}\boldsymbol{b} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{2.4}$$

$$I_2 = \frac{1}{2}\left((\mathrm{tr}\boldsymbol{b})^2 - \mathrm{tr}\left(\boldsymbol{b}^2\right)\right) = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \tag{2.5}$$

$$I_3 = \det\boldsymbol{b} = J^2 = \left(\lambda_1\lambda_2\lambda_3\right)^2. \tag{2.6}$$

If W is given in terms of the principal stretches ($\lambda_i,\ i = 1, 2, 3$), then the Cauchy stress tensor has the form

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i = \sum_{i=1}^{3} \frac{\lambda_i}{J} \frac{\partial W}{\partial \lambda_i} \boldsymbol{n}_i \otimes \boldsymbol{n}_i, \tag{2.7}$$

where σ_i are the principal stresses and $\boldsymbol{n}_i$ denote the unit vectors along the principal directions in the spatial configuration.

If the material is *incompressible*, i.e. $J = \lambda_1\lambda_2\lambda_3 = 1$ (and thus $I_3 = 1$), then the hydrostatic part of the Cauchy stress tensor, namely $\boldsymbol{p}$, cannot be determined from the constitutive equation. Consequently, the hyperelastic constitutive relation defines only the deviatoric part of the Cauchy stress tensor. Therefore, the total Cauchy stress can be written as

$$\boldsymbol{\sigma} = \boldsymbol{s} + p\boldsymbol{I} = \operatorname{dev}\left[2\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\boldsymbol{b} - 2\frac{\partial W}{\partial I_2}\boldsymbol{b}^2\right] + p\boldsymbol{I}. \tag{2.8}$$

If the strain energy function is defined in terms of the principal stretches, then the incompressibility constraint implies that $W(\lambda_1, \lambda_2, \lambda_3)$ reduces to $W\left(\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1}\right)$ and therefore the total stress becomes

$$\boldsymbol{\sigma} = \boldsymbol{s} + p\boldsymbol{I} = \operatorname{dev}\left[\sum_{i=1}^{2} \lambda_i \frac{\partial W}{\partial \lambda_i} \boldsymbol{n}_i \otimes \boldsymbol{n}_i\right] + p\boldsymbol{I}. \tag{2.9}$$

It should be emphasized that expressions (2.8) and (2.9) are still valid without applying the $\operatorname{dev}[\bullet]$ operator on the first term on the right hand sides. But in that case, the first terms would also consist hydrostatic component. Therefore, the scalar p would not be the hydrostatic part of the total stress $\boldsymbol{\sigma}$.

2.2 Solution for uniaxial extension

The motion for the uniaxial extension of an incompressible solid is defined as

$$x_1 = \lambda X_1, \qquad x_2 = \lambda^{-1/2} X_2, \qquad x_3 = \lambda^{-1/2} X_3, \tag{2.10}$$

where X_K ($K = 1, 2, 3$) are the Lagrangean coordinates, whereas x_k ($k = 1, 2, 3$) are the Eulerian coordinates. The deformation gradient has the simple form

$$\boldsymbol{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix} \qquad \text{with} \qquad J = \det\boldsymbol{F} = 1. \tag{2.11}$$

Consequently, the left Cauchy-Green deformation tensor and its principal scalar invariants can be written as

$$\boldsymbol{b} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}, \qquad I_1 = \lambda^2 + \frac{2}{\lambda}, \qquad I_2 = 2\lambda + \frac{1}{\lambda^2}, \qquad I_3 = 1. \tag{2.12}$$

Substituting into (2.8) we obtain the form

$$\boldsymbol{\sigma} = \boldsymbol{s} + p\boldsymbol{I} = \frac{2}{\lambda^2}\left(\frac{\partial W}{\partial I_2} + \lambda\frac{\partial W}{\partial I_1}\right)\left(\lambda^3 - 1\right)\begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} + p\boldsymbol{I}. \tag{2.13}$$

The hydrostatic stress p can be obtained from the zero transverse stress constraint $\sigma_{22} = \sigma_{33} = 0$ as

$$p = \frac{2}{3\lambda^2}\left(\frac{\partial W}{\partial I_2} + \lambda\frac{\partial W}{\partial I_1}\right)\left(\lambda^3 - 1\right). \tag{2.14}$$

Consequently, the total stress becomes

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{with} \qquad \sigma = \frac{2}{\lambda^2}\left(\frac{\partial W}{\partial I_2} + \lambda\frac{\partial W}{\partial I_1}\right)\left(\lambda^3 - 1\right). \tag{2.15}$$

Similar procedure can be done for the case when the strain energy function is given in terms of the principal stretches. In this case, the strain energy function $W(\lambda_1, \lambda_2, \lambda_3)$ has to be replaced with $W\left(\lambda, \lambda^{-1/2}, \lambda^{-1/2}\right)$. Since the principal directions in the spatial configuration coincide with the bases vectors $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$, it follows that Eq. (2.9) reduces to

$$\boldsymbol{\sigma} = \boldsymbol{s} + p\boldsymbol{I} = \operatorname{dev}\left[\lambda\frac{\partial W}{\partial \lambda}\mathbf{e}_1 \otimes \mathbf{e}_1\right] + p\mathbf{I} = \lambda\frac{\partial W}{\partial \lambda}\begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} + p\mathbf{I}. \tag{2.16}$$

Eliminating the hydrostatic part p yields

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{with} \qquad \sigma = \lambda\frac{\partial W}{\partial \lambda}. \tag{2.17}$$

Finally, the uniaxial extension of an incompressible hyperelastic solid is characterized by the scalar stress solution σ defined in (2.15) or in (2.17) depending on the particular form of the strain energy function. Since the principal invariants of $\boldsymbol{b}$ are expressed in terms of the stretch λ, it follows that expression (2.17) can be used for all of the isotropic hyperelastic solids.

2.3 Formulation of finite-strain viscoelasticity

The finite-strain viscoelastic approach, which is frequently termed as *"visco-hyperelasticity"* in the literature, can be imagined as the extension of the three-dimensional small-strain viscoelastic formulation defined in (1.95) and (1.96) by replacing the small-strain stress tensor $\boldsymbol{\sigma}$ with the second Piola-Kirchhoff stress tensor $\boldsymbol{S}$. Consequently, the deviatoric $\left(\boldsymbol{S}^D\right)$ and hydrostatic $\left(\boldsymbol{S}^H\right)$ parts of the second Piola-Kirchhoff stress tensor can be formulated as

$$\boldsymbol{S}^D(t) = \boldsymbol{S}_0^D(t) - \sum_{i=1}^{N_G} \frac{g_i}{\tau_i} \int_0^t \boldsymbol{S}_0^D(t-s) \exp\left[-\frac{s}{\tau_i}\right] \mathrm{d}s, \tag{2.18}$$

$$\boldsymbol{S}^H(t) = \boldsymbol{S}_0^H(t) - \sum_{i=1}^{N_K} \frac{k_i}{\tau_i} \int_0^t \boldsymbol{S}_0^H(t) \cdot \exp\left[-\frac{s}{\tau_i}\right] \mathrm{d}s. \tag{2.19}$$

Thus, the total second Piola-Kirchhoff stress is calculated as $\boldsymbol{S}(t) = \boldsymbol{S}^D(t) + \boldsymbol{S}^H(t)$. The instantaneous and the long-term hyperelastic responses are related as

$$\boldsymbol{S}_\infty^D = \left(1 - \sum_{i=1}^{N_G} g_i\right) \boldsymbol{S}_0^D, \qquad \boldsymbol{S}_\infty^H = \left(1 - \sum_{i=1}^{N_K} k_i\right) \boldsymbol{S}_0^H. \tag{2.20}$$

If the material is *incompressible*, then the hydrostatic part $\boldsymbol{S}^H(t)$ cannot be determined from the constitutive equation. It can be obtained from the boundary conditions. For incompressible solids the Cauchy stress $(\boldsymbol{\sigma})$ and the Kirchhoff stress $(\boldsymbol{\tau})$ are identical and they can be calculated from the second Piola-Kirchhoff stress according to Eq. (2.2). Consequently, the deviatoric part of the Cauchy stress for incompressible visco-hyperelastic solid is expressed as

$$\boldsymbol{s}(t) = \operatorname{dev}\left[\boldsymbol{F}(t)\, \boldsymbol{S}^D(t)\, \boldsymbol{F}^T(t)\right], \tag{2.21}$$

$$\boldsymbol{s}(t) = \operatorname{dev}\left[\boldsymbol{F}(t)\, \boldsymbol{S}_0^D(t)\, \boldsymbol{F}^T(t)\right] \tag{2.22}$$

$$-\operatorname{dev}\left[\boldsymbol{F}(t) \left(\sum_{i=1}^{N} \frac{g_i}{\tau_i} \int_0^t \mathrm{S}_0^D(t-s) \exp\left[-\frac{s}{\tau_i}\right] \mathrm{d}s\right) \boldsymbol{F}^T(t)\right], \tag{2.23}$$

where $N = N_G$ is used for simplicity. In order to be consistent with the deviatoric and hydrostatic split of the total stress, we have to use the deviatoric operator in the expression above, because the push-forward operation may results in a non-deviatoric stress tensor. Observe that

$$\boldsymbol{s}_0(t) = \operatorname{dev}\left[\boldsymbol{F}(t)\,\boldsymbol{S}_0^D(t)\,\boldsymbol{F}^T(t)\right]. \tag{2.24}$$

In addition, the deviatoric second Piola-Kirchhoff stress tensor $\mathbf{S}_0^D(t-s)$ behind the integral in (2.22) can be calculated from the deviatoric Cauchy stress $\mathbf{s}_0(t-s)$ by applying the inverse transformation rule in Eq. (2.2). Consequently, we arrive at the form

$$\begin{aligned}\boldsymbol{s}(t) &= \boldsymbol{s}_0(t)\\ &- \operatorname{dev}\left[\boldsymbol{F}(t)\left(\sum_{i=1}^{N}\frac{g_i}{\tau_i}\int_0^t \boldsymbol{F}^{-1}(t-s)\,\boldsymbol{s}_0(t-s)\,\boldsymbol{F}^{-T}(t-s)\exp\left[-\frac{s}{\tau_i}\right]\mathrm{d}s\right)\boldsymbol{F}^T(t)\right].\end{aligned} \tag{2.25}$$

Denote $\boldsymbol{f}(t-s) = \boldsymbol{F}(t)\,\boldsymbol{F}^{-1}(t-s)$ the relative deformation gradient between configurations corresponding to instant time $(t-s)$ and t (see Figure 2.1). Using this notation, the expression reduces to

$$\boldsymbol{s}(t) = \boldsymbol{s}_0(t) - \operatorname{dev}\left[\sum_{i=1}^{N}\frac{g_i}{\tau_i}\int_0^t \boldsymbol{f}(t-s)\,\boldsymbol{s}_0(t-s)\,\boldsymbol{f}^T(t-s)\exp\left[-\frac{s}{\tau_i}\right]\mathrm{d}s\right]. \tag{2.26}$$

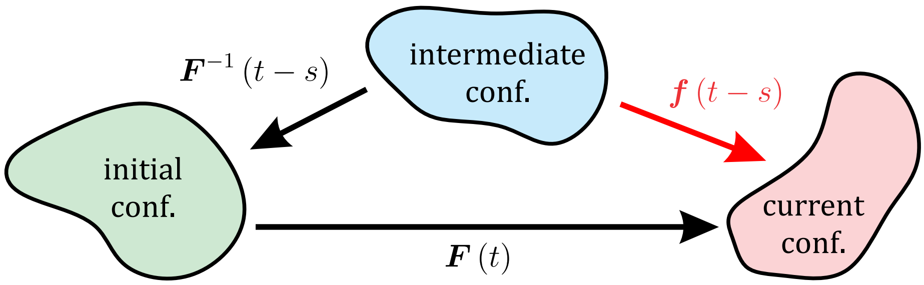


Figure 2.1: Illustration of the meaning of the relative deformation gradient.

Note that in ABAQUS, the quantity $\boldsymbol{f}(t-s)$ is denoted by $\boldsymbol{F}_t^{-1}(t-s)$.
Finally, the Cauchy stress is expressed as

$$\boldsymbol{\sigma}(t) = \boldsymbol{s}_0(t) - \operatorname{dev}\left[\sum_{i=1}^{N}\frac{g_i}{\tau_i}\int_0^t \boldsymbol{f}(t-s)\,\boldsymbol{s}_0(t-s)\,\boldsymbol{f}^T(t-s)\exp\left[-\frac{s}{\tau_i}\right]\mathrm{d}s\right] + \boldsymbol{p}(t). \tag{2.27}$$

This is the form what is also implemented in ABAQUS. Introducing the deviatoric stress tensor

$$\boldsymbol{s}_i(t) = \operatorname{dev}\left[\sum_{i=1}^{N}\frac{g_i}{\tau_i}\int_0^t \boldsymbol{f}(t-s)\,\boldsymbol{s}_0(t-s)\,\boldsymbol{f}^T(t-s)\exp\left[-\frac{s}{\tau_i}\right]\mathrm{d}s\right] \tag{2.28}$$

corresponding to the ith term allows us to rewrite the final expression in the form

$$\boldsymbol{\sigma}(t) = \boldsymbol{s}_0(t) - \sum_{i=1}^{N} \boldsymbol{s}_i(t) + \boldsymbol{p}(t). \tag{2.29}$$

2.4 Solution for uniaxial extension

2.4.1 General formula

The uniaxial extension of an incompressible solid is characterized by the deformation gradient

$$\boldsymbol{F}(t) = \begin{bmatrix} \lambda(t) & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda(t)}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda(t)}} \end{bmatrix}, \qquad \boldsymbol{F}^{-1}(t) = \begin{bmatrix} \frac{1}{\lambda(t)} & 0 & 0 \\ 0 & \sqrt{\lambda(t)} & 0 \\ 0 & 0 & \sqrt{\lambda(t)} \end{bmatrix}, \tag{2.30}$$

where the stretch along the loading direction is denoted by $\lambda(t)$. It clearly follows that the relative deformation gradient becomes

$$\boldsymbol{f}(t-s) = \boldsymbol{f}^T(t-s) = \begin{bmatrix} \frac{\lambda(t)}{\lambda(t-s)} & 0 & 0 \\ 0 & \sqrt{\frac{\lambda(t-s)}{\lambda(t)}} & 0 \\ 0 & 0 & \sqrt{\frac{\lambda(t-s)}{\lambda(t)}} \end{bmatrix}. \tag{2.31}$$

The instantaneous hyperelastic response in uniaxial extension is given by the relation (2.17) with $\sigma_0(t)$. Thus, one can observe that

$$\boldsymbol{f}(t-s)\,\boldsymbol{s}_0(t-s)\,\boldsymbol{f}^T(t-s) = \begin{bmatrix} \frac{2}{3}\frac{\lambda^2(t)}{\lambda^2(t-s)} & 0 & 0 \\ 0 & -\frac{1}{3}\frac{\lambda(t-s)}{\lambda(t)} & 0 \\ 0 & 0 & -\frac{1}{3}\frac{\lambda(t-s)}{\lambda(t)} \end{bmatrix} \sigma_0(t-s). \tag{2.32}$$

It should be emphasized it is not a deviatoric stress tensor. Substituting (2.32) into (2.28) gives

$$\boldsymbol{s}_i(t) = \frac{g_i}{\tau_i}\int_0^t \frac{2\lambda^3(t) + \lambda^3(t-s)}{\lambda(t)\,\lambda^2(t-s)} \sigma_0(\mathrm{t-s}) \exp\left[-\frac{s}{\tau_i}\right] \mathrm{d}s \cdot \begin{bmatrix} \frac{2}{9} & 0 & 0 \\ 0 & -\frac{1}{9} & 0 \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}. \tag{2.33}$$

The undetermined hydrostatic stress $p(t)$ can be obtained from the zero transverse stress constraint in (2.29) as

$$p(t) = \frac{1}{3}\sigma_0(t) - \frac{1}{9}\sum_{i=1}^{N}\frac{g_i}{\tau_i}\int_0^t \frac{2\lambda^3(t)+\lambda^3(t-s)}{\lambda(t)\lambda^2(t-s)}\sigma_0(\mathrm{t}-\mathrm{s})\exp\left[-\frac{s}{\tau_i}\right]\mathrm{d}s. \tag{2.34}$$

Substituting this solution into (2.29) leads us to the solution

$$\boldsymbol{\sigma}(t) = \begin{bmatrix} \sigma(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.35}$$

with

$$\sigma(t) = \sigma_0(t) - \frac{1}{3}\sum_{i=1}^{N}\frac{g_i}{\tau_i}\int_0^t \frac{2\lambda^3(t)+\lambda^3(t-s)}{\lambda(t)\lambda^2(t-s)}\sigma_0(\mathrm{t}-\mathrm{s})\exp\left[-\frac{s}{\tau_i}\right]\mathrm{d}s. \tag{2.36}$$

In order to simplify the representation, it is convenient to introduce the following notation:

$$\sigma(t) = \sigma_0(t) - \sum_{i=1}^{N}\sigma_i \qquad \text{with} \qquad \sigma_i = \frac{g_i}{\tau_i}\left(\frac{2\lambda^2(t)}{3}\cdot A_i + \frac{1}{3\lambda(t)}\cdot B_i\right), \tag{2.37}$$

where

$$A_i = \int_0^t \frac{\sigma_0(t-s)}{\lambda^2(t-s)}\exp\left[-\frac{s}{\tau_i}\right]\mathrm{d}s \qquad \text{and} \qquad B_i = \int_0^t \lambda(t-s)\,\sigma_0(\mathrm{t}-\mathrm{s})\exp\left[-\frac{s}{\tau_i}\right]\mathrm{d}s. \tag{2.38}$$

Consequently, closed-form stress solution for a given hyperelastic material model can be obtained if the integrals A_i and B_i in (2.38) can be evaluated in closed-forms.

2.4.2 Pure hyperelastic responses in uniaxial extension

This report focuses on the following widely-used incompressible isotropic hyperelastic models: Neo-Hookean (NH), Mooney-Rivlin (MR), Yeoh (YE), Ogden (OG). The corresponding strain energy functions are defined as

$$W^{\mathrm{NH}}(I_1) = C_{10}(I_1-3), \tag{2.39}$$

$$W^{\mathrm{MR}}(I_1, I_2) = C_{10}(I_1-3) + C_{01}(I_2-3), \tag{2.40}$$

$$W^{\mathrm{YE}}\left(I_1\right) = C_{10}\left(I_1-3\right) + C_{20}\left(I_1-3\right)^2 + C_{30}\left(I_1-3\right)^3, \tag{2.41}$$

$$W^{\mathrm{OG}}\left(\lambda_1,\lambda_2,\lambda_3\right) = \sum_{k=1}^{K}\frac{2\mu_k}{\alpha_k^2}\left(\lambda_1^{\alpha_k}+\lambda_2^{\alpha_k}+\lambda_2^{\alpha_k}-3\right). \tag{2.42}$$

The Ogden's strain energy potential (2.42) is the form what is implemented in ABAQUS. It differs from the original representation in the parameters μ_k. The ABAQUS's parameters are related to the original ones according to $\mu_k = \alpha_k\mu_k^{\mathrm{Ogden}}/2$, where superscript Ogden refers to the parameter proposed by Ogden in his original work.
The stress solutions of the investigated hyperelastic models:

$$\sigma^{\mathrm{NH}} = 2C_{10}\left(\lambda^2-\lambda^{-1}\right), \tag{2.43}$$

$$\sigma^{\mathrm{MR}} = 2\left(C_{01}+C_{10}\lambda\right)\left(\lambda-\lambda^{-2}\right), \tag{2.44}$$

$$\sigma^{\mathrm{YE}} = 2\left(1-\lambda^{-3}\right)\left(C_{10}\lambda^2+2C_{20}\left(\lambda^4-3\lambda^2+2\lambda\right)+3C_{30}\left(\lambda-1\right)^4\left(\lambda+2\right)^2\right), \tag{2.45}$$

$$\sigma^{\mathrm{OG}} = \sum_{k=1}^{K}\frac{2\mu_k}{\alpha_k}\left(\lambda^{\alpha_k}-\lambda^{-\alpha_k/2}\right). \tag{2.46}$$

2.4.3 Stress solution for uniaxial extension with constant true strain-rate

In this case, the applied stretch history is given by $\lambda\left(t\right) = \exp\left[\dot{\varepsilon}t\right]$, where the true strain-rate $\dot{\varepsilon}$ is constant. In order to simplify the representations, the following abbreviations are used in the followings: $\lambda = \lambda\left(t\right)$, $\varepsilon = \ln\lambda$ and let $\varepsilon_i = \dot{\varepsilon}\tau_i$. The closed-form solutions of the integrals in (2.38) are listed in the followings for the investigated hyperelastic models. The following solutions were obtained using Wolfram Mathematica software.
Neo-Hookean model:

$$A_i^{\mathrm{NH}} = \frac{2C_{10}\tau_i}{1-3\varepsilon_i}\left(3\mathrm{e}^{-t/\tau_i}\varepsilon_i-\mathrm{e}^{-3\varepsilon}-3\varepsilon_i+1\right), \tag{2.47}$$

$$B_i^{\mathrm{NH}} = \frac{2C_{10}\tau_i}{1+3\varepsilon_i}\left(3\mathrm{e}^{-t/\tau_i}\varepsilon_i + \mathrm{e}^{3\varepsilon} - 3\varepsilon_i - 1\right). \tag{2.48}$$

Mooney-Rivlin's model:

$$A_i^{\mathrm{MR}} = 2C_{10}\tau_i\left(1 - \frac{\mathrm{e}^{-3\varepsilon} - 3\varepsilon_i\mathrm{e}^{-t/\tau_i}}{1-3\varepsilon_i}\right) \tag{2.49}$$

$$+\,2C_{01}\tau_i\left(\frac{\mathrm{e}^{-\varepsilon}}{1-\varepsilon_i} - \frac{\mathrm{e}^{-4\varepsilon}}{1-4\varepsilon_i} + \frac{3\varepsilon_i\mathrm{e}^{-t/\tau_i}}{(1-\varepsilon_i)(1-4\varepsilon_i)}\right), \tag{2.50}$$

$$B_i^{\mathrm{MR}} = 2C_{10}\tau_i\left(\frac{\mathrm{e}^{3\varepsilon} + 3\varepsilon_i\mathrm{e}^{-t/\tau_i}}{1+3\varepsilon_i} - 1\right) \tag{2.51}$$

$$+\,2C_{01}\tau_i\left(\frac{\mathrm{e}^{-\varepsilon}}{\varepsilon_i - 1} + \frac{\mathrm{e}^{2\varepsilon}}{1+2\varepsilon_i} - \frac{3\varepsilon_i\mathrm{e}^{-t/\tau_i}}{(1-\varepsilon_i)(1+2\varepsilon_i)}\right). \tag{2.52}$$

Yeoh's model:

$$\begin{aligned}
A_i^{\mathrm{YE}} \;=\;& \frac{6C_{10}\tau_i\varepsilon_i\mathrm{e}^{-t/\tau_i}}{1-3\varepsilon_i} - \frac{72C_{20}\tau_i\varepsilon_i^3(4\varepsilon_i-3)\,\mathrm{e}^{-t/\tau_i}}{(1-\varepsilon_i)(1+2\varepsilon_i)(1-3\varepsilon_i)(1-4\varepsilon_i)}\\
&-\frac{6480C_{30}\tau_i\varepsilon_i^5(4\varepsilon_i^2+3\varepsilon_i-3)\,\mathrm{e}^{-t/\tau_i}}{(\varepsilon_i^2-1)(2\varepsilon_i+1)(3\varepsilon_i-1)(16\varepsilon_i^2-1)(5\varepsilon_i-1)}\\
&+2\tau_i(C_{10}-6C_{20}+27C_{30}) && (2.53)\\
&+4\tau_i(C_{20}-9C_{30})\left(\frac{\mathrm{e}^{-\varepsilon}}{1-\varepsilon_i} + \frac{\mathrm{e}^{2\varepsilon}}{1+2\varepsilon_i} - \frac{2\mathrm{e}^{-4\varepsilon}}{1-4\varepsilon_i}\right)\\
&+6\tau_iC_{30}\left(\frac{3\mathrm{e}^{\varepsilon}}{1+\varepsilon_i} + \frac{\mathrm{e}^{4\varepsilon}}{1+4\varepsilon_i} - \frac{4\mathrm{e}^{-5\varepsilon}}{1-5\varepsilon_i}\right) && (2.54)\\
&-2\tau_i(C_{10}-6C_{20}+27C_{30})\frac{\mathrm{e}^{-3\varepsilon}}{1-3\varepsilon_i}. && (2.55)
\end{aligned}$$

$$\begin{aligned}
B_i^{\mathrm{YE}} \;=\;& \frac{6C_{10}\tau_i\varepsilon_i\mathrm{e}^{-t/\tau_i}}{1-3\varepsilon_i} + \frac{72C_{20}\tau_i\varepsilon_i^3(5\varepsilon_i+3)\,\mathrm{e}^{-t/\tau_i}}{(1-\varepsilon_i)(1+2\varepsilon_i)(1+3\varepsilon_i)(1+5\varepsilon_i)}\\
&+\frac{6480C_{30}\tau_i\varepsilon_i^5(14\varepsilon_i^2+15\varepsilon_i+3)\,\mathrm{e}^{-t/\tau_i}}{(\varepsilon_i-1)(4\varepsilon_i^2-1)(3\varepsilon_i+1)(4\varepsilon_i+1)(5\varepsilon_i+1)(7\varepsilon_i+1)}\\
&-2\tau_i(C_{10}-6C_{20}+27C_{30}) && (2.56)\\
&-4\tau_i(C_{20}-9C_{30})\left(\frac{2\mathrm{e}^{-\varepsilon}}{1-\varepsilon_i} - \frac{\mathrm{e}^{2\varepsilon}}{1+2\varepsilon_i} - \frac{\mathrm{e}^{5\varepsilon}}{1+5\varepsilon_i}\right)\\
&-6\tau_iC_{30}\left(\frac{4\mathrm{e}^{-2\varepsilon}}{1-2\varepsilon_i} - \frac{3\mathrm{e}^{4\varepsilon}}{1+4\varepsilon_i} - \frac{\mathrm{e}^{-7\varepsilon}}{1+7\varepsilon_i}\right) && (2.57)\\
&+2\tau_i(C_{10}-6C_{20}+27C_{30})\frac{\mathrm{e}^{3\varepsilon}}{1+3\varepsilon_i}. && (2.58)
\end{aligned}$$

Ogden's model:

$$A_i^{\mathrm{OG}} = \sum_{k=1}^{K} \frac{2\mu_k \tau_i}{\alpha_k} \left(\frac{3\alpha_k \varepsilon_i \mathrm{e}^{-t/\tau_i}}{\left(1 - \varepsilon_i \left(2 - \alpha_k\right)\right)\left(2 - \varepsilon_i \left(4 + \alpha_k\right)\right)} \right) \tag{2.59}$$

$$+ \sum_{k=1}^{K} \frac{2\mu_k \tau_i}{\alpha_k} \left(-\frac{2\mathrm{e}^{-\varepsilon(2+\alpha_k/2)}}{2 - \varepsilon_i \left(4 + \alpha_k\right)} \right), \tag{2.60}$$

$$B_i^{\mathrm{OG}} = \sum_{k=1}^{K} \frac{2\mu_k \tau_i}{\alpha_k} \left(\frac{3\alpha_k \varepsilon_i \mathrm{e}^{-t/\tau_i}}{\left(1 + \varepsilon_i \left(1 + \alpha_k\right)\right)\left(2 + \varepsilon_i \left(2 - \alpha_k\right)\right)} \right) \tag{2.61}$$

$$+ \sum_{k=1}^{K} \frac{2\mu_k \tau_i}{\alpha_k} \left(\frac{\mathrm{e}^{\varepsilon(\alpha_k+1)}}{1 + \varepsilon_i \left(\alpha_k + 1\right)} - \frac{2\mathrm{e}^{\varepsilon(1-\alpha_k/2)}}{2 + \varepsilon_i \left(2 - \alpha_k\right)} \right). \tag{2.62}$$

2.4.4 Stress solution for uniaxial extension with constant engineering strain-rate

The crosshead's speed in a uniaxial extension experiment is usually constant, which is the default setting in the controlling software. In this case, the applied stretch history is defined as $\lambda(t) = 1 + \dot{\varepsilon} \cdot t$, where $\dot{\varepsilon}$ is constant. Note that here $\dot{\varepsilon}$ denotes the engineering strain-rate, whereas in the preceding subsection $\dot{\varepsilon}$ was the true strain-rate. The closed-form solutions for the integrals in (2.38) are listed in the followings for the investigated hyperelastic models.
Neo-Hookean model:

$$A_i^{\mathrm{NH}} = \frac{C_{10}\tau_i}{\varepsilon_i^2} \left(\frac{\lambda + \varepsilon_i + 2\left(\lambda\varepsilon_i\right)^2}{\lambda^2} - \frac{1 + \varepsilon_i \left(1 + 2\varepsilon_i\right)}{\mathrm{e}^{t/\tau_i}} + \frac{\mathrm{Ei}\left[\frac{1}{\varepsilon_i}\right] - \mathrm{Ei}\left[\frac{\lambda}{\varepsilon_i}\right]}{\varepsilon_i \mathrm{e}^{\lambda/\varepsilon_i}} \right), \tag{2.63}$$

$$B_i^{\mathrm{NH}} = 2C_{10}\varepsilon_i\tau_i \left(\frac{t}{\tau_i} \left(3 + \frac{t}{\tau_i}\varepsilon_i \left(2 + \lambda\right) \right) - 3\lambda\left(\lambda - 2\varepsilon_i\right) - 6\varepsilon_i^2 + \frac{3 + 6\varepsilon_i\left(\varepsilon_i - 1\right)}{\mathrm{e}^{t/\tau_i}} \right), \tag{2.64}$$

where $\mathrm{Ei}[x]$ denotes the exponential integral function

$$\mathrm{Ei}[x] = -\int_{-x}^{\infty} \frac{1}{t}\mathrm{e}^{-t}\mathrm{d}t. \tag{2.65}$$

Mooney-Rivlin's model:

$$A_i^{\mathrm{MR}} = \left(\frac{C_{01}}{3\varepsilon_i^2\dot{\varepsilon}} + \frac{C_{10}}{\varepsilon_i\dot{\varepsilon}} \right) \left(\frac{1}{\lambda} - \frac{2\varepsilon_i^2 + \varepsilon_i + 1}{\mathrm{e}^{t/\tau_i}} \right) + \frac{C_{01}}{3\dot{\varepsilon}} \cdot \frac{\lambda + 2\varepsilon_i}{\lambda^3\varepsilon_i} + \frac{C_{10}}{\dot{\varepsilon}} \left(\frac{1}{\lambda^2} + 2\varepsilon_i \right)$$

$$+ \frac{C_{01}\left(1 - 6\varepsilon_i^3\right) + 3C_{10}\varepsilon_i}{3\dot{\varepsilon}\varepsilon_i^3} \cdot \frac{\mathrm{Ei}\left[\frac{1}{\varepsilon_i}\right] - \mathrm{Ei}\left[\frac{\lambda}{\varepsilon_i}\right]}{\mathrm{e}^{\lambda/\varepsilon_i}}, \tag{2.66}$$

$$\begin{aligned} B_i^{\mathrm{MR}} &= 2C_{10}\tau_i\left(\lambda^3-1\right)-2\tau_i\left(C_{01}-3C_{10}\varepsilon_i\right)\frac{\varepsilon_i^2+(1-\varepsilon_i)^2}{\mathrm{e}^{t/\tau_i}} \\ &\quad+\left(2C_{01}\tau_i-6C_{10}\varepsilon_i\tau_i\right)\left((\lambda-\varepsilon_i)^2+\varepsilon_i^2\right)+\frac{2C_{01}\tau_i}{\varepsilon_i}\cdot\frac{\mathrm{Ei}\left[\frac{1}{\varepsilon_i}\right]-\mathrm{Ei}\left[\frac{\lambda}{\varepsilon_i}\right]}{\mathrm{e}^{\lambda/\varepsilon_i}}. \end{aligned} \tag{2.67}$$

Yeoh's model:

$$\begin{aligned} A_i^{\mathrm{YE}} &= 4\tau_i(\lambda-1)^2(C_{20}+18C_{30}\varepsilon_i(\varepsilon_i-1))+6C_{30}\tau_i\left((\lambda-1)^4+\frac{1}{\lambda^4\varepsilon_i}\right) \\ &\quad-24C_{30}\varepsilon_i(\lambda-1)^3(\varepsilon_i-1) \qquad (2.68) \\ &\quad-2(\lambda-1)\tau_i\left(4C_{20}(\varepsilon_i-1)+3C_{30}\left(24\varepsilon_i^3-24\varepsilon_i^2+5\right)\right) \\ &\quad+2\tau_i\left(C_{10}+4C_{20}\left(\varepsilon_i^2-\varepsilon_i-1\right)+3C_{30}\left(24\varepsilon_i^4-24\varepsilon_i^3+5\varepsilon_i+7\right)\right) \\ &\quad+(\lambda+\varepsilon_i)\left(+\frac{\varepsilon_i\tau_i(3C_{10}\varepsilon_i-C_{20}(18\varepsilon_i-4))+C_{30}\tau_i\left(81\varepsilon_i^2-36\varepsilon_i+3\right)}{3\lambda^2\varepsilon_i^4}\right) \\ &\quad-\frac{\left(144\varepsilon_i^8-144\varepsilon_i^7+30\varepsilon_i^5+42\varepsilon_i^4+9\varepsilon_i^3+17\varepsilon_i^2-11\varepsilon_i+1\right)\left(C_{30}\tau_i\right)}{\varepsilon_i^4\mathrm{e}^{t/\tau}} \\ &\quad-\frac{\tau_i}{3\varepsilon_i^3\mathrm{e}^{t/\tau}}\left(3C_{10}\left(2\varepsilon_i^2+\varepsilon_i+1\right)\varepsilon_i^2+2C_{20}\left(12\varepsilon_i^5-12\varepsilon_i^4-12\varepsilon_i^3-5\varepsilon_i^2-7\varepsilon_i+2\right)\right) \\ &\quad+\tau_i\left(\frac{C_{10}-6C_{20}+27C_{30}}{\varepsilon_i^3}+\frac{4(C_{20}-9C_{30})}{3\varepsilon_i^4}\right)\frac{\mathrm{Ei}\left[\frac{1}{\varepsilon_i}\right]-\mathrm{Ei}\left[\frac{\lambda}{\varepsilon_i}\right]}{\mathrm{e}^{\lambda/\varepsilon_i}} \\ &\quad+\tau_i\left(-\frac{4(C_{20}-9C_{30})}{\varepsilon_i}+\frac{C_{30}}{\varepsilon_i^5}\right)\frac{\mathrm{Ei}\left[\frac{1}{\varepsilon_i}\right]-\mathrm{Ei}\left[\frac{\lambda}{\varepsilon_i}\right]}{\mathrm{e}^{\lambda/\varepsilon_i}} \qquad (2.69) \\ &\quad+\frac{8C_{20}\varepsilon_i\tau_i-C_{30}\tau_i(72\varepsilon_i-6)}{3\lambda^3\varepsilon_i^2}, \qquad (2.70) \end{aligned}$$

$$
\begin{aligned}
B_i^{\mathrm{YE}} \;=\;& \frac{24C_{30}\tau_i}{\lambda\varepsilon_i}+\frac{24C_{30}\tau_i}{\varepsilon_i \mathrm{e}^{t/\tau}}\left(1260\varepsilon_i^8-1260\varepsilon_i^7+450\varepsilon_i^6-48\varepsilon_i^5-6\varepsilon_i^4+\varepsilon_i^2+2\varepsilon_i-1\right)\\
&+\frac{2\tau_i}{\mathrm{e}^{t/\tau}}\left(3C_{10}\left(2\varepsilon_i^3-2\varepsilon_i^2+\varepsilon_i\right)+4C_{20}\left(60\varepsilon_i^5-60\varepsilon_i^4+21\varepsilon_i^3-2\varepsilon_i^2-\varepsilon_i-1\right)\right)\\
&+2(\lambda-1)^5\tau_i\left(2C_{20}+9C_{30}\left(14\varepsilon_i^2-14\varepsilon_i+5\right)\right)\\
&+6C_{30}(\lambda-1)^7\tau_i-42C_{30}(\lambda-1)^6(\varepsilon_i-1)\tau_i\\
&-2(\lambda-1)^4\tau_i\left(10C_{20}(\varepsilon_i-1)+3C_{30}\left(210\varepsilon_i^3-210\varepsilon_i^2+75\varepsilon_i-8\right)\right)\\
&+2(\lambda-1)^3\tau_i\left(C_{10}+2\left(C_{20}\left(20\varepsilon_i^2-20\varepsilon_i+7\right)\right)\right)\\
&+2(\lambda-1)^3\tau_i\left(C_{10}+2\left(6C_{30}\left(210\varepsilon_i^4-210\varepsilon_i^3+75\varepsilon_i^2-8\varepsilon_i-1\right)\right)\right)\\
&-6C_{10}(\lambda-1)^2(\varepsilon_i-1)\tau_i+2C_{10}(\lambda-1)\left(6\varepsilon_i^2-6\varepsilon_i+3\right)\tau_i\\
&-22(\lambda-1)^2\tau_i\left(C_{20}\left(60\varepsilon_i^3-60\varepsilon_i^2+21\varepsilon_i-2\right)\right)\\
&-22(\lambda-1)^2\tau_i\left(18C_{30}\left(210\varepsilon_i^4-210\varepsilon_i^3+75\varepsilon_i^2-8\varepsilon_i-1\right)\varepsilon_i\right)\\
&+8(\lambda-1)\tau_i\left(C_{20}\left(60\varepsilon_i^4-60\varepsilon_i^3+21\varepsilon_i^2-2\varepsilon_i-1\right)\right)\\
&+8(\lambda-1)\tau_i\left(3C_{30}\left(1260\varepsilon_i^6-1260\varepsilon_i^5+450\varepsilon_i^4-48\varepsilon_i^3-6\varepsilon_i^2+1\right)\right)\\
&-8C_{20}\left(60\varepsilon_i^5-60\varepsilon_i^4+21\varepsilon_i^3-2\varepsilon_i^2-\varepsilon_i-1\right)\tau_i\\
&-6\tau_i\left(C_{10}\left(2\varepsilon_i^2-2\varepsilon_i+1\right)\varepsilon_i\right)\\
&-6\tau_i\left(4C_{30}\left(1260\varepsilon_i^7-1260\varepsilon_i^6+450\varepsilon_i^5-48\varepsilon_i^4-6\varepsilon_i^3+\varepsilon_i+2\right)\right)\\
&+8\tau_i(\varepsilon_i(C_{20}-9C_{30})+3C_{30})\left(\frac{\mathrm{Ei}\left[\frac{1}{\varepsilon_i}\right]-\mathrm{Ei}\left[\frac{\lambda}{\varepsilon_i}\right]}{\varepsilon_i^2\mathrm{e}^{\lambda/\varepsilon_i}}\right).
\end{aligned}
\tag{2.71}
$$

Ogden's model:

$$
\begin{aligned}
A_i^{\mathrm{OG}} \;=\;& \sum_{k=1}^{K}\frac{2\mu_k\tau_i}{\alpha_k\varepsilon_i\mathrm{e}^{\lambda/\varepsilon_i}}\left(\mathrm{E}_{(2-\alpha_k)}\left[-\frac{1}{\varepsilon_i}\right]-\lambda^{\alpha_k-1}\mathrm{E}_{(2-\alpha_k)}\left[-\frac{\lambda}{\varepsilon_i}\right]\right)\\
&-\sum_{k=1}^{K}\frac{2\mu_k\tau_i}{\alpha_k\varepsilon_i\mathrm{e}^{\lambda/\varepsilon_i}}\left(\mathrm{E}_{(2+\alpha_k/2)}\left[-\frac{1}{\varepsilon_i}\right]-\lambda^{-\alpha_k/2-1}\mathrm{E}_{(2+\alpha_k/2)}\left[-\frac{\lambda}{\varepsilon_i}\right]\right),
\end{aligned}
\tag{2.72}
$$

$$
\begin{aligned}
B_i^{\mathrm{OG}} \;=\;& \sum_{k=1}^{K}\frac{2\mu_k\tau_i}{\alpha_k\varepsilon_i\mathrm{e}^{\lambda/\varepsilon_i}}\left(\mathrm{E}_{(-1-\alpha_k)}\left[-\frac{1}{\varepsilon_i}\right]-\lambda^{\alpha_k+2}\mathrm{E}_{(-1-\alpha_k)}\left[-\frac{\lambda}{\varepsilon_i}\right]\right)\\
&-\sum_{k=1}^{K}\frac{2\mu_k\tau_i}{\alpha_k\varepsilon_i\mathrm{e}^{\lambda/\varepsilon_i}}\left(\mathrm{E}_{(\alpha_k/2-1)}\left[-\frac{1}{\varepsilon_i}\right]-\lambda^{2-\alpha_k/2}\mathrm{E}_{(\alpha_k/2-1)}\left[-\frac{\lambda}{\varepsilon_i}\right]\right),
\end{aligned}
\tag{2.73}
$$

where $\mathrm{E}_n\left[x\right]$ denotes the exponential integral function

$$
\mathrm{E}_n\left[x\right]=\int_1^{\infty}\frac{1}{t^x}\mathrm{e}^{-xt}\mathrm{d}t. \tag{2.74}
$$

Chapter 3

Harmonic loading

3.1 General solution

The mechanical characteristics of viscoelastic materials when subjected to harmonic stress or strain inputs is an important part of the theory of viscoelasticity. To investigate the underlying mechanism, we consider the response of the material cube under an applied harmonic strain of frequency ω as expressed by

$$\varepsilon\left(t\right)=\varepsilon_0\mathrm{sin}\omega t \qquad \text{or by} \qquad \varepsilon\left(t\right)=\varepsilon_0\mathrm{cos}\omega t. \tag{3.1}$$

Mathematically, it is advantageous to combine these two functions by assuming the strain input is given in the complex form

$$\varepsilon\left(t\right)=\varepsilon_0\left(\mathrm{cos}\omega t+i\cdot\mathrm{sin}\omega t\right)=\varepsilon_0 e^{i\omega t}, \tag{3.2}$$

where $i=\sqrt{-1}$. In practice, we apply only the real or imaginary part. However, it is easier to manipulate with the algebra associated with the exponential function. The resulting stress response has the same frequency, ω, as the imposed strain. Therefore, the stress can be expressed as

$$\sigma\left(t\right)=\sigma^* e^{i\omega t}, \tag{3.3}$$

where σ^* is a complex quantity. The response consists of two parts: i) a transient response that decays exponentially with time ii) a steady-state response with the frequency ω. We consider only the steady-state response in the followings and the transient part is neglected. σ^* can be then expressed as

$$\sigma^*=\varepsilon_0 E^*\left(i\omega\right), \tag{3.4}$$

where $E^*(i\omega)$ is the complex modulus, which can be written as

$$E^*(i\omega) = E'(\omega) + iE''(\omega) = |E^*(i\omega)|\, e^{i\omega t}. \tag{3.5}$$

The real part, $E'(\omega)$, is associated with the amount of energy stored in the material and is called the *storage modulus*. The imaginary part, $E''(\omega)$, relates to the energy dissipated per cycle and is called the *loss modulus*. Figure 3.1 provides a graphical illustration of the two moduli.

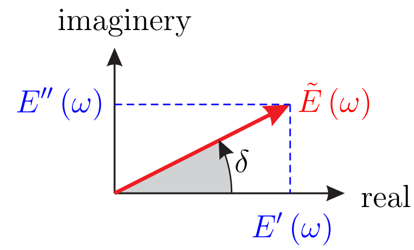


Figure 3.1: Illustration of the storage and loss moduli in the complex plane.

Therefore, the stress response can be given by

$$\sigma(t) = \varepsilon_0 E^*(i\omega)\, e^{i\omega t} = \varepsilon_0 \left(E'(\omega) + iE''(\omega)\right) e^{i\omega t}. \tag{3.6}$$

Let us define the absolute modulus

$$\tilde{E}(\omega) = \sqrt{\left(E'(\omega)\right)^2 + \left(E''(\omega)\right)^2} \tag{3.7}$$

and the *loss angle*

$$\tan\delta = \frac{E''(\omega)}{E'(\omega)}. \tag{3.8}$$

Then the stress solution can be expressed as

$$\sigma(t) = \varepsilon_0 E^*(i\omega)\, e^{i\omega t} = E^*(i\omega)\, \varepsilon(t) = \varepsilon_0 \tilde{E}(\omega)\, e^{i\delta} e^{i\omega t} = \varepsilon_0 \tilde{E}(\omega)\, e^{i(\omega t+\delta)} = \sigma_0 e^{i(\omega t+\delta)}. \tag{3.9}$$

The peak value of the solution is

$$\sigma_0 = \varepsilon_0 \tilde{E}(\omega). \tag{3.10}$$

Note that the strain lags behind the stress by the loss angle δ as illustrated in Figure 3.2.

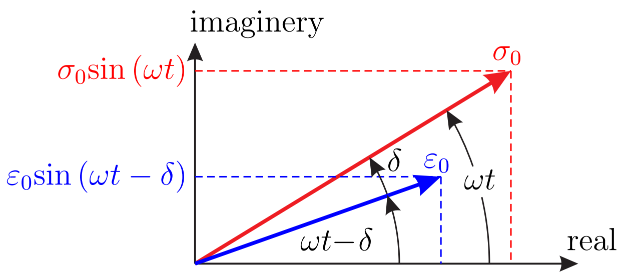


Figure 3.2: Illustration of the strain and stress functions.

The component of the stress in phase with the strain is $\sigma_0 \cos\delta$ as shown in Figure 3.3. Consequently, the storage and loss moduli may be expressed as

$$E'(\omega) = \frac{\sigma_0}{\varepsilon_0}\cos\delta, \qquad E''(\omega) = \frac{\sigma_0}{\varepsilon_0}\sin\delta. \tag{3.11}$$

Note that the loss angle is frequency-dependent, thus, $\delta = \delta(\omega)$.

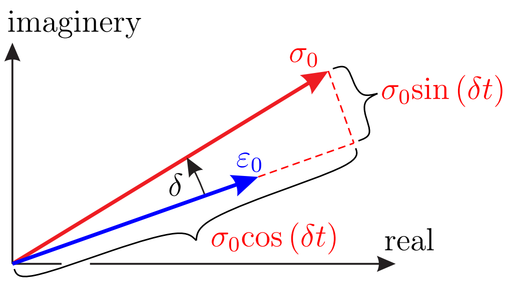


Figure 3.3: Decomposition of the stress.

Let us consider only the *real* part of the general strain input:

$$\varepsilon(t) = \Re\left[\varepsilon_0 e^{i\omega t}\right] = \Re\left[\varepsilon_0 (\cos\omega t + i\cdot\sin\omega t)\right] = \varepsilon_0 \cos\omega t. \tag{3.12}$$

Then the corresponding stress solution becomes

$$\sigma(t) = \Re\left[\varepsilon_0 E^*(i\omega) e^{i\omega t}\right] = \Re\left[\varepsilon_0 (E'(\omega) + iE''(\omega))(\cos\omega t + i\cdot\sin\omega t)\right] \tag{3.13}$$

$$= \varepsilon_0 \Re\left[E'(\omega)\cos\omega t + iE''(\omega)\cos\omega t + E'(\omega)\, i\cdot\sin\omega t + iE''(\omega)\, i\cdot\sin\omega t\right] \tag{3.14}$$

$$= \varepsilon_0 \left(E'(\omega)\cos\omega t - E''(\omega)\sin\omega t\right). \tag{3.15}$$

Consequently, the total stress solution is the sum of two terms: $\varepsilon_0 E'(\omega)\cos\omega t$, which is in phase with the strain input and $\varepsilon_0 E''(\omega)\sin\omega t$, which is in out of phase with $\varepsilon(t)$ due to the sin function. The stress solution and its decomposition is illustrated in Figure 3.4 for the following parameters: $\varepsilon_0 = 4$, $E' = 1$, $E'' = \sqrt{3}$, $\omega = 1$. In this case:

$$\tan\delta = \frac{E''(\omega)}{E'(\omega)} = \frac{\sqrt{3}}{1} \quad \rightarrow \quad \delta = \frac{\pi}{3} = 60^\circ. \tag{3.16}$$

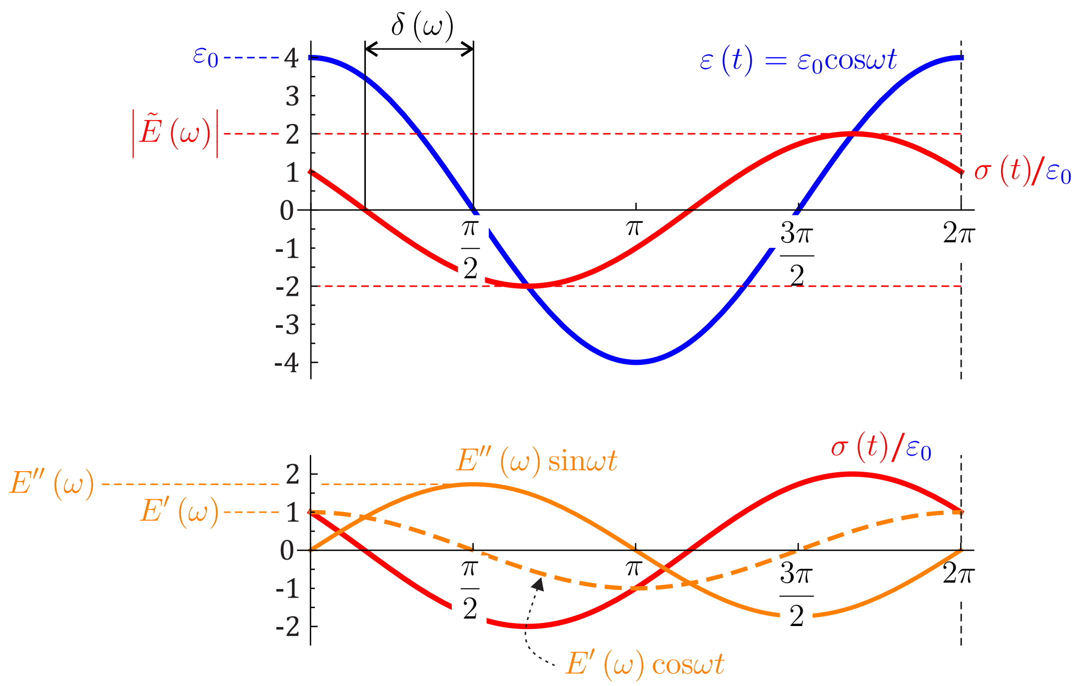


Figure 3.4: Illustration of the stress solution for strain input $\varepsilon_0 \cos\omega t$.

If stress data is recorded in a real experiment for cosine strain input then we can determine the complex modulus, storage modulus and loss modulus. Note that the value of the moduli at a single frequency is obtained. In order to obtain the moduli as a function of frequency a series of experiment must be performed.
Using the relations

$$E'(\omega) = \tilde{E}(\omega)\cos\delta \qquad \text{and} \qquad E''(\omega) = \tilde{E}(\omega)\sin\delta \tag{3.17}$$

we can formulate the stress solution as

$$\sigma(t) = \varepsilon_0 \tilde{E}(\omega)\left(\cos\delta\cos\omega t - \sin\delta\sin\omega t\right) = \varepsilon_0 \tilde{E}(\omega)\cos(\omega t + \delta)\,. \tag{3.18}$$

3.2 Interconversion formula

If the relaxation modulus is expressed with a Prony-series, then the frequency-dependent storage and loss moduli can be directly calculated with the following interconversion formulas:

$$E'(\omega) = E_\infty + \sum_{i=1}^{N} \frac{\tau_i^2\omega^2}{\tau_i^2\omega^2+1}E_i = E_0\left(1 - \sum_{i=1}^{N} e_i + \sum_{i=1}^{N} \frac{\tau_i^2\omega^2}{\tau_i^2\omega^2+1}e_i\right), \tag{3.19}$$

$$E''(\omega) = \sum_{i=1}^{N} \frac{\tau_i\omega}{\tau_i^2\omega^2+1}E_i = E_0 \sum_{i=1}^{N} \frac{\tau_i\omega}{\tau_i^2\omega^2+1}e_i. \tag{3.20}$$

The *loss angle* has the form

$$\tan\delta = \frac{E''(\omega)}{E'(\omega)} = \frac{\sum_{i=1}^{N} \frac{\tau_i \omega}{\tau_i^2 \omega^2 + 1} e_i}{1 - \sum_{i=1}^{N} e_i + \sum_{i=1}^{N} \frac{\tau_i^2 \omega^2}{\tau_i^2 \omega^2 + 1} e_i}. \tag{3.21}$$

Observe the following properties:

$$\lim_{\omega \to 0} E' = E_\infty, \qquad \lim_{\omega \to \infty} E' = E_0, \qquad \lim_{\omega \to 0} E'' = 0, \qquad \lim_{\omega \to \infty} E'' = 0. \tag{3.22}$$

The storage and loss moduli and the loss angle corresponding to the example shown in Figure 1.49 are plotted in Figure 3.5.

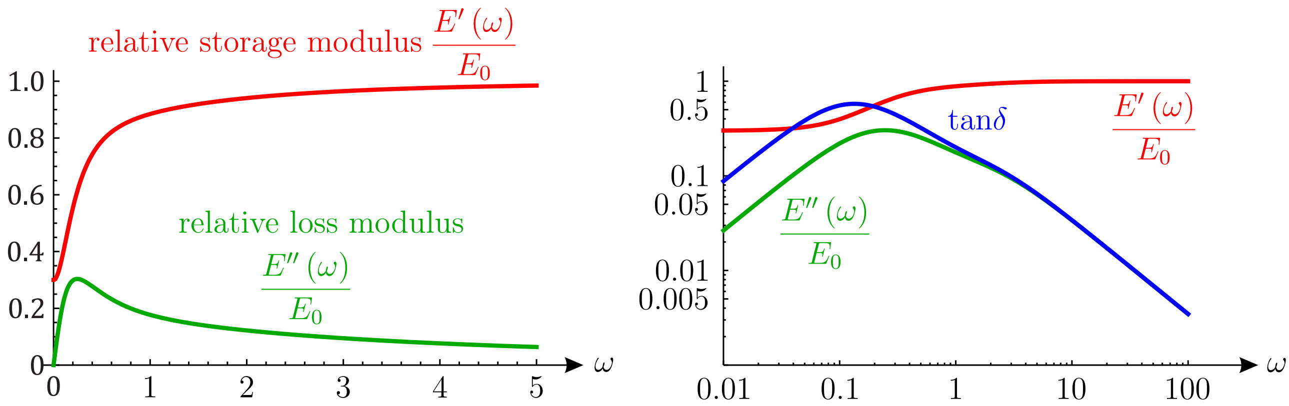


Figure 3.5: Illustration for the storage and loss moduli and loss angle.

Chapter 4

Examples

4.1 Fitting of Prony series in Python

The measured relative relaxation modulus $e(t) = E(t)/E_0$ of a rubber-cork composite material is shown in Figure 4.1 using a linear-linear and a log-log plot. The experimental data points are listed in Table 4.1. In this data, the the initial part of the ramp loading is neglected and ideal sudden strain input is assumed. The primary focus is on the fitting algorithm of the Prony series. For this task, we write a simple Python code. Note that the stress in the material is still decreasing at the end of the measurement data. Consequently, the last data point-pair cannot be considered as a relaxed configuration.

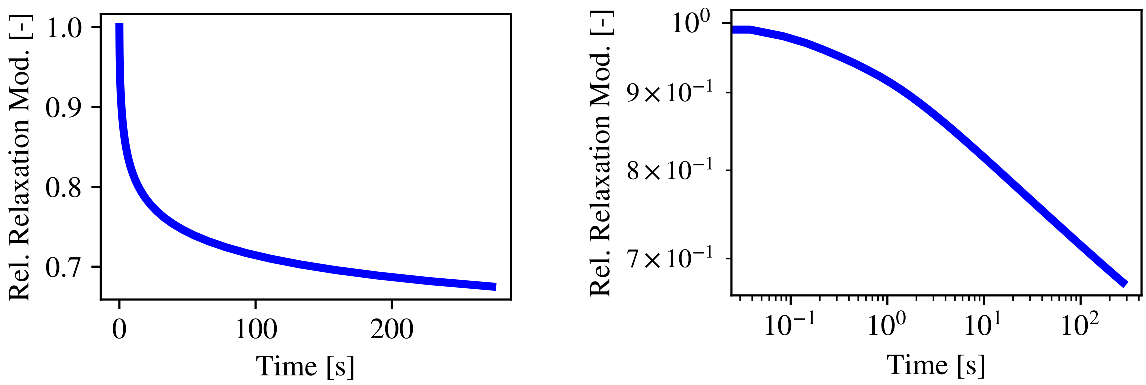


Figure 4.1: Measured relative relative relaxation modulus $e(t)$. Left: linear-linear plot; Right: log-log plot.

Table 4.1: Measurement data.

t_i [s]	e_i [−]	t_i [s]	e_i [−]
0	1	9.7	0.817254754
0.038	0.989974453	11.55	0.809078269
0.084	0.980046278	13.76	0.800980807
0.144	0.970064156	16.376	0.792961063
0.218	0.960295576	19.46	0.785026743
0.318	0.950622186	23.152	0.777176461
0.454	0.940990733	27.51	0.769402286
0.622	0.931514451	32.702	0.761706808
0.838	0.922187488	38.924	0.754082014
1.092	0.912951862	46.266	0.74653873
1.394	0.903792017	55.074	0.739069861
1.75	0.894728627	65.63	0.731673858
2.164	0.885754759	78.116	0.724355552
2.658	0.876896917	93.032	0.717104995
3.232	0.868110453	111.144	0.709933908
3.92	0.859422787	132.67	0.702833996
4.72	0.850806172	158.542	0.695804441
5.672	0.842291272	189.898	0.68884612
6.79	0.833853539	227.53	0.681954264
8.118	0.825510262	273.734	0.675134274

First, we demonstrate the fitting process for the 1st-order Prony series, which is identical to the relative relaxation modulus corresponding to the standard linear solid model. The relative relaxation modulus for this case is defined as

$$e\left(t\right)=\frac{E\left(t\right)}{E_0}=1-e_1\left(1-\exp\left[-\frac{t}{\tau_1}\right]\right). \tag{4.1}$$

The corresponding Python function is shown in Figure 4.2, where `e1` and `t1` are the parameters e_1 and τ_1. The function `PR1` calculates the relative relaxation modulus at instant time represented by the variable `time`.

```
def PR1 (time,pars):
    e1 = pars[0]
    t1 = pars[1]
    ee = e1*(1-exp(-time/t1))
    return 1 - ee
```

Figure 4.2: Python function for the 1st-order relative relaxation modulus.

We use the Sum of Squared Relative Differences for the parameter-fitting. Therefore, the quality function is defined as

$$Q = \sum_{i=1}^{n} \left(\frac{e_i^{\text{exp}} - e_i^{\text{sim}} \left(t_i^{\text{exp}} \right)}{e_i^{\text{exp}}} \right)^2 = \sum_{i=1}^{n} \left(1 - e_i^{\text{sim}} \left(t_i^{\text{exp}} \right) / e_i^{\text{exp}} \right)^2 . \tag{4.2}$$

The corresponding Python function is shown in Figure 4.3.

```python
def q1(pars):
    simdata = PR1(data[:,0], pars)
    q=sum((1-simdata[0:]/data[0:,1])**2)
    return (q);
```

Figure 4.3: Python function for the quality function.

The parameter-fitting is performed using the `scipy minimize` function with the Nelder-Mead algorithm. Some additional options are used to fine-tune the minimization routine. The corresponding Python code is shown in Figure 4.4. We do not apply bounds for the parameters in this simple case.

```python
x1 = np.array([einf/2,tmax/2])
res1 = minimize(q1, x1, method='Nelder-Mead',
                options={'disp': True, 'maxiter':1e10, 'maxfev':1e10, 'ftol':
  ↪1e-10,'xatol':1e-10})
print(res1)
parPR1=[res1.x[0],res1.x[1]]
```

Figure 4.4: Python code for the minimization task.

The obtained parameters and the quality function are

$$e_1 = 0.27373494, \qquad \tau_1 = 6.86590866 \text{ s}, \qquad Q = 0.060544466646294716. \tag{4.3}$$

The comparison of the experimental data and the fitted function is visualized in Figure 4.5, where the dashed red curve is the fitted function. The absolute difference (error) between the experimental data and the fitted function is shown in Figure 4.6. One can conclude that the first-order model is unable to represent the measured characteristics of the relaxation with good accuracy.

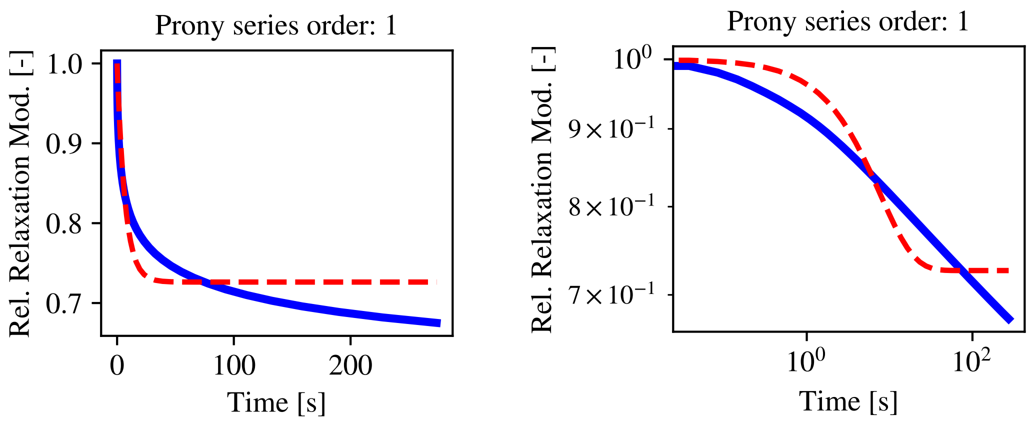


Figure 4.5: Comparison of the measurement data and the fitted relative relaxation modulus. $N = 1$.

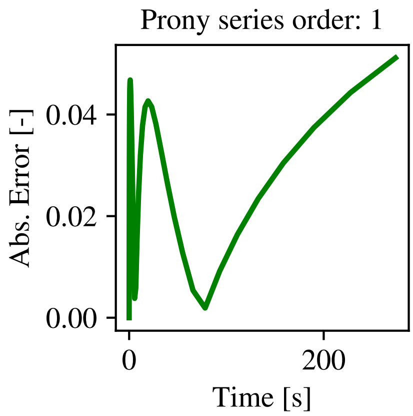


Figure 4.6: Absolute difference (error) between the experimental data and the fitted function. $N = 1$.

To increase the accuracy (i.e. reduce the quality function), the parameter-fitting was performed for higher-order models. The relative relaxation modulus for the generalized Maxwell model is given by

$$e(t) = 1 - \sum_{i=1}^{N} e_i \left(1 - \exp\left[-\frac{t}{\tau_i}\right]\right). \tag{4.4}$$

The results of the parameter-fittings can be sensitive to the starting points. There are recommendations how to choose the starting points for the minimization routine. In this example we apply the following definitions for the starting points:

$$e_i = i\frac{1-e^*}{N+1}, \qquad \tau_i = i\frac{\tau^*}{N+1}, \tag{4.5}$$

where e^* is the last experimental data for the relative relaxation modulus and τ^* is the maximum

Table 4.2: Starting points in the minimization routine.

Prony's order	Starting values	
1	$e_1 = 0.162433,$	$\tau_1 = 136.867$ s
2	$e_1 = 0.108289,$	$\tau_1 = 91.2447$ s
	$e_2 = 0.216577,$	$\tau_2 = 182.489$ s
3	$e_1 = 0.0812164,$	$\tau_1 = 68.4335$ s
	$e_2 = 0.162433,$	$\tau_2 = 136.867$ s
	$e_3 = 0.243649,$	$\tau_3 = 205.301$ s
4	$e_1 = 0.0649731,$	$\tau_1 = 54.7468$ s
	$e_1 = 0.129946,$	$\tau_2 = 109.494$ s
	$e_1 = 0.194919,$	$\tau_3 = 164.24$ s
	$e_1 = 0.259893,$	$\tau_4 = 218.987$ s
5	$e_1 = 0.0541443,$	$\tau_1 = 45.6223$ s
	$e_2 = 0.108289,$	$\tau_2 = 91.2447$ s
	$e_3 = 0.162433,$	$\tau_3 = 136.867$ s
	$e_4 = 0.216577,$	$\tau_4 = 182.489$ s
	$e_5 = 0.270721,$	$\tau_5 = 228.112$ s
6	$e_1 = 0.0464094,$	$\tau_1 = 39.1049$ s
	$e_2 = 0.0928188,$	$\tau_2 = 78.2097$ s
	$e_3 = 0.139228,$	$\tau_3 = 117.315$ s
	$e_4 = 0.185638,$	$\tau_4 = 156.419$ s
	$e_5 = 0.232047,$	$\tau_5 = 195.524$ s
	$e_6 = 0.278456,$	$\tau_6 = 234.629$ s

time value in the experimental data. In the present example:

$$e^* = 0.675134274, \qquad \tau^* = 273.734 \text{ s}. \tag{4.6}$$

This approach found to be efficient for this example. The starting points corresponding to each model are listed in Table 4.2. The highest model order we consider is $N = 6$.
We applied the constraints that all parameters must be positive. This requirement was achieved by using bounds in the minimize function. The minimum value for the parameters was set to 10^{-10}. The model predictions for the higher-order models are presented in Figure 4.7-4.7. The fitted material parameters are listed in Table 4.3. The absolute errors are shown in Figure 4.12-4.12. The comparison of the errors of the fitted functions is demonstrated in Figure 4.15.

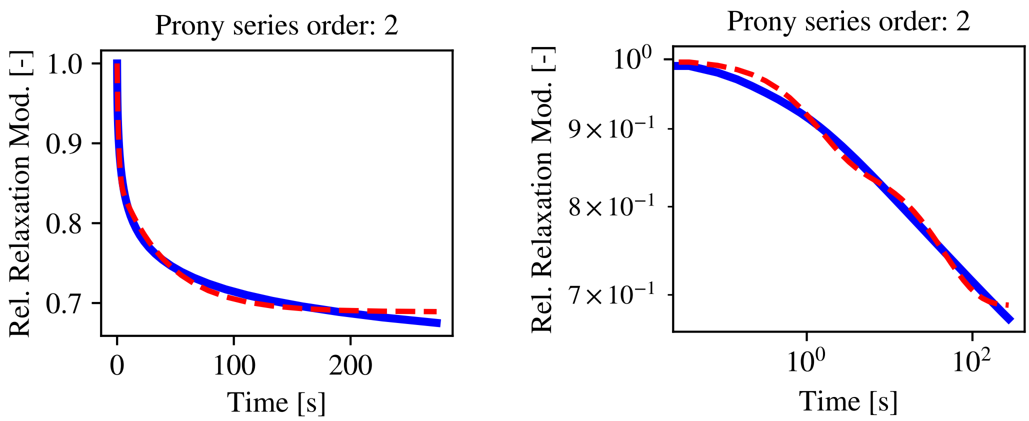


Figure 4.7: Comparison of the measurement data and the fitted relative relaxation modulus. $N = 2$.

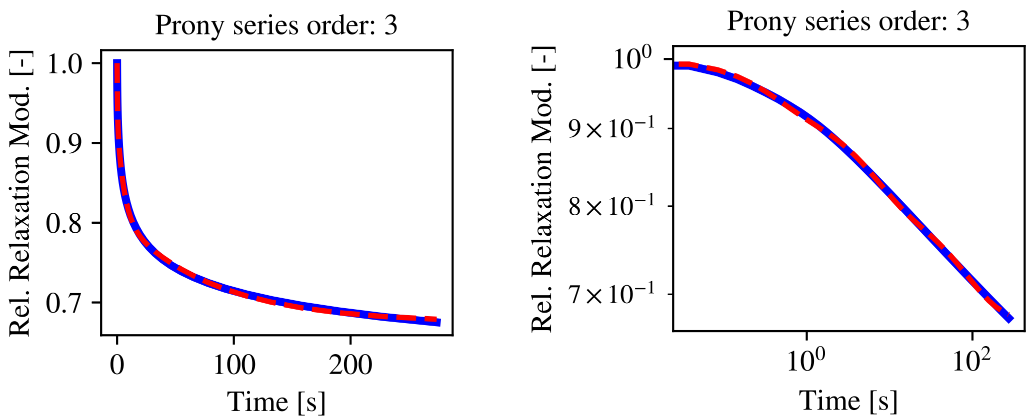


Figure 4.8: Comparison of the measurement data and the fitted relative relaxation modulus. $N = 3$.

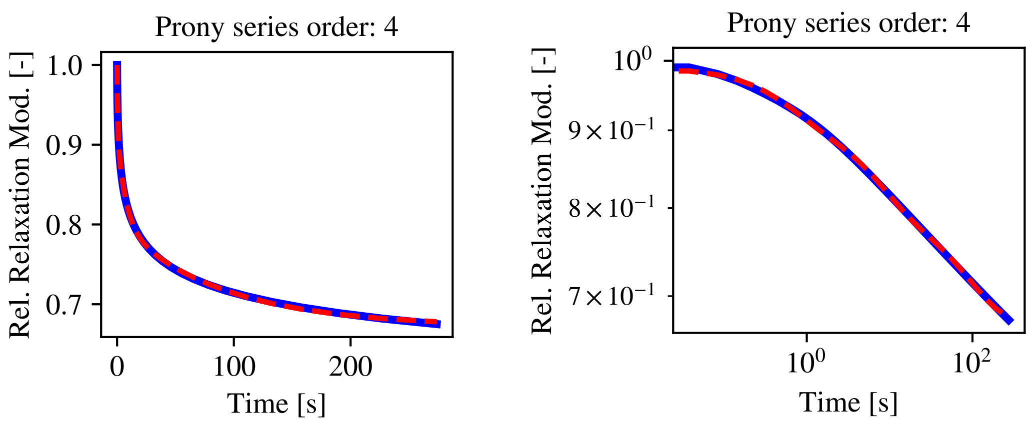


Figure 4.9: Comparison of the measurement data and the fitted relative relaxation modulus. $N = 4$.

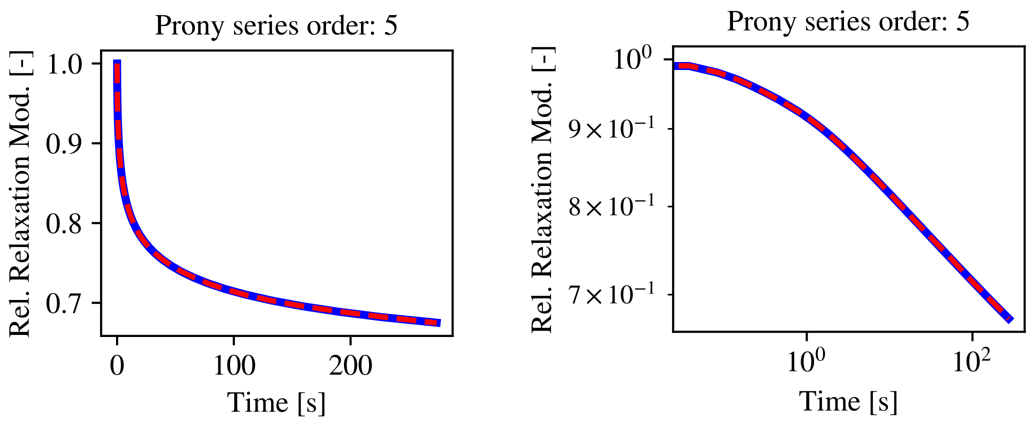


Figure 4.10: Comparison of the measurement data and the fitted relative relaxation modulus. $N = 5$.

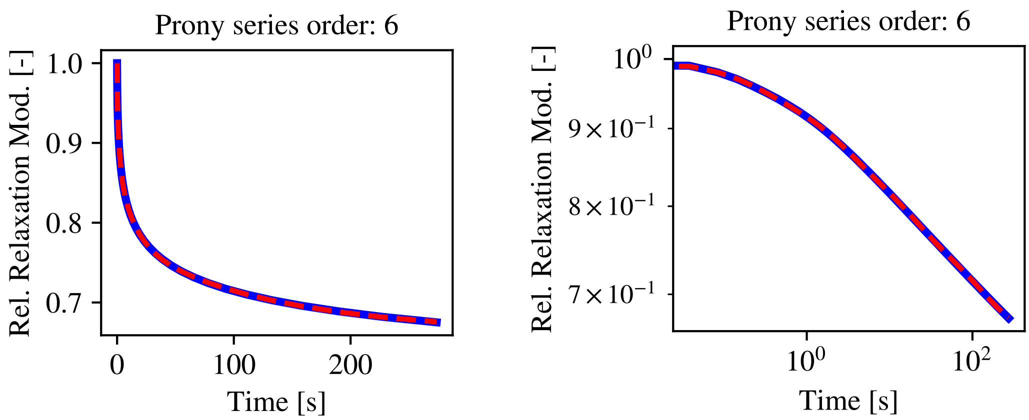


Figure 4.11: Comparison of the measurement data and the fitted relative relaxation modulus. $N = 6$.

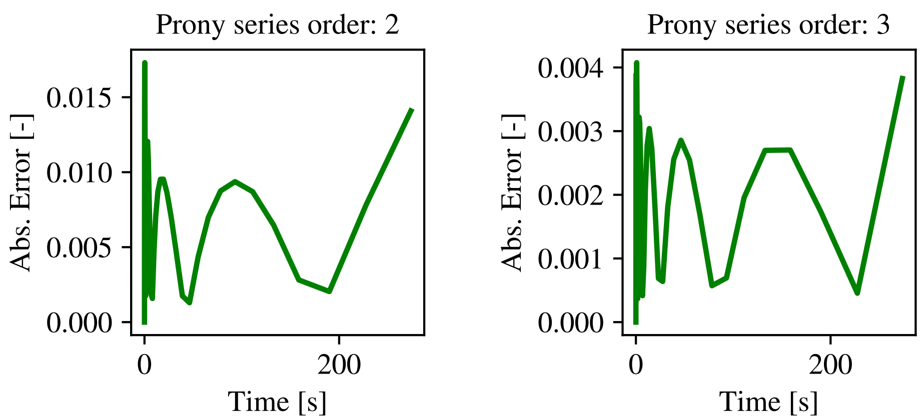


Figure 4.12: Absolute difference (error) between the experimental data and the fitted function. $N = 2$ & 3.

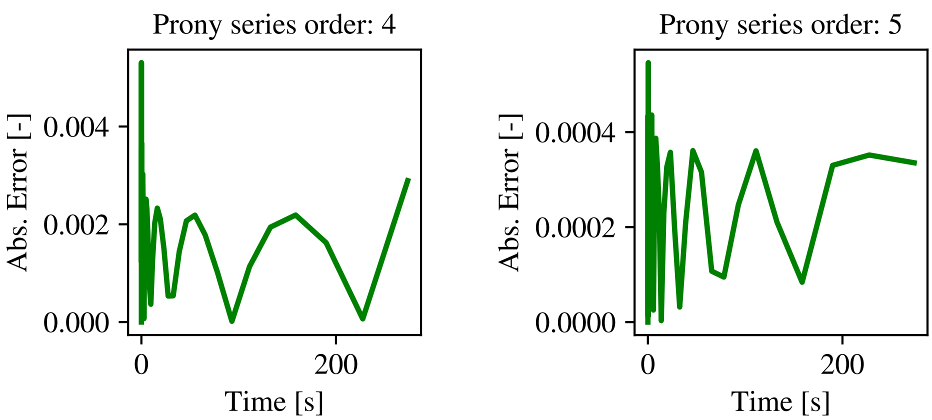


Figure 4.13: Absolute difference (error) between the experimental data and the fitted function. $N = 4$ & 5.

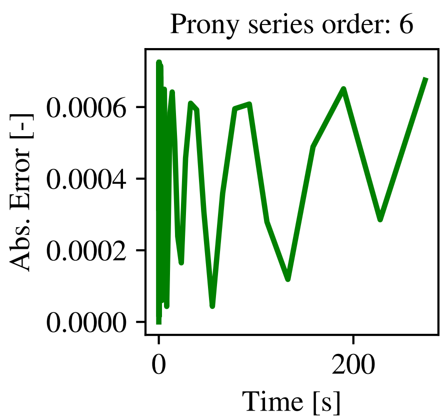


Figure 4.14: Absolute difference (error) between the experimental data and the fitted function. $N = 6$.

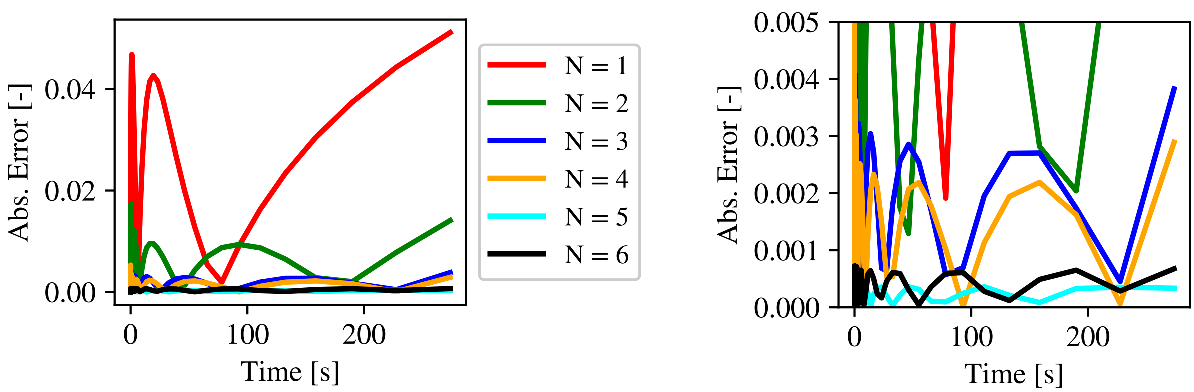


Figure 4.15: Comparison of the absolute errors. The right figure shows a magnified domain.

The variation of the quality function with respect to the model order is shown in Figure 4.16 and listed in Table 4.3. We can conclude that the accuracy can be increased with higher-order

Table 4.3: Fitted material parameters.

N	Fitted parameters	Quality	e_∞
1	$e_1 = 0.27373, \quad \tau_1 = 6.86591$ s	0.0605444	0.7262
2	$e_1 = 0.14430, \quad \tau_1 = 1.32287$ s $e_2 = 0.16675, \quad \tau_2 = 42.60461$ s	0.004446	0.6889
3	$e_1 = 0.13285, \quad \tau_1 = 80.63069$ s $e_2 = 0.07239, \quad \tau_2 = 0.37906$ s $e_3 = 0.12025, \quad \tau_3 = 5.84572$ s	0.0003332	0.6745
4	$e_1 = 0.01020, \quad \tau_1 = 10^{-10}$ s $e_1 = 0.12864, \quad \tau_2 = 88.39924$ s $e_1 = 0.07433, \quad \tau_3 = 0.61359$ s $e_1 = 0.11461, \quad \tau_4 = 7.13053$ s	0.0002319	0.6722
5	$e_1 = 0.08266, \quad \tau_1 = 8.37315$ s $e_2 = 0.06735, \quad \tau_2 = 1.57941$ s $e_3 = 0.29992, \quad \tau_3 = 1735.75973$ s $e_4 = 0.09092, \quad \tau_4 = 50.39002$ s $e_5 = 0.04091, \quad \tau_5 = 0.18332$ s	0.000005	0.4182
6	$e_1 = 0.04586, \quad \tau_1 = 0.23581$ s $e_2 = 0.08064, \quad \tau_2 = 2.21715$ s $e_3 = 0.00227, \quad \tau_3 = 10^{-10}$ s $e_4 = 0.09084, \quad \tau_4 = 13.92305$ s $e_5 = 0.03223, \quad \tau_5 = 867.25265$ s $e_6 = 0.10469, \quad \tau_6 = 110.68341$ s	0.000014	0.6434

models. However, the 5th-order model provides a slightly better accuracy than the 6th-order model in this example. This phenomenon illustrate that the fitting process can be sensitive to the initial points. Theoretically, the higher-order models should provide higher accuracy. Let us change the initial values in the 6th-order model to

$$e_i = i\frac{1 - e^*}{N + 2}, \qquad \tau_i = i\frac{\tau^*}{N + 2}. \tag{4.7}$$

The same fitting algorithm then finds the following parameters:

$$e_1 = 0.04586, \quad \tau_1 = 0.23581 \text{ s}, \quad e_2 = 0.08064, \quad \tau_2 = 2.21715 \text{ s}, \tag{4.8}$$

$$e_3 = 0.00227, \quad \tau_3 = 10^{-10} \text{ s}, \quad e_4 = 0.09084, \quad \tau_4 = 13.92305 \text{ s}, \tag{4.9}$$

$$e_5 = 0.03223, \quad \tau_5 = 867.25265 \text{ s}, \quad e_6 = 0.10469, \quad \tau_6 = 110.68341 \text{ s} \tag{4.10}$$

with quality function 0.000002. This fitting is now better than the result of 5th-order model as shown in Figure 4.17.

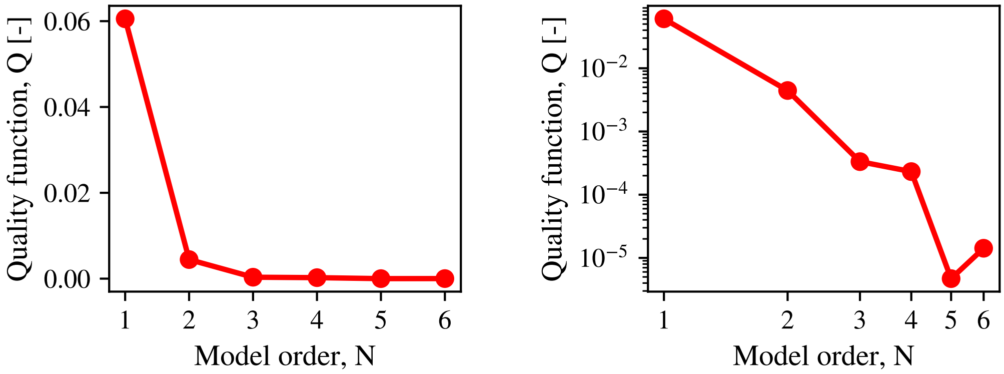


Figure 4.16: Variation of the quality function with respect to the model order. Linear-linear and log-log plots.

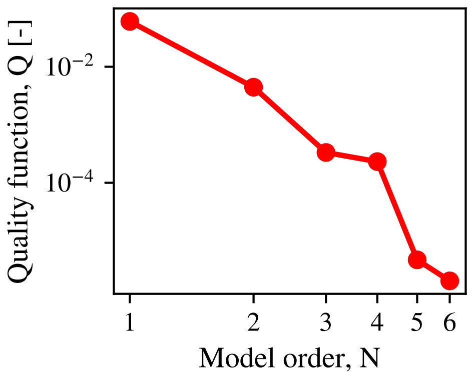


Figure 4.17: Variation of the quality function with respect to the model order. The quality of the 6th-order model corresponds to the modified initial points.

The comparison of the fitted relaxation functions is shown in Figure 4.18. The left figure shows the model responses in the time domain corresponding to the experimental data, whereas the right figure contains the model solutions beyond the maximum experimental time value too. One can conclude that the behavior of the 5th-order model shows different characteristics beyond the maximum time value. The long-term relaxation modulus corresponding to the 5th-order model is significantly different than the long-term moduli of the other models. The comparison in a log-log plot is presented in Figure 4.19.

The long-term relative modulus is calculated as

$$e_\infty = 1 - \sum_{i=1}^{N} e_i. \tag{4.11}$$

The numerical values are listed in Table 4.3.

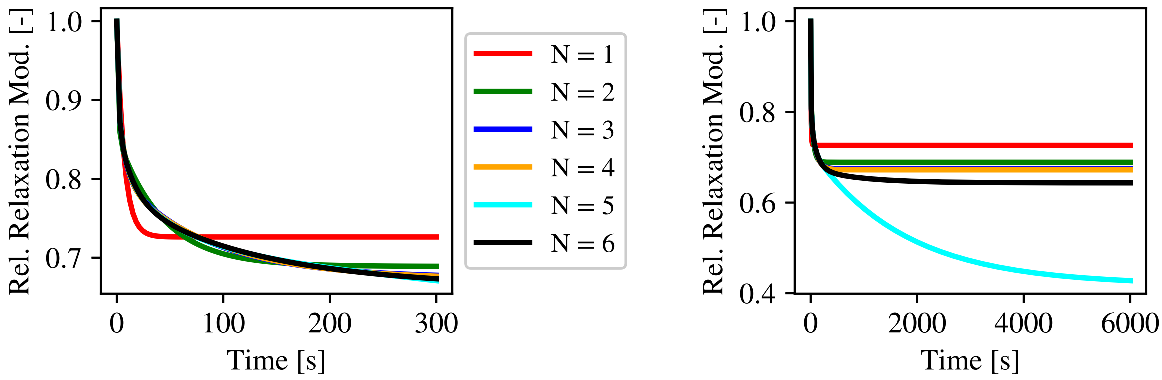


Figure 4.18: Comparison of the fitted relaxation functions.

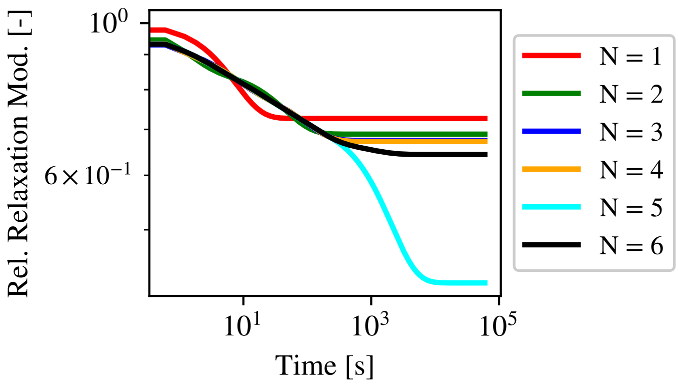


Figure 4.19: Comparison of the fitted relaxation functions in a log-log plot.

It is possible to set the long-term relative relaxation modulus to a desired value. In this case, we need to modify the relaxation data: we have to subtract the desired long-term relative relaxation modulus from the original data. Then, a modified relaxation function, $\tilde{e}(t)$, needs to be fitted to the modified data. The modified relaxation function is defined as

$$e(t) = e_\infty + \sum_{i=1}^{N} e_i \mathrm{exp}\left[-\frac{t}{\tau_i}\right] \quad \rightarrow \quad \tilde{e}(t) = \sum_{i=1}^{N} e_i \mathrm{exp}\left[-\frac{t}{\tau_i}\right]. \tag{4.12}$$

To demonstrate the use of this strategy we fit a 3rd-order model to the measurement data with desired long term moduli $e_\infty = 0.6$. The modified data is shown in Figure 4.20 with red curve.

The fitted parameters are

$$e_1 = 0.1143, \quad e_2 = 0.1214, \quad e_3 = 0.1466, \tag{4.13}$$

$$\tau_1 = 1.4881 \text{ s}, \quad \tau_2 = 21.0485 \text{ s}, \quad \tau_3 = 391.9842 \text{ s}. \tag{4.14}$$

Note that the actual long-term modulus is $1 - e_1 - e_2 - e_3 = 0.6175$, which is slightly different than the desired value of 0.6, but increasing the model order we can reduce the resulting error between the desired value and the model prediction. The fitted function is illustrated in Figure 4.21.

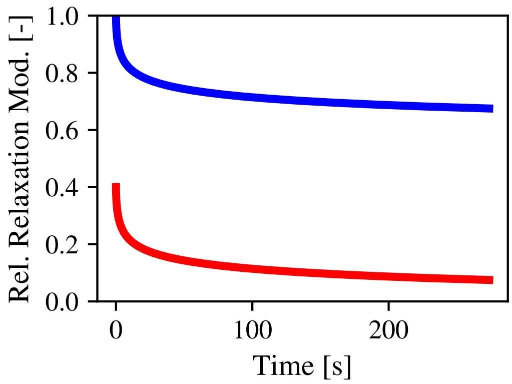
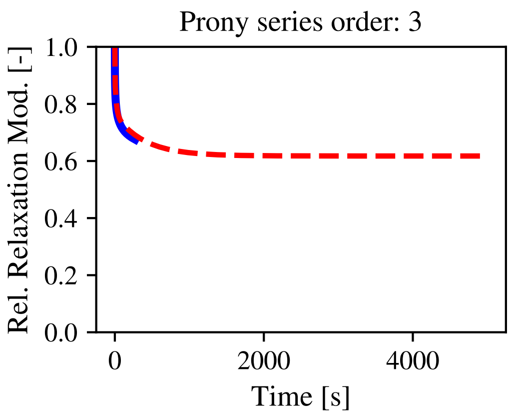


Figure 4.20: Illustration of the modified data. Blue curve: original data. Red curve: modified data.



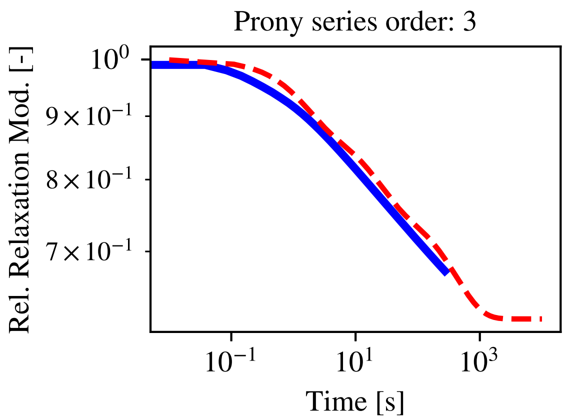


Figure 4.21: Comparison of the measurement data and the fitted relative relaxation modulus with desired long-term value. $N = 3$.

An alternative strategy is to enforce the desired long-term relative modulus via the definition of the relaxation function. If we set e_3 to $0.6 - e_1 - e_2$ and we re-run the parameter-fitting task only for the variables e_1, e_2, τ_1, τ_2, τ_3 then the resulting long-term modulus will be exactly the desired value of 0.6. The fitted parameters are

$$e_1 = 0.1384, \quad \tau_1 = 12.0082 \text{ s}, \quad e_2 = 0.0974, \quad \tau_2 = 0.6186 \text{ s}, \quad \tau_3 = 311.1279 \text{ s}. \tag{4.15}$$

Consequently, $e_3 = 0.6 - e_1 - e_2 = 0.3641$. The fitted function is illustrated in Figure 4.22. We can see that the long-term relative modulus is exactly the same as the desired value (0.6).

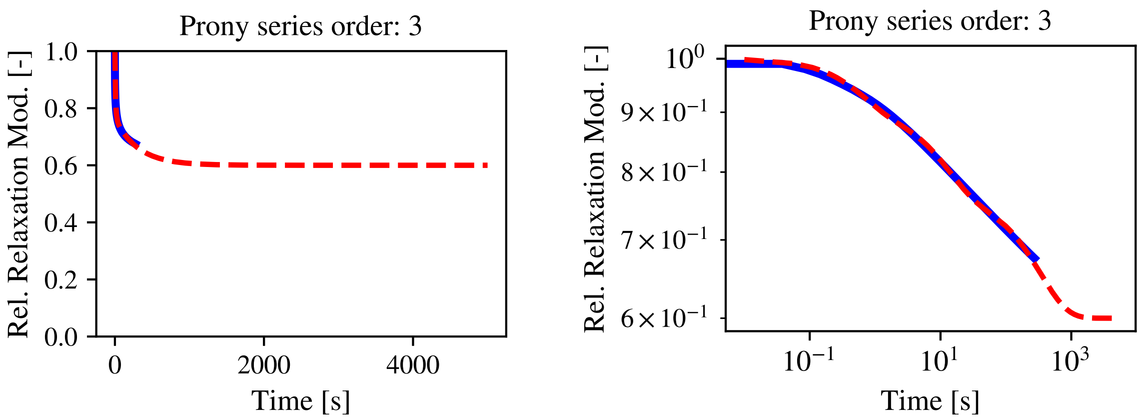


Figure 4.22: Comparison of the measurement data and the fitted relative relaxation modulus with desired long-term value. $N = 3$.

4.2 Fitting of Prony series in Abaqus

Prony's series were fitted to the same experimental data using the Abaqus built-in calibration procedure. First, the experimental data must be imported. In the Edit Material dialog box we need to select the Elasticity/Viscoelasticity option as shown in Figure 4.23. The default setting uses frequency domain with parameters as shown in Figure 4.24. We need to change the domain from Frequency to Time and the "Relaxation test data" option need to be selected from the drop-down list (see Figure 4.25). Then, under "Test data", the "Shear Test Data" must be selected. Note that we will fit a relaxation function corresponding to the Young's modulus.

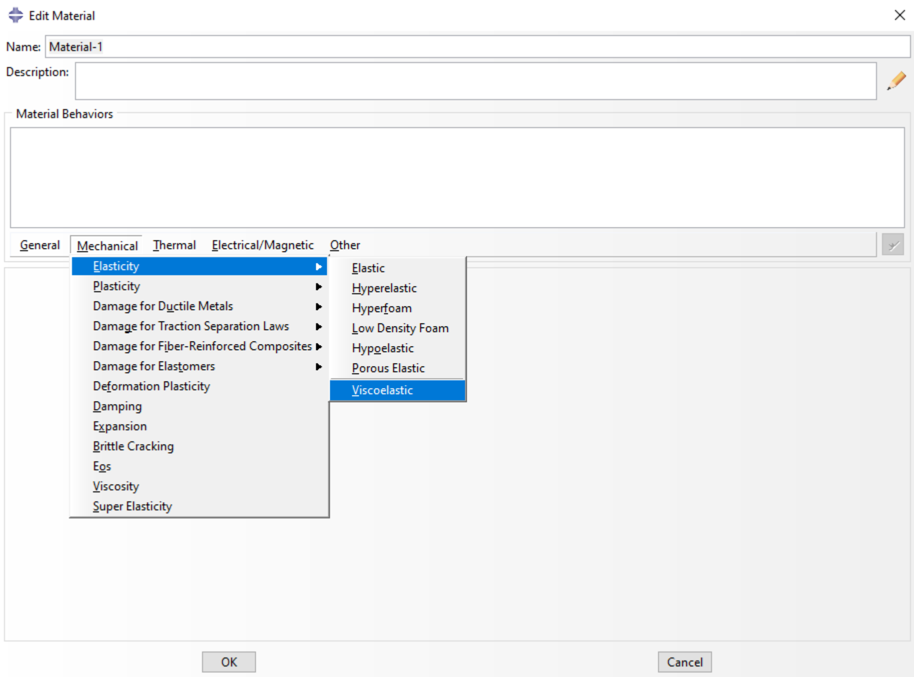


Figure 4.23: Viscoelastic material model in Abaqus CAE.

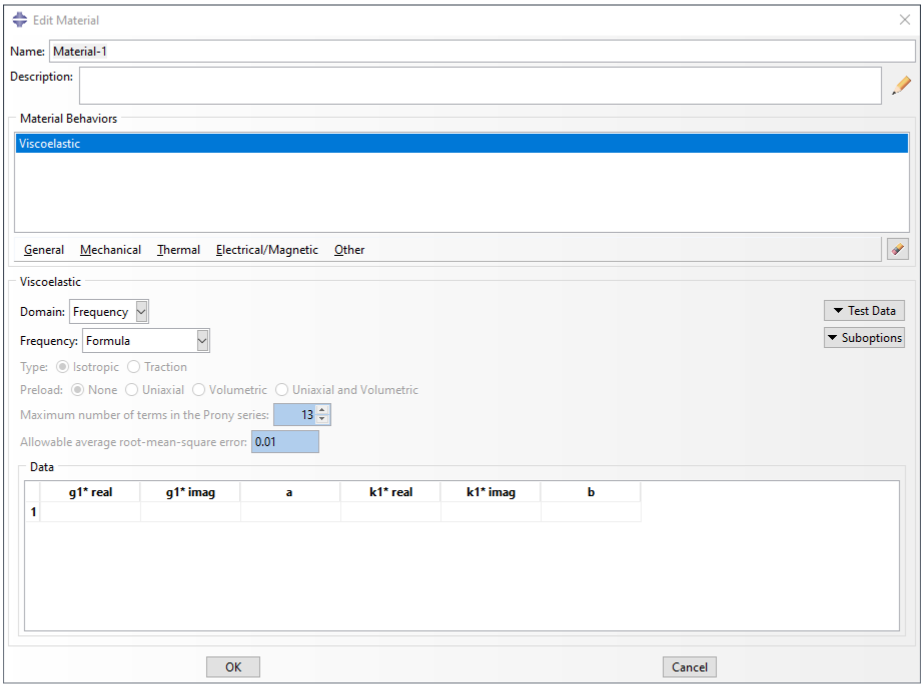


Figure 4.24: Default settings.

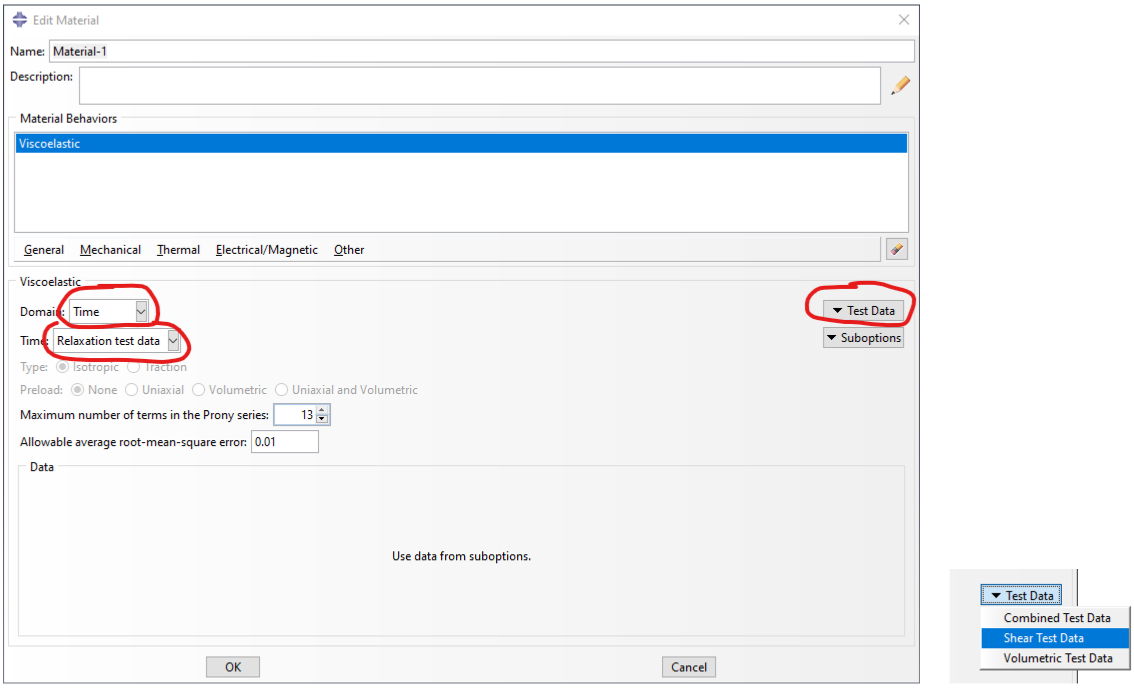


Figure 4.25: Using test data option.

We can insert the experimental data in the new window as shown in Figure 4.26. The time value in the first data pair is 0 in our case. This row must be deleted, because the software accepts only positive values for time. The "Long-term normalized shear compliance or modulus" value is unset by default, but we will change it later.

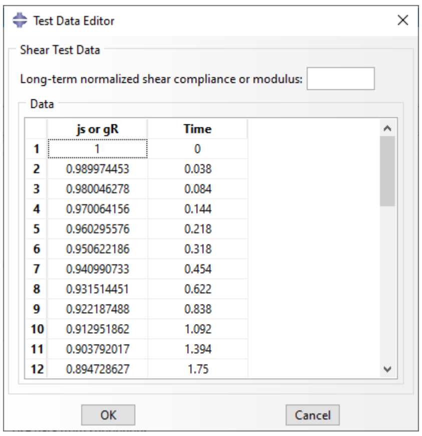


Figure 4.26: Importing the relaxation test data.

After the test data is imported, we need to set two values: "Maximum number of terms in the Prony series" and "Allowable average root-mean-square error". The default values are 13 and 0.01. We will discuss their effect later in this report. We accept the default values. It is important to emphasize that we need to define an elastic material too. However, its definition has no direct effect on the relative relaxation modulus. For simplicity, we select Hooke's law with Young's modulus of 1 and Poisson's ratio of 0.3 (see Figure 4.27). If the "Moduli time scale (for viscoelasticity)" option is set to "Instantaneous" then the elastic material model represents the instantaneous response of the corresponding viscoelastic model. We use this option (see Figure 4.27). However, it is noted that it has no effect on the relative relaxation modulus.

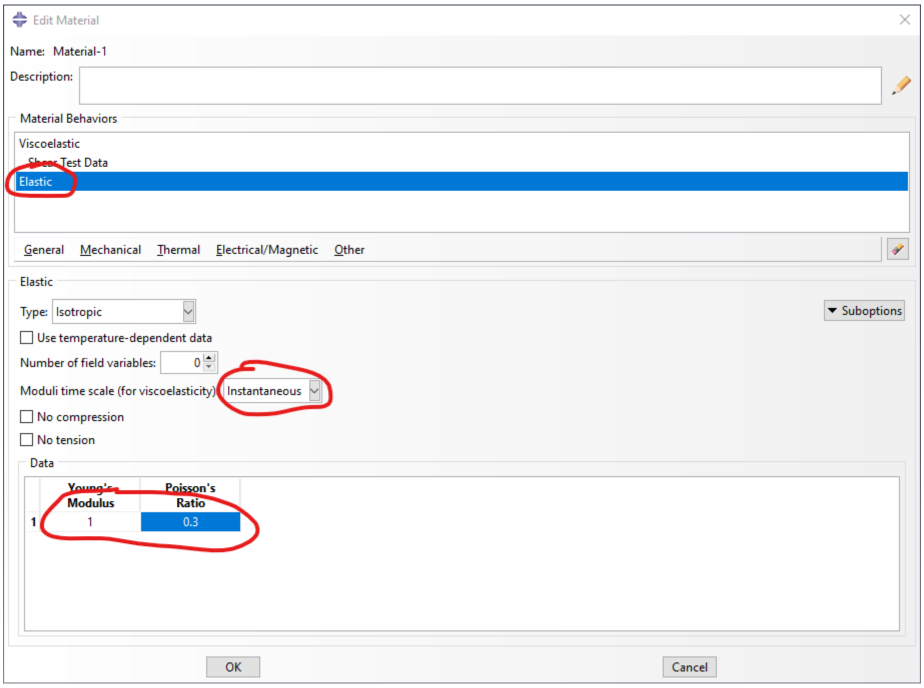


Figure 4.27: Definition of the elastic material model.

Once the definition of the material model is done we can use the "Evaluate ..." option in the Material Manager dialog box (see Figure 4.28). The new window shows the available input data. Here, we have only one data set. The maximum time in the test data is shown in the time period box. We can overwrite this value if needed. In addition, we can chose the response functions we want to plot. Here we select only the "Stress Relaxation" response mode (Figure 4.29).

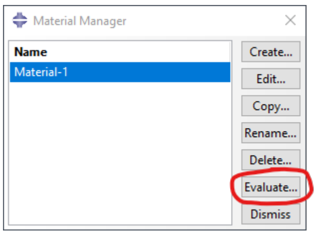


Figure 4.28: Evaluate option.

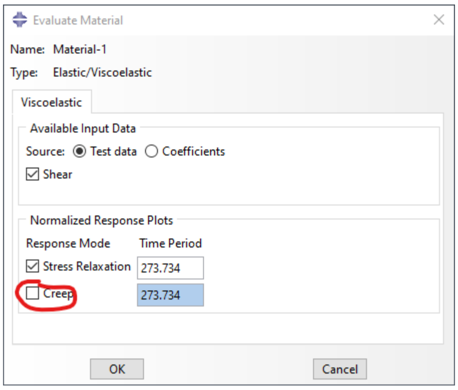


Figure 4.29: Evaluate Material Dialog Box.

By clicking the OK button, the software starts the minimization task in the background. When the software is done with the calculation it reports the obtained values in the "Material Parameters ans Stability Limit Information" windows (see Figure 4.30). The reported values are

$$G(1) = 0.13614, \qquad TAU(1) = 1.1380, \tag{4.16}$$

$$G(2) = 0.17145, \qquad TAU(2) = 37.308. \tag{4.17}$$

In addition to the fitted parameters, Abaqus compares the fitted function and the test data in a plot (see Figure 4.31).

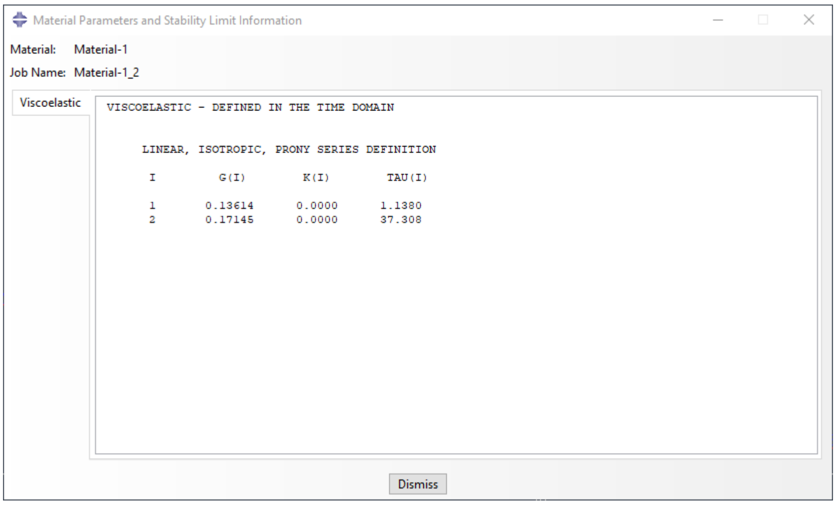


Figure 4.30: Obtained parameters.

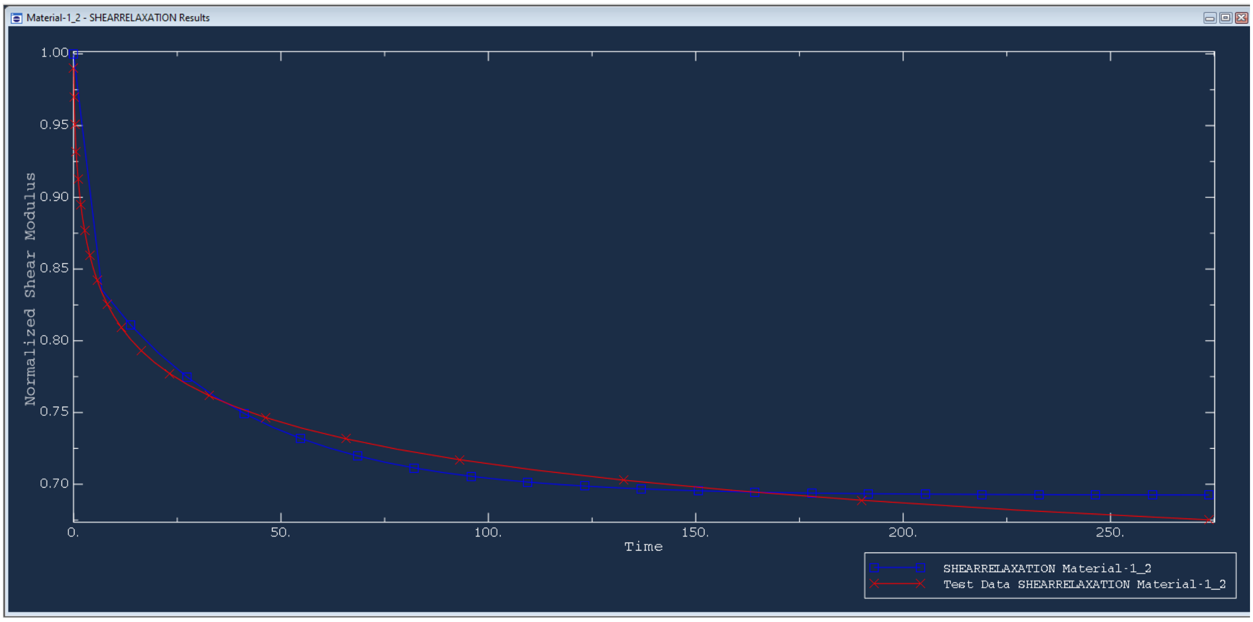


Figure 4.31: Viscoelastic

We can conclude that the fitted 2nd-order model provides higher accuracy than the requested value of 0.01 for the root-mean-square error (see Figure 4.25). The ".dat" file contains additional information about the parameter-fitting process. The corresponding content of the ".dat" file is shown in Figure 4.32.

```
*VISCOELASTIC, TIME=RELAXATION TEST DATA, ERRTOL=0.01, NMAX=13
*SHEAR TEST DATA
   VISCOELASTIC SHEAR      TEST DATA: NO CONVERGENCE FOR N =  1
   NO. OF ITERATIONS =  9     ROOT-MEAN-SQUARE ERROR (PERCENTAGE) =   3.17
   GP(TI)  0.2667
   TI        5.908

   VISCOELASTIC SHEAR      TEST DATA: CONVERGENCE FOR N =  2
   NO. OF ITERATIONS = 13     ROOT-MEAN-SQUARE ERROR (PERCENTAGE) =   0.88
   GP(TI)  0.1361      0.1715
   TI        1.138       37.31
```

Figure 4.32: Results of the fitting in the dat file.

The ".dat" file shows that the first-order model provides 3.17 root-mean-square error, therefore, it is not accepted and a higher-order model is tested. The error for the second-order model is 0.88%, which is smaller than the requested accuracy of 1%. Consequently, the software terminates the calculation and the second-order model is accepted. The definition of the root-mean-square error (RMSE):

$$RMSE = \sqrt{\frac{\sum_{i=1}^{T}\left(e_i^{\text{exp}} - e_i^{\text{sim}}\left[t_i^{\text{exp}}\right]\right)^2}{T}}, \tag{4.18}$$

where T denotes the number of data points. Note that we used the Sum of Squared Relative Differences (see Eq. (4.2)) in the Python code. If we reduce the value of the "Allowable average root-mean-square error" from the default value of 0.01 to a smaller value, than we can enforce the software to use higher-order models in the parameter-fitting process in order to reach the desired accuracy. We set the error value to 0.0001 and re-ran the evaluation process. The new results are shown in Figure 4.33.

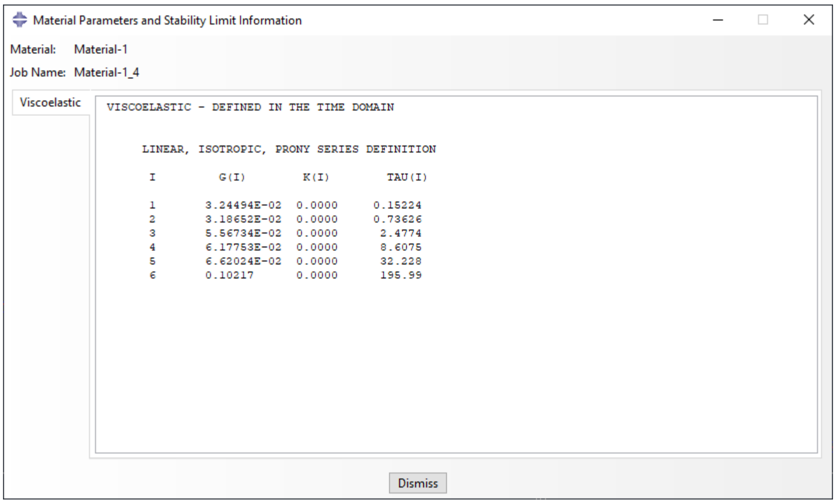


Figure 4.33: Obtained parameters.

We can conclude that the 6th-order model shown in the figure provides the accuracy we requested. The "*.dat" file contains the results for the other models (see Figure 4.34). The fitted parameters and the corresponding errors are listed in Table 4.4. The table contains a column, where the long-term relative moduli (e_∞) are also calculated. Furthermore, the Sum of Squared Relative Differences (Q) are also included. Note that the long-term relative modulus are different for each model, because we have not set its value to a desired one (see Figure 4.26).

```
*VISCOELASTIC, TIME=RELAXATION TEST DATA, ERRTOL=0.0001, NMAX=13
*SHEAR TEST DATA
   VISCOELASTIC SHEAR      TEST DATA: NO CONVERGENCE FOR N =  1
   NO. OF ITERATIONS =  9     ROOT-MEAN-SQUARE ERROR (PERCENTAGE) =   3.17
   GP(TI)  0.2667
   TI        5.908

   VISCOELASTIC SHEAR      TEST DATA: NO CONVERGENCE FOR N =  2
   NO. OF ITERATIONS = 13     ROOT-MEAN-SQUARE ERROR (PERCENTAGE) =   0.88
   GP(TI)  0.1361      0.1715
   TI        1.138       37.31

   VISCOELASTIC SHEAR      TEST DATA: NO CONVERGENCE FOR N =  3
   NO. OF ITERATIONS = 13     ROOT-MEAN-SQUARE ERROR (PERCENTAGE) =   0.24
   GP(TI)  6.8128E-02  0.1201      0.1354
   TI      0.3420       5.286       75.21

   VISCOELASTIC SHEAR      TEST DATA: NO CONVERGENCE FOR N =  4
   NO. OF ITERATIONS = 14     ROOT-MEAN-SQUARE ERROR (PERCENTAGE) =   0.06
   GP(TI)  4.4647E-02  7.9791E-02  9.3903E-02  0.1169
   TI      0.1999       2.004       13.09       117.5

   VISCOELASTIC SHEAR      TEST DATA: NO CONVERGENCE FOR N =  5
   NO. OF ITERATIONS = 16     ROOT-MEAN-SQUARE ERROR (PERCENTAGE) =   0.01
   GP(TI)  3.6298E-02  5.1130E-02  7.1846E-02  7.6689E-02  0.1077
   TI      0.1646       1.125       4.848       22.47       157.7

   VISCOELASTIC SHEAR      TEST DATA: CONVERGENCE FOR N =  6
   NO. OF ITERATIONS = 23     ROOT-MEAN-SQUARE ERROR (PERCENTAGE) =   0.00
   GP(TI)  3.2449E-02  3.1865E-02  5.5673E-02  6.1775E-02  6.6202E-02  0.1022

   TI      0.1522      0.7363       2.477       8.608       32.23       196.0
```

Figure 4.34: Results of the fitting in the dat file.

For illustration purposes, we plot the fitted relaxation functions for the first three models in Figure 4.35-4.37 using our Python code.

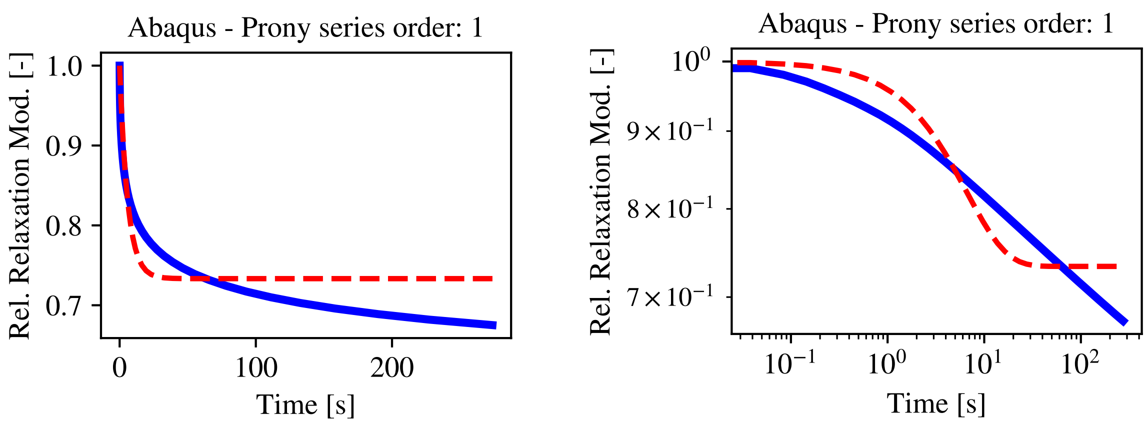


Figure 4.35: Comparison of the measurement data and the fitted (by Abaqus) relative relaxation modulus. $N = 1$.

Table 4.4: Fitted material parameters.

N	Fitted parameters	RMSE [%]	e_∞	Q
1	$e_1 = 0.2667,\quad \tau_1 = 5.908$ s	3.17	0.733	0.062061503
2	$e_1 = 0.1361,\quad \tau_1 = 1.138$ s $e_2 = 0.1715,\quad \tau_2 = 37.31$ s	0.88	0.692	0.004614584
3	$e_1 = 0.068128,\quad \tau_1 = 0.3420$ s $e_2 = 0.1201,\quad \tau_2 = 5.286$ s $e_3 = 0.1354,\quad \tau_3 = 75.21$ s	0.24	0.676	0.00034568
4	$e_1 = 0.044647,\quad \tau_1 = 10.1999$ s $e_1 = 0.079791,\quad \tau_2 = 2.004$ s $e_1 = 0.093903,\quad \tau_3 = 13.09$ s $e_1 = 0.1169,\quad \tau_4 = 117.5$ s	0.06	0.664	1.8968×10^{-5}
5	$e_1 = 0.036298,\quad \tau_1 = 0.1646$ s $e_2 = 0.051130,\quad \tau_2 = 1.125$ s $e_3 = 0.071846,\quad \tau_3 = 4.848$ s $e_4 = 0.076689,\quad \tau_4 = 22.47$ s $e_5 = 0.1077,\quad \tau_5 = 157.7$ s	0.01	0.656	1.09511×10^{-6}
6	$e_1 = 0.032449,\quad \tau_1 = 0.1522$ s $e_2 = 0.031865,\quad \tau_2 = 0.7363$ s $e_3 = 0.055673,\quad \tau_3 = 2.477$ s $e_4 = 0.061775,\quad \tau_4 = 8.608$ s $e_5 = 0.066202,\quad \tau_5 = 32.23$ s $e_6 = 0.1022,\quad \tau_6 = 196.0$ s	0.00	0.649	1.207747×10^{-7}

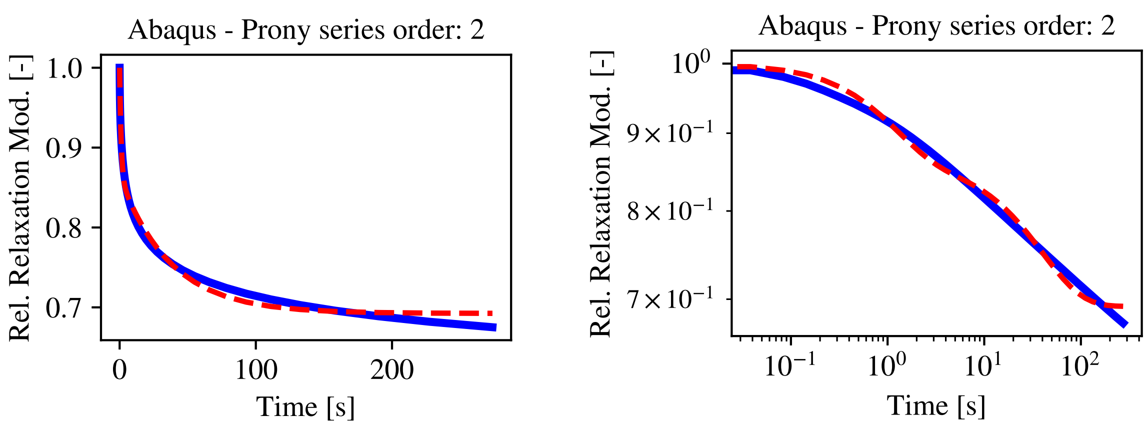


Figure 4.36: Comparison of the measurement data and the fitted (by Abaqus) relative relaxation modulus. $N = 2$.

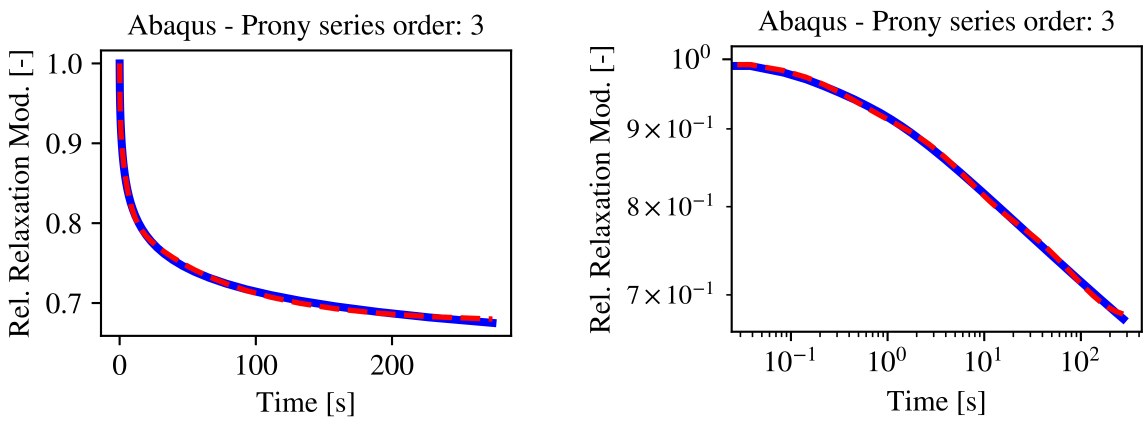


Figure 4.37: Comparison of the measurement data and the fitted (by Abaqus) relative relaxation modulus. $N = 3$.

The long-term relative modulus for the third-order model is $1 - 0.068128 - 0.1201 - 0.1354 = 0.676$. We can enforce the software to fit a relaxation function with a prescribed long-term relative modulus. For this reason we have to set the desired long-term relative modulus in the Test Dada dialog box. We set it to 0.6 (see Figure 4.38). By performing the parameter-fitting process we get the following parameters:

$$e_1 = 0.090955, \quad e_2 = 0.1405, \quad e_3 = 0.1685, \tag{4.19}$$

$$\tau_1 = 0.5384 \text{ s}, \quad \tau_2 = 10.61 \text{ s}, \quad \tau_3 = 294.1 \text{ s}. \tag{4.20}$$

Note that, $1 - e_1 - e_2 - e_3 = 0.6$. The comparison of the fitted relaxation function and the test data is visualized in Figure 4.39.

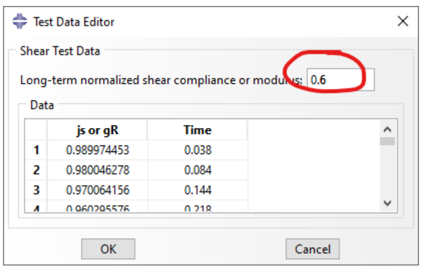


Figure 4.38: Prescribed value for the long-term relative modulus.

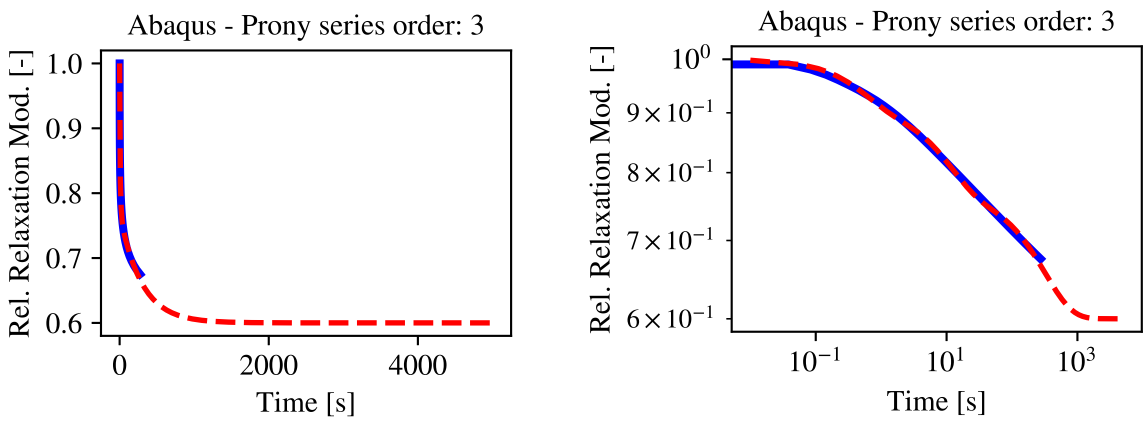


Figure 4.39: Comparison of the measurement data and the fitted (by Abaqus) relative relaxation modulus with desired long-term value. $N = 3$.

Chapter 5

Abaqus / Materials: Linear viscoelasticity

The content of this section was copied from the Abaqus online documentation from the following link: Linear Viscoelasticity. The webpage was accessed on 02/22/23. The purpose of this section is to provide a direct access to the official description in this report, without having to open the corresponding website on the Abaqus Online Documentation.

5.1 Time domain viscoelasticity

The time domain viscoelastic material model:

- describes isotropic rate-dependent material behavior for materials in which dissipative losses primarily caused by "viscous" (internal damping) effects must be modeled in the time domain;
- assumes that the shear (deviatoric) and volumetric behaviors are independent in multiaxial stress states (except when used for an elastomeric foam);
- can be used only in conjunction with Linear elastic behavior, Hyperelastic behavior of rubberlike materials, or Hyperelastic behavior in elastomeric foams to define the continuum elastic material properties;
- can be used in Abaqus/Explicit with Linear elastic traction-separation behavior;
- is active only during a transient static analysis (Quasi-static analysis), a transient implicit dynamic analysis (Implicit dynamic analysis using direct integration), an explicit dynamic analysis (Explicit dynamic analysis), a steady-state transport analysis (Steady-state transport analysis), a fully coupled temperature-displacement analysis (Fully coupled thermal-stress analysis), a fully coupled thermal-electrical-structural analysis (Fully coupled thermal-electrical-structural analysis), or a transient (consolidation) coupled pore fluid diffusion and stress analysis (Coupled pore fluid diffusion and stress analysis);
- can be used in large-strain problems;

- can be calibrated using time-dependent creep test data, time-dependent relaxation test data, or frequency-dependent cyclic test data; and
- can be used to couple viscous dissipation with the temperature field in a fully coupled temperature-displacement analysis (Fully coupled thermal-stress analysis) or a fully coupled thermal-electrical-structural analysis (Fully coupled thermal-electrical-structural analysis).

The following topics are discussed:

- Defining the shear behavior
- Defining the volumetric behavior
- Defining viscoelastic behavior for traction-separation elasticity in Abaqus/Explicit
- Temperature effects
- Numerical implementation
- Determination of viscoelastic material parameters
- Defining the rate-independent part of the material response
- Material response in different analysis procedures
- Material options
- Elements
- Output

Products: Abaqus/StandardAbaqus/ExplicitAbaqus/CAE

5.1.1 Defining the shear behavior

Time domain viscoelasticity is available in Abaqus for small-strain applications where the rate-independent elastic response can be defined with a linear elastic material model and for large-strain applications where the rate-independent elastic response must be defined with a hyperelastic or hyperfoam material model.

5.1.1.1 Small strain

Consider a shear test at small strain in which a time varying shear strain, $\gamma(t)$, is applied to the material. The response is the shear stress $\tau(t)$. The viscoelastic material model defines $\tau(t)$ as

$$\tau(t) = \int_0^t G_R(t-s)\,\dot{\gamma}(s)\,ds,$$

where $G_R(t)$ is the time-dependent "shear relaxation modulus" that characterizes the material's response. This constitutive behavior can be illustrated by considering a relaxation test in which a strain γ is suddenly applied to a specimen and then held constant for a long time. The beginning of the experiment, when the strain is suddenly applied, is taken as zero time, so that

$$\tau(t) = \int_0^t G_R(t-s)\,\dot{\gamma}(s)\,ds = G_R(t)\,\gamma \quad (\text{since } \dot{\gamma} = 0 \text{ for } t > 0),$$

where γ is the fixed strain. The viscoelastic material model is "long-term elastic" in the sense that, after having been subjected to a constant strain for a very long time, the response settles down to a constant stress; i.e., $G_R(t) \to G_\infty$ as $t \to \infty$.

The shear relaxation modulus can be written in dimensionless form:

$$g_R(t) = G_R(t)/G_0,$$

where $G_0 = G_R(0)$ is the instantaneous shear modulus, so that the expression for the stress takes the form

$$\tau(t) = G_0 \int_0^t g_R(t-s)\,\dot{\gamma}(s)\,ds.$$

The dimensionless relaxation function has the limiting values $g_R(0) = 1$ and $g_R(\infty) = G_\infty/G_0$.

5.1.1.2 Anisotropic elasticity in Abaqus/Explicit

The equation for the shear stress can be transformed by using integration by parts:

$$\tau(t) = G_0 \left(\gamma + \int_0^t \dot{g}_R(s)\,\gamma(t-s)\,ds \right).$$

It is convenient to write this equation in the form

$$\tau(t) = \tau_0(t) + \int_0^t \dot{g}_R(s)\,\tau_0(t-s)\,ds,$$

where $\tau_0(t)$ is the instantaneous shear stress at time t. This can be generalized to multi-dimensions as

$$\boldsymbol{\tau}(t) = \boldsymbol{\tau}_0(t) + \int_0^t \dot{g}_R(s)\,\boldsymbol{\tau}_0(t-s)\,ds,$$

where $\boldsymbol{\tau}(t)$ is the deviatoric part of the stress tensor and $\boldsymbol{\tau}_0(t)$ is the deviatoric part of the instantaneous stress tensor. Here the viscoelasticity is assumed to be isotropic; i.e., the relaxation function is independent of the loading direction.

This form allows a straightforward generalization to anisotropic elastic deformations, where the deviatoric part of instantaneous stress tensor is computed as $\boldsymbol{\tau}_0(t) = \overline{\boldsymbol{D}}_0 : \boldsymbol{e}$. Here $\overline{\boldsymbol{D}}_0$ is the instantaneous deviatoric elasticity tensor, and $\boldsymbol{e}$ is the deviatoric part of the strain tensor.

5.1.1.3 Large strain

The above form also allows a straightforward generalization to nonlinear elastic deformations, where the deviatoric part of the instantaneous stress $\boldsymbol{\tau}_0(t)$ is computed using a hyperelastic strain enery potential. This generalization yields a linear viscoelasticity model, in the sense that the dimensionless stress relaxation function is independent of the magnitude of the deformation. In the above equation the instantaneous stress, $\boldsymbol{\tau}_0$, applied at time $t-s$ influences the stress, $\boldsymbol{\tau}$, at time t. Therefore, to create a proper finite-strain formulation, it is necessary to map the stress that existed in the configuration at time $t-s$ into the configuration at time t. In Abaqus this is done by means of the "standard-push-forward" transformation with the relative deformation gradient $\boldsymbol{F}_{t-s}(t)$:

$$\mathbf{F}_{t-s}(t) = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}(t-s)},$$

which results in the following hereditary integral:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \operatorname{dev}\left[\int_0^t \dot{g}_R(s)\, \overline{\mathbf{F}}_t^{-1}(t-s) \cdot \boldsymbol{\tau}_0(t-s) \cdot \overline{\mathbf{F}}_t^{-T}(t-s)\, ds\right],$$

where $\boldsymbol{\tau}$ is the deviatoric part of the Kirchhoff stress. The finite-strain theory is described in more detail in Finite-strain viscoelasticity.

5.1.2 Defining the volumetric behavior

The volumetric behavior can be written in a form that is similar to the shear behavior:

$$p(t) = -\mathrm{K}_0 \int_0^t k_R(t-s)\, \dot{\varepsilon}^{vol}(s)\, ds,$$

where p is the hydrostatic pressure, K_0 is the instantaneous elastic bulk modulus, $k_R(t)$ is the dimensionless bulk relaxation modulus, and ε^{vol} is the volume strain.

The above expansion applies to small as well as finite strain since the volume strains are generally small and there is no need to map the pressure from time $t-s$ to time t.

5.1.3 Defining viscoelastic behavior for traction-separation elasticity in Abaqus/Explicit

Time domain viscoelasticity can be used in Abaqus/Explicit to model rate-dependent behavior of cohesive elements with traction-separation elasticity (Defining elasticity in terms of tractions and separations for cohesive elements). In this case the evolution equation for the normal and two shear nominal tractions take the form:

$$t_n(t) = t_n^0(t) + \int_0^t \dot{k}_R(s)\, t_n^0(t-s)\, ds,$$

$$t_s(t) = t_s^0(t) + \int_0^t \dot{g}_R(s)\, t_s^0(t-s)\, ds,$$

$$t_t(t) = t_t^0(t) + \int_0^t \dot{g}_R(s)\, t_t^0(t-s)\, ds,$$

where $t_n^0(t)$, $t_s^0(t)$, and $t_t^0(t)$ are the instantaneous nominal tractions at time t in the normal and the two local shear directions, respectively. The functions $g_R(t)$ and $k_R(t)$ now represent the dimensionless shear and normal relaxation moduli, respectively. Note the close similarity between the viscoelastic formulation for the continuum elastic response discussed in the previous sections and the formulation for cohesive behavior with traction-separation elasticity after reinterpreting shear and bulk relaxation as shear and normal relaxation.

For the case of uncoupled traction elasticity, the viscoelastic normal and shear behaviors are assumed to be independent. The normal relaxation modulus is defined as

$$k_R(t) = K_{nn}(t)/K_{nn}^0,$$

where K_{nn}^0 is the instantaneous normal moduli. The shear relaxation modulus is assumed to be isotropic and, therefore, independent of the local shear directions:

$$g_R(t) = K_{ss}(t)/K_{ss}^0 = K_{tt}(t)/K_{tt}^0,$$

where K_{ss}^0 and K_{tt}^0 are the instantaneous shear moduli.

For the case of coupled traction-separation elasticity the normal and shear relaxation moduli must be the same, $g_R(t) = k_R(t)$, and you must use the same relaxation data for both behaviors.

5.1.4 Temperature effects

The effect of temperature, θ, on the material behavior is introduced through the dependence of the instantaneous stress, τ_0, on temperature and through a reduced time concept. The expression for the linear-elastic shear stress is rewritten as

$$\tau(t) = G_0(\theta) \int_0^t g_R(\xi(t) - \xi(s))\, \dot{\gamma}(s)\, ds,$$

where the instantaneous shear modulus G_0 is temperature dependent and $\xi(t)$ is the reduced time, defined by

$$\xi(t) = \int_0^t \frac{ds}{A(\theta(s))},$$

where $A(\theta(t))$ is a shift function at time t. This reduced time concept for temperature dependence is usually referred to as thermorheologically simple (TRS) temperature dependence. Often the shift function is approximated by the Williams-Landel-Ferry (WLF) form. See Thermorheologically simple temperature effects below, for a description of the WLF and other forms of the shift function available in Abaqus.
The reduced time concept is also used for the volumetric behavior, the large-strain formulation, and the traction-separation formulation.

5.1.5 Numerical implementation

Abaqus assumes that the viscoelastic material is defined by a Prony series expansion of the dimensionless relaxation modulus:

$$g_R(t) = 1 - \sum_{i=1}^{N} \bar{g}_i^P \left(1 - e^{-t/\tau_i^G}\right),$$

where N, $\bar{g}_i^P$, and τ_i^G, $i = 1, 2, ...N$, are material constants. For linear isotropic elasticity, substitution in the small-strain expression for the shear stress yields

$$\tau(t) = G_0 \left(\gamma - \sum_{i=1}^{N} \gamma_i\right),$$

where

$$\gamma_i = \frac{\bar{g}_i^P}{\tau_i^G} \int_0^t e^{-s/\tau_i^G} \gamma(t - s)\, ds.$$

The γ_i are interpreted as state variables that control the stress relaxation, and

$$\gamma^{cr} = \sum_{i=1}^{N} \gamma_i$$

is the “creep” strain: the difference between the total mechanical strain and the instantaneous elastic strain (the stress divided by the instantaneous elastic modulus). In Abaqus/Standard γ^{cr} is available as the creep strain output variable CE (Abaqus/Standard output variable identifiers).

A similar Prony series expansion is used for the volumetric response, which is valid for both small- and finite-strain applications:

$$p = -K_0 \left(\varepsilon^{vol} - \sum_{i=1}^{N} \varepsilon_i^{vol} \right),$$

where

$$\varepsilon_i^{vol} = \frac{\bar{k}_i^P}{\tau_i^K} \int_0^t e^{-s/\tau_i^K} \varepsilon^{vol} (t-s)\, ds.$$

Abaqus assumes that $\tau_i^G = \tau_i^K = \tau_i$.

For linear anisotropic elasticity, the Prony series expansion, in combination with the generalized small-strain expression for the deviatoric stress, yields

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 - \sum_{i=1}^{N} \boldsymbol{\tau}_i,$$

where

$$\boldsymbol{\tau}_i = \frac{\bar{g}_i^P}{\tau_i^G} \int_0^t e^{-s/\tau_i^G} \boldsymbol{\tau}_0 (t-s) \; ds.$$

The $\boldsymbol{\tau}_i$ are interpreted as state variables that control the stress relaxation.

For finite strains, the Prony series expansion, in combination with the finite-strain expression for the shear stress, produces the following expression for the deviatoric stress:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 - \sum_{i=1}^{N} \operatorname{dev} (\boldsymbol{\tau}_i),$$

where

$$\boldsymbol{\tau}_i = \frac{\bar{g}_i^P}{\tau_i^G} \int_0^t e^{-s/\tau_i^G} \overline{\mathbf{F}}_t^{-1} (t-s) \cdot \boldsymbol{\tau}_0 (t-s) \cdot \overline{\mathbf{F}}_t^{-T} (t-s)\, ds.$$

The $\boldsymbol{\tau}_i$ are interpreted as state variables that control the stress relaxation.

For traction-separation elasticity, the Prony series expansion yields

$$\mathbf{t} = \begin{Bmatrix} t_n \\ t_s \\ t_t \end{Bmatrix} = \begin{Bmatrix} t_n^0 \\ t_s^0 \\ t_t^0 \end{Bmatrix} - \sum_{i=1}^{N} \begin{Bmatrix} t_n^i \\ t_s^i \\ t_t^i \end{Bmatrix} = \mathbf{t}^0 - \sum_{i=1}^{N} \mathbf{t}^i,$$

where

$$t_n^i = \frac{\bar{k}_i^P}{\tau_i^K} \int_0^t e^{-s/\tau_i^K} t_n^0\,(t-s)\;ds,$$

$$t_s^i = \frac{\bar{g}_i^P}{\tau_i^G} \int_0^t e^{-s/\tau_i^G} t_s^0\,(t-s)\;ds,$$

$$t_t^i = \frac{\bar{g}_i^P}{\tau_i^G} \int_0^t e^{-s/\tau_i^G} t_t^0\,(t-s)\;ds.$$

The $\boldsymbol{t}_i$ are interpreted as state variables that control the relaxation of the traction stresses.
If the instantaneous material behavior is defined by linear elasticity or hyperelasticity, the bulk and shear behavior can be defined independently. However, if the instantaneous behavior is defined by the hyperfoam model, the deviatoric and volumetric constitutive behavior are coupled and it is mandatory to use the same relaxation data for both behaviors. For linear anisotropic elasticity, the same relaxation data should be used for both behaviors when the elasticity definition is such that the deviatoric and volumetric response is coupled. Similarly, for coupled traction-separation elasticity you must use the same relaxation data for the normal and shear behaviors.
In all of the above expressions temperature dependence is readily introduced by replacing e^{-s/τ_i^G} by $e^{-\xi(s)/\tau_i^G}$ and e^{-s/τ_i^K} by $e^{-\xi(s)/\tau_i^K}$.

5.1.6 Determination of viscoelastic material parameters

The above equations are used to model the time-dependent shear and volumetric behavior of a viscoelastic material. The relaxation parameters can be defined in one of four ways: direct specification of the Prony series parameters, inclusion of creep test data, inclusion of relaxation test data, or inclusion of frequency-dependent data obtained from sinusoidal oscillation experiments. Temperature effects are included in the same manner regardless of the method used to define the viscoelastic material.
Abaqus/CAE allows you to evaluate the behavior of viscoelastic materials by automatically creating response curves based on experimental test data or coefficients. A viscoelastic material can be evaluated only if it is defined in the time domain and includes hyperelastic and/or elastic material data. See Evaluating hyperelastic, hyperfoam and viscoelastic material behavior.

5.1.6.1 Direct specification

The Prony series parameters $\bar{g}_i^P$, $\bar{k}_i^P$, and τ_i can be defined directly for each term in the Prony series. There is no restriction on the number of terms that can be used. If a relaxation time is associated with only one of the two moduli, leave the other one blank or enter a zero. The data should be given in ascending order of the relaxation time. The number of lines of data given defines the number of terms in the Prony series, N. If this model is used in conjunction with the hyperfoam material model, the two modulus ratios have to be the same ($\bar{g}_i^P = \bar{k}_i^P$).
Input File Usage:
`*VISCOELASTIC, TIME=PRONY`
The data line is repeated as often as needed to define the first, second, third, etc. terms in the Prony series.
Abaqus/CAE Usage:
Property module: material editor: **Mechanical** → **Elasticity** → **Viscoelastic: Domain: Time** and **Time: Prony**
Enter as many rows of data in the table as needed to define the first, second, third, etc. terms in the Prony series.

5.1.6.2 Creep test data

If creep test data are specified, Abaqus will calculate the terms in the Prony series automatically. The normalized shear and bulk compliances are defined as

$$j_S(t) = G_0 J_S(t) \qquad \text{and} \qquad j_K(t) = K_0 J_K(t),$$

where $J_S(t) = \gamma(t)/\tau_0$ is the shear compliance, $\gamma(t)$ is the total shear strain, and τ_0 is the constant shear stress in a shear creep test; $J_K(t) = \varepsilon^{vol}(t)/p_0$ is the volumetric compliance, $\varepsilon^{vol}(t)$ is the total volumetric strain, and p_0 is the constant pressure in a volumetric creep test. At time $t = 0$, $j_S(0) = j_K(0) = 1$.
The creep data are converted to relaxation data through the convolution integrals

$$\int_0^t g_R(s)\, j_S(t-s)\, ds = t \qquad \text{and} \qquad \int_0^t k_R(s)\, j_K(t-s)\, ds = t.$$

Abaqus then uses the normalized shear modulus $g_R(t)$ and normalized bulk modulus $k_R(t)$ in a nonlinear least-squares fit to determine the Prony series parameters.

Using the shear and volumetric test data consecutively
The shear test data and volumetric test data can be used consecutively to define the normalized shear and bulk compliances as functions of time. A separate least-squares fit is performed on each data set; and the two derived sets of Prony series parameters, $\left(\bar{g}_i^P, \tau_i^G\right)$ and $\left(\bar{k}_i^P, \tau_i^K\right)$, are merged into one set of parameters, $\left(\bar{g}_i^P, \bar{k}_i^P, \tau_i\right)$.
Input File Usage:

Use the following three options. The first option is required. Only one of the second and third options is required.

`*VISCOELASTIC, TIME=CREEP TEST DATA`

`*SHEAR TEST DATA`

`*VOLUMETRIC TEST DATA`

Abaqus/CAE Usage:

Property module: material editor: **Mechanical** → **Elasticity** → **Viscoelastic: Domain: Time** and **Time: Creep test data**

In addition, select one or both of the following:

Test Data → **Shear Test Data**

Test Data → **Volumetric Test Data**

Using the combined test data

Alternatively, the combined test data can be used to specify the normalized shear and bulk compliances simultaneously as functions of time. A single least-squares fit is performed on the combined set of test data to determine one set of Prony series parameters, $\left(\bar{g}_i^P, \bar{k}_i^P, \tau_i\right)$.

Input File Usage:

Use both of the following options:

`*VISCOELASTIC, TIME=CREEP TEST DATA`

`*COMBINED TEST DATA`

Abaqus/CAE Usage:

Property module: material editor: **Mechanical** → **Elasticity** → **Viscoelastic: Domain: Time, Time: Creep test data**, and

Test Data → **Combined Test Data**

5.1.6.3 Relaxation test data

As with creep test data, Abaqus will calculate the Prony series parameters automatically from relaxation test data.

Using the shear and volumetric test data consecutively

Again, the shear test data and volumetric test data can be used consecutively to define the relaxation moduli as functions of time. A separate nonlinear least-squares fit is performed on each data set; and the two derived sets of Prony series parameters, $\left(\bar{g}_i^P, \tau_i^G\right)$ and $\left(\bar{k}_i^P, \tau_i^K\right)$, are merged into one set of parameters, $\left(\bar{g}_i^P, \bar{k}_i^P, \tau_i\right)$.

Input File Usage:

Use the following three options. The first option is required. Only one of the second and third options is required.

`*VISCOELASTIC, TIME=RELAXATION TEST DATA`

`*SHEAR TEST DATA`

`*VOLUMETRIC TEST DATA`

Abaqus/CAE Usage:

Property module: material editor: **Mechanical** → **Elasticity** → **Viscoelastic: Domain: Time** and **Time: Relaxation test data**
In addition, select one or both of the following:
Test Data → **Shear Test Data**
Test Data → **Volumetric Test Data**

Using the combined test data
Alternatively, the combined test data can be used to specify the relaxation moduli simultaneously as functions of time. A single least-squares fit is performed on the combined set of test data to determine one set of Prony series parameters, $\left(\bar{g}_i^P, \bar{k}_i^P, \tau_i\right)$.
Input File Usage:
Use both of the following options:
`*VISCOELASTIC, TIME=RELAXATION TEST DATA`
`*COMBINED TEST DATA`
Abaqus/CAE Usage:
Property module: material editor: **Mechanical** → **Elasticity** → **Viscoelastic: Domain: Time, Time: Relaxation test data**, and
Test Data → **Combined Test Data**

5.1.6.4 Frequency-dependent test data

The Prony series terms can also be calibrated using frequency-dependent test data. In this case Abaqus uses analytical expressions that relate the Prony series relaxation functions to the storage and loss moduli. The expressions for the shear moduli, obtained by converting the Prony series terms from the time domain to the frequency domain by making use of Fourier transforms, can be written as follows:

$$G_s\left(\omega\right) = G_0\left[1 - \sum_{i=1}^{N} \bar{g}_i^P\right] + G_0 \sum_{i=1}^{N} \frac{\bar{g}_i^P \tau_i^2 \omega^2}{1 + \tau_i^2 \omega^2},$$

$$G_\ell\left(\omega\right) = G_0 \sum_{i=1}^{N} \frac{\bar{g}_i^P \tau_i \omega}{1 + \tau_i^2 \omega^2},$$

where $G_s\left(\omega\right)$ is the storage modulus, $G_l\left(\omega\right)$ is the loss modulus, ω is the angular frequency, and N is the number of terms in the Prony series. These expressions are used in a nonlinear least-squares fit to determine the Prony series parameters from the storage and loss moduli cyclic test data obtained at M frequencies by minimizing the error function χ^2:

$$\chi^2 = \sum_{i=1}^{M} \frac{1}{G_\infty^2}\left[\left(G_s - \overline{G}_s\right)_i^2 + \left(G_\ell - \overline{G}_\ell\right)_i^2\right],$$

where $\overline{G}_s$ and $\overline{G}_l$ are the test data and G_0 and G_∞, respectively, are the instantaneous and long-term shear moduli. The expressions for the bulk moduli, $K_s(\omega)$ and $K_l(\omega)$, are written analogously.

The frequency domain data are defined in tabular form by giving the real and imaginary parts of ωg^* and ωk^* - where ω is the circular frequency - as functions of frequency in cycles per time. $g^*(\omega)$ is the Fourier transform of the nondimensional shear relaxation function $g(t) = G_R(t)/G_\infty - 1$. Given the frequency-dependent storage and loss moduli $G_s(\omega)$, $G_l(\omega)$, $K_s(\omega)$, and $K_l(\omega)$, the real and imaginary parts of ωg^* and ωk^* are then given as

$$\omega\Re(g^*) = G_\ell/G_\infty, \qquad \omega\Im(g^*) = 1 - G_s/G_\infty,$$

$$\omega\Re(k^*) = \mathrm{K}_\ell/\mathrm{K}_\infty, \qquad \omega\Im(k^*) = 1 - \mathrm{K}_s/\mathrm{K}_\infty,$$

where G_∞ and K_∞ are the long-term shear and bulk moduli determined from the elastic or hyperelastic properties.

Input File Usage:

`*VISCOELASTIC, TIME=FREQUENCY DATA`

Abaqus/CAE Usage:

Property module: material editor: **Mechanical** → **Elasticity** → **Viscoelastic: Domain: Time** and **Time: Frequency data**

5.1.6.5 Calibrating the Prony series parameters

You can specify two optional parameters related to the calibration of Prony series parameters for viscoelastic materials: the error tolerance and N_{max}. The error tolerance is the allowable average root-mean-square error of data points in the least-squares fit, and its default value is 0.01. N_{max} is the maximum number of terms N in the Prony series, and its default (and maximum) value is 13. Abaqus will perform the least-squares fit from $N = 1$ to $N = N_{max}$ until convergence is achieved for the lowest N with respect to the error tolerance.

The following are some guidelines for determining the number of terms in the Prony series from test data. Based on these guidelines, you can choose N_{max}.

- There should be enough data pairs for determining all the parameters in the Prony series terms. Thus, assuming that N is the number of Prony series terms, there should be a total of at least $2N$ data points in shear test data, $2N$ data points in volumetric test data, $3N$ data points in combined test data, and $2N$ data points in the frequency domain.

- The number of terms in the Prony series should be typically not more than the number of logarithmic "decades" spanned by the test data. The number of logarithmic "decades" is defined as $\log_{10}(t_{max}/t_{min})$, where t_{max} and t_{min} are the maximum and minimum time in the test data, respectively.

Input File Usage:
`*VISCOELASTIC, ERRTOL=error_tolerance, NMAX=`N_{max}
Abaqus/CAE Usage:
Property module: material editor: **Mechanical** → **Elasticity** → **Viscoelastic: Domain: Time; Time: Creep test data, Relaxation test data, or Frequency data; Maximum number of terms in the Prony series:** N_{max}; and **Allowable average root-mean-square error:** *error_ tolerance*

5.1.6.6 Thermorheologically simple temperature effects

Regardless of the method used to define the viscoelastic behavior, thermo-rheologically simple temperature effects can be included by specifying the method used to define the shift function. Abaqus supports the following forms of the shift function: the Williams-Landel-Ferry (WLF) form, the Arrhenius form, and user-defined forms.
Thermo-rheologically simple temperature effects can also be included in the definition of equation of state models with viscous shear behavior (see Viscous shear behavior).

Williams-Landel-Ferry (WLF) form
The shift function can be defined by the Williams-Landel-Ferry (WLF) approximation, which takes the form:

$$\log_{10}(A) = -\frac{C_1\,(\theta - \theta_0)}{C_2 + (\theta - \theta_0)},$$

where θ_0 is the reference temperature at which the relaxation data are given; θ is the temperature of interest; and C_1, C_2 are calibration constants obtained at this temperature. If $\theta \leq \theta_0 - C_2$, deformation changes will be elastic, based on the instantaneous moduli.
For more information on the WLF equation, see Viscoelasticity.
Input File Usage:
`*TRS, DEFINITION=WLF`
Abaqus/CAE Usage:
Property module: material editor: **Mechanical** → **Elasticity** → **Viscoelastic: Domain: Time, Time: any method,** and **SuboptionsTrs: Shift function: WLF**

Arrhenius form
The Arrhenius shift function is commonly used for semi-crystalline polymers. It takes the form

$$\ln(A) = \frac{E_0}{R}\left(\frac{1}{\theta - \theta^Z} - \frac{1}{\theta_0 - \theta^Z}\right),$$

where E_0 is the activation energy, R is the universal gas constant, θ^Z is the absolute zero in the temperature scale being used, θ_0 is the reference temperature at which the relaxation data are given, and θ is the temperature of interest.

Input File Usage:
Use the following option to define the Arrhenius shift function:
`*TRS, DEFINITION=ARRHENIUS`
In addition, use the *PHYSICAL CONSTANTS option to specify the universal gas constant and absolute zero.
Abaqus/CAE Usage:
The Arrhenius shift function is not supported in Abaqus/CAE.

User-defined form
The shift function can be specified alternatively in user subroutines UTRS in Abaqus/Standard and VUTRS in Abaqus/Explicit.
Input File Usage:
`*TRS, DEFINITION=USER`
Abaqus/CAE Usage:
Property module: material editor: **Mechanical** → **Elasticity** → **Viscoelastic: Domain: Time, Time: any method**, and **Suboptions** → **Trs: Shift function: User subroutine UTRS**

5.1.7 Defining the rate-independent part of the material response

In all cases elastic moduli must be specified to define the rate-independent part of the material behavior. Small-strain linear elastic behavior is defined by an elastic material model (Linear elastic behavior), and large-deformation behavior is defined by a hyperelastic (Hyperelastic behavior of rubberlike materials) or hyperfoam (Hyperelastic behavior in elastomeric foams) material model. The rate-independent elasticity for any of these models can be defined in terms of either instantaneous elastic moduli or long-term elastic moduli. The choice of defining the elasticity in terms of instantaneous or long-term moduli is a matter of convenience only; it does not have an effect on the solution.
The effective relaxation moduli are obtained by multiplying the instantaneous elastic moduli with the dimensionless relaxation functions as described below.

5.1.7.1 Linear elastic isotropic materials

For linear elastic isotropic material behavior

$$G_R(t) = G_0 \left(1 - \sum_{k=1}^{N} \bar{g}_k^P \left(1 - e^{-t/\tau_k}\right)\right)$$

and

$$\mathrm{K}_R(t) = \mathrm{K}_0 \left(1 - \sum_{k=1}^{N} \bar{k}_k^P \left(1 - e^{-t/\tau_k}\right)\right),$$

where G_0 and K_0 are the instantaneous shear and bulk moduli determined from the values of the user-defined instantaneous elastic moduli E_0 and ν_0 : $G_0 = E_0/2\,(1+\nu_0)$ and $K_0 = E_0/3\,(1-2\nu_0)$.
If long-term elastic moduli are defined, the instantaneous moduli are determined from

$$G_\infty = G_0\left(1-\sum_{k=1}^{N}\bar{g}_k^P\right) \qquad \text{and} \qquad K_\infty = K_0\left(1-\sum_{k=1}^{N}\bar{k}_k^P\right).$$

5.1.7.2 Linear elastic anisotropic materials

For linear elastic anisotropic material behavior the relaxation coefficients are applied to the elastic moduli as

$$\overline{\mathbf{D}}_R\,(t) = \overline{\mathbf{D}}_0\left(1-\sum_{k=1}^{N}\bar{g}_k^P\left(1-e^{-t/\tau_k}\right)\right)$$

and

$$\mathrm{K}_R\,(t) = \mathrm{K}_0\left(1-\sum_{k=1}^{N}\bar{k}_k^P\left(1-e^{-t/\tau_k}\right)\right),$$

where $\overline{\boldsymbol{D}}_0$ and K_0 are the instantaneous deviatoric elasticity tensor and bulk moduli determined from the values of the user-defined instantaneous elastic moduli $\boldsymbol{D}_0$. If both shear and bulk relaxation coefficients are specified and they are unequal, Abaqus issues an error message if the elastic moduli $\boldsymbol{D}_0$ is such that the deviatoric and volumetric response is coupled.
If long-term elastic moduli are defined, the instantaneous moduli are determined from

$$\overline{\mathbf{D}}_\infty = \overline{\mathbf{D}}_0\left(1-\sum_{k=1}^{N}\bar{g}_k^P\right) \qquad \text{and} \qquad K_\infty = K_0\left(1-\sum_{k=1}^{N}\bar{k}_k^P\right).$$

5.1.7.3 Hyperelastic materials

For hyperelastic material behavior the relaxation coefficients are applied either to the constants that define the energy function or directly to the energy function. For the polynomial function and its particular cases (reduced polynomial, Mooney-Rivlin, neo-Hookean, and Yeoh)

$$C_{ij}^R\,(t) = C_{ij}^0\left(1-\sum_{k=1}^{N}\bar{g}_k^P\left(1-e^{-t/\tau_k}\right)\right),$$

for the Ogden function

$$\mu_i^R\,(t) = \mu_i^0\left(1-\sum_{k=1}^{N}\bar{g}_k^P\left(1-e^{-t/\tau_k}\right)\right),$$

for the Arruda-Boyce and Van der Waals functions

$$\mu^R(t) = \mu^0 \left(1 - \sum_{k=1}^{N} \bar{g}_k^P \left(1 - e^{-t/\tau_k}\right)\right),$$

and for the Marlow function

$$U_{dev}^R(t) = U_{dev}^0 \left(1 - \sum_{k=1}^{N} \bar{g}_k^P \left(1 - e^{-t/\tau_k}\right)\right).$$

For the coefficients governing the compressible behavior of the polynomial models and the Ogden model

$$D_i^R(t) = D_i^0 / \left(1 - \sum_{k=1}^{N} \bar{k}_k^P \left(1 - e^{-t/\tau_k}\right)\right),$$

for the Arruda-Boyce and Van der Waals functions

$$D^R(t) = D^0 / \left(1 - \sum_{k=1}^{N} \bar{k}_k^P \left(1 - e^{-t/\tau_k}\right)\right),$$

and for the Marlow function

$$U_{vol}^R(t) = U_{vol}^0 \left(1 - \sum_{k=1}^{N} \bar{k}_k^P \left(1 - e^{-t/\tau_k}\right)\right).$$

If long-term elastic moduli are defined, the instantaneous moduli are determined from

$$C_{ij}^\infty = C_{ij}^0 \left(1 - \sum_{k=1}^{N} \bar{g}_k^P\right), \quad \text{or} \quad \mu_i^\infty = \mu_i^0 \left(1 - \sum_{k=1}^{N} \bar{g}_k^P\right), \quad \text{or} \quad \mu^\infty = \mu^0 \left(1 - \sum_{k=1}^{N} \bar{g}_k^P\right),$$

while the instantaneous bulk compliance moduli are obtained from

$$D_i^\infty = D_i^0 / \left(1 - \sum_{k=1}^{N} \bar{k}_k^P\right), \quad \text{or} \quad D^\infty = D^0 / \left(1 - \sum_{k=1}^{N} \bar{k}_k^P\right);$$

for the Marlow functions we have

$$U_{dev}^\infty = U_{dev}^0 \left(1 - \sum_{k=1}^{N} \bar{g}_k^P\right) \quad \text{and} \quad U_{vol}^\infty = U_{vol}^0 \left(1 - \sum_{k=1}^{N} \bar{k}_k^P\right).$$

5.1.7.4 Mullins effect

If long-term moduli are defined for the underlying hyperelastic behavior, the instantaneous value of the parameter m in Mullins effect is determined from

$$m^{\infty} = m^{0}\left(1 - \sum_{k=1}^{N} \bar{g}_{k}^{P}\right).$$

5.1.7.5 Elastomeric foams

For elastomeric foam material behavior the instantaneous shear and bulk relaxation coefficients are assumed to be equal and are applied to the material constants μ_i in the energy function:

$$\mu_i^R(t) = \mu_i^0\left(1 - \sum_{k=1}^{N} \bar{g}_k^P\left(1 - e^{-t/\tau_k}\right)\right) = \mu_i^0\left(1 - \sum_{k=1}^{N} \bar{k}_k^P\left(1 - e^{-t/\tau_k}\right)\right).$$

If only the shear relaxation coefficients are specified, the bulk relaxation coefficients are set equal to the shear relaxation coefficients and vice versa. If both shear and bulk relaxation coefficients are specified and they are unequal, Abaqus issues an error message.
If long-term elastic moduli are defined, the instantaneous moduli are determined from

$$\mu_i^{\infty} = \mu_i^0\left(1 - \sum_{k=1}^{N} \bar{g}_k^P\right) = \mu_i^0\left(1 - \sum_{k=1}^{N} \bar{k}_k^P\right).$$

5.1.7.6 Traction-separation elasticity

For cohesive elements with uncoupled traction-separation elastic behavior:

$$K_{nn}(t) = K_{nn}^0\left(1 - \sum_{k=1}^{N} \bar{k}_k^P\left(1 - e^{-t/\tau_k}\right)\right),$$

$$K_{ss}(t) = K_{ss}^0\left(1 - \sum_{k=1}^{N} \bar{g}_k^P\left(1 - e^{-t/\tau_k}\right)\right),$$

and

$$K_{tt}(t) = K_{tt}^0\left(1 - \sum_{k=1}^{N} \bar{g}_k^P\left(1 - e^{-t/\tau_k}\right)\right),$$

where K_{nn}^0 is the instantaneous normal modulus and K_{ss}^0 and K_{tt}^0 are the instantaneous shear moduli. If long-term elastic moduli are defined, the instantaneous moduli are determined from

$$K_{nn}^{\infty}/K_{nn}^{0} = \left(1 - \sum_{k=1}^{N} \bar{k}_k^P\right), \qquad \text{and} \qquad K_{ss}^{\infty}/K_{ss}^{0} = K_{tt}^{\infty}/K_{tt}^{0} = \left(1 - \sum_{k=1}^{N} \bar{g}_k^P\right).$$

For cohesive elements with coupled traction-separation elastic behavior the shear and bulk relaxation coefficients must be equal:

$$\mathbf{K}(t) = \mathbf{K}^0 \left(1 - \sum_{k=1}^{N} \bar{k}_k^P \left(1 - e^{-t/\tau_k}\right)\right) = \mathbf{K}^0 \left(1 - \sum_{k=1}^{N} \bar{g}_k^P \left(1 - e^{-t/\tau_k}\right)\right),$$

where $\boldsymbol{K}^0$ is the user-defined instantaneous elasticity matrix. If long-term elastic moduli are defined, the instantaneous moduli are determined from

$$\mathbf{K}^{\infty} = \mathbf{K}^0 \left(1 - \sum_{k=1}^{N} \bar{k}_k^P\right) = \mathbf{K}^0 \left(1 - \sum_{k=1}^{N} \bar{g}_k^P\right).$$

5.1.8 Material response in different analysis procedures

The time-domain viscoelastic material model is active during the following procedures:

- transient static analysis (Quasi-static analysis),
- transient implicit dynamic analysis (Implicit dynamic analysis using direct integration),
- explicit dynamic analysis (Explicit dynamic analysis),
- steady-state transport analysis (Steady-state transport analysis),
- fully coupled temperature-displacement analysis (Fully coupled thermal-stress analysis),
- fully coupled thermal-electrical-structural analysis (Fully coupled thermal-electrical-structural analysis), and
- transient (consolidation) coupled pore fluid diffusion and stress analysis (Coupled pore fluid diffusion and stress analysis).

Viscoelastic material response is always ignored in a static analysis. It can also be ignored in a coupled temperature-displacement analysis, a coupled thermal-electrical-structural analysis, or a soils consolidation analysis by specifying that no creep or viscoelastic response is occurring during the step even if creep or viscoelastic material properties are defined (see Fully coupled thermal-stress analysis or Coupled pore fluid diffusion and stress analysis). In these cases it is assumed that the loading is applied instantaneously, so that the resulting response corresponds to an elastic solution based on instantaneous elastic moduli.
Abaqus/Standard also provides the option to obtain the fully relaxed long-term elastic solution directly in a static or steady-state transport analysis without having to perform a transient

analysis. The long-term value is used for this purpose. The viscous damping stresses (the internal stresses associated with each of the Prony-series terms) are increased gradually from their values at the beginning of the step to their long-term values at the end of the step if the long-term value is specified.

5.1.9 Material options

The viscoelastic material model must be combined with an elastic material model. It is used with the isotropic linear elasticity model (Linear elastic behavior) to define classical, linear, small-strain, viscoelastic behavior or with the hyperelastic (Hyperelastic behavior of rubber-like materials) or hyperfoam (Hyperelastic behavior in elastomeric foams) models to define large-deformation, nonlinear, viscoelastic behavior. It can also be used with anisotropic linear elasticity and with traction-separation elastic behavior in Abaqus/Explicit. The elastic properties defined for these models can be temperature dependent.
Viscoelasticity cannot be combined with any of the plasticity models. See Combining material behaviors for more details.

5.1.10 Elements

The time domain viscoelastic material model can be used with any stress/displacement, coupled temperature-displacement, or thermal-electrical-structural element in Abaqus.

5.1.11 Output

In addition to the standard output identifiers available in Abaqus (Abaqus/Standard output variable identifiers and Abaqus/Explicit output variable identifiers), the following variables have special meaning in Abaqus/Standard if viscoelasticity is defined:
EE: Elastic strain corresponding to the stress state at time t and the instantaneous elastic material properties.
CE: Equivalent creep strain defined as the difference between the total strain and the elastic strain.

5.1.11.1 Considerations for steady-state transport analysis

When a steady-state transport analysis (Steady-state transport analysis) is combined with large-strain viscoelasticity, the viscous dissipation, CENER, is computed as the energy dissipated per revolution as a material point is transported around its streamline; that is,

$$W_{cr} = \oint \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}.$$

Consequently, all the material points in a given streamline report the same value for CENER, and other derived quantities such as ELCD and ALLCD also have the meaning of dissipation per revolution. The recoverable elastic strain energy density, SENER, is approximated as

$$W_{el} = W_{el}^t + W_{cr}^t + \Delta W - W_{cr},$$

where ΔW is the incremental energy input and t is the time at the beginning of the current increment. Since two different units are used in the quantities appearing in the above equation, a proper meaning cannot be assigned to quantities such as SENER, ELSE, ALLSE, and ALLIE.

5.1.11.2 Considerations for large-strain viscoelasticity

In Abaqus/Standard the viscous energy dissipated is computed only approximately for large-strain viscoelasticity.
Abaqus/Explicit does not compute the viscous dissipation for performance reasons for the case of large-strain viscoelasticity. Instead, the contribution of viscous dissipation is included in the strain energy output, SENER; and CENER is output as zero. Consequently, special care must be exercised when interpreting strain energy results of large-strain viscoelastic materials in Abaqus/Explicit since they include viscous dissipation effects.

5.2 Frequency domain viscoelasticity

The frequency domain viscoelastic material model:

- describes frequency-dependent material behavior in small steady-state harmonic oscillations for those materials in which dissipative losses caused by "viscous" (internal damping) effects must be modeled in the frequency domain;
- assumes that the shear (deviatoric) and volumetric behaviors are independent in multiaxial stress states;
- can be used in large-strain problems;
- can be used only in conjunction with Linear elastic behavior, Hyperelastic behavior of rubberlike materials, and Hyperelastic behavior in elastomeric foams to define the long-term elastic material properties;
- can be used in conjunction with the elastic-damage gasket behavior (Defining a nonlinear elastic model with damage) to define the effective thickness-direction storage and loss moduli for gasket elements; and
- is active only during the direct-solution steady-state dynamic (Direct-solution steady-state dynamic analysis), the subspace-based steady-state dynamic (Subspace-based steady-state dynamic analysis), the natural frequency extraction (Natural frequency extraction), and the complex eigenvalue extraction (Complex eigenvalue extraction) procedures.

The following topics are discussed:

- Defining the shear behavior

- Defining the volumetric behavior
- Large-strain viscoelasticity
- Determination of viscoelastic material parameters
- Conversion of frequency-dependent elastic moduli
- Direct specification of storage and loss moduli for large-strain viscoelasticity
- Defining the rate-independent part of the material behavior
- Material options
- Elements

Products: Abaqus/Standard Abaqus/CAE

5.2.1 Defining the shear behavior

Consider a shear test at small strain, in which a harmonically varying shear strain γ is applied:

$$\gamma(t) = \gamma_0 \exp(i\omega t),$$

where γ_0 is the amplitude, $i = \sqrt{-1}$, ω is the circular frequency, and t is time. We assume that the specimen has been oscillating for a very long time so that a steady-state solution is obtained. The solution for the shear stress then has the form

$$\tau(t) = \left(G_s(\omega) + iG_\ell(\omega)\right)\gamma_0 \exp(i\omega t),$$

where G_s and G_l are the shear storage and loss moduli. These moduli can be expressed in terms of the (complex) Fourier transform $g^*(\omega)$ of the nondimensional shear relaxation function $g(t) = G_R(t)/G_\infty - 1$:

$$G_s(\omega) = G_\infty\left(1 - \omega\Im\left(g^*\right)\right), \qquad G_\ell(\omega) = G_\infty\left(\omega\Re\left(g^*\right)\right),$$

where $G_R(t)$ is the time-dependent shear relaxation modulus, $\Re(g^*)$ and $\Im(g^*)$ are the real and imaginary parts of $g^*(\omega)$, and G_∞ is the long-term shear modulus. See Frequency domain viscoelasticity for details.
The above equation states that the material responds to steady-state harmonic strain with a stress of magnitude $G_s\gamma_0$ that is in phase with the strain and a stress of magnitude $G_l\gamma_0$ that lags the excitation by 90°. Hence, we can regard the factor

$$G^*(\omega) = G_s(\omega) + iG_\ell(\omega)$$

as the complex, frequency-dependent shear modulus of the steadily vibrating material. The absolute magnitude of the stress response is

$$|\tau| = \sqrt{G_s^2(\omega) + G_\ell^2(\omega)}\ |\gamma_0|,$$

and the phase lag of the stress response is

$$\phi = \arctan\left(\frac{G_\ell(\omega)}{G_s(\omega)}\right).$$

Measurements of $|\tau|$ and ϕ as functions of frequency in an experiment can, thus, be used to define G_s and G_l and, thus, $\Re(g^*)$ and $\Im(g^*)$ as functions of frequency.
Unless stated otherwise explicitly, all modulus measurements are assumed to be "true" quantities.

5.2.2 Defining the volumetric behavior

In multiaxial stress states Abaqus/Standard assumes that the frequency dependence of the shear (deviatoric) and volumetric behaviors are independent. The volumetric behavior is defined by the bulk storage and loss moduli $K_s(\omega)$ and $K_l(\omega)$. Similar to the shear moduli, these moduli can also be expressed in terms of the (complex) Fourier transform $k^*(\omega)$ of the nondimensional bulk relaxation function $k(t)$:

$$K_s(w) = K_\infty\left(1 - \omega\Im\left(k^*\right)\right), \qquad K_\ell(w) = K_\infty\left(\omega\Re\left(k^*\right)\right),$$

where K_∞ is the long-term elastic bulk modulus.

5.2.3 Large-strain viscoelasticity

The linearized vibrations can also be associated with an elastomeric material whose long-term (elastic) response is nonlinear and involves finite strains (a hyperelastic material). We can retain the simplicity of the steady-state small-amplitude vibration response analysis in this case by assuming that the linear expression for the shear stress still governs the system, except that now the long-term shear modulus G_∞ can vary with the amount of static prestrain $\bar{\gamma}$:

$$G_\infty = G_\infty(\bar{\gamma}).$$

The essential simplification implied by this assumption is that the frequency-dependent part of the material's response, defined by the Fourier transform $g^*(\omega)$ of the relaxation function, is not affected by the magnitude of the prestrain. Thus, strain and frequency effects are separated, which is a reasonable approximation for many materials.

Another implication of the above assumption is that the anisotropy of the viscoelastic moduli has the same strain dependence as the anisotropy of the long-term elastic moduli. Hence, the viscoelastic behavior in all deformed states can be characterized by measuring the (isotropic) viscoelastic moduli in the undeformed state.
In situations where the above assumptions are not reasonable, the data can be specified based on measurements at the prestrain level about which the steady-state dynamic response is desired. In this case you must measure G_s, G_l, and G_∞ (likewise K_s, K_l, and K_∞) at the prestrain level of interest. Alternatively, the viscoelastic data can be given directly in terms of uniaxial and volumetric storage and loss moduli that may be specified as functions of frequency and prestrain (see Direct specification of storage and loss moduli for large-strain viscoelasticity below.)
The generalization of these concepts to arbitrary three-dimensional deformations is provided in Abaqus/Standard by assuming that the frequency-dependent material behavior has two independent components: one associated with shear (deviatoric) straining and the other associated with volumetric straining. In the general case of a compressible material, the model is, therefore, defined for kinematically small perturbations about a predeformed state as

$$\frac{1}{J}\Delta^{\nabla}(JS) = \left(1 + i\omega g^{*}\right) C^{S}\Big|_0 : \Delta e + Q\big|_0 \Delta\varepsilon^{\mathrm{vol}},$$

and

$$\Delta p = -Q\big|_0 : \Delta e - \left(1 + i\omega k^{*}\right) K\Big|_0 \Delta\varepsilon^{\mathrm{vol}},$$

where

S

is the deviatoric stress, $S = \sigma + pI$;

p

is the equivalent pressure stress, $p = -\frac{1}{3}\mathrm{trace}\,(\sigma)$;

$\Delta^{\nabla}(JS)$

is the part of the stress increment caused by incremental straining (as distinct from the part of the stress increment caused by incremental rotation of the preexisting stress with respect to the coordinate system);

J

is the ratio of current to original volume;

Δe

is the (small) incremental deviatoric strain, $\Delta e = \Delta\varepsilon - \frac{1}{3}\Delta\varepsilon^{vol} I$;

$\dot{e}$

is the deviatoric strain rate, $\dot{e} = \dot{\varepsilon} - \frac{1}{3}\dot{\varepsilon}^{vol} I$;

$\Delta\varepsilon^{vol}$

is the (small) incremental volumetric strain, $\Delta\varepsilon^{vol} = \mathrm{trace}\,(\Delta\varepsilon)$;

$\dot{\varepsilon}^{vol}$

is the rate of volumetric strain, $\dot{\varepsilon}^{vol} = \text{trace}(\dot{\varepsilon})$;

$C^S|_0$

is the deviatoric tangent elasticity matrix of the material in its predeformed state (for example, C_{1212} is the tangent shear modulus of the prestrained material);

$Q|_0$

is the volumetric strain-rate/deviatoric stress-rate tangent elasticity matrix of the material in its predeformed state; and

$K|_0$

is the tangent bulk modulus of the predeformed material.

For a fully incompressible material only the deviatoric terms in the first constitutive equation above remain and the viscoelastic behavior is completely defined by $g^*(\omega)$.

5.2.4 Determination of viscoelastic material parameters

The dissipative part of the material behavior is defined by giving the real and imaginary parts of g^* and k^* (for compressible materials) as functions of frequency. The moduli can be defined as functions of the frequency in one of three ways: by a power law, by tabular input, or by a Prony series expression for the shear and bulk relaxation moduli.

5.2.4.1 Power law frequency dependence

The frequency dependence can be defined by the power law formulae

$$g^*(\omega) = g_1^* f^{-a} \qquad \text{and} \qquad k^*(\omega) = k_1^* f^{-b},$$

where a and b are real constants, g_1^* and k_1^* are complex constants, and $f = \omega/2\pi$ is the frequency in cycles per time.

Input File Usage:

`*VISCOELASTIC, FREQUENCY=FORMULA`

Abaqus/CAE Usage:

Property module: material editor: **Mechanical** → **Elasticity** → **Viscoelastic: Domain: Frequency and Frequency: Formula**

5.2.4.2 Tabular frequency dependence

The frequency domain response can alternatively be defined in tabular form by giving the real and imaginary parts of ωg^* and ωk^* - where ω is the circular frequency - as functions of frequency in cycles per time. Given the frequency-dependent storage and loss moduli $G_s(\omega)$, $G_l(\omega)$, $K_s(\omega)$, and $K_l(\omega)$, the real and imaginary parts of ωg^* and ωk^* are then given as

$$\omega\Re\left(g^*\right) = G_\ell / G_\infty, \quad \omega\Im\left(g^*\right) = 1 - G_s / G_\infty, \quad \omega\Re\left(k^*\right) = K_\ell / K_\infty, \quad \omega\Im\left(k^*\right) = 1 - K_s / K_\infty,$$

where G_∞ and K_∞ are the long-term shear and bulk moduli determined from the elastic or hyperelastic properties.
Abaqus provides an alternative approach for specifying the viscoelastic properties of hyperelastic and hyperfoam materials. This approach involves the direct (tabular) specification of storage and loss moduli from uniaxial and volumetric tests, as functions of excitation frequency and a measure of the level of pre-strain. The level of pre-strain refers to the level of elastic deformation at the base state about which the steady-state harmonic response is desired. This approach is discussed in Direct specification of storage and loss moduli for large-strain viscoelasticity below.
Input File Usage:
`*VISCOELASTIC, FREQUENCY=TABULAR`
Abaqus/CAE Usage:
Property module: material editor: **Mechanical** → **Elasticity** → **Viscoelastic: Domain: Frequency and Frequency: Tabular**

5.2.4.3 Prony series parameters

The frequency dependence can also be obtained from a time domain Prony series description of the dimensionless shear and bulk relaxation moduli:

$$g_R(t) = 1 - \sum_{i=1}^{N} \bar{g}_i^P \left(1 - e^{-t/\tau_i}\right),$$

$$k_R(t) = 1 - \sum_{i=1}^{N} \bar{k}_i^P \left(1 - e^{-t/\tau_i}\right),$$

where N, $\bar{g}_i^P$, $\bar{k}_i^P$, and τ_i, $i = 1, 2, ...N$, are material constants. Using Fourier transforms, the expression for the time-dependent shear modulus can be written in the frequency domain as follows:

$$G_s(\omega) = G_0\left[1 - \sum_{i=1}^{N} \bar{g}_i^P\right] + G_0 \sum_{i=1}^{N} \frac{\bar{g}_i^P \tau_i^2 \omega^2}{1 + \tau_i^2 \omega^2},$$

$$G_\ell(\omega) = G_0 \sum_{i=1}^{N} \frac{\bar{g}_i^P \tau_i \omega}{1 + \tau_i^2 \omega^2},$$

where $G_s(\omega)$ is the storage modulus, $G_l(\omega)$ is the loss modulus, ω is the angular frequency, and N is the number of terms in the Prony series. The expressions for the bulk moduli, $K_s(\omega)$ and $K_l(\omega)$, are written analogously. Abaqus/Standard will automatically perform the conversion from the time domain to the frequency domain. The Prony series parameters $\bar{g}_i^P$, $\bar{k}_i^P$, τ_i can be defined in one of three ways: direct specification of the Prony series parameters, inclusion of creep test data, or inclusion of relaxation test data. If creep test data or relaxation test data are specified, Abaqus/Standard will determine the Prony series parameters in a nonlinear least-squares fit. A detailed description of the calibration of Prony series terms is provided in Time domain viscoelasticity.

For the test data you can specify the normalized shear and bulk data separately as functions of time or specify the normalized shear and bulk data simultaneously. A nonlinear least-squares fit is performed to determine the Prony series parameters, $\left(\bar{g}_i^P, \bar{k}_i^P, \tau_i\right)$.

Input File Usage:

Use one of the following options to specify Prony data, creep test data, or relaxation test data:

```
*VISCOELASTIC, FREQUENCY=PRONY
*VISCOELASTIC, FREQUENCY=CREEP TEST DATA
*VISCOELASTIC, FREQUENCY=RELAXATION TEST DATA
```

Use one or both of the following options to specify the normalized shear and bulk data separately as functions of time:

```
*SHEAR TEST DATA
*VOLUMETRIC TEST DATA
```

Use the following option to specify the normalized shear and bulk data simultaneously:

```
*COMBINED TEST DATA
```

Abaqus/CAE Usage:

Property module: material editor: **Mechanical** → **Elasticity** → **Viscoelastic: Domain: Frequency** and **Frequency: Prony, Creep test data,** or **Relaxation test data**

Use one or both of the following options to specify the normalized shear and bulk data separately as functions of time:

Test Data → **Shear Test Data**

Test Data → **Volumetric Test Data**

Use the following option to specify the normalized shear and bulk data simultaneously:

Test Data → **Combined Test Data**

5.2.4.4 Thermorheologically simple temperature effects in frequency domain viscoelasticity

You can include thermorheologically simple temperature effects in frequency domain viscoelasticity. In this case the reduced angular frequency, ω_r, is used to obtain the frequency-dependent material moduli. The reduced angular frequency is computed as

$$\omega_r = A\left(\theta\right)\omega,$$

where $A\left(\theta\right)$ and θ denote the shift function and temperature, respectively. Abaqus/Standard supports the following forms of the shift function: the Williams-Landel-Ferry (WLF) form, the Arrhenius form, and user-defined forms (see Thermorheologically simple temperature effects).

5.2.5 Conversion of frequency-dependent elastic moduli

For some cases of small straining of isotropic viscoelastic materials, the material data are provided as frequency-dependent uniaxial storage and loss moduli, $E_s(\omega)$ and $E_l(\omega)$, and bulk moduli, $K_s(\omega)$ and $K_l(\omega)$. In that case the data must be converted to obtain the frequency-dependent shear storage and loss moduli $G_s(\omega)$ and $G_l(\omega)$.
The complex shear modulus is obtained as a function of the complex uniaxial and bulk moduli with the expression

$$G^* = \frac{3K^* E^*}{9K^* - E^*}.$$

Replacing the complex moduli by the appropriate storage and loss moduli, this expression transforms into

$$G_s + iG_\ell = \frac{3\left(K_s + iK_\ell\right)\left(E_s + iE_\ell\right)}{9\left(K_s + iK_\ell\right) - \left(E_s + iE_\ell\right)}.$$

After some algebra one obtains

$$G_s = 3\frac{9E_s\left(K_s^2 + K_\ell^2\right) - K_s\left(E_s^2 + E_\ell^2\right)}{\left(9K_s - E_s\right)^2 + \left(9K_\ell - E_\ell\right)^2}, \quad G_\ell = 3\frac{9E_\ell\left(K_s^2 + K_\ell^2\right) - K_\ell\left(E_s^2 + E_\ell^2\right)}{\left(9K_s - E_s\right)^2 + \left(9K_\ell - E_\ell\right)^2}.$$

5.2.5.1 Shear strain only

In many cases the viscous behavior is associated only with deviatoric straining, so that the bulk modulus is real and constant: $K_s = K_\infty$ and $K_l = 0$. For this case the expressions for the shear moduli simplify to

$$G_s = 3K_\infty\frac{9E_s K_\infty - E_s^2 - E_\ell^2}{\left(9K_\infty - E_s\right)^2 + E_\ell^2}, \qquad G_\ell = 3K_\infty\frac{9E_\ell K_\infty}{\left(9K_\infty - E_s\right)^2 + E_\ell^2}.$$

5.2.5.2 Incompressible materials

If the bulk modulus is very large compared to the shear modulus, the material can be considered to be incompressible and the expressions simplify further to

$$G_s = E_s / 3, \qquad G_\ell = E_\ell / 3.$$

5.2.6 Direct specification of storage and loss moduli for large-strain viscoelasticity

For large-strain viscoelasticity Abaqus allows direct specification of storage and loss moduli from uniaxial and volumetric tests. This approach can be used when the assumption of the independence of viscoelastic properties on the pre-strain level is too restrictive.

You specify the storage and loss moduli directly as tabular functions of frequency, and you specify the level of pre-strain at the base state about which the steady-state dynamic response is desired. For uniaxial test data the measure of pre-strain is the uniaxial nominal strain; for volumetric test data the measure of pre-strain is the volume ratio. Abaqus internally converts the data that you specify to ratios of shear/bulk storage and loss moduli to the corresponding long-term elastic moduli. Subsequently, the basic formulation described in Large-strain viscoelasticity above is used.

For a general three-dimensional stress state it is assumed that the deviatoric part of the viscoelastic response depends on the level of pre-strain through the first invariant of the deviatoric left Cauchy-Green strain tensor (see Hyperelastic material behavior for a definition of this quantity), while the volumetric part depends on the pre-strain through the volume ratio. A consequence of these assumptions is that for the uniaxial case, data can be specified from a uniaxial-tension preload state or from a uniaxial-compression preload state but not both.

The storage and loss moduli that you specify are assumed to be nominal quantities.

Input File Usage:

Use the following option to specify only the uniaxial storage and loss moduli:

`*VISCOELASTIC, PRELOAD=UNIAXIAL`

You can also use the following option to specify the volumetric (bulk) storage and loss moduli:

`*VISCOELASTIC, PRELOAD=VOLUMETRIC`

Abaqus/CAE Usage:

Property module: material editor: **Mechanical** → **Elasticity** → **Viscoelastic: Domain: Frequency** and **Frequency: Tabular**

Use the following option to specify only the uniaxial storage and loss moduli:

Type: Isotropic or **Traction: Preload: Uniaxial**

Use the following option to specify only the volumetric storage and loss moduli:

Type: Isotropic: Preload: Volumetric

Use the following option to specify both uniaxial and volumetric moduli:

Type: Isotropic: Preload: Uniaxial and Volumetric

5.2.7 Defining the rate-independent part of the material behavior

In all cases elastic moduli must be specified to define the rate-independent part of the material behavior. The elastic behavior is defined by an elastic, hyperelastic, or hyperfoam material model. Since the frequency domain viscoelastic material model is developed around the long-term elastic moduli, the rate-independent elasticity must be defined in terms of long-term elastic moduli. This implies that the response in any analysis procedure other than a direct-solution steady-state dynamic analysis (such as a static preloading analysis) corresponds to the fully relaxed long-term elastic solution.

5.2.8 Material options

The viscoelastic material model must be combined with the isotropic linear elasticity model to define classical, linear, small-strain, viscoelastic behavior. It is combined with the hyperelastic or hyperfoam model to define large-deformation, nonlinear, viscoelastic behavior. The long-term elastic properties defined for these models can be temperature dependent.
Viscoelasticity cannot be combined with any of the plasticity models. See Combining material behaviors for more details.

5.2.9 Elements

The frequency domain viscoelastic material model can be used with any stress/displacement element in Abaqus/Standard.

Chapter 6

Abaqus / Theory: Viscoelasticity

The content of this section was copied from the Abaqus online documentation from the following link: Viscoelasticity. The webpage was accessed on 03/09/23. The purpose of this section is to provide a direct access to the official description in this report, without having to open the corresponding website on the Abaqus Online Documentation.

6.1 Viscoelasticity

The response of a viscoelastic material includes both elastic (instantaneous) and viscous (time-dependent) behavior. The instantaneous elastic response of the material is followed by creep when subjected to a fixed applied stress, and stress-relaxation when subjected to a fixed applied strain.

The following topics are discussed:

- Reduced states of stress
- Automatic time stepping procedure

Products: Abaqus/Standard, Abaqus/Explicit

The basic hereditary integral formulation for linear isotropic viscoelasticity is

$$\boldsymbol{\sigma}(t) = \int_0^t 2G\left(\tau - \tau'\right)\dot{\mathbf{e}}\,dt' + \mathbf{I}\int_0^t K\left(\tau - \tau'\right)\dot{\phi}\,dt'\,.$$

Here $\mathbf{e}$ and ϕ are the mechanical deviatoric and volumetric strains; K is the bulk modulus and G is the shear modulus, which are functions of the reduced time τ; and $\cdot$ denotes differentiation with respect to t'.

The reduced time is related to the actual time through the integral differential equation

$$\tau = \int_0^t \frac{dt'}{A_\theta\left(\theta\left(t'\right)\right)}\ , \qquad \frac{d\tau}{dt} = \frac{1}{A_\theta\left(\theta\left(t\right)\right)},$$

where θ is the temperature and A_θ is the shift function. (Hence, if $A_\theta = 1$, $\tau = t$.) A commonly used shift function is the Williams-Landel-Ferry (WLF) equation, which has the following form:

$$-\log A_\theta = h(\theta) = \frac{C_1^g \left(\theta - \theta_g\right)}{C_2^g + \left(\theta - \theta_g\right)},$$

where C_1^g and C_2^g are constants and θ_g is the "glass" transition temperature. This is the temperature at which, in principle, the behavior of the material changes from glassy to rubbery. If $\theta < \theta_g - C_2^g$, deformation changes will be elastic. C_1^g and C_2^g were once thought to be "universal" constants whose values were obtained at θ_g, but these constants have been shown to vary slightly from polymer to polymer.
Abaqus allows the WLF equation to be used with any convenient temperature, other than the glass transition temperature, as the reference temperature. The form of the equation remains the same, but the constants are different. Namely,

$$-\log A_\theta = h(\theta) = \frac{C_1 \left(\theta - \theta_0\right)}{C_2 + \left(\theta - \theta_0\right)},$$

where θ_0 is the reference temperature at which the relaxation data are given, and C_1 and C_2 are the calibration constants at the reference temperature. The "universal" constants C_1^g and C_2^g are related to C_1 and C_2 as follows:

$$\begin{aligned} C_1 &= \frac{C_1^g}{1 + \left(\theta_0 - \theta_g\right)/C_2^g}, \\ C_2 &= C_2^g + \theta_0 - \theta_g\,. \end{aligned}$$

Other forms of $h(\theta)$ are also used, such as a power series in $\theta - \theta_0$. Abaqus allows a general definition of the shift function with user subroutine UTRS.
The relaxation functions $K(t)$ and $G(t)$ can be defined individually in terms of a series of exponentials known as the Prony series:

$$K(\tau) = K_\infty + \sum_{i=1}^{n_K} K_i e^{-\tau/\tau_i^K} \qquad G(\tau) = G_\infty + \sum_{i=1}^{n_G} G_i e^{-\tau/\tau_i^G},$$

where K_∞ and G_∞ represent the long-term bulk and shear moduli. In general, the relaxation times τ_i^K and τ_i^G need not equal each other; however, Abaqus assumes that $\tau_i = \tau_i^K = \tau_i^G$. On the other hand, the number of terms in bulk and shear, n_K and n_G, need not equal each other. In fact, in many practical cases it can be assumed that $n_K = 0$. Hence, we now concentrate on the deviatoric behavior. The equations for the volumetric terms can be derived in an analogous way.
The deviatoric integral equation is

$$\begin{aligned}\mathbf{S} &= \int_0^t 2\left(G_\infty + \sum_{i=1}^{n_G} G_i e^{(\tau'-\tau)/\tau_i}\right)\dot{\mathbf{e}}\,dt' \\ &= \int_0^\tau 2\left(G_\infty + \sum_{i=1}^{n_G} G_i e^{(\tau'-\tau)/\tau_i}\right)\frac{d\mathbf{e}}{d\tau'}\,d\tau'.\end{aligned}$$

We rewrite this equation in the form

$$\mathbf{S} = 2G_0\left(\mathbf{e} - \sum_{i=1}^{n}\alpha_i\mathbf{e}_i\right), \qquad \text{(EQ1)}$$

where $G_0 = G_\infty + \Sigma_{i=1}^n G_i$ is the instantaneous shear modulus, $\alpha_i = G_i/G_0$ is the relative modulus of term i, and

$$\mathbf{e}_i = \int_0^\tau \left(1 - e^{(\tau'-\tau)/\tau_i}\right)\frac{d\mathbf{e}}{d\tau'}\,d\tau' \qquad \text{(EQ2)}$$

is the viscous (creep) strain in each term of the series. For finite element analysis this equation must be integrated over a finite increment of time. To perform this integration, we will assume that during the increment $\mathbf{e}$ varies linearly with τ; hence, $d\mathbf{e}/d\tau' = \Delta e/\Delta\tau'$. To use this relation, we break up (EQ2) into two parts:

$$\begin{aligned}\mathbf{e}_i^{n+1} &= \int_0^{\tau^n}\left(1 - e^{(\tau'-\tau^{n+1})/\tau_i}\right)\frac{d\mathbf{e}}{d\tau'}\,d\tau' \\ &\quad + \int_{\tau^n}^{\tau^{n+1}}\left(1 - e^{(\tau'-\tau^{n+1})/\tau_i}\right)\frac{d\mathbf{e}}{d\tau'}\,d\tau'.\end{aligned}$$

Now observe that

$$1 - e^{(\tau'-\tau^{n+1})/\tau_i} = 1 - e^{-\Delta\tau/\tau_i} + e^{-\Delta\tau/\tau_i}\left(1 - e^{(\tau'-\tau^n)/\tau_i}\right).$$

Use of this expression and the approximation for $d\mathbf{e}/d\tau'$ during the increment yields

$$\begin{aligned}\mathbf{e}_i^{n+1} &= \left(1 - e^{-\Delta\tau/\tau_i}\right)\int_0^{\tau^n}\frac{d\mathbf{e}}{d\tau'}\,d\tau' \\ &\quad + e^{-\Delta\tau/\tau_i}\int_0^{\tau^n}\left(1 - e^{(\tau'-\tau^n)/\tau_i}\right)\frac{d\mathbf{e}}{d\tau'}\,d\tau' \\ &\quad + \frac{\Delta\mathbf{e}}{\Delta\tau}\int_{\tau^n}^{\tau^{n+1}}\left(1 - e^{(\tau'-\tau^{n+1})/\tau_i}\right)d\tau'.\end{aligned}$$

The first and last integrals in this expression are readily evaluated, whereas from (EQ2) follows that the second integral represents the viscous strain in the ith term at the beginning of the increment. Hence, the change in the ith viscous strain is

$$\begin{aligned}\Delta \mathbf{e}_i &= \left(1-e^{-\Delta\tau/\tau_i}\right)\mathbf{e}^n + \left(e^{-\Delta\tau/\tau_i}-1\right)\mathbf{e}_i^n + \left(\Delta\tau - \tau_i\left(1-e^{-\Delta\tau/\tau_i}\right)\right)\frac{\Delta \mathbf{e}}{\Delta\tau} \\ &= \frac{\tau_i}{\Delta\tau}\left(\frac{\Delta\tau}{\tau_i}+e^{-\Delta\tau/\tau_i}-1\right)\Delta\mathbf{e} + \left(1-e^{-\Delta\tau/\tau_i}\right)\left(\mathbf{e}^n-\mathbf{e}_i^n\right).\end{aligned} \tag{EQ3}$$

If $\Delta\tau/\tau_i$ approaches zero, this expression can be approximated by

$$\Delta \mathbf{e}_i = \frac{\Delta\tau}{\tau_i}\left(\frac{1}{2}\Delta\mathbf{e}+\mathbf{e}^n-\mathbf{e}_i^n\right). \tag{EQ4}$$

The last form is used in the computations if $\Delta\tau/\tau_i < 10^{-7}$.
Hence, in an increment, (EQ3) or (EQ4) is used to calculate the new value of the viscous strains. (EQ1) is then used subsequently to obtain the new value of the stresses.
The tangent modulus is readily derived from these equations by differentiating the deviatoric stress increment, which is

$$\Delta\mathbf{S} = 2G_0\left(\Delta\mathbf{e} - \sum_{i=1}^{n_G}\alpha_i\left(\mathbf{e}_i^{n+1}-\mathbf{e}_i^n\right)\right)$$

with respect to the deviatoric strain increment $\Delta \boldsymbol{e}$. Since the equations are linear, the modulus depends only on the reduced time step:

$$G^T = \begin{cases} G_0\left[1-\sum_{i=1}^{n}\alpha_i\frac{\tau_i}{\Delta\tau}\left(\frac{\Delta\tau}{\tau_i}+e^{-\Delta\tau/\tau_i}-1\right)\right] & \text{if } \Delta\tau/\tau_i > 10^{-7} \\ G_0\left[1-\sum_{i=1}^{n}\frac{1}{2}\alpha_i\frac{\Delta\tau}{\tau_i}\right] & \text{if } \Delta\tau/\tau_i < 10^{-7}\end{cases}$$

Finally, one needs a relation between the reduced time increment, $\Delta\tau$, and the actual time increment, Δt. To do this, we observe that A ϑ varies very nonlinearly with temperature; hence, any direct approximation of A_θ is likely to lead to large errors. On the other hand, $h(\theta)$ will generally be a smoothly varying function of temperature that is well approximated by a linear function of temperature over an increment. If we further assume that incrementally the temperature θ is a linear function of time t, one finds the relation

$$h(\theta) = -\ln A_\theta(\theta(t)) = a + bt$$

or

$$A_\theta^{-1}(\theta(t)) = e^{a+bt}$$

with

$$
\begin{aligned}
a &= \frac{1}{\Delta t}\left[t^{n+1}h\left(\theta^{n}\right) - t^{n}h\left(\theta^{n+1}\right)\right] \\
b &= \frac{1}{\Delta t}\left[h\left(\theta^{n+1}\right) - h\left(\theta^{n}\right)\right].
\end{aligned}
$$

This yields the relation

$$
\begin{aligned}
\Delta\tau &= \int_{t^n}^{t^{n+1}} e^{a+bt}\,dt \\
&= \frac{1}{b}\left(e^{a+bt^{n+1}} - e^{a+bt^{n}}\right).
\end{aligned}
$$

This expression can also be written as

$$
\Delta\tau = \frac{A_\theta^{-1}\left(\theta^{n+1}\right) - A_\theta^{-1}\left(\theta^{n}\right)}{h\left(\theta^{n+1}\right) - h\left(\theta^{n}\right)}\Delta t\,.
$$

6.1.1 Reduced states of stress

So far, we have discussed full triaxial stress states. If the stress state is reduced (i.e., plane stress or uniaxial stress), the equations derived here cannot be used directly because only the total stress state is reduced, not the individual terms in the series. Therefore, we use the following procedure.

For plane stress let the third component be the zero stress component. At the beginning of the increment we presumably know the volumetric elastic strain ϕ_e^n, the volumetric viscous strain ϕ_c^n, and the volumetric viscous strains ϕ_i^n associated with the Prony series. The total volumetric strain can be obtained by adding together the elastic volumetric strain and the volumetric viscous strain

$$
\phi^n = \phi_e^n + \phi_c^n\,. \quad \text{(EQ5)}
$$

The deviatoric strain in the 3-direction follows from the relation $\phi = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$, which yields:

$$
e_3^n = \varepsilon_3^n - \frac{1}{3}\phi^n = \frac{2}{3}\phi^n - \varepsilon_1^n - \varepsilon_2^n\,.
$$

The out-of-plane deviatoric stress at the end of the increment is

$$
s_3^{n+1} = 2G_0\left(e_3^{n+1} - \sum_{i=1}^{n_G}\alpha_i^G e_{3i}^{n+1}\right).
$$

Substituting (EQ3) for e_{3i}^{n+1}, letting $e_3^{n+1} = e_3^n + \Delta e_3$, and collecting terms gives

$$s_3^{n+1} = 2G^T \Delta e_3 + \, 2G_0 e_3^n \left[1 - \sum_{i=1}^{n_G} \alpha_i^G \left(1 - e^{-\Delta\tau/\tau_i} \right) \right] - \, 2G_0 \sum_{i=1}^{n_G} \alpha_i^G e^{-\Delta\tau/\tau_i} e_{3i}^n \,. \tag{EQ6}$$

The hydrostatic stress is derived similarly as

$$-p^{n+1} = K^T \Delta\phi + \, K_0 \phi^n \left[1 - \sum_{i=1}^{n_K} \alpha_i^K \left(1 - e^{-\Delta\tau/\tau_i} \right) \right] - \, K_0 \sum_{i=1}^{n_K} \alpha_i^K e^{-\Delta\tau/\tau_i} \phi_i^n \,. \tag{EQ7}$$

We can write these equations in the form

$$\begin{aligned} s_3^{n+1} &= 2G^T \Delta e_3 + \bar{s}_3 \\ -p^{n+1} &= K^T \Delta\phi - \bar{p} \,. \end{aligned}$$

In the third direction the deviatoric stress minus the hydrostatic pressure is zero; hence,

$$2G^T \Delta e_3 + K^T \Delta\phi + \bar{s}_3 - \bar{p} = 0 \,. \tag{EQ8}$$

Since $\Delta e_3 = \frac{2}{3}\Delta\phi - \Delta\varepsilon_1 - \Delta\varepsilon_2$, it follows that

$$\left(K^T + \frac{4}{3} G^T \right) \Delta\phi = 2G^T \left(\Delta\varepsilon_1 + \Delta\varepsilon_2 \right) - \bar{s}_3 + \bar{p},$$

from which $\Delta\phi$ can be solved. One can then also calculate Δe_1 and Δe_2, and with (EQ3) or (EQ4) one can update the deviatoric viscous strains $\boldsymbol{e}_i^{n+1}$. The volumetric strains ϕ_i^{n+1} are obtained with a relation similar to (EQ3).

For uniaxial stress states a similar procedure is used. As before, ϕ^n follows from (EQ5) and e_3^n and e_2^n follow from $\varepsilon_1 + 2\varepsilon_3 = \phi$:

$$e_3^n = e_2^n = \varepsilon_3^n - \tfrac{1}{3}\phi^n = \tfrac{1}{6}\phi^n - \tfrac{1}{2}\varepsilon_1^n \,. \tag{EQ9}$$

(EQ6) and (EQ7) can be used to calculate $\bar{\boldsymbol{s}}_3$ and $\bar{p}$, which again leads to (EQ8). Applying (EQ9) for Δe_3,

$$\left(K^T + \frac{1}{3} G^T \right) \Delta\phi = G^T \Delta\varepsilon_1 - \bar{s}_3 + \bar{p}.$$

After this, one can follow the same procedure as for plane stress.

6.1.2 Automatic time stepping procedure

To create an automatic time stepping procedure in Abaqus/Standard, we want to compare viscous strain rates at the beginning and the end of the increment. The strain rates in the individual terms at the beginning of the increment can be obtained directly by taking the limit of the incremental strain:

$$\begin{aligned}\dot{\mathbf{e}}_i^n &= \lim_{\Delta t \to 0} \frac{\mathbf{e}_i^{n+1} - \mathbf{e}_i^n}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\Delta \tau}{\Delta t \tau_i} \left(\frac{1}{2} \Delta \mathbf{e} + \mathbf{e}^n - \mathbf{e}_i^n \right) \\ &= \frac{\mathbf{e}^n - \mathbf{e}_i^n}{A_\theta \left(\theta^n\right) \tau_i} .\end{aligned}$$

A similar expression can be derived for the strain rate at the end of the increment:

$$\dot{\mathbf{e}}_i^{n+1} = \frac{\mathbf{e}^{n+1} - \mathbf{e}_i^{n+1}}{A_\theta \left(\theta^{n+1}\right) \tau_i} .$$

If we use these expressions to calculate a difference in estimated total viscous strain increment, one finds

$$\begin{aligned}\Delta \bar{\mathbf{e}}_V &= \Delta t \sum_{i=1}^{n_G} \alpha_i^G \left(\dot{\mathbf{e}}_i^{n+1} - \dot{\mathbf{e}}_i^n \right) \\ &= \Delta t \sum_{i=1}^{n_G} \frac{\alpha_i^G}{\tau_i} \left(\frac{\mathbf{e}^{n+1} - \mathbf{e}_i^{n+1}}{A_\theta \left(\theta^n\right)} - \frac{\mathbf{e}^n - \mathbf{e}_i^n}{A_\theta \left(\theta^n\right)} \right) .\end{aligned}$$

This expression is readily evaluated. A similar expression can be calculated for volumetric strain $\Delta \bar{\phi}_V$, and from these two quantities a suitable scalar measure can be constructed; for example,

$$\Delta \varepsilon^{\text{est}} = \sqrt{\frac{2}{3} \Delta \bar{\mathbf{e}}_V : \Delta \bar{\mathbf{e}}_V + \frac{1}{3} \Delta \bar{\phi}_V^2} .$$

Comparison with the user-specified strain increment tolerance allows construction of an automatic time stepping scheme.

6.2 Finite-strain viscoelasticity

The finite-strain viscoelasticity theory implemented in Abaqus is a time domain generalization of either the hyperelastic or the hyperfoam constitutive models.
The following topics are discussed:

- Integral formulation
- Implementation
- Integration of the hydrostatic stress
- Integration of the deviatoric stress
- Rate equation
- Cauchy versus Kirchhoff stress

Products: Abaqus/Standard, Abaqus/Explicit

6.2.1 Integral formulation

It is assumed that the instantaneous response of the material follows from the hyperelastic constitutive equations:

$$\boldsymbol{\tau}_0(t) = \boldsymbol{\tau}_0^D\left(\overline{\mathbf{F}}(t)\right) + \boldsymbol{\tau}_0^H(J(t))$$

for a compressible material and

$$\boldsymbol{\tau}_0(t) = \boldsymbol{\tau}_0^D\left(\overline{\mathbf{F}}(t)\right) + \boldsymbol{\tau}_0^H(t)$$

for an incompressible material. In the above, $\boldsymbol{\tau}_0^D$ and $\boldsymbol{\tau}_0^H$ are, respectively, the deviatoric and the hydrostatic parts of the instantaneous Kirchhoff stress $\boldsymbol{\tau}_0$. $\overline{\boldsymbol{F}}$ is the "distortion gradient" related to the deformation gradient $\boldsymbol{F}$ by

$$\overline{\mathbf{F}} = \frac{\mathbf{F}}{J^{\frac{1}{3}}},$$

where

$$J = \det(\mathbf{F})$$

is the volume change.

Using integration by parts and a variable transformation, the basic hereditary integral formulation for linear isotropic viscoelasticity can be written in the form

$$\boldsymbol{\sigma}(t) = 2G_0\mathbf{e}(t) + \int_0^{\tau} 2\dot{G}(\tau')\,\mathbf{e}(t-t')\,d\tau' + \mathbf{I}\left(K_0\phi(t) + \int_0^{\tau} \dot{K}(\tau')\,\phi(t-t')\,d\tau'\right)$$

or entirely in terms of stress quantities,

$$\boldsymbol{\sigma}(t) = \mathbf{S}_0(t) + \int_0^{\tau} \frac{\dot{G}(\tau')}{G_0}\mathbf{S}_0(t-t')\,d\tau' + \mathbf{I}\left(p_0(t) + \int_0^{\tau} \frac{\dot{K}(\tau')}{K_0}p(t-t')\,d\tau'\right),$$

where τ is the reduced time, $\dot{G}(\tau') = dG(\tau')/d\tau'$, and $\dot{K}(\tau') = dK(\tau')/d\tau'$. G_0 and K_0 are the instantaneous small-strain shear and bulk moduli, and $G(t)$ and $K(t)$ are the time-dependent small-strain shear and bulk relaxation moduli. Recall that the reduced time represents a shift in time with temperature and is related to the actual time through the differential equation

$$d\tau' = \frac{dt'}{A_\theta(\theta(t'))},$$

where θ is the temperature and A_θ is the shift function.
Using the volumetric/deviatoric-split hereditary integral in the reference configuration for large strain (hyperelastic) materials, and then using a standard push-forward operator (see Simo, 1987), one obtains the following set of equations in the current configuration:

$$\begin{aligned}\boldsymbol{\tau}^D(t) &= \boldsymbol{\tau}_0^D(t) + \mathrm{dev}\left[\int_0^\tau \frac{\dot{G}(\tau')}{G_0}\overline{\mathbf{F}}_t^{-1}(t-t')\cdot\boldsymbol{\tau}_0^D(t-t')\cdot\overline{\mathbf{F}}_t^{-T}(t-t')\,d\tau'\right],\\ \boldsymbol{\tau}^H(t) &= \boldsymbol{\tau}_0^H(t) + \int_0^\tau \frac{\dot{K}(\tau')}{K_0}\boldsymbol{\tau}_0^H(t-t')\,d\tau'.\end{aligned} \quad \text{(EQ1')}$$

where $\mathrm{dev}(\bullet) = (\bullet) - \frac{1}{3}((\bullet):\boldsymbol{I})\boldsymbol{I}$ and $\overline{\boldsymbol{F}}_t(t-t')$ is the distortional deformation gradient of the state at $t-t'$ relative to the state at t. A transformation is performed on the stress relating the state at time $t-t'$ to the state at time t.

6.2.2 Implementation

As in small-strain viscoelasticity, we represent the relaxation moduli in terms of the Prony series

$$G(\tau) = G_0\left(g_\infty + \sum_{i=1}^{N_G} g_i e^{-\tau/\tau_i^G}\right), \quad \text{(EQ2')}$$

$$K(\tau) = K_0\left(k_\infty + \sum_{i=1}^{N_K} k_i e^{-\tau/\tau_i^K}\right), \quad \text{(EQ3')}$$

where g_i and k_i are the relative moduli of terms i. Note that $g_\infty + \Sigma_{i=1}^{N_G} g_i = k_\infty + \Sigma_{i=1}^{N_K} k_i = 1$. Abaqus assumes that the relaxation times $\tau_i = \tau_i^G = \tau_i^K$ are the same so that from here on, we will sum on N terms for both bulk and shear behavior. In reality, the number of nonzero terms in bulk and shear, N_K and N_G, need not be equal, unless the instantaneous behavior is based on the hyperfoam model. In the latter case, the two deformation modes are closely related and are then assumed to relax equally and simultaneously.
Substituting (EQ2') and (EQ3') in (EQ1'), we obtain

$$\begin{aligned}\boldsymbol{\tau}^{D}(t) &= \boldsymbol{\tau}_0^{D}(t) - \operatorname{dev}\left[\sum_{i=1}^{N}\frac{g_i}{\tau_i}\int_0^{\tau}\overline{\mathbf{F}}_t^{-1}(t-t')\cdot\boldsymbol{\tau}_0^{D}(t-t')\cdot\overline{\mathbf{F}}_t^{-T}(t-t')\,e^{-\frac{\tau'}{\tau_i}}d\tau'\right],\\ \boldsymbol{\tau}^{H}(t) &= \boldsymbol{\tau}_0^{H}(t) - \sum_{i=1}^{N}\frac{k_i}{\tau_i}\int_0^{\tau}\boldsymbol{\tau}_0^{H}(t-t')\,e^{-\frac{\tau'}{\tau_i}}d\tau'.\end{aligned} \qquad \text{(EQ4')}$$

Next, we introduce the internal stresses, associated with each term of the series

$$\boldsymbol{\tau}_i^{D}(t) \equiv \tfrac{g_i}{\tau_i}\int_0^{\tau}\overline{\mathbf{F}}_t^{-1}(t-t')\cdot\boldsymbol{\tau}_0^{D}(t-t')\cdot\overline{\mathbf{F}}_t^{-T}(t-t')\,e^{-\frac{\tau'}{\tau_i}}d\tau', \qquad \text{(EQ5')}$$

$$\boldsymbol{\tau}_i^{H}(t) \equiv \tfrac{k_i}{\tau_i}\int_0^{\tau}\boldsymbol{\tau}_0^{H}(t-t')\,e^{-\frac{\tau'}{\tau_i}}d\tau'. \qquad \text{(EQ6')}$$

These stresses are stored at each material point and are integrated forward in time. We will assume that the solution is known at time t, and we need to construct the solution at time $t+\Delta t$.

6.2.3 Integration of the hydrostatic stress

The internal hydrostatic stresses $\boldsymbol{\tau}_i^H$ at time $t+\Delta t$ follow from

$$\boldsymbol{\tau}_i^{H}(t+\Delta t) = \frac{k_i}{\tau_i}\int_0^{\tau+\Delta\tau}\boldsymbol{\tau}_0^{H}\left(t+\Delta t-t'\right)e^{-\frac{\tau'}{\tau_i}}d\tau'.$$

With $\bar{\tau} = \tau' - \Delta\tau$ and $\bar{t} = t' - \Delta t$, it follows that

$$\begin{aligned}\boldsymbol{\tau}_i^{H}(t+\Delta t) =\ & \frac{k_i}{\tau_i}e^{-\frac{\Delta\tau}{\tau_i}}\int_{-\Delta\tau}^{\tau}\boldsymbol{\tau}_0^{H}(t-\bar{t})\,e^{-\frac{\bar{\tau}}{\tau_i}}d\bar{\tau}\\ =\ & \frac{k_i}{\tau_i}e^{-\frac{\Delta\tau}{\tau_i}}\int_{-\Delta\tau}^{0}\boldsymbol{\tau}_0^{H}(t-\bar{t})\,e^{-\frac{\bar{\tau}}{\tau_i}}d\bar{\tau} + \frac{k_i}{\tau_i}e^{-\frac{\Delta\tau}{\tau_i}}\int_{0}^{\tau}\boldsymbol{\tau}_0^{H}(t-\bar{t})\,e^{-\frac{\bar{\tau}}{\tau_i}}d\bar{\tau},\end{aligned}$$

which yields with (EQ6')

$$\boldsymbol{\tau}_i^{H}(t+\Delta t) = \tfrac{k_i}{\tau_i}e^{-\frac{\Delta\tau}{\tau_i}}\int_{-\Delta\tau}^{0}\boldsymbol{\tau}_0^{H}(t-\bar{t})\,e^{-\frac{\bar{\tau}}{\tau_i}}d\bar{\tau} + e^{-\frac{\Delta\tau}{\tau_i}}\boldsymbol{\tau}_i^{H}(t)\,. \qquad \text{(EQ7')}$$

To integrate the first integral in (EQ7'), we assume that $\boldsymbol{\tau}_0^H(t-\bar{t})$ varies linearly with the reduced time $\bar{\tau}$ over the increment

$$\boldsymbol{\tau}_0^{H}(t-\bar{t}) = \left(1+\tfrac{\bar{\tau}}{\Delta\tau}\right)\boldsymbol{\tau}_0^{H}(t) - \tfrac{\bar{\tau}}{\Delta\tau}\boldsymbol{\tau}_0^{H}(t+\Delta t) \qquad -\Delta\tau \le \bar{\tau} \le 0. \qquad \text{(EQ8')}$$

Substituting into (EQ7') yields

$$
\begin{aligned}
\boldsymbol{\tau}_i^H(t+\Delta t) = \; & \left[\frac{k_i}{\tau_i} e^{-\frac{\Delta\tau}{\tau_i}} \int_{-\Delta\tau}^{0} -\frac{\bar{\tau}}{\Delta\tau} e^{-\frac{\bar{\tau}}{\tau_i}} d\bar{\tau}\right] \boldsymbol{\tau}_0^H(t+\Delta t) \\
& + \left[\frac{k_i}{\tau_i} e^{-\frac{\Delta\tau}{\tau_i}} \int_{-\Delta\tau}^{0} \left(1+\frac{\bar{\tau}}{\Delta\tau}\right) e^{-\frac{\bar{\tau}}{\tau_i}} d\bar{\tau}\right] \boldsymbol{\tau}_0^H(t) + e^{-\frac{\Delta\tau}{\tau_i}} \boldsymbol{\tau}_i^H(t) .
\end{aligned}
$$

The integrals are readily evaluated, providing the solution at the end of the increment

$$
\begin{aligned}
\boldsymbol{\tau}_i^H(t+\Delta t) \; = \; & \left[1-\frac{\tau_i}{\Delta\tau}\left(1-e^{-\frac{\Delta\tau}{\tau_i}}\right)\right] k_i \boldsymbol{\tau}_0^H(t+\Delta t) \\
& + \left[\frac{\tau_i}{\Delta\tau}\left(1-e^{-\frac{\Delta\tau}{\tau_i}}\right)-e^{-\frac{\Delta\tau}{\tau_i}}\right] k_i \boldsymbol{\tau}_0^H(t) + e^{-\frac{\Delta\tau}{\tau_i}} \boldsymbol{\tau}_i^H(t)
\end{aligned}
$$

or, in a slightly different form

$$
\boldsymbol{\tau}_i^H(t+\Delta t) = \alpha_i k_i \boldsymbol{\tau}_0^H(t+\Delta t) + \beta_i k_i \boldsymbol{\tau}_0^H(t) + \gamma_i \boldsymbol{\tau}_i^H(t) , \tag{EQ9'}
$$

with

$$
\gamma_i = e^{-\frac{\Delta\tau}{\tau_i}} , \quad \alpha_i = 1-\frac{\tau_i}{\Delta\tau}(1-\gamma_i) , \quad \beta_i = \frac{\tau_i}{\Delta\tau}(1-\gamma_i)-\gamma_i .
$$

Observe that for $\Delta t = \Delta\tau = 0$, $\gamma_i = 1$ and $\alpha_i = \beta_i = 0$. For $\Delta t = \Delta\tau = \infty$, $\alpha_i = 1$ and $\gamma_i = \beta_i = 0$.

6.2.4 Integration of the deviatoric stress

The internal deviatoric stresses $\boldsymbol{\tau}_i^D$ at time $t+\Delta t$ follow from

$$
\begin{aligned}
& \boldsymbol{\tau}_i^D(t+\Delta t) = \\
& \quad \frac{g_i}{\tau_i} \int_0^{\tau+\Delta\tau} \overline{\mathbf{F}}_{t+\Delta t}^{-1}(t+\Delta t-t') \cdot \boldsymbol{\tau}_0^D(t+\Delta t-t') \cdot \overline{\mathbf{F}}_{t+\Delta t}^{-T}(t+\Delta t-t') \, e^{-\frac{\tau'}{\tau_i}} d\tau' .
\end{aligned} \tag{EQ10'}
$$

Observe that

$$
\overline{\mathbf{F}}_{t+\Delta t}(t-t') = \overline{\mathbf{F}}_t(t-t') \cdot \overline{\mathbf{F}}_{t+\Delta t}(\tau) ,
$$

and the inverse of this is

$$
\overline{\mathbf{F}}_{t+\Delta t}^{-1}(t-t') = \overline{\mathbf{F}}_{t+\Delta t}^{-1}(t) \cdot \overline{\mathbf{F}}_t^{-1}(t-t') = \overline{\mathbf{F}}_t(t+\Delta t) \cdot \overline{\mathbf{F}}_t^{-1}(t-t') ,
$$

which - when substituted into (EQ10') with $\Delta\overline{\boldsymbol{F}} = \boldsymbol{F}_t(t+\Delta t)$ and $\Delta\overline{\boldsymbol{F}}^{-1} = \boldsymbol{F}_{t+\Delta t}(t)$ - gives

$$\boldsymbol{\tau}_i^D\ (t+\Delta t) = \frac{g_i}{\tau_i}\Delta\overline{\mathbf{F}}\cdot\int_0^{\tau+\Delta\tau}\overline{\mathbf{F}}_t^{-1}\left(t+\Delta t-t'\right)\cdot\boldsymbol{\tau}_0^D\left(t+\Delta t-t'\right)\cdot\overline{\mathbf{F}}_t^{-T}\left(t+\Delta t-t'\right)e^{-\frac{\tau'}{\tau_i}}d\tau'\cdot\Delta\overline{\mathbf{F}}^T.$$

With $\bar{\tau}=\tau'-\Delta\tau$ and $\bar{t}=t'-\Delta t$, it follows:

$$\begin{aligned}
\boldsymbol{\tau}_i^D(t+\Delta t) &= \frac{g_i}{\tau_i}e^{-\frac{\Delta\tau}{\tau_i}}\Delta\overline{\mathbf{F}}\cdot\int_{-\Delta\tau}^{\tau}\overline{\mathbf{F}}_t^{-1}(t-\bar{t})\cdot\boldsymbol{\tau}_0^D(t-\bar{t})\cdot\overline{\mathbf{F}}_t^{-T}(t-\bar{t})e^{-\frac{\bar{\tau}}{\tau_i}}d\bar{\tau}\cdot\Delta\overline{\mathbf{F}}^T\\
&= \frac{g_i}{\tau_i}e^{-\frac{\Delta\tau}{\tau_i}}\int_{-\Delta\tau}^{0}\Delta\overline{\mathbf{F}}\cdot\overline{\mathbf{F}}_t^{-1}(t-\bar{t})\cdot\boldsymbol{\tau}_0^D(t-\bar{t})\cdot\overline{\mathbf{F}}_t^{-T}(t-\bar{t})\cdot\Delta\overline{\mathbf{F}}^T e^{-\frac{\bar{\tau}}{\tau_i}}d\bar{\tau}\\
&+ \frac{g_i}{\tau_i}e^{-\frac{\Delta\tau}{\tau_i}}\Delta\overline{\mathbf{F}}\cdot\int_0^{\tau}\overline{\mathbf{F}}_t^{-1}(t-\bar{t})\cdot\boldsymbol{\tau}_0^D(t-\bar{t})\cdot\overline{\mathbf{F}}_t^{-T}(t-\bar{t})e^{-\frac{\bar{\tau}}{\tau_i}}d\bar{\tau}\cdot\Delta\overline{\mathbf{F}}^T.
\end{aligned}$$

(EQ11')

Now introduce the variable

$$\widehat{\boldsymbol{\tau}}_0^D(t-\bar{t})\equiv\Delta\overline{\mathbf{F}}\cdot\overline{\mathbf{F}}_t^{-1}(t-\bar{t})\cdot\boldsymbol{\tau}_0^D(t-\bar{t})\cdot\overline{\mathbf{F}}_t^{-T}(t-\bar{t})\cdot\Delta\overline{\mathbf{F}}^T. \quad \text{(EQ12')}$$

Note that

$$\widehat{\boldsymbol{\tau}}_0^D(t)=\Delta\overline{\mathbf{F}}\cdot\overline{\mathbf{F}}_t^{-1}(t)\cdot\boldsymbol{\tau}_0^D(t)\cdot\overline{\mathbf{F}}_t^{-T}(t)\cdot\Delta\overline{\mathbf{F}}^T=\Delta\overline{\mathbf{F}}\cdot\boldsymbol{\tau}_0^D(t)\cdot\Delta\overline{\mathbf{F}}^T \quad \text{(EQ13')}$$

and

$$\widehat{\boldsymbol{\tau}}_0^D(t+\Delta t)=\Delta\overline{\mathbf{F}}\cdot\overline{\mathbf{F}}_t^{-1}(t+\Delta t)\cdot\boldsymbol{\tau}_0^D(t+\Delta t)\cdot\overline{\mathbf{F}}_t^{-T}(t+\Delta t)\cdot\Delta\overline{\mathbf{F}}^T=\boldsymbol{\tau}_0^D(t+\Delta t).$$

(EQ14')

Then we can also introduce

$$\widehat{\boldsymbol{\tau}}_i^D(t)=\Delta\overline{\mathbf{F}}\cdot\boldsymbol{\tau}_i^D(t)\cdot\Delta\overline{\mathbf{F}}^T. \quad \text{(EQ15')}$$

Substitution of (EQ5'), (EQ12'), and (EQ15') into (EQ11') yields

$$\boldsymbol{\tau}_i^D(t+\Delta t)=\frac{g_i}{\tau_i}e^{-\frac{\Delta\tau}{\tau_i}}\int_{-\Delta\tau}^{0}\widehat{\boldsymbol{\tau}}_0^D(t-\bar{t})e^{-\frac{\bar{\tau}}{\tau_i}}d\bar{\tau}+e^{-\frac{\Delta\tau}{\tau_i}}\widehat{\boldsymbol{\tau}}_i^D(t). \quad \text{(EQ16')}$$

To integrate the first integral in (EQ16'), we assume that $\widehat{\boldsymbol{\tau}}_0^D(t-\bar{t})$ varies linearly with the reduced time $\bar{\tau}$ over the increment:

$$\widehat{\boldsymbol{\tau}}_0^D(t-\bar{t})=\left(1+\frac{\bar{\tau}}{\Delta\tau}\right)\widehat{\boldsymbol{\tau}}_0^D(t)-\frac{\bar{\tau}}{\Delta\tau}\widehat{\boldsymbol{\tau}}_0^D(t+\Delta t) \qquad -\Delta\tau\le\bar{\tau}\le 0,$$

which with (EQ14') becomes

$$\widehat{\boldsymbol{\tau}}_0^D(t-\bar{t}) = \left(1+\tfrac{\bar{\tau}}{\Delta\tau}\right)\widehat{\boldsymbol{\tau}}_0^D(t) - \tfrac{\bar{\tau}}{\Delta\tau}\boldsymbol{\tau}_0^D(t+\Delta t) \qquad -\Delta\tau \le \bar{\tau} \le 0. \tag{EQ17'}$$

(EQ16') and (EQ17') for the deviatoric stress have exactly the same form as (EQ7') and (EQ8') for the hydrostatic stress. Hence, after integration we obtain

$$\boldsymbol{\tau}_i^D(t+\Delta t) = \alpha_i g_i \boldsymbol{\tau}_0^D(t+\Delta t) + \beta_i g_i \widehat{\boldsymbol{\tau}}_0^D(t) + \gamma_i \widehat{\boldsymbol{\tau}}_i^D(t) \tag{EQ18'}$$

with

$$\gamma_i = e^{-\frac{\Delta\tau}{\tau_i}}, \quad \alpha_i = 1 - \frac{\tau_i}{\Delta\tau}(1-\gamma_i), \quad \beta_i = \frac{\tau_i}{\Delta\tau}(1-\gamma_i) - \gamma_i.$$

(EQ13'), (EQ15'), and (EQ18'), thus, provide a straightforward integration scheme. The total stress at the end of the increment becomes

$$\boldsymbol{\tau}(t+\Delta t) = \boldsymbol{\tau}_0(t+\Delta t) - \sum_{i=1}^N \boldsymbol{\tau}_i^D(t+\Delta t) - \sum_{i=1}^N \boldsymbol{\tau}_i^H(t+\Delta t), \tag{EQ19'}$$

which with (EQ9') and (EQ18') can also be written as

$$\begin{aligned}\boldsymbol{\tau}(t+\Delta t) &= \left(1-\sum_{i=1}^N \alpha_i g_i\right)\boldsymbol{\tau}_0^D(t+\Delta t) + \left(1-\sum_{i=1}^N \alpha_i k_i\right)\boldsymbol{\tau}_0^H(t+\Delta t)\\ &\quad -\sum_{i=1}^N \beta_i g_i \operatorname{dev}\left(\widehat{\boldsymbol{\tau}}_0^D(t)\right) - \sum_{i=1}^N \beta_i k_i \boldsymbol{\tau}_0^H(t) - \sum_{i=1}^N \gamma_i \operatorname{dev}\left(\widehat{\boldsymbol{\tau}}_i^D(t)\right) - \sum_{i=1}^N \gamma_i \boldsymbol{\tau}_i^H(t).\end{aligned} \tag{EQ20'}$$

6.2.5 Rate equation

To solve the system of nonlinear equations generated by the constitutive equations, we need to generate the corotational constitutive rate equations. From (EQ20') it follows

$$\begin{aligned}\check{\boldsymbol{\tau}}(t+\Delta t) &= \left(1-\sum_{i=1}^N \alpha_i g_i\right)\check{\boldsymbol{\tau}}_0^D(t+\Delta t) + \left(1-\sum_{i=1}^N \alpha_i k_i\right)\check{\boldsymbol{\tau}}_0^H(t+\Delta t)\\ &- \sum_{i=1}^N \beta_i g_i \operatorname{dev}\left(\check{\widehat{\boldsymbol{\tau}}}_0^D(t)\right) - \sum_{i=1}^N \gamma_i \operatorname{dev}\left(\check{\widehat{\boldsymbol{\tau}}}_i^D(t)\right),\end{aligned} \tag{EQ21'}$$

where $\check{\boldsymbol{\tau}}(t)$ is the corotational (Jaumann) stress rate. The rate form of the above equation is used to compute the Jacobian.

6.2.6 Cauchy versus Kirchhoff stress

All equations have been worked out in terms of the Kirchhoff stress. However, the implementation in Abaqus uses the Cauchy stress. To transform to Cauchy stress, we use the relations

$$\begin{aligned}\mathbf{S}(t) &= \boldsymbol{\tau}^D(t)/J(t),\\ p(t) &= -\frac{1}{3}\mathbf{I}:\boldsymbol{\tau}^H(t)/J(t).\end{aligned}$$

With $\Delta J \equiv J(t+\Delta t)/J(t)$, this allows us to write (EQ9'), (EQ13'), (EQ15'), (EQ18'), and (EQ19') in the following form:

$$\begin{aligned}p_i(t+\Delta t) &= \alpha_i k_i p_0(t+\Delta t) + \frac{\beta_i k_i p_0(t) + \gamma_i p_i(t)}{\Delta J},\\ \widehat{\mathbf{S}}_0(t) &= \Delta\overline{\mathbf{F}}\cdot\mathbf{S}_0(t)\cdot\Delta\overline{\mathbf{F}}^T,\\ \widehat{\mathbf{S}}_i(t) &= \Delta\overline{\mathbf{F}}\cdot\mathbf{S}_i(t)\cdot\Delta\overline{\mathbf{F}}^T,\\ \mathbf{S}_i(t+\Delta t) &= \alpha_i g_i \mathbf{S}_0(t+\Delta t) + \frac{\beta_i g_i \widehat{\mathbf{S}}_0(t) + \gamma_i \widehat{\mathbf{S}}_i(t)}{\Delta J},\end{aligned}$$

$$\boldsymbol{\sigma}(t+\Delta t) = \boldsymbol{\sigma}_0(t+\Delta t) - \sum_{i=1}^{N}\mathrm{dev}\left(\mathbf{S}_i(t+\Delta t)\right) + \sum_{i=1}^{N} p_i(t+\Delta t)\,\mathbf{I}.$$

The virtual work and rate of virtual work equations are written with respect to the current volume. Therefore, the corotational stress rates are rates of Kirchhoff stress mapped into the current configuration and transformed in the same way as the stresses themselves.

This set of equations - combined with the expressions for α_i, β_i, and γ_i - describe the full implementation of the hyper-viscoelasticity model in a displacement formulation.

The rate equations can be written in a form similar to Hyperelastic material behavior. Introduce

$$\mathbf{C}_v^S = \left(1 - \sum_{i=1}^{N}\alpha_i g_i\right)\mathbf{C}_0^S$$

and

$$K_v = \left(1 - \sum_{i=1}^{N}\alpha_i k_i\right)K_0,$$

where $\boldsymbol{C}_0^S$ and K_0 are the instantaneous moduli, corresponding to $\boldsymbol{C}^S$ and K of Hyperelastic material behavior. Thus, all rate equations can be obtained by substitution of $\boldsymbol{C}_0^S$ by $\boldsymbol{C}_v^S$ and K_0 by K_v.

6.3 Frequency domain viscoelasticity

The frequency domain viscoelastic material model describes frequency-dependent material behavior in small steady-state harmonic oscillations for those materials in which dissipative losses caused by "viscous" (internal damping) effects must be modeled in the frequency domain.

Products: Abaqus/Standard

Many applications of elastomers involve dynamic loading in the form of steady-state vibration, and often in such cases the dissipative losses in the material (the "viscous" part of the material's viscoelastic behavior) must be modeled to obtain useful results. In most problems of this class the structure is first preloaded statically, and this preloading generally involves large deformation of the elastomers. The response to that preloading is computed on the basis of purely elastic behavior in the elastomeric parts of the model - that is, we assume that the preloading is applied for a sufficiently long time so that any viscous response in the material has time to decay away.

The dynamic analysis problem in this case is, therefore, to investigate the dynamic, viscoelastic response about a predeformed elastic state. In some such cases we can reasonably assume that the vibration amplitude is sufficiently small that both the kinematic and material response in the dynamic phase of the problem can be treated as linear perturbations about the predeformed state. The small amplitude viscoelastic vibration capability provided in Abaqus/Standard, which is described in Morman and Nagtegaal (1983) and uses the procedure described in Direct steady-state dynamic analysis, is based on such a linearization. Its appropriateness to a particular application will depend on the magnitude of the vibration with respect to possible kinematic nonlinearities (the additional strains and rotations that occur during the dynamic loading must be small enough so that the linearization of the kinematics is reasonable) and with respect to possible nonlinearities in the material response, and on the particular constitutive assumptions incorporated in the viscoelastic model described in this section—in particular, the assumption of separation of prestrain and time effects described below.

In Hyperelastic material behavior it is shown that the rate of change of the true (Cauchy) stress in an elastomeric material with a strain energy potential is given by

$$d\left(J\mathbf{S}\right) = J\left(\mathbf{C}^{S} : d\mathbf{e} + \mathbf{Q}\, d\varepsilon^{\mathrm{vol}} + d\boldsymbol{\Omega}\cdot\mathbf{S} - \mathbf{S}\cdot d\boldsymbol{\Omega}\right) \quad \text{(EQ1*)}$$

for the deviatoric part of the stress and

$$dp = -\mathbf{Q} : d\mathbf{e} - K\, d\varepsilon^{\mathrm{vol}} \quad \text{(EQ2*)}$$

for the equivalent pressure stress in a compressible material. The various quantities in these equations are defined in Hyperelastic material behavior. For a fully incompressible material all components of $\boldsymbol{Q}$ are zero and the equivalent pressure stress is defined only by the loading of the structure, so that the second equation is not applicable.

For small viscoelastic vibrations about a predeformed state we linearize the additional motions that occur during the vibration so that the differential of a quantity in (EQ1*) and (EQ2*) can be interpreted as the additional incremental value,

$$d\left(f\right) \rightarrow \Delta\left(f\right) \stackrel{\text{def}}{=} f\Big|_t - f|_0,$$

for any quantity f, where $f\,|_t$ is the current value of f at some time during the vibration and $f\,|_0$ is the reference value of f; that is, $f\,|_0$ is the value of f at the end of the static (long term) preloading, about which f is fluctuating during the vibration.
The incremental elastic constitutive behavior for small added motions defined by this interpretation of (EQ1*) and (EQ2*) is now generalized to include viscous dissipation as well as elastic response in the material, following Lianis (1965), to give

$$\Delta\left(J\mathbf{S}\right) = J\left(\mathbf{C}^S\Big|_0 : \Delta\mathbf{e} + \mathbf{Q}\Big|_0 \Delta\varepsilon^{\text{vol}} + \Delta\boldsymbol{\Omega}\cdot\mathbf{S}\Big|_0 - \mathbf{S}\Big|_0 \cdot\Delta\boldsymbol{\Omega} + \int_0^t \boldsymbol{\Phi}\left(|_0, t-\tau\right) : \dot{\mathbf{e}}\left(\tau\right)\, d\tau\right),$$

and, for a compressible material,

$$\Delta p = -\mathbf{Q}\Big|_0 : \Delta\mathbf{e} - K\Big|_0 \Delta\varepsilon^{\text{vol}} - \int_0^t \kappa\left(|_0, t-\tau\right)\dot{\varepsilon}^{\text{vol}}\left(\tau\right)\, d\tau.$$

In these expressions $f\left(|_0, t-\tau\right)$ is meant to indicate that f depends on the elastic predeformation that has occurred prior to the small dynamic vibrations (the state at $t = 0$) and is evaluated at time $t-\tau$, between the start of the vibrations and the current time, t. $\boldsymbol{\Phi}$ and κ are the functions that define the viscous part of the material's response: the notation is intended to imply that these are functions of the elastic predeformation and time. $\dot{f} \stackrel{\text{def}}{=} df/dt$ is the time rate of change of a quantity.
The definitions of the viscous parts of the behavior, $\boldsymbol{\Phi}$ and κ, provided in Abaqus are simplified by assuming that this viscous behavior exhibits separation of time and prestrain effects; that is, that

$$\boldsymbol{\Phi}\left(|_0, t-\tau\right) = g\left(t-\tau\right)\mathbf{C}^S|_0$$

and

$$\kappa\left(|_0, t-\tau\right) = k\left(t-\tau\right)K\big|_0,$$

where $\boldsymbol{C}^S\,|_0$ and $K\,|_0$ are the "effective elasticity" of the material in its predeformed state, prior to the vibration. This assumption simply means that measurements of the viscous behavior during small motions of the material about a predeformed state depend only on the predeformation to the extent that the effective elasticity of the material also depends on that predeformation. There is experimental evidence that this simplification is appropriate for some practical materials (see Morman's (1979) discussion). With this assumption the definition of the viscous part of the material's behavior is reduced to finding the scalar functions of time, g and k (only g for fully incompressible materials), and the constitutive response to small perturbations is simplified to

$$\Delta(J\mathbf{S}) = J\left(\mathbf{C}^S\Big|_0 : \left\{\Delta\mathbf{e} + \int_0^t g(t-\tau)\,\dot{\mathbf{e}}(\tau)\,d\tau\right\} + \mathbf{Q}\Big|_0 \Delta\varepsilon^{\text{vol}} + \Delta\boldsymbol{\Omega}\cdot\mathbf{S}\Big|_0 - \mathbf{S}\Big|_0 \cdot\Delta\boldsymbol{\Omega}\right),$$

and, for compressible materials,

$$\Delta p = -\mathbf{Q}|_0 : \Delta\mathbf{e} - K|_0\left(\Delta\varepsilon^{\text{vol}} + \int_0^t k(t-\tau)\,\dot{\varepsilon}^{\text{vol}}(\tau)\,d\tau\right).$$

In Abaqus this model is provided only for the direct-solution and subspace-based steady-state dynamic analysis procedures, in which we assume that the dynamic response is steady-state harmonic vibration, so that we can write

$$f|_t - f|_0 = (\Re(\Delta f) + i\,\Im(\Delta f))\,\exp(i\omega t),$$

where $\Re(\Delta f) + i\Im(\Delta f)$ is the complex amplitude of a variable f.
Defining the Fourier transforms of the viscous relaxation functions $g(t)$ and $k(t)$ as

$$\Re(g) + i\,\Im(g) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} g(t)\,\exp(-i\omega t)dt,$$

and

$$\Re(k) + i\,\Im(k) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} k(t)\,\exp(-i\omega t)dt,$$

allows the constitutive model to be written for such harmonic motions in the linear form

$$\Re\left(\Delta(J\mathbf{S})\right) = J\left(\begin{aligned}&\mathbf{C}^S\big|_0 : \{(1-\omega\,\Im(g))\,\Re(\Delta\mathbf{e}) - \omega\,\Re(g)\,\Im(\Delta\mathbf{e})\}\\ &+\mathbf{Q}\Big|_0 \Re\left(\Delta\varepsilon^{\text{vol}}\right) + \Re(\Delta\boldsymbol{\Omega})\cdot\mathbf{S}\Big|_0 - \mathbf{S}\Big|_0\cdot\Re(\Delta\boldsymbol{\Omega})\end{aligned}\right),$$

and

$$\Im\left(\Delta\left(J\mathbf{S}\right)\right) = J\left(\mathbf{C}^S\Big|_0 : \{\omega\,\Re(g)\,\Re(\Delta\mathbf{e}) + (1-\omega\,\Im(g))\;\Im(\Delta\mathbf{e})\} \right.$$

$$\left. +\mathbf{Q}\Big|_0 \Im\left(\Delta\varepsilon^{\mathrm{vol}}\right) + \Im\left(\Delta\boldsymbol{\Omega}\right)\cdot\mathbf{S}\Big|_0 - \mathbf{S}\Big|_0 \cdot\Im\left(\Delta\boldsymbol{\Omega}\right) \right);$$

and, for compressible materials,

$$\Re(\Delta p) = -\mathbf{Q}|_0 : \Re(\Delta\mathbf{e}) - K|_0 : \left\{(1-\omega\,\Im(k))\;\Re\left(\Delta\varepsilon^{\mathrm{vol}}\right) - \omega\,\Re(k)\,\Im\left(\Delta\varepsilon^{\mathrm{vol}}\right)\right\},$$

and

$$\Im(\Delta p) = -\mathbf{Q}|_0 : \Im(\Delta\mathbf{e}) - K|_0 : \left\{\omega\,\Re(k)\,\Re\left(\Delta\varepsilon^{\mathrm{vol}}\right) + (1-\omega\,\Im(k))\;\Im\left(\Delta\varepsilon^{\mathrm{vol}}\right)\right\}.$$

The viscous behavior of the material is, thus, reduced to defining $\Re(g)$, $\Im(g)$, $\Re(k)$, and $\Im(k)$ as functions of frequency.
When the pure displacement formulation is used for a compressible material, the virtual work equation for dynamic response is

$$\delta W_I - \delta W_D - \delta W_E = 0, \quad \text{(EQ3*)}$$

where

$$\delta W_I = \int_V \boldsymbol{\sigma} : \delta\mathbf{D}\,dV$$

is the internal virtual work,

$$\delta W_D = -\int_{V^0} \rho^0 \delta\mathbf{u}\cdot\ddot{\mathbf{u}}\,dV^0$$

is the virtual work of the d'Alembert forces (ρ^0 is the mass density of the material in the original configuration), and

$$\delta W_E = \int_S \delta\mathbf{u}\cdot\mathbf{p}\,dS + \int_V \delta\mathbf{u}\cdot\mathbf{f}\,dV$$

is the virtual work of externally prescribed surface tractions $\boldsymbol{p}$ per current surface area and body forces $\boldsymbol{f}$ per current volume.
For the linearized perturbations considered here we recast (EQ3*) in incremental form, giving

$$\Delta\delta W_I - \Delta\delta W_D - \Delta\delta W_E = 0, \quad \text{(EQ4*)}$$

where $\Delta\delta W_I$ is obtained from Equation 12 with the interpretation $d(f) \rightarrow \Delta(f)$;

$$\Delta\delta W_D \stackrel{\text{def}}{=} \delta W_D$$

and

$$\begin{aligned}\Delta\delta W_E = & \int_S \delta\mathbf{u} \cdot (\mathbf{P}_u \cdot \Delta\mathbf{u} + \mathbf{p}_p \Delta p)\, dS \\ & + \int_V \delta\mathbf{u} \cdot \left(\mathbf{F}_u \cdot \Delta\mathbf{u} + \mathbf{f}_f \Delta f\right) dV,\end{aligned}$$

where

$$\mathbf{P}_u = \frac{1}{J_A} \frac{\partial(J_A \mathbf{p})}{\partial\mathbf{u}},$$

in which

$$J_A = \left|\frac{dA}{dA_0}\right|$$

is the ratio of current to reference surface area;

$$\mathbf{p}_p = \frac{\partial\mathbf{p}}{\partial p};$$

$$\mathbf{F}_u = \frac{1}{J} \frac{\partial(J\mathbf{f})}{\partial\mathbf{u}};$$

$$\mathbf{f}_f = \frac{\partial\mathbf{f}}{\partial f};$$

and p and f are the externally prescribed tractions, so that Δp and Δf are externally prescribed traction increments. Note that $\Delta\delta W_E$ includes terms dependent on $\Delta\boldsymbol{u}$: these give rise to the "load stiffness matrix" when the finite element interpolations are introduced.
When the motion is harmonic we can recast these quantities as

$$\begin{aligned}\Delta\delta W_I = \int_V \Bigg(& \left\lfloor \Re(\delta\mathbf{e})\ \Im(\delta\mathbf{e})\ \Re\left(\delta\varepsilon^{\text{vol}}\right)\ \Im\left(\delta\varepsilon^{\text{vol}}\right) \right\rfloor \\ & : \begin{bmatrix} \Re\left(\widetilde{\mathbf{C}}\right) & \Im\left(\widetilde{\mathbf{C}}\right) & \mathbf{Q}|_0 & 0 \\ \Im\left(\widetilde{\mathbf{C}}\right) & -\Re\left(\widetilde{\mathbf{C}}\right) & 0 & -\mathbf{Q}|_0 \\ \mathbf{Q}|_0 & 0 & \Re\left(\widetilde{K}\right) & \Im\left(\widetilde{K}\right) \\ 0 & -\mathbf{Q}|_0 & \Im\left(\widetilde{K}\right) & -\Re\left(\widetilde{K}\right) \end{bmatrix} : \left\{ \begin{matrix} \Re(\Delta\mathbf{e}) \\ \Im(\Delta\mathbf{e}) \\ \Re\left(\Delta\varepsilon^{\text{vol}}\right) \\ \Im\left(\Delta\varepsilon^{\text{vol}}\right) \end{matrix} \right\}\end{aligned}$$

$$
\left.-\mathbf{S}_0 : \left[2\lfloor \Re(\delta\mathbf{e})\ \Im(\delta\mathbf{e})\rfloor : \begin{Bmatrix} \Re(\Delta\mathbf{e}) \\ \Im(\Delta\mathbf{e}) \end{Bmatrix} - \lfloor \Re(\delta\mathbf{L})\ \Im(\delta\mathbf{L})\rfloor : \begin{Bmatrix} \Re(\Delta\mathbf{L}) \\ \Im(\Delta\mathbf{L}) \end{Bmatrix}\right]\right)\, dV,
$$

where

$$
\begin{aligned}
\Re\left(\widetilde{\mathbf{C}}\right) &= \left(1-\omega\,\Im(g)\right)\mathbf{C}^S\big|_0,\\
\Im\left(\widetilde{\mathbf{C}}\right) &= -\omega\,\Re(g)\mathbf{C}^S\big|_0,\\
\Re\left(\widetilde{K}\right) &= \left(1-\omega\,\Im(k)\right)K|_0,\\
\Im\left(\widetilde{K}\right) &= -\omega\,\Re(k)K|_0,
\end{aligned}
$$

$$
\Delta\delta W_D = -\omega^2\int_{V^0}\rho^0\lfloor \Re(\delta\mathbf{u})\ \Im(\delta\mathbf{u})\rfloor \cdot \begin{Bmatrix} \Re(\Delta\mathbf{u}) \\ \Im(\Delta\mathbf{u}) \end{Bmatrix} dV^0,
$$

and

$$
\begin{aligned}
\Delta\delta W_E = &\int_S \lfloor \Re(\delta\mathbf{u})\ \Im(\delta\mathbf{u})\rfloor \cdot \begin{bmatrix} \mathbf{P}_u & 0 \\ 0 & \mathbf{P}_u \end{bmatrix}\cdot \begin{Bmatrix} \Re(\Delta\mathbf{u}) \\ \Im(\Delta\mathbf{u}) \end{Bmatrix} dS \\
&+ \int_V \lfloor \Re(\delta\mathbf{u})\ \Im(\delta\mathbf{u})\rfloor \cdot \begin{bmatrix} \mathbf{F}_u & 0 \\ 0 & \mathbf{F}_u \end{bmatrix}\cdot \begin{Bmatrix} \Re(\Delta\mathbf{u}) \\ \Im(\Delta\mathbf{u}) \end{Bmatrix} dV \\
&+ \int_S \lfloor \Re(\delta\mathbf{u})\ \Im(\delta\mathbf{u})\rfloor \cdot \mathbf{p}_p \begin{Bmatrix} \Re(\Delta p) \\ \Im(\Delta p) \end{Bmatrix} dS + \int_V \lfloor \Re(\delta\mathbf{u})\ \Im(\delta\mathbf{u})\rfloor \cdot \mathbf{f}_f \begin{Bmatrix} \Re(\Delta f) \\ \Im(\Delta f) \end{Bmatrix} dV.
\end{aligned}
$$

In these expressions $\Re(\delta\boldsymbol{u})$ and $\Im(\delta\boldsymbol{u})$ are understood to be independent variations. Thus, when the finite element displacement interpolations are introduced into (EQ4*), we obtain a linear, frequency-dependent system that can be solved at each frequency for the real and imaginary parts of the nodal degrees of freedom of the model. In like fashion, the augmented variational principles for almost incompressible behavior and for fully incompressible behavior described in Hyperelastic material behavior can be used to obtain linear, frequency-dependent systems for harmonic viscoelastic vibration problems. The steady-state dynamic analysis procedure based on modal superposition cannot be used because the viscous behavior assumed does not correspond to a small amount of Rayleigh damping, which is a requirement for steady-state harmonic response based on modal superposition.