Defines

In [1]:
```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from matplotlib import rcParams
from scipy.optimize import minimize
import sympy as sp
rcParams['font.family'] = 'serif'

# Create the DataFrame
data = {
    "Eng. Strain [%]": [0, 30, 60, 90, 120, 150, 180, 210, 240, 270, 300],
    "Eng. Stress [MPa] (Uniaxial)": [0, 6, 8, 9, 11, 12, 14, 15, 16, 17, 19],
    "Eng. Stress [MPa] (Equibiaxial)": [0, 10, 17, 22, 28, 37, 44, 52, 60, 70, 82],
    "Eng. Stress [MPa] (Planar)": [0, 7, 10, 12, 14, 16, 17, 19, 21, 23, 24],
}

df = pd.DataFrame(data)
df
```

Out[1]:

	Eng. Strain [%]	Eng. Stress [MPa] (Uniaxial)	Eng. Stress [MPa] (Equibiaxial)	Eng. Stress [MPa] (Planar)
0	0	0	0	0
1	30	6	10	7
2	60	8	17	10
3	90	9	22	12
4	120	11	28	14
5	150	12	37	16
6	180	14	44	17
7	210	15	52	19
8	240	16	60	21
9	270	17	70	23
10	300	19	82	24

In [2]:
```
print(df.to_latex(index=False))
```

```
\begin{tabular}{rrrr}
\toprule
Eng. Strain [%] & Eng. Stress [MPa] (Uniaxial) & Eng. Stress [MPa] (Equibiaxial) & Eng. Stress
[MPa] (Planar) \\
\midrule
0 & 0 & 0 & 0 \\
30 & 6 & 10 & 7 \\
60 & 8 & 17 & 10 \\
90 & 9 & 22 & 12 \\
120 & 11 & 28 & 14 \\
150 & 12 & 37 & 16 \\
180 & 14 & 44 & 17 \\
210 & 15 & 52 & 19 \\
240 & 16 & 60 & 21 \\
270 & 17 & 70 & 23 \\
300 & 19 & 82 & 24 \\
\bottomrule
\end{tabular}
```

In [3]:

```
def W(λ1, λ2, λ3, mu1, mu2, a1, a2):
    ret  = 2*mu1/a1**2 * (λ1**a1 + λ2**a1 + λ3**a1 - 3)
    ret += 2*mu2/a2**2 * (λ1**a2 + λ2**a2 + λ3**a2 - 3)
    return ret

def P_OG_uniaxial(λ, mu1, mu2, a1, a2):
    ret  = 2*mu1/a1 * (λ**(a1-1) - λ**(-a1/2 - 1))
    ret += 2*mu2/a2 * (λ**(a2-1) - λ**(-a2/2 - 1))
    return ret

def P_OG_equibiaxial(λ, mu1, mu2, a1, a2):
    ret  = 2*mu1/a1 * (λ**(a1-1) - λ**(-2*a1 - 1))
    ret += 2*mu2/a2 * (λ**(a2-1) - λ**(-2*a2 - 1))
    return ret

def P_OG_planar(λ, mu1, mu2, a1, a2):
    ret  = 2*mu1/a1 * (λ**(a1-1) - λ**(-a1 - 1))
    ret += 2*mu2/a2 * (λ**(a2-1) - λ**(-a2 - 1))
    return ret
```

Tasks

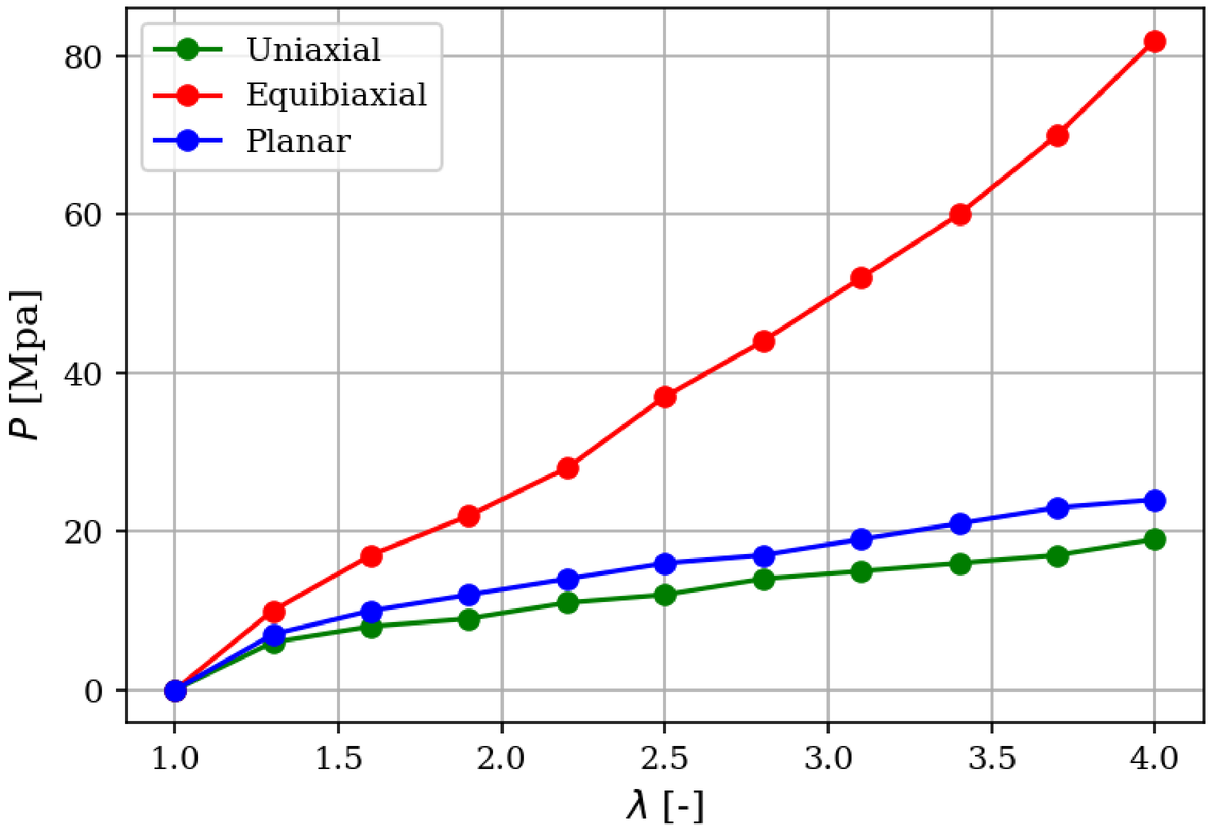
Task 1. - Plotting experimental data

In [4]:

```
ε = df["Eng. Strain [%]"].to_numpy()/100
λ = ε + 1

P_uniaxial    = df["Eng. Stress [MPa] (Uniaxial)"   ].to_numpy()
P_equibiaxial = df["Eng. Stress [MPa] (Equibiaxial)"].to_numpy()
P_planar      = df["Eng. Stress [MPa] (Planar)"     ].to_numpy()


plt.figure(figsize=(6,4),dpi=150)
plt.plot(λ,P_uniaxial,marker='o',label='Uniaxial',color='green')
plt.plot(λ,P_equibiaxial,marker='o',label='Equibiaxial',color='red')
plt.plot(λ,P_planar,marker='o',label='Planar',color='blue')
plt.grid()
plt.xlabel(r'$\lambda$ [-]',fontsize=12)
```

```
plt.ylabel(r'$P$ [Mpa]',fontsize=12)
plt.legend()
```

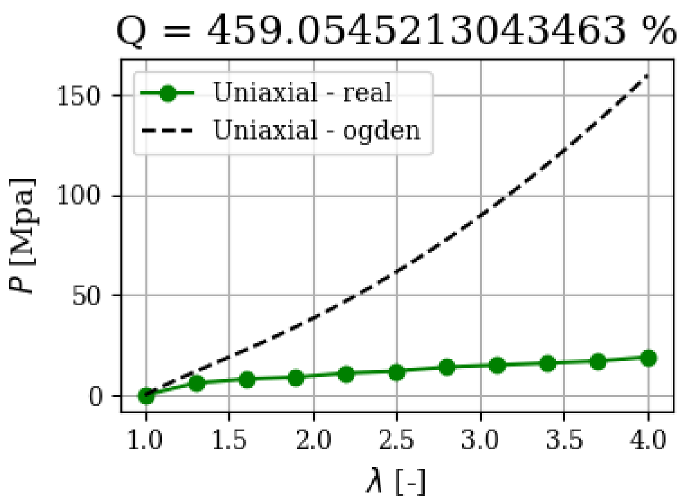
Out[4]: <matplotlib.legend.Legend at 0x1e8eceaab50>



Task 2. - Fitting model only to the unaxial data

In [5]:
```
def Q_uniaxial(params):
    mu1, mu2, a1, a2 = params
    # [:1] to avoid 0 divide
    # the model provide P=0 for λ=0
    pred = P_OG_uniaxial(λ[1:],mu1, mu2, a1, a2)
    real = P_uniaxial[1:]
    n = P_uniaxial.shape[0]

    return (np.sum(((real - pred)/real)**2)/n)**(1/2)
```

In [6]:
```
params = 10, 5, 3, 3

plt.figure(figsize=(4,3),dpi=100)
plt.plot(λ,P_uniaxial,marker='o',label='Uniaxial - real',color='green')
λ_range = np.linspace(1,4,500)
plt.plot(λ_range,P_OG_uniaxial(λ_range,*params),color='black',label='Uniaxial - ogden',ls='--'
plt.grid()
plt.xlabel(r'$\lambda$ [-]',fontsize=12)
plt.ylabel(r'$P$ [Mpa]',fontsize=12)
plt.legend()
plt.title(f'Q = {Q_uniaxial(params)*100} %',fontsize=16)
plt.tight_layout()
plt.show()
```

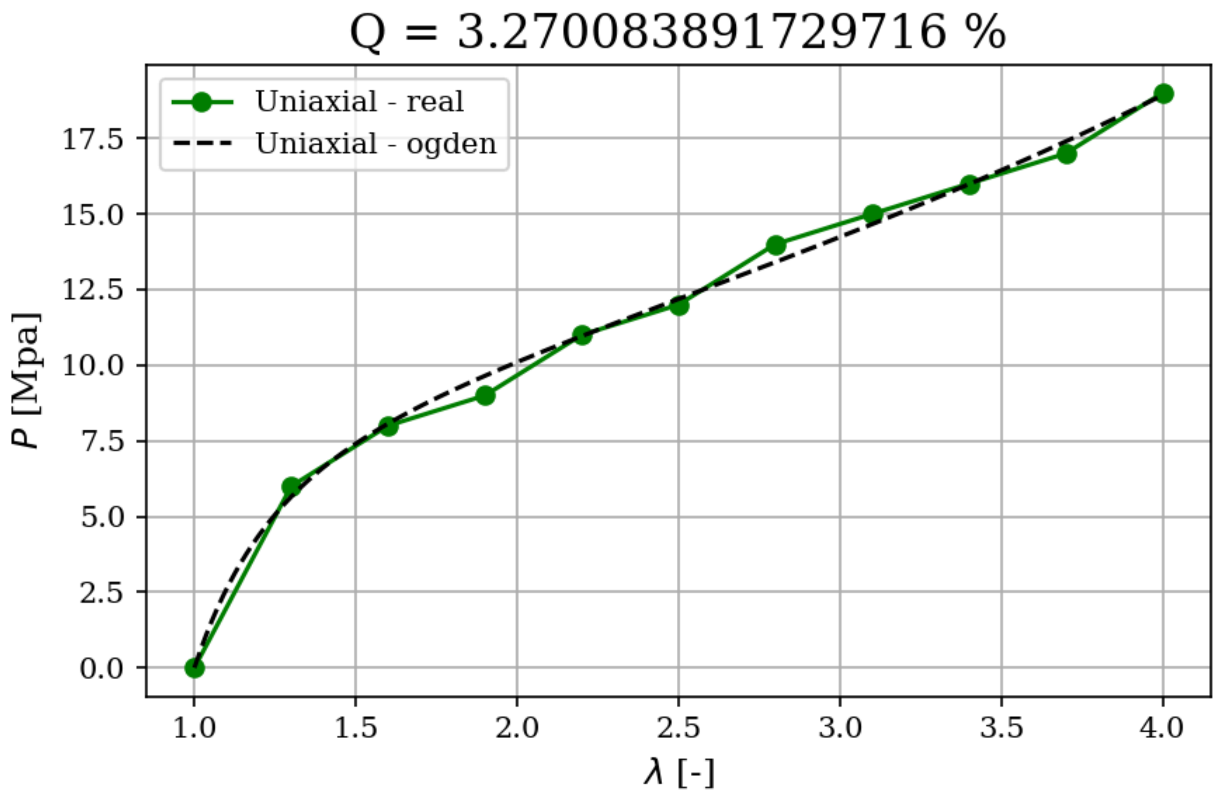


In [7]:
```
def constraint(params):
    mu1, mu2, a1, a2 = params
    return mu1 + mu2 - 1

constraints = [
    {'type': 'ineq', 'fun': constraint},
]

result = minimize(
    Q_uniaxial,
    params,
    method='BFGS',
    constraints=constraints)

print(f'Q_uniaxial = {result.fun*100} [%]')
print(f'params = {result.x}')
```

```
Q_uniaxial = 3.270083891729716 [%]
params = [10.71992575  0.04956515 -3.2772214   4.57758998]
```

```
C:\Users\Zsoci\AppData\Local\Packages\PythonSoftwareFoundation.Python.3.11_qbz5n2kfra8p0\LocalC
ache\local-packages\Python311\site-packages\scipy\optimize\_minimize.py:576: RuntimeWarning: Me
thod BFGS cannot handle constraints.
  warn('Method %s cannot handle constraints.' % method,
```

In [8]:
```
params = result.x

plt.figure(figsize=(6,4),dpi=150)
plt.plot(λ,P_uniaxial,marker='o',label='Uniaxial - real',color='green')
λ_range = np.linspace(1,4,500)
plt.plot(λ_range,P_OG_uniaxial(λ_range,*params),color='black',label='Uniaxial - ogden',ls='--'
plt.grid()
plt.xlabel(r'$\lambda$ [-]',fontsize=12)
plt.ylabel(r'$P$ [Mpa]',fontsize=12)
plt.legend()
plt.title(f'Q = {Q_uniaxial(params)*100} %',fontsize=16)
plt.tight_layout()
plt.show()
```



Task 3. - Equibiaxial and planar quality functions

In [9]:
```
def Q_equibiaxial(params):
    mu1, mu2, a1, a2 = params
    # [:1] to avoid 0 divide
    # the model provide P=0 for λ=0
    pred = P_OG_equibiaxial(λ[1:],mu1, mu2, a1, a2)
    real = P_equibiaxial[1:]
    n = P_equibiaxial.shape[0]

    return (np.sum(((real - pred)/real)**2)/n)**(1/2)

def Q_planar(params):
    mu1, mu2, a1, a2 = params
    # [:1] to avoid 0 divide
    # the model provide P=0 for λ=0
    pred = P_OG_planar(λ[1:],mu1, mu2, a1, a2)
    real = P_planar[1:]
    n = P_planar.shape[0]

    return (np.sum(((real - pred)/real)**2)/n)**(1/2)
```

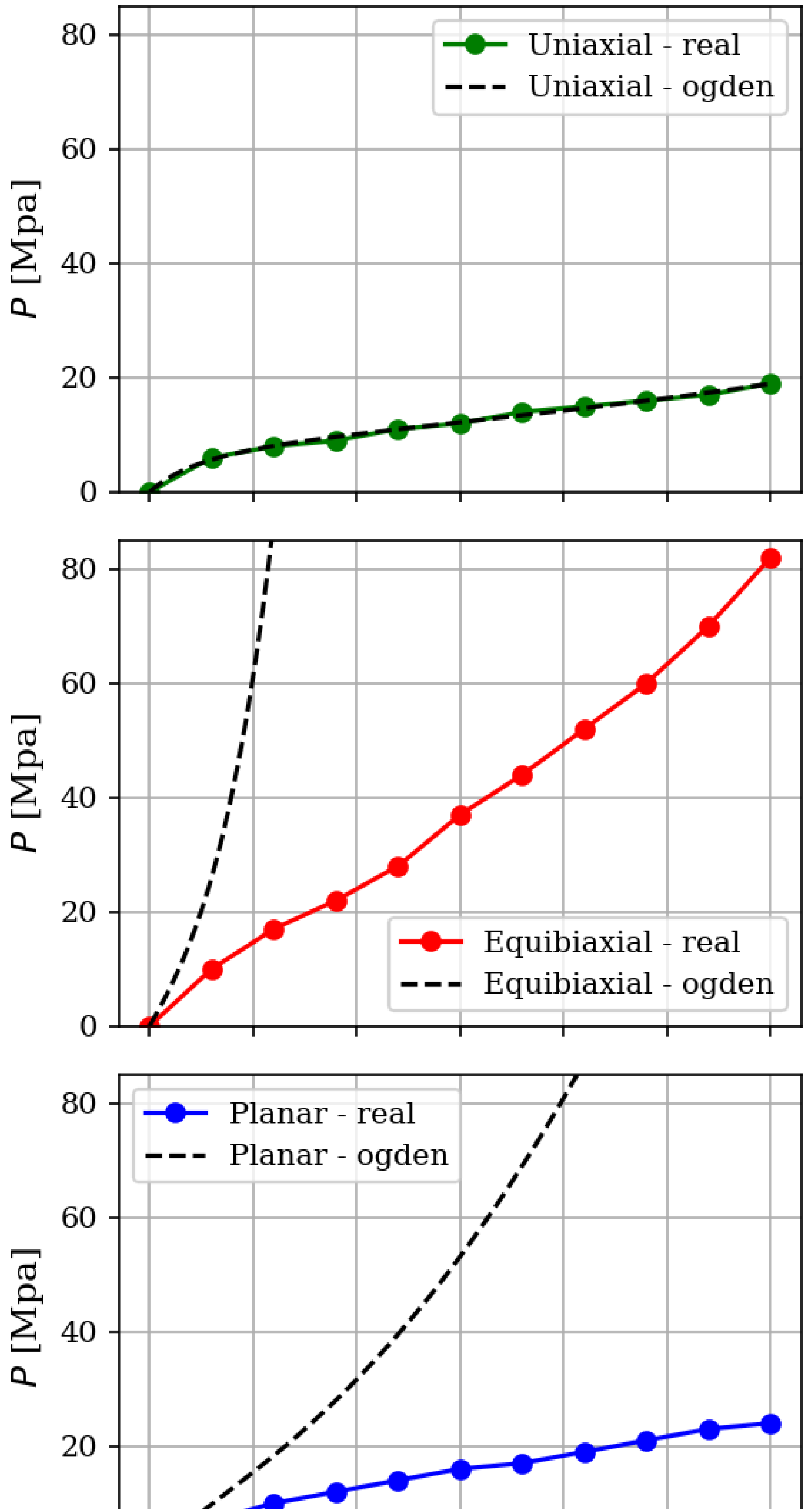
In [10]:
```
print(f'Q_equibiaxial = {Q_equibiaxial(params)*100} %')
print(f'Q_planar = {Q_planar(params)*100} %')
```

```
Q_equibiaxial = 7703.947754246242 %
Q_planar = 306.5815214494574 %
```

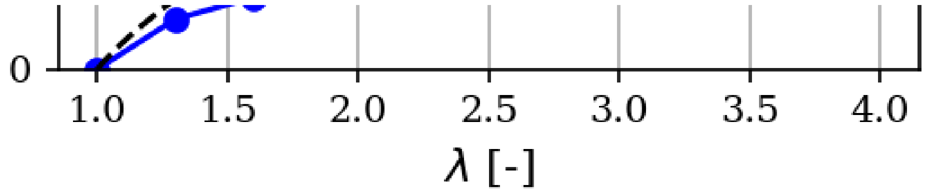
Task 4. - Plotting solutions

In [11]:
```
fig, ax = plt.subplots(3,figsize=(4,8),dpi=150,sharex=True,sharey=True)
```

```
λ_range = np.linspace(1,4,500)
ax[0].plot(λ,P_uniaxial,marker='o',label='Uniaxial - real',color='green')
ax[0].plot(λ_range,P_OG_uniaxial(λ_range,*params),color='black',label='Uniaxial - ogden',ls='-
ax[0].grid()
ax[0].set_ylim(0,85)
ax[0].legend()
ax[0].set_ylabel(r'$P$ [Mpa]',fontsize=12)
ax[1].plot(λ,P_equibiaxial,marker='o',label='Equibiaxial - real',color='red')
ax[1].plot(λ_range,P_OG_equibiaxial(λ_range,*params),color='black',label='Equibiaxial - ogden'
ax[1].grid()
ax[1].set_ylim(0,85)
ax[1].legend()
ax[1].set_ylabel(r'$P$ [Mpa]',fontsize=12)
ax[2].plot(λ,P_planar,marker='o',label='Planar - real',color='blue')
ax[2].plot(λ_range,P_OG_planar(λ_range,*params),color='black',label='Planar - ogden',ls='--')
ax[2].grid()
ax[2].set_ylim(0,85)
ax[2].legend()
ax[2].set_ylabel(r'$P$ [Mpa]',fontsize=12)
ax[2].set_xlabel(r'$\lambda$ [-]',fontsize=12)
plt.tight_layout()
plt.show()
```



Uniaxial - real
Uniaxial - ogden
Equibiaxial - real
Equibiaxial - ogden
Planar - real
Planar - ogden
P [Mpa]
0
20
40
60
80



Task 5. - Simultanious fitting to all measurements

In [12]:
```
def Q(params):
    return (Q_uniaxial(params) + Q_equibiaxial(params) + Q_planar(params))/3
```

In [13]:
```
def constraint(params):
    mu1, mu2, a1, a2 = params
    return mu1 + mu2 - 1

constraints = [
    {'type': 'ineq', 'fun': constraint},
]

result = minimize(
    Q,
    params,
    method='BFGS',
    constraints=constraints,
)

print(f'Q = {result.fun*100} [%]')
params = result.x
print(f'params = {params}')
```

```
Q = 4.204375992236135 [%]
params = [ 3.18596983  4.62213918 -1.48228961  1.77455216]
```

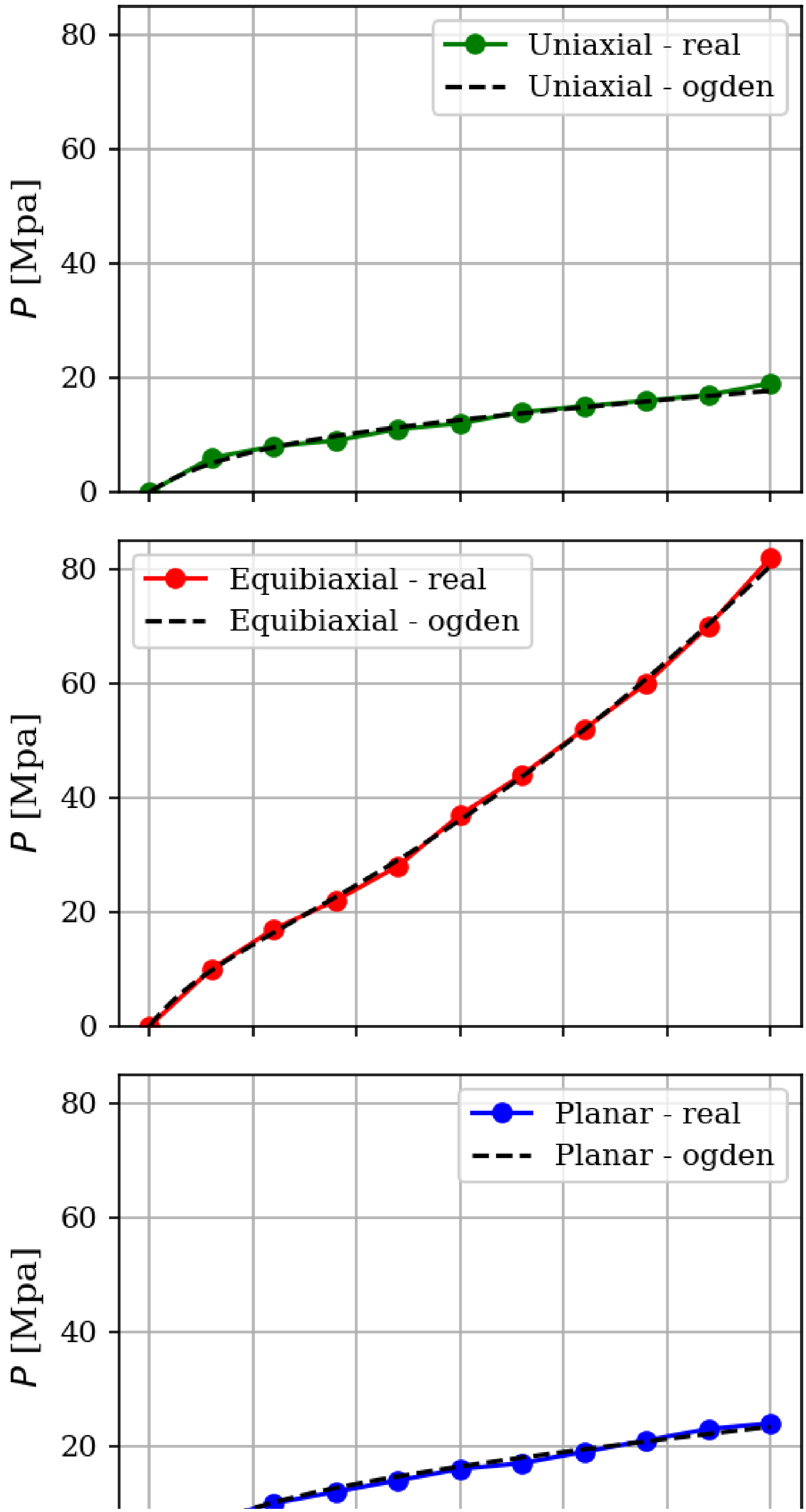
In [14]:
```
print(Q_uniaxial(params)*100,'[%]')
print(Q_equibiaxial(params)*100,'[%]')
print(Q_planar(params)*100,'[%]')
```

```
6.414037738579216 [%]
2.196516422902834 [%]
4.002573815226355 [%]
```

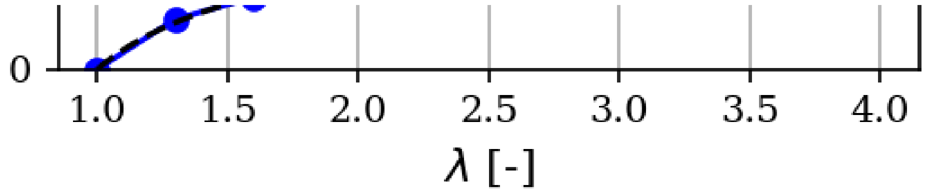
Task 6. - Plotting solutions

In [15]:
```
fig, ax = plt.subplots(3,figsize=(4,8),dpi=150,sharex=True,sharey=True)
λ_range = np.linspace(1,4,500)
ax[0].plot(λ,P_uniaxial,marker='o',label='Uniaxial - real',color='green')
ax[0].plot(λ_range,P_OG_uniaxial(λ_range,*params),color='black',label='Uniaxial - ogden',ls='-
ax[0].grid()
ax[0].set_ylim(0,85)
ax[0].legend()
ax[0].set_ylabel(r'$P$ [Mpa]',fontsize=12)
ax[1].plot(λ,P_equibiaxial,marker='o',label='Equibiaxial - real',color='red')
ax[1].plot(λ_range,P_OG_equibiaxial(λ_range,*params),color='black',label='Equibiaxial - ogden'
ax[1].grid()
ax[1].set_ylim(0,85)
ax[1].legend()
ax[1].set_ylabel(r'$P$ [Mpa]',fontsize=12)
ax[2].plot(λ,P_planar,marker='o',label='Planar - real',color='blue')
ax[2].plot(λ_range,P_OG_planar(λ_range,*params),color='black',label='Planar - ogden',ls='--')
```

```
ax[2].grid()
ax[2].set_ylim(0,85)
ax[2].legend()
ax[2].set_ylabel(r'$P$ [Mpa]',fontsize=12)
ax[2].set_xlabel(r'$\lambda$ [-]',fontsize=12)
plt.tight_layout()
plt.show()
```



Uniaxial - real
Uniaxial - ogden
Equibiaxial - real
Equibiaxial - ogden
Planar - real
Planar - ogden
P [Mpa]
P [Mpa]
P [Mpa]
0
20
40
60
80



Task 7. - Determine of k

The displacement field is given in the with the components of the material displacement field, like

$$\boldsymbol{U} = \begin{bmatrix} 5 - 3kX_1 \\ 1 + 0.3X_1 + 0.5X_2 \\ 3 \end{bmatrix}$$

.

From this we can determine the displacement gradient as

$$\boldsymbol{K} = \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}\boldsymbol{X}} = \begin{bmatrix} -3k & 0 & 0 \\ 0.3 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

.

The deformation gradient can be calculated as

$$\boldsymbol{F} = \boldsymbol{K} + \boldsymbol{I} = \begin{bmatrix} 1 - 3k & 0 & 0 \\ 0.3 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

.

The volume change can be calculated as the determinant of the deformation gradient

$$J = \det \boldsymbol{F} = (1 - 3k) \cdot 1.5$$

If there is incompressible case, the volume change should be equal with 1, therefore:

$$(1 - 3k) \cdot 1.5 = 1$$

$$k = \frac{1}{9}$$

In [16]:
```
k = 1/9

F = np.array([
    [1-3*k,   0, 0],
    [  0.3, 1.5, 0],
    [    0,   0, 1]
])

print(F)
```

```
[[0.66666667 0.         0.        ]
 [0.3        1.5        0.        ]
 [0.         0.         1.        ]]
```

We can see that $\boldsymbol{F}$ is dependent from the material coordinates, which means the deformation is isotropic.

Task 8. - Principal streches

We know that by the deformation gradient we can determine the left Cauchy-Green deformation tensor as

$$\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$$

In [17]:
```
b = F@F.T

sp.print_latex(sp.Matrix(b))
```

```
\left[\begin{matrix}0.444444444444445 & 0.2 & 0\\0.2 & 2.34 & 0\\0 & 0 & 1.0\end{matrix}\right]
```

The eigendecomposition of the Cauchy-Green tensor looks like

$$\boldsymbol{b} = \sum_{i=1}^{3} \mu_i \cdot \boldsymbol{n_i} \otimes \boldsymbol{n_i} = \sum_{i=1}^{3} \lambda_i^2 \cdot \boldsymbol{n_i} \otimes \boldsymbol{n_i}$$

,

where λ_i are the principal streches and $\boldsymbol{n_i}$ are the principal Eulerian directions

In [18]:
```
(μ1,μ2,μ3), (n1,n2,n3) = np.linalg.eig(b)

(λ1,λ2,λ3) = (μ1**(1/2),μ2**(1/2),μ3**(1/2))


print(λ1,λ2,λ3)
sp.print_latex(sp.Matrix(n1).T.evalf(6))
sp.print_latex(sp.Matrix(n2).T.evalf(6))
sp.print_latex(sp.Matrix(n3).T.evalf(6))
```

```
0.650824305915314 1.5365129896210752 1.0
\left[\begin{matrix}-0.994599 & -0.103797 & 0\end{matrix}\right]
\left[\begin{matrix}0.103797 & -0.994599 & 0\end{matrix}\right]
\left[\begin{matrix}0 & 0 & 1.0\end{matrix}\right]
```

Task 9. - Cauchy stress tensor and Mises Equivalent stress

In the coordinate system of the principal Eulerian directions, the Cauchy stress tensor can be expressed as

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \cdot \boldsymbol{n_i} \otimes \boldsymbol{n_i}$$

where the principal stress components can be calculated from the principal streches as

$$\sigma_i = \frac{1}{J} \lambda_i \frac{\partial W}{\partial \lambda_i} = \lambda_i \frac{\partial W}{\partial \lambda_i}.$$

The second term of the product can be calculated from the given strain energy potential function as

$$\frac{\partial W}{\partial \lambda_i} = \sum_{i=1}^{2} \frac{2\mu_k}{\alpha_k^2} \alpha_k \lambda_i^{\alpha_k - 1} = \sum_{i=1}^{2} \frac{2\mu_k}{\alpha_k} \lambda_i^{\alpha_k - 1}$$

In [19]:
```
mu1, mu2, a1, a2 = params

dWdλ1 = 2*mu1/a1 * λ1**(a1-1) + 2*mu2/a2 * λ1**(a2-1)
dWdλ2 = 2*mu1/a1 * λ2**(a1-1) + 2*mu2/a2 * λ2**(a2-1)
dWdλ3 = 2*mu1/a1 * λ3**(a1-1) + 2*mu2/a2 * λ3**(a2-1)

# tmp = np.array([
#     [λ1*dWdλ1,0,0],
#     [0,λ2*dWdλ2,0],
#     [0,0,λ3*dWdλ3]
# ])

tmp = λ1*dWdλ1 * np.outer(n1,n1) + λ2*dWdλ2 * np.outer(n2,n2) + λ3*dWdλ3 * np.outer(n3,n3)


s = tmp - 1/3*np.trace(tmp)*np.eye(3)
p = -s[2,2] * np.eye(3)
sig = s + p

sp.print_latex(sp.Matrix(s).evalf(5))
sp.print_latex(sp.Matrix(p).evalf(5))
sp.print_latex(sp.Matrix(sig).evalf(5))
```

```
\left[\begin{matrix}-6.9058 & -1.5056 & 0\\-1.5056 & 7.3637 & 0\\0 & 0 & -0.45787\end{matrix}\r
ight]
\left[\begin{matrix}0.45787 & 0 & 0\\0 & 0.45787 & 0\\0 & 0 & 0.45787\end{matrix}\right]
\left[\begin{matrix}-6.4479 & -1.5056 & 0\\-1.5056 & 7.8215 & 0\\0 & 0 & 0\end{matrix}\right]
```

The Mises equivalent stress can be expressed as

$$\sigma_{vM} = \sqrt{\frac{3}{2}\boldsymbol{s} : \boldsymbol{s}}$$

In [20]:
```
sig_vM = np.sqrt(3/2 * np.sum(s**2))

sig_vM
```

Out[20]:
```
12.648525122821695
```

In [21]:
```
params
```

Out[21]:
```
array([ 3.18596983,  4.62213918, -1.48228961,  1.77455216])
```