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Planned Syllabus

Week	Topic	HW
1 (9/3)	Introduction, Tensor algebra & analysis	
2 (9/10)	Description of deformation	
3 (9/17)	Description of deformation / BME Sport Day (educational break)	
4 (9/24)	Canceled	
5 (10/1)	Description of deformation	
6 (10/8)	Stress measures	
7 (10/15)	Hyperelasticity	HW 1
8 (10/22)	1st Mid-term exam/test	
9 (10/29)	Hyperelasticity	
10 (11/5)	Hyperelasticity	
11 (11/12)	Velocity, acceleration, time derivatives	
12 (11/19)	Objectivity	
13 (11/26)	Fundamental principles	
14 (12/3)	2nd Mid-term exam/test	HW 2
Ratake week	Retake exams/tests 1 & 2	

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1. **Tensor Algebra and Analysis**
    1. Tensor Algebra
    2. Tensor Analysis
2. **Continuum Mechanics**
    1. **Description of deformation**
        1. Continuum concept
        2. Configurations
        3. Motion
        4. Displacement
        5. Deformation gradient
        6. Stretch ratio
        7. Standard 3D deformation and strain definitions
        8. 1D deformation and strain measures
        9. Linearization of deformation
        10. Area changes
        11. Volume changes
        12. Angle changes
        13. Isochoric/Volumetric split of the deformation gradient
        14. Finite (large) rotations
        15. Polar decomposition theorem
        16. 3D logarithmic strain
        17. Generalized strain measures
    2. **Stress measures**
        1. 1D Engineering and Trues stress definitions
        2. Body and surface forces
        3. Cauchy stress tensor
        4. 1st and 2nd Piola–Kirchhoff stress tensors
        5. Other stress tensors
        6. Normal and shear stresses
        7. Deviatoric and hydrostatic stress
        8. Stress triaxiality
        9. Princpial stresses, principal invariants
    3. **Velocity, acceleration, time derivatives**
        1. Velocity and acceleration
        2. Material time derivative
        3. Velocity gradient, rate of deformation, spin
        4. Rates of basic quantities
    4. **Objectivity**
        1. Superposed rigid body motion
        2. Transformation of basic quantities
        3. Objective rates
        4. Hypoelasticity
    5. **Fundamental principles**
        1. Integral Theorems
        2. Conservation of mass
        3. Reynolds' transport theorem
        4. Balance of linear and angular momentums
        5. Principle of virtual work
        6. Balance of mechanical energy
3. **Hyperelasticity**
    1. Introduction
    2. Various forms of the Hooke's law
    3. Overview of scalar-valued tensor functions
    4. Hyperelasticity
        1. General form for the stress
        2. Isotropic case
        3. Incompressible isotropic case
        4. Standard incompressible hyperelastic models
        5. Standard homogeneous deformations
        6. Parameter fitting
        7. Slightly compressible hyperelastic models
        8. Drucker stability

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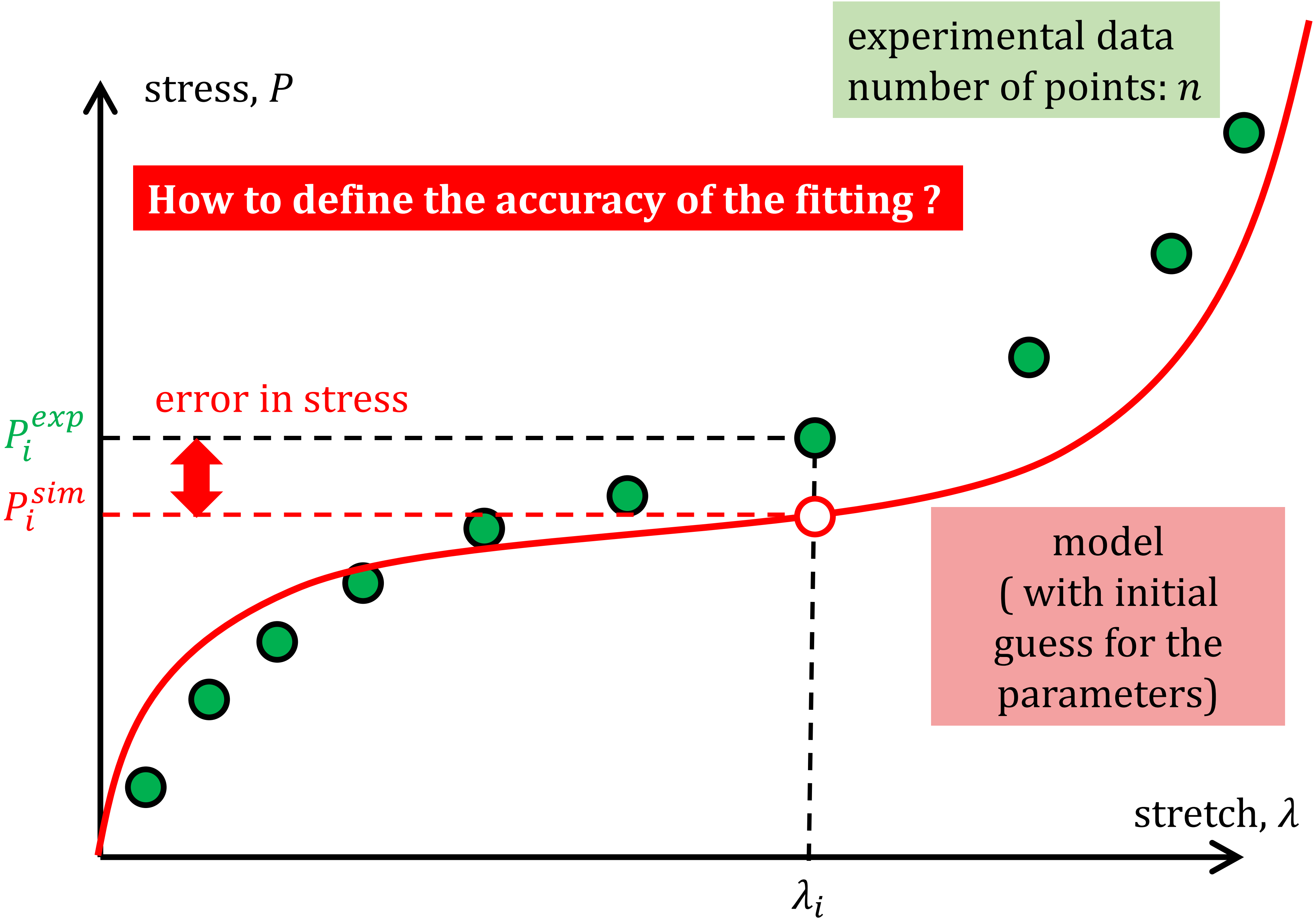
3.4.6 Parameter fitting



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3.4.6 Parameter fitting

● Basic definitions for the „**quality function**”:

Sum of absolute differences

$$Q = \sum_{i=1}^{n} \left| P_i^{exp} - P_i^{sim} \right| \qquad [\mathrm{Pa}]$$

$$Q = \sum_{i=1}^{n} \left| P_i^{exp} - P^{sim}(\lambda_i) \right|$$

Sum of squared differences

$$Q = \sum_{i=1}^{n} \left(P_i^{exp} - P_i^{sim} \right)^2 \qquad [\mathrm{Pa}^2]$$

$$Q = \sum_{i=1}^{n} \left(P_i^{exp} - P^{sim}(\lambda_i) \right)^2$$

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Root Mean Squared Relative Error

$$Q=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{P_i^{exp}-P_i^{sim}}{P_i^{exp}}\right)^2}$$

$$Q=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{P_i^{exp}-P^{sim}(\lambda_i)}{P_i^{exp}}\right)^2}\qquad [1]\qquad [\%]$$

Coefficient of determination

$$R^2=1-\frac{\sum_{i=1}^{n}\left(P_i^{exp}-P_i^{sim}\left(\lambda_i^{exp}\right)\right)^2}{\sum_{i=1}^{n}\left(P_i^{exp}-P\right)^2}\qquad [1]\qquad [\%]$$

$$P=\frac{1}{n}\sum_{i=1}^{n}P_i^{exp}$$

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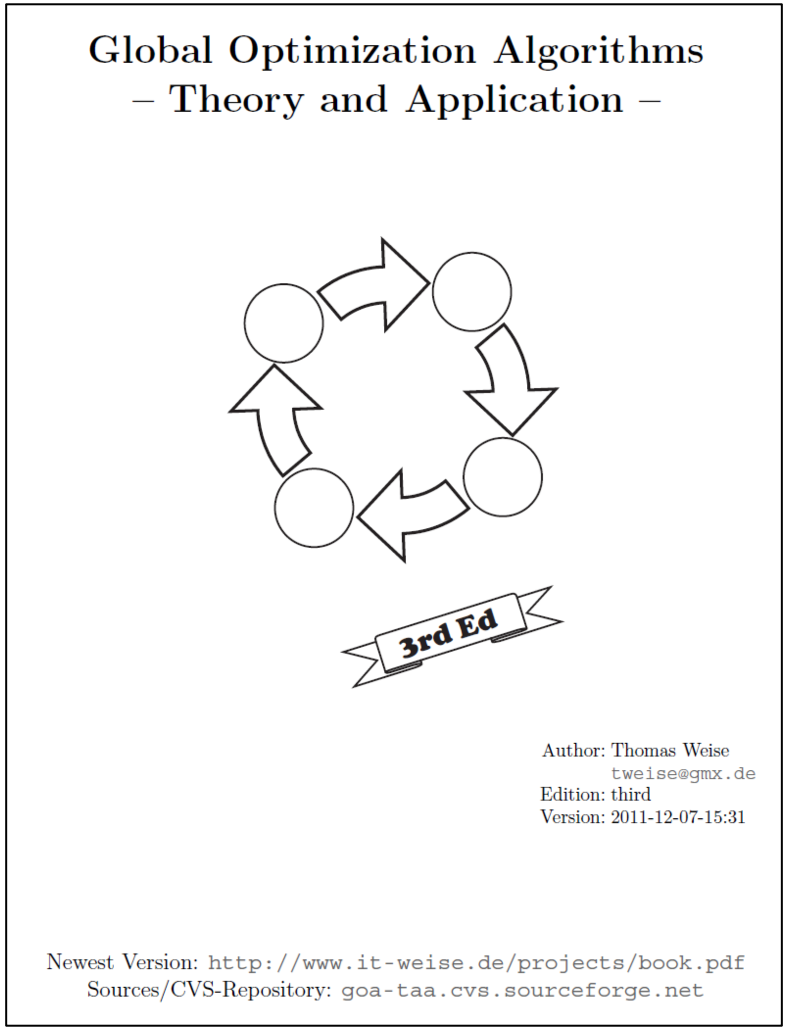
● Fitting algorithm

Goal: find the global minimum of Q

Different optimization routines exist!

Excellent book:


Global Optimization Algorithms
– Theory and Application –

3rd Ed

Author: Thomas Weise
tweise@gmx.de
Edition: third
Version: 2011-12-07-15:31

Newest Version: http://www.it-weise.de/projects/book.pdf
Sources/CVS-Repository: goa-taa.cvs.sourceforge.net

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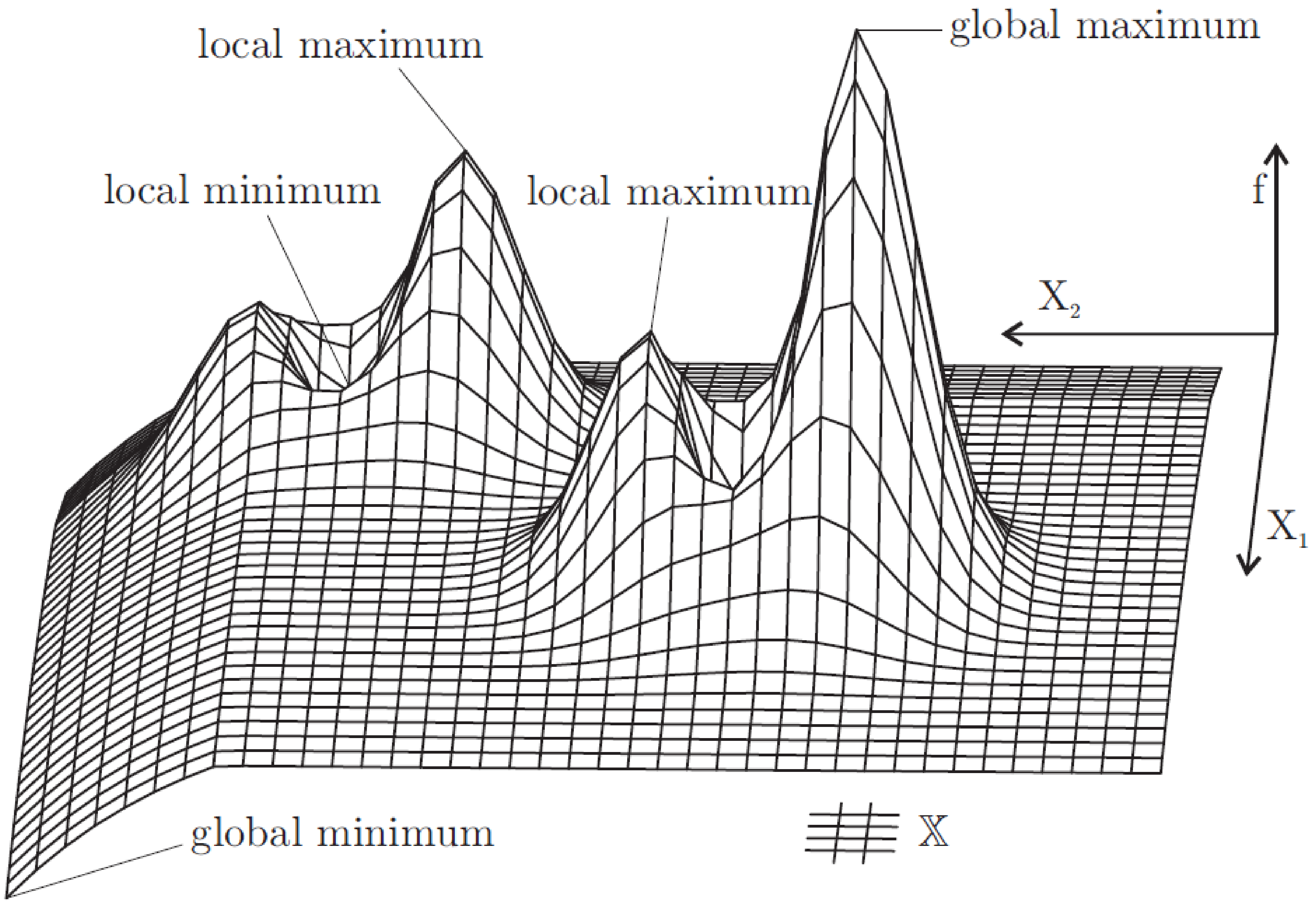

Figure 1.2: Global and local optima of a two-dimensional function.

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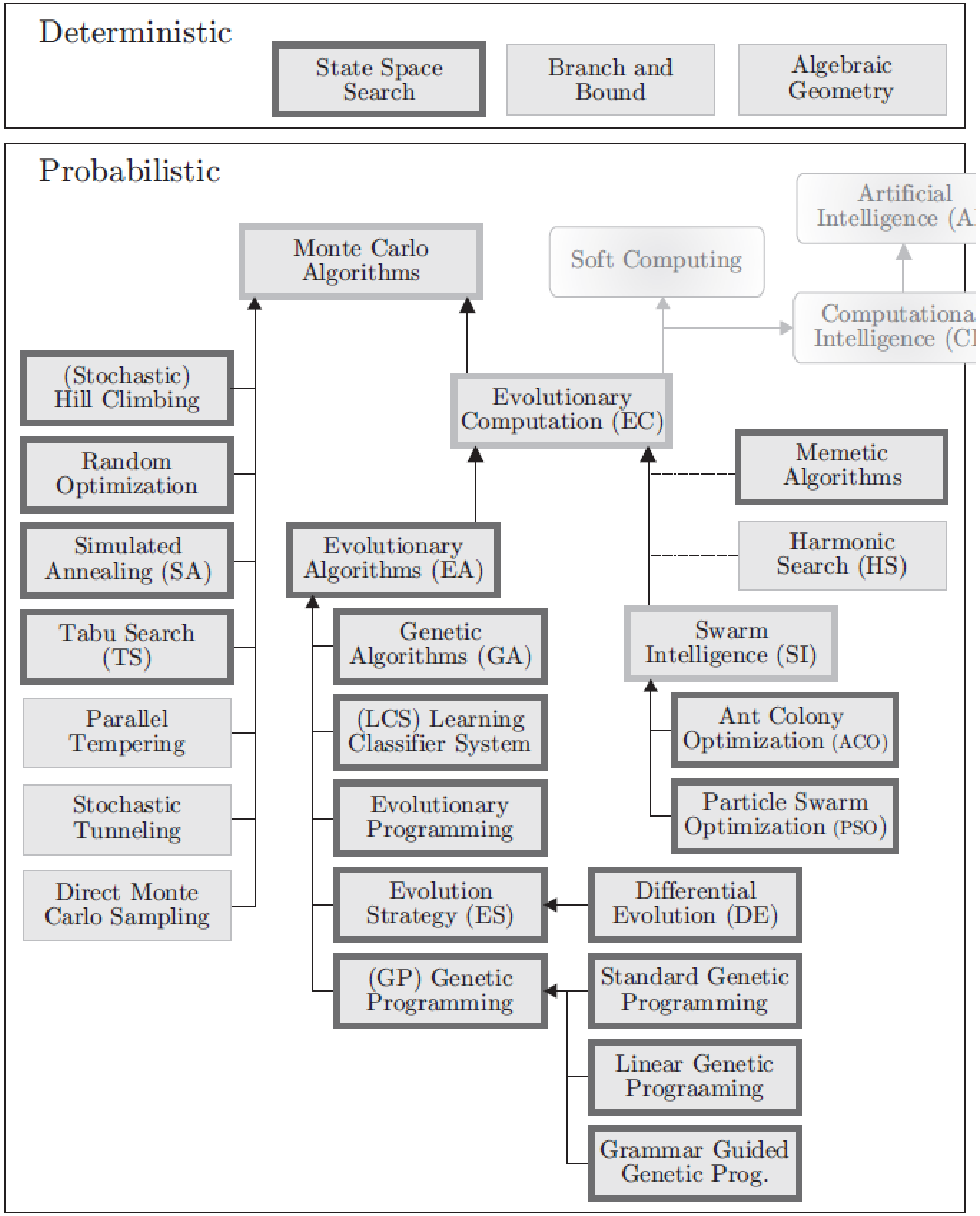

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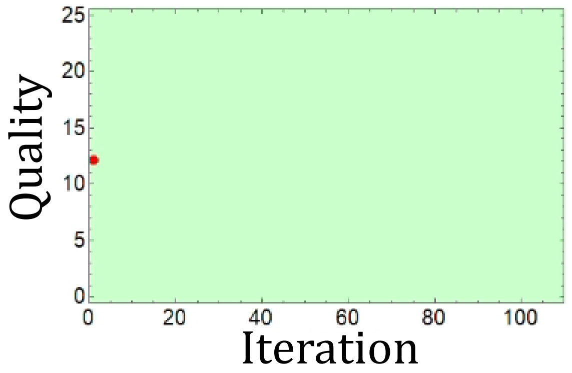
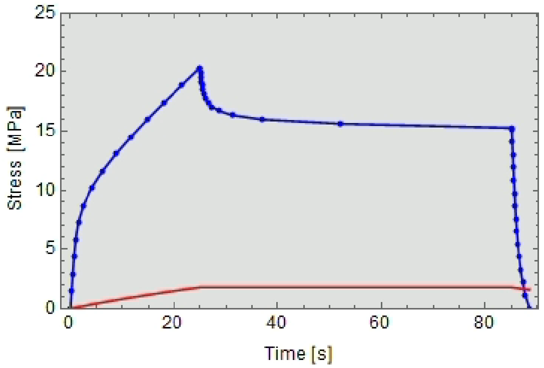
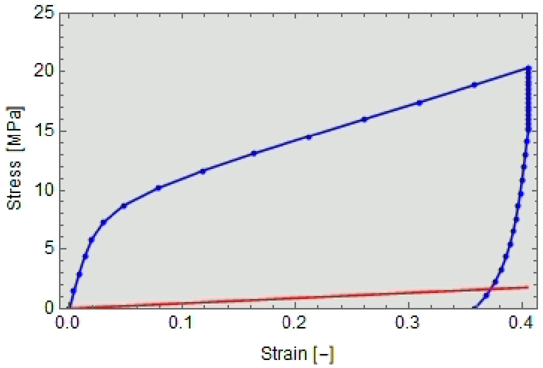
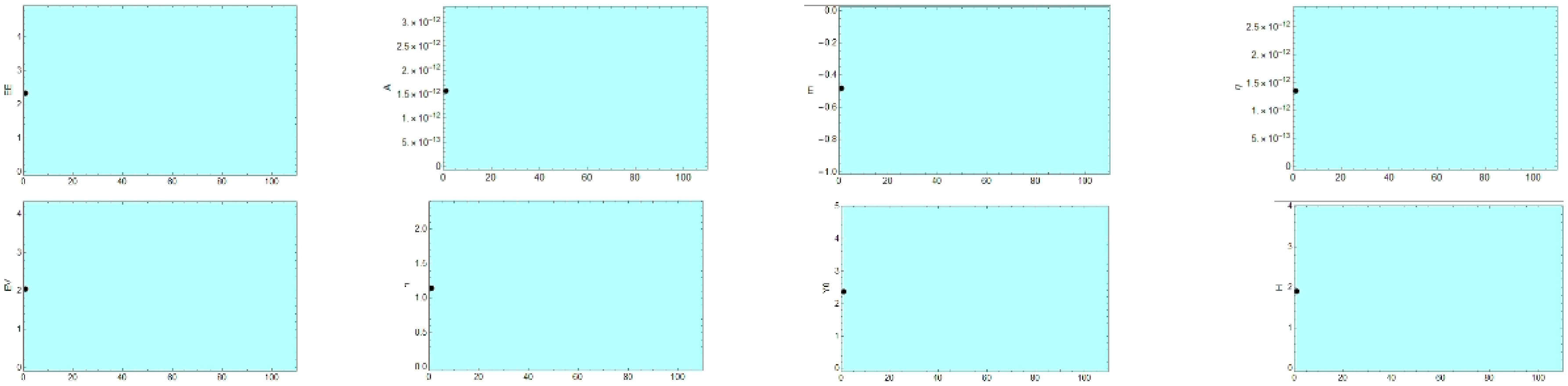
Example of fitting for a viscoelastic/viscoplastic model *(8 material parameters)*









media1.mp4

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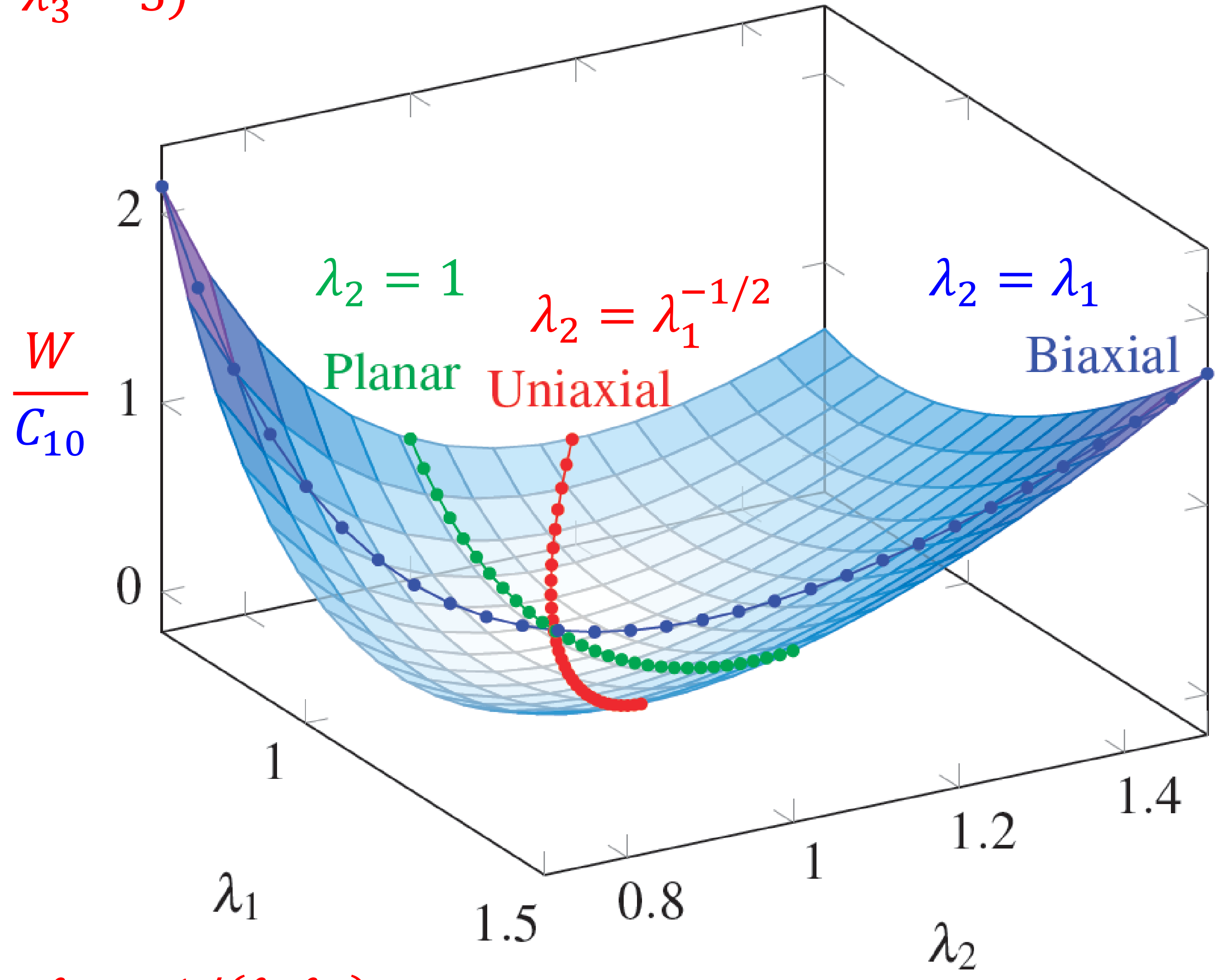
Illustration of the neo-Hookean's strain energy function as a surface in incompressible case. The three basic loading paths are shown.

$$W = C_{10}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$$



Incompressible case: $\lambda_3 = 1/(\lambda_1 \lambda_2)$

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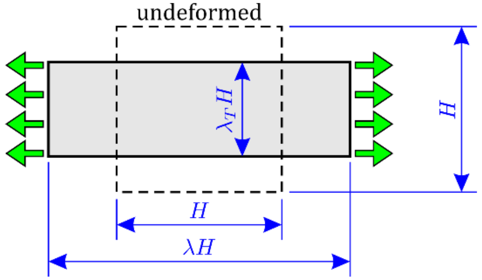
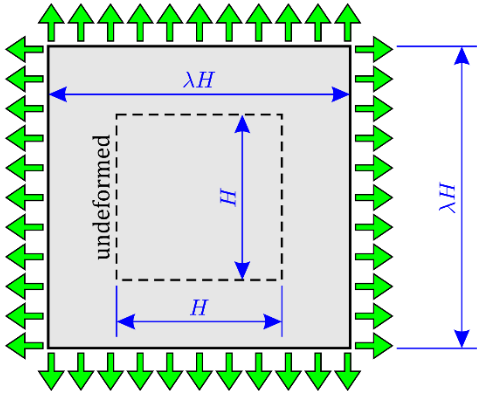
• Demonstration example

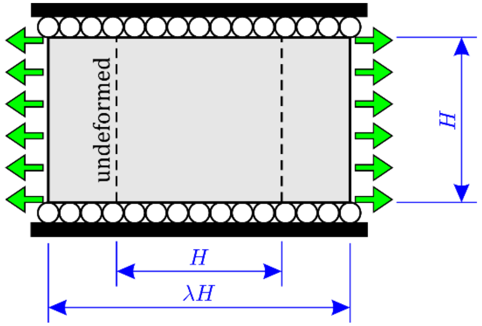
L. R. G. Treloar: Stress-strain data for vulcanised rubber under various types of deformation. Trans. Faraday Soc., 1944, 40, 59-70.

Experimental data for

- Uniaxial tension (UT)
- Equibiaxial tension (ET)
- Pure shear (PS)







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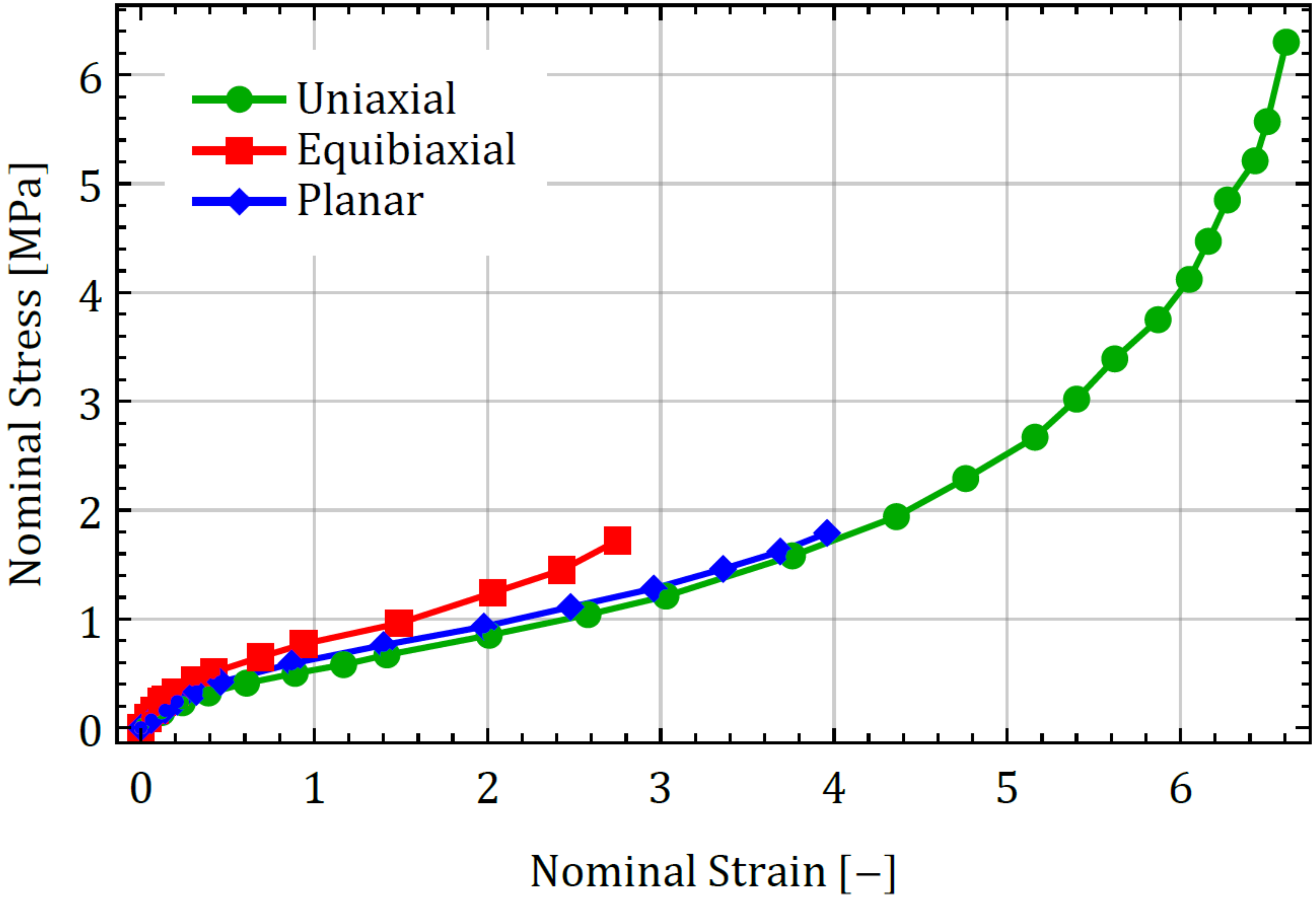

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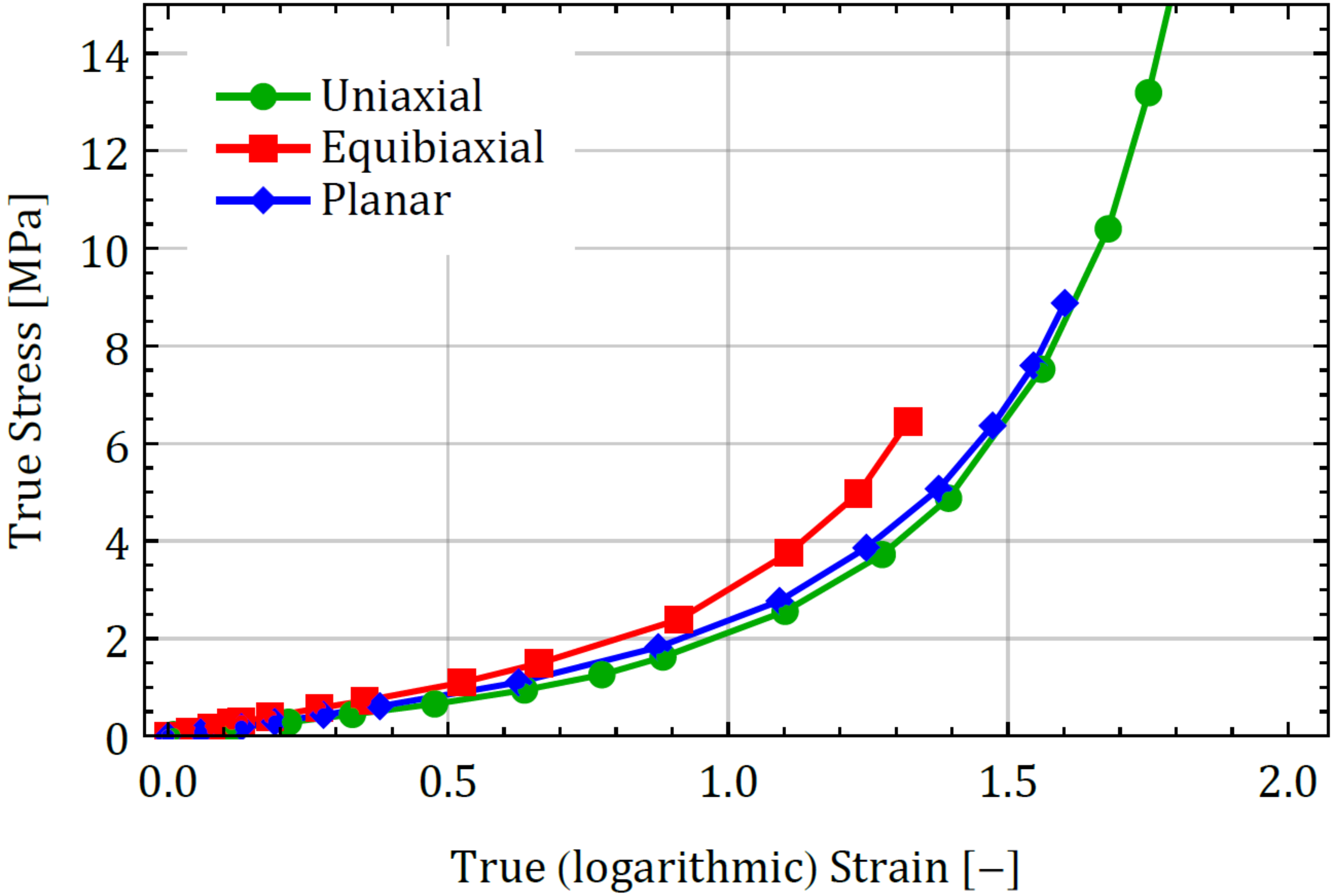

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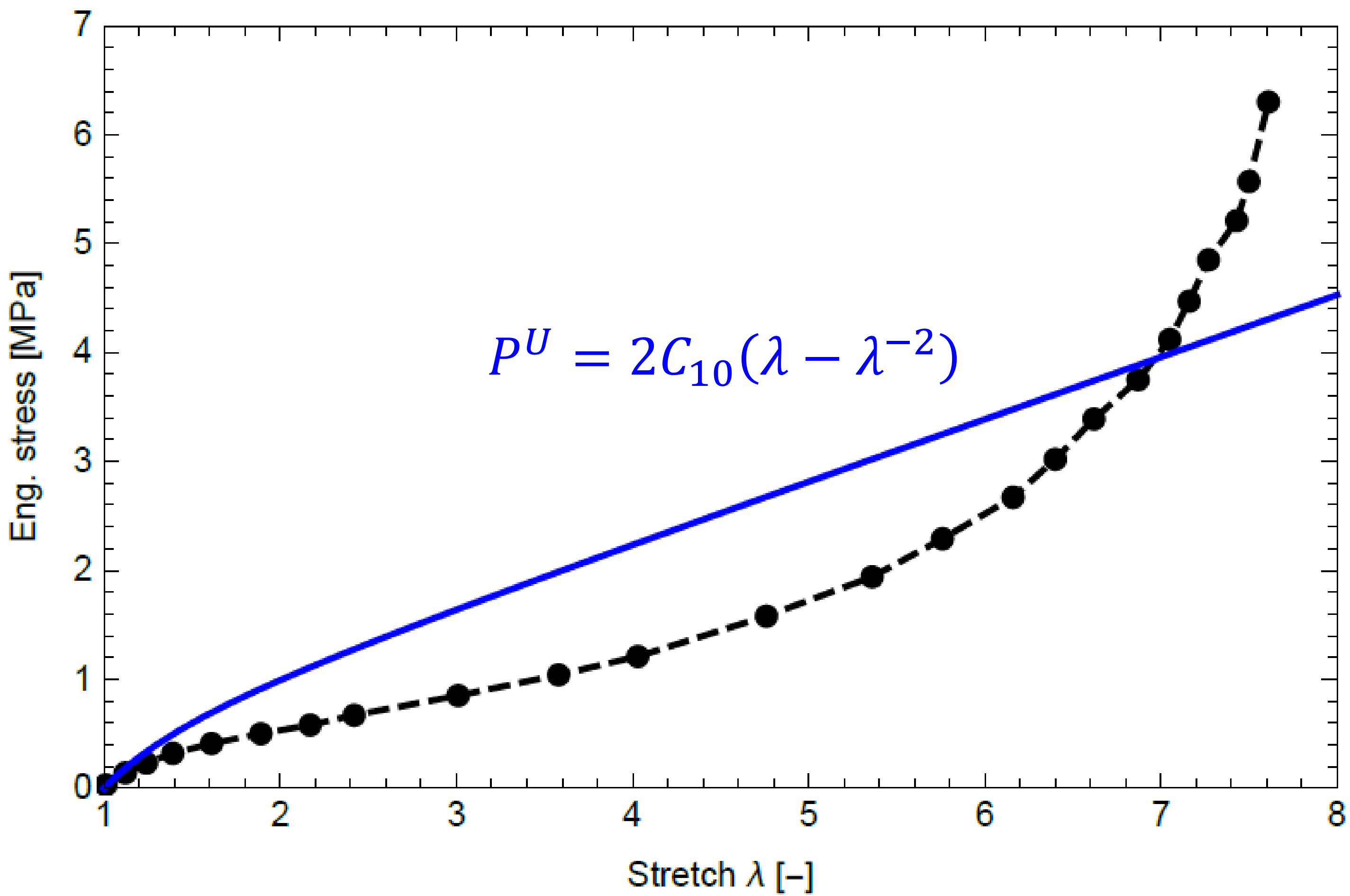
Test data: Uniaxial extension

Best fit for the Neo-Hookean model.

Quality function: **Sum of squared differences**



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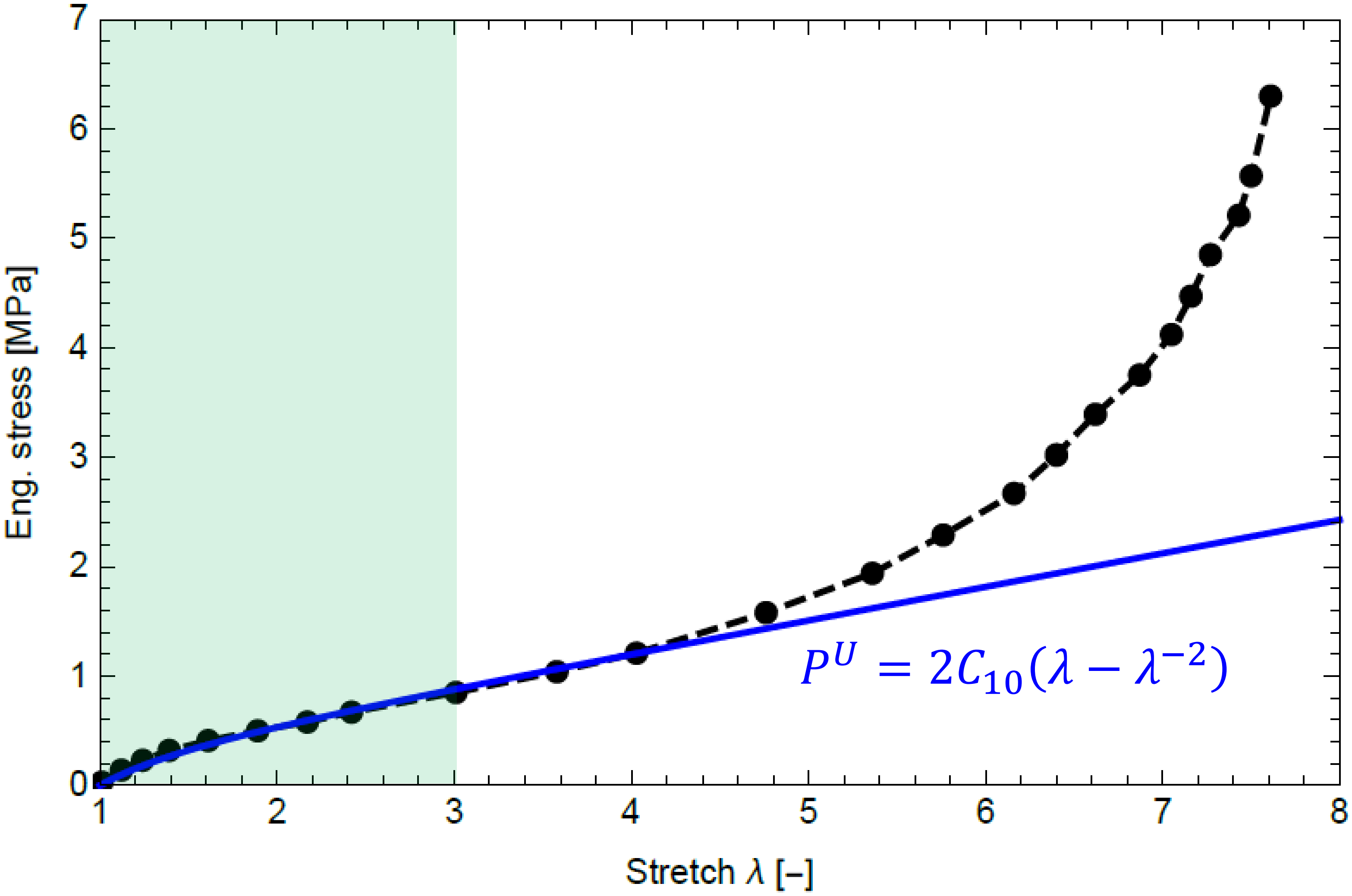
Test data: Uniaxial extension

Best fit for the Neo-Hookean model for $\lambda < 3$



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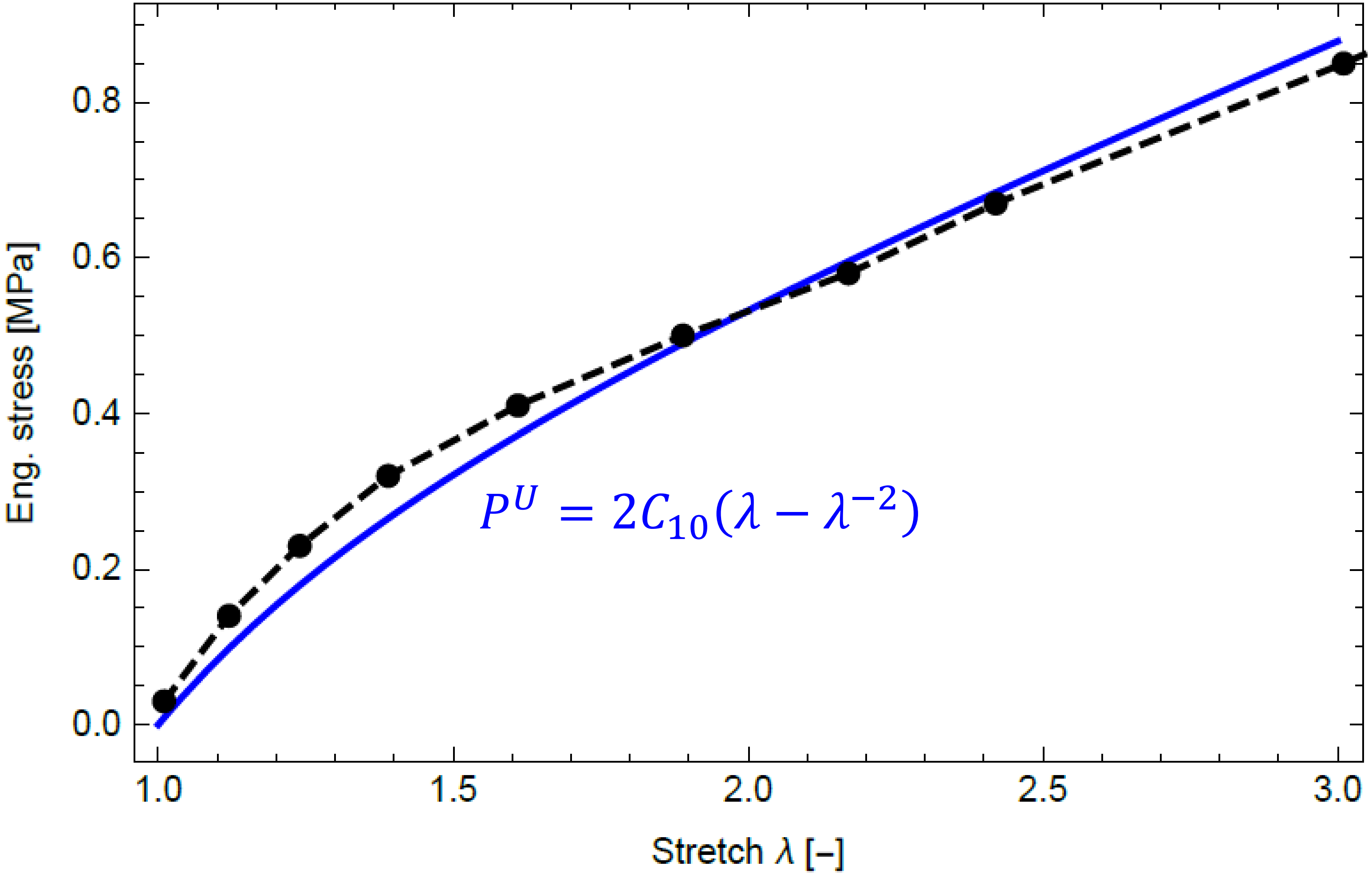

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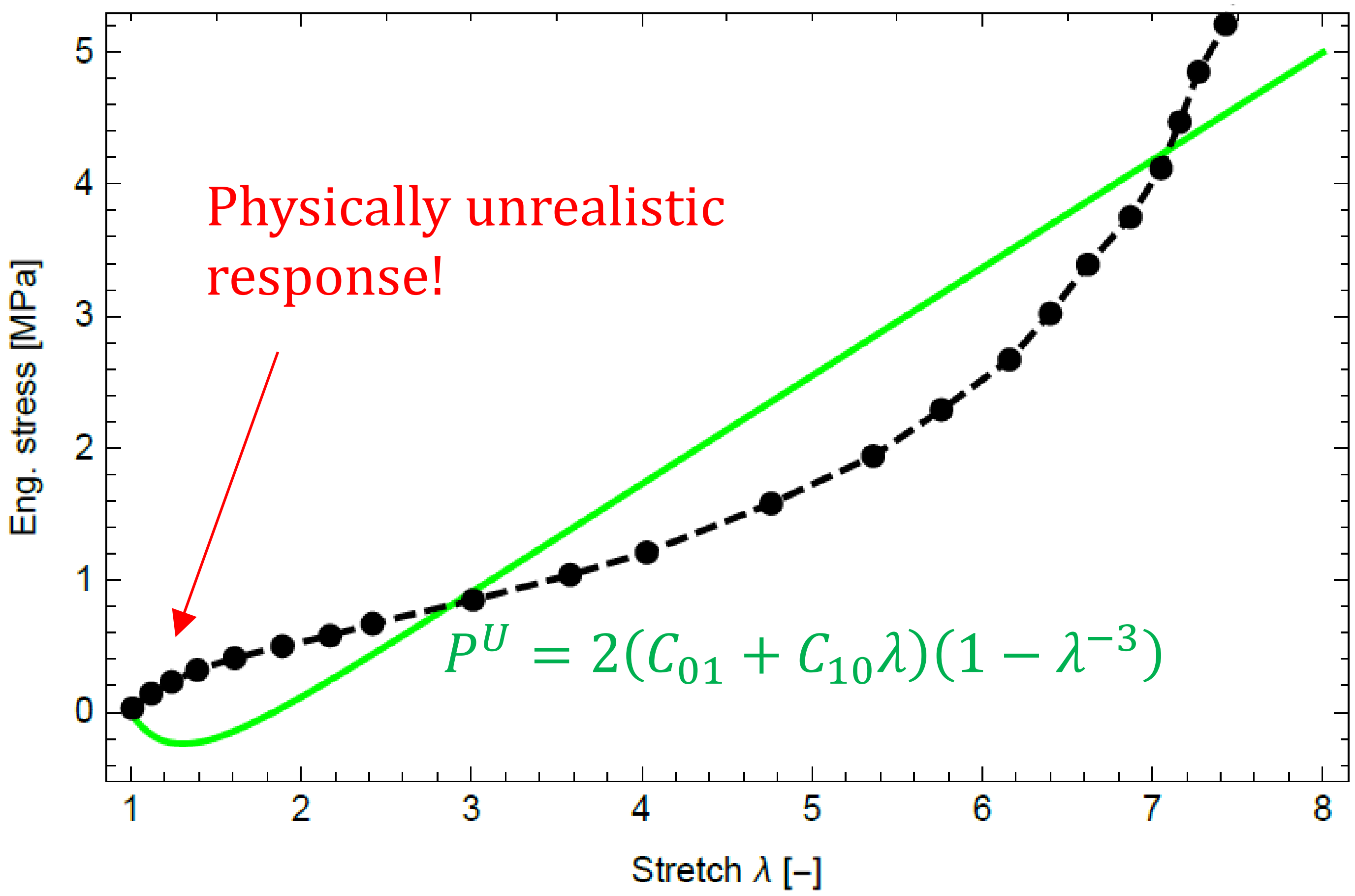
Test data: Uniaxial extension

Best fit for the Mooney-Rivlin model



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$$P^U = 2(C_{01} + C_{10}\lambda)(1 - \lambda^{-3})$$

The apparent (or current) elastic modulus in uniaxial loading is calculated as

$$P = 2C_{10}\left(\lambda - \lambda^{-2}\right) + 2C_{01}\left(1 - \lambda^{-3}\right),$$

$$\frac{\partial P}{\partial \lambda} = 2C_{10}\left(1 + 2\lambda^{-3}\right) + 6C_{01}\lambda^{-4},$$

$$E(\lambda) = \frac{2}{\lambda^2}\left(C_{01}\left(2 + \lambda^3\right) + C_{10}\left(\lambda + 2\lambda^4\right)\right).$$

Its value about the undeformed configuration defines the ground-state (or initial) Young's modulus:

$$E_0 = E(\lambda = 1) = 6\left(C_{10} + C_{01}\right).$$

Constraint: positive Young's modulus

$$C_{01} + C_{10} > 0$$

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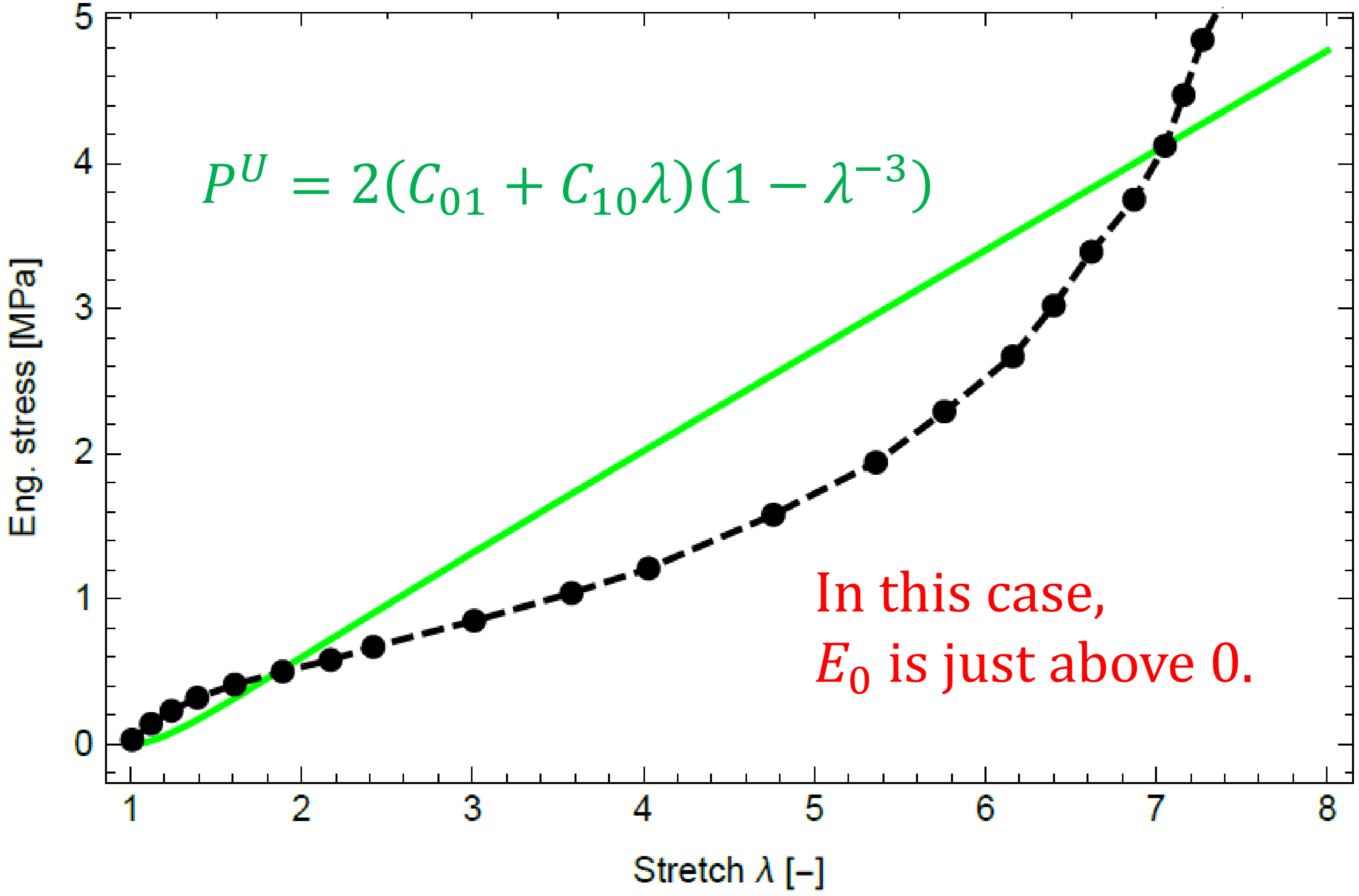
Test data: Uniaxial extension

Best fit for the Mooney-Rivlin model

+ ***constraint*** $C_{01} + C_{10} > 0$



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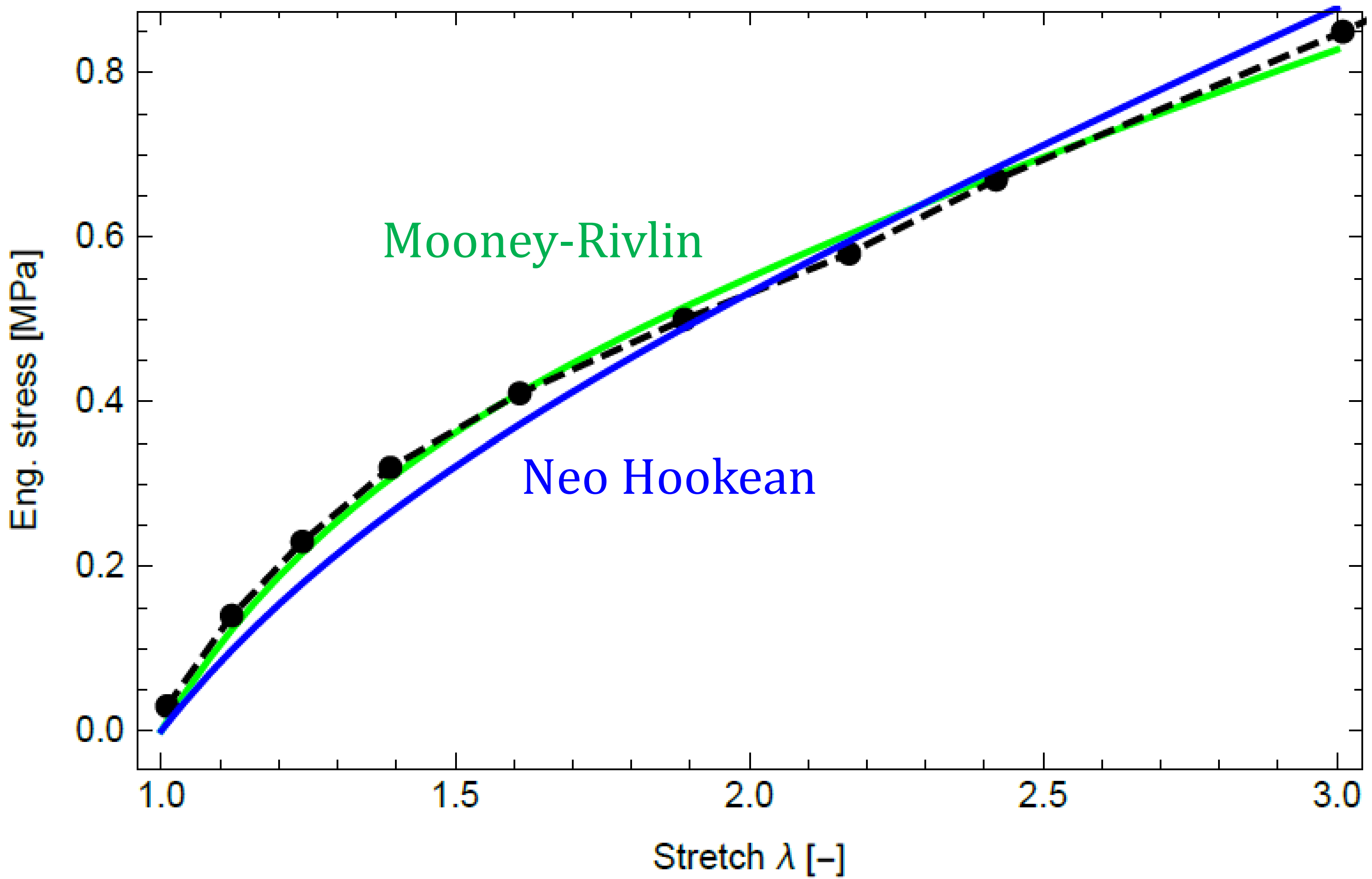
Test data: Uniaxial extension

Best fit for the Mooney-Rivlin model
+ constraint $C_{01} + C_{10} > 0$ for $\lambda < 3$



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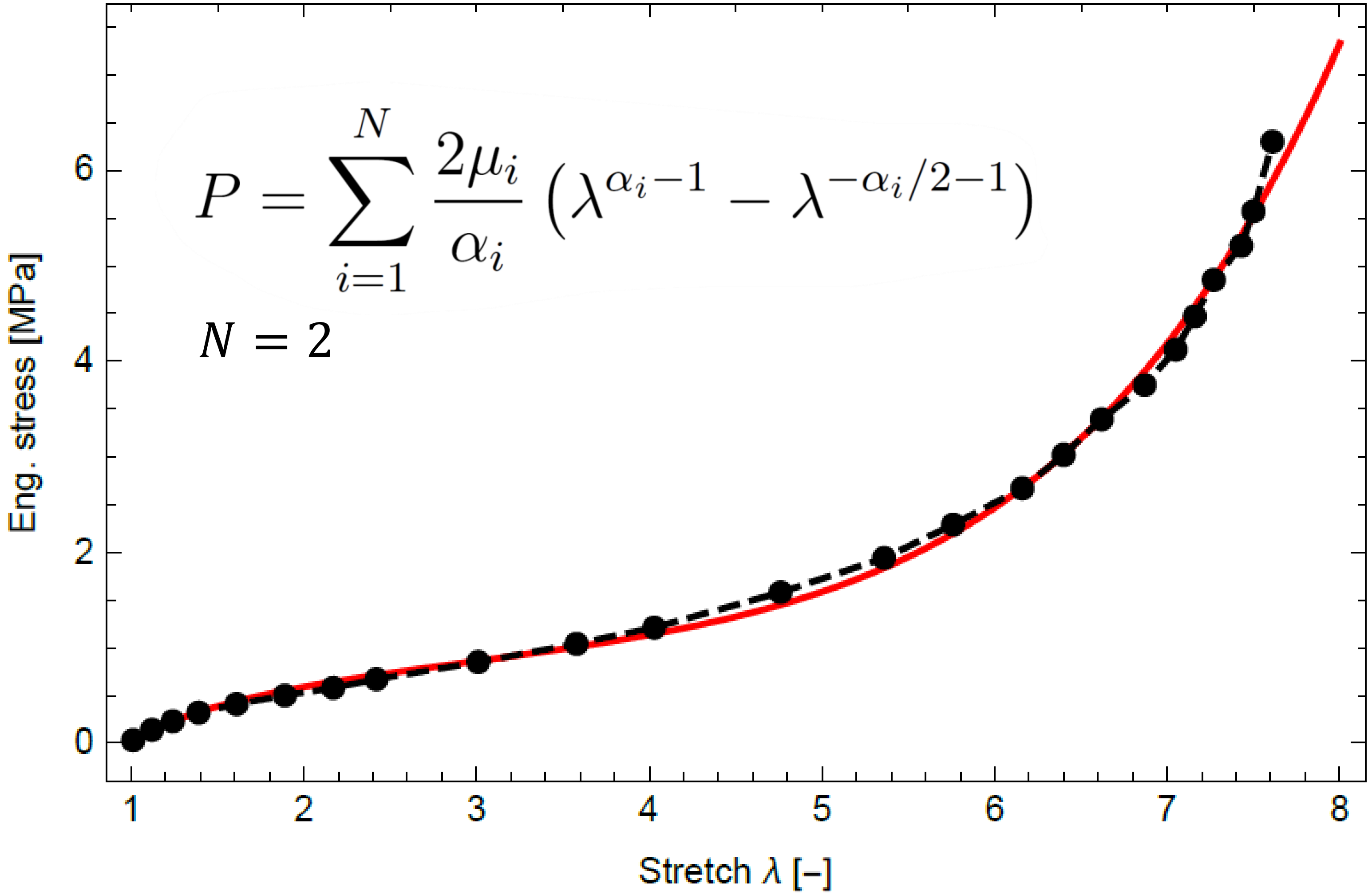
Test data: Uniaxial extension

Best fit for the 2nd-order Ogden model



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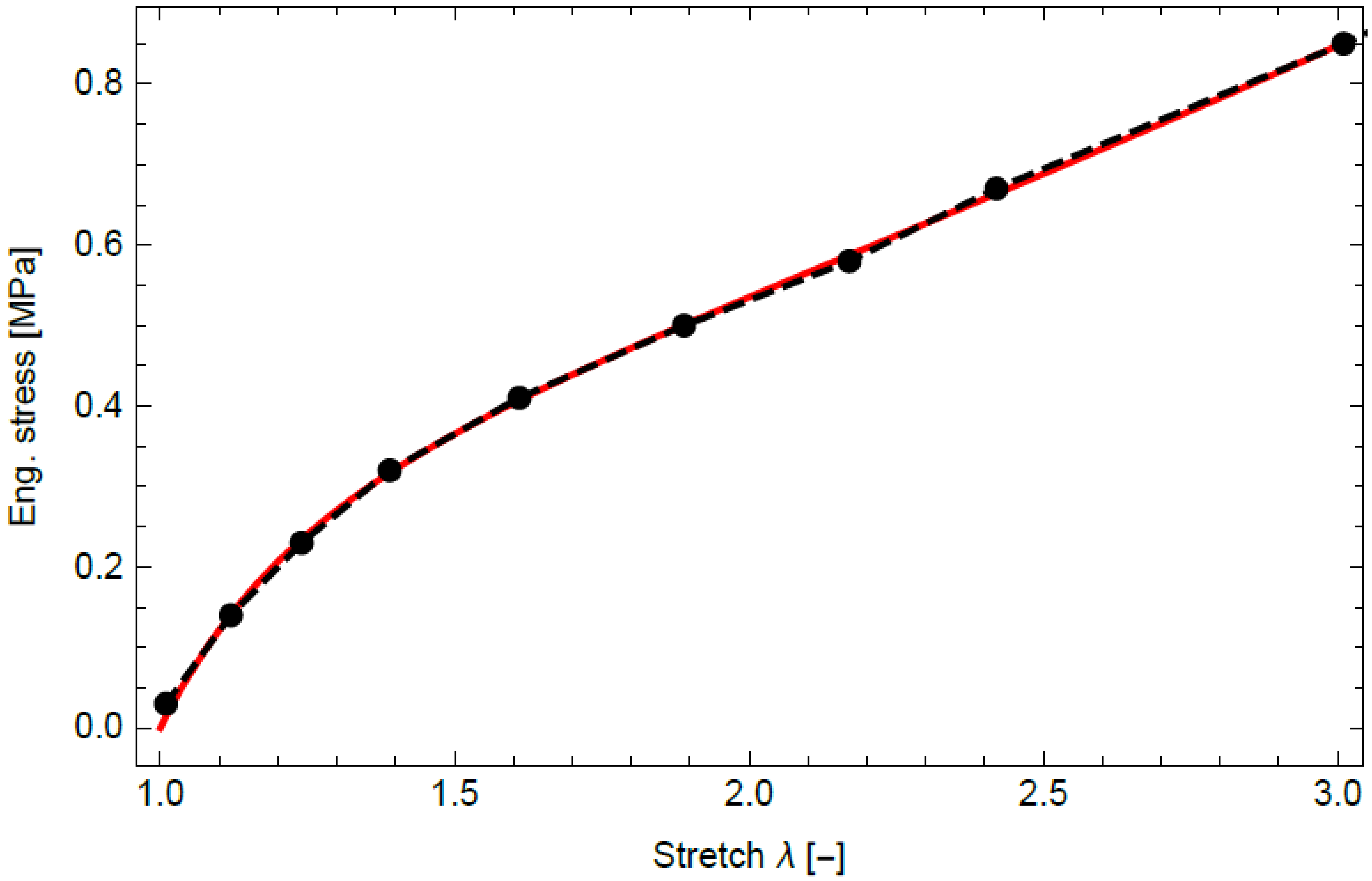
Test data: Uniaxial extension

Best fit for the 2nd-order Ogden model
for $\lambda < 3$



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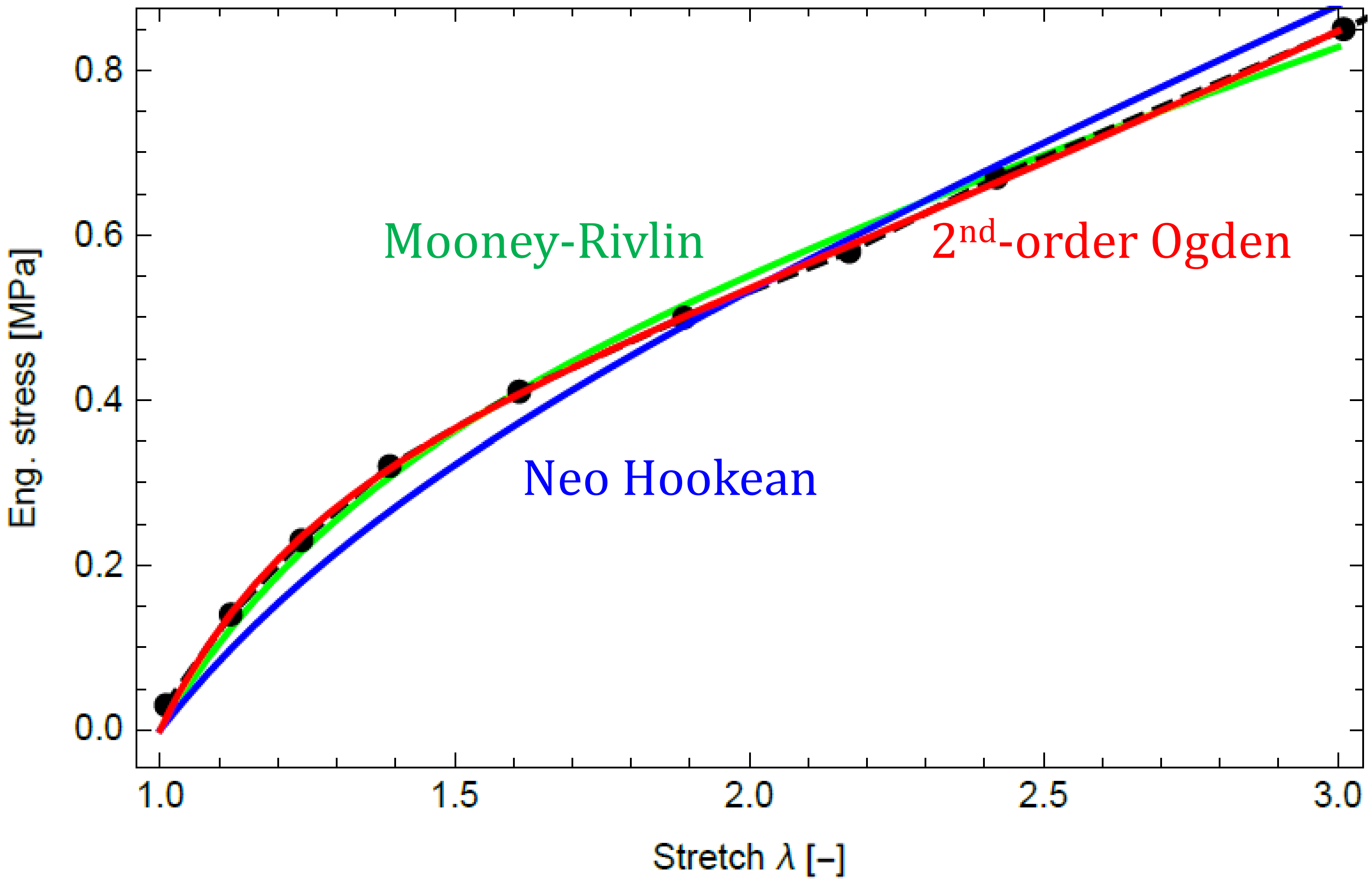
Best fit for the 2nd-order Ogden model for $\lambda < 3$



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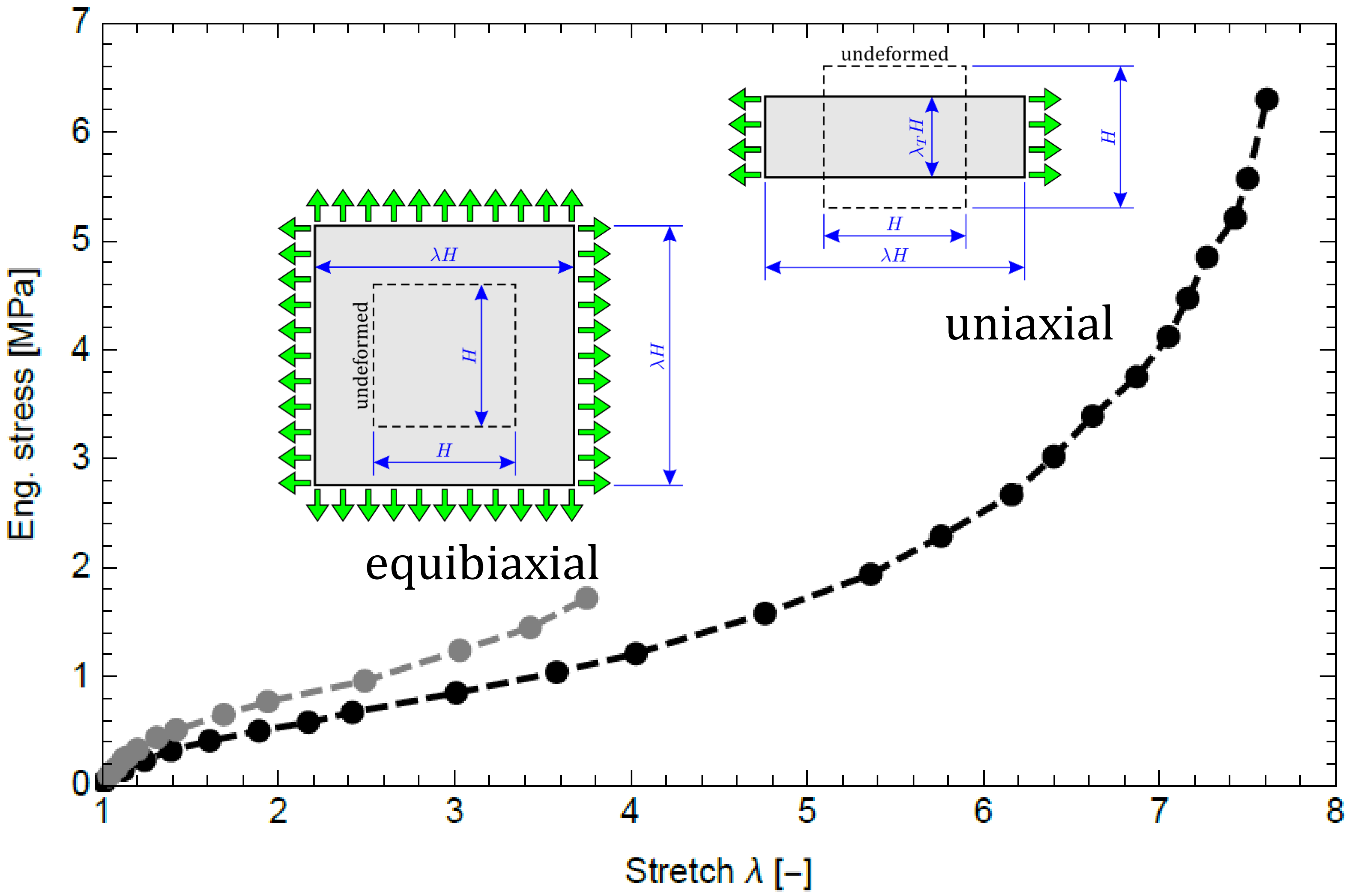
Test data: Equibiaxial extension



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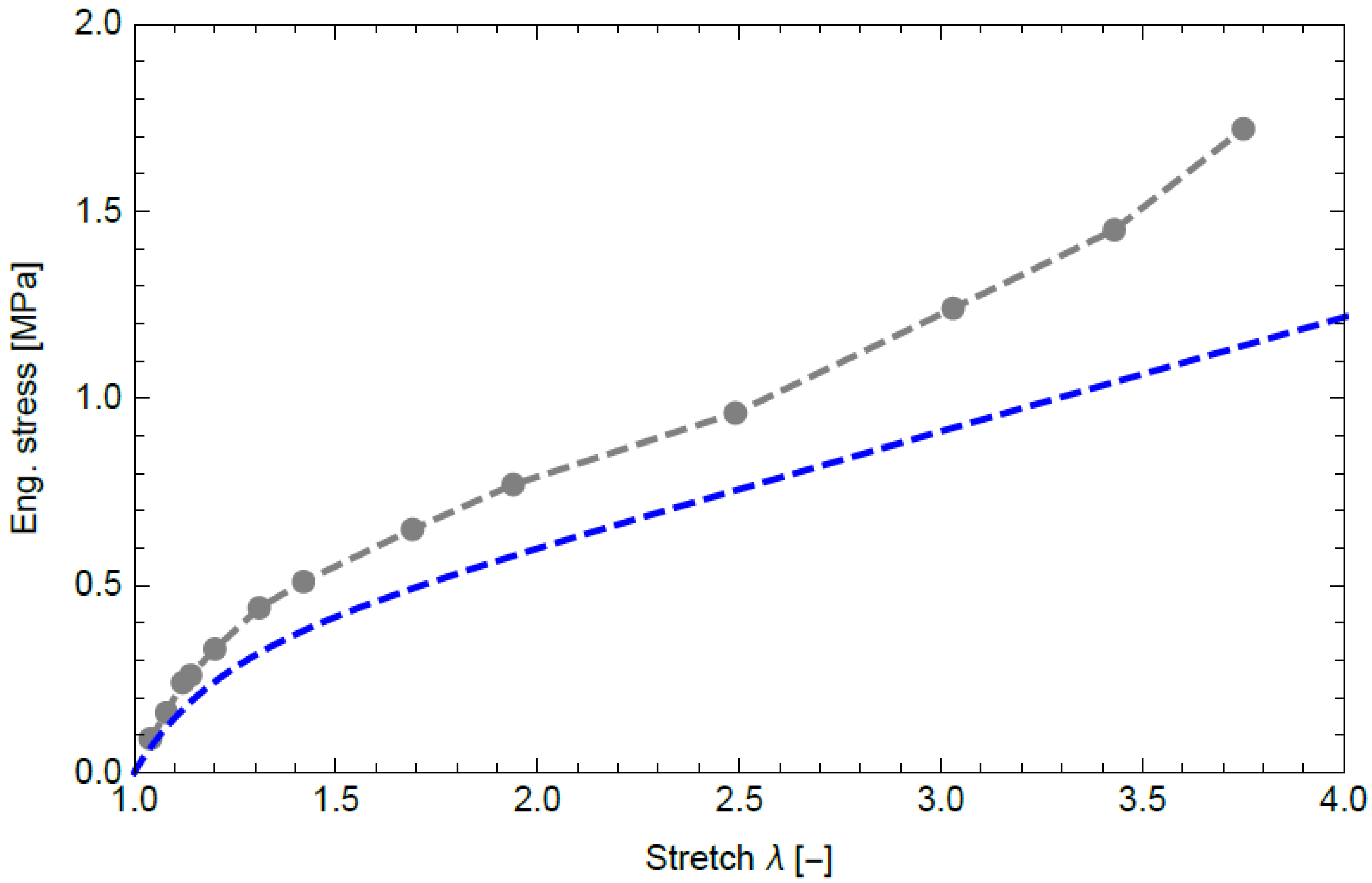
Test data: Equibiaxial extension

Equibiaxial model response using the parameters from the uniaxial fit.

Neo Hookean model.



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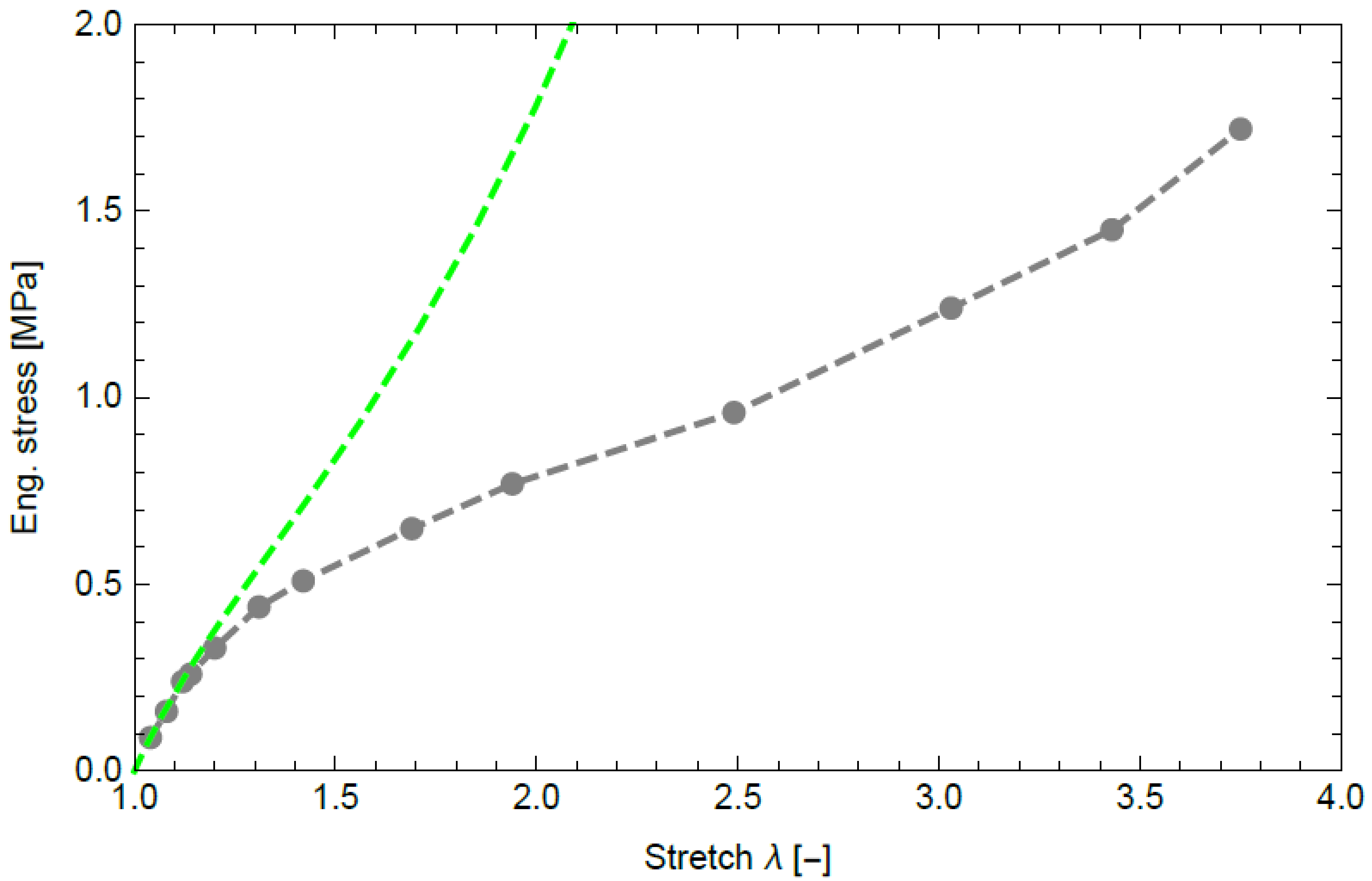
Test data: Equibiaxial extension

Equibiaxial model response using the parameters from the uniaxial fit.

Mooney-Rivlin model.



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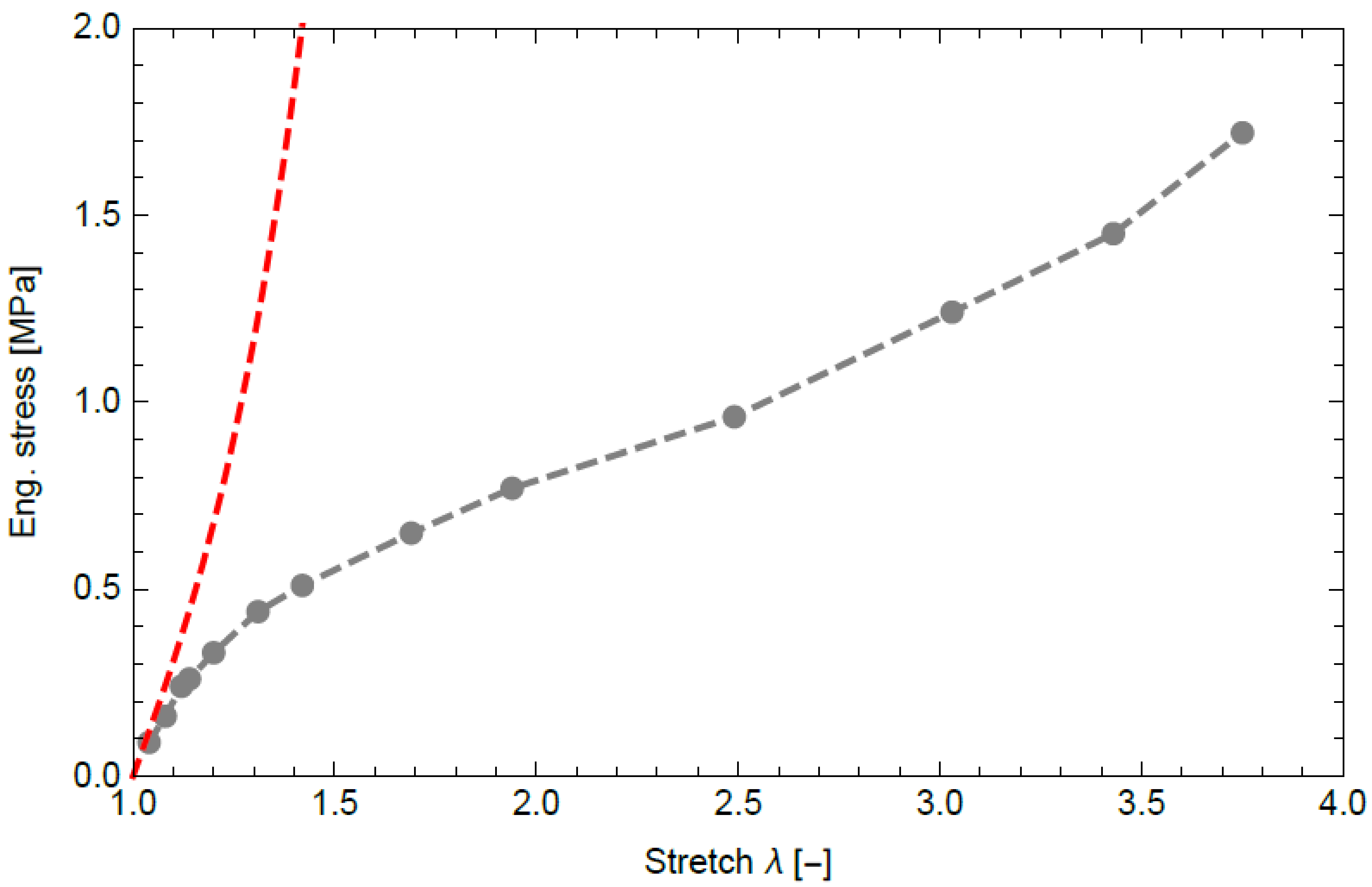
Test data: Equibiaxial extension

Equibiaxial model response using the parameters from the uniaxial fit.
2nd-order Ogden model.



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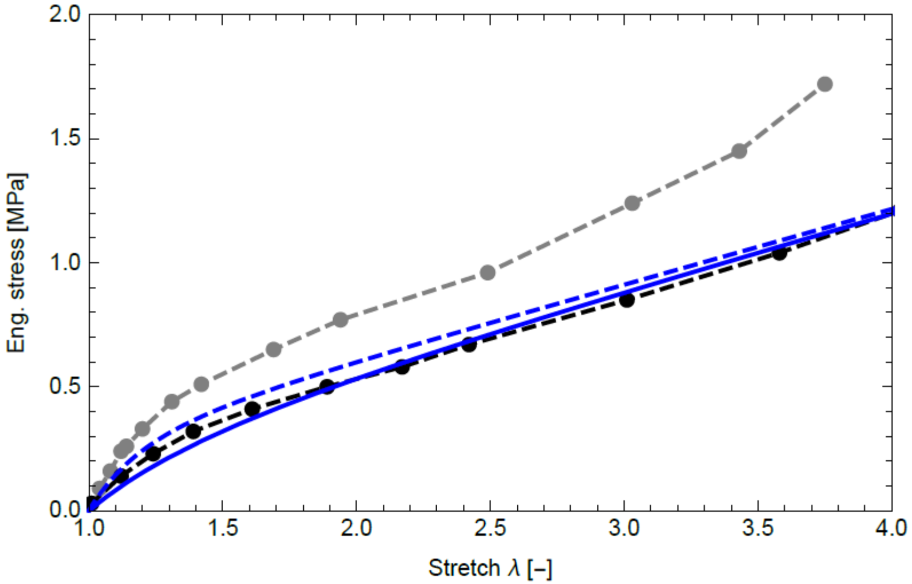
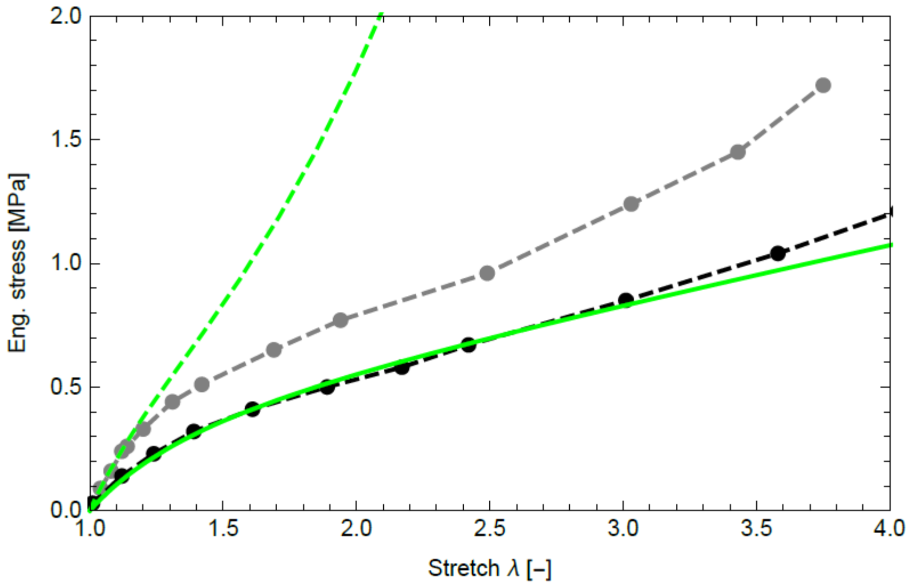
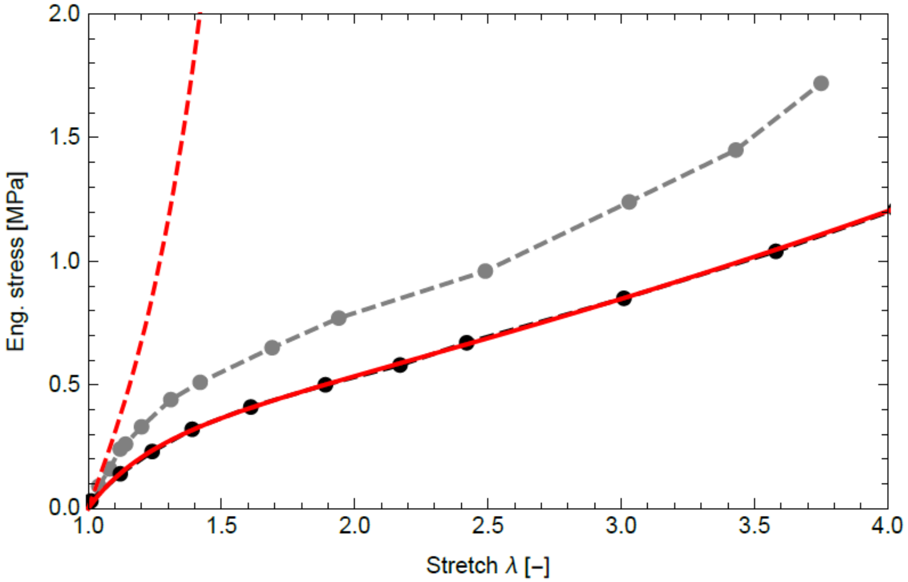


Neo-Hookean



Mooney-Rivlin



2nd-order Ogden model

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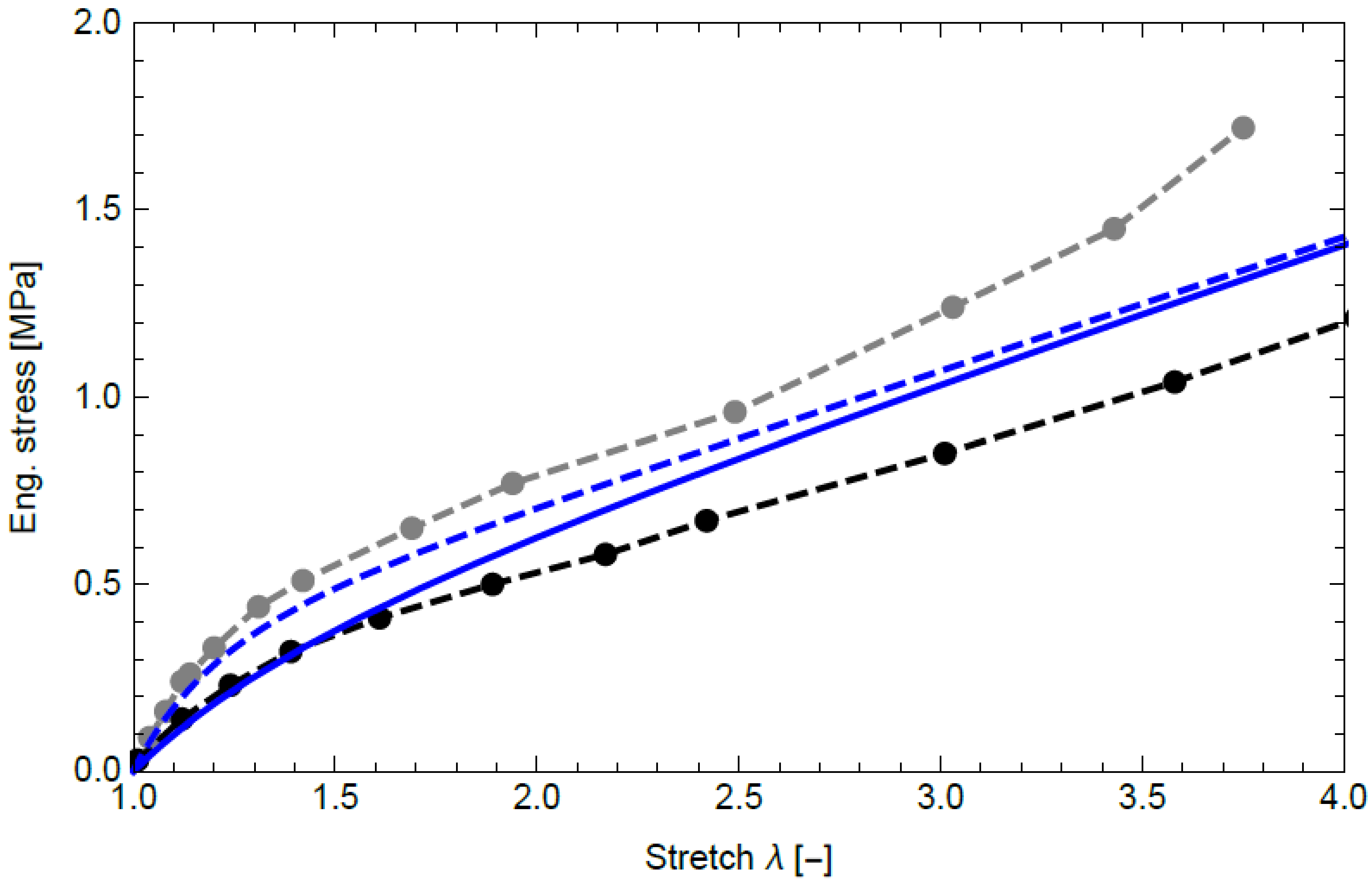
Test data: Equibiaxial + Uniaxial extension

Fitting simultaneously to both exp. data.

Neo Hookean model.



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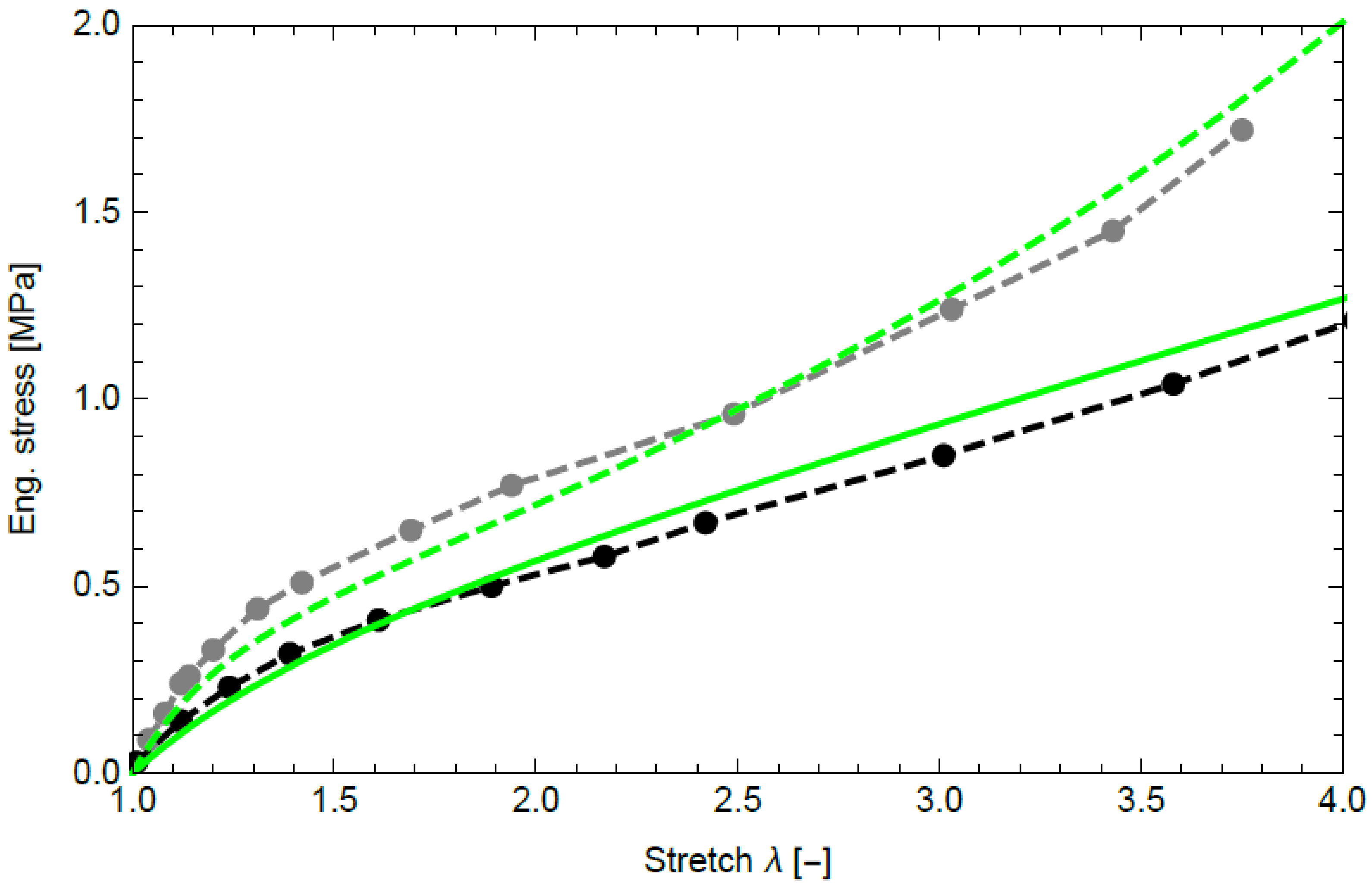
Test data: Equibiaxial + Uniaxial extension

Fitting simultaneously to both exp. data.

Mooney-Rivlin model.



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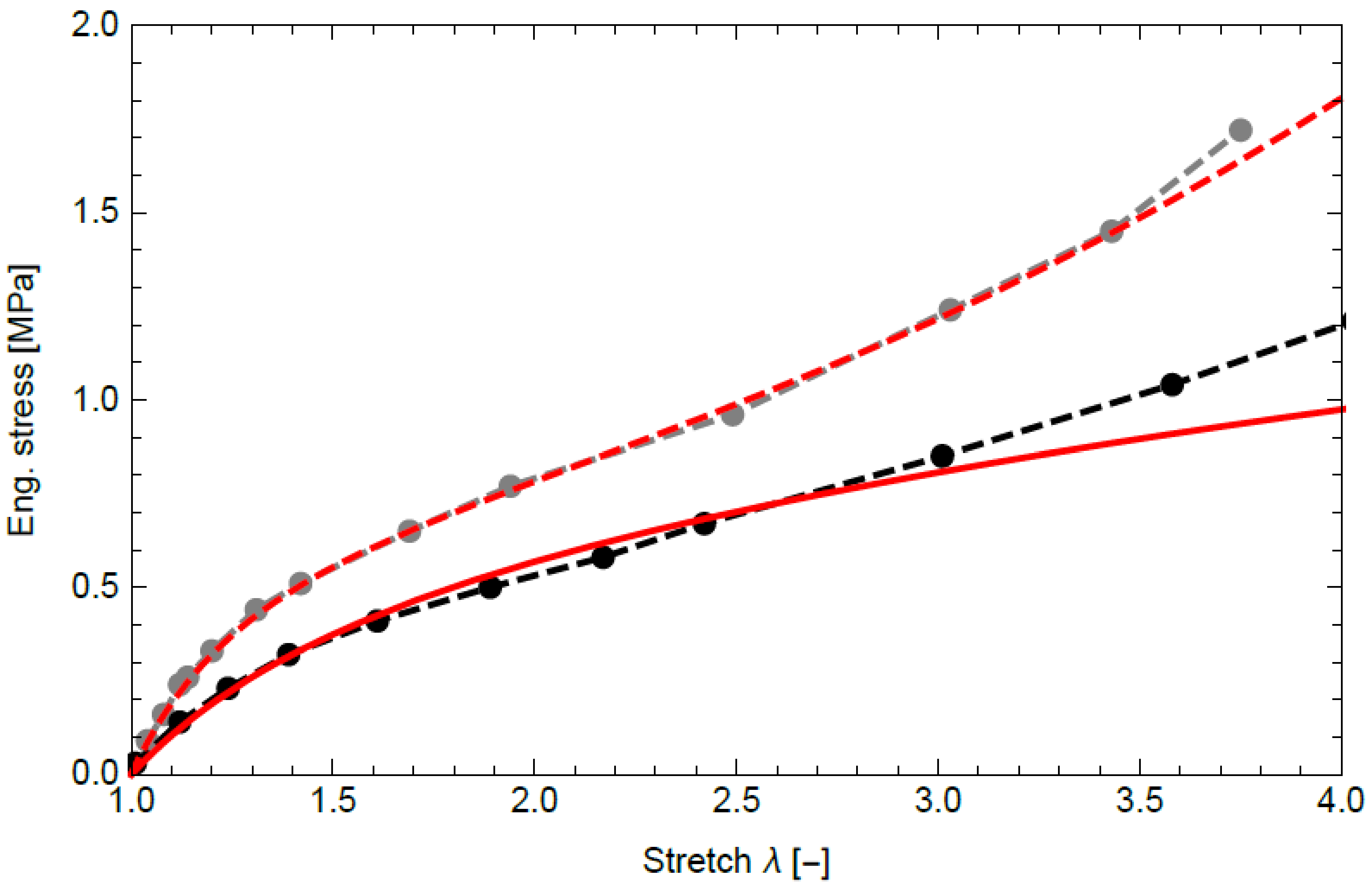
Test data: Equibiaxial + Uniaxial extension

Fitting simultaneously to both exp. data.

2nd-order Ogden model.



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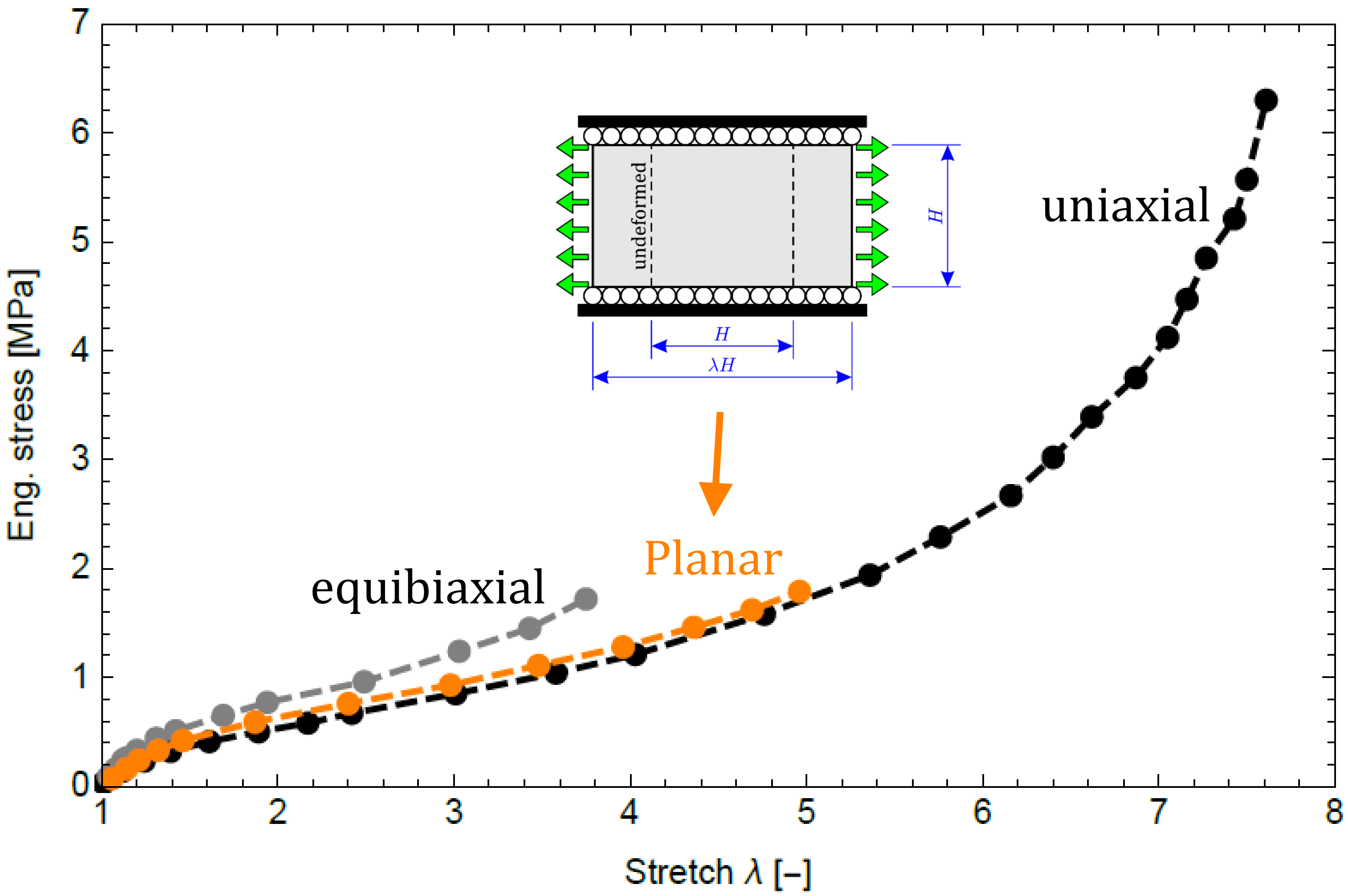
Test data: Planar



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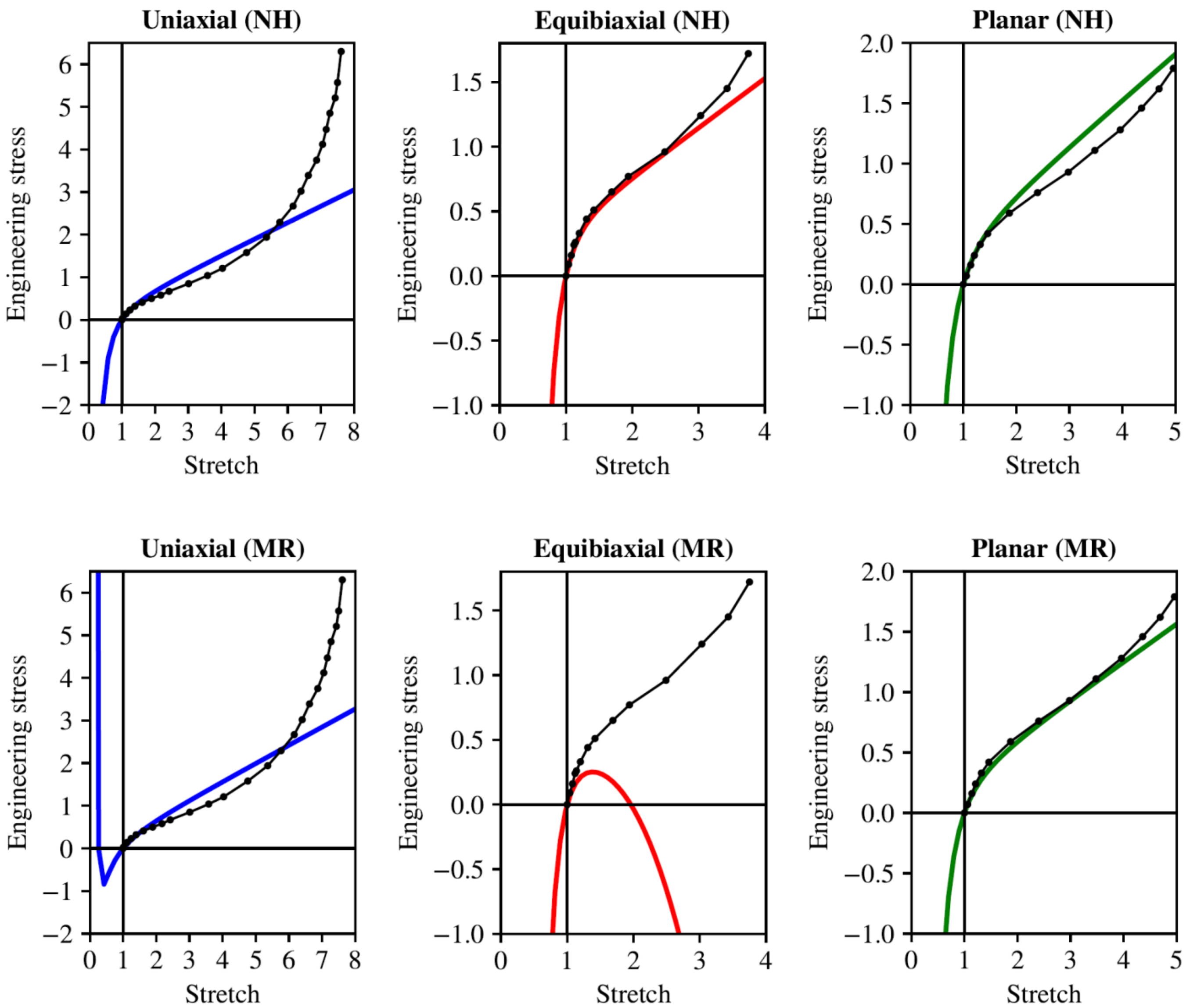
New fittings using the **Root Mean Squared Relative Error**

Fitting only to the uniaxial data.

Neo-Hookean

Mooney-Rivlin



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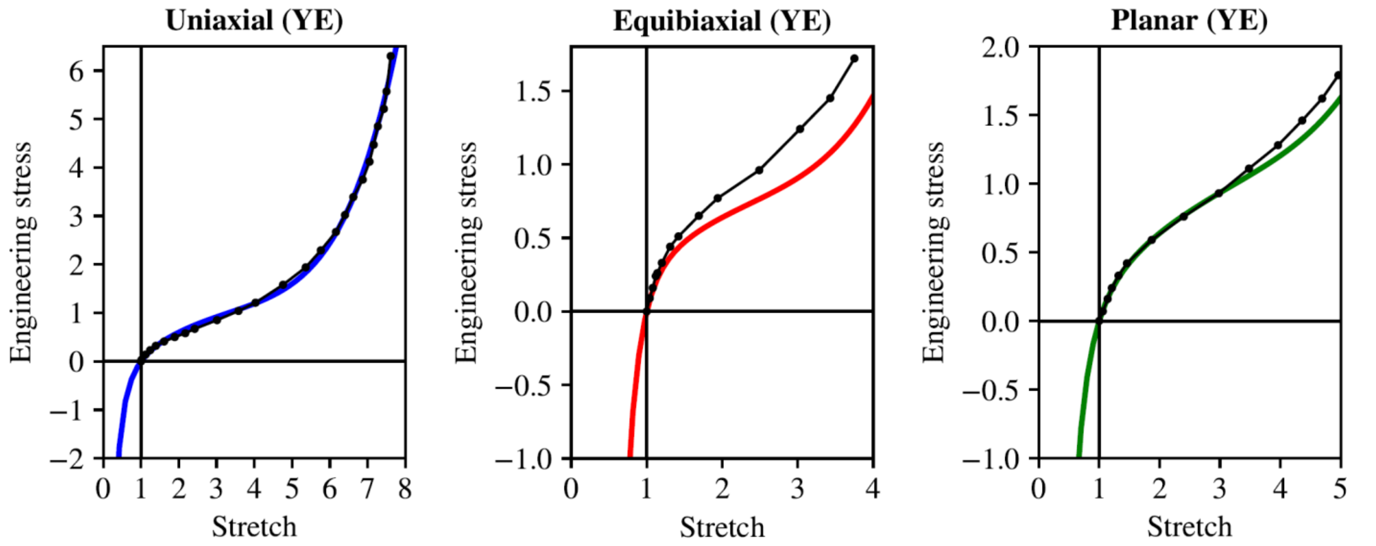
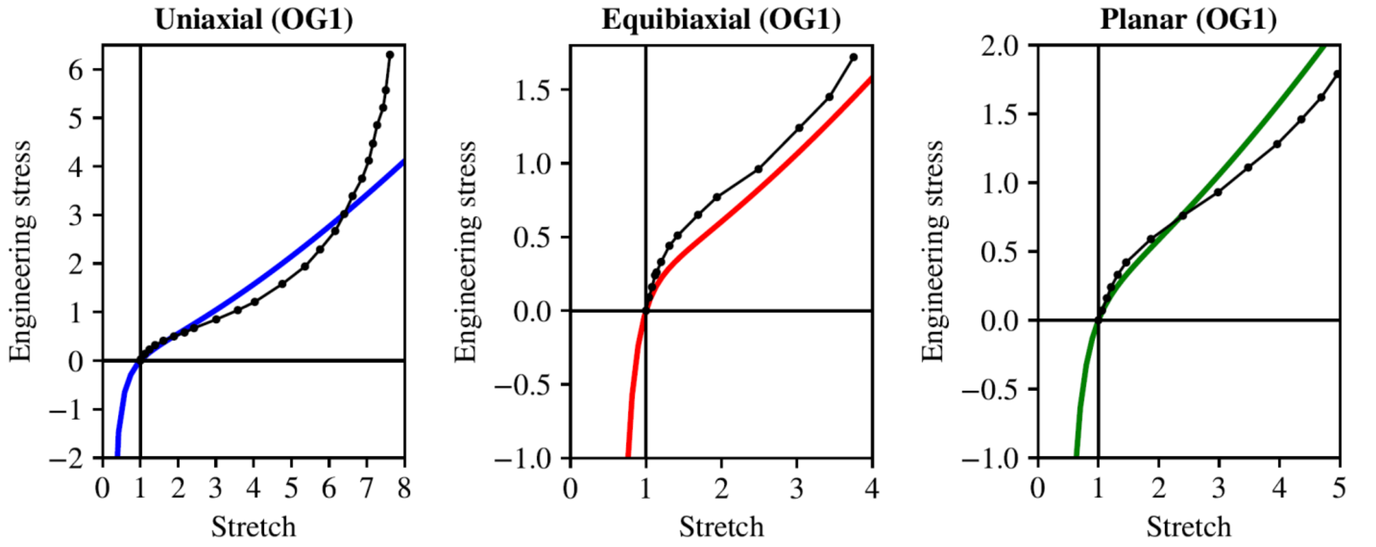
Yeoh:



Ogden, N=1:



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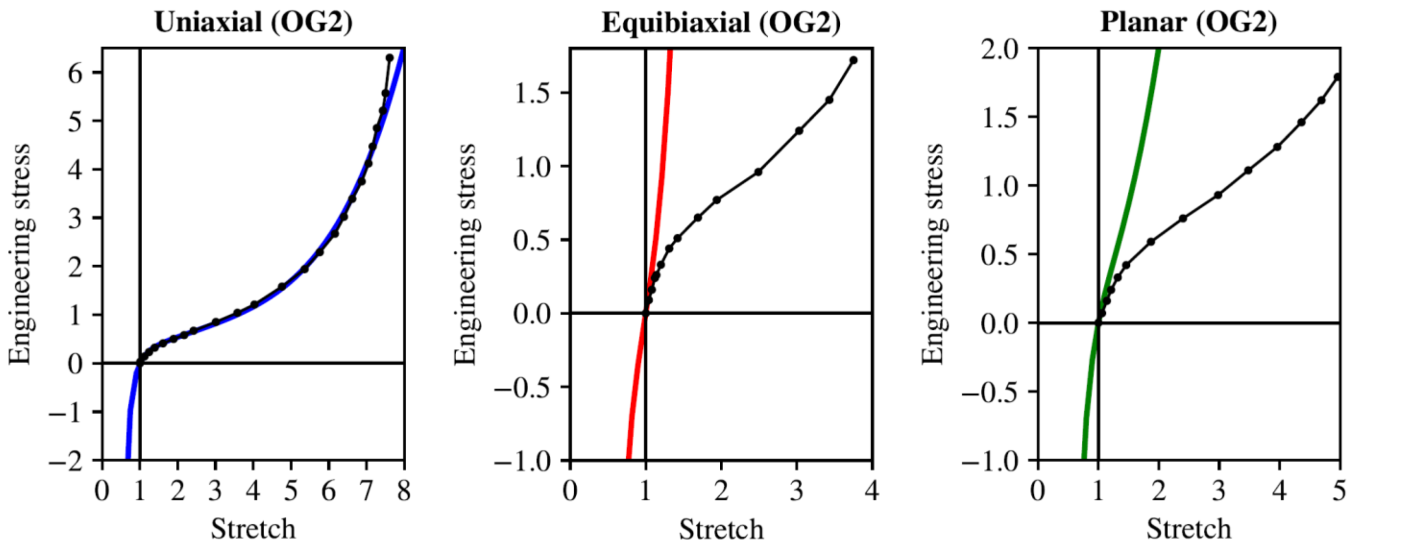
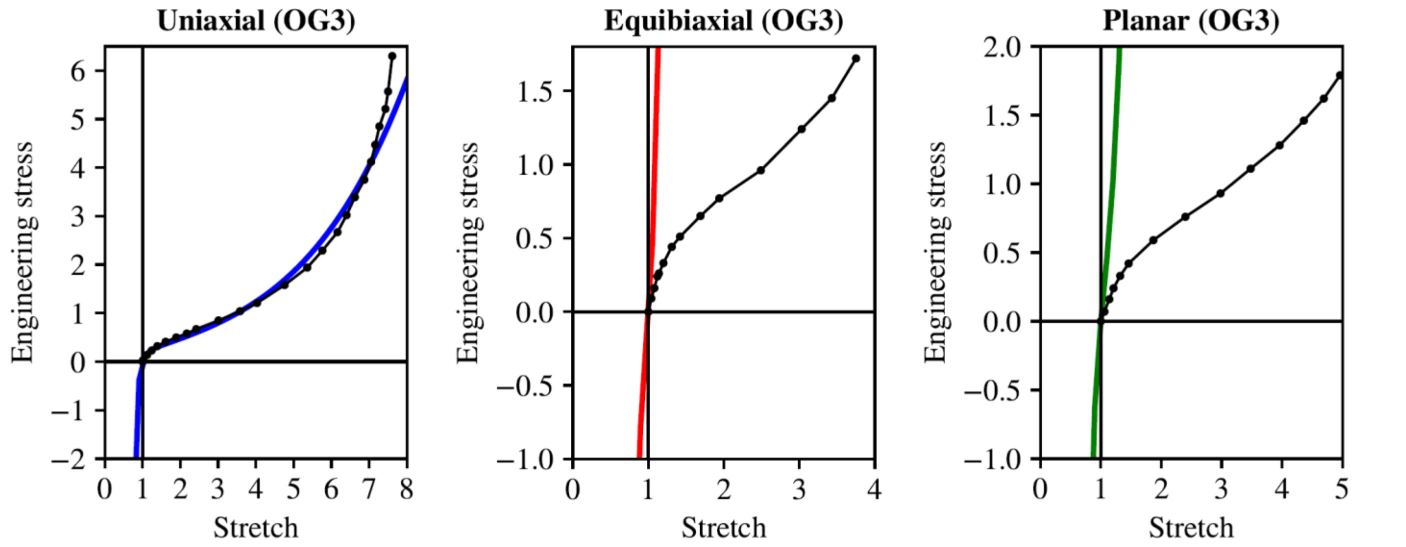
Ogden, N=2:



Ogden, N=3:



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3.4.7 Slightly compressible hyperelastic models

In most cases, the implementations of the compressible versions of hyperelastic materials are based on the dilatational/distortional decomposition of the deformation gradient. For an isochoric (volume-preserving) deformation we have the constraint

$$J = \det \boldsymbol{F} = 1. \tag{1.1}$$

The deformation gradient $\boldsymbol{F}$ can be multiplicatively decomposed into an isochoric and volumetric part as

$$\boldsymbol{F} = \boldsymbol{F}_{\text{iso}} \boldsymbol{F}_{\text{vol}} = \boldsymbol{F}_{\text{vol}} \boldsymbol{F}_{\text{iso}}, \tag{1.2}$$

where

$$\boldsymbol{F}_{\text{iso}} = J^{-1/3} \boldsymbol{F}, \qquad \boldsymbol{F}_{\text{vol}} = J^{1/3} \boldsymbol{I}, \qquad [\boldsymbol{F}_{\text{vol}}] = J^{1/3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.3}$$

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It follows that

$$\det \boldsymbol{F}_{\text{iso}} = 1, \qquad \det \boldsymbol{F}_{\text{vol}} = \det \boldsymbol{F} = J. \tag{1.4}$$

The isochoric part is often denoted as

$$\boldsymbol{F}_{\text{iso}} \equiv \overline{\boldsymbol{F}} = J^{-1/3}\boldsymbol{F}. \tag{1.5}$$

We use this notation in the followings. The modified principal stretches are defined as

$$\overline{\lambda}_1 = J^{-1/3}\lambda_1, \qquad \overline{\lambda}_2 = J^{-1/3}\lambda_2, \qquad \overline{\lambda}_3 = J^{-1/3}\lambda_3, \tag{1.6}$$

$$\overline{\lambda}_i = J^{-1/3}\lambda_i \quad \text{with} \quad i = 1, 2, 3. \tag{1.7}$$

One can conclude that

$$\det \overline{\boldsymbol{F}} = \overline{\lambda}_1\overline{\lambda}_2\overline{\lambda}_3 = 1. \tag{1.8}$$

The modified left and right Cauchy–Green deformation tensors are defined as

$$\overline{\boldsymbol{b}} = \overline{\boldsymbol{F}}\,\overline{\boldsymbol{F}}^T = J^{-2/3}\boldsymbol{F}\boldsymbol{F}^T = J^{-2/3}\boldsymbol{b} \qquad \rightarrow \qquad \boldsymbol{b} = J^{2/3}\overline{\boldsymbol{b}}, \tag{1.9}$$

$$\overline{\boldsymbol{C}} = \overline{\boldsymbol{F}}^T\overline{\boldsymbol{F}} = J^{-2/3}\boldsymbol{F}^T\boldsymbol{F} = J^{-2/3}\boldsymbol{C} \qquad \rightarrow \qquad \boldsymbol{C} = J^{2/3}\overline{\boldsymbol{C}}. \tag{1.10}$$

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The modified principal invariants become

$$\overline{I}_1 = \mathrm{tr}\overline{\boldsymbol{C}} = \mathrm{tr}\overline{\boldsymbol{b}} = J^{-2/3} I_1, \tag{1.11}$$

$$\overline{I}_2 = \frac{1}{2}\left(\overline{I}_1^2 - \mathrm{tr}\overline{\boldsymbol{C}}^2\right) = \frac{1}{2}\left(\overline{I}_1^2 - \mathrm{tr}\overline{\boldsymbol{b}}^2\right) = J^{-4/3} I_2, \tag{1.12}$$

$$\overline{I}_3 = \det\overline{\boldsymbol{C}} = \det\overline{\boldsymbol{b}} = 1. \tag{1.13}$$

The modified principal invariants can be also expressed with the modified principal stretches:

$$\overline{I}_1 = \overline{\lambda}_1^2 + \overline{\lambda}_2^2 + \overline{\lambda}_3^2 = J^{-2/3}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \tag{1.14}$$

$$\overline{I}_2 = \overline{\lambda}_1^2\overline{\lambda}_2^2 + \overline{\lambda}_1^2\overline{\lambda}_3^2 + \overline{\lambda}_2^2\overline{\lambda}_3^2 = J^{-4/3}\left(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2\right), \tag{1.15}$$

$$\overline{I}_3 = 1. \tag{1.16}$$

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The strain energy (per unit reference volume) can be also decoupled into the sum of a distortional and a volumetric contributions:

$$U = U_{\text{iso}} + U_{\text{vol}}, \tag{1.17}$$

where the volumetric part (U_{vol}) is dependent only on the volume ratio $J = I_3^{1/2}$. The distortional part does not depend on the volume ratio. Therefore, the distortional part can be expressed with the help of the modified principal invariants. Consequently, the strain energy can be written as

$$U = U\left(I_1, I_2, I_3\right) = U_{\text{iso}}\left(\overline{I}_1, \overline{I}_2\right) + U_{\text{vol}}\left(I_3\right), \tag{1.18}$$

or

$$U = U_{\text{iso}}\left(\overline{I}_1, \overline{I}_2\right) + U_{\text{vol}}\left(J\right). \tag{1.19}$$

For simplicity we introduce the following notation:

$$\overline{U} = U_{\text{iso}}\left(\overline{I}_1, \overline{I}_2\right), \tag{1.20}$$

$$U = \overline{U}\left(\overline{I}_1, \overline{I}_2\right) + U_{\text{vol}}\left(J\right). \tag{1.21}$$

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This form implies that the 2nd-Piola–Kirchhoff stress tensor can be decoupled into the sum of an isochoric and volumetric stress contributions:

$$\boldsymbol{S} = 2\frac{\partial U}{\partial \boldsymbol{C}} = 2\frac{\partial \overline{U}}{\partial \boldsymbol{C}} + 2\frac{\partial U_{\text{vol}}}{\partial \boldsymbol{C}} = \boldsymbol{S}_{\text{iso}} + \boldsymbol{S}_{\text{vol}}, \tag{1.22}$$

where

$$\boldsymbol{S}_{\text{iso}} = 2\frac{\partial \overline{U}}{\partial \boldsymbol{C}}, \qquad \boldsymbol{S}_{\text{vol}} = 2\frac{\partial U_{\text{vol}}}{\partial \boldsymbol{C}}. \tag{1.23}$$

We can reformulate the expressions with the help of the modified deformation tensors as

$$\boldsymbol{\sigma} = \frac{2}{J}\left(\left(U_{,1} + I_1 U_{,2}\right)\boldsymbol{b} - U_{,2}\boldsymbol{b}^2 + I_3 U_{,3}\boldsymbol{I}\right), \tag{1.24}$$

where

$$U_{,1} = \frac{\partial U}{\partial I_1}$$

$$U_{,2} = \frac{\partial U}{\partial I_2}$$

$$U_{,3} = \frac{\partial U}{\partial I_3}$$

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$$U_{,1}=\frac{\partial U}{\partial I_1}=\frac{\partial U}{\partial \bar{I}_1}\frac{\partial \bar{I}_1}{\partial I_1}=J^{-2/3}\frac{\partial U}{\partial \bar{I}_1}=J^{-2/3}\frac{\partial \bar{U}}{\partial \bar{I}_1}=J^{-2/3}\bar{U}_{,\bar{1}}, \tag{1.25}$$

$$U_{,2}=\frac{\partial U}{\partial I_2}=\frac{\partial U}{\partial \bar{I}_2}\frac{\partial \bar{I}_2}{\partial I_2}=J^{-4/3}\frac{\partial U}{\partial \bar{I}_2}=J^{-4/3}\frac{\partial \bar{U}}{\partial \bar{I}_2}=J^{-4/3}\bar{U}_{,\bar{2}}, \tag{1.26}$$

$$U_{,3}=\frac{\partial U}{\partial I_3}=\frac{\partial U}{\partial \bar{I}_1}\frac{\partial \bar{I}_1}{\partial I_3}+\frac{\partial U}{\partial \bar{I}_2}\frac{\partial \bar{I}_2}{\partial I_3}+\frac{\partial U}{\partial J}\frac{\partial J}{\partial I_3} \tag{1.27}$$

$$=\frac{\partial U}{\partial \bar{I}_1}\frac{\partial \bar{I}_1}{\partial J}\frac{\partial J}{\partial I_3}+\frac{\partial U}{\partial \bar{I}_2}\frac{\partial \bar{I}_2}{\partial J}\frac{\partial J}{\partial I_3}+\frac{\partial U}{\partial J}\frac{\partial J}{\partial I_3} \tag{1.28}$$

$$=\left(-\frac{2}{3}J^{-5/3}I_1\frac{\partial U}{\partial \bar{I}_1}-\frac{4}{3}J^{-7/3}I_2\frac{\partial U}{\partial \bar{I}_2}+\frac{\partial U}{\partial J}\right)\frac{\partial J}{\partial I_3} \tag{1.29}$$

$$=\left(-\frac{2}{3}J^{-1}\bar{I}_1\frac{\partial U}{\partial \bar{I}_1}-\frac{4}{3}J^{-1}\bar{I}_2\frac{\partial U}{\partial \bar{I}_2}+\frac{\partial U}{\partial J}\right)\frac{1}{2J} \tag{1.30}$$

$$=-\frac{1}{3J^2}\bar{I}_1\bar{U}_{,\bar{1}}-\frac{2}{3J^2}\bar{I}_2\bar{U}_{,\bar{2}}+\frac{1}{2J}\frac{\partial U_{\mathrm{vol}}}{\partial J} \tag{1.31}$$

$$=\frac{1}{3J^2}\left(-\bar{I}_1\bar{U}_{,\bar{1}}-2\bar{I}_2\bar{U}_{,\bar{2}}+\frac{3}{2}J\frac{\partial U_{\mathrm{vol}}}{\partial J}\right) \tag{1.32}$$

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where

$$\overline{U}_{,\overline{1}} = \frac{\partial \overline{U}}{\partial \overline{I}_1}, \qquad \overline{U}_{,\overline{2}} = \frac{\partial \overline{U}}{\partial \overline{I}_2}. \tag{1.33}$$

In addition, we can easily found that

$$U_{,J} = \frac{\partial U}{\partial J} = \frac{\partial U}{\partial I_3}\frac{\partial I_3}{\partial J} = U_{,3} 2J \qquad \rightarrow \qquad U_{,3} = \frac{1}{2J} U_{,J}. \tag{1.34}$$

Consequently, the Cauchy stress tensor can be written as

$$\boldsymbol{\sigma} = \frac{2}{J}\left(\left(\overline{U}_{,\overline{1}} + \overline{I}_1 \overline{U}_{,\overline{2}}\right)\overline{\boldsymbol{b}} - \overline{U}_{,\overline{2}}\overline{\boldsymbol{b}}^2 + \frac{1}{3}\left(-\overline{I}_1\overline{U}_{,\overline{1}} - 2\overline{I}_2\overline{U}_{,\overline{2}} + \frac{3}{2}J\frac{\partial U_{\mathrm{vol}}}{\partial J}\right)\boldsymbol{I}\right), \tag{1.35}$$

$$\boldsymbol{\sigma} = \frac{2}{J}\left[\left(\overline{U}_{,\overline{1}} + \overline{I}_1 \overline{U}_{,\overline{2}}\right)\overline{\boldsymbol{b}} - \overline{U}_{,\overline{2}}\overline{\boldsymbol{b}}^2 - \frac{1}{3}\left(\overline{I}_1\overline{U}_{,\overline{1}} + 2\overline{I}_2\overline{U}_{,\overline{2}}\right)\boldsymbol{I}\right] + \frac{\partial U_{\mathrm{vol}}}{\partial J}\boldsymbol{I}. \tag{1.36}$$

The form above can be decoupled into volumetric and distortional components as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\mathrm{iso}} + \boldsymbol{\sigma}_{\mathrm{vol}}, \tag{1.37}$$

$$\boldsymbol{\sigma}_{\mathrm{iso}} = \frac{2}{J}\left[\left(\overline{U}_{,\overline{1}} + \overline{I}_1 \overline{U}_{,\overline{2}}\right)\overline{\boldsymbol{b}} - \overline{U}_{,\overline{2}}\overline{\boldsymbol{b}}^2 - \frac{1}{3}\left(\overline{I}_1\overline{U}_{,\overline{1}} + 2\overline{I}_2\overline{U}_{,\overline{2}}\right)\boldsymbol{I}\right], \tag{1.38}$$

$$\boldsymbol{\sigma}_{\mathrm{vol}} = \frac{\partial U_{\mathrm{vol}}}{\partial J}\boldsymbol{I}. \tag{1.39}$$

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Volumetric dilatation

The deformation gradient and the left Cauchy–Green deformation tensors for volumetric dilatation are

$$[\boldsymbol{F}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \qquad [\boldsymbol{b}] = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}. \tag{1.40}$$

The volume ratio is $J = \det F = \lambda^3$. Therefore, the modified left Cauchy–Green deformation tensor becomes the second-order identity tensor:

$$[\bar{\boldsymbol{b}}] = J^{-2/3} \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix} = \lambda^{-2} \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\boldsymbol{I}]. \tag{1.41}$$

The principal scalar invariants:

$$\bar{I}_1 = \mathrm{tr}\bar{\boldsymbol{b}} = 3,$$

$$\bar{I}_2 = \frac{1}{2}\left(\bar{I}_1^2 - \mathrm{tr}\bar{\boldsymbol{b}}^2\right) = 3,$$

$$\bar{I}_3 = \det\bar{\boldsymbol{b}} = 1.$$

$$\boxed{\boldsymbol{\sigma}_{\mathrm{iso}} = \boldsymbol{0}, \qquad \boldsymbol{\sigma}_{\mathrm{vol}} = \frac{\partial U_{\mathrm{vol}}}{\partial J}\boldsymbol{I}, \qquad \boldsymbol{\sigma} = \frac{\partial U_{\mathrm{vol}}}{\partial J}\boldsymbol{I}.}$$

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Slightly Compressible Models in Abaqus

The compressible hyperelastic models (except the hyperfoam model, which is not investigated in this report) proposed for nearly-incompressible materials in Abaqus are based on the incompressible versions but extended with a volumetric term as illustrated by the following expression

$$U_{\text{compressible}}\left(\bar{I}_1, \bar{I}_2, J\right) = U_{\text{incompressible}}\left(\bar{I}_1, \bar{I}_2\right) + U_{\text{vol}}\left(J\right). \tag{1.47}$$

Consequently, the principal invariants in the incompressible strain energy potential must be replaced with the modified invariants and, in addition, a volumetric term must be added to the strain energy (per unit reference volume). The form of the volumetric part of the strain energy potential depends on the particular hyperelastic model.

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The compressible version of the **neo-Hookean** model is

$$U = C_{10}\left(\bar{I}_1 - 3\right) + \frac{1}{D_1}\left(J-1\right)^2 .$$

The compressible version of the **Mooney–Rivlin** model is

$$U = C_{10}\left(\bar{I}_1 - 3\right) + C_{01}\left(\bar{I}_2 - 3\right) + \frac{1}{D_1}\left(J-1\right)^2 .$$

The compressible version of the **Yeoh** model is

$$U = C_{10}\left(\bar{I}_1 - 3\right) + C_{20}\left(\bar{I}_1 - 3\right)^2 + C_{30}\left(\bar{I}_1 - 3\right)^3 + \frac{1}{D_1}\left(J-1\right)^2 + \frac{1}{D_2}\left(J-1\right)^4 + \frac{1}{D_3}\left(J-1\right)^6 .$$

The compressible version of the **Ogden** model is

$$U = \sum_{i=1}^{N} \frac{2\mu_i}{\alpha_i^2}\left(\bar{\lambda}_1^{\alpha_i} + \bar{\lambda}_2^{\alpha_i} + \bar{\lambda}_3^{\alpha_i} - 3\right) + \sum_{i=1}^{N} \frac{1}{D_i}\left(J-1\right)^{2i} .$$

Only for slightly compressible problems !

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Compressible neo Hookean model

$$U = C_{10}\left(\bar{I}_1 - 3\right) + \frac{1}{D_1}\left(J-1\right)^2 .$$

This form is rewritten (for simplicity) as

$$U = \overline{U}\left(\bar{I}_1, \bar{I}_2\right) + U_{\text{vol}}\left(J\right) = \frac{G}{2}(\bar{I}_1 - 3) + \frac{K}{2}(J-1)^2.$$

Consequently, the following new parameters are introduced:

$$G = 2C_{10}, \qquad K = \frac{2}{D_1}.$$

$$\boldsymbol{\sigma} = \frac{2}{J}\overline{U}_{,\bar{1}}\bar{\boldsymbol{b}} - \frac{2}{3J}\bar{I}_1\overline{U}_{,\bar{1}}\boldsymbol{I} + U_{,J}\boldsymbol{I},$$

$$\frac{\partial U}{\partial \bar{I}_1} = \frac{G}{2} \qquad \frac{\partial U}{\partial J} = K(J-1)$$

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- Simplest hyperelastic model: **neo-Hookean** (Rivlin 1948).
- Due to its simplicity, it is widely-used in the industry and even in Hollywood too.
- Available in FE software.

Stable Neo-Hookean Flesh Simulation

BREANNAN SMITH, FERNANDO DE GOES, and THEODORE KIM, Pixar Animation Studios

Fig. 1. **Left**: Thirteen skeletal bones drive a hexahedral lattice with 45,809 elements and 156,078 degrees of freedom. **Center**: A quasi-static simulation with our new Neo-Hookean model and a Poisson's ratio of $\nu = 0.488$. Wrinkles and bulges emerge from our model's excellent volume-preserving properties. An average time step with our model took 13.7 Newton iterations, 5,860 Conjugate Gradient (CG) iterations, and 25.6 seconds. **Right**: The same simulation with corotational elasticity and $\nu = 0.488$. The model fails to preserve volume and instead collapses the trapezius and forms a spurious fold around the shoulder blade. The artifacts persist across all values of ν. An average time step with this model took 17.9 Newton iterations, 16,183 CG iterations, and 46.6 seconds. ©Disney/Pixar.

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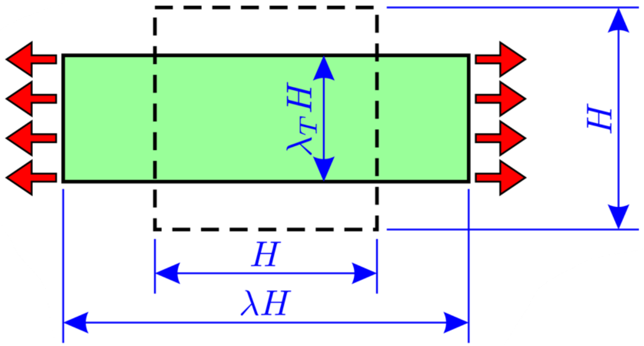
Stress solution in uniaxial loading:



$$[\boldsymbol{b}] = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda_T^2 & 0 \\ 0 & 0 & \lambda_T^2 \end{bmatrix}$$

$$J = \lambda \lambda_T^2$$

$$I_1 = \lambda^2 + 2\lambda_T^2$$

$$\bar{\boldsymbol{b}} = J^{-2/3}\boldsymbol{b}$$

$$\sigma_{11} = \frac{2}{3} G (\lambda \lambda_T^2)^{-5/3} (\lambda^2 - \lambda_T^2) + K(\lambda \lambda_T^2 - 1)$$

$$\sigma_{22} = \sigma_{33} = \frac{1}{3} G (\lambda \lambda_T^2)^{-5/3} (\lambda_T^2 - \lambda^2) + K(\lambda \lambda_T^2 - 1)$$

$$\sigma_{22} = 0$$

Cannot be solved analytically for $\lambda_T = \lambda_T(\lambda)$

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Linearization of these stress components are formulated as

$$\tilde{\sigma}_{11} = \left(\frac{\partial \sigma_{11}}{\partial \lambda} \bigg|_{\lambda=1, \lambda_T=1} \right) (\lambda - 1) + \left(\frac{\partial \sigma_{11}}{\partial \lambda_T} \bigg|_{\lambda=1, \lambda_T=1} \right) (\lambda_T - 1),$$

$$\tilde{\sigma}_{22} = \left(\frac{\partial \sigma_{22}}{\partial \lambda} \bigg|_{\lambda=1, \lambda_T=1} \right) (\lambda - 1) + \left(\frac{\partial \sigma_{22}}{\partial \lambda_T} \bigg|_{\lambda=1, \lambda_T=1} \right) (\lambda_T - 1).$$

Evaluation of the partial derivatives on the right-hand sides gives the solutions:

$$\tilde{\sigma}_{11} = \frac{4}{3} G (e - e_T) + K (e + 2 e_T),$$

$$\tilde{\sigma}_{22} = \frac{2}{3} G (e_T - e) + K (e + 2 e_T),$$

where the engineering strains $e = \lambda - 1$ and $e_T = \lambda_T - 1$ are introduced for simplicity. The zero transverse stress constraint $\tilde{\sigma}_{22} = 0$ is solved for the transverse engineering strain, giving:

$$e_T = -\frac{3K - 2G}{6K + 2G} e = -\nu_0 e.$$

Ground-state Poisson's ratio:

$$\nu_0 = \frac{3K/G - 2}{6K/G + 2} = \frac{1}{2} (1 - C_{10} D_1)$$

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Ground-state Young's modulus:

$$\tilde{\sigma}_{11} = \frac{9KG}{3K+G} e = E_0 e \qquad E_0 = \frac{9KG}{3K+G} e$$

Ground-state shear modulus:

$$G_0 = \frac{E_0}{2(1+\nu_0)} = G$$

Ground-state bulk modulus:

$$K_0 = \frac{E_0}{3(1-2\nu_0)} = K$$

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Example: uniaxial compression

Material parameters: $G = 2, K = 6$

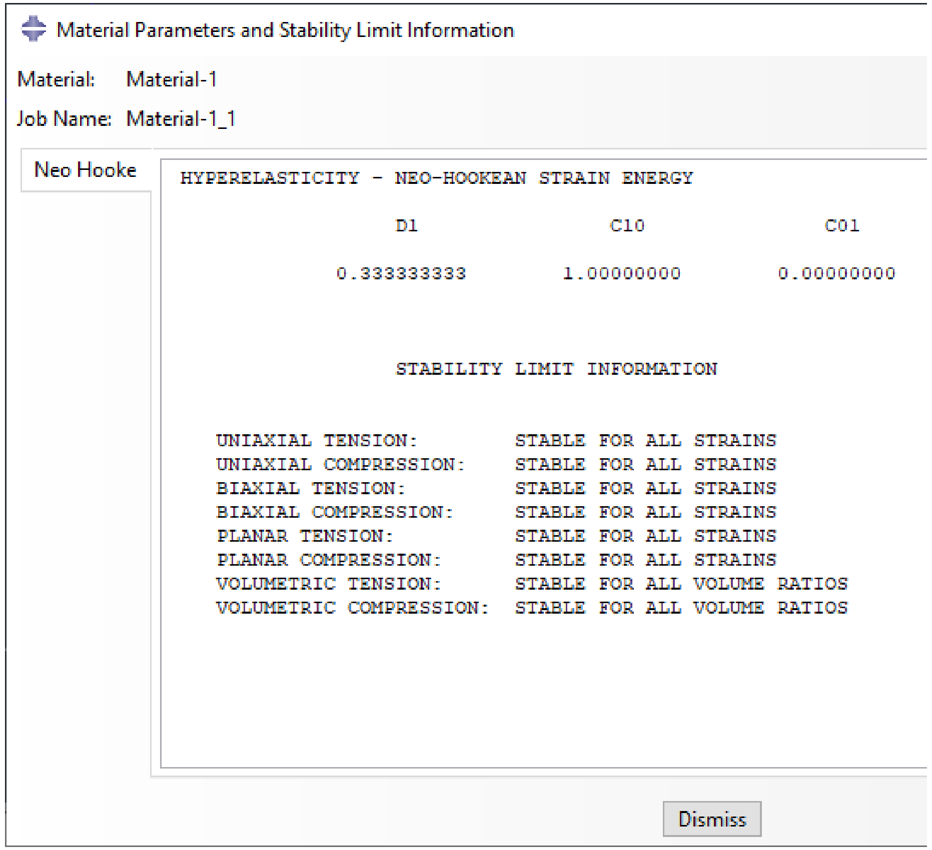
Ground-state Poisson's ratio: $\nu_0 = 0.35$

Drucker's stability check: stable for all strains

Drucker's stability criterion is a strong condition on the incremental internal energy of a material which states that the incremental internal energy can only increase.

The Drucker stability condition requires that the change in the stress $\mathrm{d}\boldsymbol{\sigma}$ following from any infinitesimal change in the logarithmic strain $\mathrm{d}\boldsymbol{\varepsilon}$ satisfies the inequality:

$$\mathrm{d}\boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\varepsilon} \geq 0$$



$$C_{10} = G/2 = 1$$

$$D_1 = 2/K = 1/3$$

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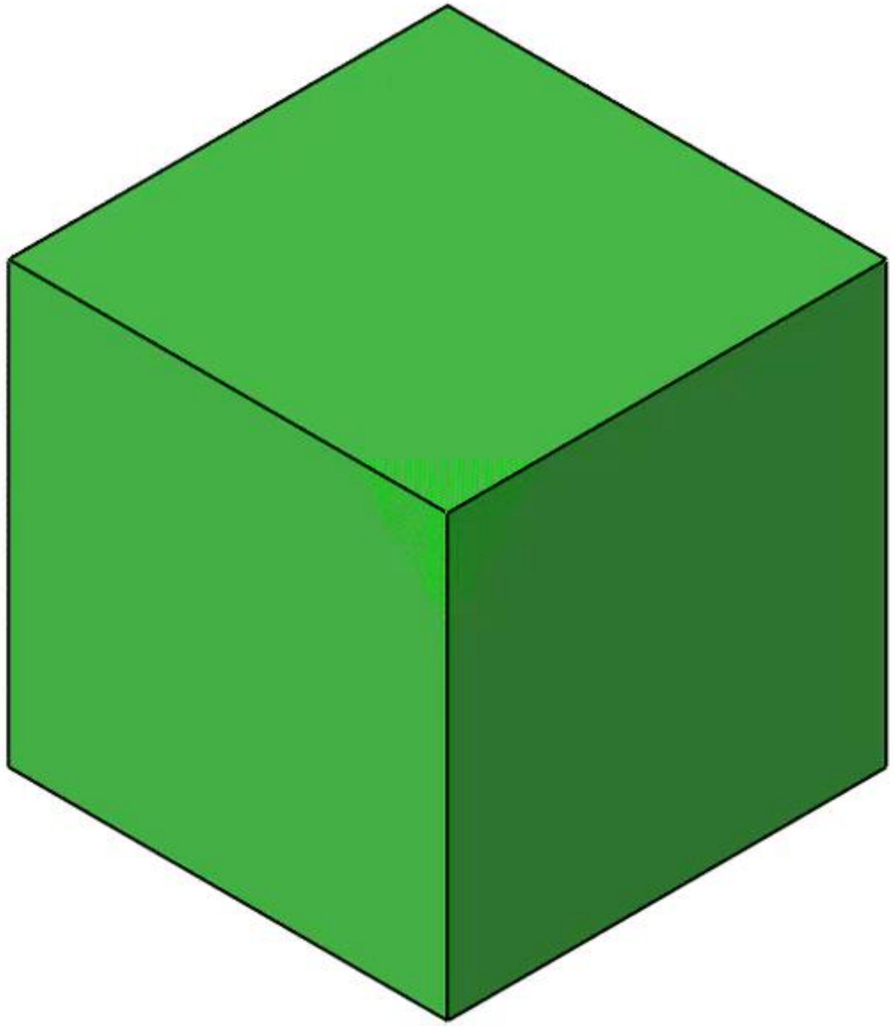
Expected behavior: Increasing lateral (transverse) deformation.



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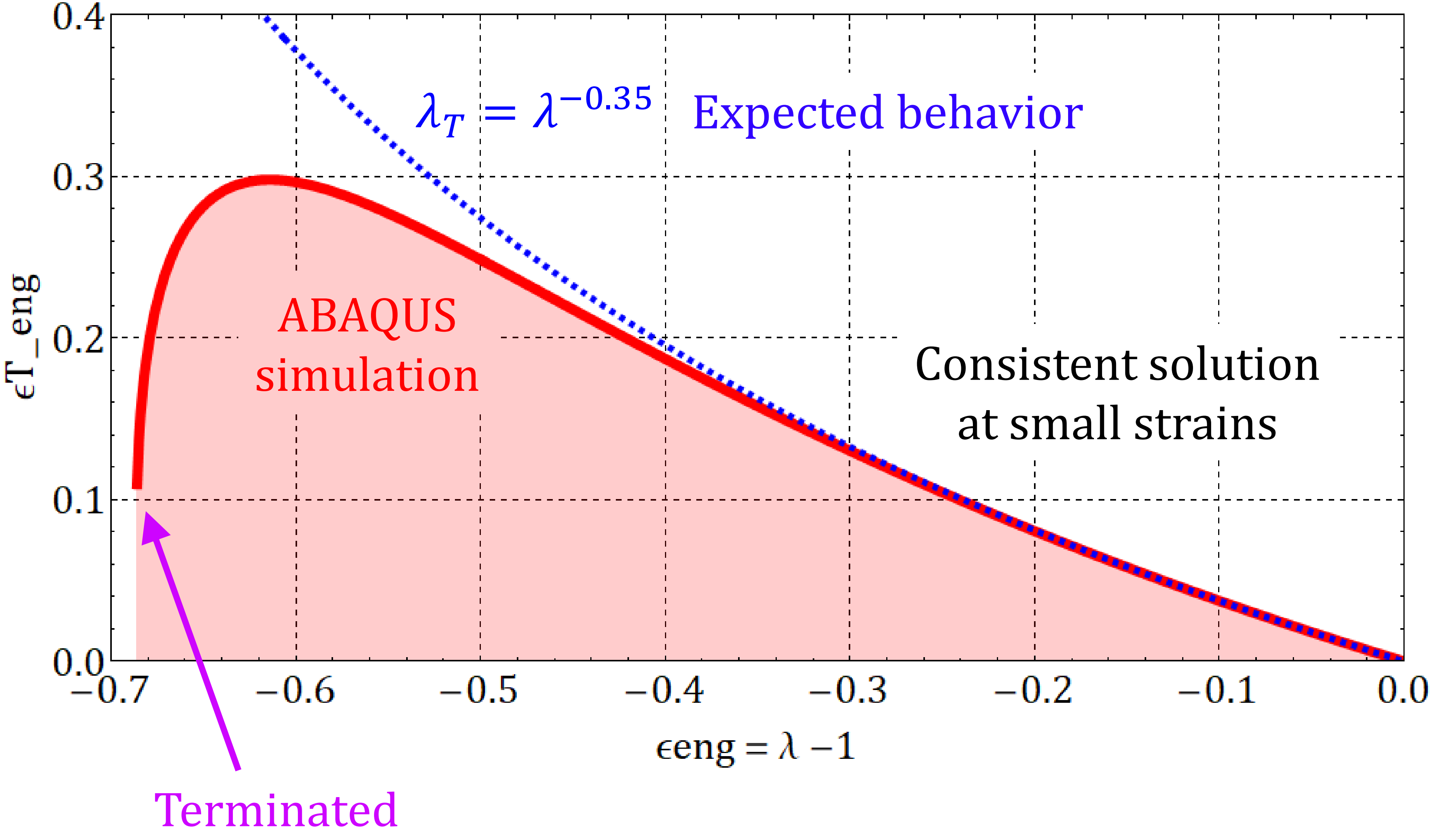
Unrealistic phenomenon:



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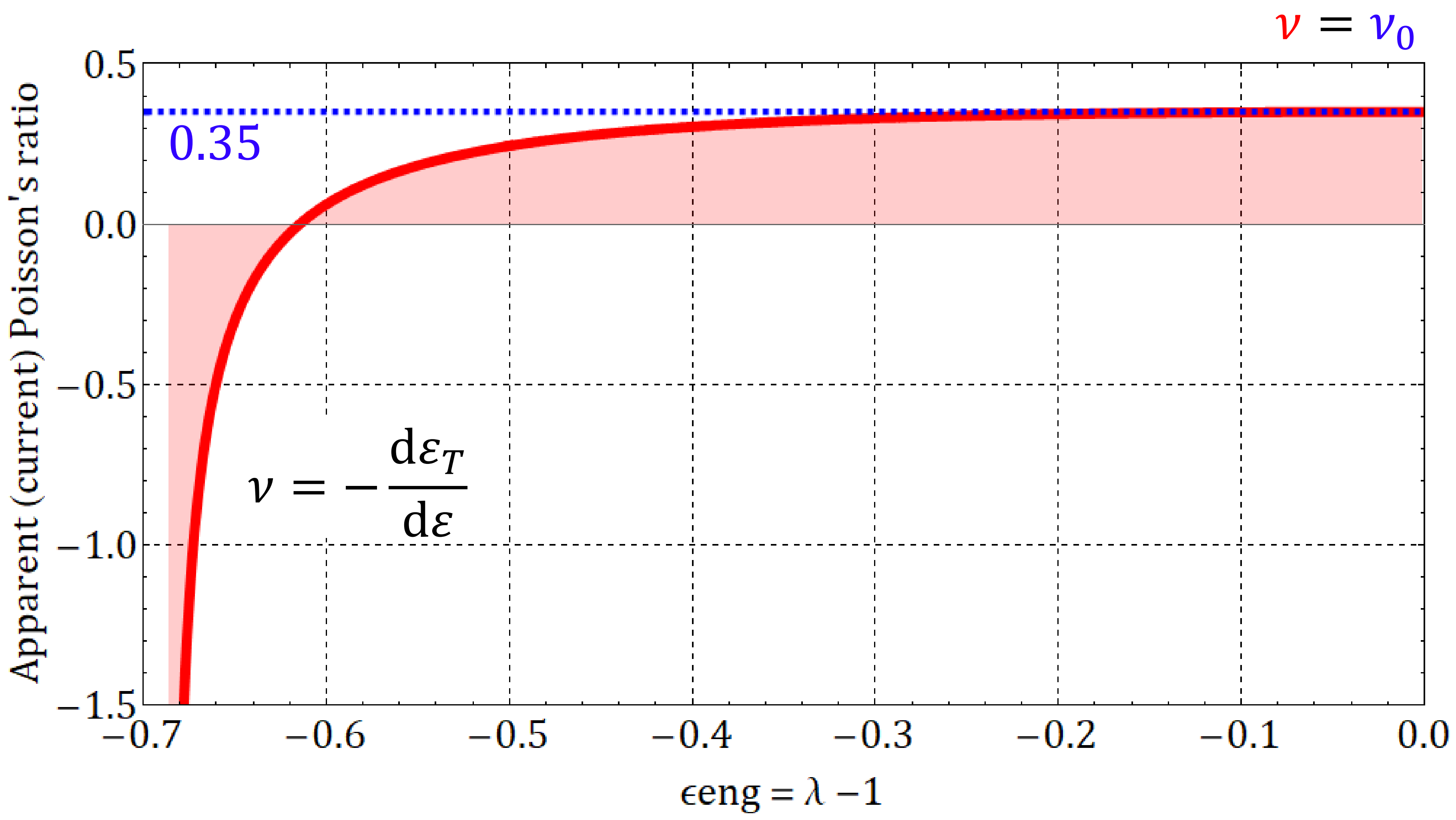
Apparent (current) Poisson's ratio:



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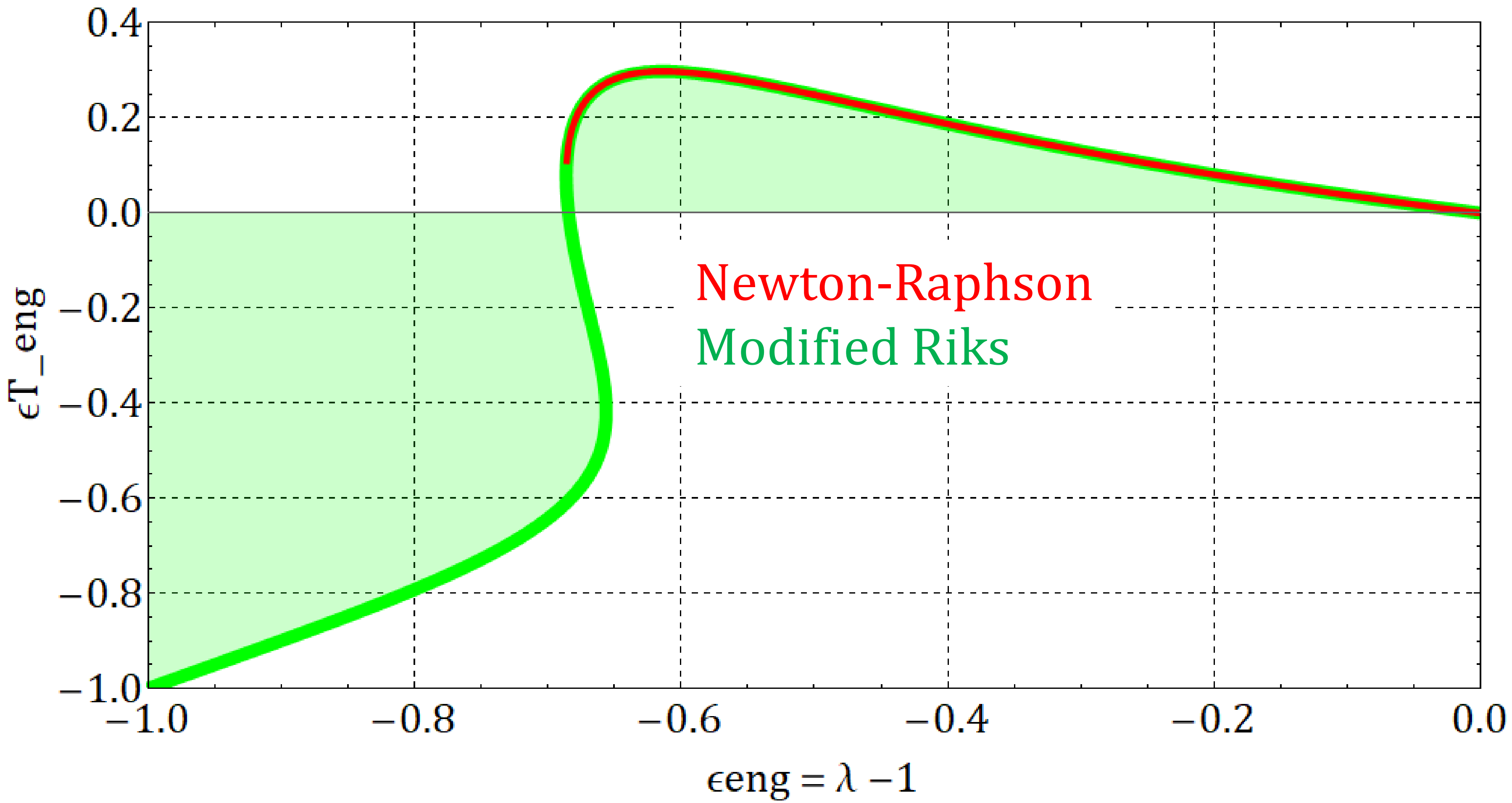
Arc-length method:



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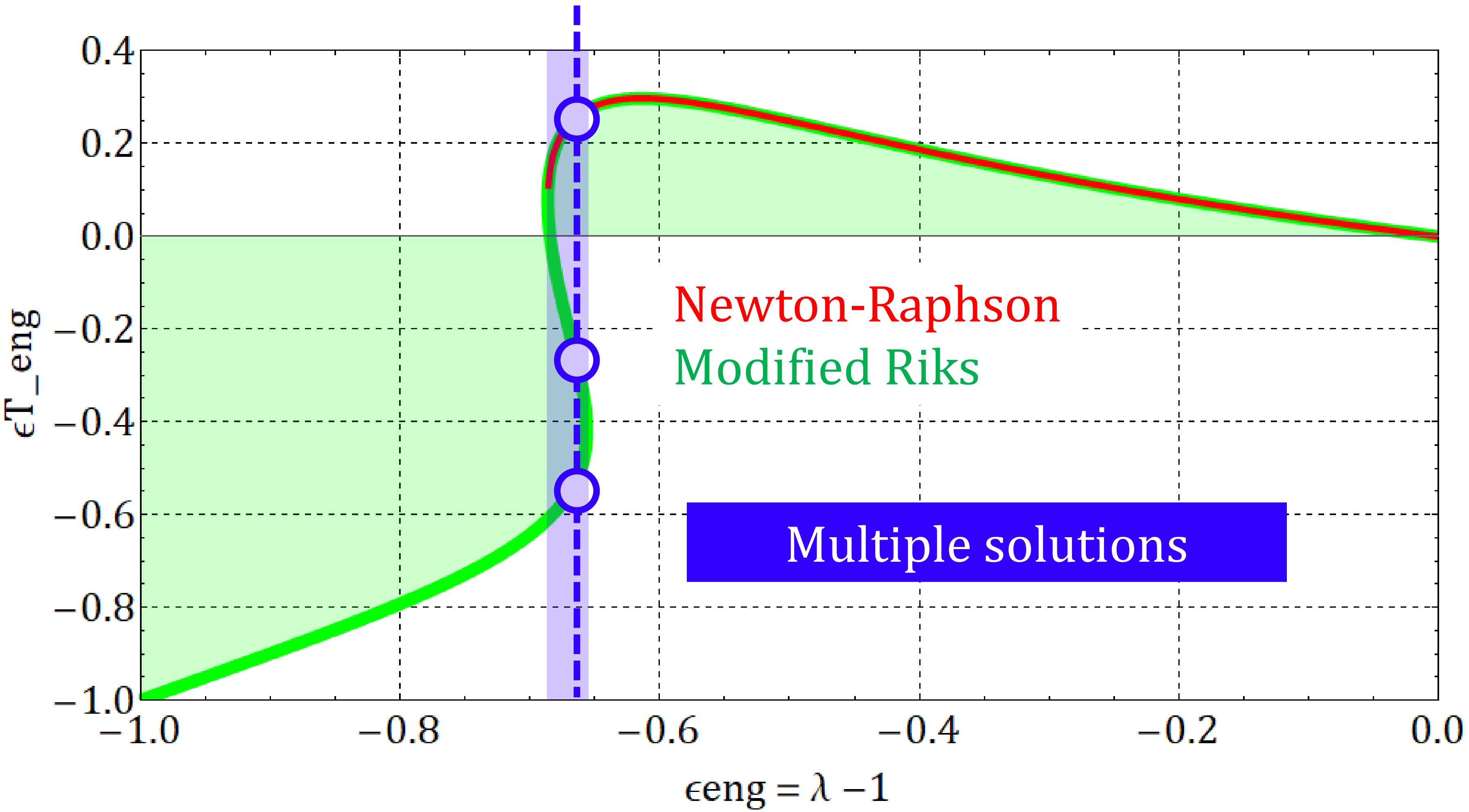
Arc-length method:



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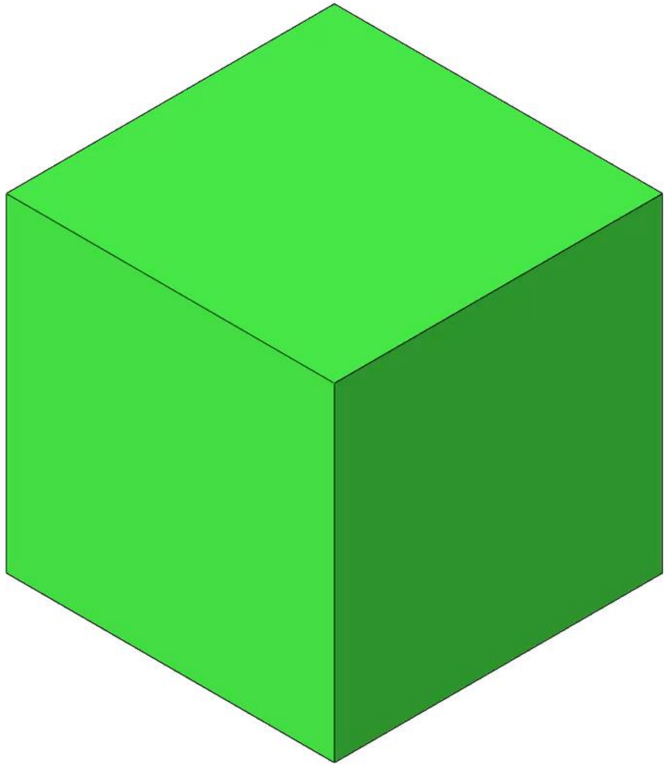

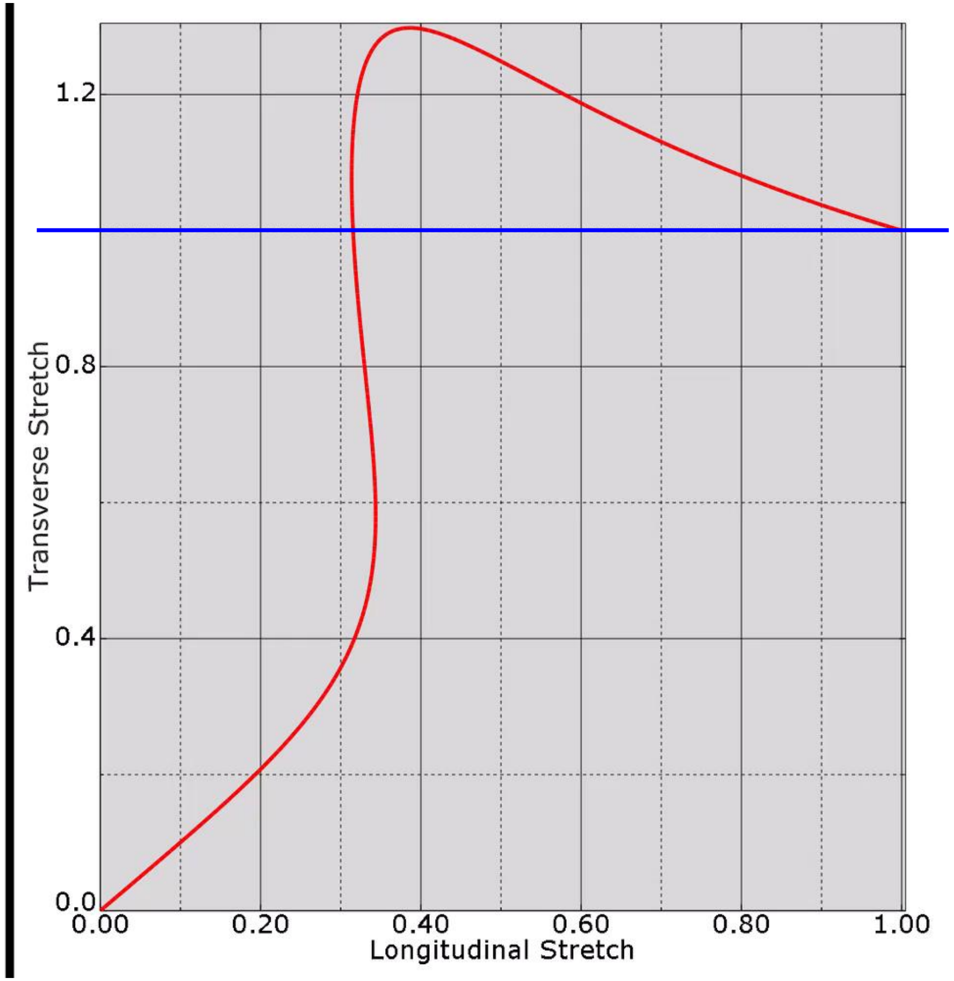

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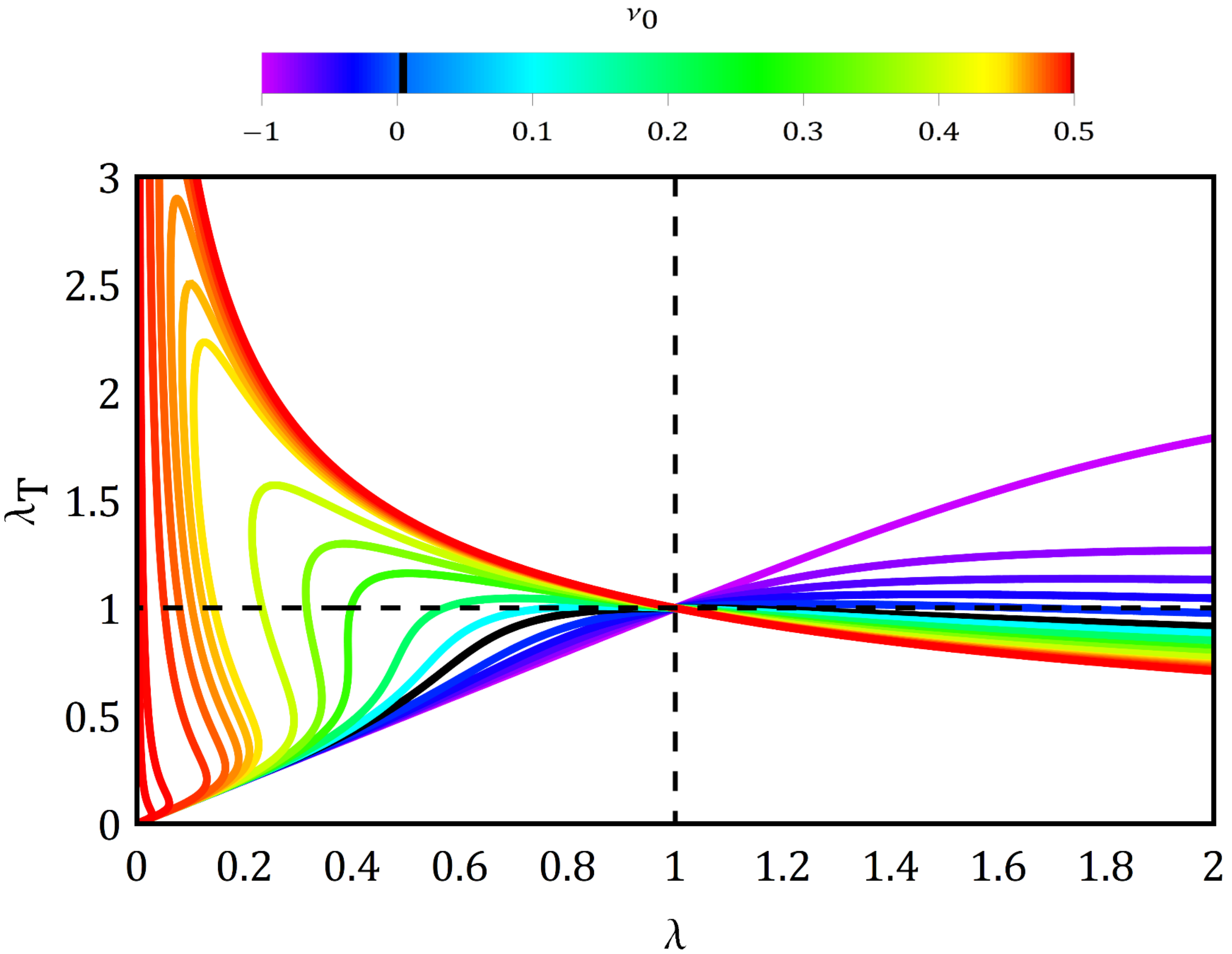
Multiple solutions exist above certain values of ν_0 in compression!
Zero transverse stretch in complete compression.



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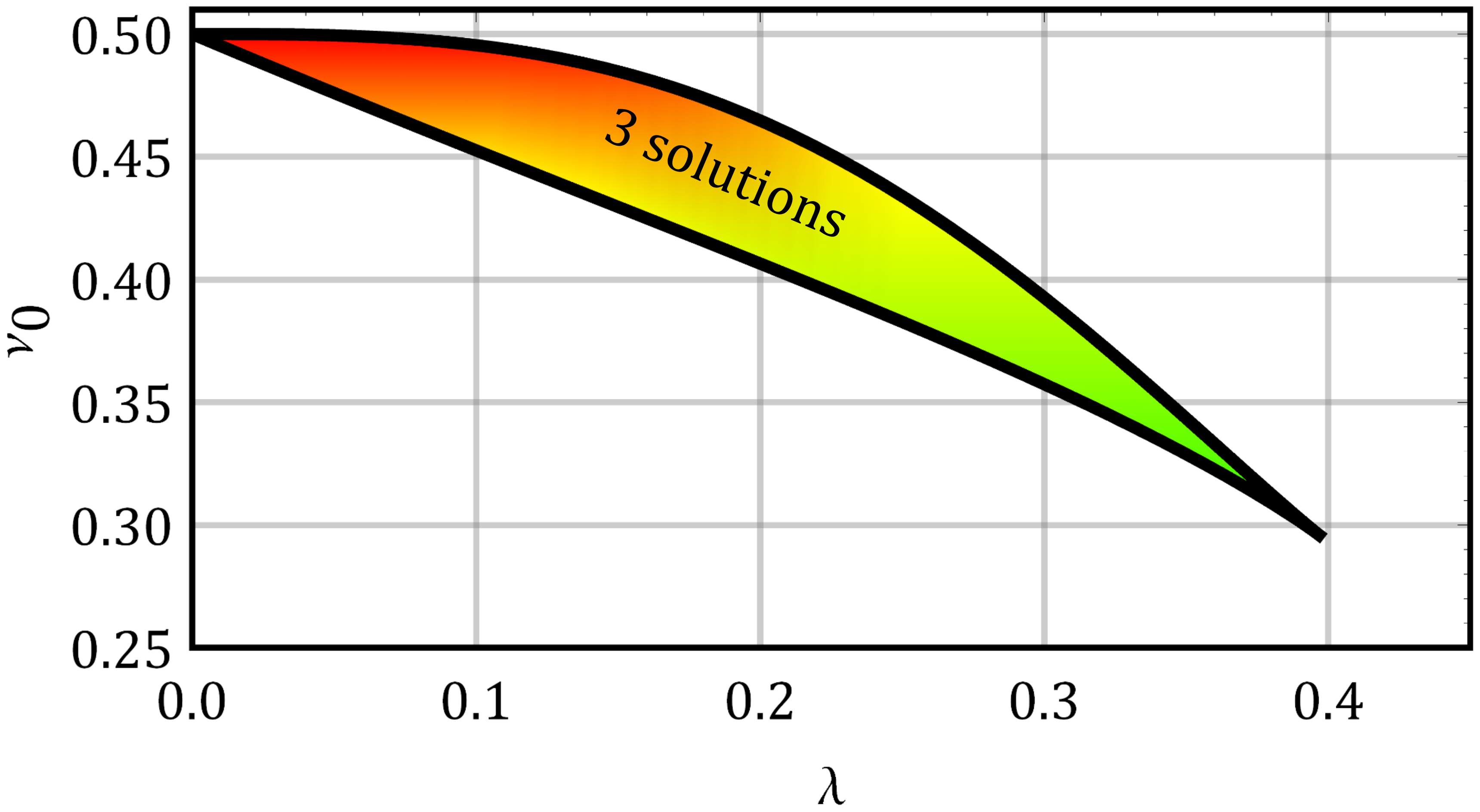
Domain for the multiple solutions:



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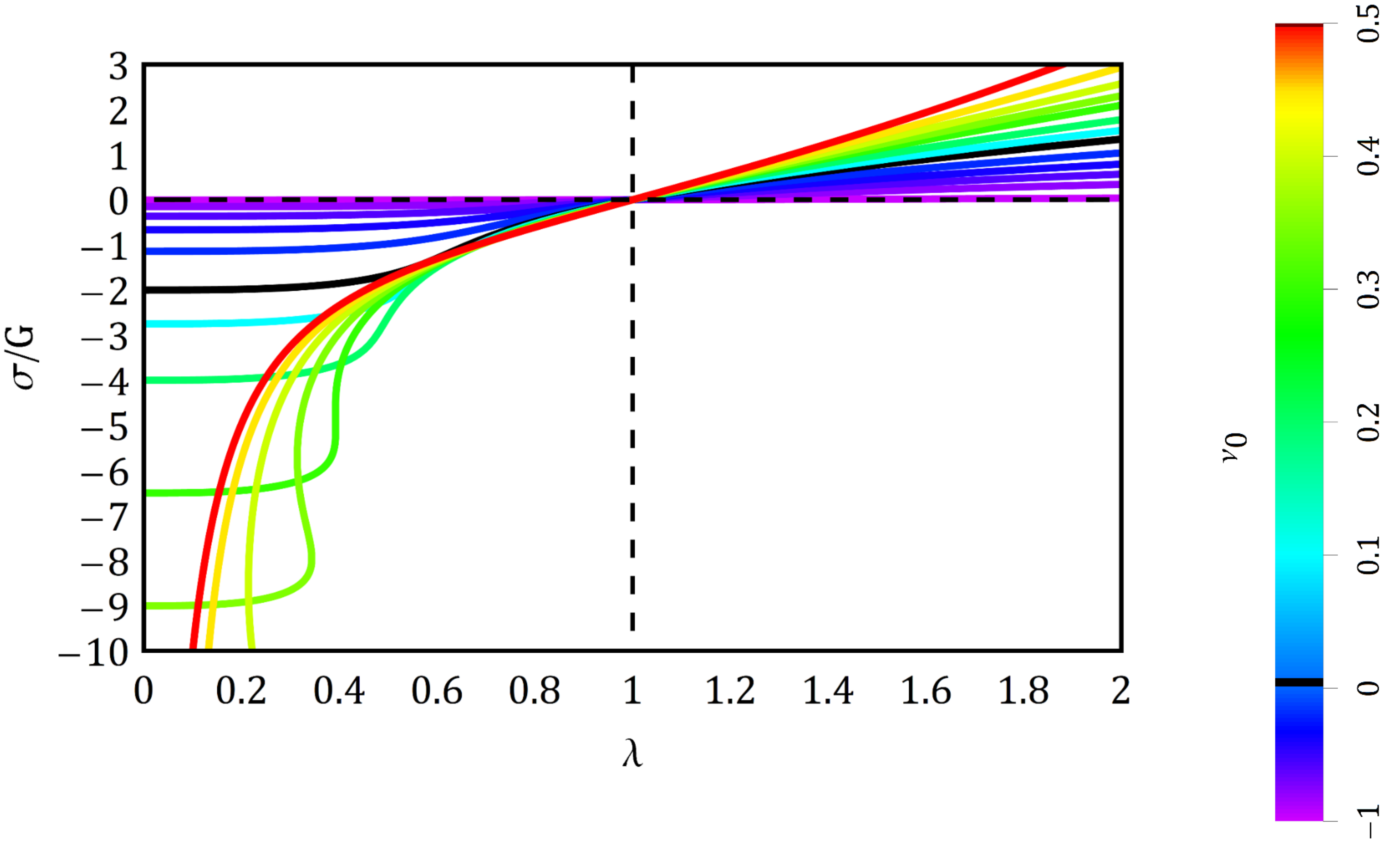
True stress solution: σ / C_{10}

$$\sigma(0) = -3K$$



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More details:

https://link.springer.com/article/10.1007/s11012-022-01633-2

Meccanica (2023) 58:217–232
https://doi.org/10.1007/s11012-022-01633-2

Analysis of the compressible, isotropic, neo-Hookean hyperelastic model

Attila Kossa · Megan T. Valentine · Robert M. McMeeking

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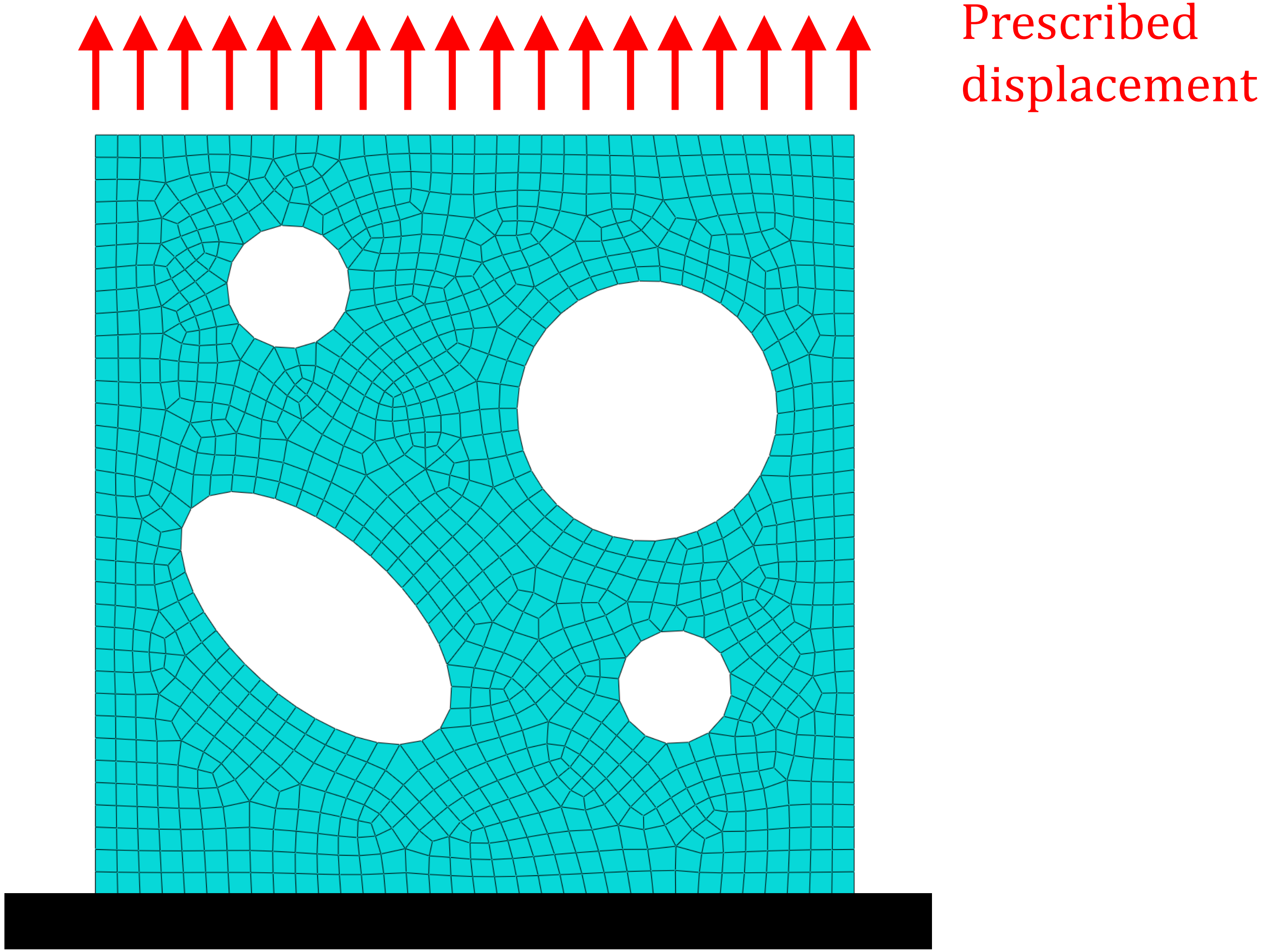
Effect of the compressibility



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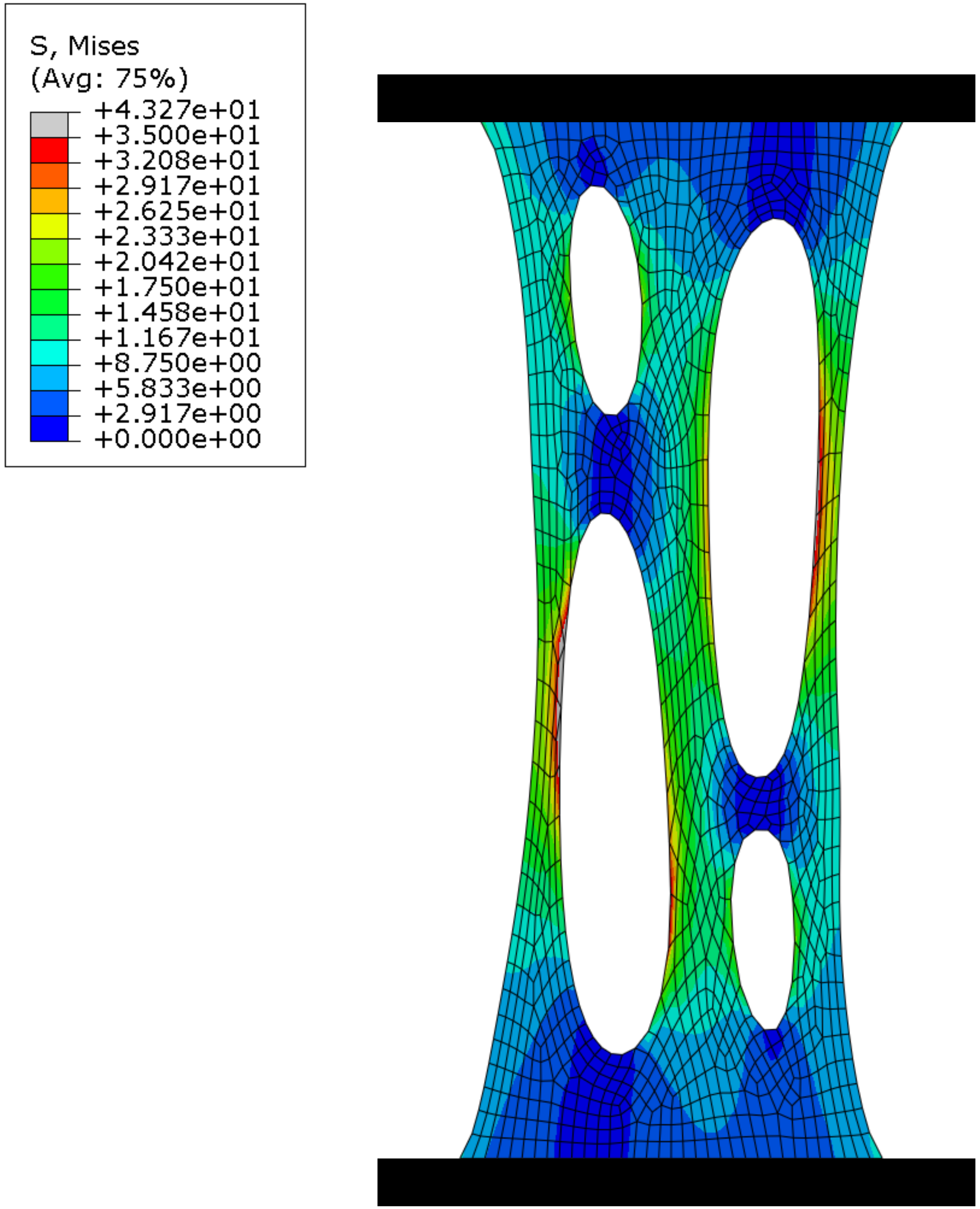

$$\nu_0 = 0.4999999$$

$$\frac{K_0}{G_0} = 5\,000\,000$$

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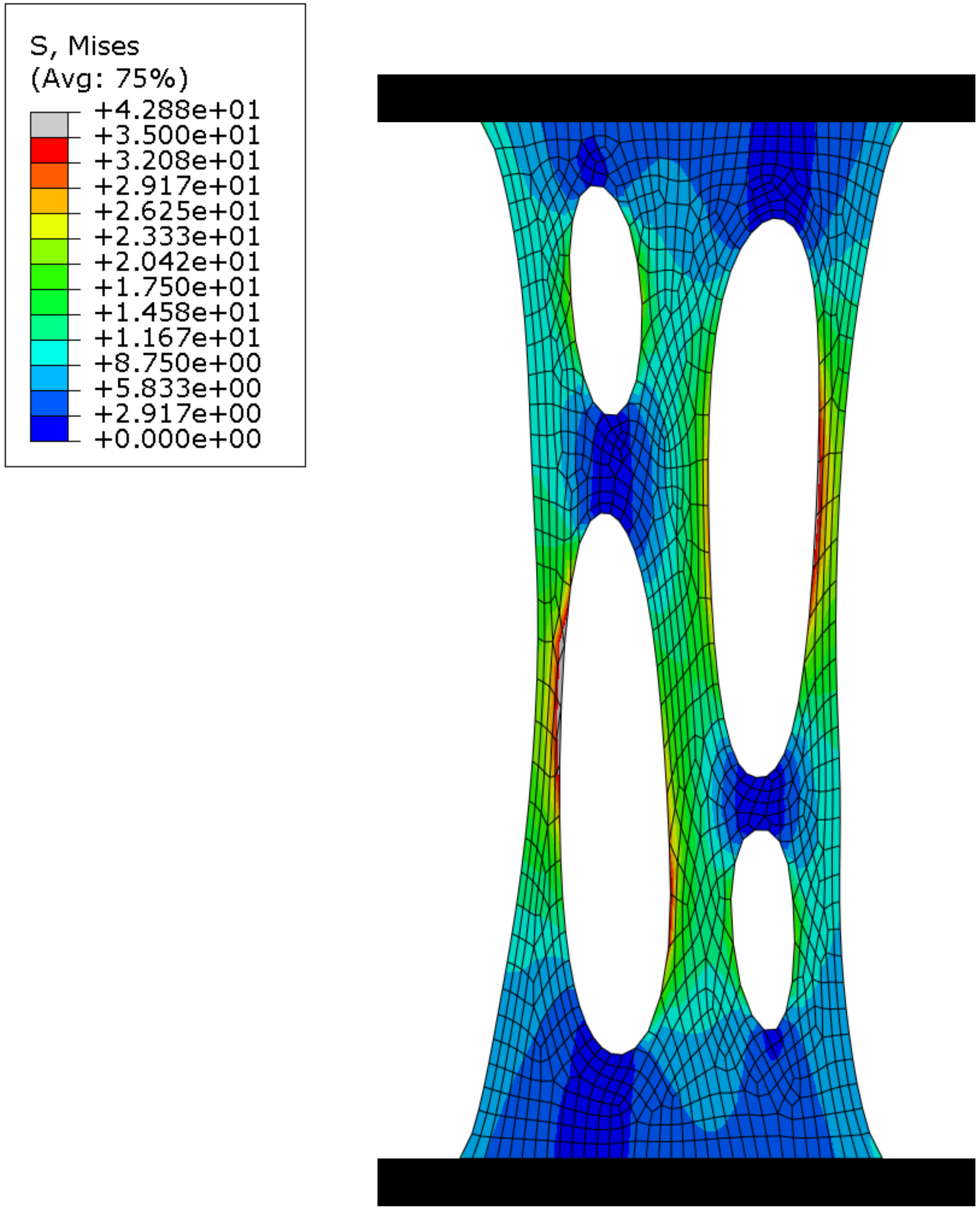

$$\nu_0 = 0.4995$$

$$\frac{K_0}{G_0} = 1\,000$$

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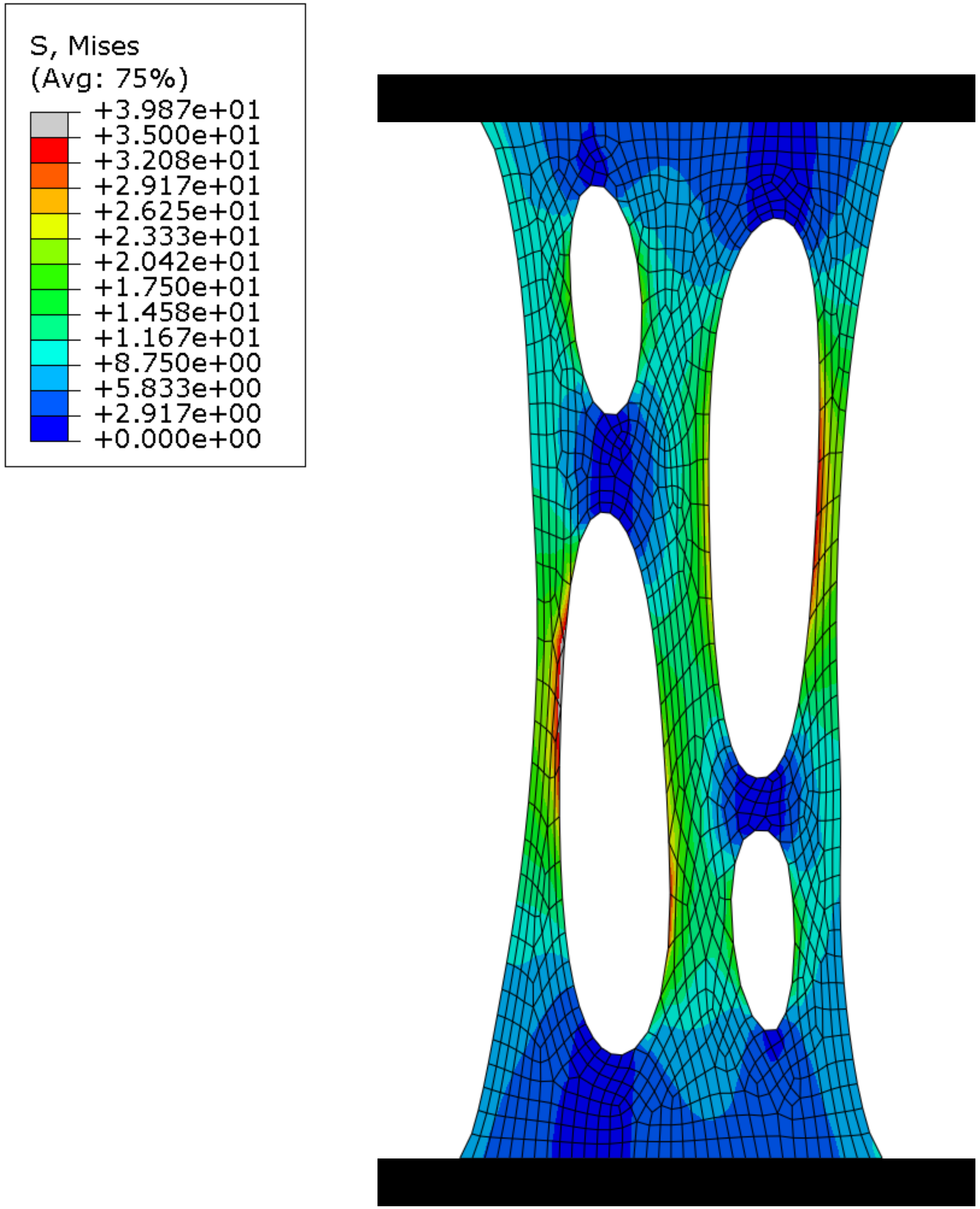

$$\nu_0 = 0.495$$

$$\frac{K_0}{G_0} = 100$$

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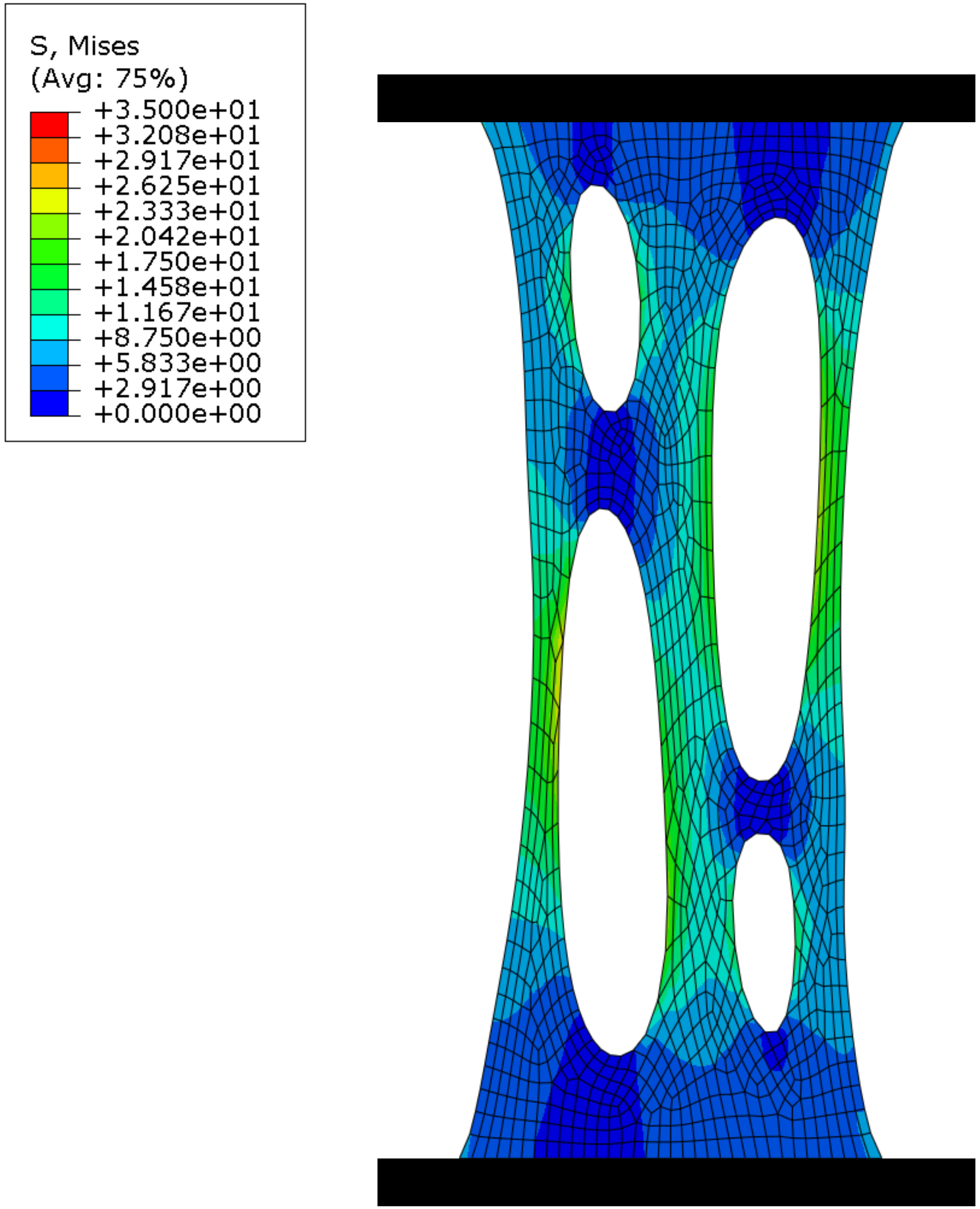

$$\nu_0 = 0.452$$

$$\frac{K_0}{G_0} = 10$$

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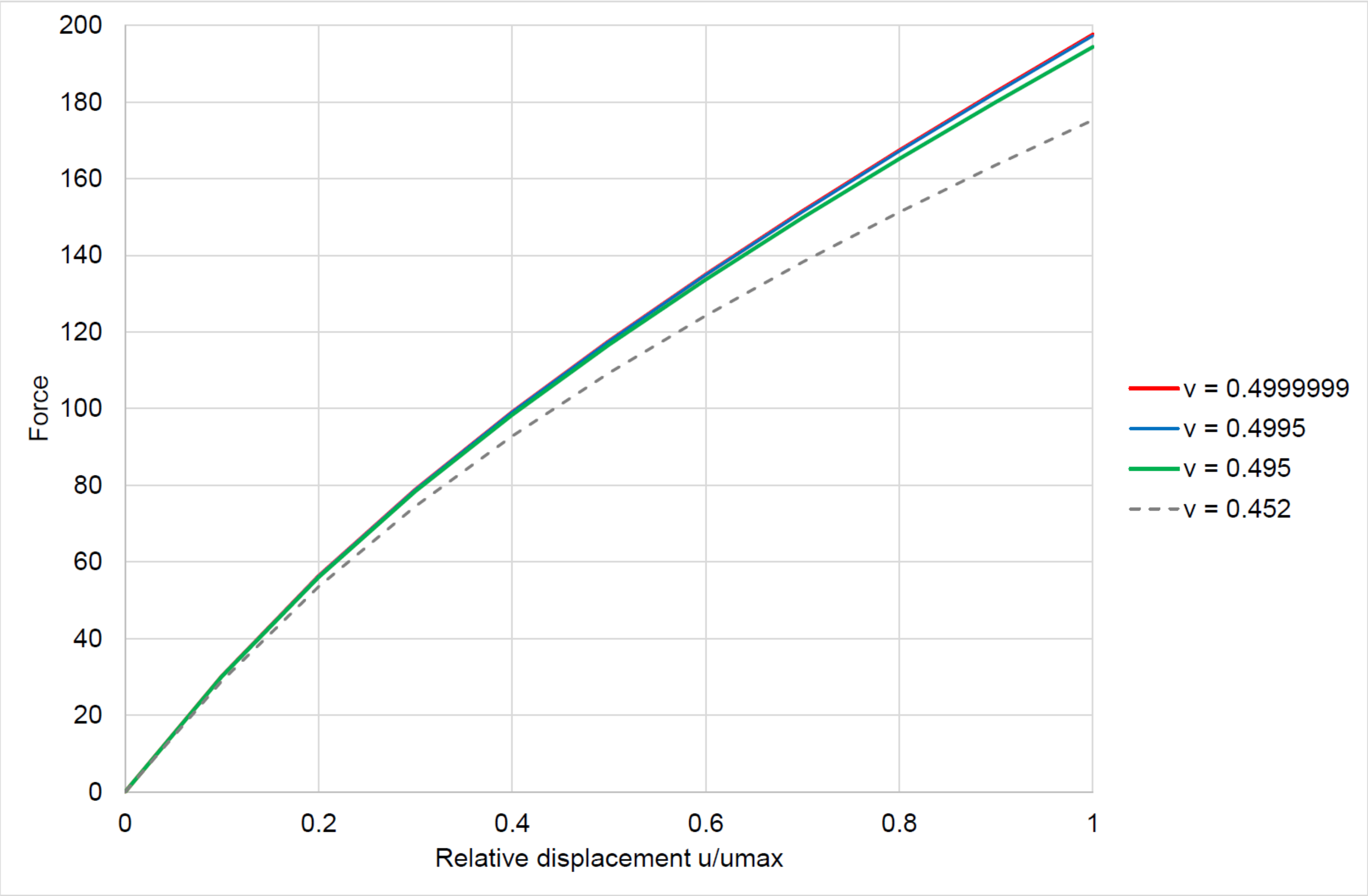

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Large volumetric compressibility

ABAQUS: Hyperfoam

$$\boxed{W = \sum_{i=1}^{N} \frac{2\bar{\mu}_i}{\alpha_i^2} \left(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3 + \frac{1}{\beta_i} \left(J^{-\alpha_i \beta_i} - 1 \right) \right)}.$$

ANSYS: Ogden compressible foam

$$\boxed{W = \sum_{i=1}^{N} \frac{\tilde{\mu}_i}{\alpha_i} \left(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3 + \frac{1}{\beta_i} \left(J^{-\alpha_i \beta_i} - 1 \right) \right)}.$$

$$\tilde{\mu}_i = \frac{2\bar{\mu}_i}{\alpha_i}.$$

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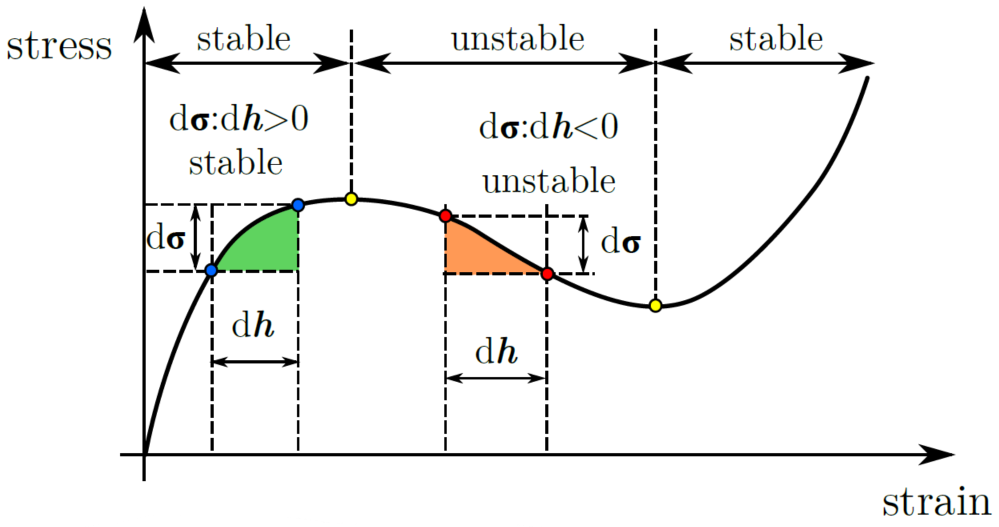
Drucker stability check

One way to examine if a model with a given set of material parameters is stable is to check is Drucker's stability. According to Drucker's postulate, an infinitesimal increment in the applied loading must result in increasing strain energy density, or in other words increasing strain must lead to increasing stresses and no strain softening is allowed during a loading mode. Using the notation in Abaqus, the Drucker stability criterion requires that the change of the Kirchhoff stress $\mathrm{d}\boldsymbol{\tau}$ following an infinitesimal change in the spatial logarithmic strain (or spatial Hencky strain) $\mathrm{d}\boldsymbol{h}$ must hold

$$\mathrm{d}\boldsymbol{\tau} : \mathrm{d}\boldsymbol{h} > 0,$$

where $\boldsymbol{h} = \ln \boldsymbol{V}$. The concept of Drucker stability is illustrated in Figure 5.1.



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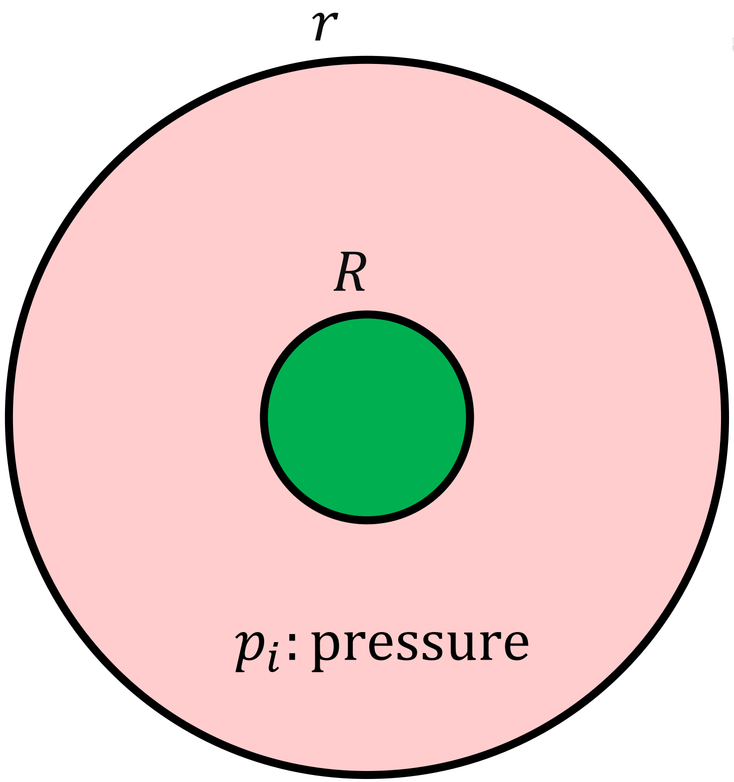
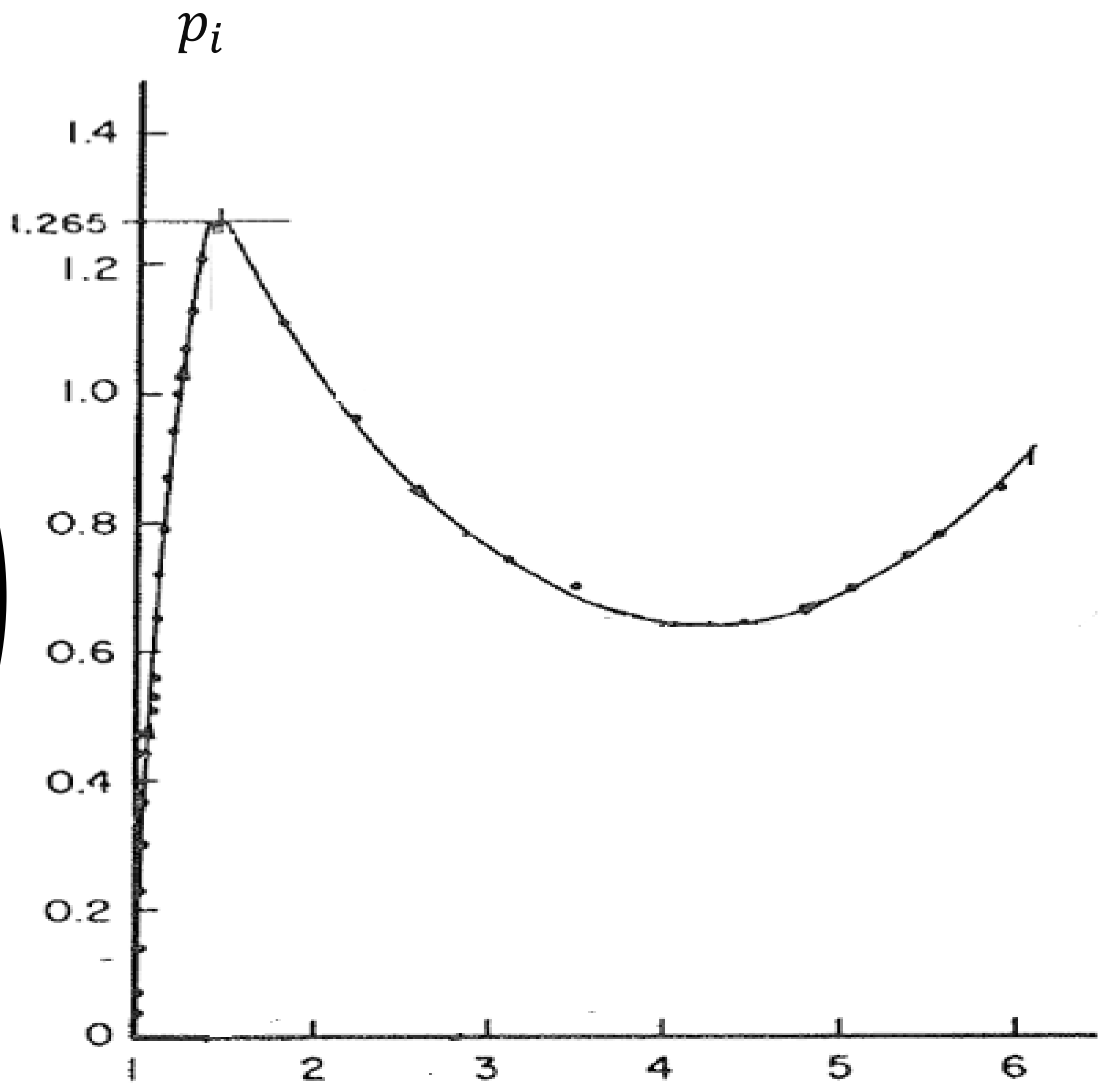
Example: balloon inflation

Experiment





Circumferential stretch $\lambda = r/R$

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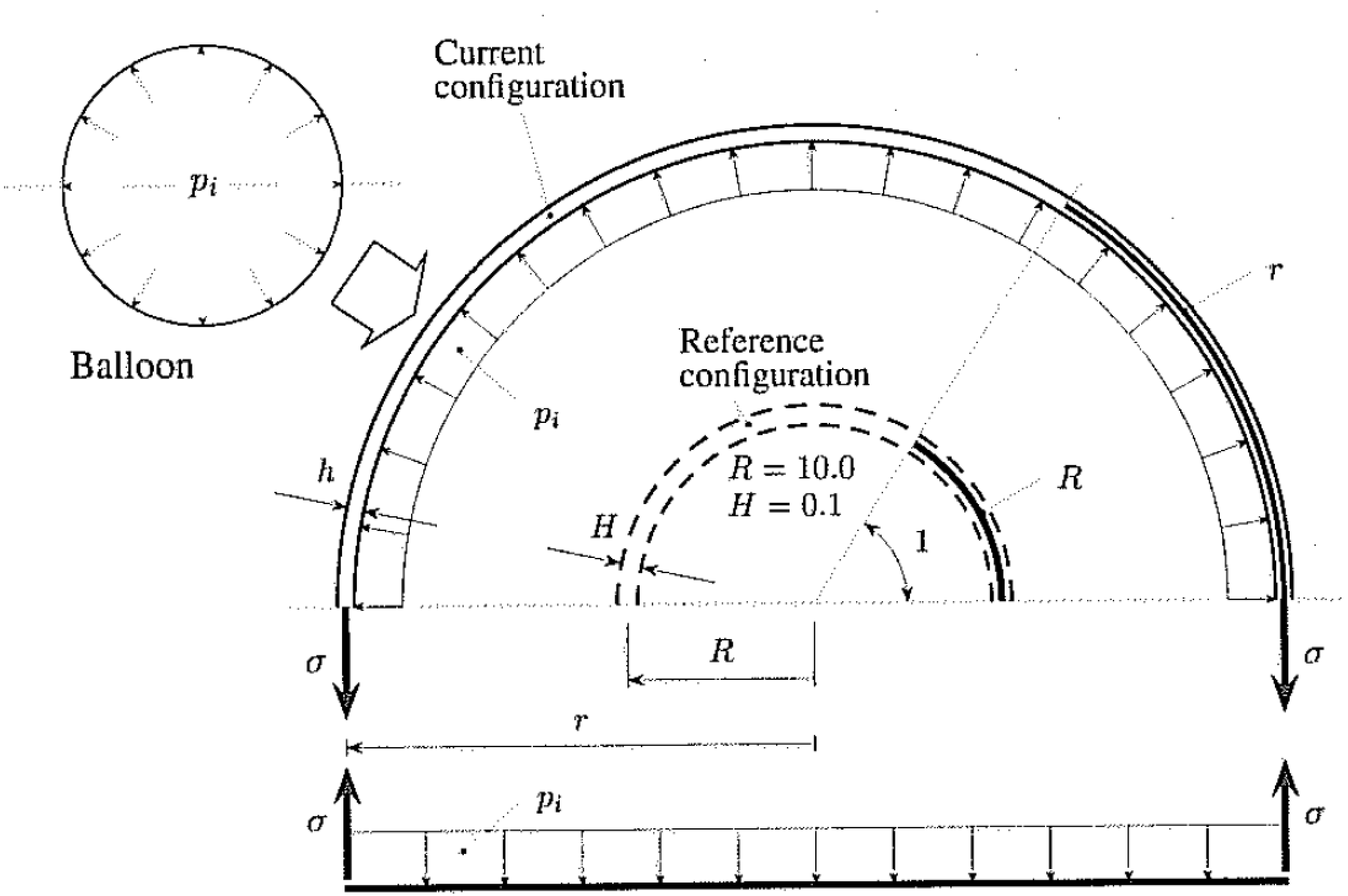
Example: balloon inflation

Equilibrium: $2r\pi h\sigma = r^2\pi p_i \rightarrow p_i = 2\frac{h}{r}\sigma$



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Example: balloon inflation

Incompressibility:

$$4R^2\pi H = 4r^2\pi h \;\rightarrow h = \frac{H}{(r/R)^2} = \frac{H}{\lambda^2}$$

Thus:

$$p_i = 2\frac{1}{r}\frac{H}{\lambda^2}\sigma = 2\frac{H}{R}\frac{1}{\lambda^3}\sigma$$

Depends on the particular hyperelastic model

$$p_i = f(\text{geom}, \lambda, mat.par.)$$

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Example: balloon inflation

Ogden's solution for equibiaxial case:

$$\sigma = \sum_{k=1}^{K} \frac{2\mu_k}{\alpha_k} \left(\lambda^{\alpha_k} - \lambda^{-2\alpha_k}\right)$$

$$p_i = 2\frac{H}{R}\frac{1}{\lambda^3} \sum_{k=1}^{K} \frac{2\mu_k}{\alpha_k} \left(\lambda^{\alpha_k} - \lambda^{-2\alpha_k}\right)$$

$$\boxed{p_i = 2\frac{H}{R} \sum_{k=1}^{K} \frac{2\mu_k}{\alpha_k} \left(\lambda^{\alpha_k - 3} - \lambda^{-2\alpha_k - 3}\right)}$$

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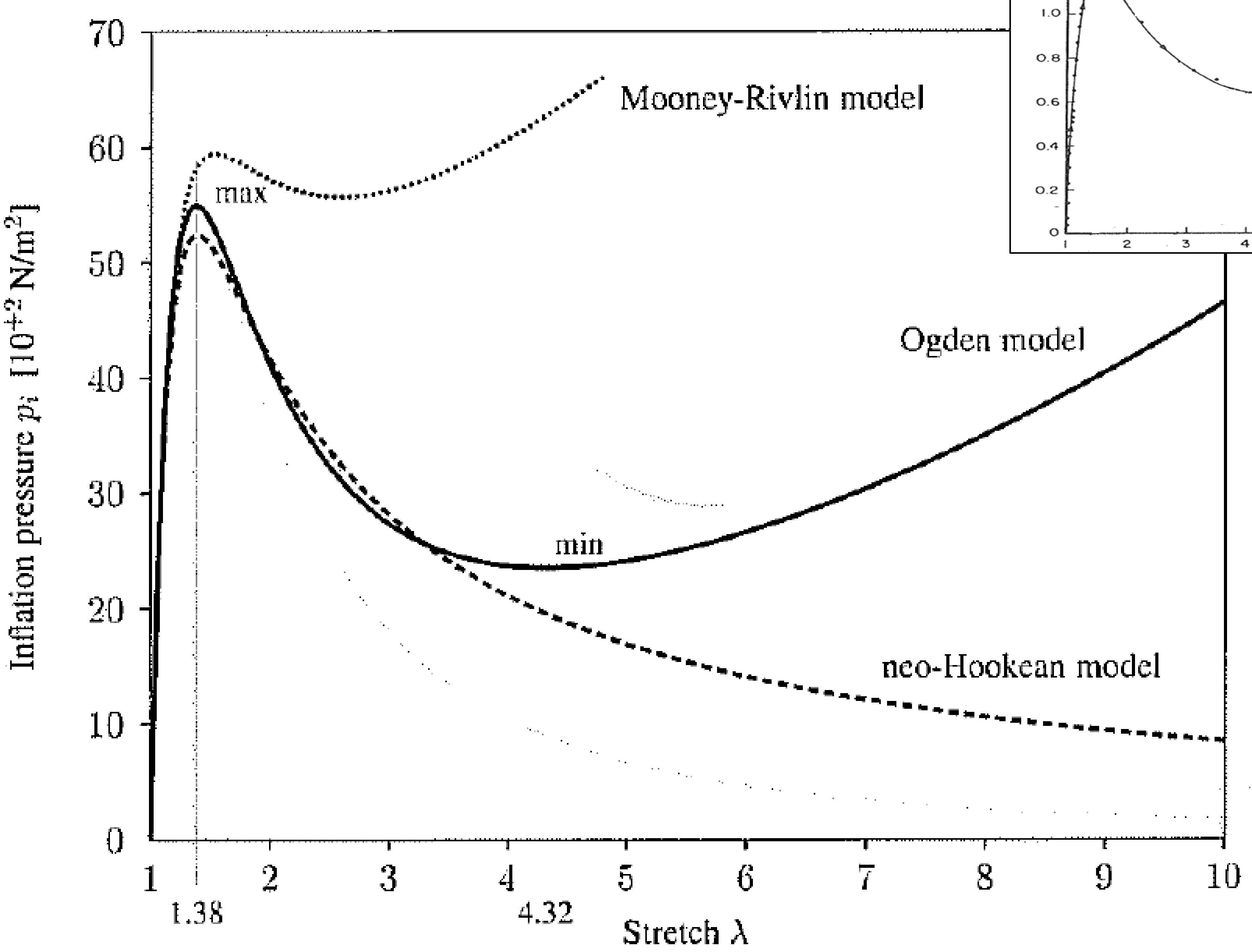
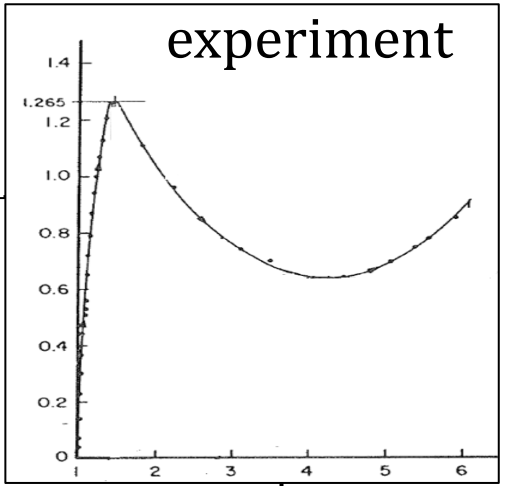
Example: balloon inflation





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