# Finite elastic-plastic deformations (BMEGEMMDKPL) II. Homework

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### Loading

We have an L mm  $\times$  L mm  $\times$  L mm brick element (L=1 mm), which node numbering is shown on Figure 1.

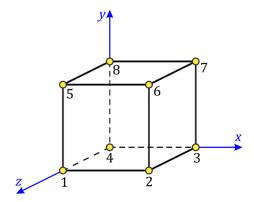


Figure 1: Nodal layout of the brick element.

We have a prescribed displacements on the upper nodes, defined by displacement vector as

$$[U_1] = [U_2] = [U_3] = [U_4] = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T,$$
 (1)

$$[U_5] = [U_6] = [U_7] = [U_8] = \begin{bmatrix} u_x & u_y & 0 \end{bmatrix}^T,$$
 (2)

where the time evolution of  $u_x$  and  $u_y$  and their time derivatives are shown on Figure 2.

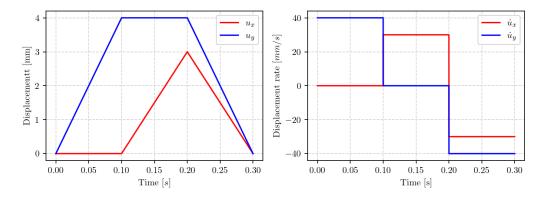


Figure 2: The parameters of the loading

From the displacement vectors we can determine the deformations gradient regarding to that motion.

$$[F] = \begin{bmatrix} 1 & u_x/L & 0 \\ 0 & 1 + u_y/L & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
 (3)

For the further calculations we will need the velocity gradiend, which can be calculated from the deformation gradient as

$$[\boldsymbol{l}] = [\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}] = \begin{bmatrix} 0 & \dot{u_x} & 0 \\ 0 & \dot{u_y} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\frac{u_x}{1+u_y} & 0 \\ 0 & \frac{1}{1+u_y} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{1+u_y} \begin{bmatrix} 0 & \dot{u_x} & 0 \\ 0 & \dot{u_y} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
(4)

The rate of deformation (d) is the symmetric and the spin tensor (w) is the skew symmetric part of the velocity gradient:

$$\boldsymbol{d} = \frac{1}{2}(\boldsymbol{l} + \boldsymbol{l}^T) \quad \& \quad \boldsymbol{w} = \frac{1}{2}(\boldsymbol{l} - \boldsymbol{l}^T). \tag{5}$$

#### Material behaviour

This element has an elastic-plastic behaviour. We have finite deformations, therefore we need to use hypoelastic-plastic modelling approach, which is based on the additive decomposition of the rate of deformation tensor into elastic and plastic part

$$\boldsymbol{d} = \boldsymbol{d}^e + \boldsymbol{d}^p. \tag{6}$$

Following the ABAQUS formulation, the elastic behaviour can be defined by the Hooke's law as follows

$$\overset{\circ}{\boldsymbol{\sigma}} = \frac{E}{1+\nu} \left( \boldsymbol{d}^e + \frac{\nu}{1-2\nu} (\operatorname{tr} \boldsymbol{d}^e) \boldsymbol{I} \right) := \mathbb{D} : \boldsymbol{d}^e, \tag{7}$$

where the  $\overset{\circ}{\sigma}$  denotes the Jaumann rate of the Cauchy stress tensor. The discrete form of the Hooke's law in the rotated coordinate system looks like

$$\tilde{\boldsymbol{\sigma}}_{n+1} = \tilde{\boldsymbol{\sigma}}_n + \mathbb{D}\Delta \tilde{\boldsymbol{\varepsilon}}^e \tag{8}$$

To be able to rotate the quantities back and forth, we need to determine the Q rotation tensor in each step, which can be calculated from the w spin tensor as

$$\dot{\mathbf{Q}} = \mathbf{w}\mathbf{Q},\tag{9}$$

where we can conclude that we will need time integration to determine the values. We will use the Hughes-Winget algorithm as it is required, which is defined as

$$\mathbf{Q}_{n+1} = \left(\mathbf{I} - \frac{1}{2}\mathbf{w}_{n+1/2}\Delta t\right)^{-1} \left(\mathbf{I} + \frac{1}{2}\mathbf{w}_{n+1/2}\Delta t\right) \mathbf{Q}_n. \tag{10}$$

The values of the elastic material parameters are the followings

$$E = 200 \text{ GPa} \quad \& \quad \nu = 0.3$$
 (11)

Furthermore the characteristic of the Yield stress is defined by the following Swift model:

$$Y(\lambda) = a(\varepsilon_0 + \lambda)^n$$
, where  $a = 400 \text{ MPa}$ ,  $\varepsilon_0 = 0.05$ ,  $n = 0.25$ . (12)

#### Solution

My implementation of the radial-retirn algorithm for this hypoelastic-plastic model in Python language is the following:

```
Q = np.tile(I, (N, 1, 1))
  sig_tilde = np.tile(0, (N, 1, 1))
  sig = np.tile(0, (N, 1, 1))
  Y = np.ones(N) * Y_F(0)
   _lambda = np.zeros(N)
6
   for n in range(N-1):
7
       dt = t[n+1] - t[n]
       w_{mid} = (w[n+1] + w[n])/2
       d_{mid} = (d[n+1] + d[n])/2
       # Hughes-Winget
       Q[n+1] = inv(I - 1/2*w_mid*dt) @ (I + 1/2*w_mid*dt) @ Q[n]
13
       Q_{mid} = Q[n+1]
14
       deps_tilde = Q_mid.T @ d_mid* dt @ Q_mid
16
17
       # Step 3: trial stress
18
       sig_trial_tilde = sig_tilde[n] + D(deps_tilde)
19
       s_trial_tilde = dev(sig_trial_tilde)
20
       # Step 4: trial yield function
22
       F_{trial} = (3/2)**(1/2) * norm(s_{trial_tilde}) - Y[n]
23
24
25
       if F_trial <= 0:</pre>
           # Step 6: Elastic increment
26
           sig_tilde[n+1] = sig_trial_tilde
27
           Y[n+1] = Y[n]
28
           _{lambda[n+1]} = _{lambda[n]}
29
       else:
30
           # Step 7: Accumulated plastic strain increment
31
           def eq(d_lambda):
                ret = (3/2)**(1/2) * norm(s_trial_tilde)
                ret -= 3*G*d_lambda
34
                ret -= Y_F(_lambda[n]+d_lambda)
35
                return ret
36
           d_{lambda} = (root(eq, 1e-8).x)[0]
37
           # Step 8: Plastic strain increment and yield stress update
39
           deps_p_tilde = (3/2)**(1/2) * d_lambda * s_trial_tilde /
40
               norm(s_trial_tilde)
           _{lambda[n+1]} = _{lambda[n]} + d_{lambda}
41
           Y[n+1] = Y_F(_lambda[n+1])
42
43
           # Step 9: Update the rotated stress
           sig_tilde[n+1] = sig_trial_tilde - D(deps_p_tilde)
45
46
       # Step 10: Compute the stress tensor
47
       sig[n+1] = Q[n+1] @ sig_tilde[n+1] @ Q[n+1].T
48
```

# 5 increments per step

The values of the plotted quantities are also attached as a CSV file.

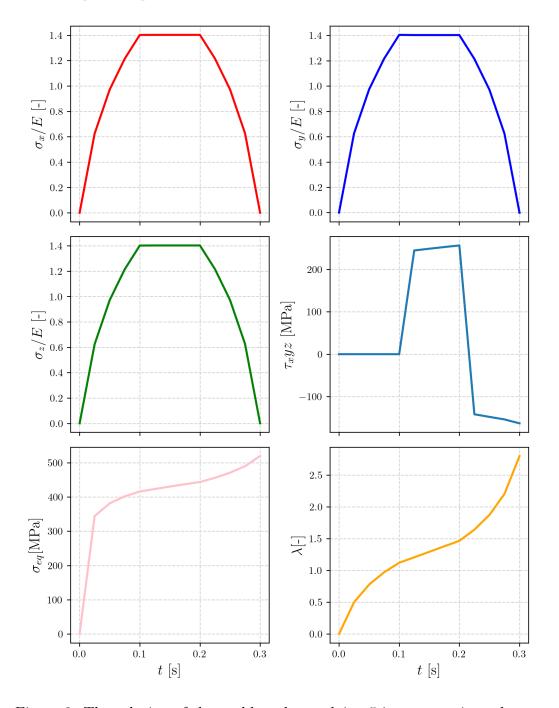


Figure 3: The solution of the problem, by applying 5 increments in each step.

# 1000 increments per step

The values of the plotted quantities are also attached as a CSV file.

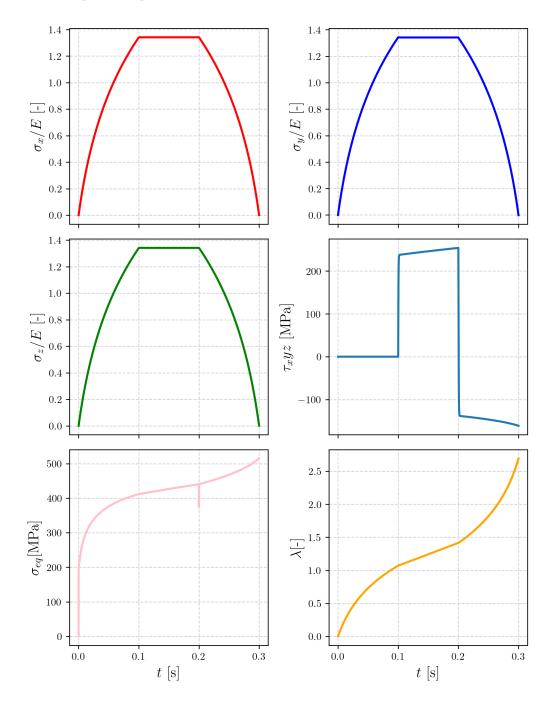


Figure 4: The solution of the problem, by applying 1000 increments in each step.

# References

You can find the detailed code on  $\underline{\text{GitHub}}.$