Basic concepts:

Definition 1 *$(\Omega, \mathcal{A})$ is a measurable space, if Ω nonempty, $\mathcal{A}$ is a sigma-algebra over Ω. That is*

- $\Omega \in \mathcal{A}$
- $A \in \mathcal{A}$ *then* $\Omega \setminus A \in \mathcal{A}$
- $A_1, \ldots, A_n, \ldots \in \mathcal{A}$ *then* $\bigcup_n A_n \in \mathcal{A}$.

Definition 2 *Let $P : \mathcal{A} \to [0, 1]$ a set function such that*

- $P(\Omega) = 1$
- *for arbitrary (A_n) disjoint sets of $\mathcal{A}$: $P(\overset{*}{\bigcup} A_n) = \sum_n P(A_n)$*

then P is called a probability measure.

Definition 3 *The triplet $(\Omega, \mathcal{A}, P)$ is called a probability (Kolmogorov) space if $\Omega \neq \emptyset$, $\mathcal{A}$ is a sigma-algebra over Ω and P is a probability measure.*

- the possible outcome of an experiment: elemental event $\omega \in \Omega$
- sample space Ω, consists of ω's
- subsets of Ω (which are elements of $\mathcal{A}$) are called events $(A, B, C, \ldots)$
- an A event occurs, if any of the containing ω's occurs

Example 1 *Dice throwing: $\Omega = \{1, 2, 3, 4, 5, 6\}$. If A means that we get an even number, then $A = \{2, 4, 6\}$.*
Coin tossing, twice: $\Omega = \{HH, HT, TH, TT\}$. $A = \{HT, HH\}$ event means that the first is heads.
Toss a coin as long as we get heads: $\Omega = \{H, TH, TTH, TTTH, \ldots\}$.

Events:

- special events: Ω certain event, $\varnothing$ impossible event
- operations with events (usual set operations): e.g. $A \cup B$ (A or B occurs, or both of them), $A \cap B$ (A and B occur), $\overline{A}$ (opposite of A)
- $A \backslash B = A \cap \overline{B}$; $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (de Morgan); $\overline{\overline{A}} = A$ (examples: dice, coin)

Probability $P(A)$: nonnegative for all A; for exclusive events $(A \cap B = 0)$ $P(A \cup B) = P(A) + P(B)$ (additivity); $P(\Omega) = 1$; $(\Omega, \mathcal{A}, P)$ probability space.

Properties:

1. additivity for n events: $A_1, \ldots, A_n$ pairwise exclusive events: $P(A_1 \cup \ldots \cup A_n) = P(A_1) + \ldots + P(A_n)$. Proof: with induction, use $P(\varnothing) = 0$, $\Omega = \Omega \cup \varnothing$ and additivity.
2. $P(A \backslash B) = P(A) - P(A \cap B)$. Proof: $A = (A \cap B) \cup (A \backslash B)$ decomposition and additivity

Definition 4 *Discrete probability field: $\Omega = \{\omega_1, \omega_2, \ldots\}$ (finite or countable infinite), $\mathcal{A} = 2^{\Omega}$. Let denote $p_i = P(\omega_i)$. $\sum p_i = 1$, $P(A) = \sum\limits_{i:\omega_i \in A} p_i$.*

The classical probability models

Definition 5 (The classical probability space) *$\{\Omega, \mathcal{A}, P\}$. Here Ω is discrete and finite. If $A \in \mathcal{A}$ define $P(A) = \frac{|A|}{|\Omega|}$.*

1. **Maxwell-Boltzmann statistics**. Suppose there is n object and N boxes and suppose that all boxes and object are different. The number of ways we can distribute the object in the boxes is N^n. Here
$$\Omega = \{(a_1, \ldots, a_n), 1 \le a_j \le N, j = 1, \ldots, n\}$$
and $|\Omega| = N^n$. Now let $A_{k,i} = \{$ the i^{th} box contains k objects$\}$. What is $P(A)$?
$$P(A_{k,i}) = \frac{\binom{n}{k}(N-1)^{n-k}}{N^n}$$
Note that the probability is independent of index i. Translating this question into the language of our model : $A_{k,i} = \bigcup_{1 \le j_1 < j_2 < \ldots < j_k \le n} \{(a_1, \ldots, a_n), a_{j_1} = \ldots = a_{j_k} = i$ and the others differ $\}$. Note that
$$|\{(a_1, \ldots, a_n), a_{j_1} = \ldots = a_{j_k} = i \text{ and the others differ } \}| = (N-1)^{n-k}$$
and the sets in the union are disjoint, hence
$$|\bigcup_{1 \le j_1 < j_2 < \ldots < j_k \le n} \{(a_1, \ldots, a_n), a_{j_1} = \ldots = a_{j_k} = i \text{ and the others differ } \}| =$$
$$= \sum_{1 \le j_1 < j_2 < \ldots < j_k \le n} (N-1)^{n-k} = \binom{n}{k}(N-1)^{n-k}.$$

2. **Bose-Einstein statistics**. Suppose that in the previous example the object are indistinguishable. This means that given two setups where box i has exactly the same number of objects in the first setup than in the second setup, the setups are the same. This gives as an equivalence relation on Ω (HW:prove it!). Denote by $\tilde{\Omega} = \Omega/\sim$. Thus
$$\tilde{\Omega} = \{(b_1, \ldots, b_N) : 0 \le b_j \le n \ \sum_{j=1}^{N} b_j\}$$
and $|\tilde{\Omega}| = \binom{N+n-1}{N-1}$, hence
$$P(A_{i,k}) = \frac{\binom{N-1+(n-k)-1}{N-2}}{\binom{N+n-1}{N-1}}$$