1. Compute the expectation of a standard normal variable.
2. Compute the variance of a standard normal variable.
3. Define $\xi = \sigma\zeta + \mu$, where ζ is standard normal variable. Compute the expectation, variance and find the density function of ξ.
4. Define $\xi = \sigma\zeta + \mu$, where ζ is standard normal variable. Compute $\mathbb{B}[e^{\xi}]$.
5. Let W_t be a Brownian Motion. Prove that the following processes are also BM:

$$B_t = -W_t,$$

$$B_t = W_{t+h} - W_h,$$

$$B_t = \frac{1}{\sqrt{c}} W_{ct},$$

$$B_t = t \cdot W_{\frac{1}{t}}.$$

6. Let $\Omega = [0,1], B_i = [\frac{i}{4}, \frac{i+1}{4}], \mathcal{G} = \sigma(B_0, B_1, B_2, B_3), \mathcal{F} = \sigma([0, \frac{1}{2}], (\frac{1}{2}, 1])$ and $\xi(\omega) = \omega$. Compute $\mathbb{E}[\xi|\mathcal{F}], \mathbb{E}[\xi|\mathcal{G}], \mathbb{E}[\mathbb{E}[\xi|\mathcal{F}]|\mathcal{G}]$ and $\mathbb{E}[\mathbb{E}[\xi|\mathcal{G}]|\mathcal{F}]$.
7. Let $f_{\xi,\eta}(x,y) = x + y$, if $x, y \in [0,1]$ an 0 otherwise. Compute $\mathbb{E}[\xi|\eta]$.
8. Prove that $(X_n)_{n\geq 0}$ is a martingale, where $\mathcal{F}_n$ is the natural filtration, $X_0 = 1$ and X_n is $2 \cdot X_n$ with probability $\frac{1}{2}$ and 0 with probability $\frac{1}{2}$.
9. Let $(X_n)_{n\geq 0}$ be a sequence of independent, identically distributed $Bernoulli(p)$ random variables and let $S_n = \sum_{j=0}^{n} X_j$. Prove that

$$Z_n = (\frac{q}{p})^{2S_n - n}$$

is a martingale with respect to the natural filtration.

10. Let $\Omega = [0,1]$ and

$$I_n^+ = [1 - \frac{1}{2^n}, 1 - \frac{1}{2^n} + \frac{1}{2^{n+2}}],$$

$$I_n^- = [1 - \frac{1}{2^n} + \frac{1}{2^{n+2}}, 1 - \frac{1}{2^n} + \frac{1}{2^{n+1}}].$$

Let ξ_n be 1 on I_n^+, -1 on I_n^- and 0 otherwise. Let $\mathcal{G}_n$ be the natural filtration of the sequence. Prove that $M_n = \sum_{i=1}^{n} \xi_i$, however the martingale differences are not independent.

11. Prove that the Brownian Motion is almost surely not differentiable at 0.
12. Simulate the Brownian Motion on a computer!
13. Compute the following stochastic integral, using the definition:

$$\int_0^t 1 ds.$$

14. Compute the following stochastic integral:

$$\int_0^t W_s ds.$$

(Hint: use that $W_{t_k} = \frac{1}{2}(W_{t_{k+1}} + W_{t_k}) - \frac{1}{2}(W_{t_{k+1}} - W_{t_k})$.)

15. Compute the following stochastic integral for $t = \frac{1}{2}, \frac{2}{3}, 1$, using the definition:

$$\int_0^t f_s ds,$$

where $f_t = 5 \cdot \mathbb{1}_{[1,\frac{1}{3}]} + W_{\frac{1}{3}} \cdot \mathbb{1}_{(\frac{1}{3},\frac{2}{3}]} + (W_{\frac{1}{2}})^2 \cdot \mathbb{1}_{(\frac{2}{3},1]}$.

16. Prove that

$$\int_0^\infty e^{-nt} dW_t \xrightarrow{L^p} 0.$$

(Hint: use Ito isometry.)

17. Show that $Y_t = t^2 W_t^3$ satisfies the SDE

$$dY_t = (2\frac{Y_t}{t} + 3(t^4 Y_t)^{\frac{1}{3}})dt + 3(tY_t)^{\frac{2}{3}} dW_t.$$

18. Define $B_t = W_t - tW_1$ for $t \in [0,1]$, where W_t is the standard Brownian Motion. Then $B_0 = B_1 = 0$. Compute the covariance $cov(B_t, B_s)$. (Also look up the Brownian Bridge!)

19. Let X_t be an Ito process (i.e. $dX_t = a_t dt + b_t dW_t$), and suppose that $X_t > 0$. Let $Z_t = ln(X_t)$. Compute the stochastic differential dZ_t.

20. Find the equation solved by the process $x_t = sin(W_t)$. (Hint: use Ito formula to get dX_t and then try to express the terms with X_t.)

21. Find the equation solved by X^2, where X is the O-U process $dX_t = \mu X_t dt + \sigma dW_t$. (Hint: use the product rule!)

22. Consider the stochastic process

$$Y_t = exp((t+1)W_t - \frac{t^3}{6} - \frac{t^2}{2}/t), t > 0,$$

where W_t is a Brownian Motion. Compute the stochastic differential dY_t and prove that Y_t satisfies the stochastic differential equation

$$dY_t = Y_t \frac{(lnY_t) + \frac{t^3}{6} + \frac{t^2}{2} + \frac{t}{2}/\frac{1}{2}}{t+1}.$$

23. Consider the stochastic process $X_t = e^{W_t + \frac{t}{2}}$, where W_t is a Brownian motion. Show that X_t satisfies the SDE $dX_t = X_t dt + X_t dW_t, X_0 = 1$. Consider the function $F(x)$ for positive numbers, and check that the derivatives satisfy the following equations:

$$F(x)(F(x) - 1) = xF'(x)$$

$$F^2(x)(F(x) - 1) = xF'(x) + \frac{x^2}{2}F''(x)$$

Calculate the stochastic differential dY_t for $Y_t = F(X_t)$ and conclude that Y_t satisfies the SDE

$$dY_t = Y_t^2(Y_t - 1)dt + Y_t(Y_t - 1)dW_t, Y_0 = \frac{1}{2}.$$

The End.