Discrete random variables

Random variables: in many cases, we are interested in a quantifying result. A random variable is a measurable function $X : \Omega \to \mathbb{R}$.

Example 1 *Throwing a die:*

1. *X means the number of pips (the result of our throw), $\Omega = \{1, 2, 3, 4, 5, 6\}$, $X(i) = i$ (the range is 1,2,3,4,5,6).*
2. *X is the number of the first throw, when we get a 6. $\Omega = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \times \ldots$, the range of X is $\{1, 2, \ldots\}$.*

Definition 1 (Discrete random variables) *X is a discrete random variable, if the range is at most countably infinite.*

Example 2 *Degenerate distribution: $X(\omega_i) = c$ for all ω_i, it follows that $P(X = c) = 1$.*

Example 3 *Bernoulli variable: X equals to 1, if an event A occurs (with probability p), and equals to 0 otherwise. Itt follows that $P(X = 1) = p$ and $P(X = 0) = 1 - p$.*

Example 4 *Binomial distribution: sampling with replacement, X means the number of substandard units in the n-element sample. $P(X = k) = \binom{n}{k} \left(\frac{M}{N}\right)^k \left(1 - \frac{M}{N}\right)^{n-k} \quad (k = 0, \ldots, n)$*

Proposition 1 *If X is a discrete random variable, and $f : \mathbb{R} \to \mathbb{R}$ function, then $f(X)$ is also a discrete random variable.*

Example 5 *X is the length of the manufactured product in inches. Suppose that $P(X = 18) = \ldots = P(X = 22) = \frac{1}{5}$, and the ideal length is 20 inches. The distribution of $|d - 20|$ (absolute difference) is $P(d = 0) = \frac{1}{5}$, $P(d = 1) = P(d = 2) = \frac{2}{5}$.*

In this case, $A_i = \{\omega : X(\omega) = x_i\}$ $(i = 1, 2, \ldots)$ is a complete system of events.

Definition 2 *The **conditional distribution of** X **given** A is $q_i = P(X = x_i|A)$. This is a real distribution, because $\sum\limits_i q_i = \sum\limits_i P(X = x_i|A) = \sum\limits_i \frac{P(X=x_i \cap A)}{P(A)} = 1$.*

Definition 3 (independence of random variables) *Random variables X and Y are independent, if the equality $P(\{X = x_i\} \cap \{Y = y_k\}) = P(\{X = x_i\})P(\{Y = y_k\})$ holds for all i and k. In other words, the complete systems of events corresponding to the variables X, Y are independent.*

Remark 1 *The degenerate random variable is independent of any random variable.*

Definition 4 (Expectation and variance)

- *Expectation of a random variable X: $EX = \sum_i x_i P(X = x_i)$*
- *Variance of a random variable X: $D^2X = EX^2 - E^2X = \sum_i x_i^2 P(X = x_i) - (\sum_i x_i P(X = x_i))^2$*

Definition 5 (Binomial distribution.) *Another realization of sampling with replacement: independent experiments under the same conditions. Let event A mean that one single experiment is successful, and $P(A) = p$. The number of experiments we make is n (fixed). X random variable means the number of successful experiments. In this case, the distribution of X is Binomial(n, p). X can be written as $X_1 + X_2 + \ldots + X_n$, where X_i is the indicator of event A in the ith experiment, i.e. a Bernoulli random variable. They are independent of each other.*

Definition 6 (Pascal distribution.) *Suppose independent experiments under the same conditions. The trials continue until event A occurs, where $P(A) = p$. Let X denote the number of the first successful experiment. In this case, the distribution can be described as $p_k = P(X = k) = p(1-p)^{k-1} \quad (k = 1, 2, \ldots)$*

Remark 2 *It is easy to proof that the distribution we got above is probability distribution, because $p_1 + p_2 + \ldots = 1$.*

Definition 7 (Poisson distribution.) *If a random variable X has the following property, $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ ($k = 0, 1, \ldots$; $\lambda > 0$ parameter), then the distribution of X is Poisson distribution.*

Proposition 2 *Suppose that n, p are the parameters of the binomial distribution. If $n \to \infty$ on condition that $np \to \lambda$, then the limit distribution exists and it is the Poisson(λ) distribution.*

An important application is the Poisson process, which counts the number of events occur in a given time interval. In other words, the number of points (events) in interval $[a, b)$ denoted by the random variable $X_{a,b}$ has Poisson($\lambda(b - a)$) distribution, if the process is ((proof))

1. homogeneous: the distribution of $X_{a,a+t}$ depends only on t
2. increments are independent: $X_{a,b}$ and $X_{b,c}$ are independent variables if $a < b < c$
3. $0 < P(X_{a,b} = 0) < 1$

In practice, several processes can be modeled using Poisson processes. For example, the number of car accidents, number of storms, system failures.

Expected value of discrete random variables

In gambling, the exact wins are unknown in advance, but the average can be calculated. The mean amount of win gives us the answer, whether a game is fair or not, i.e, the expected amount of win is equal to the price of the lottery ticket, for example.

Example 6 *Suppose a dice throw game, where the prize is the numbers obtained, which has a mean value $\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \ldots + \frac{1}{6} \cdot 6 = \frac{21}{6}$. And if the dice is not regular, for example, on one side there is another 1 instead of 6, the expected value is $\frac{2}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \ldots + \frac{1}{6} \cdot 5 = \frac{16}{6}$.*

Definition 8 *Suppose a discrete random variable given by the probabilities (distribution) $p_i = P(X = x_i)$ $i = 1, 2, \ldots$, and the range of X is $x_1, x_2, \ldots$. The expected value of random variable X is defined as $E(X) = p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots$ (if the series is absolutely convergent).*

Example 7 *The expected value of the degenerate distribution is $E(X) = c \cdot P(X = c) = c$.*

Example 8 *The expected value of a Bernoulli random variable X is $E(X) = 1 \cdot P(X = 1) + 0 \cdot P(X = 0) = p$. Remember that X is the indicator of event A, which has a probability of p, and X equals to 1, if A occurs, and 0 if not.*

Example 9 *Let X be a random variable, which has a uniform distribution on numbers $x_1, \ldots, x_n$, i.e., they have the same probability $\frac{1}{n}$. $E(X) = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$, the mean value.*

Properties:

1. If $X \geq 0$ and $E(X) < \infty$, then $E(X) \geq 0$.
2. If $E(X)$ is finite, then $E(aX + b) = aE(X) + b$, where $a, b \in \mathbb{R}$.
3. If X is a nonnegative random variable with integer values, then $E(X) = P(X \geq 1) + P(X \geq 2) + \ldots$.

Proposition 3 *Let X, Y be arbitrary random variables with finite expected value. If so, $E(X + Y) = E(X) + E(Y)$. It can be proved by induction that $E(X_1 + \ldots + X_n) = E(X_1) + \ldots + E(X_n)$.*

It is possible to apply this statement to calculate the expected value of a binomial random variable easily. Let this variable be X. In another approach, this is the sum of n independent Bernoulli variables, $X = X_1 + \ldots + X_n$, where X_i is the indicator variable of the ith trial. So $E(X) = E(X_1) + \ldots + E(X_n) = nE(X_1) = np$.