Proposition 1 (Properties of a (probability) measure)

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,
- $A_n \subset A_{n+1}$, $A_n \in \mathcal{A}$ *for* $n \geq 1$, *then* $\lim_n P(A_n) = P(\bigcup_n A_n)$ *(lower semi continuity),*
- $A_n \supset A_{n+1}$, $A_n \in \mathcal{A}$ *for* $n \geq 1$, *then* $\lim_n P(A_n) = P(\bigcap_n A_n)$ *(upper semi continuity).*

Theorem 1 (Poincaré-formula) $P(A_1 \cup \ldots \cup A_n) = \sum\limits_{i=1}^{n} (-1)^{i+1} S_i^{(n)}$, *where* $S_i^{(n)} = \sum\limits_{1 \leq j_1 < \ldots < j_i \leq n} P(A_{j_1} \ldots A_{j_i})$

Applicatons: If each event and the i-element intersections of events have equal probability, then $P(A_1 \cup \ldots \cup A_n) = \sum\limits_{i=1}^{n} (-1)^{i+1} \binom{n}{i} P(A_1 \cap \ldots \cap A_i)$. In other words, $P(\overline{A_1} \cap \ldots \cap \overline{A_n}) = 1 - P(A_1 \cup \ldots \cup A_n) = \sum\limits_{i=0}^{n} (-1)^i S_i^{(n)}$ (let S_0 be 1).

Example 1 *If we throw a dice k times, how large is the probability that all the possible numbers will occur? Solution:* $A_i = \{$*we did not get number* $i\}$. $P(\overline{A_1} \cap \ldots \cap \overline{A_6}) = \sum\limits_{i=0}^{6} (-1)^i \binom{6}{i} P(A_1 \cap \ldots \cap A_i) = \sum\limits_{i=0}^{6} (-1)^i \binom{6}{i} \left(\frac{6-i}{6}\right)^k$

Definition 1 (conditional probability) $P(A|B) = \frac{P(AB)}{P(B)}$ *is the conditional probability of A given B. In other words, if B is known to occur, then what is the probability of A. We assume that $P(B) > 0$.*

Example 2 *Dice throwing,* $A = \{$*we get an even number*$\}$, $B = \{$*we get greater than 3*$\}$. $P(A|B) = \frac{2}{3}$.

Definition 2 (Finite or countably infinite partition of sample space) $A_1, A_2, \ldots, A_n$ ($n = \infty$ *also permitted), if they are pairwise disjoint events, measurable and the union is the entire sample space* Ω. *Remark:* $P(A_1) + \ldots + P(A_n) = 1$*; we mostly looking at finite partitions. (collectively exhaustive events)*

Theorem 2 (Law of total probability) *Let $B_1, B_2, \ldots$ be a countably infinite partition, each with positive probability, and A an arbitrary event.* $P(A) = \sum\limits_i P(A|B_i)P(B_i)$.

Proof: $A = (A \cap B_1) \cup (A \cap B_2) \cup \ldots$ is a partition to disjoint elements. $P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots$ and $P(A \cap B_i) = P(A|B_i)P(B_i)$ gives the solution. □

Example 3 *The probability of color-blindness for men is* 0.01 *and for women* 0.001. *How large is the probability that a randomly chosen person is color-blind?*

Theorem 3 (Bayes theorem) *Let $B_1, \ldots, B_n$ be a finite partition of sample space with positive probabilities and A an arbitrary event with also positive probability.* $P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum\limits_i P(A|B_i)P(B_i)}$.

Example 4 *How large is the probability that a randomly chosen person is a man, if he/she is color-blind?*

Definition 3 (Independence of events) *Events A and B are independent, if* $P(A \cap B) = P(A)P(B)$.

Example 5 *Suppose a card drawing from a pack of French playing cards. The following events: A : the card is Diamonds, B : the card is Ace. Are they independent?*

Properties:

1. if A and B are disjoint, they are only independent in the trivial case ($P(A) = P(B) = 0$)
2. if A and B are independent, their complements are also independent

Generalization: n events are independent, if $P(A_{i_1} \cap \ldots \cap A_{i_k}) = P(A_{i_1}) \cdot \ldots \cdot P(A_{i_k})$ is true for all $1 \leq i_1 < \ldots < i_k \leq n$ indices and $2 \leq k \leq n$.

Remark 1 *It is not enough to require the above mentioned product property for only $k = 2$. In this case we call it pairwise independence.*

The geometric probability space

Let $\Omega \subset \mathbb{R}^N$ for some $N \in \mathbb{N}$ and suppose we are given a "measure" (most of the time the Lebesgue measure or the standard volume) $\lambda : 2^{R^N} \supset\to [0, \infty]$ such that $\lambda(\Omega) < \infty$. Suppose $A \subset \Omega$ (measurable) then $P(A) = \frac{\lambda(A)}{\lambda(\Omega)}$.

Example 6 *A shooter is shooting at a circle shaped target with diameter $2R$, which is divided into 4 subsections the following way. The most inner section is a circle of radius circle $R/10$, the second section is a ring around the most inner circle with radii $R/10$ and $3R/10$, the third section is a ring with radii $3R/10$ and $6R/10$, the fourth section is also a ring with radii $6R/10$ and R. The section are 4,3,2,1 points worth going from inside out. Assuming that we have no apriori knowledge about the shooting skill of the shooter (meaning that the shot is distributed uniformly on the disc) what is the probability that he hits the most inner circle? What is the probability that he makes 8 point with 3 shots?*