

## Homework 2

### Instructions

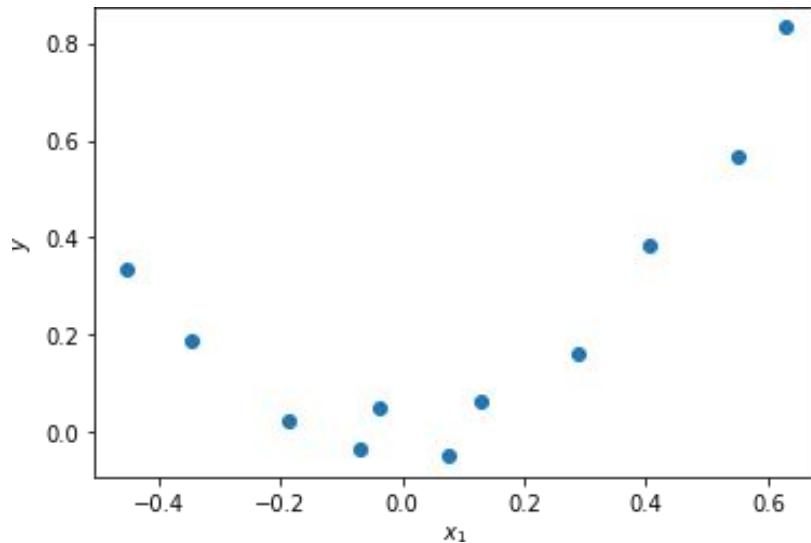
This homework contains **5** concepts and **8** programming questions. In MS word or a similar text editor, write down the problem number and your answer for each problem. Combine all answers for concept questions in a single PDF file. Export/print the Jupyter notebook as a PDF file including the code you implemented and the outputs of the program. **Make sure all plots and outputs are visible in the PDF.**

Combine all answers into a single PDF named andrewID\_hw2.pdf and submit it to Gradescope before the due date. Refer to the syllabus for late homework policy. Please assign each question a page by using the “Assign Questions and Pages” feature in Gradescope.

Here is a breakdown of the points for questions:

Name	Points
Concept 1	4
Concept 2	2
Concept 3	2
Concept 4	2
Concept 5	2
M2_L1_p1	6
M2_L1_p2	6
M2_L1_p3	6
M2_L2_p1	6
M2_L2_p2	6
M2_L2_p3	6
M2_HW1	36
M2_HW2	36
<b>Total</b>	120
Bonus	6

### Problem 1



a) Consider fitting a second order LLS regression model to the following data:

What is the size (height and width) of the design matrix  $X$ ?

What is the size (height and width) of the parameter vector  $w$ ?

What is the size of the product  $X'X$ ?

$X$  has shape(11, 3)

$W$  has shape(3,1)

$X'X$  has shape (3,3)

b) Now consider a dataset with 1000 points, for the same second order model:

What is the size (height and width) of the design matrix  $X$ ?

What is the size (height and width) of the parameter vector  $w$ ?

What is the size of the product  $X'X$ ?

$X$  has shape(1000, 3)

$W$  has shape(3,1)

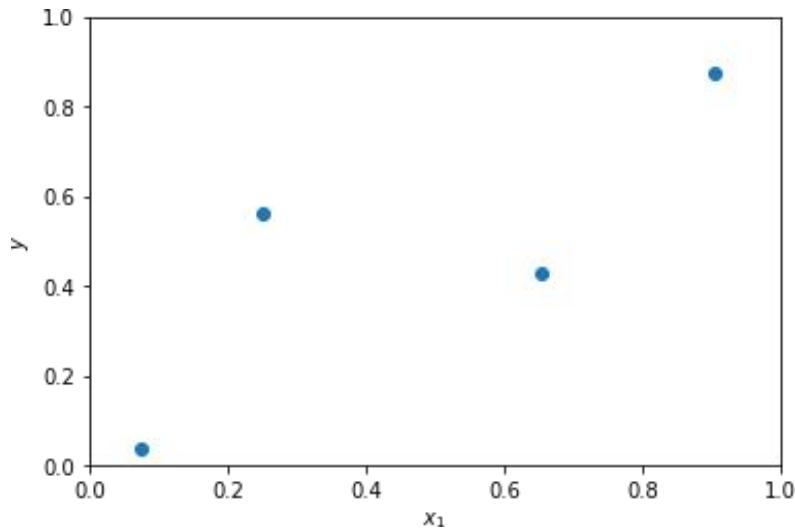
$X'X$  has shape (3,3)

c) Which of the following statements is true:

1. The model complexity is determined by the number of data points used for training
2. The model complexity is determined by the number of model parameters

Statement 2 is correct

## Problem 2



Consider the four points shown in the following figure:

If we use polynomial regression, what is the smallest order polynomial to perfectly interpolate these points? (multiple choice)

1. 1st order -  $w_1x_1 + w_2$
2. 2nd order -  $w_1x_1^2 + w_2x_1 + w_3$
3. 3rd order -  $w_1x_1^3 + w_2x_1^2 + w_3x_1 + w_4$
4. 4th order -  $w_1x_1^4 + w_2x_1^3 + w_3x_1^2 + w_4x_1 + w$

The smallest order is third order as the curve

### Problem 3

Imagine you are hiking to the top of a mountain, but you get lost off trail, and it gets dark. All you have is a dim flashlight, but it is enough for you to see your feet and which way the mountain slopes. Since you took AIML for engineers you know you can use gradient descent to find the extrema of the mountain. Which of the following formulations will lead you to the top of the mountain?

$$1. \quad x_{new} = x_{old} - \eta \cdot \frac{\partial \text{obj}}{\partial x}$$

$$2. \quad x_{new} = x_{old} + \eta \cdot \frac{\partial \text{obj}}{\partial x}$$

The first second one would lead to the top as Xnew increases when the slope is positive.

#### **Problem 4**

You are implementing batch gradient descent but notice that your solution is taking a long time to minimize the objective function due to oscillations. Which of the following approaches are likely to help resolve this issue? (Multiple choice, select all that are true)

1. Increase the batch size
2. Switch to stochastic gradient descent
3. Reduce the learning rate

2 and 3 would both help.

Stochastic gradient descent only select one random point instead of the entire data set when doing calculation, which would be much faster.

If the learning rate is too high, it may oscillate around the minimum instead of converging.

### Problem 5

The function for elastic net regularization in sklearn has a L1\_ratio parameter which describes the weight of the L1 and L2 penalty terms relative to each other. The  $\alpha$  parameter describes the strength of the regularization terms. The formulation of the objective function follows:

$$\|Xw - y\|_2^2 + \alpha \cdot L1_{ratio} \cdot \|w\|_1 + \alpha \cdot (1 - L1_{ratio}) \cdot \|w\|_2^2$$

If the L1\_ratio term is 0, what is the formulation equivalent to?

3.LLS with L2 regularization

1. Standard LLS
2. LLS with L1 regularization
3. LLS with L2 regularization
4. LLS with L1 and L2 regularization

# M2-L1 Problem 1 (5 points)

In this question you will perform linear least squares regression on a very small dataset of 3 points. First, load and plot the data by running the following cell.

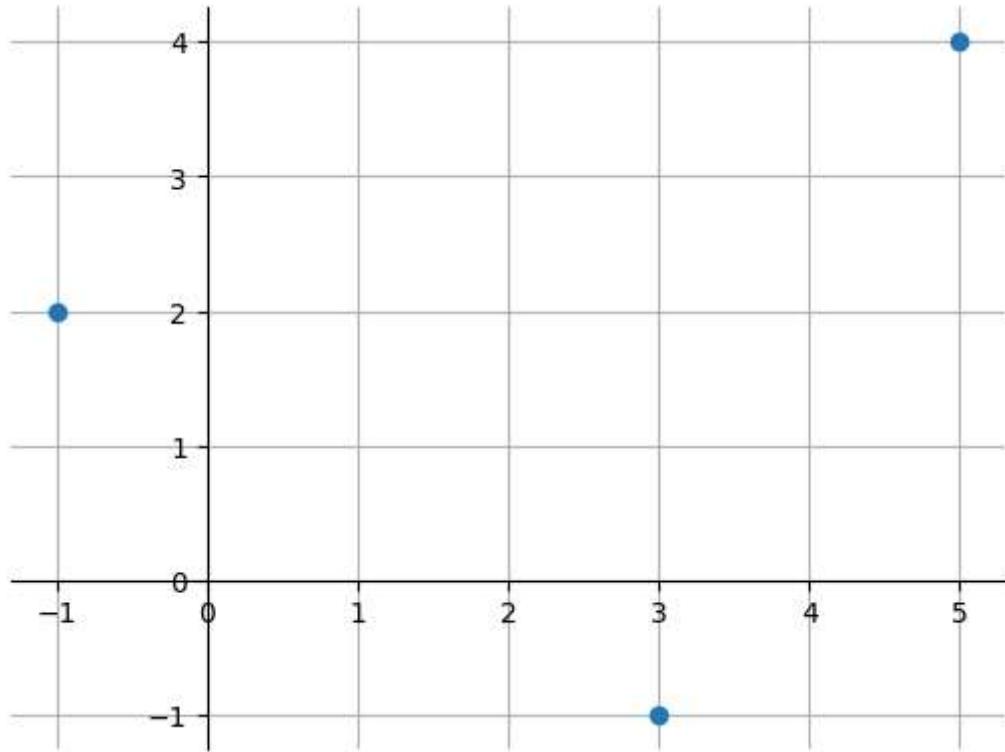
The variables provided are:

- $x$ : 3x1 input data
- $y$ : 3x1 output data

```
In [1]: import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns

x = np.array([[ -1,  3,  5]]).T
y = np.array([[ 2, -1,  4]]).T

fig, ax = plt.subplots()
plt.plot(x, y, 'o')
ax.spines['left'].set_position(('data', 0))
ax.spines['bottom'].set_position(('data', 0))
sns.despine()
plt.grid()
plt.show()
```



## Construct a design matrix

For 1-D linear regression, the design matrix must contain not only a column of input  $x$ -values, but also a 'bias column' -- a column of ones (to allow the regression line to have an intercept).

The next step is to construct the design matrix  $X$  by concatenating a column of ones to the given input  $x$ . This has been done for you below:

```
In [2]: bias = np.ones_like(x)
X = np.concatenate([x,bias],1)
print("Design Matrix:\n",X)
```

Design Matrix:

```
[[ -1  1]
 [ 3  1]
 [ 5  1]]
```

## Solving for regression coefficients

Now that we have the design matrix  $X$  and the output  $y$ , we can solve for the coefficients  $w$  such that  $Xw \approx y$  using:

$$w = (X' X)^{-1} X' y$$

Note that you can use the following in Python:

- `@` for matrix multiplication
- `np.linalg.inv(A)` for inversion of matrix  $A$
- `A.T` for transpose of a matrix  $A$
- `b.reshape(-1,1)` to treat 1D array  $b$  as a column (you will need to do this for  $y$ )

Your line's slope should be  $\approx 0.18$  and your y-intercept should be  $\approx 1.25$ .

```
In [3]: # YOUR CODE GOES HERE
# Get coefficients w
w = np.linalg.inv(X.T@X)@X.T@y
print("Linear Coefficients:\n", w)
```

Linear Coefficients:

```
[[0.17857143]
 [1.25      ]]
```

## Making predictions

Now that we have the coefficients, we can make predictions on new data with the model.

Do the following steps:

- [Given] Sample 40 points on the interval [-3,7], such as by using `np.linspace()` (Append `.reshape(-1,1)` to convert to a column)
- [Given] Create a design matrix by adding a column of ones as done previously
- Make a prediction by multiplying your new design matrix by `w`. You can do matrix multiplication with the `@` symbol
- [Given] Add a line to the plot showing these predictions

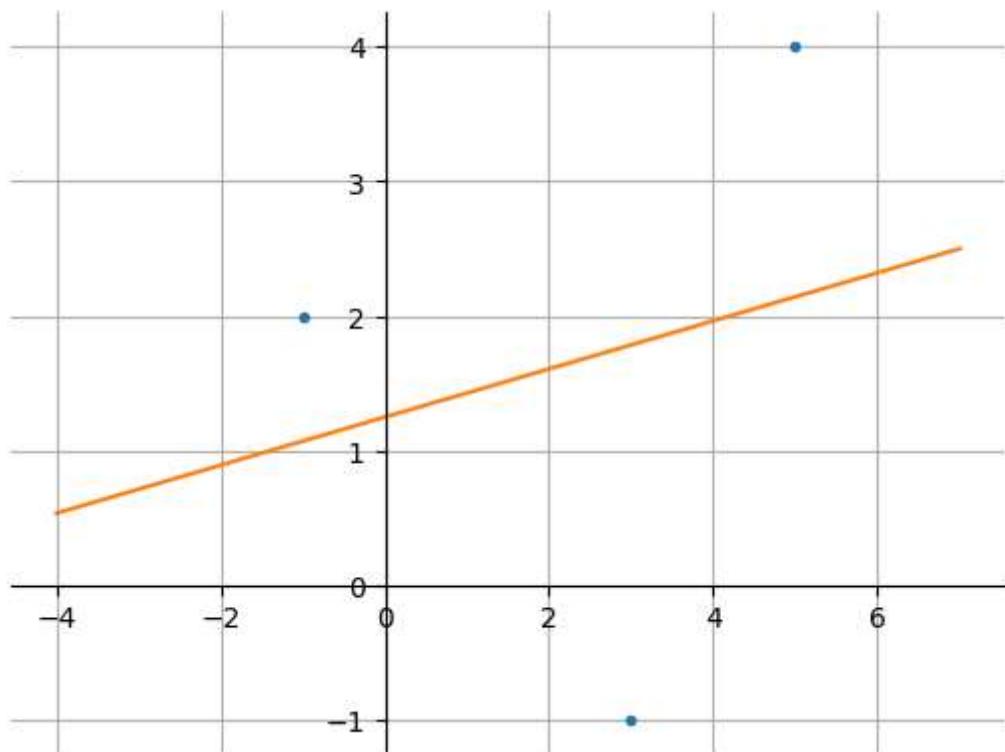
In [4]:

```
n = 40
x_test = np.linspace(-4,7,n).reshape(-1,1)
bias_test = np.ones_like(x_test)
X_test = np.concatenate([x_test, bias_test], 1)

# YOUR CODE GOES HERE
# Predict y_test

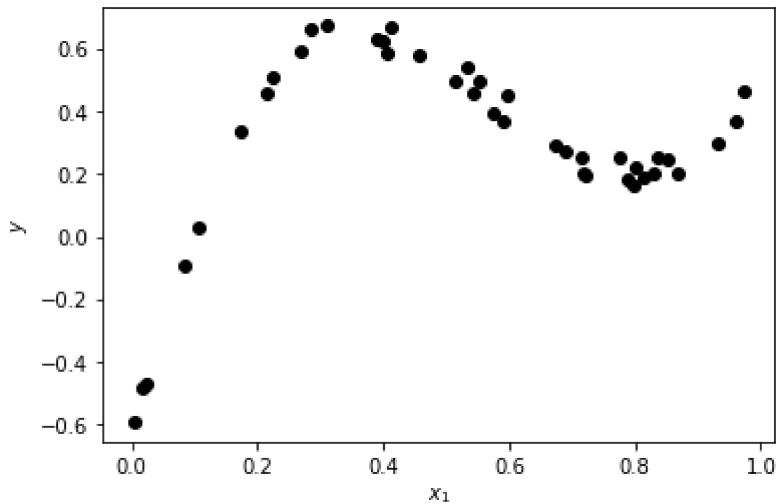
y_test = X_test@w

fig, ax = plt.subplots()
plt.plot(x, y, '.')
plt.plot(x_test, y_test)
ax.spines['left'].set_position(('data', 0))
ax.spines['bottom'].set_position(('data', 0))
sns.despine()
plt.grid()
plt.show()
```



# M2-L1 Problem 2 (5 points)

Consider the following data:



We will perform these steps:

1. Load the data
2. Generate a design matrix
3. Solve for the regression coefficients
4. Create and plot a curve using these coefficients

First, we demonstrate the above for a linear (1st order) model. First, run each cell in the demo and follow along with each step. Your job is then to make 2nd and 3rd order models for the same data using similar techniques.

## Demonstration: Linear fit

First, we load the data in to x and y:

```
In [1]: import numpy as np
import matplotlib.pyplot as plt

x = np.array([0.2855, 0.0033, 0.8307, 0.9606, 0.8153, 0.5539, 0.5152, 0.7761,
    0.5763, 0.2697, 0.6744, 0.7998, 0.1052, 0.8674, 0.598, 0.3985,
    0.0171, 0.1732, 0.7976, 0.4137, 0.7161, 0.7225, 0.3892, 0.0834,
    0.9733, 0.3097, 0.8509, 0.0226, 0.6901, 0.2235, 0.5914, 0.5436,
    0.7189, 0.4558, 0.8366, 0.534, 0.214, 0.9314, 0.4065, 0.788])

y = np.array([ 0.6603, -0.5925, 0.2045, 0.3698, 0.191, 0.496, 0.4986,
    0.2516, 0.3905, 0.5932, 0.2924, 0.219, 0.0287, 0.2024,
    0.4489, 0.6237, -0.4857, 0.3384, 0.162, 0.6694, 0.2539,
    0.1936, 0.6322, -0.0953, 0.4632, 0.6721, 0.2464, -0.4672,
```

```
0.2746,  0.5087,  0.3691,  0.4559,  0.2021,  0.5797,  0.2531,
0.5417,  0.4577,  0.2952,  0.5856,  0.1818])
```

Now we generate a linear design matrix for the data:

```
In [2]: # Function to get a Linear design matrix for 1D data
def get_linear_design_matrix(x):
    x = x.reshape(-1, 1)                      # Turn x into a column array
    columns = [x, np.ones_like(x)]            # Linear design matrix has a column of x and 1s
    X = np.concatenate(columns, axis=1)        # Combine each column horizontally to make X
    return X
```

```
In [3]: x = get_linear_design_matrix(x)
print("First four rows of X:")
print(X[:4,:])
```

First four rows of X:

```
[[0.2855 1.]
 [0.0033 1.]
 [0.8307 1.]
 [0.9606 1.]]
```

Now that we have the design matrix  $X$  and the output  $y$ , we can solve for the coefficients  $w$  such that  $Xw \approx y$  using:

$$w = (X' X)^{-1} X' y$$

Note the use of the following in Python:

- `@` for matrix multiplication
- `np.linalg.inv(A)` for inversion of matrix `A`
- `A.T` for transpose of a matrix `A`
- `b.reshape(-1,1)` to treat 1D array `b` as a column

```
In [4]: # Get coefficients
w = np.linalg.inv(X.T @ X) @ X.T @ y.reshape(-1,1)
print("Linear Coefficients:", w.flatten())
```

Linear Coefficients: [0.26080739 0.16441038]

Next, we write a plotting function to plot data and our least squares regression on the same axes. To apply our regression we can call the

`get_linear_design_matrix()` function on a new set of `x` values and multiply the result by our coefficients `w`.

```
In [5]: def plot_data_with_regression(x_data, y_data, x_reg, y_reg):
    plt.figure()

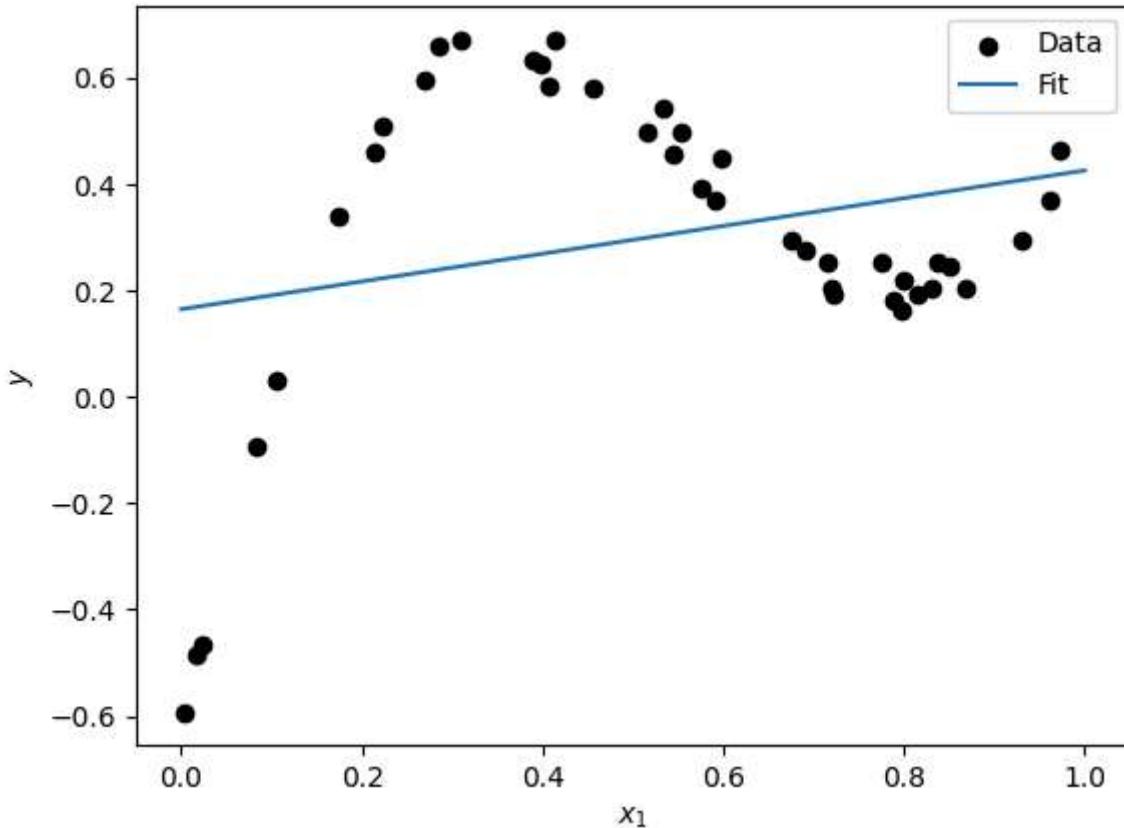
    plt.scatter(x_data, y_data, label="Data", c="black")
    plt.plot(x_reg, y_reg, label="Fit")

    plt.legend()
    plt.xlabel(r"\$x_1\$")
```

```
plt.ylabel(r"$y$")
plt.show()
```

```
In [6]: x_fit = np.linspace(0, 1, 100)           # x values for fit line
y_fit = get_linear_design_matrix(x_fit) @ w    # y values for fit line

plot_data_with_regression(x, y, x_fit, y_fit)
```



## Your turn: Second order polynomial

Now you will solve the same problem, but with a quadratic polynomial instead of linear. We need to write a new function to generate a design matrix.

Replace the commented code below:

```
In [7]: def get_quadratic_design_matrix(x):
    # YOUR CODE GOES HERE
    # GENERATE A DESIGN MATRIX WITH 2ND ORDER FEATURES: X
    columns = np.array([[x**2], [x], [np.ones_like(x)]])
    X = np.concatenate(columns, axis=0)
    return X.T
```

```
In [8]: X = get_quadratic_design_matrix(x)
print("First four rows of X:")
print(X[:4, :])
```

First four rows of X:

```
[[8.1510250e-02 2.8550000e-01 1.0000000e+00]
 [1.0890000e-05 3.3000000e-03 1.0000000e+00]
 [6.9006249e-01 8.3070000e-01 1.0000000e+00]
 [9.2275236e-01 9.6060000e-01 1.0000000e+00]]
```

Compute the `w` coefficients with a pseudo-inverse as in the example:

```
In [9]: # YOUR CODE GOES HERE
# COMPUTE COEFFICIENTS w

w2 = np.linalg.inv(X.T @ X) @ X.T @ y.reshape(-1,1)

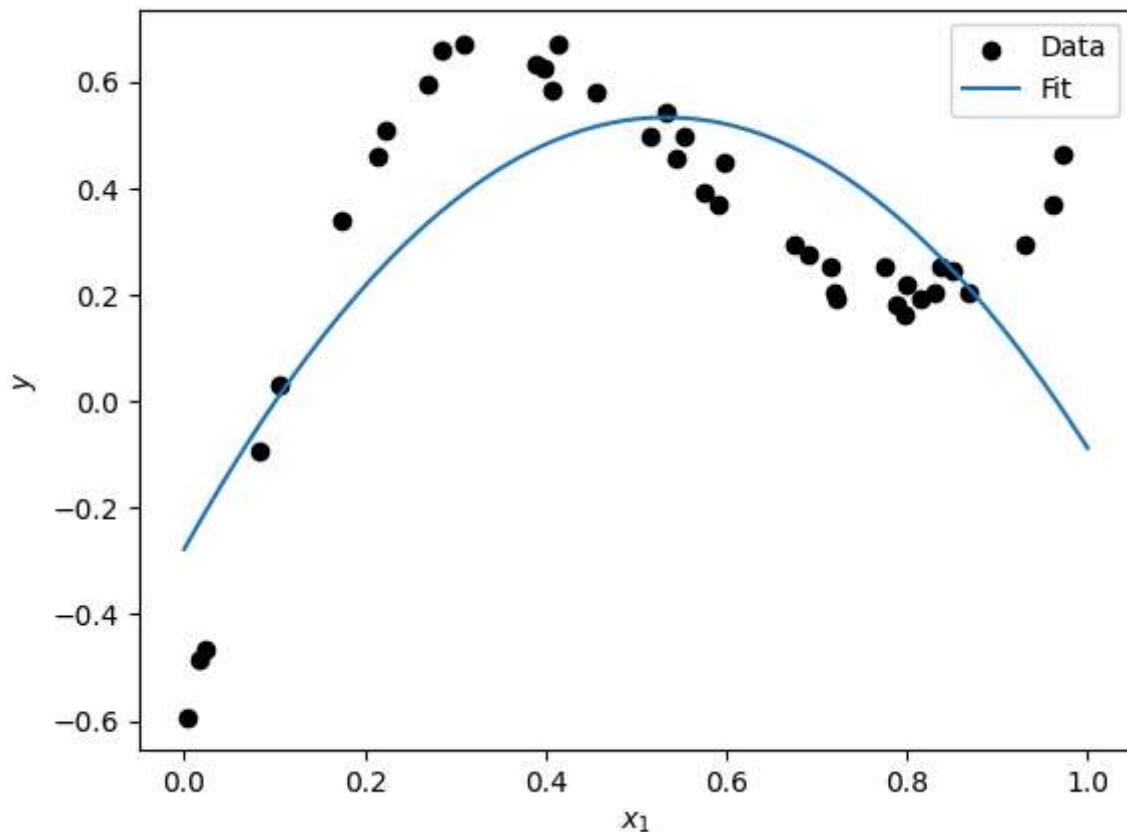
print("Quadratic Coefficients:", w2.flatten())
```

Quadratic Coefficients: [0.26080739 0.16441038]

Now plot by generating a regression curve and calling the `plot_data_with_regression()` function:

```
In [10]: # YOUR CODE GOES HERE
# PLOT

x_fit = np.linspace(0, 1, 100)                      # x values for fit line
y_fit = get_quadratic_design_matrix(x_fit) @ w2      # y values for fit line
plot_data_with_regression(x,y,x_fit,y_fit)
```



## 3rd Order Polynomial

Here we go through the same steps as above, but this time we perform a cubic polynomial fit. Once again, fill in the necessary code:

```
In [11]: def get_cubic_design_matrix(x):
    # YOUR CODE GOES HERE
    # GENERATE A DESIGN MATRIX WITH 3RD ORDER FEATURES: X
    columns = np.array([[x**3],[x**2],[x],[np.ones_like(x)]])
    X = np.concatenate(columns, axis=0)
    return X.T
```

```
In [12]: X = get_cubic_design_matrix(x)
print("First four rows of X:")
print(X[:4,:])
```

First four rows of X:

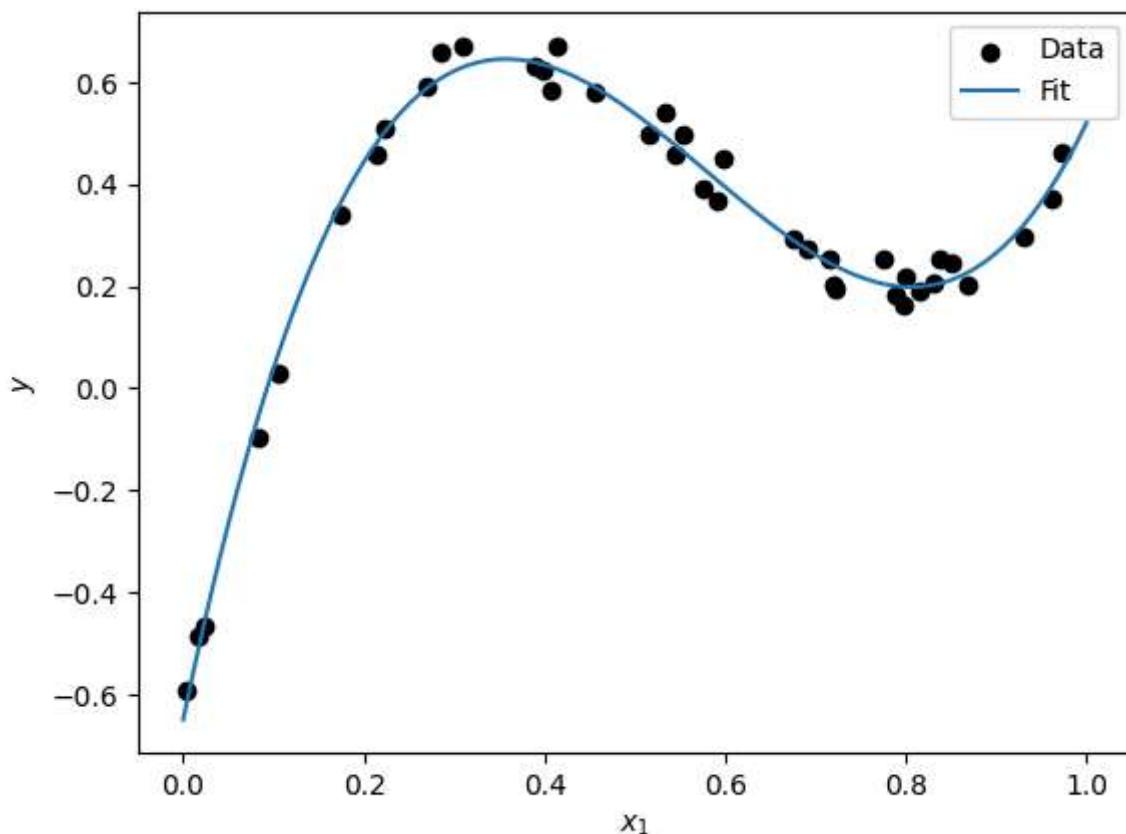
```
[[2.32711764e-02 8.15102500e-02 2.85500000e-01 1.00000000e+00]
 [3.59370000e-08 1.08900000e-05 3.30000000e-03 1.00000000e+00]
 [5.73234910e-01 6.90062490e-01 8.30700000e-01 1.00000000e+00]
 [8.86395917e-01 9.22752360e-01 9.60600000e-01 1.00000000e+00]]
```

```
In [13]: # YOUR CODE GOES HERE
# COMPUTE COEFFICIENTS w
w3 = np.linalg.inv(X.T @ X) @ X.T @ y.reshape(-1,1)
print("Cubic Coefficients:", w3.flatten())
```

Cubic Coefficients: [0.26080739 0.16441038]

```
In [14]: # YOUR CODE GOES HERE
# PLOT

x_fit = np.linspace(0, 1, 100)           # x values for fit line
y_fit = get_cubic_design_matrix(x_fit) @ w3 # y values for fit line
plot_data_with_regression(x,y,x_fit,y_fit)
```



The Kernel crashed while executing code in the current cell or a previous cell.

Please review the code in the cell(s) to identify a possible cause of the failure.

Click [here](https://aka.ms/vscodeJupyterKernelCrash) for more info.

View Jupyter [log](command:jupyter.viewOutput) for further details.

# M2-L1 Problem 3 (5 points)

In this question you will perform regression on 2D data. A linear fit will be demonstrated, and afterward you will extend the code to perform a second-order fit. First, run the Setup cells.

## Setup

### Generating data

```
In [2]: # Generating data for the problem
import numpy as np
import matplotlib.pyplot as plt

def gaussian2d(A, mx, my, sx, sy):
    F = lambda xy: A*np.exp(-((xy[:,0]-mx)**2/(2*sx*sx)
                             + (xy[:,1]-my)**2/(2*sy*sy)))
    return F

def get_data_function():
    f1 = gaussian2d(A=0.7, mx = 0.25,my=0.25,sx=0.25,sy=0.25)
    f2 = gaussian2d(A=0.7, mx = 0.75,my=0.75,sx=0.25,sy=0.45)
    f = lambda xy: f1(xy) + f2(xy)
    return f

np.random.seed(0)
x = np.random.rand(60,2)
f = get_data_function()
y = f(x)
```

### Function for 3D plotting

```
In [3]: from mpl_toolkits.mplot3d import Axes3D
from matplotlib import cm

# Writing a 3D Plotting function. Inputs data points and regression function
def plot_data_with_regression(x_data, y_data, regfun=None):
    plt.figure(figsize=(8,8))
    fig = plt.gcf()
    ax = fig.add_subplot(111, projection='3d')
    ax.scatter(x_data[:,0], x_data[:,1], 0*y_data, s=13, c=y_data, zorder=-1, cmap="cool")
    ax.scatter(x_data[:,0], x_data[:,1], y_data, s=20, c="black", zorder=-1)
    for i in range(len(y_data)):
        ax.plot([x_data[i,0],x_data[i,0]],[x_data[i,1],x_data[i,1]],[0,y_data[i]], 'r')

    ax.set_xlabel('\n' + r"$x_1$")
    ax.set_ylabel('\n' + r"$x_2$")
    ax.set_zlabel('\n'+r"$y$")
```

```

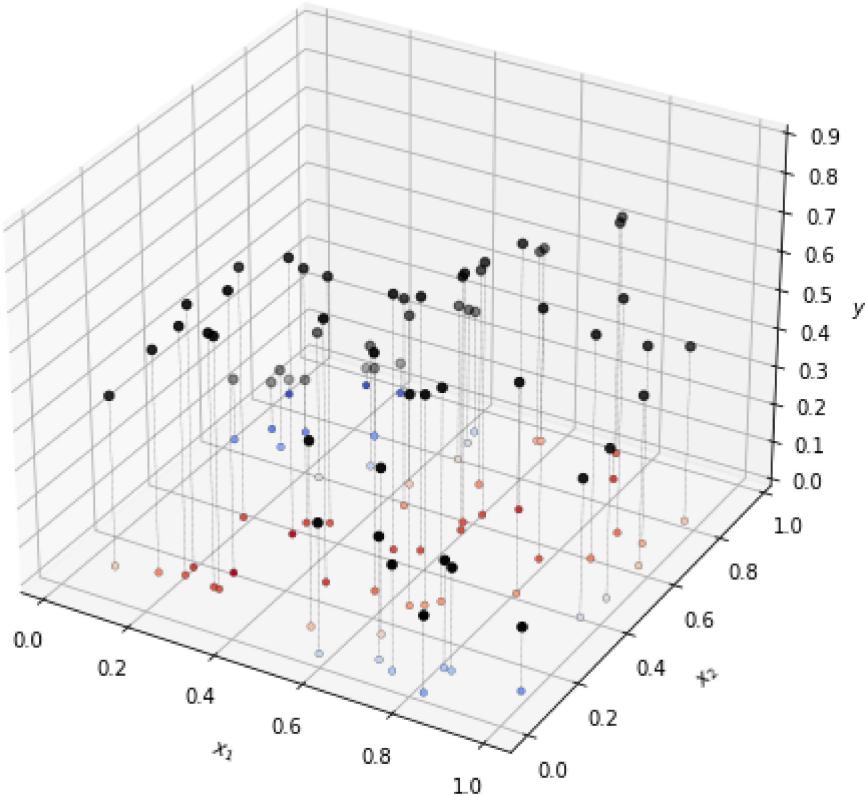
ax.set_zlim(0,0.9)

if regfun is not None:
    vals = np.linspace(0, 1, 100)
    x1grid, x2grid = np.meshgrid(vals, vals)
    y = regfun(np.concatenate((x1grid.reshape(-1,1),x2grid.reshape(-1,1)),1)).r
    ax.plot_surface(x1grid, x2grid, y.reshape(x1grid.shape), alpha = 0.8, cmap
plt.show()

```

## Data visualized

In [4]: `plot_data_with_regression(x,y)`



## Demonstration: 2D Linear Regression

First, I generate a design matrix within a function called

`get_linear_design_matrix()`

In [5]: `def get_linear_design_matrix(x):
 x1 = x[:,0].reshape(-1, 1)
 x2 = x[:,1].reshape(-1, 1)
 columns = [x1, x2, np.ones_like(x1)] # Linear design matrix has a column of x
 X = np.concatenate(columns, axis=1) # Combine each column horizontally to make
 return X`

```
In [6]: X = get_linear_design_matrix(x)
print("First four rows of X:")
print(X[:4,:])
```

First four rows of X:

```
[[0.5488135  0.71518937 1.      ]
 [0.60276338  0.54488318 1.      ]
 [0.4236548   0.64589411 1.      ]
 [0.43758721  0.891773   1.      ]]
```

Next, get the coefficients of the regression:

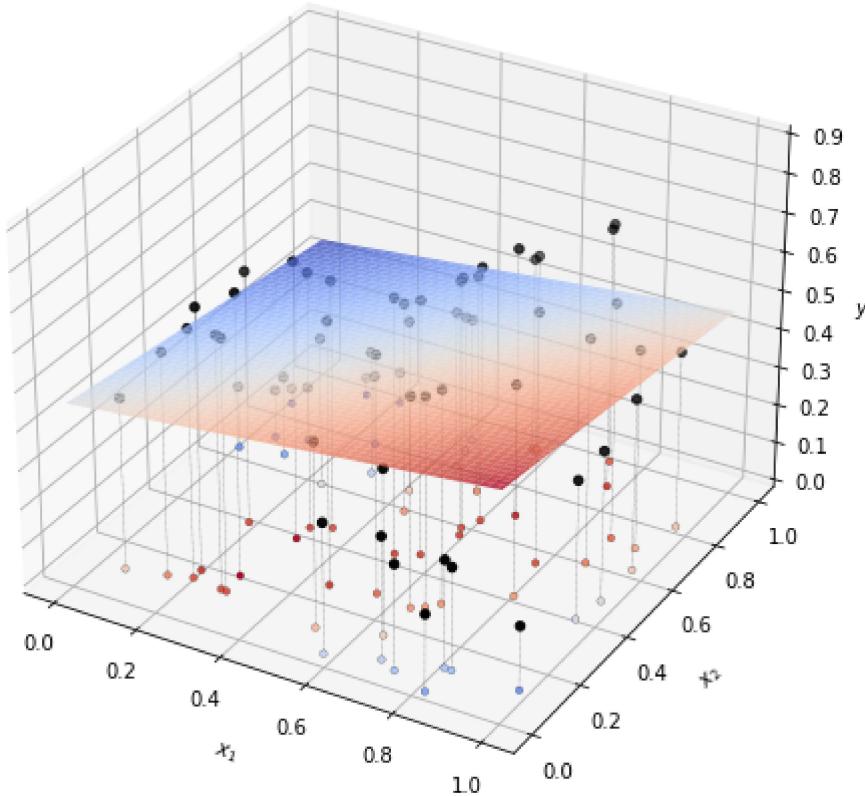
```
In [7]: # Get coefficients
w1 = np.linalg.inv(X.T @ X) @ X.T @ y.reshape(-1,1)
print("Linear Coefficients:", w1.flatten())
```

Linear Coefficients: [ 0.11233939 -0.10638434 0.45720916]

Finally, we plot the result. Here, `plot_data_with_regression()` takes as input the x input data, y output data, and a function which performs the desired regression. Therefore I first define said regression function, and plug it in as an argument to the plotting function:

```
In [8]: def do_2d_linear_regression(x):
    y_fit = get_linear_design_matrix(x) @ w1
    return y_fit

plot_data_with_regression(x, y, do_2d_linear_regression)
```



## Your Turn: 2D Quadratic Regression

The linear regression results are clearly not a great fit. You will see if a 2nd order fit can do any better. Fill in the missing code below to generate a quadratic design matrix and plot the results:

```
In [9]: def get_quadratic_design_matrix(x):
    x1 = x[:,0].reshape(-1, 1)
    x2 = x[:,1].reshape(-1, 1)

    # YOUR CODE GOES HERE
    # 2ND ORDER, 2-D DESIGN MATRIX NEEDS 6 TOTAL COLUMNS
    columns = np.array([[x1**2], [x2**2], [(x1*x2)], [x1], [x2], [np.ones_like(x1)]])
    X = np.concatenate(columns, axis=0)
    return X.T[0]
```

```
In [10]: X = get_quadratic_design_matrix(x)
print("First four rows of X:")
print(X.shape)
print(X[:4,:])
```

First four rows of X:

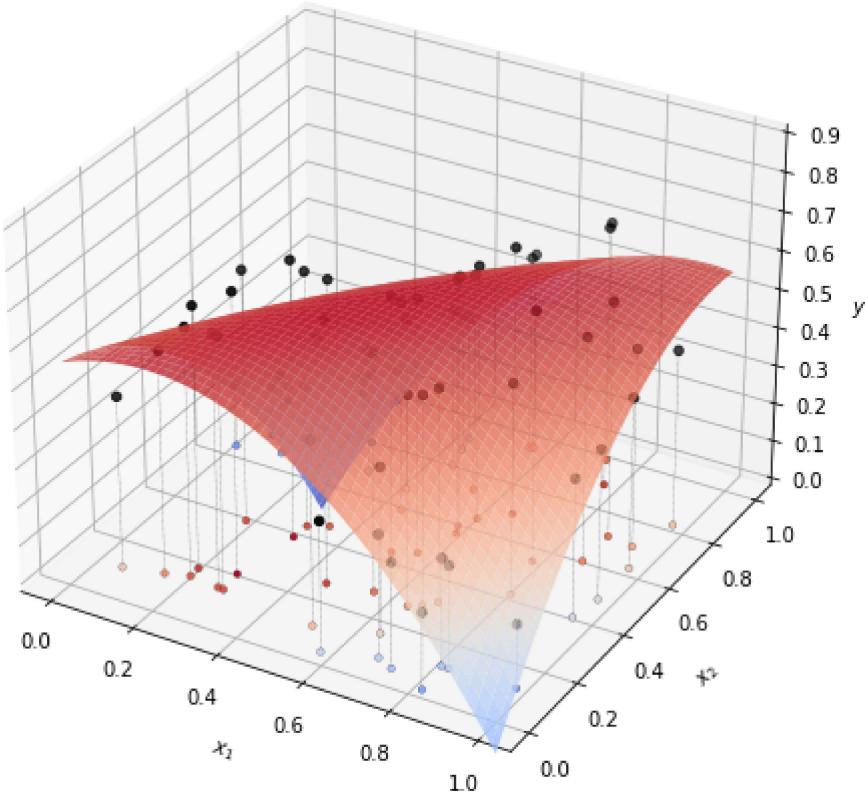
```
(60, 6)
[[0.30119626 0.51149583 0.39250558 0.5488135 0.71518937 1.
 [0.36332369 0.29689768 0.32843563 0.60276338 0.54488318 1.
 [0.17948339 0.41717921 0.27363614 0.4236548 0.64589411 1.
 [0.19148257 0.79525908 0.39022846 0.43758721 0.891773 1.
 ]]
```

```
In [11]: # Get coefficients
w2 = np.linalg.inv(X.T @ X) @ X.T @ y.reshape(-1,1)
print("Quadratic Coefficients:", w2.flatten())
```

Quadratic Coefficients: [-1.09949493 -0.78655383 1.62592273 0.44193704 -0.17753776 0.55677679]

```
In [12]: def do_2d_quadratic_regression(x):
    y_fit = get_quadratic_design_matrix(x) @ w2
    return y_fit

plot_data_with_regression(x, y, do_2d_quadratic_regression)
```



# Problem 1 (30 points)

## Problem Description

In this problem you will implement polynomial linear least squares regression on two datasets, with and without regularization. Additionally, you will use gradient descent to optimize for the model parameters.

Fill out the notebook as instructed, making the requested plots and printing necessary values.

*You are welcome to use any of the code provided in the lecture activities.*

### Summary of deliverables:

#### Results:

- Print fitted model parameters  $w$  for the 4 models requested without regularization
- Print fitted model parameters  $w$  for the 2 models requested *with*  $L_2$  regularization
- Print fitted model parameters  $w$  for the one model solved via gradient descent

#### Plots:

- 2 plots of each dataset along with the ground truth function
- 4 plots of the fitted function along with the respective data and ground truth function for LLS without regularization
- 2 plots of the fitted function along with the respective data and ground truth function for LLS with  $L_2$  regularization
- 1 plot of the fitted function along with the respective data and ground truth function for LLS with  $L_2$  regularization solved via gradient descent

#### Discussion:

- Discussion of challenges fitting complex models to small datasets
- Discussion of difference between the  $L_2$  regularized model versus the standard model
- Discussion of whether gradient descent could get stuck in a local minimum
- Discussion of gradient descent results versus closed form results

#### Imports and Utility Functions:

```
In [2]: import numpy as np
import matplotlib.pyplot as plt

def gt_function():
    xt = np.linspace(0,1,101)
    yt = np.sin(2 *np.pi*xt)
    return xt, yt

def plot_data(x,y,xt,yt,title = None):
    # Provide title as a string e.g. 'string'
    plt.plot(x,y,'bo',label = 'Data')
    plt.plot(xt,yt,'g-', label = 'Ground Truth')
    plt.xlabel('x')
    plt.ylabel('y')
    plt.grid()
    plt.legend()
    if title:
        plt.title(title)
    plt.show()

def plot_model(x,y,xt,yt,xr,yr,title = None):
    # Provide title as a string e.g. 'string'
    plt.plot(x,y,'bo',label = 'Data')
    plt.plot(xt,yt,'g-', label = 'Ground Truth')
    plt.plot(xr,yr,'r-', label = 'Fitted Function')
    plt.xlabel('x')
    plt.ylabel('y')
    plt.grid()
    plt.legend()
    if title:
        plt.title(title)
    plt.show()
```

## Load and visualize the data

The data is contained in `d10.npy` and `d100.npy` and can be loaded with `np.load()`.

Store the data as:

- `x10` and `x100` (the first column of the data)
- `y10` and `y100` (the second column of the data)

Generate the ground truth function  $f(x) = \sin(2\pi x)$  using `xt, yt = gt_function()`.

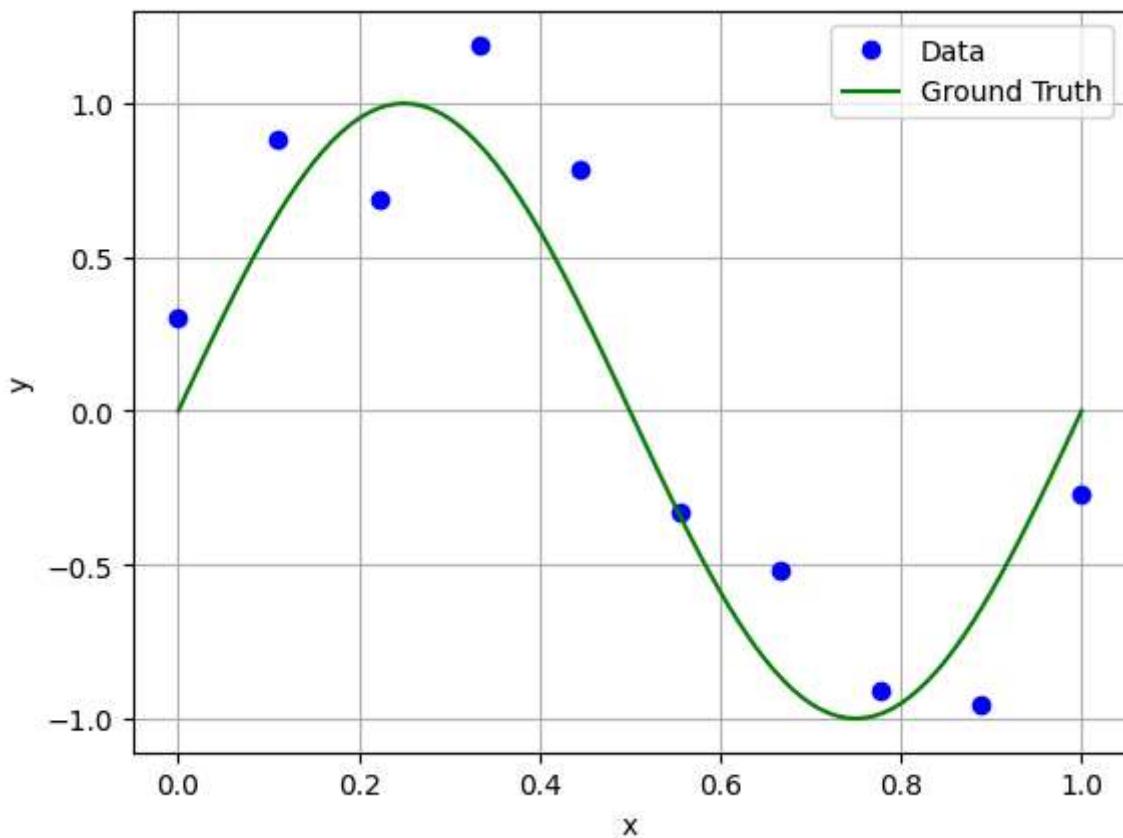
Then visualize the each dataset with `plotxy(x,y,xt,yt,title)` with an appropriate title. You should generate two plots.

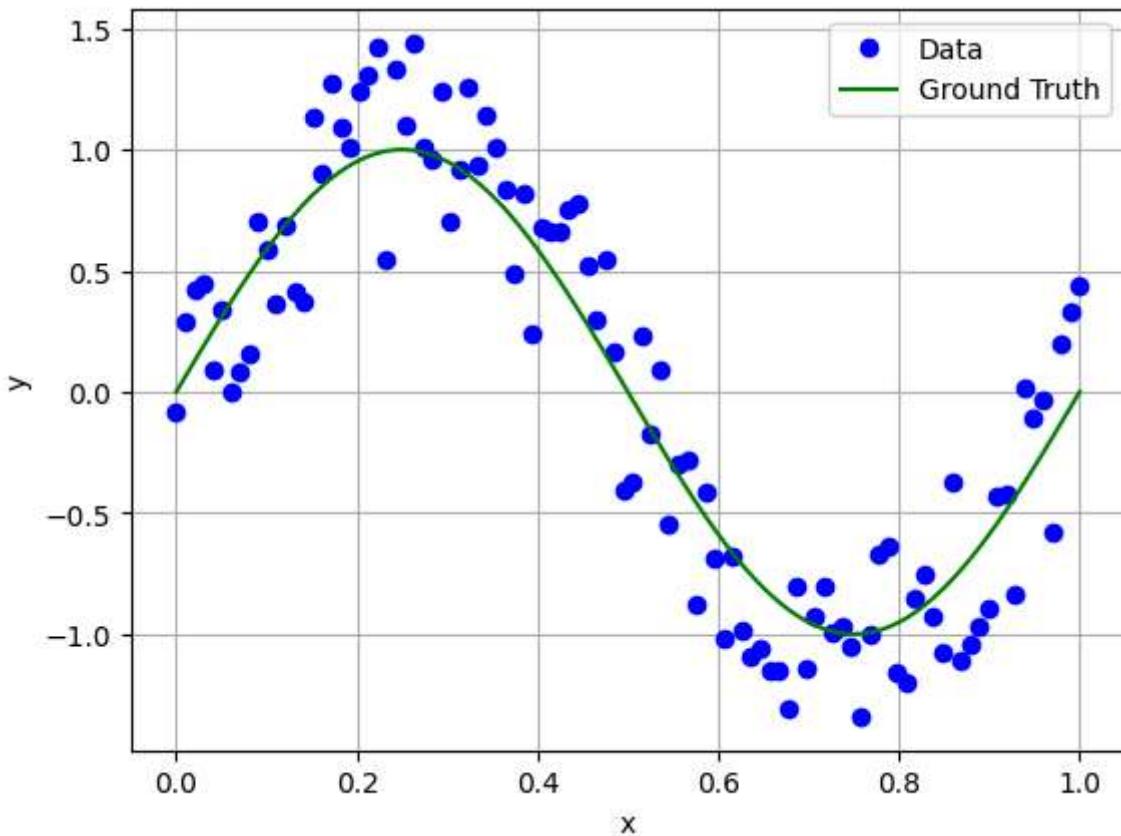
```
In [3]: # YOUR CODE GOES HERE
d10 = np.load('d10.npy')
```

```
d100 = np.load('d100.npy')

x10,y10 = d10[:,0],d10[:,1]
x100,y100 = d100[:,0],d100[:,1]

xt,yt = gt_function()
plot_data(x10,y10,xt,yt,title=None)
plot_data(x100,y100,xt,yt,title=None)
```





## Implement polynomial linear regression

Now you will implement polynomial linear least squares regression without regularization using the closed form solution from lecture to compute the model parameters. You will consider the following 4 cases:

1. Data: data10.txt, Model: 2nd order polynomial (highest power of  $x$  in your regression model = 2)
2. Data: data100.txt, Model: 2nd order polynomial (highest power of  $x$  in your regression model = 2)
3. Data: data10.txt, Model: 9th order polynomial (highest power of  $x$  in your regression model = 9)
4. Data: data100.txt, Model: 9th order polynomial (highest power of  $x$  in your regression model = 9)

For each model:

- Print the learned model parameters `w`
- Use the model parameters `w` to predict `yr` values over a range of `x` values given by `xr = np.linspace(0,1,101)`
- Plot the data, ground truth function, and regressed model using `plot_model(x,y,xt,yt,xr,yr,title)` with an appropriate title.

```
In [4]: # YOUR CODE GOES HERE
```

```
def get_quadratic_design_matrix(x):
    # YOUR CODE GOES HERE
    # GENERATE A DESIGN MATRIX WITH 2ND ORDER FEATURES: X
    columns = np.array([[x**2],[x],[np.ones_like(x)]])
    X = np.concatenate(columns, axis=0)
    return X.T

def get_9th_design_matrix(x):
    # YOUR CODE GOES HERE
    # GENERATE A DESIGN MATRIX WITH 2ND ORDER FEATURES: X
    columns = np.array([(x**9),(x**8),(x**7),(x**6),(x**5),(x**4),(x**3),(x**2)])
    X = np.concatenate(columns, axis=0)
    return X.T

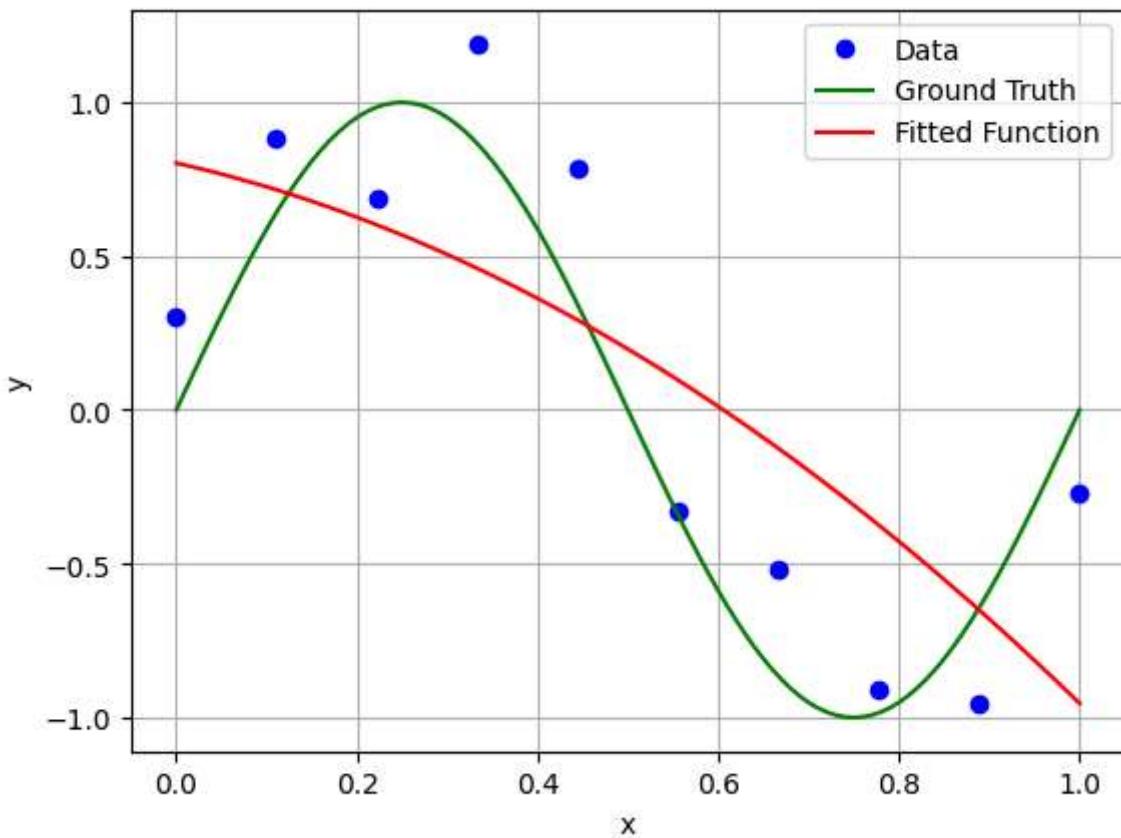
X_10_2 = get_quadratic_design_matrix(x10)
X_100_2 = get_quadratic_design_matrix(x100)
w_10_2 = np.linalg.inv(X_10_2.T @ X_10_2) @ X_10_2.T @ y10.reshape(-1,1)
w_100_2 = np.linalg.inv(X_100_2.T @ X_100_2) @ X_100_2.T @ y100.reshape(-1,1)
print(w_10_2)
print(w_100_2)

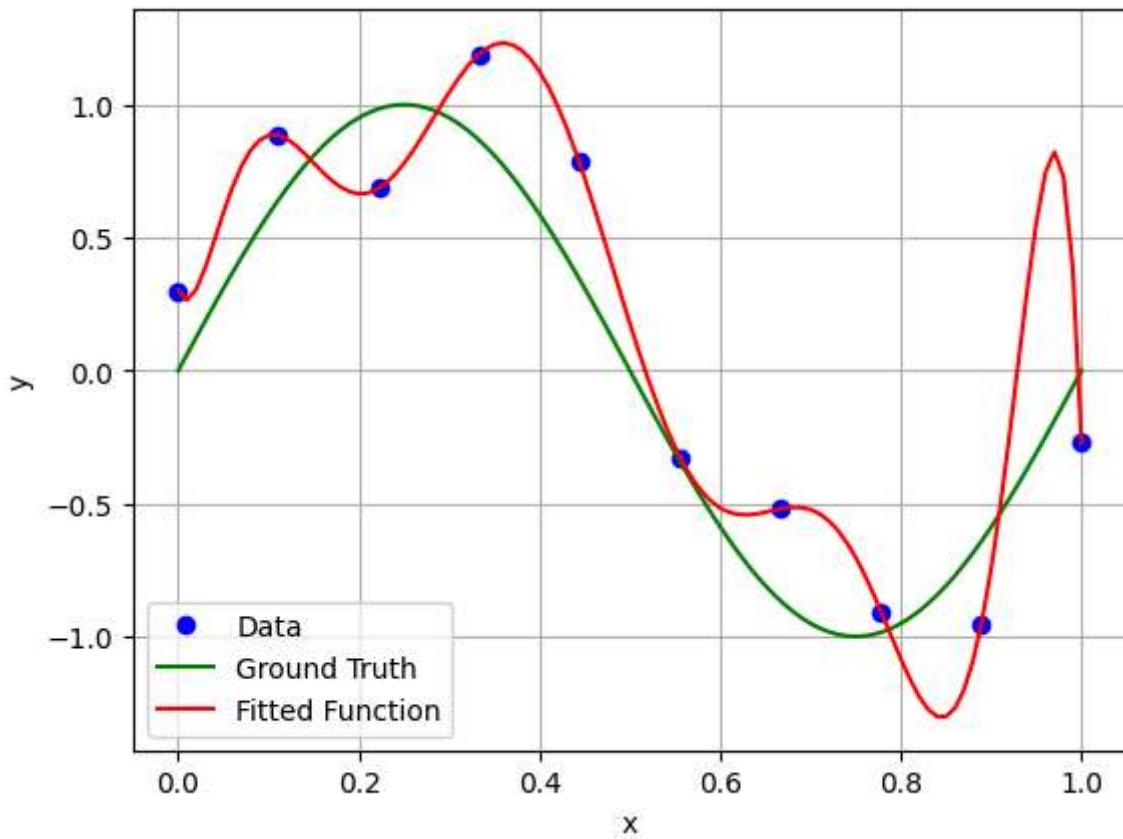
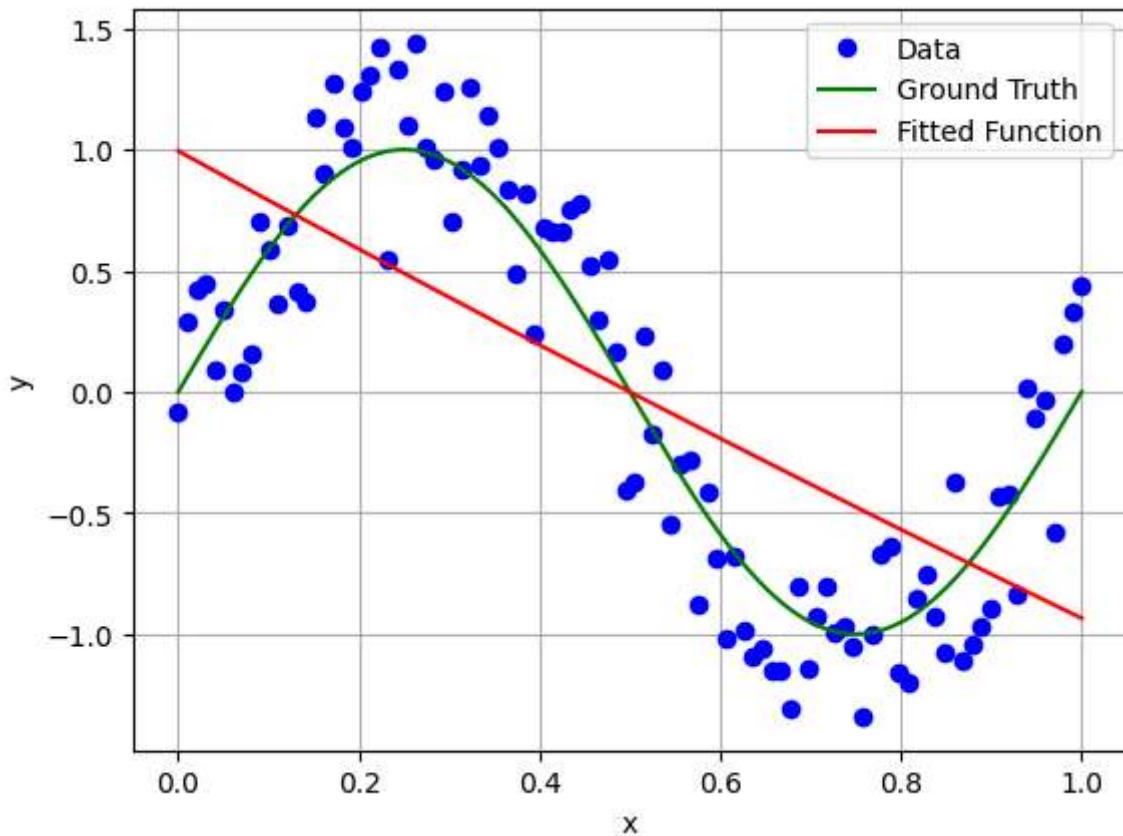
X_10_9 = get_9th_design_matrix(x10)
X_100_9 = get_9th_design_matrix(x100)
w_10_9 = np.linalg.inv(X_10_9.T @ X_10_9) @ X_10_9.T @ y10.reshape(-1,1)
w_100_9 = np.linalg.inv(X_100_9.T @ X_100_9) @ X_100_9.T @ y100.reshape(-1,1)
print(w_10_9)
print(w_100_9)

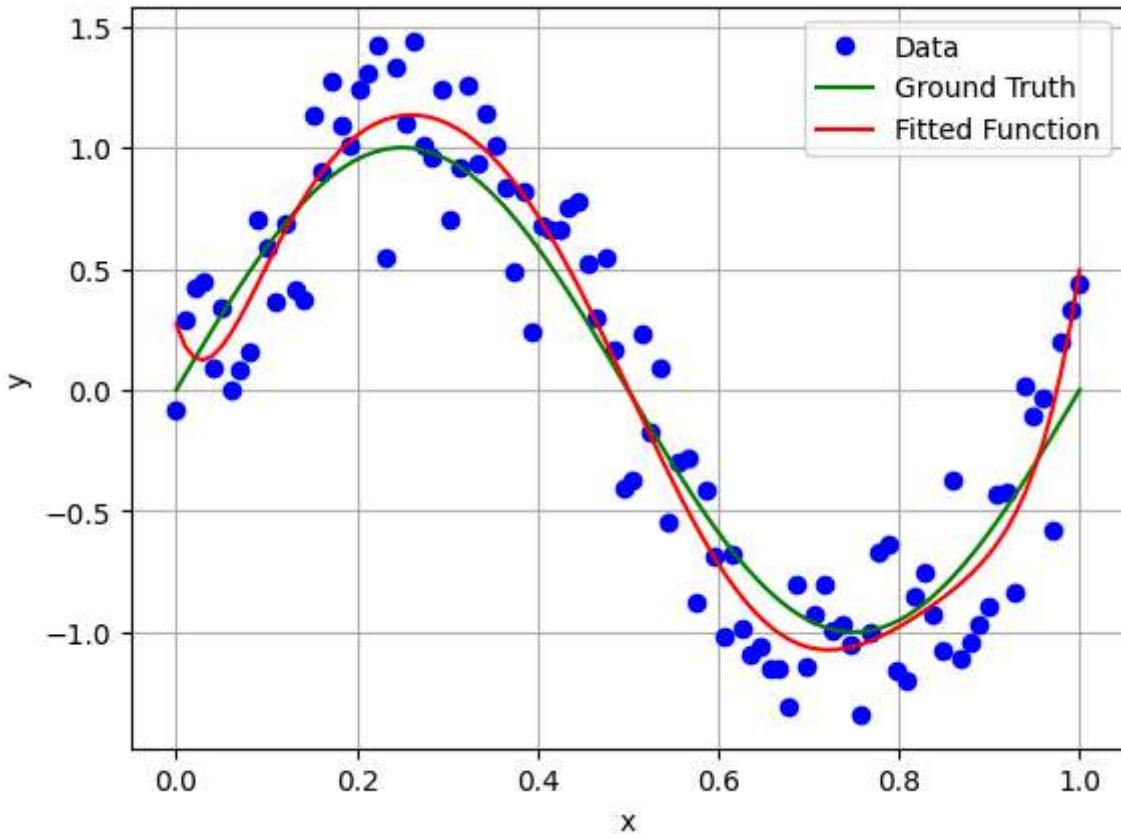
xr = np.linspace(0,1,101)
yr_10_2 = get_quadratic_design_matrix(xr) @ w_10_2
yr_100_2 = get_quadratic_design_matrix(xr) @ w_100_2
yr_10_9 = get_9th_design_matrix(xr) @ w_10_9
yr_100_9 = get_9th_design_matrix(xr) @ w_100_9

plot_model(x10,y10,xt,yt,xr,yr_10_2,title = None)
plot_model(x100,y100,xt,yt,xr,yr_100_2,title = None)
plot_model(x10,y10,xt,yt,xr,yr_10_9,title = None)
plot_model(x100,y100,xt,yt,xr,yr_100_9,title = None)
```

```
[[ -1.09384447]
 [ -0.66283292]
 [  0.80276877]]
[[  0.12128272]
 [ -2.04993418]
 [  0.99435046]]
[[-3.91495548e+04]
 [ 1.67474612e+05]
 [-2.96002347e+05]
 [ 2.78781821e+05]
 [-1.50327937e+05]
 [ 4.63121529e+04]
 [-7.65098243e+03]
 [ 5.70103270e+02]
 [-8.43835656e+00]
 [ 2.99999530e-01]]
[[-2.21757782e+03]
 [ 1.15792903e+04]
 [-2.48687620e+04]
 [ 2.86930080e+04]
 [-1.95215334e+04]
 [ 8.13290444e+03]
 [-2.07020969e+03]
 [ 2.84954674e+02]
 [-1.18520402e+01]
 [ 2.71583131e-01]]
```







## Discussion:

When the sample size (number of data points) is small, what issues or tendencies do you see with complex models?

*Your answer goes here*

When sample size is too small, it can cause overfitting and ill-Conditioned design matrix in a complex model

## Implement polynomial linear regression with $L_2$ regularization

You will repeat the previous section, but this time using  $L_2$  regularization. Your regularization term should be  $\lambda w' \mathbb{I}_m w$ , where  $\lambda = e^{-10}$ , and  $\mathbb{I}_m$  is the modified identity matrix that masks out the bias term from regularization.

You will consider only two cases:

1. Data: data10.txt, Model: 9th order polynomial (highest power of  $x$  in your regression model = 9)

2. Data: data100.txt, Model: 9th order polynomial (highest power of  $x$  in your regression model = 9)

For each model:

- Print the learned model parameters `w`
- Use the model parameters `w` to predict `yr` values over a range of `x` values given by `xr = np.linspace(0,1,101)`
- Plot the data, ground truth function, and regressed model using `plot_model(x,y,xt,yt,xr,yr,title)` with an appropriate title.

In [5]: # YOUR CODE GOES HERE

```
lamda = np.exp(-10)
Im_10 = np.identity(10)
Im_10[-1,-1] = 0

reg_10_9 = lamda * Im_10
reg_100_9 = lamda * Im_10

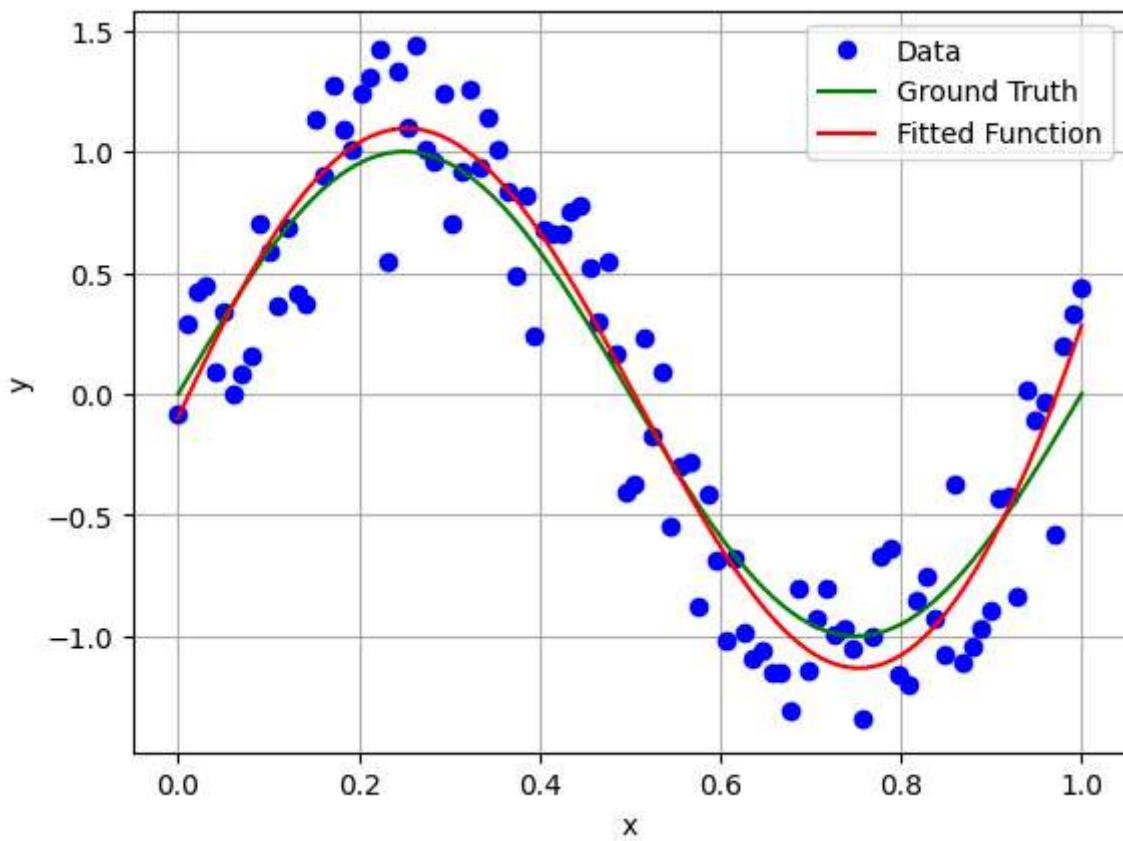
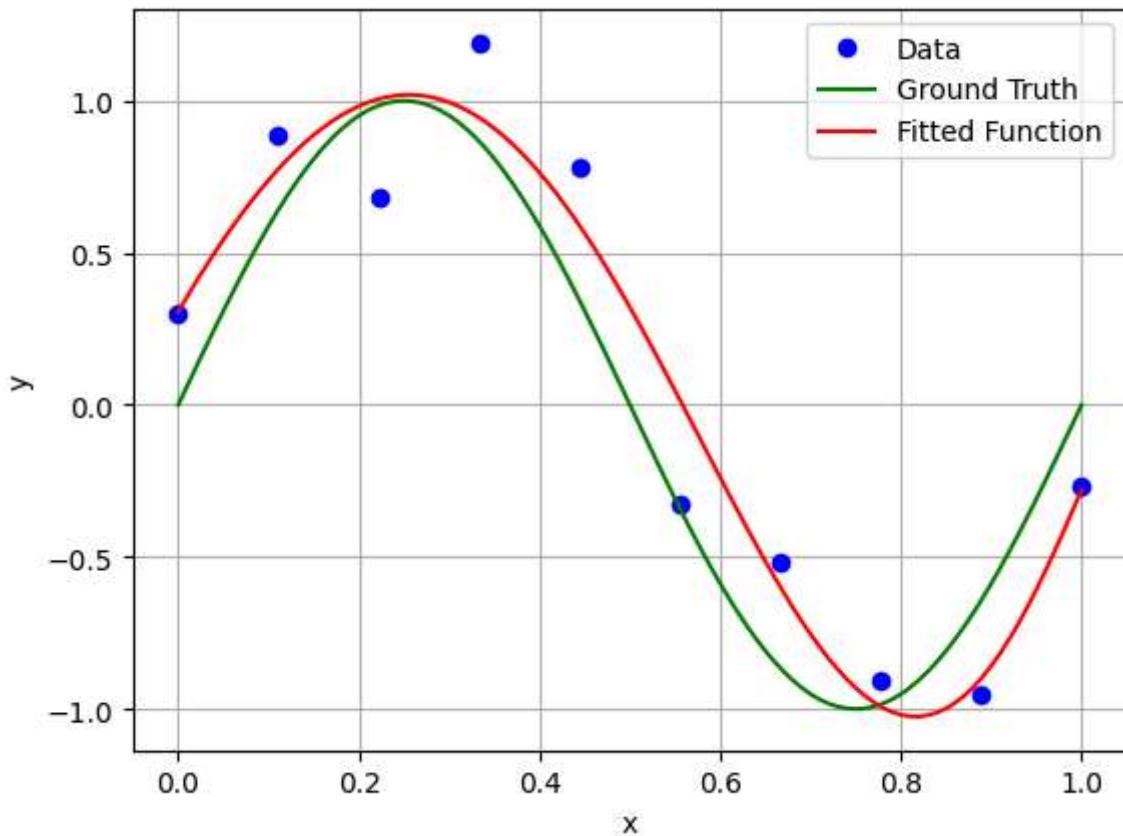
w_10_9_L2 = np.linalg.inv(X_10_9.T @ X_10_9 + reg_10_9) @ X_10_9.T @ y10.reshape(-1)
w_100_9_L2 = np.linalg.inv(X_100_9.T @ X_100_9 + reg_100_9) @ X_100_9.T @ y100.reshape(-1)

print(w_10_9_L2)
print(w_100_9_L2)

yr_10_9_L2 = get_9th_design_matrix(xr) @ w_10_9_L2
yr_100_9_L2 = get_9th_design_matrix(xr) @ w_100_9_L2

plot_model(x10,y10,xt,yt,xr,yr_10_9_L2,title = None)
plot_model(x100,y100,xt,yt,xr,yr_100_9_L2,title = None)
```

```
[[ -3.5338486 ]
 [-1.84896842]
 [ 2.29997278]
 [ 6.47141056]
 [ 6.80582425]
 [ 0.25321228]
 [-9.30526316]
 [-6.83132438]
 [ 5.10343135]
 [ 0.30633599]]
[[ 17.53893754]
 [-22.00647044]
 [-21.06911983]
 [ 7.03985659]
 [ 33.82816171]
 [ 20.79696185]
 [-38.38285113]
 [ -5.43884905]
 [ 8.07411308]
 [ -0.0974112 ]]
```



Discussion:

What differences between the regularized and standard 9th order models fit to `d10` do you notice? How does regularization affect the fitted function?

*Your answer goes here*

With L2 regularization, it is obvious that the overfitting issue is mitigated, resulting in smoother curve.

## LLS with $L_2$ regularization and gradient descent

For complex models, the size of  $X'X$  can be large, making matrix inversion computationally demanding. Instead, one can use gradient descent to compute  $w$ . In our notes, we derived the gradient descent approach both for unregularized as well as  $L_2$  regularized linear regression. The formula for the gradient descent approach with  $L_2$  regularization is:

$$\frac{\partial \text{obj}}{\partial w} = X'Xw - X'y + \lambda \mathbb{I}_m w$$

$$w^{new} \leftarrow w^{cur} - \alpha \frac{\partial \text{obj}}{\partial w}$$

In this problem, could gradient descent get stuck in a local minimum? Explain why / why not?

*Your answer goes here*

No it can't, as the objective is a quadratic function of  $w$  in nature, which only contain 1 global minimum. And gradient descent will always get to the minimum with enough iterations.

You will consider just a single case in the following question:

1. Data: `data10.txt`, Model: 9th order polynomial.

Starting with a weight vector of zeros as the initial guess, and  $\lambda = e^{-10}$ ,  $\alpha = 0.075$ , apply 50000 iterations of gradient descent to find the optimal model parameters. In practice, when you train your own models you will have to determine these parameters yourself!

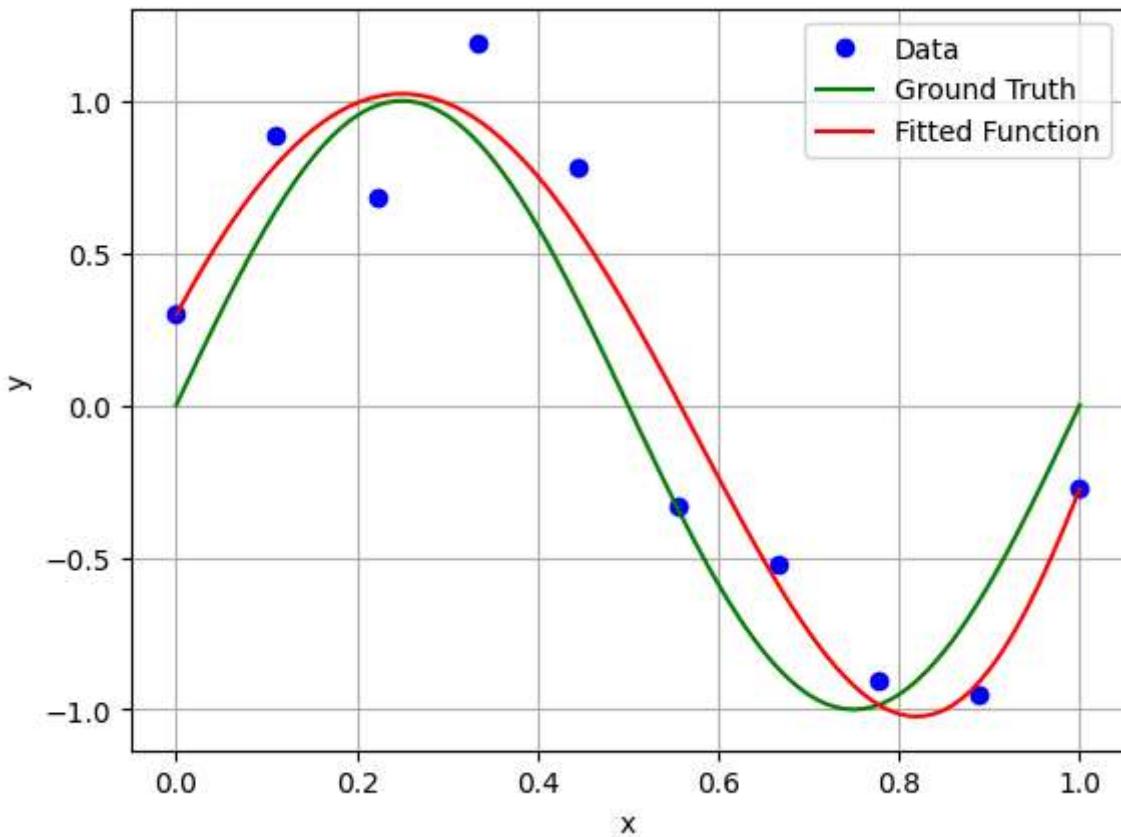
For the trained model:

- Print the learned model parameters `w`
- Use the model parameters `w` to predict `yr` values over a range of `x` values given by `xr = np.linspace(0,1,101)`

- Plot the data, ground truth function, and regressed model using `plot_model(x,y,xt,yt,xr,yr,title)` with an appropriate title.

```
In [6]: # YOUR CODE GOES HERE
n = 50000
alpha = 0.075
w_cur = np.zeros(10).reshape(10,1)
for i in range(n):
    grad = (X_10_9.T @ X_10_9 @ w_cur) - X_10_9.T @ y10.reshape(-1,1) + lamda * Im_
    w_cur = w_cur - alpha*grad
print(w_cur)
yr_10_9_grad = get_9th_design_matrix(xr) @ w_cur
print(yr_10_9_grad.shape)
plot_model(x10,y10,xt,yt,xr,yr_10_9_grad,title = None)
```

```
[[ -3.85457875]
 [-0.40351789]
 [ 2.81217121]
 [ 4.9128699 ]
 [ 4.63900381]
 [ 0.854452  ]
 [-5.76006075]
 [-9.34031661]
 [ 5.56763369]
 [ 0.29561747]]
(101, 1)
```



## Discussion:

Visually compare the result you just obtained to the same 9th order polynomial model with  $L_2$  regularization where you solved for  $w$  directly in the previous section. They should be very similar. Comment on whether gradient descent has converged.

*Your answer goes here*

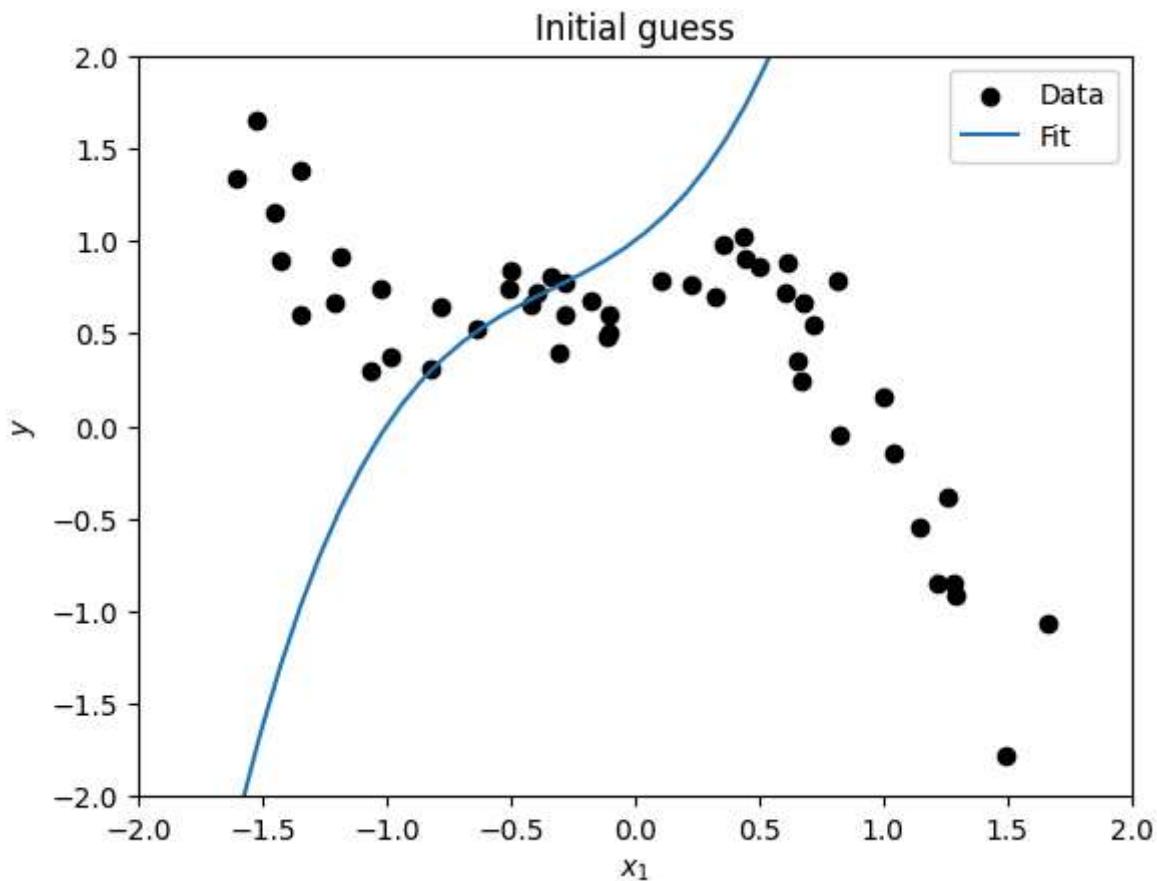
Gradient descent has converged as the resulting plot is almost identical to the direct solution. The curve appears to be smooth instead of spiky.

```
In [53]: def grad(w):
    # YOUR CODE GOES HERE
    # Return the gradient of the objective function
    w_grad = 2*X.T @ (X@w - y.reshape(-1,1))
    return w_grad
```

## Initial guess

Let's define an initial guess for  $w$  and visualize the resulting curve:

```
In [48]: w = np.array([[1,1,1,1]]).T
yreg = Xreg @ w
plot_data_with_regression(x,y,xreg, yreg, "Initial guess")
```



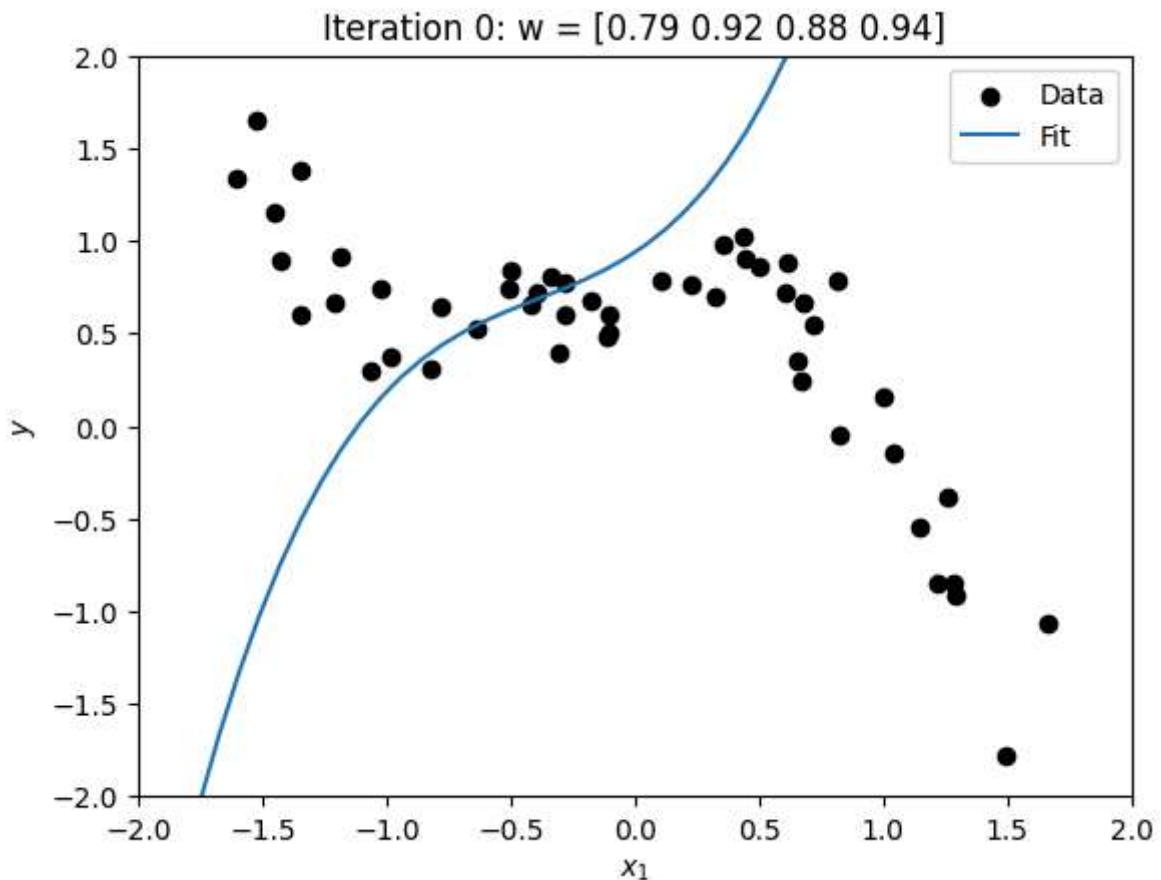
## Gradient Descent Loop

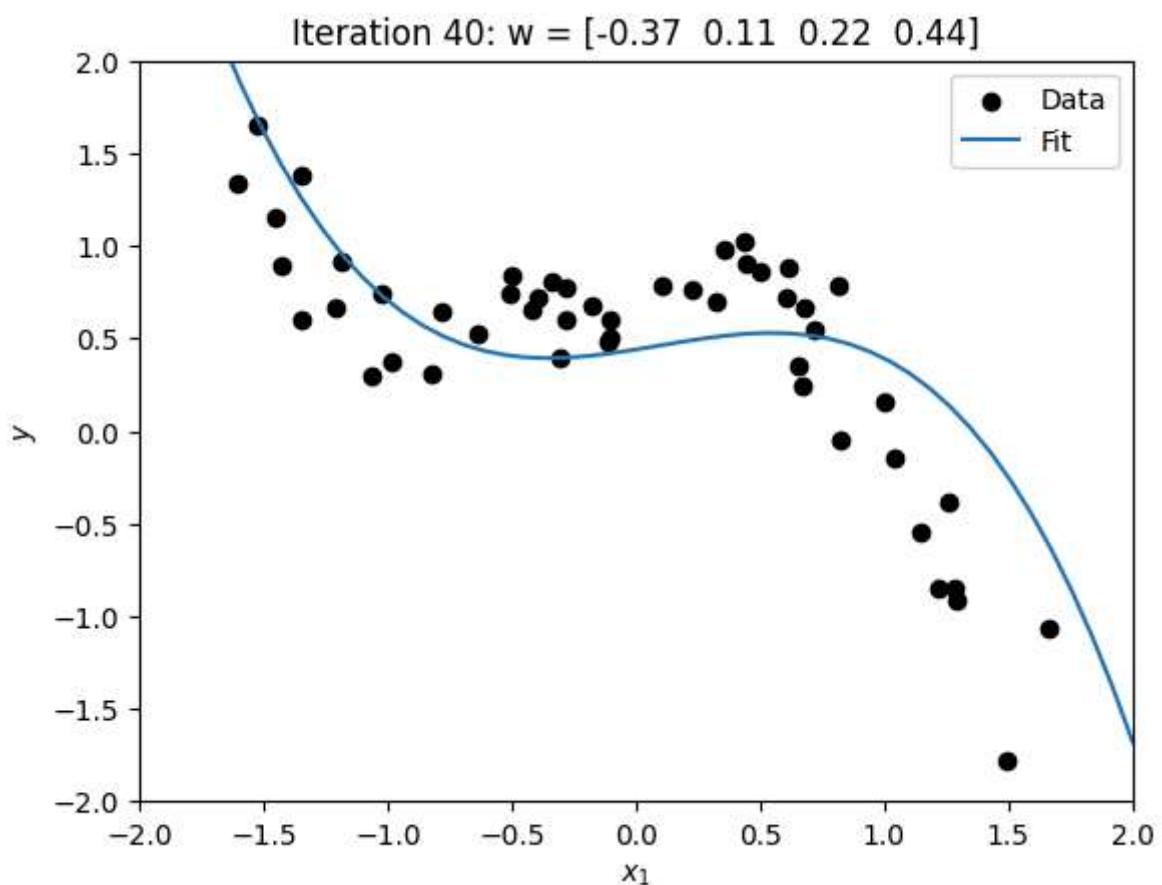
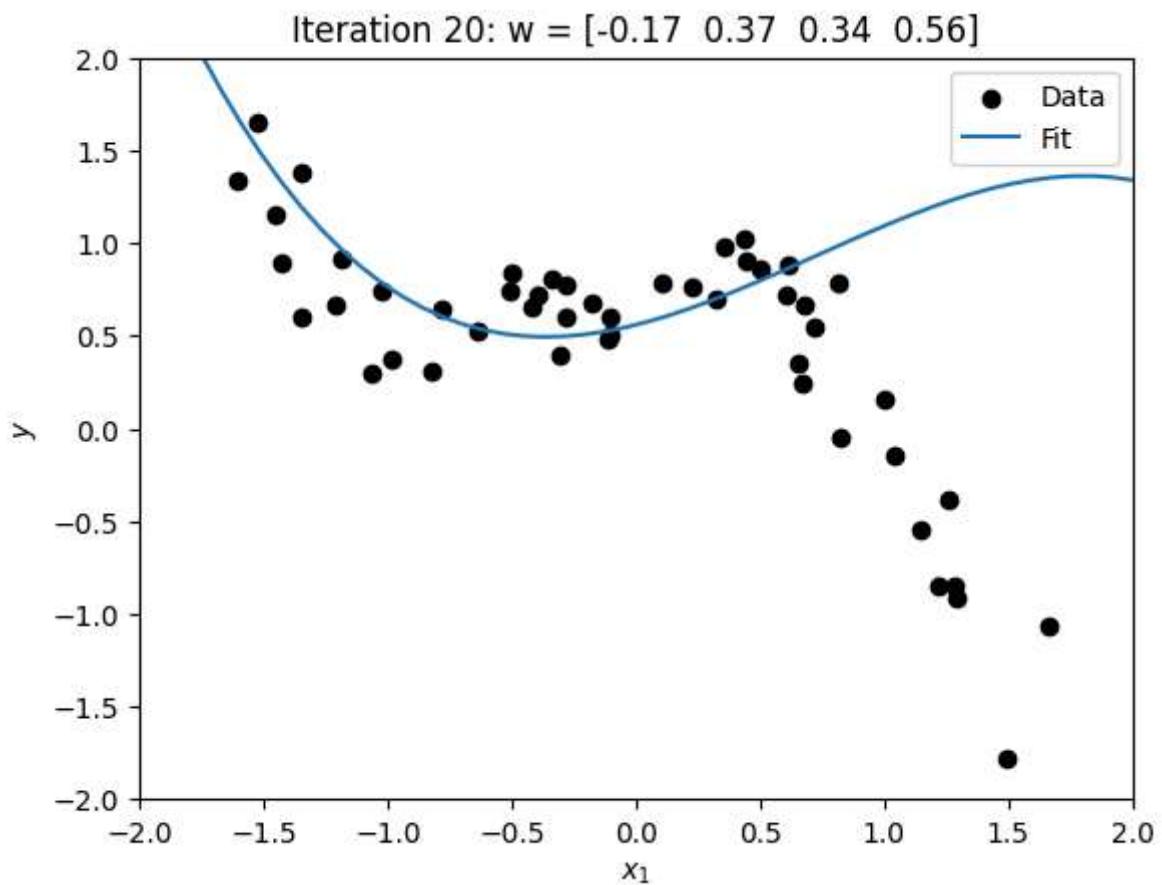
Now complete the gradient descent loop below. You should call your gradient function, update  $w$  by subtracting  $\eta$  (the step size) times the gradient.

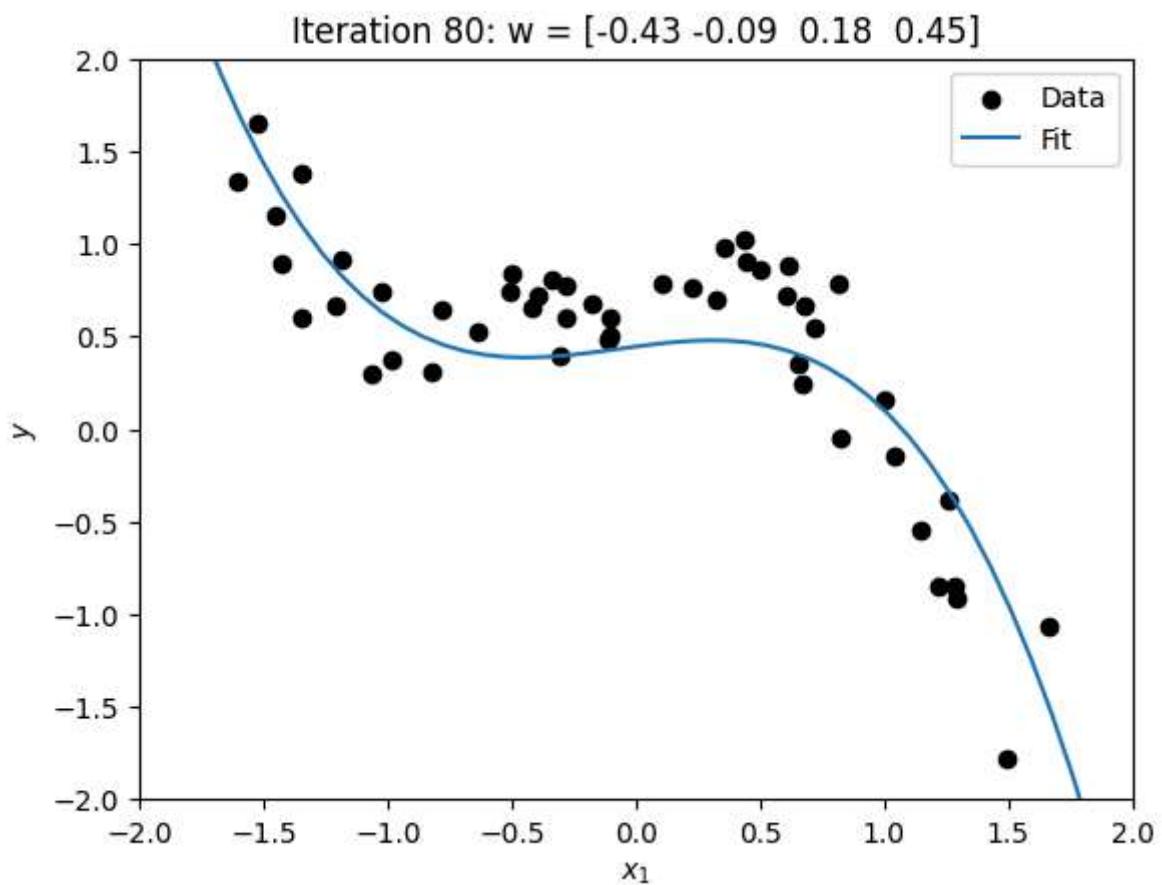
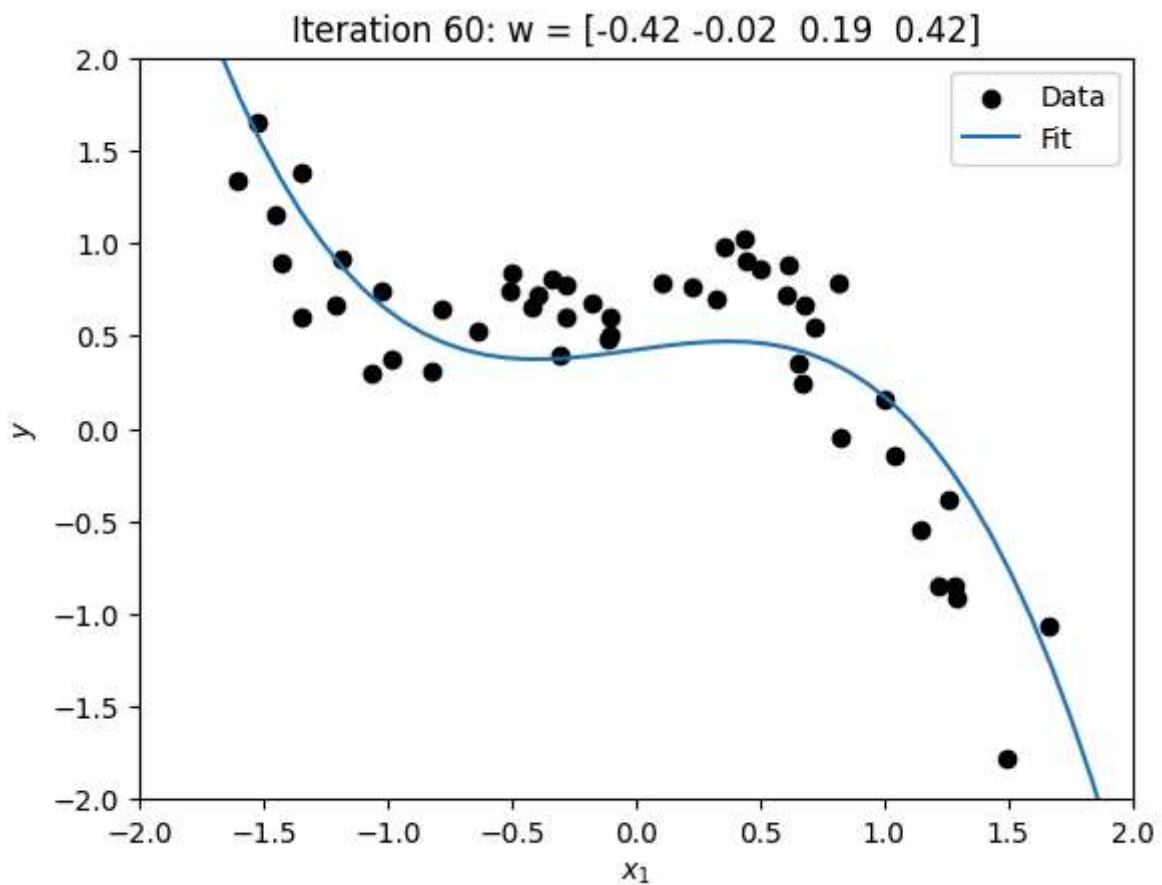
Every several iterations, the result is plotted. You should observe the gradual improvement of the model.

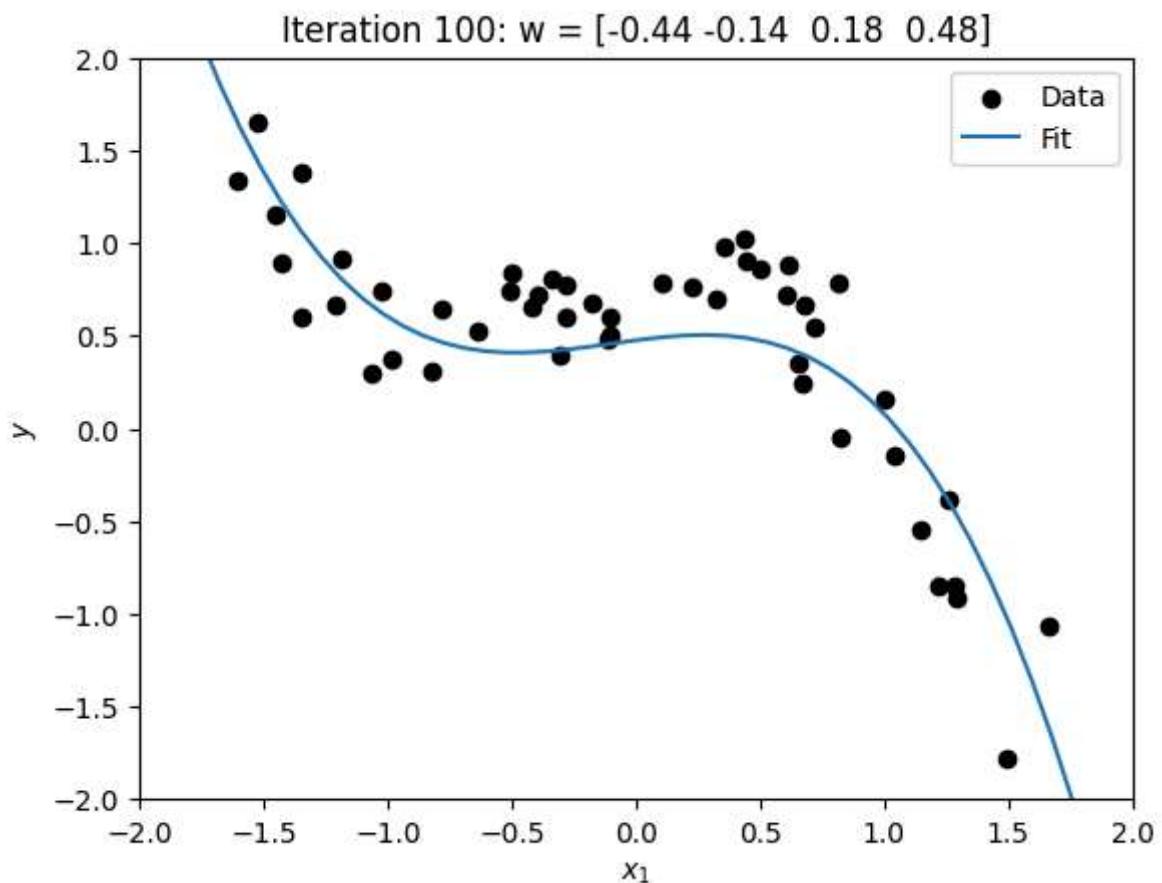
```
In [54]: eta = 0.00025
for i in range(101):

    # YOUR CODE GOES HERE
    # Update w
    w = w - eta*grad(w)
    if 0 == i%20:
        yreg = Xreg @ w
        plot_data_with_regression(x,y,xreg, yreg, f"Iteration {i}: w = {np.round(w, 3)}")
```

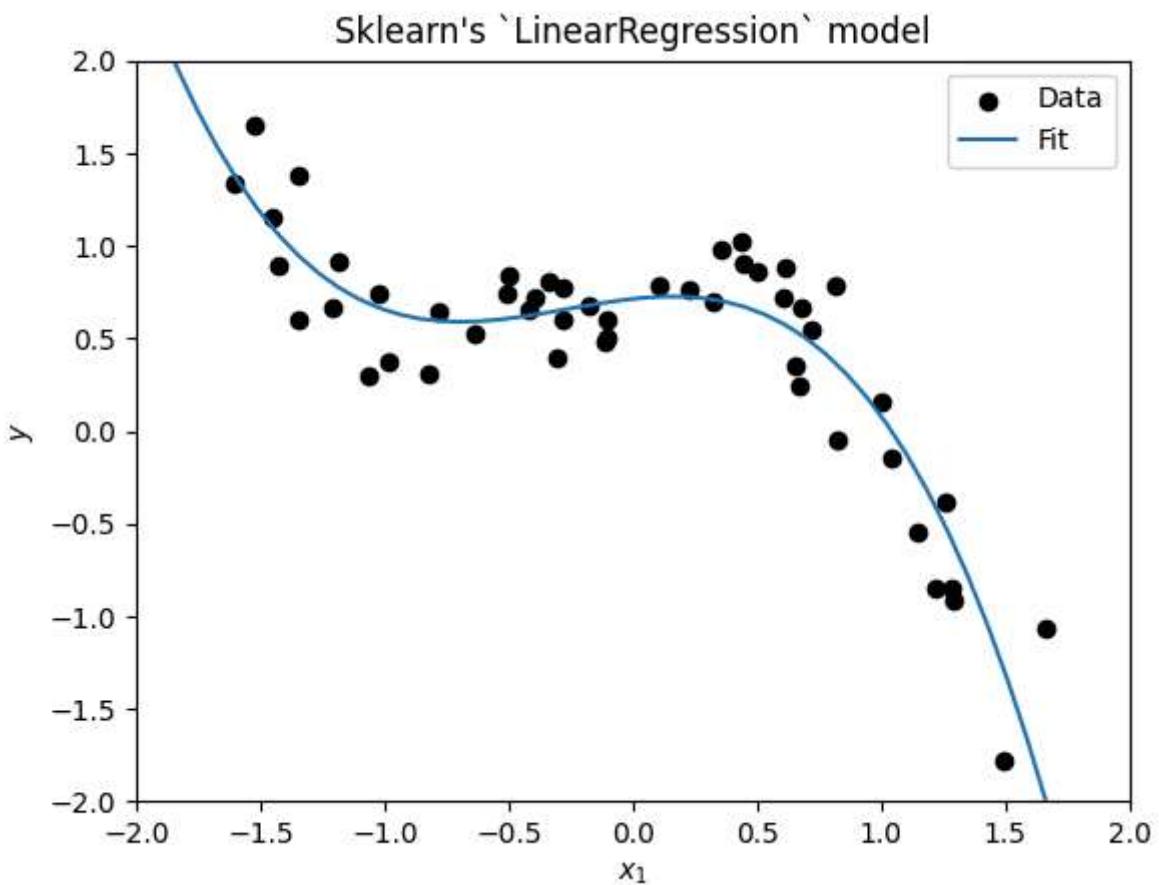








In [ ]:



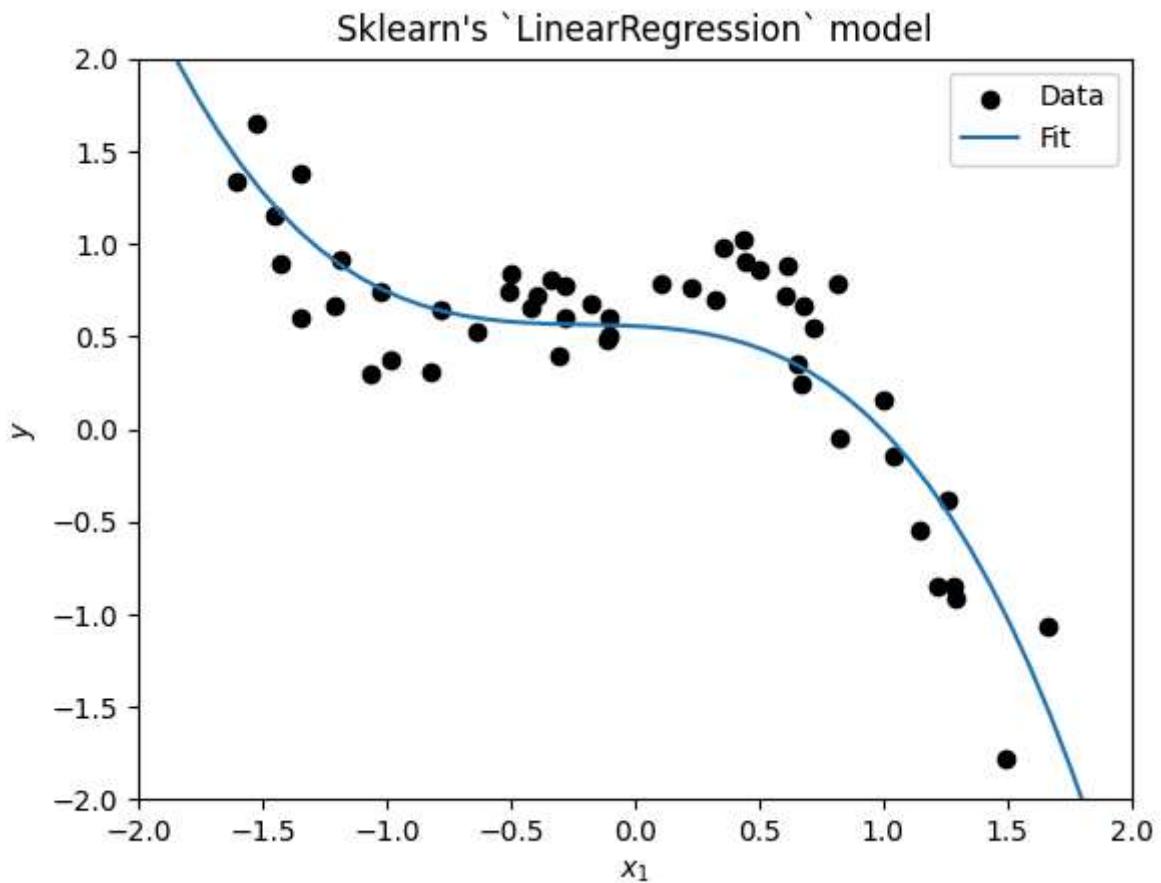
## Using SGD

Now use stochastic gradient descent to solve the same problem and make a similar plot, but for a `SGDRegressor` model instead of `LinearRegression`:

```
In [4]: # YOUR CODE GOES HERE

model = SGDRegressor()
model.fit(X,y)

yreg = model.predict(Xreg)
plot_data_with_regression(x, y, xreg, yreg, "Sklearn's `LinearRegression` model")
```



In [ ]:

# M2-L2 Problem 3 (5 points)

When flow is directed across a pin fin heat sink, increasing fluid velocity can improve the heat transfer, making the heat sink more effective.

You have been given a dataset containing measurements for such a scenario, which contains the following:

- Input: Reynolds Number of air flowing past the heat sink
- Output: Heat transfer coefficient of the heat sink, in  $\text{W}/(\text{m}^2 \text{ K})$

Your job is to train a model on this data to predict the heat transfer coefficient, given Reynolds number as input. You will use a high-order polynomial

Start by loading the data in the following cell:

```
In [1]: import numpy as np
import matplotlib.pyplot as plt

def plot_data_with_regression(x_data, y_data, x_reg, y_reg, title=""):
    plt.figure()

    plt.scatter(x_data.flatten(), y_data.flatten(), label="Data", c="black")
    plt.plot(x_reg.flatten(), y_reg.flatten(), label="Fit")

    plt.legend(loc="upper left")
    plt.xlabel(r"$Re / 1000$")
    plt.ylabel(r"$h, \text{W/m}^2 \text{ K}$")
    plt.xlim(0,6)
    plt.ylim(50,200)
    plt.title(title)
    plt.show()

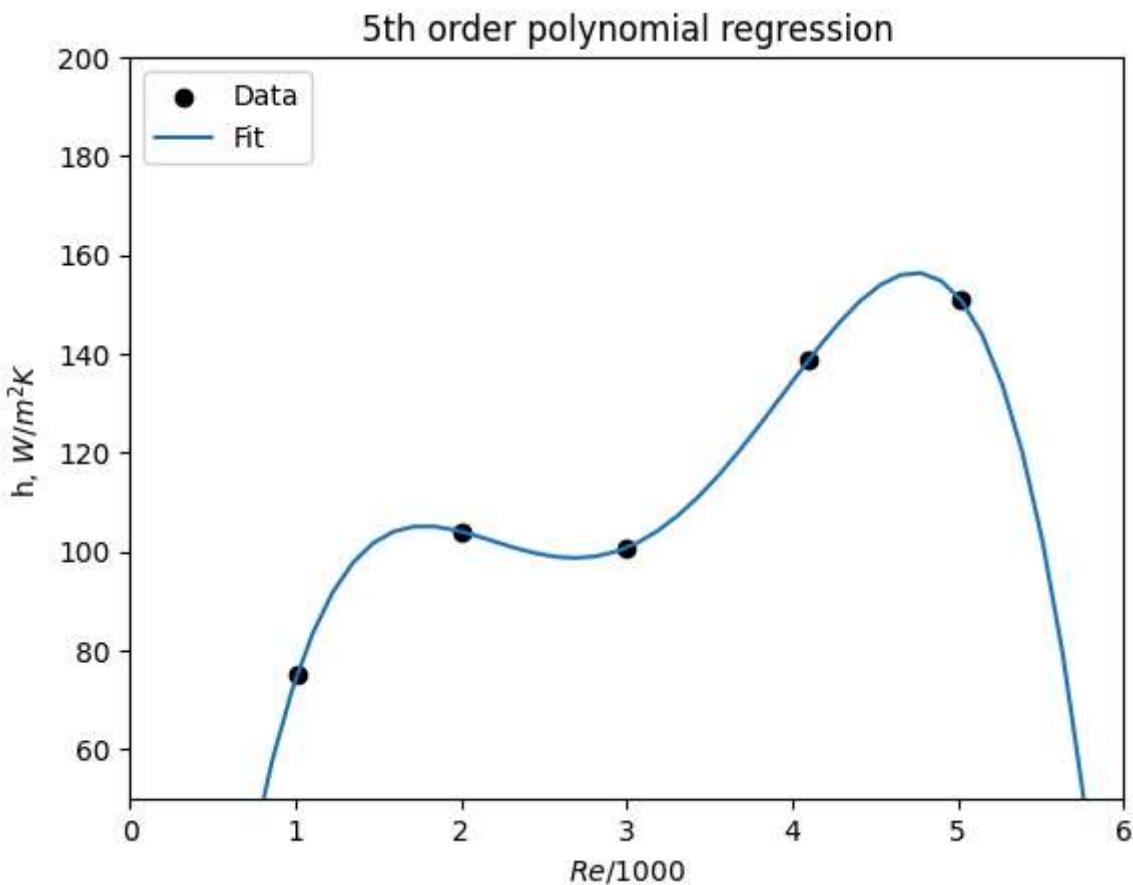
deg = 5
x = np.array([1.010, 2.000, 2.990, 4.100, 5.020])
y = np.array([75.1, 104.0, 100.6, 138.8, 150.8])
X = np.vander(x, deg)

xreg = np.linspace(0,6)
Xreg = np.vander(xreg, deg)
```

## Least Squares Regression

As we have done for previous problems, we can do least squares regression by computing the pseudo-inverse of the design matrix. Notice how the model performs beyond the training data.

```
In [2]: w = np.linalg.inv(X.T @ X) @ X.T @ y.reshape(-1,1)
yreg = Xreg @ w
plot_data_with_regression(x, y, xreg, yreg, "5th order polynomial regression")
```



## L2 Regularization

Notice that the plot above reveals that our fifth-order model is overfitting to the data. Let's try applying L2 regularization to fix this. In the lecture, the closed-form solution to least squares with L2 regularization was:

$$w = (X'X + \lambda I_m)^{-1} X'y$$

where  $I_m$  is the identity matrix, but with zero in the bias row/column instead of 1;  $\lambda$  is regularization strength;  $X'$  is the design matrix and  $y$  column vector output.

Complete the function below to compute this  $w$  for a given lambda:

```
In [5]: def get_regularized_w(L):
    I_m = np.eye(deg)
    I_m[-1,-1] = 0

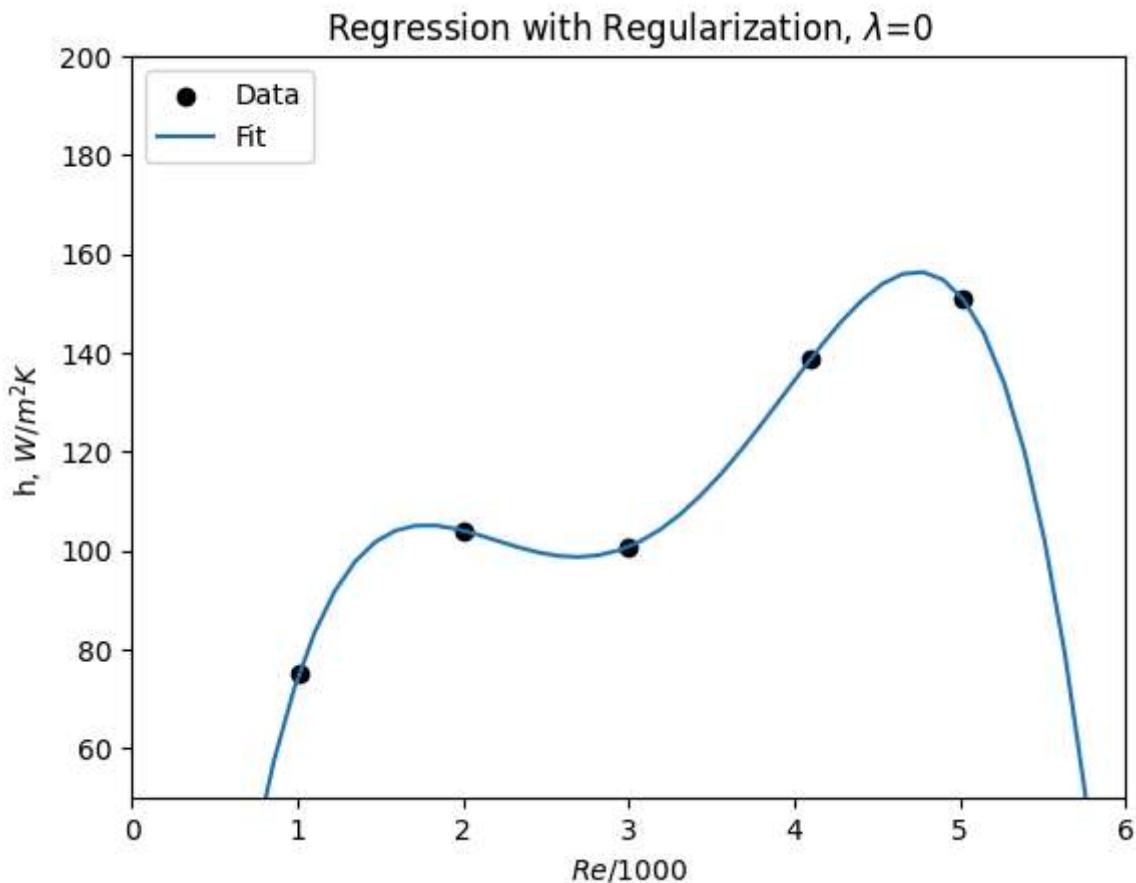
    # YOUR CODE GOES HERE
    # return regularized w
```

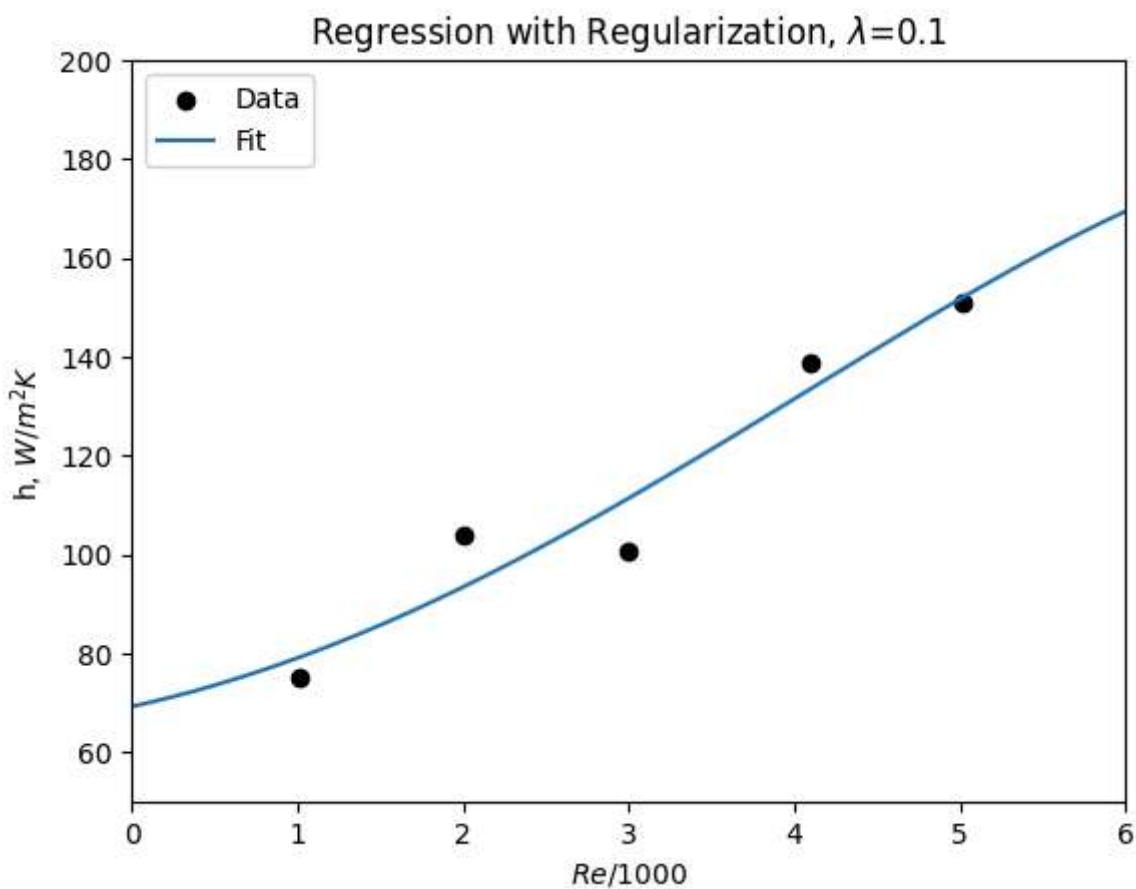
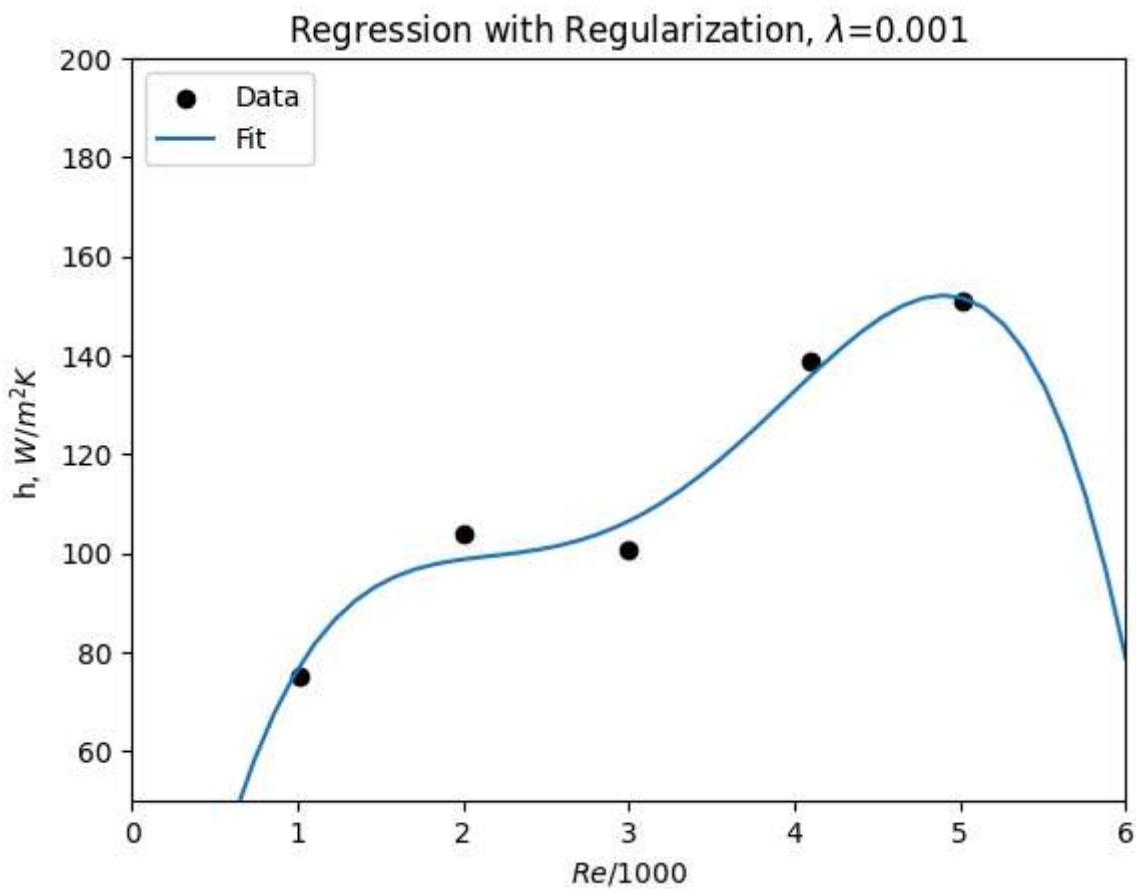
```
w = np.linalg.inv(X.T@X + L*I_m)@X.T@y
return w
```

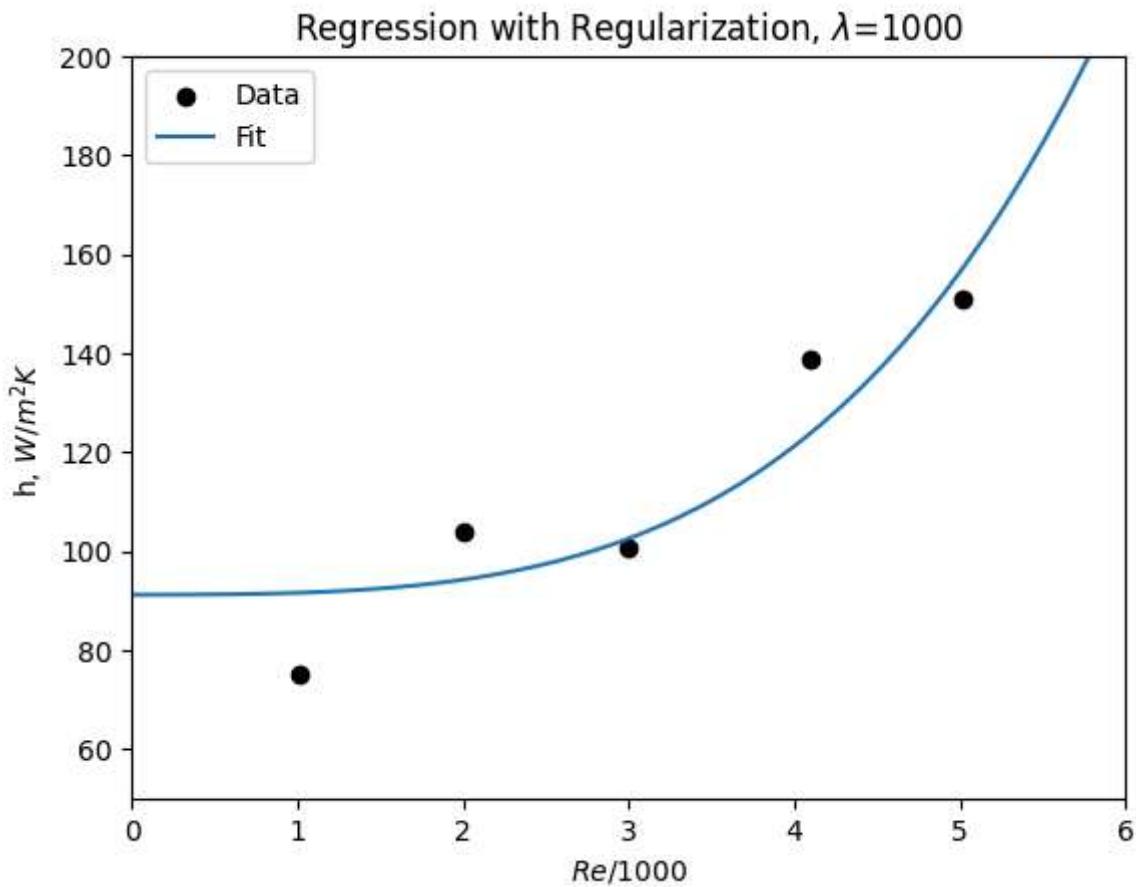
## Testing different lambda values

With the above function written, we can compute  $w$  for some different values of  $\lambda$  and decide which is qualitatively best.

```
In [6]: for L in [0,.001,0.1,1000]:
    w = get_regularized_w(L)
    yreg = Xreg @ w
    plot_data_with_regression(x, y, xreg, yreg,
                               f'Regression with Regularization, $\lambda$={L}')
```







## Model Selection

Which value of lambda appears to yield the "best" model?

0.1 seems to be the best as it does not over fit and provide good accuracy.

# Porblem 2 (30 points)

## Problem Description

In this problem you will use linear least squares to fit a linear function to a 3D temperature field, with  $x,y,z$  locations and an associated temperature  $T$ .

Fill out the notebook as instructed, making the requested plots and printing necessary values.

*You are welcome to use any of the code provided in the lecture activities.*

### Summary of deliverables:

#### Results:

- Predicted temperature  $T(5,5,5)$  using a hand-coded LLS squares model with a linear function
- Direction of travel from  $(5,5,5)$  to experience the greatest decrease in temperature

#### Discussion:

- Reasoning for how we can use our fitted function to determine the direction of greatest decrease in temperature

### Imports and Utility Functions:

```
In [25]: import numpy as np
```

## Load the data

The data is contained in `tempfield.npy` and can be loaded with `np.load(tempfield.npy)`. The first three columns correspond to the `x`, `y`, and `z` locations of the data points, and the 4th column corresponds to the temperature `T` at the respective point. Store the data as you see fit.

```
In [26]: # YOUR CODE GOES HERE  
tempfield = np.load('tempfield.npy')  
print(tempfield)
```

```
[[ 0  0  0 10]
 [ 8  6  1 15]
 [ 5  2  8 20]
 [ 8  2  6 22]
 [ 5  1  2 16]
 [ 3  3  3 23]
 [ 9  8  2 18]
 [ 3  6  5 19]
 [ 4  6  9 25]
 [ 1  8  2 20]
 [ 1  1  2 28]
 [ 6  4  2 27]]
```

## LLS Regression in 3D

Now fit a linear function to the data using the closed form of LLS regression. Use your fitted function to report the predicted temperature at  $x = 5, y = 5, z = 5$ . You are free to add regularization to your model, but this is not required and will not be graded.

```
In [59]: # YOUR CODE GOES HERE
x = tempfield[:,0]
y = tempfield[:,1]
z = tempfield[:,2]
t = tempfield[:,3]

import matplotlib.pyplot as plt
from sklearn.linear_model import LinearRegression

X = np.vstack([x,y,z]).T
model = LinearRegression()
reg = model.fit(X,t)
Xreg = np.array([[5,5,5]])
Treg = model.predict(Xreg)
print(Treg)
```

[21.38907338]

## Gradient Intuition

Using the function you fit in the previous part, in which direction should one move from the point  $p = (5,5,5)$  to experience the largest decrease in temperature in the immediate neighborhood of the point? Report the specific direction, along with your reasoning.

*Your answer goes here*

```
In [60]: coord = np.array([[4,5,5],[5,4,5],[5,5,4],[6,5,5],[5,6,5],[5,5,6]])
for i in coord:
    i = i.reshape(1,i.size)
    temp = model.predict(i)
    print(temp)
    print(Treg - temp)
    print("")
```

```
[21.52328607]
[-0.13421269]
```

```
[21.27165476]
[0.11741862]
```

```
[20.66229964]
[0.72677374]
```

```
[21.25486069]
[0.13421269]
```

```
[21.50649201]
[-0.11741862]
```

```
[22.11584712]
[-0.72677374]
```

Given by the results above, going toward -Z direction would result in the greatest temperature drop.